Behrouz Barati B

Homework 1

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Problem 1.

Let $f(x) = \sin^2(x)$, $x_0 = 0$, $x = 1^\circ = \dfrac{\pi}{180}$.

Using $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$:

$$f(x) = \sin^2(x), \quad f'(x) = \sin(2x), \quad f''(x) = 2\cos(2x), \quad f'''(x) = -4\sin(2x), \quad f^{(4)}(x) = -8\cos(2x).$$

At $x_0 = 0$:

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f'''(0) = 0.$$

The third-degree Taylor polynomial:

$$P_3(x) = 0 + 0 + \frac{2}{2!}\,x^2 + 0 = x^2.$$

This confirms $\sin^2(x) \approx x^2$ is $P_3(x)$.

By Taylor's Theorem, $\exists\,\xi$ between 0 and x:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}\,x^4 = \frac{-8\cos(2\xi)}{24}\,x^4 = -\frac{\cos(2\xi)}{3}\,x^4.$$

Since $|\cos(2\xi)| \le 1$:

$$|R_3(x)| \le \frac{x^4}{3}.$$

At $x = \dfrac{\pi}{180}$:

$$\left|\sin^2\!\left(\frac{\pi}{180}\right) - \left(\frac{\pi}{180}\right)^2\right| \le \frac{1}{3}\left(\frac{\pi}{180}\right)^4 = \frac{\pi^4}{3\cdot 180^4}$$

$$\boxed{\left|\sin^2(1^\circ) - (1^\circ)^2\right| \le \frac{\pi^4}{3\cdot 180^4} \approx 3.09\times 10^{-8}}$$

Problem 2.

Claim: $|\sin^2(x) - \sin^2(y)| \le 2|x-y|$ for all $x, y \in \mathbb{R}$.

Proof. Let $f(t) = \sin^2(t)$. Then f is differentiable on $\mathbb{R}$ with

$$f'(t) = 2\sin(t)\cos(t).$$

$\forall\, t \in \mathbb{R}$:

$$|f'(t)| = 2|\sin t|\,|\cos t| \le 2 \cdot 1 \cdot 1 = 2.$$

Let $x, y \in \mathbb{R}$ with $x \neq y$ (the case $x = y$ is trivial). By the Mean Value Theorem, $\exists\, c$ between x and y:

$$f(x) - f(y) = f'(c)(x - y).$$

Taking absolute values:

$$|\sin^2(x) - \sin^2(y)| = |f'(c)|\,|x - y| \le 2\,|x - y|. \qquad \square$$

Problem 3.

$f(x) = (2 - x)^{-1}$, $x_0 = 0$.

Using the hint:

$$\frac{1}{2 - x} = \frac{1}{2} \cdot \frac{1}{1 - (x/2)}.$$

The geometric series $\dfrac{1}{1 - r} = \displaystyle\sum_{k=0}^{\infty} r^k$ for $|r| < 1$, with $r = x/2$:

$$\frac{1}{2 - x} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}, \qquad |x| < 2.$$

The nth Taylor polynomial:

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{x^k}{2^{k+1}} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \cdots + \frac{x^n}{2^{n+1}}}$$

Error on $[0, 1]$: The remainder is the tail of the geometric series. For $x \in [0, 1]$:

$$|f(x) - P_n(x)| = \sum_{k=n+1}^{\infty} \frac{x^k}{2^{k+1}} = \frac{1}{2} \cdot \frac{(x/2)^{n+1}}{1 - x/2}.$$

Since $x/2 \in [0, 1/2]$ on $[0, 1]$, we have $1/(1 - x/2) \le 2$, and $(x/2)^{n+1} \le (1/2)^{n+1}$:

$$|f(x) - P_n(x)| \le \frac{1}{2} \cdot \frac{(1/2)^{n+1}}{1/2} = \frac{1}{2^{n+1}}.$$

Verification at $x = 1$: $f(1) = 1$ and $P_n(1) = \sum_{k=0}^{n} 1/2^{k+1} = 1 - 1/2^{n+1}$, so $|f(1) - P_n(1)| = 1/2^{n+1}$ exactly.

Finding n:

$$\frac{1}{2^{n+1}} \le 10^{-6} \iff 2^{n+1} \ge 10^6 \iff n + 1 \ge \log_2(10^6) = \frac{6 \ln 10}{\ln 2} \approx 19.932 \implies n + 1 \ge 20.$$

$$\boxed{n = 19}$$

Check: $1/2^{20} = 9.54 \times 10^{-7} < 10^{-6}, \quad 1/2^{19} = 1.91 \times 10^{-6} > 10^{-6}$.

Problem 4.

$E = 133 + 0.921 - 10\pi + 6e - \dfrac{3}{62}$.

Exact value (to 7 significant digits):

$$10\pi = 31.41593\ldots, \quad 6e = 16.30969\ldots, \quad 3/62 = 0.04838710\ldots$$

$$E = 133 + 0.921 - 31.41593 + 16.30969 - 0.04839 = 118.7664$$

(a) Four-digit rounding arithmetic.

Constants: $\mathrm{fl}_R(\pi) = 3.142, \quad \mathrm{fl}_R(e) = 2.718$.

$$\begin{aligned}
\text{Step 1:}\quad & 133.0 + 0.9210 = 133.921 = 0.133921 \times 10^3 \xrightarrow{\mathrm{fl}_R} 0.1339 \times 10^3 = 133.9 \\
\text{Step 2:}\quad & \mathrm{fl}_R(10\pi) = 10.00 \times 3.142 = 31.42 \\
& 133.9 - 31.42 = 102.48 = 0.10248 \times 10^3 \xrightarrow{\mathrm{fl}_R} 0.1025 \times 10^3 = 102.5 \\
\text{Step 3:}\quad & \mathrm{fl}_R(6e) = 6.000 \times 2.718 = 16.308 \xrightarrow{\mathrm{fl}_R} 16.31 \\
& 102.5 + 16.31 = 118.81 = 0.11881 \times 10^3 \xrightarrow{\mathrm{fl}_R} 0.1188 \times 10^3 = 118.8 \\
\text{Step 4:}\quad & \mathrm{fl}_R(3/62) = 0.048387\ldots \xrightarrow{\mathrm{fl}_R} 0.04839 \\
& 118.8 - 0.04839 = 118.752\ldots \xrightarrow{\mathrm{fl}_R} 0.1188 \times 10^3 = 118.8
\end{aligned}$$

$$\hat{y}_R = 118.8, \qquad \text{Abs. error} = |118.8 - 118.7664| = \boxed{0.0336}, \qquad \text{Rel. error} = \frac{0.0336}{118.7664} = \boxed{2.83 \times 10^{-4}}$$

(b) Four-digit chopping arithmetic.

Constants: $\mathrm{fl}_C(\pi) = 3.141, \quad \mathrm{fl}_C(e) = 2.718$.

$$\begin{aligned}
\text{Step 1:}\quad & 133.0 + 0.9210 = 133.921 \xrightarrow{\mathrm{fl}_C} 0.1339 \times 10^3 = 133.9 \\
\text{Step 2:}\quad & \mathrm{fl}_C(10\pi) = 10.00 \times 3.141 = 31.41 \\
& 133.9 - 31.41 = 102.49 = 0.10249 \times 10^3 \xrightarrow{\mathrm{fl}_C} 0.1024 \times 10^3 = 102.4 \\
\text{Step 3:}\quad & \mathrm{fl}_C(6e) = 6.000 \times 2.718 = 16.308 \xrightarrow{\mathrm{fl}_C} 16.30 \\
& 102.4 + 16.30 = 118.70 \xrightarrow{\mathrm{fl}_C} 0.1187 \times 10^3 = 118.7 \\
\text{Step 4:}\quad & \mathrm{fl}_C(3/62) = 0.048387\ldots \xrightarrow{\mathrm{fl}_C} 0.04838 \\
& 118.7 - 0.04838 = 118.652\ldots \xrightarrow{\mathrm{fl}_C} 0.1186 \times 10^3 = 118.6
\end{aligned}$$

$$\hat{y}_C = 118.6, \qquad \text{Abs. error} = |118.6 - 118.7664| = \boxed{0.1664}, \qquad \text{Rel. error} = \frac{0.1664}{118.7664} = \boxed{1.40 \times 10^{-3}}$$

Problem 5.

Approximate e^{-5} using the degree-9 Taylor polynomial with three-digit chopping.

True value: $e^{-5} \approx 6.74 \times 10^{-3}$.

Terms $5^i/i!$ and their three-digit chopped values:

i	$5^i/i!$ (exact)	fl_C	sign in (a)
0	1.00000	1.00	+
1	5.00000	5.00	−
2	12.5000	12.5	+
3	20.8333	20.8	−
4	26.0417	26.0	+
5	26.0417	26.0	−
6	21.7014	21.7	+
7	15.5010	15.5	−
8	9.68810	9.68	+
9	5.38228	5.38	−

(a) $e^{-5} \approx \displaystyle\sum_{i=0}^{9} \frac{(-1)^i 5^i}{i!}$

From left to right, applying fl_C after each step:

i	Operation	Result	$\mathrm{fl}_C(S)$
0	$S = +1.00$	1.00	1.00
1	$S = 1.00 - 5.00$	-4.00	-4.00
2	$S = -4.00 + 12.5$	8.50	8.50
3	$S = 8.50 - 20.8$	-12.3	-12.3
4	$S = -12.3 + 26.0$	13.7	13.7
5	$S = 13.7 - 26.0$	-12.3	-12.3
6	$S = -12.3 + 21.7$	9.40	9.40
7	$S = 9.40 - 15.5$	-6.10	-6.10
8	$S = -6.10 + 9.68$	3.58	3.58
9	$S = 3.58 - 5.38$	-1.80	-1.80

$$\boxed{e^{-5} \approx -1.80 \quad \text{(Part (a))}}$$

(b) $e^{-5} = \dfrac{1}{e^5} \approx \left(\displaystyle\sum_{i=0}^{9} \frac{5^i}{i!}\right)^{-1}$

i	Operation	Result	$\text{fl}_C(S)$
0	$S = 1.00$	1.00	1.00
1	$S = 1.00 + 5.00$	6.00	6.00
2	$S = 6.00 + 12.5$	18.5	18.5
3	$S = 18.5 + 20.8$	39.3	39.3
4	$S = 39.3 + 26.0$	65.3	65.3
5	$S = 65.3 + 26.0$	91.3	91.3
6	$S = 91.3 + 21.7$	113.	113.
7	$S = 113. + 15.5$	128.5	128.
8	$S = 128. + 9.68$	137.7	137.
9	$S = 137. + 5.38$	142.4	142.

$$\frac{1}{142} = 0.00704225\ldots \xrightarrow{\text{fl}_C} 0.00704$$

$$\boxed{e^{-5} \approx 0.00704 \quad \text{(Part (b))}}$$

Comparison.

Method	Result	Abs. Error	Rel. Error
(a) Alternating	-1.80	1.807	≈ 268
(b) Reciprocal	0.00704	3.02×10^{-4}	0.0448

Method (b) is far more accurate: In (a), the alternating series adds and subtracts terms as large as 26.0 to produce a result of order 10^{-3}. Each subtraction of nearly equal numbers destroys significant digits. In (b), all terms are positive: no cancellation occurs.

Problem 6.

Recall: $\alpha_n \to \alpha$ with rate $O(1/n^p)$ if $|\alpha_n - \alpha| \le K/n^p$ for large n; find the largest such p.

(a) $\lim_{n\to\infty} \sin^2(1/n) = 0.$

$\sin(u) = u - u^3/6 + \cdots \implies \sin^2(u) = u^2 - u^4/3 + \cdots$

$$\sin^2(1/n) = \frac{1}{n^2} - \frac{1}{3n^4} + \cdots \implies |\alpha_n - 0| \le K \cdot \frac{1}{n^2} \implies \boxed{O(1/n^2)}$$

(b) $\lim_{n\to\infty} n^4[1 - \cos(1/n^2)] = \frac{1}{2}.$

$1 - \cos(u) = u^2/2 - u^4/24 + \cdots$ With $u = 1/n^2$:

$$n^4[1 - \cos(1/n^2)] = n^4\left[\frac{1}{2n^4} - \frac{1}{24n^8} + \cdots\right] = \frac{1}{2} - \frac{1}{24n^4} + \cdots \implies \left|\alpha_n - \frac{1}{2}\right| \le \frac{K}{n^4} \implies \boxed{O(1/n^4)}$$

(c) $\lim_{n\to\infty} \sin(1/n^3) = 0.$

$$\sin(1/n^3) = \frac{1}{n^3} - \frac{1}{6n^9} + \cdots \implies |\alpha_n - 0| \le \frac{K}{n^3} \implies \boxed{O(1/n^3)}$$

(d) $\lim_{n\to\infty} [\sin(1/n^2)]^2 = 0.$

$\sin(1/n^2) = 1/n^2 - 1/(6n^6) + \cdots$ Squaring:

$$[\sin(1/n^2)]^2 = \frac{1}{n^4} - \frac{1}{3n^8} + \cdots \implies |\alpha_n - 0| \le \frac{K}{n^4} \implies \boxed{O(1/n^4)}$$

(e) $\lim_{n\to\infty} [\ln((n+1)^2) - \ln(n^2)] = 0.$

$$\ln((n+1)^2) - \ln(n^2) = 2\ln\left(1 + \frac{1}{n}\right) = \frac{2}{n} - \frac{1}{n^2} + \frac{2}{3n^3} - \cdots \implies |\alpha_n - 0| \le \frac{K}{n} \implies \boxed{O(1/n)}$$