ENGR233 Sample Midterm

Applied Advanced Calculus (Concordia University)

ENGR-233: Applied Advanced Calculus Winter 2016

Midterm sample 2 solutions

Problem 1. Find the parametric equation of the line of intersection of two planes:

P_1: $x+3y+5z=0$ and P_2: $x+y-3z=6$.

Solution. Let us eliminate the variables x and y from the equations of the planes. Subtract the second equation from the first one:

$$2y+8z=-6; \quad y=-4z=3.$$

Now express x in terms of y, z from the first equation:

$$x=-y+3z+6=4z+3+3z+6=7z+9 \ .$$

So, the parametric equations of the line are

$$x=7t+9, \ y=-4t-3, \ z=t \ .$$

Problem 2. Position vector of a moving particle is given by $\boldsymbol{r}(t)=\left(2t^2-5t+2, 2t^2+1, (t+1)^2\right)$.

(a) At what time(s) does the particle pass the yz -plane?

(b) What are the particle (i) coordinates, (ii) velocity, (iii) speed, and ((iv) acceleration at $t=1$?

Solution. (a) We have to solve the equation $2t^2-5t+2=0$; ;
$t_1=\frac{1}{2,}$ $t_2=2$.

(b) (i) Position $\boldsymbol{r}(1)=(-1,3,4)$; (ii) Velocity $\boldsymbol{r}'(t)=(4t-5,4t,2t+2)$, so $\boldsymbol{r}'(1)=(-1,4,4)$; (iii) Speed $\boldsymbol{r}'(1)=\sqrt{1+16+16}=\sqrt{33}$;
(iv) Acceleration $\boldsymbol{r}''(t)=(4,4,2)$, so $\boldsymbol{r}''(1)=(4,4,2)$

Problem 3. Find the directional derivative of $F(x,y,z)=7y^2e^{-x}+3z^2$ in the direction $\boldsymbol{u}=(3,6,-2)$ at the point $(0,1,7)$.

Solution. $\frac{\partial F}{\partial x}=-7\mathrm{y}^2e^{-x}$, $\frac{\partial F}{\partial y}=14ye^{-x}$, $\frac{\partial F}{\partial z}=6\mathrm{z}$; so,

$\frac{\partial F}{\partial x}(0,1,7)=-7,\ \frac{\partial F}{\partial y}(0,1,7)=14,\ \frac{\partial F}{\partial z}(0,1,7)=42$. Next,
$\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}=\frac{(3,6,-2)}{\sqrt{9+36+4}}=\frac{(3,6,-2)}{7}=(\frac{3}{7},\frac{6}{7},-\frac{2}{7})$.

Hence, $D_{\boldsymbol{u}}F=\frac{3}{7}\cdot(-7)+\frac{6}{7}\cdot 14+-\frac{2}{7}\cdot 42=-3+12-12=-3$.

Problem 4. Let
$\boldsymbol{F}(x,y,z)=\left(x(x^2+y^2+z^2-1),y(x^2+y^2+z^2-1),z(x^2+y^2+z^2-1)\right)$.
(a) Find $\nabla\cdot\boldsymbol{F}$; (b) Find $\|\boldsymbol{r}\|$ such that $\nabla\cdot\boldsymbol{F}=0$ where $\boldsymbol{r}=(x,y,z)$.

Solution. (a)

$$\nabla\cdot\boldsymbol{F}=(x^2+y^2+z^2-1)+x\cdot 2x+(x^2+y^2+z^2-1)+y\cdot 2y$$

$$+(x^2+y^2+z^2-1)+z\cdot 2\mathrm{z}=5(x^2+y^2+z^2)-3 \ .$$

(b) $\nabla\cdot\boldsymbol{F}=0$ if $5(x^2+y^2+z^2)=3;\quad r^2=x^2+y^2+z^2=\frac{3}{5}$; $r=\sqrt{\frac{3}{5}}$.

Problem 5. Let $\boldsymbol{F}(x,y,z)=\left(-y(x^2+y^2)^m,x(x^2+y^2)^m,0\right)$.
(a) Find $\nabla\times\boldsymbol{F}$; (b) Find m such that $\nabla\times\boldsymbol{F}=0$ for $x^2+y^2>0$.

Solution. (a)

$$\nabla\times\boldsymbol{F}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y(x^2+y^2)^m & x(x^2+y^2)^m & 0 \end{vmatrix}$$

$$=0\cdot\boldsymbol{i}+0\cdot\boldsymbol{j}+\big[(x^2+y^2)^m+x\cdot m(x^2+y^2)^{m-1}\cdot 2x$$
$$+(x^2+y^2)^m+y\cdot m(x^2+y^2)^{m-1}\cdot 2y\big]\boldsymbol{k}$$
$$=\left[2(x^2+y^2)^m+2m(x^2+y^2)^{m-1}(x^2++y^2)\right]\boldsymbol{k}=(2m+2)(x^2+y^2)^m\boldsymbol{k} \ .$$

(b) $\nabla\cdot\boldsymbol{F}=0$ if $2+2m=0$, i.e. $m=-1$.

Problem 6. Find the work done by the force $\boldsymbol{F}(x,y,z)=(xyz,-\cos(yz),xz)$ moving a particle along a **line segment** from a point $P(1,1,1)$ to a point $Q(-2,1,3)$. **Hint:** find the parametric equation of a line connecting P and Q , then evaluate the integral.

Solution. The segment connecting the points P and Q has the parametric equation $\boldsymbol{r}(t)=(1,1,1)+t(-3,0,2)=(1-3t,1,1+2t)$ $(0\le t\le 2)$; the velocity vector is $\boldsymbol{r}'(t)=(-3,0,2)$. The field along this segment, $\boldsymbol{F}(\boldsymbol{r}(t))=\big((1-3t)(1+2t),-\cos(1+2t),(1-3t)(1+2t)\big)$. The work along this segment,

$$W=\int_C \boldsymbol{F}\cdot d\boldsymbol{r}=\int_0^1 \boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\,dt$$
$$=\int_0^1 \big(-3(1-3t)(1+2t)+2(1-3t)(1+2t)\big)\,dt=-\int_0^1 (1-3t)(1+2t)\,dt$$
$$=-\int_0^1 (1-t-6t^2)\,dt=-1+\frac{1}{2}+2=\frac{3}{2} \ .$$

Problem 7. Let $\boldsymbol{F}(x,y,z)=(y,x+z,y)$.
(a) Show that $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ is independent of the path;

(b) compute the integral for any path C from the point $A(2,1,4)$ to the point $B(8,3,1)$.

Solution. (a) $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1-1=0;$ $\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=1-1=0$; $\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=0-0=0$. So, the integral is path-independent.

(b) Let us find the potential function $\phi(x,y,z)$. It satisfies equations $\frac{\partial \phi}{\partial x}=y$, $\frac{\partial \phi}{\partial y}=x+z$, $\frac{\partial \phi}{\partial z}=y$. So,

$$\phi=\int y\,dx=sy+g(y,z) \ ;$$

$$\frac{\partial \phi}{\partial y}=x+\frac{\partial g}{\partial y}=x+z \ ; \quad \frac{\partial g}{\partial y}=z; \ \ g=yz+h(z); \ \ \phi=xy+yz+h(z).$$

$$\frac{\partial \phi}{\partial z}=y+h(z)=y; \ \ h(z)=0; \ \ \phi(x,y,z)=xy+yz \ .$$

Now, the work

$$W=\int_A^B \boldsymbol{F}\cdot d\,\boldsymbol{r}=\phi(B)-\phi(A)=\phi(8,3,1)-\phi(2,1,4)=27-6=21.$$