Exercise Series 2 Solution

Exercise 1

a) We need to compute recursively backwards the cost-to-go for every possible state at every stage n

$J_N\left(x_N\right) = \left(x_N\right)^2$

$J_n\left(x_n\right) = min\left(x_n^2 + u_n^2\right) + J_{n+1}\left(x_n + u_n\right)$

Stage $N = 3$

Possible states x_3=0,1,2,3,4,5.

For each terminal state $J_3\left(x_3\right) = \left(x_3\right)^2$.

State x_3	Cost-to-go
$x_3 = 0$	$J_3 = 0$
$x_3 = 1$	$J_3 = 1$
$x_3 = 2$	$J_3 = 4$
$x_3 = 3$	$J_3 = 9$
$x_3 = 4$	$J_3 = 16$
$x_3 = 5$	$J_3 = 25$

Stage $N = 2$

Possible states x_2=0,1,2,3,4,5.

The cost-to-got is $J_2 = \min_{u_2}\left\{\left(x_2^2 + u_2^2\right) + J_3(x_3 = x_2 + u_2)\right\}$

State x_2	Possible control	Cost-to-go
$x_2 = 0$	u_2=0,1,2,3,4,5	$\min J_2 = 0$ when $u_2 = 0$
$x_2 = 1$	u_2=-1,0,1,2,3,4	$\min J_2 = 2$ when $u_2 = -1$ or $u_2 = 0$
$x_2 = 2$	u_2=-2,-1,0,1,2,3	$\min J_2 = 6$ when $u_2 = -1$
$x_2 = 3$	u_2=-3,-2,-1,0,1,2	$\min J_2 = 14$ when $u_2 = -1$ or $u_2 = -2$
$x_2 = 4$	u_2=-4,-3,-2,-1,0,1	$\min J_2 = 24$ when $u_2 = -2$
$x_2 = 5$	u_2=-5,-4,-3,-2,-1,0	$\min J_2 = 38$ when $u_2 = -2$ or $u_2 = -3$

Stage $N = 1$

Possible states x_1=0,1,2,3,4,5.

The cost-to-got is $J_1 = \min_{u_1}\left\{\left(x_1^2 + u_1^2\right) + J_2(x_2 = x_1 + u_1)\right\}$

State x_1	Possible control	Cost-to-go
$x_1 = 0$	u_1=0,1,2,3,4,5	$\min J_1 = 0$ when $u_1 = 0$
$x_1 = 1$	u_1=-1,0,1,2,3,4	$\min J_1 = 2$ when $u_1 = -1$
$x_1 = 2$	u_1=-2,-1,0,1,2,3	$\min J_1 = 7$ when $u_1 = -1$
$x_1 = 3$	u_1=-3,-2,-1,0,1,2	$\min J_1 = 15$ when $u_1 = -2$
$x_1 = 4$	u_1=-4,-3,-2,-1,0,1	$\min J_1 = 26$ when $u_1 = -2$
$x_1 = 5$	u_1=-5,-4,-3,-2,-1,0	$\min J_1 = 40$ when $u_1 = -3$

Stage $N = 0$

Possible states x_0=5

The cost-to-got is $J_0 = \min_{u_0}\left\{\left(x_0^2 + u_0^2\right) + J_1(x_1 = x_0 + u_0)\right\}$

State x_0	Possible control	Cost-to-go
$x_0 = 5$	u_2=-5,-4,-3,-2,-1,0	$\min J_0 = 41$ when $u_0 = -3$

There are therefore 2 possible optimal control sequences:

1. $(u_0 = -3,\ u_1 = -1,\ u_2 = -1)$ with states $x_0 = 5$, $x_1 = 2$, $x_2 = 1$ and $x_3 = 0$ or
2. $(u_0 = -3,\ u_1 = -1,\ u_2 = 0)$ with states $x_0 = 5$, $x_1 = 2$, $x_2 = 1$ and $x_3 = 1$.

In both cases the optimal cost is $J^* = J_0 = 41$

b)
In this case we have as a constraint $x_3 = 5$.

Stage $N = 3$
Possible states x_3=5.
For each terminal state $J_3(x_3) = (x_3)^2$.

State x_3	Cost-to-go
$x_3 = 5$	$J_3 = 25$

Stage $N = 2$
Possible states x_2=0,1,2,3,4,5.
In this case, the control needs to bring x_2 to $x_3 = 5$, no choice.

State x_2	Possible control	Cost-to-go
$x_2 = 0$	u_2=5	$\min J_2 = 50$ when $u_2 = 5$
$x_2 = 1$	u_2=4	$\min J_2 = 42$ when $u_2 = 4$
$x_2 = 2$	u_2=3	$\min J_2 = 38$ when $u_2 = 3$
$x_2 = 3$	u_2=2	$\min J_2 = 38$ when $u_2 = 2$
$x_2 = 4$	u_2=1	$\min J_2 = 42$ when $u_2 = 1$
$x_2 = 5$	u_2=0	$\min J_2 = 50$ when $u_2 = 0$

Stage $N = 1$
Possible states x_1=0,1,2,3,4,5.

State x_1	Possible control	Cost-to-go
$x_1 = 0$	u_1=0,1,2,3,4,5	$\min J_1 = 42$ when $u_1 = 2$
$x_1 = 1$	u_1=-1,0,1,2,3,4	$\min J_1 = 40$ when $u_1 = 1$
$x_1 = 2$	u_1=-2,-1,0,1,2,3	$\min J_1 = 42$ when $u_1 = 0$
$x_1 = 3$	u_1=-3,-2,-1,0,1,2	$\min J_1 = 47$ when $u_1 = 0$
$x_1 = 4$	u_1=-4,-3,-2,-1,0,1	$\min J_1 = 55$ when $u_1 = -1$
$x_1 = 5$	u_1=-5,-4,-3,-2,-1,0	$\min J_1 = 67$ when $u_1 = -2$

Stage $N = 0$
Possible states x_0=5.

State $x_0 = 5$	Possible control	Cost-to-go
$x_0 = 5$	u_0=-5,-4,-3,-2,-1,0	$\min J_0 = 76$ when $u_0 = -3$ or $u_0 = -2$

There are therefore 2 possible optimal control sequences:
1. $(u_0 = -3,\ u_1 = 0,\ u_2 = 3)$ with states $x_0 = 5$, $x_1 = 2$, $x_2 = 2$ and $x_3 = 5$ or
2. $(u_0 = -2,\ u_1 = 0,\ u_2 = 2)$ with states $x_0 = 5$, $x_1 = 3$, $x_2 = 3$ and $x_3 = 5$.
In both cases the optimal cost is $J^* = J_0 = 76$

c)

In this case we need to compute the expectation and we won't find an optimal control sequence (since the system is not deterministic) but rather an optimal policy for each stage.

Stage $N = 3$

Possible states x_3=0,1,2,3,4,5.

For each terminal state $J_3(x_3) = (x_3)^2$.

State x_3	Cost-to-go
$x_3 = 0$	$J_3 = 0$
$x_3 = 1$	$J_3 = 1$
$x_3 = 2$	$J_3 = 4$
$x_3 = 3$	$J_3 = 9$
$x_3 = 4$	$J_3 = 16$
$x_3 = 5$	$J_3 = 25$

Stage $N = 2$

Possible states x_2=0,1,2,3,4,5.

$J_2 = minE\left\{\left(x_2^2 + u_2^2\right) + J_3\right\}$:

State x_2	Possible control	Cost-to-go
$x_2 = 0$	u_2=0,1,2,3,4,5	min $J_2 = 0$ when $u_2 = 0$
$x_2 = 1$	u_2=-1,0,1,2,3,4	min $J_2 = 2$ when $u_2 = 0$ or $u_2 = -1$
$x_2 = 2$	u_2=-2,-1,0,1,2,3	min $J_2 = 7$ when $u_2 = -2$
$x_2 = 3$	u_2=-3,-2,-1,0,1,2	min $J_2 = 15$ when $u_2 = -2$ or $u_2 = -1$
$x_2 = 4$	u_2=-4,-3,-2,-1,0,1	min $J_2 = 25$ when $u_2 = -2$
$x_2 = 5$	u_2=-5,-4,-3,-2,-1,0	min $J_2 = 39$ when $u_2 = -3$

Stage $N = 1$

Possible states x_1=0,1,2,3,4,5.

$J_1 = minE\left\{\left(x_1^2 + u_1^2\right) + J_2\left(x_1 + u_1\right)\right\}$:

State x_1	Possible control	Cost-to-go
$x_1 = 0$	u_2=0,1,2,3,4,5	min $J_1 = 0$ when $u_1 = 0$
$x_1 = 1$	u_2=-1,0,1,2,3,4	min $J_1 = 2$ when $u_1 = -1$
$x_1 = 2$	u_2=-2,-1,0,1,2,3	min $J_1 = 8$ when $u_1 = -2$
$x_1 = 3$	u_2=-3,-2,-1,0,1,2	min $J_1 = 16.5$ when $u_1 = -2$
$x_1 = 4$	u_2=-4,-3,-2,-1,0,1	min $J_1 = 28.5$ when $u_1 = -3$ or $u_1 = -2$
$x_1 = 5$	u_2=-5,-4,-3,-2,-1,0	min $J_1 = 42.5$ when $u_1 = -3$

Stage $N = 0$

Possible states x_0=5.

State x_0	Possible control	Cost-to-go
$x_0 = 5$	u_0=-5,-4,-3,-2,-1,0	min $J_0 = 43.25$ when $u_0 = -3$

Therefore the optimal policy is as follows $0 = -3$, if $x_1 = 1$ then $u_1 = -1$ else if $x_1 = 3$ then $u_1 = -2$. Then if $x_2 = 0$ then $u_2 = 0$ else if $x_2 = 2$ then $u_2 = -2$. The optimal cost is $J^* = J_0 = 43.25$

Exercise 2

When all the λ of the system satisfy that $\mid \lambda \mid< 1$, then the system is stable.

And when the matrix M=[B AB A^2 B] is full rank, then the system is controllable.

Q1)

For system (a), $\lambda = 1.61803399, -0.61903399, 1.5,$

For system (b), $\lambda = 1.61803399, -0.61903399, 1.5,$

For system (c), $\lambda = 0.5, -0.5, 0.5,$
For system (d), $\lambda = 0.70380158, 0.24007567, -0.44387725,$
Therefore, a and b are unstable, c and d are stable.
Q2)
rank(M) of a = 2 < 3, rank(M) of b,c,d = 3. Therefore, a is not controllable and b,c,d are controllable.
Q3) We can find control law for those controllable systems. Therefore, we can find control laws for b,c and d system.

Exercise 3
See the attached code file *Linear Quadratic Regulators solution. ipynb*

Exercise 4
Q1)
The linearization equation is:

$$\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{v}_{n+1} \\ \tilde{\theta}_{n+1} \\ \tilde{\omega}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & \dfrac{\Delta t m_p g}{m_c} & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & \dfrac{\Delta t\,(m_p + m_c)\,g}{m_c l} & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_n \\ \tilde{v}_n \\ \tilde{\theta}_n \\ \tilde{\omega}_n \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\Delta t}{m_c} \\ 0 \\ \dfrac{\Delta t}{m_c l} \end{bmatrix} \tilde{f}_n$$

Q2)
A possible cost function is (written in the new variables from the linearized system):

$$\min \tilde{x}_N^T Q_N \tilde{x}_N + \sum_{n=0}^{N-1} \tilde{x}_n^T Q_n \tilde{x}_n + \tilde{f}_n^T R_n \tilde{f}_n$$

$$s.t. \quad \tilde{x}_{n+1} = A\tilde{x}_n + B\tilde{f}_n$$

We can do backward Riccati equation from N to 0 to minimize the cost and get a sequence of optimal gains K_n. The optimal controller is $\tilde{f}_n = K_n \tilde{x}_n$. Rewritten in the original coordinates we find

$$f_n = K_n x_n - K_n \bar{x} \tag{1}$$

Q3)
For the initial conditions x=0.2, v=0.1, $\theta = \pi - 0.2$ and $\omega = 0$, we can see the states will be converge and the system stabilizes.

We can change the cost function, for example the response speed is faster when the control cost R becomes smaller. When R=0.001, the system achieves stability faster than R = 0.1.

For initial conditions x=0.2, v=0.1, $\theta = 0.2$ and $\omega = 0$, the controller does not work and the system is unstable. Indeed, we start the system far from the linearization point, thus the linearized dynamics is not approximately correct anymore and the controller we found does not work.

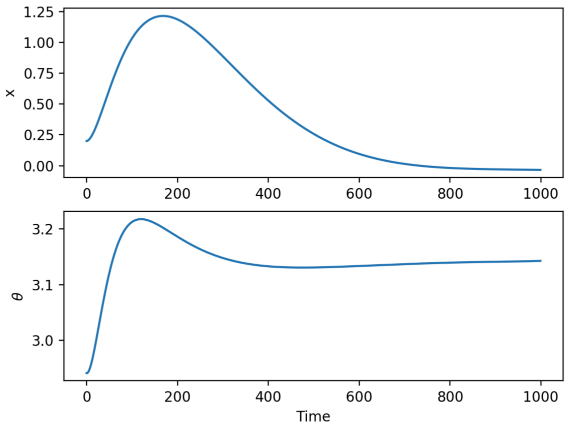

Figure 1: *The states of the system.*

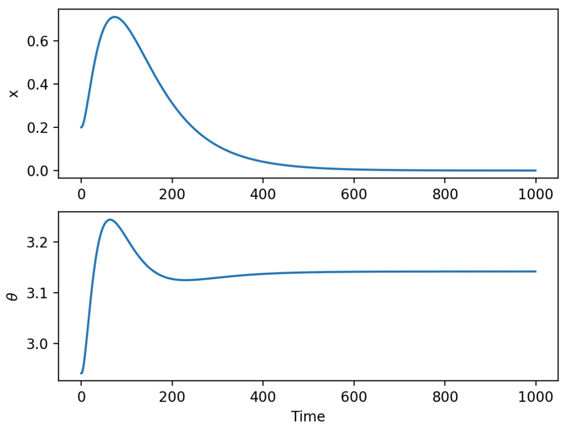

Figure 2: *The states of the system.*

Exercise 5
Q1)
The optimal control problem:

$$\min \sum_{n=0}^{N-1} (x_n - \overline{x})^T Q_n (x_n - \overline{x}) + u_n^T R_n u_n$$

$$s.t. \qquad x_0 = x_{initial} = (0,0,0,0)$$

$$x_{n+1} = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} x_n + \begin{bmatrix} 0 & 0 \\ \Delta t & 0 \\ 0 & 0 \\ 0 & \Delta t \end{bmatrix} u_n$$

$$-5 \le u_n \le 5$$

where $\bar{x} = (5, 0, 2, 0)$ is the goal state.