Magnet Analysis 1A
Rose 4/22/2008

Name KEY

Anti-differentiation by the Method of Substitution

$\frac{d}{dx}\left[\sin\left(x^2\right)\right] = \cos(x^2)(2x)$

So... $\int \cos\left(x^2\right)2x\,dx = \sin(x^2) + C$

$\frac{d}{dx}\left[\sqrt{5x^3+1}\right] = \frac{1}{2}(5x^3+1)^{-\frac{1}{2}}(15x^2) = \frac{15x^2}{2\sqrt{5x^3+1}}$

So... $\int \frac{x^2}{\sqrt{5x^3+1}}dx = \frac{2}{15}\int \frac{15}{2}\frac{x^2}{2\sqrt{5x^3+1}}dx = \frac{2}{15}\sqrt{5x^3+1} + C$

$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x)$

Make the change of variable substitution, $u = g(x)$

So... $\int F'(g(x))g'(x)dx = \int F'(u)\,du = F(u)+C = F(g(x))+C$

1. $\int x^3\cos\left(x^4+2\right)dx = \frac{1}{4}\int 4x^3\cos(x^4+2)\,dx$

$u = x^4+2$
$du = 4x^3\,dx$

$= \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2)+C$

2. $\int \sin^3 x\cos x\,dx$

$u = \sin x$
$du = \cos x\,dx$

$\int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$

3. $\int \sqrt{2x+1}\, dx =$

Method #1 $u = 2x+1$, $du = 2dx$

$\frac{1}{2}\int 2\sqrt{2x+1}\, dx$

$= \frac{1}{2}\int u^{\frac{1}{2}} du = \frac{1}{2}\cdot\frac{2}{3} u^{\frac{3}{2}} + C$

$= \boxed{\frac{1}{3}(2x+1)^{3/2} + C}$

Method #2 $u = 2x+1$, $du = 2dx$, $\frac{du}{2} = dx$

$\int u^{\frac{1}{2}} \cdot \frac{du}{2}$

$= \frac{1}{2}\int u^{\frac{1}{2}} du$

etc.

Method #3 $u = 2x+1$, $\frac{du}{dx} = 2$, $du = 2dx$

etc.

4. $\int \frac{x}{\sqrt{1-4x^2}}\, dx =$ $-\frac{1}{8}\int \frac{-8x}{\sqrt{1-4x^2}}\, dx$

$u = 1-4x^2$
$du = -8x\, dx$

$= -\frac{1}{8}\int u^{-\frac{1}{2}} du = -\frac{1}{8}\cdot 2u^{\frac{1}{2}} + C$

$= \boxed{-\frac{1}{4}\sqrt{1-4x^2} + C}$

5. $\int \cos(5x) dx =$ $\frac{1}{5}\int 5\cos(5x)\, dx$

$u = 5x$
$du = 5dx$

$= \boxed{\frac{1}{5}\sin 5x + C}$

6. $\int \tan x\, dx =$ $\int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u}$

$u = \cos x$
$du = -\sin x\, dx$

$= -\ln|u| + C = -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \boxed{\ln|\sec x| + C}$

7. $\int \cos^3 x\, dx$ $= \int \cos x \cos^2 x\, dx = \int \cos x(1-\sin^2 x)\, dx$

$= \int(\cos x - \sin^2 x \cos x)\, dx$

$= \int \cos x\, dx - \int u^2 du = \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$

$u = \sin x$
$du = \cos x\, dx$

8. $\int \frac{x^2}{x^3+5}\, dx$ $= \frac{1}{3}\int \frac{3x^2}{x^3+5} dx$

$u = x^3+5$
$du = 3x^2 dx$

$= \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|x^3+5| + C$

9. $\int \frac{x^2+2}{x^2+1}\,dx = \int \frac{x^2+1+1}{x^2+1}\,dx = \int \left(1 + \frac{1}{x^2+1}\right)dx$

$= \boxed{x + \arctan x + C}$

10. $\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\ln x)^2 + C}$

$u = \ln x$
$du = \frac{1}{x}\,dx$

11. $\int \frac{dx}{x\sqrt{\ln x}} = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C$

$= \boxed{2\sqrt{\ln x} + C}$

$u = \ln x$
$du = \frac{1}{x}\,dx$

12. $\int \cos x \cos(\sin x)\,dx = \int \cos u\,du$

$= \sin u + C = \boxed{\sin(\sin x) + C}$

$u = \sin x$
$du = \cos x\,dx$

13. $\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C$

$= \boxed{\ln|e^x+1| + C}$

$u = e^x + 1$
$du = e^x\,dx$

14. $\int \frac{\arctan x}{1+x^2}\,dx = \int u\,du$

$= \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\arctan^2 x + C}$

$u = \arctan x$
$du = \frac{1}{x^2+1}$

15. $\int 5^x\,dx = \frac{1}{\ln 5}\int \ln 5 \cdot 5^x\,dx = \frac{1}{\ln 5}\int du$ $\quad u = 5^x$, $du = \ln 5 \cdot 5^x\,dx$

$= \frac{1}{\ln 5}u + C = \boxed{\frac{1}{\ln 5}\cdot 5^x + C}$

16. $\int \tan^2 x \sec^2 x\,dx$ $\quad u = \tan x$, $du = \sec^2 x\,dx$

$\int u^2\,du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\tan^3 x + C}$

17. $\int \frac{10\sqrt{x}}{\left(1+x^{3/2}\right)^2} = 10\cdot\frac{2}{3}\int \frac{\frac{3}{2}\sqrt{x}}{(1+x^{3/2})^2}\,dx$ $\quad u = 1 + x^{3/2}$, $du = \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{20}{3}\int u^{-2}\,du = \frac{20}{3}(-1)\,u^{-1} + C$

$= \boxed{\frac{-20}{3(1+x^{3/2})} + C}$

18. $\int \sin(10x)e^{\sin^2(5x)}\,dx$ $\quad u = \sin^2(5x)$, $du = 2\sin(5x)\cos(5x)\,dx$, $du = \sin(10x)\,dx$

$= \int e^u\,du = e^u + C$

$= \boxed{e^{\sin^2(5x)} + C}$

19. $\int \frac{x^3+1}{x^2+4}\,dx = \int\left(x - \frac{4x}{x^2+4} + \frac{1}{x^2+4}\right)dx$

$$\begin{array}{r|l} x^2+4 & \overline{x^3+0x^2+0x+1} \quad x \\ & \underline{x^3 \qquad\quad +4x} \\ & \qquad\qquad -4x+1 \end{array}$$

$\int x\,dx - 2\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$

$\frac{1}{2}x^2 - 2\ln|x^2+4| + \int \frac{1}{4(\frac{x^2}{4}+1)}\,dx$

$+ \frac{1}{4}\int \frac{1}{(\frac{x}{2})^2+1}\,dx$ $\quad u = \frac{x}{2}$, $du = \frac{1}{2}\,dx$

$+ \frac{1}{2}\int \frac{du}{u^2+1}$

$\boxed{\frac{1}{2}x^2 - 2\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$

20. $\int \frac{1}{\sqrt{1-4x^2}}\,dx$

$u = 2x$

$du = 2\,dx$

$\int \frac{1}{\sqrt{1-(2x)^2}}\,dx$

$\frac{1}{2}\int \frac{2\,dx}{\sqrt{1-(2x)^2}}$

$\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$

$= \frac{1}{2}\arcsin u + C$

$= \frac{1}{2}\arcsin(2x) + C$

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21. $\int \frac{x^4}{x^2+1}\,dx$

$$\begin{array}{r} x^2 - 1 \\ x^2+1 \,\big)\, x^4 \\ \underline{x^4 + x^2} \\ -x^2 \\ \underline{-x^2 - 1} \\ 1 \end{array}$$

$\int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx$

$= \frac{x^3}{3} - x + \arctan x + C$

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22. $\int \frac{1}{x\sqrt{x^2-16}}\,dx$

$u = \frac{x}{4}$

$du = \frac{1}{4}\,dx$

$\int \frac{dx}{x\sqrt{16\left(\frac{x^2}{16}-1\right)}}$

$\frac{1}{4}\int \frac{dx}{x\sqrt{\left(\frac{x}{4}\right)^2-1}}$

$\frac{1}{4}\int \frac{\left(\frac{1}{4}dx\right)}{\left(\frac{x}{4}\right)\sqrt{\left(\frac{x}{4}\right)^2-1}}$

$\frac{1}{4}\int \frac{du}{u\sqrt{u^2-1}}$

$= \frac{1}{4}\operatorname{arcsec}|u| + C \quad = \frac{1}{4}\operatorname{arcsec}\left|\frac{x}{4}\right| + C$

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23. $\int \frac{2}{\sqrt{-x^2+4x}}\,dx$

$u = x - 2$

$du = dx$

$w = \frac{u}{2}$

$dw = \frac{1}{2}\,du$

$2\int \frac{dx}{\sqrt{-(x^2-4x+4)+4}}$

$2\int \frac{dx}{\sqrt{4-(x-2)^2}}$

$2\int \frac{du}{\sqrt{4-u^2}}$

$2\int \frac{du}{\sqrt{4\left(1-\frac{u^2}{4}\right)}}$

$\int \frac{du}{\sqrt{1-\left(\frac{u}{2}\right)^2}}$

$2\int \frac{\frac{1}{2}du}{\sqrt{1-\left(\frac{u}{2}\right)^2}}$

$2\int \frac{dw}{\sqrt{1-w^2}}$

$= 2\arcsin w + C$

$2\arcsin \frac{u}{2} + C$

$2\arcsin\left(\frac{x-2}{2}\right) + C$

24. $\displaystyle\int \frac{1}{x\sqrt{x^4-4}}\,dx$

$$\int \frac{1}{x\sqrt{(x^2)^2-4}}\,dx$$

$$\frac{1}{2}\int \frac{2x\,dx}{x^2\sqrt{(x^2)^2-4}}$$

$$\frac{1}{2}\int \frac{du}{u\sqrt{u^2-4}}$$

$$\frac{1}{2}\int \frac{du}{u\sqrt{4\left(\frac{u^2}{4}-1\right)}}$$

$$\frac{1}{4}\int \frac{du}{u\sqrt{\left(\frac{u}{2}\right)^2-1}}$$

$$\frac{1}{4}\int \frac{\frac{1}{2}\,du}{\frac{u}{2}\sqrt{\left(\frac{u}{2}\right)^2-1}}$$

$$\frac{1}{4}\int \frac{dw}{w\sqrt{w^2-1}}$$

$$= \frac{1}{4}\operatorname{arcsec}|w| + C$$

$$\frac{1}{4}\operatorname{arcsec}\left|\frac{u}{2}\right| + C$$

$$\frac{1}{4}\operatorname{arcsec}\left|\frac{x^2}{2}\right| + C$$

$$\frac{1}{4}\operatorname{arcsec}\left(\frac{x^2}{2}\right) + C$$

$u = x^2$

$du = 2x\,dx$

$w = \frac{u}{2}$

$dw = \frac{1}{2}\,du$

25. $\displaystyle\int \frac{dx}{x^2-4x+7}$

$$\int \frac{dx}{(x^2-4x+4)+7-4}$$

$$\int \frac{dx}{(x-2)^2+3}$$

$$\int \frac{du}{u^2+3}$$

$$\int \frac{du}{3\left(\frac{u^2}{3}+1\right)}$$

$$\frac{1}{3}\int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$$

$$\frac{1}{3}\cdot\sqrt{3}\int \frac{\frac{1}{\sqrt{3}}\,du}{\left(\frac{u}{\sqrt{3}}\right)+1}$$

$$\frac{\sqrt{3}}{3}\int \frac{dw}{w^2+1}$$

$$= \frac{\sqrt{3}}{3}\arctan w + C$$

$$= \frac{\sqrt{3}}{3}\arctan\left(\frac{u}{\sqrt{3}}\right) + C = \frac{\sqrt{3}}{3}\arctan\left(\frac{x-2}{\sqrt{3}}\right) + C$$

$u = x-2$

$du = dx$

$w = \frac{u}{\sqrt{3}}$

$dw = \frac{1}{\sqrt{3}}\,du$

26. $\displaystyle\int \frac{dx}{3x^2+12x+25}$

$$\int \frac{dx}{3(x^2+4x+4)+25-12}$$

$$\int \frac{dx}{3(x+2)^2+13}$$

$$\int \frac{dx}{\left[\sqrt{3}(x+2)\right]^2+13}$$

$$\frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\,dx}{\left[\sqrt{3}(x+2)\right]^2+13}$$

$$\frac{1}{\sqrt{3}}\int \frac{du}{u^2+13}$$

$$\frac{1}{\sqrt{3}}\int \frac{du}{13\left(\frac{u^2}{13}+1\right)}$$

$$\frac{1}{13\sqrt{3}}\int \frac{du}{\left(\frac{u}{\sqrt{13}}\right)^2+1}$$

$$\frac{\sqrt{13}}{13\sqrt{3}}\int \frac{\frac{1}{\sqrt{13}}\,du}{\left(\frac{u}{\sqrt{13}}\right)^2+1}$$

$$\frac{1}{\sqrt{39}}\int \frac{dw}{w^2+1}$$

$$\arctan w + C$$

$$\arctan\left(\frac{u}{\sqrt{13}}\right) + C$$

$$\arctan\left(\frac{\sqrt{3}}{\sqrt{13}}(x+2)\right) + C$$

$u = \sqrt{3}(x+2)$

$du = \sqrt{3}\,dx$

$w = \frac{u}{\sqrt{13}}$

$dw = \frac{1}{\sqrt{13}}\,du$

Other Trigonometric Anti-differentiation Techniques

8. $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx = \frac{1}{2}\int dx - \frac{1}{4}\int 2\cos 2x\, dx$

$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$

9. $\int \cos^4 x\, dx$

$= \int (\cos^2 x)^2\, dx$

$\int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$\frac{1}{4}\int (1 + 2\cos 2x + \cos^2(2x))\, dx$

$= \frac{1}{4}\int dx + \frac{1}{4}\int 2\cos 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}\, dx$

$\rightarrow \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x\, dx$

$\frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{1}{32}\int 4\cos 4x\, dx$

$\boxed{\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C}$

10. $\int \sec x\, dx$ $\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$\int \frac{du}{u} = \boxed{\ln|\sec x + \tan x| + C}$

11. $\int \csc x\, dx$ $\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx$

$u = \csc x + \cot x$

$du = -\csc x \cot x - \cot^2 x$

$= -\int \frac{du}{u} = -\ln|\csc x + \cot x| + C$

$= \ln\left|\frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}\right| = \boxed{\ln\left|\frac{\sin x}{1+\cos x}\right| + C}$

"Miscellaneous Substitution"

12. $\int \frac{1}{1+\sqrt{3x}}\,dx = \int \frac{1}{u}\left(\frac{2\sqrt{3x}}{3}\,du\right)$

$u = 1+\sqrt{3x}$

$\sqrt{3x} = u-1$

$du = \frac{3}{2\sqrt{3x}}\,dx$

$dx = \frac{2\sqrt{3x}}{3}\,du$

$= \frac{2}{3}\int \frac{\sqrt{3x}}{u}\,du$

$= \frac{2}{3}\int \frac{u-1}{u}\,du$

$= \frac{2}{3}\int \left(1-\frac{1}{u}\right)du$

$\frac{2}{3}u - \frac{2}{3}\ln|u| + C$

$\boxed{\frac{2}{3}(1+\sqrt{3x}) - \frac{2}{3}\ln\left|1+\sqrt{3x}\right| + C}$

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13. $\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx = \int \frac{u+1}{u}\left(3x^{2/3}\,du\right)$

$u = \sqrt[3]{x}-1$

$\sqrt[3]{x} = u+1$

$du = \frac{1}{3x^{2/3}}\,dx$

$dx = 3x^{2/3}\,du$

$= 3\int \frac{u+1}{u}(u+1)^2\,du$

$= 3\int \frac{(u+1)^3}{u}\,du$

$= 3\int \frac{u^3+3u^2+3u+1}{u}\,du$

$3\int\left(u^2+3u+3+\frac{1}{u}\right)du$

$= u^3 + \frac{3}{2}u^2 + 3u + \ln|u| + C$

$= \boxed{(\sqrt[3]{x}-1)^3 + \frac{3}{2}(\sqrt[3]{x}-1)^2 + 3(\sqrt[3]{x}-1) + \ln|\sqrt[3]{x}-1| + C}$

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14. $\int \frac{\sqrt{x-2}}{x+1}\,dx = \int \frac{u}{u^2+3}(2u\,du)$

$u = \sqrt{x-2}$

$u^2 = x-2$

$x = u^2+2$

$du = \frac{1}{2\sqrt{x-2}}\,dx$

$dx = 2\sqrt{x-2}\,du$

$dx = 2u\,du$

$= 2\int \frac{u^2}{u^2+3}\,du$

$= 2\int \frac{u^2+3-3}{u^2+3}\,du$

$= 2\int du - 6\int \frac{du}{u^2+3}$

$= 2u - 6\int \frac{du}{3\left(\frac{u^2}{3}+1\right)}$

$2u - 2\int \frac{du}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$

$w = \frac{u}{\sqrt{3}}$

$dw = \frac{1}{\sqrt{3}}\,du$

$2u - 2\sqrt{3}\int \frac{\frac{1}{\sqrt{3}}\,du}{\left(\frac{u}{\sqrt{3}}\right)^2+1}$

$2u - 2\sqrt{3}\int \frac{dw}{w^2+1}$

$2u - 2\sqrt{3}\arctan w + C$

$\boxed{2\sqrt{x-2} - 2\sqrt{3}\arctan\sqrt{\frac{x-2}{3}} + C}$

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15. $\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2u\,du}{u(1+u^2)}$

$u = \sqrt{x}$

$x = u^2$

$du = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,du$

$dx = 2u\,du$

$= 2\int \frac{du}{1+u^2}$

$= 2\arctan u + C$

$= \boxed{2\arctan\sqrt{x} + C}$