Convex Optimization

Stephen Boyd

Electrical Engineering
Computer Science
Management Science and Engineering
Institute for Computational Mathematics & Engineering
Stanford University

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Outline

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Optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & g_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is (vector) variable to be chosen
- f_0 is the *objective function*, to be minimized
- $f_1, \ldots, f_m$ are the *inequality constraint functions*
- $g_1, \ldots, g_p$ are the *equality constraint functions*

- variations: maximize objective, multiple objectives, ...

Finding good (or best) actions

- x represents some *action*, *e.g.*,
  - trades in a portfolio
  - airplane control surface deflections
  - schedule or assignment
  - resource allocation
  - transmitted signal
- constraints limit actions or impose conditions on outcome
- the smaller the objective $f_0(x)$, the better
  - total cost (or negative profit)
  - deviation from desired or target outcome
  - risk
  - fuel use

Engineering design

- x represents a design (of a circuit, device, structure, ...)
- constraints come from
  - manufacturing process
  - performance requirements
- objective $f_0(x)$ is combination of cost, weight, power, ...

Finding good models

- x represents the *parameters* in a model
- constraints impose requirements on model parameters (*e.g.*, nonnegativity)
- objective $f_0(x)$ is the prediction error on some observed data (and possibly a term that penalizes model complexity)

Inversion

- x is something we want to estimate/reconstruct, given some measurement y
- constraints come from prior knowledge about x
- objective $f_0(x)$ measures deviation between predicted and actual measurements

Worst-case analysis (pessimization)

- variables are actions or parameters out of our control
  (and possibly under the control of an adversary)
- constraints limit the possible values of the parameters
- minimizing $-f_0(x)$ finds *worst possible parameter values*

- if the worst possible value of $f_0(x)$ is tolerable, you're OK
- it's good to know what the worst possible scenario can be

Optimization-based models

- model an entity as taking actions that solve an optimization problem
  - an individual makes choices that maximize expected utility
  - an organism acts to maximize its reproductive success
  - reaction rates in a cell maximize growth
  - currents in a circuit minimize total power

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- (except the last) these are *very crude* models
- and yet, they often work very well

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- **the bad news**: most optimization problems are *intractable*
  i.e., *we cannot solve them*
- **an exception**: *convex optimization problems are tractable*
  i.e., we (generally) *can* solve them

Outline

Convex optimization

convex optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- equality constraints are linear
- $f_0, \ldots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

  i.e., f_i have nonnegative (upward) curvature

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  - get **global solution** (and optimality certificate)
  - polynomial complexity
- conceptual unification of many methods
- **lots of applications** (many more than previously thought)

Application areas

- machine learning, statistics
- finance
- supply chain, revenue management, advertising
- control
- signal and image processing, vision
- networking
- circuit design
- combinatorial optimization
- quantum mechanics
- flux-based analysis

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- if you succeed, you can (usually) solve it (numerically)

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- some tricks:
  - change of variables
  - approximation of true objective, constraints
  - *relaxation*: ignore terms or constraints you can't handle

Outline

Radiation treatment planning

- radiation beams with intensities x_j are directed at patient
- radiation dose y_i received in voxel i
- $y = Ax$
- $A \in \mathbf{R}^{m \times n}$ comes from beam geometry, physics
- goal is to choose x to deliver prescribed radiation dose d_i
  - $d_i = 0$ for non-tumor voxels
  - $d_i > 0$ for tumor voxels
- $y = d$ not possible, so we'll need to compromise
- typical problem has $n = 10^3$ beams, $m = 10^6$ voxels

Radiation treatment planning via convex optimization

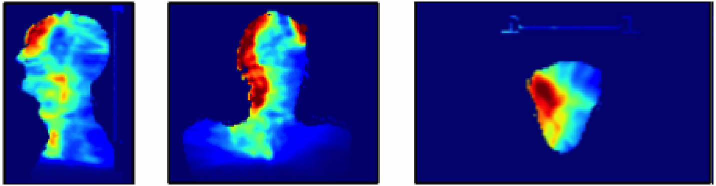
$$\begin{array}{ll} \text{minimize} & \sum_i f_i(y_i) \\ \text{subject to} & x \geq 0, \quad y = Ax \end{array}$$

- variables $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$
- objective terms are

$$f_i(y_i) = w_i^{\text{over}}(y_i - d_i)_+ + w_i^{\text{under}}(d_i - y_i)_+$$

- w_i^{over} and w_i^{under} are positive weights
- *i.e.*, we charge linearly for over- and under-dosing
- a convex optimization problem

Example



- real patient case with $n = 360$ beams, $m = 360000$ voxels
- optimization-based plan essentially the same as plan used

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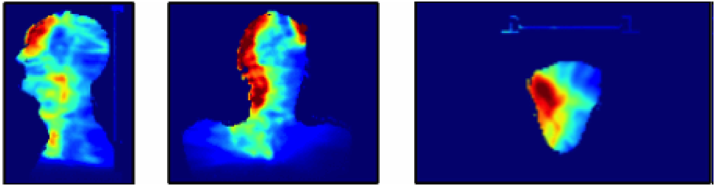

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- optimization-based plan essentially the same as plan used
- (but we computed the plan in a few seconds, not many hours)

Image in-painting

- guess pixel values in obscured/corrupted parts of image
- *total variation in-painting*: choose pixel values $x_{ij} \in \mathbf{R}^3$ to minimize *total variation*

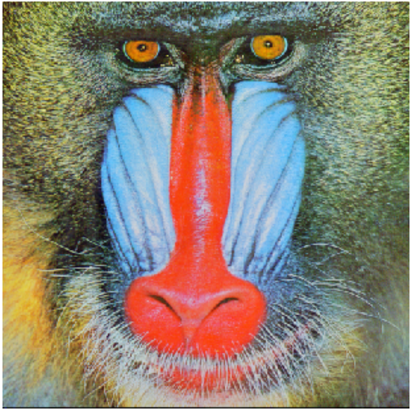
$$\mathrm{TV}(x) = \sum_{ij} \left\| \begin{bmatrix} x_{i+1,j} - x_{ij} \\ x_{i,j+1} - x_{ij} \end{bmatrix} \right\|_2$$

- a convex problem

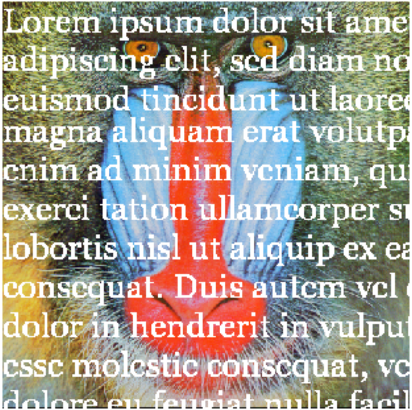
Example

512 $\times$ 512 color image ($n \approx 800000$ variables)

Original



Corrupted



Example

Original

Recovered

Support vector machine

- goal: predict a Boolean outcome from a set of n features
  - *e.g.*, spam filter, fraud detection, customer purchase

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- linear predictor: $\hat{b} = \mathrm{sign}(w^T a - v)$
  - $w \in \mathbf{R}^n$ is weight vector; $v \in \mathbf{R}$ is threshold

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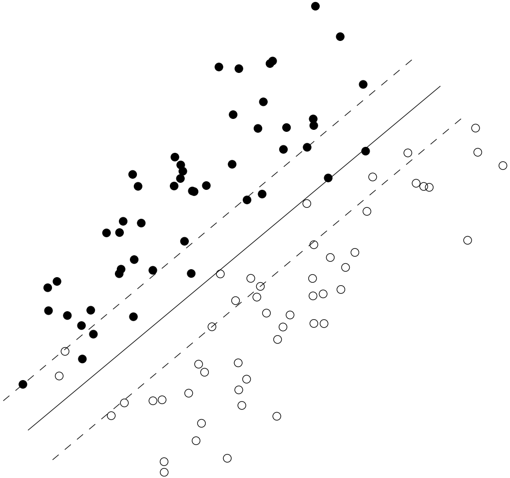
- SVM: choose w, v to minimize (convex) objective

$$(1/m) \sum_{i=1}^{m} \left(1 - b_i(w^T a_i - v)\right)_+ + (\lambda/2)\|w\|_2^2$$

where $\lambda > 0$ is parameter

SVM

$$w^T z - v = 0 \text{ (solid)}; \quad |w^T z - v| = 1 \text{ (dashed)}$$



Sparsity via ℓ_1 regularization

- adding ℓ_1-norm regularization

$$\lambda \|x\|_1 = \lambda(|x_1| + |x_2| + \cdots + |x_n|)$$

  to objective results in **sparse** x

- $\lambda > 0$ controls trade-off of sparsity versus main objective
- **preserves convexity, hence tractability**
- used for many years, in many fields
  - sparse design
  - feature selection in machine learning (lasso, SVM, ...)
  - total variation reconstruction in signal processing
  - compressed sensing

Lasso

- regression problem with ℓ_1 regularization:

$$\text{minimize} \quad (1/2)\|Ax - b\|_2^2 + \lambda\|x\|_1$$

  with $A \in \mathbf{R}^{m \times n}$

- useful even when $n \gg m$ (!!); does **feature selection**

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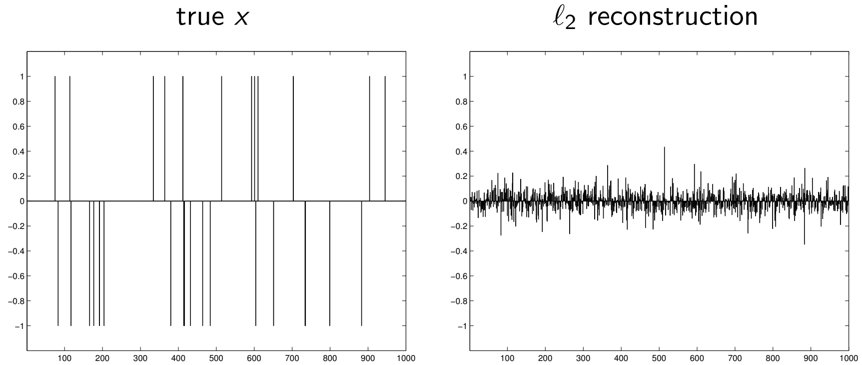
- lasso, ridge regression have **same computational cost**

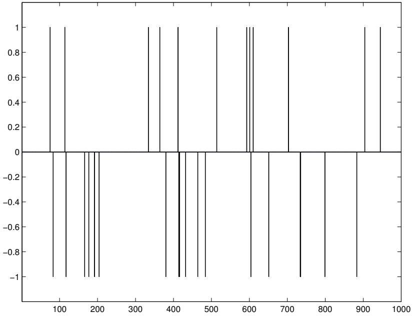
Example

- $m = 200$ examples, $n = 1000$ features
- examples are noisy linear measurements of true x
- true x is sparse (30 nonzeros)

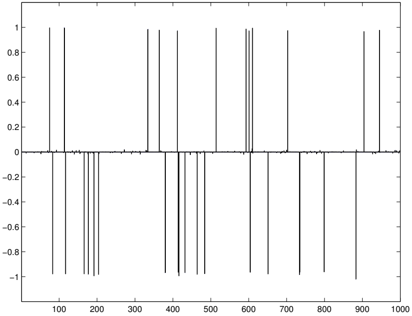


Example

true x



ℓ_1 (lasso) reconstruction



State of the art — Medium scale solvers

- 1000s–10000s variables, constraints
- reliably solved by interior-point methods on single machine
- exploit problem sparsity
- not quite a technology, but getting there

State of the art — Modeling languages

- (new) high level language support for convex optimization
  - describe problem in high level language
  - description is automatically transformed to cone problem
  - solved by standard solver, transformed back to original form

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- enables rapid prototyping (for small and medium problems)
- ideal for teaching (can do a lot with short scripts)

CVXPY

- parser/solver written in Python (S. Diamond, 2013)
- SVM: minimize

$$(1/m)\sum_{i=1}^{m}\left(1-b_i(w^Ta_i-v)\right)_+ + (\lambda/2)\|w\|_2^2$$

- CVXPY specification:

```
w = Variable(n); v = Variable()   # weight, offset
losses = pos(1-mul_elemwise(b, A*w-v))
L = (1/m)*sum_entries(losses)   # avg. loss
obj = Minimize(L+(lambda/2)*sum_squares(w))
Problem(obj).solve()
```

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Large-scale distributed optimization

- *large-scale* optimization problems arise in many applications
  - machine learning/statistics with huge datasets
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  - image, video processing

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- we'll use *distributed optimization*
  - split variables/constraints/objective terms among a set of *agents/processors/devices*
  - agents coordinate to solve large problem, by passing relatively small messages
  - can target modern large-scale computing platforms
  - long history, going back to 1950s

Consensus optimization

- want to solve problem with N objective terms

$$\text{minimize} \quad \sum_{i=1}^{N} f_i(x)$$

e.g., f_i is the loss function for ith block of training data

- consensus form:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} f_i(x_i) \\ \text{subject to} & x_i - z = 0 \end{array}$$

  - x_i are **local variables**
  - z is the **global variable**
  - $x_i - z = 0$ are **consistency** or **consensus** constraints

Consensus optimization via ADMM

with $\overline{x}^k = (1/N)\sum_{i=1}^N x_i^k$ (average over local variables)

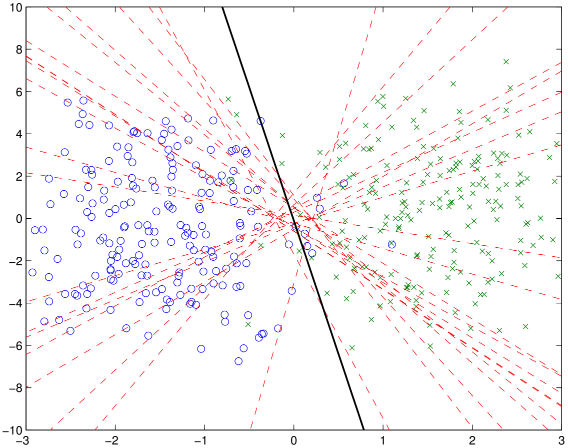
$$\begin{array}{rcl} x_i^{k+1} & := & \operatorname*{argmin}_{x_i} \left(f_i(x_i) + (\rho/2)\|x_i - \overline{x}^k + u_i^k\|_2^2 \right) \\ u_i^{k+1} & := & u_i^k + (x_i^{k+1} - \overline{x}^{k+1}) \end{array}$$

- get **global** minimum, under very general conditions
- u^k is running sum of inconsistencies (PI control)
- minimizations carried out independently and in parallel
- coordination is via averaging of local variables x_i

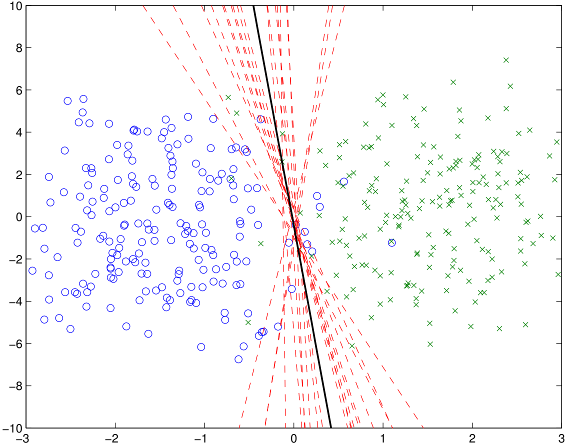
Example — Consensus SVM

- baby problem with $n = 2$, $m = 400$ to illustrate
- examples split into $N = 20$ groups, in worst possible way: each group contains only positive or negative examples

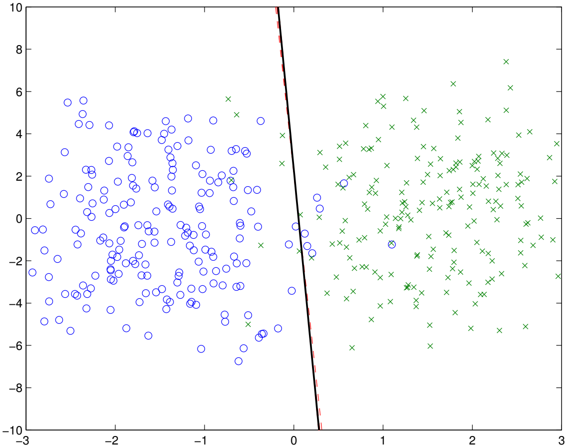
Iteration 1



Iteration 5



Iteration 40



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Summary

- convex optimization problems **arise in many applications**
- convex optimization problems **can be solved effectively**
  - small problems at microsecond/millisecond time scales
  - medium-scale problems using general purpose methods
  - arbitrary-scale problems using distributed optimization
- high level language support makes prototyping easy

References

many researchers have worked on the topics covered

- *Convex Optimization* (Boyd & Vandenberghe)
- *CVXPY: A Pyhton-embedded modeling language for convex optimization* (Diamond & Boyd)
- *Distributed optimization and statistical learning via the alternating direction method of multipliers*
(Boyd, Parikh, Chu, Peleato, & Eckstein)

all available (with code) on-line