Optimal Aircraft Design Decisions under Uncertainty via Robust Signomial Programming

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TODO: Complete rework to frame around AC design.

Signomial programming is useful in multidisciplinary non-convex optimization problems such as aircraft design. The formulation and solution of robust signomial programs (RSPs) would be beneficial since many parameters involved in these problems are prone to uncertainty, and can have significant effects on solution performance and feasibility. This paper proposes an approximate solution for an RSP leveraging an existing approximate robust geometric programming (RGP) formulation developed by Saab. The method is based on solving a sequence of RGPs, where each GP is a local approximation of the SP. Moreover, this paper also discusses the trade-off between robustness and optimality by implementing RSPs on a simple aircraft problem, and demonstrates how robust optimization affects aircraft design decisions.

Nomenclature

J	Jacobian Matrix
f	Residual value vector
x	Variable value vector
F	Force, N
m	Mass, kg
Δx	Variable displacement vector
α	Acceleration, m/s^2

Subscript

i	Variable number

I. Introduction

Robust optimization methods provide tractable methods to capture uncertainty in design.

The advantages of robust optimization methods over stochastic optimization methods for optimization under uncertainty are summarized in,[1] and

TODO: Motivate the use of robust opt. for aircraft design over traditional methods, and stochastic/UQ.

Geometric programming is a method of log-convex optimization for which robust formulations exist. However, the stringent mathematical requirements of a Geometric Program (GP) limits its application to non-log-convex problems. The Signomial Program (SP) is the difference-of-log-convex extension of the GP which can be applied to solve this larger set of problems, albeit with the loss of some mathematical guarantees compared to the GP. In this paper, we propose a tractable Robust Signomial Program (RSP) which we solve as a sequential Robust Geometric Program (RGP), allowing us to implement robustness in non-log-convex problems.

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We implement the RSP formulation on a simple aircraft design problem with 19 free variables, 12 uncertain parameters and 17 constraints to demonstrate its potential. We believe that aircraft design problems can especially benefit from robustness. Oftentimes, aerospace engineers will implement margins in the design process to account for uncertainties in parameters that a design may be sensitive to, without explicit knowledge of the trade-off between robustness and optimality.[2]

TODO: More depth/references as to methods for UQ/RO in aircraft design.

A robust aircraft design formulation will allow designers to allocate margin more effectively to obtain better-performing designs with feasibility guarantees.

I.A. Geometric Programming

A **geometric program in posynomial form** is a log-convex optimization problem of the form:

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{u}) \\ \text{subject to} & f_i(\mathbf{u}) \le 1, i = 1, ..., m_p \\ & h_i(\mathbf{u}) = 1, i = 1, ..., m_e \end{array} \tag{1}$$

where each f_i is a *posynomial*, each h_i is a *monomial*, m_p is the number of posynomials, and m_e is the number of monomials. A monomial $h(\mathbf{u})$ is a function of the form:

$$h_i(\mathbf{u}) = e^{b_i}\prod_{j=1}^{n} u_j{}^{a_{ij}}$$

where a_{ij} is the j^{th} component of a row vector $\mathbf{a_i}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_i is in $\mathbb{R}$. A posynomial $f(\mathbf{u})$ is the sum of $K \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{u}) = \sum_{k=1}^{K} e^{b_{ikj}} \prod_{j=1}^{n} u_j{}^{a_{ikj}}$$

where a_{ikj} is the j^{th} component of a row vector $\mathbf{a_{ik}}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_{ik} is in $\mathbb{R}$.[3]

A logarithmic change of the variables $x_j = \log(u_j)$ would turn a monomial into *the exponential of an affine function* and a posynomial into *the sum of exponentials of affine functions*. A transformed monomial $h_i(\mathbf{x})$ is a function of the form:

$$h_i(\mathbf{x}) = e^{\mathbf{a_i}\mathbf{x}+b_i}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A transformed posynomial $f_i(\mathbf{x})$ is the sum of $K_i \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{x}) = \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}\mathbf{x}+b_{ik}}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A geometric program with transformed constraints is a **geometric program in exponential form**.

The positive nature of exponential functions restricts the space spanned by posynomials and limits the applications of GPs to certain classes of problems. The limited applicability of GPs has motivated the introduction of signomials.

I.B. Signomial Programming

A *signomial* can be defined as the difference between two posynomials, consequently, an SP is a non-log-convex optimization problem of the form:

$$\begin{array}{lll} \text{minimize} & f_0(\mathbf{x}) & \\ \text{subject to} & f_i(\mathbf{x}) - g_i(\mathbf{x}) & \le 0, i = 1, ..., m \end{array} \tag{2}$$

where f_i and g_i are both posynomials, and $\mathbf{x}$ is a column vector in $\mathbb{R}^n$.

Reliably solving an SP to a local optimum has been described in[3] and.[4] A common solution heuristic involves solving an SP as a sequence of GPs, where each GP is a local approximation of the SP. Although it is a powerful tool, applications involving SPs are usually prone to uncertainties that have a significant effect on the solution.

I.C. Overview

TODO: General optimization techniques for aircraft design

Signomial programming can cover constraints that might be neither linear, convex, nor log-covex and, hence, it can be used to model problems that cannot be formulated by standard optimization tools such as linear or geometric programs. Although global optimal solutions are not guaranteed, however, signomial programming is a powerful tool that is currently being used in modeling and solving complex aircraft designs quickly and reliably as in[5][6].[7] Other interesting applications for SPs such as in network flow problems are being investigated.

TODO: Motivate the use of robust opt. for aircraft design over traditional methods, and stochastic/UQ.

TODO: More depth/references as to methods for UQ/RO in aircraft design.

TODO: Sections and outline

II. Robust Signomial Programming

The parameters in an aircraft design problem are usually prone to uncertainties. Robust signomial programming (RSP) assumes that uncertainties are known to belong to an uncertainty set, and solves the design problem for the worst case scenario to find the best solution that is feasible to all possible realizations from the uncertainty set. This section introduces RSPs and derives the intractable formulation of an RSP.

A SP in its **exponential form** is as follows:

$$\begin{array}{ll} \min & \sum_{k=1}^{K_0} e^{\mathbf{a_{0k}}\mathbf{x}+b_{0k}} \\ \text{s.t.} & \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}\mathbf{x}+b_{ik}} - \sum_{k=1}^{K_i} e^{\mathbf{c_{ik}}\mathbf{x}+d_{ik}} \leq 0 \quad \forall i \in 1,...,m \end{array} \tag{3}$$

Let $\mathbf{A}$ and $\mathbf{C}$ be the exponents with components $\mathbf{a_{ik}}$ and $\mathbf{c_{ik}}$ respectively, and $\mathbf{b}$ and $\mathbf{d}$ be the coefficients with components b_{ik} and d_{ik} respectively.

TODO by Ali: Continue RSP definition

III. Robust Geometric Programming

TODO by Ali: A Brief review of RGP

IV. Approach to Solving RSPs

This section presents a heuristic algorithm to safely solve a robust signomial program based on our previous discussion on robust geometric programming.

IV.A. General RSP Solver

As mentioned before, a common heuristic algorithm to solve an SP is by sequentially solving local GP approximations, but the solution is not guaranteed to be globally optimal. Our approach to solve an RSP is based on sequentially solving local RGP approximations. Below we provide a step-by-step algorithm to solve an RSP:

1. Choose an initial guess x_0
2. Repeat
   (a) Find the local GP approximation of the SP at x_i.
   (b) Find the RGP formulation of the GP.
   (c) Solve the RGP to obtain x_{i+1}.
   (d) If $x_{i+1} \approx x_i$: break

Similar to an SP, a good initial guess would lead to faster convergence and possibly a better solution. A quick candidate is the deterministic solution of the uncertain SP, which will certainly lead to a faster convergence.

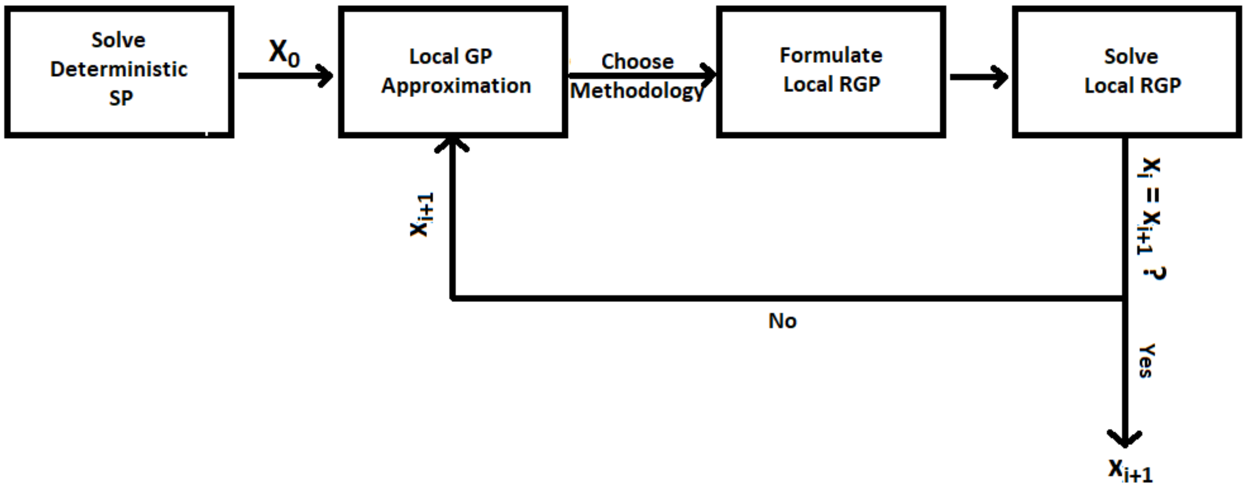


Figure 1: A block diagram showing the steps of solving an RSP

Any of the previously mentioned methodologies can be used to formulate the local RGP approximation. However, Depending on the RGP formulation chosen to solve an RSP, the last two blocks in Figure **??** are tweaked slightly for a faster convergence.

IV.B. Best Pairs RSP Solver

If the Best Pairs methodology is exploited, then the above algorithm would change so that each iteration would solve the local RGP approximation and choose the best permutation for each large posynomial. The modified algorithm would become as follows:

1. Choose an initial guess x_0
2. Repeat
   (a) Find the local GP approximation of the SP at x_i.
   (b) $\forall (i,j) \in \mathbf{P}$, select the new permutations $\phi \in \hat{\mathcal{P}}_{i,j}$ such that ϕ minimizes $\sum_{k=1}^{|S_{i,j}|/2} \max\limits_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi_{2k-1}}} + e^{\mathcal{L}_{i,j}^{\phi_{2k}}} \right\} \Big|_{\mathbf{x}_i}$
   (c) Solve the approximate tractable counterparts of the local GP in (**??**), and let $\mathbf{x}_{i+1}$ be the solution
   (d) If $x_{i+1} \approx x_i$: break

IV.C. Linearized Perturbations RSP Solver

On the other hand, if the Linearized Perturbations formulation is to be used, then we can avoid solving a signomial program at each iteration by first approximating the original SP constraints locally, and in the same loop approximating the robustified possibly signomial constraints locally, thus solving a GP at each iteration instead of an SP. The algorithm would then become as follows:

1. Choose an initial guess x_0
2. Repeat
   (a) Find the local GP approximation of the SP at x_i.
   (b) Robustify the constraints of the local GP approximation using the Linearized Perturbations methodology.
   (c) Find the local GP approximation of the resulting local SP at x_i.
   (d) Solve the local GP approximation in step c to obtain x_{i+1}
   (e) If $x_{i+1} \approx x_i$: break

V. Models

We implemented the RSP formulation ideas above on a simple aircraft design problem, with 12 uncertain variables, and a single signomial constraint. In the simple aircraft problem, we conduct an aerostructural optimization of a wing and fuselage given a payload and range requirement. A short overview of the model follows.

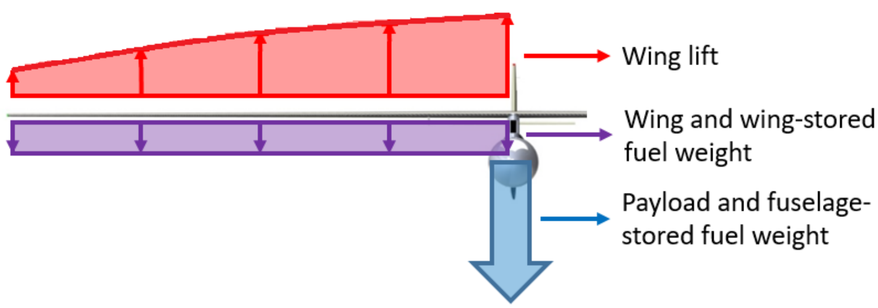
V.A. Lift, Weight, Drag and Thrust

The aircraft is assumed to be in steady, level flight. As a result, we can assume that the thrust generated by the aircraft is equal to the drag, and the lift generated by the wing is equal to the total weight.

The drag is the sum of the wing (induced and profile) drag, and the fuselage drag, which is linearly proportional to the fuel volume in the fuselage. The aircraft model does not assume limitations on thrust, but instead assumes constant thrust specific fuel consumption, at 0.6 lbs for each lb*hour of thrust, with associated uncertainty.

The weight of the aircraft is the sum of the payload weight, wing weight, and the fuselage weight, as shown in Figure 2. Lift is generated by the wing, which is described by an aspect ratio AR and surface area S.

Figure 2: Wing lift is equal and opposite to the wing weight, payload weight, and total fuel weight.



V.B. Wing Structure

The wing structure model is based on a simple beam model with a distributed lift load, and a point mass in the center representing the fuselage, as shown in Figure 2.

V.C. Fuel Volume

The fuel in the aircraft can be stored either in the wing or the fuselage. The signomial constraint in the optimization appears in the fuel volume model, as shown in Equation 6:

$$V_f \leq V_{f_{wing}} + V_{f_{fuse}} \tag{4}$$

where $V_{f_{wing}}$ and $V_{f_{fuse}}$ represent the fuel volume available in the wing and the fuselage respectively. They are each represented by the following monomials.

$$V_{f_{wing}} <= 9e^{-4} \frac{S^{1.5}\tau}{A^{0.5}} \tag{5}$$

$$V_{f_{fuse}} <= 10 \times CDA_0 \; m \tag{6}$$

Note that the monomials above are represented with inequalities, to be compatible with the RSP formulation.

V.D. Takeoff constraints

We specify that the aircraft has to be able to takeoff at a speed of V_{min} without exceeding the aircraft stall lift coefficient $C_{L_{max}}$, both of which are specified with an associated uncertainty.

VI. Uncertainties and Sets

The uncertainties for the different constants in the problem have been determined considering the parameters in aircraft design that often have the largest uncertainty. These uncertainties are listed in Table 2.

Table 1: Constants and Uncertainties (increasing order)

Constant	Description	Value	% Uncert. (3σ)
$S_{wetratio}$	wetted area ratio	2.075	3
e	span efficiency	0.92	3
μ	viscosity of air	1.775e-5 $kg/(ms)$	4
ρ	air density	1.23 kg/m^3	5
$C_{L_{max}}$	stall lift coefficient	1.6	5
k	fuselage form factor	1.17	10
τ	airfoil thickness ratio	0.12	10
N_{ult}	ultimate load factor	3.3	15
V_{min}	takeoff speed	25 m/s	20
W_0	payload weight	6250 N	20
$W_{w_{coeff1}}$	wing weight coefficient 1	2e-5 $1/m$	20
$W_{w_{coeff2}}$	wing weight coefficient 2	60 N/m^2	20

The parameter uncertainties reflect aerospace engineering intuition. The wing weight coefficients $W_{w_{coeff1}}$ and $W_{w_{coeff2}}$, and the ultimate load factor N_{ult} have large 3σs because build quality of aircraft components often difficult to quantify with a large degree of certainty. The payload weight (W_0) has a large uncertainty, because it is valuable if the aircraft has the flexibility to accommodate larger payloads. Parameters that engineers take to be physical constants (μ, ρ) and those that can be determined/manufactured with a relatively high degree of accuracy ($S_{wetratio}$, e) have relatively low deviations. Parameters that require testing to determine ($C_{L_{max}}$, V_{min}) have a level of uncertainty that reflects the expected variance of the parameters.

VII. Simple Aircraft Design

We finally implemented our RSP heuristic algorithm on a simple aircraft design problem. In this problem, we conduct an aerostructural optimization of a wing and fuselage given a payload and a range requirement.

VII.A. Model Description

Below we briefly describe the different sub-models of our simple aircraft design, and then specify the different uncertainties in this model.

VII.A.1. Lift, Weight, Drag and Thrust

This aircraft is assumed to be in a steady, level flight. Accordingly, we can assume that the thrust generated by the aircraft is equal to the drag, and the lift generated by the wing is equal to the total weight.

Both the wing and the fuselage contribute to the total drag in our model. The wing drag is divided into induced and profile drag, while the fuselage drag is proportional to the fuel volume in the fuselage. Note that the model does not assume limitations on thrust, but instead assumes constant thrust specific fuel consumption. Moreover, the model assumes constant airfoil thickness to chord ratio.

The weight of the aircraft is the sum of the payload weight, wing weight, and the fuselage weight, as shown in Figure 2. Lift is generated by the wing, which is described by an aspect ratio A and surface area S.

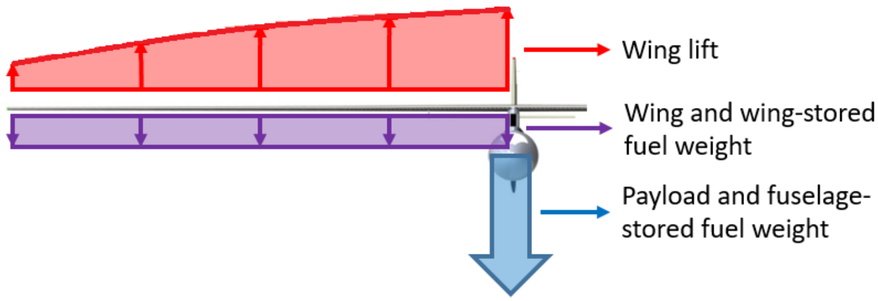


Figure 3: Wing lift is equal and opposite to the wing weight, payload weight, and total fuel weight.

VII.A.2. Wing Structure

The wing structure model is based on a simple beam model with a distributed lift load, and a point mass in the center representing the fuselage, as shown in Figure 2.

VII.A.3. Fuel Volume

The fuel in the aircraft can be stored either in the wing or in the fuselage. The signomial constraint in the optimization appears in the fuel volume model as in Equation 6:

$$V_f \leq V_{f_{wing}} + V_{f_{fuse}} \tag{7}$$

where $V_{f_{wing}}$ and $V_{f_{fuse}}$ represent the fuel volume available in the wing and the fuselage respectively. They are each represented by the following monomial constraints.

$$V_{f_{wing}} <= 3e^{-2} \frac{S^{1.5}\tau}{A^{0.5}} \tag{8}$$

$$V_{f_{fuse}} <= 10 \times CDA_0 \tag{9}$$

Where S is the total wing area, τ is the airfoil thickness ratio, A is the wing aspect ratio, and CDA_0 is the fuselage drag area. Note that the monomials above are represented with inequalities, to be compatible with the RSP formulation.

VII.A.4. Takeoff constraints

We specify that the aircraft has to be able to takeoff at a speed of V_{min} without exceeding the aircraft stall lift coefficient $C_{L_{max}}$, both of which are specified with an associated uncertainty.

VII.A.5. Uncertainties

The uncertainties for the different constants in the problem have been determined considering the parameters in aircraft design that often have the largest uncertainty. These uncertainties are listed in Table 2.

Uncertain Parameter	Value	Description
$S_{wetratio}$	$2.075 \pm 3\%$	wetted area ratio
e	$0.920 \pm 3\%$	span efficiency
μ	$1.775e^{-5}$ $[kg/(ms)]$ $\pm 4\%$	viscosity of air
ρ	1.230 $[kg/m^3]$ $\pm 5\%$	air density
$C_{L_{max}}$	$1.600 \pm 5\%$	stall lift coefficient
k	$1.170 \pm 10\%$	fuselage form factor
τ	$0.120 \pm 10\%$	airfoil thickness ratio
N_{ult}	$3.300 \pm 15\%$	ultimate load factor
V_{min}	25.00 $[m/s]$ $\pm 20\%$	takeoff speed
W_0	6250 $[N]$ $\pm 20\%$	payload weight
$W_{w_{coeff1}}$	$2.000e^{-5}$ $[\frac{1}{m}]$ $\pm 20\%$	Wing Weight Coefficient 1
$W_{w_{coeff2}}$	60.00 $[Pa]$ $\pm 20\%$	Wing Weight Coefficient 2

Table 2: Uncertain parameters in the signomial simple aircraft design.

The parameter uncertainties reflect aerospace engineering intuition. The wing weight coefficients $W_{w_{coeff1}}$ and $W_{w_{coeff2}}$, and the ultimate load factor N_{ult} have large uncertainty because the build quality of aircraft components is often difficult to quantify with a large degree of certainty. The payload weight (W_0) has a large uncertainty, because it is valuable if the aircraft has the flexibility to accommodate larger payloads. Parameters that engineers take to be physical constants (μ, ρ) and those that can be determined/manufactured with a relatively high degree of accuracy ($S_{wetratio}$, e) have relatively low deviations. Parameters that require testing to determine ($C_{L_{max}}$, V_{min}) have a level of uncertainty that reflects the expected variance of the parameters.

VII.B. Optimization Results

The problem is optimized for different sizes of box and elliptical uncertainty sets by varying the parameter Γ as defined in Appendix VII.A. The design variables are then fixed for each solution so that the design can be simulated for 1000 different realizations of the uncertain parameters in table 2 to examine average design performance.

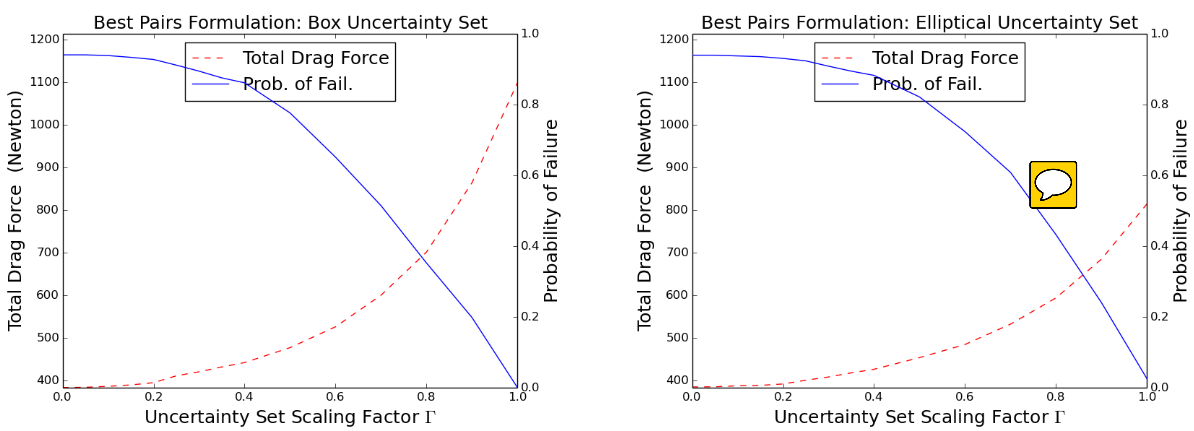


Figure 4: Performance of the optimal robust signomial simple aircraft, using the Best Pairs formulation, as a function of Γ for different uncertainty sets.

We can see from Figure 3 that probability of failure goes to zero as Γ increases. Obviously, it is worth using elliptical uncertainty sets for this aircraft design problem as the performance is significantly better than that of a box uncertainty set, despite the increase in complexity. Moreover, using margins would in the best case be as good as using a box uncertaintyset, and therefore will lead to an inferior performance.

Figure 4 compares the different methodologies in terms of run times, number of constraints, and average performance. The Best Pairs and Linearized Perturbations achieves good performance, however the Best Pairs methodology needs the most number of constraints, while the Linearized perturbations requires the most setup and solve time. The Simple Conservative formulation is significantly faster than the other formulations and requires the least number of additional constraints.

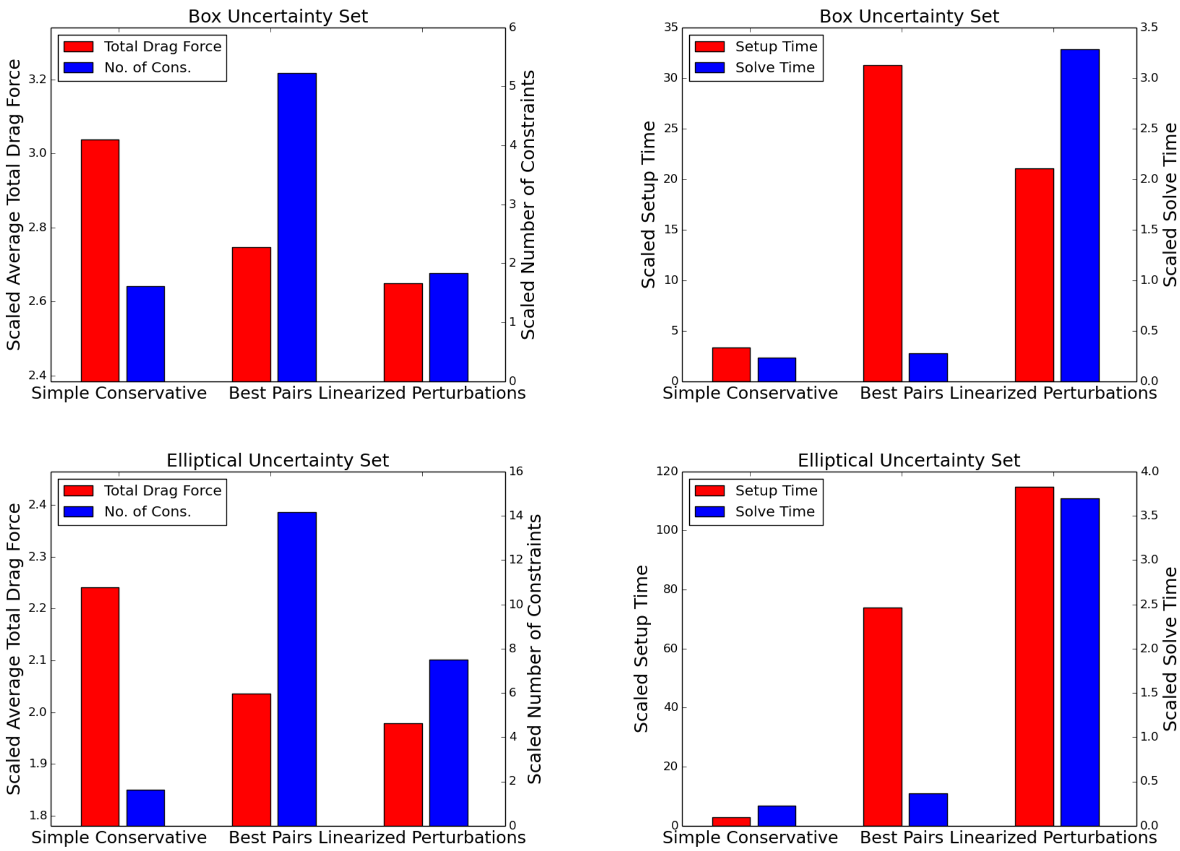


Figure 5: Robust signomial simple aircraft design results relative to the deterministic design problem.

VII.C. The Effect of Robustness

To further demonstrate the capabilities of robust SPs in aircraft design, we performed the optimization of the aircraft with no uncertainty and ellipsoidal uncertainty ($\Gamma = 1$) for two more objective functions, and plotted the results on spider plots. Spider plots are useful because they allow engineers to find non-dominated solutions among the solutions that lie on the Pareto frontier of potential objective functions. The objective functions chosen for this analysis were fuel burn over lift-to-drag ratio ($\frac{W_f}{L/D}$), drag (D), and fuel burn (W_f).

In the spider plots in Figures 5 and 6, none of the solutions are non-dominated for both the no uncertainty and ellipsoidal uncertainty cases. In a GP, it is not possible that one of the solutions is non-dominated since the solutions are globally optimal. But since there is no guarantee in optimality for SPs, it is possible to find non-dominated solutions if the obtained solution is a local optimum.

In the case where there is no non-dominated solution such as this one, we take the internal areas of the triangle formed by each optimization to be the figure of merit. The smaller the area in the triangle, the higher the performance of the proposed solution. In the no uncertainty case shown in Figure 5, the red solution with the objective of $\frac{W_f}{L/D}$ has the smallest internal area. In the ellipsoidal case shown in Figure 6, the blue solution with the objective of D has the smallest internal area.

This is an interesting result, because the presence of an uncertainty set is shown to affect the efficacy of

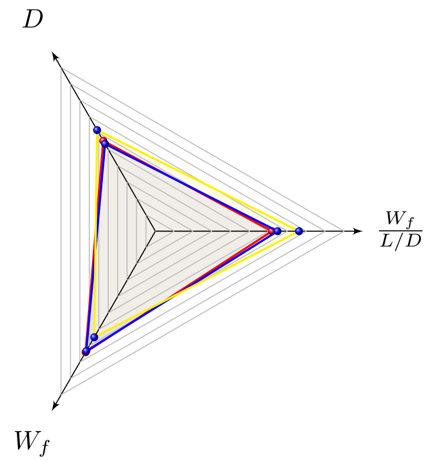


Figure 6: Design optimization of the aircraft with no uncertainty for 3 different objective functions. The red, blue and yellow correspond to $\frac{W_f}{L/D}$, D, W_f and objectives respectively.

different objective functions to obtain solutions with the best overall performance. The differences between the objective functions in the simple aircraft design problem are minute, because the different potential objectives have a high degree of coupling. It is likely that, if the three objective functions didn't have high degree of coupling, that the internal areas of the solution triangles may differ more significantly.

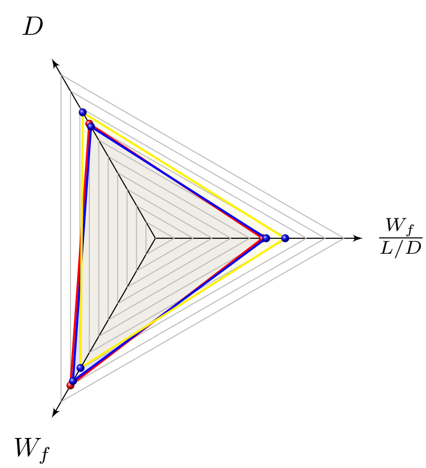


Figure 7: Design optimization of the aircraft with ellipsoidal uncertainty for 3 different objective functions. The red, blue and yellow correspond to $\frac{W_f}{L/D}$, D, W_f and objectives respectively.

VIII. Conclusion

We have developed and applied a tractable glsrsp formulation to a simple aircraft model, and then discussed the benefits of having robust solutions. RSP formulations extend the tractable approximate RGP framework developed by Saab to non-log-convex problems, and are a valuable contribution to the fields of robust optimization and difference-of-convex programming.

RSPs have a wide variety of potential applications in engineering design. Within the Hoburg Research Group in the Aerospace Computational Design Lab, during the past year we have developed a commercial aircraft design SP that has between 1700 and 8000 variables, depending on the whether it is a single-point, or multi-point optimization. We expect that using RO in this conceptual aircraft design will result in designs that are more robust with respect to uncertainties in operational parameters, such as payload mass and range, as well as uncertain constants.

Appendix

VIII.A. Robust Linear Programming: A Quick Review

As mentioned earlier, robust linear programming will be used to formulate an approximate robust geometric program.

Consider the system of linear constraints

$$\mathbb{A}\mathbf{x} + \mathbf{b} \leq 0$$

where

$$\begin{aligned} &\mathbb{A} \text{ is } m \times n \\ &\mathbf{x} \text{ is } n \times 1 \\ &\mathbf{b} \text{ is } m \times 1 \end{aligned}$$

where that data is uncertain and is given by equations (**??**) and (**??**).

VIII.A.1. Box Uncertainty Set

If the perturbation set $\mathcal{Z}$ given in equation (**??**) is a box uncertainty set, i.e. $\|\zeta\|_\infty \leq \Gamma$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \textstyle\sum_{l=1}^{L} |-b_i^l - \mathbf{a}_i^l \mathbf{x}| + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \tag{10}$$

If only b is uncertain, i.e. $A^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (9) will become

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \textstyle\sum_{l=1}^{L} |b_i^l| \leq 0 \tag{11}$$

which is a linear constraint
On the other hand, if A is uncertain, then equation (9) is equivalent to the following set of linear constraints

$$\begin{aligned} \Gamma \textstyle\sum_{l=1}^{L} w_i^l + \mathbf{a}_i^0 \mathbf{x} + b_i^0 &\leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} &\leq w_i^l \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} &\leq w_i^l \quad \forall l \in 1, ..., L \end{aligned} \tag{12}$$

VIII.A.2. Elliptical Uncertainty Set

Briefly, if the perturbation set $\mathcal{Z}$ is an elliptical, i.e. $\sum_{l=1}^{L} \frac{\zeta_l^2}{\sigma_l^2} \leq \Gamma^2$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \sqrt{\textstyle\sum_{l=1}^{L} \sigma_l^2 (-b_i^l - \mathbf{a}_i^l \mathbf{x})^2} + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \tag{13}$$

which is a second order conic constraint.
If only b is uncertain, i.e. $\mathbb{A}^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (12) will become

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \sqrt{\textstyle\sum_{l=1}^{L} \sigma_l^2 (b_i^l)^2} \leq 0 \tag{14}$$

which is a linear constraint.

VIII.A.3. Norm-1 Uncertainty Sets

Briefly, if the perturbation set represented by $\mathcal{Z}$ is a norm-1 uncertainty set, i.e. $\|\zeta\|_1 \leq \Gamma$, then the robust constraint is

$$\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \max_{l=1,..,L} |b_i^l| \leq 0 \tag{15}$$

when $\mathbb{A}^l = 0$, and

$$\begin{aligned} \Gamma w_i + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} \leq w_i \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} \leq w_i \quad \forall l \in 1, ..., L \end{aligned} \tag{16}$$

if $\mathbb{A}^l \neq 0$
Note that for this type of uncertainty, the robust constraints are linear.

Acknowledgments

A place to recognize others.

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