FAIRPORTFOLIO ~ A SIMPLE AND STABLE OPTIMAL PORTFOLIO

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ABSTRACT. This paper describes a method of creating a stable, optimal portfolio based on an arbitrary list of assets. This method employs multiple simplifications and stabilisations that have lead

The model was practically tested vs other existing models and resulted in significantly better results (the test data is to be provided later or left as an exercise to the interested). The result is achieved via simplications of multiple types.

1. INTRODUCTION

Portfolio optimization is about finding weights for each asset that one wants to include in a portfolio. One of longest standing theoretical results in this field is due to Markiwitz[1], which minimizes the portfolio variance using the expected return and variance of each asset.

In this paper, we are also ultimately going to minimize portfolio risk after applying the following 3 simplifications:

(1) consider only 2nd moments

(2) homogenize the 2nd moments

(3) algebraicly reduce dimension

We assume that we have chosen a list of n assets that we want to invest into. Us choosing these assets implies that we believe that each of these assets has a long term positive return.

2. ONLY 2ND MOMENTS

The distribution of returns of any asset can be fully described using moments.

The 1st moment is the expected return and is the hardest to predict. An investor should only choose assets that are believed to have positive 1st moment.

The 2nd moments are the covariances between the assets, σ_{ij}, including the variances of each asset σ_{ii}.

Moments of degree 3 and higher have ever larger relative errors of measurement due to financial data being highly noisy.

For numerical and qualitative stability, we should not include high error parameters in our optimization. This leaves us only with 2nd moments. Assuming that an investor has chosen assets with positive 1st moment, optimizing based on 2nd moments only is most stable.

3. Homogeneous 2nd moments

To further stablilize our portfolio, we homogenize the variances across all assets and time:

$$\sigma_{ii} = \sigma$$

An investor could choose any model to predict the variance of an asset for the following time period and pre-scale the asset such that all assets have approx. the same variance (or volatility) over any time period. One simple model to achieve that is to take the rolling window average variance, which works quite well.

This reduces our variable space to the cross-variances, σ_{ij} with $i \neq j$

4. Algebraicly reduce dimension

Since all the weights of the portfolio sum to 1, we can reduce one dimension as follows:

$$w_n = 1 - \sum_{i<n} w_i$$

This is not a statistical reduction of dimension, rather an algebraic one. This guarantees increase in optimization stability by removing $n-1$ parameters σ_{in} with $i < n$.

5. Minimal Variance Portfolio

Using simplification (1), our portfolio risk is equal to the portfolio variance V

$$V = w^T \cdot C \cdot w = (w_1 ... w_n) \cdot C \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

with w being the asset weights and C being the covariance matrix, which thanks to simplification (2), looks as follows

$$C = \begin{pmatrix} \sigma & \dots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{1n} & \dots & \sigma \end{pmatrix}$$

We simplify algebraicly

$$V = (w_1 \dots w_n) \cdot \begin{pmatrix} \sigma & \dots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{1n} & \dots & \sigma \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$= \left(\sum_{i<n} w_i \sigma_{1i} + w_n \sigma_{1n} \dots \sum_{i<n} w_i \sigma_{ni} + w_n \sigma_{nn} \right) \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$= \sum_{j<n} w_j \sum_{i<n} w_i \sigma_{ji} + w_n \sum_{i<n} w_i \sigma_{n1} + \sum_{j} w_j w_n \sigma_{jn}$$

$$= \underbrace{\sum_{i,j<n} w_i w_j \sigma_{ij}}_{A} + \underbrace{2 \sum_{j<n} w_j w_n \sigma_{jn}}_{B} + \underbrace{w_n^2 \sigma_{nn}}_{C}$$

To find the w_k that minimizes V, we will solve for

$$\frac{\partial V}{\partial w_k} = 0$$

for all $k < n$. This is equivalent (and slightly simpler) to solving

$$0 = \frac{1}{2} \cdot \frac{\partial V}{\partial w_k} = \frac{1}{2} \cdot \frac{\partial (A + B + C)}{\partial w_k}$$

Using simplication (3) and $\frac{\partial w_n}{\partial w_k} = -1$ for $k < n$, we get
For A

$$\frac{1}{2} \cdot \frac{\partial A}{\partial w_k} = \sum_{i<n} w_i \sigma_{ki}$$

For B

$$\frac{1}{2} \cdot \frac{\partial B}{\partial w_k} = \sigma_{kn} \frac{\partial}{\partial w_k} (w_k w_n) + \sum_{k \neq j < n} w_j \sigma_{jn} \frac{\partial w_n}{\partial w_k}$$

$$= \sigma_{kn} (w_n - w_k) - \sum_{k \neq j < n} w_j \sigma_{jn}$$

$$= \sigma_{kn} - \sigma_{kn} w_k - \sigma_{kn} \sum_{i<n} w_i - \sum_{k \neq j < n} w_j \sigma_{jn}$$

$$= \sigma_{kn} - \sigma_{kn} \sum_{i<n} w_i - \sum_{j<n} w_j \sigma_{jn}$$

$$= \sigma_{kn} - \sum_{i<n} w_i (\sigma_{in} + \sigma_{kn})$$

For C

$$\frac{1}{2} \cdot \frac{\partial C}{\partial w_k} = -w_n \sigma_{nn} = -\sigma_{nn} + \sigma_{nn} \sum_{i<n} w_i$$

Putting it together, we get

$$\frac{1}{2} \cdot \frac{\partial V}{\partial w_k} = \frac{1}{2} \cdot \frac{\partial (A+B+C)}{\partial w_k}$$

$$= \sigma_{kn} - \sigma_{nn} + \sum_{i<n} w_i (\sigma_{nn} - \sigma_{in} - \sigma_{kn} + \sigma_{ki})$$

We can put previous formula into matrix form as follows

$$0 = \frac{\partial V}{\partial w_k} \Leftrightarrow \hat{S}\hat{w} = \hat{b}$$

with

$$\hat{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \end{pmatrix}$$

$$\hat{b} = \begin{pmatrix} \sigma - \sigma_{1n} \\ \vdots \\ \sigma - \sigma_{n-1,n} \end{pmatrix}$$

and $\hat{S}$ the matrix containing s_{ik} with

$$s_{ik} = \sigma - \sigma_{in} - \sigma_{kn} + \sigma_{ki} = s_{ki}$$

$$s_{ii} = 2(\sigma - \sigma_{in})$$

Now we are left with solving a linear equation system

$$\hat{S}\hat{w} = \hat{b} \Leftrightarrow \hat{w} = \hat{S} \setminus \hat{b}$$

which gives us our optimal weights w.

6. Conclusion

Separating the task of optimising the portfolio from homogenizing the variance of each asset across time and the different assets enables a significantly more stable portfolio. Using shrinkage to estimate the covariances also plays helps this stability.

Reducing the optimisation dimension helps further.

The described model was found to perform significantly better than other tested models in 2012. This data will be provided at some point and is left as an excercise for the interested reader until then.

7. Notes

7.1. **stable covariance.** To get a stable covariance matrix, one should use returns over multiple time periods (e.g. if daily data is available, use 10 day rolling returns). Additionally, use shrinkage to stabilize the covariance (e.g. a shrinkage factor of 0.1 has worked well for me).

References

[1] Markowitz, H. (1952). *Portfolio selection.*

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