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https://github.com/1ncompleteness/Numerical-Analysis

Homework 2

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Problem 1.

Aitken's Δ^2 Method:

$$\hat{p}_n = \frac{p_{n+2}\, p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n}$$

(a) Substitute $p_n = x$, $p_{n+1} = g(x)$, $p_{n+2} = g(g(x))$:

$$\boxed{G(x) = \frac{g(g(x)) \cdot x - [g(x)]^2}{g(g(x)) - 2g(x) + x}}$$

(b) $g(x) = x + x^2$, fixed point $p = 0$. Then $g'(x) = 1 + 2x$, so $g'(0) = 1$.

$\forall\, p_0 > 0$: $g(p_0) = p_0 + p_0^2 > p_0$, so the iterates are strictly increasing. Since $|g'(0)| = 1$, the fixed point is not attracting.

$$\boxed{\text{Diverges}}$$

(c) Compute $g(g(x))$ with $g(x) = x + x^2$:

$$\begin{aligned} g(g(x)) &= g(x + x^2) = (x + x^2) + (x + x^2)^2 \\ &= x + x^2 + x^2 + 2x^3 + x^4 = x + 2x^2 + 2x^3 + x^4 \end{aligned}$$

Numerator of $G(x)$:

$$\begin{aligned} g(g(x)) \cdot x - [g(x)]^2 &= x(x + 2x^2 + 2x^3 + x^4) - (x + x^2)^2 \\ &= (x^2 + 2x^3 + 2x^4 + x^5) - (x^2 + 2x^3 + x^4) = x^4 + x^5 = x^4(1 + x) \end{aligned}$$

Denominator of $G(x)$:

$$g(g(x)) - 2g(x) + x = (x + 2x^2 + 2x^3 + x^4) - 2(x + x^2) + x = 2x^3 + x^4 = x^3(2 + x)$$

$$\boxed{G(x) = \frac{x^4(1 + x)}{x^3(2 + x)} = \frac{x(1 + x)}{2 + x}}$$

(d) Using the quotient rule on $G(x) = \dfrac{x + x^2}{x + 2}$:

$$G'(x) = \frac{(1 + 2x)(x + 2) - (x + x^2)(1)}{(x + 2)^2} = \frac{x^2 + 4x + 2}{(x + 2)^2}$$

$$G'(0) = \frac{0+0+2}{(0+2)^2} = \frac{2}{4} = \frac{1}{2}$$

Since $|G'(0)| = \frac{1}{2} < 1$, the fixed-point iteration $x = G(x)$ converges to $p = 0$.

$$\boxed{\text{Converges (linear, rate } \tfrac{1}{2})}$$

Problem 2.

$f(x) = x^3 + x - 4$ on $[1, 4]$.

$f(1) = 1 + 1 - 4 = -2 < 0, \quad f(4) = 64 + 4 - 4 = 64 > 0.$

Bisection needs n iterations where:

$$\frac{b-a}{2^n} < 10^{-5} \implies \frac{3}{2^n} < 10^{-5} \implies 2^n > 3 \times 10^5$$

$2^{18} = 262144 < 300000, \quad 2^{19} = 524288 > 300000.$

$$\boxed{N = 19}$$

Problem 3.

(a) The three missing lines are:

Arrow	Code
1st (if branch)	`b = p`
2nd (else branch)	`a = p`
3rd (after iter)	`p = (a + b)/2`

(b) Iteration with $f(x) = \tan x - x$, $a = 4.3$, $b = 4.6$:

$f(4.3) \approx -2.014, \quad f(4.6) \approx 4.260 \quad$ (opposite signs).

iter	a	b	p	$\lvert f(p)\rvert$
0	4.3000	4.6000	4.4500	0.7267
1	4.4500	4.6000	4.5250	0.7489
2	4.4500	4.5250	4.4875	0.1161
3	4.4875	4.5250	4.5063	0.2759
4	4.4875	4.5063	4.4969	0.0711
5	4.4875	4.4969	4.4922	0.0245
6	4.4922	4.4969	4.4945	0.0228
7	4.4922	4.4945	4.4934	0.0010

At iteration 7: $|f(p)| = 0.0010 < 0.01$. The loop terminates.

$$\boxed{\text{7 iterations}}$$

(c) The approximation is:

$$\boxed{p \approx 4.4934}$$

Problem 4.

Target: $p^* = 7^{1/5} \approx 1.47577$. For each method $p_n = g(p_{n-1})$, compute $|g'(p^*)|$:

(a) $g(x) = \left(\dfrac{7}{x}\right)^{1/4}$:

$$g'(x) = -\frac{1}{4} \cdot \frac{(7/x)^{1/4}}{x} \implies g'(p^*) = -\frac{1}{4} \cdot \frac{p^*}{p^*} = -\frac{1}{4} \implies |g'(p^*)| = 0.25$$

(b) $g(x) = x - \dfrac{x^5 - 7}{x^2}$:

$$g'(x) = 1 - \frac{5x^4 \cdot x^2 - (x^5 - 7) \cdot 2x}{x^4} = 1 - 3x^2 - \frac{14}{x^3}$$

$$g'(p^*) = 1 - 5 \cdot 7^{2/5} \approx -9.89 \implies |g'(p^*)| \approx 9.89 > 1 \quad \text{(diverges)}$$

(c) $g(x) = x - \dfrac{x^5 - 7}{5x^4}$ (Newton's method for $f(x) = x^5 - 7$):

$$g'(x) = 1 - \frac{5x^4 \cdot 5x^4 - (x^5 - 7) \cdot 20x^3}{25x^8} = \frac{4(x^5 - 7)}{5x^5}$$

$$g'(p^*) = \frac{4(7 - 7)}{5 \cdot 7} = 0 \quad \text{(quadratic convergence)}$$

(d) $g(x) = x - \dfrac{x^5 - 7}{12}$:

$$g'(x) = 1 - \frac{5x^4}{12} \implies g'(p^*) = 1 - \frac{5 \cdot 7^{4/5}}{12} \approx -0.976 \implies |g'(p^*)| \approx 0.976$$

Ranking (fastest to slowest):

Rank	Method	$\lvert g'(p^*)\rvert$	Convergence
1	(c) Newton	0	Quadratic
2	(a) $(7/x)^{1/4}$	1/4	Linear
3	(d) $x - (x^5 - 7)/12$	0.976	Linear
4	(b) $x - (x^5 - 7)/x^2$	9.89	Diverges

$$\boxed{\text{(c)} \succ \text{(a)} \succ \text{(d)} \succ \text{(b)}}$$

Problem 5.

$3x^2 - e^x = 0 \implies x^2 = \dfrac{e^x}{3} \implies x = \dfrac{e^{x/2}}{\sqrt{3}}$.

Let $g(x) = \dfrac{e^{x/2}}{\sqrt{3}}$ on $[a,b] = [0,1]$.

Condition 1: g maps $[0,1]$ into $[0,1]$.

g is increasing on $[0,1]$, so:

$$g(0) = \frac{1}{\sqrt{3}} \approx 0.5774, \qquad g(1) = \frac{e^{1/2}}{\sqrt{3}} \approx 0.9519$$

$g([0,1]) = [0.5774,\ 0.9519] \subset [0,1]$. ✓

Condition 2: $|g'(x)| < 1$ on $(0,1)$.

$$g'(x) = \frac{e^{x/2}}{2\sqrt{3}} \implies \max_{[0,1]} |g'(x)| = g'(1) = \frac{e^{1/2}}{2\sqrt{3}} \approx 0.476 < 1 \quad ✓$$

g has a unique fixed point in $[0,1]$.

Iteration from $p_0 = 0.5$:

n	p_n	$\lvert p_n - p_{n-1}\rvert$
0	0.5000	—
1	0.7413	0.2413
2	0.8364	0.0951
3	0.8771	0.0407
4	0.8952	0.0180
5	0.9033	0.0081
6	0.9070	0.0037
7	0.9086	0.0017
8	0.9094	0.0008

$|p_8 - p_7| \approx 7.57 \times 10^{-4} < 10^{-3}$.

$$\boxed{p \approx 0.9094}$$

Problem 6.

Let $g(x) = \dfrac{x}{2} + \dfrac{1}{x}$ on $[1,2]$.

Condition 1: g maps $[1,2]$ into $[1,2]$.

$g'(x) = \dfrac{1}{2} - \dfrac{1}{x^2}$. Setting $g'(x) = 0$: $x = \sqrt{2}$.

Since $g''(x) = 2/x^3 > 0$, the minimum of g on $[1, 2]$ is at $x = \sqrt{2}$:

$$g(\sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.414$$

At the endpoints:

$$g(1) = \frac{1}{2} + 1 = \frac{3}{2} = 1.5, \qquad g(2) = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$$

$g([1, 2]) = [\sqrt{2},\ 3/2] \subset [1, 2]$. ✓

Condition 2: $|g'(x)| < 1$ on $(1, 2)$.

$$g'(x) = \frac{1}{2} - \frac{1}{x^2} = \frac{x^2 - 2}{2x^2}$$

$\forall\, x \in [1, 2]$: $|x^2 - 2| \leq \max(|1 - 2|, |4 - 2|) = 2$ and $2x^2 \geq 2$, so:

$$|g'(x)| = \frac{|x^2 - 2|}{2x^2} \leq \frac{2}{2 \cdot 1} = 1$$

For inequality on $(1, 2)$: $|x^2 - 2| < 2$ when $x \in (1, 2)$ and $2x^2 > 2$ when $x > 1$, so $|g'(x)| < 1$ on $(1, 2)$. More precisely:

$$\max_{x \in [1,2]} |g'(x)| = |g'(1)| = \left| \frac{1 - 2}{2} \right| = \frac{1}{2} < 1 \quad ✓$$

Fixed point: $g(p) = p \implies \dfrac{p}{2} + \dfrac{1}{p} = p \implies \dfrac{1}{p} = \dfrac{p}{2} \implies p^2 = 2 \implies p = \sqrt{2}$.

$\{x_n\}$ converges to the unique fixed point $p = \sqrt{2}$ for any $x_0 \in [1, 2]$.

$$\boxed{x_n \to \sqrt{2} \text{ for all } x_0 \in [1, 2]}$$

Problem 7.

Distance from (x, x^2) to $(1, 0)$:

$$[d(x)]^2 = (x - 1)^2 + (x^2)^2 = (x - 1)^2 + x^4$$

Minimize by setting $\dfrac{d}{dx}[d(x)]^2 = 0$:

$$2(x - 1) + 4x^3 = 0 \implies f(x) = 4x^3 + 2x - 2 = 0$$

$f'(x) = 12x^2 + 2$.

Newton's method with $x_0 = 1$:

$$x_{n+1} = x_n - \frac{4x_n^3 + 2x_n - 2}{12x_n^2 + 2}$$

n	x_n	$f(x_n)$	$\lvert x_n - x_{n-1}\rvert$
0	1.000000	4.000000	—
1	0.714286	0.886297	2.857×10^{-1}
2	0.605169	0.096859	1.091×10^{-1}
3	0.590022	0.001652	1.515×10^{-2}
4	0.589755	0.000001	2.674×10^{-4}
5	0.589755	0.000000	8.202×10^{-8}

$|x_5 - x_4| \approx 8.2 \times 10^{-8} < 10^{-4}$.

$x \approx 0.5898, \quad y = x^2 \approx 0.3478$.

$$\boxed{\text{Closest point: } (0.5898,\ 0.3478)}$$