DIVISOR SUMS ALONG SUMS OF TWO BIQUADRATES

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Abstract. We establish power saving asymptotics for the sum of the divisor function along a binary quartic form, improving on work of Daniel. The proof involves an application of a recent two dimensional delta method due to Li, Rydin-Myerson, and Vishe and an application of GL_2 level aspect subconvexity for dihedral forms.

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1. Introduction and statement of results

Let

$$d_k(n) = \sum_{d_1 \dots d_k = n} 1$$

be the k-fold divisor function, and write $d(n) = d_2(n)$ (which we'll refer to as the divisor function). It is often of interest to estimate sums of the form

$$\sum_{n<X} a(n) d_k(n) \tag{1.1}$$

for sequences $a : \mathbb{N} \to \mathbb{C}$ of arithmetic interest. Beyond their own intrinsic interest, questions of estimating (1.1) are relevant to the study of sums over primes

$$\sum_{p<X} a(p),$$

a connection most clearly seen through an identity of Linnik (see [IK04, Proposition 13.2]) that

$$\sum_{k \geqslant 1} \frac{(-1)^k}{k} d'_k(n) = \begin{cases} 1/k & n = p^k \\ 0 & \text{else,} \end{cases}$$

where $d'_k(n)$ is the number of ways to write n as the product of k integers greater than 1. d'_k is closely related to the usual d_k for we have

$$d'_k(n) = \sum_{0 \leqslant \ell \leqslant k} (-1)^{k-\ell} \binom{k}{\ell} d_\ell(n).$$

In this paper, we obtain power saving asymptotics for sums of the divisor function $d(n) = d_2(n)$ along values of a binary quartic form, showing the following result.

Theorem 1. There exists $\delta > 0$ such that for $N \geqslant 1$, we have that

$$\sum_{\substack{m,n \in \mathbb{Z} \\ 0 < m^4 + n^4 \leqslant N}} d(m^4 + n^4) = \kappa N^{\frac{1}{2}} (c_{-1} \log N + c_0) + O(N^{\frac{1}{2} - \delta}), \tag{1.2}$$

where

$$\kappa = \int_{\mathbb{R}^2} \mathbb{1}_{x_1^4 + x_2^4 \leqslant 1} \, dx_1 \, dx_2,$$

and c_{-1}, c_0 are such that

$$\sum_{q \geqslant 1} \frac{\rho(q)}{q^{s+1}} = \frac{c_{-1}}{s-1} + c_0 + O(|s-1|)$$

for $\operatorname{Re} s > 1$, where $\rho(q) = \#\{x_1, x_2 \in \mathbb{Z}/q\mathbb{Z} : q | x_1^4 + x_2^4\}$.

This improves on work of Daniel [Dan99], who showed asymptotics for (1.2) with a saving in the remainder of $(\log N)^{-1+o(1)}$, but was unable to handle the case of two divisors of equal size, whose contribution to (1.2) is $\asymp N$. See §1.1 for further discussion of Daniel's argument, a sketch of our argument, and a discussion of the relevant bottlenecks.

We have not sought to make δ explicit here for our exponent is almost certain to be extremely poor and not reflect any barrier of note. Furthermore, if we were to work it out in Theorem 1, the precise exponent obtained would be an artifact of the sharp cutoff and the loss from sieving out nonsplit primes in our passage to algebraic integers. We have made an exponent explicit in Theorem 2, which may be more useful in applications. We shall now begin preparations for the statement of our main theorem, from which Theorem 1 follows.

The power saving in Theorem 2 is dependent on the strength of subconvex bounds for GL_2 L-functions in the level aspect and t-aspect, so we shall express our exponent in terms of this.

Precisely, let

$$\delta_{\mathrm{sc}}^* = \frac{1}{4} - \inf\left\{\theta : |L(1/2,\pi)| \leqslant C(\pi)^{\theta}\right\}, \tag{1.3}$$

where π ranges over all $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ cuspidal automorphic representations (with not necessarily trivial central character), and $C(\pi)$ denotes the analytic conductor. In our case, we shall require bounds in the t-aspect and level aspect simultaneously (though for a power-saving in the problem itself, any bound in the level aspect which is simultaneously subconvex in the level aspect and no worse than polynomial in the t-aspect would suffice).

That $\delta_{\mathrm{sc}}^* > 0$ follows, for example, from work of Michel and Venkatesh [MV10] on subconvexity for general GL_2 L-functions, though they do not provide an explicit exponent.

Throughout this paper, we will fix some $\delta_{\mathrm{sc}}^* \geqslant \delta_{\mathrm{sc}} > 0$

The best explicit exponent we can find applicable to us is due to Han Wu [Wu22], whose result implies that we may take $\delta_{\mathrm{sc}} = 1/224$.

We now state the main theorem, which takes place over $K = \mathbb{Q}(\zeta)$ with ζ a primitive 8th root of unity. We refer the reader to §2 for notation not yet defined.

Theorem 2. Let $M, \Omega \geqslant 1$ and suppose that $\Phi^{\infty} \in C_c^{\infty}((K_{\infty} \setminus K_{\infty}^0)^2)$ and for $\beta_1', \beta_2' \in \mathcal{O}_K/M\mathcal{O}_K$, write

$$\Phi^f(\beta_1, \beta_2) = \mathbb{1}_{\substack{\beta_1 \equiv \beta_1' \,(M) \\ \beta_2 \equiv \beta_2' \,(M)}}.$$

Suppose furthermore that Φ^{∞} satisfies the following properties:

(1) For all places $v|\infty$ of K and $(x_1^{\infty}, x_2^{\infty}) \in \mathrm{supp}(\Phi^{\infty})$, we have that

$$\frac{1}{\Omega} \ll |x_1^{\infty}|_v, |x_2^{\infty}|_v \ll \Omega. \tag{1.4}$$

(2) For all differential operators $P \in \mathcal{D}(K_{\infty}^2)$ of order k, we have that

$$\|P\Phi^{\infty}\|_{\infty} \ll_P \Omega^k.$$

Write $\boldsymbol{\ell} : \mathcal{O}_K \to \mathbb{Z}^2$ to denote the linear map

$$\boldsymbol{\ell}(n_0 + n_1\zeta + n_2\zeta^2 + n_3\zeta^3) = (n_3, n_2).$$

Then, for any $X_1, X_2 \geqslant 1$, we have that

$$\sum_{\substack{\alpha_1,\alpha_2\in\mathcal{O}_K \\ \boldsymbol{\ell}(\alpha_1\alpha_2)=0}} \Phi^\infty\left(\frac{\alpha_1}{X_1}, \frac{\alpha_2}{X_2}\right)\Phi^f(\alpha_1,\alpha_2) = X_1^2X_2^2\sigma_\infty \prod_p \sigma_p$$
$$+ O\left((\Omega M)^{O(1)} X_1^2 X_2^2 \min\left((X_1X_2)^{-\eta_{\mathrm{sc}}+o(1)}, \frac{X_1}{X_2}, \frac{X_2}{X_1}\right)\right),$$

where

$$\sigma_\infty = \sigma_\infty(\Phi^\infty, \boldsymbol{\ell}) := \int_{K_\infty^2} \Phi^\infty(x_1^\infty, x_2^\infty)\delta(\boldsymbol{\ell}(x_1^\infty x_2^\infty))\, dx_1^\infty\, dx_2^\infty,$$
$$\sigma_p = \sigma_p(\Phi^f, \boldsymbol{\ell}) := \int_{\mathcal{O}_{K,p}^2} \Phi^f(x_1^p, x_2^p)\delta(\boldsymbol{\ell}(x_1^p x_2^p))\, dx_1^p\, dx_2^p$$

and

$$\eta_{\mathrm{sc}} = \frac{8\delta_{\mathrm{sc}}/9}{49/6 - 32\delta_{\mathrm{sc}}/9}.$$

With Han Wu's $\delta_{\mathrm{sc}} = 1/224$, we record that

$$\eta_{\mathrm{sc}} = \frac{1}{2054}.$$

1.1. **Sketch of proof and discussion of argument.** We end this section with a discussion of Daniel's methods in [Dan99], the bottlenecks and limitations in his method, along with a sketch of our proofs of Theorem 1 and Theorem 2. As is typical with such sketches, we will avoid discussing the technicalities of analytic number theory such as smoothing and non-coprimality; we will only discuss the top dyadic ranges which are typically the main difficulty.

Taking the sum in Theorem 1, opening up the divisor function, and splitting the divisors into dyadic intervals, the problem of estimating the left-hand side of (1.2) reduces to estimating

$$\sum_{d\sim D} \sum_{\substack{m,n\sim N^{1/4} \\ d\mid m^4+n^4}} 1 \tag{1.5}$$

for scales $D \ll N^{1/2}$.

Daniel [Dan99], using lattice point methods of Greaves [Gre71, Gre92], was able to handle divisors of size $D \ll N^{1/2}\log^{-A} N$. The idea is that the inner sum in (1.5) ought to approximately be

$$\frac{\rho(d)}{d^2}N^{1/2} \approx \frac{N^{1/2}}{D},$$

which Daniel shows on average for $d \sim D$, bounding by $O(N^{1/4}\sqrt{D}(\log N)^{O(1)})$ the size of

$$\sum_{d\sim D}\Bigg|\sum_{\substack{n_1,n_2\sim N^{1/4}\\ d|n_1^4+n_2^4}} 1 - \frac{\rho(d)}{d^2}N^{1/2}\Bigg|. \tag{1.6}$$

Daniel obtains such a bound by splitting into arithmetic progressions modulo d, interpreting the count over n_1, n_2 as an average of lattice point counts, and nontrivially estimating the count on average by using the homogeneity of $x_1^4 + x_2^4$ to show that the lattices on average do not have a shortest vector of size smaller than one would expect given the covolume.

We remark that Friedlander–Iwaniec in [FI10, Theorem 22.20] obtain the same level of distribution of $1/2$ as Daniel with Poisson summation, in which case the homogeneity plays a related role, this time providing more cancellation in the resulting exponential sums than one would expect for an inhomogeneous polynomial.

The large moduli are treated similarly, but at this point, there is a strict limit to the saving one can obtain. Focusing on the case of $D \asymp N^{1/2}$, the inner sum in (1.5) has 1 term on average and so one cannot hope for a bound on (1.6) better than $N^{1/2}$, and indeed, this is the bound Daniel obtains. Summing over the $\log\log N$ scales $N^{1/2}\log^{-A} N \ll D \ll N^{1/2}$, this gives Daniel an asymptotic for (1.2) with remainder term $O(N^{1/2}\log\log N)$.

To obtain anything more than a saving of $(\log N)^{1-o(1)}$ over the main term, it is necessary then to obtain estimates for the sum (1.6) without the absolute values. It is this barrier we will refer to as the limit of the hyperbola method: that point at which the sequence is too sparse to allow for further gains without exploiting cancellation in more than 1 of the divisors.

One direct way to seek to breach this barrier, following either Daniel or Friedlander-Iwaniec's treatments, is to seek equidistribution of roots of the congruence $x_1^4 + x_2^4 \equiv 0(q)$ as q varies (beyond the equidistribution already implied by the homogeneity). Such equidistribution of roots of a polynomial congruence to a varying modulus is a notoriously difficult problem. Is only known for irreducible single variable quadratic polynomials.

Incidentally, the case of single variable quadratic polynomials is the only case of the hyperbola method barrier having been breached, which was first done by Hooley [Hoo63] who obtained power saving asymptotics for sums of $d(n^2 + a)$. Every aspect of the single variable case turns out to be intimately tied to GL_2 automorphic forms (see [Sar84, TT13], for example), but no such interpretation appears to exist for any other natural case of either polynomial root equidistribution or for the divisor sum problem (though see [Wel22]).

Our work represents the first case of this hyperbola method barrier being breached for polynomial sequences since Hooley [Hoo63]. We shall now sketch our argument. As we said before, as is typical with such sketches, the symbol "=" should not be interpreted too literally for it shall hide many nongeneric ranges and phenomena we deal with carefully in the full proof.

We proceed with the estimation of the sum

$$\sum_{d\sim N^{1/2}}\sum_{\substack{m,n\sim N^{1/4}\\ d|m^4+n^4}} 1=\sum_{\substack{a_1a_2=m^4+n^4\\ a_1,a_2\sim X^2\\ m,n\sim X}} 1$$

"as a whole", treating a_1,a_2 equally (we've relabelled $X\asymp N^{1/4}$ for future compatibility with Theorem 2). We pass to number fields with the observation that $N_{K/\mathbb{Q}}(m+n\zeta)=m^4+n^4$, and therefore, by unique factorization[1], there is (modulo minor technicalities on split primes[2]) a bijection between factorizations into rational integers $a_1a_2=m^4+n^4$ of m^4+n^4 and factorizations $\mathfrak{a}_1\mathfrak{a}_2=(m+n\zeta)$ of $(m+n\zeta)$ into integral ideals $\mathfrak{a}_1,\mathfrak{a}_2\subset\mathcal{O}_K$, with $\mathfrak{a}_i\mapsto N\mathfrak{a}_i=a_i$ giving one direction of the correspondence.

Writing $\mathfrak{a}_i=(\alpha_i)$ for some $\alpha_i\in\mathcal{O}_K$ by picking some fundamental domain for $\mathcal{O}_K/\mathcal{O}_K^\times$ and summing over the α_i, detecting if $\alpha_1\alpha_2$ is of the form $m+n\zeta$, we obtain that

$$\sum_{\substack{a_1a_2=m^4+n^4\\ a_1,a_2\sim X^2\\ m,n\sim X}} 1=\sum_{\alpha_1,\alpha_2\sim X^{1/2}}\mathbb{1}_{\boldsymbol{\ell}(\alpha_1\alpha_2)=0}.$$

The estimation of the right-hand side of the above is the content of Theorem 2, whose proof we now sketch. We detect the condition $\boldsymbol{\ell}(\alpha_1\alpha_2)=0$ with a perturbation of a two dimensional delta method due to [LMV24], the statement of which amounts to

$$\mathbb{1}_{\boldsymbol{\ell}(\alpha_1\alpha_2)=0}\approx-\frac{L^2}{X^{2/3}}\sum_{q\sim X^{2/3}L}\mathbb{1}_{q|\boldsymbol{\ell}(\alpha_1\alpha_2)}+\frac{L^2}{X^{2/3}}\sum_{\substack{d\ll X^{1/3}/L\\ \mathbf{c}\in\mathbb{Z}^2\\ |\mathbf{c}|d\sim X^{1/3}/L\\ (c_1,c_2)=1}}\frac{dL^{1/2}}{X^{2/3}}\sum_{q\sim X^{2/3}/(dL^{1/2})}\mathbb{1}_{\substack{dq|\det(\mathbf{c},\boldsymbol{\ell}(\alpha_1\alpha_2)),\\ d|\boldsymbol{\ell}(\alpha_1\alpha_2)}}$$

where L is a small power of X to be chosen (note here that $|\boldsymbol{\ell}(\alpha_1\alpha_2)|\ll X$). Note that the conductor of the conditions on the first sum is $\ll X^{2/3}L$ and on the second $\ll X^{2/3}/L^{1/2}$; the purpose of the introduction of the parameter L is to shift difficulty from the second sum onto the first sum for reasons that will soon be clear.

After applying Poisson summation in α_1,α_2, we are reduced to bounding, in the generic case (so apart from main terms coming from zero frequencies)

$$\Sigma_1=\frac{1}{L^3}\sum_{q\sim X^{2/3}L}\sum_{\alpha_1,\alpha_2\sim X^{1/6}L}S_1(\alpha_1,\alpha_2;q),$$

where

$$S_1(\alpha_1,\alpha_2;q)=\frac{1}{q^3}\sum_{\substack{\beta_1,\beta_2\in\mathcal{O}_K/(q)\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv0\,(q)}}e_q\left(\mathrm{Tr}\,\frac{\alpha_1\beta_1+\alpha_2\beta_2}{\delta_K}\right)$$

[1] Finiteness of the class group would suffice in practice

[2] In the case at hand, this holds as stated so long as $(m,n)=1$

turns out to be multiplicative in q and satisfy

$$S_1(\alpha_1, \alpha_2; p) = -1 + \#\{x\,(p) : n_0 + \cdots + n_3x^3 \equiv 0\,(p)\} + O\left(\frac{1}{p}\right) \tag{1.7}$$

(we'll write $\alpha_1\alpha_2 = n_0 + \cdots + n_3\zeta^3$ from now on), and

$$\Sigma_2 = X^{2/3}L^{5/2}\frac{L^2}{X^{2/3}} \sum_{\substack{|\mathbf{c}|d\sim X^{1/3}/L \\ (c_1,c_2)=1}} \frac{dL^{1/2}}{X^{2/3}} \sum_{q\sim X^{2/3}/(dL^{1/2})} \sum_{\alpha_1,\alpha_2\sim X^{1/6}/L^{1/2}} S_1(\alpha_1,\alpha_2;d)\mathbb{1}_{q|n_0c_1^3-n_1c_1^2c_2+n_2c_1c_2^2-n_3c_2^3}.$$

At this point, we wish to bound $\max(|\Sigma_1|, |\Sigma_2|)$, and the trivial bounds (bounding each term by $X^{o(1)}$ on average) on Σ_1, Σ_2 are $X^2L^6, X^2L^{-3/2}$, respectively, so when $L = 1$, we are right at the boundary.

We are able to go beyond the trivial bound for Σ_1 by noting that because of (1.7), we have that

$$\sum_q \frac{S_1(\alpha_1,\alpha_2;q)}{q^s} \approx \frac{L_{k_\alpha}(s)}{\zeta(s)} = L(s, F_\alpha), \tag{1.8}$$

where k_α is the splitting field over $\mathbb{Q}$ of $f_\alpha(x) = n_0 + \cdots + n_3x^3$, and F_α is a weight 1 cusp form of level $\mathrm{Disc}(n_0 + \cdots + n_3\zeta^3) \ll X^{4/3}L^2$. Therefore, subconvexity for $L(s, F_\alpha)$ is essentially equivalent to obtaining power saving cancellation in

$$\sum_{q\sim X^{2/3}L} S_1(\alpha_1, \alpha_2; q).$$

Such subconvex bounds are known so we obtain the desired cancellation. We thus show that there exists some $\delta > 0$ for which $|\Sigma_1| \ll X^{2-\delta}L^{O(1)}$. Putting this together with the bound $|\Sigma_2| \ll X^2L^{-3/2}$ and taking L a sufficiently small power of X depending on δ, we obtain that $|\Sigma_1| + |\Sigma_2| \ll X^{2-\eta}$ for some $\eta > 0$, as desired.

We end this section with some remarks on what more our methods should yield, in principle. With some work handling the class group, it should be possible to prove Theorem 1 with $m^4 + n^4$ replaced by any irreducible binary quartic form.

Furthermore, note that at no point in our proof did we truly use the convolution structure of the divisor function: all we do is Poisson in both variables simultaneously. Therefore, we expect that Theorem 1 should also hold with $d(n)$ replaced by the Fourier coefficients of any GL_2 automorphic form (perhaps assuming Ramanujan), with the substitute for unique factorization functoriality of the base change lift to the splitting field of the binary quartic. The presence of a main term should then depend on the cuspidality of the automorphic form after lifting.

2. Notation

We fix $\delta_K = 4\zeta^3$ a generator of the different ideal, and take for $\alpha, \beta \in K$,

$$\langle \alpha, \beta \rangle := \mathrm{Tr}\left(\frac{\alpha\beta}{\delta_K}\right),$$

the trace pairing. Note that $\alpha = x_0 + x_1\zeta + x_2\zeta^2 + x_3\zeta^3 \in K$ satisfy $\langle \alpha, 1 \rangle = x_3$. We write $\psi(\alpha) = e(\langle \alpha, 1 \rangle)$ and let $\psi_\varphi(\alpha) = \psi(\alpha/\varphi)$.

We let $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$, $K_p = K \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and $\mathcal{O}_{K,p} = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The measure we take on K_∞ and K_p will be the Haar measure, normalized so that $\mathrm{vol}(\mathcal{O}_{K,p}) = 1$. The norm can be extended as a function from $K_\infty \to \mathbb{R}_{\geqslant 0}$ and $\langle \cdot, \cdot \rangle$ can be extended to K_∞, K_p as well.

We let $|-|_\infty$ and $|-|_p$ be defined on K_∞ and K_p as

$$|x|_\infty := \left(\prod_{v|\infty} |x_v|_v\right)^{1/[K:\mathbb{Q}]}$$

$$|x|_p := \left(\prod_{v|p} |x_v|_v\right)^{1/[K:\mathbb{Q}]}.$$

On K, we have that $|x|_\infty = |N(x)|^{1/[K:\mathbb{Q}]}$ and similarly at p. With this, we write $K_\infty^0 = \{x \in K_\infty : |x|_\infty = 0\}$.

We also write $|x|_{\sup} = \sup_{v|\infty} |\sigma_v(x)|$. Clearly, we have $|x|_\infty \leqslant |x|_{\sup}$.

For a vector $\mathbf{x} = (x_1, x_2)$, we write $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$. We call a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$ *primitive* if $(x_1, x_2) = 1$. We also write $\mathbf{x}^\perp$ to denote $(-x_2, x_1)$.

Also, throughout this paper, implied constants are allowed to be of size $\Omega^{O(1)}$, which will not be mentioned throughout the proof otherwise. For instance, we may say that for (x_1^∞, x_2^∞) in the support of Φ^∞, by the condition (1.4), we have

$$|x_j^\infty|_\infty \asymp |x_j^\infty|_{\sup}.$$

On the other hand, the statement $\exp(\Omega) \ll 1$ does not comply with our convention here.

3. The unbalanced case

In this section, we shall quickly show the main theorem with a remainder of

$$\ll M^{O(1)} X_1^2 X_2^2 \min\left(\frac{X_1}{X_2}, \frac{X_2}{X_1}\right).$$

We shall suppose without loss of generality that $X_1 \geqslant X_2$ throughout this section, so we wish to show a remainder of $\ll M^{O(1)} X_1 X_2^3$.

We proceed by summing over α_2 first, which reduces to counting points of magnitude $\ll X_1$ in a codimension 2 sublattice of $\mathbb{Z}^4$ (up to congruence conditions modulo M, the cost of which can be absorbed into the $M^{O(1)}$ factor in the remainder).

This codimension 2 lattice has shortest vector of length $\ll |\alpha_2|_{\sup}$, so Poisson summation or basic geometry of numbers estimates yields an accurate estimate for this inner sum over α_2. Executing the sum over α_2 of these main terms then yields the desired result.

Write

$$\Lambda(\alpha_2) = \{\alpha_1 \in \mathcal{O}_K : \boldsymbol{\ell}(\alpha_1\alpha_2) = 0\},$$

and let $\operatorname{covol}(\Lambda(\alpha_2))$ denote the covolume of Λ in its $\mathbb{R}$-span, a plane which we denote $W(\alpha_2) \subset K_\infty$.

We record that the bound (1.4) implies that α_2 for which $\Phi^\infty(-, \alpha_2/X_2) \not\equiv 0$ satisfy

$$|\alpha_2|_{\sup} \asymp |\alpha_2|_\infty \asymp \sqrt{\operatorname{covol}(\Lambda(\alpha_2))}.$$

Therefore, it follows that

$$\sum_{\substack{\alpha_1,\alpha_2\\ \boldsymbol{\ell}(\alpha_1\alpha_2)=0}} \Phi^\infty\left(\frac{\alpha_1}{X_1}, \frac{\alpha_2}{X_2}\right)\Phi^f(\alpha_1,\alpha_2)$$

$$= \sum_{\alpha_2} \sum_{\lambda_0 \in \Lambda(\alpha_2)/M\Lambda(\alpha_2)} \Phi^f(\lambda_0, \alpha_2) \sum_{\lambda \in \lambda_0 + M\Lambda(\alpha_2)} \Phi^\infty\left(\frac{\lambda}{X_1}, \frac{\alpha_2}{X_2}\right).$$

Then, standard lattice point counting estimates (bounding the remainder by the volume of the boundary, see for example [Dav51]) imply that

$$\sum_{\lambda \in \lambda_0 + M\Lambda(\alpha_2)} \Phi^\infty\left(\frac{\lambda}{X_1}, \frac{\alpha_2}{X_2}\right) = \frac{1}{\operatorname{covol}(M\Lambda(\alpha_2))} X_1^2$$

$$\int_{W(\alpha_2)} \Phi^\infty\left(x_1^\infty, \frac{\alpha_2}{X_2}\right) dx_1^\infty + O\left(\frac{X_1}{\sqrt{\operatorname{covol}(M\Lambda(\alpha_2))}}\right),$$

and so

$$\sum_{\substack{\alpha_1,\alpha_2\\ \boldsymbol{\ell}(\alpha_1\alpha_2)=0}} \Phi^\infty\left(\frac{\alpha_1}{X_1}, \frac{\alpha_2}{X_2}\right)\Phi^f(\alpha_1,\alpha_2)$$

$$= \sum_{\alpha_2}\left(\frac{1}{M^2} \sum_{\lambda_0 \in \Lambda(\alpha_2)/M\Lambda(\alpha_2)} \Phi^f(\lambda_0,\alpha_2)\right) \int_{W(\alpha_2)} \Phi^\infty\left(x_1^\infty, \frac{\alpha_2}{X_2}\right) dx_1^\infty + O(MX_1X_2^3).$$

Now, note that

$$\int_{W(\alpha_2)} \Phi^\infty\left(x_1^\infty, \frac{\alpha_2}{X_2}\right) dx_1^\infty = X_2^{-2} \int_{K_\infty} \Phi^\infty\left(x_1^\infty, \frac{\alpha_2}{X_2}\right)\delta(\boldsymbol{\ell}(x_1^\infty\alpha_2))\, dx_1^\infty,$$

so the desired result follows upon summing over α_2 and applying Poisson summation.

4. Setup for proof of Theorem 2 in the balanced case

Throughout, we shall now let $X = X_1X_2$.

For the remainder of the paper, we restrict ourselves to the balanced case

$$X^{-\frac{1}{20}} \ll \frac{X_1}{X_2}, \frac{X_2}{X_1} \ll X^{\frac{1}{20}}, \tag{4.1}$$

for the contents of §3 imply the desired result (in light of the fact that $\delta_{\mathrm{sc}} \leqslant 1/4$, which implies $\eta_{\mathrm{sc}} \leqslant 4/131$, less than the saving from §3 if (4.1) does not hold).

We begin with some reductions to simplify notation for the remainder of the paper.

By Fourier inversion, we have that

$$\Phi^\infty(x_1^\infty, x_2^\infty) = \int_{K_\infty^2} \widehat{\Phi^\infty}(y_1^\infty, y_2^\infty)\psi(-x_1^\infty y_1^\infty - x_2^\infty y_2^\infty)\, dy_1^\infty\, dy_2^\infty,$$

so the decay of $\widehat{\Phi^\infty}$ implied by the derivative bounds we have supposed implies that we may suppose (at an acceptable cost absorbed into the $\Omega^{O(1)}$ factor in the final remainder term) that $\Phi^\infty(x_1^\infty, x_2^\infty) = \phi_1(x_1^\infty)\phi_2(x_2^\infty)$ for some $\phi_1, \phi_2 \in C_c^\infty(K_\infty \setminus K_\infty^0)$ upon performing a dyadic partition of unity on α_1, α_2.

We are therefore reduced to showing that

$$\Sigma = \sum_{\substack{\alpha_1,\alpha_2\in\mathcal{O}_K \\ \boldsymbol{\ell}(\alpha_1\alpha_2)=0 \\ \alpha_1\equiv\beta_1'(M) \\ \alpha_2\equiv\beta_2'(M)}} \phi_1\left(\frac{\alpha_1}{X_1}\right)\phi_2\left(\frac{\alpha_2}{X_2}\right) = X^2\sigma_\infty \prod_p \sigma_p + O((M\Omega)^{O(1)}X^{2-\eta_{\mathrm{sc}}}),$$

with

$$\sigma_\infty = \int_{K_\infty^2} \phi_1(x_1^\infty)\phi_2(x_2^\infty)\delta(\boldsymbol{\ell}(x_1^\infty x_2^\infty))\, dx_1^\infty\, dx_2^\infty,$$

$$\sigma_p = \int_{\mathcal{O}_{K,p}} \mathbb{1}_{\substack{\beta_1\equiv\beta_1'\,(M) \\ \beta_2\equiv\beta_2'\,(M)}}\delta(\boldsymbol{\ell}(\beta_1\beta_2))\, d\beta_1\, d\beta_2.$$

Our proof begins with an application of the 2-dimensional δ-method to detect the condition $\boldsymbol{\ell} = 0$, followed by an application of Poisson summation.

For the application of the δ-method, we fix two even $\omega_1, \omega_2 \in C_c^\infty(\mathbb{R} \setminus \{0\})$ with

$$1 = \int_{\mathbb{R}^2} \omega_1(|\mathbf{x}|)\, d\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}} \omega_2(x)\, dx.$$

These will remain fixed throughout the argument, with no dependence on any of the other parameters.

The parameters we will introduce at this point which we shall not fix at the moment are $1 \leqslant L \ll X^{\frac{1}{10}}$ and

$$D := \frac{X^{\frac{1}{3}}}{L}.$$

Then, an application of the 2-dimensional δ-method and Poisson summation yields the following.

Proposition 4.1. *We have that*

$$\Sigma = -2\Sigma_1^{\{\}} + \Sigma_2^{\{\}} - 2\Sigma_1^{\{1,2\}} + \Sigma_2^{\{1,2\}} + \sum_{j\leqslant 2}(-2\Sigma_1^{\{j\}} + \Sigma_2^{\{j\}}),$$

where for $S \subset \{1,2\}$, we write

$$\Sigma_1^S = \frac{X^4}{D^2} \sum_{\substack{q \geqslant 1 \\ M|q}} \frac{1}{q^5} \sum_{\substack{\alpha_1,\alpha_2 \\ \alpha_j = 0 \iff j \notin S}} (S_1 I_1)(\alpha_1, \alpha_2; q), \tag{4.2}$$

with

$$S_1(\alpha_1,\alpha_2;q) = \frac{1}{q^3} \sum_{\substack{\beta_1,\beta_2 \in \mathcal{O}_K/q\mathcal{O}_K \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (q/(q,M)) \\ \beta_1 \equiv \beta_1' ((q,M)) \\ \beta_2 \equiv \beta_2' ((q,M))}} \psi_q(\alpha_1\beta_1 + \alpha_2\beta_2), \tag{4.3}$$

$$I_1(\alpha_1,\alpha_2;q) = \int_{K_\infty^2} \omega_1\left(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|MX}{qD}\right) \phi_1(x_1^\infty)\phi_2(x_2^\infty) \psi\left(-\frac{X_1 x_1^\infty \alpha_1}{q} - \frac{X_2 x_2^\infty \alpha_2}{q}\right) dx_1^\infty \, dx_2^\infty, \tag{4.4}$$

and

$$\Sigma_2^S = \frac{X^4}{D^2} \sum_{\substack{d \geqslant 1 \\ \mathbf{c} \in \mathbb{Z}^2 \\ (c_1,c_2)=1}} \frac{1}{d^5} \omega_1\left(\frac{|\mathbf{c}|d}{D}\right) \frac{d}{\sqrt{DX}} \sum_{\substack{q \geqslant 1 \\ M|q}} \frac{1}{q^5} \sum_{\substack{\alpha_1,\alpha_2 \\ \alpha_i = 0 \iff i \notin S}} (S_2 I_2)(\alpha_1,\alpha_2,\mathbf{c};d,q),$$

where

$$S_2(\alpha_1,\alpha_2,\mathbf{c};d,q) = \frac{1}{d^3q^3} \sum_{\substack{\beta_1,\beta_2 \in \mathcal{O}_K/dq\mathcal{O}_K \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (d) \\ \det(\mathbf{c},\boldsymbol{\ell}(\beta_1\beta_2)) \equiv 0 (dq/(q,M)) \\ \beta_1 \equiv \beta_1' ((dq,M)) \\ \beta_2 \equiv \beta_2' ((dq,M))}} \psi_{dq}(\alpha_1\beta_1 + \alpha_2\beta_2), \tag{4.5}$$

$$\begin{aligned} I_2(\alpha_1,\alpha_2;\mathbf{c},d,q) = \int_{K_\infty^2} \left(\omega_2\left(\frac{dq}{M\sqrt{DX}}\right) - \omega_2\left(\frac{M\sqrt{X}\det(\mathbf{c},\boldsymbol{\ell}(x_1^\infty x_2^\infty))}{q\sqrt{D}}\right)\right) \phi_1(x_1^\infty)\phi_2(x_2^\infty) \\ \psi\left(-\frac{X_1x_1^\infty\alpha_1}{dq} - \frac{X_2x_2^\infty\alpha_2}{dq}\right) dx_1^\infty \, dx_2^\infty. \end{aligned} \tag{4.6}$$

4.1. **Proof of Theorem 2.** In this subsection, we shall prove Theorem 2 assuming bounds we will show in §13 and §12 for Σ_2^S and Σ_1^S, respectively. The remainder of this paper will then be devoted to showing these bounds. We refer the reader to the next subsection, §4.2, for a explicit listing of the main propositions and what they bound, which may make the present subsection easier to read.

Putting together Propositions 13.6 and 12.6 to estimate $\Sigma_2^{\{\}}, \Sigma_1^{\{\}}$, we obtain that

$$-2\Sigma_1^{\{\}} + \Sigma_2^{\{\}} = X^2\sigma_\infty \prod_p \sigma_p + O\left((M\Omega)^{O(1)} X^{2+o(1)} \left(\frac{L^4}{X^{\frac{1}{3}}} + \frac{L^2}{X^{\frac{2}{3}}}\right)\right).$$

By Proposition 13.1, we have

$$\Sigma_2^{\{1,2\}} \ll (M\Omega)^{O(1)} X^{2+o(1)} \left(\frac{1}{L^{\frac{3}{2}}} + \frac{L^{\frac{2}{7}}}{X^{\frac{2}{21}}} \right), \tag{4.7}$$

and by Proposition 12.1, we have that

$$\Sigma_1^{\{1,2\}} \ll (M\Omega)^{O(1)} X^{2+o(1)} \left(\frac{L^{\frac{20}{3} - \frac{32}{9}\delta_{\mathrm{sc}}}}{X^{\frac{16}{27}\delta_{\mathrm{sc}}}} + \frac{L^{\frac{47}{6}}}{X^{\frac{7}{36}}} \right). \tag{4.8}$$

At this point, we optimize to match the first two terms in (4.8) and (4.7), taking

$$L = X^{\frac{16\delta_{\mathrm{sc}}/27}{49/6 - 32\delta_{\mathrm{sc}}/9}}.$$

Putting this together (along with the bound $\delta_{\mathrm{sc}} \leqslant 1/4$) with Propositions 12.4 and 13.3, which imply that that

$$\Sigma_1^{\{j\}} \ll \frac{X^{3+o(1)}}{DX_2^2} \ll X^{2-1/10+o(1)},$$

$$\Sigma_2^{\{j\}} \ll X^{2+o(1)} \frac{X^{\frac{3}{2}}}{X_2^3 D} \ll X^{2-1/20+o(1)},$$

yields that

$$\sum_{j \leqslant 2} (-2\Sigma_1^{\{j\}} + \Sigma_2^{\{j\}}) - 2\Sigma_1^{\{1,2\}} + \Sigma_2^{\{1,2\}} \ll (M\Omega)^{O(1)} X^{2 - \frac{8\delta_{\mathrm{sc}}/9}{49/6 - 32\delta_{\mathrm{sc}}/9} + o(1)},$$

from which Theorem 2 follows.

4.2. **Structure of paper.** The remainder of the paper is organized as follows.

- §5 contains certain minor lemmas which will be used at various points throughout the proof, as well as the statement of Poisson summation we shall use.
- §6 contains a statement and proof of the two-dimensional delta method we use, providing a perturbation of the method of [LMV24] for our purposes.
- §7 contains the application of the delta method followed by Poisson summation that leads to the decomposition of Proposition 4.1.
- §8 contains estimates on S_1 that will be used to handle the nonzero frequencies Σ_1^{12}.
- §9 contains upper bounds on S_2 germane to the estimation of the nonzero frequency contribution $\Sigma_2^{\{1,2\}}$, as well as a related estimate, Proposition 9.2, which will play a significant role in the
- §10 contains upper bounds on both the oscillatory integrals and their derivatives which will be used to bounding the nonzero frequency contributions.
- §11 contains preparatory lemmas leading up to the computation of the constants in front of the main terms of $\Sigma_j^{\{\}}$ and the secondary main term of $\Sigma_1^{\{\}}$ (which leads to the main term of Theorem 2)
- §12.1 contains a bound on the nonzero frequency contribution $\Sigma_1^{\{1,2\}}$ in Proposition 12.1. It is in this section that we apply subconvexity.

§12.2 contains a bound on the partial zero frequency contribution $\Sigma_1^{\{j\}}$ in Proposition 12.4.

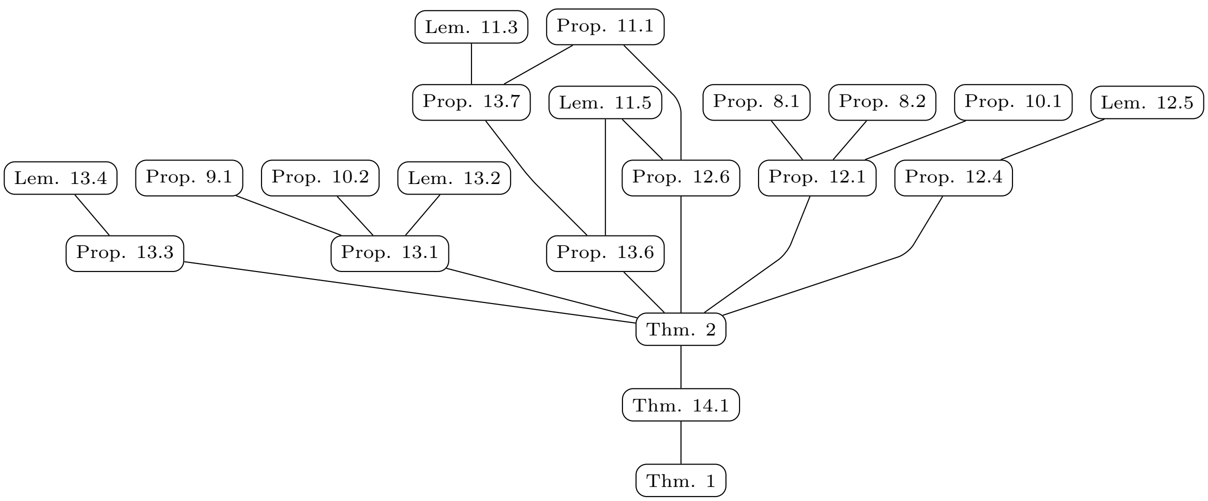
§12.3 contains an asymptotic for $\Sigma_1^{\{\}}$ in Proposition 12.6, giving a main term of size X^3/D and a secondary main term of size X^2, the former of which cancels with the main term of $\Sigma_2^{\{\}}$.

§13.1 contains a bound on the nonzero frequency contribution $\Sigma_1^{\{1,2\}}$ in Proposition 13.1.

§13.2 contains a bound on the partial zero frequency contribution $\Sigma_2^{\{j\}}$ in Proposition 13.3.

§13.3 contains an asymptotic for $\Sigma_2^{\{\}}$ in Proposition 13.6, matching the appropriate constant in front of X^3/D coming from Proposition 12.6.

We provide below a dependency graph containing most of the main key bounds and their dependencies for ease of reading and tracking.



5. Preliminary lemmas

5.1. **Factorization in a rank 2 submodule of $\mathcal{O}_K$.** In this section, we shall prove some basic facts about the factorization of elements in the orthogonal complement of $\boldsymbol{\ell}(\mathcal{O}_K)$ in $\mathcal{O}_K$, given by $\mathbb{Z} \oplus \mathbb{Z}\zeta$.

Lemma 5.1. *Take p a rational prime, $x, y \in \mathbb{Z}$, and $\mathfrak{p}_1, \mathfrak{p}_2 | p$ prime ideals in $\mathcal{O}_K$ such that $\mathfrak{p}_1, \mathfrak{p}_2 | x + y\zeta$. Then, at least one of the following holds:*

(1) $p | x, y$.
(2) $\mathfrak{p}_1 = \mathfrak{p}_2$, $N\mathfrak{p}_1 = p$.

In particular, if $p \nmid (x, y)$, we have

$$p^m | N(x + y\zeta) \iff \exists \mathfrak{p} | p \text{ s.t. } N\mathfrak{p} = p, \mathfrak{p}^m | x + y\zeta.$$

Proof. Suppose now that $p \nmid (x, y)$. Without loss of generality, suppose that $p \nmid y$, which we can harmlessly scale to be 1.

The condition that $\mathfrak{p}_1, \mathfrak{p}_2 | x + \zeta$ implies that $I = (p, x + \zeta) \subset \mathfrak{p}_1, \mathfrak{p}_2$. Then, we have that $\mathcal{O}_K/I$ is isomorphic to

$$(\mathcal{O}_K/p\mathcal{O}_K)/(x+\zeta) \cong \mathbb{Z}[\zeta]/(p, x+\zeta) \cong \mathbb{Z}[T]/(p, T^4+1, T+x) \cong (\mathbb{Z}/p\mathbb{Z})/(x^4+1).$$

It follows that $N(I) = p$ and that $\mathfrak{p}_1 = \mathfrak{p}_2 = I$, from which the desired result follows. $\square$

Lemma 5.2. *For p a rational prime and $k \geqslant 1$, we have that*

$$\sum_{x(p^k)} N((x+\zeta, p^k)) \ll p^k.$$

Proof. This follows by noting that for $k \geqslant 1$, $N((x+\zeta, p^k)) = (x^4+1, p^k)$, at which point the desired result follows from Hensel's lemma. $\square$

5.2. **Poisson summation over K.** The statement of Poisson summation we shall use is the following

Proposition 5.3. *Suppose that $w \in \mathcal{S}(K_\infty)$. Then, for any $\varphi \in \mathcal{O}_K \setminus \{0\}$, $g_\varphi : \mathcal{O}_K/(\varphi) \to \mathbb{C}$ and $X > 0$, we have that*

$$\sum_{\alpha \in \mathcal{O}_K} g_\varphi(\alpha) w\left(\frac{\alpha}{X}\right) = \frac{X^4}{\sqrt{N(\varphi)}} \sum_{\alpha \in \mathcal{O}_K} \hat{g}_\varphi(\alpha) \hat{w}\left(\frac{\alpha X}{\varphi}\right)$$

where the transforms $\hat{w}, \hat{g}_\varphi$ are defined as

$$\hat{w}(y) = \int_{K_\infty} w(x^\infty) e(-\langle x^\infty, y\rangle)\, dx^\infty,$$

$$\hat{g}_\varphi(\alpha) = \frac{1}{\sqrt{N(\varphi)}} \sum_{\beta \in \mathcal{O}_K/(\varphi)} g_\varphi(\beta)\psi_\varphi(\alpha\beta).$$

Note that this simply amounts to Poisson summation for sums over a sublattice of $\mathbb{Z}^4$ with coordinates $n_i \in \mathbb{Z}$ such that $\alpha = n_0 + \cdots + n_3\zeta^3$. In the case we are concerned with, we will only ever be considering $\varphi \in \mathbb{Z}$, so the correspondence is even more plain.

We elect to work over K rather than return to sums over $\mathbb{Z}^8$ because doing so motivates maneuvers in our exponential sum evaluation which might have otherwise been opaque.

6. The two dimensional δ-method

In this section, we show the expansion of the two dimensional δ-symbol we require. We follow along the lines of [LMV24], though we shall not require expanding into additive characters, the lack thereof simplifies the proof. We shall also require a certain perturbation of the parameters in [LMV24] so we need to show the following version of their result ourselves.

Theorem 6.1. *Suppose that* $\mathbf{n} \in \mathbb{Z}^2$ *with* $|n_1|, |n_2| < X$. *Take* $1 \ll D \ll X^{1/2}$. *Let* $\omega_1 \in C_c^\infty(\mathbb{R}_{>0})$. *Let* $\omega_2 \in C_c^\infty(\mathbb{R})$ *be even with* $\omega_2(0) = 0$ *such that* ω_2 *is compactly supported away from* 0. *Further suppose that*

$$\int_{\mathbb{R}^2} \omega_1(|\mathbf{x}|)\, d\mathbf{x} = \int_{\mathbb{R}_{>0}} \omega_2(x)\, dx = 1. \tag{6.1}$$

Then, for any $A > 0$, *we have that*

$$\begin{aligned}\mathbb{1}_{\mathbf{n}=0} = \frac{1}{D^2} \sum_{d \geqslant 1} \sum_{\substack{\mathbf{c} \in \mathbb{Z}^2 \\ (c_1,c_2)=1}} \omega_1\left(\frac{|\mathbf{c}|d}{D}\right) \frac{d}{\sqrt{DX}} \sum_{q \geqslant 1} \left(\omega_2\left(\frac{dq}{\sqrt{DX}}\right) - \omega_2\left(\frac{\det(\mathbf{c},\mathbf{n})}{q\sqrt{DX}}\right)\right) \mathbb{1}_{\substack{\det(\mathbf{c},\mathbf{n}) \equiv 0 (dq) \\ \mathbf{n} \equiv 0 (d)}} \\ - \frac{2}{D^2} \sum_{q \geqslant 1} \omega_1\left(\frac{|\mathbf{n}|}{qD}\right) \mathbb{1}_{\mathbf{n} \equiv 0 (q)} + O_{A,\omega_1,\omega_2}(D^{-A}),\end{aligned}$$

As an ingredient to the proof of Theorem 6.1, we shall require the following simple version of the usual δ-method of Duke-Friedlander-Iwaniec [DFI93].

Proposition 6.2. *Suppose that* $\omega \in C_c^\infty(\mathbb{R})$ *is even with* $\omega(0) = 0$ *and* $\hat{\omega}(0) = 2$, *such that* ω *is compactly supported away from* 0. *Then, for any* $A > 0$, $Q \geqslant 1$, *and* $n \in \mathbb{Z}$, *we have that*

$$\mathbb{1}_{n=0} = \frac{1}{Q} \sum_{q \geqslant 1} \left(\omega\left(\frac{q}{Q}\right) - \omega\left(\frac{n}{qQ}\right)\right) \mathbb{1}_{n \equiv 0 (q)} + O_{A,\omega}(Q^{-A}).$$

Proof of Theorem 6.1. We begin by noting for all $\mathbf{n} \neq 0$, there is an involution of

$$\{(\mathbf{c}, d, q) : n = \mathbf{c}dq, \mathbf{c} \text{ primitive}, d, q \geqslant 1\}$$

given by $(\mathbf{c}, d, q) \mapsto (\mathbf{c}, q, d)$. Therefore, for $\mathbf{n} \neq 0$, we have

$$\begin{aligned}0 &= \sum_{\substack{\mathbf{c} \text{ primitive} \\ \langle \mathbf{c},\mathbf{n}\rangle > 0}} \mathbb{1}_{\det(\mathbf{c},\mathbf{n})=0} \sum_{dq\mathbf{c}=\mathbf{n}} \left(\omega_1\left(\frac{|\mathbf{c}|d}{D}\right) - \omega_1\left(\frac{|\mathbf{n}|}{qD}\right)\right) \\ &= -\sum_{q|\mathbf{n}} \omega_1\left(\frac{|\mathbf{n}|}{qD}\right) + \sum_{\substack{\mathbf{c} \text{ primitive} \\ \langle \mathbf{c},\mathbf{n}\rangle > 0}} \sum_{d|\mathbf{n}} \mathbb{1}_{\det(\mathbf{c},\mathbf{n}/d)=0} \omega_1\left(\frac{|\mathbf{c}|d}{D}\right) \qquad (6.2) \\ &= -\sum_{q|\mathbf{n}} \omega_1\left(\frac{|\mathbf{n}|}{qD}\right) + \frac{1}{2} \sum_{\mathbf{c} \text{ primitive}} \sum_{d|\mathbf{n}} \mathbb{1}_{\det(\mathbf{c},\mathbf{n}/d)=0} \omega_1\left(\frac{|\mathbf{c}|d}{D}\right).\end{aligned}$$

At this point, we apply Theorem 6.2 with $Q = \sqrt{DX}/d, \omega = \omega_2$ to expand out $\mathbb{1}_{\det(\mathbf{c},\mathbf{n}/d)=0}$ and obtain that (6.2) equals

$$\begin{aligned} 0 = & -\sum_{q\geqslant 1}\mathbb{1}_{\mathbf{n}\equiv 0(q)}\omega_1\left(\frac{|\mathbf{n}|}{qD}\right) \\ & + \frac{1}{2}\sum_{d|\mathbf{n}}\sum_{\mathbf{c}\text{ primitive}}\omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\frac{1}{Q}\sum_{q\geqslant 1}\left(\omega_2\left(\frac{q}{Q}\right)-\omega_2\left(\frac{\det(\mathbf{c},\mathbf{n}/d)}{qQ}\right)\right)\mathbb{1}_{\det(\mathbf{c},\mathbf{n}/d)\equiv 0(q)} + O(D^{-A}) \end{aligned} \tag{6.3}$$

for any $A > 0$. Here we use that $d \ll D \ll X^{1/2}$ to obtain the error term $O(D^{-A})$. This completes the proof for $\mathbf{n} \neq 0$.

For $\mathbf{n} = 0$, the right-hand side of (6.3) equals

$$\sum_{d\geqslant 1}\sum_{\mathbf{c}\text{ primitive}}\omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\frac{1}{Q}\sum_{q\geqslant 1}\omega_2\left(\frac{q}{Q}\right) = D^2 + O(D^{-A} + Q^{-A})$$

by Poisson summation and the normalizations (6.1). The desired result follows. □

7. Proof of Proposition 4.1

In this section, we briefly discuss the application of the two dimensional delta method and the applications of Poisson summation that lead to the decomposition in Proposition 4.1. Recall that we are estimating

$$\Sigma = \sum_{\substack{\alpha_1,\alpha_2\in\mathcal{O}_K \\ \boldsymbol{\ell}(\alpha_1\alpha_2)=0 \\ \alpha_1\equiv\beta_1'(M) \\ \alpha_2\equiv\beta_2'(M)}}\phi_1\left(\frac{\alpha_1}{X_1}\right)\phi_2\left(\frac{\alpha_2}{X_2}\right).$$

Applying Theorem 6.1 to decompose $\mathbb{1}[\boldsymbol{\ell}(\alpha_1\alpha_2) = 0]$ (with X, D in Theorem 6.1 taken to be $\asymp X_1X_2$, D respectively, and any fixed choice of ω_1, ω_2), we obtain that

$$\Sigma = -2\Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 = \frac{1}{D^2}\sum_{q\geqslant 1}\sum_{\substack{\alpha_1,\alpha_2\in\mathcal{O}_K \\ \alpha_1\equiv\beta_1'(M) \\ \alpha_2\equiv\beta_2'(M)}}\mathbb{1}_{\boldsymbol{\ell}(\alpha_1\alpha_2)\equiv 0\,(q)}\omega_1\left(\frac{|\boldsymbol{\ell}(\alpha_1\alpha_2)|}{qD}\right),$$

$$\Sigma_2 = \frac{1}{D^2} \sum_{\substack{d\geqslant 1\\ \mathbf{c}\in\mathbb{Z}^2\\ (c_1,c_2)=1}} \omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\frac{d}{\sqrt{DX}}\sum_{q\geqslant 1}$$
$$\sum_{\substack{\alpha_1,\alpha_2\in\mathcal{O}_K\\ \alpha_1\equiv\beta_1'\,(M)\\ \alpha_2\equiv\beta_2'\,(M)}}\left(\omega_2\left(\frac{dq}{\sqrt{DX}}\right)-\omega_2\left(\frac{\det(\mathbf{c},\boldsymbol{\ell}(\alpha_1\alpha_2))}{q\sqrt{DX}}\right)\right)\mathbb{1}_{\substack{\det(\mathbf{c},\boldsymbol{\ell}(\alpha_1\alpha_2))\equiv 0\,(dq)\\ \boldsymbol{\ell}(\alpha_1\alpha_2)\equiv 0\,(d)}}.$$

Then, applying Poisson summation, Proposition 5.3, to both the sums over α_1, α_2 yields that

$$\Sigma_1 = \frac{1}{D^2}\sum_{\substack{q\geqslant 1\\ M|q}}\frac{1}{q^5}\sum_{\alpha_1,\alpha_2\in\mathcal{O}_K} S_1(\alpha_1,\alpha_2;q)I_1(\alpha_1,\alpha_2;q),$$

and

$$\Sigma_2 = \frac{1}{D^2}\sum_{\substack{d\geqslant 1\\ \mathbf{c}\in\mathbb{Z}^2\\ (c_1,c_2)=1}}\frac{1}{d^5}\omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\frac{d}{\sqrt{DX}}\sum_{\substack{q\geqslant 1\\ M|q}}\frac{1}{q^5}\sum_{\alpha_1,\alpha_2\in\mathcal{O}_K} S_2(\alpha_1,\alpha_2;\mathbf{c},d,q)I_2(\alpha_1,\alpha_2;\mathbf{c},d,q),$$

where the S_j and I_j were defined in (4.3)-(4.6). Note that we have rewritten qM as q above. Splitting based on which of the α_j are 0 yields the desired result.

8. Estimating S_1

In this seciton, we provide estimates for $S_1(\alpha_1,\alpha_2;q)$ when $\alpha_1\alpha_2\neq 0$. It is here that we obtain what will turn out to be cusp for Fourier coefficients we mentioned in our sketch at (1.8). This will be exploited in §12.1 when we apply subconvexity.

The estimation of $S_1(\alpha_1,\alpha_2;q)$ reduces to the case of $q=p^k$ for $k\geqslant 1$, for $S_1(\alpha_1,\alpha_2;q)$ is a multiplicative function in q. These cases we resolve with the following collection of results.

The first is a result we will be applying in the general case

Proposition 8.1. *Suppose that p is prime such that $p\nmid M$. Then, we have that*

$$S_1(\alpha_1,\alpha_2;p) = a_{\alpha_1\alpha_2}(p) + r_{\alpha_1\alpha_2}(p)$$

where

$$a_{n_0+n_1\zeta+n_2\zeta^2+n_3\zeta^3}(p) = -1 - \mathbb{1}_{p|n_3} + \#\left\{x(p) : n_0+n_1x+n_2x^2+n_3x^3\equiv 0(p)\right\}$$

and $r_{\alpha_1\alpha_2}(p)$ satisfies

$$r_{\alpha_1\alpha_2}(p) \ll \frac{1}{p} + \mathbb{1}_{p|N(\alpha_1\alpha_2)} + p\mathbb{1}_{p^2|N(\alpha_1\alpha_2)} + p^3\mathbb{1}_{p|\alpha_1\alpha_2}.$$

In all other cases, we can afford to show the following upper bounds.

Proposition 8.2. *Suppose that p is prime and $k \geqslant 1$. Then, we have that*

$$S_1(\alpha_1, \alpha_2; p^k) \ll (q, M)^2 p^{k-1} \sum_{0 \leqslant s \leqslant k} p^{3s} \mathbb{1}_{p^s | M^2 \alpha_1 \alpha_2}.$$

The proofs of Propositions 8.1 and 8.2 begin with the observation that by orthogonality,

$$S_1(\alpha_1, \alpha_2; q) = \frac{1}{(q, M)^8} \sum_{\gamma_1, \gamma_2 ((q,M))} \psi_{(q,M)}(-\gamma_1 \beta_1' - \gamma_2 \beta_2') S_1^* \left(\alpha_1 + \frac{q}{(q, M)} \gamma_1, \alpha_2 + \frac{q}{(q, M)} \gamma_2; q \right),$$

where

$$S_1^*(\alpha_1, \alpha_2; q) = \frac{1}{q^3} \sum_{\substack{\beta_1, \beta_2 (q) \\ \boldsymbol{\ell}(\beta_1 \beta_2) \equiv 0 (q/(q,M))}} \psi_q(\alpha_1 \beta_1 + \alpha_2 \beta_2).$$

Then, by a couple more applications of orthogonality, we have

$$\begin{aligned} S_1^*(\alpha_1, \alpha_2; q) &= \frac{1}{q^5} \sum_{x, y\, (q)} \sum_{\beta_1, \beta_2\, (q)} \psi_q(\alpha_1 \beta_1 + \alpha_2 \beta_2 - (q, M)(x + y\zeta) \beta_1 \beta_2) \\ &= \frac{1}{q} \sum_{x, y\, (q)} \sum_{\substack{\beta_1\, (q) \\ (q,M)(x+y\zeta)\beta_1 \equiv \alpha_2\, (q)}} \psi_q(\alpha_1 \beta_1) \\ &= \frac{(q, M)^2}{q} \sum_{\substack{x, y\, (q) \\ (q,M) | x, y}} \sum_{\substack{\beta_1\, (q) \\ (x+y\zeta)\beta_1 \equiv \alpha_2\, (q)}} \psi_q(\alpha_1 \beta_1). \end{aligned} \tag{8.1}$$

Armed with this, we shall now show the two propositions.

Proof of Proposition 8.1. Because $p \nmid M$, $S_1^* = S_1$, so we shall proceed starting from (8.1). Scaling β_1 by $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ and x, y by t^{-1} and averaging over t, we obtain

$$\begin{aligned} S_1(\alpha_1, \alpha_2; p) =& \frac{1}{p(p-1)} \sum_{\substack{\beta_1\, (p) \\ x, y\, (p) \\ (x+y\zeta)\beta_1 \equiv \alpha_2 (p)}} \sum_{\substack{t\, (p) \\ (t,p)=1}} \psi_p(t \alpha_1 \beta_1) \qquad (8.2) \\ =& \frac{1}{(p-1)} \sum_{\substack{\beta_1\, (p) \\ x, y\, (p) \\ \langle \alpha_1, \beta_1 \rangle \equiv 0\, (p) \\ (x+y\zeta)\beta_1 \equiv \alpha_2\, (p)}} 1 - \frac{1}{p(p-1)} \sum_{\substack{\beta_1\, (p) \\ x, y\, (p) \\ (x+y\zeta)\beta_1 \equiv \alpha_2\, (p)}} 1. \end{aligned}$$

We claim that the second sum is

$$\frac{1}{p(p-1)} \sum_{\substack{\beta_1\, (p) \\ x, y\, (p) \\ (x+y\zeta)\beta_1 \equiv \alpha_2\, (p)}} 1 = 1 + O\left(\frac{1}{p} + \mathbb{1}_{p | N(\alpha_2)} + p^2 \mathbb{1}_{p | \alpha_2} \right). \tag{8.3}$$

To see this precisely, note that the contribution of $x \equiv y \equiv 0(p)$ is given by

$$\frac{1}{p(p-1)} \sum_{\beta_1\,(p)} \mathbb{1}_{p|\alpha_2} \ll p^2 \mathbb{1}_{p|\alpha_2}.$$

On the other hand, the contribution of $x^4 + y^4 \equiv 0\,(p)$ with $(x,y) \not\equiv (0,0)\,(p)$ is given by

$$\frac{1}{p(p-1)} \sum_{\substack{\beta_1\,(p) \\ x,y\,(p),(xy,p)=1 \\ (x/y)^4+1\equiv 0\,(p) \\ (x+y\zeta)\beta_1\equiv\alpha_2\,(p)}} 1 \ll \mathbb{1}_{p|N(\alpha_2)}.$$

Since $(x + y\zeta, p) = 1 \Leftrightarrow x^4 + y^4 \not\equiv 0\,(p)$, we have

$$\begin{aligned}\frac{1}{p(p-1)} \sum_{\substack{\beta_1\,(p) \\ x,y\,(p) \\ (x+y\zeta)\beta_1\equiv\alpha_2\,(p)}} 1 &= \frac{1}{p(p-1)} \sum_{\substack{x,y\,(p) \\ x^4+y^4\not\equiv 0\,(p)}} 1 + O\Big(\mathbb{1}_{p|N(\alpha_2)} + p^2\mathbb{1}_{p|\alpha_2}\Big) \\ &= 1 + O\Big(\frac{1}{p} + \mathbb{1}_{p|N(\alpha_2)} + p^2\mathbb{1}_{p|\alpha_2}\Big).\end{aligned}$$

It remains now to estimate the first sum in (8.2), which is given by

$$\frac{1}{p-1} \sum_{\substack{\beta_1\,(p) \\ x,y\,(p) \\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p) \\ (x+y\zeta)\beta_1\equiv\alpha_2\,(p)}} 1. \tag{8.4}$$

The contribution of $y \equiv 0\,(p)$ to (8.4) is

$$\begin{aligned}\frac{1}{p-1} \sum_{\substack{\beta_1\,(p) \\ x\,(p) \\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p) \\ x\beta_1\equiv\alpha_2\,(p)}} 1 &= \frac{1}{p-1} \sum_{\substack{\beta_1\,(p) \\ x\,(p),(x,p)=1 \\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p) \\ x\beta_1\equiv\alpha_2\,(p)}} 1 + \frac{1}{p-1} \sum_{\substack{\beta_1\,(p) \\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p)}} \mathbb{1}_{p|\alpha_2} \\ &= \frac{1}{p-1} \sum_{\substack{x\,(p),(x,p)=1 \\ \langle\alpha_1\alpha_2,x\rangle\equiv 0\,(p)}} 1 + O\left(p^3\mathbb{1}_{p|\alpha_2}\right) = \mathbb{1}_{p|\langle\alpha_1,\alpha_2\rangle} + O\left(p^3\mathbb{1}_{p|\alpha_2}\right).\end{aligned}$$

Hence we have

$$\frac{1}{p-1}\sum_{\substack{\beta_1\,(p)\\ x,y\,(p)\\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p)\\ (x+y\zeta)\beta_1\equiv\alpha_2\,(p)}} 1 = \frac{1}{p-1}\sum_{\substack{\beta_1\,(p)\\ x,y\,(p),(y,p)=1\\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p)\\ (x+y\zeta)\beta_1\equiv\alpha_2\,(p)}} 1 + \mathbb{1}_{p|\langle\alpha_1,\alpha_2\rangle} + O\left(p^3\mathbb{1}_{p|\alpha_2}\right)$$

$$= \sum_{\substack{\beta_1\,(p)\\ x\,(p)\\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p)\\ (x+\zeta)\beta_1\equiv\alpha_2\,(p)}} 1 + \mathbb{1}_{p|\langle\alpha_1,\alpha_2\rangle} + O\bigg(p^3\mathbb{1}_{p|\alpha_2}\bigg), \tag{8.5}$$

At this point, we suppose without loss of generality that $N((\alpha_2,p))|N((\alpha_1,p))$. Then, the contribution of x with $(x+\zeta,p)>1 \Leftrightarrow x^4+1\equiv 0(p)$ is

$$\sum_{\substack{\beta_1\,(p)\\ x\,(p)\\ x^4+1\equiv 0\,(p)\\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p)\\ (x+\zeta)\beta_1\equiv\alpha_2\,(p)}} 1 \leqslant 4p\mathbb{1}_{p|N(\alpha_2)} \leqslant 4p\mathbb{1}_{p^2|N(\alpha_1\alpha_2)}. \tag{8.6}$$

As for the remaining contribution, we have that

$$\sum_{\substack{\beta_1\,(p)\\ x\,(p)\\ (x+\zeta,p)=1\\ \langle\alpha_1,\beta_1\rangle\equiv 0\,(p)\\ (x+\zeta)\beta_1\equiv\alpha_2\,(p)}} 1 = \sum_{\substack{x\,(p)\\ (x+\zeta,p)=1\\ \langle\alpha_1\alpha_2,\overline{x+\zeta}\rangle\equiv 0\,(p)}} 1 = \sum_{\substack{u,v,w\,(p)\\ x\,(p)\\ (x+\zeta,p)=1\\ (x+\zeta)(u+v\zeta+w\zeta^2)\equiv\alpha_1\alpha_2\,(p)}} 1. \tag{8.7}$$

The contribution of $\alpha_1\alpha_2\equiv 0(p)$ is then at most

$$\leqslant p\mathbb{1}_{p|\alpha_1\alpha_2}, \tag{8.8}$$

so we shall from now on focus on the case of $p\nmid\alpha_1\alpha_2$. In this case, we have that (8.7) equals

$$\sum_{\substack{u,v,w\,(p)\\ x\,(p),(x+\zeta,p)=1\\ (x+\zeta)(u+v\zeta+w\zeta^2)\equiv\alpha_1\alpha_2\,(p)}} 1$$
$$= \mathbb{1}_{p\nmid N(\alpha_1\alpha_2)}\sum_{\substack{u,v,w,x\,(p)\\ (x+\zeta)(u+v\zeta+w\zeta^2)\equiv\alpha_1\alpha_2\,(p)}} 1 + O\bigg(\mathbb{1}_{p|N(\alpha_1\alpha_2)}\sum_{\substack{u,v,w,x\,(p)\\ (x+\zeta)(u+v\zeta+w\zeta^2)\equiv\alpha_1\alpha_2\,(p)}} 1\bigg). \tag{8.9}$$

Note that

$$\sum_{\substack{u,v,w,x\,(p)\\ (x+\zeta)(u+v\zeta+w\zeta^2)\equiv\alpha_1\alpha_2\,(p)}} 1 = \#\big\{x(p) : n_0+n_1x+n_2x^2+n_3x^3\equiv 0\,(p)\big\} \leqslant 4 \tag{8.10}$$

if we write $\alpha_1\alpha_2 = n_0 + n_1\zeta + n_2\zeta^2 + n_3\zeta^3$, with the upper bound of 4 following from the assumption that $p \nmid \alpha_1\alpha_2$. Therefore, it follows that (8.9) equals

$$\#\{x(p) : n_0 + n_1 x + n_2 x^2 + n_3 x^3 \equiv 0\,(p)\} + O(\mathbb{1}_{p|N(\alpha_1\alpha_2)}).$$

Combining (8.2), (8.3), (8.5), (8.6), (8.7), (8.8), (8.9), and (8.10), the desired result follows. □

Proof of Proposition 8.2. We shall show that

$$S_1^*(\alpha_1, \alpha_2; p^k) \ll (q, M)^2 p^{k-1} \sum_{0 \leqslant s \leqslant k} p^{3s} \mathbb{1}_{p^s|\alpha_1\alpha_2}$$

from which the desired result follows upon noting the implications of the divisibilities when the $\tilde{\alpha}_i$ are considered (specifically, $p^s|\tilde{\alpha}_1\tilde{\alpha}_2 \implies p^s|M^2\alpha_1\alpha_2$).

It can be checked that without loss of generality, we may suppose that $(q, M) = 1$, for by (8.1), we are reduced to this case at the cost of a factor of $(q, M)^5$.

By symmetry, $S_1(\alpha_1, \alpha_2; p^k) = S_1(\alpha_2, \alpha_1; p^k)$, so we may suppose without loss of generality that

$$p^a|\alpha_2 \quad \Rightarrow \quad p^a|\alpha_1 \text{ for any } a \geqslant 1. \tag{8.11}$$

With (8.11) in place, we start again with (8.1). We wish to bound

$$\frac{1}{p^k} \sum_{x,y(p^k)} \sum_{\substack{\beta_1(p^k) \\ (x+y\zeta)\beta_1 \equiv \alpha_2(p^k)}} \psi_{p^k}(\alpha_1\beta_1). \tag{8.12}$$

As in the proof of Proposition 8.1, we average over dilations of β_1 by $(\mathbb{Z}/p^k\mathbb{Z})^\times$. We may also pull out the factors of p from (x, y), after which we obtain that (8.12) equals

$$p^{3k}\mathbb{1}_{p^k|\alpha_2} + \frac{p}{p-1} \sum_{\substack{0 \leqslant s < k \\ p^s|\alpha_2}} \sum_{\eta \in \{0,1\}} (-1)^\eta \frac{1}{p^{k+\eta}} \sum_{\substack{x,y(p^{k-s}) \\ (x,y,p)=1}} \sum_{\substack{\beta_1(p^k) \\ \langle \alpha_1, \beta_1 \rangle \equiv 0(p^{k-\eta}) \\ (x+y\zeta)\beta_1 \equiv \alpha_2 p^{-s}(p^{k-s})}} 1.$$

With (8.11), this is

$$\begin{aligned} &\leqslant p^{3k}\mathbb{1}_{p^k|\alpha_1,\alpha_2} + \frac{p}{p-1} \sum_{\substack{0 \leqslant s < k \\ p^s|\alpha_1,\alpha_2}} \frac{1}{p^k} \sum_{\substack{x,y(p^{k-s}) \\ (x,y,p)=1}} \sum_{\substack{\beta_1(p^k) \\ \langle \alpha_1 p^{-s}, \beta_1 \rangle \equiv 0(p^{k-s}) \\ (x+y\zeta)\beta_1 \equiv \alpha_2 p^{-s}(p^{k-s})}} 1 \\ &= p^{3k}\mathbb{1}_{p^{2k}|\alpha_1\alpha_2} + \frac{p}{p-1} \sum_{\substack{0 \leqslant s < k \\ p^s|\alpha_1,\alpha_2}} \frac{p^{4s}}{p^k} \sum_{\substack{x,y(p^{k-s}) \\ (x,y,p)=1}} \sum_{\substack{\beta_1(p^{k-s}) \\ \langle \alpha_1 p^{-s}, \beta_1 \rangle \equiv 0(p^{k-s}) \\ (x+y\zeta)\beta_1 \equiv \alpha_2 p^{-s}(p^{k-s})}} 1. \end{aligned} \tag{8.13}$$

We shall now bound the contribution of some fixed $0 \leqslant s < k$ to (8.13), which is

$$\begin{aligned}\frac{p^{4s}}{p^k} \sum_{\substack{x,y(p^{k-s}) \\ (x,y,p)=1}} \sum_{\substack{\beta_1(p^{k-s}) \\ \langle \alpha_1 p^{-s}, \beta_1\rangle \equiv 0\,(p^{k-s}) \\ (x+y\zeta)\beta_1 \equiv \alpha_2 p^{-s}\,(p^{k-s})}} 1 \\ \leqslant p^{3s} \sum_{x(p^{k-s})} \sum_{\substack{\beta_1(p^{k-s}) \\ \langle \alpha_1 p^{-s}, \beta_1\rangle \equiv 0\,(p^{k-s}) \\ (x+\zeta)\beta_1 \equiv \alpha_2 p^{-s}\,(p^{k-s})}} 1 + p^{3s} \sum_{y(p^{k-s})} \sum_{\substack{\beta_1(p^{k-s}) \\ \langle \alpha_1 p^{-s}, \beta_1\rangle \equiv 0\,(p^{k-s}) \\ (1+y\zeta)\beta_1 \equiv \alpha_2 p^{-s}\,(p^{k-s})}} 1,\end{aligned} \tag{8.14}$$

where the second bound has followed by separating the cases $(y,p) = 1$ and $(x,p) = 1$ and rescaling. It thus suffice to bound only the first sum on the RHS of (8.14)

$$p^{3s} \sum_{x(p^{k-s})} \sum_{\substack{\beta_1(p^{k-s}) \\ \langle \alpha_1 p^{-s}, \beta_1\rangle \equiv 0\,(p^{k-s}) \\ (x+\zeta)\beta_1 \equiv \alpha_2 p^{-s}\,(p^{k-s})}} 1, \tag{8.15}$$

for the treatment of its counterpart is identical.

We shall split based on the power $p^h || (x^4+1, p^{k-s})$ for some $h \leqslant k-s$. For a given h, we have that the condition $(x+\zeta)\beta_1 \equiv \alpha_2 p^{-s}\,(p^{k-s})$ determines β_1 for a given x up to p^h possibilities. Furthermore, the conditions in the sum over β_1 imply that

$$\begin{aligned}&\left\langle \frac{\alpha_1\alpha_2 p^{-2s}}{x+\zeta}, \beta_1 \right\rangle \equiv 0\,(p^{k-s-h}) \\ &\qquad \implies f_{\alpha_1\alpha_2 p^{-2s}}(x) = \langle \alpha_1\alpha_2 p^{-2s}, (x+\zeta^3)(x+\zeta^5)(x+\zeta^7)\rangle \equiv 0\,(p^{k-s}).\end{aligned}$$

Therefore, the contribution of some $0 \leqslant h \leqslant k-s$ to (8.15) is at most

$$p^{3s} \sum_{x(p^{k-s})} \mathbb{1}[f_{\alpha_1\alpha_2 p^{-2s}}(x) \equiv 0(p^{k-s})]. \tag{8.16}$$

If $s = 0$, this is $\leqslant p^k \mathbb{1}_{p^k|\alpha_1\alpha_2} + p^{k-1}$, which is acceptable, so we'll suppose $s \geqslant 1$ from now on.

As above, we have two cases: either $p^{k-s}|\alpha_1\alpha_2 p^{-2s}$, whose contribution we may bound by

$$p^{k+2s}\mathbb{1}_{p^{k+s}|\alpha_1\alpha_2} \leqslant p^{k-1}p^{2s+1}\mathbb{1}_{p^{2s+1}|\alpha_1\alpha_2} \leqslant p^{k-1}p^{3s}\mathbb{1}_{p^s|\alpha_1\alpha_2},$$

which is acceptable, or $p^{k-s} \nmid \alpha_1\alpha_2 p^{-2s}$, in which case the polynomial has at most $3p^{k-s-1}$ roots modulo p^{k-s} and the contribution to (8.16) is at most

$$p^{k-1+2s}\mathbb{1}_{p^{2s}|\alpha_1\alpha_2} = p^{k-1}p^{2s}\mathbb{1}_{p^{2s}|\alpha_1\alpha_2} \leqslant p^{k-1}p^{3s}\mathbb{1}_{p^s|\alpha_1\alpha_2},$$

which is also admissible. The desired result follows. $\square$

9. Bounding S_2

Since we will choose the parameter D in a way that shifts the burden of the trivial bound from Σ_2 onto Σ_1 (which enjoys a saving from subconvexity), we require only an upper bound for S_2 for $\alpha_1\alpha_2 \neq 0$. We shall show a somewhat less complete bound than we did in §8, opting to use the divisor bound from the sums over d, q in §13.1 to yield the final bound on $\Sigma_2^{\{1,2\}}$.

We recall the definition of S_2 in (4.5), that

$$S_2(\alpha_1, \alpha_2; \mathbf{c}, d, q) = \frac{1}{d^3q^3} \sum_{\substack{\beta_1,\beta_2 \in \mathcal{O}_K/(dq) \\ \det(\mathbf{c}, \boldsymbol{\ell}(\beta_1\beta_2)) \equiv 0 (dq/(q,M)) \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (d) \\ \beta_1 \equiv \beta_1' ((dq,M)) \\ \beta_2 \equiv \beta_2' ((dq,M))}} \psi_{dq}(\alpha_1\beta_1 + \alpha_2\beta_2).$$

We begin by using orthogonality to detect $\beta_i \equiv \beta_i'((dq, M))$, which yields that

$$S_2(\alpha_1, \alpha_2; \mathbf{c}, d, q) \;=\; \frac{1}{(dq, M)^8} \sum_{\gamma_1,\gamma_2((dq,M))} \psi_{(dq,M)}(-\gamma_1\beta_1' - \gamma_2\beta_2') S_2^*(\tilde{\alpha}_1, \tilde{\alpha}_2; \mathbf{c}, d, q) \tag{9.1}$$

where

$$S_2^*(\alpha_1, \alpha_2; \mathbf{c}, d, q) = \frac{1}{d^3q^3} \sum_{\substack{\beta_1,\beta_2 \in \mathcal{O}_K/(dq) \\ \det(\mathbf{c}, \boldsymbol{\ell}(\beta_1\beta_2)) \equiv 0\, (dq/(q,M)) \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (d)}} \psi_{dq}(\alpha_1\beta_1 + \alpha_2\beta_2),$$

with

$$\tilde{\alpha}_i = \alpha_i + \frac{dq}{(dq, M)}\gamma_i.$$

As we do not care for losses of $O(M^{O(1)})$, we will be fixing γ_1, γ_2 from now on and just bounding S_2^*. Write

$$\gamma_{\mathbf{c}}(x, y, z; q) := (q, M)x(c_2 - c_1\zeta) + q(y + z\zeta).$$

By orthogonality, we have

$$\begin{aligned} S_2^*(\alpha_1, \alpha_2; \mathbf{c}, d, q) &= \frac{1}{d^6q^4} \sum_{\substack{x\,(dq) \\ y,z(d)}} \sum_{\beta_1,\beta_2 \in \mathcal{O}_K/(dq)} \psi_{dq}(\alpha_1\beta_1 + \alpha_2\beta_2 - \gamma_{\mathbf{c}}(x, y, z; q)\beta_1\beta_2) \\ &= \frac{1}{d^2} \sum_{\substack{x(dq) \\ y,z(d)}} \sum_{\substack{\beta_1 \in \mathcal{O}_K/(dq) \\ \beta_1\gamma_{\mathbf{c}}(x,y,z;q) \equiv \alpha_2 (dq)}} \psi_{dq}(\alpha_1\beta_1). \end{aligned} \tag{9.2}$$

We shall be showing the following bound on S_2^*.

Proposition 9.1. *We have that*

$$|S_2^*(\alpha_1,\alpha_2;\mathbf{c},d,q)| \ll (dq)^{o(1)} \sum_{\substack{g'h'|d\\ g''h''|q\\ g'g''|\alpha_1\alpha_2\\ h'h''|N(\alpha_1),N(\alpha_2/g)}} \frac{g^4h}{(dq)^2}\sum_{x,y,z(dq)}\sum_{r|dq}\frac{1}{r}\mathbb{1}[Y_{\mathbf{c}}(\alpha_1\alpha_2,x,y,z;q)\equiv 0\,(dq/r)],$$

where we write $g=g'g'',h=h'h''$ *and*

$$Y_{\mathbf{c}}(\alpha,x,y,z;q)=\langle\alpha,\gamma_{\mathbf{c}}(x,y,z;q)^*\rangle.$$

Here $\gamma^*=N(\gamma)/\gamma$ *for* $\gamma\in\mathcal{O}_K$ *as before, so* $Y_{\mathbf{c}}(\alpha,x,y,z;q)$ *is a cubic form in* x,y,z*. In particular, we have*

$$|S_2(\alpha_1,\alpha_2;\mathbf{c},d,q)| \ll (dq)^{o(1)} \sum_{\substack{g'h'|d\\ g''h''|q\\ g'g''|M\alpha_1\alpha_2\\ h'^2h''^2|M^2N(\frac{\alpha_1\alpha_2}{g'g''})}} \frac{g^4h}{(dq)^2}\sum_{x,y,z(dq)}\sum_{r|dq}\frac{1}{r}\mathbb{1}[MY_{\mathbf{c}}(\alpha_1\alpha_2,x,y,z;q)\equiv 0\,(dq/r)].$$

Proof. As S_2^* is multiplicative in d,q, we may suppose throughout the proof that q and d are both powers of some prime p.

Scaling and averaging over $(t,dq)=1$ and then applying orthogonality, we obtain that

$$S_2^*(\alpha_1,\alpha_2;\mathbf{c},d,q)=\frac{dq}{\varphi(dq)}\sum_{r|dq}\frac{\mu(r)}{(dq)^2r}\sum_{x,y,z(dq)}\sum_{\substack{\beta_1(dq)\\ \langle\alpha_1,\beta_1\rangle\equiv 0(dq/r)\\ \gamma_{\mathbf{c}}(x,y,z;q)\beta_1\equiv\alpha_2(dq)}}1.$$

Note that

$$\begin{cases}\langle\alpha_1,\beta_1\rangle\equiv 0\,(dq/r)\\ \gamma_{\mathbf{c}}(x,y,z;q)\beta_1\equiv\alpha_2\,(dq)\end{cases}\implies \langle\alpha_1\alpha_2,\gamma_{\mathbf{c}}(x,y,z;q)^*\rangle\equiv 0\,(dq/r). \tag{9.3}$$

By the symmetry of α_1,α_2 in S_2, we shall suppose without loss of generality that with g^* the maximal power of p dividing α_1, with $g^*|\alpha_2$ and

$$v_p(N(\alpha_1))\geqslant v_p(N(\alpha_2/g^*)). \tag{9.4}$$

Suppose that $(\gamma_{\mathbf{c}}(x,y,z;q),dq)=(\eta)g$ for some $g\geqslant 1$ and some primitive (not divisible by any rational prime) $\eta\in\mathcal{O}_K$ with $N(\eta)=h$, say. Then, the congruence condition on α_2 implies that $g\eta|\alpha_2$, which implies $\eta|\alpha_2/g^*$. So by (9.4), we have $h^2|N(\alpha_1\alpha_2/g)$. We furthermore have that $hg|dq$, and we may split so that $h=h'h''$ and $g=g'g''$ with $h'g'|d,h''g''|q$.

Combined with (9.3), we obtain that

$$|S_2^*(\alpha_1,\alpha_2,\mathbf{c},d,q)| \ll \frac{dq}{\varphi(dq)} \sum_{\substack{h'g'|d\\ h''g''|q\\ g'g''|\alpha_1\alpha_2\\ h'h''|N(\alpha_1),N(\alpha_2/g)}} \frac{g^4h}{(dq)^2} \sum_{\substack{x,y,z(dq)\\ g|\gamma_{\mathbf{c}}(x,y,z;q)}} \sum_{r|dq} \frac{1}{r}\mathbb{1}[\langle \alpha_1\alpha_2,\gamma_{\mathbf{c}}(x,y,z;q)^*\rangle \equiv 0(dq/r)]$$

with the notation $g=g'g''$ and $h=h'h''$, as desired. $\square$

We now consider the 0 frequency case $\alpha_1=\alpha_2=0$. We will renormalize for clarity at this point, taking

$$\tilde{N}_2(\mathbf{c},d;q)=\frac{1}{d^3q^4}S_2(0,0;d,q)=\frac{1}{d^6q^7}\sum_{\substack{\beta_1,\beta_2\in\mathcal{O}_K/(dq)\\ \det(\mathbf{c},\boldsymbol{\ell}(\beta_1\beta_2))\equiv 0\,(dq/(q,M))\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(d)\\ \beta_i\equiv\beta_i'\,((dq,M))}} 1.$$

Notice that $\tilde{N}_2(\mathbf{c},d,q)$ is multiplicative in d and q. Recall the notation that we take $m_p=v_p(M)$.

Proposition 9.2. *Let $\mathbf{c}=(c_1,c_2)$ with c_1,c_2 being positive integers coprime with each other. Let p be prime and take integers $h\geqslant 0$ and $k\geqslant m_p$. Then we have the bound*

$$\tilde{N}_2(\mathbf{c},p^h;p^k)\ll (h+k+1)p^{2m_p}.$$

Write $p^\ell=(c_1^4+c_2^4,p^{h+k+1})$. Then we have

$$|\tilde{N}_2(\mathbf{c},p^h;p^{k+1})-\tilde{N}_2(\mathbf{c},p^h;p^k)|\ll (h+k+1)p^{4m_p-2h-3k-3+\ell}. \tag{9.5}$$

Proof. Write $m=m_p$ for simplicity. We first prove the second statement. From (9.1) to (9.2), we have that

$$\tilde{N}_2(\mathbf{c},p^h;p^{k+1})=\frac{1}{p^{4m+5h+4k+4}}\sum_{\gamma\,(p^m)}\psi_{p^m}(-\gamma\beta_2')\sum_{\substack{x\,(p^{h+k+1})\\ \gamma_{\mathbf{c}}(x,y,z;p^{k+1})}}\sum_{y,z\,(p^h)}\sum_{\substack{\beta\,(p^{h+k+1})\\ \beta\equiv p^{h+k+1-m}\gamma\,(p^{h+k+1})\\ \beta\equiv\beta_1'\,(p^m)}} 1. \tag{9.6}$$

Notice that the contribution of $p|x$ to above is given by

$$\frac{1}{p^{4m+5h+4k+4}}\sum_{\gamma\,(p^m)}\psi_{p^m}(-\gamma\beta_2')\sum_{\substack{x\,(p^{h+k})\\ \gamma_{\mathbf{c}}(px,y,z;p^{k+1})}}\sum_{y,z\,(p^h)}\sum_{\substack{\beta\,(p^{h+k+1})\\ \beta\equiv p^{h+k+1-m}\gamma\,(p^{h+k+1})\\ \beta\equiv\beta_1'\,(p^m)}} 1$$

$$=\frac{1}{p^{4m+5h+4k}}\sum_{\gamma\,(p^m)}\psi_{p^m}(-\gamma\beta_2')\sum_{\substack{x\,(p^{h+k})\\ \gamma_{\mathbf{c}}(x,y,z;p^k)}}\sum_{y,z\,(p^h)}\sum_{\substack{\beta\,(p^{h+k})\\ \beta\equiv p^{h+k-m}\gamma\,(p^{h+k})\\ \beta\equiv\beta_1'\,(p^m)}} 1=\tilde{N}_2(\mathbf{c},p^h;p^k).$$

Hence the difference is given by

$$\begin{aligned}D(\mathbf{c},p^h;p^k) &:= \tilde{N}_2(\mathbf{c},p^h;p^{k+1}) - \tilde{N}_2(\mathbf{c},p^h;p^k)\\ &= \frac{1}{p^{4m+5h+4k+4}} \sum_{\gamma\,(p^m)} \psi_{p^m}(-\gamma\beta_2') \sideset{}{^*}\sum_{\substack{x\,(p^{h+k+1})\\ \gamma_{\mathbf{c}}(x,y,z;p^{k+1})\beta\equiv p^{h+k+1-m}\gamma\,(p^{h+k+1})\\ \beta\equiv\beta_1'\,(p^m)}} \sum_{y,z\,(p^h)} \sum_{\beta(p^{h+k+1})} 1.\end{aligned}$$

Note that γ is completely determined by the other variables, giving us

$$D(\mathbf{c},p^h;p^k) \ll \frac{1}{p^{4m+5h+4k+4}} \sideset{}{^*}\sum_{\substack{x\,(p^{h+k+1})\\ \gamma_{\mathbf{c}}(x,y,z;p^{k+1})\beta\equiv 0\,(p^{h+k+1-m})}} \sum_{y,z\,(p^h)} \sum_{\beta\,(p^{h+k+1})} 1,$$

which is equal to

$$\frac{1}{p^{3h+4k+4-m}} \sideset{}{^*}\sum_{\substack{x\,(p^{h+k+1-m})\\ p^m x(c_2-c_1\zeta)\beta\equiv 0\,(p^{h+k+1-m})}} \sum_{\beta\,(p^{h+k+1-m})} 1 \tag{9.7}$$

if $m > h$, and

$$\frac{1}{p^{5h+4k+4-3m}} \sideset{}{^*}\sum_{\substack{x\,(p^{h+k+1-m})\\ \gamma_{\mathbf{c}}(x,y,z;p^{k+1})\beta\equiv 0\,(p^{h+k+1-m})}} \sum_{y,z\,(p^{h-m})} \sum_{\beta\,(p^{h+k+1-m})} 1. \tag{9.8}$$

if $m \leqslant h$.

For $m > h$, counting the number of β, Lemma 5.1 implies that the sum in (9.7) is

$$\begin{aligned}&\leqslant \frac{1}{p^{2h+3k+3}} \sum_{\substack{\beta\,(p^{h+k+1-m})\\ (c_2-c_1\zeta)\beta\equiv 0\,(p^{\max\{0,h+k+1-2m\}})}} 1 = \frac{1}{p^{2h+3k+3-4m}} N((c_2-c_1\zeta, p^{\max\{0,h+k+1-2m\}}))\\ &= \frac{1}{p^{2h+3k+3-4m}} (c_1^4+c_2^4, p^{\max\{0,h+k+1-2m\}}) \leqslant p^{4m-2h-3k-3+\ell}. \end{aligned}\tag{9.9}$$

For $m \leqslant h$, counting the number of β, (9.8) is equal to

$$\frac{1}{p^{5h+4k+4-3m}} \sideset{}{^*}\sum_{x\,(p^{h+k+1-m})} \sum_{y,z\,(p^{h-m})} N((\gamma_{\mathbf{c}}(x,y,z;p^{k+1}), p^{h+k+1-m})).$$

Case 1: Suppose $(c_2,p)=1$, then we can perform a change of variable $x \mapsto \overline{c_2}(x - p^{k+1-m}y)$ and $z \mapsto z - c_1\overline{c_2}y$ to get

$$D(\mathbf{c},p^h;p^k) \ll \frac{1}{p^{4h+4k+4-2m}} \sideset{}{^*}\sum_{x\,(p^{h+k+1-m})} \sum_{z\,(p^{h-m})} N((p^m x + (p^{k+1}z - c_1\overline{c_2}p^m x)\zeta, p^{h+k+1-m})).$$

Write $c_1 = p^b c_1'$ for some $b \geqslant 0$ and $(c_1',p)=1$.

If $b \geqslant 1$, we have

$$D(\mathbf{c}, p^h; p^k) \ll p^{2m-4h-4k-4} \sideset{}{^*}\sum_{x\,(p^{h+k+1-m})} \sum_{z\,(p^{h-m})} p^{4m} \ll p^{4m-2h-3k-3}. \tag{9.10}$$

On the other hand, if $b = 0$, we have

$$D(\mathbf{c}, p^h; p^k) \ll \frac{1}{p^{4h+4k+4-6m}} \sideset{}{^*}\sum_{x\,(p^{h+k+1-m})} \sum_{g=0}^{h-m} \sideset{}{^*}\sum_{z\,(p^{h-m-g})} N((x + (p^{k+g+1-m}z - \overline{c_2}c_1x)\zeta, p^{h+k+1-2m})).$$

Lemma 5.1 then implies that this is equal to

$$\sum_{g=0}^{h-m} \frac{1}{p^{4h+4k+4-6m}} \sideset{}{^*}\sum_{x(p^{h+k+1-m})} \sideset{}{^*}\sum_{z\,(p^{h-m-g})} (x^4 + (p^{k+g+1-m}z - \overline{c_2}c_1x)^4, p^{h+k+1-2m}).$$

Expanding the fourth power, Hensel's lemma (Lemma 5.2) then implies that

$$\begin{aligned} D(\mathbf{c}, p^h; p^k) &\ll \sum_{g=0}^{h-m} \frac{h+k+1}{p^{4h+4k+4-6m}} \sideset{}{^*}\sum_{z\,(p^{h-m-g})} p^{h+k+1-m+\min\{\ell, k+g+1-m\}} \\ &\ll (h+k+1)p^{4m-2h-3k-3+\ell}. \end{aligned} \tag{9.11}$$

<u>Case 2:</u> Suppose $p|c_2$, then $(c_1, p) = 1$ as $(c_1, c_2) = 1$. Write $p^b || c_2$ for some $b \geqslant 1$. Performing a change of variable $x \mapsto \overline{c_1}(x + p^{k+1-m}z)$ and $y \mapsto y - \overline{c_1}c_2z$ yields

$$\begin{aligned} D(\mathbf{c}, p^h; p^k) &\ll \frac{1}{p^{4h+4k+4-2m}} \sideset{}{^*}\sum_{x\,(p^{h+k+1-m})} \sum_{y\,(p^{h-m})} N((\overline{c_1}c_2p^mx + p^{k+1}y - p^mx\zeta, p^{h+k+1-m})) \\ &= \frac{1}{p^{4h+4k+4-2m}} \sideset{}{^*}\sum_{x(p^{h+k+1-m})} \sum_{y(p^{h-m})} p^{4m} \ll p^{4m-2h-3k-3}. \end{aligned} \tag{9.12}$$

Observe that (9.9), (9.10) and (9.12) are all smaller than (9.11). Hence (9.11) holds in all cases.

Finally, we have to show the bound for $\tilde{N}_2$. Starting again from (9.6), with γ completely determined by the other variables, we have

$$\begin{aligned} \tilde{N}_2(\mathbf{c}, p^h; p^m) &\leqslant \frac{1}{p^{8m+5h}} \sum_{\substack{x\,(p^{h+m}) \\ (p^mx(c_2-c_1\zeta)+y+z\zeta)\beta\equiv 0\,(p^h)}} \sum_{y,z\,(p^h)} \sum_{\beta\,(p^{h+m})} 1 \\ &= \frac{1}{p^{3m+5h}} \sum_{\substack{x,y,z\,(p^h) \\ (p^mx(c_2-c_1\zeta)+y+z\zeta)\beta\equiv 0\,(p^h)}} \sum_{\beta\,(p^h)} 1. \end{aligned}$$

Performing change of variables $y \mapsto y - c_2 p^m x$ and $z \mapsto z + c_1 p^m x$ gives us

$$\tilde{N}_2(\mathbf{c}, p^h; p^m) \leqslant \frac{1}{p^{3m+4h}} \sum_{\substack{y,z\,(p^h) \\ (y+z\zeta)\beta \equiv 0\,(p^h)}} \sum_{\beta\,(p^h)} 1 = \frac{1}{p^{3m+4h}} \sum_{y,z\,(p^h)} N((y+z\zeta, p^h)).$$

Pulling out the powers of p from y and z, we have

$$\begin{aligned}\tilde{N}_2(\mathbf{c}, p^h; p^m) &\leqslant \frac{1}{p^{3m+4h}} \sum_{0\leqslant s,t\leqslant h} \sideset{}{^*}\sum_{y\,(p^{h-s})} \sideset{}{^*}\sum_{z\,(p^{h-t})} N((p^s y + p^t z\zeta, p^h)) \\ &\leqslant \frac{1}{p^{3m+4h}} \sum_{\substack{0\leqslant s,t\leqslant h \\ s\neq t}} p^{2h-s-t+4\min\{s,t\}} + \frac{1}{p^{3m+4h}} \sum_{0\leqslant s\leqslant h} \sideset{}{^*}\sum_{y,z\,(p^{h-s})} p^{4s} N(y+z\zeta, p^{h-s}).\end{aligned}$$

Note that

$$\frac{1}{p^{3m+4h}} \sum_{\substack{0\leqslant s,t\leqslant h \\ s\neq t}} p^{2h-s-t+4\min\{s,t\}} \leqslant \frac{2}{p^{3m+2h}} \sum_{0\leqslant s<t\leqslant h} p^{2s} \ll \frac{h+1}{p^{3m}}.$$

On the other hand, a change of variable $y \mapsto yz$ gives

$$\frac{1}{p^{3m+4h}} \sum_{0\leqslant s\leqslant h} p^{4s} \sideset{}{^*}\sum_{y,z\,(p^{h-s})} N(y+z\zeta, p^{h-s}) \leqslant \sum_{0\leqslant s\leqslant h} \frac{1}{p^{3m+3h-3s}} \sideset{}{^*}\sum_{y\,(p^{h-s})} N(y+\zeta, p^{h-s}).$$

Lemma 5.2 gives us that this is $\ll (h+1)p^{-3m}$. Hence we have the bound

$$\tilde{N}_2(\mathbf{c}, p^h; p^m) \ll \frac{h+1}{p^{3m}}.$$

Together with (9.11) which holds for all cases, we have for $k \geqslant m$,

$$\tilde{N}_2(\mathbf{c}, p^h; p^k) = \tilde{N}_2(\mathbf{c}, p^h; p^m) + \sum_{j=m}^{k-1} D(\mathbf{c}, p^h; p^j) \ll (h+k+1)p^{2m}.$$

This concludes the proof. $\square$

10. Oscillatory integral estimation

In this section, we carry out the estimation of the oscillatory integrals I_1, I_2 defined in (4.4), (4.6), respectively. Their treatments will be very similar to the applications of orthogonality that went into the treatments of the exponential sums S_1, S_2 in §8, 9, though matters will be simpler in this archimedean analogue for we only seek an upper bound.

10.1. **Estimation of I_1.** The main result of this subsection is the following.

Proposition 10.1. *For any $A, k \geqslant 0$, we have that*

$$q^k \frac{d^k}{dq^k} I_1(\alpha_1, \alpha_2; q) \ll \frac{1}{T} \mathbb{1}_{|\tilde{\alpha}_1|_{\sup}, |\tilde{\alpha}_2|_{\sup} \ll T^{-1+o(1)}} + |\tilde{\alpha}_1 \tilde{\alpha}_2|_{\sup}^{-\frac{5}{6}} T^{-13/3+o(1)} (1 + |\tilde{\alpha}_1|_{\sup} + |\tilde{\alpha}_2|_{\sup})^{-A}, \tag{10.1}$$

where we write

$$\tilde{\alpha}_i = \frac{\alpha_i D X_i}{MX} \quad \text{and} \quad T = \frac{MX}{qD}.$$

Proof. Differentiating under the integral, it is sufficient to bound, for any $\omega^* \in C_c^\infty(\mathbb{R}^2 \backslash \{0\})$ and $\Phi_1, \Phi_2 \in C_c^\infty(K_\infty \setminus \{0\})$, the integral

$$I_T(\omega^*, \Phi_1, \Phi_2) := \int_{K_\infty^2} \omega^*(T \boldsymbol{\ell}(x_1^\infty x_2^\infty)) \Phi_1(x_1^\infty) \Phi_2(x_2^\infty) e(-T(\langle x_1^\infty, \tilde{\alpha}_1 \rangle + \langle x_2^\infty, \tilde{\alpha}_2 \rangle)) \, dx_1^\infty \, dx_2^\infty. \tag{10.2}$$

Repeated integration by parts implies that

$$I_T(\omega^*, \Phi_1, \Phi_2) \ll (1 + |\tilde{\alpha}_1|_{\sup} + |\tilde{\alpha}_2|_{\sup})^{-A},$$

This immediately gives the statement if $T \ll X^\varepsilon$. So we'll suppose from now on that $T \gg X^\varepsilon$ and $|\tilde{\alpha}_i|_{\sup} \ll T^{o(1)}$.

We shall suppose WLOG that $|\tilde{\alpha}_1|_{\sup} \leqslant |\tilde{\alpha}_2|_{\sup}$. Recalling $\boldsymbol{\ell}(x_\infty) = (\langle x_\infty, 1 \rangle, \langle x_\infty, \zeta \rangle)$, Fourier inversion implies that (10.2) equals

$$\begin{aligned} &\int_{\mathbb{R}^2} \widehat{\omega^*}(y_1, y_2) \int_{K_\infty^2} \Phi_1(x_1^\infty) \Phi_2(x_2^\infty) \\ &\qquad \cdot e(-T(\langle x_1^\infty, \tilde{\alpha}_1 \rangle + \langle x_2^\infty, \tilde{\alpha}_2 \rangle - \langle x_1^\infty x_2^\infty, y_1 + y_2 \zeta \rangle)) \, dx_1^\infty \, dx_2^\infty \, dy_1 \, dy_2 \\ &= \int_{\mathbb{R}^2} \widehat{\omega^*}(y_1, y_2) \int_{K_\infty} \Phi_1(x_1^\infty) \widehat{\Phi}_2(T(\tilde{\alpha}_2 - x_1^\infty(y_1 + y_2 \zeta))) e(-T \langle x_1^\infty, \tilde{\alpha}_1 \rangle) \, dx_1^\infty \, dy_1 \, dy_2. \end{aligned}$$

We suppose from now on that

$$|\tilde{\alpha}_2|_{\sup} \gg \frac{1}{T^{1-2\varepsilon}}, \tag{10.3}$$

for otherwise, the trivial bound may be absorbed into the first term in (10.1). Let $\eta_\infty = T(\tilde{\alpha}_2 - x_1^\infty(y_1 + y_2 \zeta))$ so that by the change of variables

$$dx_1^\infty = \frac{d\eta_\infty}{T^4(y_1^4 + y_2^4)},$$

we have that $I_T(\omega^*, \Phi_1, \Phi_2)$ is equal to

$$\frac{1}{T^4} \int_{\mathbb{R}^2} \frac{\widehat{\omega^*}(y_1, y_2)}{y_1^4 + y_2^4} \int_{K_\infty} \widehat{\Phi}_2(\eta_\infty) \Phi_1\left(\frac{\tilde{\alpha}_2 - \eta_\infty / T}{y_1 + y_2 \zeta} \right) e\left(- \left\langle \frac{T \tilde{\alpha}_2 - \eta_\infty}{y_1 + y_2 \zeta}, \tilde{\alpha}_1 \right\rangle \right) d\eta_\infty.$$

The contribution of $|\eta_\infty|_{\sup} > T^\varepsilon$ is $\ll T^{-A}$ from the decay of $\widehat{\Phi}_2$. Also, from the support of Φ_1 and the fact that both embeddings of $y_1 + y_2 \zeta$ are $\asymp |y_1^4 + y_2^4|^{1/4} \asymp |\mathbf{y}|$ in magnitude, we have that in the support of the integrand above with $|\eta_\infty|_{\sup} \leqslant T^\varepsilon$ and (10.3),

$$1 \asymp \left|\frac{\tilde{\alpha}_2 - \eta_\infty/T}{y_1 + y_2\zeta}\right|_{\sup} \asymp \left|\frac{\tilde{\alpha}_2}{y_1 + y_2\zeta}\right|_{\sup} \asymp \frac{|\tilde{\alpha}_2|_{\sup}}{|\mathbf{y}|}.$$

In particular, it suffices to bound for $|\eta_\infty|_{\sup} \leqslant T^\varepsilon$, some inert function $\Psi : \mathbb{R}\times\mathbb{R}\times K_\infty \to \mathbb{C}$ compactly supported away from 0 on the third coordinate, and some $\tilde{\alpha}_i'$[3] satisfying

$$|\tilde{\alpha}_i'|_{\sup} \asymp |\tilde{\alpha}_i|_{\sup} \asymp |\tilde{\alpha}_i'|_\infty^{\frac{1}{4}} \tag{10.4}$$

the quantity

$$\frac{|\tilde{\alpha}_2'|_{\sup}^{-2}}{T^4}\int_{|\mathbf{y}|\asymp 1} \Psi\left(y_1, y_2, \frac{\tilde{\alpha}_2'/|\tilde{\alpha}_2'|_{\sup}}{y_1 + y_2\zeta}\right) e\left(-T|\tilde{\alpha}_1'|_{\sup}\left\langle \frac{\tilde{\alpha}_1'\tilde{\alpha}_2'|\tilde{\alpha}_1'|_{\sup}^{-1}|\tilde{\alpha}_2'|_{\sup}^{-1}}{y_1 + y_2\zeta}, 1\right\rangle\right) dy_1\, dy_2. \tag{10.5}$$

It now remains to execute the y_1, y_2 integrals. Taking any fixed $\phi \in C_c^\infty((1,2))$ with

$$\int_{\mathbb{R}} \phi(t)\, dt = 1,$$

applying a change of variables $(y_1, y_2) \to (y_1/t, y_2/t)$, and integrating against $\phi(t)\, dt$ yields that (10.5) equals

$$\frac{|\tilde{\alpha}_2'|_{\sup}^{-2}}{T^4}\int_{\mathbb{R}} \frac{\phi(t)}{t^2}\int_{|\mathbf{y}|\asymp 1} \Psi\left(y_1/t, y_2/t, t\frac{\tilde{\alpha}_2'/|\tilde{\alpha}_2'|_{\sup}}{y_1 + y_2\zeta}\right) e\left(-Tt|\tilde{\alpha}_1'|_{\sup}\left\langle \frac{\tilde{\alpha}_i'\tilde{\alpha}_2'|\tilde{\alpha}_1'|_{\sup}^{-1}|\tilde{\alpha}_2'|_{\sup}^{-1}}{y_1 + y_2\zeta}, 1\right\rangle\right) dy_1\, dy_2\, dt.$$

Executing the t-integral and noting the decay of the Fourier transform in t along with a change of variable $y_2 \mapsto -y_2$ reduces us to bounding

$$\frac{|\tilde{\alpha}_2|_{\sup}^{-2}}{T^4}\int_{|\mathbf{y}|\asymp 1} \mathbb{1}\left[u_0 y_2^3 + u_1 y_1 y_2^2 + u_2 y_1^2 y_2 + u_3 y_1^3 \leqslant |\tilde{\alpha}_1'|_{\sup}\frac{T^\varepsilon}{T}\right] dy_1\, dy_2 \tag{10.6}$$

where we write

$$\frac{\tilde{\alpha}_1'\tilde{\alpha}_2'}{|\tilde{\alpha}_1'|_{\sup}|\tilde{\alpha}_2'|_{\sup}} = u_0 + u_1\zeta + u_2\zeta^2 + u_3\zeta^3.$$

The bounds (10.4) imply that $\max_{0\leqslant i\leqslant 3}(|u_i|) \gg 1$ and so we obtain that (10.6) is

$$\ll \frac{1}{T^5}|\tilde{\alpha}_2'|_{\sup}^{-2}|\tilde{\alpha}_1'|_{\sup}^{\frac{1}{3}} T^{\frac{2}{3}+\frac{\varepsilon}{3}} \ll \frac{1}{T^5}|\tilde{\alpha}_1\tilde{\alpha}_2|_{\sup}^{-\frac{5}{6}} T^{\frac{2}{3}+\frac{\varepsilon}{3}}.$$

The desired result follows.

□

[3] This is just $\tilde{\alpha}_i - \eta_\infty/T$

10.2. **Estimation of I_2.** The estimate for I_2 we show is as follows.

Proposition 10.2. *For $|\mathbf{c}|d \asymp D$, we have that $I_2 = 0$ unless $q \ll \sqrt{DX}/d$, in which case we have the bound*

$$
\begin{aligned}
&I_2(\alpha_1, \alpha_2; \mathbf{c}, d, q) \\
&\ll_A \Big(1 + |\tilde{\alpha}_1|_{\sup} + |\tilde{\alpha}_2|_{\sup}\Big)^{-A} \left(\frac{1}{T}\mathbb{1}_{|\tilde{\alpha}_1|_{\sup}, |\tilde{\alpha}_2|_{\sup} \ll T^{-1+o(1)}} + \frac{1}{T^4}|\tilde{\alpha}_1|_{\sup}^{-3/2}|\tilde{\alpha}_2|_{\sup}^{-3/2}\right)T^{o(1)},
\end{aligned}
\tag{10.7}
$$

where

$$T = \frac{M\sqrt{DX}}{dq} \qquad \textit{and} \qquad \tilde{\alpha}_i = \frac{\alpha_i X_i}{M\sqrt{DX}}.$$

Note that we can do better in the generic case: with a bit more work, we can show a result that is $\ll 1/T^5$ on average (say, at the generic range $\alpha_i \asymp \sqrt{DX}/X_i$). However, the size of I_2 at $\alpha_i \asymp \sqrt{DX}/X_i$ with

$$\langle \alpha_1\alpha_2, (c_2 - c_1\zeta)^{-1}\rangle \ll \frac{1}{T}|\mathbf{c}|^{-1}D$$

can be $\gg 1/T^4$ (note that $|\mathbf{c}|^{-1}D$ is the trivial bound for the above quantity). Exploiting this is unnecessary, as it turns out, for there are fewer terms as T grows.

Proof. We have that

$$I_2(\alpha_1, \alpha_2; \mathbf{c}, d, q) = \omega_2\left(\frac{1}{T}\right)\hat{\phi}_1\Big(T\tilde{\alpha}_1\Big)\hat{\phi}_2\Big(T\tilde{\alpha}_2\Big) - I,$$

where

$$I := \int_{K_\infty^2} \omega_2\left(\frac{dT\det(\mathbf{c}, \boldsymbol{\ell}(x_1^\infty x_2^\infty))}{D}\right)\phi_1(x_1^\infty)\phi_2(x_2^\infty)e\Big(-T\langle x_1^\infty, \tilde{\alpha}_1\rangle - T\langle x_2^\infty, \tilde{\alpha}_2\rangle\Big)\,dx_1^\infty\,dx_2^\infty,$$

The first term may be absorbed into the bound (10.7), so it remains to bound I, which we shall trivially rewrite for clarity, with $\tilde{\mathbf{c}} = \mathbf{c}d/D$ as

$$\int_{K_\infty^2} \omega_2(T\det(\tilde{\mathbf{c}}, \boldsymbol{\ell}(x_1^\infty x_2^\infty)))\phi_1(x_1^\infty)\phi_2(x_2^\infty)e(-T(\langle x_1^\infty, \tilde{\alpha}_1\rangle \quad + \quad \langle x_2^\infty, \tilde{\alpha}_2\rangle))\,dx_1^\infty\,dx_2^\infty.$$

Since $|\tilde{\mathbf{c}}| \asymp 1$, repeated integration by parts gives us $I \ll (1+|\tilde{\alpha}_1|_{\sup}+|\tilde{\alpha}_2|_{\sup})^{-A}$. We shall suppose from now on that $|\tilde{\alpha}_i|_{\sup} \ll X^{o(1)}$. As before, we begin with Fourier inversion. Noting that

$$\det(\tilde{\mathbf{c}}, \boldsymbol{\ell}(x_1^\infty x_2^\infty)) = \langle \tilde{c}_2 - \tilde{c}_1\zeta, x_1^\infty x_2^\infty\rangle,$$

we have

$$
\begin{aligned}
I &= \int_{\mathbb{R}} \widehat{\omega}_2(y) \int_{K_\infty^2} \phi_1(x_1^\infty)\phi_2(x_2^\infty)e(-T(\langle x_1^\infty, \tilde{\alpha}_1\rangle + \langle x_2^\infty, \tilde{\alpha}_2\rangle + \langle x_1^\infty x_2^\infty, y(\tilde{c}_2 - \tilde{c}_1\zeta)\rangle))\,dx_1^\infty\,dx_2^\infty \\
&= \int_{\mathbb{R}} \widehat{\omega}_2(y) \int_{K_\infty} \phi_1(x_1^\infty)\hat{\phi}_2(T(\tilde{\alpha}_2 + y(\tilde{c}_2 - \tilde{c}_1\zeta)x_1^\infty))e(-T\langle x_1^\infty, \tilde{\alpha}_1\rangle)\,dx_1^\infty.
\end{aligned}
$$

Similar to before, we make the change of variables

$$\eta_\infty = T(\tilde{\alpha}_2 + y(\tilde{c}_2 - \tilde{c}_1\zeta)x_1^\infty)$$

so that

$$I = \frac{1}{T^4(\tilde{c}_1^4 + \tilde{c}_2^4)} \int_{\mathbb{R}} \frac{\widehat{\omega}_2(y)}{y^4} \int_{K_\infty} \phi_1\left(\frac{-\tilde{\alpha}_2 + \eta_\infty/T}{y(\tilde{c}_2 - \tilde{c}_1\zeta)}\right)\hat{\phi}_2(\eta_\infty) e\left(-T\left\langle \frac{-\tilde{\alpha}_2 + \eta_\infty/T}{y(\tilde{c}_2 - \tilde{c}_1\zeta)}, \tilde{\alpha}_1\right\rangle\right) d\eta_\infty\, dy.$$

As in the proof of Proposition 10.1, we may restrict to the case of $|\eta_\infty|_{\sup} < T^\varepsilon$, $|\tilde{\alpha}_2|_{\sup} \gg T^{2\varepsilon-1}$ and fix an η_∞. Then, it is forced that $|y| \asymp |\tilde{\alpha}_2|_{\sup}$. A trivial bound then gives us

$$I \ll (1 + |\tilde{\alpha}_1|_{\sup} + |\tilde{\alpha}_2|_{\sup})^{-A} T^{-4+o(1)} |\tilde{\alpha}_2|_{\sup}^{-3}.$$

Noticing that the same bound holds with $\tilde{\alpha}_1$ by swapping the roles of α_1 and α_2, the desired result follows.

$\square$

11. Local density estimates

In this section, we record a number of facts about local densities, their averages, and associated objects, which will be of use in the following two subsections when analyzing zero frequencies.

11.1. **Singular series estimates.** Write $V_q = \{(\beta_1, \beta_2) \in (\mathcal{O}_K/(q))^2 : \beta_i \equiv \beta_i'(M) \text{ for } i = 1, 2\}$, and for $d, q \geqslant 1$, $\mathbf{c} = (c_1, c_2) \in \mathbb{Z}^2$ with $(c_1, c_2) = 1$, let

$$\tilde{N}_1(q) = \frac{1}{q^6} \sum_{\substack{(\beta_1,\beta_2)\in V_q \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (q/(q,M))}} 1, \qquad \tilde{N}_2(\mathbf{c}, d; q) = \frac{1}{d^6 q^7} \sum_{\substack{(\beta_1,\beta_2)\in V_{dq} \\ \det(\mathbf{c}, \boldsymbol{\ell}(\beta_1\beta_2)) \equiv 0 (dq/(q,M)) \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (d)}} 1.$$

Define the associated Dirichlet series

$$\mathcal{D}_1(s) = \sum_{\substack{q \geqslant 1 \\ M|q}} \frac{\tilde{N}_1(q)}{q^s}, \qquad \mathcal{D}_2(s; \mathbf{c}, d) = \sum_{\substack{q \geqslant 1 \\ M|q}} \frac{\tilde{N}_2(\mathbf{c}, d; q)}{q^s}.$$

We also write $\mathcal{D}_1^*(s) = \mathcal{D}_1(s)/\zeta(s)$, $\mathcal{D}_2^*(s; \mathbf{c}, d) = \mathcal{D}_2(s; \mathbf{c}, d)/\zeta(s)$ so that

$$\mathcal{D}_1^*(s) = \sum_{\substack{q \geqslant 1 \\ M|q}} \frac{N_1^*(q)}{q^s}, \quad \text{where} \quad N_1^*(q) = \sum_{b|q/M} \mu(b) \tilde{N}_1(q/b) \tag{11.1}$$

and

$$\mathcal{D}_2^*(s; \mathbf{c}, d) = \sum_{\substack{q \geqslant 1 \\ M|q}} \frac{N_2^*(q; \mathbf{c}, d)}{q^s}, \quad \text{where} \quad N_2^*(q; \mathbf{c}, d) = \sum_{b|q/M} \mu(b) \tilde{N}_2(q/b; \mathbf{c}, d).$$

The absolute convergence of $\mathcal{D}_1(s)$, $\mathcal{D}_2(s; \mathbf{c}, d)$ for $\operatorname{Re} s > 1$ is clear. Further analytic properties of these two relevant to us are contained in the following two propositions.

Proposition 11.1. *We have that*

$$N_1^*(q) \ll \frac{M^4}{q^{2-o(1)}}. \tag{11.2}$$

In particular, we have that $\mathcal{D}_1^*(s)$ is absolutely convergent for $\operatorname{Re} s > -2$ so that $\mathcal{D}_1(s)$ has a meromorphic continuation to $\operatorname{Re} s > -2$ with

$$\mathcal{D}_1^*(0) = M^2 \prod_p \lim_{k\to\infty} \frac{1}{p^{6k}} \sum_{\substack{\beta_1,\beta_2 (p^k) \\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0 (p^k) \\ \beta_1 \equiv \beta_1' (p^{m_p}) \\ \beta_2 \equiv \beta_2' (p^{m_p})}} 1 = M^2 \prod_p \sigma_p. \tag{11.3}$$

Furthermore, for $\sigma > \sigma_0 > -2$, we have that $|\mathcal{D}_1^*(s)| \ll_{\sigma_0} 1$.

In the computation of the local densities, we will encounter the following p-adic integral,

$$\begin{aligned} \sigma_p(\mathbf{c}, d) &= \int_{\mathcal{O}_{K,p}^2} \mathbb{1}_{\substack{(\beta_1,\beta_2)\in V_{p^k} \\ d|\boldsymbol{\ell}(\beta_1\beta_2)}} \delta(\det(\mathbf{c}, \boldsymbol{\ell}(\beta_1\beta_2)))\, d\beta_1\, d\beta_2 \\ &= \lim_{k\to\infty} \frac{1}{p^{7k}} \sum_{\substack{\beta_1,\beta_2 \in V_{p^k} \\ p^k \mid \det(\mathbf{c},\boldsymbol{\ell}(\beta_1\beta_2)) \\ p^{d_p} | \boldsymbol{\ell}(\beta_1\beta_2)}} 1. \end{aligned} \tag{11.4}$$

Here, as before, $d_p = v_p(d)$.

Proposition 11.2. *For $\mathbf{c}$ primitive and $d \geqslant 1$, $\mathcal{D}_2^*(s; \mathbf{c}, d)$ has an analytic continuation to $\operatorname{Re} s > -2$. Furthermore, for $\delta > 0$, all $\operatorname{Re} s > -2 + \delta$ satisfy the bound*

$$\mathcal{D}_2^*(s; \mathbf{c}, d) \ll_\delta (|\mathbf{c}|d)^{o(1)} (c_1^4 + c_2^4, d). \tag{11.5}$$

Furthermore, we have that

$$\mathcal{D}_2^*(0; \mathbf{c}, d) = M \prod_p \sigma_p(\mathbf{c}, d). \tag{11.6}$$

Lemma 11.3. *We have*

$$\sigma_p(\mathbf{c}, d) = \tau_p(\mathbf{c}d), \tag{11.7}$$

where

$$\tau_p(\mathbf{v}) := \frac{1}{(v_1, v_2, p^\infty)} \sum_{k \geqslant 0} S_p(\mathbf{v}; k)$$

for $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, and

$$S_p(\mathbf{v}; k) := \frac{1}{p^{9k + 8\max\{0, m_p - k\}}} \sum_{\beta_1,\beta_2 \in V_{p^k}} \sum_{w\, (p^k)} \sum_{\substack{\mathbf{a}=(a_1,a_2)\, (p^k) \\ (a_1,a_2,p)=1}} e_{p^k}(\langle \boldsymbol{\ell}(\beta_1\beta_2) - \mathbf{v}w, \mathbf{a}\rangle) \tag{11.8}$$

satisfies

$$S_p(\mathbf{v}; k) \ll (k+1)(v_1^4 + v_2^4, p^k) p^{-3k}. \tag{11.9}$$

Proof of Proposition 11.1. The absolute convergence follows immediately from the bound (11.2), which itself follows from Hensel's lemma.

Now, for the product of local factors, note that $\mathcal{D}_1^*(0)$, due to the absolute convergence implied by (11.2), satisfies

$$\mathcal{D}_1^*(0) = \prod_p \Bigg(\sum_{k \geqslant v_p(M)} N_1^*(p^k) \Bigg) = \prod_p \lim_{k\to\infty} \tilde{N}_1(p^k) = \prod_p \lim_{k\to\infty} \frac{1}{p^{6k}} \sum_{\substack{\beta_1,\beta_2 (p^k) \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (p^{k-v_p(M)})}} 1$$

$$= M^2 \prod_p \lim_{k\to\infty} \frac{1}{p^{6k}} \sum_{\substack{\beta_1,\beta_2 (p^k) \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv 0 (p^k)}} 1 = M^2 \prod_p \sigma_p,$$

as desired. □

Proof of Proposition 11.2. All of these straightforwardly follow from the bounds of Proposition 9.2, for (9.5) implies the absolute convergence of $\mathcal{D}_2^*(s; \mathbf{c}, d)$ for $\operatorname{Re} s > -2$. The bound (11.5) also follows from (9.5).

(11.6), that $\mathcal{D}_2^*(0; \mathbf{c}, d)$ is a product of local densities, follows identically to the proof of (11.3). □

Proof of Lemma 11.3. What follows can be naturally written in terms of p-adic integrals, though we shall proceed equivalently takign limits of counts in V_{p^k}.

By the limit characterization of $\sigma_p(\mathbf{c}, d)$ in (11.4), we can perform a shift of k to $d_p + k$ to write

$$\sigma_p(\mathbf{c}, d) = \lim_{k\to\infty} \frac{1}{p^{7(d_p+k)}} \sum_{\substack{\beta_1,\beta_2 \in V_{p^{d_p+k}} \\ p^{d_p+k} \mid \det(\mathbf{c}, \boldsymbol{\ell}(\beta_1\beta_2)) \\ p^{d_p} \mid \boldsymbol{\ell}(\beta_1\beta_2)}} 1.$$

Taking $d\mathbf{u} = \boldsymbol{\ell}(\beta_1\beta_2)$, we have

$$\sigma_p(\mathbf{c}, d) = \lim_{k\to\infty} \frac{1}{p^{7(d_p+k)}} \sum_{\substack{\beta_1,\beta_2 \in V_{p^{d_p+k}} \\ \boldsymbol{\ell}(\beta_1\beta_2) \equiv d\mathbf{u}\, (p^{d_p+k}) \\ c_1u_2 \equiv c_2u_1\, (p^k)}} \sum_{\mathbf{u}\, (p^k)} 1.$$

Take integers a, b such that $ac_1 + bc_2 = 1$, which exist since $(c_1, c_2) = 1$. Performing the change of variables

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ -c_2 & c_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_1 & -b \\ c_2 & a \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

yields that

$$\begin{aligned}\sigma_p(\mathbf{c},d) &= \lim_{k\to\infty}\frac{1}{p^{7(d_p+k)}}\sum_{\substack{\beta_1,\beta_2\in V_{p^{d_p+k}}}}\ \sum_{\substack{w_1,w_2\,(p^k)\\ \langle\beta_1\beta_2,1\rangle\equiv d(c_1w_1-bw_2)\,(p^{d_p+k})\\ \langle\beta_1\beta_2,\zeta\rangle\equiv d(c_2w_1+aw_2)\,(p^{d_p+k})\\ w_2\equiv 0\,(p^k)}} 1\\
&= \lim_{k\to\infty}\frac{1}{p^{7(d_p+k)}}\sum_{\substack{\beta_1,\beta_2\in V_{p^{d_p+k}}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv\mathbf{c}dw\,(p^{d_p+k})}}\ \sum_{w\,(p^k)} 1.\end{aligned}$$

Reindexing w, we conclude

$$\sigma_p(\mathbf{c},d) = \lim_{k\to\infty}\frac{1}{p^{8d_p+7k}}\sum_{\substack{\beta_1,\beta_2\in V_{p^{d_p+k}}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv\mathbf{c}dw\,(p^{d_p+k})}}\ \sum_{w\,(p^{d_p+k})} 1 = \lim_{k\to\infty}\frac{1}{p^{d_p+7k}}\sum_{\substack{\beta_1,\beta_2\in V_{p^k}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv\mathbf{c}dw\,(p^k)}}\ \sum_{w\,(p^k)} 1.$$

To complete the proof of (11.7), we use the telescopic nature of $S(\mathbf{c}d;p^k)$. Precisely, for $k\geqslant 1$, we have

$$\begin{aligned}S_p(\mathbf{c}d;k) &= \sum_{j=0,1}\frac{(-1)^j}{p^{9k+8\max\{0,m_p-k\}}}\sum_{\beta_1,\beta_2\in V_{p^k}}\ \sum_{w\,(p^k)}\ \sum_{\mathbf{a}\,(p^{k-j})} e_{p^{k-j}}(\langle\boldsymbol{\ell}(\beta_1\beta_2)-\mathbf{c}dw,\mathbf{a}\rangle)\\
&= \sum_{j=0,1}\frac{(-1)^j}{p^{9k-9j+8(\max\{0,m_p-k\}+j\mathbb{1}_{k\leqslant m_p})}}\sum_{\beta_1,\beta_2\in V_{p^{k-j}}}\ \sum_{w\,(p^{k-j})}\\
&\qquad\sum_{\mathbf{a}\,(p^{k-j})} e_{p^{k-j}}(\langle\boldsymbol{\ell}(\beta_1\beta_2)-\mathbf{c}dw,\mathbf{a}\rangle).\end{aligned}$$

With

$$p^{9k+8\max\{0,m_p-k\}} = p^{9k+9-9+8(\max\{0,m_p-k-1\}+\mathbb{1}_{k+1\leqslant m_p})},$$

the telescopic sum leads to

$$\begin{aligned}\sum_{k\geqslant 0} S_p(\mathbf{c}d;k) &= \lim_{k\to\infty}\frac{1}{p^{9k}}\sum_{\beta_1,\beta_2\in V_{p^k}}\ \sum_{w\,(p^k)}\ \sum_{\mathbf{a}\,(p^k)} e_{p^k}(\langle\boldsymbol{\ell}(\beta_1\beta_2)-\mathbf{c}dw,\mathbf{a}\rangle)\\
&= \lim_{k\to\infty}\frac{1}{p^{7k}}\sum_{\substack{\beta_1,\beta_2\in V_{p^k}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv\mathbf{c}dw\,(p^k)}}\ \sum_{w\,(p^k)} 1 = p^{d_p}\sigma_p(\mathbf{c},d),\end{aligned}$$

so we obtain (11.7).

For the bound (11.9), write $p^h = (v_1^4 + v_2^4, p^k)$ and write $\eta = \min\{m_p, k\}$. Recalling that $\boldsymbol{\ell}(\beta) = (\langle \beta, 1\rangle, \langle \beta, \zeta\rangle)$ and detecting $\beta_2 \equiv \beta_2'$ with additive characters, we have

$$S_p(\mathbf{v}; k) = \frac{1}{p^{9k+4\eta}} \sum_{\gamma \in \mathcal{O}_K/(p^\eta)} \sum_{\substack{\beta_1, \beta_2 \in \mathcal{O}_K/(p^k) \\ \beta_1 \equiv \beta_1' \, (p^\eta)}} \sum_{w \, (p^k)} \sum_{\substack{a_1, a_2 \, (p^k) \\ (a_1, a_2, p) = 1}} e_{p^k}(\langle \beta_1\beta_2, a_1 + a_2\zeta\rangle + p^{k-\eta}\langle \beta_2' - \beta_2, \gamma\rangle - a_1v_1w - a_2v_2w).$$

Summing over β_2 and w, we get

$$S_p(\mathbf{v}; k) = \frac{1}{p^{4(k+\eta)}} \sum_{\gamma \in \mathcal{O}_K/(p^\eta)} \psi_{p^\eta}(\gamma\beta_2') \sum_{\substack{\beta \in \mathcal{O}_K/(p^k) \\ \beta \equiv \beta_1' \, (p^\eta) \\ (a_1 + a_2\zeta)\beta \equiv p^{k-\eta}\gamma \, (p^k) \\ a_1v_1 \equiv -a_2v_2 \, (p^k)}} \sum_{\substack{a_1, a_2 \, (p^k) \\ (a_1, a_2, p) = 1}} 1.$$

With γ completely determined by the other variables, we have

$$|S_p(\mathbf{v}; k)| \leqslant \frac{1}{p^{4(k+\eta)}} \sum_{\beta \in \mathcal{O}_K/(p^k)} \sum_{\substack{a_1, a_2 \, (p^k) \\ (a_1, a_2, p) = 1 \\ (a_1 + a_2\zeta)\beta \equiv 0 \, (p^{k-\eta}) \\ a_1v_1 \equiv -a_2v_2 \, (p^k)}} 1 \leqslant \frac{1}{p^{4k}} \sum_{\substack{a_1, a_2 \, (p^k) \\ (a_1, a_2, p) = 1 \\ a_1v_1 \equiv -a_2v_2 \, (p^k)}} N((a_1 + a_2\zeta, p^k)).$$

Write $v_1 = p^s v_1'$ and $v_2 = p^t v_2'$ with $(v_1'v_2', p) = 1$. We separate into four cases.

Case 1: $s, t \geqslant k$. Then

$$|S_p(\mathbf{v}; k)| \leqslant \frac{1}{p^{4k}} \sum_{\substack{a_1, a_2 \, (p^k) \\ (a_1, a_2, p) = 1}} N((a_1 + a_2\zeta, p^k)) \leqslant p^{-2k}$$

by Lemma 5.2.

Case 2: $s = t \leqslant k$. In this case, the congruence condition with $(a_1, a_2, p) = 1$ forces $(a_1a_2, p) = 1$. Hence we have

$$\begin{aligned} |S_p(\mathbf{v}; k)| &\leqslant \frac{1}{p^{4k}} \sideset{}{^*}\sum_{a_1, a_2 \, (p^k)} N((a_1 + a_2\zeta, p^k)) \\ &= \frac{1}{p^{4k}} \sideset{}{^*}\sum_{a_1 \, (p^k)} \sideset{}{^*}\sum_{z \, (p^s)} N((a_1 + (-v_1'\overline{v_2'}a_1 + zp^{k-s})\zeta, p^k)) \\ &= \frac{1}{p^{4k}} \sum_{d=0}^{s} \sideset{}{^*}\sum_{a_1 \, (p^k)} \sideset{}{^*}\sum_{z \, (p^d)} N((a_1 + (-v_1'\overline{v_2'}a_1 + zp^{k-d})\zeta, p^k)). \end{aligned}$$

Lemma 5.1 implies that this is equal to

$$\frac{1}{p^{4k}} \sum_{d=0}^{s} \sideset{}{^*}\sum_{a_1 \, (p^k)} \sideset{}{^*}\sum_{z \, (p^d)} (a_1^4 + (-v_1'\overline{v_2'}a_1 + zp^{k-d})^4, p^k).$$

Applying a change of variable $z \mapsto a_1 z$, this is equal to

$$\frac{1}{p^{4k}} \sum_{d=0}^{s} \varphi(p^d) \sum_{a_1\ (p^k)}^{*} (a_1^4 + (-v_1'\overline{v_2'}a_1 + p^{k-d})^4, p^k).$$

Lemma 5.2 then gives us the bound

$$|S_p(\mathbf{v}; k)| \leqslant (s+1)p^{s+\ell-3k},$$

with ℓ defined by $p^\ell || (v_1'^4 + v_2'^4)$.

Case 3: $t < s < k$. In this case, the congruence condition with $(a_1, a_2, p) = 1$ forces $(a_1, p) = 1$ and $p^{s-t} || a_2$. Hence we have

$$|S_p(\mathbf{v}; k)| \leqslant \frac{1}{p^{4k}} \sum_{\substack{a_1\ (p^k) \\ a_1 v_1' \equiv -a_2 v_2'\ (p^{k-s})}}^{*} \sum_{a_2\ (p^{k-s+t})}^{*} N((a_1 + a_2 p^{s-t}\zeta, p^k)) \leqslant p^{t-3k}.$$

Case 4: $s < t < k$. This case is symmetric to Case 3 and we have the bound

$$|S_p(\mathbf{v}; k)| \leqslant p^{s-3k}.$$

Combining all four cases, we obtain the desired bound (11.9). □

It will be useful in later sections to write, for $q \geqslant 1$,

$$S(\mathbf{v}; q) = \prod_{p^k || q} S_p(\mathbf{v}; k). \tag{11.10}$$

11.2. **Archimedean local densities.** We shall require the following integral computation in §13.3, and can be thought of as amounting roughly to an archimedean analogue of some of the content of Lemma 11.3.

Lemma 11.4. *For $\omega \in C_c^\infty(\mathbb{R}_{>0})$, $\Phi \in C_c^\infty(K_\infty^2 \setminus \{(0,0)\})$, we have that*

$$\int_{\mathbb{R}^2} \omega(|\mathbf{v}|) \int_{K_\infty^2} \Phi(x_1^\infty, x_2^\infty) \delta(\langle \mathbf{v}, \boldsymbol{\ell}(x_1^\infty x_2^\infty) \rangle)\, dx_1^\infty\, dx_2^\infty\, d\mathbf{v}$$
$$= 2\tilde{\omega}(1) \int_{K_\infty^2} \Phi(x_1^\infty, x_2^\infty) |\boldsymbol{\ell}(x_1^\infty x_2^\infty)|^{-1}\, dx_1^\infty\, dx_2^\infty.$$

Proof. It will be sufficient to show that for any $\mathbf{w} \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have

$$\int_{\mathbb{R}^2} \omega(|\mathbf{v}|) \delta(\langle \mathbf{v}, \mathbf{w} \rangle)\, d\mathbf{v} = 2\tilde{\omega}(1)|w|^{-1}. \tag{11.11}$$

First, note that this is invariant under the action of $\mathrm{SO}(2)$, so we may suppose without loss of generality that $\mathbf{w} = (|w|, 0)$. Letting $u = (u_1, u_2)$, we obtain that the LHS of (11.11) is

$$|\mathbf{w}|^{-1} \int_{\mathbb{R}^2} \omega(|u|) \delta(u_1)\, du = |w|^{-1} \int_{\mathbb{R}} \omega(u_2)\, du_2 = 2\tilde{\omega}(1)|w|^{-1},$$

as desired. □

At a couple points, in our computation of the zero frequencies in §12.3, §13.3, we will encounter the following Riesz-type integrals.

Lemma 11.5. *Suppose that $w \in C_c^\infty(\mathbb{R}^n)$ and that $F : \mathbb{R}^n \to \mathbb{R}^m$ is such that 0 is a regular value of $F|_{\operatorname{supp} w}$[4]. Then, we have that the function $U(s)$, defined for $\operatorname{Re} s > -m$ as*

$$U(s) = \int_{\mathbb{R}^n} w(x)|F(x)|^s \, dx,$$

has a meromorphic continuation to $\mathbb{C}$ with simple poles at $-m-2k$ for $k \geqslant 0$. Furthermore, we have that

$$\operatorname*{res}_{s=-m-2k} U(s) = \frac{2\pi^{m/2}}{4^k k! \Gamma(m/2+k)} \int_{\mathbb{R}^n} \Delta_F^k w(x) \delta(F(x)) \, dx,$$

where Δ_F satisfies $\Delta_F(\phi \circ F) = (\Delta_{\mathbb{R}^m} \phi) \circ F$. Furthermore, for any $k \geqslant 0$, $-m-2k-2 < \sigma_1 < \sigma_2 < -m-2k$, and $\sup_{|x| \leqslant 1} |F(x)| \leqslant 1$, we have that s with $\operatorname{Re} s \in [\sigma_1, \sigma_2]$ satisfy

$$U(s) \ll_{\sigma_1,\sigma_2} 1.$$

Proof. If $m > n$, the regularity condition implies that F is nonzero on $\operatorname{supp} w$ from which the result follows trivially, so suppose $m \leqslant n$ from now on.

Let $K = \operatorname{supp} w$, and let $Z = K \cap F^{-1}(0)$. Because 0 is a regular value of F on K, there exists a neighborhood Ω of Z on which dF is surjective at every point.

By compactness, there also exists a neighborhood $0 \in V \subset \mathbb{R}^m$ such that $K \cap F^{-1}(V) \subset \Omega$.

Now, take $\chi \in C_c^\infty(\mathbb{R}^m)$ such that for some $\varepsilon > 0$, $\mathbb{1}_{|\cdot|<\varepsilon} \leqslant \chi \leqslant \mathbb{1}_V$, and take $w_0 = w \cdot (\chi \circ F)$, $w_1 = w - w_0$,

$$U_j(s) = \int_{\mathbb{R}^n} w_j(x)|F(x)|^s \, dx.$$

We have that $|F| \gg 1$ on $\operatorname{supp} w_1$, so U_1 is entire.

Now, since dF is surjective on $\operatorname{supp} w_0$, we have that

$$U_0(s) = \int_{\mathbb{R}^n} w_0(x)|F(x)|^s \, dx = \int_{\mathbb{R}^m} |y|^s g(y) \, dy.$$

where $g(y)\,dy = F_*(w_0(x)\,dx)$ (with $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$).

To proceed, note that

$$U_0(s) = \int_0^\infty r^{s+m-1} A(r) \, dr \tag{11.12}$$

where

$$A(r) = \int_{|y|=1} g(yr) \, dy.$$

We record that the surface area of the $(m-1)$-sphere equals

$$\int_{|y|=1} 1 \, dy = \frac{2\pi^{m/2}}{\Gamma(m/2)}.$$

[4]For any a in $\operatorname{supp} w$ such that $F(a) = 0$, the matrix $(\frac{\partial}{\partial x_i} F_j(x)|_{x=a})$ has full rank, here $F|_{\operatorname{supp} w} = (F_j)$.

Now, Taylor expanding the radial $A(r)$, we have that

$$A(r) = \sum_{k \leqslant N} \frac{2\pi^{m/2}}{4^k k! \Gamma(m/2+k)} \Delta^k g(0) r^{2k} + O(r^{2N+2}).$$

The desired result follows for $\sigma > -m - 2N$ upon executing the integral (11.12), noting that

$$\Delta^k g(0) = \int_{\mathbb{R}^n} \Delta_F^k w(x) \delta(F(x))\, dx$$

by repeated integration by parts.

□

12. Estimation of Σ_1

In this section, we carry out the estimation of Σ_1^S for all $S \subseteq \{1, 2\}$. The technical core of this paper is contained in the estimation of the nonzero frequencies Σ_1^{12}, where we utilize level aspect subconvexity for GL_2 L-functions to bound cubic Dedekind zeta functions as the discriminant of the cubic field grows.

12.1. **Estimation of the nonzero frequency contribution.** In this section, we carry out the technical heart of our proof, applying subconvexity to gain a saving from the sum over q in Σ_1^{12}.

Our main result this subsection is as follows.

Proposition 12.1. *We have the bound*

$$\Sigma_1^{12} \ll X^{2+o(1)} M^8 (X^{-\frac{16}{27}\delta_{sc}} L^{\frac{20}{3} - \frac{32}{9}\delta_{sc}} + X^{-\frac{7}{36}} L^{\frac{47}{6}}).$$

The main technical inputs to our proof of Proposition 12.1 are as follows. The first is on the sums of the coefficients $a_\alpha(q)$ over squarefree q (which were defined in Proposition 8.1). It is at this point that we apply the GL_2 subconvexity bound mentioned in §1 upon noting that these coefficients are essentially Fourier coefficients of a weight 1 dihedral form.

Proposition 12.2. *Suppose that $\alpha \in \mathcal{O}_K$ is such that $\mathrm{Gal}(f_\alpha) = S_3$. Also, suppose that $Q \gg Q' \geqslant 1$. Then, for any Schwartz function ϕ satisfying*

$$x^j \phi^{(j)}(x) \ll_j Q^j$$

for any $j \geqslant 0$, and any $M \neq 0$ and fixed $w \in C_c^\infty((1,2))$, we have that

$$\sum_{(q, N(\alpha)M)=1} \mu(q)^2 a_\alpha(q) \phi(q) w\left(\frac{q}{Q'}\right) \ll Q^{\frac{1}{2}} \|\alpha\|_{\sup}^{1-4\delta_{sc}} \left(\frac{Q}{Q'}\right)^{\frac{3}{4}} (QM\|\alpha\|_{\sup})^{o(1)}.$$

Proof. Take k_α the splitting field of f_α, and let F be the dihedral weight 1 cusp form of level $\mathrm{Disc}(f_\alpha) \ll \|\alpha\|_{\sup}^4$ such that

$$\zeta(s) L(s, F_\alpha) = \zeta_{k_\alpha}(s).$$

Suppose that

$$L(s,F_\alpha)=\sum_{q\geqslant 1}\frac{\lambda_{F_\alpha}(q)}{q^s}.$$

The key to Proposition 12.2 is that the coefficients $a_\alpha(q)$ are, apart from finitely many primes, equal to $\lambda_{F_\alpha}(q)$. We'll make this more precise below.

Write $n_3=\langle\alpha,1\rangle$. We begin by noting that for $p\nmid \mathrm{Disc}(f_\alpha)n_3N(\alpha)$, we have that $a_\alpha(p)=\lambda_{F_\alpha}(p)$. We also have that $|a_\alpha(p)|\leqslant 2$ for all $p\nmid\alpha$ and that $|\lambda_{F_\alpha}(p^k)|\leqslant k+1$ for all p, k. It follows that for $\mathrm{Re}(s)>1/2$, we have

$$\sideset{}{^\flat}\sum_{(q,N(\alpha)M)=1}\frac{\mu(q)^2a_\alpha(q)}{q^s}=L(s,F_\alpha)\prod_{p\mid n_3\mathrm{Disc}(f_\alpha)N(\alpha)M}\left(1+\frac{O(1)}{p^s}\right)\prod_p\left(1+\frac{O(1)}{p^{2s}}\right).\quad(12.1)$$

Applying Mellin inversion, moving the contour to $\sigma=1/2+1/\log Q$, and bounding the products over primes in (12.1) by $X^{o(1)}$, we obtain that

$$\sum_{(q,N(\alpha))=1}a_\alpha(q)\phi(q)w\left(\frac{q}{Q'}\right)$$
$$\ll (Q\|\alpha\|_{\sup}M)^{o(1)}\int_{\mathbb{R}}|L(\sigma+it,F_\alpha)|\left|\int_{\mathbb{R}}\phi(x)w\left(\frac{x}{Q'}\right)x^{\sigma+it}\frac{dx}{x}\right|dt.\quad(12.2)$$

Repeated integration by parts implies that for $1/4\leqslant\mathrm{Re}(s)\leqslant 3/4$, we have

$$\left|\int_{\mathbb{R}}\phi(x)w\left(\frac{x}{Q'}\right)x^s\frac{dx}{x}\right|\ll_A Q'^{\,\mathrm{Re}\,s}\left(1+\frac{Q'|\operatorname{Im}s|}{Q}\right)^{-A}.$$

It follows from the subconvex bound (1.3) that (12.2) is at most

$$\frac{Q}{Q'}Q'^{\frac12}\left(\frac{Q}{Q'}\|\alpha\|_{\sup}^4\right)^{1/4-\delta_{\mathrm{sc}}}(Q\|\alpha\|_{\sup})^{o(1)},$$

as desired. □

Our next ingredient is a bound on the number of α for which $\mathrm{Gal}(f_\alpha)\neq S_3$. Precisely, we show the following.

Proposition 12.3. *We have that*

$$\sum_{\substack{\alpha\in\mathcal{O}_K\\ \|\alpha\|_{\sup}<X}}\mathbb{1}_{\mathrm{Gal}(f_\alpha)\neq S_3}\ll X^3\log^2 X.$$

Proof. The case of $\deg f_\alpha<3$ can clearly be disposed of, so we may suppose that $\deg f_\alpha=3$ from now on.

We begin by disposing of the case that $|\mathrm{Gal}(f_\alpha)|<3$, the case of f_α reducible over $\mathbb{Q}$. We shall bound this contribution by summing over splittings of the polynomial.

In particular, we have that

$$\sum_{\substack{\alpha\in\mathcal{O}_K\\ \|\alpha\|_{\sup}<X}} \mathbb{1}_{|\operatorname{Gal}(f_\alpha)|<3} = \sum_{\substack{x,y,u,v,w\in\mathbb{Z}\\ wy\neq 0\\ \|(x+y\zeta)(u+v\zeta+w\zeta^2)\|_{\sup}\ll X}} 1 = \sum_{\substack{x,y,u,v,w\in\mathbb{Z}\\ uwxy\neq 0\\ \|(x+y\zeta)(u+v\zeta+w\zeta^2)\|_{\sup}\ll X}} 1 + O(X^3) \tag{12.3}$$

Note that the bound on $\|-\|_{\sup}$ in the last two sums forces $ux, wy \ll X$. For any fixed x, y, u, w, looking at the ζ^2-coefficient implies the bound forces that

$$|vy + wx| \ll X.$$

That $y \neq 0$ implies that the second inequality can hold for at most $O(X)$-many $v \in \mathbb{Z}$. Therefore, we have that (12.3) is

$$\ll \sum_{\substack{x,y,u,w\ll X\\ uwxy\neq 0\\ ux,wy\ll X}} X + O(X^3) \ll X^3\log^2 X.$$

It remains therefore to bound the size of

$$\sum_{\substack{\alpha\in\mathcal{O}_K\\ \|\alpha\|_{\sup}<X}} \mathbb{1}_{\operatorname{Gal}(f_\alpha)=A_3}.$$

Note that a polynomial with Galois group A_3 has square discriminant, so it suffices to bound

$$\sum_{\substack{|n_0|,\ldots,|n_3|\ll X\\ n_3\neq 0}} \sum_m \mathbb{1}_{n_1^2n_2^2-4n_1^3n_3-4n_0n_2^3-27n_0^2n_3^2+18n_0n_1n_2n_3=m^2}, \tag{12.4}$$

Letting $A = -27n_0^2, B = 18n_0n_1n_2 - 4n_1^3, C = n_1^2n_2^2 - 4n_0n_2^3$. The number of $(n_i)_{0\leqslant i\leqslant 2}$ with $ABC = 0$ can be checked to be $\ll X^2$, so we'll suppose from now on that $ABC \neq 0$. The equation in the indicator function then may be rewritten as

$$m^2 = An_3^2 + Bn_3 + C$$

so completing the square yields that

$$(2An_3 + B)^2 - 4Am^2 = B^2 - 4AC.$$

Letting $U = 2An_3 + B, V = 2m$, this is equivalent to

$$U^2 + 3(3n_0V)^2 = B^2 - 4AC. \tag{12.5}$$

The number of n_0, n_1, n_2 with $B^2 - 4AC = 0$ is at most $O(X^2)$. If $B^2 - 4AC \neq 0$, the number of n_3 satisfying (12.5) is at most $d(|B^2 - 4AC|)$. It follows that (12.4) is at most

$$\sum_{\substack{|n_0|,|n_1|,|n_2|\ll X\\ B^2-4AC\neq 0}} d(|B^2 - 4AC|) \ll X^3\log X,$$

from which the desired result follows.

□

We may now prove Proposition 12.1. Applying a dyadic partition of unity and splitting $|\alpha_1|_\infty, |\alpha_2|_\infty$ into dyadic intervals, we have that for some fixed $\phi \in C_c^\infty((1,2))$,

$$\Sigma_1^{\{1,2\}} \ll X^{o(1)}(\Sigma_{\text{gen}} + \Sigma_{\text{bad}}) + O(X^{-A}),$$

where

$$\Sigma_{\text{gen}} = \sup_{\substack{1 \ll Q' \ll X/D \\ Y_1 \ll X^{o(1)}(MX_2/D) \\ Y_2 \ll X^{o(1)}(MX_1/D)}} \left| \frac{D^3}{X} \sum_{\substack{q \geqslant 1 \\ M|q}} \phi\left(\frac{q}{Q'}\right) \frac{(X/D)^5}{q^5} \sum_{\substack{\alpha_1,\alpha_2 \neq 0 \\ \text{Gal}(f_{\alpha_1\alpha_2}) = S_3 \\ |\alpha_i|_{\sup} \asymp Y_i \\ |\alpha_i|_{\sup} \ll X^{o(1)} X_{3-i}/D}} S_1(\alpha_1, \alpha_2; q) I_1(\alpha_1, \alpha_2; q) \right|,$$

$$\Sigma_{\text{bad}} = \sup_{\substack{1 \ll Q' \ll X/D \\ Y_1 \ll X^{o(1)}(MX_2/D) \\ Y_2 \ll X^{o(1)}(MX_1/D)}} \left| \frac{D^3}{X} \sum_{\substack{q \geqslant 1 \\ M|q}} \phi\left(\frac{q}{Q'}\right) \frac{(X/D)^5}{q^5} \sum_{\substack{\alpha_1,\alpha_2 \neq 0 \\ \text{Gal}(f_{\alpha_1\alpha_2}) \neq S_3 \\ |\alpha_i|_{\sup} \asymp Y_i \\ |\alpha_i|_{\sup} \ll X^{o(1)} X_{3-i}/D}} S_1(\alpha_1, \alpha_2; q) I_1(\alpha_1, \alpha_2; q) \right|.$$

All implied constants in what follows in this section will be allowed to have an $M^{O(1)}$ factor (similarly to the assumption we made in §2 regarding $\Omega^{O(1)}$ throughout the paper). We begin by bounding

$$\sum_{M|q} \frac{1}{q^5} \phi\left(\frac{q}{Q'}\right) S_1(\alpha_1, \alpha_2; q) I_1(\alpha_1, \alpha_2; q)$$

We split up $q = q_1 q_2 q_3$ for pairwise coprime q_i with $q_1 q_2$ squarefree, $(q_1, N(\alpha_1\alpha_2)) = 1$ and q_3 squarefull so that by Propositions 8.1 and the multiplicativity of $S_1(\alpha_1, \alpha_2; q)$ in q, we have that

$$\sum_q \frac{1}{q^5} \phi\left(\frac{q}{Q'}\right) S_1(\alpha_1, \alpha_2; q) I_1(\alpha_1, \alpha_2; q) = \sum_{\substack{q_2, q_3 \\ (q_2,q_3)=1 \\ M|q_2 q_3}} r_{\alpha_1\alpha_2}(q_2) S_1(\alpha_1, \alpha_2; q_3)$$
$$\sum_{(q_1, MN(\alpha_1\alpha_2))=1} \mu(q_1)^2 a_{\alpha_1\alpha_2}(q_1) \phi\left(\frac{q_1 q_2}{Q'}\right) \left(\frac{X/D}{q_1 q_2}\right)^5 I(\alpha_1, \alpha_2; q_1 q_2). \quad (12.6)$$

For Q' large, we apply Proposition 12.2 with Q, Q' taken to be Q'/q_2, $(X/D)/q_2$, respectively, when $\alpha_1\alpha_2$ is such that $\text{Gal}(f_{\alpha_1\alpha_2}) = S_3$. In that case, applying the oscillatory integral estimate Proposition 10.1, we have that the inner sum in (12.6) satisfies

$$\begin{aligned}\sum_{q_1} \mu(q_1)^2 a_{\alpha_1\alpha_2}(q_1)\phi\left(\frac{q_1q_2}{Q'}\right)\left(\frac{X/D}{q_1q_2}\right)^5 I(\alpha_1,\alpha_2;q_1q_2) \\ \ll X^{o(1)}\sqrt{\frac{X/D}{q_2}}\|\alpha_1\alpha_2\|_{\sup}^{1-4\delta_{\mathrm{sc}}} T^{\frac{3}{4}}\left(\frac{Y_1Y_2}{X/D^2}\right)^{-\frac{5}{3}} T^{\frac{2}{3}} \\ \ll X^{o(1)}\sqrt{\frac{X/D}{q_2q_3}}\left(\frac{X}{D^2}\right)^{1-4\delta_{\mathrm{sc}}}\left(\frac{Y_1Y_2}{X/D^2}\right)^{-\frac{5}{3}} T^{\frac{17}{12}}, \quad (12.7)\end{aligned}$$

where we write

$$T = \frac{X/D}{Q'} \asymp \frac{X/D}{q_1q_2}.$$

We also record that by Proposition 10.1, we have the trivial bound

$$\sum_{q_1} \mu(q_1)^2 a_{\alpha_1\alpha_2}(q_1)\phi\left(\frac{q_1q_2}{Q'}\right)\left(\frac{X/D}{q_1q_2}\right)^5 I(\alpha_1,\alpha_2;q_1q_2) \ll \frac{Q'}{q_2}\left(\frac{Y_1Y_2}{X/D^2}\right)^{-\frac{5}{3}} T^{\frac{2}{3}} X^{o(1)}. \quad (12.8)$$

At this point, applying Propositions 8.1, 8.2 and dropping the condition $M|q_2q_3$, we have that (12.6) is at most

$$\sum_{\substack{q_2q_3\ll Q' \\ (q_2,q_3)=1 \\ q_2'|q_2^2 \\ q_2'|N(\alpha_1\alpha_2) \\ q_3'|q_3 \\ q_3'|M^2\alpha_1\alpha_2}} \frac{q_2'}{\mathrm{rad}(q_2)} q_3'^3 \left|\sum_{(q_1,MN(\alpha_1\alpha_2))=1} \mu(q_1)^2 a_{\alpha_1\alpha_2}(q_1)\phi\left(\frac{q_1q_2}{Q'}\right)\left(\frac{X/D}{q_1q_2}\right)^5 I(\alpha_1,\alpha_2;q_1q_2)\right|.$$

It can be checked that since $\#\{\alpha\in\mathcal{O}_K : |\alpha|_{\sup} < Y, \beta|\alpha\} \ll Y^4/N(\beta)$ for any $\beta\in\mathcal{O}_K\backslash\{0\}$, we have by a divisor bound that

$$\sum_{\substack{\alpha_1,\alpha_2\neq 0 \\ |\alpha_i|_{\sup}\asymp Y_i \\ |\alpha_i|_{\sup}\ll X^{o(1)}X_{3-i}/D}} \mathbb{1}_{q_2'|N(\alpha_1\alpha_2)} \ll X^{o(1)}\frac{1}{q_2'}Y_1^4Y_2^4.$$

By the bound

$$\sum_{n<N}\frac{1}{\mathrm{rad}(n)} \ll N^{o(1)},$$

we obtain by combining (12.7) and (12.8) that

$$\Sigma_{\mathrm{gen}} \ll X^{o(1)}M^8 \sup_{T\gg 1}\min\left(\frac{D^3}{X}\frac{X}{D}\frac{X^4}{D^8}T^{-\frac{1}{3}}, \frac{D^3}{X}\left(\frac{X}{D}\right)^{\frac{1}{2}}\left(\frac{X}{D^2}\right)^{1-4\delta_{\mathrm{sc}}}\frac{X^4}{D^8}T^{\frac{5}{12}}\right).$$

The supremum is attained at

$$T \asymp \frac{D^2}{X^{\frac{2}{3}}}\left(\frac{X}{D^2}\right)^{\frac{16}{3}\delta_{\mathrm{sc}}} = L^{-2+\frac{32}{3}\delta_{\mathrm{sc}}}X^{\frac{16}{9}\delta_{\mathrm{sc}}},$$

yielding that

$$\Sigma_{\mathrm{gen}} \ll X^{o(1)}X^{2-\frac{16}{27}\delta_{\mathrm{sc}}}L^{\frac{20}{3}-\frac{32}{9}\delta_{\mathrm{sc}}}M^8.$$

It remains to bound Σ_{bad}. We begin with Cauchy-Schwarz, so that

$$\sum_{q\geqslant 1}\phi\left(\frac{q}{Q'}\right)\frac{(X/D)^5}{q^5}\sum_{\substack{\alpha_1,\alpha_2\neq 0\\ \mathrm{Gal}(f_{\alpha_1\alpha_2})\neq S_3\\ |\alpha_i|_\infty\asymp Y_i\\ |\alpha_i|_{\sup}\ll X^{o(1)}X_{3-i}/D}} S_1(\alpha_1,\alpha_2;q)I_1(\alpha_1,\alpha_2;q)$$

$$\ll X^{o(1)}\left(\frac{Y_1Y_2}{X/D^2}\right)^{-\frac{3}{2}}T^{\frac{2}{3}}\sum_{q\geqslant 1}\phi\left(\frac{q}{Q'}\right)\sum_{q_1,q_2|q}\sum_{\substack{\alpha\neq 0\\ \mathrm{Gal}(f_\alpha)\neq S_3\\ |\alpha|_\infty\asymp Y_1Y_2\\ |\alpha|_{\sup}\ll X^{o(1)}X/D^2}} q_1^3q_2\mathbb{1}_{\substack{q_1|\alpha\\ q_2|N(\alpha/q_2)}}$$

$$\ll X^{o(1)}\frac{X/D}{T}\left(\frac{Y_1Y_2}{X/D^2}\right)^{-\frac{3}{2}}T^{\frac{2}{3}}\sum_{q_1\ll Q'}q_1^2\sum_{q_2\geqslant 1}\sum_{\substack{\alpha\neq 0\\ \mathrm{Gal}(f_\alpha)\neq S_3\\ |q_1\alpha|_\infty\asymp Y_1Y_2\\ |q_1\alpha|_{\sup}\ll X^{o(1)}X/D^2\\ \nexists p \text{ s.t.} p|\alpha}} \mathbb{1}_{q_2|N(\alpha)}.$$

By a divisor bound and Proposition 12.3, we have that this is

$$\ll X^{o(1)}\frac{X}{D}\left(\frac{Y_1Y_2}{X/D^2}\right)^{-\frac{5}{3}}T^{-1/3}\min\left(\left(\frac{X}{D^2}\right)^3, Y_1^4Y_2^4\right)$$

$$\ll X^{o(1)}\frac{X}{D}\left(\frac{Y_1Y_2}{X/D^2}\right)^{-\frac{5}{3}}(Y_1Y_2)^{\frac{5}{3}}\left(\frac{X}{D^2}\right)^{\frac{7}{4}}T^{-1/3} \ll \frac{X^{\frac{53}{12}+o(1)}}{D^{\frac{47}{6}}} = X^{2-\frac{7}{36}}L^{\frac{47}{6}}.$$

Proposition 12.1 follows.

12.2. **Estimation of Σ_1^1, Σ_1^2: the partial zero frequencies.** In this section, we estimate Σ_1^1, Σ_1^2. Precisely, we show

Proposition 12.4. *Suppose that $j \in \{1,2\}$ and that $q/X_2 \gg X^\varepsilon$. Then, we have that*

$$\Sigma_1^j \ll M^6X^{2+o(1)}\frac{X}{DX_j^2}.$$

For the proof, we shall require a minor lemma on the number of $x, y(q)$ for which $(x + y\zeta, q)$ has large norm, with averaging over q. Specifically, we show

Lemma 12.5. *For any $Q, B \geqslant 1$, we have that*

$$\sum_{Q<q\leqslant 2Q} \sum_{\substack{x,y(q) \\ N((x+y\zeta,q))>B}} 1 \ll \frac{Q^3}{\sqrt{B}}.$$

Proof. We shall sum over $g = (x, y, q)$ and $\mathfrak{h} = (x/g + y\zeta/g, q/g)$ separately, noting that

$$\#\{x, y\,(q) : g\mathfrak{h}|x + y\zeta\} = \frac{q^2}{g^2 N\mathfrak{h}}.$$

It follows that

$$\sum_{Q<q\leqslant 2Q} \sum_{\substack{x,y\,(q) \\ N((x+y\zeta,q))>B}} 1 \leqslant \sum_{\substack{g\geqslant 1 \\ \mathfrak{h}\subset\mathcal{O}_K \\ g^4 N\mathfrak{h}>B}} \sum_{\substack{Q<q\leqslant 2Q \\ gN\mathfrak{h}|q}} \frac{q^2}{gN\mathfrak{h}} \leqslant Q^3 \sum_{\substack{g\geqslant 1 \\ \mathfrak{h}\subset\mathcal{O}_K \\ g^4 N\mathfrak{h}>B}} \frac{1}{g^3 (N\mathfrak{h})^2} \ll \frac{Q^3}{\sqrt{B}},$$

as desired. ∎

Armed with this, we can now quickly prove Proposition 12.4.

Proof of Proposition 12.4. Throughout the proof, fix $\varepsilon > 0$ sufficiently small.

Without loss of generality, suppose that $j = 2$. When $M|q$, by orthogonality, we have

$$S_1(0, \alpha_2; q) = \frac{1}{M^8} \sum_{\gamma_1,\gamma_2\in\mathcal{O}_K/(M)} \psi_M(\beta_1'\gamma_1 + \beta_2'\gamma_2) \frac{1}{q} \sum_{x,y(q)} G_M(\alpha_2, x, y, \gamma_1, \gamma_2; q), \tag{12.9}$$

where

$$G_M(\alpha_2, x, y, \gamma_1, \gamma_2; q) = \sum_{\substack{\beta_1\in\mathcal{O}_K/(q) \\ (x+y\zeta)\beta_1\equiv(\alpha_2+\gamma_2 q/M)\,(q)}} \psi_M(\gamma_1\beta_1).$$

All we shall require of G_M is that it is $M(x + y\zeta, q)$-periodic in α_2 and that it satisfies the bound

$$G_M(\alpha_2, x, y, \gamma_1, \gamma_2; q) \leqslant N((x + y\zeta, q))\mathbb{1}_{(x+y\zeta,q)|M\alpha_2}. \tag{12.10}$$

Opening up the definition of I_1 and inserting (12.9) into the definition of Σ_1^2, we obtain that

$$\Sigma_1^2 = \frac{1}{M^8} \sum_{\gamma_1,\gamma_2\in\mathcal{O}_K/(M)} \psi_M(\beta_1'\gamma_1 + \beta_2'\gamma_2)\Sigma_1^2(\gamma_1, \gamma_2), \tag{12.11}$$

where

$$\Sigma_1^2(\gamma_1, \gamma_2) := \frac{X^4}{D^2} \sum_{\substack{q\geqslant 1 \\ M|q}} \frac{1}{q^5} \sum_{\alpha_2\neq 0} \frac{1}{q} \sum_{x,y(q)} G_M(\alpha_2, x, y, \gamma_1, \gamma_2; q)$$

$$\int_{K_\infty} \phi_1(x_1^\infty) \int_{K_\infty} \omega_1\left(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|X}{qD}\right) \phi_2(x_2^\infty) e\left(\frac{X_2}{q}\langle x_2^\infty, \alpha_2\rangle\right) dx_1^\infty\, dx_2^\infty.$$

In preparation for what will follow, note that the terms on the RHS of (12.11) are supported on $q \ll X/D$ and, up to an error of $O_\varepsilon(X^{-100})$, $|\alpha_2|_{\sup} \leqslant (q/X_2)X^\varepsilon$, for by repeated integration by parts, we have

$$\iint_{K_\infty^2} \omega_1\left(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|X}{qD}\right)\phi_1(x_1^\infty)\phi_2(x_2^\infty)e\left(\frac{X_2}{q}\langle x_2^\infty, \alpha_2\rangle\right) dx_1^\infty\, dx_2^\infty \ll_A \left(\frac{q}{X/D}\right)^2\left(1+\frac{|\alpha_2|_{\sup}}{q/X_2}\right)^{-A}.$$

The nonzeroness of α_2 also implies that the contribution of $q \leqslant X_2X^{-\varepsilon}$ is $O_\varepsilon(X^{-100})$.

We shall discard those x, y with $(x+y\zeta, q)$ of norm greater than $B_q = (q/X_2)^4M^{-4}X^{-2\varepsilon}$ so that we may apply Poisson summation in α_2 and deal only with the zero-frequency.

To this end, note that the contribution of x, y with $N((x+y\zeta, q)) > B_q$ to (12.11) is at most

$$\ll_\varepsilon X^{-100} + M^4X^{5\varepsilon}\frac{X^4}{D^2}\sum_{q\ll X/D}\frac{1}{q^5}q\left(\frac{q}{X_2}\right)^4\left(\frac{q}{X/D}\right)^2\frac{1}{\sqrt{(q/X_2)^4}} \ll_\varepsilon M^6X^{5\varepsilon}\frac{X^3}{DX_2^2}.$$

Therefore, we have that

$$\Sigma_1^2(\gamma_1,\gamma_2) = \frac{X^4}{D^2}\sum_{X_2X^{-\varepsilon}\leqslant q\ll X/D}\frac{1}{q^6}\sum_{\substack{x,y(q)\\ N((x+y\zeta,q))\leqslant B_q}}\sum_{\alpha_2\neq 0} G_M(\alpha_2,x,y,\gamma_1,\gamma_2;q)$$
$$\int_{K_\infty}\phi_1(x_1^\infty)\int_{K_\infty}\omega_1\left(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|X}{qD}\right)\phi_2(x_2^\infty)e\left(\frac{X_2}{q}\langle x_2^\infty,\alpha_2\rangle\right)dx_2^\infty\, dx_1^\infty + O_\varepsilon\left(M^6\frac{X^{3+5\varepsilon}}{DX_2^2}\right). \tag{12.12}$$

We can reinsert the contribution of $\alpha_2 = 0$ at this point (critically, the restriction $q \geqslant X_2X^{-\varepsilon}$ has been made at this point, for otherwise the cost would be unacceptable) by Lemma 12.5 and the bound (12.10) at the cost of an additional remainder of size

$$\ll_\varepsilon \frac{X^4}{D^2}\sum_{X_2X^{-\varepsilon}\leqslant q\ll X/D}\frac{1}{q^6}q^2\left(\frac{q}{X/D}\right)^2\sqrt{B_q} \ll_\varepsilon \frac{X^{3+\varepsilon}}{X_2^2D}.$$

It follows that

$$\Sigma_1^2(\gamma_1,\gamma_2) = \frac{X^4}{D^2}\sum_{X_2X^{-\varepsilon}\leqslant q\ll X/D}\frac{1}{q^6}\sum_{\substack{x,y(q)\\ N((x+y\zeta,q))\leqslant B_q}}\sum_{\alpha_2} G_M(\alpha_2,x,y,\gamma_1,\gamma_2;q)$$
$$\int_{K_\infty}\phi_1(x_1^\infty)\int_{K_\infty}\omega_1\left(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|X}{qD}\right)\phi_2(x_2^\infty)e\left(\frac{X_2}{q}\langle x_2^\infty,\alpha_2\rangle\right)dx_2^\infty\, dx_1^\infty$$
$$+ O_\varepsilon\left(M^6\frac{X^{3+5\varepsilon}}{DX_2^2}+\frac{X^{2+\varepsilon}}{X_2}\right).$$

In what remains, we shall fix x, y, x_1^∞, q for which $N(M(x+y\zeta, q)) \leqslant M^4 B_q$, as is implied by the bounds we have put in thus far, and show a bound of $O(X^{-100})$ on

$$\sum_{\alpha_2} G_M(\alpha_2, x, y, \gamma_1, \gamma_2; q) \int_{K_\infty} \omega_1\bigg(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|X}{qD}\bigg)\phi_2(x_2^\infty) e\bigg(\frac{X_2}{q}\langle x_2^\infty, \alpha_2\rangle\bigg)\, dx_2^\infty. \tag{12.13}$$

Let $(\kappa) = M(x + y\zeta, q)$, and suppose that $|\kappa|_{\sup} \ll |N(\kappa)|^{1/4}$. Then, the fact that G_M is κ-periodic in α_2 implies that by Proposition 5.3 (Poisson summation), we have that (12.13) equals

$$\bigg(\frac{q}{X_2}\bigg)^4 \sum_{\alpha_2}\bigg(\sum_{\beta\in\mathcal{O}_K/(\kappa)} G_M(\beta)\psi\bigg(-\frac{\alpha_2\beta}{\kappa}\bigg)\bigg)\omega_1\bigg(\frac{X}{DX_2}\bigg|\boldsymbol{\ell}\bigg(\frac{x_1^\infty\alpha_2}{\kappa}\bigg)\bigg|\bigg)\phi_2\bigg(\frac{q\alpha_2}{\kappa X_2}\bigg),$$

where we're omitting the dependence of G_M on x, y, γ_1, γ_2, q. Now, note that the compact support of ϕ_2 along with the fact that for α_2 nonzero,

$$\bigg|\frac{q\alpha_2}{\kappa X_2}\bigg|_{\sup} \gg \frac{q}{X_2|N(\kappa)|^{1/4}} \geqslant \frac{q}{X_2(M^4B_q)^{1/4}} \gg X^{\varepsilon/2},$$

and so we obtain that (12.13) vanishes for $X \gg_\varepsilon 1$. In particular, we have

$$\sum_{\alpha_2} G_M(\alpha_2, x, y, \gamma_1, \gamma_2; q) \int_{K_\infty} \omega_1\bigg(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|X}{qD}\bigg)\phi_2(x_2^\infty) e\bigg(\frac{X_2}{q}\langle x_2^\infty, \alpha_2\rangle\bigg)\, dx_2^\infty \ll_\varepsilon X^{-100}. \tag{12.14}$$

Taking $\varepsilon \to 0$ sufficiently slowly yields the desired result upon collecting (12.11), (12.12), (12.13), and (12.14). $\square$

12.3. **Estimation of $\Sigma_1^{\{\}}$: the full-zero frequencies.** We begin with the treatment of the full-zero frequencies $\Sigma_1^{\{\}}$. The main result of this subsection is as follows.

Proposition 12.6. *We have*

$$\Sigma_1^{\{\}} = -\frac{1}{2}X^2\sigma_\infty\prod_p \sigma_p + \frac{X^3}{D}M^{-1}\mathcal{D}_1^*(1)\tilde{\omega}_1(1)\int_{K_\infty^2}\phi_1(x_1^\infty)\phi_2(x_2^\infty)|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|^{-1}\, dx_1^\infty\, dx_2^\infty + O(M^4DX^{1+o(1)}).$$

The remainder of this section is devoted to the proof of Proposition 12.6. We can open up the definition of I_1 from (4.2) so that

$$\Sigma_1^{\{\}} = \frac{X^4}{D^2}\sum_{\substack{q\geqslant 1\\ M|q}}\frac{1}{q^5}S_1(0,0;q)\int_{K_\infty^2}\phi_1(x_1^\infty)\phi_2(x_2^\infty)\omega_1\bigg(\frac{|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|MX}{qD}\bigg)\, dx_1^\infty\, dx_2^\infty.$$

By Mellin inversion, we have that

$$\Sigma_1^{\{\}} = \frac{X^4}{D^2}\frac{1}{2\pi i}\int_{(3)}\mathcal{D}_1(s+2)\tilde{\omega}_1(-s)U_1(s)\bigg(\frac{MX}{D}\bigg)^s\, ds, \tag{12.15}$$

where for $\mathrm{Re}(s) > -2$,

$$U_1(s) = \int_{K_\infty^2} \phi_1(x_1^\infty)\phi_2(x_2^\infty)|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|^s \, dx_1^\infty \, dx_2^\infty. \tag{12.16}$$

All we shall require, coming from Lemma 11.5, is that $U_1(s)$ has a meromorphic continuation to $\mathrm{Re}\, s > -3$, in which region it has a pole only at $s = -2$ with

$$\operatorname*{res}_{s=-2} U_1(s) = 2\pi \int_{K_\infty^2} \phi_1(x_1^\infty)\phi_2(x_2^\infty)\delta(\boldsymbol{\ell}(x_1^\infty x_2^\infty)) \, dx_1^\infty \, dx_2^\infty, \tag{12.17}$$

and that $|U_1(s)| \ll |s+2|^{-1}$ for $\mathrm{Re}\, s > -3$.

Therefore, applying Propositions 11.1 and moving the contour in (12.15) to $\mathrm{Re}(s) = -3 + 1/\log X$ yields that

$$\begin{aligned}
\Sigma_1^{\{\}} &= \frac{X^4}{D^2}\frac{1}{2\pi i}\int_{(3)} \mathcal{D}_1(s+2)\tilde{\omega}_1(-s)U_1(s)\left(\frac{MX}{D}\right)^s ds \\
&= \frac{X^3}{D}M^{-1}\mathcal{D}_1^*(1)\tilde{\omega}_1(1)U_1(-1) + X^2M^{-2}\mathcal{D}_1^*(0)\zeta(0)\tilde{\omega}_1(2)\operatorname*{res}_{s=-2} U_1(s) \\
&\qquad + \frac{X^4}{D^2}\frac{1}{2\pi i}\int_{(-3+1/\log X)} \zeta(s+2)\mathcal{D}_1^*(s+2)\tilde{\omega}_1(-s)U_1(s)\left(\frac{MX}{D}\right)^s ds.
\end{aligned}$$

Recalling (11.3), (12.17), that $2\pi\tilde{\omega}_1(2) = 1$ by (6.1), and that $\zeta(0) = -1/2$, Proposition 12.6 follows from bounds on U_1, ζ in vertical strips.

13. Estimation of Σ_2

In this section, we carry out the estimation of Σ_2^S.

13.1. **Estimation of Σ_2^{12}: nonzero frequencies.**

We prove in this subsection the following bound on Σ_2^{12}.

Proposition 13.1. *We have that*

$$\Sigma_2^{12} \ll X^{2+o(1)}\left(\frac{D^{3/2}}{\sqrt{X}} + \frac{1}{D^{2/7}}\right)M^4.$$

We also record the following bound on incomplete norm forms satisfying a divisibility condition.

Lemma 13.2. *For any $Y, b \geqslant 1$, we have that*

$$\sum_{\substack{|\alpha|_{\sup} < Y \\ \langle \alpha, 1\rangle = 0 \\ b \mid N(\alpha)}} 1 \ll b^{o(1)}\frac{Y^3}{b^{3/4}}.$$

Proof. If $b > Y^4$, the result is clear, so let's suppose $b \leqslant Y^4$ from now on. Take r a prime in $[Y/b^{1/4}, 10Y(1+\log b)/b^{1/4}]$ coprime to b. Then, we have that

$$\sum_{\substack{|\alpha|_{\sup}<Y\\ \langle\alpha,1\rangle=0\\ b|N(\alpha)}} 1 \leqslant \sum_{\substack{\mathfrak{b}\subset\mathcal{O}_K\\ N\mathfrak{b}=b}} \sum_{\substack{\beta\in\mathcal{O}_K/(r)\\ \langle\beta,1\rangle\equiv 0(r)}} \sum_{\substack{|\alpha|_{\sup}<Y\\ \mathfrak{b}|\alpha\\ \alpha\equiv\beta(r)}} 1.$$

Note that the two congruence conditions, due to the coprimality of b, r, are together equivalent to a congruence condition modulo $\mathfrak{b}r$. Since $N(\mathfrak{b}r) = br^4 \ll Y^4 b^{o(1)}$, we may bound the above by

$$\ll b^{o(1)} r^3 \frac{Y^4}{r^4 b} \ll b^{o(1)} \frac{Y^3}{b^{3/4}},$$

as desired. □

We are now ready to proceed with the proof of Proposition 13.1, to which we shall devote the remainder of this subsection. We begin by splitting the ranges of q, d, $|\alpha_1|_{\sup}$, $|\alpha_2|_{\sup}$ into dyadic intervals. Applying Propositions 9.1 and 10.2, we have that

$$\Sigma_2^{12} \ll X^{-A} + X^{o(1)} \sup_{\substack{D_1\ll D\\ Q\ll M\sqrt{XD}/D_1\\ A_1\ll X^{o(1)}\sqrt{XD}/X_1\\ A_2\ll X^{o(1)}\sqrt{XD}/X_2\\ G'H'\ll D_1\\ G''H''\ll Q}} \mathcal{S}(D_1, Q, A_1A_2, G', H', G'', H''),$$

where

$$\begin{aligned}\mathcal{S} &:= \mathcal{S}(D_1, Q, A, G', H', G'', H'')\\ &:= \frac{X^4}{D^2}\frac{\sqrt{DX}}{D_1Q}\frac{D_1}{\sqrt{DX}}\left(\frac{1}{\sqrt{DX}}\right)^5\left(\frac{A}{D}\right)^{-\frac{3}{2}} G^4H \sum_{\substack{g'\asymp G'\\ h'\asymp H'\\ g''\asymp G''\\ h''\asymp H''}} \sum_{\substack{|\mathbf{c}|\asymp D/D_1\\ (c_1,c_2)=1}} \sum_{\substack{d\asymp D_1\\ g'h'|d}} \sum_{\substack{q\asymp Q\\ g''h''|q}}\\ &\qquad \sum_{r|dq}\frac{1}{r}\frac{1}{(D_1Q)^2} \sum_{\substack{x,y,z\ll D_1Q\\ g|\gamma_{\mathbf{c}}(x,y,z;q)\\ y<z \text{ if } |c_1|\leqslant|c_2|\\ z<y \text{ if } |c_2|<|c_1|}} \sum_{\substack{|\alpha|_{\sup}\ll AM\\ g|\alpha\\ h^2|N(\alpha/g)}} \mathbb{1}_{dq/r|Y_{\mathbf{c}}(\alpha,x,y,z;q)},\end{aligned}$$

with the notations $g = g'g''$, $h = h'h''$, $G = G'G''$, $H = H'H''$ and $\gamma_{\mathbf{c}}$, $Y_{\mathbf{c}}$ as in Proposition 9.1. Here we first extend the x, y, z-sum mod dq to any interval of size D_1Q containing $[d_1q, 2d_1q]$ for any $d \asymp D_1$ and $q \asymp Q$. Since we can take two such intervals that are disjoint from each other, we can impose the restriction $y < z$ or $z < y$ depending on the size of c_1, c_2. We then extend the summations to $x, y, z \ll D_1Q$ under such restrictions. We fix some choice of $D_1, Q, A_1, A_2, G', H', G'', H''$ satisfying the constraints in the supremum from now on and will omit their dependence of $\mathcal{S}$ on these parameters. Here, we have replaced $\alpha_1\alpha_2$ with α', and replaced α' with $\alpha = M\alpha'$ (at which point we have dropped the condition $M|\alpha$).

At this point, we may split (perhaps not uniquely) $r = uv$ for some $u|d$ and $v|q$. Writing $d_1 = d/u$, $q_1 = q/v$ followed by a change of variable $\alpha \mapsto \alpha g$, we obtain that

$$\mathcal{S} \leqslant \frac{X^{3/2}}{D^3}\frac{G^4H}{A^{3/2}D_1^2Q^3} \sum_{\substack{g'\asymp G'\\ h'\asymp H'\\ g''\asymp G''\\ h''\asymp H''}} \sum_{\substack{|\alpha|_{\sup}\ll AM/G\\ h^2|N(\alpha)}} \sum_{\substack{|\mathbf{c}|\asymp D/D_1\\ (c_1,c_2)=1}} \sum_{\substack{x,y,z\ll D_1Q\\ g|\gamma_{\mathbf{c}}(x,y,z;q)\\ y<z \text{ if } |c_1|\leqslant|c_2|\\ z<y \text{ if } |c_2|<|c_1|}} \sum_{\substack{q_1v\asymp Q\\ g''h''|q_1v}} \frac{1}{v}\mathbb{1}_{q_1|Y_{\mathbf{c}}(\alpha,x)} \sum_{\substack{d_1u\asymp D_1\\ g'h'|d_1u}} \frac{1}{u}\mathbb{1}_{d_1|Y_{\mathbf{c}}(\alpha,x,y,z;q)}.$$

Here, we have written $Y_{\mathbf{c}}(\alpha, x)$ to denote $Y_{\mathbf{c}}(\alpha, x, 0, 0; q)$, which doesn't depend on q. We will also let $Y_{\mathbf{c}}(\alpha) = Y_{\mathbf{c}}(\alpha, 1)$ and observe that $Y_{\mathbf{c}}(\alpha, x) = x^3 Y_{\mathbf{c}}(\alpha)$. We may clearly suppose from now on that $H' \ll D_1$ and $H'' \ll Q$. Splitting according to whether $Y_{\mathbf{c}}(\alpha)$ is nonzero or not yields that

$$\mathcal{S} \ll \mathcal{S}_1 + \mathcal{S}_2,$$

where

$$\mathcal{S}_1 = \frac{X^{3/2}}{D^3}\frac{G^4H}{A^{3/2}D_1^2Q^3} \sum_{\substack{g'\asymp G'\\ h'\asymp H'\\ g''\asymp G''\\ h''\asymp H''}} \sum_{\substack{|\mathbf{c}|\asymp D/D_1\\ (c_1,c_2)=1}} \sum_{\substack{|\alpha|_{\sup}\ll AM/G\\ h^2|MN(\alpha)\\ Y_{\mathbf{c}}(\alpha)\neq 0}} \sum_{\substack{x,y,z\ll D_1Q\\ g|\gamma_{\mathbf{c}}(x,y,z;q)\\ y<z \text{ if } |c_1|\leqslant|c_2|\\ z<y \text{ if } |c_2|<|c_1|}} \sum_{\substack{q_1v\asymp Q\\ g''h''|q_1v}} \frac{1}{v}\mathbb{1}_{q_1|Y_{\mathbf{c}}(\alpha,x)} \sum_{\substack{d_1u\asymp D_1\\ g'h'|d_1u}} \frac{1}{u}\mathbb{1}_{d_1|Y_{\mathbf{c}}(\alpha,x,y,z;q)},$$

$$\mathcal{S}_2 = \frac{X^{3/2}}{D^3}\frac{G^4H}{A^{3/2}D_1^2Q^3} \sum_{\substack{g'\asymp G'\\ h'\asymp H'\\ g''\asymp G''\\ h''\asymp H''}} \sum_{\substack{|\mathbf{c}|\asymp D/D_1\\ (c_1,c_2)=1}} \sum_{\substack{|\alpha|_{\sup}\ll AM/G\\ h^2|N(\alpha)\\ Y_{\mathbf{c}}(\alpha)=0}} \sum_{\substack{x,y,z\ll D_1Q\\ g|\gamma_{\mathbf{c}}(x,y,z;q)\\ y<z \text{ if } |c_1|\leqslant|c_2|\\ z<y \text{ if } |c_2|<|c_1|}} \sum_{\substack{q\asymp Q\\ g''h''|q}} \sum_{\substack{d_1u\asymp D_1\\ g'h'|d_1u}} \frac{1}{u}\mathbb{1}_{d_1|Y_{\mathbf{c}}(\alpha,x,y,z;q)}.$$

We start with a bound on $\mathcal{S}_1$, which we begin by discarding the d_1 and y, z conditions, so that

$$\mathcal{S}_1 \ll \frac{X^{3/2}}{D^3}\frac{G^4H}{A^{3/2}D_1Q^3}\sum_{\substack{g'\asymp G'\\ h'\asymp H'\\ g''\asymp G''\\ h''\asymp H''}}\frac{1}{g'h'}\sum_{\substack{|\mathbf{c}|\asymp D/D_1\\ (c_1,c_2)=1}}\sum_{\substack{|\alpha|_{\sup}\ll AM/G\\ h^2|MN(\alpha)\\ Y_{\mathbf{c}}(\alpha)\neq 0}}\sum_{\substack{x,y,z\ll D_1Q\\ g|\gamma_{\mathbf{c}}(x,y,z;q)}}\sum_{\substack{q_1v\asymp Q\\ g''h''|q_1v}}\frac{1}{v}\mathbb{1}_{q_1|x^3Y_{\mathbf{c}}(\alpha)}.$$

We then further decompose $g'' = g'''w$ for some $g'''|q_1$ and $w|v$. Letting $q_2 = q_1/g'''$, we obtain that

$$\begin{aligned}\mathcal{S}_1 &\ll \frac{X^{3/2}}{D^3}\frac{G^4H}{A^{3/2}D_1^2H'Q^3}\sum_{\substack{w\ll G''\\ h\asymp H}}\frac{1}{w}\sum_{\substack{|\mathbf{c}|\asymp D/D_1\\ (c_1,c_2)=1}}\sum_{\substack{|\alpha|_{\sup}\ll AM/G\\ h^2|N(\alpha)\\ Y_{\mathbf{c}}(\alpha)\neq 0}}\\ &\qquad\sum_{x,y,z\ll D_1Q}\sum_{v\ll Q/w}\frac{1}{v}\sum_{g'''q_2\asymp Q/(vw)}\mathbb{1}_{g'''q_2|x^3Y_{\mathbf{c}}(\alpha)}\\ &\ll \frac{X^{3/2}}{D^3}\frac{G^4H}{A^{3/2}D_1H'Q^3}\left(\frac{D}{D_1}\right)^2\left(\frac{AM}{G}\right)^4\frac{1}{H}(D_1Q)^3 \ll M^4X^{\frac{3}{2}+o(1)}D^{\frac{3}{2}},\end{aligned}$$

which is the first term in the statement of Proposition 13.1.

Now, we turn to the degenerate case $\mathcal{S}_2$. We begin by similarly applying the divisor bound to the sum over d_1, this time making no effort to make use of the condition $g'h'|d_1u$. It can be checked that as a polynomial in q, $Y_{\mathbf{c}}(\alpha, x, y, z; q)$ can only be identically 0 if $c_1y + c_2z = 0$ and $Y_{\mathbf{c}}(\alpha) = 0$. By the condition on y, z, the first of these cannot hold, and hence $Y_{\mathbf{c}}(\alpha, x, y, z; q) \neq 0$. Thus, we sum over d_1 to obtain

$$\mathcal{S}_2 \ll \frac{X^{3/2}}{D^3}\frac{G^4H}{A^{3/2}D_1^2Q^3}\sum_{\substack{g'\asymp G'\\ h'\asymp H'\\ g''\asymp G''\\ h''\asymp H''}}\sum_{\substack{|\mathbf{c}|\asymp D/D_1\\ (c_1,c_2)=1}}\sum_{\substack{|\alpha|_{\sup}\ll AM/G\\ h^2|MN(\alpha)\\ Y_{\mathbf{c}}(\alpha)=0}}\sum_{\substack{x,y,z\ll D_1Q\\ y<z \text{ if } |c_1|\leqslant|c_2|\\ z<y \text{ if } |c_2|<|c_1|}}\sum_{\substack{q\asymp Q\\ g''h''|q\\ g|\gamma_{\mathbf{c}}(x,y,z;q)}} 1.$$

With $\gamma_{\mathbf{c}}(x, y, z; q) = c_2x + qy + (-c_1x + qz)\zeta$, the divisibility condition on $\gamma_{\mathbf{c}}(x, y, z; q)$ is satisfied a $\ll (g, q)/g^2$-fraction of the time, and so

$$\sum_{\substack{q\asymp Q\\ g''h''|q}}\sum_{\substack{x,y,z\ll D_1Q\\ g|\gamma_{\mathbf{c}}(x,y,z;q)}} \ll Q^{1+o(1)}\frac{(g,h'')}{g^2h''}(D_1Q)^3.$$

Hence we arrive at

$$\mathcal{S}_2 \ll \frac{X^{3/2+o(1)}}{A^{3/2}D^3} D_1 G^2 HQ \sum_{\substack{g' \asymp G' \\ h' \asymp H' \\ g'' \asymp G'' \\ h'' \asymp H''}} \frac{(g, h'')}{h''} \sum_{\substack{|\mathbf{c}| \asymp D/D_1 \\ (c_1, c_2)=1}} \sum_{\substack{|\alpha|_{\sup} \ll AM/G \\ h^2 | N(\alpha) \\ Y_{\mathbf{c}}(\alpha)=0}} 1$$

$$\ll \frac{X^{3/2+o(1)}}{D^3} \frac{D_1 G^3 H' Q}{A^{3/2}} \sum_{h \asymp H} \sum_{\substack{|\mathbf{c}| \asymp D/D_1 \\ (c_1, c_2)=1}} \sum_{\substack{|\alpha|_{\sup} \ll AM/G \\ h^2 | N(\alpha) \\ Y_{\mathbf{c}}(\alpha)=0}} 1.$$

Note that α with $Y_{\mathbf{c}}(\alpha) = 0$ are precisely those α of the form $(c_2 - c_1\zeta)(m_0 + m_1\zeta + m_2\zeta^2)$, the second factor of which we call τ. Furthermore, we have that $|\alpha|_{\sup} \ll AM/G$ implies that $|m_i| \ll AD_1M/(GD)$. Summing now over $\tau = m_0 + m_1\zeta + m_2\zeta^2$ yields that

$$\mathcal{S}_2 \ll \frac{X^{3/2+o(1)}}{A^{3/2}D^3} D_1 G^3 H' Q \sum_{h \asymp H} \sum_{\substack{|\mathbf{c}| \asymp D/D_1 \\ (c_1, c_2)=1}} \sum_{\substack{|\tau|_{\sup} \ll AD_1M/(GD) \\ \langle \tau, 1 \rangle = 0 \\ h^2 | M(c_1^4 + c_2^4) N(\tau)}} 1. \tag{13.1}$$

At this point, we shall gather several different bounds. First, we note that the divisor bound applied to the sum over h (weakening the congruence condition to $h | (c_1^4 + c_2^4) N(\tau)$) yields that

$$\mathcal{S}_2 \ll \frac{X^{3/2+o(1)}}{A^{3/2}D^3} D_1 G^3 H' Q \left(\frac{D}{D_1}\right)^2 \left(\frac{AD_1M}{GD}\right)^3 \ll \frac{X^{2+o(1)} D_1 H'}{D^2} \ll M^3 \frac{X^{2+o(1)} D_1^2}{D^2}. \tag{13.2}$$

This is nearly sufficient, apart from the extreme cases when $D_1, H' \approx D$. To deal with these cases, we'll apply Lemma 13.2, which yields that

$$\sum_{\substack{|\tau|_{\sup} \ll AD_1M/(GD) \\ \langle \tau, 1 \rangle = 0 \\ h^2 | (c_1^4 + c_2^4) N(\tau)}} 1 \leqslant \sum_{\substack{|\tau|_{\sup} \ll AD_1M/(GD) \\ \langle \tau, 1 \rangle = 0 \\ h^2/(c_1^4 + c_2^4, h^2) | N(\tau)}} 1 \ll X^{o(1)} (h^2, c_1^4 + c_2^4)^{\frac{3}{4}} \frac{(AD_1M/(GD))^3}{h^{3/2}}$$

$$\ll X^{o(1)} (D/D_1)^3 \frac{(AD_1M/(GD))^3}{h^{3/2}} \ll X^{o(1)} M^3 \frac{A^3/G^3}{h^{3/2}}.$$

Plugged into (13.1) yields that

$$\mathcal{S}_2 \ll \frac{X^{3/2+o(1)}}{A^{3/2}D^3} D_1 G^3 H' Q H \left(\frac{D}{D_1}\right)^2 \frac{A^3}{G^3 H^{3/2}} \ll M^3 \frac{X^{2+o(1)} D}{D_1^2/\sqrt{H'}} \ll M^3 \frac{X^{2+o(1)} D}{D_1^{3/2}}.$$

Combined with (13.2), we obtain that

$$\mathcal{S}_2 \ll M^4 \frac{X^{2+o(1)}}{D^{\frac{2}{7}}},$$

and the desired result follows.

13.2. **Estimation of Σ_2^1, Σ_2^2: the partial zero frequencies.** We prove in this subsection the following bound on the partial zero frequencies Σ_2^j.

Proposition 13.3. *We have that for $j \in \{1,2\}$,*

$$\Sigma_2^j \ll X^{2+o(1)} M^6 \frac{X^{3/2}}{X_j^3 \sqrt{D}}.$$

We will require the following estimate:

Lemma 13.4. *For any D_1, Q, C, B, we have that*

$$\sum_{\substack{d \asymp D_1 \\ q \asymp Q}} \sum_{\substack{|\mathbf{c}| \asymp C \\ (c_1,c_2)=1}} \sum_{\substack{x(dq) \\ y,z(d) \\ N((x(c_2-c_1\zeta)+q(y+z\zeta),dq))>B}} 1 \ll \frac{C^2 D_1^4 Q^2}{B^{\frac{1}{4}}} (CD_1Q)^{o(1)}.$$

This follows immediately from the following result.

Lemma 13.5. *We have that for any $g|dq$, $h|(dq/g)^4$, and $(c_1, c_2) = 1$,*

$$\#\{x(dq), y, z(d) : g||x(c_2 - c_1\zeta) + q(y + z\zeta), h|N((x(c_2 - c_1\zeta) + q(y + z\zeta))/g)\} \ll \frac{d^3 q}{gh}(h, c_1^4 + c_2^4)(dq)^{o(1)}.$$

Proof. By multiplicativity, we may suppose that d, q are prime powers (of p, say). If $d = 1$, we have

$$\#\big\{x(q) : g||x(c_2 - c_1\zeta), h|x^4(c_1^4 + c_2^4)/g^4\big\} = \frac{q}{g}\mathbb{1}_{h|c_1^4+c_2^4} \leqslant \frac{q}{gh}(h, c_1^4 + c_2^4),$$

from which the desired result is clear, and if $q = 1$, we have that

$$\#\big\{y, z(d) : g|y + z\zeta, h|(y^4 + z^4)/g^4\big\} = \#\big\{y_1, z_1(d/g) : (y_1, z_1, p) = 1, h|y_1^4 + z_1^4\big\} \leqslant 4\frac{d}{gh}.$$

Therefore, we may suppose from now on that d, q are powers of p greater than 1. We can split based on the power of p dividing x. The contribution of $q|x$ reduces to the case of $q = 1$, and the contribution more generally of $(x, q) = q_1$ reduces to the case of $(x, q) = 1$ (with q replaced by q/q_1). We'll therefore suppose from now on that $(x, q) = 1$.

It can then be checked by Hensel's lemma that for any $w(d)$, we have

$$\#\big\{y, z(d) : (c_2 + qy)^4 + (-c_1 + qz)^4 \equiv c_1^4 + c_2^4 + wq(dq)\big\} \ll d$$

(in fact, for odd p, it is equal to d). The desired result follows. □

Proof of Lemma 13.4. Letting $(x(c_2 - c_1\zeta) + q(y + z\zeta), dq) = g\mathfrak{h}$ for some $\mathfrak{h}$ of norm h not contained in an ideal generated by a rational integer, we have that

$$\sum_{\substack{d \asymp D_1 \\ q \asymp Q}} \sum_{\substack{|\mathbf{c}| \asymp C \\ (c_1,c_2)=1}} \sum_{\substack{x(dq) \\ y,z(d) \\ N((x(c_2-c_1\zeta)+q(y+z\zeta),dq))>B}} 1 \leqslant \sum_{\substack{g \geqslant 1 \\ \mathfrak{h} \subset \mathcal{O}_K \\ g^4 N\mathfrak{h} > B}} \sum_{\substack{d \asymp D_1 \\ q \asymp Q \\ gh|dq}} \sum_{\substack{|\mathbf{c}| \asymp C \\ (c_1,c_2)=1}} \sum_{\substack{x(dq) \\ y,z(d) \\ g||x(c_2-c_1\zeta)+q(y+z\zeta) \\ \mathfrak{h}|(x(c_2-c_1\zeta)+q(y+z\zeta))/g}} 1.$$

Applying Lemma 13.5, we obtain that this is

$$(D_1Q)^{o(1)} \sum_{\substack{g,h\geqslant 1\\ g^4h>B}} \sum_{\substack{d\asymp D_1\\ q\asymp Q\\ gh|dq}} \sum_{\substack{|\mathbf{c}|\asymp C\\ (c_1,c_2)=1}} \frac{d^3q}{gh}(h, c_1^4+c_2^4) \ll (D_1Q)^{o(1)} D_1^4Q^2 \sum_{\substack{|\mathbf{c}|\asymp C\\ (c_1,c_2)=1}} \sum_{\substack{g,h\geqslant 1\\ g^4h>B}} \frac{(h, c_1^4+c_2^4)}{g^2h^2}$$

$$\ll (CD_1Q)^{o(1)} \frac{D_1^4Q^2}{B^{\frac14}} \sum_{\substack{|\mathbf{c}|\asymp C\\ (c_1,c_2)=1}} \sum_{h\geqslant 1} \frac{(h, c_1^4+c_2^4)}{h^{\frac74}} \ll (CD_1Q)^{o(1)} \frac{C^2D_1^4Q^2}{B^{\frac14}},$$

as desired. $\square$

The remainder of this subsection is dedicated to the proof of Proposition 13.3. We shall prove it in the case $j=1$, for the other case is identical.

Throughout the proof, fix $\varepsilon > 0$ (which at the end we will send slowly to 0).

Then, as in the proof of Proposition 12.4, we have that

$$\Sigma_2^1 = \frac{1}{M^8} \sum_{\gamma_1,\gamma_2\in\mathcal{O}_K/(M)} \psi_M(\beta_1'\gamma_1 + \beta_2'\gamma_2)\Sigma_2^1(\gamma_1,\gamma_2),$$

where

$$\Sigma_2^1(\gamma_1,\gamma_2) = \frac{X^4}{D^2} \sum_{\substack{d\geqslant 1\\ \mathbf{c}\in\mathbb{Z}^2\\ (c_1,c_2)=1}} \frac{1}{d^5}\omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\frac{d}{\sqrt{DX}} \sum_{\substack{q\geqslant 1\\ M|q}} \frac{1}{q^5} \sum_{\alpha_2\neq 0} \frac{1}{d^2} \sum_{\substack{x(dq)\\ y,z(d)}} G_M(\alpha_2)$$

$$\int_{K_\infty^2} \phi_1(x_1^\infty)\phi_2(x_2^\infty)\left(\omega_2\left(\frac{dq}{M\sqrt{DX}}\right) - \omega_2\left(\frac{M\sqrt{X}\det(\mathbf{c},\boldsymbol{\ell}(x_1^\infty x_2^\infty))}{q\sqrt{D}}\right)\right) e\left(\frac{X_2}{dq}\langle x_2^\infty,\alpha_2\rangle\right) dx_1^\infty\, dx_2^\infty, \tag{13.3}$$

with

$$G_M(\alpha_2) = G_M(\alpha_2,x,y,z,\gamma_1,\gamma_2;d,q) = \sum_{\substack{\beta_2\in\mathcal{O}_K/(dq)\\ \gamma_{\mathbf{c}}(x,y,z;q)\beta_2\equiv(\alpha_2+\frac{dq}{M}\gamma_1)\ (dq)}} \psi_M(\gamma_2\beta_2).$$

We'll omit the dependence of $\gamma_{\mathbf{c}}$ on q from now on for convenience. Similar to earlier, we have that G_M is $M(\gamma_{\mathbf{c}}(x,y,z),dq)$-periodic and

$$|G_M(\alpha_2,x,y,z,\gamma_1,\gamma_2;d,q)| \leqslant N((\gamma_{\mathbf{c}}(x,y,z),dq))\mathbb{1}_{(\gamma_{\mathbf{c}}(x,y,z),dq)|M\alpha_2}.$$

We have that the terms in (13.3) are supported on $d \ll D, dq \ll \sqrt{DX}$. We shall focus on $d \sim D_1$, $q \sim Q$ for some $D_1 \ll D$, $Q \ll \sqrt{DX}/D_1$. It can also be checked that we have

$$\int_{K_\infty^2} \phi_1(x_1^\infty)\phi_2(x_2^\infty)\left(\omega_2\left(\frac{dq}{M\sqrt{DX}}\right)-\omega_2\left(\frac{M\sqrt{X}\det(\mathbf{c},\boldsymbol{\ell}(x_1^\infty x_2^\infty))}{q\sqrt{D}}\right)\right)e\left(\frac{X_2}{dq}\langle x_2^\infty,\alpha_2\rangle\right)dx_1^\infty\,dx_2^\infty$$
$$\ll_A \frac{dq}{\sqrt{DX}}\left(1+\frac{|\alpha_2|_{\sup}}{dq/X_2}\right)^{-A}.$$

Therefore, we may suppose that $D_1Q \gg X_2X^{-\varepsilon}$ at a cost of $O(X^{-100})$.

The same treatment, discarding those x, y, z with $N((\gamma_{\mathbf{c}}(x,y,z),dq)) > B = (D_1Q/X_2)^4M^{-4}X^{-\varepsilon}$ and then reinserting the contribution of $\alpha_2 = 0$ at a cost of

$$\begin{aligned}\ll \frac{X^{4+o(1)}}{D^2}D_1\frac{1}{D_1^5}\left(\frac{D}{D_1}\right)^2\frac{D_1}{\sqrt{DX}}Q\frac{1}{Q^5}\left(\frac{D_1QM}{X_2}\right)^4 D_1Q\frac{D_1Q}{\sqrt{DX}}\frac{1}{B^{1/4}}+&\\ \frac{X^4}{D^2}D_1\frac{1}{D_1^5}\left(\frac{D}{D_1}\right)^2\frac{D_1}{\sqrt{DX}}Q\frac{1}{Q^5}D_1Q\frac{D_1Q}{\sqrt{DX}}B^{3/4}&\\ \ll \frac{X^{4+O(\varepsilon)}M^6D_1Q^2}{X_2^4DXB^{1/4}}+\frac{X^{3+O(\varepsilon)}}{DD_1^3Q^2}B^{3/4} \ll \frac{X^{3+O(\varepsilon)}M^6}{X_2^3}\frac{Q}{D} \ll \frac{X^{7/2+O(\varepsilon)}}{X_2^3\sqrt{D}}.&\end{aligned}$$

It remains to bound, for $d \sim D_1$, $q \sim Q$, $\mathbf{c}$ primitive, and x, y, z such that with $(\kappa) = M(\gamma_{\mathbf{c}}(x,y,z),dq)$ of norm $\leqslant BM^4$ (with $|\kappa|_{\sup} \ll |N(\kappa)|^{1/4}$), the sum

$$\sum_{\alpha_2} G_M(\alpha_2)\int_{K_\infty}\phi_2(x_2^\infty)\left(\omega_2\left(\frac{dq}{M\sqrt{DX}}\right)-\omega_2\left(\frac{M\sqrt{X}\det(\mathbf{c},\boldsymbol{\ell}(x_1^\infty x_2^\infty))}{q\sqrt{D}}\right)\right)e\left(\frac{X_2}{dq}\langle x_2^\infty,\alpha_2\rangle\right)dx_2^\infty.$$

By Poisson summation (recalling that G_M is κ-periodic), we have that this equals

$$\sum_{\alpha_2}\left(\sum_{\beta(\kappa)}G_M(\beta)\psi\left(-\frac{\alpha_2\beta}{\kappa}\right)\right)\phi_2\left(\frac{dq\alpha_2}{\kappa X_2}\right)\left(\omega_2\left(\frac{dq}{M\sqrt{DX}}\right)-\omega_2\left(\frac{dM\sqrt{X}}{X_2\sqrt{D}}\det\left(\mathbf{c},\boldsymbol{\ell}\left(\frac{x_1^\infty\alpha_2}{\kappa}\right)\right)\right)\right).$$

Now, note that for α nonzero, we have that

$$\left|\frac{dq\alpha_2}{\kappa X_2}\right|_{\sup} \gg \frac{dq}{X_2|\kappa|_{\sup}} \gg X^{\varepsilon/2},$$

from which the desired result follows from the compact support away from 0 of ϕ_2.

13.3. **Estimation of $\Sigma_2^{\{\}}$: the full-zero frequencies.** We begin with the treatment of the full-zero frequencies $\Sigma_2^{\{\}}$ defined in Proposition 4.1, which contributes a main term of size X^3/D, larger than the "true" main term of Theorem 2 which has size X^2.

This is to cancel with the same illusory main term that arises in the evaluation of $\Sigma_1^{\{\}}$ along with the true main term. We show the following.

Proposition 13.6. *We have*

$$\Sigma_2^{\{\}} = 2\tilde{\omega}_1(1)\frac{X^3}{D}U_1(-1)M^{-1}\mathcal{D}_1^*(1) + O\left(\frac{X^{2+o(1)}}{D^2} + \frac{X^{3+o(1)}}{D^4}\right).$$

Recall that $\mathcal{D}_1^*(s)$ is defined in (11.1) and $U_1(s)$ is defined in (12.16).

Our proof will amount to two maneuvers. The first is to show that $\Sigma_2^{\{\}}$ is a smooth sum over $\mathbb{Z}^2$ of a product of local densities. The second is to, with a good remainder term, estimate this smooth sum of a product of local densities by expanding the product into a convergent Dirichlet series whose coefficients are periodic and applying Poisson summation (as it will turn out, we do not have to truncate the Dirichlet series).

13.3.1. *Extracting a sum of products of local densities.* Recalling the definition of S_2 in (4.5) and I_2 in (4.6), we have that

$$\Sigma_2^{\{\}} = \frac{X^4}{D^2}\sum_{\substack{d\geqslant 1\\ \mathbf{c}\in\mathbb{Z}^2\\ (c_1,c_2)=1}}\frac{1}{d^5}\omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\frac{d}{\sqrt{DX}}\Sigma_2^{\{\}}(\mathbf{c},d), \tag{13.4}$$

where

$$\Sigma_2^{\{\}}(\mathbf{c},d) = \sum_{\substack{q\geqslant 1\\ M|q}}\frac{1}{q^5}S_2(0,0;\mathbf{c},d,q)I_2(0,0;\mathbf{c},d,q) = \sum_{\substack{q\geqslant 1\\ M|q}}\frac{1}{d^3q^8}\sum_{\substack{(\beta_1,\beta_2)\in V_{dq}\\ \det(\mathbf{c},\boldsymbol{\ell}(\beta_1\beta_2))\equiv 0\ (dq/M)\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0 (d)}} I_2(0,0;\mathbf{c},d,q),$$

Opening up the definition of I_2 and applying Mellin inversion yields that

$$\begin{aligned}&\Sigma_2^{\{\}}(\mathbf{c},d)\\ &= d^3\frac{1}{2\pi i}\int_{(3)}\left(\left(\frac{M\sqrt{DX}}{d}\right)^s\hat{\phi}_1(0)\hat{\phi}_2(0)\tilde{\omega}_2(s) - \left(M\sqrt{\frac{X}{D}}\right)^s\tilde{\omega}_2(-s)U_2(s;\mathbf{c})\right)\mathcal{D}_2(s+1;\mathbf{c},d)\,ds,\end{aligned} \tag{13.5}$$

where $\mathcal{D}_2$ was defined in §11.1, and

$$U_2(s;\mathbf{c}) = \int_{K_\infty^2}\phi_1(x_1^\infty)\phi_2(x_2^\infty)|\det(\mathbf{c},\boldsymbol{\ell}(x_1^\infty x_2^\infty))|^s\,dx_1^\infty\,dx_2^\infty.$$

Note

$$\begin{aligned}&\left(\left(\frac{M\sqrt{DX}}{d}\right)^s\hat{\phi}_1(0)\hat{\phi}_2(0)\tilde{\omega}_2(s) - \left(M\sqrt{\frac{X}{D}}\right)^s\tilde{\omega}_2(-s)U_2(s;\mathbf{c})\right)\Bigg|_{s=0}\\ &\qquad\qquad = \tilde{\omega}_2(0)(\hat{\phi}_1(0)\hat{\phi}_2(0) - U_2(0;\mathbf{c})) = 0,\end{aligned}$$

for $U_2(0;\mathbf{c}) = \hat{\phi}_1(0)\hat{\phi}_2(0)$ and that $\mathcal{D}_2(s+1;\mathbf{c},d)$ has only a simple pole at $s=0$ by Proposition 11.2. Therefore, the residue of the integrand of (13.5) at $s=0$ is 0.

Therefore, the only pole to the left of $\operatorname{Re} s > -3$ is at $s = -1$ by Lemma 11.5 (with $F(x) = \det(\mathbf{c}/|\mathbf{c}|, \boldsymbol{\ell}(x))$ for $x \in K_\infty^2$), at which we obtain a residue of

$$\begin{aligned}
&\operatorname*{res}_{s=-1}\left[\left(\left(\frac{M\sqrt{DX}}{d}\right)^s \hat{\phi}_1(0)\hat{\phi}_2(0)\tilde{\omega}_2(s) - \left(M\sqrt{\frac{X}{D}}\right)^s \tilde{\omega}_2(-s)U_2(s;\mathbf{c})\right)\mathcal{D}_2(s+1;\mathbf{c},d)\right] \\
&= -\zeta(0)\tilde{\omega}_2(1)\frac{2\sqrt{\pi}}{\Gamma(1/2)}\frac{1}{M}\sqrt{\frac{D}{X}}\mathcal{D}_2^*(0;\mathbf{c},d)\int_{K_\infty^2}\phi_1(x_1^\infty)\phi_2(x_2^\infty)\delta(\det(\mathbf{c},\boldsymbol{\ell}(x_1^\infty x_2^\infty)))\,dx_1^\infty\,dx_2^\infty.
\end{aligned}$$

By Proposition 11.2, this is equal to

$$\sqrt{\frac{D}{X}}\sigma_\infty(\mathbf{c})\prod_p \sigma_p(\mathbf{c},d),$$

recalling that

$$\begin{aligned}
\sigma_\infty(\mathbf{c}) &= \int_{K_\infty^2}\phi_1(x_1^\infty)\phi_2(x_2^\infty)\delta(\det(\mathbf{c},\boldsymbol{\ell}(x_1^\infty x_2^\infty)))\,dx_1^\infty\,dx_2^\infty, \\
\sigma_p(\mathbf{c},d) &= \int_{\mathcal{O}_{K,p}^2}\mathbb{1}_{\substack{(\beta_1,\beta_2)\in V_{p^{m_p}} \\ d|\boldsymbol{\ell}(\beta_1\beta_2)}}\delta(\det(\mathbf{c},\boldsymbol{\ell}(\beta_1\beta_2)))\,d\beta_1\,d\beta_2.
\end{aligned}$$

In particular, moving the contour in (13.5) to $\operatorname{Re} s = -3+\delta$ for $\delta \to 0$ sufficiently slowly in terms of X, picking up the poles of $\zeta(s+1)$ at $s = 0$ (which cancels with the difference which vanishes) and $U_2(s)$ at $s = -1$ (by Lemma 11.5), we obtain that

$$\Sigma_2^{\{\}}(\mathbf{c},d) = d^3\sqrt{\frac{D}{X}}\sigma_\infty(\mathbf{c})\prod_p \sigma_p(\mathbf{c},d) + O(d^6(DX)^{-\frac{3}{2}}(|\mathbf{c}|d)^{o(1)}).$$

Substituting this into (13.4), we obtain that

$$\begin{aligned}
\Sigma_2^{\{\}} &= \frac{X^3}{D^2}\sum_{\substack{d\geqslant 1 \\ \mathbf{c}\in\mathbb{Z}^2 \\ (c_1,c_2)=1}}\frac{1}{d}\omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\sigma_\infty(\mathbf{c})\prod_p \sigma_p(\mathbf{c},d) \\
&\qquad + O\left(\frac{X^{4+o(1)}}{D^2}\sum_{d\ll D}\frac{1}{d^5}\left(\frac{D}{d}\right)^2\frac{d}{\sqrt{DX}}d^6(DX)^{-\frac{3}{2}}\right) \\
&= \frac{X^3}{D^2}\sum_{\substack{d\geqslant 1 \\ \mathbf{c}\in\mathbb{Z}^2 \\ (c_1,c_2)=1}}\frac{1}{d}\omega_1\left(\frac{|\mathbf{c}|d}{D}\right)\sigma_\infty(\mathbf{c})\prod_p \sigma_p(\mathbf{c},d) + O\left(\frac{X^{2+o(1)}}{D}\right).
\end{aligned}$$

We shall now show that this is a sum over $\mathbf{v} = \mathbf{c}d$, so that we may sum smoothly over $\mathbb{Z}^2$ and drop the primitivity condition. We start by noting that by a change of variable,

$$\sigma_\infty(\mathbf{c}) = \frac{d}{D}\sigma_\infty\left(\frac{\mathbf{c}d}{D}\right).$$

Slightly more work is required at the non-archimedean places, which is contained in Lemma 11.3.

Applying (11.7) in Lemma 11.3, we obtain that

$$\Sigma_2^{\{\}} = \frac{X^3}{D^3} \sum_{\mathbf{v}\in\mathbb{Z}^2} \sigma_\infty\left(\frac{\mathbf{v}}{D}\right) \prod_p \sigma_p(\mathbf{v})\omega_1\left(\frac{|\mathbf{v}|}{D}\right) + O\left(\frac{X^{2+o(1)}}{D}\right). \tag{13.6}$$

13.3.2. *Summing the products of local factors.* The main result of this subsection is the following.

Proposition 13.7. *We have that*

$$\sum_{\mathbf{v}\in\mathbb{Z}^2} \sigma_\infty\left(\frac{\mathbf{v}}{D}\right) \prod_p \sigma_p(\mathbf{v})\omega_1\left(\frac{|\mathbf{v}|}{D}\right)$$
$$= 2D^2M^{-1}\tilde{\omega}_1(1)\mathcal{D}_1^*(1) \int_{K_\infty^2} \phi_1(x_1^\infty)\phi_2(x_2^\infty)|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|^{-1}\, dx_1^\infty\, dx_2^\infty + O\left(\frac{1}{D^{1-o(1)}}\right).$$

It's clear that from this and (13.6), Proposition 13.6 follows

Expanding out with (11.7) (with absolute convergence following from the bounds (11.9)), we have that

$$\sum_{\mathbf{v}\in\mathbb{Z}^2} \sigma_\infty\left(\frac{\mathbf{v}}{D}\right) \prod_p \sigma_p(\mathbf{v})\omega_1\left(\frac{|\mathbf{v}|}{D}\right) = \sum_{q\geqslant 1} \sum_{\mathbf{v}\in\mathbb{Z}^2} S(\mathbf{v};q)\sigma_\infty\left(\frac{\mathbf{v}}{D}\right)\omega_1\left(\frac{|\mathbf{v}|}{D}\right).$$

Applying Poisson summation, we obtain that this equals

$$D^2 \sum_{q\geqslant 1} \frac{1}{q^2} \sum_{\mathbf{w}\in\mathbb{Z}^2} \hat{S}(\mathbf{w};q) \int_{\mathbb{R}^2} \sigma_\infty(\mathbf{v})\omega_1(|\mathbf{v}|)e\left(-\frac{D}{q}\langle \mathbf{v},\mathbf{w}\rangle\right) d\mathbf{v}, \tag{13.7}$$

where

$$\hat{S}(\mathbf{w};q) := \sum_{\mathbf{v}(q)} S(\mathbf{v};q)e_q(-\langle \mathbf{v},\mathbf{w}\rangle).$$

The evaluation of $\hat{S}(\mathbf{w};q)$ required is contained in the following lemma.

Lemma 13.8. *We have that*

$$\hat{S}(\mathbf{0};q) = \frac{q}{M^2}N_1^*(qM).$$

Furthermore, for $\mathbf{w} \neq 0$*, we have that*

$$\hat{S}(\mathbf{w};q) \ll M^{O(1)} \frac{(w_1^4 + w_2^4, q)}{q^{2-o(1)}}. \tag{13.8}$$

We are now ready to prove Proposition 13.7

Proof of Proposition 13.7 assuming Lemma 13.8. The contribution of $\mathbf{w} = 0$ to (13.7) is, by Lemma 13.8 and Lemma 11.4, equal to

$$D^2 \int_{\mathbb{R}^2} \sigma_\infty(\mathbf{v})\omega_1(|\mathbf{v}|)\, d\mathbf{v} \sum_{q \geqslant 1} \frac{\hat{S}(\mathbf{0}; q)}{q^2} = 2\tilde{\omega}(1)U_1(-1)D^2M^{-1} \sum_{\substack{q\geqslant 1 \\ M|q}} \frac{N_1^*(q)}{q}$$
$$= 2\tilde{\omega}(1)U_1(-1)D^2M^{-1}\mathcal{D}_1^*(1),$$

matching the main term in Proposition 13.7. It remains to bound the contribution of the nonzero frequencies. First, note that by repeated integration by parts, we have that

$$\int_{\mathbb{R}^2} \sigma_\infty(\mathbf{v})\omega_1(|\mathbf{v}|)e\left(-\frac{D}{q}\langle v, w\rangle\right) \ll_A \left(1 + \frac{D|\mathbf{w}|}{q}\right)^{-A},$$

so by the bound (13.8), we are reduced to bounding by $M^{O(1)}/D^{1-o(1)}$ the quantity

$$M^{O(1)}D^2 \sum_q \frac{1}{q^4} \sum_{\mathbf{w}\neq 0} (w_1^4 + w_2^4, q)\left(1 + \frac{D|\mathbf{w}|}{q}\right)^{-A}$$
$$\ll M^{O(1)}D^2 \sum_{\substack{Q=2^k \\ k\geqslant 0}} \frac{1}{Q^4} \sum_{\mathbf{w}\neq 0} \left(1 + \frac{D|\mathbf{w}|}{Q}\right)^{-A} \sum_{q\sim Q} (w_1^4 + w_2^4, q)$$
$$\ll M^{O(1)}D^2 \sum_{\substack{Q=2^k \\ k\geqslant 0}} \frac{1}{Q^3} \sum_{|\mathbf{w}|\neq 0} |\mathbf{w}|^{o(1)} \left(1+\frac{D|\mathbf{w}|}{Q}\right)^{-A} \ll M^{O(1)} \sum_{\substack{Q=2^k \\ Q\gg D^{1-o(1)}}} \frac{1}{Q^{1-o(1)}} \ll \frac{1}{D^{1-o(1)}},$$

as desired. □

Proof of Lemma 13.8. By (11.10), we have

$$\hat{S}(\mathbf{w}; q) = \prod_{p^k || q} \hat{S}_p(\mathbf{w}; k), \tag{13.9}$$

where

$$\hat{S}_p(\mathbf{w}; k) = \sum_{\mathbf{v}(p^k)} S_p(\mathbf{v}; k)e_{p^k}(-\langle \mathbf{v}, \mathbf{w}\rangle).$$

Let p be a prime and $k \geqslant 0$. Recall the definition of $S_p(\mathbf{v}; k)$ from (11.8). For $k = 0$, we have

$$\hat{S}_p(\mathbf{0}; 0) = p^{-8m_p} = p^{-2m_p}N_1^*(p^{m_p}). \tag{13.10}$$

Technically $N_1^*(M)$ contains the condition $\boldsymbol{\ell}(\beta_1'\beta_2') \equiv 0\,(M)$, but the reader is reminded that this is assumed throughout the entire paper.

For $k \geqslant 1$, write $\eta = \max\{0, m_p - k\}$. We have that

$$\begin{aligned}
\hat{S}_p(\mathbf{0}; k) &= \frac{1}{p^{9k+8\eta}} \sum_{\mathbf{v}\,(p^k)} \sum_{\substack{\mathbf{a}=(a_1,a_2)\,(p^k)\\(a_1,a_2,p)=1}} \sum_{w\,(p^k)} \sum_{(\beta_1,\beta_2)\in V_{p^k}} e_{p^k}(\langle \boldsymbol{\ell}(\beta_1\beta_2) - \mathbf{v}w, \mathbf{a}\rangle) \\
&= \frac{1}{p^{9k+8\eta}} \sum_{j=0,1} (-1)^j \sum_{\mathbf{v}\,(p^k)} \sum_{\mathbf{a}\,(p^{k-j})} \sum_{w\,(p^k)} \sum_{(\beta_1,\beta_2)\in V_{p^k}} e_{p^{k-j}}(\langle \boldsymbol{\ell}(\beta_1\beta_2) - \mathbf{v}w, \mathbf{a}\rangle) \\
&= \sum_{j=0,1} \frac{(-1)^j}{p^{7k+2j+8\eta}} \sum_{\mathbf{v}\,(p^k)} \sum_{w\,(p^k)} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^k}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv \mathbf{v}w\,(p^{k-j})}} 1 \\
&= \sum_{j=0,1} \frac{(-1)^j}{p^{7k-j+8(\eta-j\mathbb{1}_{k>m_p})}} \sum_{\mathbf{v}\,(p^{k-j})} \sum_{w\,(p^{k-j})} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k-j}}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv \mathbf{v}w\,(p^{k-j})}} 1.
\end{aligned}$$

Here in the last step $\mathbb{1}_{k>m_p}$ appears because of scaling β_1, β_2 restricted to $\beta_i \equiv \beta_i'\,(p^{m_p})$. The reader is reminded of this fact throughout the proof. Pulling out the powers of p from w, we obtain

$$\begin{aligned}
\hat{S}_p(\mathbf{0}; k) &= \sum_{j=0,1} \sum_{r=0}^{k-j} \frac{(-1)^j}{p^{7k-j+8(\eta-j\mathbb{1}_{k>m_p})}} \sum_{\mathbf{v}\,(p^{k-j})} \sideset{}{^*}\sum_{w\,(p^{k-j-r})} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k-j}}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv \mathbf{v}wp^r\,(p^{k-j})}} 1 \\
&= \sum_{j=0,1} \sum_{r=0}^{k-j} \frac{(-1)^j \varphi(p^{k-j-r})}{p^{7k-j-2r+8(\eta-j\mathbb{1}_{k>m_p})}} \sum_{\mathbf{v}\,(p^{k-j-r})} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k-j}}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv \mathbf{v}p^r\,(p^{k-j})}} 1 \\
&= \sum_{j=0,1} \sum_{r=0}^{k-j} \frac{(-1)^j \varphi(p^{k-j-r})}{p^{7k-j-2r+8(\eta-j\mathbb{1}_{k>m_p})}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k-j}}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^r)}} 1.
\end{aligned}$$

Notice that the contribution of $r = k$ is given by

$$\frac{1}{p^{5k+8\eta}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^k}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^k)}} 1 = \frac{1}{p^{5k+8\eta+8\min\{m_p,k\}}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k+m_p}}\\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^k)}} 1 = p^{k-2m_p}\tilde{N}_1(p^{k+m_p}).$$

Similarly, the contribution of $r = k-1$ is given by

$$\begin{aligned}
&\frac{p-1}{p^{5k+2+8\eta}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^k} \\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^{k-1})}} 1 - \frac{1}{p^{5k+1+8(\eta-\mathbb{1}_{k>m_p})}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k-1}} \\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^{k-1})}} 1 \\
&= -\frac{1}{p^{5k+2+8(\eta-\mathbb{1}_{k>m_p})}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k-1}} \\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^{k-1})}} 1 \\
&= -\frac{1}{p^{5k+2+8(\eta-\mathbb{1}_{k>m_p}+\min\{m_p,k-1\})}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k+m_p-1}} \\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^{k-1})}} 1 = -p^{k-2m_p}\tilde{N}_1(p^{k+m_p-1}),
\end{aligned}$$

and for $r \leqslant k-2$, the contribution is

$$\frac{p-1}{p^{6k-r+1+8\eta}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^k} \\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^{r})}} 1 - \frac{p-1}{p^{6k-r+1+8(\eta-\mathbb{1}_{k>m_p})}} \sum_{\substack{(\beta_1,\beta_2)\in V_{p^{k-1}} \\ \boldsymbol{\ell}(\beta_1\beta_2)\equiv 0\,(p^{r})}} 1 = 0.$$

Hence we conclude for $k \geqslant 1$,

$$\hat{S}_p(\mathbf{0};k) = p^{k-2m_p} N_1^*(p^{k+m_p}). \tag{13.11}$$

Inserting (13.10) and (13.11) into (13.9), we obtain the first part of the lemma.

It remains to prove (13.8). By (13.9), it suffices to prove a local bound for $\hat{S}_p(\mathbf{w};k)$ when $\mathbf{w} \neq 0$. The case $k=0$ is trivial, so assume $k \geqslant 1$ and write $\eta_0 = \min\{m_p, k\}$.

Proceeding as in the proof of (11.9) (starting from (11.8), detecting $\beta_2 \equiv \beta_2'$ by additive characters, and summing over β_2), inserting the resulting expression for $S_p(\mathbf{v};k)$ into the definition of $\hat{S}_p(\mathbf{w};k)$ and summing over $\mathbf{v}(p^k)$ yields that

$$|\hat{S}_p(\mathbf{w};k)| \leqslant \frac{p^{O(\eta_0)}}{p^{3k}} \sum_{u\,(p^k)} \sum_{\substack{a_1,a_2\,(p^k) \\ (a_1,a_2,p)=1 \\ u(a_1,a_2)\equiv -\mathbf{w}\,(p^k)}} N((a_1+a_2\zeta, p^k)). \tag{13.12}$$

By Lemma 5.1, we have $N((a_1+a_2\zeta,p^k)) = (a_1^4+a_2^4,p^k)$. For each u with $p^r || u$, the congruence $u(a_1,a_2) \equiv -\mathbf{w}\,(p^k)$ has $\leqslant p^{2r}$ solutions $(a_1,a_2)\,(p^k)$. Furthermore, for any such solution, we have

$$w_1^4 + w_2^4 \equiv u^4(a_1^4+a_2^4)\,(p^k),$$

so $(a_1^4+a_2^4,p^k) \leqslant (w_1^4+w_2^4,p^k)/p^{4r}$. Since there are $\ll p^{k-r}$ choices of $u\,(p^k)$ with $p^r||u$, we conclude from (13.12) that

$$|\hat{S}_p(\mathbf{w};k)| \ll \frac{p^{O(\eta_0)}}{p^{3k}} \sum_{r=0}^{k} p^{k-r}\cdot p^{2r}\cdot \frac{(w_1^4+w_2^4,p^k)}{p^{4r}} \ll \frac{p^{O(\eta_0)}}{p^{2k}}(w_1^4+w_2^4,p^k).$$

Multiplying over primes gives (13.8). □

14. Proof of Theorem 1

In this section, we give a proof of Theorem 1.

It will actually follow relatively quickly from the more useful statement below, in whose proof we will carry out the passage from a sum over binary quartic form to a restricted sum over a quartic number field where we apply Theorem 2.

Theorem 14.1. *There exists $\delta > 0$ such that for $Q, X \geqslant 1$, $q_0 \geqslant 1$, $a_0 \in \mathbb{Z}/q_0\mathbb{Z}$, convex $\mathcal{R} \subset [0,1]^2$ with $\partial\mathcal{R}$ bounded and $0 \notin \mathcal{R}$, we have*

$$\Bigg| \sum_{\substack{q \leqslant Q \\ q \equiv a_0 (q_0)}} \Bigg(\sum_{\substack{m,n \in X\mathcal{R} \\ q | m^4+n^4}} 1 - \frac{\rho(q)}{q^2} X^2 \operatorname{vol}\mathcal{R} \Bigg) \Bigg| \ll q_0^{O(1)} X^{2-\delta} \Bigg(1 + \Big(\frac{Q}{X^2} \Big)^{O(1)} \Bigg),$$

where $\rho(q) = \#\big\{x_1, x_2 \in \mathbb{Z}/q\mathbb{Z} : x_1^4 + x_2^4 \equiv 0(q)\big\}$.

Proof of Theorem 1 assuming Theorem 14.1. We will be brief in this proof, for the bulk of the computations were already carried out by Daniel [Dan99].

We have that

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ 0 < n_1^4 + n_2^4 \leqslant N}} d(n_1^4 + n_2^4) = 4 \sum_{\substack{n_1, n_2 \geqslant 1 \\ n_1^4 + n_2^4 \leqslant N}} d(n_1^4 + n_2^4) + O(N^{1/4}).$$

Opening up the divisor function and summing over $d_1 d_2 = n_1^4 + n_2^4$, we obtain that the above equals

$$8 \sum_{d_1 \leqslant \sqrt{N}} \sum_{\substack{d_1^2 \leqslant n_1^4 + n_2^4 \leqslant N \\ d_1 | n_1^4 + n_2^4}} 1.$$

By Theorem 14.1, the desired result follows upon noting that (see, e.g., [Dan99, §7])

$$\sum_{d \leqslant D} \frac{\rho(d)}{d^2} = c_{-1} \log D + c_0 + O(D^{-1/4}).$$

$\square$

The remainder of this section is devoted to the proof of Theorem 14.1. We begin with some reductions. Existing level of distribution results for binary forms, namely [Dan99, Lemma 3.3], implies that

$$\begin{aligned} \Bigg| \sum_{\substack{q \leqslant Q \\ q \equiv a_0 (q_0)}} \Bigg(\sum_{\substack{m,n \in X\mathcal{R} \\ (m,n) \neq (0,0) \\ q | m^4+n^4}} 1 - \frac{\rho(q)}{q^2} X^2 \operatorname{vol}\mathcal{R} \Bigg) \Bigg| &\leqslant \sum_{q \leqslant Q} \Bigg| \sum_{\substack{m,n \in X\mathcal{R} \\ (m,n) \neq (0,0) \\ q | m^4+n^4}} 1 - \frac{\rho(q)}{q^2} X^2 \operatorname{vol}\mathcal{R} \Bigg| \\ &\ll (X\sqrt{Q} + Q) \log^{O(1)} X, \end{aligned}$$

which is an acceptable $O(X^{2-1/200+o(1)})$ when $Q \leqslant X^{2-1/100}$. Note too that we may suppose that $Q \leqslant X^{2+1/100}$, for the contribution of $q \geqslant X^{2+1/100}$ is acceptable from the divisor bound by virtue of the term $(Q/X^2)^{O(1)}$.

Now, write

$$R(Q,X;\Psi,W,a_0,q_0) := \sum_{q\equiv a_0(q_0)} \Psi\left(\frac{q}{Q}\right)\Big(S_q(X;W) - M_q(X;W)\Big),$$

where

$$S_q = S_q(X;W) := \sum_{\substack{(m,n)\neq(0,0)\\ q\mid m^4+n^4}} W\left(\frac{m}{X},\frac{n}{X}\right),$$

$$M_q = M_q(X;W) := \frac{\rho(q)}{q^2}X^2\int_{\mathbb{R}^2} W(x,y)\,dx\,dy.$$

We now apply a dyadic partition of unity to the sum over q. Fix some $\Psi \in C_c^\infty((3/2,5/2))$ satisfying

$$\sum_{k\in\mathbb{Z}} \Psi\left(\frac{x}{2^k}\right) = 1$$

for all $x \in \mathbb{R}_{>0}$. Then, we are reduced to showing that

$$R(Q,X;\Psi,\mathbb{1}_{\mathcal{R}},a_0,q_0) \ll q_0^{O(1)}X^{2-\delta}$$

for any scale $X^{2-1/100} \ll Q \ll X^{2+1/100}$.

Now, fix some nonzero $\phi \in C_c^\infty((0,1)^2)$, write $\phi_\Omega(x) = \Omega^2\phi(\Omega x)$, and let $W_{\mathcal{R}} = \mathbb{1}_{\mathcal{R}_1} * \phi_\Omega$, where

$$\mathcal{R}_1 = \{x \in \mathcal{R} : \operatorname{dist}(x,\partial\mathcal{R}) > 2/\Omega\}.$$

Then, by a divisor bound, we have that

$$R(Q,X;\Psi,\mathbb{1}_{\mathcal{R}},a_0,q_0) = R(Q,X;\Psi,W_{\mathcal{R}},a_0,q_0) + O\left(\frac{X^{2+o(1)}}{\Omega}\right),$$

so Theorem 14.1 will actually follow from the following statement.

Proposition 14.2. *There exist $\delta > 0$ with the following property. Let $X,\Omega,q_0 \geqslant 1$,*

$$X^{2-1/100} \ll Q \ll X^{2+1/100},$$

and let $W \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ satisfy

1. *For all $x \in \operatorname{supp}(W)$ we have $\Omega^{-1} \ll |x| \ll \Omega$.*
2. *For all $j_1,j_2 \geqslant 0$, we have*

$$\left\|\frac{\partial^{j_1+j_2}}{\partial x^{j_1}\partial y^{j_2}}W\right\|_\infty \ll_{j_1,j_2} \Omega^{j_1+j_2}.$$

Then, we have that

$$R(Q,X;\Psi,W,a_0,q_0) \ll (q_0\Omega)^{O(1)}X^{2-\delta}.$$

In the remainder of this section, we shall prove Proposition 14.2, beginning with preparation for the application of Theorem 2.

We start by analyzing the condition $q|m^4+n^4$. If $(m,n)=1$, we would have a correspondence $\left\{q|m^4+n^4\right\} \iff \{\mathfrak{q}|m+n\zeta\}$, but we must deal with the possibility of primes that aren't totally split dividing both m and n.

To this end, note that

$$S_q(X;W) = \sum_{d\geqslant 1} \sum_{\substack{(m',n')=1\\ q|d^4(m'^4+n'^4)}} W\left(\frac{dm'}{X},\frac{dn'}{X}\right) = \sum_{\substack{d\geqslant 1\\ r|d^4,q\\ (r,q/r)=1}} \sum_{\substack{(m',n')=1\\ q/r|m'^4+n'^4}} W\left(\frac{dm'}{X},\frac{dn'}{X}\right).$$

Letting $q/r=a_1$, we obtain that

$$\sum_{q\equiv a_0(q_0)} \Psi\left(\frac{q}{Q}\right) S_q(X;W) = \sum_{\substack{d\geqslant 1\\ r|d^4}} \sum_{\substack{(m',n')=1\\ a_1|m'^4+n'^4\\ (a_1,r)=1}} \Psi\left(\frac{a_1 r}{Q}\right) W\left(\frac{dm'}{X},\frac{dn'}{X}\right).$$

We can discard the contribution of $d>D$ (to be specified, but it will be a small power of X ultimately) at the cost of a remainder $O(X^{2+o(1)}/D)$, so

$$\sum_{q\equiv a_0(q_0)} \Psi\left(\frac{q}{Q}\right) S_q(X;W) = \sum_{\substack{d\leqslant D\\ r|d^4,q}} \sum_{\substack{(m',n')=1\\ a_1|m'^4+n'^4\\ (a_1,r)=1\\ a_1 r\equiv a_0(q_0)}} \Psi\left(\frac{a_1 r}{Q}\right) W\left(\frac{dm'}{X},\frac{dn'}{X}\right) + O\left(\frac{X^{2+o(1)}}{D}\right)$$

We are now set for our passage to K. We have that there exists a bijection

$$\left\{a_1|m'^4+n'^4\right\} \leftrightarrow \left\{\mathfrak{a}_1|(m'+n'\zeta)\right\}$$

given by $N\mathfrak{a}_1 \leftrightarrow \mathfrak{a}_1$. Therefore, we have that

$$\begin{aligned}
&\sum_{q\equiv a_0(q_0)} \Psi\left(\frac{q}{Q}\right) S_q \\
&= \sum_{\substack{d\leqslant D\\ r|d^4}} \sum_{\substack{\alpha'\in\mathcal{O}_K\\ \boldsymbol{\ell}(\alpha')=0\\ (\langle\alpha',\zeta^3\rangle,\langle\alpha',\zeta^2\rangle)=1}} \sum_{\substack{\mathfrak{a}_1|(\alpha')\\ (N\mathfrak{a}_1,r)=1\\ rN\mathfrak{a}_1\equiv a_0(q_0)}} \Psi\left(\frac{rN\mathfrak{a}_1}{Q}\right) W\left(\frac{d\langle\alpha',\zeta^3\rangle}{X}, \frac{d\langle\alpha',\zeta^2\rangle}{X}\right) + O\left(\frac{X^{2+o(1)}}{D}\right) \\
&= \sum_{\substack{d\leqslant D\\ r|d^4}} \sum_{h\geqslant 1} \mu(h) \sum_{\substack{\alpha\in\mathcal{O}_K\\ \boldsymbol{\ell}(\alpha)=0\\ h|\alpha}} \sum_{\substack{\mathfrak{a}_1|(\alpha)\\ (N\mathfrak{a}_1,r)=1\\ rN\mathfrak{a}_1\equiv a_0(q_0)}} \Psi\left(\frac{rN\mathfrak{a}_1}{Q}\right) W\left(\frac{d\langle\alpha,\zeta^3\rangle}{X}, \frac{d\langle\alpha,\zeta^2\rangle}{X}\right) + O\left(\frac{X^{2+o(1)}}{D}\right) \\
&= \sum_{\substack{dh\leqslant D\\ r|d^4}} \mu(h) \sum_{\substack{\alpha\in\mathcal{O}_K\\ \boldsymbol{\ell}(\alpha)=0\\ h|\alpha}} \sum_{\substack{\mathfrak{a}_1|(\alpha)\\ (N\mathfrak{a}_1,r)=1\\ rN\mathfrak{a}_1\equiv a_0(q_0)}} \Psi\left(\frac{rN\mathfrak{a}_1}{Q}\right) W\left(\frac{d\langle\alpha,\zeta^3\rangle}{X}, \frac{d\langle\alpha,\zeta^2\rangle}{X}\right) + O\left(\frac{X^{2+o(1)}}{D}\right).
\end{aligned}$$

At this point, we wish to move from our sum over ideals to a sum over $\mathcal{O}_K$. To do this, take $u : K_\infty^\times \to \mathbb{R}$ given by

$$u(\alpha) = \log|\alpha|_{v_1} - \log|\alpha|_{v_2}$$

for the two complex places v_1, v_2 of K so that $u(\varepsilon_0) \neq 0$ and pick any $\xi \in C_c^\infty(\mathbb{R})$ such that

$$\#\mu_K \sum_{k\in\mathbb{Z}} \xi(t - u(\varepsilon_0^k)) = \#\mu_K \sum_{k\in\mathbb{Z}} \xi(t - ku(\varepsilon_0)) = 1$$

for all $t \in \mathbb{R}$. Taking $\Xi(x) = \xi(u(x))$, we obtain that

$$\sum_{\varepsilon\in\mathcal{O}_K^\times} \Xi(\varepsilon x) = 1$$

for all $x \in K_\infty^\times$.

Applying this, we obtain that

$$\sum_{q\equiv a_0(q_0)} \Psi\left(\frac{q}{Q}\right) S_q = \sum_{\substack{dh\leqslant D\\ r|d^4}} \mu(h) \Sigma_{d,h,r}(X;W) + O\left(\frac{X^{2+o(1)}}{D}\right),$$

where

$$\Sigma_{d,h,r}(X;W) = \sum_{\substack{\alpha_1,\alpha_2\in\mathcal{O}_K\\ \boldsymbol{\ell}(\alpha_1\alpha_2)=0}} \Phi^\infty\left(\frac{\alpha_1}{X_1}, \frac{\alpha_2}{X_2}\right) \Phi^f_{h,r}(\alpha_1,\alpha_2),$$

where $\Phi^\infty \in C_c^\infty(K_\infty^2)$, $\Phi^f_{h,r} : (\mathcal{O}_K/M\mathcal{O}_K)^2 \to \mathbb{C}$ are given by

$$\Phi^\infty(x_1^\infty, x_2^\infty) = \Xi(x_1^\infty)\Psi(N(x_1^\infty))W(\langle x_1^\infty x_2^\infty, \zeta^3\rangle, \langle x_1^\infty x_2^\infty, \zeta^2\rangle)\phi(|\boldsymbol{\ell}(x_1^\infty x_2^\infty)|^2),$$

$$\Phi^f_{h,r}(\beta_1,\beta_2) = \mathbb{1}_{\substack{N(\beta_1)r\equiv a_0(q_0),\\ (\beta_1,r)=1\\ h\mid\beta_1\beta_2}}$$

where we write $X_1 = X_1(r) = (Q/r)^{1/4}$, $X_2(d,r) = (X/d)/X_1$, $M = M(h,r) = hq_0r$, and $\phi \in C_c^\infty(\mathbb{R})$ is any fixed bump function with $\phi(0) = 1$.

It can be checked that the conditions of Theorem 2 hold with $M, \Omega, X_1, X_2, \boldsymbol{\ell}$ in the theorem as we have them here, X taken to be X/d. Therefore, we obtain that

$$\Sigma_{d,h,r} = \frac{1}{d^2}X^2\sigma_\infty \prod_p \sigma_p(h,r) + O\Big((\Omega D)^{O(1)}X^{2-\eta_{\mathrm{sc}}+o(1)}\Big),$$

where

$$\sigma_\infty = \int_{K_\infty^2} \Phi^\infty(x_1^\infty, x_2^\infty)\delta(\boldsymbol{\ell}(x_1^\infty x_2^\infty))\, dx_1^\infty\, dx_2^\infty,$$
$$\sigma_p(h,r) = \int_{\mathcal{O}_{K,p}^2} \Phi^f_{h,r}(x_1^p, x_2^p)\delta(\boldsymbol{\ell}(x_1^p x_2^p))\, dx_1^p\, dx_2^p.$$

We obtain that

$$\sum_q \Psi\left(\frac{q}{Q}\right)S_q(X;W) = X^2\sigma_\infty \sum_{\substack{dh\leqslant D\\ r\mid d^4}} \frac{\mu(h)}{d^2}\prod_p \sigma_p(h,r) + O\left(X^{2+o(1)}\left(\frac{1}{D} + \frac{(\Omega D)^{O(1)}}{X^{\eta_{\mathrm{sc}}}}\right)\right).$$

It remains to evaluate the sum of products of local factors, and show that it lines up with

$$\sum_q \Psi\left(\frac{q}{Q}\right)M_q(X;W).$$

We capture all our local computation in the following lemma, from which Theorem 14.1 follows upon taking D a sufficiently small power of X.

Lemma 14.3. *We have that*

$$\sigma_\infty \sum_{dh\leqslant D}\frac{\mu(h)}{d^2}\sum_{r\mid d^4}\prod_p \sigma_p(h,r) = \left(\int_{\mathbb{R}^2} W(x,y)\,dx\,dy\right)\sum_{q\equiv a_0(q_0)}\Psi\left(\frac{q}{Q}\right)\frac{\rho(q)}{q^2} + O\left(\frac{1}{D^{1-o(1)}}\right).$$

Proof. First, note that with $Y = Q^{20}$, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^2} W(x,y)\,dx\,dy\right)&\sum_{q\equiv a_0(q_0)}\Psi\left(\frac{q}{Q}\right)\frac{\rho(q)}{q^2}\\ &= \frac{1}{Y^2}\sum_q \Psi\left(\frac{q}{Q}\right)\sum_{\substack{m,n\\ m^4+n^4\equiv 0(q)}} W\left(\frac{m}{Y},\frac{n}{Y}\right) + O\left(\frac{1}{Y^2}\right)\\ &= \frac{1}{Y^2}\sum_q \Psi\left(\frac{q}{Q}\right)S_q(Y;W) + O\left(\frac{1}{Y^2}\right).\end{aligned}$$

We may now perform the exact same maneuvers leading up to 14 (with the same choice of D) and apply Theorem 2 to yield the desired result. □

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