MIDTERM 1 2018, questions and answers

Applied Ordinary Differential Equations (Concordia University)

ENGR 213 –ORDINARY DIFFERENTIAL EQUATIONS
FALL 2018 Date: October 19, 2018

Midterm Test 1 Section XX

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Name and Surname: ID:

1. (*4 marks*) Solve the following initial value problem for the first order differential equation $\frac{dy}{dx}=\frac{3x^2y}{1+x^3}$, y(1)=2.

Solution

$$\frac{dy}{y}-\frac{3x^2}{1+x^3}dx=0\to\int\frac{dy}{y}-\int\frac{3x^2}{1+x^3}dx=C\to\ln y-\ln(1+x^3)=C\to\ln y=\ln(1+x^3)+\ln C,$$

$\to y=C(1+x^3).$

When $x=1,\ y=2\to 2=C(1+1)\to C=1$

$y=(1+x^3)$ Particular solution.

2. (*4 marks*) Solve the following differential equation of the first order $\frac{dy}{dx}+2xy=-xy^4$.

Solution

The transformation $u=y^{-3}$, $\frac{du}{dx}=-3y^{-4}\frac{dy}{dx}$, $\frac{dy}{dx}=-\frac{1}{3}y^4\frac{du}{dx}$

reduces the equation to

$$-\frac{1}{3}y^4\frac{du}{dx}+2xy=-xy^4,\quad -\frac{1}{3}\frac{du}{dx}+\frac{2x}{y^3}=-x,\quad \frac{du}{dx}-6xu=3x$$

Therefore $\mu(x)=e^{\int P(x)dx}=e^{-\int 6xdx}=e^{-3x^2}$, is the integrating factor.

Then,

$$\frac{d}{dx}\left[e^{-3x^2}u\right]=3xe^{-3x^2}.$$

$$\int\frac{d}{dx}\left[e^{-3x^2}u\right]dx=\int 3xe^{-3x^2}dx$$

$$e^{-3x^2}u=-\frac{1}{2}e^{-3x^2}+C$$

Solution $\to y^{-3}=-\frac{1}{2}+Ce^{3x^2}$

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3. (4 *marks*) Solve the differential equation
$(4x^3+3x^2+3y-1)dx+(3x+2y+1)dy=0$.

Solution

$$\frac{\partial M}{\partial y}=3=\frac{\partial N}{\partial x}$$ and the equation is exact.

$$\mu(x,y)=\int_0^x(4x^3+3x^2+3y)dx=x^4+x^3+3xy+g(y),$$

$$\frac{\partial \mu}{\partial y}=3x+g'(y)=N(x,y)=3x+2y+1\to g'(y)=2y+1\to g(y)=y^2+y,$$

Solution $\to x^4+x^3+3xy+y^2+y=C.$

4. (*4 marks*) Solve $(x^3+y^3)\,dx-3xy^2\,dy=0, \quad y(1)=1.$

Solution

The equation is homogeneous of degree 3.

We use the substitution y=vx, dy=vdx + xdv, → 1 mark

$x^3\left[(1+v^3)dx-3v^2(vdx+xdv)\right]=0.$ The variables are separable.

$$x^3\left[(1+v^3)-3v^3\right]dx-3v^2x^4dv=0, \quad (1-2v^3)dx-3v^2xdv=0,$$

$$\frac{dx}{x}=\frac{3v^2}{1-2v^3}dv$$

$$\ln x=\frac{1}{2}\ln(1-2v^3)+\ln C, \quad 2\ln x=\ln(1-2v^3)+\ln C,$$

$$\ln\left[\frac{x^2}{1-2v^3}\right]=\ln C, \quad \frac{x^2}{1-2v^3}=C, \quad \frac{x^2}{1-2\left(\frac{y}{x}\right)^3}=C.$$

Solution: $x^3-2y^3=Cx.$

But y(1)=1 thus C=-1 then $x^3-2y^3=-x$ is the particular solution.

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5. (*4 marks*) In a certain culture of bacteria the rate of increase is proportional to the number present. If it is found that the number triples in 4 hours, how many may be expected at the end of 12 hours?

Hint: Assume that $x = x_0$ *at time* $t = 0$.

Solution

$$\frac{dx}{dt} = kx \to \frac{dx}{x} = kdt \to ln\,x = kt + ln\,C_1 \to x = Ce^{kt}.$$

Assuming that $x = x_0$ at time $t = 0 \to C = x_0 \to x = x_0 e^{kt}$.

At time $t = 4,\ x = 3x_0 \to 3x_0 = x_0 e^{4k} \to e^{4k} = 3.$

When $t = 12,\ x = x_0 e^{12k} = x_0 (e^{4k})^3 = x_0 (3)^3 = 27x_0.$

that is, there are 27 times the original number.