Assignment 1

Estimation Theory

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1 Problem Statement

- Line Fitting
- Polynomial Fitting
- Comparing Mean and Max Estimators in DC Estimation

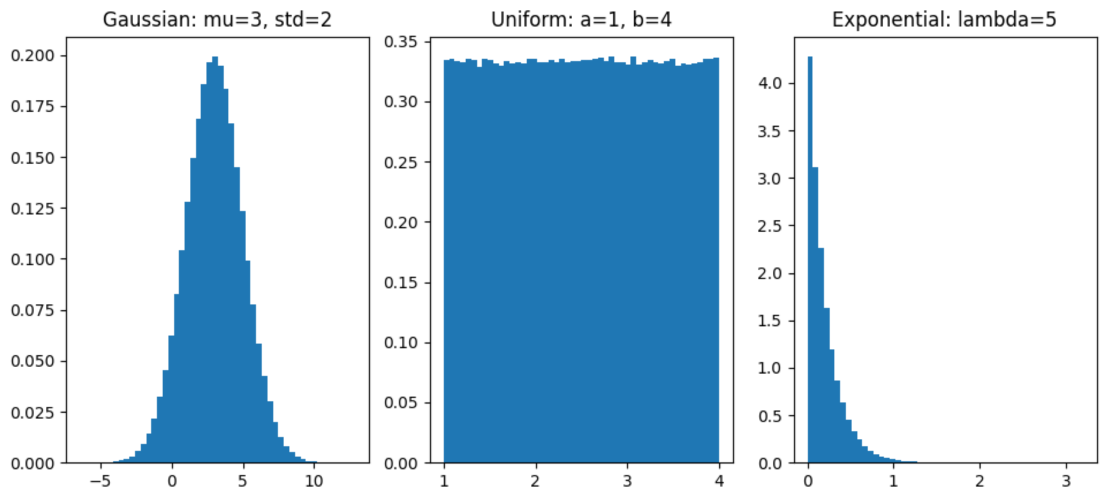
2 Code

2.1 Helper Functions

Random Variables:

```python
class RandomVariable:
    def __init__(self) -> None:
        pass

    def uniform(self, a=0, b=1):
        assert b > a, "Upper Bound can't be lower!"
        return random.uniform(a, b)

    def exponential(self, l=2):
        # Monte Carlo Inversion
        return - np.log(self.uniform()) / l

    def gaussian(self, mu=0, sigma=1):
        # Box Muller
        u = self.uniform()
        v = self.uniform()
        r = np.sqrt(-2 * np.log(u))
        theta = 2 * np.pi * v
        X = r * np.cos(theta)
        return sigma*X + mu

    def distribution(self, samples, bins=10):
        plt.hist(samples, bins=bins, density=True)
        plt.title("Distribution Function")

    def show_plot(self):
        plt.show()
```

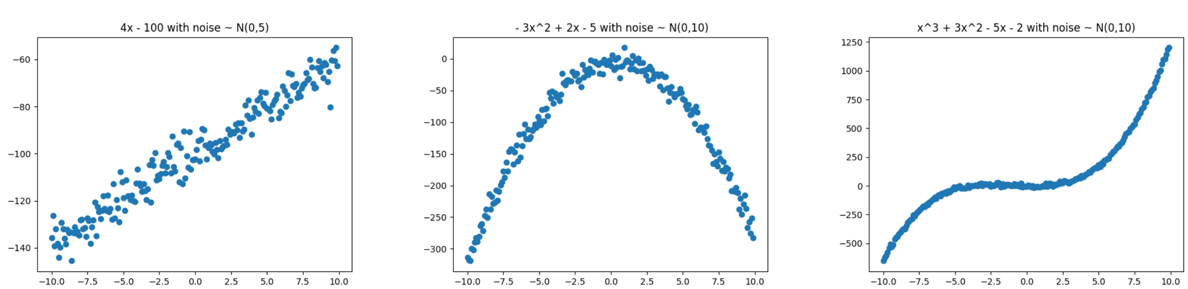
Distributions were generated using Monte Carlo Inversion / Box Muller Transform from a uniform random variable.



Polynomials:
Generates polynomial from a list of params (coefficients) and also generates noisy (gaussian) polynomials.

```python
rv = RandomVariable()

def infer_polynomial(x: int | float | np.ndarray, params: list):
    cexp = 1
    val = 0
    for p in params[::-1]:
        val += p * cexp
        cexp *= x
    return val

def generate_noisy_polynomial(x: int | float | np.ndarray, params: list, mu=0, sigma=20):
    y = infer_polynomial(x, params)
    y = [_y + rv.gaussian(mu=mu, sigma=sigma) for _y in y]
    return np.array(y)
```

Plots of few polynomials with noise.



Curve Fitting:

```
def get_vander(x, degree):
    vander = []
    for i in range(degree+1):
        vander.append(x**i)
    return np.vstack(vander).T

def fit_polynomial(x, y, degree):
    H = get_vander(x, degree)
    params = np.linalg.inv(H.T @ H) @ H.T @ y
    return params[::-1]

def fit_line(x, y):
    H = np.vstack([np.ones_like(x), x]).T
    params = np.linalg.inv(H.T @ H) @ H.T @ y
    return params[::-1]

def get_fim(x, est_params, est_sigma):
    degree = len(est_params) - 1
    I = np.zeros((degree + 1, degree + 1))

    for i in range(degree + 1):
        for j in range(degree + 1):
            grad_i = x ** i
            grad_j = x ** j
            I[i, j] = np.sum(grad_i * grad_j) / est_sigma**2

    return I
```

Contains code for getting the Vandermonde Matrix (get_vander) and fitting polynomial:

$$H = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} \tag{1}$$

$$\theta = (H^T H)^{-1} H^T y = H^\dagger y \tag{2}$$

2.2 Main Code

Line Fitting

```python
import numpy as np
import matplotlib.pyplot as plt
from utils.stochastic import RandomVariable
from utils.polynomial import infer_polynomial, generate_noisy_polynomial, pretty_polynomial
from utils.fit import fit_line, get_fim

rv = RandomVariable()

if __name__ == "__main__":
    # data setup
    x = np.arange(-10, 10, 0.1)
    opt_params =  [4, -5]

    print("Optimal Params: ", pretty_polynomial(opt_params))
    y = generate_noisy_polynomial(x, opt_params, sigma=1) # N(mu=0, sigma=1)

    # estimation
    estimated_params = fit_line(x, y)
    print("Estimated Params: ", pretty_polynomial(estimated_params))

    # fim and covariance
    fim = get_fim(x, estimated_params, 1)
    cov = np.linalg.inv(fim)
    crlb = np.diag(cov)
    print("CRLB:\n", crlb)

    plt.scatter(x, y)
    plt.plot(x, infer_polynomial(x, estimated_params), color="orange", label="estimated")
    plt.show()
```

Polynomial Fitting

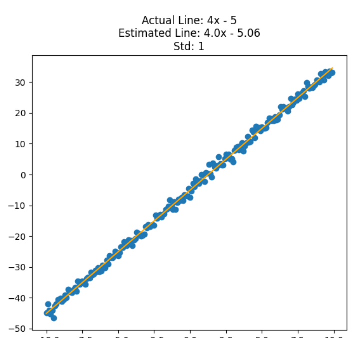
```python
import numpy as np
import matplotlib.pyplot as plt
from utils.stochastic import RandomVariable
from utils.polynomial import infer_polynomial, generate_noisy_polynomial, pretty_polynomial
from utils.fit import fit_polynomial, get_fim

rv = RandomVariable()

if __name__ == "__main__":
    # data setup
    x = np.arange(-10, 10, 0.1)
    degree = 4

    # generating some curve to estimate
    opt_params =  [rv.uniform(-1, 1) for _ in range(degree+1)]
    sigma = 10
    # opt_params = [0.5, 2, 1]
    # degree = len(opt_params)-1

    print("Optimal Params: ", pretty_polynomial(opt_params))
    y = generate_noisy_polynomial(x, opt_params, sigma=sigma) # N(mu=0, sigma=1)

    # estimation
    estimated_params = fit_polynomial(x, y, degree)
    print("Estimated Params: ", pretty_polynomial(estimated_params))

    # fim and covariance
    fim = get_fim(x, estimated_params, 1)
    cov = np.linalg.inv(fim)
    crlb = np.diag(cov)
    print("CRLB:\n", crlb)

    plt.scatter(x, y)
    plt.plot(x, infer_polynomial(x, estimated_params), color="orange", label="estimated")
    plt.title(f"Actual Line: {pretty_polynomial(opt_params)}\nEstimated Line:
{pretty_polynomial(estimated_params)}\nStd: {sigma}")
    plt.show()
```

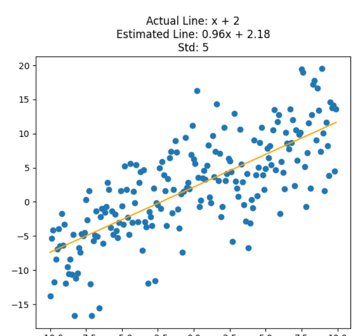
Comparing Mean and Max Estimators

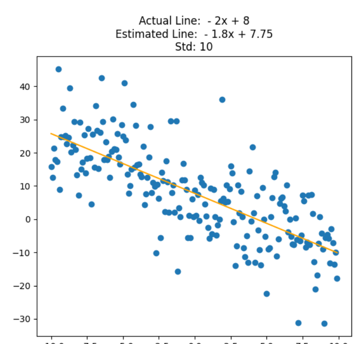
```python
class EstimatorComparison:
    def __init__(self) -> None:
        pass

    def T(self, X):
        if self.noise_distribution == "gaussian":
            return np.sum(X)
        elif self.noise_distribution == "dependent_uniform":
            return np.max(X)

    def g(self, T):
        if self.noise_distribution == "gaussian":
            return T / self.N
        elif self.noise_distribution == "dependent_uniform":
            return (self.N + 1) / (2 * self.N) * T

    def mvue(self, n_readings, noise_distribution="gaussian"):
        self.noise_distribution = noise_distribution
        self.N = n_readings

        return lambda x: self.g(self.T(x))

    def compare_estimators(self, estimator1, estimator2, noise, dc_val, n_readings=1000, n_iters=1000):
        estimates = []
        estimators = [estimator1, estimator2]

        for _ in range(n_iters):
            dc = np.ones(n_readings) * dc_val
            dc = [d + noise() for d in dc]

            estimates.append((estimator1(dc), estimator2(dc)))

        plt.figure(figsize=(12, 6))
        plt.suptitle(f"Noise Type: {noise.__name__}")

        for i in range(2):
            plt.subplot(1, 2, i+1)
            plot_data = [e[i] for e in estimates]
            plt.hist(plot_data, bins=int(np.sqrt(n_iters)), color="orange")
            plt.title(f"Mean: {np.mean(plot_data):.2f}, Var: {np.var(plot_data):.4f}")
            plt.legend()

        plt.show()
```

3 Results

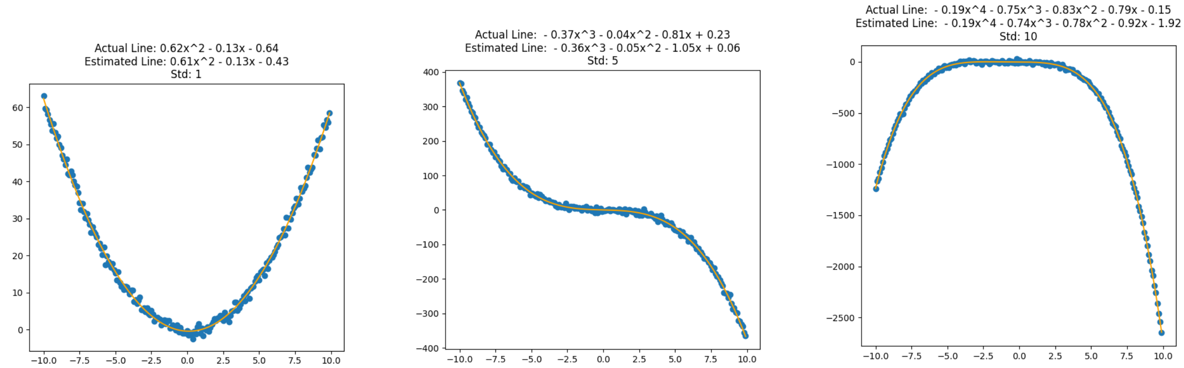
Line Fitting







Polynomial Fitting



Mean vs Max Estimators

Parameter Dependent Uniform - `U(0,2*DC)`

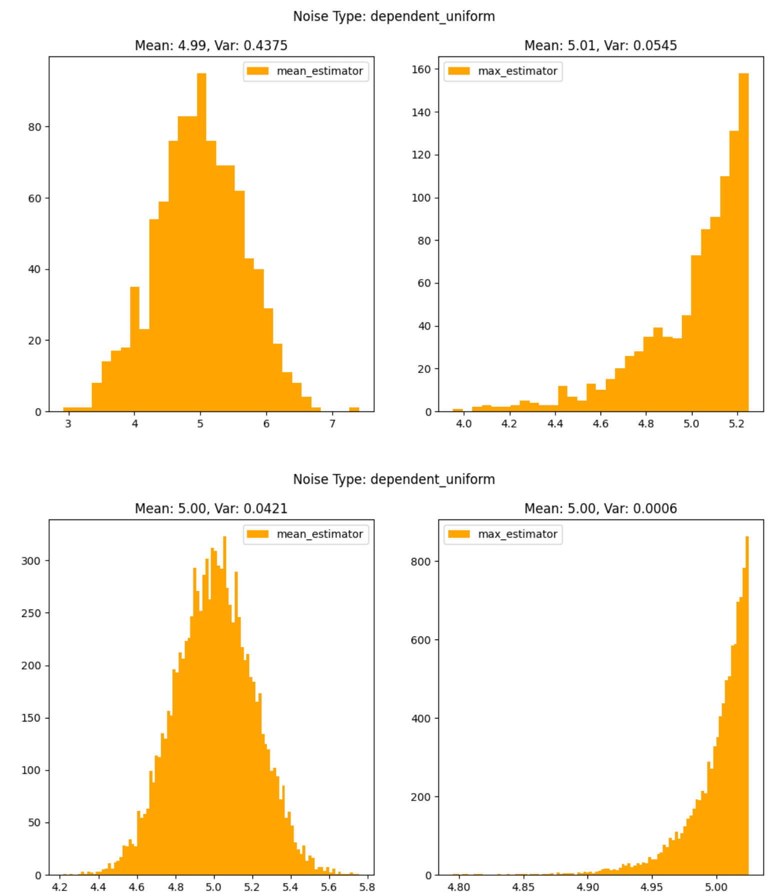


Figure 1: (a) `n_readings=20`, (b) `n_readings=2000`

Gaussian - `N(DC,DC)`

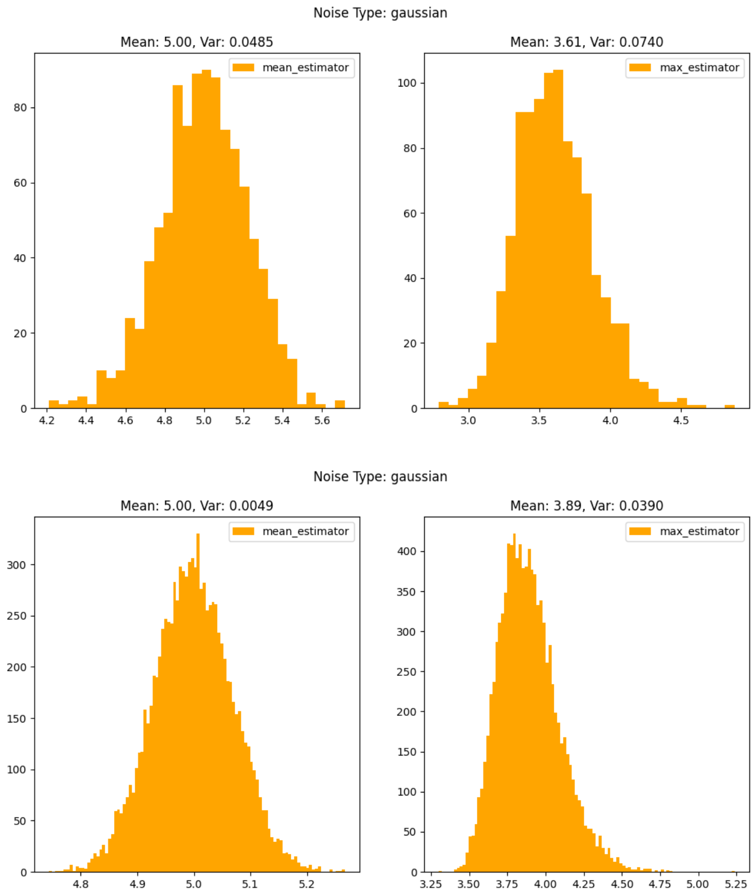


Figure 2: (a) `n_readings=20`, (b) `n_readings=2000`

4 Discussion

- I generated different types of random variables and was able to verify Law of Large Numbers when I varied number of samples as it more closely approximated the real distribution.
- Using the random variable I had made before I modeled the input data (with gaussian noise)
- Then for the first two parts, the equations for polynomial regression were used to get estimations for the polynomial coefficients

$$H = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix}$$

$$\theta = (H^T H)^{-1} H^T y = H^{\dagger} y$$

- The polynomial corresponding to the theta value was then plotted and the Fisher Information Matrix and Cramer Rao Lower Bound was calculated.

$$CRLB = I^{-1}(\theta)$$

- For comparing the mean and max estimators, first a "dependent" uniform random variable was created with noise:

$$x[n] = \theta + w[n]$$

$$w[n] - U(-DC, DC)$$

- It was interesting to see that for the dependent uniform, the mean estimator followed a bell-like curve and the max estimator was similar to a curve of x^n.

- The max estimator was more precise than the mean estimator in case of uniform distribution.

$$var(\theta_{mean}) = \frac{\theta^2}{3N}$$
$$var(\theta_{max}) = \frac{\theta^2}{N(N+2)}$$

- Same max estimator seems to have a lot of bias in the case of Gaussian Noise, this could be due to the fact that Gaussian has a smaller Kurtosis value. Also calculating the correct $g(\max_{1 \leq i \leq N} x[i])$ for this would require to know $F_X(x)$ for the gaussian.