Lennart Edsberg

Numerisk Analys, Matematik, KTH

Lab 1 (CL1-GA)

Numerical experiments with the Harmonic Oscillator

This lab involves an investigation of the three numerical methods 1) Symplectic Euler (SE), 2) Implicit Midpoint Method (IMM) and 3) Verlet's method (VM) applied to the ODE-system corresponding to the Harmonic Oscillator. Follow the guidelines below and show for each method 1) solution trajetories, 2) phase portrait and 3) energy trajectory.

As a start, download the file `lab1demo.m` from Canvas and run the MATLAB-program. After each graph shown in the graphical window there is a pause in the program. To continue running the program go to the command window, press the return button and look at the next graph and so on until the program has come to an end. If you make a copy of this file you can have that as a starting point for this Lab by making modifications to a version where the three methods SE, IMM and VM are programmed.

The Harmonic Oscillator is an ODE

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad x(0) = 1, \frac{dx}{dt}(0) = 0, \tag{1}$$

where initial values have been given and where ω is a constant parameter. In this lab you test your program with the ω-value $\omega = 2$ (just as in the demoprogram). The variable x denotes the position and its derivative $\dot{x}$ the velocity of the oscillating particle.

The analytic (exact) solutions of (1) and its derivative are

$$x(t) = cos(\omega t), \quad \frac{dx}{dt}(t) = -\omega sin(\omega t) \tag{2}$$

With the oscillating particle there is also an expression for the total energy, being the sum of the kinetic and the potential energy, also called the Hamiltonian function $H(t)$

$$H(t) = \dot{x}(t)^2 + \omega^2 x(t)^2, \qquad (\dot{x}(t) = \frac{dx}{dt}(t)) \tag{3}$$

By inserting (2) into (3) we obtain

$$H(t) = \omega^2$$

hence the total energy is preserved as a function of time. In the `lab1demo` you first see a graph of the solution $x(t), \dot{x}(t)$ as functions of t (called trajectories), then a graph of the phase portrait, i.e. $\dot{x}(t)$ as a function of $x(t)$ (in this case an ellipse) and finally $H(t)$ as a function of t.

When you continue to run the `lab1demo` you see the result of using Explicit and Implicit Euler on the Harmonic Oscillator. Neither of the two methods give acceptable approximations of the trajectories, the phase portrait or the energy preservation. We have now come to *your task* of this lab, namely to investigate how the three methods Symplectic Euler, Implicit Midpoint Method and (NOT Compulsary) Verlet's method work on this problem.

For SE and IMM it is necessary to first rewrite (1) as a system of two first order ODEs by defining two help variables $u_1(t) = x(t), u_2(t) = \dot{x}(t)$:
On *component form* the ODE-system can be formulated as:

$$\dot{u}_1 = u_2, \quad u_1(0) = 1$$

$$\dot{u}_2 = -\omega^2 u_1, \quad u_2(0) = 0 \tag{4}$$

As an alternative the system can be formulated on *vector form*:

$$\dot{\mathbf{u}} = A\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}_0 \tag{5}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad \mathbf{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

1) (Compulsary) Symplectic Euler is formulated from the component form:

$$u_{1,k} = u_{1,k-1} + hu_{2,k-1}, \quad u_{1,0} = 1$$
$$u_{2,k} = u_{2,k-1} - h\omega^2 u_{1,k}, \quad u_{2,0} = 0 \tag{7}$$

where $k = 1, 2, \ldots, N$. In your lab use $h = 0.1$, $N = 100$, the same values as in the demolab. Plot graphs of trajectories, phase portrait and energy as function of time. Notice the behaviour of the phase portrait and the energy!

2) (Compulsary) The Implicit Midpoint Method formulated for the vector form is

$$\mathbf{u}_k = \mathbf{u}_{k-1} + hA(\frac{\mathbf{u}_k + \mathbf{u}_{k-1}}{2}), \tag{8}$$

where $k = 1, 2, \ldots, N$, $h = 0.1$ and $N = 100$. Plot the same graphs as in 1).

3) (NOT compulsary) Verlet's method is formulated from the ODE formulation

$$\frac{d^2u_1}{dt^2} + \omega^2 u_1 = 0$$
$$u_2 = \frac{du_1}{dt}$$

using central difference approximations of the derivatives:

$$u_{1,k+1} = 2u_{1,k} - u_{1,k-1} - h^2\omega^2 u_{1,k}, \quad u_{1,0} = 1$$
$$u_{2,k} = \frac{(u_{1,k+1} - u_{1,k-1})}{2h}, \quad u_{2,0} = 0$$

To start the timestepping we here need one more initial value:

$$u_{1,1} = u_{1,0} + hu_{2,0} - \frac{h^2}{2}\omega^2 u_{1,0}$$

where $k = 1, 2, \ldots, N - 1$, $h = 0.1$ and $N = 100$. Plot the same graphs as in 1).