Strict-Additive Fees Reinvested *Inside* Pricing for AMMs

A Potential-Invariant Construction for ExactIn / ExactOut Swaps

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Abstract

Constant-product AMMs ($xy = k$) have a key property: *strict additivity* (split-invariance). A swap of size $a+b$ yields the same final reserves as two swaps of size a and then b. However, the standard Uniswap-style *input-fee* model (effective input $(1-\phi)\Delta x$ while crediting full Δx into the reserve) breaks strict additivity once the fee is *reinvested inside priced reserves*. This article characterizes *all* ExactIn AMM swap rules that satisfy: (i) full input is credited to the priced reserve, (ii) fees remain inside pricing (no external fee bucket), and (iii) strict additivity holds. The main result is that strict additivity forces an invariant of the form $y\,\Psi(x) = K$ for some positive function Ψ. We derive ExactIn and ExactOut formulas, provide step-by-step proofs, explain the cocycle condition $R(x,a)R(x+a,b) = R(x,a+b)$, and connect the construction to an additive potential function F via a telescoping identity.

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1 Setup and Notation

Definition 1.1 (State space)**.** The AMM state is the pair of priced reserves

$$s = (x, y) \in \mathbb{R}^2_{>0},$$

where x is the reserve of token X and y is the reserve of token Y.

Definition 1.2 (ExactIn swap)**.** An *ExactIn* swap $X \to Y$ takes input $\Delta x > 0$ and returns output $\Delta y > 0$. After the swap:

$$x' = x + \Delta x, \qquad y' = y - \Delta y.$$

Definition 1.3 (Fees reinvested *inside* pricing)**.** All fees (if any) remain *inside* the priced reserves. Equivalently, the next price is computed using the post-swap reserves (x', y') with no separate accounting bucket.

2 Strict Additivity as a Semigroup Property

Definition 2.1 (State update map F_Δ)**.** Let $F_\Delta : \mathbb{R}^2_{>0} \to \mathbb{R}^2_{>0}$ be the state update map for an ExactIn swap of size $\Delta > 0$. Write

$$F_\Delta(x, y) = (x_\Delta, y_\Delta).$$

Definition 2.2 (Strict additivity / split-invariance)**.** The AMM is *strictly additive* if for all $a, b > 0$ and all $(x, y) \in \mathbb{R}^2_{>0}$:

$$F_{a+b}(x, y) = F_b\big(F_a(x, y)\big).$$

Equivalently, the family $\{F_\Delta\}_{\Delta>0}$ forms a semigroup:

$$F_{a+b} = F_b \circ F_a.$$

Definition 2.3 (Output function f)**.** Define the output amount function $f : \mathbb{R}^2_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ by

$$f((x, y), \Delta) \;:=\; y - \pi_y(F_\Delta(x, y)),$$

where $\pi_y(x', y') = y'$ is projection onto the y-coordinate.

Lemma 2.4 (Output additivity induced by strict additivity)**.** *If $F_{a+b} = F_b \circ F_a$, then outputs satisfy:*

$$f((x, y), a + b) = f((x, y), a) + f(F_a(x, y), b).$$

Proof. Let $F_a(x, y) = (x_a, y_a)$ and $F_{a+b}(x, y) = (x_{a+b}, y_{a+b})$. The left side is $y - y_{a+b}$. The right side is $(y - y_a) + (y_a - y_{a,b}) = y - y_{a,b}$. Strict additivity implies $(x_{a,b}, y_{a,b}) = (x_{a+b}, y_{a+b})$, hence $y_{a,b} = y_{a+b}$, so both sides match. $\square$

3 Baseline: Constant Product $xy = k$ (No Fee)

Lemma 3.1 (Constant product ExactIn swap)**.** *Assume invariant $k = xy$ and full input credit $x' = x + \Delta$. Then the post-swap reserve is uniquely:*

$$y' = \frac{xy}{x + \Delta} = y\frac{x}{x + \Delta},$$

and the output is

$$\Delta y = y - y' = y\frac{\Delta}{x + \Delta}.$$

Proof. From $x'y' = xy$ and $x' = x + \Delta$ we obtain $y' = xy/(x + \Delta)$. Subtracting from y gives the output formula. □

Lemma 3.2 (Strict additivity holds for constant product)**.** *For constant product swaps, $F_{a+b} = F_b \circ F_a$.*

Proof. After input a, we have $y_1 = xy/(x + a)$ and state $(x + a, y_1)$. After input b, final $y_2 = (x+a)y_1/(x+a+b) = xy/(x+a+b)$. A single swap of size $a+b$ yields $y_S = xy/(x+a+b)$, hence $y_2 = y_S$. □

4 Impossibility: Exact $xy = k$ After Every Swap Leaves No Fee Freedom

Theorem 4.1 (No nonzero fee with exact $xy = k$ and full credit)**.** *Assume:*

(i) Full input credit: $x' = x + \Delta$,

(ii) Exact curve preservation: $x'y' = xy$ after every swap.

Then the swap output is uniquely forced and there is no parameter to apply a nonzero fee while staying on the same curve. In particular, any mechanism that charges a fee and reinvests it inside pricing *must change the invariant away from $xy = k$ (unless fee=0).*

Proof. Condition (ii) with (i) forces $y' = xy/(x+\Delta)$ uniquely, leaving no free degree of freedom to alter trader output while still satisfying the same invariant. □

5 Standard Uniswap-Style Input Fee Breaks Strict Additivity (Inside Pricing)

Definition 5.1 (Standard input fee semantics)**.** Let $\phi \in (0, 1)$ be the fee rate and $\alpha := 1 - \phi \in (0, 1)$. The output is computed using *effective input* $\alpha\Delta$:

$$\Delta y(\Delta) = \frac{\alpha\Delta\, y}{x + \alpha\Delta}.$$

But the pool credits the *full* input into reserves:

$$x' = x + \Delta, \qquad y' = y - \Delta y(\Delta).$$

Theorem 5.2 (Strict additivity fails for standard input fee)**.** *For any $a, b > 0$ and any $\alpha \in (0, 1)$, the standard input-fee swap is* not *strictly additive:*

$$\Delta y(a + b) \;>\; \Delta y(a) + \Delta y\big(b \mid \textit{after } a\big).$$

Equivalently, splitting changes the final state.

Proof. Let $\Delta = a + b$.

Single swap final. Compute:

$$y_S = y - \Delta y(a+b) = y - \frac{\alpha(a+b)y}{x+\alpha(a+b)} = y\frac{x}{x+\alpha(a+b)}.$$

Split swap final. First swap of a gives:

$$y_1 = y - \frac{\alpha a y}{x+\alpha a} = y\frac{x}{x+\alpha a}, \qquad x_1 = x+a.$$

Second swap of b from (x_1, y_1) yields:

$$y_2 = y_1 - \frac{\alpha b\, y_1}{x_1+\alpha b} = y_1\left(1 - \frac{\alpha b}{x+a+\alpha b}\right) = y_1\frac{x+a}{x+a+\alpha b}.$$

Substitute y_1:

$$y_2 = y\frac{x}{x+\alpha a}\cdot\frac{x+a}{x+a+\alpha b}.$$

Compare y_2 and y_S. Consider:

$$\frac{y_2}{y_S} = \frac{\frac{x}{x+\alpha a}\cdot\frac{x+a}{x+a+\alpha b}}{\frac{x}{x+\alpha(a+b)}} = \frac{(x+a)(x+\alpha(a+b))}{(x+\alpha a)(x+a+\alpha b)}.$$

Let $N = (x+a)(x+\alpha(a+b))$ and $D = (x+\alpha a)(x+a+\alpha b)$.

Expand N:

$$N = (x+a)(x+\alpha a+\alpha b) = x^2 + ax + \alpha ax + \alpha a^2 + \alpha bx + \alpha ab.$$

Expand D:

$$D = (x+\alpha a)(x+a+\alpha b) = x^2 + ax + \alpha ax + \alpha a^2 + \alpha bx + \alpha^2 ab.$$

Subtract:

$$N - D = \alpha ab - \alpha^2 ab = \alpha ab(1-\alpha) > 0.$$

Hence $N > D$ so $y_2/y_S > 1$, thus $y_2 > y_S$. Therefore split output is smaller:

$$\Delta y_{\text{split}} = y - y_2 < y - y_S = \Delta y_{\text{single}}.$$

Strict additivity fails. □

Remark 5.3. This does *not* create a splitting advantage for the trader (it is subadditive), but it violates strict additivity, so the final state depends on how the trade is split.

6 The Reinvested-Within-Pricing Strict-Additive Family

6.1 A general "constant-product times factor" form

Define the constant-product (no-fee) final reserve:

$$y_{\text{cp}}(x,\Delta) := y\frac{x}{x+\Delta}.$$

We allow a reinvestment factor $R(x,\Delta) \geq 1$ (keeps extra Y inside pricing compared to CP):

$$\boxed{y'(x,\Delta) = y\frac{x}{x+\Delta}\,R(x,\Delta).}$$

Then the state update map is:

$$F_\Delta(x,y) = \left(x+\Delta,\ y\frac{x}{x+\Delta}R(x,\Delta)\right).$$

6.2 The cocycle condition

Theorem 6.1 (Strict additivity $\Leftrightarrow$ cocycle equation)**.** *The above mechanism is strictly additive iff for all $x, a, b > 0$:*

$$\boxed{R(x, a)\, R(x + a, b) = R(x, a + b).}$$

Proof. Start with (x, y).

After a:

$$y_1 = y\frac{x}{x + a}\, R(x, a), \qquad x_1 = x + a.$$

After b:

$$y_2 = y_1\frac{x_1}{x_1 + b}\, R(x_1, b) = \left(y\frac{x}{x + a}R(x, a)\right)\frac{x + a}{x + a + b}R(x + a, b).$$

Cancel $(x + a)$:

$$y_2 = y\frac{x}{x + a + b}\, R(x, a)R(x + a, b).$$

Single swap of $a + b$ yields:

$$y_S = y\frac{x}{x + a + b}\, R(x, a + b).$$

Strict additivity requires $y_2 = y_S$ for all $y > 0$, hence $R(x, a)R(x + a, b) = R(x, a + b)$. $\square$

6.3 Solving the cocycle equation via a potential ratio

Theorem 6.2 (General solution: ratio of a potential)**.** *The cocycle equation holds iff there exists a function $G : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that:*

$$\boxed{R(x, \Delta) = \frac{G(x)}{G(x + \Delta)}.}$$

Proof. (*Sufficiency*) If $R(x, \Delta) = G(x)/G(x + \Delta)$ then

$$R(x, a)R(x + a, b) = \frac{G(x)}{G(x + a)} \cdot \frac{G(x + a)}{G(x + a + b)} = \frac{G(x)}{G(x + a + b)} = R(x, a + b).$$

(*Necessity*) A standard endpoint-potential construction exists whenever the cocycle identity holds (e.g. fix x_0 and define $G(x) = 1/R(x_0, x - x_0)$ for $x \geq x_0$, then derive the ratio representation). $\square$

6.4 Cleaner form: separable invariant $y\Psi(x) = K$

Define:

$$\boxed{\Psi(x) := x\, G(x).}$$

Then:

$$\frac{x}{x + \Delta} \cdot \frac{G(x)}{G(x + \Delta)} = \frac{xG(x)}{(x + \Delta)G(x + \Delta)} = \frac{\Psi(x)}{\Psi(x + \Delta)}.$$

So the strictly additive reinvested-within-pricing update becomes:

Definition 6.3 (Strictly additive reinvested-within-pricing AMM family)**.** Pick any positive function $\Psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Define

$$\boxed{F_\Delta(x, y) = \left(x + \Delta,\;\; y \cdot \frac{\Psi(x)}{\Psi(x + \Delta)}\right).}$$

Theorem 6.4 (Invariant)**.** *For the above family,*

$$\boxed{K := y\,\Psi(x) \quad \textit{is invariant under all swaps.}}$$

Proof. Compute:

$$y'\Psi(x') = \left(y\frac{\Psi(x)}{\Psi(x+\Delta)}\right)\Psi(x+\Delta) = y\Psi(x).$$

□

Remark 6.5. Constant product is the special case $\Psi(x) = x$, giving $K = xy$.

7 ExactIn Output and "Reinvested Fee" Accounting

Proposition 7.1 (ExactIn output)**.** *For the* Ψ*-invariant AMM,*

$$\boxed{\Delta y = f((x,y),\Delta) = y - y' = y\left(1 - \frac{\Psi(x)}{\Psi(x+\Delta)}\right).}$$

Proof. By definition, $y' = y\,\Psi(x)/\Psi(x+\Delta)$, so subtract from y. □

7.1 How much fee is reinvested inside pricing?

Use constant product (no-fee) as a baseline:

$$y_{\text{cp}} = y\frac{x}{x+\Delta}.$$

Your mechanism yields:

$$y' = y\frac{\Psi(x)}{\Psi(x+\Delta)}.$$

Define the reinvested-within-pricing retained Y relative to constant product:

$$\boxed{\text{fee}_Y((x,y),\Delta) := y' - y_{\text{cp}} = y\left(\frac{\Psi(x)}{\Psi(x+\Delta)} - \frac{x}{x+\Delta}\right).}$$

For ExactIn with full credit $x' = x + \Delta$, there is no extra X beyond the trader's input:

$$\boxed{\text{fee}_X((x,y),\Delta) := 0.}$$

8 ExactOut Derivation (General Ψ and Power Family)

8.1 ExactOut for general Ψ

Theorem 8.1 (ExactOut input formula)**.** *Given a target output* Δy *with* $0 < \Delta y < y$*, the required input for the* Ψ*-invariant AMM is:*

$$\boxed{\Delta x = \Psi^{-1}\left(\Psi(x)\frac{y}{y-\Delta y}\right) - x.}$$

Proof. ExactOut means final reserve $y' = y - \Delta y$. The invariant gives:

$$(y-\Delta y)\Psi(x') = y\Psi(x).$$

Solve for $\Psi(x')$:

$$\Psi(x') = \Psi(x)\frac{y}{y - \Delta y}.$$

Apply Ψ^{-1}:

$$x' = \Psi^{-1}\left(\Psi(x)\frac{y}{y - \Delta y}\right).$$

Finally, $\Delta x = x' - x$. □

8.2 Power family $\Psi(x) = x^\alpha$

Definition 8.2 (Power invariant)**.** Let $\Psi(x) = x^\alpha$ for $\alpha \in (0, 1]$. Then the invariant is:

$$x^\alpha y = K.$$

Proposition 8.3 (ExactIn for power family)**.**

$$\boxed{y' = y\left(\frac{x}{x + \Delta}\right)^\alpha, \qquad \Delta y = y\left(1 - \left(\frac{x}{x + \Delta}\right)^\alpha\right).}$$

Proof. Substitute $\Psi(x) = x^\alpha$ into $y' = y\,\Psi(x)/\Psi(x + \Delta)$. □

Proposition 8.4 (ExactOut for power family)**.** *For* $0 < \Delta y < y$,

$$\boxed{\Delta x = x\left(\left(\frac{y}{y - \Delta y}\right)^{1/\alpha} - 1\right).}$$

Proof. From invariant:

$$(y - \Delta y)(x')^\alpha = yx^\alpha \;\Rightarrow\; (x')^\alpha = x^\alpha \frac{y}{y - \Delta y}.$$

Take power $1/\alpha$:

$$x' = x\left(\frac{y}{y - \Delta y}\right)^{1/\alpha}.$$

Then $\Delta x = x' - x$. □

9 Worked Numerical Examples (Paper-Checkable)

Take:

$$x = 1000, \quad y = 1000, \quad \Delta = 100, \quad \alpha = 0.997, \quad \Psi(x) = x^\alpha.$$

9.1 Constant product baseline

$$y_{\text{cp}} = 1000 \cdot \frac{1000}{1100} = 909.0909090909, \qquad \Delta y_{\text{cp}} = 90.9090909091.$$

9.2 Reinvested-within-pricing power AMM

$$y' = 1000\left(\frac{1000}{1100}\right)^{0.997} \approx 909.3508831104,$$

$$\Delta y = 1000 - 909.3508831104 \approx 90.6491168896.$$

9.3 Reinvested fee retained in Y (vs constant product)

$$\mathrm{fee}_Y = y' - y_{\mathrm{cp}} \approx 909.3508831104 - 909.0909090909 \approx 0.2599740195.$$

9.4 Split invariance check: $40 + 60$

First trade $a = 40$:

$$y_1 = 1000 \left(\frac{1000}{1040}\right)^{0.997} \approx 961.6516048672.$$

Second trade $b = 60$:

$$y_2 = y_1 \left(\frac{1040}{1100}\right)^{0.997} \approx 909.3508831104.$$

This matches the one-shot y' (up to rounding), confirming strict additivity.

10 What is the Function F? (Additive Potential and Telescoping)

10.1 Additive potential fee construction

Definition 10.1 (Additive potential F)**.** Let $V \geq 0$ be a cumulative scalar (e.g. cumulative volume in an epoch). Choose a nondecreasing function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $F(0) = 0$. Define the fee charged for a trade of size Δ at volume V by:

$$\boxed{f(\Delta; V) := F(V + \Delta) - F(V).}$$

Theorem 10.2 (Strict additivity by telescoping)**.** *For $\Delta = a + b$:*

$$\boxed{f(a; V) + f(b; V + a) = f(a + b; V).}$$

Proof. Compute:

$$f(a; V) = F(V + a) - F(V), \qquad f(b; V + a) = F(V + a + b) - F(V + a).$$

Sum:

$$(F(V + a) - F(V)) + (F(V + a + b) - F(V + a)) = F(V + a + b) - F(V),$$

which equals $f(a + b; V)$. □

10.2 Relation to multiplicative potential

If $R(x, \Delta) = G(x)/G(x + \Delta)$, then taking logs gives an additive difference:

$$\ln R(x, \Delta) = \ln G(x) - \ln G(x + \Delta),$$

which is the same "endpoint-only" (path-independent) structure as $F(V + \Delta) - F(V)$.

11 Summary of Design Rules

- If you insist on staying *exactly* on $xy = k$ after each swap and credit full input, then there is no room for a nonzero fee (Theorem 4.1).
- Standard Uniswap-style input fees break strict additivity when fees are reinvested within priced reserves (Theorem 5.2).

- If you require full input credit + reinvestment within pricing + strict additivity, then swaps must preserve an invariant $y\Psi(x) = K$ and the state update is:

$$F_\Delta(x, y) = \left(x + \Delta,\ y\frac{\Psi(x)}{\Psi(x + \Delta)}\right).$$

- ExactIn output:

$$\Delta y = y\left(1 - \frac{\Psi(x)}{\Psi(x + \Delta)}\right).$$

- ExactOut input:

$$\Delta x = \Psi^{-1}\left(\Psi(x)\frac{y}{y - \Delta y}\right) - x.$$

References and External Docs (Links)

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6. "Additive Fee Mechanisms for Constant-Product AMMs" (your PDF; used as contrast to the inside-pricing track).