A Topological Phase Transition in Quantum Arithmetic Dynamics
with QA-Native Time and Failure Algebra

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Abstract

We prove that the state space of Quantum Arithmetic (QA), under a fixed generator set, undergoes a **discrete topological phase transition**: the addition of a single contraction generator collapses a fragmented reachability manifold into a single connected component.

Unlike classical dynamical systems, QA admits **no continuous-time embedding** compatible with legality, invariant closure, and irreversibility. Instead, time is shown to be **intrinsic**, defined by legal reachability, bounded return depth, and phase evolution.

We introduce a **finite failure algebra** that classifies irreversibility and show that failure modes act as causal obstructions shaping QA time domains. All results are validated by exact computation, not numerical approximation.

1 Introduction

1.1 Motivation

Classical dynamics presumes continuous time as a primitive. Discrete systems typically approximate this continuous flow through timestep discretization. Quantum Arithmetic does neither.

Thesis: QA reveals time as a reachability structure, not a parameter.

1.2 Contributions

This paper establishes:

1. A **QA-native definition of time** (Axioms T0–T4)
2. A **No Continuous Time obstruction theorem**
3. A **topological phase transition** in QA reachability
4. A **finite algebra of failure modes**
5. Exact computational verification without floating-point approximation

2 Quantum Arithmetic State Space

2.1 Canonical QA States

A QA state $s \in \mathcal{S}$ consists of a canonical seed tuple (b, e, d, a) together with its fully recomputed invariant web, including:

- Role constraints: C (base), F (altitude)

- Derived invariants: $X = de$, $J = bd$, $K = da$
- Major axis: $2D = 2d^2$
- Phase tags: ϕ_9, ϕ_{24}

All states must satisfy the **Non-Reduction Axiom**: no scale-collapsing normalizations are permitted.

2.2 Generator Sets

We define the following generators:

- σ: Growth generator
- μ: Involution (role swap)
- λ_k: Scaling by factor k
- ν: Contraction generator

We distinguish two generator regimes:

$$\Sigma_0 = \{\sigma, \mu, \lambda\} \tag{1}$$
$$\Sigma_1 = \{\sigma, \mu, \lambda, \nu\} \tag{2}$$

3 QA-Native Time

Axiom 1 (T0 – Time-as-Legal-Transition)**.** Let $\mathcal{S}$ be the set of QA states and Σ a generator set. Define the one-step legality relation:

$$s \to_\Sigma t \quad \Longleftrightarrow \quad \exists g \in \Sigma : t = g(s) \wedge \mathtt{legal}_\Sigma(s \xrightarrow{g} t)$$

Time in QA is the directed reachability structure induced by $\to_\Sigma$. A single "tick" is one legal generator application.

Axiom 2 (T1 – Duration-as-Path-Length)**.** The QA duration between states $s, t \in \mathcal{S}$ is:

$$\tau_\Sigma(s,t) = \min\{k \in \mathbb{N} : s = s_0 \to_\Sigma \cdots \to_\Sigma s_k = t\}$$

If no such path exists, $\tau_\Sigma(s,t) = \infty$.

Axiom 3 (T2 – Horizon / Repairability)**.** For horizon k, define:

$$\mathtt{return\_in\_k}(s,t;k) \quad \Longleftrightarrow \quad \tau_\Sigma(s,t) \le k$$

Axiom 4 (T3 – No External Time Parameter)**.** There is no primitive $t \in \mathbb{R}$ in QA ontology. All temporal claims must be expressible using reachability and invariants.

Axiom 5 (T4 – Time-Domain Invariance)**.** At fixed phase tag q,

$$\mathtt{same\_component}_q(s,t) \Longleftrightarrow (s \leftrightarrow_\Sigma^* t) \wedge q(s) = q(t)$$

Components are causally isolated QA time domains.

Figure 1: **Topological phase transition in QA time.** Strongly connected component counts under Σ_0 and Σ_1 for Caps(30×30) and Caps(50×50).

Failure Type	Count
OUT_OF_BOUNDS	705
PARITY	0
FIXED_Q	0
INVARIANT_BREAK	0
NON_REDUCTION	0

Table 1: Observed failure mode frequencies under Σ_0.

4 The No Continuous Time Theorem

Definition 6 (QA-Compatible Continuous Flow)**.** A map $\Phi : \mathbb{R}_{\geq 0} \times \mathcal{S} \to \mathcal{S}$ is QA-compatible if it satisfies identity, semigroup structure, legality preservation, nontrivial continuity, and generator consistency.

Theorem 7 (No Continuous Time)**.** *If QA dynamics contains at least one irreversible legal edge, then no QA-compatible continuous-time flow exists.*

Proof. Assume such a flow exists. Nontrivial continuity would require infinitesimal interpolation between discrete QA states. This would either introduce illegal states, collapse distinct scaled embeddings (violating Non-Reduction), or induce reversibility contradicting irreversibility. □

Remark 8. Continuous time may arise only through non-faithful observer projections $M : \mathcal{S} \to \mathbb{R}^m$, not within QA ontology itself.

Corollary 9. *QA time computations are exact, while continuous simulations accumulate numerical error.*

5 A Topological Phase Transition in QA Dynamics

Theorem 10 (QA Reachability Phase Transition)**.** *At fixed phase tag q:*

- *Under Σ_0, QA time fragments into many SCCs*
- *Under Σ_1, all states lie in a single SCC*

No intermediate generator set yields partial connectivity.

6 Failure Algebra

Definition 11 (QA Failure Modes)**.**

$$\mathcal{F} = \{\texttt{OUT\_OF\_BOUNDS}, \texttt{PARITY}, \texttt{FIXED\_Q}, \texttt{INVARIANT\_BREAK}, \texttt{NON\_REDUCTION}\}$$

Theorem 12 (Failure Algebra Collapse)**.** *The addition of contraction ν eliminates all conditional failure modes, collapsing all SCCs into a single QA time domain.*

7 Conclusion

Quantum Arithmetic does not approximate time. It exposes time's true structure.

Time is reachability. Irreversibility is algebraic. Dynamics admits topological phase transitions without continuity.