Multiview Geometry, 3D reconstruction: Theories & scan3D Engine design.

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lemma. For $\forall$ camera model. $\exists$ invertible H s.t. for $\forall P \in \mathbb{R}_{3\times 4}$. $PH = (I|0)$

In case of finite camera. $P = (M|T)$. M is full-rank. Define $H = \left(\begin{array}{c|c} M^{-1} & -M^{-1}T \\ \hline \vec{0} & 1 \end{array}\right)$.

then H is invertible & $PH = (I|0)$.

In case of infinite camera. $P = (M|T)$. $|M| = 0$, $\text{rank}(P) = 3 \Rightarrow \text{Null}(P) = 1$

$\Rightarrow \exists \vec{l} \neq \vec{0}$ s.t. $P \cdot \vec{l} = \vec{0}$. suppose $\vec{l} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$, then $\tilde{P} \cdot \vec{l} \neq 0$ for $\tilde{P} \triangleq \left(\begin{array}{c} P \\ \hline c_1 c_2 c_3 c_4 \end{array}\right)$

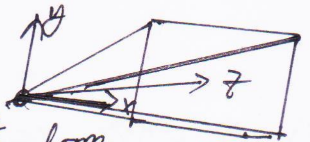
for $\forall \vec{y}$, $\tilde{P} \cdot \vec{y} = \vec{0} \Rightarrow P\vec{y} = \vec{0} \Rightarrow \vec{y} = \alpha \cdot \vec{l}$ for $\alpha \in \mathbb{R} \Rightarrow \alpha = 0$ as $\tilde{P} \cdot (\alpha \vec{l}) \neq 0$.

$\Rightarrow \text{rank}(\tilde{P}) = 4 \Rightarrow \exists \tilde{P}^{-1}$ s.t. $P\tilde{P}^{-1} = (I|0)$ // (remark: H is not necessarily unique
so for (P, P') $\overset{H}{\text{we have}}$ $\{(I|0), \tilde{P}\}$ with different $\tilde{P}$)

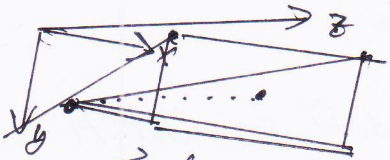
Definition: Camera Projection model.

$\vec{x} \sim K(R|T)\vec{X}$ $\vec{X} \in \mathbb{R}_4$. $\vec{x} \in \mathbb{R}_3$ is homogeneous form. $K = \begin{pmatrix} f_x & 0 & u \\ 0 & -f_y & v \\ 0 & 0 & 1 \end{pmatrix}$, $|R| = 1$

general form: $\vec{x} \sim P\vec{X}$ where $P = (A|a_4)$ & $\text{rank}(P) = 3$.



$\vec{X}$-form



$\vec{x}$-form

Def: Fundamental matrix

$\exists F \in \mathbb{R}_{3\times 3}$ s.t. $\vec{x}'^T F \vec{x} = 0$ for $\vec{x}', \vec{x}$ is matched stereo pts. on two camera image.

$\Leftrightarrow$ F is a fundamental matrix.

lemma. Existence of F (special case).

$P_1 = (I|0)$, $P_2 = (M|m) \Rightarrow F = [m]_\times M$

$\begin{cases} \vec{x}_1 \sim (I|0)\vec{X} \\ \vec{x}_2 \sim (M|m)\vec{X} \end{cases} \Rightarrow \vec{x}_2 \sim (M|m)\begin{pmatrix} a\vec{x}_1 \\ 1 \end{pmatrix}$, $a \in \mathbb{R}$ (assumption of $\vec{X}$ being finite pt).

$\Rightarrow \vec{x}_2^T [m]_\times M \vec{x}_1 = 0 \Rightarrow F = [m]_\times M$ (infinite case is same).

(general case).

$P_1 = (A|a_4)$, $P_2 = (B|b_4)$.

by previous lemma, $\exists H^{-1}$ s.t. $P_1 H = (I|0)$, $P_2 H = (M|m)$ ~~for (M|m) 6~~

then for any $\vec{X}$, $\vec{X}' \triangleq H^{-1}\vec{X} \Rightarrow F = [m]_\times M$ by above special case.

lemma: F ~~uniqueness up to~~ value remains by projectivity ($F_{P,P'} = F_{PH,\, P'H}$ given invertible H.) ~~(not only up to)~~

Pf. transform $\begin{cases} P \text{ to } PH \text{ by } H \\ P' \text{ to } P'H \end{cases}$ not affect value of F as F's elements freedom are all completed solely by the matched pts. while matched pts are not affected by H. (Also, by this, F is unique given by same pair of (P, P'))

~~But for any (P,P') & $(\tilde{P},\tilde{P}')$ pairs giving same F, $\exists ?\, H$ s.t. $\begin{cases} PH = \tilde{P} \\ P'H = \tilde{P}' \end{cases}$? $\{(P,P')H = (\tilde{P},\tilde{P}')\}$? (only up to projectivity?). most importantly, given a F, what is the relation of its all corresponding (P,P') pairs?~~ $\{F\} \xrightarrow{\text{relation}} \{(P,P')\}$

Most importantly, given F, what are the possible ~~pairs~~ projection pairs corresponding? precisely, given (P,P'), $(\tilde{P},\tilde{P}')$ gives out same F, what are relation between them?

note that ~~given~~ $\exists\, H$ & $\tilde{H}$ s.t. $(P,P')H = ((I|0), P'')$, $(\tilde{P},\tilde{P}')\tilde{H} = ((I|0), \tilde{P}'')$

now suppose $P'' = (M|m)$, $\tilde{P}'' = (\tilde{M}|\tilde{m})$ $\Rightarrow$ $F \sim [m]_\times M \sim [\tilde{m}]_\times \tilde{M}$,

note that F must be rank 2. $\Rightarrow$ $m \sim \tilde{m}$ $\Leftrightarrow$ $[m]_\times (M - k\tilde{M}) = 0_{3\times3}$ for some $k \in \mathbb{R}$

$\Leftrightarrow$ $M - k\tilde{M} = m \cdot v^T$ for $v \in \mathbb{R}^3$

$\Rightarrow$ $\tilde{P}'' = (k^{-1}(M - m\cdot v^T) \,|\, \lambda m)$, $k, \lambda \in \mathbb{R}$. Also, if $H \triangleq \left(\begin{array}{c|c} k^{-1} I & 0 \\ \hline -k^{-1} v^T & \lambda \end{array}\right) \Rightarrow \begin{cases} \tilde{P}'' = P''H \\ (I|0)H \sim (I|0) \end{cases}$

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lemma. given $F = [m]_\times M$, (P,P') pair $= ((I|0), (k^{-1}(M - m\cdot v^T) \,|\, \lambda m)) \cdot H$, for some $H^{-1} \in \mathbb{R}^{4\times4}$

While we wanna know how to construct back (P,P') in ~~consistent~~ correspondent form explicitly

i.e. $(P,P') \longleftrightarrow ((I|0), (f_1(F) \,|\, f_2(F)))$ for some function f from F

Problem arise like what λ, v, k should we use. to obtain valid (P,P') pair. Are those parameters really free of constraints?

~~trivially, v cannot be chosen freely, as. $M - k\tilde{M} = m \cdot n^T = 0$ if $n \in m^\perp$~~ ($m \neq 0$ trivially for example).

~~to safe~~ note that P' must be in rank 3. this has restriction on M or F

Observe that $\vec{e}'^T F = 0$, $\vec{e}'$ is epipole in 2^{nd} image $\Rightarrow (P, P') = (I|0, (A|k\vec{e}'))$
for $A \in \mathbb{R}_{3\times3}$, $\vec{e}' \in \mathbb{R}_3$

$\Rightarrow$ note $\vec{X}^T P'^T F P \vec{X} = 0 \ \forall \vec{X} \Leftrightarrow P'^T F P$ is skew-sym

$$\Leftrightarrow (A|k\vec{e}')^T F (I|0) = \begin{pmatrix} A^T F & \vec{0} \\ \vec{0}^T & 0 \end{pmatrix} \text{ is skew-sym.}$$

choose $A^T = F^T S^T$, S is skew symmetric. to maintain the property.

$A = SF$ (possible case).

Also, $(SF | k\vec{e}')$ has rank 3 necessarily. while SF is rank-2 now

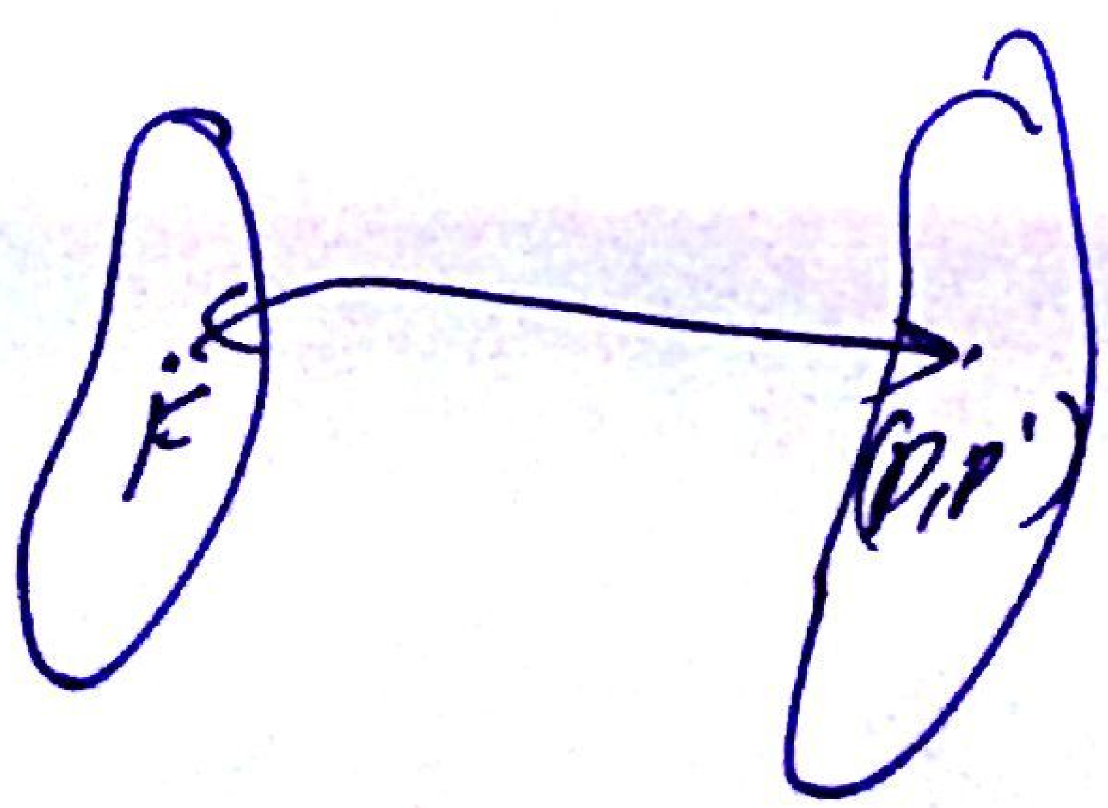
SF's columns must lin. ind to $\vec{e}'$ $\Rightarrow$ SF columns must be on same plane

and perp to $\vec{e}'$, a good choice is $S = [e]_\times$.

$\Rightarrow$ $P' = ([\vec{e}']_\times F + \vec{e}' \vec{v}^T | \lambda \vec{e}')$

take it as equivalent class representation element.

$(F)_{s \in \mathbb{R}} \longleftrightarrow (P, P'), \vec{H} \in \mathbb{R}_{3\times3}$ $\quad P = (I|0)$, $P' = ([\vec{e}']_\times F + \vec{e}' \vec{v}^T | \lambda \vec{e}'), \lambda \in \mathbb{R}$.



Calibration & extraction of pose from F

$P \triangleq K_1(I|0)$, $P' \triangleq K_2(R|T)$. note $\tilde{P} = P\begin{pmatrix}K_1^{-1} & \vec{0} \\ \vec{0}^T & 1\end{pmatrix} = (I|0)$. $\tilde{P}' = K_2(R|T)\begin{pmatrix}K_1^{-1} & \vec{0} \\ \vec{0}^T & 1\end{pmatrix} = (K_2RK_1^{-1}|K_2T)$

by previous lemma, we have $[K_2T]_\times K_2RK_1^{-1} = F$,

note that $[KT]_\times = |K| \cdot K^{-T}[T]_\times K^{-1} \Rightarrow F \sim K_2^{-T}[T]_\times RK_1^{-1} \sim [\vec{e}']_\times K_2RK_1^{-1}$

if R is known. & F is known. $\because$ $FK_1 \sim [\vec{e}']_\times K_2R$.

suppose furthermore K has constraint assumption on its elements. and having more # of images, we can estimate out K's value.

~~After know K, we can find T by finding $[\vec{e}']$ $\vec{e}'$ of F~~

In case of knowing K_1 & K_2, we can compute $\tilde{F} = K_2^T F K_1$, $\tilde{F} \sim [T]_\times R$

$\tilde{F}$ is rank 2, with svd, we can find out $\vec{T} \sim \vec{u}_3$ of $\tilde{F}$ (left last singular vector of $\tilde{F}$)

note that $[\vec{u}_3]_\times(\vec{u}_1\ \vec{u}_2\ \vec{u}_3) = (\vec{u}_2\ -\vec{u}_1\ \vec{u}_3) = (\vec{u}_1\ \vec{u}_2\ \vec{u}_3)\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$.

$\Rightarrow [T]_\times \sim U\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}U^T$, but how to know R?

$\tilde{F} \sim UDV^T \sim U\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}U^TUMV^T$ for some $M \in \mathbb{R}_{3\times3}$

$RR^T = R^TR = I \Leftrightarrow (UMV^T)(UMV^T)^T = UMM^TU^T = VM^TMV^T = I$

$\Leftrightarrow |M| = 1$, $MM^T = M^TM = I$.

note that $D \sim \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$. $\because$ $\tilde{F}^T\tilde{F} \sim R^T[T]_\times^T[T]_\times R \sim R^TU\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}U^TR$

$\Rightarrow \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} \sim \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix} \triangleq M \Leftrightarrow M = \begin{pmatrix}0&1&0\\-1&0&0\\g&h&i\end{pmatrix}$

$MM^T = I \Leftrightarrow g, h = 0 \Leftrightarrow M = \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix} \Leftrightarrow R \sim U\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}V^T$
$|R| = 1$

Thm: $\tilde{F} = U\,\mathrm{diag}(1,1,0)V^T$

$\Leftrightarrow [T]_\times \sim \pm U\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}U^T$, $R = U\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}V^T$ $|R| = 1$

$= U\begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}V^T$

∘ Constraints on E

$$\begin{cases} |E| = 0 \\ 2EE^TE - \mathrm{trace}(EE^T)E = 0 \end{cases} \quad \text{or} \quad \begin{cases} e_1 \cdot e_2 \times e_3 = 0 \\ \|e_1 \times e_2\|^2 + \|e_2 \times e_3\|^2 + \|e_3 \times e_1\|^2 \\ = \frac{1}{4}(\|e_1\|^2 + \|e_2\|^2 + \|e_3\|^2)^2 \end{cases}$$

for $E = \begin{pmatrix} - e_1 - \\ - e_2 - \\ - e_3 - \end{pmatrix}$

Property of conic

conic $\triangleq \{(x,y) \mid ax^2 + bxy + cy^2 + dx + ey + f = 0\}$

in homo. conic $\triangleq \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; a\left(\frac{x}{z}\right)^2 + b\frac{xy}{z^2} + c\left(\frac{y}{z}\right)^2 + d\left(\frac{x}{z}\right) + e\left(\frac{y}{z}\right) + f = 0 \right\}$

in matrix form, $\vec{x}^T C \vec{x} = 0$, $\vec{x} \triangleq \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $C = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$

5 pts determine C as 5 pts form full rank matrix

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lemma l_x is tangent to $C \Leftrightarrow l = Cx$

$\forall x \in C$, $l^T x = x^T C x = 0 \Rightarrow x \in l$

suppose $\exists y, x \in C$ s.t. $l^T x = l^T y = 0$ i.e. $\begin{cases} x^T C x = y^T C y = 0 \\ x^T C y = 0 \end{cases}$

$\Rightarrow x^T C (x - y) = 0$

$\begin{cases} x^T C = 0 \Rightarrow \forall x,\; x^T C x = 0 \Rightarrow C \text{ is skew\_sym} \\ x^T C \neq 0 \Rightarrow x = y \Rightarrow l = Cx \text{ is the tangent.} \end{cases}$

Dual conic

$x^T C = 0 \Rightarrow x^T C x = 0, \forall x \Rightarrow C = 0_{3\times 3} \Rightarrow C$ is a pt-conic

Now, we know $\forall x \in C$, Cx is tangent at x on C

$l = Cx \Leftrightarrow C^{-1} l = x$ (assume C is full rank)

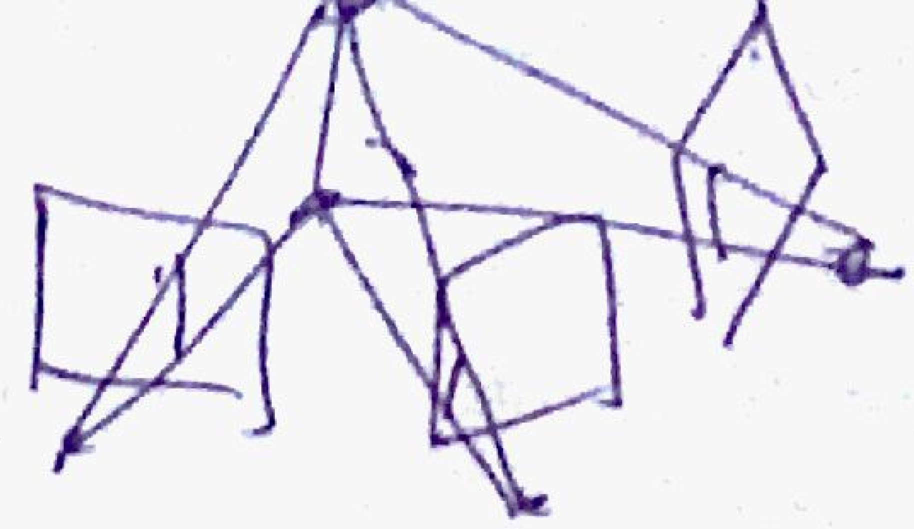
$\Leftrightarrow x^T C x = 0 \Leftrightarrow (C^{-1} l)^T C (C^{-1} l) = 0$

$\Leftrightarrow l^T C^{-T} l = 0$

Then l is ~~atom~~ tangent to conic $\Leftrightarrow l^T C^{-T} l = 0$

In case C is not full rank $\Rightarrow$ def: C is degenerated.

Three view geometry

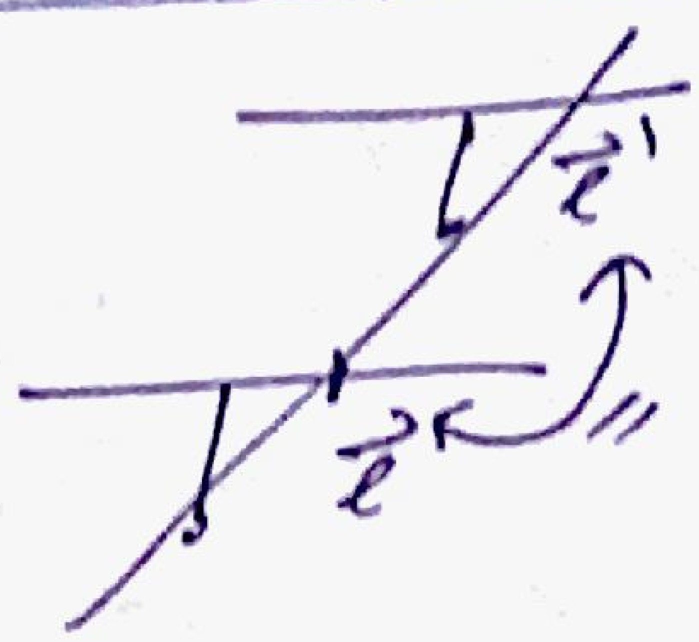


Deviation of Trifocal Tensor from incidence of lines:

$P = (I|0)$, $P' = (A|a_4)$, $P'' = (B|b_4)$

Fundamental matrix in special motion

Pure translation with same camera param.



$\vec{e} = \vec{e}'$

$$\left.\begin{aligned} P &= K(I|0) \\ P' &= K(I|t) \end{aligned}\right\} \Rightarrow F = [\vec{e}']_\times K K^{-1} = [\vec{e}']_\times$$

In case $t \parallel$ x-axis

$\vec{e}'$ in finity $= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$F = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]_\times = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{cases} z_1 \vec{x}_1 = K(I|0)X \\ z_2 \vec{x}_2 = K(I|t)X \end{cases} \Rightarrow z_2 \vec{x}_2 = K(I|t)\begin{pmatrix} z_1 K^{-1} \vec{x}_1 \\ 1 \end{pmatrix}$$

$$= z_1 \vec{x} + Kt$$

$$\Rightarrow \cancel{z_2 = z} \quad \vec{x}_2 = \vec{x}_1 + \frac{Kt}{z_1}$$

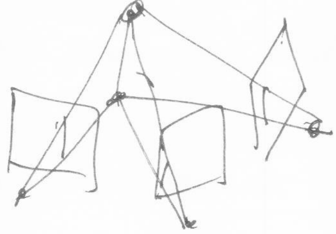
Insight to general motion, consider $\forall$ motion with one pose.

Imagine pure rotation (i.e. $\vec{x}' = H\vec{x}$, $\cancel{H = K_2 R K_1^{-1}}$

$H = K_2 R K_1^{-1}$

$$\Rightarrow \vec{x}_2' = K_2 R K_1^{-1} \vec{x}_1 + \frac{K_2 t}{z}$$

Three view geometry



Derivation of Trifocal tensor from incidence of lines:

$P=(I|0)$, $P'=(A|a_4)$, $P''=(B|b_4)$

Planes back-projected from lines:

$$\begin{cases} \pi_1^T X = 0 \\ \cancel{x} \sim PX \Rightarrow 0 \sim l^T x \sim l^T P X \end{cases} \Rightarrow \pi_1 \sim P^T l$$

$$\begin{cases} \pi \sim P^T l = \begin{pmatrix} l \\ 0 \end{pmatrix} \\ \pi' \sim P'^T l' = \begin{pmatrix} A^T l' \\ a_4^T l' \end{pmatrix} \\ \pi'' \sim P''^T l'' = \begin{pmatrix} B^T l'' \\ b_4^T l'' \end{pmatrix} \end{cases}$$

$\rightarrow$ while 3D space of line only has two basis element.

for $M^T = \begin{pmatrix} \pi_1^T \\ \pi'^T \\ \pi''^T \end{pmatrix}$, $M^T X = 0$, $\dim(Null(M^T)) = 2$

$\Leftrightarrow$ rank $(M) = 2$

By rank - nullity thm

$\hookrightarrow M = \begin{pmatrix} l & A^T l' & B^T l'' \\ 0 & a_4^T l' & b_4^T l'' \end{pmatrix}$ has rank 2 $\Leftrightarrow$ $l \sim (b_4^T l'')(A^T l') - (B^T l'')(a_4^T l')$

$\sim \cancel{l''^T b_4 A^T) l}$

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \sim l''^T \begin{pmatrix} b_4 a_1^T - b_1 a_4^T \\ b_4 a_2^T - b_2 a_4^T \\ b_4 a_3^T - b_3 a_4^T \end{pmatrix} l'$$

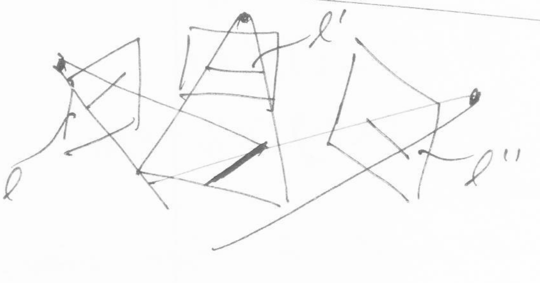
$$\sim l'^T \begin{pmatrix} a_1 b_4^T - a_4 b_1^T \\ a_2 b_4^T - a_4 b_2^T \\ a_3 b_4^T - a_4 b_3^T \end{pmatrix} l''$$

$\hookrightarrow$ Denote $a_i b_4^T - a_4 b_i^T \triangleq T_i$

$$\Rightarrow l \sim l'^T \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} l''$$

this is the special case of derivation of T when $P = (I|0)$

line transfer by homography



$\exists H \in \mathbb{R}^{3\times3}$ s.t. $x'' \sim Hx$

$\Rightarrow 0 = l''^T x'' \sim l''^T H x$

$\Rightarrow l^T \cancel{\sim l''^T H} \sim l''^T H$

$\Rightarrow l \cancel{\sim H^T l''} \sim H^T l''$

while $l \sim l'^T \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} l'' \Leftrightarrow H^T \sim l'^T \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}^T \Leftrightarrow (h_1 h_2 h_3) \sim [T_1^T, T_2^T, T_3^T] l'$ $\boxed{}$

Denote H ~~from~~ between image-i and j be H_{ij}

$\Rightarrow H_{13} \sim [T_1^T, T_2^T, T_3^T] l'$

Similarly, we have $l^T \sim l'^T H_{12} \Leftrightarrow l \sim H_{12}^T l' \Leftrightarrow H_{12}^T \sim l''^T \begin{bmatrix} T_1^T \\ T_2^T \\ T_3^T \end{bmatrix} \Leftrightarrow H_{12} = (h_1, h_2, h_3) \sim [T_1, T_2, T_3] l''$ $\boxed{}$

remark: up to now, we only have line transfer from 1 to 3 view

1. 1 to 2 view

important geometry relations ~~by trifoc~~ in terms of trifocal tensor.

- point-line-line

$x^T l = 0$ & $l_i \sim l'^T T_i l''$

$\Rightarrow \sum_{i=1}^{3} x_i \cdot l_i = 0 \Rightarrow l'^T \left(\sum_{i=1} x_i T_i\right) l'' = 0$ □

- point-line-point

$x'' \sim H_{13} x$ & $H_{13} \sim [T_1^T, T_2^T, T_3^T] l'$

$\Rightarrow x'' \sim [T_1^T, T_2^T, T_3^T] l' x$

$\Rightarrow 0 = [x'']_\times \left(\sum_i x_i T_i^T\right) l'$ or $l'^T \left(\sum_i x_i T_i^T\right) [x'']_\times = 0^T$ □

- point-point-point

note that $l'^T \left(\sum_i x_i T_i^T\right) [x'']_\times = 0$ & $l' = [x']_\times y'$ for x', y on the line.

but $y'^T [x']_\times \left(\sum_i x_i T_i^T\right) [x'']_\times = 0$ for $\forall y'$ forms line intersect x'

$\Rightarrow [x']_\times \left(\sum_i x_i T_i^T\right) [x'']_\times = 0_{3\times 3}$ □

remark: above relations not necessarily guarantee incidence of geometric obj. i.e. pt lines. e.g. in plp case.

summarize

- line-line-line : $l' [T_1, T_2, T_3] l'' [l]_\times = 0^T$
- point-line-line : $l'^T \left(\sum x_i T_i\right) l'' = 0$
- point-line-point : $l'^T \left(\sum_i x_i T_i\right) [x'']_\times = 0^T$
- point-point-line : $[x']_\times \left(\sum x_i T_i\right) l'' = 0$
- point-point-point : $[x']_\times \left(\sum_i x_i T_i\right) [x'']_\times = 0_{3\times 3}$

note that for a point in first view, it has corresponding epipolar lines in other views. what is the relationship of tensor and the epip. lines?

note that we have point-line-line correspondings., what if the second image line is the epip. line of first image pt. we will have any ~~image~~ line in 3rd image fulfil the constraint condition that is $l'^T \left(\sum x_i T_i\right) = 0^T$ if l' is the epipolar line in 2nd image, similarly for l'' is epip lines in 3rd image. i.e. $\left(\sum x_i T_i\right) l'' = 0$.

$\Rightarrow$ left null vector of $\left(\sum x_i T_i\right) \sim l'$

right null vector of $\left(\sum x_i T_i\right) \sim l''$ □

After know epip. line, we ~~can~~ also wanna know relation of trifocal tensor & epipoles.

note that different points in 1^{st} image generates different epip. line in 2^{nd} and 3^{rd} image. That's why we can use different produced epip. lines to intersect out epipoles. So what point should we choose?

One ~~convenient~~ convenient choice is $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$ to generate epip. lines.

Then we have another thm.

Thm

e' is given by left null vectors intersection of T_i, $i=1,2,3$

e'' is given by right null vectors intersection of T_i, $i=1,2,3$

Properties of T_i

for $a_i \neq \alpha a_4 \cdot s \in \mathbb{R}$

- T_i is rank-2 as $T_i \triangleq a_i b_4^T - a_4 b_i^T$ = linear comb. of $\{a_i, a_4\}$
- right null vector of $T_i = b_4 \times b_i$, left null-vector of $T_i = a_i \times a_4$
- $\sum x_i T_i$ has rank-2 due to T_i's special structure.

Relation of fundamental matrix & trifocal tensor.

note that ~~$x_2^T F x_1 = 0 \Leftrightarrow x$~~

~~$x''^T F_{21} x' = 0$ & $l'^T x' = 0$ (l' is the epip. line of x' on 2^{nd} image)~~

~~$\Rightarrow x''^T F_{21} = l'^T \Leftrightarrow l' = F$~~

$x'^T F_{21} x = 0$ & $x'^T l' = 0$ (l' is the epipline of x on 2^{nd} image).

$\Rightarrow F_{21} x \simeq l' \simeq [e']_\times \cdot x' \simeq [e']_\times \cdot [T_1, T_2, T_3] l'' \cdot x$ (from ppl corresponding)

$\Rightarrow F_{21} \sim [e']_\times [T_1, T_2, T_3] l''$.

note that l'' shd be carefully chosen to not lie in null space of $[T_i]$

i.e. l'' shd be $\perp$ to null space of $\forall T_i$

~~$\because T_i = b$~~

$\because T_i (b_4 \times b_i) = 0$ & T_i is rank-2.

choose $l'' = b_4 \sim e''$ $\Rightarrow$ $F_{21} \sim [e']_\times [T_1, T_2, T_3] e''$

similarly, $\Rightarrow$ $F_{31} \sim [e'']_\times [T_1^T, T_2^T, T_3^T] e'$ ∎

After knowing the fundamental matrix, it's time to recover projection matrix P, P', P'', if $P = (I|0)$. P' can be trivially chosen as $([T_1, T_2, T_3]e''|e')$ as stated in previous pages' thm. then (P, P') will have fundamental matrix $[e']_\times [T_1, T_2, T_3] e''$. While you may think that $P'' = ([T_1^T, T_2^T, T_3^T]e'' | e'')$ but it is not correct to derive like that as. (P, P', P'') may ~~be~~ not given back same $[T_i]_i$ tensors, they are not necessity consistent.

~~From previous thm. $P'' =$~~

Suppose $P = (I|0)$, $P' = ([T_1, T_2, T_3]e''/e')$, $P = (b_1 b_2 b_3 / b_4)$

~~now. $T_i = a_4 b_i^T - a_i b_4^T = e' b_i^T - T_i e'' b_4^T$~~

$\because T_i = a_i b_4^T - a_4 b_i^T \Rightarrow T_i = T_i e'' b_4^T - a_4 b_i^T \Rightarrow T_i(e'' b_4^T - I) = a_4 b_i^T$

$\Rightarrow T_i(e'' b_4^T - I) = e' b_i^T$

note that $b_4 \sim e''$ & e' can be chosen as $e'^T e' = 1$

$\Rightarrow e'^T T_i (e'' b_4^T - I) = b_i^T \Rightarrow b_i = (b_4 e''^T - I) T_i^T e'$

choose $b_4 = e''$ then $b_i = (e'' e''^T - I) T_i^T e'$

$$\Rightarrow \begin{cases} P'' = ((e'' e''^T - I)[T_1^T, T_2^T, T_3^T] e' | e'') & \text{(one solution)} \\ P' = ([T_1, T_2, T_3] e'' | e') & , e'^T e' = 1 \\ P = (I|0) \end{cases}$$

Special notes.

suppose $P = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $P' = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = B$

what is the epipoles in two image?

$$e^i = \det \begin{pmatrix} a_i \\ B \end{pmatrix} \quad e'^j = \det \begin{pmatrix} A \\ b_j \end{pmatrix}$$

you may be surpised why it has such beautiful formula

note that $e_i = a_i C_1'$ &, $BC_1' = 0$, C_1' &. $\det \begin{pmatrix} a_i \\ B \end{pmatrix}$ has some relations

~~note that~~ $\cancel{\begin{pmatrix} B_{14} B_{13} \; B_{12} B_{11} \\ B \end{pmatrix} \; \mathbb{I} = 0 \qquad B_{ij} = |(\sim b_{ij})|}$

$$\cancel{\Rightarrow \det \begin{pmatrix} B_{14} \; -B_{13} \; B_{12} \; -B_{11} \\ B \end{pmatrix} = B_{14}B_{11} - B_{13}B_{12} + B_{12}B_{13} - B_{11}B_{14} = 0}$$

$\triangleq M$

$\Rightarrow$ ~~By rank 3 of B, we have rank-3 of M.~~

note.

note. $\begin{pmatrix} \vec{m} \\ B \end{pmatrix} C_1' = 0$ for any $\vec{m} \in C_1'^{\perp}$

& $\begin{pmatrix} \vec{m} \\ B \end{pmatrix}$ has rank-3 $\Rightarrow \left| \begin{pmatrix} \vec{m} \\ B \end{pmatrix} \right| = 0 \quad \vec{m} \cdot \begin{pmatrix} |B_{11}| \\ -|B_{12}| \\ |B_{13}| \\ -|B_{14}| \end{pmatrix} = 0$. for $\forall \vec{m}$

$$\Rightarrow \quad C_1 \sim \begin{pmatrix} -|B_{11}| \\ |B_{12}| \\ -|B_{13}| \\ |B_{14}| \end{pmatrix}$$

$$\Rightarrow \quad e \sim A \begin{pmatrix} -|B_{11}| \\ |B_{12}| \\ -|B_{13}| \\ |B_{14}| \end{pmatrix} \Rightarrow e_i \sim \det \begin{pmatrix} a_i \\ B \end{pmatrix}$$

$\boxtimes$

Intr calibration problem

it divides into 2 categories — object pattern calibration &.
non-obj pattern calibration.

object pattern calibration

homography — chessboard calibration — planar assumption

$\forall \vec{\Xi} \in \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, & $\begin{pmatrix} \vec{x} \\ 1 \end{pmatrix} \sim K(R|T)\vec{\Xi}$ · $R \in SO(3)$, $T \in \mathbb{R}^3$
· K is intrin matrix.

(assumpt reference frame determined by its plane normal)

assumpt we defined reference world points $\vec{\Xi}$ on chessboard &

their corresponding matched pts on sensor screen $\vec{x}$, Also note

that $\begin{pmatrix} \vec{x} \\ 1 \end{pmatrix} \sim K(R|T)\vec{\Xi} \to \begin{pmatrix} \vec{x} \\ 1 \end{pmatrix} \sim \underbrace{K(\vec{R_1}\,\vec{R_2}\,\vec{T})}_{\triangleq H} \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ 1 \end{pmatrix} \to 0 = \left[\begin{pmatrix} \vec{x} \\ 1 \end{pmatrix}\right]_\times H \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ 1 \end{pmatrix}$

H

H has dof = 8, which requires 8 eq^n to solve the linear system, while each matched pts provide one eq^n. (at least 8 world pt to find H), while K is embeded into H & it requires several images to find K, below explains why:

$H = sK(\vec{R}_1\ \vec{R}_2\ \vec{T})$ for some $s \in \mathbb{R}$.

$$\begin{cases} \frac{1}{s} K^{-1} \vec{H}_i = \vec{R}_i \quad i = 1, 2 \\ \frac{1}{s} K^{-1} \vec{H}_3 = \vec{T} \end{cases} \quad \& \quad \begin{array}{l} \vec{R}_1 \perp \vec{R}_2 \\ |\vec{R}_1| = |\vec{R}_2| \end{array} \Rightarrow \begin{array}{l} \vec{H}_j^T K^{-T} K^{-1} \vec{H}_i = 0, \ i \neq j \\ \vec{H}_1^T K^{-T} K^{-1} \vec{H}_1 = \vec{H}_2^T K^{-T} K^{-1} \vec{H}_2 \end{array}$$

$$K^{-T} K^{-1} = \begin{pmatrix} \frac{1}{f_x^2} & 0 & \frac{-u}{f_x^2} \\ 0 & \frac{1}{f_y^2} & \frac{-v}{f_y^2} \\ \frac{-u}{f_x^2} & \frac{-v}{f_y^2} & \frac{u^2}{f_x^2} + \frac{v^2}{f_y^2} + 1 \end{pmatrix} \triangleq \begin{pmatrix} a & 0 & ac \\ 0 & b & bd \\ * & b* & e \end{pmatrix}$$

$$\Rightarrow \vec{h}^T \cdot \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = 0 \rightarrow$$ whole system with dof = 4 (one internal constraint minus 1 dof)

while for each image, there are only two eq^n. to means dof.

$2K \geq 4$, at minimum, it requires 2 images. for numerical accuracy,

8 images are required.

Of course, there are other calibration methods, but the general

~~prompt~~ principle is same.

<u>auto calibration without strong assumptions</u>

$$\begin{cases} F = K^{-T} \mathcal{E} K^{-1} \\ \mathcal{E} = [T]_\times R \end{cases} \Rightarrow K^T F K = \mathcal{E} \Rightarrow K^T F K K^T F^T K = \mathcal{E}\mathcal{E}^T = [T]_\times [T]_\times^T$$

(eliminated R)

$$\Rightarrow \underbrace{K^T F K K^T F^T K = [T]_\times [T]_\times^T}$$

rank 2 & symmetric.

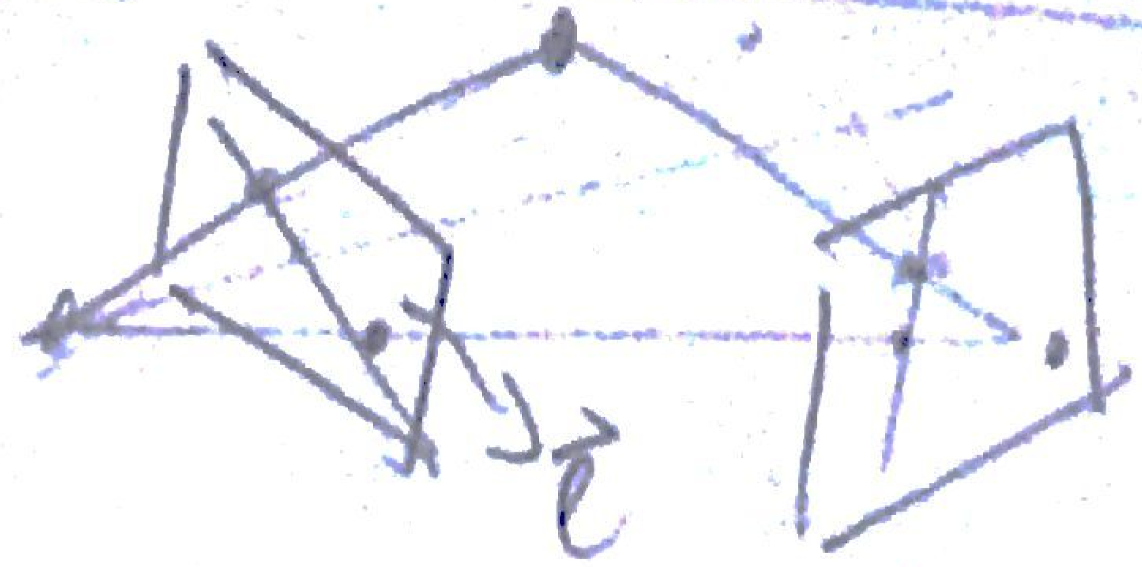
two nonzero singular values are same,

→ This derive certain high degree Eq^n.

With enough images (different F)

the unknowns in K can be solved

Remark: Kruppa's Eq.



$$K^{-T}[T]_\times R K^{-1}\vec{e}' = \vec{e}'P = 0$$

$$\vec{e}' = \lambda K \vec{T}$$

$$\vec{T} = \lambda' K^{-1}\vec{e}'$$

$$[\vec{T}]_\times = \lambda' \det(K^{-1}) K^T [\vec{e}']_\times K$$

$$\Rightarrow F K K^T F^T = \gamma [\vec{e}']_\times K ([\vec{e}']_\times)^T$$

two Eq^{n} for each system to remove dof

at least 3 images to find K.
(diff F)

[1,1]

Existence of symmetric fundamental matrix in spinning assumption

$F^T = F \Leftrightarrow [\vec{T}]_\times = -R^T[\vec{T}]_\times R$, where $R = (\theta, \vec{r})$ in Rodrigues notation

$R = I + \sin\theta[\vec{r}]_\times + (1-\cos\theta)[\vec{r}]_\times^2$

& $\vec{r} \perp \vec{T}$

spinning assumption

$$\Leftrightarrow [\vec{T}]_\times = -\left(I + \sin\theta[\vec{r}]_\times + (1-\cos\theta)[\vec{r}]_\times^2\right)[\vec{T}]_\times R$$

$$= -[\vec{T}]_\times R + \sin\theta|\vec{r}|^2\vec{T} + (1-\cos\theta)|\vec{r}|^2\vec{T}$$

$$\boxed{\begin{array}{l} [\vec{r}]_\times[\vec{T}]_\times \vec{y} = -|\vec{r}|^2\vec{T}, \forall \vec{y} \\ \vec{y} = a\vec{r} + b\vec{T} + c\,\vec{r}\times\vec{T} \\ L.H.S = -|\vec{r}|^2\vec{T} = R.H.S \end{array}}$$

$\forall \vec{y}$, $[\vec{T}]_\times \vec{y} = -[\vec{T}]_\times R\vec{y} + m\vec{T}$, $m = (1+\sin\theta-\cos\theta)|\vec{r}|^2$

$[\vec{T}]_\times(I+R)\vec{y} = m\vec{T}$

$[\vec{T}]_\times \tilde{y} = m\vec{T}$, $\tilde{y} \triangleq (I+R)\vec{y}$

$\Rightarrow \vec{T} \perp \vec{T} \Rightarrow$ contradiction $\Rightarrow$ symmetric F NOT EXIST

3rd parametrization

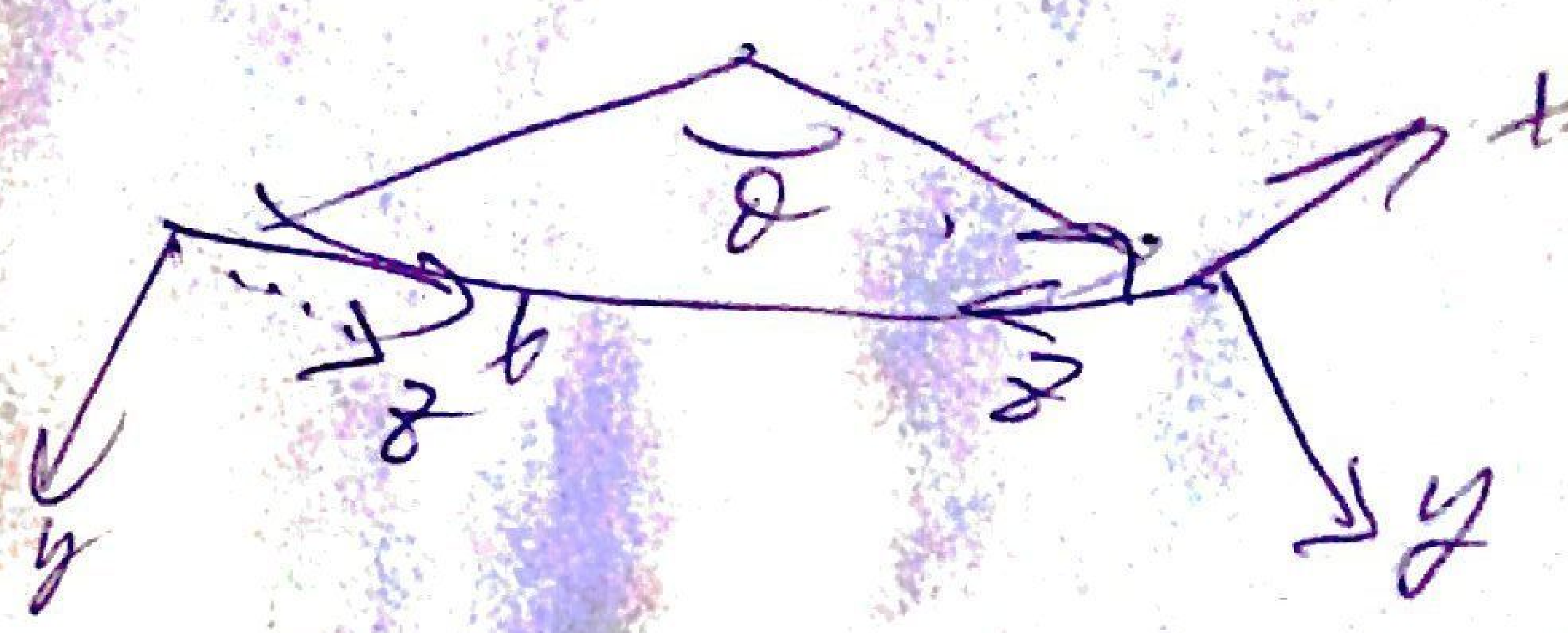
$$\begin{pmatrix} \vec{x} \\ 1 \end{pmatrix} \sim (K R^{-1} \bar{R} R \mid \bar{R} R^{-1} T) \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix}$$

$[M u]_\times = (M^{-1})^T [u]_\times M^{-1}$

$G^T F G = d [R^{-1} T]_\times R^{-1} \bar{R} R$

$= d R^T [T]_\times \bar{R} R$

$$T = \begin{pmatrix} -\cos\frac{\theta}{2} \\ 0 \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad \bar{R} = R\left(\theta, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$



$\boxed{R}$

$\boxed{\text{for strong assumption}}$

the building is just rotated from x-axis.

$$R = R\left(a, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) \quad (\text{dof} = 1).$$

for weak assumption.

$$R = R_z R_y R_x$$

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos a_x & \sin a_x \\ 0 & -\sin a_x & \cos a_x \end{pmatrix} \quad R_y = \begin{pmatrix} \cos a_y & 0 & -\sin a_y \\ 0 & 1 & 0 \\ \sin a_y & 0 & \cos a_y \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$