KTH-CBH

SIMULATION METHODS IN MEDICAL ENGINEERING (CM2014)

LABORATORY REPORT

FEA LAB

GROUP 18

FEBRUARY 23, 2024

GROUP MEMBERS:
YIZHI DONG YIHAN XIAO

DATE PERFORMED:
FEB , 2024

Contents

1 Introduction of Question

In this laboratory, the task to solve is the determination of the a fishing rod's deflection when a fish is hooked, using Finite Element Method together with Matlab. The rod will be simplified as a beam.

The fishing rod is $2\ \mathrm{m}$ with a radius of $5\ \mathrm{cm}$. The cross-section of the fishing rod is circular with inertia (I) of $\frac{\pi r^4}{4}$. The Young's modulus (E) is assumed to be $100\mathrm{MPa}$.

For the objectives, the undeformed and deformed rod will be compared and the deflection of each node will be calculated [1]. The version of Matlab used in this lab is Matlab 2022b.

2 Background

2.1 General Beam Unit Analysis

The analysis begins with the force analysis of one beam. The points analyzed are the two ends of the beam (two nodes). The schematic diagram is shown in figure 1[1].

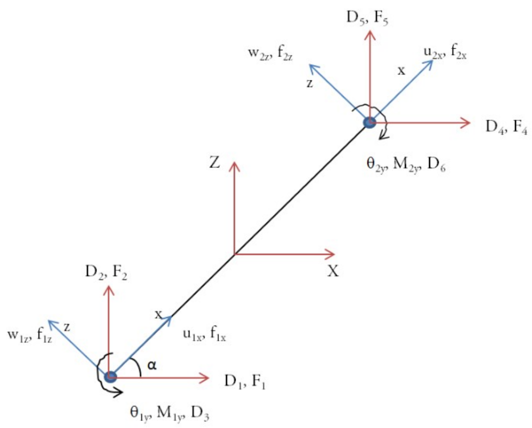

Figure 1: Variables for one beam element

According to the diagram, derived from Hooke's law, the force equilibrium is shown as below:

$$f_{1x} = \frac{EA}{L}u_{1x} - \frac{EA}{L}u_{2x} \tag{1}$$
$$f_{1z} = \frac{12EI}{L^3}w_{1z} + \frac{6EI}{L^2}\theta_{1y} - \frac{12EI}{L^3}w_{2z} + \frac{6EI}{L^2}\theta_{2y} \tag{2}$$
$$M_{1y} = \frac{6EI}{L^3}w_{1z} + \frac{4EI}{L}\theta_{1y} - \frac{6EI}{L^2}w_{2z} + \frac{2EI}{L}\theta_{2y} \tag{3}$$
$$f_{2x} = -\frac{EA}{L}u_{1x} + \frac{EA}{L}u_{2x} \tag{4}$$
$$f_{2z} = -\frac{12EI}{L^3}w_{1z} - \frac{6EI}{L^2}\theta_{1y} + \frac{12EI}{L^3}w_{2z} - \frac{6EI}{L^2}\theta_{2y} \tag{5}$$
$$M_{1y} = \frac{6EI}{L^3}w_{1z} + \frac{2EI}{L}\theta_{1y} - \frac{6EI}{L^2}w_{2z} + \frac{4EI}{L}\theta_{2y} \tag{6}$$

From the equations, the stiffness matrix for one element is derived:

$$k_e = \begin{pmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix}$$

where E is the Young's Modulus, A is the area of the cross section of the rod, I is the rotation Inertia of the rod, and L is the length of each beam unit. And the relationship between the stiffness matrix and deflection d_e is $k_e d_e = f_e$, where f_e is force vector in local coordinate vector.

To make this element in the global coordinate system, a transformation matrix T is needed. In figure 1, the relationship of the local coordinate system and global coordinate system is shown in the equation below:

$$u_{1x} = D_1 \cos\alpha + D_2 \cos\beta = \left\{ \cos\beta = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha \right\} = D_1 \cos\alpha + D_2 \sin\alpha \tag{7}$$

$$w_{1z} = -D_1 \sin\alpha + D_2 \cos\alpha \tag{8}$$

$$\theta_{1y} = D_3 \tag{9}$$

D_e is the deflection in global coordinate system. According to the equations, the transformation matrix T is defined as:

$$T = \begin{pmatrix} cos\alpha & sin\alpha & 0 & 0 & 0 & 0 \\ -sin\alpha & cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos\alpha & sin\alpha & 0 \\ 0 & 0 & 0 & -sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

So, the transformation equation should be $d_e T = D_e$. And it's the same with the force, $f_e T = F_e$. With these equations, the deflection of one element can be solved in the global coordinate system.

$$F_e = T^{-1} f_e \tag{10}$$

$$\rightarrow F_e = k_e T D_e \tag{11}$$

$$\rightarrow F_e = T^{-1} k_e T D_e = K_e D_e \tag{12}$$

From Equation (12), the global stiffness matrix K_e can be represented as $T^{-1} k_e T$.

2.2 Finite Element Analysis

The beam is divide into 4 elements (beam with same length) for analysis in this lab. The visualization and angles are shown in figure 2[1].

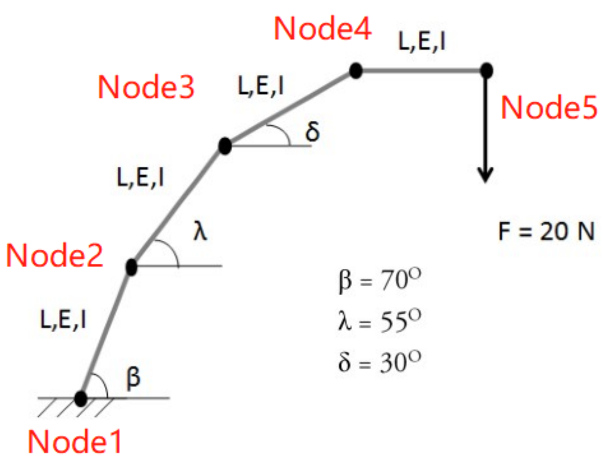

Figure 2: Divided beam with angles

For the nth element, the transformation to global coordinate system can be expressed as below:

$$K_{en} = T_n^{-1} \cdot k_{en} \cdot T_n$$

For a single beam unit, the matrix and the vectors have the relationship as followings (K_{en} is a 6x6 matrix, and $K_{an}, K_{bn}, K_{cn}, K_{dn}$ are all 3x3 matrix):

$$K_{en} = \begin{pmatrix} K_{an} & K_{bn} \\ K_{cn} & K_{dn} \end{pmatrix}$$

And the global stiffness matrix is to add the 4 K_{en}.

$$\begin{pmatrix} K_{a1} & K_{b1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ K_{c1} & K_{d1}+K_{a2} & K_{b2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & K_{c2} & K_{d2}+K_{a3} & K_{b3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & K_{c3} & K_{d3}+K_{a4} & K_{b4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & K_{c4} & K_{d4} \end{pmatrix}$$

Because the first node is fixed, the first row and column of K_e can be ignored (turned to 0 or deleted). So the final equation can be expressed as:

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & K_{d1}+K_{a2} & K_{b2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & K_{c2} & K_{d2}+K_{a3} & K_{b3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & K_{c3} & K_{d3}+K_{a4} & K_{b4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & K_{c4} & K_{d4} \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix}$$

D_n is global deformation 3x1 vector $(d_x, d_z, d_\theta)^T$ for the nth node. Similarly, F_n is global force 3x1 vectors of each node of a beam unit for Node n. F_2, F_3 and F_4 are all zero vector, because they aren't exerted any external force, and F_5 is $(0, -20, 0)$.

3 Method

The core of this Finite Element Analysis is based on the formula derived in background part:

$$K_e \cdot D_e = F_e$$

$$D_e = K_e^{-1} \cdot F_e$$

Since the deformation vector D_e is wanted from simulation, establishing the global stiffness matrix K_e is needed, as well as finding the correct boundary condition to fill global force vector F_e.

3.1 Construct local stiffness matrices

Since the fish rod is simplified as a beam with four beam unit, in order to obtain the global stiffness matrix K_e, we can first construct local stiffness matrix k_e for each beam unit. The stiffness matrix is calculated in the general beam unit analysis above.

Constants (E, A, I, L, etc.) and the local stiffness matrix are calculated with the MATLAB code below.

```
% Radius
r = 0.05; % m
% Young's Modulus
E = 100e6; % Pa
% Rotational Inertia
I = pi * r^4 / 4; % kg/m^2
% Cross-section area
A = pi * r^2; % m^2
```

```
% Length of the rod
L = 2; % m
% Length of each segment
l = L / 4;

% Local Stiffness matrix
k_ul = blkdiag(E*A/l, [12*E*I/l^3 6*E*I/l^2; 6*E*I/l^2 4*E*I/l]);
k_ur = blkdiag(-E*A/l, [-12*E*I/l^3 6*E*I/l^2; -6*E*I/l^2 2*E*I/l]);
k_ll = blkdiag(-E*A/l, [-12*E*I/l^3 -6*E*I/l^2; 6*E*I/l^2 2*E*I/l]);
k_lr = blkdiag(E*A/l, [12*E*I/l^3 -6*E*I/l^2; -6*E*I/l^2 4*E*I/l]);
k_e = [k_ul k_ur; k_ll k_lr];
```

3.2 Construct transformation matrix

Next step is to create a transformation matrix to transform local stiffness matrix k_e to global stiffness matrix K_e. The transformation matrix is defined in the background part, and the angle α is the degree of each beam unit in global coordinate system.

The transformation matrix is calculated with the MATLAB codes below:

```
% Initial Angels
beta = deg2rad(70); % rad
lambda = deg2rad(55); % rad
delta = deg2rad(30); % rad

% Transformation matrix
Angle = [beta, lambda, delta, 0];
T = axisT(Angle(i+1));

function T_mat = axisT(angle)
    axisMat = [cos(angle) sin(angle); -sin(angle) cos(angle)];
    T_mat = blkdiag(axisMat, 1, axisMat, 1);
end
```

3.3 Calculate global stiffness matrix

For each local force vector and global force vector, four K_e is transformed to global coordinate system for 4 different angles. The conversion can be done with the formula below:

$$K_{en} = T_n^{-1} \cdot k_{en} \cdot T_n$$

Since local stiffness matrix k_e and the transformation matrix T are known, the global stiffness matrix K_e ($K_{e1} + K_{e2} + K_{e3} + K_{e4}$) can be obtained by with the MATLAB codes below:

```
Angle = [beta, lambda, delta, 0];

for i = 0:3
    % Transformation matrix
    T = axisT(Angle(i+1));
    % Axis transform
    K_e = inv(T)*k_e*T;
    % Global stiffness matrix
    K = K + blkdiag(zeros(i*3), K_e, zeros((3-i)*3));
end
```

3.4 Calculating the global deformation matrix

After having the transformation matrix, the force can also be transformed to the global coordinate system. Also, because the first node is fixed, the related columns and rows in the matrix and vectors can be ignored. To get the deflection, in Matlab the this step is very simple.

```
% Because the first point is fixed
K_e(:,1:3) = 0;
K_e(1:3,:) = 0;

% Global force vector
F_e = [0; 0; zeros(11, 1); -20; 0];
% The only load is the vertical force applied to node 5

% Solve Global deformation matrix
D_e = K_e\F_e
```

4 Results

The result is based on the calculation of D_e and nodes' coordinates in the Method part. The following table (table 1) shows the original and changed coordinates of nodes, as well as the deflections. The solved vector D_e is $(NaN, NaN, NaN, 0.0064, -0.0023, -0.0266, 0.0220, -0.0133, -0.0485, 0.0361, -0.0377, -0.0631, 0.0361, -0.0710, -0.0682)$.

Node	Original Coordinates [m]	Deformed Coordinates [m]	Deflection [m]	d_θ [rad]
1	(0, 0)	(0, 0)	(0, 0)	0
2	(0.1710, 0.4698)	(0.1774, 0.4675)	(0.0064, -0.0023)	-0.0266
3	(0.4578, 0.8794)	(0.4798, 0.8662)	(0.0220, -0.0133)	-0.0485
4	(0.8908, 1.1294)	(0.9269, 1.0917)	(0.0361, -0.0377)	-0.0631
5	(1.3908, 1.1294)	(1.4269, 1.0584)	(0.0361, -0.0710)	-0.0682

Table 1: Results of the script

A plot of the undeformed and deformed fishing rod is shown in figure 3, and the version with numerical coordinates is also demonstrated.

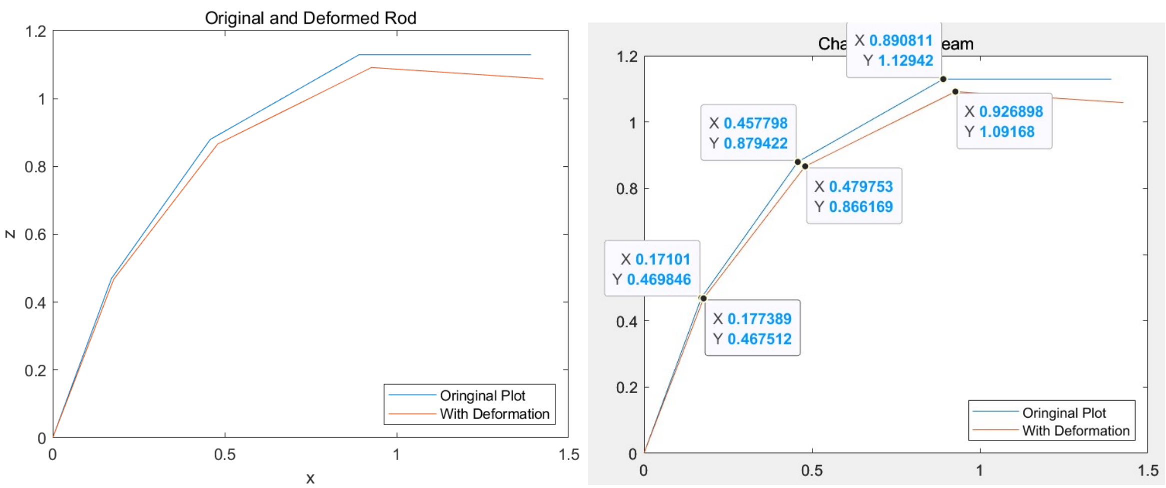

Figure 3: Plot of the undeformed and deformed fishing rod

According to the plots, except the first node, the deformed fishing rod is below the original rod, because the force of the fish on the fifth node. And the deflection increases when node is approaching where the force is exerted.

The code for calculating the coordinates and plotting the figure is shown as below:

```
%% Plot

%Initialization of the coordinates vector
X_0 = [0]; Z_0 = [0];
x_i = 0; z_i = 0;

%calculate the original coordinates
for i = 1:4
    x_i = x_i + l*cos(Angle(i));
    z_i = z_i + l*sin(Angle(i));
    X_0 = [X_0, x_i];
    Z_0 = [Z_0, z_i];
end

%calculate the deformed coordinates
X_1 = X_0; Z_1 = Z_0;

for i = 1:4
    X_1(i+1) = X_0(i+1) + D_e(3*i+1);
    Z_1(i+1) = Z_0(i+1) + D_e(3*i+2);
end

%plot the comparison
plot(X_0, Z_0)
hold on
plot(X_1, Z_1)
title("Change of the Beam")
legend(["Oringinal Plot", "With Deformation"], 'Location', 'southeast')
```

5 Discussion

From the plot of results, one impressive point is that the rod is not that smooth, because the rod is only divided into 4 elements. If it's divided into more elements, the rod may deform more naturally, closed to real case as a smooth curve. But this may cause the increase cost of performance and time of calculation.

However, the fishing rod model is still greatly simplified. In a real-world situation the material and cross-sectional area of each node would be uneven, and the angle at each node needs to be obtained rather than being defined in this simulation. Moreover, the Young's Modulus is not constant for real material, and the fishing rod can break when the load is too heavy. Also, the gravity forces applied to the rod need to be considered, so the problem will become much more complicated.

6 Conclusion

In this lab, the analysis of general beam unit is discussed, and the basic steps of Finite Element Analysis are applied to the fishing rod example, and the deformation of the fishing rod is successfully obtained. However, this is only a simple demo of FEA, for real-world problems the simulation would be more complicated.

References

[1] Lindgren, Natalia. *CM2014 Finite Element Analysis Computer Lab Instructions*. 2024.

[2] Lindgren, Natalia. *CM2014: FEA Computer Laboratory* Lecture Note.