Optimal Aircraft Design Decisions under Uncertainty via Robust Signomial Programming

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Aircraft design benefits greatly from optimization under uncertainty, since design feasibility and objectives can have large sensitivities to uncertain parameters. The traditional, mathematically non-rigorous methods of capturing uncertainty do not adequeately explain the tradeoffs between feasibility and optimality, and require prior engineering knowledge which may not be available for new aerospace vehicle concepts. Signomial programs (SPs) are difference-of-convex extensions of geometric programs (GPs), and have demonstrated potential in the solution of multidisciplinary non-convex optimization problems such as aircraft design [3]. The formulation and solution of robust signomial programs (RSPs) would be beneficial since many parameters involved in these problems are prone to uncertainty, and can have significant effects on solution performance and feasibility. This paper proposes an approximate solution method for an RSP leveraging an existing approximate robust geometric programming (RGP) formulation developed by Saab [1]. The method is based on solving a sequence of RGPs, where each GP is a local approximation of the SP. The paper then discusses the trade-off between robustness and optimality in aircraft design by implementing RSPs on a simple aircraft problem, and demonstrates how robustness requirements affect aircraft design decisions.

Nomenclature

CEG	Convex Engineering Group
DC	difference-of-convex
GP	geometric program
LHS	left hand side
MDO	multidisciplinary design optimization
NLP	nonlinear program
SP	signomial program
RGP	robust geometric program
RHS	right hand side
RO	robust optimization
RSP	robust signomial program
SO	stochastic optimization

I. Introduction

Aircraft design exists in a niche of design problems where "failure is not an option"[a]. This is remarkable since aircraft design problems are rife with uncertainty about technological capabilities, environmental factors, manufacturing quality and the state of markets and regulatory agencies. Since the program risk of

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[a]Quoting Gene Kranz, the mission director of Apollo 13.

aircraft design problems is high, optimization under uncertainty for aircraft presents a lot of low hanging fruit, since its goal is to be able to provide designs that are robust to realizations of uncertainty in the real world.

Zang et al.[2] succinctly describe the categories of benefits for optimization under uncertainty for aircraft. These are the following:

- *Confidence in analysis tools will increase.* The uptake of new design tools in the aerospace industry has been low due to heavy reliance on legacy design methods and prior experience when faced with risky design propositions, even in the design of novel configurations where the understanding of the design tradespaces is lacking. Robustness will increase confidence in analysis tools because of its ability to better capture the effects of technological uncertainty on the potential benefits of new configurations.

- *Design cycle time, cost, and risk will be reduced.* Design cycle costs of aerospace vehicles have been increasing. This has to do many factors, such as the growth of requirements (add here and cite)... Aircraft design and development is costly, so the ability to handle uncertainty in the conceptual design process is critical for the long-term success of an aircraft, helping reduce the program risk.

- *System performance will increase while ensuring that reliability requirements are met.* The effectiveness of an aircraft depends heavily on its ability to deliver on performance, which is dependent on assumptions about the current technological environment and the ability to produce vehicles of a certain quality.

- *Designs will be more robust.* The ability to provide designs with feasibility and performance guarantees will mean that designs and products will be more robust to uncertainties in manufacturing quality, environmental factors, technology level and markets.

In economics, the idea that risk is related to profit is well understood and leveraged. In aerospace engineering however, we often forget that there is no such thing as a free lunch, and that the consequence of risk-aversity is often performance that is left on the table. Good conceptual design in the aerospace industry hedges against program risk, the Robust Optimization (RO) frameworks proposed in this paper will give aerospace engineers the ability to rigorously trade robustness and the performance penalties that result from it.

I.A. Approaches to optimization under uncertainty

Faced with the challenge of developing general Nonlinear Program (NLP)s that can incorporate uncertainty, the aerospace field has developed a number of mathematically non-rigorous methods to design under uncertainty. Oftentimes, aerospace engineers will implement *margins* in the design process to account for uncertainties in parameters that a design's feasibility may be sensitive to, such as material properties or maximum lift coefficient. Another traditional method of adding robustness is through *multi-mission design* [3], which ensures that the aircraft is able to handle multiple kinds of missions in the presence of no uncertainty. This is a type of *finitely adaptive* optimization geared to ensure objective performance in off-nominal operations.

The weaknesses of these non-rigorous methods are many. They provide no quantitative measures of robustness or reliability [2]. Furthermore, they rely on the expertise of an experienced engineer to guide the design process, without explicit knowledge of the trade-off between robustness and optimality [4]. This is a dangerous proposition especially in the conceptual design phase of new configurations, since prior information and expertise is not available. In these scenarios, it is especially important to go back to fundamental physics and use rigorous mathematics to explore the design space [3].

There are two rigorous approaches to solving design optimization problems under uncertainty, which are Stochastic Optimization (SO) and RO. Stochastic optimization[b] deals with probability distributions of uncertain parameters by propagating these uncertainties through the physics of a design problem to ensure constraint feasibility with certain probabilistic guarantees. The goal of SO is minimize the expectation of an objective function [5]. Robust optimization takes a different approach, instead choosing to make designs immune to uncertainties in parameters as long as the parameter values come from within the defined uncertainty set. RO minimizes the worst-case objective outcome from a defined uncertainty set.

[b]Note that stochastic optimization is an overloaded term, and exists in two contexts in the literature. The first is the solution of deterministic problems with stochastic search space exploration. The other is the solution of problems of stochastic uncertainty. We explore the latter.

I.B. Advantages of robust over stochastic optimization

The formulation of stochastic models as RO problems has many advantages over general stochastic optimization methods, as summarized in [6], and fall into three categories, which are tractability, conservativeness and flexibility. RO is more tractable than SO due to the nature of uncertainty propagation. General stochastic methods involve the propagation of uncertainties throughout a model to determine their effects on constraint feasibility and the objective function. This requires the integration of the product of probability distributions with potential outcomes, and since the integration of continuous functions is difficult this is often achieved through a discretization of the uncertainty into possible scenarios. The propagation of parameter scenarios results in a combinatorial explosion of possible outcomes which need to be evaluated to determine constraint satisfaction and the distribution of the objective.

Few problems can be addressed purely through stochastic optimization (eg. the recourse problem as shown in [7],[8], and energy planning problem such as in [9]). It is arguable that the methods used are still limited by the combinatorial explosion of possible outcomes. RO has to deal with a somewhat related problem, which is the issue of an infinite number of possible realizations of constraints within a given constraint set. However, this is easily tackled by considering the worst case robust counterpart of each constraint, which results in many kinds of optimization problems having tractable robust formulations [6].

Although RO problems solve problems with uncertainty, RO formulations result in solutions that are deterministically immune [6] to all possible realizations of parameters in an uncertainty set, which is defined as conservativeness. SO formulations provide no such guarantees. RO also does not require distributional information about uncertain parameters as SO does, and therefore can better address problems where there is a lack of experience or data. It is arguable that RO leaves a lot on the table by not taking advantage of distributional information, however there is a body of research on distributionally robust optimization [10] which seeks to leverage existing data.

There is significantly greater flexibility in the formulation of robust versus stochastic models since the methods proposed are more general. It is important to highlight that, although both RO and SO seek to address the problem of optimization under uncertainty, they solve fundamentally different problems. In an ideal world where we have a problem that is tractable with global optimality for both methods, the two different approaches would result in different solutions.

I.C. Geometric and signomial programming for engineering design

Geometric programming[c] is a method of log-convex optimization that has been developed to solve problems in engineering design [11]. Although theory of the Geometric Program (GP) has existed since the 1960's, GPs have recently experienced a resurgence due to the advent of polynomial-time interior point methods [12] and improvements in computing. They have been applied to a range of engineering design problems with success. For a non-exhaustive list of examples, please refer to [13].

GPs have been effective in aircraft conceptual design ([14], [15]). However, the stringent mathematical requirements of a GP limits its application to non-log-convex problems. The Signomial Program (SP) is the difference-of-log-convex extension of the GP which can be applied to solve this larger set of problems, albeit with the loss of some mathematical guarantees compared to the GP [16]. Aircraft pose some of the most challenging design problems (cite York here), and signomial programming has been used to great effect in modeling and designing complex aircraft at a conceptual level quickly and reliably as in [3] [17] [16]. Other interesting applications for SPs such as in network flow problems are being investigated.

Robust formulations exist for solving geometric programs with parametric uncertainty [1]. We posit that the creation of a robust signomial programming framework to capture uncertainty in engineering design, and specifically aircraft design, will allow us to have more confidence in the results of the conceptual design phase, reduce program risk, and increase overall system performance.

I.D. Contributions

In this paper, we propose a tractable Robust Signomial Program (RSP) which we solve as a sequential Robust Geometric Program (RGP), allowing us to implement robustness in non-log-convex problems such as aircraft design. We extend the RGP framework developed by Saab [1] to SPs. We implement the RSP formulation on a simple aircraft design problem with several hundred variables as defined in [18] (TODO:update). We

[c]Programming refers to the mathematical formulation of an optimization problem.

demonstrate the benefits of robust optimization both in ensuring design feasibility and performance in off-nominal conditions. We further explore the benefits of RO in multiobjective optimization.

II. Mathematical Background

II.A. Robust Optimization

Given a general problem under parametric uncertainty, we can define the set of possible realizations of uncertain vector of parameters u in the uncertainty set $\mathcal{U}$. This allows us to define the problem under uncertainty below.

$$\begin{aligned}\text{minimize } & f_0(x) \\ \text{s.t. } & f_i(x,u) \leq 0,\ \forall u \in \mathcal{U},\ i = 1,\ldots,n\end{aligned}$$

This problem is infinite-dimensional, since it is possible to formulate an infinite number of constraints with the countably infinite number of possible realizations of $u \in \mathcal{U}$. To circumvent this issue, we can define the following robust formulation of the uncertain problem below.

$$\begin{aligned}\text{minimize } & f_0(x) \\ \text{s.t. } & \max_{u \in \mathcal{U}} f_i(x,u) \leq 0,\ i = 1,\ldots,n\end{aligned}$$

This formulation hedges against the worst-case realization of the uncertainty in the defined uncertainty set. This is often posed by creating an uncertainty set to contain all possible realizations of the uncertainty we are concerned about, usually through a norm,

$$\begin{aligned}\text{minimize } & f_0(x) \\ \text{s.t. } & \max_{u} f_i(x,u) \leq 0,\ i = 1,\ldots,n \\ & \|u\| \leq \Gamma\end{aligned}$$

where Γ is defined by the user as an uncertainty bound.

II.B. Geometric Programming

A **geometric program in posynomial form** is a log-convex optimization problem of the form:

$$\begin{aligned}\text{minimize} \quad & f_0\left(\mathbf{u}\right) \\ \text{subject to} \quad & f_i\left(\mathbf{u}\right) \leq 1, i = 1, ..., m_p \\ & h_i\left(\mathbf{u}\right) = 1, i = 1, ..., m_e\end{aligned} \tag{1}$$

where each f_i is a *posynomial*, each h_i is a *monomial*, m_p is the number of posynomials, and m_e is the number of monomials. A monomial $h(\mathbf{u})$ is a function of the form:

$$h_i(\mathbf{u}) = e^{b_i} \prod_{j=1}^{n} {u_j}^{a_{ij}}$$

where a_{ij} is the j^{th} component of a row vector $\mathbf{a_i}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_i is in $\mathbb{R}$. A posynomial $f(\mathbf{u})$ is the sum of $K \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{u}) = \sum_{k=1}^{K} e^{b_{ikj}} \prod_{j=1}^{n} {u_j}^{a_{ikj}}$$

where a_{ikj} is the j^{th} component of a row vector $\mathbf{a_{ik}}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_{ik} is in $\mathbb{R}$ [13].
A logarithmic change of the variables $x_j = \log(u_j)$ would turn a monomial into *the exponential of an affine function* and a posynomial into *the sum of exponentials of affine functions*. A transformed monomial $h_i(\mathbf{x})$ is a function of the form:

$$h_i(\mathbf{x}) = e^{\mathbf{a_i x} + b_i}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A transformed posynomial $f_i(\mathbf{x})$ is the sum of $K_i \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{x}) = \sum_{k=1}^{K_i} e^{\mathbf{a_{ik} x} + b_{ik}}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A geometric program with transformed constraints is a **geometric program in exponential form**.

The positive nature of exponential functions restricts the space spanned by posynomials and limits the applications of GPs to certain classes of problems. The limited applicability of GPs has motivated the introduction of signomials.

II.C. Signomial Programming

A *signomial* can be defined as the difference between two posynomials, consequently, an SP is a non-log-convex optimization problem of the form:

$$\begin{array}{lll} \text{minimize} & f_0(\mathbf{x}) & \\ \text{subject to} & f_i(\mathbf{x}) - g_i(\mathbf{x}) & \leq 0, i = 1, ..., m \end{array} \tag{2}$$

where f_i and g_i are both posynomials, and $\mathbf{x}$ is a column vector in $\mathbb{R}^n$.

Reliably solving an SP to a local optimum has been described in [13] and [19]. A common solution heuristic involves solving an SP as a sequence of GPs, where each GP is a local approximation of the SP. Although it is a powerful tool, applications involving SPs are usually prone to uncertainties that have a significant effect on the solution.

III. Robust Signomial Programming

Robust signomial programming (RSP) assumes that parameter uncertainties belong to an uncertainty set, and solves the design problem to find the best solution as shown in Figure 1. This section introduces RSPs and derives the intractable formulation of an RSP.

An SP in its **exponential form** is as follows:

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{s.t.} & \sum_{k=1}^{K_i} e^{\mathbf{a_{ik} x} + b_{ik}} - \sum_{k=1}^{G_i} e^{\mathbf{c_{ik} x} + d_{ik}} \leq 0 \quad \forall i \in 1, ..., m \end{array} \tag{3}$$

Let $\mathbf{a_{ik}}$ and $\mathbf{c_{ik}}$ be the $((i-1) \times m + k)^{th}$ rows of the exponents matrices $\mathbf{A}$ and $\mathbf{C}$ respectively, and b_{ik} and d_{ik} be the $((i-1) \times m + k)^{th}$ elements of the coefficients vectors $\mathbf{b}$ and $\mathbf{d}$ respectively.

The data ($\mathbf{A}$, $\mathbf{C}$, $\mathbf{b}$, $\mathbf{d}$) is assumed uncertain and living in an uncertainty set $\mathcal{U}$, where $\mathcal{U}$ is parametrized affinely by a perturbation vector ζ as follows:

$$\mathcal{U} = \left\{ [\mathbf{A}; \mathbf{C}; \mathbf{b}; \mathbf{d}] = \left[\mathbf{A}^0; \mathbf{C}^0; \mathbf{b}^0\ \mathbf{d}^0\right] + \sum_{l=1}^{L} \zeta_l \left[\mathbf{A}^l; \mathbf{C}^l; \mathbf{b}^l; \mathbf{d}^l\right] \right\} \tag{4}$$

where $\mathbf{A}^0$, $\mathbf{C}^0$, $\mathbf{b}^0$, and $\mathbf{d}^0$ are the nominal exponents and coefficients, $\left\{\mathbf{A}^l\right\}_{l=1}^{L}$, $\left\{\mathbf{C}^l\right\}_{l=1}^{L}$, $\left\{\mathbf{b}^l\right\}_{l=1}^{L}$, and $\left\{\mathbf{d}^l\right\}_{l=1}^{L}$ are the basic shifts of the exponents and coefficients, and ζ_l is the l^{th} component of ζ belonging to a conic perturbation set $\mathcal{Z} \in \mathbb{R}^L$ paramaterized by $\mathbf{F}$, $\mathbf{G}$, $\mathbf{h}$ and $\mathbf{K}$ such that

$$\mathcal{Z} = \left\{\zeta \in \mathbb{R}^L : \exists \mathbf{u} \in \mathbb{R}^k : \mathbf{F}\zeta + \mathbf{G}\mathbf{u} + \mathbf{h} \in \mathbf{K}\right\} \tag{5}$$

where $\mathbf{K}$ is a regular cone in $\mathbb{R}^N$ with a non-empty interior if it is not polyhedral, $\mathbf{F} \in \mathbb{R}^{N \times L}$, $\mathbf{G} \in \mathbb{R}^{N \times k}$, and $\mathbf{h} \in \mathbb{R}^N$.

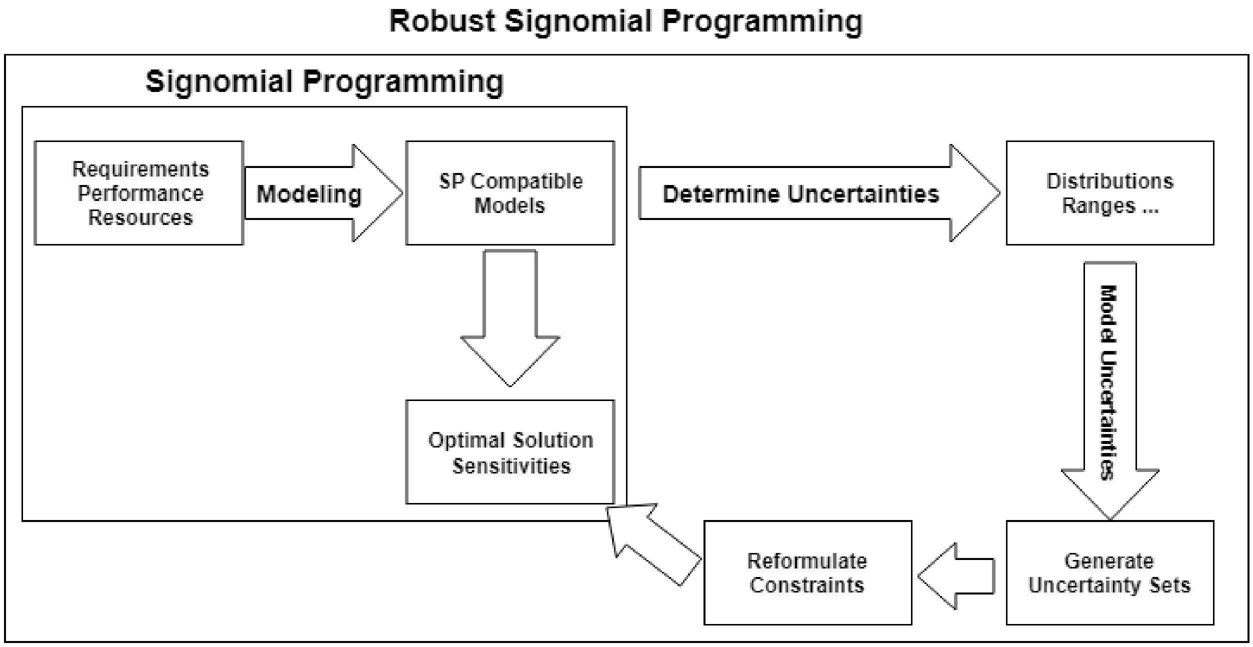


Figure 1: A block diagram showing the difference between the design process using an SP and an RSP.

As mentioned earlier, there should exist a formulation immune to uncertainty in the system's data. Accordingly, the robust counterpart of the uncertain signomial program in (3) is:

$$
\begin{array}{ll}
\min & f_0\left(\mathbf{x}\right) \\
\text{s.t.} & \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)} - \sum_{k=1}^{G_i} e^{\mathbf{c_{ik}}(\zeta)\mathbf{x}+d_{ik}(\zeta)} \leq 0 \quad \forall i \in 1,...,m \quad \forall \zeta \in \mathcal{Z}
\end{array} \tag{6}
$$

These constraints state that the robust optimal solution should be feasible for all possible realizations of the perturbation vector ζ. However, the above is a semi-infinite optimization problem, i.e. an optimization problem with finite number of variables and infinite number of constraints. Such a problem is intractable using current solvers and so an equivalent finite set of constraints is usually derived:

$$
\begin{array}{lll}
\min & f_0\left(\mathbf{x}\right) & \\
\text{subject to} & \max\limits_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)} - \sum_{k=1}^{G_i} e^{\mathbf{c_{ik}}(\zeta)\mathbf{x}+d_{ik}(\zeta)} \right\} & \leq 1 \quad \forall i \in 1,...,m
\end{array} \tag{7}
$$

The optimization problem in (7) is intractable. In the following sections, a heuristic approach to solving RSPs approximately as a sequential RGP will be presented. As our approach is based on Robust Geometric Programming, a brief review of the subject will follow based on [1].

IV. Robust Geometric Programming

In this section we will briefly review the approximation of an RGP as a tractable optimization problem as presented in [1] and [20].

The robust counterparts of an uncertain geometric program is:

$$
\begin{array}{lll}
\min & f_0\left(\mathbf{x}\right) & \\
\text{subject to} & \max\limits_{\zeta \in \mathcal{Z}} \left\{ \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)} \right\} & \leq 1 \quad \forall i \in 1,...,m
\end{array} \tag{8}
$$

IV.A. Two Term Formulation

V. Approach to Solving RSPs

This section presents a heuristic algorithm to safely solve a RSP based on our previous discussion on robust geometric programming.

V.A. General RSP Solver

As mentioned before, a common heuristic algorithm to solve an SP is by sequentially solving local GP approximations, but the solution is not guaranteed to be globally optimal. Our approach to solve an RSP is based on sequentially solving local RGP approximations. Below we provide a step-by-step algorithm to solve an RSP:

1. Choose an initial guess x_0
2. Repeat
   (a) Find the local GP approximation of the SP at x_i.
   (b) Find the RGP formulation of the GP.
   (c) Solve the RGP to obtain x_{i+1}.
   (d) If $x_{i+1} \approx x_i$: break

Similar to an SP, a good initial guess would lead to faster convergence and possibly a better solution. A quick candidate is the deterministic solution of the uncertain SP, which will certainly lead to a faster convergence.

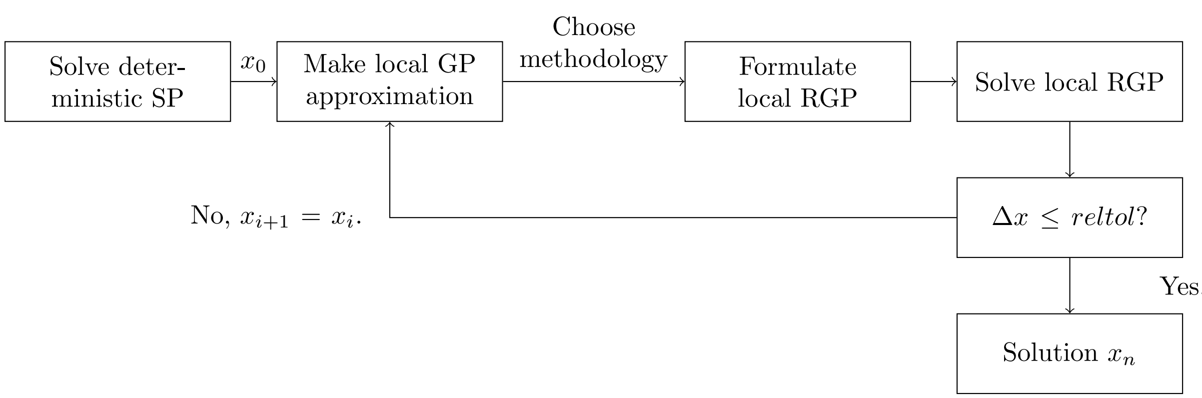


Figure 2: A block diagram showing the steps of solving an RSP.

Any of the previously mentioned methodologies can be used to formulate the local RGP approximation. However, Depending on the RGP formulation chosen to solve an RSP, the last formulation and solution bloks in Figure 2 are tweaked slightly for a faster convergence.

V.B. Best Pairs RSP Solver

If the Best Pairs methodology is exploited, then the above algorithm would change so that each iteration would solve the local RGP approximation and choose the best permutation for each large posynomial. The modified algorithm would become as follows:

1. Choose an initial guess x_0
2. Repeat
   (a) Find the local GP approximation of the SP at x_i.

(b) $\forall(i,j) \in \mathbf{P}$, select the new permutations $\phi \in \hat{\mathcal{P}}_{i,j}$ such that ϕ minimizes $\sum_{k=1}^{|S_{i,j}|/2} \max_{\zeta \in \mathcal{Z}} \left\{ e^{\mathcal{L}_{i,j}^{\phi_{2k-1}}} + e^{\mathcal{L}_{i,j}^{\phi_{2k}}} \right\} \Big|_{\mathbf{x}_i}$

(c) Solve the approximate tractable counterparts of the local GP in (**??**), and let $\mathbf{x}_{i+1}$ be the solution

(d) If $x_{i+1} \approx x_i$: break

V.C. Linearized Perturbations RSP Solver

On the other hand, if the Linearized Perturbations formulation is to be used, then we can avoid solving a signomial program at each iteration by first approximating the original SP constraints locally, and in the same loop approximating the robustified possibly signomial constraints locally, thus solving a GP at each iteration instead of an SP. The algorithm would then become as follows:

1. Choose an initial guess x_0
2. Repeat
   (a) Find the local GP approximation of the SP at x_i.
   (b) Robustify the constraints of the local GP approximation using the Linearized Perturbations methodology.
   (c) Find the local GP approximation of the resulting local SP at x_i.
   (d) Solve the local GP approximation in step c to obtain x_{i+1}
   (e) If $x_{i+1} \approx x_i$: break

VI. Models

We implemented the RSP formulation ideas above on a simple aircraft design problem as defined in [18]. We conduct optimization of a wing, fuselage, engine given a payload and range requirement.

A short overview of the model

VI.A. Flight Profile

The flight trajectory of the aircraft is optimized

VI.B. Atmosphere Model

The atmosphere model

VI.C. Wing Model

VI.D. Fuselage Model

VI.E. Engine Model

The aircraft is powered by a naturally aspirated piston engine. This means that it is subject to the

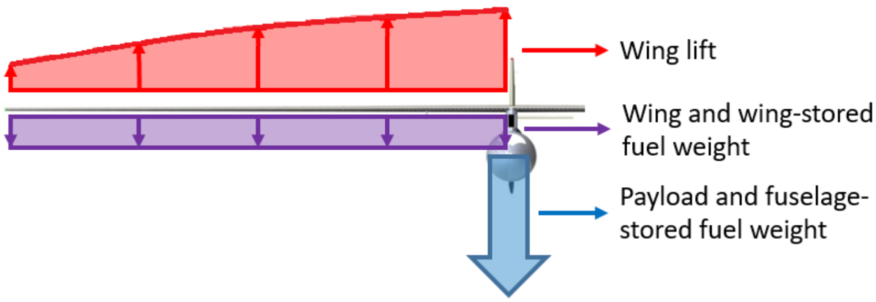
VI.F. Lift, Weight, Drag and Thrust

The weight of the aircraft is the sum of the payload weight, wing weight, and the fuselage weight, as shown in Figure 3. Lift is generated by the wing, which is described by an aspect ratio AR and surface area S.

VI.G. Wing Structure

The wing structure model is based on a simple beam model with a distributed lift load, and a point mass in the center representing the fuselage, as shown in Figure 3.

Figure 3: Wing lift is equal and opposite to the wing weight, payload weight, and total fuel weight.



VI.H. Fuel Volume

The fuel in the aircraft can be stored either in the wing or the fuselage. The signomial constraint in the optimization appears in the fuel volume model, as shown in Equation 9:

$$V_f \leq V_{f_{wing}} + V_{f_{fuse}} \tag{9}$$

where $V_{f_{wing}}$ and $V_{f_{fuse}}$ represent the fuel volume available in the wing and the fuselage respectively. They are each represented by the following monomials.

$$V_{f_{wing}} \leq 9 \times 10^{-4} \frac{S^{1.5}\tau}{A^{0.5}} \tag{10}$$

$$V_{f_{fuse}} \leq 10 \times CDA_0 \ m \tag{11}$$

Note that the monomials above are represented with inequalities, to be compatible with the RSP formulation.

VI.I. Takeoff constraints

We specify that the aircraft has to be able to takeoff at a speed of V_{min} without exceeding the aircraft stall lift coefficient $C_{L_{max}}$, both of which are specified with an associated uncertainty.

VII. Uncertainties and Sets

The uncertainties for the different constants in the problem have been determined considering the parameters in aircraft design that often have the largest uncertainty. These uncertainties are listed in Table 1.

The parameter uncertainties reflect aerospace engineering intuition. The wing weight coefficients $W_{w_{coeff,i}}$ and $W_{w_{coeff,ii}}$, and the ultimate load factor N_{ult} have large 3σs because build quality of aircraft components often difficult to quantify with a large degree of certainty. The payload weight (W_0) has a large uncertainty, because it is valuable if the aircraft has the flexibility to accommodate larger payloads. Parameters that engineers take to be physical constants (μ, ρ) and those that can be determined/manufactured with a relatively high degree of accuracy ($S_{wetratio}$, e) have relatively low deviations. Parameters that require testing to determine ($C_{L_{max}}$, V_{min}) have a level of uncertainty that reflects the expected variance of the parameters.

VIII. Results

We finally implemented our RSP heuristic algorithm on a simple aircraft design problem. In this problem, we conduct an aerostructural optimization of a wing and fuselage given a payload and a range requirement.

Table 1: Constants and Uncertainties (increasing order)

Constant	Description	Value	Units	% Uncert. (3σ)
$S_{wetratio}$	wetted area ratio	2.075	-	3
e	span efficiency	0.92	-	3
μ	viscosity of air	1.78×10^{-5}	$kg/(ms)$	4
ρ	air density	1.23	kg/m^3	5
$C_{L_{max}}$	stall lift coefficient	1.6	-	5
k	fuselage form factor	1.17	-	10
τ	airfoil thickness ratio	0.12	-	10
N_{ult}	ultimate load factor	3.3	-	15
V_{min}	takeoff speed	25	m/s	20
W_0	payload weight	6250	N	20
$W_{w_{coeff,i}}$	wing weight coefficient 1	2×10^{-5}	$1/m$	20
$W_{w_{coeff,ii}}$	wing weight coefficient 2	60	N/m^2	20

VIII.A. Optimization Results

The problem is optimized for different sizes of box and elliptical uncertainty sets by varying the parameter Γ as defined in Appendix IX.A. The design variables are then fixed for each solution so that the design can be simulated for 1000 different realizations of the uncertain parameters in Table 1 to examine average design performance.

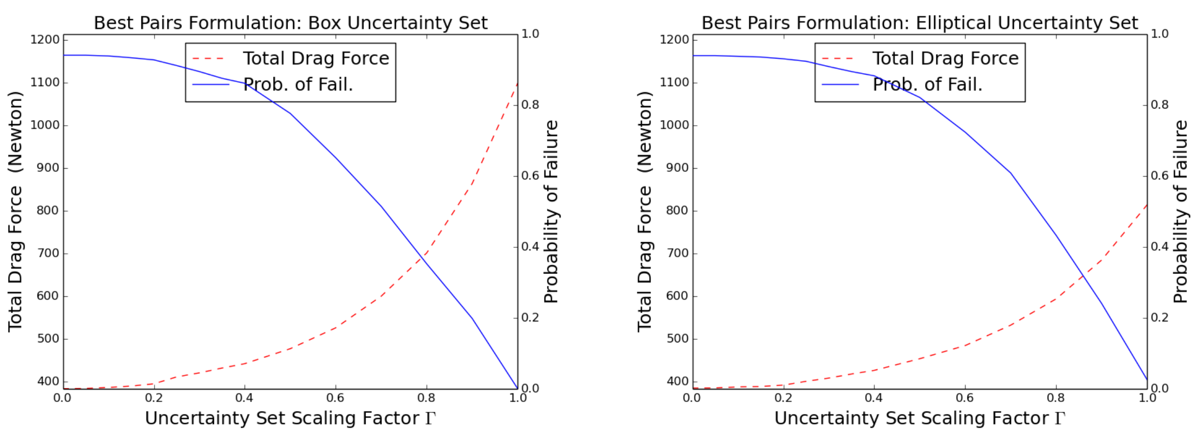


Figure 4: Performance of the optimal robust signomial simple aircraft, using the Best Pairs formulation, as a function of Γ for different uncertainty sets.

We can see from Figure 4 that probability of failure goes to zero as Γ increases. Obviously, it is worth using elliptical uncertainty sets for this aircraft design problem as the performance is significantly better than that of a box uncertainty set, despite the increase in complexity. Moreover, using margins would in the best case be as good as using a box uncertaintyset, and therefore will lead to an inferior performance.

Figure 5 compares the different methodologies in terms of run times, number of constraints, and average performance. The Best Pairs and Linearized Perturbations achieves good performance, however the Best Pairs methodology needs the most number of constraints, while the Linearized perturbations requires the most setup and solve time. The Simple Conservative formulation is significantly faster than the other formulations and requires the least number of additional constraints.

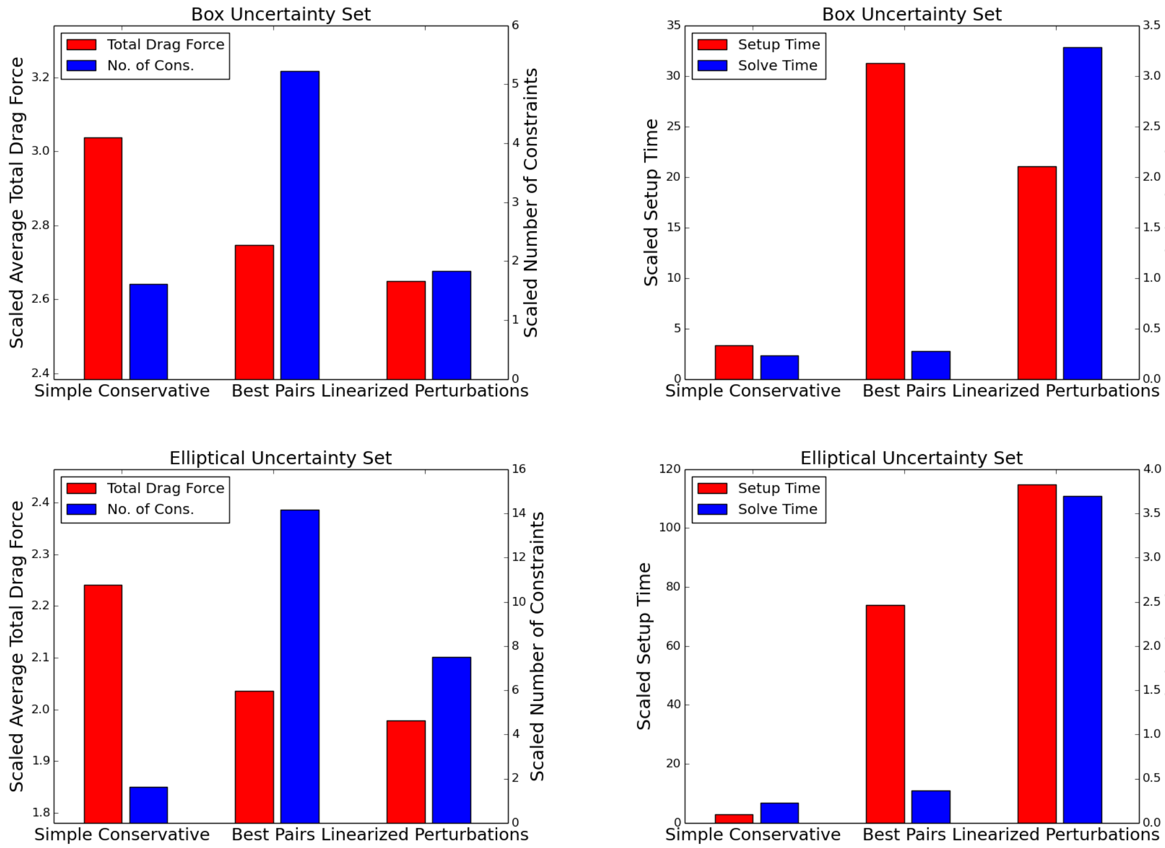


Figure 5: Robust signomial simple aircraft design results relative to the deterministic design problem.

VIII.B. The Effect of Robustness

One of the benefits of convex and difference-of-convex optimization methods is the ability to optimize for different objectives [3]. For the aircraft model in question, we optimized for 7 different objectives, and show the non-dimensionalized results in Table 2.

Objective	Total fuel	Time cost	Aspect ratio	Engine weight	Wing loading	Total cost	Takeoff weight
Total fuel	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Time cost	9.44	0.3	0.09	152.85	1.0	1.86	2.6
Aspect ratio	4.26	0.46	0.04	25.72	0.94	1.11	1.22
Engine weight	1.09	1.22	1.28	0.8	1.0	1.2	1.15
Wing loading	8.89	1.5	0.28	13.06	0.22	2.76	5.82
Total cost	1.68	0.51	0.39	5.75	1.0	0.71	0.92
Takeoff weight	1.33	0.94	0.33	2.13	1.0	1.01	0.85

Table 2: Non-dimensionalized variations in objective values with respect to the aircraft optimized for different objectives. Objective values were normalized by the total fuel solution.

To further demonstrate the capabilities of robust SPs in aircraft design, we performed the optimization of the aircraft with no uncertainty and ellipsoidal uncertainty ($\Gamma = 1$) for two more objective functions, and plotted the results on spider plots. Spider plots are useful because they allow engineers to see the performance of different designs in a multi-objective environment. Due to the large disparities in the potential values of design variables depending on objective as shown in Table 2, we chose to demonstrate this using four objective functions that would be expected to have a high degree of correlation and therefore yield similar aircraft designs. These were total (time and fuel) cost, total fuel, takeoff weight and mid-cruise lift-over-drag

(L/D).

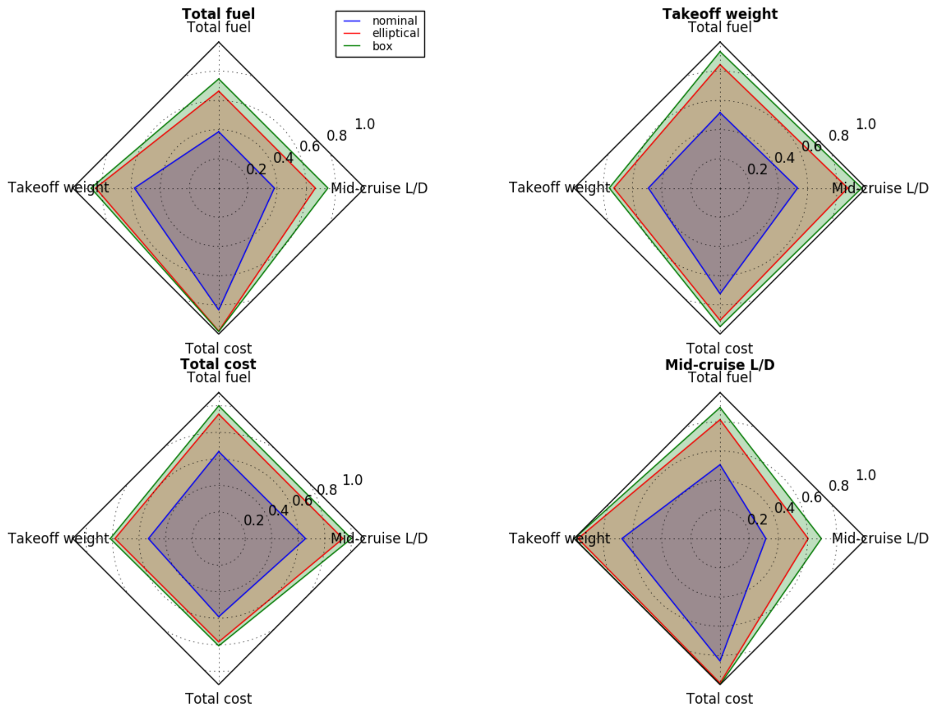


Figure 6: The spider plots of aircraft optimized for different objectives. The bolded titles are the design objectives for each plot, whereas the individual spiderwebs show the non-dimensionalized multiobjective performance of the aircraft designed under different uncertainty sets.

In the spider plots in Figure 6, it is possible to see the effect of robustness on the different performance metrics of the different aircraft. One way to envision the multi-objective performance of the aircraft is to consider the area contained within the web defined by the aircraft's performance; the smaller the web area the beter.

For example, for the nominal case, which has no uncertainty, it is possible to see that the aircraft designed for total fuel performs the best when all four objectives are considered. However, this behavior changes when uncertainty is added. An aircraft optimized for total cost (bottom left graph) with a box uncertainty set (in green) has better multiobjective performance compared to an aircraft designed for total fuel with the same uncertainties.

This is an interesting result, because the presence of an uncertainty set is shown to affect the efficacy of different objective functions to obtain solutions with the best overall performance. If the three objective functions didn't have high degree of coupling, that the internal areas of the solution triangles may differ more significantly.

VIII.C. Goal Programming

However, this assumes that we have an understanding of exactly how much risk we are willing to tolerate. This begs the question, could we have risk as the output of our model? This would suggest the following formulation:

$$
\begin{aligned}
\text{maximize } & \Gamma \\
\text{s.t. } & f_i(x,u) \le 0, i = 1, \dots, n \\
& \|u\| \le \Gamma \\
& f_0(x) \le (1+\delta) f_0^*, \ \delta \ge 0
\end{aligned} \tag{a}
$$

where f_0^* is the optimum of the original problem in Formulation **??**, δ is a fractional measure of the objective that we are willing to sacrifice for robustness, which gives $(1+\delta)f_0^*$ as the upper bound on the objective value.

We can also expand this framework to perform multivariate goal programming, by changing (a) in the formulation a to include all objectives we are interested in.

$$f_{0,j}(x) \le (1+\delta_j) f_{0,j}^*, \ \delta_j \ge 0, \ i = 1, \dots, m$$

The benefit of goal programming is that it allows us to explore multidisciplinary tradeoffs without having to enumerate the design space along each objective direction. Furthermore, in design it is not obvious whether an objective should in fact be a constraint instead. For example, it's not clear that the design of an aircraft is useful if it consumes

VIII.C.1. Changes in flight envelope

IX. Conclusion

We have developed and applied a tractable RSP formulation to a simple aircraft model, and then discussed the benefits of having robust solutions. RSP formulations extend the tractable approximate RGP framework developed by Saab to non-log-convex problems, and are a valuable contribution to the fields of robust optimization and difference-of-convex programming.

RSPs have a wide variety of potential applications in engineering design. Within the Hoburg Research Group in the Aerospace Computational Design Lab, during the past year we have developed a commercial aircraft design SP that has between 1700 and 8000 variables, depending on the whether it is a single-point, or multi-point optimization. We expect that using RO in this conceptual aircraft design will result in designs that are more robust with respect to uncertainties in operational parameters, such as payload mass and range, as well as uncertain constants.

Appendix

IX.A. Robust Linear Programming: A Quick Review

As mentioned earlier, robust linear programming will be used to formulate an approximate robust geometric program.

Consider the system of linear constraints

$$\mathbb{A}\mathbf{x} + \mathbf{b} \le 0$$

where

$$
\begin{aligned}
\mathbb{A} &\text{ is } m \times n \\
\mathbf{x} &\text{ is } n \times 1 \\
\mathbf{b} &\text{ is } m \times 1
\end{aligned}
$$

where that data is uncertain and is given by equations (4) and (5).

IX.A.1. Box Uncertainty Set

If the perturbation set $\mathcal{Z}$ given in equation (5) is a box uncertainty set, i.e. $\|\zeta\|_\infty \leq \Gamma$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \textstyle\sum_{l=1}^{L} | - b_i^l - \mathbf{a}_i^l \mathbf{x}| + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \tag{12}$$

If only b is uncertain, i.e. $A^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (12) will become

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \textstyle\sum_{l=1}^{L} |b_i^l| \leq 0 \tag{13}$$

which is a linear constraint
On the other hand, if A is uncertain, then equation (12) is equivalent to the following set of linear constraints

$$\begin{aligned} \Gamma \textstyle\sum_{l=1}^{L} w_i^l + \mathbf{a}_i^0 \mathbf{x} + b_i^0 &\leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} &\leq w_i^l \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} &\leq w_i^l \quad \forall l \in 1, ..., L \end{aligned} \tag{14}$$

IX.A.2. Elliptical Uncertainty Set

Briefly, if the perturbation set $\mathcal{Z}$ is an elliptical, i.e. $\sum_{l=1}^{L} \frac{\zeta_l^2}{\sigma_l^2} \leq \Gamma^2$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \sqrt{\textstyle\sum_{l=1}^{L} \sigma_l^2 (-b_i^l - \mathbf{a}_i^l \mathbf{x})^2} + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \tag{15}$$

which is a second order conic constraint.
If only b is uncertain, i.e. $\mathbb{A}^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (15) will become

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \sqrt{\textstyle\sum_{l=1}^{L} \sigma_l^2 (b_i^l)^2} \leq 0 \tag{16}$$

which is a linear constraint.

IX.A.3. Norm-1 Uncertainty Sets

Briefly, if the perturbation set represented by $\mathcal{Z}$ is a norm-1 uncertainty set, i.e. $\|\zeta\|_1 \leq \Gamma$, then the robust constraint is

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \max_{l=1,..,L} |b_i^l| \leq 0 \tag{17}$$

when $\mathbb{A}^l = 0$, and

$$\begin{aligned} \Gamma w_i + \mathbf{a}_i^0 \mathbf{x} + b_i^0 &\leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} &\leq w_i \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} &\leq w_i \quad \forall l \in 1, ..., L \end{aligned} \tag{18}$$

if $\mathbb{A}^l \neq 0$
Note that for this type of uncertainty, the robust constraints are linear.

Acknowledgments

A place to recognize others.

References

[1] Saab, A., Burnell, E., and Hoburg, W. W., "Robust Designs via Geometric Programming," 2018, pp. 1–23.

[2] Zang, T. a., Hemsch, M. J., Hilburger, M. W., Kenny, S. P., Luckring, J. M., Maghami, P., Padula, S. L., and Stroud, W. J., "Needs and opportunities for uncertainty-based multidisciplinary design methods for aerospace vehicles," *Nasa Tm*, Vol. 211462, No. July, 2002.

[3] York, M. A., Öztürk, B., Burnell, E., and Hoburg, W. W., "Efficient Aircraft Multidisciplinary Design Optimization and Sensitivity Analysis via Signomial Programming," 2018, pp. 1–16.

[4] Yao, W., Chen, X., Luo, W., van Tooren, M., and Guo, J., "Review of uncertainty-based multidisciplinary design optimization methods for aerospace vehicles," *Progress in Aerospace Sciences*, Vol. 47, No. 6, 2011, pp. 450 – 479.

[5] Diwekar, U., *Introduction to Applied Optimization*, Vol. 22, 2008.

[6] Bertsimas, D., Brown, D. B., and Caramanis, C., "Theory and Applications of Robust Optimization," *Society for Industrial and Applied Mathematics*, 2011.

[7] Kall, P. and Stoyan, D., "Solving stochastic programming problems with recourse including error bounds," *Math. Operationsforsch. Statist. Ser. Optim.*, Vol. 13, No. 3, 1982, pp. 431–447.

[8] Higle, J. L. and Sen, S., "Stochastic Decomposition: An Algorithm for Two-Stage Linear Programs with Recourse," *Mathematics of Operations Research*, Vol. 16, No. 3, 1991, pp. 650–669.

[9] Pereira, M. and Pinto, L., "Multi-stage stochastic optimization applied to energy planning," *Mathematical Programming*, Vol. 52, No. 1-3, 1991, pp. 359–375.

[10] Bertsimas, D., Gupta, V., and Kallus, N., "Data-Driven Robust Optimization," 2013.

[11] Duffin, R., Peterson, E., and Zener, C., *Geometric programming: theory and application*, Wiley New York, 1967.

[12] Nesterov, Y. and Nemirovski, A., *Interior-Point Polynomial Algorithms in Convex Programming*, Society for Industrial and Applied Mathematics, 1994.

[13] Boyd, S., Kim, S.-J., Vandenberghe, L., and Hassibi, A., "A tutorial on geometric programming," *Optimization and Engineering*, Vol. 8, No. 1, Oct 2007, pp. 67–127.

[14] Hoburg, W. W., Ghaoui, L. E., and Packard, A., "Aircraft Design Optimization as a Geometric Program," 2013.

[15] Burton, M. J. and Hoburg, W. W., "Solar-Electric and Gas Powered, Long-Endurance UAV Sizing via Geometric Programming," *18th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference*, , No. June, 2017, pp. 1–26.

[16] Kirschen, P. G., York, M. A., Ozturk, B., and Hoburg, W. W., "Application of Signomial Programming to Aircraft Design," *Journal of Aircraft*, Vol. 55, No. 3, 2018, pp. 965–987.

[17] Kirschen, P. G., Burnell, E., and Hoburg, W., "Signomial Programming Models for Aircraft Design," *54th AIAA Aerospace Sciences Meeting*, Feb 2016.

[18] Ozturk, B., "Conceptual Engineering Design and Optimization Methodologies using Geometric Programming," 2018.

[19] Lipp, T. and Boyd, S., "Variations and extension of the convex – concave procedure," *Optimization and Engineering*, Vol. 17, No. 2, 2016, pp. 263–287.

[20] Hsiung, K.-L., Kim, S.-J., and Boyd, S., "Tractable approximate robust geometric programming," *Optimization and Engineering*, Vol. 9, No. 2, Apr 2007, pp. 95–118.