

Robust Signomial Programs

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May 2017

Abstract

Signomial programming is useful in multidisciplinary non-convex optimization problems such as aircraft design. The formulation and solution of robust signomial programs (RSPs) would be beneficial since many parameters involved in these problems are prone to uncertainty, and can have significant effects on solution performance and feasibility. This paper proposes an approximate solution for an RSP leveraging an existing approximate robust geometric programming (RGP) formulation developed by Saab. The method is based on solving a sequence of RGPs, where each GP is a local approximation of the SP. Moreover, this paper also discusses the trade-off between robustness and optimality by implementing RSPs on a simple aircraft problem, and demonstrates how robust optimization affects aircraft design decisions.

Acronyms

AR	aspect ratio	SP	signomial program
C_D	drag coefficient	T_{flight}	flight time
C_{D_A}	fuselage drag area	W	takeoff weight
C_L	lift coefficient	W_w	wing weight
$C_{L_{max}}$	lift coefficient at stall	$W_{w_{strc}}$	wing structural weight
GP	geometric program	$W_{w_{surf}}$	wing skin weight
MDO	multidisciplinary design optimization	V	flight speed
N_{ult}		V_f	fuel volume
Re	Reynolds number	$V_{f_{fuse}}$	fuel volume in the fuselage
S	wing area	$V_{f_{wing}}$	fuel volume in the wing

1 Introduction

Robust optimization methods provide tractable methods to capture uncertainty in design. Geometric programs [GPs] are a method of log-convex optimization for which robust formulations exist. However, their stringent mathematical requirements limit their application in non-log-convex problems. In this paper, we propose a tractable robust signomial program [RSP] which we solve as a sequential robust geometric program, allowing us to implement robustness in non-log-convex problems.

We implement the RSP formulation on a simple aircraft design problem with 19 free variables, 12 uncertain parameters and 17 constraints to demonstrate its potential. We believe that aircraft design problems can especially benefit from robustness. Oftentimes, aerospace engineers will implement margins in the design process to account for uncertainties in parameters that a design may be sensitive to, without explicit knowledge of the trade-off between robustness and optimality. A robust aircraft design formulation will allow designers to allocate margin more effectively to obtain better-performing designs with feasibility guarantees.

1.1 Geometric Programs

Geometric programs are a method for log-convex optimization. They offer globally optimal solutions with no initial guesses, and allow engineers and designers to solve large-scale ($>10,000$ variables) nonlinear problems on the order of ~ 1 second, given some stringent mathematical requirements. Given the general optimization problem below:

$$\begin{aligned} & \text{minimize } f_0(\mathbf{x}) \\ & \text{subject to } f_i(\mathbf{x}) \leq 1, i = 1, \dots, m \\ & \quad g_i(\mathbf{x}) = 1, i = 1, \dots, p \end{aligned} \tag{1}$$

In a GP, f_i 's and g_i 's must have special posynomial and monomial forms below, with strictly positive coefficients: A monomial is a function of the form

$$g(\mathbf{x}) = e^{\mathbf{a}\mathbf{x}+b} \quad (2)$$

where $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

A posynomial is a function of the form

$$f(\mathbf{x}) = \sum_{k=1}^K e^{\mathbf{a}_k\mathbf{x}+b_k} \quad (3)$$

where $\mathbf{a}_k \in \mathbb{R}^n, b_k \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. A posynomial is a sum of monomials. Therefore, all monomials are also one-term posynomials.

Any monomial constraint $g(\mathbf{x}) = 1$ can be represented as $g(\mathbf{x}) \leq 1$ and $\frac{1}{g(\mathbf{x})} \leq 1$, therefore, the standard GP can be written in the inequality constrained form as follows:

$$\begin{aligned} & \text{minimize} && \sum_{k=1}^{K_0} e^{\mathbf{a}_{0k}\mathbf{x}+b_{0k}} \\ & \text{subject to} && \sum_{k=1}^{K_i} e^{\mathbf{a}_{ik}\mathbf{x}+b_{ik}} \leq 1 \quad \forall i \in 1, \dots, m \end{aligned} \quad (4)$$

A tractable approximation of robust GPs (RGPs) exists and will be discussed, and have been formulated with polyhedral and ellipsoidal uncertainty sets. RGPs allow for uncertainty in both the coefficients b_k and the exponents \mathbf{a}_k . However, the restriction on the sign of the coefficients limits GPs (and consequently RGPs) to certain classes of problems. Signomials are constraints of the same form as $f(x)$, but allow for negative coefficients in $f(x)$. To be able to find robust solutions to signomial programs [SPs], we would like to create a framework to generalize the uncertainty sets used in RGPs to RSPs.

1.2 Signomial Programs

Signomials allow us to solve non-log-convex problems as sequential geometric programs. Signomials are a difference of posynomials shown in Equation 3. The log transform of an SP is not a convex optimization problem, but instead a difference of convex optimization problem that can be written in log-space as

$$\begin{aligned} & \text{minimize} && f_0(\mathbf{x}) \\ & \text{subject to} && f_i(\mathbf{x}) - h_i(\mathbf{x}) \leq 0, i = 1, \dots, m \end{aligned} \quad (5)$$

where f_i and h_i are posynomials. This problem can be reliably solved iteratively by taking the monomial approximation of g_i with a solution or initial guess x_i , solving this GP approximation to obtain a new solution x_{i+1} , and repeating the process until some convergence parameter is satisfied in the objective function.

The SP algorithm is well-studied, reliably solving SPs with an initial guess of all 1's. SPs are guaranteed to be sub-optimal but feasible solutions to a general non-linear problem.

2 Approach to Solving RSPs

Our approach to solving the RSP is the following:

1. Solve the SP with no uncertainty, initialized by a vector of 1's. We call this solution x_0 .
2. Solve the SP with a chosen uncertainty set, using x_0 as the initial guess.

Step 2 requires the retuning of the SP algorithm as follows:

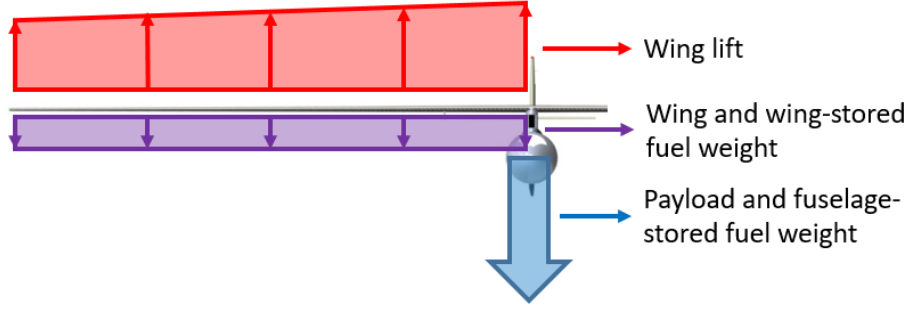
Initialize $x_0 = \mathbf{1}$

While $reltol \geq 1e - 4$:

1. Substitute x_i as the initial guess to the SP.
2. Find the RGP approximation to the SP with guess x_i .
3. Solve the RGP to obtain x_{i+1} .
4. Calculate $reltol$.

$reltol$ is the relative change in the objective function value for the GP. The RSP algorithm solves reliably for a range of $reltols$, but a value of $1e - 4$ was chosen for the models evaluated in this paper.

Figure 1: Wing lift is equal and opposite to the wing weight, payload weight, and total fuel weight.



3 RGPs

This part is commented because it repeats the Robust Geometric Programming stuff from the first paper

4 Models

We implemented the RSP formulation ideas above on a simple aircraft design problem, with 12 uncertain variables, and a single signomial constraint. In the simple aircraft problem, we conduct an aerostructural optimization of a wing and fuselage given a payload and range requirement. A short overview of the model follows.

4.1 Lift, Weight, Drag and Thrust

The aircraft is assumed to be in steady, level flight. As a result, we can assume that the thrust generated by the aircraft is equal to the drag, and the lift generated by the wing is equal to the total weight.

The drag is the sum of the wing (induced and profile) drag, and the fuselage drag, which is linearly proportional to the fuel volume in the fuselage. The aircraft model does not assume limitations on thrust, but instead assumes constant thrust specific fuel consumption, at 0.6 lbs for each lb*hour of thrust, with associated uncertainty.

The weight of the aircraft is the sum of the payload weight, wing weight, and the fuselage weight, as shown in Figure 1. Lift is generated by the wing, which is described by an aspect ratio AR and surface area S .

4.2 Wing Structure

The wing structure model is based on a simple beam model with a distributed lift load, and a point mass in the center representing the fuselage, as shown in Figure 1.

4.3 Fuel Volume

The fuel in the aircraft can be stored either in the wing or the fuselage. The signomial constraint in the optimization appears in the fuel volume model, as shown in Equation 6:

$$V_f \leq V_{f_{wing}} + V_{f_{fuse}} \quad (6)$$

where $V_{f_{wing}}$ and $V_{f_{fuse}}$ represent the fuel volume available in the wing and the fuselage respectively. They are each represented by the following monomials.

$$V_{f_{wing}} \leq 9e^{-4} \frac{S^{1.5} \tau}{A^{0.5}} \quad (7)$$

$$V_{f_{fuse}} \leq 10 \times CDA_0 m \quad (8)$$

Note that the monomials above are represented with inequalities, to be compatible with the RSP formulation.

Table 1: Constants and Uncertainties (increasing order)

Constant	Description	Value	% Uncert. (3σ)
$S_{wetratio}$	wetted area ratio	2.075	3
e	span efficiency	0.92	3
μ	viscosity of air	$1.775\text{e-}5 \text{ kg}/(\text{ms})$	4
ρ	air density	$1.23 \text{ kg}/\text{m}^3$	5
$C_{L_{max}}$	stall lift coefficient	1.6	5
k	fuselage form factor	1.17	10
τ	airfoil thickness ratio	0.12	10
N_{ult}	ultimate load factor	3.3	15
V_{min}	takeoff speed	25 m/s	20
W_0	payload weight	6250 N	20
$W_{w_{coeff1}}$	wing weight coefficient 1	$2\text{e-}5 \text{ 1}/\text{m}$	20
$W_{w_{coeff2}}$	wing weight coefficient 2	$60 \text{ N}/\text{m}^2$	20

4.4 Takeoff constraints

We specify that the aircraft has to be able to takeoff at a speed of V_{min} without exceeding the aircraft stall lift coefficient $C_{L_{max}}$, both of which are specified with an associated uncertainty.

5 Uncertainties and sets

The uncertainties for the different constants in the problem have been determined considering the parameters in aircraft design that often have the largest uncertainty. These uncertainties are listed in Table 1.

The parameter uncertainties reflect aerospace engineering intuition. The wing weight coefficients $W_{w_{coeff1}}$ and $W_{w_{coeff2}}$, and the ultimate load factor N_{ult} have large 3σ s because build quality of aircraft components often difficult to quantify with a large degree of certainty. The payload weight (W_0) has a large uncertainty, because it is valuable if the aircraft has the flexibility to accommodate larger payloads. Parameters that engineers take to be physical constants (μ , ρ) and those that can be determined/manufactured with a relatively high degree of accuracy ($S_{wetratio}$, e) have relatively low deviations. Parameters that require testing to determine ($C_{L_{max}}$, V_{min}) have a level of uncertainty that reflects the expected variance of the parameters.

6 Results

6.1 Determining the appropriate Γ

In this section, we optimize the aircraft configuration for a given payload and range with no uncertainty, box uncertainty and ellipsoidal uncertainty, and attempt to determine the value of Γ that gives the best trade-off between robustness and optimality for the RO formulations. The problems have been solved with box uncertainty and ellipsoidal uncertainty for a range of Γ . For each of the robust solutions, both the estimated probability of failure and the objective cost have been plotted in Figures 2 and 3 respectively.

We define the probability of failure to be the probability of constraint violation in 1000 samples of the uncertain parameters from a normal distribution with the specified 3σ uncertainties. For this problem, we allow for less than 2% probability of failure, shown by the magenta line in Figure 2, and consider half-integer values of Γ . This corresponds to a Γ of 2.5 for both the box uncertainty and the ellipsoidal uncertainty, resulting in 1.2% and 1.3% probability of failure respectively.

We know that elliptical uncertainty sets are less conservative than box uncertainty sets, especially when we have a large number of uncertain parameters per constraint. The results we got are somehow expected, and this would show how designing using uncertainty sets is better than using margins which are - in the best case - somehow similar to box uncertainty sets.

6.2 Optimization results

At the chosen values of Γ , the results of the optimization are shown in Table 2.

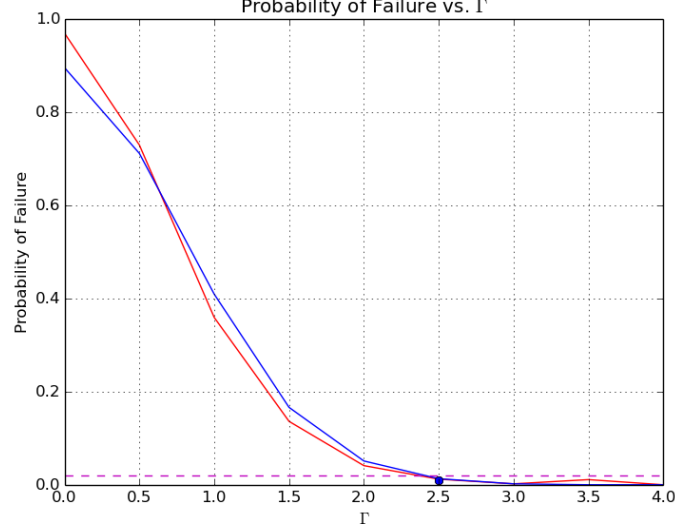


Figure 2: The probability of failure vs. Γ . Note the high ($\geq 90\%$) probability of failure for the non-robust ($\Gamma = 0$) solutions. The difference in the probability of failure for the non-robust cases is due to subtle numerical errors in the evaluation of the solution obtained for the ellipsoidal uncertainty set.

Table 2: SP Aircraft Optimization Results

Free variable	No Uncert.	Box [$\Gamma = 2.5$]	Ellipsoidal [$\Gamma = 2.5$]
L/D	23.8	16.41	17.3
AR	12.0	5.016	6.56
Re	4.76e6	1.16e7	9.55e6
$S(m^2)$	21.6	72.21	58.9
$V(m/s)$	51.0	47.73	48.6
$T_{flight}(hr)$	17.46	18.1	17.2
W_w	2440	6278	5470
$W_{w_{strc}}(N)$	1210	1570	1650
$W_{w_{surf}}(N)$	1230	1592	3820
$V_{f_{avail}}(m^3)$	0.566	1.794	1.48
$V_{f_{fuse}}(m^3)$	0.461	0.891	0.903
$V_{f_{wing}}(m^3)$	0.105	0.9027	0.582
$CD A_0(m^2)$	0.0461	0.0891	0.0903
E[Objective]	No Uncert.	Box [$\Gamma = 2.5$]	Ellipsoidal [$\Gamma = 2.5$]
$W_{fuel}(N)$	4430	10858	9299
P[failure]	No Uncert.	Box [$\Gamma = 2.0$]	Ellipsoidal [$\Gamma = 2.5$]
%	92	1.2	1.3

6.3 Comparing results for different objective functions

To further demonstrate the capabilities of robust SPs in aircraft design, we performed the optimization of the aircraft with no uncertainty and ellipsoidal uncertainty ($\Gamma = 2.5$) for two more objective functions, and plotted the results on spider plots. Spider plots are useful because they allow engineers to find non-dominated solutions among the solutions that lie on the Pareto frontier of potential objective functions. The objective functions chosen for this analysis were fuel burn over lift-to-drag ratio ($\frac{W_f}{L/D}$), drag (D), and fuel burn (W_f).

In the spider plots in Figures 4 and 5, none of the solutions are non-dominated for both the no uncertainty and ellipsoidal uncertainty cases. In a GP, it is not possible that one of the solutions is non-dominated since the solutions are globally optimal. But since there is no guarantee in optimality for SPs, it is possible to find non-dominated solutions if the obtained solution is a local optimum.

In the case where there is no non-dominated solution such as this one, we take the internal areas of the triangle formed by each optimization to be the figure of merit. The smaller the area in the triangle, the higher the performance of the proposed solution. In the no uncertainty case shown in Figure 4, the red solution with the objective of $\frac{W_f}{L/D}$ has the smallest internal area. In the ellipsoidal case shown in Figure 5, the blue solution with

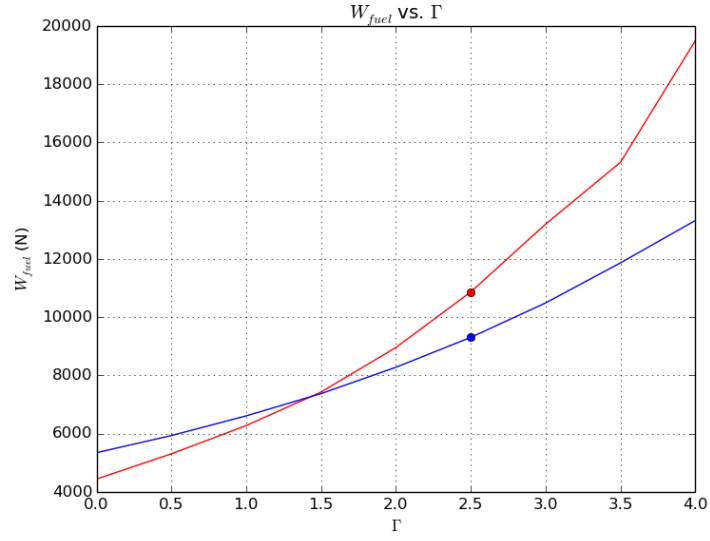
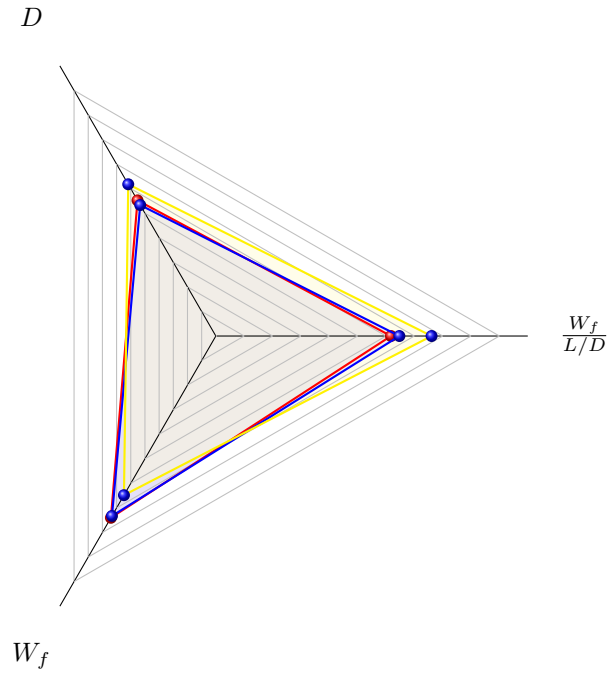


Figure 3: Fuel weight vs. Γ demonstrates the deterioration of the objective as we protect against larger variation of parameters.

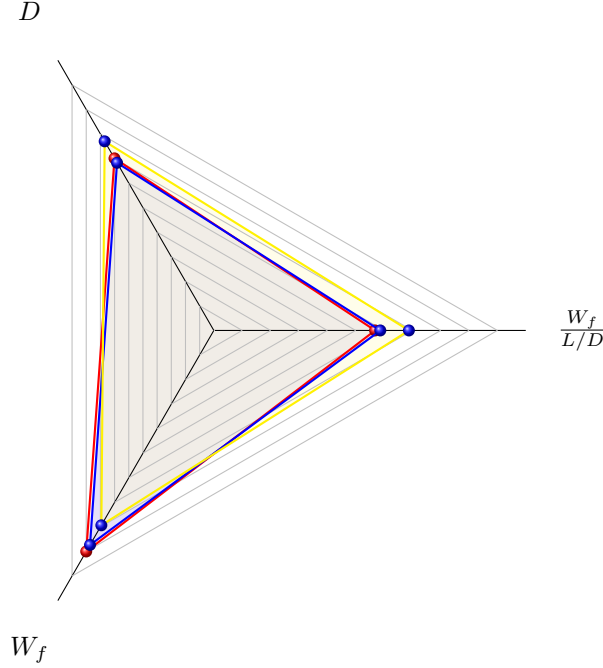
Figure 4: Design optimization of the aircraft with no uncertainty for 3 different objective functions. The red, blue and yellow correspond to $\frac{W_f}{L/D}$, D , W_f and objectives respectively.



the objective of D has the smallest internal area.

This is an interesting result, because the presence of an uncertainty set is shown to affect the efficacy of different objective functions to obtain solutions with the best overall performance. The differences between the objective functions in the simple aircraft design problem are minute, because the different potential objectives have a high degree of coupling. It is likely that, if the three objective functions didn't have high degree of coupling, that the internal areas of the solution triangles may differ more significantly.

Figure 5: Design optimization of the aircraft with ellipsoidal uncertainty ($\Gamma = 2.5$) for 3 different objective functions. The red, blue and yellow correspond to $\frac{W_f}{L/D}$, D , W_f and objectives respectively.



7 Conclusion, and Future Work

We have developed and applied a tractable RSP formulation to a simple aircraft model, and then discussed the benefits of having robust solutions. RSP formulations extend the tractable approximate RGP framework developed by Saab to non-log-convex problems, and are a valuable contribution to the fields of robust optimization and difference-of-convex programming.

RSPs have a wide variety of potential applications in engineering design. Within the Hoburg Research Group in the Aerospace Computational Design Lab, during the past year we have developed a commercial aircraft design SP that has between 1700 and 8000 variables, depending on the whether it is a single-point, or multi-point optimization. We expect that using RO in this conceptual aircraft design will result in designs that are more robust with respect to uncertainties in operational parameters, such as payload mass and range, as well as uncertain constants.

Appendices

A Robust Linear Programming: A Quick Review

As mentioned earlier, robust linear programming will be used to formulate an approximate robust geometric program.

Consider the system of linear constraints

$$\mathbf{Ax} + \mathbf{b} \leq 0$$

where

$$\begin{aligned}\mathbf{A} & \text{ is } m \times n \\ \mathbf{x} & \text{ is } n \times 1 \\ \mathbf{b} & \text{ is } m \times 1\end{aligned}$$

where that data is uncertain and is given by equations (??) and (??).

A.1 Box Uncertainty Set

If the perturbation set \mathcal{Z} given in equation (??) is a box uncertainty set, i.e. $\|\zeta\|_\infty \leq \Gamma$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \sum_{l=1}^L | -b_i^l - \mathbf{a}_i^l \mathbf{x} | + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \quad (9)$$

If only b is uncertain, i.e. $A^l = 0 \quad \forall l = 1, 2, \dots, L$, then equation (9) will become

$$\sum_{l=1}^L \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \sum_{l=1}^L |b_i^l| \leq 0 \quad (10)$$

which is a linear constraint

On the other hand, if A is uncertain, then equation (9) is equivalent to the following set of linear constraints

$$\begin{aligned}\Gamma \sum_{l=1}^L w_i^l + \mathbf{a}_i^0 \mathbf{x} + b_i^0 & \leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} & \leq w_i^l \quad \forall l \in 1, \dots, L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} & \leq w_i^l \quad \forall l \in 1, \dots, L\end{aligned} \quad (11)$$

A.2 Elliptical Uncertainty Set

Briefly, if the perturbation set \mathcal{Z} is an elliptical, i.e. $\sum_{l=1}^L \frac{\zeta_l^2}{\sigma_l^2} \leq \Gamma^2$, then the robust formulation of the i^{th} constraint is equivalent to

$$\Gamma \sqrt{\sum_{l=1}^L \sigma_l^2 (-b_i^l - \mathbf{a}_i^l \mathbf{x})^2} + \mathbf{a}_i^0 \mathbf{x} + b_i^0 \leq 0 \quad (12)$$

which is a second order conic constraint.

If only b is uncertain, i.e. $\mathbf{A}^l = 0 \quad \forall l = 1, 2, \dots, L$, then equation (12) will become

$$\sum_{l=1}^L \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \sqrt{\sum_{l=1}^L \sigma_l^2 (b_i^l)^2} \leq 0 \quad (13)$$

which is a linear constraint.

A.3 Norm-1 Uncertainty Sets

Briefly, if the perturbation set represented by \mathcal{Z} is a norm-1 uncertainty set, i.e. $\|\zeta\|_1 \leq \Gamma$, then the robust constraint is

$$\sum_{l=1}^L \mathbf{a}_i^0 \mathbf{x} + b_i^0 + \Gamma \max_{l=1, \dots, L} |b_i^l| \leq 0 \quad (14)$$

when $\mathbf{A}^l = 0$, and

$$\begin{aligned}\Gamma w_i + \mathbf{a}_i^0 \mathbf{x} + b_i^0 & \leq 0 \\ -b_i^l - \mathbf{a}_i^l \mathbf{x} & \leq w_i \quad \forall l \in 1, \dots, L \\ b_i^l + \mathbf{a}_i^l \mathbf{x} & \leq w_i \quad \forall l \in 1, \dots, L\end{aligned} \quad (15)$$

if $\mathbf{A}^l \neq 0$

Note that for this type of uncertainty, the robust constraints are linear.