Analysis 1A

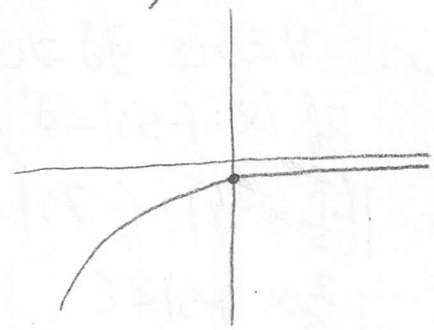
Name KEY

Date

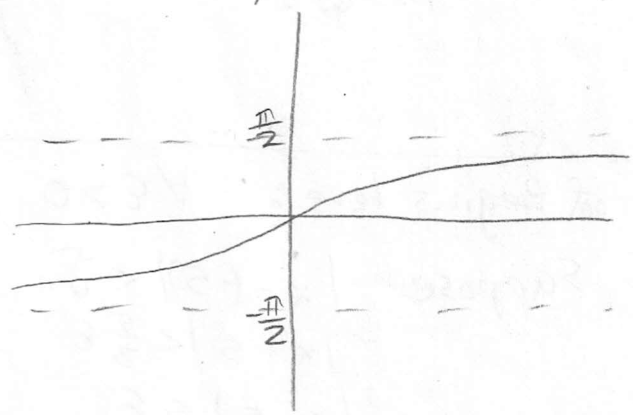
Review for Limits Test

1. Find and justify $\lim_{x \to a} \arctan\left(-e^{-x}\right)$ when…

$y = -e^{-x}$



$y = \arctan x$



$a = 0$

$$\lim_{x \to 0} \arctan(-e^{-x})$$
$$= \arctan\left(\lim_{x \to 0} -e^{-x}\right)$$
$$= \arctan(-1) = -\frac{\pi}{4}$$

$a = \infty$

$$\lim_{x \to \infty} \arctan(-e^{-x})$$
$$= \arctan\left(\lim_{x \to \infty} -e^{-x}\right)$$
$$= \arctan(0) = 0$$

$a = -\infty$

$$\lim_{x \to -\infty} \arctan(-e^{-x})$$
$$= \arctan\left(\lim_{x \to -\infty} -e^{-x}\right)$$
$$= \arctan(-\infty) = -\frac{\pi}{2}$$

2. Find analytically: $\lim_{\Delta x \to 0} \dfrac{\sin\left[\frac{\pi}{6} + \Delta x\right] - \frac{1}{2}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\sin\frac{\pi}{6}\cos\Delta x + \cos\frac{\pi}{6}\sin\Delta x - \frac{1}{2}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\frac{1}{2}\cos\Delta x + \frac{\sqrt{3}}{2}\sin\Delta x - \frac{1}{2}}{\Delta x}$$
$$= \frac{1}{2}\left[\lim_{\Delta x \to 0} \frac{\cos\Delta x - 1}{\Delta x}\right] + \frac{\sqrt{3}}{2}\left[\lim_{\Delta x \to 0} \frac{\sin\Delta x}{\Delta x}\right]$$
$$= \frac{1}{2}(0) + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}$$

3. Prove using an ε, δ proof that $\lim\limits_{x \to -5}\left(\frac{3}{5}x - 4\right) = -7$. First make a graph of the function, labelling ε and δ on your picture. Please show your work as you find δ, but clearly mark the point at which your proof begins by writing something like "proof begins here."

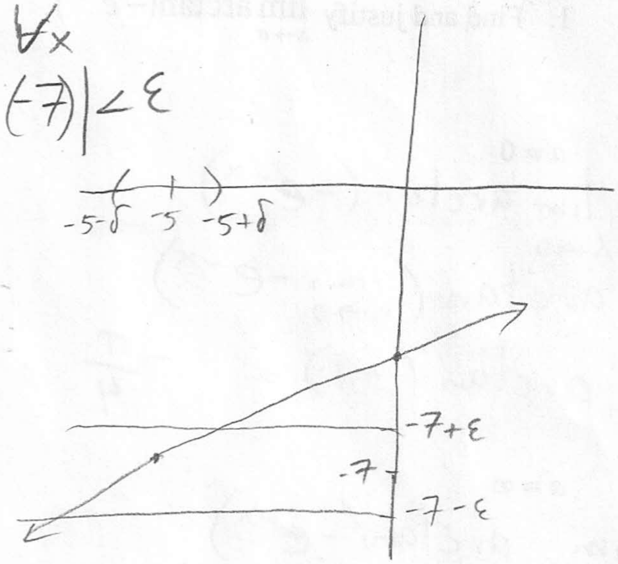
Show: $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x$
if $|x - (-5)| < \delta$, then $\left|\left(\frac{3}{5}x - 4\right) - (-7)\right| < \varepsilon$

Want: $\left|\left(\frac{3}{5}x - 4\right) - (-7)\right| < \varepsilon$

$\left|\frac{3}{5}x + 3\right| < \varepsilon$

$\frac{3}{5}|x + 5| < \varepsilon$

$|x - (-5)| < \frac{5}{3}\varepsilon$ Set $\delta = \frac{5}{3}\varepsilon$



Proof Begins here: $\forall \varepsilon > 0$, set $\delta = \frac{5}{3}\varepsilon$

Suppose $|x - (-5)| < \delta$

$|x + 5| < \frac{5}{3}\varepsilon$

$\frac{3}{5}|x + 5| < \varepsilon$

$\left|\frac{3}{5}x + 3\right| < \varepsilon$

$\left|\left(\frac{3}{5}x - 4\right) - (-7)\right| < \varepsilon$

4. Determine all values of the constant a such that $\lim\limits_{x \to 0} f(x)$ exists where

$$f(x) = \begin{cases} a^2 - 2, & x < 0 \\ \dfrac{ax}{\tan x}, & x \geq 0 \end{cases}$$

$\lim\limits_{x \to 0} f(x)$ exists iff $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (a^2 - 2) = a^2 - 2$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{ax}{\tan x} = a\left(\lim\limits_{x \to 0^+} \frac{x}{\tan x}\right) = a \cdot 1 = a$

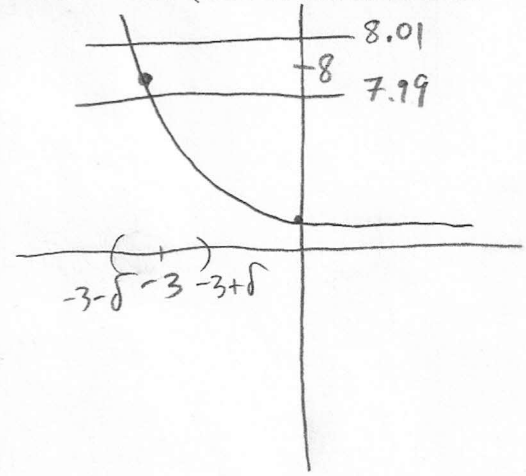
So $a^2 - 2 = a$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$\boxed{a = 2, -1}$

5. In proving that $\lim_{x\to -3} 2^{-x} = 8$, find the largest possible δ, given an $\varepsilon = .01$. (You do not need to do the proof; just find the δ.)



For what values of x is

$$7.99 < 2^{-x} < 8.01$$

$$\log_2(7.99) < -x < \log_2(8.01)$$

$$-\log_2(7.99) > x > -\log_2(8.01)$$

$$-\log_2(8.01) < x < -\log_2(7.99)$$

$$\underbrace{3-\log_2(8.01)}_{\text{tiny negative \#}} < x+3 < \underbrace{3-\log_2(7.99)}_{\text{tiny positive \#}}$$

Choose the smaller of $\log_2(8.01)-3$ and $3-\log_2(7.99)$

smaller, so

$$\delta = \log_2(8.01)-3$$

6. Prove $\lim_{x\to 0} |\tan x \sec x| \cos\left(\frac{1}{x}\right) = 0$. (Hint: use the Squeeze Theorem.)

$$-1 \le \cos\left(\tfrac{1}{x}\right) \le 1$$

$$-|\tan x \sec x| \le |\tan x \sec x| \cos\left(\tfrac{1}{x}\right) \le |\tan x \sec x|$$

$$\lim_{x\to 0} |\tan x \sec x| = \lim_{x\to 0} \left|\frac{\sin x}{\cos^2 x}\right| = 0$$

$$\lim_{x\to 0} -|\tan x \sec x| = 0 \text{ also.}$$

So by the Squeeze Theorem, $\lim_{x\to 0} |\tan x \sec x| \cos\left(\frac{1}{x}\right) = 0$

7. $\lim_{x\to 1^+} \frac{1-\sqrt[3]{x}}{x-1} = \lim_{x\to 1^+} \frac{-(\cancel{\sqrt[3]{x}-1})}{(\cancel{\sqrt[3]{x}-1})(x^{2/3}+\sqrt[3]{x}+1)}$

$$= \lim_{x\to 1^+} \frac{-1}{x^{2/3}+\sqrt[3]{x}+1} = \boxed{-\frac{1}{3}}$$

8. $\lim_{x\to 2}\frac{\ln\left((x-1)^{3x^2}\right)}{\ln(x-1)}=$ $\lim_{x\to 2}\frac{3x^2\,\cancel{\ln(x-1)}}{\cancel{\ln(x-1)}}=\lim_{x\to 2}3x^2=12$

9. Rederive $\lim_{x\to 0}\frac{\cos x-1}{x}\left(\frac{\cos x+1}{\cos x+1}\right)=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}$

$=\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{-\sin x}{\cos x+1}\right)$

$=1\cdot\left(\frac{0}{2}\right)=0$

10. Determine constants b and c so that $f(x)=\begin{cases} x+1 & 1<x<3 \\ x^2+bx+c & |x-2|\ge 1\end{cases}$ is continuous everywhere.

$f(x)=\begin{cases} x^2+bx+c, & x\le 1 \\ x+1, & 1<x<3 \\ x^2+bx+c, & x\ge 3\end{cases}$

@ $x=1$

$1+b+c=1+1$

$b+c=1$

@ $x=3$

$9+3b+c=4$

$3b+c=-5$

$b+c=1$

$2b=-6$

$\boxed{b=-3 \quad c=4}$

11. $\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2-x}=\left(\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}\right)$

$\lim_{x\to\infty}\frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$

$\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{\frac{1}{x^2}}\sqrt{x^2+x+1}+\sqrt{\frac{1}{x^2}}\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}}$

$=\frac{2}{\sqrt{1}+\sqrt{1}}=\frac{2}{2}=\boxed{1}$

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More Review for Limits Test

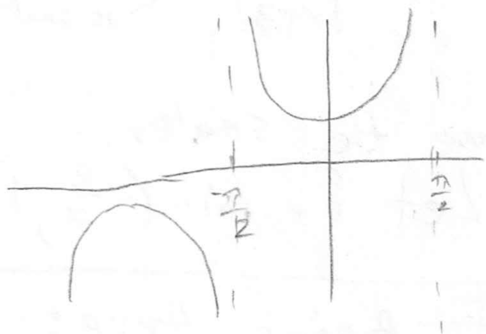
~~Review for Second Test on Limits / Differentiation~~

1. $\lim\limits_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\right) = \lim\limits_{x\to 1} \arcsin\left(\dfrac{1-\sqrt{x}}{1-x}\left(\dfrac{1+\sqrt{x}}{1+\sqrt{x}}\right)\right)$

$= \lim\limits_{x\to 1} \arcsin\left(\dfrac{1-x}{(1-x)(1+\sqrt{x})}\right) = \lim\limits_{x\to 1} \arcsin\left(\dfrac{1}{1+\sqrt{x}}\right)$

$= \arcsin\left(\dfrac{1}{2}\right) = \boxed{\dfrac{\pi}{6}}$

2. $\lim\limits_{x\to\left(\frac{-\pi}{2}\right)^-} \dfrac{\sec x}{x} = \dfrac{\lim\limits_{x\to -\frac{\pi}{2}^-} \sec x}{\lim\limits_{x\to -\frac{\pi}{2}^-} x}$



$= \dfrac{-\infty}{-\frac{\pi}{2}} = \infty$

3. $\lim\limits_{x\to 2} \dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\left(\dfrac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right) = \lim\limits_{x\to 2} \dfrac{(\sqrt{6-x}-2)(\sqrt{3-x}+1)}{(3-x)-1}$

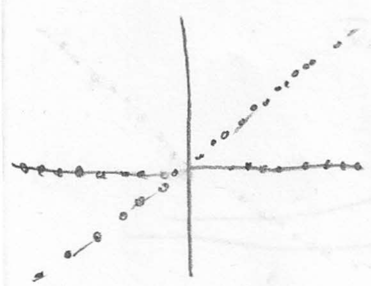
$= \lim\limits_{x\to 2} \dfrac{(\sqrt{6-x}-2)(\sqrt{3-x}+1)}{2-x}\left(\dfrac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\right)$

$= \lim\limits_{x\to 2} \dfrac{\boxed{(6-x)-4}(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)} = \lim\limits_{x\to 2} \dfrac{\sqrt{3-x}+1}{\sqrt{6-x}+2}$

$= \boxed{\dfrac{1}{2}}$

4. Let $f(x)=\begin{cases} x, & x \text{ is irrational} \\ 0, & x \text{ is rational} \end{cases}$. Show that $\lim_{x\to 0} f(x)=0$. (Hint: use the Squeeze Theorem.)

$f(x)$ $\quad -|x| \le f(x) \le |x|$ for all x



$\lim_{x\to 0} -|x| = 0$

$\lim_{x\to 0} |x| = 0$

So by the Squeeze Thm, $\lim_{x\to 0} f(x) = 0$

5. $\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)\left(\dfrac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}\right)$

$= \lim_{x\to\infty} \dfrac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x}$

$= \lim_{x\to\infty} \dfrac{x}{\sqrt{9x^2+x}+3x}$

$= \lim_{x\to\infty} \dfrac{\frac{x}{x}}{\sqrt{\frac{9x^2}{x^2}+\frac{x}{x^2}}+\frac{3x}{x}}$

$\to \lim_{x\to\infty} \dfrac{1}{\sqrt{9+\frac{1}{x}}+3}$

$= \dfrac{1}{6}$

6. Find and justify all horizontal and vertical asymptotes of $f(x)=\dfrac{x^3+1}{x^3+x}$. Make a graph.

$f(x) = \dfrac{x^3+1}{x^3+x} = \dfrac{x^3+1}{x(x^2+1)}$

$\lim_{x\to\infty} \dfrac{x^3+1}{x^3+x} = \lim_{x\to\infty} \dfrac{\frac{x^3}{x^3}+\frac{1}{x^3}}{\frac{x^3}{x^3}+\frac{x}{x^3}} = 1$ so $y=1$ is a horizontal asymptote for f

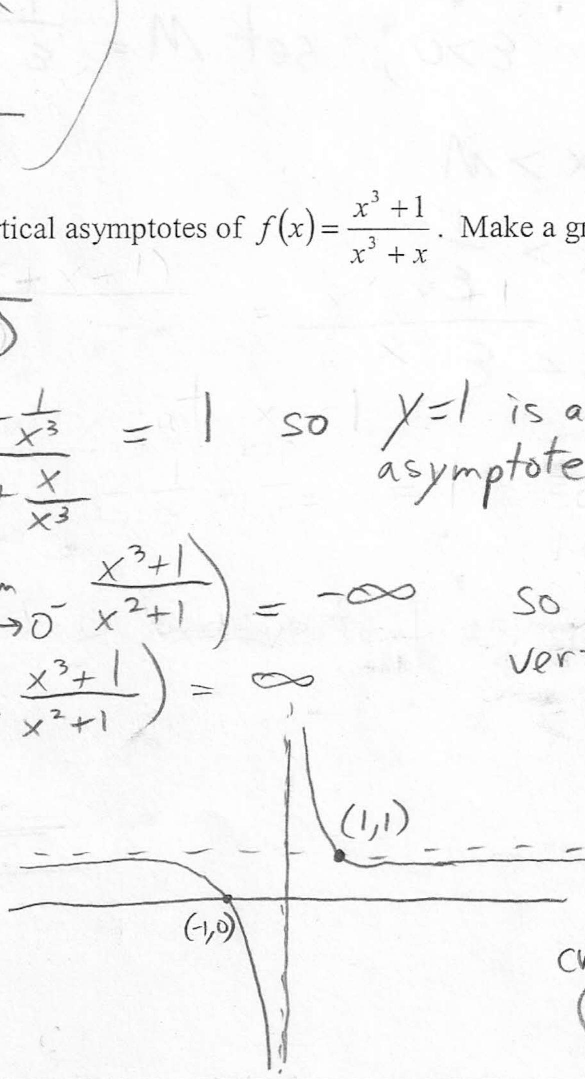
$\lim_{x\to 0^-} \dfrac{x^3+1}{x(x^2+1)} = \left(\lim_{x\to 0^-}\dfrac{1}{x}\right)\left(\lim_{x\to 0^-}\dfrac{x^3+1}{x^2+1}\right) = -\infty$

$\lim_{x\to 0^+} \dfrac{x^3+1}{x(x^2+1)} = \left(\lim_{x\to 0^+}\dfrac{1}{x}\right)\left(\lim_{x\to 0^+}\dfrac{x^3+1}{x^2+1}\right) = \infty$

So $x=0$ is a vertical asymptote for f

$$\begin{array}{r} 1 \\ x^3+x \,\big)\, x^3+0x^2+0x+1 \\ \underline{x^3 \qquad\quad +x} \\ -x+1 \end{array}$$

$\boxed{f(x) = 1 + \dfrac{-x+1}{x^3+x}}$



crosses $y=1$ @ $(1,1)$

7. Find and justify all horizontal and vertical asymptotes of $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$. Make a graph.

what happens @ $x=\frac{5}{3}$?

$\boxed{f(0)=\frac{-1}{5}}$

$$\lim_{x\to\frac{5}{3}^-} f(x)$$

$$=\lim_{x\to\frac{5}{3}^-}\frac{\sqrt{2x^2+1}}{3\left(x-\frac{5}{3}\right)}=\left(\lim_{x\to\frac{5}{3}^-}\sqrt{2x^2+1}\right)\left(\lim_{x\to\frac{5}{3}^-}\frac{1}{3\left(x-\frac{5}{3}\right)}\right)$$

$$=\left(\sqrt{\frac{59}{9}}\right)(-\infty)=-\infty$$

So vert asymptote @ $x=5/3$

$$\lim_{x\to\frac{5}{3}^+} f(x)=\left(\lim_{x\to\frac{5}{3}^+}\sqrt{2x^2+1}\right)\left(\lim_{x\to\frac{5}{3}^+}\frac{1}{3\left(x-\frac{5}{3}\right)}\right)=\sqrt{\frac{59}{9}}\cdot\infty=\infty$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)=\lim_{x\to\infty}\frac{\sqrt{\frac{1}{x^2}}\sqrt{2x^2+1}}{3-\frac{5}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}=\frac{\sqrt{2}}{3}$$

so horizontal asymptote @ $y=\frac{\sqrt{2}}{3}$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)=\lim_{x\to-\infty}\frac{-\sqrt{\frac{1}{x^2}}\sqrt{2x^2+1}}{3-\frac{5}{x}}$$

$$=\lim_{x\to-\infty}\frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}=\frac{-\sqrt{2}}{3}$$

so horizontal asymptote @ $y=\frac{-\sqrt{2}}{3}$

$$\frac{\sqrt{2x^2+1}}{3x-5}=\frac{\pm\sqrt{2}}{3}$$

$$3\sqrt{2x^2+1}=\pm\sqrt{2}(3x-5)$$

$$9(2x^2+1)=2(9x^2-30x+25)$$

$$18x^2+9=18x^2-60x+50$$

$$60x=41$$

at $x=\frac{41}{60}$ f crosses its horiz asymptote

3