15.094 Homework 1

Due: 03/03/2021

1 Robust Reformulation

Consider the following constraint on x, where z is an uncertain parameter:

$$(2+z)x \leq 1.$$

This constraint is equivalent to the following formulation, when there is no uncertainty in z.

$$(2+z)x + s = 1,$$
$$s \geq 0.$$

Now think of the robust counterparts of these two sets of constraints. The uncertainty set is the interval $\{z : |z| \leq 1\}$.

1.1

Find the feasible sets of the above formulations. Are they equal? What can you say about robust formulation of constraints?

Solution: Feasible set of the first formulation is $\{x : x \leq \frac{1}{3}\}$. Feasible set of the second formulation is $\{x : x = 0\}$. This shows that even if the deterministic versions are equal, robust reformulations may not be.

1.2

Now, consider the following reformulation:

$$(2+z)x + s(z) = 1, \quad \forall z : |z| \leq 1,$$
$$s(z) \geq 0, \quad \forall z : |z| \leq 1.$$

Here the difference is that the slack variable introduced is an arbitrary function of uncertain parameter z. Find the feasible set, and compare it with the previous result.

Solution: Since $s(z)$ is now a function of z, we can plug $s(z) = 1-(2+z)x$ into the nonnegativity constraint, thereby getting $1-(2+z)x \geq 0, \quad \forall z : |z| \leq 1$. Now this constraint is equivalent to the first formulation in 1.1

2 MI/LP Modeling

Model the following using LP/MILP compatible big-M constraints. There is no need to indicate values for M, but please indicate the subset of real numbers to which variables belong (eg. integers $\mathbb{Z}$ etc.).

Note that the solutions are only one way to represent the constraints. There may be others.

2.1

$|2x_1 - 3x_2| \geq 6$

Solution: $z \in \{0, 1\},\ 2x_1 - 3x_2 + Mz \geq 6,\ 2x_1 - 3x_2 \leq 6 + M(1 - z)$.

2.2

$|x_1 + 2x_2| \leq 4$ (Do not use integer variables here.)

Solution: $x_1 + 2x_2 \leq 4, -x_1 - 2x_2 \leq 4$.

2.3

x_1 and x_2 are integer variables. If $x_1 \leq 4$, $x_2 \geq 5$.

Solution: $x_1, x_2 \in \mathbb{Z},\ z \in \{0, 1\},\ Mz \geq 4 - x_1,\ x_2 \geq 5 + Mz$.

2.4

x_1, x_2 are continuous. Either $x_1 + 4x_2 \leq 7$ or $2x_1 + 3x_2 \geq 4$ are satisfied, but not both.

Solution:

$$
\begin{aligned}
z &\in \{0, 1\} \\
x_1 + 4x_2 &\leq 7 + Mz \\
x_1 + 4x_2 + M(1 - z) &\geq 7 \\
2x_1 + 3x_2 + M(1 - z) &\geq 4 \\
2x_1 + 3x_2 &\leq 4 + Mz
\end{aligned}
$$

2.5

x is integer, but not equal to 3.

Solution: $x \in \mathbb{Z},\ z \in \{0, 1\},\ x + Mz \geq 4 + \epsilon,\ x\ \ \leq 2 + M(1 - z)$.

3 Computational Tools

3.1 Writing a robust knapsack problem.

Using your newly acquired knowledge of JuMPeR, please write a robust version of the binary knapsack problem under ellipsoidal uncertainty.

$$\begin{aligned} \text{maximize } & \mathbf{c}'\mathbf{x} \\ \text{s.t. } & (\mathbf{a}+\tilde{\mathbf{a}})'\mathbf{x} \le b, \ \forall\ \tilde{a}: \ ||\tilde{a}||_2 \le \rho \\ & x_i \in \{0,1\}, \ i = 1,\ldots,n \end{aligned}$$

Please use $n = 20,\ b = 4$, and randomly generated $a_i,\ c_i \in [0,1]$. Include your code, as well as a plot of the optima for a relevant range of ρ. Assuming you were using this problem to maximize the revenue of a cargo aircraft subject to weight capacity, please use your intuition and knowledge of Gaussian probabilities to suggest a ρ that provides a good tradeoff between robustness and optimality.

Solution: Please check out the example code, in `HW1_RobustKnapsack.ipynb`.

As far as the value of ρ, we wanted to see intuition about what the uncertainty in weight comes from. For example, we could say that our uncertainty comes purely from measurement error. If our scale is accurate (zero-mean error) but has error with standard deviation 0.01, it is reasonable to assume a relatively low ρ. An ellipsoid with $\rho = 0.045$, while small, contains the case where all 20 a's are perturbed by at most one standard deviation, 0.01. It would be a reasonable choice.

However, if we suspect the uncertainties come from suppliers under-reporting weights, then we may have to be more conservative. In future lectures, we will cover how we may use data to infer uncertainty sets to be used in RO.