NOTES FOR LINEAR ALGEBRA 341, SPRING 2025

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1. INTRODUCTION TO VECTORS IN $\mathbb{R}^n$

A vector $\vec{x}$ (or n-dimensional vector) is an ordered n-tuple of numbers.

$$\vec{x} = [x_1, x_2, \ldots, x_n] \text{ (row vector)}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ (column vector)}$$

We can represent a vector as either a row or column vector as shown above. In general, we use either of them depending on what we are trying to do with vectors.

For $\mathbb{R}^n$, it can be expressed as the set:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \ : \ x_1, \ldots, x_n \in \mathbb{R} \right\}$$

For $n = 1$, we define the real number space as:

$$\mathbb{R}^1 = \mathbb{R}$$

where a vector in $\mathbb{R}^1$ is written as:

$$[x_1] \to x_1$$

For $n = 2$, we define the two-dimensional real vector space as:

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ : \ x_1, x_2 \in \mathbb{R} \right\}$$

A vector in $\mathbb{R}^2$ can be visualized as an arrow originating from the point $(0, 0)$ and terminating at the point (x_1, x_2). The vector is expressed in matrix notation as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are equal if and only if each corresponding component is equal. Formally, if:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

then we say that $\mathbf{x} = \mathbf{y}$ if and only if:

$$x_j = y_j \quad \text{for } j = 1, 2, \ldots, n.$$

2. Vector Spaces

For this class, let we will refer to a vector space using notation V.

Let V be a field (a set of numbers that define the scalars used in multiplication)

$$V^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in V \text{ for } i = 1, 2, \ldots, n\}$$

Here we have that V^n is a vector space over V. In vector spaces, there are only two operations we can perform.

1) Vector Addition

Commutativity: For all $\mathbf{v}, \mathbf{w} \in V$,

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}.$$

Associativity: For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

Additive Identity (Zero Vector): There exists a special vector $\mathbf{0} \in V$ such that for every $\mathbf{v} \in V$,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}.$$

Additive Inverse: For every $\mathbf{v} \in V$, there exists another vector $-\mathbf{v} \in V$ such that:

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}.$$

2) Scalar Multiplication

Multiplicative Identity: For every $\mathbf{v} \in V$,

$$1 \cdot \mathbf{v} = \mathbf{v}.$$

Associativity with Scalars: For all real numbers $a, b \in \mathbb{R}$ and $\mathbf{v} \in V$,

$$a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}.$$

Distributivity Over Scalar Addition: For all real numbers $a, b \in \mathbb{R}$ and $\mathbf{v} \in V$,

$$(a + b) \cdot \mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}.$$

Distributivity Over Vector Addition: For all real numbers $a \in \mathbb{R}$ and vectors $\mathbf{v}, \mathbf{w} \in V$,

$$a \cdot (\mathbf{v} + \mathbf{w}) = a \cdot \mathbf{v} + a \cdot \mathbf{w}.$$

Examples of Vector Spaces

An easy example of a vector space is $\mathbb{R}^2$

Here are other important examples of vector spaces:

1. The set of polynomials $\mathbb{R}[x]$

This is the set of all polynomials with real coefficients. Each element is a polynomial function, such as:

$$f(x) = 3x^2 - 2x + 5.$$

Vector addition corresponds to adding polynomials term by term, and scalar multiplication corresponds to multiplying every term by a real number.

2. The set of polynomials of degree at most k, denoted $P_k(x)$

This is a subset of $\mathbb{R}[x]$ where every polynomial has degree at most k. That is:

$$P_k(x) = \{f \in \mathbb{R}[x] \mid \deg f \leq k\}.$$

For example, if $k = 2$, the set consists of all quadratic polynomials like $ax^2 + bx + c$.

3. The set of $a \times b$ matrices, denoted $M_{a,b}$

This is the set of all matrices with a rows and b columns, where each entry is a real number. A typical matrix in $M_{2,3}$ (2 rows, 3 columns) looks like:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Vector addition corresponds to matrix addition, and scalar multiplication means multiplying every entry by a real number.

4. The set of k-times differentiable functions, denoted $C^k(x)$

This is the set of all functions that can be differentiated at least k times, where the kth derivative exists and is continuous. If $k = 1$, the functions must be at least once differentiable.

5. The set of infinitely differentiable functions, denoted $C^\infty(x)$

This is the set of all functions that can be differentiated infinitely many times, such as:

$$e^x, \quad \sin x, \quad \cos x.$$

These functions have derivatives of all orders, and their derivatives never become undefined.