Optimal Aircraft Design Decisions under Uncertainty via Robust Signomial Programming

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Aircraft design benefits greatly from optimization under uncertainty, since design feasibility and performance can have large sensitivities to uncertain parameters. The traditional, mathematically non-rigorous methods of capturing uncertainty do not adequately explain the trade-offs between feasibility and optimality, and require prior engineering knowledge which may not be available for novel aerospace vehicle concepts. Signomial programs (SPs) are difference-of-convex extensions of geometric programs (GPs), and have demonstrated potential in the solution of multidisciplinary non-convex optimization problems such as aircraft design [1]. The formulation and solution of robust signomial programs (RSPs) would be beneficial since it would allow for conceptual engineering design that captures parametric uncertainty. This paper proposes an approximate solution method for an RSP leveraging an existing approximate robust geometric programming (RGP) formulation developed by Saab [2]. The method is based on solving a sequence of RGPs, where each RGP is a local approximation of the Robust Signomial Program (RSP). The paper then discusses the trade-off between robustness and optimality in aircraft design by implementing RSPs on a simple aircraft problem, and demonstrates how robustness requirements affect aircraft design decisions.

Nomenclature

CEG	Convex Engineering Group
GP	geometric program
LHS	left hand side
MDO	multidisciplinary design optimization
NLP	nonlinear program
SP	signomial program
RGP	robust geometric program
RHS	right hand side
RO	robust optimization
RSP	robust signomial program
SO	stochastic optimization

I. Introduction

Aircraft design exists in a niche of design problems where "failure is not an option"[a]. This is remarkable since aircraft design problems are rife with uncertainty about technological capabilities, environmental factors, manufacturing quality and the state of markets and regulatory agencies. Optimization under uncertainty for aircraft presents low hanging fruit, since the program risk of aircraft design problems is high and

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[a]Quoting Gene Kranz, the mission director of Apollo 13.

the goal of these methods is to be able to provide designs that are robust to realizations of uncertainty in the real world.

Zang et al.[3] succinctly describe the categories of benefits for optimization under uncertainty for aircraft. These are the following:

- *Confidence in analysis tools will increase.* The uptake of new design tools in the aerospace industry has been low due to heavy reliance on legacy design methods and prior experience when faced with risky design propositions, even in the design of novel configurations where the understanding of the design tradespaces is lacking. Robustness will increase confidence in analysis tools because of its ability to better capture the effects of technological uncertainty on the potential benefits of new configurations.

- *Design cycle time, cost, and risk will be reduced.* Design cycle costs as well as the engineering hours per aircraft have been increasing [4]. Aircraft design and development is costly, so the ability to handle uncertainty in the conceptual design process is critical for the long-term success of an aircraft, helping reduce the program risk.

- *System performance will increase while ensuring that reliability requirements are met.* The effectiveness of an aircraft depends heavily on its ability to deliver on performance, which is dependent on assumptions about the current technological environment and the ability to produce vehicles of a certain quality.

- *Designs will be more robust.* The ability to provide designs with feasibility and performance guarantees will mean that designs and products will be more robust to uncertainties in manufacturing quality, environmental factors, technology level and markets.

In economics, the idea that risk is related to profit is well understood and leveraged. In aerospace engineering however we often forget that there is no such thing as a free lunch, and that the consequence of risk aversity necessarily is performance that is left on the table. Considering that conceptual design in the aerospace industry hedges against program risk, the Robust Optimization (RO) frameworks proposed in this paper will give aerospace engineers the ability to rigorously trade robustness and the performance penalties that result from it.

I.A. Approaches to optimization under uncertainty

Faced with the challenge of developing general nonlinear programs that can incorporate uncertainty, the aerospace field has developed a number of mathematically non-rigorous methods to design under uncertainty. Oftentimes, aerospace engineers will implement *margins* in the design process to account for uncertainties in parameters that a design's feasibility may be sensitive to, such as material properties or maximum lift coefficient. Another traditional method of adding robustness is through *multi-mission design* [1], which ensures that the aircraft is able to handle multiple kinds of missions in the presence of no uncertainty. This is a type of *finitely adaptive* optimization geared to ensure objective performance in off-nominal operations.

The weaknesses of these non-rigorous methods are many. They provide no quantitative measures of robustness or reliability [3]. Furthermore, they rely on the expertise of an experienced engineer to guide the design process, without explicit knowledge of the trade-off between robustness and optimality [5]. This is a dangerous proposition especially in the conceptual design phase of new configurations, since prior information and expertise is not available. In these scenarios, it is especially important to go back to fundamental physics and use rigorous mathematics to explore the design space [1].

There are two rigorous approaches to solving design optimization problems under uncertainty, which are Stochastic Optimization (SO) and RO. Stochastic optimization[b] deals with probability distributions of uncertain parameters by propagating these uncertainties through the physics of a design problem to ensure constraint feasibility with certain probabilistic guarantees. The goal of SO is predominantly to minimize the expectation of an objective function, or to optimize on some desired characteristic of the probability density function of the objective [6]. Robust optimization takes a different approach, instead choosing to make designs immune to uncertainties in parameters as long as the parameter values come from within the

[b]Note that stochastic optimization is an overloaded term, and exists in three contexts in the literature. The first is the solution of deterministic problems with stochastic search space exploration. The second is the solution of simulatios, often partial differential equations, with uncertain parameters. The final is the solution of design optimization problems with stochastic parameters. We explore the third interpretation.

defined uncertainty set. As such, RO minimizes the worst-case objective outcome from a defined uncertainty set.

I.B. Advantages of robust over stochastic optimization

The formulation of stochastic models as RO problems has many advantages over general stochastic optimization methods, as explored in detail in [7], and fall into three categories, which are tractability, conservativeness and flexibility. RO is more tractable than SO due to the nature of uncertainty propagation. General stochastic methods involve the propagation of uncertainties throughout a model to determine their effects on constraint feasibility and the objective function. This requires the integration of the product of probability distributions with potential outcomes, and since the integration of continuous functions is difficult this is often achieved through a combination of high-dimensional quadrature and discretizations of the uncertainty into possible scenarios. The propagation of parameter scenarios results in a combinatorial explosion of possible outcomes which need to be evaluated to determine constraint satisfaction and the distribution of the objective.

Few problems can be addressed purely through stochastic optimization (eg. the recourse problem as shown in [8],[9], and energy planning problem such as in [10]). It is arguable that the methods used are still limited by the combinatorial explosion of possible outcomes. RO has to deal with a somewhat related problem, which is the issue of an infinite number of possible realizations of constraints within a given constraint set. However, this is easily tackled by considering the worst case robust counterpart of each constraint, which results in many kinds of optimization problems having tractable robust formulations [7].

Although RO problems solve problems with uncertainty, RO formulations result in solutions that are deterministically immune to all possible realizations of parameters in an uncertainty set [7], which is defined as conservativeness. SO formulations provide no such guarantees. RO also does not require distributional information about uncertain parameters as SO does, and therefore can better address problems where there is a lack of experience or data. It is arguable that RO leaves a lot on the table by not taking advantage of distributional information, however there is a body of research on distributionally robust optimization [11] which seeks to leverage existing data.

There is significantly greater flexibility in the formulation of robust versus stochastic models since the methods proposed are more general. It is important to highlight that, although both RO and SO seek to address the problem of optimization under uncertainty, they solve fundamentally different problems. In an ideal world where we have a problem that is tractable with global optimality for both methods, the two different approaches would result in different solutions.

I.C. Geometric and signomial programming for engineering design

Geometric programming[c] is a method of log-convex optimization that has been developed to solve problems in engineering design [12]. Although theory of the Geometric Program (GP) has existed since the 1960's, GPs have recently experienced a resurgence due to the advent of polynomial-time interior point methods [13] and improvements in computing. They have been applied to a range of engineering design problems with success. For a non-exhaustive list of examples, please refer to [14].

GPs have been effective in aircraft conceptual design ([15], [16]). However, the stringent mathematical requirements of a GP limits its application to non-log-convex problems. The Signomial Program (SP) is the difference-of-log-convex extension of the GP which can be applied to solve this larger set of problems, albeit with the loss of some mathematical guarantees compared to the GP [17]. Aircraft pose some of the most challenging design problems [1], and signomial programming has been used to great effect in modeling and designing complex aircraft at a conceptual level quickly and reliably as in [1], [18] and [17]. Other interesting applications for SPs such as in network flow problems are being investigated.

Robust formulations exist for solving geometric programs with parametric uncertainty [2]. We posit that the creation of a robust signomial programming framework to capture uncertainty in engineering design, and specifically aircraft design, will allow us to have more confidence in the results of the conceptual design phase, reduce program risk, and increase overall system performance.

[c]Programming refers to the mathematical formulation of an optimization problem.

I.D. Contributions

In this paper, we propose a tractable RSP which we solve as a sequential Robust Geometric Program (RGP), allowing us to implement robustness in non-log-convex problems such as aircraft design. We extend the RGP framework developed by Saab [2] to SPs. We implement the RSP formulation on a simple aircraft design problem with several hundred variables as defined in [19]. We demonstrate the benefits of robust optimization both in ensuring design feasibility and performance using Monte Carlo (MC) simulations of the uncertain parameters. We further explore the benefits of RO in multiobjective optimization.

II. Mathematical Background

II.A. Robust Optimization

Given a general optimization problem under parametric uncertainty, we can define the set of possible realizations of uncertain vector of parameters u in the uncertainty set $\mathcal{U}$. This allows us to define the problem under uncertainty below.

$$\begin{aligned} &\min \; f_0(x) \\ &\text{s.t.} \; f_i(x,u) \leq 0, \; \forall u \in \mathcal{U}, \; i = 1, \ldots, n \end{aligned}$$

This problem is infinite-dimensional, since it is possible to formulate an infinite number of constraints with the countably infinite number of possible realizations of $u \in \mathcal{U}$. To circumvent this issue, we can define the following robust formulation of the uncertain problem below.

$$\begin{aligned} &\min \; f_0(x) \\ &\text{s.t.} \; \max_{u \in \mathcal{U}} f_i(x,u) \leq 0, \; i = 1, \ldots, n \end{aligned}$$

This formulation hedges against the worst-case realization of the uncertainty in the defined uncertainty set. This is often posed by creating an uncertainty set to contain all possible realizations of the uncertainty we are concerned about, usually through a norm,

$$\begin{aligned} &\min \; f_0(x) \\ &\text{s.t.} \; \max_{u} f_i(x,u) \leq 0, \; i = 1, \ldots, n \\ &\qquad \|u\| \leq \Gamma \end{aligned} \tag{1}$$

where Γ is defined by the user as an uncertainty bound.

II.B. Geometric Programming

A **geometric program in posynomial form** is a log-convex optimization problem of the form:

$$\begin{array}{ll} \min & f_0(\mathbf{u}) \\ \text{s.t.} & f_i(\mathbf{u}) \leq 1, i = 1, ..., m_p \\ & h_i(\mathbf{u}) = 1, i = 1, ..., m_e \end{array} \tag{2}$$

where each f_i is a *posynomial*, each h_i is a *monomial*, m_p is the number of posynomials, and m_e is the number of monomials. A monomial $h(\mathbf{u})$ is a function of the form:

$$h_i(\mathbf{u}) = e^{b_i} \textstyle\prod_{j=1}^{n} {u_j}^{a_{ij}}$$

where a_{ij} is the j^{th} component of a row vector $\mathbf{a_i}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_i is in $\mathbb{R}$. An example of a monomial is the lift equation, $L = \frac{1}{2}\rho V^2 C_L S$. A posynomial $f(\mathbf{u})$ is the sum of $K \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{u}) = \sum_{k=1}^{K} e^{b_{ikj}} \prod_{j=1}^{n} {u_j}^{a_{ikj}}$$

where a_{ikj} is the j^{th} component of a row vector $\mathbf{a_{ik}}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_{ik} is in $\mathbb{R}$ [14]. The stagnation pressure definition is a good example of a posynomial: $P_t = P + \frac{1}{2}\rho V^2$.

A logarithmic change of the variables $x_j = \log(u_j)$ would turn a monomial into *the exponential of an affine function* and a posynomial into *the sum of exponentials of affine functions.* A transformed monomial $h_i(\mathbf{x})$ is a function of the form:

$$h_i(\mathbf{x}) = e^{\mathbf{a_i}\mathbf{x}+b_i}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A transformed posynomial $f_i(\mathbf{x})$ is the sum of $K_i \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{x}) = \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}\mathbf{x}+b_{ik}}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A geometric program with transformed constraints is a **geometric program in exponential form**.

The positivity of exponential functions restricts the space spanned by posynomials and limits GPs to certain classes of problems. However, since many engineering problems of interest have purely positive quantities GPs are quite applicable, and certain variable transformations can make problems with negative quantities tractable. The restriction of posynomials to the *less-than-side of inequalities* is a more significant barrier, and motivates the introduction of signomials.

II.C. Signomial Programming

A *signomial* can be defined as the difference between two posynomials, consequently, a SP is a non-log-convex optimization problem of the form:

$$\begin{aligned} &\text{minimize} && f_0(\mathbf{x}) \\ &\text{subject to} && f_i(\mathbf{x}) - g_i(\mathbf{x}) \quad \leq 0, i = 1, ..., m \end{aligned} \tag{3}$$

where f_i and g_i are both posynomials, and $\mathbf{x}$ is a column vector in $\mathbb{R}^n$.

Reliably solving an SP to a local optimum has been described in [14] and [20]. A common solution heuristic involves solving an SP as a sequence of GPs, where each GP is a local approximation of the SP. Although it is a powerful tool, applications involving SPs are usually prone to uncertainties that have a significant effect on the solution.

III. Robust Signomial Programming

As a preview of the following sections, robust signomial programming assumes that parameter uncertainties belong to an uncertainty set, and solves a reformulated design problem to find the best solution, through a process as shown in Figure 1. As long as the original optimization problem is SP-compatible, a tractable robust formulation of the problem exists, making this method general. We derive the intractable formulation of a RSP below.

A SP in its **exponential form** is as follows:

$$\begin{aligned} &\min && f_0\left(\mathbf{x}\right) \\ &\text{s.t.} && \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}\mathbf{x}+b_{ik}} - \sum_{k=1}^{G_i} e^{\mathbf{c_{ik}}\mathbf{x}+d_{ik}} \leq 0 \quad \forall i \in 1, ..., m \end{aligned} \tag{4}$$

where the constraints are represented as difference-of-posynomials in exponential form. Let $\mathbf{a_{ik}}$ and $\mathbf{c_{ik}}$ be the $((i-1) \times m + k)^{th}$ rows of the exponents matrices $\mathbf{A}$ and $\mathbf{C}$ respectively, and b_{ik} and d_{ik} be the $((i-1) \times m + k)^{th}$ elements of the coefficients vectors $\mathbf{b}$ and $\mathbf{d}$ respectively.

The data ($\mathbf{A}$, $\mathbf{C}$, $\mathbf{b}$, $\mathbf{d}$) is assumed to be uncertain and living in an uncertainty set $\mathcal{U}$, where $\mathcal{U}$ is parametrized affinely by a perturbation vector ζ:

$$\mathcal{U} = \left\{ [\mathbf{A}; \mathbf{C}; \mathbf{b}; \mathbf{d}] = \left[\mathbf{A}^0; \mathbf{C}^0; \mathbf{b}^0\ \mathbf{d}^0\right] + \sum_{l=1}^{L} \zeta_l \left[\mathbf{A}^l; \mathbf{C}^l; \mathbf{b}^l; \mathbf{d}^l\right] \right\} \tag{5}$$

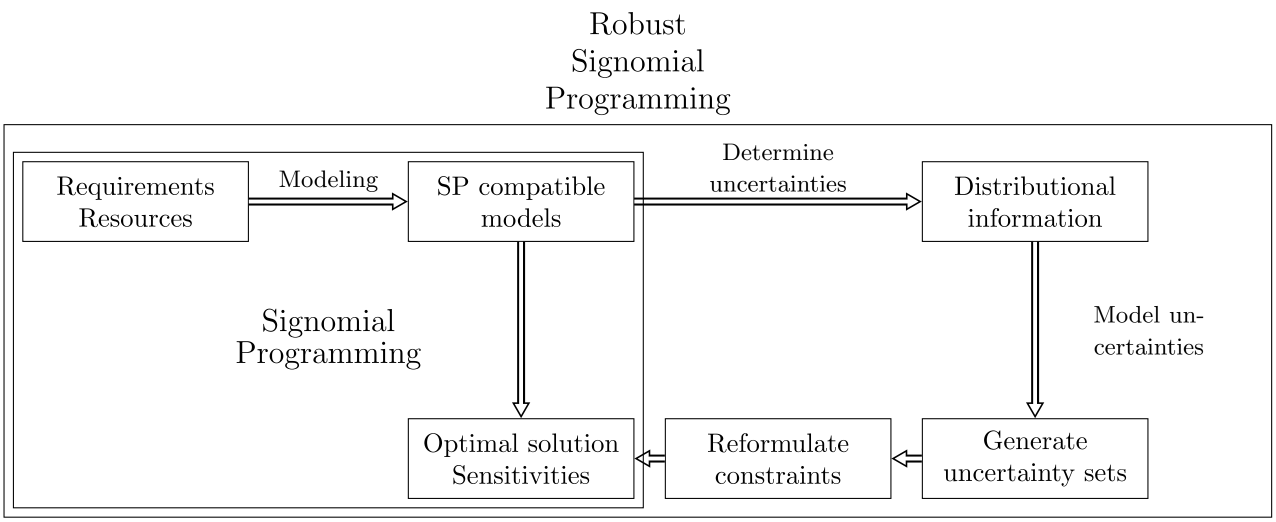


Figure 1: A block diagram showing the difference between the design process using a SP and a RSP.

where $\mathbf{A}^0$, $\mathbf{C}^0$, $\mathbf{b}^0$, and $\mathbf{d}^0$ are the nominal exponents and coefficients, $\left\{\mathbf{A}^l\right\}_{l=1}^{L}$, $\left\{\mathbf{C}^l\right\}_{l=1}^{L}$, $\left\{\mathbf{b}^l\right\}_{l=1}^{L}$, and $\left\{\mathbf{d}^l\right\}_{l=1}^{L}$ are the basic shifts of the exponents and coefficients, and ζ_l is the l^{th} component of ζ belonging to a perturbation set $\mathcal{Z} \in \mathbb{R}^L$ such that

$$\mathcal{Z} = \left\{\zeta \in \mathbb{R}^L : \|\zeta\| \leq \Gamma\right\} \tag{6}$$

As mentioned earlier, there should exist a formulation immune to uncertainty in the system's data. Accordingly, the robust counterpart of the uncertain SP in (4) is:

$$\begin{array}{lll} \min & f_0\left(\mathbf{x}\right) & \\ \text{subject to} & \max\limits_{\zeta \in \mathcal{Z}} \left\{\sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)} - \sum_{k=1}^{G_i} e^{\mathbf{c_{ik}}(\zeta)\mathbf{x}+d_{ik}(\zeta)}\right\} & \leq 1 \quad \forall i \in 1,...,m \end{array} \tag{7}$$

The optimization problem in (7) is intractable using current solvers, therefore, a heuristic approach to solving RSPs approximately as a sequential RGP will be presented in the following sections. As our approach is based on robust geometric programming, a brief review of the subject will follow based on [2].

IV. Robust Geometric Programming

This section presents a brief review of the approximation of an RGP as a tractable optimization problem as discussed in [2]. The robust counterpart of an uncertain geometric program is:

$$\begin{array}{lll} \min & f_0\left(\mathbf{x}\right) & \\ \text{subject to} & \max\limits_{\zeta \in \mathcal{Z}} \left\{\sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)}\right\} & \leq 1 \quad \forall i \in 1,...,m \end{array} \tag{8}$$

which is Co-NP hard in its natural posynomial form [**?**]. We will present three approximate formulations of a RGP.

IV.A. Simple Conservative Formulation

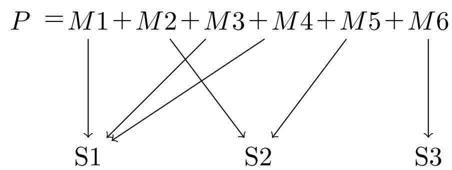
One way to approach the intractability in (8) is to replace each constraint by a tractable approximation. Replacing the max-of-sum in (8) by the sum-of-max will lead to the following formulation

$$\begin{array}{lll} \min & f_0\left(\mathbf{x}\right) & \\ \text{subject to} & \sum_{k=1}^{K_i} \max\limits_{\zeta \in \mathcal{Z}} \left\{e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)}\right\} & \leq 1 \quad \forall i \in 1,...,m \end{array} \tag{9}$$

Maximizing a monomial term is equivalent to maximizing an affine function, therefore (9) is tractable.

IV.B. Equivalent Intermediate Formulation

This formulation is equivalent to the formulation in (8), but with smaller, easier to handle posynomial constraints. By the properties of inequalities, the posynomial P in posynomial inequality $M \geq P$ can be divided into an equivalent set of smaller posynomials based on the dependence between its monomial terms. Figure 2 shows how a constraint can be represented as an equivalent set of smaller posynomial constraints.



$$max\{P\} \leq 1 \iff \begin{array}{rl} t_1 + t_2 + t3 & \leq 1 \\ max\{S_1\} = max\{M_1 + M_3 + M_4\} & \leq t_1 \\ max\{S_2\} = max\{M_2 + M_5\} & \leq t_2 \\ max\{S_3\} = max\{M_6\} & \leq t_3 \end{array}$$

Figure 2: Partitioning of a large posynomial into smaller posynomials requires the addition of auxiliary variables. S_i are posynomials with independent sets of variables.

The posynomial constraints are categorized into three sets: large posynomials, two-term posynomials and monomials, represented by $S1$, $S2$ and $S3$ respectively. Monomials are tractable, and two-term posynomials can be well approximated using piecewise-linear functions [21]. We implement the following two tractable approximations for large posynomials.

IV.B.1. Linearized Perturbations Formulation

If the exponents are known and certain, then large posynomial constraints can be approximated as signomial constraints. The exponential perturbations in each posynomial are linearized using a modified least squares method, and then the posynomial is robustified using techniques from robust linear programming. The resulting set of constraints is SP compatible, therefore, a robust geometric program can be approximated as a signomial program.

IV.B.2. Best Pairs Formulation

If the exponents are also uncertain, then large posynomials can't be approximated as an SP, and further simplification is needed. This formulation aims to maximize each pair of monomials in each posynomial, while finding the best combination of monomials that gives the least conservative solution. [2] provides a descent algorithm to find locally optimal combinations of the monomials, and shows how the uncertain geometric program can be approximated as a geometric program for polyhedral uncertainty, and a conic optimization problem for elliptical uncertainty with uncertain exponents. For a detailed description of the above formulations refer to [2]. An algorithm for solving an RSP based on the above formulations is provided in the next section.

V. Approach to Solving Robust Signomial Programs

This section presents a heuristic algorithm to safely solve a RSP based on our previous discussion on robust geometric programming.

V.A. General RSP Solver

As mentioned before, a common heuristic algorithm to solve a SP is by sequentially solving local GP approximations, but the solution is not guaranteed to be globally optimal. Our approach to solve a RSP is based on sequentially solving local RGP approximations. Below we provide a step-by-step algorithm to solve a RSP:

1. Choose an initial guess x_0.
2. Repeat:
   (a) Find the local GP approximation of the SP at x_i.
   (b) Find the RGP formulation of the GP.
   (c) Solve the RGP to obtain x_{i+1}.
   (d) If $x_{i+1} \approx x_i$: break

Similar to a SP, a good initial guess would lead to faster convergence and possibly a better solution. A quick candidate is the deterministic solution of the uncertain SP, which will certainly lead to a faster convergence.

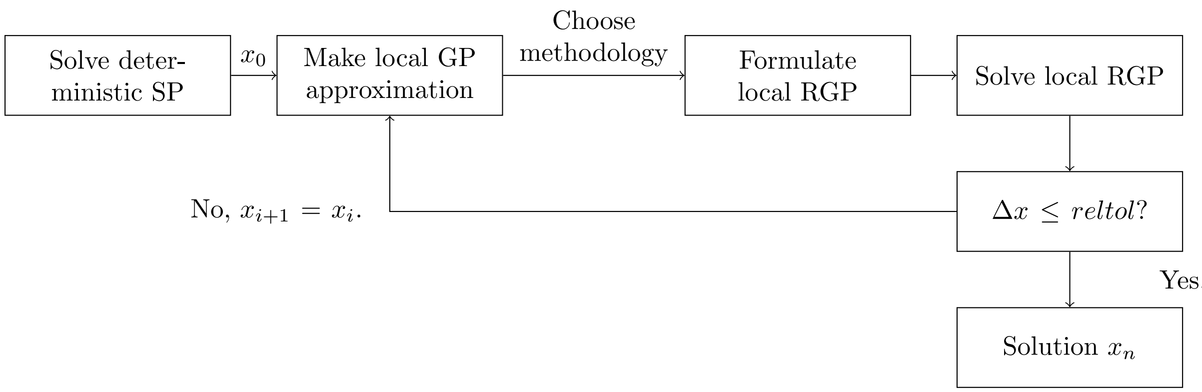


Figure 3: A block diagram showing the steps of solving an RSP.

Any of the previously mentioned methodologies can be used to formulate the local RGP approximation. However, depending on the RGP formulation chosen to solve a RSP, the last formulation and solution blocks in Figure 3 are adjusted for a faster rate of convergence.

V.B. Best Pairs RSP Solver

If the Best Pairs methodology is exploited, then the above algorithm would change so that each iteration would solve the local RGP approximation and choose the best permutation for each large posynomial. The modified algorithm would become as follows:

1. Choose an initial guess x_0.
2. Repeat:
   (a) Find the local GP approximation of the SP at x_i.
   (b) For each large posynomial constraint, select the new permutation ϕ such that ϕ minimizes the robust large constraint evaluated at x_i.
   (c) Solve the approximate tractable counterparts of the local GP in (8), and let $\mathbf{x}_{i+1}$ be the solution.
   (d) If $x_{i+1} \approx x_i$: break.

V.C. Linearized Perturbations RSP Solver

On the other hand, if the Linearized Perturbations formulation is to be used, then we can avoid solving a SP at each iteration by first approximating the original SP constraints locally, and in the same loop approximating the robustified possibly signomial constraints locally, thus solving a GP at each iteration instead of an SP. The algorithm would then become as follows:

1. Choose an initial guess x_0.

2. Repeat:
   (a) Find the local GP approximation of the SP at x_i.
   (b) Robustify the constraints of the local GP approximation using the Linearized Perturbations methodology.
   (c) Find the local GP approximation of the resulting local SP at x_i.
   (d) Solve the local GP approximation in step c to obtain x_{i+1}
   (e) If $x_{i+1} \approx x_i$: break.

VI. Models

We implemented the RSP formulation above on a simple unmanned, gas-powered aircraft design problem that is systematically developed in [19]. We optimize a wing, fuselage, and engine given a payload and range requirement. The nominal model has 154 variables and 184 constraints, a common level of sparsity for GP and SP models. A short qualitative overview of the model follows; for more detailed information, please refer to [19]. The uncertainties associated with the parameters will be described in Section VII.

VI.A. Flight Profile

The flight profile models have been borrowed from [1]. Within the model, the trajectory of the aircraft is optimized over five steady flight segments, although we are restricted to modeling only climb segments and therefore the stored gravitational potential energy of the aircraft is not captured.

VI.B. Atmosphere

The atmosphere model is taken from [22], and considers changes in density and dynamic viscosity with altitude, for a standard atmosphere.

VI.C. Aircraft

The aircraft is modeled as a wing, fuselage and engine system. The aircraft is assumed to be in steady flight, so that the thrust power is equal to the sum of the drag power and rate of change of potential energy of the aircraft, and the lift is equal to the total weight, ignoring the vertical component of thrust in climb. Its total weight is the sum of its components. The aircraft has to be able to takeoff at specified minimum speed without stalling as well. Its component models are detailed below.

VI.C.1. Wing

Lift is generated by the wing as a function of its geometry and freestream conditions. The wing structure model is based on a simple beam model with a distributed lift load, and a point mass in the center representing the fuselage. Wing fuel volume is modeled as a fraction of the internal volume available in the wing. Its drag is approximated simply as a sum of the induced and profile drags, the latter of which is estimated using a form factor. The weight of the wing is the sum of skin and spar weights.

VI.C.2. Fuselage

The fuselage is assumed to only contain fuel. The fuselage drag is exactly proportional to its frontal area. The fuselage is assumed not to contain any structural members, and so its weight consists only of skin weight.

VI.C.3. Engine

The aircraft is powered by a naturally aspirated piston engine. It is subject to power lapse at lower air densities at higher altitudes. Its weight is modeled using a posynomial fit of existing engines. Its brake specific fuel consumption (BSFC) is modeled as a function of maximum thrust at a given altitude.

VI.D. Source of non-log-convexity: fuel volume

We have already detailed the fuel models in the wing and fuselage sections, but it is noteworthy that the signomial constraint in the optimization appears in the aircraft total fuel volume constraint, as shown in Equation 10:

$$V_f \leq V_{f_{wing}} + V_{f_{fuse}} \tag{10}$$

The signomial constraints makes the problem non-log-convex, which means that the solution methods detailed by Saab [2] need to be extended to accommodate this optimization problem.

VII. Uncertainties and Sets

The uncertainties for the different parameters in the problem have been determined considering the parameters in aircraft design that often have the largest uncertainty. These uncertainties, given by three times the coefficient of variation (CV)[d], are listed in Table 1. Since for the rest of this work all standard deviations (σ) are normalized by the means of the parameters, we will use 3σ to represent 3CV.

Table 1: Parameters and Uncertainties (increasing order)

Parameters	Description	Value	Units	% Uncert. (3σ)
$S_{wetratio}$	wetted area ratio	2.075	-	3
e	span efficiency	0.92	-	3
μ	viscosity of air	1.78×10^{-5}	$kg/(ms)$	4
ρ	air density	1.23	kg/m^3	5
$C_{L_{max}}$	stall lift coefficient	1.6	-	5
k	fuselage form factor	1.17	-	10
τ	airfoil thickness ratio	0.12	-	10
N_{ult}	ultimate load factor	3.3	-	15
V_{min}	takeoff speed	30	m/s	20
W_0	payload weight	6250	N	20
$W_{w_{coeff,i}}$	wing weight coefficient 1	2×10^{-5}	$1/m$	20
$W_{w_{coeff,ii}}$	wing weight coefficient 2	60	N/m^2	20

The parameter uncertainties reflect aerospace engineering intuition. The wing weight coefficients $W_{w_{coeff,i}}$ and $W_{w_{coeff,ii}}$, and the ultimate load factor N_{ult} have large 3σs because the build quality of aircraft components is often difficult to quantify with a large degree of certainty. The payload weight (W_0) has a large uncertainty for similar reasons, since it is often developed concurrently with the aircraft. Parameters that engineers take to be physical constants (μ, ρ) and those that can be determined or manufactured with a relatively high degree of accuracy ($S_{wetratio}$, e) have relatively low deviations. Parameters that require testing to determine ($C_{L_{max}}$, V_{min}) have a level of uncertainty that reflects the expected variance of empirical studies. However, note that these quantities are ultimately picked by the designer and the level of conservatism in the design will be greatly affected by the chosen 3σs.

VIII. Results

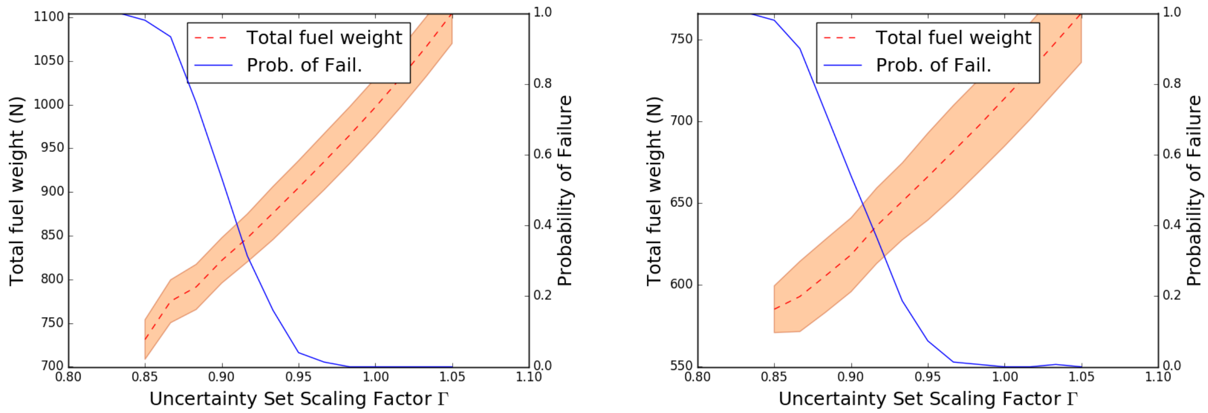
We implement our RSP heuristic algorithm on the aforementioned conceptual aircraft design problem. Our nominal objective function is total fuel consumption, which is to be minimized given a payload and a range requirement.

[d]The CV is defined as follows: $\text{CV} = \frac{\sigma}{|\mu|}$, where σ is the standard deviation and μ is the mean of the parameter.

VIII.A. Mitigation of probability of failure

The problem is solved for different sizes of box and elliptical uncertainty sets by varying the parameter Γ, as defined in Appendix X.A. Mathematically, for box uncertainty, Γ is a measure of the centered width in logspace of the defined parameter uncertainty, normalized by the standard deviation of the parameter. For elliptical uncertainty, it is the maximum diameter of the Euclidian norm ball of u_i, which is the number of standard deviations of perturbation of each ith parameter from its nominal value. Intuitively, it is a measure of how much risk is being hedged against. $\Gamma = 0$ implies that all of the parameters take their nominal values with zero uncertainty, and larger Γ protects against more parameter uncertainty, where a box uncertainty is more conservative than elliptical uncertainty.

The design variables are then fixed for each solution so that the design can be simulated for different realizations of the uncertain parameters in Table 1 to examine average design performance. In this MC scheme, the random variables are simulated from independent and identically distributed 3σ truncated Gaussians. We simulate from the truncated Gaussian since this makes it possible to confirm mathematically that for $\Gamma = 1$, all simulations of uncertain parameters are feasible. Designs for each solution in Figure 4 are simulated with the same set of uncertainty realizations for consistency.



(a) Box Uncertainty Set

(b) Elliptical Uncertainty Set

Figure 4: Simulated performance of the optimal robust aircraft, using the Best Pairs formulation, as a function of Γ for different uncertainty sets. The dashed line and the band represent the mean and standard deviation of the performance of aircraft designed for different Γ, and simulated with 100 MC samples of uncertain parameters.

We define the probability of failure of a design is the probability that any constraint in the design optimization problem is violated in a MC simulation. As expected, Figure 4 shows that probability of failure goes to zero as Γ increases. It is noteworthy that, for the nominal problem ($\Gamma = 0$) and for uncertainty up to $\Gamma \leq 0.8$, none of the 100 simulated uncertain parameters result in feasible solutions. Under uncertain parameters, the aircraft designed for the average case would almost surely fail to complete its mission. That being said, it is necessary to sacrifice performance to achieve a high degree (3σ) of reliability as in the solution for $\Gamma = 1$.

Moreover, using margins would in the best case be as good as using a box uncertainty set, and therefore will lead to a more conservative solution with inferior performance.

Figure 5 compares the different methodologies in terms of setup and run times. Since the setup time of the nominal problem is minimal, we have normalized the results by the run time of the nominal problem for comparison. The bottom axis ranks the methods by their level of conservativeness (Best Pairs and Simple Conservative formulations being the least and most conservative respectively), and the elliptical formulations are less conservative than the box formulations **??**. For the box uncertainty, solution times increase for increasing levels of conservativeness, whereas for the elliptical uncertainty they decrease.

Table 2: SP Aircraft Optimization Results: TODO update

Free variable	No Uncert.	Box [$\Gamma = 1.0$]	Ellipsoidal [$\Gamma = 1.0$]
L/D	23.8	16.41	31.7
AR	12.0	5.016	12.0
Re	4.76×10^6	1.16×10^7	3.73×10^6
$S(m^2)$	21.6	72.21	61.1
$V(m/s)$	51.0	47.73	33.4
$T_{flight}(hr)$	17.46	18.1	25.0
W_w	2440	6278	6450
$W_{w_{strc}}(N)$	1210	1570	2420
$W_{w_{surf}}(N)$	1230	1592	4032
$V_{f_{avail}}(m^3)$	0.566	1.794	0.121
$V_{f_{fuse}}(m^3)$	0.461	0.891	0
$V_{f_{wing}}(m^3)$	0.105	0.9027	0.121
$CDA_0(m^2)$	0.0461	0.0891	0
E[Objective]	No Uncert.	Box [$\Gamma = 1.0$]	Ellipsoidal [$\Gamma = 1.0$]
W_{fuel} (N)	4430	10858	974
P[failure]	No Uncert.	Box [$\Gamma = 1.0$]	Ellipsoidal [$\Gamma = 1.0$]
%	100	0	0

VIII.B. The Effect of Robustness on Multiobjective Performance

One of the benefits of convex and difference-of-convex optimization methods is the ability to optimize for different objectives [1]. As a demonstration, we optimized the aircraft without uncertainty for 7 different objectives, and show the non-dimensionalized results in Table 3.

Objective	Total fuel	Time cost	Aspect ratio	Engine weight	Wing loading	Total cost	Takeoff weight
Total fuel	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Time cost	9.44	0.3	0.09	152.85	1.0	1.86	2.6
Aspect ratio	4.26	0.46	0.04	25.72	0.94	1.11	1.22
Engine weight	1.09	1.22	1.28	0.8	1.0	1.2	1.15
Wing loading	8.89	1.5	0.28	13.06	0.22	2.76	5.82
Total cost	1.68	0.51	0.39	5.75	1.0	0.71	0.92
Takeoff weight	1.33	0.94	0.33	2.13	1.0	1.01	0.85

Table 3: Non-dimensionalized variations in objective values with respect to the aircraft optimized for different objectives. Objective values were normalized by the total fuel solution.

Since the model is physics based, it is able to accommodate a range of objectives, even ones that are not often considered such as aspect ratio. The resulting aircraft also differ drastically with respect to performance and design variables. As the most extreme example, the aircraft optimized for time cost has 150 times the engine weight as the aircraft optimized for total fuel, since a huge amount of power is required to fly fast.

Aside from this caricature example, we further demonstrate the capabilities of RSPs in multiobjective design by considering a more realistic scenario. We perform the optimization of the aircraft with no uncertainty and ellipsoidal uncertainty ($\Gamma = 1$) for 4 different objective functions, and plot the results on spider plots. Spider plots are useful because they allow engineers to see the performance of different designs in a multi-objective environment. One way to envision the multi-objective performance of the aircraft is to consider the area contained within the web defined by the aircraft's performance; the smaller the web area the better. Due to the large disparities in the potential values of design variables depending on objective as shown in Table 3, we chose to demonstrate this using four objective functions that would be expected

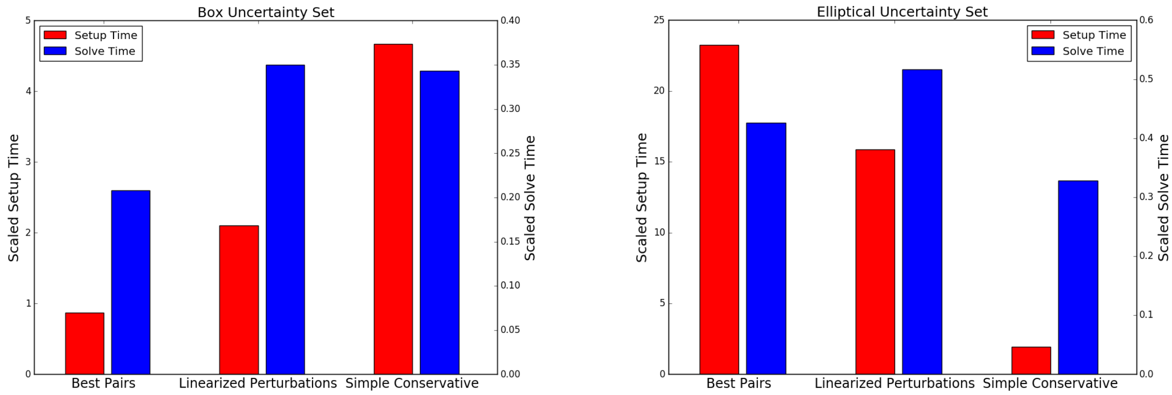


Figure 5: Robust signomial simple aircraft solution and setup times, normalized by the nominal problem solution time. Note that the problems with box uncertainty have much lower setup time costs versus those with elliptical uncertainty.

to have a high degree of correlation and therefore yield similar aircraft designs. These were total (time and fuel) cost, total fuel, takeoff weight and mid-cruise lift-over-drag (L/D).

In the spider plots in Figure 6, it is possible to see the effect of robustness on the different worst-case performance metrics of the different aircraft. For example, for the nominal case, it is possible to see that the aircraft designed for total fuel performs the best when all four objectives are considered, assuming that all objectives are weighted equally. However, this behavior changes when uncertainty is considered. An aircraft optimized for total cost (bottom left graph) with a box uncertainty set (in green) has better worst-case multiobjective performance compared to an aircraft designed for total fuel with the same uncertainties.

This is an interesting result, because the presence of an uncertainty set is shown to affect the efficacy of different objective functions to obtain solutions with the best overall performance. If the three objective functions didn't have high degree of coupling, that the internal areas of the solution triangles may differ more significantly. This analysis could also be performed for the mean performance of the aircraft determined through MC simulation, but this demonstration limits its scope to the worst-case analysis.

VIII.C. Risk minimization problems

All of the previous multi-objective analyses have assumed that we have an understanding of exactly how much risk we are willing to tolerate. However, it would also make sense if risk was the objective of our model. This would suggest the following formulation:

$$\begin{aligned} \max\ & \Gamma \\ \text{s.t. } & f_i(x,u) \leq 0, i = 1,\ldots,n \\ & \|u\| \leq \Gamma \\ & f_0(x) \leq (1+\delta) f_0^*,\ \delta \geq 0 \end{aligned} \tag{a}$$

where f_0^* is the optimum of the nominal problem in Formulation 1, δ is a fractional measure of the objective that we are willing to sacrifice for robustness, which gives $(1+\delta)f_0^*$ as the upper bound on the objective value. Intuitively, this is a form of goal programming, where we specify the exact maximum worst-case value of an objective we can tolerate so that the program risk is acceptable, but in the meanwhile maximize the total size of the uncertainty we can handle. We call the method of minimizing an objective for set Γ *the* Γ *method*, while we call the goal programming approach, which maximizes Γ for a set δ, *the* δ *method*.

The problem in Formulation a is not equivalent to the problem in Formulation 1, but should yield the same result. As a proof of concept, we perform the same probability of failure analysis as in Figure 7, but have the objective bound be an input to the optimization problem and the Γ of the uncertainty set be maximized.

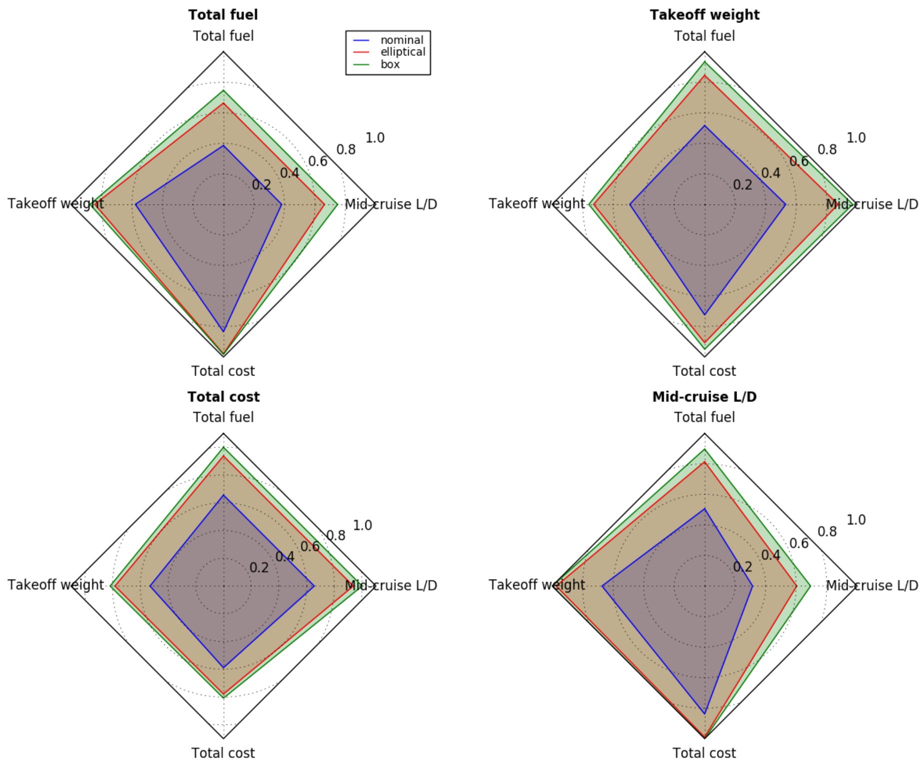
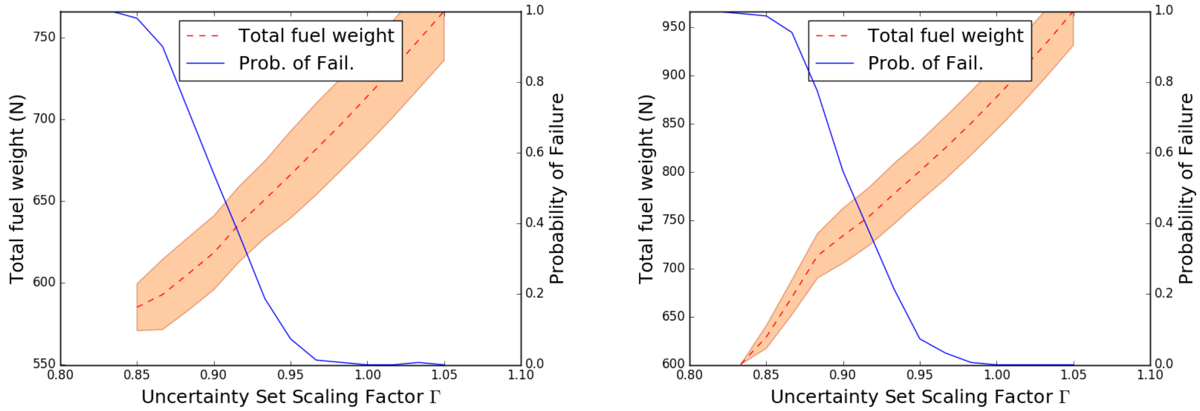


Figure 6: The spider plots of aircraft optimized for different objectives. The bolded titles are the design objectives for each plot, whereas the individual spiderwebs show the non-dimensionalized multiobjective performance of the aircraft designed under different uncertainty sets.



(a) Elliptical Uncertainty, Γ method

(b) Elliptical Uncertainty, δ method

Figure 7: Simulated performance of the optimal robust aircraft, using the Best Pairs formulation for the Γ and δ methods. The dashed line and the band represent the mean and standard deviation of the performance of aircraft designed for different Γ, and simulated with 100 MC samples of uncertain parameters.

As expected, we get identical results from the outputs of the two formulations (to confirm). We can also expand this framework to perform multivariate goal programming, by changing (a) in the formulation a to include all objectives we are interested in.

$$f_{0,j}(x) \leq (1 + \delta_j) f^*_{0,j}, \ \delta_j \geq 0, \ j = 1, \ldots, m$$

The benefit of goal programming is that it allows us to explore multidisciplinary tradeoffs without having to enumerate the design space along each objective direction. In design it is not obvious whether an objective should in fact be a constraint instead. The most fundamental choice that an engineer can make in design is what the objective function is, and it is often the case that there are many potential objectives that are conflicting. The term multiobjective optimization is misleading because you can only optimize for one objective at once, and the design is going to be influenced by how engineers weight different objectives. Risk minimization makes these implicit decisions explicit, empowers engineers to choose how much performance they would be willing to sacrifice with respect to optimal but fragile designs to obtain robust designs, and gives a prediction of the margin of error for every parameter that they have to make designs feasible.

IX. Potential Future Work or Studies

There are a myriad of potential extensions to signomial programming under uncertainty. In the spirit of helping reduce program risk in aerospace design, the authors make a few observations and recommendations.

In this study, we do not discriminate between the kinds of constraints violated. However, it would be possible to rank the severity of constraint violations so as to penalize some (eg. structural safety) more heavily than others (maximum range constraint). This would inject further realism into the design under uncertainty since some violations contribute to program risk more strongly than others.

Another potentially valuable extension to the proposed framework is the concurrent implementation of multiple sets to contain the uncertain parameters, with the purpose of restricting uncertain outcomes further. One example of this would be to impose an l1-norm on the integer number of uncertain parameters as well as an l2-norm on the overall size of uncertainty set This method can be used to set the total size of the uncertainty set in a Euclidian sense, but then also to restrict the stochasticity to a subset of all of the uncertain parameters, thereby somewhat restricting nature. This also turns the problem into an integer robust optimization problem which poses interesting computational challenges.

With respect to interesting studies, RO opens up many possibilities to discover and analyze the benefits of adaptable architectures in aircraft design versus more traditional point designs when faced with parametric uncertainty. Some examples of these are modular designs, morphing designs, adaptively manufactured designs and aircraft families. It is likely that these types of engineered robustness become more effective at reducing program risk in presence of uncertainty, since they are more likely to deliver value under adverse stochastic outcomes.

In situations where there is data available to aid design, RO can help explore the design space while taking into account the stochasticity and noise in the data. This opens up an array of potential trade studies where engineers can learn about the exposure of designs to the sparsity and spread of data and attempt to gather data which best reduces the uncertainty in the performance of optimal designs.

X. Conclusion

We have developed and applied a tractable RSP formulation to a simple aircraft model, and then discussed the benefits of having robust solutions. Our RSP formulations extend the tractable approximate RGP framework developed by Saab [2] to non-log-convex problems, and are valuable contributions to the fields of robust optimization and difference-of-convex programming.

RSPs have a wide variety of potential applications in engineering design. We expect that using RO in conceptual aircraft design will result in systems that are more robust with respect to uncertainties in operational parameters, such as payload mass and range, as well as uncertain environmental and manufacturing parameters. By making designs immune to all realizations of uncertainty in a set, engineers can trade off robustness and optimality within the context of an optimization framework in a tractable manner.

RO has the potential to change current aerospace design paradigms by introducing mathematical rigor to design under uncertainty. Current aerospace conceptual design practices still rely heavily on the expertise of established engineers even in absence of prior experience exploring the design trade space. RO is compatible for use alongside physics based models that are deprived of or lacking in data, and so can bring quantitative measures of design reliability to the table and steer the field of aerospace design towards physics-based tools and methods.

Appendix

X.A. Robust Linear Programming: A Quick Review

As mentioned earlier, robust linear programming will be used to formulate an approximate robust geometric program.

Consider the system of linear constraints

$$\mathbb{A}\mathbf{x} + \mathbf{b} \leq 0$$

where

$$\mathbb{A} \text{ is } m \times n$$
$$\mathbf{x} \text{ is } n \times 1$$
$$\mathbf{b} \text{ is } m \times 1$$

where that data is uncertain and is given by equations (5) and (6).

X.A.1. Box Uncertainty Set

If the perturbation set $\mathcal{Z}$ given in equation (6) is a box uncertainty set, i.e. $\|\zeta\|_\infty \leq \Gamma$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \textstyle\sum_{l=1}^{L} |-b_i^l - \mathbf{a}_i^l \mathbf{x}| + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \tag{11}$$

If only b is uncertain, i.e. $A^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (11) will become

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \textstyle\sum_{l=1}^{L} |b_i^l| \leq 0 \tag{12}$$

which is a linear constraint
On the other hand, if A is uncertain, then equation (11) is equivalent to the following set of linear constraints

$$\begin{aligned} \Gamma\textstyle\sum_{l=1}^{L} w_i^l + \mathbf{a}_i^0\mathbf{x} + b_i^0 \leq 0 \\ -b_i^l - \mathbf{a}_i^l\mathbf{x} \leq w_i^l \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l\mathbf{x} \leq w_i^l \quad \forall l \in 1, ..., L \end{aligned} \tag{13}$$

X.A.2. Elliptical Uncertainty Set

Briefly, if the perturbation set $\mathcal{Z}$ is an elliptical, i.e. $\sum_{l=1}^{L}\frac{\zeta_l^2}{\sigma_l^2} \leq \Gamma^2$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma\sqrt{\textstyle\sum_{l=1}^{L}\sigma_l^2(-b_i^l - \mathbf{a}_i^l\mathbf{x})^2} + \mathbf{a}_i^0\mathbf{x} + b_i^0 \leq 0 \tag{14}$$

which is a second order conic constraint.
If only b is uncertain, i.e. $\mathbb{A}^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (14) will become

$$\textstyle\sum_{l=1}^{L}\mathbf{a}_i^0\mathbf{x} + b_i^0 + \Gamma\sqrt{\textstyle\sum_{l=1}^{L}\sigma_l^2(b_i^l)^2} \leq 0 \tag{15}$$

which is a linear constraint.

X.A.3. Norm-1 Uncertainty Sets

Briefly, if the perturbation set represented by $\mathcal{Z}$ is a norm-1 uncertainty set, i.e. $\|\zeta\|_1 \leq \Gamma$, then the robust constraint is

$$\textstyle\sum_{l=1}^{L}\mathbf{a}_i^0\mathbf{x} + b_i^0 + \Gamma\max_{l=1,..,L}|b_i^l| \leq 0 \tag{16}$$

when $\mathbb{A}^l = 0$, and

$$\begin{aligned} \Gamma w_i + \mathbf{a}_i^0\mathbf{x} + b_i^0 \leq 0 \\ -b_i^l - \mathbf{a}_i^l\mathbf{x} \leq w_i \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l\mathbf{x} \leq w_i \quad \forall l \in 1, ..., L \end{aligned} \tag{17}$$

if $\mathbb{A}^l \neq 0$
Note that for this type of uncertainty, the robust constraints are linear.

Acknowledgments

A place to recognize others.

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