# ELEC 221 HW5

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November 10, 2022

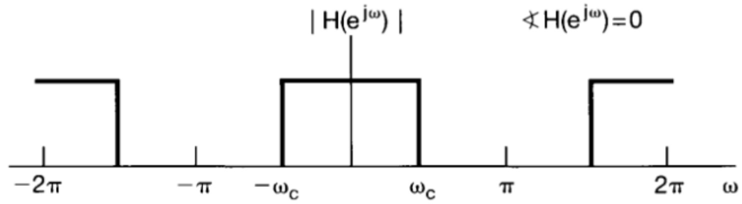
## HW5.6: Signal Reframing

Consider an ideal discrete-time lowpass filter with impulse response and for which the frequency response is that shown in the figure below. Let us consider obtaining a new filter with impulse response $h_1[n]$ and frequency response $H_1(e^{j\omega})$ as follows:

$$h_1[n] = \begin{cases} h[n/2], & \text{n even} \\ 0, & \text{n odd} \end{cases}$$

This corresponds to inserting a sequence value of zero between each sequence value of $h[n]$. Determine and sketch $H_1(e^{j\omega})$ and state the class of ideal filters to which it belongs (e.g., lowpass, highpass, bandpass, multiband, etc.)

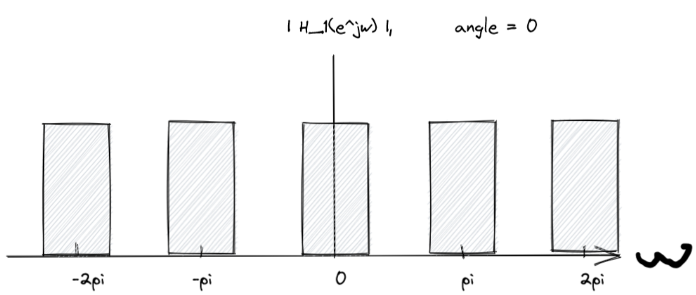
Figure 1: Ideal discrete-time l;lowpass Filter




### HW5.6 Solution

This filter is essentially an upsampled version of the original filter with width $\omega_c$, where we are sampling twice as frequent as with the original signal.
Thus, for $H_1(e^{j\omega})$, we can expect a compression by a factor of 2 in the frequency domain, and a periodic extension about $\pi$, since the sampling rate has been doubled.

Figure 2: Frequency Response of new filter



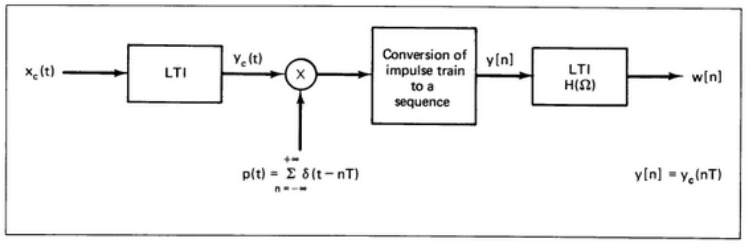

This filter is a multiband filter, as it allows bands $-\frac{\omega_c}{2} < \omega < \frac{\omega_c}{2}$ and $\pi - \frac{\omega_c}{2} < \omega < \pi + \frac{\omega_c}{2}$ to pass.

# HW5.7: Interpolation

The figure shows a system consisting of a continuous-time linear time-invariant system followed by a sampler, conversion to a sequence, and a discrete-time linear time-invariant system. The continuous-time LTI system is causal and satisfies the LCCDE:

$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t)$$

Figure 3: System




The input is a unit impulse $\delta(t)$
(a) Determine $y_c(t)$
(b) Determine the frequency response $H(\Omega)$ and the impulse $h[n]$ such that $w[n] = \delta[n]$.
Upload a pdf or an image.

## HW5.7 Solution

### a)

This LCCDE corresponds to a frequency response of

$$H_{LTI}(j\omega) = \frac{1}{1 + j\omega}$$

so, we have

$$\begin{aligned}
&Y(j\omega) = X(j\omega)H_{LTI}(j\omega) \\
&Y(j\omega) = \mathcal{F}(\delta(t))H_{LTI}(j\omega) \\
&Y(j\omega) = H_{LTI}(j\omega) \\
&\Rightarrow y_c(t) = \mathcal{F}^{-1}(H_{LTI}(j\omega)) \\
&y_c(t) = \mathcal{F}^{-1}(\frac{1}{1 + j\omega}) \\
&y_c(t) = e^{-t}u(t), \text{ using TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS from Oppenheim p.328}
\end{aligned}$$

### b)

We want to find frequency response $H(\Omega)$ and corresponding impulse response $h[n]$ such that $w[n] = \delta[n]$.

Looking at Figure 3, we take $y_c(t)$ and multiply it by $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$.
In the frequency domain, we have:

$$
\begin{aligned}
&y[n] = y_c(nT) \\
&y[n] = e^{-nT} u(nT) \\
&Y(e^{j\omega}) = \mathcal{F}(e^{-nT} u(nT)) \\
&\text{using } a^n u[n],\ |a| < 1\ \rightarrow\ \frac{1}{1 - ae^{-j\omega}} \\
&\text{Take } a^n = e^{-nT} = (e^{-T})^n \\
&\Rightarrow Y(e^{j\omega}) = \mathcal{F}(e^{-nT} u(nT)) = \frac{1}{1 - e^{-T} e^{-j\omega}}
\end{aligned}
$$

And now we have the spectra of the input into the second LTI system. We want to find $H(\Omega)$ such that:

$$Y(e^{j\omega}) H(\Omega) = W(\Omega)$$

where

$$W(\Omega) \xrightarrow{\mathcal{F}^{-1}} \delta[n]$$

$$
\begin{aligned}
&\Rightarrow W(\Omega) = 1 = Y(e^{j\omega}) H(\Omega) = \frac{H(\Omega)}{1 - e^{-T} e^{-j\omega}} \\
&\Rightarrow H(\Omega) = 1 - e^{-T} e^{-j\omega} \\
&\Rightarrow h[n] = \delta[n] - e^{-T} \delta[n-1]
\end{aligned}
$$

So we have $H(\Omega) = 1 - e^{-T} e^{-j\omega}$ and $h[n] = \delta[n] - e^{-T} \delta[n-1]$.

## HW5.8: Sampling

Consider the system in Figure 1

Figure 4:

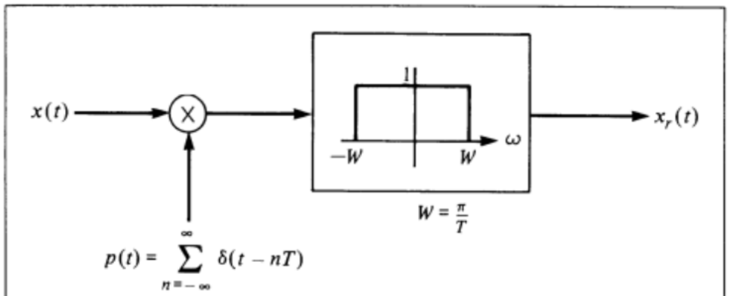



Figure 1

Given the Fourier transform of $x_r(t)$ in Figure 4, sketch the Fourier transform of two different signals $x(t)$ that could have generated $x_r(t)$:

Figure 5:

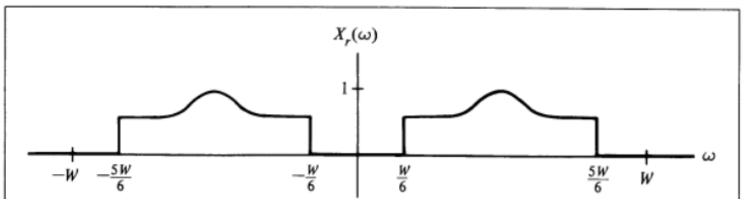



Figure 2

### HW5.8 Solution

This system is functionally a system that samples a signal, turns it into a DT sequence, and then applies a lowpass filter at $\omega_c = W = \frac{\pi}{T_s} = \frac{1}{2}\omega_s$ in order to remove the periodic extensions of sampling function $P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s)$

The effects of aliasing tell us that any frequency components present in the input signal that exist in the range

$$\frac{\omega_s}{2} < |\omega| < \omega_s$$

will get mirrored about $\frac{\omega_s}{2}$ in the frequency domain.

Therefore, we could have two signals we'll call $x_1(t)$ and $x_2(t)$ with spectra $X_1(j\omega)$ and $X_2(j\omega)$ that end up as the same result due to aliasing where the first one has the same exact spectra as $x_r(t)$ and the second has the spectra of $x_r(t)$ mirrored about $\omega = \frac{\omega_s}{2}$.
that will have the same fourier spectra as $x_r(t)$, once passed through the sampling system.

Figure 6: Frequency spectra of $X_1(j\omega)$

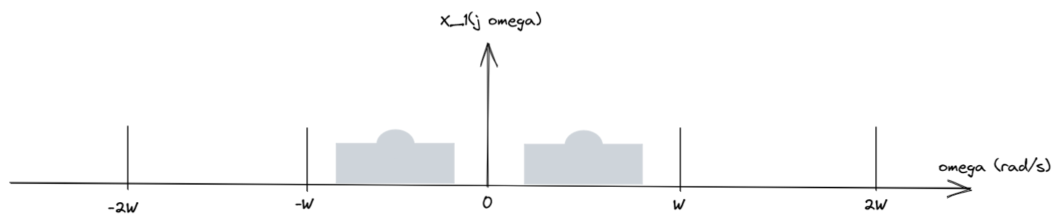
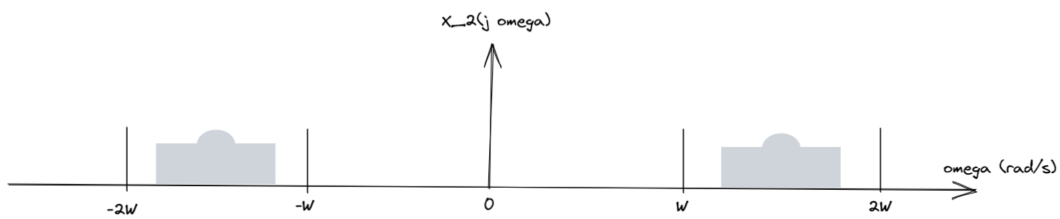



Figure 7: Frequency spectra of $X_2(j\omega)$




NOTE: The peak in the diagrams is of height 1.