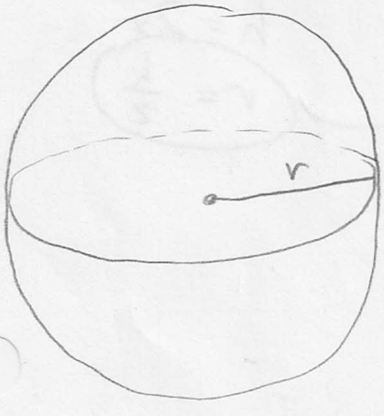
Magnet Analysis 1A
Rose

Name KEY

Introduction to Related Rates

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius. In a related rates problem, the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

1. Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?



Know: $\frac{dV}{dt} = 100 \text{cm}^3/\text{s}$

Want: $\left.\frac{dr}{dt}\right|_{r=25}$

$V = \frac{4}{3}\pi r^3$

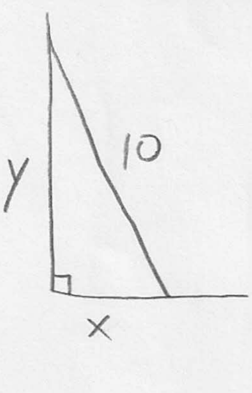
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$

$\frac{dr}{dt} = \frac{100}{4\pi r^2} = \frac{25}{\pi r^2}$

$\boxed{\left.\frac{dr}{dt}\right|_{r=25} = \frac{1}{25\pi} \text{ cm/s}}$

2. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



Given: $\frac{dx}{dt} = 1 \text{ ft/s}$

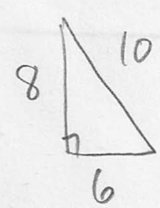
Want: $\left.\frac{dy}{dt}\right|_{x=6}$

$x^2 + y^2 = 100$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

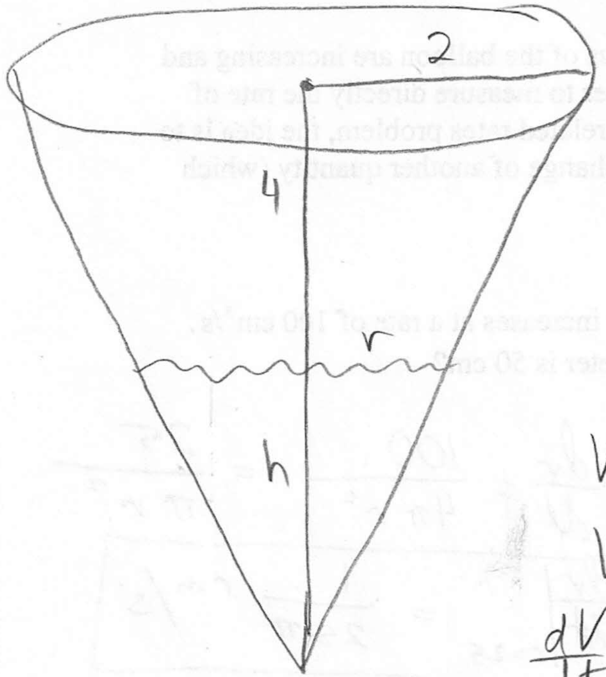
$\frac{dy}{dt} = \frac{-x \frac{dx}{dt}}{y}$

$\frac{dy}{dt} = -\frac{x}{y}$



$\boxed{\left.\frac{dy}{dt}\right|_{x=6} = \frac{-6}{8} = -\frac{3}{4} \text{ ft/s}}$

3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $100\ \text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.



Given: $\frac{dV}{dt} = 100\ m^3/min$

Want: $\left.\frac{dh}{dt}\right|_{h=3} = ?$

$V = \frac{1}{3}\pi r^2 h$

Also $\frac{r}{h} = \frac{2}{4}$

$h = 2r$

$r = \frac{h}{2}$

$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$V = \frac{\pi}{12} h^3$

$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt}$

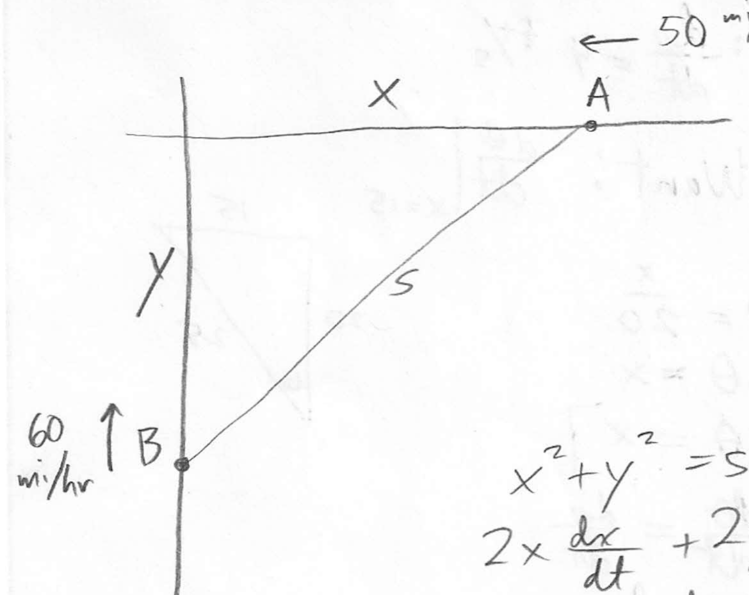
$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4\frac{dV}{dt}}{\pi h^2}$

$\frac{dh}{dt} = \frac{400}{\pi h^2}$

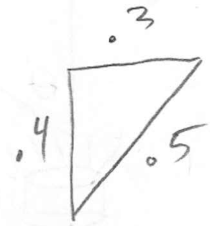
$$\boxed{\left.\frac{dh}{dt}\right|_{h=3} = \frac{400}{9\pi}\ m/min}$$

4. Car A is traveling west at 50 mi/hr and car B is traveling north at 60mi/hr. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection? Which car gets to the intersection first and by how many seconds do the cars miss crashing into each other?



$$\left.\begin{aligned}\frac{dx}{dt} &= -50 \text{ mi/hr}\\ \frac{dy}{dt} &= -60 \text{ mi/hr}\end{aligned}\right\} \text{Given}$$

Want: $\left.\dfrac{ds}{dt}\right|_{\substack{x=.3\\y=.4}}$



$$x^2+y^2=s^2$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2s\frac{ds}{dt}$$

$$\frac{ds}{dt}=\frac{x\frac{dx}{dt}+y\frac{dy}{dt}}{s}=\frac{-50x-60y}{s}$$

$$\left.\frac{ds}{dt}\right|_{\substack{x=.3\\y=.4}}=\frac{-50(.3)-60(.4)}{.5}=\frac{-150-240}{5}=\frac{-390}{5}=-78 \text{ mi/hr}$$

time of A from $x=.3$ to intersection $=\dfrac{.3}{50}=\dfrac{3}{500}$ hr

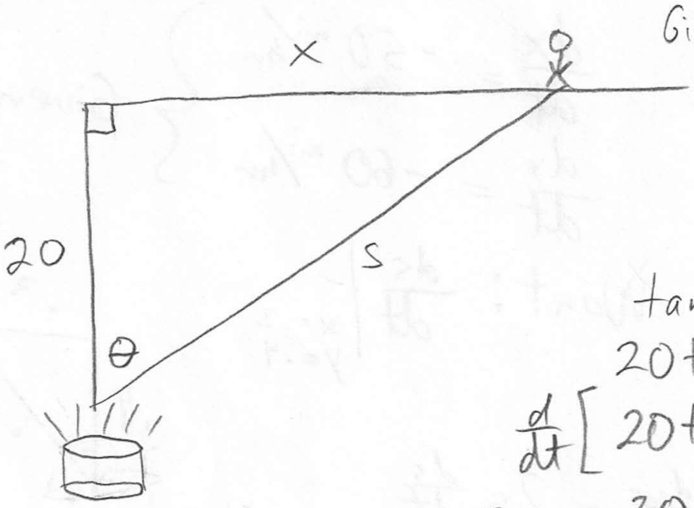
time of B from $x=.4$ to " $=\dfrac{.4}{60}=\dfrac{1}{150}$ hr

So A gets to the intersection first.

They miss each other by $\dfrac{1}{150}-\dfrac{3}{500}=\dfrac{1 \text{ hr}}{1500}\left(\dfrac{60 \text{ min}}{1 \text{ hr}}\right)\left(\dfrac{60 \text{ sec}}{1 \text{ min}}\right)$

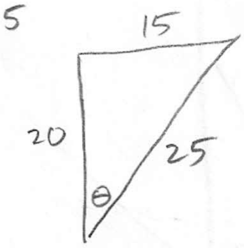
$= 2.4$ seconds

5. A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15ft from the point on the path closest to the searchlight? And how fast is he moving away from the searchlight at this same moment?



Given: $\frac{dx}{dt} = 4$ ft/s

Want: $\left.\frac{d\theta}{dt}\right|_{x=15}$



$$\tan\theta = \frac{x}{20}$$

$$20\tan\theta = x$$

$$\frac{d}{dt}\left[20\tan\theta = x\right]$$

$$20\sec^2\theta\,\frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta\,\frac{dx}{dt}}{20} = \frac{\cos^2\theta\cdot 4}{20} = \frac{\cos^2\theta}{5}$$

$$\left.\frac{d\theta}{dt}\right|_{x=15} = \frac{\left(\frac{4}{5}\right)^2}{5} = \frac{16}{125}\ \text{rad/sec}$$

Want: $\left.\frac{ds}{dt}\right|_{x=15}$

$$x^2 + 400 = s^2$$

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

$$\frac{ds}{dt} = \frac{x\cdot\frac{dx}{dt}}{s} = \frac{4x}{s}$$

$$\left.\frac{ds}{dt}\right|_{x=15} = \frac{4\cdot 15}{25} = \frac{12}{5}\ \text{ft/sec}$$