Optimal Aircraft Design Decisions under Uncertainty via Robust Signomial Programming

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Aircraft design benefits from optimization under uncertainty, since design feasibility and performance can have large sensitivities to uncertain parameters. Legacy methods of capturing uncertainty do not adequately explain the trade-offs between feasibility and optimality, and require prior engineering knowledge which may not be available for novel aerospace vehicle concepts. This paper proposes a solution method for engineering design optimization problems under uncertainty using robust signomial programs (RSPs). The method transforms stochastic optimization problems to deterministic problems by considering the worst-case robust counterpart of each design constraint. The formulation leverages an existing approximate robust geometric programming (RGP) formulation and extends it by allowing difference-of-log-convex constraints that appear in many design problems. Signomial programs have demonstrated potential in the solution of multidisciplinary non-convex optimization problems such as aircraft design, and the formulation of robust signomial programs (RSPs) allows for conceptual engineering design that captures parametric uncertainty with probabilistic guarantees of design feasibility. The paper details a method based on solving a sequence of RGPs, where each RGP is a local approximation of the RSP. Then it explores the trade-off between robustness and optimality rigorously by implementing RSPs on an unmanned aircraft design problem, and evaluates the effect of robustness requirements on aircraft design decisions.

Nomenclature

CEG	Convex Engineering Group
GP	geometric program
LHS	left hand side
MDO	multidisciplinary design optimization
NLP	nonlinear program
SP	signomial program
RGP	robust geometric program

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RHS	right hand side
RO	robust optimization
RSP	robust signomial program
SO	stochastic optimization

I. Introduction

Aircraft design exists in a niche of design problems where “failure is not an option"*. This is remarkable since aircraft design problems are rife with uncertainty about technological capabilities, environmental factors, manufacturing quality and the future state of markets and regulatory agencies. Optimization under uncertainty seeks to provide designs that are robust to realizations of uncertainty in the real world and can reduce the high risk of aerospace programs.

Optimization has become ubiquitous in the design of engineered systems, and especially aerospace systems, in the late 20th and 21st centuries as computing has improved dramatically and as designs have continued to approach the limits of the second law of thermodynamics. Optimization under uncertainty has been identified by academia and industry as an area of opportunity in multiple review papers ([1], [2]), and we elaborate on three potential benefits from [1] below:

- *Confidence in analysis tools will increase.* The uptake of new design tools in the aerospace industry has been low due to heavy reliance on legacy design methods and prior experience when faced with risky design propositions, and notably in the design of novel configurations where understanding of the design tradespaces is lacking. Robustness will increase confidence in analysis tools because it appropriately captures the effects of technological uncertainty on the potential benefits of new configurations.
- *Designs will be more robust.* The ability to provide designs with feasibility guarantees will mean that designs and products will be more robust to uncertainties in manufacturing quality, environmental factors, technology level and markets, and better able to handle off-nominal operating conditions.
- *System performance will increase while ensuring that reliability requirements are met.* Design under uncertainty will allow for a better understanding of the trade-off between risk and performance. As a result, it will allow for designs that are less conservative than traditional designs while meeting the same reliability requirements.

In economics, the idea that risk is related to profit is well understood and leveraged. In aerospace engineering however we often forget that risk aversity necessarily results in lower performance. Considering that conceptual design in the aerospace industry hedges against program risk, the tractable Robust Optimization (RO) frameworks proposed in

*Quoting Gene Kranz, the mission director of Apollo 13.

this paper will give aerospace engineers the ability to rigorously trade-off robustness to uncertainty with the performance penalties that result.

A. Approaches to optimization under uncertainty

Faced with the challenge of finding designs that can handle uncertainty, the aerospace field has developed a number of methods to design under uncertainty. Oftentimes, aerospace engineers will implement *margins* in the design process to account for uncertainties in parameters that a design's feasibility may be sensitive to, such as material properties or maximum lift coefficient. Another traditional method of adding robustness is through *multi-mission design* [3], which ensures that the aircraft is able to handle multiple kinds of missions in the presence of no uncertainty. This is a type of *finitely adaptive* optimization geared to ensure objective performance in off-nominal operations.

These methods have several weaknesses. They provide no quantitative measures of robustness or reliability [1]. They rely on the expertise of an experienced engineer to guide the design process, without explicit knowledge of the trade-off between robustness and optimality [2]. This is a dangerous proposition especially in the conceptual design phase of new configurations, since prior information and expertise is not available. In these scenarios, it is especially important to go back to fundamental physics and use rigorous mathematics to explore the design space [3]. Furthermore, the legacy methods are often too conservative, ruling out potentially beneficial technologies and configurations due to the inability to adequately trade off performance and risk.

There are two rigorous approaches to solving design optimization problems under uncertainty, which are Stochastic Optimization (SO) and RO. Note that stochastic optimization is an overloaded term, and exists in at least two contexts in the literature. The first is the solution of deterministic problems with stochastic search space exploration. The second is the solution of design optimization problems with stochastic parameters, which is the focus of this paper. In this context, SO problems deal with uncertainty by propagating the probability distributions of uncertain parameters through the physics of a design problem to ensure constraint feasibility with certain probability. The predominant goal of SO is to minimize some characteristics, for example moments or risk measures, of the probability density function of the quantity of interest [4]. In contrast, RO takes a different approach, instead choosing to make designs immune to uncertainties in parameters as long as the parameter values come from within a defined uncertainty set. As such, RO avoids the need to propagate entire probability distributions by minimizing the worst-case objective outcome of a design for a given set over the uncertain parameters.

B. Comparison of robust and stochastic optimization methods for conceptual design

Both RO and SO have relative advantages in implementation. This paper will argue specifically that the formulation of conceptual engineering design problems under uncertainty as RO problems has advantages over SO formulations (a more mathematical programming centric comparison is made in [5]).

1. Generality and tractability

In the context of engineering, we claim that an optimization method is general when it can be used to solve a range of problems of interest. On the other hand, tractability describes whether or not the problems are solved to a satisfactory optimum with reasonable computational time. Optimization under uncertainty is a difficult task that puts these two desirable subjective traits at odds with each other.

SO has the advantage of generality. SO methods are easily applicable to black box models or input-output systems. They require little knowledge, if any, about the constraints in the system of interest. RO methods are less general, since they require the design objective and constraints to be explicit and cast in a form that has a worst-case counterpart. Thus models for RO have to be transparent, and RO cannot be applied to black box models without significant prior data manipulation at a minimum. A mitigating factor is that many classes of conceptual engineering design problems can be cast or approximated in a form that is compatible with robust optimization, such as linear, quadratic, semidefinite and geometric programs.

On the other hand, RO is more tractable than SO due to the difference in method of uncertainty propagation. As aforementioned, stochastic methods involve the propagation of probability densities throughout a model to determine their effects on constraint feasibility and the objective function. This requires the integration of the product of probability distributions with potential outcomes, and since the integration of continuous functions is difficult this is often achieved through a combination of high-dimensional quadrature and discretizations of the uncertainty into possible scenarios. The propagation of parameter scenarios results in a combinatorial explosion of possible outcomes which need to be evaluated to determine constraint satisfaction and the distribution of the objective. Few problems can be addressed purely through stochastic optimization (eg. the recourse problem as shown in [6],[7], and energy planning problem such as in [8]), and even these are limited by combinatorics and costly system evaluations. Furthermore, they require problem-specific approximations, so that generality is compromised. Robust versions of tractable optimization problems are not guaranteed to be tractable, but in practice the aforementioned classes of optimization problems have tractable robust formulations [5]. In RO, there are no separate optimization and evaluation loops by construction, and thus RO problems can be solved optimally many orders of magnitude faster than SO problems of the same form [5].

Conceptual design optimization values generality, because engineers would like to apply methods for optimization under uncertainty without significant mathematical groundwork, and tractability, because fast solution times are critical to reduce program risk early on in the design process when more aspects of the design are fluid. From this perspective, the relative intractability of SO-based approaches makes them unreliable for conceptual design, since significant time is needed both to develop problem-specific tractable formulations, and to find satisfactory optima. Furthermore, many engineering design problems such as aircraft design are approximable by optimization forms that have tractable robust counterparts, making RO better suited to conceptual design.

2. Use of data

SO problems generally require complete knowledge of the probability distribution of parameters. RO requires only 'modest assumptions about distributions, such as a known mean and bounded support' [9]. Since RO does not require as much information about uncertain parameters as SO does, it can better address conceptual design problems where there is a lack of experience, or sparse and noisy data [5]. It is arguable that RO leaves a lot on the table by not taking advantage of distributional information, however there is a growing body of research on distributionally robust optimization [10] which seeks to leverage existing data.

3. Conservativeness

Although RO problems solve problems with uncertainty, RO formulations result in solutions that are *deterministically immune* to all possible realizations of parameters in an uncertainty set [5], which is defined as conservativeness. There is extensive literature on RO methods that offer differing levels of conservativeness [11], depending on the kind of uncertainty set considered. SO formulations provide no guarantees of conservativeness, since the solution methods rely on randomized algorithms [12]. Conservativeness is an advantage in conceptual design, since we would like to have some guarantees that a design can handle uncertainty in parameters.

4. Stochasticity

The solution of RO problems is deterministic, meaning that different instances of a design problem with the same parameters will result in the same solution. This is not the case with SO, since the optimum depends on realizations of random variables. This is not satisfactory from an engineering perspective, since a design can be sensitive to issues in sampling schemes. Furthermore, optimization runs over the same parameters may result in different solutions. In the context of engineering design determinism makes RO superior with respect to SO.

It is important to highlight that, although both RO and SO seek to address the problem of optimization under uncertainty, they solve fundamentally different problems. In an ideal world where we have a problem that is tractable and globally optimal for both methods, the two different approaches would result in different solutions.

C. Geometric and signomial programming for engineering design

Geometric programming† is a method of log-convex optimization that has been developed to solve problems in engineering design [13]. Although theory of the Geometric Program (GP) has existed since the 1960's, GPs have recently experienced a resurgence due to the advent of polynomial-time interior point methods [14] and improvements in computing. They have been applied to a range of engineering design problems with success. For a non-exhaustive list of examples, please refer to [15].

GPs have been effective in aircraft conceptual design ([16], [17]). However, the stringent mathematical requirements

†Programming refers to the mathematical formulation of an optimization problem.

of a GP limits its application to non-log-convex problems. The Signomial Program (SP) is the difference-of-log-convex extension of the GP which can be applied to solve this larger set of problems, albeit with the loss of some mathematical guarantees compared to the GP [18]. Aircraft pose some of the most challenging design problems [3], and signomial programming has been used to great effect in modeling and designing complex aircraft at a conceptual level quickly and reliably as in [3], [19] and [18]. Other interesting applications for SPs such as in network flow problems are being investigated.

Robust formulations exist for solving geometric programs with parametric uncertainty [20]. The creation of a robust signomial programming framework to capture uncertainty in engineering design, and specifically aircraft design, will allow us to have more confidence in the results of the conceptual design phase, reduce program risk, and increase overall system performance.

D. Contributions

This paper proposes a tractable Robust Signomial Program (RSP) which we solve as a sequential Robust Geometric Program (RGP), allowing us to implement robustness in non-log-convex problems such as aircraft design. We extend the RGP framework developed by Saab [20] to SPs. We implement the RSP formulation on a simple aircraft design problem with several hundred variables as defined in [21]. The benefits of robust optimization are demonstrated both in ensuring design feasibility and performance using Monte Carlo (MC) simulations of the uncertain parameters. We further explore the benefits of RO in multiobjective optimization, and propose a goal programming RSP formulation for risk minimization problems.

II. Mathematical Background

A. Robust Optimization

Given a general optimization problem under parametric uncertainty, we can define the set of possible realizations of uncertain vector of parameters u in the uncertainty set $\mathcal{U}$. This allows us to define the problem under uncertainty below.

$$\begin{aligned} &\min\ f_0(x) \\ &\text{s.t. } f_i(x,u) \leq 0,\ \forall u \in \mathcal{U},\ i = 1,\dots,n \end{aligned}$$

This problem is infinite-dimensional, since it is possible to formulate an infinite number of constraints with the countably infinite number of possible realizations of $u \in \mathcal{U}$. To circumvent this issue, we can define the following robust formulation of the uncertain problem below.

$$
\begin{aligned}
&\min\ f_0(x)\\
&\text{s.t.} \max_{u\in\mathcal{U}} f_i(x,u) \leq 0,\ i = 1,\ldots,n
\end{aligned}
$$

This formulation hedges against the worst-case realization of the uncertainty in the defined uncertainty set. This is often posed by creating an uncertainty set to contain all possible realizations of the uncertainty we are concerned about, usually through a norm,

$$
\begin{aligned}
&\min\ f_0(x)\\
&\text{s.t.} \max_{u} f_i(x,u) \leq 0,\ i = 1,\ldots,n\\
&\qquad \|u\| \leq \Gamma
\end{aligned}
\tag{1}
$$

where Γ is defined by the user as an uncertainty bound. The larger the Γ, the greater the size of the uncertainty set that is protected against.

B. Geometric Programming

A **geometric program in posynomial form** is a log-convex optimization problem of the form:

$$
\begin{aligned}
\min \quad & f_0\left(\mathbf{u}\right)\\
\text{s.t.} \quad & f_i\left(\mathbf{u}\right) \leq 1, i = 1, \ldots, m_p\\
& h_i\left(\mathbf{u}\right) = 1, i = 1, \ldots, m_e
\end{aligned}
\tag{2}
$$

where each f_i is a *posynomial*, each h_i is a *monomial*, m_p is the number of posynomials, and m_e is the number of monomials. A monomial $h(\mathbf{u})$ is a function of the form:

$$h_i(\mathbf{u}) = e^{b_i} \prod_{j=1}^{n} u_j{}^{a_{ij}}$$

where a_{ij} is the j^{th} component of a row vector $\mathbf{a_i}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_i is in $\mathbb{R}$. An example of a monomial is the lift equation, $L = \frac{1}{2}\rho V^2 C_L S$. A posynomial $f(\mathbf{u})$ is the sum of $K \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{u}) = \sum_{k=1}^{K} e^{b_{ikj}} \prod_{j=1}^{n} u_j{}^{a_{ikj}}$$

where a_{ikj} is the j^{th} component of a row vector $\mathbf{a_{ik}}$ in $\mathbb{R}^n$, u_j is the j^{th} component of a column vector $\mathbf{u}$ in $\mathbb{R}^n_+$, and b_{ik} is in $\mathbb{R}$ [15]. The stagnation pressure definition is a good example: $P_t = P + \frac{1}{2}\rho V^2$.

A logarithmic change of the variables $x_j = \log(u_j)$ would turn a monomial into *the exponential of an affine function* and a posynomial into *the sum of exponentials of affine functions*. A transformed monomial $h_i(\mathbf{x})$ is a function of the form:

$$h_i(\mathbf{x}) = e^{\mathbf{a_i x} + b_i}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A transformed posynomial $f_i(\mathbf{x})$ is the sum of $K_i \in \mathbb{Z}^+$ monomials:

$$f_i(\mathbf{x}) = \sum_{k=1}^{K_i} e^{\mathbf{a_{ik} x} + b_{ik}}$$

where $\mathbf{x}$ is a column vector in $\mathbb{R}^n$. A geometric program with transformed constraints is a **geometric program in exponential form**, and is a convex optimization problem.

The positivity of exponential functions restricts the space spanned by posynomials and limits GPs to certain classes of problems. However, since many engineering problems of interest have purely positive quantities GPs are quite applicable, and certain variable transformations can make problems with negative quantities tractable. The restriction of posynomials to the *less-than-side of inequalities* is a more significant barrier, and motivates the introduction of signomials.

C. Signomial Programming

A *signomial* can be defined as the difference between two posynomials. Consequently, a SP is a non-log-convex optimization problem of the form:

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) - g_i(\mathbf{x}) \quad \leq 0, i = 1, ..., m \end{aligned} \tag{3}$$

where f_i and g_i are both posynomials, and $\mathbf{x}$ is a column vector in $\mathbb{R}^n$.

Reliably solving an SP to a local optimum has been described in [15] and [22]. A common solution heuristic involves solving an SP as a sequence of GPs, where each GP is a local approximation of the SP. Although it is a powerful tool, applications involving SPs are usually prone to uncertainties that have a significant effect on the solution.

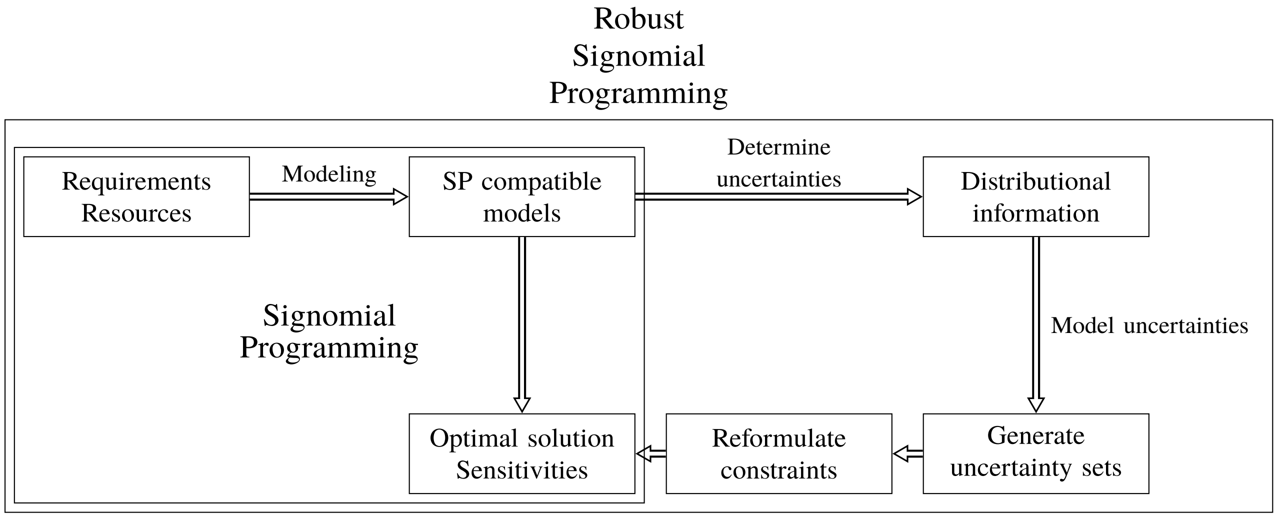


Fig. 1 A block diagram showing the difference between the design process using a SP and a RSP.

III. Robust Signomial Programming

As a preview of the following sections, robust signomial programming assumes that parameter uncertainties belong to an uncertainty set, and solves a reformulated design problem to find the best solution, through the process shown in Figure 1. As long as the original optimization problem is SP-compatible, a tractable robust formulation of the problem exists, making this method general. We derive the intractable formulation of a RSP below.

A SP in its **exponential form** is as follows:

$$\begin{aligned} \min \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & \textstyle\sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}\mathbf{x}+b_{ik}} - \sum_{k=1}^{G_i} e^{\mathbf{c_{ik}}\mathbf{x}+d_{ik}} \leq 0 \quad \forall i \in 1,\ldots,m \end{aligned} \tag{4}$$

where the constraints are represented as difference-of-posynomials in exponential form. Let $\mathbf{a_{ik}}$ and $\mathbf{c_{ik}}$ be the $((i-1)\times m + k)^{th}$ rows of the exponents matrices $\mathbf{A}$ and $\mathbf{C}$ respectively, and b_{ik} and d_{ik} be the $((i-1)\times m + k)^{th}$ elements of the coefficients vectors $\mathbf{b}$ and $\mathbf{d}$ respectively.

The data ($\mathbf{A}$, $\mathbf{C}$, $\mathbf{b}$, $\mathbf{d}$) is assumed to be uncertain and living in an uncertainty set $\mathcal{U}$, where $\mathcal{U}$ is parametrized affinely by a perturbation vector ζ:

$$\mathcal{U} = \left\{[\mathbf{A};\mathbf{C};\mathbf{b};\mathbf{d}] = \left[\mathbf{A}^0;\mathbf{C}^0;\mathbf{b}^0\ \mathbf{d}^0\right] + \textstyle\sum_{l=1}^{L} \zeta_l \left[\mathbf{A}^l;\mathbf{C}^l;\mathbf{b}^l;\mathbf{d}^l\right]\right\} \tag{5}$$

where $\mathbf{A}^0$, $\mathbf{C}^0$, $\mathbf{b}^0$, and $\mathbf{d}^0$ are the nominal exponents and coefficients, $\left\{\mathbf{A}^l\right\}_{l=1}^{L}$, $\left\{\mathbf{C}^l\right\}_{l=1}^{L}$, $\left\{\mathbf{b}^l\right\}_{l=1}^{L}$, and $\left\{\mathbf{d}^l\right\}_{l=1}^{L}$ are the basic shifts of the exponents and coefficients, and ζ_l is the l^{th} component of ζ belonging to a perturbation set $\mathcal{Z} \in \mathbb{R}^L$ such that

$$\mathcal{Z} = \left\{\zeta \in \mathbb{R}^L : \|\zeta\| \leq \Gamma\right\} \tag{6}$$

As mentioned earlier, there should exist a formulation immune to uncertainty in the system's data. Accordingly, the robust counterpart of the uncertain SP in (4) is:

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{subject to} & \max\limits_{\zeta\in\mathcal{Z}} \left\{ \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)} - \sum_{k=1}^{G_i} e^{\mathbf{c_{ik}}(\zeta)\mathbf{x}+d_{ik}(\zeta)} \right\} \quad \leq 1 \quad \forall i \in 1,\ldots,m \end{array} \tag{7}$$

The optimization problem in (7) is intractable using current solvers, therefore, a heuristic approach to solving RSPs approximately as a sequential RGP will be presented in the following sections. As our approach is based on robust geometric programming, a brief review of the subject will follow based on [20].

IV. Robust Geometric Programming

This section presents a brief review of the approximation of an RGP as a tractable optimization problem as discussed in [20]. The robust counterpart of an uncertain geometric program is:

$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{subject to} & \max\limits_{\zeta\in\mathcal{Z}} \left\{ \sum_{k=1}^{K_i} e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)} \right\} \quad \leq 1 \quad \forall i \in 1,\ldots,m \end{array} \tag{8}$$

which is Co-NP hard in its natural posynomial form [23]. We will present three approximate formulations of a RGP.

A. Simple Conservative Formulation

One way to approach the intractability in (8) is to replace each constraint by a tractable approximation. Replacing the max-of-sum in (8) by the sum-of-max will lead to the following formulation

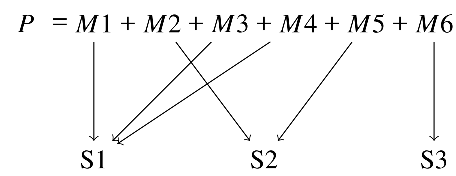
$$\begin{array}{ll} \min & f_0(\mathbf{x}) \\ \text{subject to} & \sum_{k=1}^{K_i} \max\limits_{\zeta\in\mathcal{Z}} \left\{ e^{\mathbf{a_{ik}}(\zeta)\mathbf{x}+b_{ik}(\zeta)} \right\} \quad \leq 1 \quad \forall i \in 1,\ldots,m \end{array} \tag{9}$$

Maximizing a monomial term is equivalent to maximizing an affine function, therefore (9) is tractable.

B. Equivalent Intermediate Formulation

This formulation is equivalent to the formulation in (8), but with smaller, easier to handle posynomial constraints. By the properties of inequalities, the posynomial P in posynomial inequality $M \geq P$ can be divided into an equivalent set of smaller posynomials based on the dependence between its monomial terms. Figure 2 shows how a constraint can be represented as an equivalent set of smaller posynomial constraints.

The posynomial constraints are categorized into three sets: large posynomials, two-term posynomials and monomials, represented by $S1$, $S2$ and $S3$ respectively. Monomials are tractable, and two-term posynomials can be well approximated



$$
max\{P\} \leq 1 \iff \begin{array}{lr} t_1 + t_2 + t3 & \leq 1 \\ max\{S_1\} = max\{M_1 + M_3 + M_4\} & \leq t_1 \\ max\{S_2\} = max\{M_2 + M_5\} & \leq t_2 \\ max\{S_3\} = max\{M_6\} & \leq t_3 \end{array}
$$

Fig. 2 Partitioning of a large posynomial into smaller posynomials requires the addition of auxiliary variables. S_i are posynomials with independent sets of variables.

using piecewise-linear functions [24]. We implement the following two tractable approximations for large posynomials.

1. Linearized Perturbations Formulation

If the exponents are known and certain, then large posynomial constraints can be approximated as signomial constraints. The exponential perturbations in each posynomial are linearized using a modified least squares method, and then the posynomial is robustified using techniques from robust linear programming. The resulting set of constraints is SP compatible, therefore, a robust geometric program can be approximated as a signomial program.

2. Best Pairs Formulation

If the exponents are also uncertain, then large posynomials can't be approximated as an SP, and further simplification is needed. This formulation aims to maximize each pair of monomials in each posynomial, while finding the best combination of monomials that gives the least conservative solution. [20] provides a descent algorithm to find locally optimal combinations of the monomials, and shows how the uncertain geometric program can be approximated as a geometric program for polyhedral uncertainty, and a conic optimization problem for elliptical uncertainty with uncertain exponents. For a detailed description of the above formulations refer to [20]. An algorithm for solving an RSP based on the above formulations is provided in the next section.

V. Approach to Solving Robust Signomial Programs

This section presents a heuristic algorithm to solve a RSP based on our previous discussion on robust geometric programming.

A. General RSP Solver

As mentioned before, a common heuristic algorithm to solve a SP is by sequentially solving local GP approximations, but the solution is not guaranteed to be globally optimal. Our approach to solve a RSP is based on sequentially solving

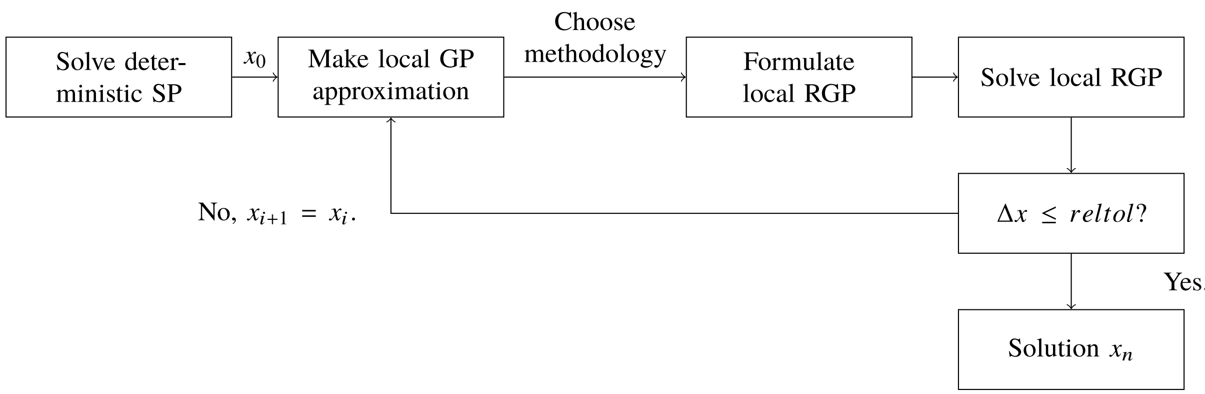


Fig. 3 A block diagram showing the steps of solving an RSP.

local RGP approximations. Below we provide a step-by-step algorithm to solve a RSP:

1) Choose an initial guess x_0.
2) Repeat:
    1) Find the local GP approximation of the SP at x_i.
    2) Find the RGP formulation of the GP.
    3) Solve the RGP to obtain x_{i+1}.
    4) If $x_{i+1} \approx x_i$: break

Similar to a SP, a good initial guess will lead to faster convergence and possibly a better solution. The deterministic solution of the uncertain SP is a good candidate x_0, and is used to speed up convergence.

Any of the previously mentioned methodologies can be used to formulate the local RGP approximation. However, depending on the RGP formulation chosen to solve a RSP, the last formulation and solution blocks in Figure 3 are adjusted for a faster rate of convergence.

B. Best Pairs RSP Solver

If the Best Pairs methodology is exploited, then the above algorithm would change so that each iteration would solve the local RGP approximation and choose the best permutation for each large posynomial. The modified algorithm would become as follows:

1) Choose an initial guess x_0.
2) Repeat:
    1) Find the local GP approximation of the SP at x_i.
    2) For each large posynomial constraint, select the new permutation ϕ such that ϕ minimizes the robust large constraint evaluated at x_i.

3) Solve the approximate tractable counterparts of the local GP in (8), and let $\mathbf{x}_{i+1}$ be the solution.

4) If $x_{i+1} \approx x_i$: break.

C. Linearized Perturbations RSP Solver

On the other hand, if the Linearized Perturbations formulation is to be used, then we can avoid solving a SP at each iteration by first approximating the original SP constraints locally, and in the same loop approximating the robustified possibly signomial constraints locally, thus solving a GP at each iteration instead of an SP. The algorithm would then become as follows:

1) Choose an initial guess x_0.
2) Repeat:
    1) Find the local GP approximation of the SP at x_i.
    2) Robustify the constraints of the local GP approximation using the Linearized Perturbations methodology.
    3) Find the local GP approximation of the resulting local SP at x_i.
    4) Solve the local GP approximation in step c to obtain x_{i+1}.
    5) If $x_{i+1} \approx x_i$: break.

VI. Models

We implement the RSP formulation above on an unmanned, gas-powered aircraft design problem that is systematically developed in [21], with the elliptical fuselage model borrowed from [17]. We optimize a wing, fuselage, and engine given a payload and range requirement. The optimization model was developed using GPkit, a Python package that provides abstractions for using GPs in engineering design [25]. The nominal model has 176 variables and 154 constraints, a common level of sparsity for GP and SP models. A short qualitative overview of the model follows; for more detailed information, please refer to [21] and [17]. The uncertainties associated with the parameters will be described in Section VII.

A. Flight Profile

The flight profile models is borrowed from [3]. Within the model, the trajectory of the aircraft is optimized over five steady flight segments, although we are restricted to modeling only climb segments and therefore the stored gravitational potential energy of the aircraft is not captured.

B. Atmosphere

The atmosphere model is taken from [26], and considers changes in density and dynamic viscosity with altitude, for a standard atmosphere.

C. Aircraft

The aircraft is modeled as a wing, fuselage and engine system. The aircraft is assumed to be in steady flight, so that the thrust power is equal to the sum of the drag power and rate of change of potential energy of the aircraft, and the lift is equal to the total weight, ignoring the vertical component of thrust in climb. Its total weight is the sum of its components. The aircraft has to be able to takeoff at specified minimum speed without stalling as well. Aircraft component models are detailed below.

1. Wing

Lift is generated by the wing as a function of its geometry and freestream conditions. The wing structure model is based on a simple beam model with a distributed lift load, and a point mass in the center representing the fuselage. Wing fuel volume is modeled as a fraction of the internal volume available in the wing. Its drag is approximated simply as a sum of the induced and profile drags, the latter of which is estimated using a form factor. The weight of the wing is the sum of skin and spar weights.

2. Fuselage

The fuselage is assumed to be ellipsoidal in shape and to contain fuel and payload. The fuselage drag is estimated using a form factor. The fuselage is assumed not to contain any structural members, and so its weight consists only of skin weight.

3. Engine

The aircraft is powered by a naturally aspirated piston engine. It is subject to power lapse at lower air densities at higher altitudes. Engine weight is modeled using a posynomial fit of existing engines. brake specific fuel consumption (BSFC) is modeled as a function of maximum thrust at a given altitude.

D. Source of non-log-convexity: fuel volume

The fuel models have been detailed in the previous sections, but it is noteworthy that the signomial constraint in the optimization appears in the aircraft total fuel volume constraint, as shown in Equation 10:

$$V_f \leq V_{f_{wing}} + V_{f_{fuse}} \tag{10}$$

The signomial constraints makes the problem non-log-convex, which means that the solution methods detailed by Saab [20] need to be extended to accommodate this optimization problem.

Table 1   Parameters and Uncertainties (increasing order)

Parameters	Description	Value	% Uncert. (3σ)
S_{wetratio}	wetted area ratio	2.075	3
e	span efficiency	0.92	3
μ	air viscosity (SL)	1.78×10^{-5} kg/(ms)	4
ρ	air density (SL)	1.23 kg/m^3	5
$C_{L_{\max}}$	stall lift coefficient	1.6	5
k	fuselage form factor	1.17	10
$C_{f_{\mathrm{fuse,ref}}}$	fuselage skin friction factor	0.455	10
ρ_{p}	payload density	1.5 kg/m^3	10
τ	airfoil thickness ratio	0.12	10
N_{ult}	ultimate load factor	3.3	15
$V_{\min}$	takeoff speed	30 m/s	20
W_{p}	payload weight	6250 N	20
$W_{\mathrm{W}_{\mathrm{coeff,strc}}}$	wing structural weight coefficient	2×10^{-5} 1/m	20
$W_{\mathrm{W}_{\mathrm{coeff,surf}}}$	wing surface weight coefficient	60 N/m^2	20

VII. Uncertainties and Sets

As aforementioned in Section I.B, one of the advantages of RO over SO is the fact that it is more effective in absence of data, since the problem has uncertainty set bounds on parameters as inputs instead of complete probability distributions. These uncertainties, given by three times the coefficient of variation (CV)[‡], are listed in Table 1. Since for the rest of this work all standard deviations (σ) are normalized by the means of the parameters, we will use 3σ to represent 3CV.

In this case of a conceptual aircraft design with no prior data, the parameter uncertainties reflect aerospace engineering intuition. The wing weight coefficients $W_{\mathrm{W}_{\mathrm{coeff,strc}}}$ and $W_{\mathrm{W}_{\mathrm{coeff,surf}}}$, and the ultimate load factor N_{ult} have large 3σs because the build quality of aircraft components is often difficult to quantify with a large degree of certainty. The payload weight and density (W_{p} and ρ_{p}) have large uncertainties for similar reasons, since the payload is often developed concurrently with the aircraft. Parameters that engineers take to be physical constants (sea level air viscosity and density, μ and ρ) and those that can be determined or manufactured with a relatively high degree of accuracy (S_{wetratio}, e) have relatively low deviations. Parameters that require testing to determine ($C_{L_{\max}}$, $C_{f_{\mathrm{fuse,ref}}}$, $V_{\min}$) have a level of uncertainty that reflects the expected variance of empirical studies. However, note that these quantities are ultimately picked by the designer using prior experience and data, and the level of conservativeness in the design will be greatly affected by the chosen 3σs.

[‡]The CV is defined as follows: CV = $\frac{\sigma}{|\mu|}$, where σ is the standard deviation and μ is the mean of the parameter.

A. Types of uncertainty sets considered

The robust design problem is solved for box and elliptical uncertainty sets, which are defined by the L∞- and L2-norms and sized by varying the parameter Γ, as defined in Appendix X.A. Intuitively, for both sets, Γ is a measure of how much risk is being hedged against. $\Gamma = 0$ implies that all of the parameters take their nominal values with zero uncertainty, and larger Γ protects against more parameter uncertainty.

Mathematically, for box uncertainty, Γ is the width of the L-∞ hypercube, whose dimension is the same as the number of uncertain parameters (14). More intuitively, it is defines the range of the set provided for each parameter, normalized by its standard deviation. It can be easy to assume that using margins and box uncertainty sets will yield the same designs, but they fundamentally function differently. Firstly, the worst case outcome in box uncertainty can come *from the interior* of the uncertainty set, instead of the corner of the hypercube considered by margins. Furthermore, there is no guarantee (and it is highly unlikely) that the chosen corner, i.e. particular allocation of margins, is the most conservative point in the uncertainty set. It is even possible that the *the wrong sign of margin* is allocated for certain parameters, since SPs are nonlinear and local sensitivities cannot be used reliably to intuit global behavior. Consider in this particular example wing thickness τ. A thicker wing is beneficial for the wing structure, but detrimental to cruise aerodynamics, so it is difficult for a designer to determine how to best allocate margin on τ. Thus for the rest of this paper the direction of margins is determined using the local sensitivities of the nominal solution, which are obtained at no extra computational cost in the solution of the terminal GP approximation of the SP. With these considerations in mind, box uncertainty is expected to be strictly more conservative and more appropriate than the use of margins in conceptual design, since (1) margins fail to capture the level of conservativeness they signal, and (2) prior information is required to allocate margin effectively.

For elliptical uncertainty, Γ is the maximum diameter of the Euclidian norm ball of u_i, which is the number of standard deviations of perturbation of each ith parameter from its nominal value. Elliptical uncertainty relies on the fact that the joint probability of multiple uncertain parameters taking values in the tails of their respective distributions is very low. So while it does not protect deterministically for all outcomes of the uncertain parameters within 3σ, it is expected to protect against uncertain outcomes less conservatively than the box uncertainty set, with little compromise in probability of failure of the design.

VIII. Results

We implement our RSP heuristic algorithm on the aforementioned conceptual aircraft design problem. Our objective function is total fuel consumption, which is to be minimized given a payload and range requirement.

A. Mitigation of probability of failure

First, the optimization problem is solved in presence of no uncertainty. It is expected that this aircraft has a high probability of failure due to its sensitivity to the outcomes of the uncertain parameters. Then, using the sign of sensitivities of the nominal solution, we assign 3σ margins for each parameter and generate a design using margins. These two solutions are compared with RO results for box and elliptical uncertainty sets at $\Gamma = 1$. From here onward we refer to aircraft designed under margins, under box uncertainty and under elliptical uncertainty as 'the margin aircraft', 'the box aircraft' and 'the elliptical aircraft' respectively.

The design variables are then fixed for each solution, and the designs are simulated for different realizations of the uncertain parameters in Table 1. This allows for statistical analysis of design performance, and an estimate of each design's probability of constraint violation, which we define as its probability of failure. In this MC scheme, the random variables are simulated from independent and identically distributed 3σ truncated Gaussians. We simulate from the truncated Gaussian since this makes it possible to confirm mathematically that for $\Gamma = 1$, all simulations of 3σ uncertain parameters are deterministically feasible for the box uncertainty set. Designs for each solution in Table 2 and Figure 4 are simulated with the same set of uncertainty realizations for consistency.

Table 2 SP Aircraft Optimization Results, for $\Gamma = 1$

Free variable	Description	Units	No Uncert.	Margins	Box	Elliptical
L/D	mean lift-to-drag ratio	-	33.6	23.6	25.1	27.7
AR	aspect ratio	-	24.6	13.3	13.0	16.3
Re	Reynolds number	-	1.54×10^6	2.65×10^6	3.03×10^6	250×10^6
S	wing planform area	m^2	13.6	32.8	32.0	28.1
V	mean flight velocity	m/s	41.6	37.3	38.9	38.4
T_{flight}	time of flight	hr	20.1	22.4	21.4	21.7
W_{w}	wing weight	N	2830	4760	4800	4480
$W_{\text{w,strc}}$	wing structural weight	N	2010	4760	2670	2620
$W_{\text{w,surf}}$	wing skin weight	N	820	2170	2120	1860
W_{fuse}	fuselage weight	N	250	314	288	279
$V_{\text{f,avail}}$	total fuel volume	m^3	0.0759	0.146	0.154	0.136
$V_{\text{f,fuse}}$	fuselage fuel volume	m^3	0.0394	0	0	0.0159
$V_{\text{f,wing}}$	wing fuel volume	m^3	0.0365	0.167	0.154	0.120
Objective metric	Description	Units	No Uncert.	Margins	Box	Elliptical
Objective	total fuel weight	N	608	1170	1240	1090
E[Objective]	expected total fuel weight	N	572	964	976	856
σ[Objective]	std. dev. of fuel weight	N	9	32	32	29
P[failure]	probability of failure	%	94	0	0	0

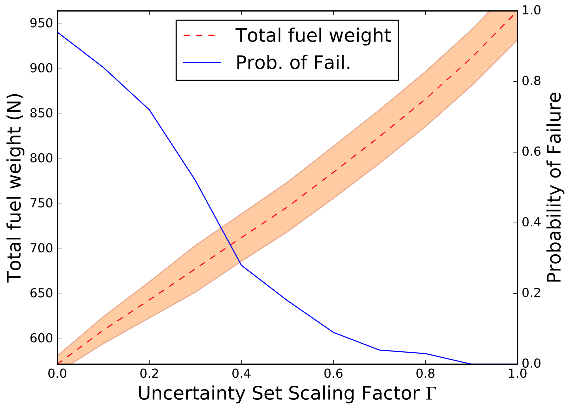
It is noteworthy in Table 2 that, for the nominal problem ($\Gamma = 0$), only 6 percent of the MC evaluations result in feasible solutions. This means that an aircraft designed for the average case would almost surely fail to satisfy the mission requirements, even with equal likelihood of favorable versus unfavorable uncertain outcomes from the symmetric truncated Gaussian. That being said, depending on the problem, it may necessary to sacrifice performance to achieve a high degree (3σ) of reliability as in the solution for $\Gamma = 1$. As shown by Figure 2, the box aircraft and the elliptical aircraft spend on average 71% and 50% more fuel respectively than the aircraft designed for the nominal case, but they also are robust to all uncertain outcomes in the 3σ set.

Table 2 also indicates that margins are not a good method of allocating uncertainty. The claim for the use of margins is that they protect against the worst case outcome of each parameter, but the results show otherwise. Since the box design at $\Gamma = 1$ is strictly more conservative (worse worst-case outcome) over the 3σ hypercube than the margin design, we see that a margin from the interior of the hypercube rather than its corner is more effective in protecting against the worst case. Furthermore, there are no probabilistic guarantees that the aircraft with margins would not fail one of the MC simulations. Given enough samples, it is almost surely true that some MC simulations will violate feasibility for the design with margins, whereas box uncertainty guarantees deterministically that the constraints are satisfied.

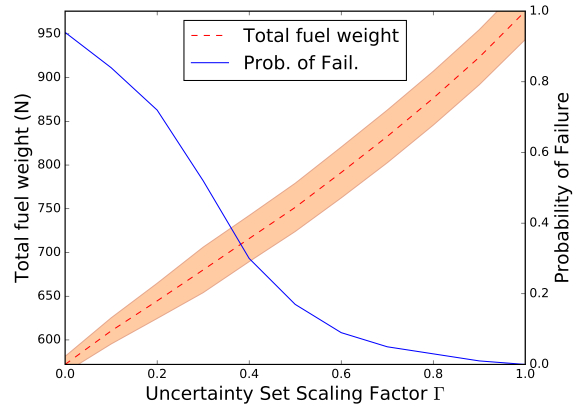
We also posited that the elliptical uncertainty, although it doesn't protect deterministically against all 3σ uncertainties, would be less conservative than the margin and box designs while not significantly sacrificing probability of failure. This was also confirmed. The elliptical design fails none of the random samples, and spends 11 and 12% less fuel on average than the margin and box aircraft respectively. The significance of this cannot be understated: the use of elliptical uncertainty results in designs that have strictly better performance outcomes, while protecting against a similar amount of risk as designs using margins or box uncertainty.

An analysis on the range $\Gamma = [0, 1]$ was performed to confirm that the trends from Table 2 hold for all Γ. Figure 4 shows that probability of failure goes monotonically towards zero as Γ increases for all three methods, where box uncertainty is more conservative than margins, and elliptical uncertainty is less conservative than the other two methods over the whole Γ domain.

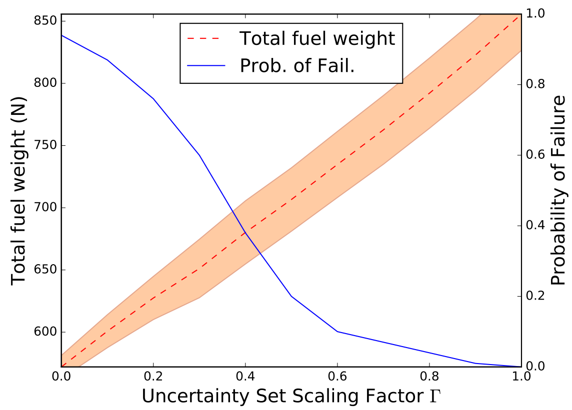
In absolute terms, the nominal SP under zero uncertainty or with margins takes just under 0.9 seconds to solve on a modern laptop computer using Mosek [27], an interior point solver that is free for academic use; the authors refer to [28] and [3] for more in-depth SP solution time analyses. Here we examine briefly in relative terms about how the different RSP methodologies compare in terms of setup and run times in Figure 5. Since the setup time of the nominal problem is minimal, we have normalized the results by the run time of the nominal problem. The bottom axis ranks the methods by their level of conservativeness, Best Pairs and Simple Conservative formulations being the least and most conservative respectively, and where the elliptical formulations are less conservative than the box formulations. For the box uncertainty, solution times are almost identical for different levels of conservativeness, whereas for the elliptical uncertainty they decrease as the method becomes more conservative.



(a) Margins



(b) Box Uncertainty Set



(c) Elliptical Uncertainty Set

Fig. 4 Simulated performance of the optimal robust aircraft, using margins, box and elliptical uncertainty sets, as a function of Γ. The robust solutions use the Best Pairs formulation. The dashed line and the band represent the mean and standard deviation of the performance of aircraft, simulated with 100 MC samples of uncertain parameters.

B. The Effect of Robustness on Multiobjective Performance

One of the benefits of convex and difference-of-convex optimization methods is the ability to optimize for different objectives [3]. As a demonstration, we optimize the aircraft without uncertainty for 8 different objectives, and show the non-dimensionalized results in Table 3. Since the model is physics based, the model can even accommodate objectives such as aspect ratio which are unintuitive and often not considered. The resulting aircraft differ drastically with respect to performance and design variables. As the most extreme example, the aircraft optimized for time cost has more than 100 times the engine weight as the aircraft optimized for total fuel, since a huge amount of power is required to fly fast. Furthermore, we can see that some more traditional objectives such as wing loading pull the design towards extreme corners of the performance envelope. This shows the importance of considering many objectives in design, and demonstrates the power of SPs in helping consider the multiobjective performance engineered systems.

Aside from this caricature example, we demonstrate the capabilities of RSPs by considering a more realistic scenario,

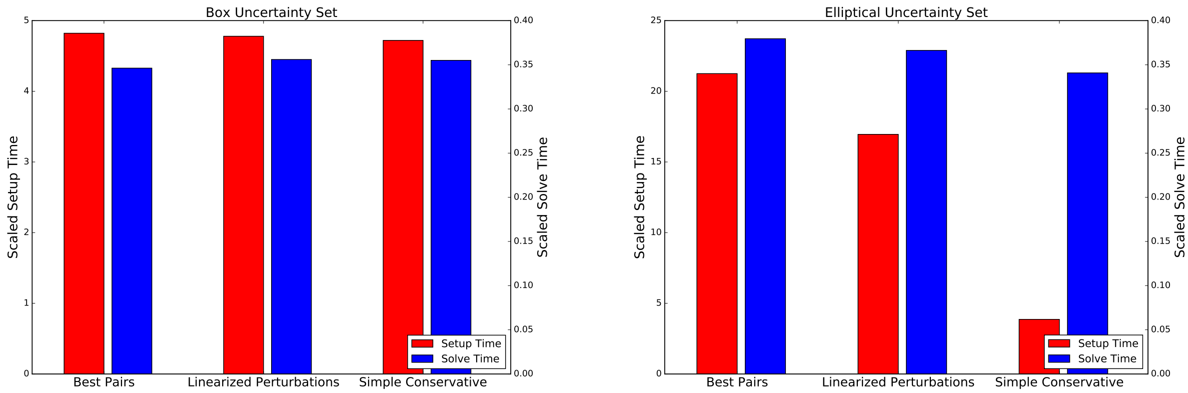


Fig. 5 Robust signomial simple aircraft solution and setup times, normalized by the nominal problem solution time, for $\Gamma = 1$. Note that the problems with box uncertainty have much lower setup time costs versus those with elliptical uncertainty, but similar solve times.

Objective	Takeoff weight	Engine weight	Total cost	Wing loading	Total fuel	Time cost	Aspect ratio	Cruise L/D
Takeoff weight	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Engine weight	1.43	0.37	1.58	0.63	0.95	1.85	3.06	1.00
Total cost	1.09	2.26	0.83	1.00	1.17	0.69	1.38	1.12
Wing loading	40.32	73.87	15.25	0.11	45.32	2.58	0.60	46.46
Total fuel	1.17	0.49	1.11	1.00	0.75	1.26	2.89	0.72
Time cost	4.63	101.82	3.24	1.00	9.95	0.40	0.40	8.37
Aspect ratio	3.91	51.28	4.01	0.37	11.59	0.82	0.06	12.31
Cruise L/D	1.34	2.67	1.14	0.74	0.97	1.21	2.69	0.58

Table 3 Non-dimensionalized variations in objective values with respect to the aircraft optimized for different objectives. Objective values are normalized by the total fuel solution.

now with uncertainty. We perform the optimization of the aircraft with no uncertainty and both box and ellipsoidal uncertainty ($\Gamma = 1$) for 4 different objective functions, and plot the results on radar plots. Radar plots are useful because they allow engineers to visualize the performance of designs in many dimensions. One way to envision the multi-objective performance of the aircraft is to consider the area of the polygon defined by the aircraft's performance as the figure of merit; the smaller the better. Due to the large disparities in the potential values of design variables depending on objective as shown in Table 3, we choose to demonstrate this using four objective functions that would be expected to have a high degree of correlation and therefore could yield a nuanced comparative analysis. These are total (time and fuel) cost, total fuel, takeoff weight and mid-cruise lift-over-drag (L/D).

Figure 6 shows the effects of robustness on the different worst-case performance metrics of the different aircraft. As expected, the box uncertainty set is strictly more conservative than the elliptical uncertainty set for all objectives. Note that the radar plots show the worst-case performance of the vehicles. This analysis can also be performed for the mean performance of the aircraft determined through MC simulation, but this demonstration limits its scope to the worst-case analysis.

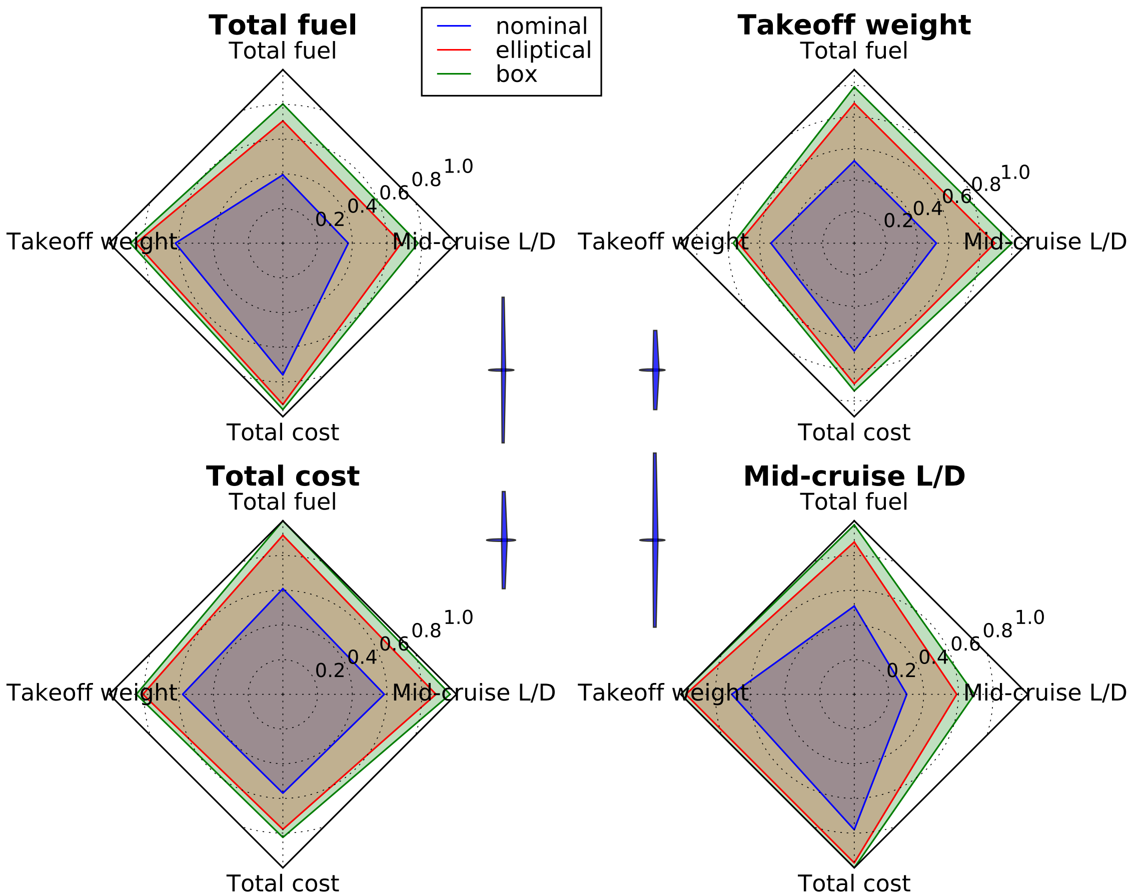
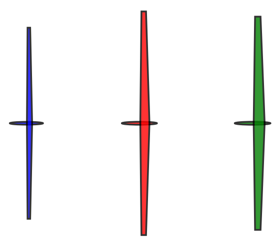
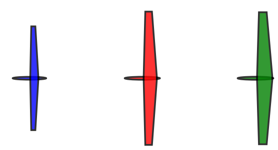


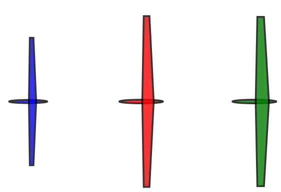
Fig. 6 The radar plots of aircraft performance, for aircraft optimized for different objectives. The bolded titles are the design objectives for each plot, whereas the individual plots show the non-dimensionalized multiobjective performance of the aircraft designed under different uncertainty sets. Nominal aircraft shown for comparison.



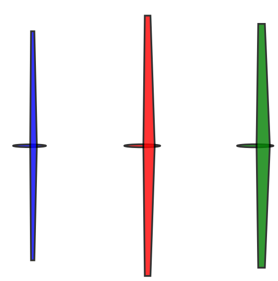
(a) Total fuel



(b) Takeoff weight



(c) Total cost



(d) Mid-cruise L/D

Fig. 7 Sketches of the aircraft drawn for corresponding radar plots. Drawn to scale for comparison.

Notably, mid-cruise lift-over-drag, which is a commonly used performance metric, produces designs that are low-performing in all other objectives, especially under robustness conditions. Based on these observations, we argue that there could be significant value left on the table if uncertainty is not considered with sufficient mathematical rigor in early phases of the engineering design process. RSPs allow engineers to capture complex tradeoffs in nonlinear optimization problems while considering uncertainty, resulting in *less conservative* solutions than solutions that implement margins and other less rigorous methods for risk mitigation. Thus RSPs improve significantly on the the paradigms of design under uncertainty in use in the aerospace industry today.

C. Risk minimization problems

All of the previous multi-objective analyses have assumed that we have an understanding of exactly the amount of uncertainty we are willing to tolerate. However, minimizing risk can also be the objective of our model. This would suggest the following formulation:

$$\begin{aligned} &\max \Gamma \\ &\quad \text{s.t. } f_i(x,u) \leq 0, i = 1,\ldots,n \\ &\quad\quad \|u\| \leq \Gamma \\ &\quad\quad f_0(x) \leq (1+\delta) f_0^*,\ \delta \geq 0 \end{aligned} \tag{11}$$

where f_0^* is the optimum of the nominal problem in Formulation 1 and δ is a fractional measure of the objective that we are willing to sacrifice for robustness, which gives $(1+\delta)f_0^*$ as the upper bound on the objective value. Intuitively, this is a form of goal programming, where we specify the exact maximum worst-case value of an objective we can tolerate with the goal of maximizing the total size of the uncertainty Γ we can handle.

The goal programming problem in Formulation 11 is clearly not equivalent to the problem in Formulation 1, but should yield the same results if both methods are optimal. To show this, we use the objective values from the probability of failure study shown in Figure 4 as the δ inputs to the goal programming model, and compare the results. The results are tabulated in Table 4. Note that the two methods were evaluated MC runs using the same 100 realizations of the uncertainty, for consistency in probability of failure results.

Firstly, note that there are no results reported for the goal program for zero uncertainty, $\Gamma = [0.00]$. Since the feasible set of this problem is a point design, the signomial program solution heuristic declares the problem infeasible after being unable to locate the singular feasible region. However we confirm that when we positively perturb the singular δ, the goal program has a non-empty feasible set and returns the same solution as the original RO method. Otherwise, the Γ values found by the goal program match exactly with the original RO problem. We confirm that both methods

Table 4 Results of original RO problem versus goal program in terms of size of uncertainty set Γ, objective penalty δ, and probability of failure. Both methods use the Best Pairs formulation under elliptical uncertainty. The designs obtained through the two different methods match.

RO form	Γ	δ	PoF	Goal form	δ	Γ	PoF
	0.00	2.5×10^{-4}	0.94		-	-	-
	0.10	0.057	0.87		0.057	0.10	0.87
	0.20	0.118	0.76		0.118	0.20	0.76
	0.30	0.183	0.60		0.183	0.30	0.60
	0.40	0.252	0.38		0.252	0.40	0.38
	0.50	0.326	0.20		0.326	0.50	0.21
	0.60	0.406	0.10		0.406	0.60	0.10
	0.70	0.492	0.07		0.492	0.70	0.07
	0.80	0.583	0.04		0.583	0.80	0.04
	0.90	0.681	0.01		0.681	0.90	0.01
	1.00	0.787	0.00		0.787	1.00	0.00

produce the same designs by examining the physical dimensions of the aircraft, and through the probability of failure found through MC simulation in Table 4. Note that there is a small discrepancy in the probability of failure, notably in the value for $\Gamma = 0.5$. This is possible because there are uncertainty realizations that can fall in or out of feasibility due to numerical precision. The interior point solvers used cannot make computations exactly [14].

We can also expand this framework to perform multivariate goal programming, by changing Formulation 11 to include all objectives we are interested in.

$$f_{0,j}(x) \leq (1+\delta_j) f_{0,j}^*, \ \delta_j \geq 0, \ j = 1, \ldots, m$$

The benefit of goal programming is that it allows us to explore multidisciplinary tradeoffs without having to enumerate the design space along each objective direction. In design it is not obvious whether an objective should in fact be a constraint instead. The most fundamental choice that an engineer can make in design is what the objective function is, and it is often the case that there are many potential objectives that are conflicting. The term multiobjective optimization is misleading because you can only optimize for one objective at once, and the design is going to be influenced by how engineers weigh different objectives. But risk is ubiquitous in engineering design problems, so goal programming allows risk to be used as a global design variable against which all objectives can weighed.

IX. Potential Future Work or Studies

There are a myriad of potential extensions to signomial programming under uncertainty. In the spirit of helping reduce program risk in aerospace design, the authors make a few observations and recommendations.

In this study, we do not discriminate between the kinds of constraints violated. However, it would be possible to rank the severity of constraint violations so as to penalize some (eg. structural safety) more heavily than others (maximum range constraint). This would inject further realism into the design under uncertainty since some violations contribute to program risk more strongly than others.

Another potentially valuable extension to the proposed framework is the concurrent implementation of multiple sets to contain the uncertain parameters, with the purpose of restricting uncertain outcomes further. One example of this would be to impose an L1-norm on the integer number of uncertain parameters as well as an L2-norm on the overall size of uncertainty set This method can be used to set the total size of the uncertainty set in a Euclidian sense, but then also to restrict the stochasticity to a subset of all of the uncertain parameters, thereby somewhat restricting nature. This also turns the problem into an integer robust optimization problem which poses interesting computational challenges.

With respect to interesting studies, RO opens up the possibility to discover and analyze the benefits of adaptable architectures in aircraft design versus more traditional point designs with mathematical rigor. Some examples of these are modular designs, morphing designs, adaptively manufactured designs and aircraft families. It is likely that these types of engineered robustness become more effective at reducing program risk in presence of uncertainty, since they are more likely to deliver value under adverse stochastic outcomes.

In situations where there is data available to aid design, RO can help explore the design space while taking into account the stochasticity and noise in the data. This opens up an array of potential trade studies where engineers can learn about the exposure of designs to the sparsity and spread of data and attempt to gather data which best reduces the uncertainty in the performance of optimal designs.

X. Conclusion

This paper has motivated the use of robust optimization in conceptual engineering design, in lieu of the mathematically non-rigorous methods of optimization under uncertainty widely used in the aerospace industry today. We have developed a tractable RSP formulation in response to a need to optimize over uncertain parameters, extending the tractable approximate RGP framework developed by Saab [20] to non-log-convex problems. This RSP formulation is a valuable contribution to the fields of robust optimization and difference-of-convex programming.

RSPs have a wide variety of potential applications in engineering design. We demonstrated using a simple aircraft design problem that using RO, and specifically RSPs in conceptual aircraft design will result in systems that are more robust with respect to uncertainties in operational parameters, such as payload mass and range, as well as uncertain environmental and manufacturing parameters. Unlike legacy methods, this robustness has probabilistic guarantees, where sets of size $\Gamma = 1$ protect against all realizations of uncertainty for a given set of parameters. Thus engineers can use robust signomial programming to trade off robustness and optimality within engineered systems in a tractable and mathematically rigorous manner.

We compared the results of designs with no uncertainty and margins with robust solutions determined through the use of box and elliptical uncertainty sets. We confirmed that designs using box uncertainty are strictly more conservative than designs using margins. This indicates that the traditional method of allocating margins by observing the local sensitivities of the nominal solution is inadequate, since it does not represent the worst-case outcomes of uncertain parameters as claimed. Furthermore, we show that box uncertainty has approximately the same expectation and standard deviation as the solution with margin, but provides probabilistic guarantees of feasibility unlike its counterpart.

We also confirmed that elliptical designs are strictly less conservative than those that would be generated through the use of box uncertainty while protecting against the same parametric uncertainties. Since designs found using robust signomial programming under elliptical uncertainty are less conservative than designs found through traditional methods, RSPs have the potential to reduce the program risk and increase the performance of designs compared to traditional methods with no sacrifice in reliability.

RO has the potential to change current aerospace design paradigms by introducing mathematical rigor to design under uncertainty. Current aerospace conceptual design practices still rely heavily on the expertise of established engineers even in absence of prior experience exploring the design space. RSPs provide new opportunities in aerospace conceptual design since they are compatible with physics based models that are deprived of or lacking in data, and bring quantitative measures of design reliability to the table.

Appendix

A. Robust Linear Programming: A Quick Review

As mentioned earlier, robust linear programming will be used to formulate an approximate robust geometric program.

Consider the system of linear constraints

$$\mathbb{A}\mathbf{x} + \mathbf{b} \leq 0$$

where

$$\mathbb{A} \text{ is } m \times n$$

$$\mathbf{x} \text{ is } n \times 1$$

$$\mathbf{b} \text{ is } m \times 1$$

where that data is uncertain and is given by equations (5) and (6).

1. Box Uncertainty Set

If the perturbation set $\mathcal{Z}$ given in equation (6) is a box uncertainty set, i.e. $\|\zeta\|_\infty \leq \Gamma$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \textstyle\sum_{l=1}^{L} |-b_i^l - \mathbf{a}_i^l \mathbf{x}| + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \tag{12}$$

If only b is uncertain, i.e. $A^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (12) will become

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \sum_{l=1}^{L} |b_i^l| \leq 0 \tag{13}$$

which is a linear constraint

On the other hand, if A is uncertain, then equation (12) is equivalent to the following set of linear constraints

$$\begin{aligned} \Gamma \textstyle\sum_{l=1}^{L} w_i^l + \mathbf{a}_i^0 \mathbf{x} + b_i^0 &\leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} &\leq w_i^l \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} &\leq w_i^l \quad \forall l \in 1, ..., L \end{aligned} \tag{14}$$

2. Elliptical Uncertainty Set

Briefly, if the perturbation set $\mathcal{Z}$ is an elliptical, i.e. $\sum_{l=1}^{L} \frac{\zeta_l^2}{\sigma_l^2} \leq \Gamma^2$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \sqrt{\textstyle\sum_{l=1}^{L} \sigma_l^2 (-b_i^l - \mathbf{a}_i^l \mathbf{x})^2} + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \tag{15}$$

which is a second order conic constraint.

If only b is uncertain, i.e. $\mathbb{A}^l = 0 \quad \forall l = 1, 2, ..., L$, then equation (15) will become

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \sqrt{\sum_{l=1}^{L} \sigma_l^2 (b_i^l)^2} \leq 0 \tag{16}$$

which is a linear constraint.

3. Norm-1 Uncertainty Sets

Briefly, if the perturbation set represented by $\mathcal{Z}$ is a norm-1 uncertainty set, i.e. $\|\zeta\|_1 \leq \Gamma$, then the robust constraint is

$$\textstyle\sum_{l=1}^{L} \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \max_{l=1,..,L} |b_i^l| \leq 0 \tag{17}$$

when $\mathbb{A}^l = 0$, and

$$\begin{aligned} \Gamma w_i + \mathbf{a}_i^0 \mathbf{x} + b_i^0 &\leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} &\leq w_i \quad \forall l \in 1, ..., L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} &\leq w_i \quad \forall l \in 1, ..., L \end{aligned} \tag{18}$$

if $\mathbb{A}^l \neq 0$

Note that for this type of uncertainty, the robust constraints are linear.

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