ECE 203 Notes

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Hi there, I want to preface this booklet by saying this is NOT a comprehensive all-you-need-to-know note document for the class ECE 203 which, by the way, is called *Signals, Information and Computation* if you did not know already. All the concepts talked about here will be related to the material discussed in class with a couple practice questions thrown in for good measure. Hope you find it useful!

*Some material is sourced from external sources.

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$E = mc^2$ is typeset in a paragraph using inline math mode—as is $E = mc^2$, and so too is $E = mc^2$.

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11 Sampling, Aliasing, Spectrum of Sampled Signals, Amplitude Quantization

11.1 Key Concepts

- Sampling converts physical signals $x(t)$, such as a voltage, into a form that is compatible with computation.
  - Collect values of $x(t)$ at distinct times $t = \ldots, -T_s, 0, T_s, 2T_s, 3T_s, \ldots$
  - The value at each time sample is approximated using B bits, or 2^B possible levels.
  - This process is called **analog-to-digital (A/D) conversion**.
  - The effects of collecting samples at distinct times and quantizing the amplitude are analyzed separately.

11.2 Sampling and Information Loss

- Many signals have identical samples.
  - Information between samples is discarded.
  - The **sampling theorem** provides a rule for selecting T_s to ensure uniqueness.

Notation

- **Continuous independent variables** are denoted using (.), for example, $x(t)$.
- **Discrete-valued independent variables** are denoted using [.], for example, $x[n]$.

11.3 Reconstruction of Sampled Signals

- Converting a sampled signal back to a continuous-valued form, such as voltage, is called **reconstruction** or **digital-to-analog (D/A) conversion**.

  - Sequentially generate a constant voltage proportional to each sample and hold it for T_s.
  - Smooth sharp transitions using a circuit that passes low frequencies and attenuates high frequencies.

11.4 Aliasing and the Sampling Theorem

- The effects of sampling in time are understood by studying the sampling of sinusoids.

11.4.1 Digital Frequency and Sampling

- Digital (discrete-time) frequency is given by:

$$\hat{f} = fT_s$$

  where f is the continuous-time frequency and T_s is the sampling interval.

11.4.2 Relationship to Sampling Frequency

- Equivalently, digital frequency can be expressed as:

$$\hat{f} = \frac{f}{f_s}$$

  where $f_s = \frac{1}{T_s}$ is the sampling frequency.

- Digital frequency is measured in units of **cycles/sample**.

11.5 Sampling of Complex Sinusoids

- Samples of a complex sinusoid correspond to points in the complex plane separated by angles $2\pi\hat{f}$ radians, or $\hat{f}$ cycles.

11.6 Reconstruction of Sampled Signals

- Reconstruction finds the simplest or lowest-frequency signal consistent with a given set of samples.

11.7 Aliasing

- Aliasing occurs when a sinusoid of one frequency appears to be a sinusoid of a different frequency.

11.7.1 Aliasing in Complex Sinusoids

- Complex sinusoids with frequencies $\hat{f} + \ell$ cycles per sample, where ℓ is an integer, are identical.
- For example, if $\hat{f} = 0.25$, then sinusoids with digital frequencies:

$$-1.75, -0.75, 0.25, 1.25, 2.25, \ldots$$

produce identical samples.

11.7.2 Aliasing in Real-Valued Sinusoids

- Real-valued sinusoids do not differentiate between negative and positive frequencies.
- As a result, sinusoids with frequencies $|\hat{f} + \ell|$, where ℓ is an integer, cannot be distinguished based on their samples.
- Example: If $\hat{f} = 0.25$, then the sinusoids with digital frequencies:

$$0.25, 0.75, 1.25, 1.75, 2.25, 2.75, \ldots$$

all appear identical.

- This non-uniqueness of digital frequency corresponds to the non-uniqueness of continuous-time frequency.

11.8 Recovering a Real-Valued Sinusoid from Digital Frequency

- The frequency f_a of a reconstructed real-valued sinusoid is obtained from the digital frequency $\hat{f}$ as follows:

11.8.1 Principal Frequency Representation

- The angle $2\pi\hat{f}$ can be sketched in the complex plane.
- Let θ be the **principal value** of $2\pi\hat{f}$, ensuring $-\pi < \theta \leq \pi$.
- The principal value represents the smallest plausible digital frequency.

11.8.2 Conversion to Continuous Frequency

- Convert the principal value θ to a real-valued sinusoid frequency using:

$$f_a = \frac{|\theta|}{2\pi T_s}$$

11.9 Sampling Theorem

- The sampling theorem guarantees that the reconstructed frequency f_a is equal to the original frequency f_0 of the sampled real-valued sinusoid.

11.9.1 Condition for Proper Sampling

- The sampling theorem requires:

$$f_s > 2f_0$$

- Equivalently, the sampling interval T_s must be small enough so that **more than two samples** are taken per period.

11.10 Spectrum of Sampled Signals

11.10.1 Complex Sinusoids and Frequency Uniqueness

- Discrete-time complex sinusoids $e^{j2\pi\hat{f}n}$ with different frequencies are not always distinct.
- Some frequencies result in identical sampled values, making them indistinguishable.

11.10.2 Frequency Folding in the Discrete Spectrum

- Two frequencies $\hat{f}$ and $\hat{f} + \ell$ (where ℓ is an integer) produce identical discrete-time sinusoids.
- To ensure uniqueness, restrict the discrete-time spectrum to:

$$-0.5 < \hat{f} \leq 0.5$$

- If a frequency f_o does not satisfy this range, add or subtract integers to find the corresponding **wrapped digital frequency**:

$$\hat{f}_1 = f_o + \ell, \quad -0.5 < \hat{f}_1 \leq 0.5$$

- This process is similar to computing the **principal value** of a phase angle to ensure a unique representation.

11.11 Discrete-Time Spectrum of a Sampled Signal

- Consider a continuous-time signal composed of multiple sinusoids:

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

- To analyze its discrete-time spectrum:

11.11.1 Using Euler's Formula

- Express each sinusoid using Euler's identity:

$$A_k \cos(2\pi f_k t + \phi_k) = \frac{A_k e^{j\phi_k}}{2} e^{j2\pi f_k t} + \frac{A_k e^{-j\phi_k}}{2} e^{-j2\pi f_k t}$$

- Sampling replaces t with $T_s n$ in each exponential term.

11.11.2 Frequency Wrapping for Sampling

- The sampled frequency is wrapped using integer shifts:

$$\hat{f}_k = f_k T_s + \ell_k$$

  ensuring $-0.5 < \hat{f}_k \leq 0.5$.

- The amplitude $\frac{A_k e^{j\phi_k}}{2}$ corresponds to the discrete-time frequency $\hat{f}_k$, which may be negative.

- Similarly, wrap the negative frequency component:

$$-\hat{f}_k = -f_k T_s - \ell_k$$

  where the amplitude $\frac{A_k e^{-j\phi_k}}{2}$ corresponds to $-\hat{f}_k$, which may be positive.

- If $\hat{f}_k$ is negative, the complex amplitude originally associated with positive f_k is now linked to negative discrete-time frequency.

11.12 Reconstruction and Continuous-Time Spectrum

- Reconstruction finds the lowest continuous-time frequencies consistent with the samples.

11.12.1 Reconstructed Signal Spectrum

- The discrete-time signal is expressed as a sum of complex sinusoids with frequencies $\hat{f}_k$ and $-\hat{f}_k$, having weights α_k and α_k^*, respectively.

- Map these frequencies to the continuous-time domain:

$$\pm f_k = \pm \hat{f}_k f_s$$

  where $f_s = \frac{1}{T_s}$ is the sampling frequency.

- The reconstructed continuous-time spectrum contains coefficients α_k and α_k^* at frequencies f_k and $-f_k$.

11.13 Example: Sampling a Sinusoid

- Consider a sinusoid:

$$x(t) = 2\cos(2\pi 8t + \pi/3)$$

  sampled at $f_s = 10$ Hz.

11.13.1 Step 1: Express in Exponential Form

- Rewrite using Euler's identity:

$$x(t) = e^{j\pi/3}e^{j2\pi 8t} + e^{-j\pi/3}e^{-j2\pi 8t}$$

11.13.2 Step 2: Compute Sampled Frequency

- Compute the sampled frequency:

$$8 \times \frac{1}{10} = 0.8$$

- Wrap it within $-0.5 < \hat{f} \leq 0.5$ by subtracting $\ell = 1$:

$$\hat{f}_1 = 0.8 - 1 = -0.2$$

- The sampled spectrum has a frequency component at $\hat{f} = -0.2$ cycles per sample.

11.13.3 Step 3: Compute Negative Frequency Component

- Similarly, for the negative frequency:

$$-8 \times \frac{1}{10} = -0.8$$

- Wrap it by adding $\ell = 1$:

$$-\hat{f}_1 = -0.8 + 1 = 0.2$$

- The spectrum at $\hat{f} = 0.2$ cycles per sample contains $e^{-j\pi/3}$.

11.13.4 Step 4: Compute Reconstructed Frequency

- Reconstructing at $f_s = 10$ Hz gives:

$$f_1 = 0.2 \times 10 = 2 \text{ Hz}, \quad -f_1 = -2 \text{ Hz}$$

with coefficients $\alpha_1 = e^{-j\pi/3}$ and $\alpha_1^* = e^{j\pi/3}$.

11.13.5 Step 5: Final Reconstructed Signal

- The reconstructed signal is:

$$z(t) = e^{-j\pi/3} e^{j2\pi 2t} + e^{j\pi/3} e^{-j2\pi 2t}$$

- Simplifying:

$$z(t) = 2\cos(2\pi 2t - \pi/3)$$

- **Observation:** Aliasing from positive to negative frequencies has altered both the frequency and phase of the sinusoid.

11.14 Amplitude Quantization in A-D Conversion

11.14.1 Key Concepts

- A B**-bit A/D converter** quantizes the samples of a continuous-time signal into 2^B discrete levels.
- With 2^B levels, there are $2^B - 1$ quantization intervals.

11.14.2 Quantization Step Size

- If the 2^B levels are spread over a range R, then the step size between quantization levels is:

$$\Delta = \frac{R}{2^B - 1}$$

- If $2^B \gg 1$, then:

$$\Delta \approx \frac{R}{2^B}$$

11.14.3 Quantization Error

- The absolute error due to rounding a continuous-amplitude signal to the nearest quantization level is at most:

$$\frac{\Delta}{2} \approx R2^{-(B+1)}$$

11.14.4 Efficient Use of Quantization Levels

- To maximize the use of available quantization levels, the input signal range should be **scaled** to match the dynamic range of the A/D converter.

11.14.5 Loss of Signal Information

- Information loss occurs when:
  - The input signal exceeds the measurable range of the A/D converter, causing **saturation**.
  - The input signal variations are smaller than the quantization step size, causing a loss of resolution.

11.14.6 Dynamic Range

- The **dynamic range**, or the ratio of the largest to the smallest measurable signal amplitudes, is proportional to:

$$2^B$$

- Each additional bit in the A/D converter increases the dynamic range by:

$$6 \text{ dB}$$

12 Music Synthesis 2 Lab

13 DFT and Computing the Spectrum of Sampled Signals

13.1 Key Concepts

- The Discrete Fourier Transform (DFT) is the discrete-time version of the Fourier series for continous-time signals.
- The DFT has a very wide range of applications in addition to computing the spectrum of discrete-time signals.
- The DFT represents a finite-duration discrete-time signal $x[n]$, $n =$ 0,1,2,...,$N - 1$ as a weighted sum of harmonically related complex sinusoids...

13.2 Discrete Fourier Transform (DFT)

The **DFT** is a mathematical tool used to analyze discrete-time signals in the **frequency domain**. It transforms a discrete-time signal $x[n]$ into its frequency components $X[k]$, representing how much of each frequency is present in the signal.

DFT Formula:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi \frac{k}{N} n}, \quad k = 0, 1, 2, \ldots, N - 1$$

Key Points about DFT:

- $X[k]$ represents the frequency content of the signal.
- The term $e^{-j2\pi \frac{k}{N} n}$ is a complex exponential, acting as a **basis function** for frequency analysis.
- Each k-th component corresponds to a frequency $\frac{k}{N}$ cycles per sample.
- The magnitude $|X[k]|$ gives the **magnitude spectrum**, indicating the strength of each frequency.

13.3 Inverse Discrete Fourier Transform (IDFT)

The **IDFT** allows us to reconstruct the original discrete-time signal $x[n]$ from its frequency domain representation $X[k]$. It essentially "reverses" the transformation performed by the DFT.

IDFT Formula:

$$x[n] = \sum_{k=0}^{N-1} \frac{1}{N} X[k] e^{j2\pi \frac{k}{N} n}, \quad n = 0, 1, 2, \ldots, N-1$$

Key Points about IDFT:

- It **recovers** the original time-domain signal $x[n]$ from the frequency coefficients $X[k]$.
- The term $e^{j2\pi \frac{k}{N} n}$ is the **inverse basis function**, reversing the effect of the DFT.
- The $\frac{1}{N}$ **factor** ensures proper scaling so that the reconstructed signal has the same amplitude as the original.

13.4 DFT & IDFT Summary

Aspect	**DFT**
Purpose	Converts $x[n]$ from time domain to frequency domain
Formula	$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$
Exponential Term	$e^{-j2\pi \frac{k}{N} n}$ (negative exponent)
Scaling Factor	None

Table 1: Summary of Discrete Fourier Transform (DFT)

Aspect	IDFT
Purpose	Converts $X[k]$ from frequency domain back to time domain
Formula	$x[n] = \sum_{k=0}^{N-1} \frac{1}{N} X[k] e^{j2\pi \frac{k}{N} n}$
Exponential Term	$e^{j2\pi \frac{k}{N} n}$ (positive exponent)
Scaling Factor	$\frac{1}{N}$ to normalize

Table 2: Summary of Inverse Discrete Fourier Transform (IDFT)

13.5 Intuition Behind DFT and IDFT

- Think of **DFT** as breaking a signal into **a sum of sinusoids (frequencies)**.
- Think of **IDFT** as **reconstructing the original signal** by summing those sinusoids back together.
- The process is like a musical equalizer: DFT separates a song into different frequencies, and IDFT combines them back into the original song.

13.6 Properties of the DFT Coefficients

1. Symmetry Properties

- **Periodic Property:** The DFT coefficients are periodic with period N:
$$X[k+N] = X[k]$$
- **Index Reversal Property:** The negative indices can be rewritten as:
$$X[-k] = X[N-k]$$
This allows us to determine negative frequency coefficients using positive frequency coefficients.
- **Conjugate Symmetry (For Real Signals):** If $x[n]$ is real, then the DFT coefficients satisfy:
$$X[-k] = X^*[k]$$
which means the spectrum is symmetric in the frequency domain.

2. Frequency Indexing

- The DFT computation generally uses frequency indices $k = 0, 1, 2, \ldots, N-1$.
- For display purposes, we often represent the spectrum using both positive and negative indices.
- If N is odd, the frequency indices are commonly written as:

$$k = -\frac{N-1}{2}, \ldots, -2, -1, 0, 1, 2, \ldots, \frac{N-1}{2}$$

3. Interpretation of DFT Coefficients

- If a real sinusoid has a frequency that is an **integer multiple** of $1/N$ cycles per sample, then its DFT representation contains only **two nonzero coefficients**.
- If a real sinusoid has a frequency that is **not** an integer multiple of $1/N$, then **all** DFT coefficients may be nonzero.
- The peak in the magnitude spectrum $|X[k]|$ occurs at a frequency k/N that is closest to the actual sinusoid frequency.

13.7 Fast Fourier Transform (FFT)

The **Fast Fourier Transform (FFT)** is an efficient algorithm for computing the DFT. Instead of directly applying the DFT formula, the FFT reduces computation time from $O(N^2)$ to $O(N \log N)$, making it significantly faster for large N.

13.8 Computing the Spectrum of Sampled Signals with the DFT

Key Concepts

- One important application of the **Discrete Fourier Transform (DFT)** is to analyze the frequency content of measured signals.
- The DFT represents a discrete-time signal $x[n]$, where $n = 0, 1, 2, \ldots, N-1$, as a sum of complex sinusoids with different frequencies.

- This can be expressed mathematically as:

$$x[n] = \sum_{k=0}^{N-1} \frac{1}{N} X[k] e^{j2\pi \frac{k}{N} n}$$

- The DFT coefficients $X[k]$ determine the signal's **frequency spectrum**.

Understanding Frequency Representation in the DFT

- Each DFT coefficient $X[k]$ corresponds to a **specific discrete-time frequency** given by:

$$\hat{f}_k = \frac{k}{N} \quad \text{(cycles per sample)}$$

- The equivalent **continuous-time frequency** is found by:

$$f_k = \hat{f}_k f_s = \frac{k}{NT_s}$$

  where f_s is the sampling frequency and T_s is the sampling interval.

- The **continuous-time and discrete-time spectra** are related by:

$$X(f_k) = \frac{1}{N} X[k]$$

Symmetry of the DFT Spectrum

- The spectrum at **negative frequencies** can be obtained using the property:

$$X[-k] = X[N-k]$$

- More specifically:
  - If N is **odd**:

$$X[-1] = X[N-1], X[-2] = X[N-2], \ldots, X\left[-\frac{N-1}{2}\right] = X\left[\frac{N+1}{2}\right]$$

  - If N is **even**:

$$X[-1] = X[N-1], X[-2] = X[N-2], \ldots, X\left[-\frac{N}{2}\right] = X\left[\frac{N}{2}\right]$$

- In MATLAB, the function **fftshift** rearranges the DFT output to show both **positive and negative frequencies** properly.

How the DFT Represents Sinusoidal Signals

- The DFT of a sampled sinusoid depends on the relationship between the signal's frequency and the sampling parameters:

$$x(t) = A\cos(2\pi f_0 t + \phi)$$

- Two cases arise:
  - **Case 1:** $f_0 T_s$ **is an integer multiple of** $1/N$ The DFT spectrum contains only **two nonzero coefficients** at $\pm f_0$ Hz, meaning the sinusoid is exactly represented in the frequency domain.
  - **Case 2:** $f_0 T_s$ **is not an integer multiple of** $1/N$ The DFT cannot perfectly align with the signal's frequency, leading to a **nonzero spread** in all DFT coefficients (spectral leakage).

14 Using Sinusoids to Detect Activity in fMRI Lab

15 DSP Systems, Impulse Response, Linearity, Time Invariance and Causality

16 Assessment 2 Review