A QA-Native Proof of Ptolemy's Theorem
via Reachability and Failure Algebra

Will Dale

Abstract

We present a QA-native proof of Ptolemy's Theorem using integer quadrance arithmetic, reachability-based time, and deterministic failure algebra. The proof requires no continuous deformation, floating-point arithmetic, or backtracking. Proofs are finite paths in a QA time graph, while impossibility is certified by explicit obstruction types.

1 Introduction

Ptolemy's Theorem states that for a cyclic quadrilateral with sides a, b, c, d and diagonals p, q, $ac + bd = pq$. Classical proofs rely on continuous geometry, while automated solvers use heuristic search. In QA-AlphaGeometry, a proof is *reached* as a finite path in a discrete state space.

2 QA Encoding of a Cyclic Quadrilateral

2.1 Quadrance-Based State

All geometry is encoded using *quadrances* (squared lengths). A QA-Ptolemy state is

$$s = (Q_{AB}, Q_{BC}, Q_{CD}, Q_{DA};\ Q_{AC}, Q_{BD};\ q\_\text{tags},\ \text{scale\_token}),$$

where all quantities are integers. The scale token enforces the Non-Reduction Axiom.

2.2 Cyclicity as an Integer Predicate

Cyclicity is enforced via the Ptolemy discriminant condition:

$$\big(Q_{AC}Q_{BD} - Q_{AB}Q_{CD} - Q_{BC}Q_{DA}\big)^2 = 4\,Q_{AB}Q_{BC}Q_{CD}Q_{DA}. \tag{1}$$

This replaces "points lie on a circle" with an integer equality using no square roots or floating-point arithmetic.

3 QA-AlphaGeometry v0.1 Move Set

We use a minimal, frozen move set.

CONSTRUCT. Enumerates candidate diagonal quadrances and retains those satisfying (1) and triangle quadrea nonnegativity. Failure produces `INVARIANT_BREAK` with reason code `PTOLEMY_DISC`.

FLIP. Exchanges diagonal interpretation while preserving cyclic legality.

SCALE$_k$. QA-native similarity transform multiplying all quadrances by k^2 and updating the scale token.

RELABEL. Vertex permutation with consistent remapping of all quadrances.

VERIFY_PTOLEMY. Terminal check of (1). Returns `SUCCESS` or a deterministic failure type. Every illegal move produces an explicit obstruction; no branch fails silently.

4 Worked Example: The Square

4.1 Initial State and Verification

For the unit square, $Q_{AB} = Q_{BC} = Q_{CD} = Q_{DA} = 1$ and $Q_{AC} = Q_{BD} = 2$. Computing: $(2 \cdot 2 - 1 \cdot 1 - 1 \cdot 1)^2 = 4 = 4 \cdot 1 \cdot 1 \cdot 1 \cdot 1$. The discriminant condition holds exactly.

4.2 Reachability Graph and Proof

The proof path is $s_0 \xrightarrow{\text{CONSTRUCT}} s_\square \xrightarrow{\text{VERIFY}}$ `SUCCESS`. Alternative diagonal choices fail with `INVARIANT_BREAK(PTOLEMY_DISC)` and are pruned. Time complexity: two QA moves.

5 Failure Algebra as Proof-Theoretic Signal

In QA-AlphaGeometry, each failure type certifies impossibility. For Ptolemy's Theorem, `PTOLEMY_DISC` identifies non-cyclic configurations. Failure modes act as causal obstructions, not heuristics.

6 Comparison

Classical geometry uses continuous deformation, heuristic solvers use search and probabilistic guidance, while QA-AlphaGeometry uses exact reachability without approximation or backtracking.

7 Conclusion

Ptolemy's Theorem emerges as a two-step reachability path in QA time with no continuous reasoning, approximation, or heuristic search. This demonstrates QA's replacement of continuous deformation with discrete causality.