



## ENGR233 Sample Midterm

Applied Advanced Calculus (Concordia University)



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## ENGR-233: Applied Advanced Calculus Winter 2016

### Midterm sample 2 solutions

**Problem 1.** Find the parametric equation of the line of intersection of two planes:

$$P_1: x+3y+5z=0 \text{ and } P_2: x+y-3z=6 .$$

**Solution.** Let us eliminate the variables  $x$  and  $y$  from the equations of the planes. Subtract the second equation from the first one:

$$2y+8z=-6; \quad y=-4z+3.$$

Now express  $x$  in terms of  $y, z$  from the first equation:

$$x=-y+3z+6=4z+3+3z+6=7z+9 .$$

So, the parametric equations of the line are

$$x=7t+9, \quad y=-4t+3, \quad z=t .$$

**Problem 2.** Position vector of a moving particle is given by

$$\mathbf{r}(t) = (2t^2 - 5t + 2, 2t^2 + 1, (t+1)^2) .$$

(a) At what time(s) does the particle pass the  $yz$ -plane?

(b) What are the particle (i) coordinates, (ii) velocity, (iii) speed, and (iv) acceleration at  $t=1$  ?

**Solution.** (a) We have to solve the equation  $2t^2 - 5t + 2 = 0$  ;  $t = \frac{1}{2}, 2$  ;

$$t_1 = \frac{1}{2}, \quad t_2 = 2 .$$

(b) (i) Position  $\mathbf{r}(1)=(-1, 3, 4)$  ; (ii) Velocity  $\mathbf{r}'(t)=(4t-5, 4t, 2t+2)$  ,  
 so  $\mathbf{r}'(1)=(-1, 4, 4)$  ; (iii) Speed  $\mathbf{r}'(1)=\sqrt{1+16+16}=\sqrt{33}$  ;  
 (iv) Acceleration  $\mathbf{r}''(t)=(4, 4, 2)$  , so  $\mathbf{r}''(1)=(4, 4, 2)$

**Problem 3.** Find the directional derivative of  $F(x, y, z)=7y^2e^{-x}+3z^2$  in the direction  $\mathbf{u}=(3, 6, -2)$  at the point  $(0, 1, 7)$  .

**Solution.**  $\frac{\partial F}{\partial x}=-7y^2e^{-x}$  ,  $\frac{\partial F}{\partial y}=14ye^{-x}$  ,  $\frac{\partial F}{\partial z}=6z$  ; so,

$$\frac{\partial F}{\partial x}(0, 1, 7)=-7, \quad \frac{\partial F}{\partial y}(0, 1, 7)=14, \quad \frac{\partial F}{\partial z}(0, 1, 7)=42 \text{ . Next,}$$

$$\frac{\mathbf{u}}{\|\mathbf{u}\|}=\frac{(3, 6, -2)}{\sqrt{9+36+4}}=\frac{(3, 6, -2)}{7}=\left(\frac{3}{7}, \frac{6}{7}, -\frac{2}{7}\right) \text{ .}$$

$$\text{Hence, } D_{\mathbf{u}}F=\frac{3}{7}\cdot(-7)+\frac{6}{7}\cdot 14+(-\frac{2}{7})\cdot 42=-3+12-12=-3 \text{ .}$$

**Problem 4.** Let

$$\mathbf{F}(x, y, z)=\left(x(x^2+y^2+z^2-1), y(x^2+y^2+z^2-1), z(x^2+y^2+z^2-1)\right) \text{ .}$$

(a) Find  $\nabla \cdot \mathbf{F}$  ; (b) Find  $\|\mathbf{r}\|$  such that  $\nabla \cdot \mathbf{F}=0$  where  $\mathbf{r}=(x, y, z)$  .

**Solution.** (a)

$$\begin{aligned} \nabla \cdot \mathbf{F} &= (x^2+y^2+z^2-1)+x \cdot 2x+(x^2+y^2+z^2-1)+y \cdot 2y \\ &+ (x^2+y^2+z^2-1)+z \cdot 2z=5(x^2+y^2+z^2)-3 \text{ .} \end{aligned}$$

$$(b) \quad \nabla \cdot \mathbf{F}=0 \text{ if } 5(x^2+y^2+z^2)=3; \quad r^2=x^2+y^2+z^2=\frac{3}{5} \text{ ; } r=\sqrt{\frac{3}{5}} \text{ .}$$

**Problem 5.** Let  $\mathbf{F}(x, y, z)=(-y(x^2+y^2)^m, x(x^2+y^2)^m, 0)$  .

(a) Find  $\nabla \times \mathbf{F}$  ; (b) Find  $m$  such that  $\nabla \times \mathbf{F}=0$  for  $x^2+y^2>0$  .

**Solution.** (a)

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y(x^2+y^2)^m & x(x^2+y^2)^m & 0 \end{vmatrix} \\ &= 0 \cdot \mathbf{i} + 0 \cdot \mathbf{j} + \left[ (x^2+y^2)^m + x \cdot m(x^2+y^2)^{m-1} \cdot 2x \right. \\ &\quad \left. + (x^2+y^2)^m + y \cdot m(x^2+y^2)^{m-1} \cdot 2y \right] \mathbf{k} \\ &= \left[ 2(x^2+y^2)^m + 2m(x^2+y^2)^{m-1}(x^2+y^2) \right] \mathbf{k} = (2m+2)(x^2+y^2)^m \mathbf{k} .\end{aligned}$$

(b)  $\nabla \cdot \mathbf{F} = 0$  if  $2+2m=0$ , i.e.  $m=-1$ .

**Problem 6.** Find the work done by the force

$\mathbf{F}(x, y, z) = (xyz, -\cos(yz), xz)$  moving a particle along a **line segment** from a point  $P(1,1,1)$  to a point  $Q(-2,1,3)$ . **Hint:** find the parametric equation of a line connecting  $P$  and  $Q$ , then evaluate the integral.

**Solution.** The segment connecting the points  $P$  and  $Q$  has the parametric equation  $\mathbf{r}(t) = (1,1,1) + t(-3,0,2) = (1-3t, 1, 1+2t)$  ( $0 \leq t \leq 1$ ); the velocity vector is  $\mathbf{r}'(t) = (-3, 0, 2)$ . The field along this segment,

$\mathbf{F}(\mathbf{r}(t)) = ((1-3t)(1+2t), -\cos(1+2t), (1-3t)(1+2t))$ . The work along this segment,

$$\begin{aligned}W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 (-3(1-3t)(1+2t) + 2(1-3t)(1+2t)) dt = - \int_0^1 (1-3t)(1+2t) dt \\ &= - \int_0^1 (1-t-6t^2) dt = -1 + \frac{1}{2} + 2 = \frac{3}{2} .\end{aligned}$$

**Problem 7.** Let  $\mathbf{F}(x, y, z) = (y, x+z, y)$ .

(a) Show that  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of the path;

(b) compute the integral for any path  $C$  from the point  $A(2,1,4)$  to the point  $B(8,3,1)$ .

**Solution.** (a)  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0$ ;  $\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 1 - 1 = 0$  ;

$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0 - 0 = 0$  . So, the integral is path-independent.

(b) Let us find the potential function  $\phi(x, y, z)$  . It satisfies equations

$\frac{\partial \phi}{\partial x} = y$  ,  $\frac{\partial \phi}{\partial y} = x + z$  ,  $\frac{\partial \phi}{\partial z} = y$  . So,

$\phi = \int y \, dx = sy + g(y, z)$  ;

$\frac{\partial \phi}{\partial y} = x + \frac{\partial g}{\partial y} = x + z$  ;  $\frac{\partial g}{\partial y} = z$  ;  $g = yz + h(z)$  ;  $\phi = xy + yz + h(z)$ .

$\frac{\partial \phi}{\partial z} = y + h'(z) = y$  ;  $h'(z) = 0$  ;  $\phi(x, y, z) = xy + yz$  .

Now, the work

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A) = \phi(8,3,1) - \phi(2,1,4) = 27 - 6 = 21.$$