Magnet Analysis 1A
Rose 6/3/2010

Name KEY

Final Exam Review

1. $\lim_{x\to 3}\frac{x^2-9}{5x-x^2-6} = \lim_{x\to 3}\frac{(x-3)(x+3)}{-(x^2-5x+6)}$

$\lim_{x\to 3}\frac{(x-3)(x+3)}{-(x-3)(x-2)} = \lim_{x\to 3}\frac{x+3}{-(x-2)} = \boxed{-6}$

2. $\lim_{x\to 0^+}\frac{e^{1+x}-e}{x} = \lim_{x\to 0^+}\frac{e(e^x-1)}{x}$

$= e\left[\lim_{x\to 0^+}\frac{e^x-e^0}{x-0}\right] = e\cdot\frac{d}{dx}\left[e^x\right]\Big|_{x=0}$

$= e\cdot e^0 = \boxed{e}$

3. $\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{6}+h\right)-\sin\frac{\pi}{6}}{h}$

$= \frac{d}{dx}\left[\sin x\right]\Big|_{x=\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right)$

$= \boxed{\frac{\sqrt{3}}{2}}$

4. $\lim_{x\to 2}\frac{x^2+2x-8}{x-2} = \lim_{x\to 2}\frac{(x+4)\cancel{(x-2)}}{\cancel{x-2}}$

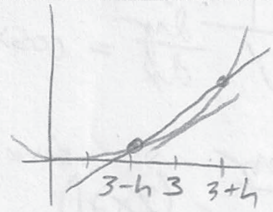
$= \boxed{6}$

5. $\lim_{x\to 0}\frac{5x}{3\sin x\cos 4x} =$

$= \frac{5}{3}\left(\lim_{x\to 0}\frac{x}{\sin x}\right)\left(\lim_{x\to 0}\frac{1}{\cos 4x}\right)$

$= \frac{5}{3}\cdot 1\cdot 1 = \boxed{\frac{5}{3}}$

6. $\lim_{h\to 0}\frac{(3+h)^2-(3-h)^2}{2h}$



$= \frac{d}{dx}\left[x^2\right]\Big|_{x=3}$

$= 2x\Big|_{x=3} = \boxed{6}$

7. $\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{x} = \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \boxed{\frac{2}{\pi}}$

8. $\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a}\frac{\cancel{\sqrt{x}-\sqrt{a}}}{\cancel{(\sqrt{x}-\sqrt{a})}(\sqrt{x}+\sqrt{a})}$

$= \lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$

Find $\frac{dy}{dx}$ for the following

9. $y=5(2x^3+1)^2+5\arctan(2x)$

$\frac{dy}{dx}=10(2x^3+1)(6x^2)+\frac{5}{1+(2x)^2}\cdot 2$

$=\boxed{60x^2(2x^3+1)+\frac{10}{1+4x^2}}$

10. $y=(2x+1)^{\sin x}$

$\ln y=\sin x\ln(2x+1)$

$\frac{1}{y}\frac{dy}{dx}=\cos x\ln(2x+1)+\sin x\cdot\frac{1}{2x+1}\cdot 2$

$\frac{dy}{dx}=(2x+1)^{\sin x}\left[\cos x\ln(2x+1)+\frac{2\sin x}{2x+1}\right]$

11. $y=4(3x+5)\cos^2 x$

$\frac{dy}{dx}=12\cos^2 x+4(3x+5)2\cos x(-\sin x)$

$=12\cos^2 x-4(3x+5)\sin 2x$

12. $y=\frac{5}{\sqrt{x^3+5}}=5(x^3+5)^{-1/2}$

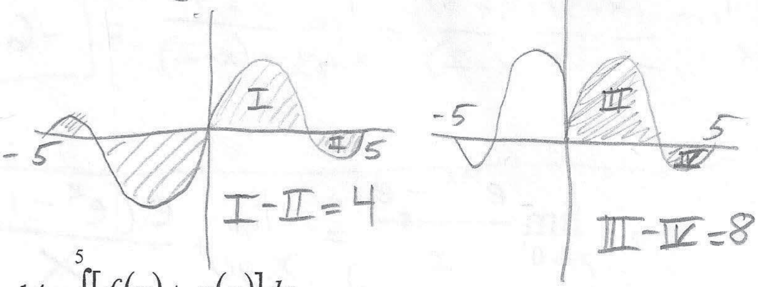
$\frac{dy}{dx}=\frac{-5}{2}(x^3+5)^{-3/2}(3x^2)$

$=\frac{-15x^2}{2(x^3+5)^{3/2}}$

13. $y=\int_{\pi}^{\cos x}\frac{dt}{(t^2+1)^{2/3}}$

$\frac{dy}{dx}=\frac{1}{(\cos^2 x+1)^{2/3}}\cdot(-\sin x)$

Given that f is an even function and $\int_0^5 f(x)dx=8$ and g is an odd function and $\int_0^5 g(x)dx=4$, evaluate each of the following if possible: g odd f even



14. $\int_{-5}^{5}[f(x)+g(x)]dx$

$=\int_{-5}^{5}f(x)dx+\int_{-5}^{5}g(x)dx$

$=16+0=\boxed{16}$

15. $\int_{-5}^{5}|g(x)|dx=2(I+II)$

So not possible to determine

16. $\int_{-5}^{5}[3f(x)+x^2]dx=3\int_{-5}^{5}f(x)dx+\int_{-5}^{5}x^2dx$

$=3(16)+\left.\frac{x^3}{3}\right]_{-5}^{5}=48+\left(\frac{125}{3}--\frac{125}{3}\right)$

$=\frac{394}{3}$

17. $\int_0^5\frac{f(x)}{g(x)}dx$

Not possib6 to determine

18. $\int_{-5}^{5}[f(x)+4]dx=\int_{-5}^{5}f(x)+4\int_{-5}^{5}dx$

$16+40$

$\boxed{56}$

19. Let $f(x) = \int_{3}^{5x^2} \frac{dt}{\sqrt[3]{t^2+1}}$

Find $f\left(\frac{1}{2}\right)$ [Use calculator]

$f\left(\frac{1}{2}\right) = \int_{3}^{\frac{5}{4}} \frac{dt}{\sqrt[3]{t^2+1}}$

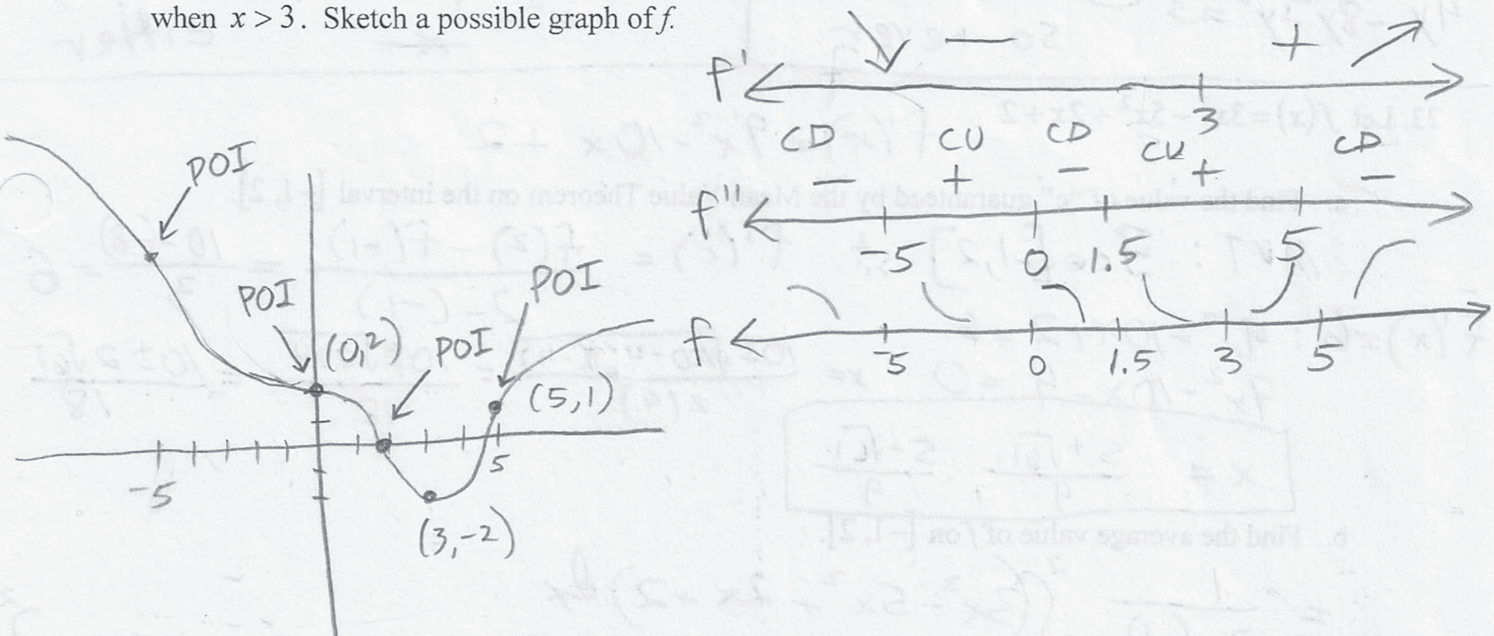
$= \boxed{-1.009}$

Find $f'(2)$

$f'(x) = \frac{10x}{\sqrt[3]{(5x^2)^2+1}} = \frac{10x}{\sqrt[3]{25x^4+1}}$

$f'(2) = \frac{20}{\sqrt[3]{401}}$

20. Suppose that f is continuous on $[-5, 5]$ and has the following properties: $f(0) = 2$, $f(3) = -2$, $f(5) = 1$, $f''(x) > 0$ on $(-5, 0)$ and $(1.5, 5)$ only, f is decreasing when $x < 3$, and f is increasing when $x > 3$. Sketch a possible graph of f.



21. Given $f(x) = \ln(3x^2+1)$, use the linearization of f at $x = 1$ to estimate $f(1.2)$

$f'(x) = \frac{1}{3x^2+1} \cdot 6x \quad f'(1) = \frac{3}{2} \quad f(1) = \ln 4$

So $y - \ln 4 = \frac{3}{2}(x-1)$ is the tangent line to f @ x = 1

So $f(1.2) \approx \frac{3}{10} + \ln 4$

22. Let $x^2 - 4xy + y^2 = 3$

a. Find $\frac{dy}{dx}$

$$2x - 4\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$2x - 4y - 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - 4x) = 4y - 2x$$

$$\frac{dy}{dx} = \frac{4y-2x}{2y-4x} = \frac{2y-x}{y-2x}$$

b. Find and justify any points on the curve where the tangent line is horizontal or vertical, if they exist.

Horizontal:

when $2y - x = 0 \Rightarrow x = 2y$

Does this ever happen?

$(2y)^2 - 4(2y)y + y^2 = 3$

$4y^2 - 8y^2 + y^2 = 3$

$\Rightarrow -3y^2 = 3$

$y^2 = -1$ $\cancel{}$

so never

Vertical:

when $y - 2x = 0 \Rightarrow y = 2x$

$x^2 - 4x(2x) + (2x)^2 = 3$

$x^2 - 8x^2 + 4x^2 = 3$

$-3x^2 = 3$

$x^2 = -1$ $\cancel{}$

so no vertical tangents either

23. Let $f(x) = 3x^3 - 5x^2 + 2x + 2$

$$f'(x) = 9x^2 - 10x + 2$$

a. Find the value of "c" guaranteed by the Mean Value Theorem on the interval $[-1, 2]$.

MVT: $\exists c \in [-1,2]$ s.t. $f'(c) = \frac{f(2) - f(-1)}{2-(-1)} = \frac{10-(-8)}{3} = 6$

$f'(x) = 6$: $9x^2 - 10x + 2 = 6$

$9x^2 - 10x - 4 = 0$ $\quad x = \frac{10 \pm \sqrt{100 - 4(9)(-4)}}{2(9)} = \frac{10 \pm \sqrt{244}}{18} = \frac{10 \pm 2\sqrt{61}}{18}$

$$\boxed{x = \frac{5+\sqrt{61}}{9},\ \frac{5-\sqrt{61}}{9}}$$

b. Find the average value of f on $[-1, 2]$.

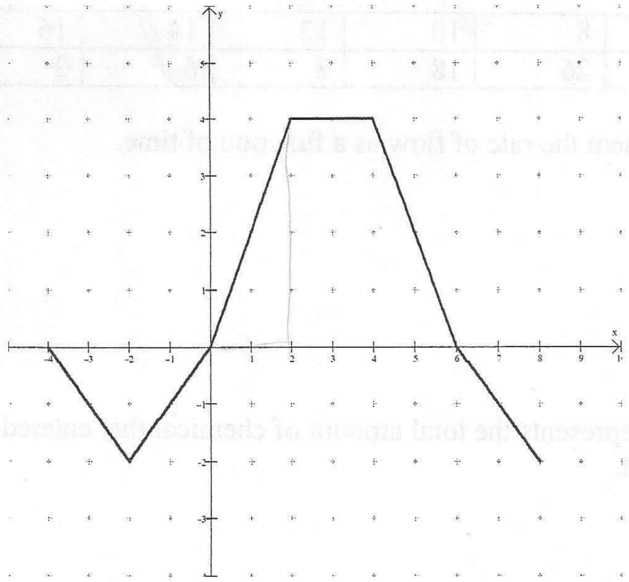
$$= \frac{1}{2-(-1)} \int_{-1}^{2} (3x^3 - 5x^2 + 2x + 2)\,dx$$

$$= \frac{1}{3}\left[\frac{3}{4}x^4 - \frac{5}{3}x^3 + x^2 + 2x\right]_{-1}^{2} = \frac{1}{36}\left[9x^4 - 20x^3 + 12x^2 + 24x\right]_{-1}^{2}$$

$$= \frac{1}{36}\left[(144 - 160 + 48 + 48) - (9 + 20 + 12 - 24)\right]$$

$$= \frac{1}{36}\left[80 - 17\right] = \frac{63}{36} = \boxed{\frac{7}{4}}$$

24. The graph of a function f is given below. Let $g(x)=\int_0^x f(t)dt$



a. Find each of the following: $g(0)$, $g(2)$, $g(-2)$

$g(0)=\int_0^0 f(t)dt=0$ $\qquad g(-2)=\int_0^{-2} f(t)dt=-\int_{-2}^{0} f(t)dt$

$g(2)=\int_0^2 f(t)dt=4$ $\qquad =-(-2)=2$

b. Find all values of x for which g has a relative minimum on the open interval $(4, 8)$. Justify your answer.

$g'(x)=f(x)$ by the FTC

Since $g'=f$ changes from neg to pos @ $x=0$, g has a rel min @ $x=0$ by the FDT.

c. Write an equation of the tangent line to the graph of g at $x=2$.

$g(2)=4$ $\qquad g'(2)=f(2)=4$

So $\boxed{y-4=4(x-2)}$

d. Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-4, 8)$. Justify your answer.

$g''(x)=f'(x)$ by the FTC

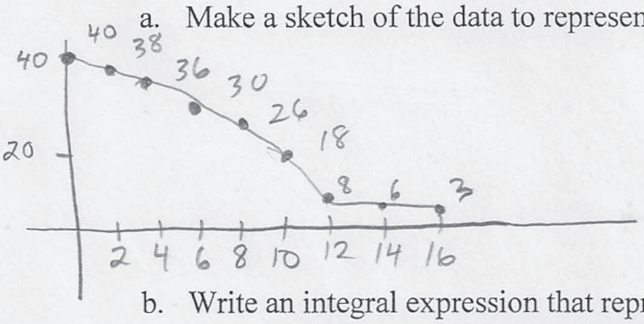
Since $f'=g''$ changes from neg to pos @ $x=-2$

g has a POI @ $x=-2$

25. Chemicals from a storage tank are leaking into a pond. The rate of flow is measured at intervals and is recorded in the table below: (*t* is measured in hours; *R(t)* in gallons per hour)

T	0	2	4	6	8	10	12	14	16
R(t)	40	38	36	30	26	18	8	6	3

a. Make a sketch of the data to represent the rate of flow as a function of time.



b. Write an integral expression that represents the total amount of chemical that entered the pond during the 16-hour period.

$\int_0^{16} R(t)\,dt$

c. Estimate the total volume of chemicals that entered the pond using LRRAM (n = 8), RRAM (n = 8), Midpoint Rule (n = 4), and Trapezoidal Rule (n = 4)

LRRAM:
$40(2) + 38(2) + 36(2) + 30(2) + 26(2) + 18(2) + 8(2) + 6(2)$
$= 404$

Midpoint
$38(4) + 30(4) + 18(4) + 6(4)$
$= 368$

RRAM:
$38(2) + 36(2) + 30(2) + 26(2) + 18(2) + 8(2) + 6(2) + 3(2)$
$= 330$

Trap
$\frac{1}{2}\cdot 4\left[40 + 2(36) + 2(26) + 2(8) + 3\right]$
$= 366$

26. The acceleration of a particle is given by $a(t) = 2 + 5\sqrt{t}\ \ ft/s^2;\ t \geq 0$. $v(0) = 6$

a. What is the velocity of the object at $t = 9$ seconds?

$v(t) = \int (2 + 5t^{1/2})\,dt = 2t + \frac{10}{3}t^{3/2} + C$

$v(0) = C = 6$ so $v(t) = 2t + \frac{10}{3}t^{3/2} + 6$

$v(9) = 18 + 90 + 6$

$\boxed{114\ ft/sec}$

b. What is the total distance covered by the object during the first 9 seconds?

$v > 0$ on $[0, 9]$ so distance = displacement

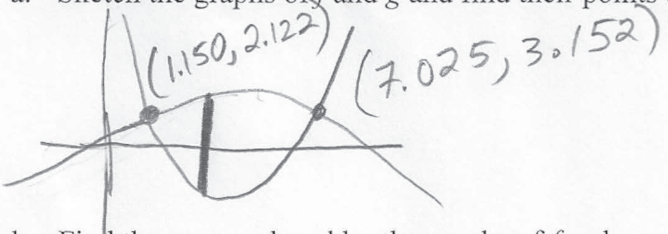
So distance travelled $= \int_0^9 v(t)\,dt = \int_0^9 \left[2t + \frac{10}{3}t^{3/2} + 6\right]dt = t^2 + \frac{4}{3}t^{5/2} + 6t\Big]_0^9$

$= 81 + \frac{4}{3}(243) + 54 = \boxed{459\ ft}$

$f'(x) = \cos\frac{x}{3}$ $\quad g(x) = 2x - 8$

27. Let $f(x) = 3\sin\left(\frac{x}{3}\right) + 1$ and $g(x) = x^2 - 8x + 10$

a. Sketch the graphs of f and g and find their points of intersection by calculator.



$a = 1.150$
$b = 7.025$

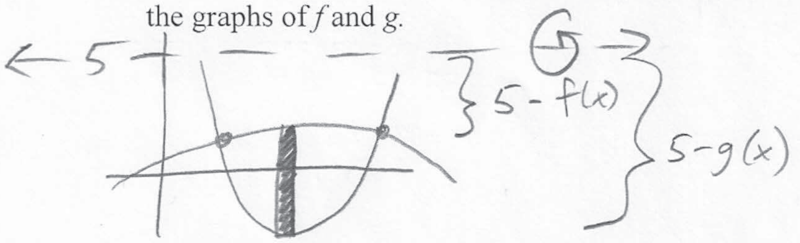
b. Find the area enclosed by the graphs of f and g.

$$\int_a^b \left[f(x) - g(x)\right] dx \approx 38.799$$

c. Find the volume of the solid whose base is the region enclosed by the graphs of f and g such that cross sections perpendicular to the x-axis are quarter circles.

$$V = \int_a^b \frac{\pi\left(f(x) - g(x)\right)^2}{4}\, dx \approx 241.793$$

d. Find the volume of the solid formed by rotating about the line $y = 5$ the region enclosed by the graphs of f and g.



$$\text{So } V = \pi \int_a^b \left[5 - g(x)\right]^2 - \left[5 - f(x)\right]^2 dx$$

$$\approx 1292.646$$

e. Find the length of the boundary of the region enclosed by the graphs of f and g.

$$\int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx + \int_a^b \sqrt{1 + \left[g'(x)\right]^2}\, dx$$

$$\int_a^b \left(\sqrt{1 + \cos^2\left(\frac{x}{3}\right)} + \sqrt{1 + (2x-8)^2}\right) dx \approx 25.384$$

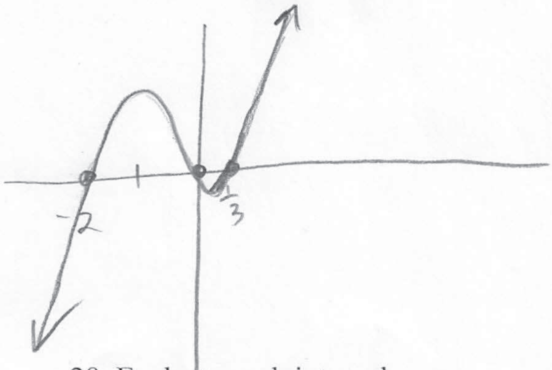
28. Set up but do not evaluate one or more integral expressions that could be used to find the area between the curve $y = 3x^3 + 5x^2 - 2x$ and the x-axis.

$y = x(3x^2 + 5x - 2)$

$x(3x - 1)(x + 2)$

Area $=$

$$\int_{-2}^{0} (3x^3 + 5x^2 - 2x)\,dx - \int_{0}^{\frac{1}{3}} (3x^3 + 5x^2 - 2x)\,dx$$



29. Evaluate each integral:

a. $\int \left(xe^{x^2-1} + \cos 3x\right)dx = \frac{1}{2}\int e^{x^2-1}(2x\,dx) + \frac{1}{3}\int \cos(3x)(3\,dx)$
$= \frac{1}{2}e^{x^2-1} + \frac{1}{3}\sin(3x) + C$

b. $\frac{1}{4}\int \left[\sin^3(4x)\cos(4x)\right]dx\ 4 = \frac{1}{4}\int u^3\,du = \frac{1}{16}u^4 + C$
$u = \sin 4x$
$du = 4\cos 4x\,dx$
$= \frac{1}{16}\sin^4(4x) + C$

c. $-\frac{1}{3}\int \left[5\sin(3x)e^{\cos 3x}\right]dx\ -3 = -\frac{5}{3}\int e^u\,du = -\frac{5}{3}e^u + C$
$u = \cos 3x$
$du = -\sin(3x) \cdot 3$
$= -\frac{5}{3} \cdot e^{\cos 3x} + C$

d. $\frac{1}{6}\int \left[x\cos(3x^2)\right]dx\ 6 = \frac{1}{6}\int \cos u\,du = \frac{1}{6}\sin u + C$
$u = 3x^2$
$du = 6x\,dx$
$= \frac{1}{6}\sin(3x^2) + C$

e. $\frac{1}{3}\int \left[x^2\sec^2(x^3)\right]dx \cdot 3 = \frac{1}{3}\int \sec^2 u\,du = \frac{1}{3}\tan u + C$
$u = x^3$
$du = 3x^2\,dx$
$= \frac{1}{3}\tan(x^3) + C$

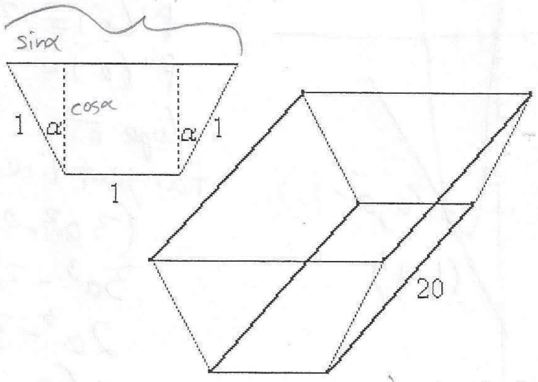
f. $\frac{1}{9}\int_0^1 \frac{5x^2\ 9}{8 + 3x^3}\,dx = \frac{5}{9}\int_{u(0)}^{u(1)} \frac{du}{u} = \frac{5}{9}\int_8^{11} \frac{du}{u} = \frac{5}{9}\ln|u|\Big]_8^{11}$
$u = 8 + 3x^3$
$du = 9x^2\,dx$

$= \frac{5}{9}(\ln 11 - \ln 8) = \boxed{\frac{5}{9}\ln\left(\frac{11}{8}\right)}$

(This is the trough page)

30. The trough in the figure below is to be made with the dimensions shown. Only the angle α can be varied. What value of α will maximize the trough's volume. $1+2\sin\alpha$

$$V(\alpha)=\left[\frac{(2\sin\alpha+1)+1}{2}\cdot\cos\alpha\right]20$$



$$V(\alpha)=20\cos\alpha(\sin\alpha+1)$$
$$V(\alpha)=20\left[\sin\alpha\cos\alpha+\cos\alpha\right]$$
$$V(\alpha)=20\left[\tfrac{1}{2}\sin 2\alpha+\cos\alpha\right]$$
$$V'(\alpha)=20\left[\tfrac{1}{2}\cos(2\alpha)\cdot 2-\sin\alpha\right]$$
$$V'(x)=20\left[\cos 2\alpha-\sin\alpha\right]$$
$$V'(\alpha)=20\left[1-2\sin^2\alpha-\sin\alpha\right]$$
$$=-20\left[2\sin^2\alpha+\sin\alpha-1\right]$$
$$V'(\alpha)=-20(2\sin\alpha-1)(\sin\alpha+1)$$

$2\sin\alpha-1>0$ $\quad$ $\sin\alpha+1>0$

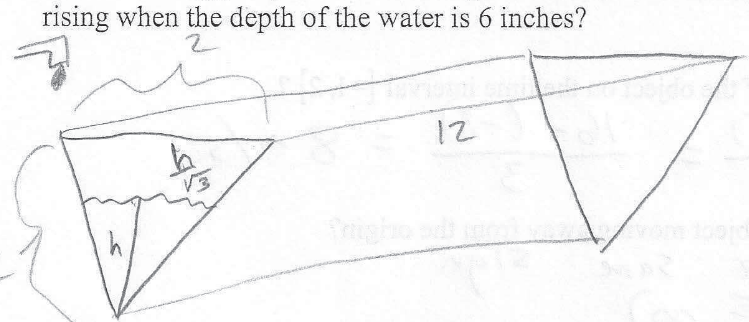
$\sin\alpha>\frac{1}{2}$ $\quad$ $\sin\alpha>-1$

$+ \quad -$

$0 \quad \frac{\pi}{6} \quad \frac{\pi}{2}$

$\boxed{\text{rel max @ } \alpha=\frac{\pi}{6}}$

31. Water trough is 12 feet long and its cross section is an equilateral triangle with sides 2 feet long. Water is being pumped into the trough at a rate of 3 cubic feet per minute. How fast is the water level rising when the depth of the water is 6 inches?

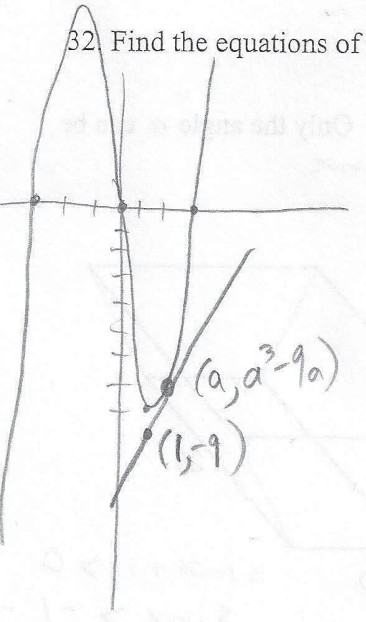


Know: $\frac{dV}{dt}=3\ ft^3/min$

Want: $\left.\frac{dh}{dt}\right|_{h=\frac{1}{2}}$

$$V(h)=\tfrac{1}{2}\left(2\tfrac{h}{\sqrt{3}}\right)h\cdot 12$$
$$V(h)=4\sqrt{3}h^2$$
$$\frac{dV}{dt}=8\sqrt{3}h\frac{dh}{dt}$$
$$\frac{dh}{dt}=\frac{3}{8\sqrt{3}h} \qquad \left.\frac{dh}{dt}\right|_{h=\frac{1}{2}}=\frac{3}{4\sqrt{3}}=\boxed{\frac{\sqrt{3}}{4}\ ft/min}$$

32. Find the equations of the tangent lines to the curve $y = x^3 - 9x$ through the point $(1, -9)$.



$y = x(x^2 - 9)$

$f(x) = x^3 - 9x$

$f'(x) = 3x^2 - 9$

$f'(a) = 3a^2 - 9$

slope of tangent line $= 3a^2 - 9 = \dfrac{a^3 - 9a - (-9)}{a - 1}$

$(3a^2 - 9)(a - 1) = a^3 - 9a + 9$

$3a^3 - 3a^2 - 9a + 9 = a^3 - 9a + 9$

$2a^3 - 3a^2 = 0$

$a^2(2a - 3) = 0$

$a = 0, \frac{3}{2}$

$a = 0, m = -9$ $\quad y + 9 = -9(x - 1)$

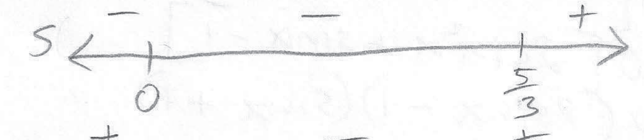
$a = \frac{3}{2}, m = \frac{-9}{4}$ $\quad y + 9 = -\frac{9}{4}(x - 1)$

33. An object moves along the x-axis according to the following position function which describes its x-coordinate (in meters) at time t seconds: $s(t) = 3t^5 - 5t^4$ Consider values of t on the interval $(-\infty, \infty)$. Make clear, labeled number lines below for position, velocity, and acceleration. Include units in your answer when appropriate.

$v(t) = 15t^4 - 20t^3$
$= 5t^3(3t - 4)$

$a(t) = 60t^3 - 60t^2$
$= 60t^2(t - 1)$

$s(t) = t^4(3t - 5)$



a. On what time intervals is the object moving to the left?

$(0, \frac{4}{3})$

b. What is the average velocity of the object on the time interval $[-1, 2]$?

$\dfrac{\Delta s}{\Delta t} = \dfrac{s(2) - s(-1)}{3} = \dfrac{16 - (-8)}{3} = 8 \text{ m/s}$

c. On what time intervals is the object moving away from the origin?

when s, v have same sign

$(0, \frac{4}{3})$ and $(\frac{5}{3}, \infty)$

d. On what time intervals is the object slowing down? (speed is a scalar)

when a, v diff sign

$(-\infty, 0)$ and $(1, \frac{4}{3})$

e. What is the total distance travelled on the interval $[-1, 2]$?

$\int_{-1}^{2} |v(t)|dt = \int_{-1}^{0} v(t)dt - \int_{0}^{4/3} v(t)dt + \int_{4/3}^{2} v(t)dt$

$= s(0) - s(-1) - [s(4/3) - s(0)] + s(2) - s(\frac{4}{3})$

$= 2s(0) - s(-1) - 2s(4/3) + s(2) = 2(0) - (-8) - 2(\frac{-256}{81}) + 16$

$= 24 + \frac{512}{81} = \boxed{\frac{2456}{81}}$

$\frac{4-1}{10} = \frac{3}{10}$ $\quad f(x) = e^{x^2}$

Using 10 subintervals, approximate $\int_1^4 e^{x^2}\,dx$ using:

(a) a right endpoint approximation (b) midpoint rule (c) trapezoid rule (d) Simpson's rule

a) $\frac{3}{10}\left[f(\frac{13}{10}) + f(\frac{16}{10}) + f(\frac{19}{10}) + f(\frac{22}{10}) + f(\frac{25}{10}) + f(\frac{28}{10}) + f(\frac{31}{10}) + f(\frac{34}{10}) + f(\frac{37}{10}) + f(\frac{40}{10})\right]$

2967339.049

b) $\frac{3}{10}\left[f(1.15) + f(1.45) + f(1.75) + f(2.05) + f(2.35) + f(2.65) + f(2.95) + f(3.25) + f(3.55) + f(3.85)\right]$

924238.846

c) $\frac{1}{2}(\frac{3}{10})\left[f(1) + 2f(1.3) + 2f(1.6) + 2f(1.9) + 2f(2.2) + 2f(2.5) + 2f(2.8) + 2f(3.1) + 2f(3.4) + 2f(3.7) + f(4)\right]$

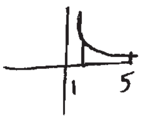
1634422.879

d) $\frac{3}{3(10)}\left[f(1) + 4f(1.3) + 2f(1.6) + 4f(1.9) + 2f(2.2) + 4f(2.5) + 2f(2.8) + 4f(3.1) + 2f(3.4) + 4f(3.7) + f(4)\right]$

1269119.257

34. In approximating $\int_1^5 \frac{1}{x}\,dx$, how large should we take n in order to guarantee that our answer is accurate to within 0.0001, using (a) left endpoint approx. (b) trap rule (c) midpoint rule (d) Simpson's rule ?

$f(x) = \frac{1}{x}$



$f'(x) = -x^{-2}$

$Max|f'(x)| = 1$

$f''(x) = 2x^{-3}$

$Max|f''(x)| = 2$

$f'''(x) = -6x^{-4}$

$Max|f'''(x)| = 6$

$f^{IV}(x) = 24x^{-5}$

$Max|f^{IV}(x)| = 24$

a) $\frac{K_1(b-a)^2}{2n} \le \frac{1}{10,000}$

$\frac{(1)(16)}{2n} \le \frac{1}{10,000}$

$\frac{8}{n} \le \frac{1}{10000}$

$\frac{n}{8} \ge 10,000$

$n \ge 80,000$

b) $\frac{K_2(b-a)^3}{12n^2} \le \frac{1}{10000}$

$\frac{2(4)^3}{12n^2} \le \frac{1}{10,000}$

$12n^2 \ge 1,280,000$

$\boxed{n \ge 327}$

c) $\frac{K_2(b-a)^3}{24n^2} \le \frac{1}{10,000}$

$\frac{2(4)^3}{24n^2} \le \frac{1}{10,000}$

$\boxed{n \ge 231}$

d) $\frac{K_4(b-a)^5}{180n^4} \le \frac{1}{10,000}$

$\frac{24(4)^5}{180n^4} \le \frac{1}{10,000}$

$n \ge 9$

35. A curve C is defined by the parametric equations $\begin{array}{l} x = t^2 \\ y = t^3 - 3t \end{array}$.

a) Show that C crosses itself at $(3, 0)$ by finding two values of t that place the curve there.

$3 = t^2$

$t = \pm\sqrt{3}$

$y(\sqrt{3}) = (\sqrt{3})^3 - 3\sqrt{3} = 0$ ✓

$y(-\sqrt{3}) = (-\sqrt{3})^3 + 3\sqrt{3} = 0$ ✓

b) Find $\frac{dy}{dx}$ and use it to give the equations of the two tangent lines to C at $(3, 0)$

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = 3t^2 - 3$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} = \boxed{\frac{3}{2}t - \frac{3}{2}t^{-1}}$

$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{3}{2}\sqrt{3} - \frac{3}{2\sqrt{3}} = \frac{3}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right)$

$\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = -\frac{3}{2}\sqrt{3} + \frac{3}{2\sqrt{3}} = -\frac{3}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right)$

$y - 0 = \frac{3}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right)(x-3)$

$y - 0 = -\frac{3}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right)(x-3)$

c) Give the coordinates of all points where C has a horizontal tangent line.

$\frac{dy}{dx} = 0$

$\frac{dx}{dt} \neq 0$

$3t^2 - 3 = 0$

$t = \pm 1$

$t=1: (1, -2)$

$t=-1: (1, 2)$

d) Give the coordinates of all points where C has a vertical tangent line.

$\frac{dx}{dt} = 0$

$\frac{dy}{dt} \neq 0$

$2t = 0$

$t = 0$

$t: 0 \quad (0, 0)$

e) Find all intervals where C is concave up or concave down.

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{3}{2} + \frac{3}{2}t^{-2}}{2t} = \frac{3}{4}t^{-1} + \frac{3}{4}t^{-3}$

$\frac{3}{4}t^{-3}(t^2+1)$

$\frac{3(t^2+1)}{4t^3}$

$- \;|\; +$ (at 0)

CU: $t: (0, \infty)$

CD: $t: (-\infty, 0)$