FAIRPORTFOLIO ~ A SIMPLE AND STABLE OPTIMAL PORTFOLIO

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Abstract. This paper describes a method of creating a stable and optimal portfolio based on an arbitrary list of assets. The method employs multiple simplifications that lead to numerical and qualitative stability of the portfolio.

Introduction

Portfolio optimization is about finding weights for each asset that one wants to include in a portfolio. One of the longest standing theoretical results in this field is due to Markowitz[1], which minimizes the portfolio variance using the expected return and variance of each asset.

In this paper, we are also ultimately going to minimize portfolio risk after applying the following 3 simplifications:

- consider only 2nd moments (simplification 1)
- homogenize variances (simplification 2)
- algebraically reduce dimension (simplification 3)

We assume that we have chosen a list of n assets that we want to invest into. Us choosing these assets implies that we believe that each of these assets has a long term positive return.

1 Only 2nd Moments

The distribution of returns of any asset can be fully described using moments.

The 1st moment is the expected return and is the hardest to predict. An investor should only choose assets that are believed to have positive 1st moment.

The 2nd moments are the covariances between the assets, σ_{ij}, including the variances of each asset σ_{ii}.

Moments of degree 3-and- higher have ever larger relative errors of measurement due to financial data being highly noisy.

For numerical and qualitative stability, we should not include high error parameters in our optimization. This leaves us only with 2nd moments.

2 Homogeneous variances

To further stablilize our portfolio, we homogenize the variances across all assets and time:

$$\sigma_{ii}(t) = \sigma$$

An investor could choose any model to predict the variance of an asset for the following time period and pre-scale the asset such that all assets have approx. the same variance (or volatility) over any time period. One simple model to achieve that is to take the rolling window average variance, which works quite well.

This reduces our parameter space to the cross-variances, σ_{ij} with $i \neq j$

3 Algebraically reduce dimension

Since all the weights of the portfolio sum to 1, we can reduce one dimension as follows:

$$w_n = 1 - \sum_{i<n} w_i$$

This is not a statistical reduction of dimension, rather an algebraic one. This guarantees increase in optimization stability by removing $n-1$ parameters σ_{in} with $i < n$.

Minimal Variance Portfolio

Using simplication 1, our portfolio risk is equal to the portfolio variance V

$$V = w^T \cdot C \cdot w = (w_1 ... w_n) \cdot C \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

with w being the asset weights and C being the covariance matrix, which thanks to simplication 2, looks as follows

$$C = \begin{pmatrix} \sigma & ... & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{1n} & ... & \sigma \end{pmatrix}$$

We simplify algebraically

$$\begin{aligned}
V &= (w_1 ... w_n) \cdot \begin{pmatrix} \sigma & ... & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{1n} & ... & \sigma \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\
&= \left(\sum_{i<n} w_i \sigma_{1i} + w_n \sigma_{1n} ... \sum_{i<n} w_i \sigma_{ni} + w_n \sigma_{nn} \right) \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \\
&= \sum_{j<n} w_j \sum_{i<n} w_i \sigma_{ji} + w_n \sum_{i<n} w_i \sigma_{n1} + \sum_j w_j w_n \sigma_{jn} \\
&= \underbrace{\sum_{i,j<n} w_i w_j \sigma_{ij}}_{A} + \underbrace{2 \sum_{j<n} w_j w_n \sigma_{jn}}_{B} + \underbrace{w_n^2 \sigma_{nn}}_{C}
\end{aligned}$$

To find the w_k that minimizes V, we will solve for

$$\frac{\partial V}{\partial w_k} = 0$$

for all $k < n$. This is equivalent (and slightly simpler) to solving

$$0 = \frac{1}{2} \cdot \frac{\partial V}{\partial w_k} = \frac{1}{2} \cdot \frac{\partial (A + B + C)}{\partial w_k}$$

Using simplication 3 and $\frac{\partial w_n}{\partial w_k} = -1$ for $k < n$, we get:

For A

$$\frac{1}{2} \cdot \frac{\partial A}{\partial w_k} = \sum_{i<n} w_i \sigma_{ki}$$

For B

$$\begin{aligned}
\frac{1}{2} \cdot \frac{\partial B}{\partial w_k} &= \sigma_{kn} \frac{\partial}{\partial w_k} (w_k w_n) + \sum_{k \neq j < n} w_j \sigma_{jn} \frac{\partial w_n}{\partial w_k} \\
&= \sigma_{kn} (w_n - w_k) - \sum_{k \neq j < n} w_j \sigma_{jn} \\
&= \sigma_{kn} - \sigma_{kn} w_k - \sigma_{kn} \sum_{i<n} w_i - \sum_{k \neq j < n} w_j \sigma_{jn} \\
&= \sigma_{kn} - \sigma_{kn} \sum_{i<n} w_i - \sum_{j<n} w_j \sigma_{jn} \\
&= \sigma_{kn} - \sum_{i<n} w_i (\sigma_{in} + \sigma_{kn})
\end{aligned}$$

For C

$$\frac{1}{2} \cdot \frac{\partial C}{\partial w_k} = -w_n \sigma_{nn} = -\sigma_{nn} + \sigma_{nn} \sum_{i<n} w_i$$

Putting it together, we get

$$\frac{1}{2} \cdot \frac{\partial V}{\partial w_k} = \frac{1}{2} \cdot \frac{\partial (A+B+C)}{\partial w_k}$$

$$= \sigma_{kn} - \sigma_{nn} + \sum_{i<n} w_i (\sigma_{nn} - \sigma_{in} - \sigma_{kn} + \sigma_{ki})$$

We can put previous formula into matrix form as follows

$$0 = \frac{\partial V}{\partial w_k} \Leftrightarrow \hat{S}\hat{w} = \hat{b}$$

with

$$\hat{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_{n-1} \end{pmatrix}$$

$$\hat{b} = \begin{pmatrix} \sigma - \sigma_{1n} \\ \vdots \\ \sigma - \sigma_{n-1,n} \end{pmatrix}$$

and $\hat{S}$ the matrix containing s_{ik} with $i \neq k$

$$s_{ik} = \sigma - \sigma_{in} - \sigma_{kn} + \sigma_{ki} = s_{ki}$$

$$s_{ii} = 2(\sigma - \sigma_{in})$$

Now we are left with solving a linear equation system

$$\hat{S}\hat{w} = \hat{b} \Leftrightarrow \hat{w} = \hat{S} \setminus \hat{b}$$

which gives us our optimal weights w.

Conclusion

Targeting portfolio risk minization whilst only considering numerically stable parameters and reducing variables as mathematically possible leads to a stable and optimal portfolio.

Performance is improved by the gain in stability vs the loss of information by disregarding highly noisy parameters (e.g. moments of degree 1 and 3-and-higher).

This method was tested practically on multiple common trading strategies (e.g.

trend following) vs. what were considered the 10 top portfolio optimization methods in 2012. This model presented significantly better historical performance whilst being simpler. The test data is not presented in this paper. This model has been since used to manage several large portfolios.

Notes

stable covariance To get a stable covariance matrix, one should use returns over multiple time periods (e.g. if daily data is available, use 10 day rolling returns). Additionally, use shrinkage to stabilize the covariance:

$$C_{\text{shrunk}} = \alpha \cdot C + (1 - \alpha) \cdot \text{diagm}(\text{diag}(C))$$

where diagm(diag(C)) is the matrix with zeros off-diagonal and the same diagonal as C. $\alpha = 0.1$ has worked well in the author's experience.

performance Performace of investment strategies is, in the author's experience, best measured using Omega.

References

[1] Markowitz, H. (1952). *Portfolio selection.*

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