Optimal Design Decisions via Robust Signomial Programming

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TODO: Complete rework to frame around AC design.

Signomial programming is useful in multidisciplinary non-convex optimization problems such as aircraft design. The formulation and solution of robust signomial programs (RSPs) would be beneficial since many parameters involved in these problems are prone to uncertainty, and can have significant effects on solution performance and feasibility. This paper proposes an approximate solution for an RSP leveraging an existing approximate robust geometric programming (RGP) formulation developed by Saab. The method is based on solving a sequence of RGPs, where each GP is a local approximation of the SP. Moreover, this paper also discusses the trade-off between robustness and optimality by implementing RSPs on a simple aircraft problem, and demonstrates how robust optimization affects aircraft design decisions.

Nomenclature

J	Jacobian Matrix
f	Residual value vector
x	Variable value vector
F	Force, N
m	Mass, kg
Δx	Variable displacement vector
α	Acceleration, m/s^2

Subscript
i	Variable number

I.   introduction

Robust optimization methods provide tractable methods to capture uncertainty in design.

TODO: Motivate the use of robust opt. for aircraft design over traditional methods, and stochastic/UQ.

Geometric programs [GPs] are a method of log-convex optimization for which robust formulations exist. However, their stringent mathematical requirements limit their application in non-log-convex problems. In this paper, we propose a tractable robust signomial program [RSP] which we solve as a sequential robust geometric program, allowing us to implement robustness in non-log-convex problems.

We implement the RSP formulation on a simple aircraft design problem with 19 free variables, 12 uncertain parameters and 17 constraints to demonstrate its potential. We believe that aircraft design problems can especially benefit from robustness. Oftentimes, aerospace engineers will implement margins in the design process to account for uncertainties in parameters that a design may be sensitive to, without explicit knowledge of the trade-off between robustness and optimality. A robust aircraft design formulation will allow designers to allocate margin more effectively to obtain better-performing designs with feasibility guarantees.

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I.A.   Geometric Programs

Geometric programs are a method for log-convex optimization. They offer globally optimal solutions with no initial guesses, and allow engineers and designers to solve large-scale (>10,000 variables) nonlinear problems on the order of ~ 1 second, given some stringent mathematical requirements. Given the general optimization problem below:

$$
\begin{aligned}
& \text{minimize } f_0(\mathbf{x}) \\
& \text{subject to } f_i(\mathbf{x}) \leq 1, i = 1, ..., m \\
& \qquad\qquad g_i(\mathbf{x}) = 1, i = 1, ..., p
\end{aligned}
\tag{1}
$$

In a GP, f_is and g_is must have special posynomial and monomial forms below, with strictly positive coefficients:

A monomial is a function of the form

$$
g(\mathbf{x}) = e^{\mathbf{a}\mathbf{x}+b}
\tag{2}
$$

where $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

A posynomial is a function of the form

$$
f(\mathbf{x}) = \sum_{k=1}^{K} e^{\mathbf{a}_k \mathbf{x}+b_k}
\tag{3}
$$

where $\mathbf{a}_k \in \mathbb{R}^n, b_k \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. A posynomial is a sum of monomials. Therefore, all monomials are also one-term posynomials.

Any monomial constraint $g(\mathbf{x}) = 1$ can be represented as $g(\mathbf{x}) \leq 1$ and $\frac{1}{g(\mathbf{x})} \leq 1$, therefore, the standard GP can be written in the inequality constrained form as follows:

$$
\begin{aligned}
& \text{minimize} \quad \sum_{k=1}^{K_0} e^{\mathbf{a}_{0k}\mathbf{x}+b_{0k}} \\
& \text{subject to} \quad \sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}} \leq 1 \quad \forall i \in 1, ..., m
\end{aligned}
\tag{4}
$$

A tractable approximation of robust GPs (RGPs) exists and will be discussed, and have been formulated with polyhedral and ellipsoidal uncertainty sets. RGPs allow for uncertainty in both the coefficients b_k and the exponents $\mathbf{a}_k$. However, the restriction on the sign of the coefficients limits GPs (and consequently RGPs) to certain classes of problems. Signomials are constraints of the same form as $f(x)$, but allow for negative coefficients in $f(x)$. To be able to find robust solutions to signomial programs [SPs], we would like to create a framework to generalize the uncertainty sets used in RGPs to RSPs.

I.B.   Signomial Programs

Signomials allow us to solve non-log-convex problems as sequential geometric programs. Signomials are a difference of posynomials shown in Equation 3. The log transform of an SP is not a convex optimization problem, but instead a difference of convex optimization problem that can be written in log-space as

$$
\begin{aligned}
& \text{minimize } f_0(\mathbf{x}) \\
& \text{subject to } f_i(\mathbf{x}) - h_i(\mathbf{x}) \leq 0, i = 1, ..., m
\end{aligned}
\tag{5}
$$

where f_i and h_i are posynomials. This problem can be reliably solved iteratively by taking the monomial approximation of g_i with a solution or initial guess x_i, solving this GP approximation to obtain a new solution x_{i+1}, and repeating the process until some convergence parameter is satisfied in the objective function.

The SP algorithm is well-studied, reliably solving SPs with an initial guess of all 1's. SPs are guaranteed to be sub-optimal but feasible solutions to a general non-linear problem.

II.   Approach to Solving RSPs

Our approach to solving the RSP is the following:

1. Solve the SP with no uncertainty, initialized by a vector of 1's. We call this solution x_0.

2. Solve the SP with a chosen uncertainty set, using x_0 as the initial guess.

Step 2 requires the retuning of the SP algorithm as follows:

Initialize $x_0 = \mathbf{1}$
While $reltol \geq 1e - 4$:

1. Substitute x_i as the initial guess to the SP.

2. Find the RGP approximation to the SP with guess x_i.

3. Solve the RGP to obtain x_{i+1}.

4. Calculate $reltol$.

$reltol$ is the relative change in the objective function value for the GP. The RSP algorithm solves reliably for a range of $reltol$s, but a value of $1e - 4$ was chosen for the models evaluated in this paper.

III.   Robust Geometric Programming

III.A.   Decoupled Form

WOLG the objective function could be assumed linear (one dimensional variable) and deprived of any data (epigraph formulation), therefore the cost will be ignored throughout this report. Moreover, the GP will be represented as follows

$$\begin{aligned}
\sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}} &\leq 1 \quad \forall i \in \mathbf{P} \\
e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}} + e^{\mathbf{a}_{i2}\mathbf{x}+b_{i2}} &\leq 1 \quad \forall i \in \mathbf{M} \\
e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}} &\leq 1 \quad \forall i \in \mathbf{N}
\end{aligned} \tag{6}$$

where

- $\mathbf{P} = \{i : \mathbf{K}_i > 2\}$

- $\mathbf{N} = \{i : \mathbf{K}_i = 2\}$

- $\mathbf{M} = \{i : \mathbf{K}_i = 1\}$

The GP in equation (6) in convex form is represented as follows

$$\begin{aligned}
\log(\sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}}) &\leq 0 \quad \forall i \in \mathbf{P} \\
\log(e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}} + e^{\mathbf{a}_{i2}\mathbf{x}+b_{i2}}) &\leq 0 \quad \forall i \in \mathbf{N} \\
\mathbf{a}_{i1}\mathbf{x} + b_{i1} &\leq 0 \quad \forall i \in \mathbf{M}
\end{aligned} \tag{7}$$

III.B.   Robust Counterpart

In robust optimization, a formulation that is immune to the uncertainty in the system's data should be derived. The data will be assumed living in an uncertainty set $\mathcal{U}$, where $\mathcal{U}$ is parameterized affinly by a perturbation vector ζ as follows

$$\mathcal{U} = \left\{ [\mathbf{A}; \mathbf{b}] = [\mathbf{A}^0; \mathbf{b}^0] + \sum_{l=1}^{L} \zeta_l [\mathbf{A}^l; \mathbf{b}^l] \right\} \tag{8}$$

where ζ belongs to some perturbation set $\mathcal{Z} \in \mathbb{R}^L$ such that

$$\mathcal{Z} = \left\{ \zeta \in \mathbb{R}^L : \exists \mathbf{u} \in \mathbb{R}^k : \mathbf{P}\zeta + \mathbf{Q}\mathbf{u} + \mathbf{p} \in \mathbf{K} \right\} \tag{9}$$

The robust counterpart of the uncertain geometric program given by equation (6) is

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$$\sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}} \leq 1 \quad \forall i \in \mathbf{P} \quad \forall \zeta \in \mathcal{Z} \qquad\qquad \max_{\zeta \in \mathcal{Z}}\left\{\sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}}\right\} \qquad \leq 1 \quad \forall i \in \mathbf{P}$$

$$e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}} + e^{\mathbf{a}_{i2}\mathbf{x}+b_{i2}} \leq 1 \quad \forall i \in \mathbf{M} \quad \forall \zeta \in \mathcal{Z} \quad (10) \qquad \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}} + e^{\mathbf{a}_{i2}\mathbf{x}+b_{i2}}\right\} \quad \leq 1 \quad \forall i \in \mathbf{M} \quad (11)$$

$$e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}} \leq 1 \quad \forall i \in \mathbf{N} \quad \forall \zeta \in \mathcal{Z}. \qquad\qquad \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}}\right\} \qquad\qquad \leq 1 \quad \forall i \in \mathbf{N}$$

The above two sets of constraints state that the robust optimal solution should be feasible for all possible realizations of the perturbation vector ζ. Unfortunately, the robust counterpart of a geometric program is intractable using current solvers. Throughout this report, an approximate formulation will be derived using robust linear programming.

III.C.   Conservative Tractable Formulation

Since it is intractable to solve a robust geometric program, this section will discuss a simple approximate formulation of the robust counterparts given by (11).

The constraints corresponding to the set $\mathbf{M}$ of constraints are linear and therefore tractable, as a result, we should deal with the sets $\mathbf{P}$ and $\mathbf{N}$ only.

The fact that $\max_{\zeta \in \mathcal{Z}}\left\{\sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}}\right\} \leq \sum_{k=1}^{K_i} \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}}\right\}$ suggests the following safe constraints for the constraints corresponding to elements in $\mathbf{P}$ and $\mathbf{N}$

$$\sum_{k=1}^{K_i} \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}}\right\} \leq 1 \tag{12}$$

Equation (12) suggests the following "conservative" formulation of the robust counterpart for the uncertain geometric program

$$\begin{aligned} \sum_{k=1}^{K_i} \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}}\right\} &\leq 1 \quad \forall i \in \mathbf{P}, \mathbf{M} \\ \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}}\right\} &\leq 1 \quad \forall i \in \mathbf{N} \end{aligned} \tag{13}$$

which is equivalent to

$$\begin{aligned} \sum_{k=1}^{K_i} t_{ik} &\leq 1 \quad \forall i \in \mathbf{P}, \mathbf{M} \\ \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}}\right\} &\leq t_{ik} \quad \forall i \in \mathbf{P}, \mathbf{M} \quad \forall k \in 1,...,K_i \\ \max_{\zeta \in \mathcal{Z}}\left\{e^{\mathbf{a}_{i1}\mathbf{x}+b_{i1}}\right\} &\leq 1 \quad \forall i \in \mathbf{N} \end{aligned} \tag{14}$$

In log-space, Equation (14) is

$$\begin{aligned} \log\left(\sum_{k=1}^{K_i} e^{s_{ik}}\right) &\leq 0 \quad \forall i \in \mathbf{P}, \mathbf{M} \\ \max_{\zeta \in \mathcal{Z}}\left\{\mathbf{a}_{ik}\mathbf{x} + b_{ik}\right\} &\leq s_{ik} \quad \forall i \in \mathbf{P}, \mathbf{M} \quad \forall k \in 1,...,\mathbf{K}_i \\ \max_{\zeta \in \mathcal{Z}}\left\{\mathbf{a}_{i1}\mathbf{x} + b_{i1}\right\} &\leq 0 \quad \forall i \in \mathbf{N} \end{aligned} \tag{15}$$

It can be observed that the convex constraints in equation (15) are deprived of data and uncertainty is only present in the linear constraints. As a result, this problem is tractable using robust linear programming.

Although this formulation might seem too conservative for some problems due to the fact that monomials are being decoupled, however, it is exact for a wide range of problems that satisfy the following criteria

- C_1: The perturbation set is independent, e.g. $\mathcal{Z} = \left\{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq \Gamma\right\}$

- C_2: The monomials in each posynomial are independent, and if dependence only exists between the $'b's$, then it should be "good" dependence, e.g. if $b_{11} = b_{11}^0 + \zeta_1$, $b_{12} = b_{12}^0 + \zeta_1 - \zeta_2$, and $b_{13} = b_{13}^0 + \zeta_2$, then b_{11} and b_{12} are dependent, but the dependence is good since the sign of the coefficients multiplied by the perturbation ζ_1 are the same. However, b_{12} and b_{13} are dependent, but in a bad way due to the fact that the signs of the coefficients multiplied by the perturbation ζ_2 are different

When C_1 and C_2 are satisfied for the ith posynomial, then

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x} + b_{ik}} \right\} = \sum_{k=1}^{K_i} \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathbf{a}_{ik}\mathbf{x} + b_{ik}} \right\}$$

and the "conservative" formulation is no longer conservative, but exact.

III.D.   Robust Two Term Posynomials

The focus now will be on modifying the methodology provided in section 3.3 so that the solution is less conservative. This section will review Boyd's work on approximating two term posynomials in log-space using piece-wise linear functions [a].

Consider the convex function $\phi(x) = \log(1 + e^x)$, then the unique best r-term piece-wise linear convex lower approximation of ϕ is

$$\underline{\phi_r} = \begin{cases} 0 & \text{if} & x \in (-\infty, x_1] \\ \underline{a_i}x + \underline{b_i} & \text{if} & x \in [x_i, x_{i+1}], i = 1, 2, .., r - 2 \\ x & \text{if} & x \in [x_{r-1}, \infty) \end{cases} \tag{16}$$

such that

- $x_1 < x_2 < ... < x_{r-1}$

- $\underline{a_0} = 0 < \underline{a_1} < \underline{a_2} < ... < \underline{a_{r-2}} < \underline{a_{r-1}} = 1$

- $\underline{a_i} + \underline{a_{r-i-1}} = 1 \quad \forall i \in \{0, 1, ..., r - 1\}$

- $\underline{b_i} = \underline{b_{r-i-1}} \quad \forall i \in \{1, ..., r - 2\}$

- $\underline{b_0} = \underline{b_{r-1}} = 0$

Moreover, $\exists \ \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{r-2} \in \mathbf{R}$ satisfying

$$x_1 < \tilde{x}_1 < x_2 < \tilde{x}_2 < ... < x_{r-2} < \tilde{x}_{r-2} < x_{r-1}$$

such that $\underline{a_i}x + \underline{b_i}$ is tangent to ϕ at $\tilde{x}_i$.
Finally, the maximum approximation error ϵ_r of this piece-wise linearization occurs at the break points $x_1, ..., x_{r-1}$. The piece-wise linearization above will be used in approximating two term posynomials using piece-wise linear functions.

Let $h = \log(e^{y_1} + e^{y_2})$ be a two term posynomial in log-space, where $y_1 = \mathbf{a_1}\mathbf{x} + b_1$ and $y_2 = \mathbf{a_2}\mathbf{x} + b_2$, then the unique best r-term piece-wise linear lower approximation is

$$\underline{h_r} = \max\{\underline{a_{r-1}}y_1 + \underline{a_0}y_2 + \underline{b_0}, \underline{a_{r-2}}y_1 + \underline{a_1}y_2 + \underline{b_1}, \underline{a_{r-3}}y_1 + \underline{a_2}y_2 + \underline{b_2}, \ ..., $$
$$\underline{a_1}y_1 + \underline{a_{r-2}}y_2 + \underline{b_{r-2}}, \underline{a_0}y_1 + \underline{a_{r-1}}y_2 + \underline{b_{r-1}}\} \tag{17}$$

while its unique best r-term piece-wise linear upper approximation is

$$\overline{h_r} = \underline{h_r} + \epsilon_r \tag{18}$$

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[a]Kan-Lin Hsiung, Seung-Jean Kim, and Stephen Boyd. Tractable approximate robust geometric programming. *Optimization and Engineering*, 9(2):95118, Apr 2007

where $\underline{a}_0, \underline{a}_1, \underline{a}_2, ..., \underline{a}_{r-2}$ and $\underline{b}_1, \underline{b}_2, ..., \underline{b}_{r-2}, \underline{a}_{r-1}$ are as given in equation (16), and ϵ_r is the maximum error between ϕ and $\underline{\phi}_r$.

As a result, we know that $\overline{h}_r \geq h$, and therefore each posynomial in the set $\mathbf{N}$ is replaced by its best r-term piece-wise linear upper approximation (a safe approximation).

This methodology needs to throw a sufficient number of constraints if a less conservative solution (better than the previous formulation) is to be achieved, however, it accounts for any dependency between the two monomials in a two term posynomial, and the problem will become tractable since a piece-wise linear constraint could be represented as a set of linear constraints.

III.E.   Robust Large Term Posynomials

After taking care of the monomials and two term posynomials in a GP, its time to deal with the posynomial constraints corresponding to the set $\mathbf{P}$.

The first step is to divide the large posynomial into smaller posynomials if possible. To do so, consider the set $\mathbf{I}_i$ associated with a large posynomial p_i, $i \in \mathbf{P}$ where

$$\mathbf{I}_i = \{1, 2, ..., K_i\} \tag{19}$$

Then define an equivalence relation $\mathcal{R}$ on the set $\mathbf{I}_i$ where

$$\mathcal{R} = \left\{ k_1 \sim k_2 \iff e^{\mathbf{a}_{ik_1}\mathbf{x}+b_{ik_1}} \text{ and } e^{\mathbf{a}_{ik_2}\mathbf{x}+b_{ik_2}} \text{ are directly or indirectly dependent} \right\} \tag{20}$$

It is clear that $\mathcal{R}$ is an equivalence relation, therefore, when applied on $\mathbf{I}_i$, $\mathbf{I}_i$ would split into equivalence classes $S_{i,1}, S_{i,2}, ..., S_{i,N_e^i}$, $N_e^i \leq K_i$, then

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}} \right\} = \sum_{j=1}^{N_e^i} \max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k \in S_{i,j}} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}} \right\}$$

and therefore, the constraints corresponding to $\mathbf{P}$ in equation (11) will be replaced by the following equivalent set of constraints

$$\begin{aligned} \sum_{j=1}^{N_e^i} t_{ij} &\leq 1 \qquad \forall i \in \mathbf{P} \\ \max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k \in S_{i,j}} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}} \right\} &\leq t_{ij} \qquad \forall i \in \mathbf{P} \qquad \forall j = 1, ..., N_e^i \end{aligned} \tag{21}$$

Let

- $\mathbf{P_i'} = \{j : |S_{i,j}| \geq 3\} \rightarrow$ intractable

- $\mathbf{M_i'} = \{j : |S_{i,j}| = 2\} \rightarrow$ tractable

- $\mathbf{N_i'} = \{j : |S_{i,j}| = 1\} \rightarrow$ tractable

Let $S_{i,j}^k$ be the kth element of $S_{i,j}$, and let $\mathcal{L}_{i,j}^k$ be the monomial $\mathbf{a}_{iS_{i,j}^k}\mathbf{x} + b_{iS_{i,j}^k}$, then (21) is equivalent to

$$\begin{aligned} \sum_{j=1}^{N_e^i} t_{ij} &\leq 1 \qquad &\forall i \in \mathbf{P} \\ \max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} e^{\mathcal{L}_{i,j}^k} \right\} &\leq t_{ij} \qquad &\forall i \in \mathbf{P} \qquad &\forall j \in \mathbf{P_i'} \\ \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^1} + e^{\mathcal{L}_{i,j}^2} \right\} &\leq t_{ij} \qquad &\forall i \in \mathbf{P} \qquad &\forall j \in \mathbf{M_i'} \\ \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^1} \right\} &\leq t_{ij} \qquad &\forall i \in \mathbf{P} \qquad &\forall j \in \mathbf{N_i'} \end{aligned} \tag{22}$$

When most of the monomials are certain, or when the the perturbation set is independent, the large posynomials would certainly be reduced into several smaller posynomials, and in some cases into monomials.

The discussion regarding large posynomial approximation will be divided into two parts

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1. Robust large posynomials with uncertain coefficients $\mathbf{b}$ and certain exponents $\mathbf{A}$

2. Robust large posynomials with uncertain coefficients and exponents.

III.F.   Uncertain Coefficients Only

The exponents in many applications regarding geometric programming are certain, therefore it is interesting to look at the problem where only the $'b's$ are uncertain and are given by

$$\mathbf{b} = \mathbf{b}^0 + \sum_{l=1}^{L} \mathbf{b}^l \zeta_l$$

where $\zeta \in \mathcal{Z}$ as given by equation (9).
Consider the second set of constraints of equation (22), and note that

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} e^{\mathbf{a}_{iS_{i,j}^k}\mathbf{x} + b_{iS_{i,j}^k}} \right\} \le t_{ij} \quad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P}_i'$$

$$\Leftrightarrow \max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} e^{\mathbf{a}_{iS_{i,j}^k}\mathbf{x} + b_{iS_{i,j}^k}^0} e^{\sum_{l=1}^{L} b_{iS_{i,j}^k}^l \zeta_l} \right\} \le t_{ij} \quad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P}_i'$$

$$\Leftrightarrow \max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} \prod_{l=1}^{L} e^{b_{iS_{i,j}^k}^l \zeta_l} e^{\mathbf{a}_{iS_j^i(k)}\mathbf{x} + b_{iS_{i,j}^k}^0} \right\} \le t_{ij} \quad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P}_i'$$

The above constraint could be thought of as an uncertain linear constraint in terms of the variables $v_{i,j}^k = e^{\mathbf{a}_{iS_{i,j}^k}\mathbf{x} + b_{iS_{i,j}^k}^0}$ for $k = 1, ..., |S_{i,j}|$ as follows

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} \left(\prod_{l=1}^{L} e^{b_{iS_{i,j}^k}^l \zeta_l} \right) v_{i,j}^k \right\} \le t_{ij} \qquad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P}_i' \tag{23}$$

Although the constraints are linear in terms of $\mathbf{v_j^i}$, the perturbations are not affine but exponential. And since it is hard to deal with exponential perturbations, then it might be more convenient to linearize the perturbations.

III.F.1.   Linearizing perturbations

Although the perturbations are not affine, however, they have some nice property which is convexity. This implies that there exists some half-space (affine function) $[\mathbf{f}_{i,j}^k]^T \zeta + g_{i,j}^k \ge \prod_{l=1}^{L} e^{b_{iS_{i,j}^k}^l \zeta_l}$. Therefore, a safe approximation of the constraints in equation (23) is

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} \left([\mathbf{f}_{i,j}^k]^T \zeta \right) v_{i,j}^k \right\} + \sum_{k=1}^{|S_{i,j}|} g_{i,j}^k v_{i,j}^k \le t_{ij} \qquad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P}_i' \tag{24}$$

To construct the half-space $[\mathbf{f}_{i,j}^k]^T \zeta + g_{i,j}^k$, and taking into consideration the fact that $-1 \le \zeta_l \le 1$ for $l = 1, ..., L$, it is suggested to follow the steps listed below

1. Find the list of vertices $\mathcal{V}$ of the unit box in $\mathbf{R}^L$

2. Find the list of values $\mathcal{O}$ of $\prod_{l=1}^{L} e^{b_{iS_{i,j}^k}^l \zeta_l}$ at the vertices $\mathcal{V}$, note that the ith vertex corresponds to the ith value.

3. Find the maximum M_k and minimum m_k of $\mathcal{O}$ and their corresponding vertices ζ_M and ζ_m.

4. Solve the least-square problem $min \sqrt{\sum_{\alpha=1}^{|\mathcal{O}|} ([\mathbf{f}_{i,j}^k]^T \mathcal{V}_\alpha + g_{i,j}^k - \mathcal{O}_\alpha)^2}$ such that $[\mathbf{f}_{i,j}^k]^T \zeta_M + g_{i,j}^k = M_k$ and $[\mathbf{f}_{i,j}^k]^T \zeta_m + g_{i,j}^k = m_k$

The half-space constructed above is a safe approximation of the intractable large posynomial.

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III.F.2. SP compatible constraint

By looking again at equation (24), it can be seen that it is now tractable using robust linear programming. Unfortunately, The resulting set of robust constraints is not always GP compatible due to the fact that some components of $\mathbf{f}_{i,j}^k$ might not be positive, as a result, an SP need to be solved.

The solution of the SP depends on some initial guess, and a global optimum is not always guaranteed, therefore, it is important to choose a good initial guess.

III.G.   Uncertain Coefficients and Exponents

Things would become harder when the exponents are also uncertain. Consider again the set of constraints

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_j^i|} e^{\mathbf{a}_{iS_{i,j}^k} \mathbf{x} + b_{iS_{i,j}^k}} \right\} \leq t_{ij} \qquad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P_i'} \tag{25}$$

Let $\phi_{i,j}(s) : S_{i,j} \to S_{i,j}$ be a bijection from $S_{i,j}$ onto $S_{i,j}$ known as a permutation on $S_{i,j}$. Note that (25) is equivalent to

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} e^{\mathbf{a}_{iS_{i,j}^{\phi_{i,j}(k)}} \mathbf{x} + b_{iS_{i,j}^{\phi_{i,j}(k)}}} \right\} \leq t_{ij} \qquad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P_i'} \tag{26}$$

The suggested methodology is to maximize each two monomials alone starting from the fact that

$$\max \{a + b + c + d\} \leq \max \{a + b\} + \max \{c + d\}$$

Let $\mathbf{a}_{iS_{i,j}^{\phi_{i,j}(k)}} \mathbf{x} + b_{iS_{i,j}^{\phi_{i,j}(k)}} = \mathcal{L}_{i,j}^{\phi(k)}$, and Assume $|S_{i,j}|$ is even, then

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} e^{\mathcal{L}_{i,j}^{\phi(k)}} \right\} \leq \sum_{k=1}^{|S_{i,j}|/2} \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi(2(k-1)+1)}} + e^{\mathcal{L}_{i,j}^{\phi(2k)}} \right\}$$

for any permutation $\phi_{i,j}$ in the set of permutation functions $\mathcal{P}_{i,j}$, Therefore

$$\max_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{|S_{i,j}|} e^{\mathcal{L}_{i,j}^{\phi(k)}} \right\} \leq \min_{\phi \in \mathcal{P}_{i,j}} \left\{ \sum_{k=1}^{|S_{i,j}|/2} \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi(2(k-1)+1)}} + e^{\mathcal{L}_{i,j}^{\phi(2k)}} \right\} \right\}$$

and

$$\min_{\phi \in \mathcal{P}_{i,j}} \left\{ \sum_{k=1}^{|S_{i,j}|/2} \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi(2(k-1)+1)}} + e^{\mathcal{L}_{i,j}^{\phi(2k)}} \right\} \right\} \leq t_{ij} \qquad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P_i'} \tag{27}$$

is a safe approximation of (26).

The size of the permutations set $\mathcal{P}_{i,j}$ is quit large for large posynomials, however, due to the fact that

$$max \{a + b\} + max \{c + d\} = max \{c + d\} + max \{a + b\} = max \{b + a\} + max \{d + c\}$$

Then some perturbations are similar in terms of the resulting safe approximation, and the permutation set could be modified to contain the "different" permutations only. Indeed, from now on, $\mathcal{P}_{i,j}$ will represent the set of "different" permutations, and not all permutations.

It is hard to find the permutation that minimizes the above expression due to the large number of possible combinations of permutations, therefore, let $\hat{\mathcal{P}}_{i,j}$ be a subset of $\mathcal{P}_{i,j}$, where the permutations are either chosen depending on the structure of the posynomial, or randomly selected. Also, note that the cardinality of $\hat{\mathcal{P}}_{i,j}$ depends on the size of the problem, and should not increase as fast as $\mathcal{P}_{i,j}$.

Using the above sets, the following lemma will be utilized to find a relatively good permutation.

Lemma III.1. *Consider the two optimization problems*

$$\min f(\mathbf{x})$$
$$s.t. \ \mathcal{S}_i(\mathbf{x}) \leq 0 \qquad i = 1, 2, ..., n$$

and

$$\min f(\mathbf{x})$$
$$s.t. \ \mathcal{T}_i(\mathbf{x}) \leq 0 \qquad i = 1, 2, ..., n$$

and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the optimal solutions of the first and second optimization problems respectively. If $\mathcal{T}_i(\mathbf{x}_1) \leq \mathcal{S}_i(\mathbf{x}_1) \quad \forall i \in \{1, 2, ..., n\}$, then $f(\mathbf{x}_2) \leq f(\mathbf{x}_1)$

Proof.

$$\mathcal{T}_i(\mathbf{x}_1) \leq \mathcal{S}_i(\mathbf{x}_1) \leq 0 \quad \forall i \in \{1, 2, ..., n\}$$

Then $\mathbf{x}_1$ is a feasible solution for the second optimization problem.
$\mathbf{x}_2$ is the optimal solution of the second problem, then

$$f(\mathbf{x}_2) \leq f(\mathbf{x}_{feasible}) \implies f(\mathbf{x}_2) \leq f(\mathbf{x}_1)$$

$\square$

Finding the least conservative solution can be done by solving a sequence of geometric programs. Start by replacing the set of large posynomial constraints by

$$\sum_{k=1}^{|S_{i,j}|/2} \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi(2(k-1)+1)}} + e^{\mathcal{L}_{i,j}^{\phi(2k)}} \right\} \leq t_{ij} \qquad \forall i \in \mathbf{P} \qquad \forall j \in \mathbf{P}'_{\mathbf{i}} \tag{28}$$

where $\phi_{i,j}$ is randomly selected from $\hat{\mathcal{P}}_{i,j}$.
(28) is a safe approximation for (26), and is equivalent to

$$\begin{aligned} \sum_{k=1}^{|S_{i,j}|/2} z_{ij}^k &\leq t_{ij} & \forall i \in \mathbf{P} & \qquad \forall j \in \mathbf{P}'_{\mathbf{i}} \\ \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi(2(k-1)+1)}} + e^{\mathcal{L}_{i,j}^{\phi(2k)}} \right\} &\leq z_{ij}^k & \forall i \in \mathbf{P} & \qquad \forall j \in \mathbf{P}'_{\mathbf{i}} \quad \forall k \in \{1, 2.., |S_{i,j}|/2\} \end{aligned} \tag{29}$$

Summing up all the work from different sections, it can be seen that all the constraints are now tractable using linear programming techniques.
The new formulation is either composed of monomials, two-term posynomials, or data-deprived large posynomials. As a result, this formulation is tractable using our knowledge from the previous sections.

The following algorithm illustrates how the permutations are chosen:

1. randomly choose the permutations $\phi_{i,j}$ for the set $\hat{\mathcal{P}}$

2. solve the new formulated optimization problem and let $\mathbf{x}_1$ be the solution

3. repeat

    (a) select the new permutations $\phi_{i,j} \in \hat{\mathcal{P}}_{i,j}$ such that $\phi_{i,j}$ minimizes $\sum_{k=1}^{|S_{i,j}|/2} \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi(2(k-1)+1)}} + e^{\mathcal{L}_{i,j}^{\phi(2k)}} \right\}\Big|_{\mathbf{x}_{i-1}}$

    (b) solve the optimization problem again with the new permutations and let $\mathbf{x}_i$ be the solution

    (c) if $\mathbf{x}_i = \mathbf{x}_{i-1}$: break

Although the solution might not be the least conservative, however, the solution is relatively good.

We are now ready to solve a robust geometric program approximately, the next section will discuss extending the RGP into RSP.

IV.   Models

We implemented the RSP formulation ideas above on a simple aircraft design problem, with 12 uncertain variables, and a single signomial constraint. In the simple aircraft problem, we conduct an aerostructural optimization of a wing and fuselage given a payload and range requirement. A short overview of the model follows.

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IV.A. Lift, Weight, Drag and Thrust

The aircraft is assumed to be in steady, level flight. As a result, we can assume that the thrust generated by the aircraft is equal to the drag, and the lift generated by the wing is equal to the total weight.

The drag is the sum of the wing (induced and profile) drag, and the fuselage drag, which is linearly proportional to the fuel volume in the fuselage. The aircraft model does not assume limitations on thrust, but instead assumes constant thrust specific fuel consumption, at 0.6 lbs for each lb*hour of thrust, with associated uncertainty.

The weight of the aircraft is the sum of the payload weight, wing weight, and the fuselage weight, as shown in Figure 1. Lift is generated by the wing, which is described by an aspect ratio AR and surface area S.

Figure 1. Wing lift is equal and opposite to the wing weight, payload weight, and total fuel weight.
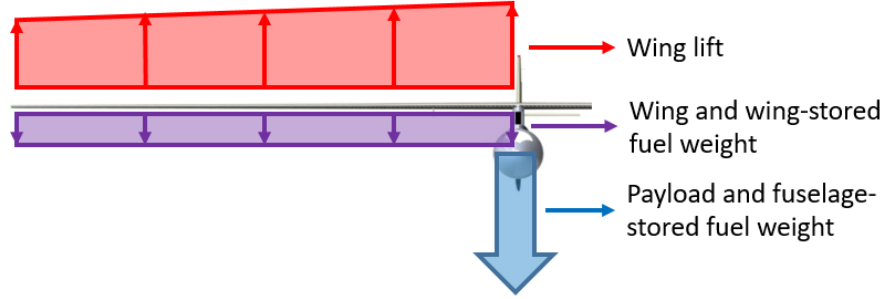
IV.B. Wing Structure

The wing structure model is based on a simple beam model with a distributed lift load, and a point mass in the center representing the fuselage, as shown in Figure 1.

IV.C. Fuel Volume

The fuel in the aircraft can be stored either in the wing or the fuselage. The signomial constraint in the optimization appears in the fuel volume model, as shown in Equation 30:

$$V_f \leq V_{f_{wing}} + V_{f_{fuse}} \tag{30}$$

where $V_{f_{wing}}$ and $V_{f_{fuse}}$ represent the fuel volume available in the wing and the fuselage respectively. They are each represented by the following monomials.

$$V_{f_{wing}} <= 9e^{-4}\frac{S^{1.5}\tau}{A^{0.5}} \tag{31}$$

$$V_{f_{fuse}} <= 10 \times CDA_0 \, m \tag{32}$$

Note that the monomials above are represented with inequalities, to be compatible with the RSP formulation.

IV.D. Takeoff constraints

We specify that the aircraft has to be able to takeoff at a speed of V_{min} without exceeding the aircraft stall lift coefficient $C_{L_{max}}$, both of which are specified with an associated uncertainty.

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V. Uncertainties and Sets

The uncertainties for the different constants in the problem have been determined considering the parameters in aircraft design that often have the largest uncertainty. These uncertainties are listed in Table 1.

Table 1. Constants and Uncertainties (increasing order)

Constant	Description	Value	% Uncert. (3σ)
$S_{wetratio}$	wetted area ratio	2.075	3
e	span efficiency	0.92	3
μ	viscosity of air	1.775e-5 $kg/(ms)$	4
ρ	air density	1.23 kg/m^3	5
$C_{L_{max}}$	stall lift coefficient	1.6	5
k	fuselage form factor	1.17	10
τ	airfoil thickness ratio	0.12	10
N_{ult}	ultimate load factor	3.3	15
V_{min}	takeoff speed	25 m/s	20
W_0	payload weight	6250 N	20
$W_{w_{coeff1}}$	wing weight coefficient 1	2e-5 $1/m$	20
$W_{w_{coeff2}}$	wing weight coefficient 2	60 N/m^2	20

The parameter uncertainties reflect aerospace engineering intuition. The wing weight coefficients $W_{w_{coeff1}}$ and $W_{w_{coeff2}}$, and the ultimate load factor N_{ult} have large 3σs because build quality of aircraft components often difficult to quantify with a large degree of certainty. The payload weight (W_0) has a large uncertainty, because it is valuable if the aircraft has the flexibility to accommodate larger payloads. Parameters that engineers take to be physical constants (μ, ρ) and those that can be determined/manufactured with a relatively high degree of accuracy ($S_{wetratio}$, e) have relatively low deviations. Parameters that require testing to determine ($C_{L_{max}}$, V_{min}) have a level of uncertainty that reflects the expected variance of the parameters.

VI. Results

VI.A. Determining the appropriate Γ

In this section, we optimize the aircraft configuration for a given payload and range with no uncertainty, box uncertainty and ellipsoidal uncertainty, and attempt to determine the value of Γ that gives the best trade-off between robustness and optimality for the RO formulations. The problems have been solved with box uncertainty and ellipsoidal uncertainty for a range of Γ. For each of the robust solutions, both the estimated probability of failure and the objective cost have been plotted in Figures 2 and 3 respectively.

We define the probability of failure to be the probability of constraint violation in 1000 samples of the uncertain parameters from a normal distribution with the specified 3σ uncertainties. For this problem, we allow for less than 2% probability of failure, shown by the magenta line in Figure 2, and consider half-integer values of Γ. This corresponds to a Γ of 2.5 for both the box uncertainty and the ellipsoidal uncertainty, resulting in 1.2% and 1.3% probability of failure respectively.

We know that elliptical uncertainty sets are less conservative than box uncertainty sets, especially when we have a large number of uncertain parameters per constraint. The results we got are somehow expected, and this would show how designing using uncertainty sets is better than using margins which are - in the best case - somehow similar to box uncertainty sets.

VI.B. Optimization results

At the chosen values of Γ, the results of the optimization are shown in Table 2.
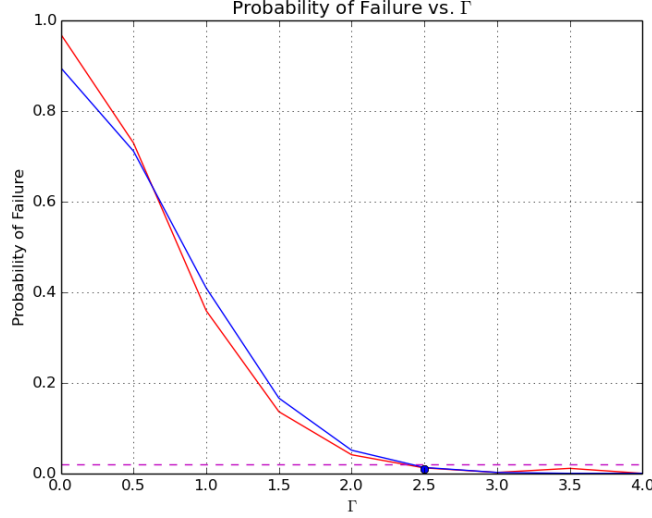
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Figure 2. The probability of failure vs. Γ. Note the high (≥ 90%) probability of failure for the non-robust (Γ = 0) solutions. The difference in the probability of failure for the non-robust cases is due to subtle numerical errors in the evaluation of the solution obtained for the ellipsoidal uncertainty set.

Table 2. SP Aircraft Optimization Results

Free variable	No Uncert.	Box [Γ = 2.5]	Ellipsoidal [Γ = 2.5]
L/D	23.8	16.41	17.3
AR	12.0	5.016	6.56
Re	4.76e6	1.16e7	9.55e6
$S(m^2)$	21.6	72.21	58.9
$V(m/s)$	51.0	47.73	48.6
$T_{flight}(hr)$	17.46	18.1	17.2
W_w	2440	6278	5470
$W_{w_{strc}}(N)$	1210	1570	1650
$W_{w_{surf}}(N)$	1230	1592	3820
$V_{f_{avail}}(m^3)$	0.566	1.794	1.48
$V_{f_{fuse}}(m^3)$	0.461	0.891	0.903
$V_{f_{wing}}(m^3)$	0.105	0.9027	0.582
$CDA_0(m^2)$	0.0461	0.0891	0.0903
E[Objective]	No Uncert.	Box [Γ = 2.5]	Ellipsoidal [Γ = 2.5]
W_{fuel} (N)	4430	10858	9299
P[failure]	No Uncert.	Box [Γ = 2.0]	Ellipsoidal [Γ = 2.5]
%	92	1.2	1.3

VI.C.   Comparing results for different objective functions

To further demonstrate the capabilities of robust SPs in aircraft design, we performed the optimization of the aircraft with no uncertainty and ellipsoidal uncertainty (Γ = 2.5) for two more objective functions, and plotted the results on spider plots. Spider plots are useful because they allow engineers to find non-dominated solutions among the solutions that lie on the Pareto frontier of potential objective functions. The objective functions chosen for this analysis were fuel burn over lift-to-drag ratio ($\frac{W_f}{L/D}$), drag (D), and fuel burn (W_f).

In the spider plots in Figures 4 and 5, none of the solutions are non-dominated for both the no uncertainty and ellipsoidal uncertainty cases. In a GP, it is not possible that one of the solutions is non-dominated since
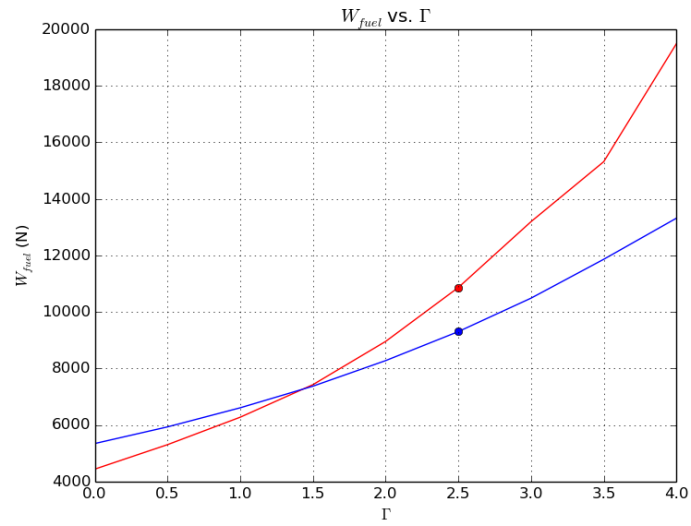
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Figure 3. Fuel weight vs. Γ demonstrates the deterioration of the objective as we protect against larger variation of parameters.

Figure 4. Design optimization of the aircraft with no uncertainty for 3 different objective functions. The red, blue and yellow correspond to $\frac{W_f}{L/D}$, D, W_f and objectives respectively.
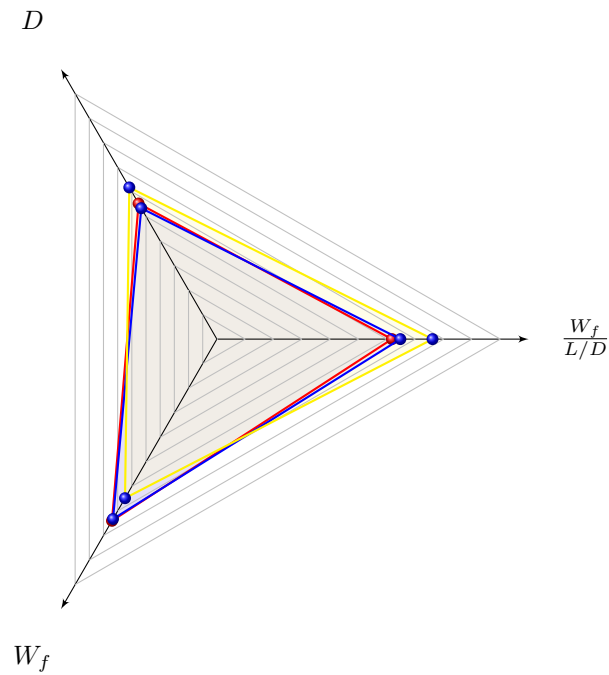
the solutions are globally optimal. But since there is no guarantee in optimality for SPs, it is possible to find non-dominated solutions if the obtained solution is a local optimum.

In the case where there is no non-dominated solution such as this one, we take the internal areas of the triangle formed by each optimization to be the figure of merit. The smaller the area in the triangle, the higher the performance of the proposed solution. In the no uncertainty case shown in Figure 4, the red solution with the objective of $\frac{W_f}{L/D}$ has the smallest internal area. In the ellipsoidal case shown in Figure 5, the blue solution with the objective of D has the smallest internal area.

This is an interesting result, because the presence of an uncertainty set is shown to affect the efficacy of different objective functions to obtain solutions with the best overall performance. The differences between the objective functions in the simple aircraft design problem are minute, because the different potential objectives have a high degree of coupling. It is likely that, if the three objective functions didn't have high degree of coupling, that the internal areas of the solution triangles may differ more significantly.

Figure 5. Design optimization of the aircraft with ellipsoidal uncertainty ($\Gamma = 2.5$) for 3 different objective functions. The red, blue and yellow correspond to $\frac{W_f}{L/D}$, D, W_f and objectives respectively.
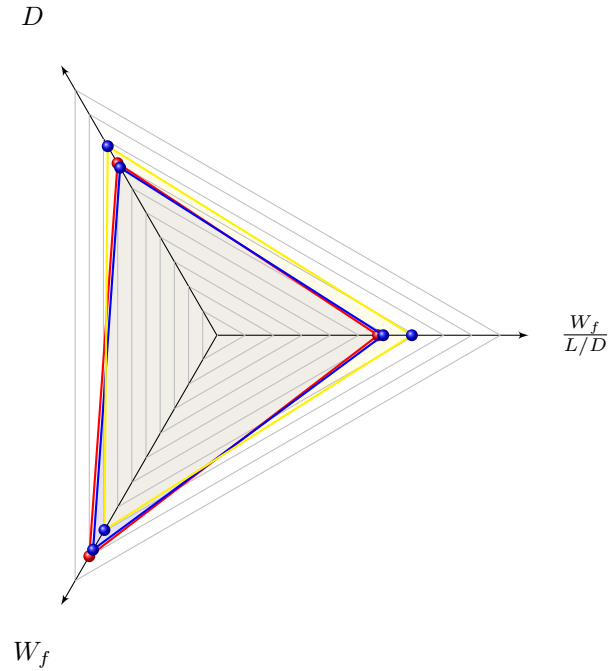
VII. Conclusion

We have developed and applied a tractable RSP formulation to a simple aircraft model, and then discussed the benefits of having robust solutions. RSP formulations extend the tractable approximate RGP framework developed by Saab to non-log-convex problems, and are a valuable contribution to the fields of robust optimization and difference-of-convex programming.

RSPs have a wide variety of potential applications in engineering design. Within the Hoburg Research Group in the Aerospace Computational Design Lab, during the past year we have developed a commercial aircraft design SP that has between 1700 and 8000 variables, depending on the whether it is a single-point, or multi-point optimization. We expect that using RO in this conceptual aircraft design will result in designs that are more robust with respect to uncertainties in operational parameters, such as payload mass and range, as well as uncertain constants.

Appendix

VII.A.   Robust Linear Programming: A Quick Review

As mentioned earlier, robust linear programming will be used to formulate an approximate robust geometric program.

Consider the system of linear constraints

$$\mathbb{A}\mathbf{x} + \mathbf{b} \le 0$$

where

$$\mathbb{A} \text{ is } m \times n$$
$$\mathbf{x} \text{ is } n \times 1$$
$$\mathbf{b} \text{ is } m \times 1$$

where that data is uncertain and is given by equations (8) and (9).

VII.A.1.   Box Uncertainty Set

If the perturbation set $\mathcal{Z}$ given in equation (9) is a box uncertainty set, i.e. $\|\zeta\|_\infty \le \Gamma$, then the robust formulation of the ith constraint is equivalent to

$$\Gamma\sum_{l=1}^{L}|-b_i^l - \mathbf{a}_i^l\mathbf{x}| + \mathbf{a}_i^0\mathbf{x} + b_i^0 \le 0 \tag{33}$$

If only b is uncertain, i.e. $A^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (33) will become

$$\sum_{l=1}^{L}\mathbf{a}_i^0\mathbf{x} + b_i^0 + \Gamma\sum_{l=1}^{L}|b_i^l| \le 0 \tag{34}$$

which is a linear constraint
On the other hand, if A is uncertain, then equation (33) is equivalent to the following set of linear constraints

$$
\begin{aligned}
\Gamma\sum_{l=1}^{L}w_i^l + \mathbf{a}_i^0\mathbf{x} + b_i^0 &\le 0 \\
-b_i^l - \mathbf{a}_i^l\mathbf{x} &\le w_i^l \quad \forall l \in 1, ..., L \\
b_i^l + \mathbf{a}_i^l\mathbf{x} &\le w_i^l \quad \forall l \in 1, ..., L
\end{aligned}
\tag{35}
$$

VII.A.2.   Elliptical Uncertainty Set

Briefly, if the perturbation set $\mathcal{Z}$ is an elliptical, i.e. $\sum_{l=1}^{L}\frac{\zeta_l^2}{\sigma_l^2} \le \Gamma^2$, then the robust formulation of the ith constraint is equivalent to

$$\Gamma\sqrt{\sum_{l=1}^{L}\sigma_l^2(-b_i^l - \mathbf{a}_i^l\mathbf{x})^2} + \mathbf{a}_i^0\mathbf{x} + b_i^0 \le 0 \tag{36}$$

which is a second order conic constraint.
If only b is uncertain, i.e. $\mathbb{A}^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (36) will become

$$\sum_{l=1}^{L}\mathbf{a}_i^0\mathbf{x} + b_i^0 + \Gamma\sqrt{\sum_{l=1}^{L}\sigma_l^2(b_i^l)^2} \le 0 \tag{37}$$

which is a linear constraint.

VII.A.3.   Norm-1 Uncertainty Sets

Briefly, if the perturbation set represented by $\mathcal{Z}$ is a norm-1 uncertainty set, i.e. $\|\zeta\|_1 \le \Gamma$, then the robust constraint is

$$\sum_{l=1}^{L}\mathbf{a}_i^0\mathbf{x} + b_i^0 + \Gamma\max_{l=1,..,L}|b_i^l| \le 0 \tag{38}$$

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when $\mathbb{A}^l = 0$, and

$$
\begin{aligned}
\Gamma w_i + \mathbf{a}_i^0 \mathbf{x} + b_i^0 &\leq 0 \\
-b_i^l - \mathbf{a}_i^l \mathbf{x} &\leq w_i \quad \forall l \in 1, ..., L \\
b_i^l + \mathbf{a}_i^l \mathbf{x} &\leq w_i \quad \forall l \in 1, ..., L
\end{aligned}
\tag{39}
$$

if $\mathbb{A}^l \neq 0$

Note that for this type of uncertainty, the robust constraints are linear.

Acknowledgments

A place to recognize others.

References

[1]Rebek, A., *Fickle Rocks*, Fink Publishing, Chesapeake, 1982.