Implementation of Quantum Verification of Matrix Products

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1 Introduction

Quantum computing has experienced a recent surge in popularity given the advancements in NISQ machines. Quantum algorithms are known to solve problems like factoring numbers and simulating natural systems more efficiently than a classical computer. Companies such as IBM, Google, and Rigetti have recently built quantum computers that allow researchers to run quantum algorithms on physical hardware. As a result, researchers have started to apply quantum algorithms to solve real-world problems such as computational genomics, machine learning, and molecule simulation. Given these developments, it has become important to quantify the **limits of today's quantum computers**, and **to evaluate the frameworks and languages** used to program them.

Several studies have experimentally evaluated quantum algorithms on quantum hardware. Mandviwalla et al. evaluated a 4-qubit implementation of Grover search on the IBM-Q quantum processor [1]. Similarly, Acasiete et al. evaluated quantum random walks on graphs with 8 to 16 vertices [2]. However, these studies do not quantify the largest possible input size and they do not investigate a heterogenous classical-quantum approach.

Similarly, extensive work has been carried out on developing quantum programming frameworks. IBM has created Qiskit, a Python framework that supports prototyping and executing quantum algorithms on simulators and quantum hardware. ORNL is currently developing QCOR, a heterogenous classical-quantum framework that aims to use quantum computers as accelerators akin to GPUs [3]. However, no substantial work has been done to evaluate the efficacy of these programming frameworks in developing quantum-based systems.

Our study aims to fill these gaps in the literature by **implementing the Quantum Verification of Matrix Products (VMP)** [4][5]. We selected this algorithm because it uses a combination of the Grover search [6], quantum random walk [7], and amplitude amplification [8] algorithms to solve the larger problem of VMP. Further, parts of the algorithm that can be executed efficiently on a classical computer, which will allow us to evaluate the heterogeneous classical-quantum model. Finally, this algorithm claims to improve upon the best known classical algorithm. Evaluating it will allow us to experimentally verify this claim.

This study implements the quantum verification of matrix products using the Qiskit framework. We evaluate this implementation using gate count, qubit count, circuit depth, and transpilation time metrics. Through this study, we hope to expand the existing suite of experimental quantum hardware evaluation, provide feedback to quantum framework authors, and suggest improvements to existing hardware.

2 Literature Review

Quantum computing places an emphasis on thinking how computation is performed physically, and achieves speedup over classical algorithms by using physical phenomenon like entanglement to perform computation. The main promise of the field is that it can offer a non-trivial speedup over classical computing. The two algorithms that are often brought up are Shor's algorithm and Grover search. Shor's algorithm provides an exponential speedup over classical algorithms for factoring numbers and is capable of cracking current RSA encryption. It achieves this by using the Quantum Fourier Transform (QFT), the quantum analogue of a Fourier transform, to perform phase estimation and order-finding [6]. Grover search provides a $O(\sqrt{N})$ time-complexity for searching over unstructured data, compared to the classical complexity of $O(N)$, where N is the size of the search space. This is carried out by performng multiple Grover iteration steps on the quantum state which constructively amplify states that correspond to search results [6].

There exist many more algorithms like superdense coding, quantum key distribution, and quantum simulation that have potential applications in scientific simulation, machine learning, and cryptography. However, most of these algorithms require a very large number of qubits to be of practical use. One of the major challenges in developing large-scale multi-qubit systems is error-correction and noise. Before we reach the holy grail of fault-tolerant quantum systems, the field is currently attempting to make use of Noisy Intermediate Quantum Computers (NISQ) to solve problems of important practical use. However, existing literature on quantum algorithms do not treat their physical implementations and do not offer resource estimates required to obtain reasonable results. Our study explores this domain by attempting to implement the quantum matrix product verification, measure resource estimates and how they scale on varying input sizes, and examine the limits of current quantum hardware and simulators.

There are two popular algorithms for quantum matrix product verification. The first algorithm, proposed by Ambainis, Buhrman, Høyer, Karpinski, and Kurur, uses amplitude amplification along with Grover search to look for a submatrix that doesn't satisfy the product [5]. This algorithm runs in $O(n^{\frac{7}{3}})$ time and improves upon the optimal classical bound provided by Freivalds [9]. The speedup is obtained because the algorithm makes use of interference to arrive at a result in a smaller number of iterations. However, metrics do not exist for the number of qubits required to implement the oracles for the quantum search algorithms used, and the resources required to carry out operations like multiplying sub-matrices. Further, little research has been done on evaluating the algorithm in a heterogenous classical-quantum setup where quantum computers are used to accelerate certain parts of the algorithm. There exists a 4-qubit physical implementation of Grover search on IBM's quantum processor [1]. This implementation tests IBM quantum computers on Grover's algorithm to investigate the impacts of different circuit and device attributes, and to highlight the current capabilities of the system. This study reports that current quantum computers are able to solve the search problem on very small data sets. This is similar to what our study intends to do, however, it does not investigate the practicality of running algorithms that use Grover search and does not comment on the composability of circuits and how it affects performance and results.

The second algorithm, proposed by Buhrman and Spalek, uses quantum random walks to speed up the verification process and runs in $O(n^{\frac{5}{3}})$ time [4]. Quantum random walks are analogous to classical walks, and have a number of applications in quantum programming tasks. For example, they are used in solving the element distinctness problem, in which the goal is to find if there exists a set of M non-distinct elements in a domain of N elements [7]. However, the papers describing these applications do not talk about implementations on quantum hardware, and restrict the analysis to theoretical concerns like correctness and time complexity. There have been attempts to run quantum random walks on quantum hardware. Balu et al. implemented an efficient physical realization of a quantum random walk using $log_2(N\ )$qubits to represent an N-point lattice [10]. Experimental evaluation was carried out on the IBM-Q five-qubit processor. To overcome resource requirements, they used a continuous time-limit quantum random walk implementation. Acasiete et al. have implemented discrete-time quantum random walks on IBM-Q, and were able to run quantum search based algorithms on graphs with 8 and 16 vertices [2]. They were able to obtain results with a 50% fidelity, and claim that the results are more efficient than equivalent classical algorithms.

There exists research on resource estimate quantification and benchmarking for some quantum algorithms. Jaques et al. implemented Grover oracles for key search on AES and LowMC encryption [11]. They lay out a formal description of the oracle, describe a reversible quantum-gate implementation of the AES encryption-decryption algorithm, and estimate the number of Clifford, T, and CNOT gates required for running circuits that can crack AES-128, AES-192, and AES-256. The project uses Q#, a quantum programming language developed by Microsoft. The project reduces the circuit depth of the Grover oracle by using internal parallelization, in which the Grover search instance is run on disjoint subsets of the input domain. However, the project does not address the largest size input that could be run on real-world quantum hardware, and they restrict their analysis to using the Q# simulator instead.

Borujeni et al. have evaluated the quantum Bayesian networks on the IBM-Q processor [12]. They set up a 4-node Bayesian network for stock prediction and run the circuit on IBM's nine quantum computers. As part of their study, they evaluate the performance of the circuits and the efficiency of the quantum gate transpilation process that maps a circuit graph to a quantum device. This study is in-line with what we aim to do with our research. However, we would also like to report on the maximum sized input we can run before we are constrained by hardware or we start getting results below a threshold.

A number of open-source frameworks exist for conducting quantum computing research. IBM provides the Qiskit framework which lets researchers quickly prototype and test algorithms on a simulator, and also run some workloads on a quantum computer, the biggest one being the IBM-Q 16-qubit processor in Melbourne. Fingerhuth et al. have compiled comparisons between Qiskit and other frameworks like Quil, XACC, and ScaffCC [13]. They comment on the programming language choice, documentation, license, and general culture around these communities. However, they do not compare these frameworks based on their performance and ability to execute on quantum hardware. LaRose has compared simulator performance and the quantum compiler of Qiskit and Rigetti [14]. However, the study does not report which algorithm was used during performance evaluation. Instead, it qualifies a benchmark based on the number of qubits used. ORNL is currently working on QCOR, which is a heterogenous framework that aims to enable developers to use quantum computers as accelerators, much like GPUs [3]. QCOR doesn't support amplitude amplification, quantum random walks, and basic circuits for performing arithmetic as of now. Support will need to be added to facilitate experimentation using the hybrid classical-quantum programming approach provided by this framework. Salm et al. has worked on a NISQ analyzer that determines the best quantum computer system to run a given workload based on the nature of the quantum algorithm [15]. They believe that this will improve developer experience by obviating the need to understand complicated mathematics to determine the best machine for running a particular quantum programming task. None of the frameworks currently have a working implementation of quantum verification of matrix products which we can use to perform benchmarking.

We believe that it is important to have estimates on how big of an input a concrete implementation of an algorithm can process. We can use such evaluation reports to gauge the current state of quantum computing and suggest areas which need more improvement. Further, we can provide valuable feedback to library authors about features that need to be added to facilitate productive quantum algorithm research. Our study contributes to the domain of experimental evaluation of quantum algorithms by attempting to implement the verification of matrix products algorithm in Qiskit. We gather resource estimates like the number of qubits, number of gates, and circuit depth.

3 Materials and Methods

This section will cover the main algorithms used in quantum VMP, implementation details, and experimental setup. This will help in understanding our design decisions and our analysis of it in 5.

3.1 Grover's algorithm

Grover's algorithm is a popular quantum search algorithm. Given an input space of N elements and an oracle U_f, Grover search can find M solution indices in $O(\sqrt{\frac{N}{M}})$ time. For simplicity, we assume that N is a power of 2.

For $M = 1$ Grover search runs in $O(\sqrt{N})$ time, which is a quadratic speedup over the classical algorithm for searching in an unstructured database which takes $O(N)$ time. Therefore, Grover search offers a significant speedup.

The algorithm works by repeatedly applying a Grover operator G to the initial state $H^{\otimes n} |0\rangle^{\otimes n}$:

$$(H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n})U_f = (2|\psi\rangle\langle\psi| - I)U_f \tag{1}$$

It consists of the oracle U_f and a phase shift operator $(2|\psi\rangle\langle\psi| - I)$ known as the diffuser. The specific characteristics of U_f are described in 3.1.1.

Each iteration can be geometrically viewed as a rotation of the state vector in a plane spanned by the uniform superposition of solutions and non-solutions. After the application of U_f, the diffuser rotates the state vector towards the superposition of solutions. The number of such iterations can be shown to be $O(\sqrt{\frac{N}{M}})$. Therefore, in order to use Grover's algorithm, one needs to know the exact number of solutions M in the search space.

The Grover operator circuit is summarized in Fig 1.

3.1.1 Grover Oracles

The oracle U_f used in Grover's algorithm can be viewed as black box that knows how to recognize solutions in the search problem. Let us say we are given a function f which takes an input $x \in N$ and returns 1 if x is a solution to the search problem, and 0 otherwise. Then, the action of U_f can be written as:

$$U_f |x\rangle \mapsto (-1)^{f(x)} |x\rangle \tag{2}$$

Note how the oracle only applies a phase shift to solutions of the search space.

The above oracle is commonly implemented by encoding f in a quantum circuit that flips a target qubit $|y\rangle$ for all inputs x that are a solution to the search problem. We obtain the phase shift by initializing the target qubit to the $|-\rangle$ state.

3.2 Amplitude Amplification

Amplitude amplification is a generalization of the Grover operator G. Instead of wrapping Hadamard gates H around the diffuser, we now use an arbitrary unitary U.

$$U(2|0\rangle\langle 0| - I)U^{\dagger}O \tag{3}$$

The oracle O behaves in the same way as described in 3.1.1. The unitary can be thought of as a quantum subroutine A that performs a series of quantum operations without making measurements.

3.3 Quantum Verification of Matrix Products

Given three square matrices A, B, and C of size n, the verification of matrix products (VMP) decides if $AB = C$. Freivalds describes a classical algorithm which can run in $O(n^2)$ time.

In this paper, we implement the recursive Grover search based quantum VMP by Ambainis et al. [5]. The algorithm proceeds by first partitioning B and C into submatrices B_i and C_i of size $n \times \sqrt{n}$ respectively. It is easy to observe that $AB = C$ iff $AB_i = C_i$ $\forall i$. Now, perform amplitude amplification over the following subroutine: pick a random vector x, classically compute $y = B_i x$ and $z = C_i x$, and verify the product $Ay = z$. The verification is done using a Grover search where the search space is the set of row indices and the oracle verifies if the inner product between the row and the vector matches the output.

The verification oracle takes $O(n)$ time. Therefore, each Grover iteration runs in $O(n^{\frac{3}{2}})$ time. We need to run $\sqrt{\frac{n}{\sqrt{n}}} = n^{\frac{1}{4}}$ iterations of amplitude amplification. Therefore, the overall running time of the algorithm is $O(n^{\frac{7}{4}})$.

The algorithm is summarized in Algorithm 1.

3.4 Implementation

We implement QVMP in the Qiskit programming framework.

Qiskit is a popular open-source quantum computing platform developed by IBM. Qiskit uses Python as the host

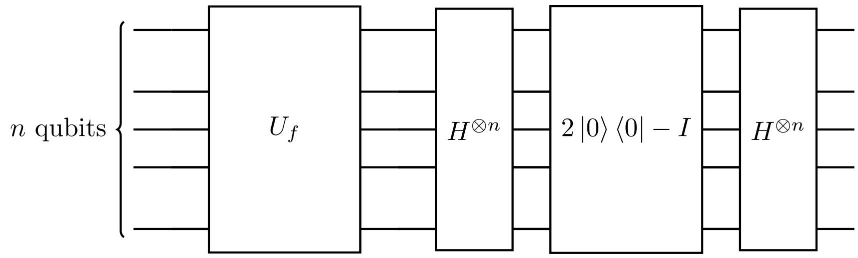


Figure 1: Grover operator circuit

Algorithm 1 Quantum VMP using Grover Search [16]

Input: $n \times n$ matrices A, B, C
Output: 1 if $AB = C$ and 0 otherwise
Procedure:

1. Partition B and C into sub-matrices of size $n \times \sqrt{n}$
2. Perform amplitude amplification for $n^{\frac{1}{4}}$ iterations using this subroutine:
   (a) Pick a random vector x of size $\sqrt{n}$
   (b) Classically compute $y = B_i x$ and $z = C_i x$
   (c) Using Grover search with $\sqrt{n}$ iterations, find a row of index j such that $(Ay! = z)_j$

language and does not restrict developers from using the various abstractions and constructs that Python provides. It lives as a DSL within Python. Qiskit has a large libary of quantum circuits and allows developers to compose different circuits together. Further, it provides a transpiler which can compile a quantum circuit into both simulator and real-device backends. Qiskit makes it easy to summarize circuit details like gate count, qubit count, and circuit depth.

We restrict our implementation to using binary matrices. The implementation, however, can be extended to support other number encodings. The difference lies in the number of qubits used.

3.5 Experimental Setup

We evaluate our implementation on a 13' MacBook Pro 2017 with 16GB RAM and a 2.5 GHz Dual-Core Intel COre i7 processor.

We focus on the following metrics for our implementation:

- Gate count
- Qubit count
- Circuit depth
- Average transpilation time

Using these metrics we will determine the practicality of using QVMP to solve real-world problems and some of the challenges associated with implementing Grover oracles.

4 Results

We report the gate counts, circuit depth, and number of qubits used when targeting Aer simulator in Tables 1 and 2. Further, we report transpilation times in Table 3. The sub-circuits used in constructing the QVMP marking oracle are summarized in figures 2 and 4. Figures 3 and 5 demonstrate the functionality of these auxiliary circuits. We show the marking oracle circuit for a 4x4 matrix in Fig 6.

5 Analysis

The QVMP oracle is a blackbox that checks if $(Ay - z)_i \neq 0$, where A, y, z, and i are as defined in the Alg 1. Qiskit does not offer automatic compilation of programming constructs like classical arrays to a quantum circuit. The programmer has to define these operations on a per-problem specific basis. The QVMP oracle is composed of the following two sub-circuits: the indexer circuit and the inner product circuit.

5.1 Indexer Circuit

The role of the indexer is to output $A[i]$, where A is an $m \times n$ matrix and i is an integer index. We encode the matrix A using $size(A)$ qubits, where $size(A)$ is the total number of elements in A. The integer index is encoded using $log_2(m)$ qubits in the binary format.

If we observe the functionality of the indexer closely, we see that it is isomorphic to a multiplexer. In other words, the indexer selects a specific row given a selector (which is the index in our case). We can implement the indexing operation either as an in-place (Fig 2) or out-of-place operation ([17]). The in-place solution requires more qubits, while the out-of-place solution increases the circuit depth and is a little more tricky to integrate with the rest of the oracle. For simplicity, we decided to go with the in-place version.

Fig 3 shows the indexer circuit in action. When the index qubits are not in a superposition state, we measure $A[i]$ with 100% probability. On the other hand, if the index bits are in a superposition, we measure multiple rows of A depending on the superposition. The index superposition lets us index multiple rows of A at a time, but we can only observe one of them when measured.

An interesting point to note is that unlike a classical mul-

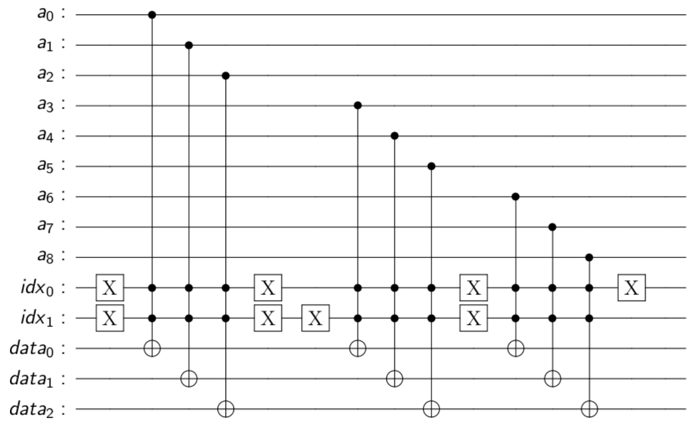
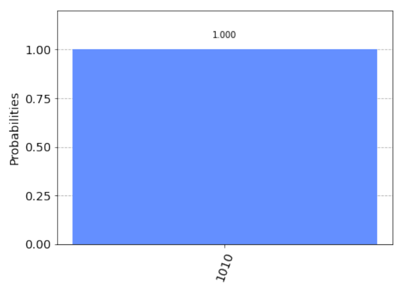
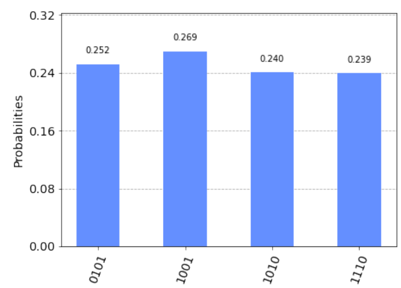


Figure 2: Indexer Circuit for a 3×3 matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



(a) Indexing the third row of A



(b) Indexing A using a superposition of the index

Figure 3: Indexer circuit functionality

tiplexer which checks for the selector bits in parallel, the quantum indexer has to check every index permutation sequentially. This pattern can be seen in the way the X gates are applied to the index qubits before applying the Tofolli gate.

5.2 Inner Product Circuit

The inner product circuit 4, as the name suggests, computes the inner product between two vectors encoded as qubits. Our specific implementation only handles binary numbers (where multiplication is defined as logical AND and addition as logical OR), but can be easily extended to support other data types.

The AND operation is done using a Toffoli gate. The OR gate is then trivially implemented using De-Morgan's law. Both of these operations are out-of-place.

5.3 QVMP Oracle

The marking oracle (Fig 6) is composed of both the indexer and the inner product circuits. We first index into A and z outputting their results into ancilla qubits. Then we compute the inner product $A_i y$. Finally, we compare the inner product with z_i. The comparator circuit is a $CNOT$ with an additional X gate applied to the target qubit.

As mentioned in Section 3.1.1, Grover's algorithm needs an oracle which performs a phase-flip on correct solutions. We can easily convert the marking oracle to a phase-flip oracle by simply sandwiching the marker qubit with XH and its inverse. This has the effect of putting the marker qubit in the $|-\rangle$ state.

Another important point to note is that the oracle undoes the operations it carries out in the workspace qubits. This is required so that the next iteration of Grover can start with a clean workspace. Qiskit simplifies the process of obtaining the inverse circuits. Every `QuantumCircuit` object has an inverse method which yields the corresponding circuit.

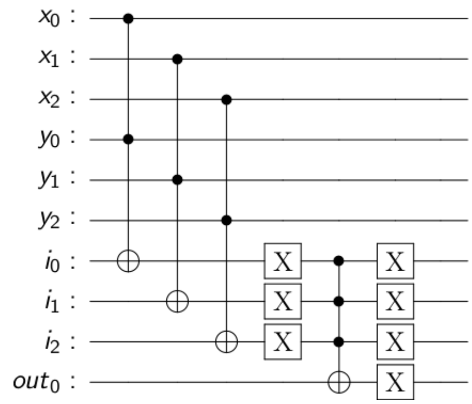
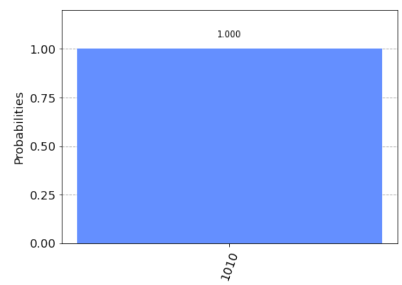


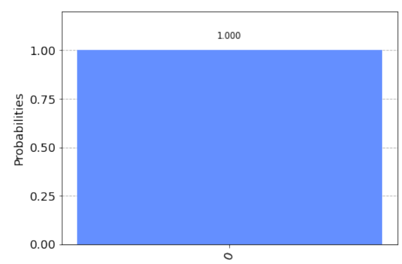
Figure 4: Inner Product Circuit for 3×1 vectors

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



(a) Inner product circuit outputting 1

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



(b) Inner product circuit outputting 0

Figure 5: Inner product circuit functionality

5.4 Overall Circuit

To complete the circuit, we need to apply Grover's search followed by amplitude amplification. Qiskit provides the `AmplificationProblem` routine which lets you describe a search problem in the form of an oracle, initial state preparation, and good states. We use this routine to build the necessary circuits. For the Grover sub-circuit G_i, the state preparation is set to a Hadamard on the objective qubits. For amplitude amplification, we pass G_i as the state preparation parameter.

Figures 1 and 2 show metrics associated with gate count, qubit count, and circuit depth for the oracle transpilation and overall circuit transpilation respectively. After transpilation, the Qiskit compiler decomposes higher-level gates to match the gate set supported by the backend. We see that the qubit count rises to more than 1000 for even small matrices of order 32. Further, transpilation onto the Aer backend added gates that we not part of the high-level design (like U_2 and U_3). This results in a large increase in the number of quantum operations performed. The transpilation time rises exponentially with an increase in the order of the matrix (reaching slightly more than a minute for an order 24 matrix), thereby slowing the development process.

5.5 Optimization Opportunities

The implementation of the QVMP oracle can be further optimized. Note that the matrix A is not used for the rest of the oracle after the index operation. This means that we can use A as the oracle workspace to finish the rest of the operations. Since quantum gates are reversible, we will be able to get back A by applying the reverse operations. This will reduce the number of qubits required at the cost of increasing circuit depth.

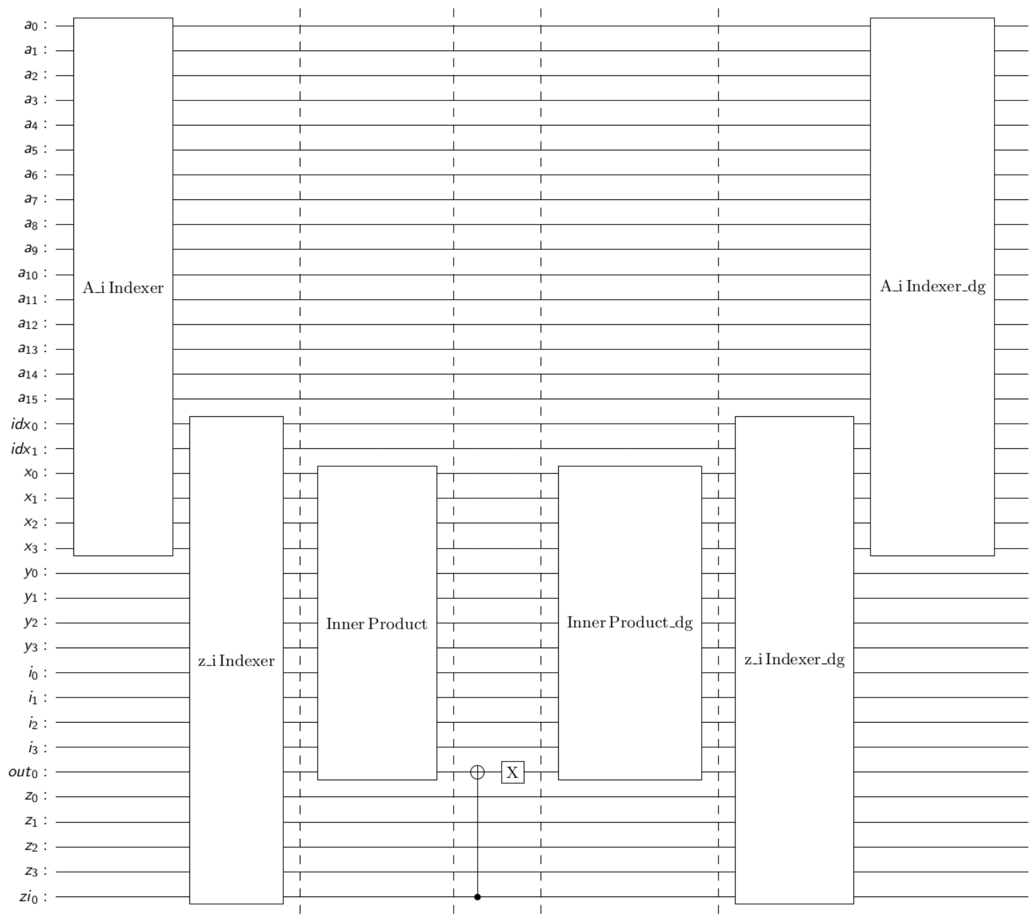


Figure 6: QVMP Marking Oracle for a 4×4 matrix

Order	ccx	cx	mcx	mcxgray	x	Total Gates	Circuit Depth	Qubit Count
2	18	1	0	0	13	32	21	15
4	8	1	42		33	84	57	36
8	16	1	144	2	73	236	177	101
16	32	1	0	546	153	732	609	326
24	48	1	0	1202	235	1486	1297	679
32	64	1	0	2114	313	2492	2241	1159
64	128	1	0	8322	633	9084	8577	4360

Table 1: VMP Marking Oracle Stats

Order	ccx	cx	h	mcx	mcxgray	u1	u2	u3	x	Total Gates	Circuit Depth	Qubit Count
2	72	4	13		4	9	116	2	49	269	94	15
4	56	7	33	294	7	15	495	5	219	1131	415	36
8	112	7	97	1008	21	15	1406	5	493	3164	1255	101
16	704	22	321		12034	45	14309	19	3286	30740	13445	326
24	1056	22	673		26466	45	29842	19	5070	63193	28581	679
32	1728	27	1153		57105	55	62542	24	8326	130960	60564	1159

Table 2: VMP Circuit Stats

Order	Time (in seconds)
2	0.24860209489997942s
4	0.8189974269000686s
8	2.3804883372999486s
16	27.484900246300004s
24	66.20581728570009s

Table 3: VMP Circuit Transpilation Stats

5.6 Challenges

Implementing Grover oracles in Qiskit is cumbersome and error-prone. While the library offers some routines for automating oracle synthesis, the tooling is only limited to integers and boolean operators. This makes it challenging to implement more complex oracles that have succinct classical descriptions. Grover oracles typically try to emulate a classical operation, and the task of implementing the oracle amounts to writing the equivalent quantum circuits. We can develop tooling that takes a classical description of a Grover oracle and automatically synthesize that into an quantum circuit. Qiskit offers the `@classical_function` decorator to do this, but as mentioned above, it is only limited to integers. We believe that there is scope to extend the synthesis to higher-level data types (like arrays and structs) and operators (like indexing). This will boost developer productivity and make it easier to iterate on quantum search problems.

Another challenge revolves around verification of correctness. Due to the non-deterministic nature of quantum computing, it is more nuanced and challenging to verify outputs. Simulators help alleviate this problem, but they do not scale very well. For the QVMP case, as highlighted before, compilation of a circuit for an order 32 square matrix is quite slow. Further, the Aer simulator can only run circuits using less than 32 qubits. This inhibits the practicality of algorithms like QVMP.

As of this writing, NISQ machines support around 50-100 qubits. From Fig 2, this means that we are only able to run QVMP for matrices of order 4 to 8. The speedup of the quantum algorithm shows up on matrices of much larger size. Quantum noise makes it harder to put QVMP to practical use. Our findings show that NISQ machines and simulators do not have the resources to support solving a VMP problem of sufficient size.

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