Homework Template

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Problem 1

Consider the following inpulse response of a discrete-time, LTI system

$$h[n] = \delta[n] + 2\delta[n-1]$$

a) What is the frequency repsonse of the system
b) What is its magnitude and phase representation
c) What would be the output if the input is $x[n] = cos(\pi n/2 + \pi/6) + sin(\pi n + \pi/3)$?

Solution

a)

The frequency response is

$$h[n] \xrightarrow{\mathcal{F}} H(e^{j\omega})$$

Thus,

$$\begin{aligned}
H(e^{j\omega}) &= \mathcal{F}(\delta[n] + 2\delta[n-1]) \\
H(e^{j\omega}) &= \mathcal{F}(\delta[n]) + \mathcal{F}(2\delta[n-1]) \\
H(e^{j\omega}) &= 1 + 2\mathcal{F}(2\delta[n-1]) \\
H(e^{j\omega}) &= 1 + 2e^{-j\omega}\mathcal{F}(\delta[n]) \\
H(e^{j\omega}) &= 1 + 2e^{-j\omega}
\end{aligned}$$

b)

$$\begin{aligned}
|H(e^{j\omega})| &= |1 + 2e^{-j\omega}| \\
|H(e^{j\omega})| &= |1 + 2cos(-\omega) + 2jsin(-\omega)| \\
|H(e^{j\omega})| &= \sqrt{(1 + 2cos(-\omega))^2 + (2sin(-\omega))^2} \\
|H(e^{j\omega})| &= \sqrt{(1 + 2cos(\omega))^2 + (2sin(\omega))^2}
\end{aligned}$$

$$\begin{aligned}
\angle H(e^{j\omega}) &= \angle 1 + 2e^{-j\omega} \\
\angle H(e^{j\omega}) &= \angle 1 + 2cos(-\omega) + 2jsin(w) \\
\angle H(e^{j\omega}) &= \angle 1 + 2cos(\omega) - 2jsin(w) \\
\angle H(e^{j\omega}) &= arctan\left(\frac{-2sin(\omega)}{1 + 2cos(\omega)}\right)
\end{aligned}$$

c)

- Input is $x[n] = cos(\pi n/2 + \pi/6) + sin(\pi n + \pi/3)$
- Output is $H(e^{j\omega})x[n]$

For $cos(\pi n/2 + \pi/6)$ term:

$$|H(e^{j\omega})| = \sqrt{(1 + 2cos(\pi/2))^2 + (2sin(\pi/2))^2}$$
$$|H(e^{j\omega})| = \sqrt{(1 + 0)^2 + (2(1))^2}$$
$$|H(e^{j\omega})| = \sqrt{5}$$

$$\angle H(e^{j\omega}) = arctan\left(\frac{-2sin(\pi/2)}{1 + 2cos(\pi/2)}\right)$$
$$\angle H(e^{j\omega}) = arctan\left(\frac{-2(1)}{1 + (0)}\right)$$
$$\angle H(e^{j\omega}) = arctan\left(-2\right)$$
$$\angle H(e^{j\omega}) = -1.107$$

For $sin(\pi n + \pi/3)$ term:

$$\omega = \pi$$
$$\angle = \pi/3$$

$$|H(e^{j\omega})| = \sqrt{(1 + 2cos(\omega))^2 + (2sin(\omega))^2}$$
$$|H(e^{j\omega})| = \sqrt{(1 + 2cos(\pi))^2 + (2sin(\pi))^2}$$
$$|H(e^{j\omega})| = \sqrt{(-1))^2 + (0)^2}$$
$$|H(e^{j\omega})| = 1$$

$$\angle H(e^{j\omega}) = arctan\left(\frac{-2sin(\pi)}{1 + 2cos(\pi)}\right)$$
$$\angle H(e^{j\omega}) = 0$$

Therefore,

$$y[n] = \sqrt{5}cos(\pi n/2 + \pi/6 - 1.107) + sin(\pi n + \pi/3)$$

Problem 2

Consider system described by

$$|H(j\omega)| = (\omega+1)u(\omega+1) - 2(\omega)u(\omega) + (\omega-1)u(\omega-1)$$

$$\angle H(j\omega) = \frac{\pi}{2}u(-\omega) - \frac{\pi}{2}u(\omega) \text{ where } \angle H(0) = 0$$

a) Sketch the magnitude and phase of $H(j\omega)$.
b) For input $x(t) = 1 + cos(t/2)$, sketch the magnitude and phase of $X(j\omega)$.
c) Find output $y(t)$.

Solution

a)

Figure 1: Magnitude of Frequency Response

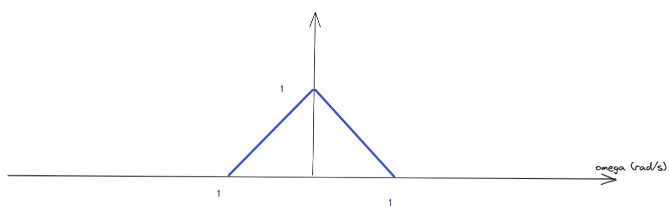
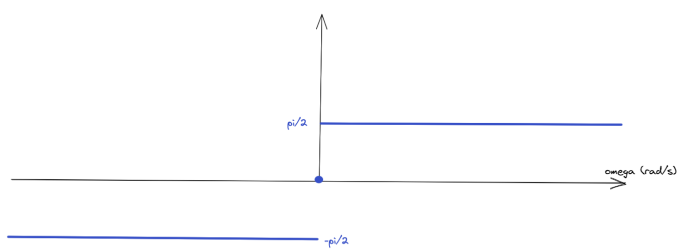


Figure 2: Angle of Frequency Response



b)

$$X(j\omega) = \mathcal{F}\left(1 + cos(t/2)\right)$$

$$X(j\omega) = \mathcal{F}\left(1 + \frac{e^{jt/2} + e^{-jt/2}}{2}\right)$$

$$X(j\omega) = \delta(\omega) + \frac{1}{2}\delta(\omega - 1/2) + \frac{1}{2}\delta(\omega + 1/2)$$

$$\Rightarrow |X(j\omega)| = \delta(\omega) + \frac{1}{2}\delta(\omega - 1/2) + \frac{1}{2}\delta(\omega + 1/2)$$
$$\Rightarrow \angle X(j\omega) = 0$$

c)

For (1) Term:

$$\omega = 0$$
$$\angle = 0$$

$$H(j\omega) = 1$$

For cos(t/2) Term:

$$\omega = 1/2$$
$$\angle = 0$$

$$H(j\omega) = 1/2$$

Therefore

$$y(t) = 1 + \frac{cos(t/2)}{2}$$

Problem 3

The frequency response of a filter is

$$H(s) = \frac{\sqrt{60}s}{s^2 + 2s\sqrt{14} + 15}$$

where $s = j\omega$.
a) Find magnitude and phase, in degrees, of $H(s)$ at frequencies 0, 10, and 100 rad/s
b) Roughly sketch $H(s)$. What type of filter is this?
c) If a signal generator that produces a biased sinusoid $x(t) = B + Acos(\omega_0 t)$ is connected to this filter, find the frequency or frequencies, ω_0 at which the filter's output is $y(t) = Acos(\omega_0 t + \theta)$, i.e. the amplitude of the sinusoid term remains the same.

a)

$$H(s) = \frac{\sqrt{60}s}{s^2 + 2s\sqrt{14} + 15}$$

$$H(j\omega) = \frac{\sqrt{60}j\omega}{(j\omega)^2 + 2j\omega\sqrt{14} + 15}$$
$$H(j\omega) = \frac{\sqrt{60}j\omega}{15 - \omega^2 + 2\omega\sqrt{14}j}$$

$$|H(j\omega)| = \frac{|\sqrt{60}j\omega|}{|15 - \omega^2 + 2\omega\sqrt{14}j|}$$
$$|H(j\omega)| = \frac{|(\sqrt{60}\omega)^2|}{|(15 - \omega^2)^2 + (2\omega\sqrt{14})^2|}$$
$$|H(j\omega)| = \frac{|60\omega|}{|225 - 30\omega^2 - \omega^4 + 56\omega^2|}$$
$$|H(j\omega)| = \frac{|60\omega|}{|225 - \omega^4 + 56\omega^2|}$$

$$\angle H(j\omega) = \angle\frac{\sqrt{60}j\omega}{(j\omega)^2 + 2j\omega\sqrt{14} + 15}$$
$$\angle H(j\omega) = \angle\sqrt{60}j\omega - \angle\left[(j\omega)^2 + 2j\omega\sqrt{14} + 15\right]$$
$$\angle H(j\omega) = \frac{\pi}{2}sgn(\omega) - \angle\left[(j\omega)^2 + 2j\omega\sqrt{14} + 15\right]$$
$$\angle H(j\omega) = \frac{\pi}{2}sgn(\omega) - arctan\left(\frac{2\sqrt{14}\omega}{15 - \omega^2}\right)$$
$$\angle H(j\omega) = \frac{\pi}{2}sgn(\omega) + arctan\left(\frac{-2\sqrt{14}\omega}{15 - \omega^2}\right)$$