1. (*4 marks*) Solve the following initial value problem for the first order differential equation $\dfrac{dy}{dx} = \dfrac{3x^2 y}{1+x^3}$,   y(1)=2.

Solution

$$\frac{dy}{y} - \frac{3x^2}{1+x^3}dx = 0 \to \int \frac{dy}{y} - \int \frac{3x^2}{1+x^3}dx = C \to \ln y - \ln(1+x^3) = C \to \ln y = \ln(1+x^3) + \ln C,$$

$$\to y = C(1+x^3).$$

When $x = 1$, $y = 2 \to 2 = C(1+1) \to C = 1$

$y = (1+x^3)$ Particular solution.

2. (*4 marks*) Solve the following differential equation of the first order

$$\frac{dy}{dx} + 2xy = -xy^4.$$

Solution

The transformation $u = y^{-3}$, $\dfrac{du}{dx} = -3y^{-4}\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = -\dfrac{1}{3}y^4\dfrac{du}{dx}$

reduces the equation to

$$-\frac{1}{3}y^4\frac{du}{dx} + 2xy = -xy^4, \quad -\frac{1}{3}\frac{du}{dx} + \frac{2x}{y^3} = -x, \quad \frac{du}{dx} - 6xu = 3x$$

Therefore $\mu(x) = e^{\int P(x)dx} = e^{-\int 6xdx} = e^{-3x^2}$, is the integrating factor.

Then,

$$\frac{d}{dx}\left[e^{-3x^2}u\right] = 3xe^{-3x^2}.$$

$$\int \frac{d}{dx}\left[e^{-3x^2}u\right]dx = \int 3xe^{-3x^2}dx$$

$$e^{-3x^2}u = -\frac{1}{2}e^{-3x^2} + C$$

Solution $\to y^{-3} = -\dfrac{1}{2} + Ce^{3x^2}$

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3. (4 *marks*) Solve the differential equation
$(4x^3 + 3x^2 + 3y - 1)dx + (3x + 2y + 1)dy = 0$.

Solution

$\dfrac{\partial M}{\partial y} = 3 = \dfrac{\partial N}{\partial x}$ and the equation is exact.

$\mu(x,y)=\displaystyle\int_{0}^{x}(4x^3 + 3x^2 + 3y)dx = x^4 + x^3 + 3xy + g(y),$

$\dfrac{\partial \mu}{\partial y} = 3x + g'(y) = N(x,y) = 3x + 2y + 1 \rightarrow g'(y) = 2y + 1 \rightarrow g(y) = y^2 + y,$

Solution $\rightarrow x^4 + x^3 + 3xy + y^2 + y = C.$

4. (*4 marks*) Solve $(x^3 + y^3)\,dx - 3xy^2\,dy = 0,$ y(1)=1.

Solution

The equation is homogeneous of degree 3.

We use the substitution y=vx, dy=vdx + xdv, → 1 mark

$x^3\left[(1 + v^3)dx - 3v^2(vdx + xdv)\right] = 0$. The variables are separable.

$x^3\left[(1 + v^3) - 3v^3\right]dx - 3v^2 x^4 dv = 0,\quad (1 - 2v^3)dx - 3v^2 x dv = 0,$

$\dfrac{dx}{x} = \dfrac{3v^2}{1 - 2v^3}dv$

$\ln x = \dfrac{1}{2}\ln(1 - 2v^3) + \ln C,\quad 2\ln x = \ln(1 - 2v^3) + \ln C,$

$\ln\left[\dfrac{x^2}{1 - 2v^3}\right] = \ln C,\quad \dfrac{x^2}{1 - 2v^3} = C,\quad \dfrac{x^2}{1 - 2\left(\dfrac{y}{x}\right)^3} = C.$

Solution: $x^3 - 2y^3 = Cx.$

But y(1)=1 thus C=-1 then $x^3 - 2y^3 = -x$ is the particular solution.

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5. (*4 marks*) In a certain culture of bacteria the rate of increase is proportional to the number present. If it is found that the number triples in 4 hours, how many may be expected at the end of 12 hours?

Hint: Assume that $x = x_0$ at time $t = 0$.

Solution

$$\frac{dx}{dt} = kx \rightarrow \frac{dx}{x} = kdt \rightarrow \ln x = kt + \ln C_1 \rightarrow x = Ce^{kt}.$$

Assuming that $x = x_0$ at time $t = 0 \rightarrow C = x_0 \rightarrow x = x_0 e^{kt}$.

At time $t = 4$, $x = 3x_0 \rightarrow 3x_0 = x_0 e^{4k} \rightarrow e^{4k} = 3$.

When $t = 12$, $x = x_0 e^{12k} = x_0 (e^{4k})^3 = x_0 (3)^3 = 27x_0$.

that is, there are 27 times the original number.

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