Transducer Sound Radiation

Learning Objectives

Planar Immersion Transducer

- on-axis near field - far field
- radiation into solid-normal incidence, plane interface
- diffraction correction

Spherically focused transducer

- on axis field
- focal spot size
- radiation into solid-normal incidence, plane interface
- diffraction correction

Learning Objectives (continued)

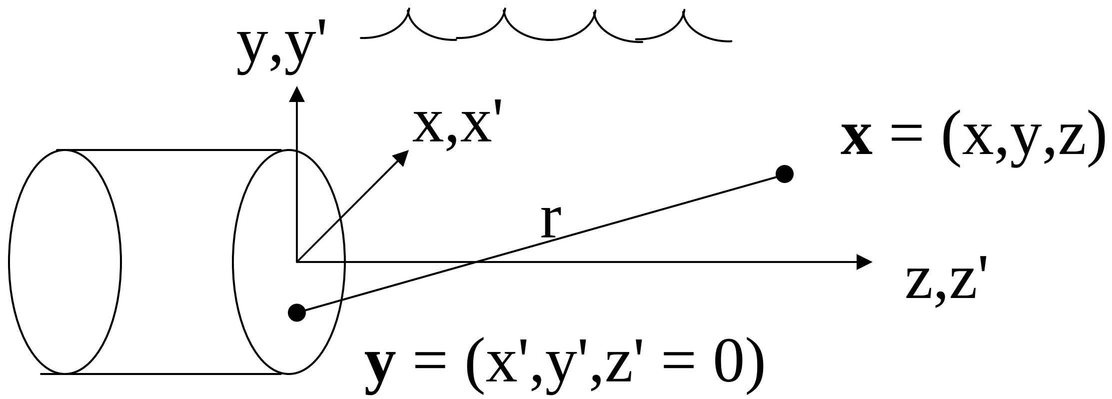
Contact P-wave transducer on a solid
- wave types present
- directivity functions

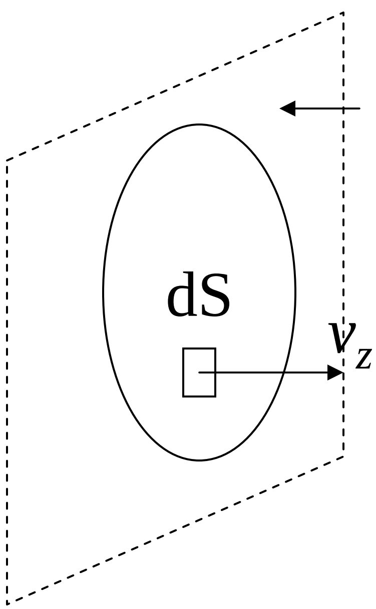
Angle beam shear wave transducer

Overview of beam theories

Ultrasonic Beam Models

Plane piston transducer radiating into a fluid





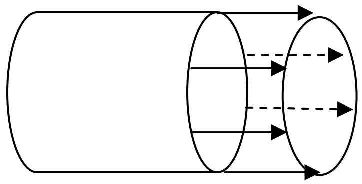
Can show that each area element that is in motion acts like a source of spherical waves:

$$dp(x,y,z,\omega) = \frac{-i\omega\rho v_z(x',y',\omega)dS}{2\pi}\frac{\exp(ikr)}{r}$$

Adding up all such sources over the face of the transducer gives the Rayleigh-Sommerfeld Integral

$$p\left(x, y, z, \omega\right) = \frac{-i\omega\rho}{2\pi} \int_S \frac{v_z\left(x', y', \omega\right)\exp\left(ikr\right)}{r} dS\left(\mathbf{y}\right)$$

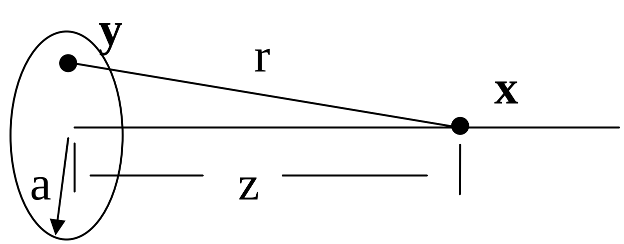
If we let $v_z(x',y',\omega) = v_0(\omega)$ (piston model)



$$\Rightarrow \quad p\left(x, y, z, \omega\right) = \frac{-i\omega\rho v_0\left(\omega\right)}{2\pi} \int_S \frac{\exp\left(ikr\right)}{r} dS\left(\mathbf{y}\right)$$

$$p(\mathbf{x},\omega)=\frac{-i\omega\rho v_0}{2\pi}\int_S \frac{\exp(ikr)}{r}\,dS(\mathbf{y})$$

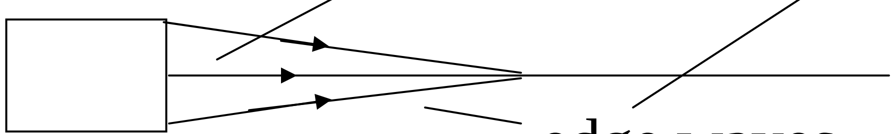
For on-axis response of a circular transducer of radius, a



It can be shown that the area element can be written as dS = r dr dϕ

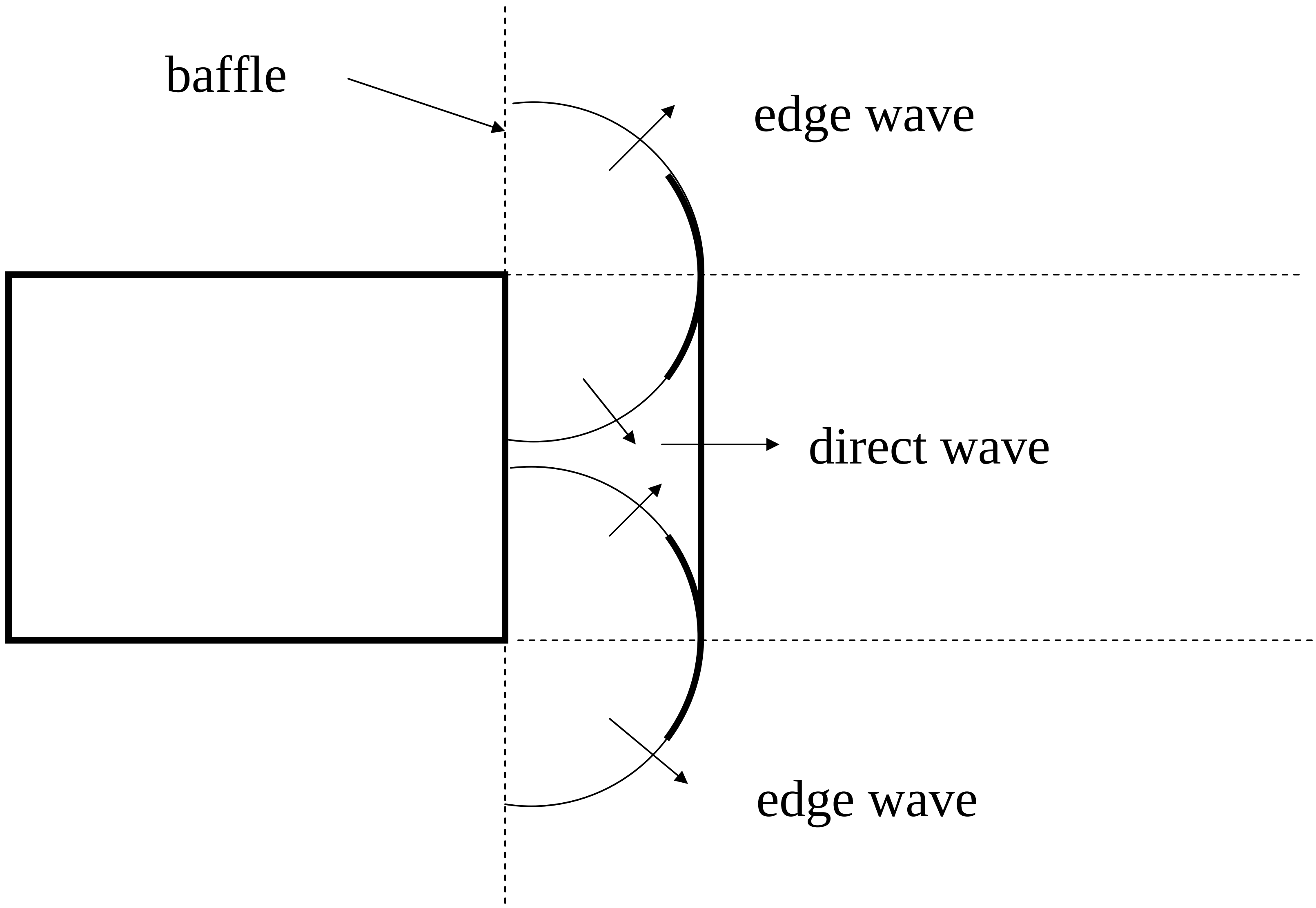
$$p(z,\omega)=\rho c v_0\left[\ \exp(ikz)-\exp\left(ik\sqrt{a^2+z^2}\right)\ \right]$$

direct wave



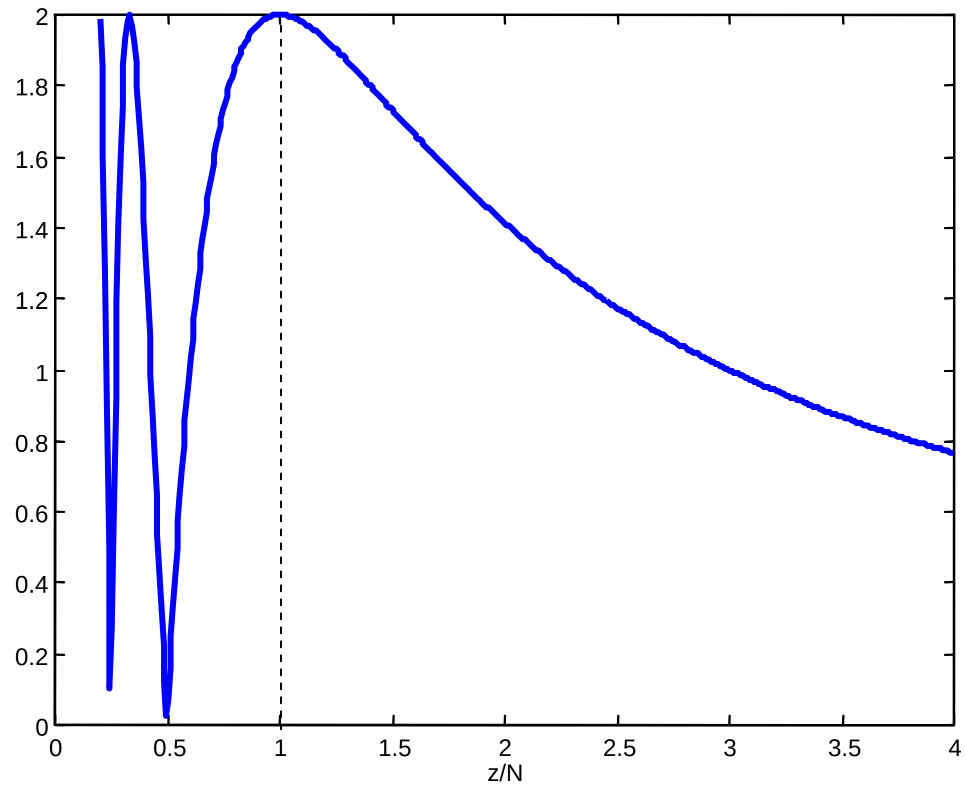
edge waves

Direct and edge waves as seen for a pulsed transducer



on-axis pressure:

$$\frac{|p|}{\rho c v_0}$$

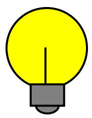


a = 6.35 mm

f= 5 MHz

c = 1500 m/s

Near field distance $N = a^2/\lambda$

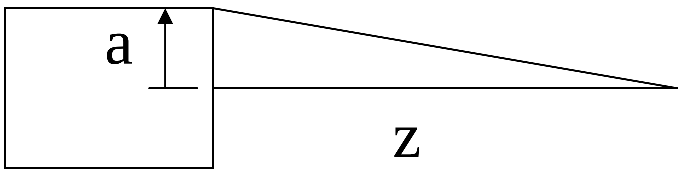


Maxima: $z = N/(2m+1)$ $m = 0,1,2,\ldots$

Minima: $z = N/2n$ $n = 1,2,3,\ldots$

Example: for a 5 MHz, 1/2 in. diameter transducer radiating into water
N = 5 in. (approx.)

Paraxial approximation : a/z <<1



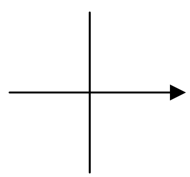
$$\sqrt{a^2 + z^2} \cong z\left(1 + \frac{a^2}{2z^2} + \ldots\right)$$

on-axis pressure:

$$p(z,\omega) = \rho c v_0 \exp(ikz)\left[1 - \exp\left(\frac{ika^2}{2z}\right)\right]$$

plane wave

$C_1(a,\omega,z)$

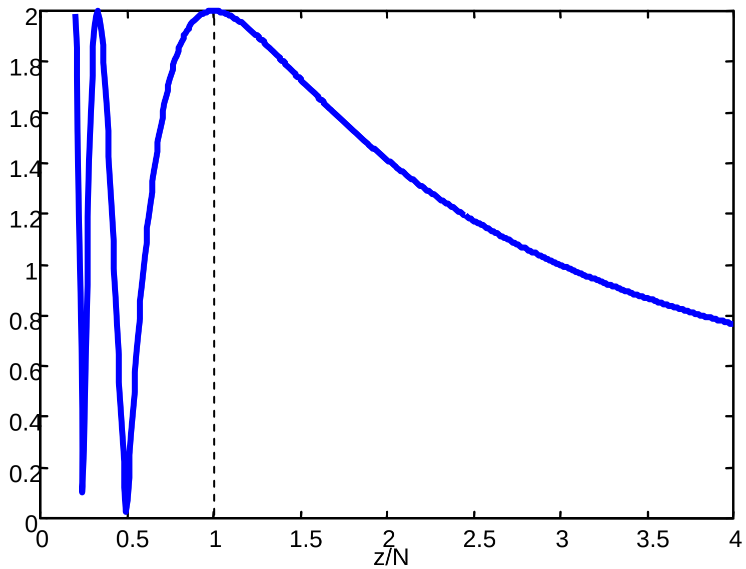


diffraction correction

on-axis pressure:

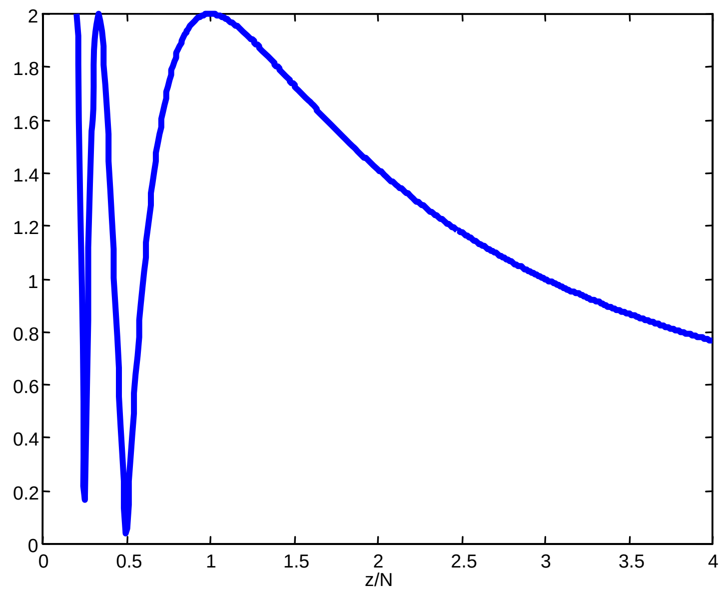
$$\frac{|p|}{\rho c v_0}$$

exact result



$$\frac{|p|}{\rho c v_0}$$

paraxial result



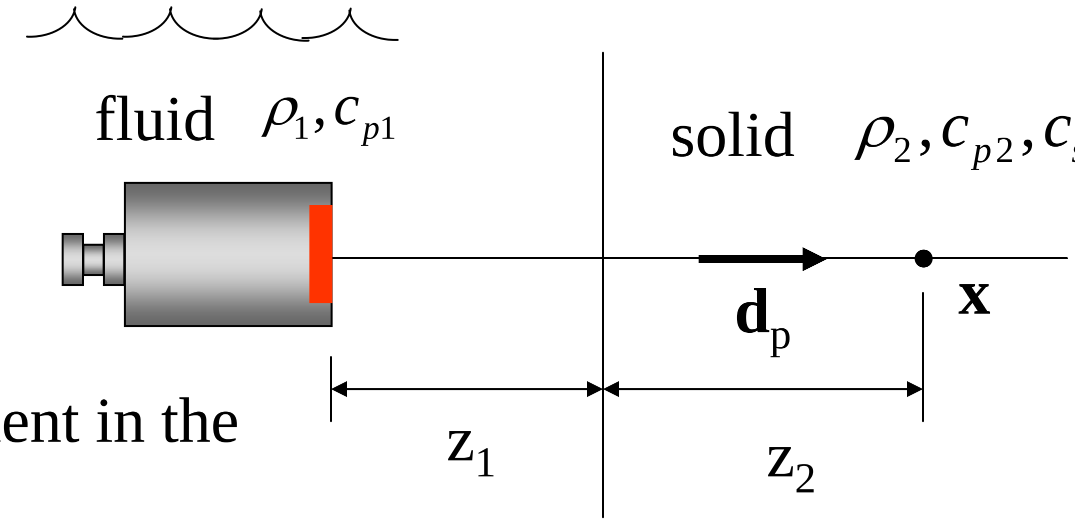
```
function p  = on_axis(zN, A,c,F)
% exact on axis pressure from a piston source
%radiating into a fluid. A is radius in mm, c the
%wavespeed of the fluid in m/sec, F the frequency in MHz,
% zN is the distance in the fluid divided by the near field
%distance a^2/lamba  (lamba is the wavelength)
al= 1000*A*F/c;    % a/lamba
ka = 2*pi*al;                % ka for the transducer
kz = ka*al*zN;
ke = 2*pi*(al^2).*sqrt(zN.^2 + (1/al)^2);
p = exp(i*kz) - exp(i*ke);
```

```
function p = par_on_axis(zN, A,c,F)
% paraxial axis pressure from a piston source
%radiating into a fluid. A is radius in mm, c the
%wavespeed of the fluid in m/sec, F the frequency in MHz,
% zN is the distance in the fluid divided by the near field
%distance a^2/lamba
al= 1000*A*F/c;    % a/lamba
ka = 2*pi*al;              % ka for the transducer
kz = ka*al*zN;
ke = ka./(2*al.*zN);
p = exp(i*kz).*(1 - exp(i*ke));
```

```
MAT> z = linspace(.2, 4,500);
MAT> p = on_axis(z,6.35,1500,5);
MAT> plot(z, abs(p))
MAT> xlabel('z/N')
```

```
MAT> p = par_on_axis(z, 6.35, 1500, 5);
MAT> plot(z, abs(p))
MAT> xlabel('z/N')
```

On-axis response at normal incidence to a plane interface (paraxial approximation)



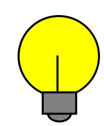
displacement in the solid

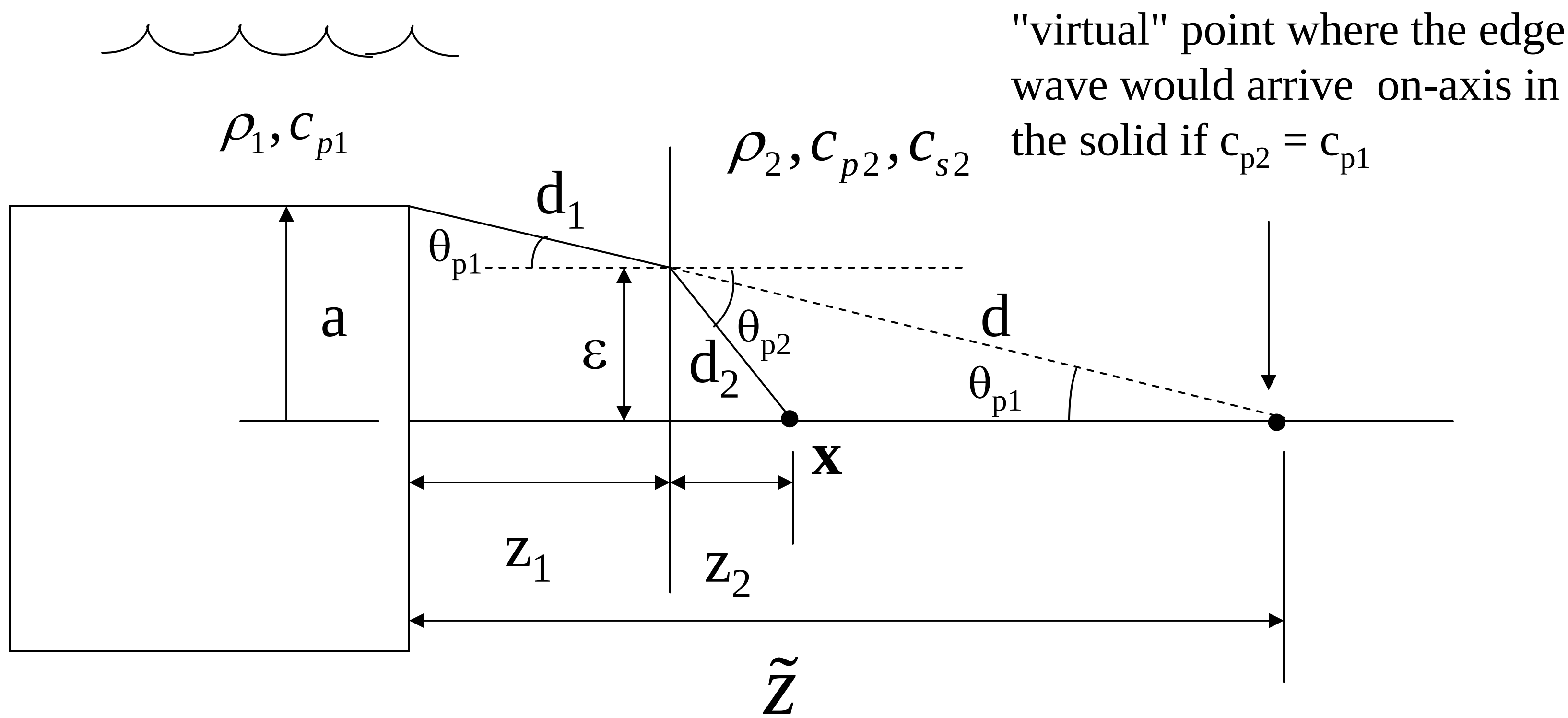
$$\mathbf{u}(\mathbf{x},\omega) = \frac{v_0}{-i\omega} T_{12}^{P;P} \mathbf{d}_p \exp\left(ik_{p1}z_1 + k_{p2}z_2\right) \underbrace{\left[1 - \exp\left(\frac{ik_{p1}a^2}{2\tilde{z}}\right)\right]}_{C(a,\omega,\tilde{z})}$$

transmission coefficient
(at normal incidence)
(velocity/velocity)

same diffraction correction
expression as for a single fluid

$$\tilde{z} = z_1 + \frac{c_{p2}}{c_{p1}} z_2$$





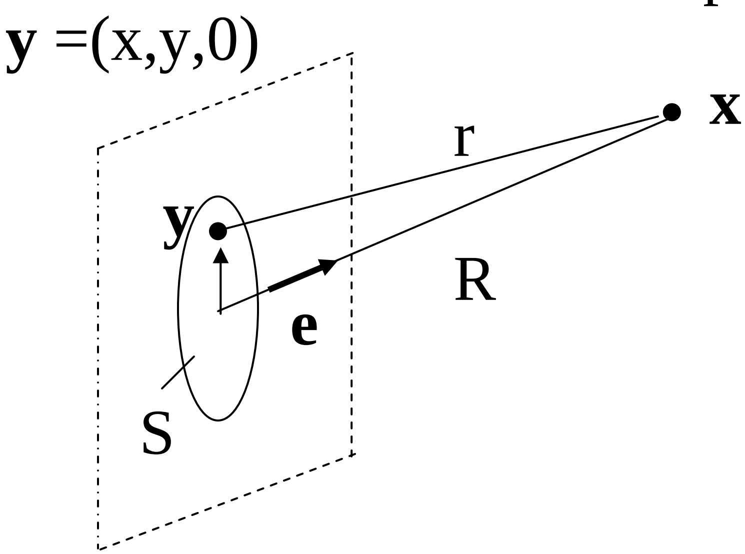
$$\varepsilon = d_2 \sin\theta_{p2} = d \sin\theta_{p1} \quad \text{so} \quad d = \frac{\sin\theta_{p2}}{\sin\theta_{p1}} d_2 = \frac{c_{p2}}{c_{p1}} d_2$$

which gives, in the paraxial approximation

$$\frac{a^2}{\tilde{z}} \cong \frac{a^2}{d_1 + d} = \frac{a^2}{d_1 + \frac{c_2}{c_1} d_2} \cong \frac{a^2}{z_1 + \frac{c_2}{c_1} z_1}$$

Far-field beam of a planar piston transducer

The far-field is usually defined as z > 3N - also called the "spherical wave region"



$$r = \sqrt{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})}$$
$$= \sqrt{(R\mathbf{e}-\mathbf{y})\cdot(R\mathbf{e}-\mathbf{y})}$$
$$\cong R\sqrt{1-2\mathbf{e}\cdot\mathbf{y}/R}$$
$$\cong R-\mathbf{e}\cdot\mathbf{y}$$

$$p(\mathbf{x},\omega) = \frac{-i\omega\rho v_0}{2\pi}\frac{\exp(ikR)}{R}\int_S \exp(-ik\mathbf{e}\cdot\mathbf{y})\,dxdy$$

Define the 2-D spatial Fourier transform of Θ , where

$$\Theta = \begin{cases} 1 & in\ S \\ 0 & otherwise \end{cases}$$

as

$$F\left(e_x, e_y, \omega\right) = \frac{1}{\left(2\pi\right)^2} \iint_S \exp\left(-ip_x x - ip_y y\right) dxdy$$

$$= \frac{1}{\left(2\pi\right)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Theta\left(x.y\right) \exp\left(-ip_x x - ip_y y\right) dxdy$$

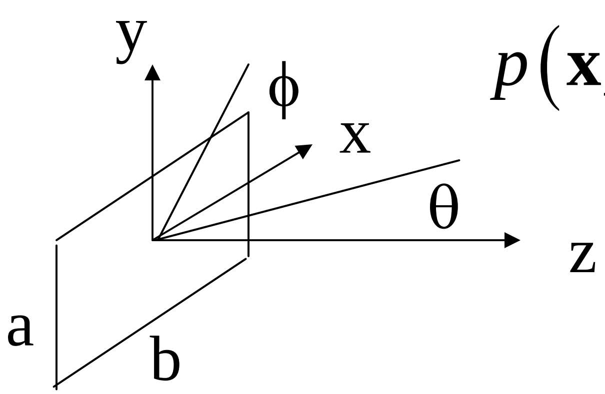
$$p_x = ke_x$$

$$p_y = ke_y$$

Then the far field pressure can be written as

$$p\left(\mathbf{x}, \omega\right) = \underbrace{-2\pi i\omega\rho v_0 F\left(e_x, e_y, \omega\right)}_{\text{angular beam profile}} \underbrace{\frac{\exp\left(ikR\right)}{R}}_{\text{spherical wave}}$$

Rectangular Piston Transducer



$$p(\mathbf{x},\omega) = -2\pi i\omega\rho v_0 F \frac{\exp(ikR)}{R}$$

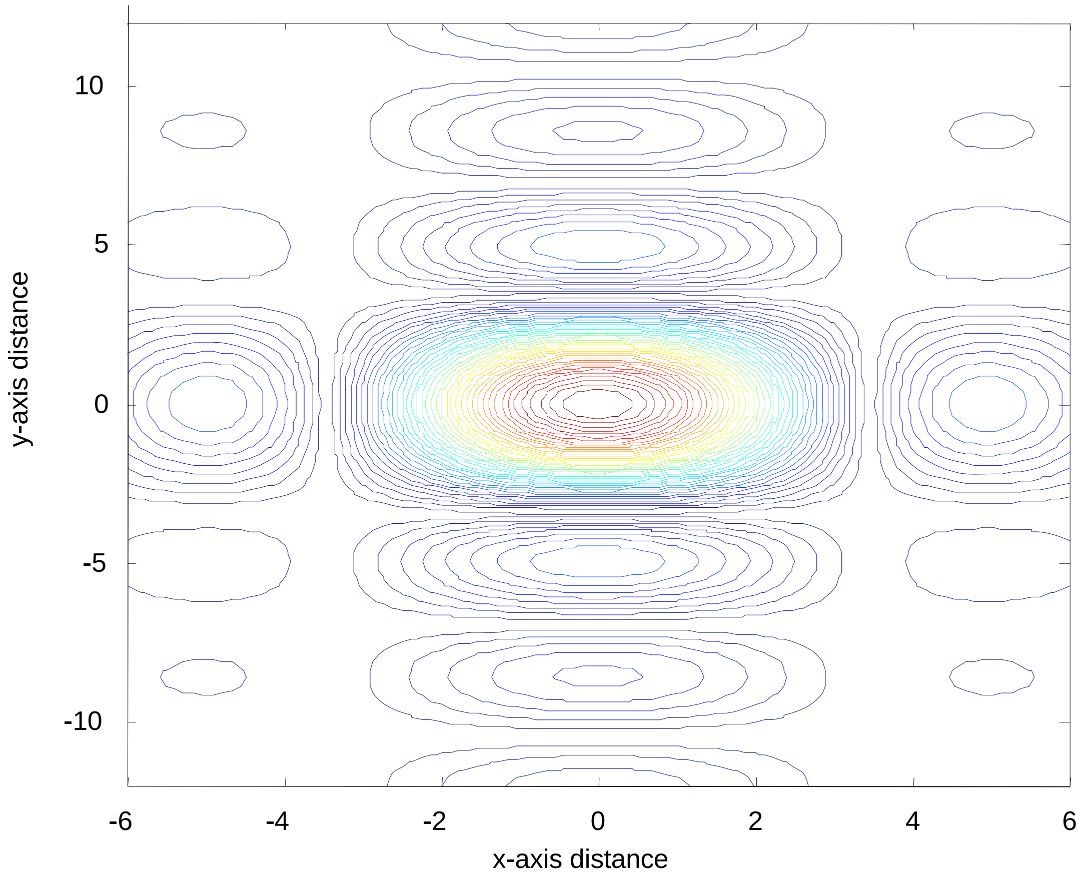
$$F(e_x, e_y, \omega) = \frac{ab}{(2\pi)^2} \frac{\sin\left(\frac{kbe_x}{2}\right)\sin\left(\frac{kae_y}{2}\right)}{\left(\frac{kbe_x}{2}\right)\left(\frac{kae_y}{2}\right)}$$

In spherical coordinates

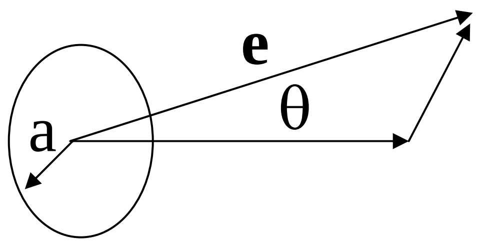
$$e_x = \sin\theta\cos\phi$$

$$e_y = \sin\theta\sin\phi$$

Example far-field pattern of a rectangular transducer



Circular Piston Transducer

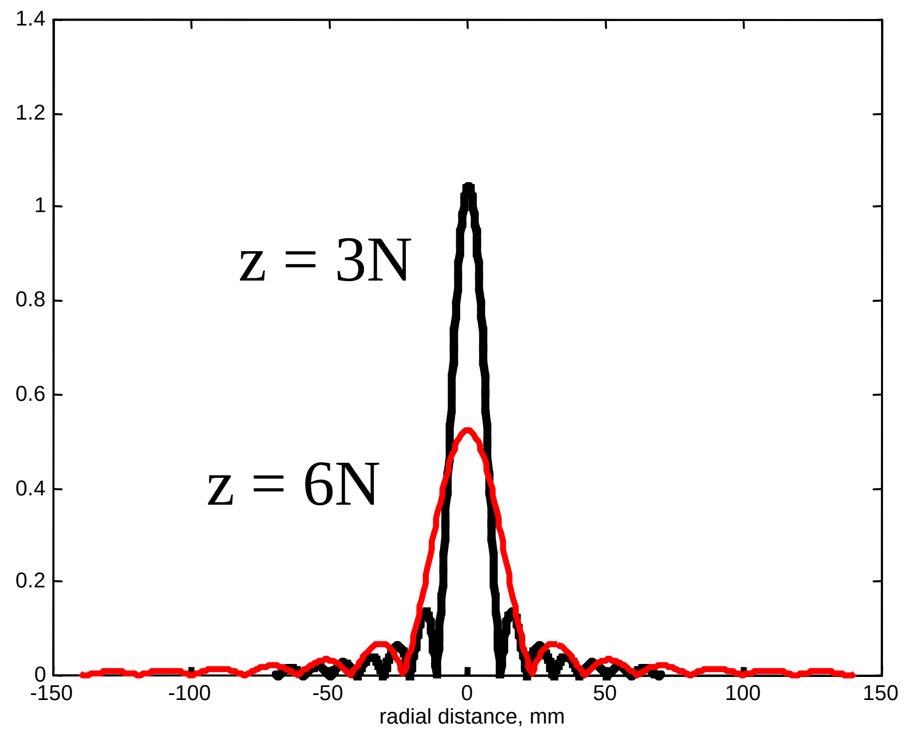


$$p(\mathbf{x},\omega) = -2\pi i\omega\rho v_0 F \frac{\exp(ikR)}{R}$$

$$F(e_x, e_y, \omega) = \frac{a^2}{2\pi}\frac{J_1(ke_\rho a)}{(ke_\rho a)}$$

$$e_\rho = \sqrt{e_x^2 + e_y^2} = \sin\theta$$

$$\frac{|p|}{\rho c v_0}$$

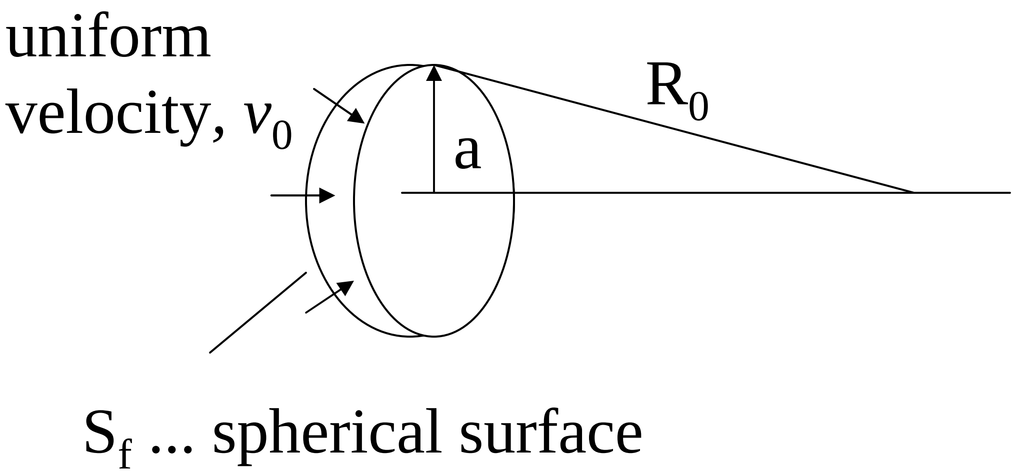


```
function  [p, rho] = far_field(ang,A, c, F, RN)
% far_field computes the normalized far field pressure
% for a circular piston (omitting the exp(ikR) phase term)
% A is the radius of the transducer in mm, c the wavespeed
%in m/sec, F the frequency in MHz, and RN is
%the normalized radial distance in near field units.
% rho is the transverse distance (normal to z)  in mm
ka = 2*pi*(1000*A*F/c);
al= 1000*A*F/c;
x = ka*sin(ang*pi/180);
rho =RN*(A*al)*sin(ang*pi/180);
p = -i*(ka/(al*RN))*besselj(1,x)./(x+eps*(x ==0));
```

```
MAT> ang = linspace(-10, 10,500);
MAT> [p,r] =far_field(ang,6.35,1500,5,3);
MAT> plot(r,abs(p), '--')
MAT> hold on
MAT> [p,r] =far_field(ang,6.35,1500,5,6);
MAT> plot(r,abs(p), 'red')
MAT> xlabel('radial distance, mm')
```

Spherically Focused Piston Transducer Radiating Into a Fluid

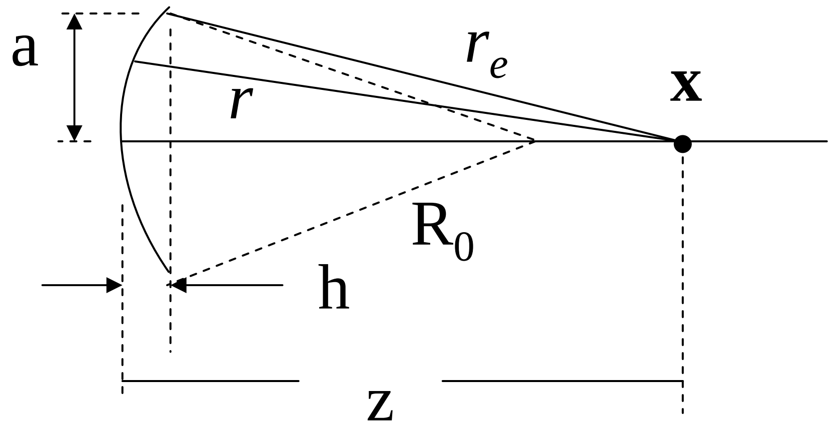
O'Neil Model



$$p(\mathbf{x},\omega) = \frac{-i\omega\rho v_0}{2\pi} \int_{S_f} \frac{\exp(ikr)}{r} \, dS(\mathbf{y})$$

For **x** on the central axis

$$dS = r\,dr\,d\phi/q_0 \qquad q_0 = 1 - z/R_0$$



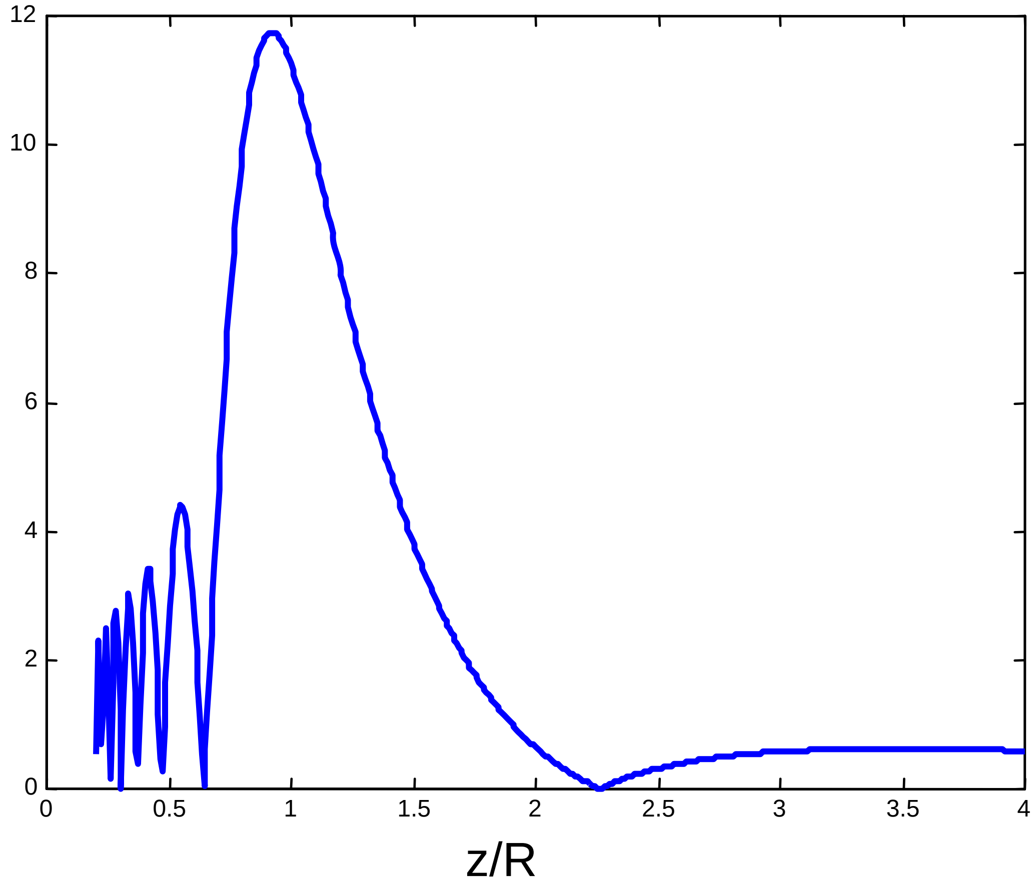
$$p(\mathbf{x},\omega) = \frac{\rho c v_0}{q_0}\left[\exp(ikz) - \exp\left(ikr_e\right)\right]$$

$$r_e = \sqrt{(z-h)^2 + a^2} \qquad h = R_0 - \sqrt{R_0^2 - a^2}$$

a = 6.35 mm
R_0 = 76.2 mm
f= 10 MHz
c = 1480 m/

on-axis pressure versus z/R_0:

True focus



$$\frac{|p|}{\rho c v_0}$$

```
function p = focused_on_axis(zR, A,c,F,R)
% on axis pressure of a spherically focused probe
% as a function of the normalized distance, zR = z/R
%A, radius of the transducer in mm. R , focal length in mm.
%c, the wave speed in m/sec, and F the frequency in MHz
al=1000*A*F/c;
ka=2*pi*al;
zN=(R/A)*(1/al)*(zR);
kz=ka*al*zN;
kR=2000*pi*F*R/c;
kh=kR-sqrt(kR^2-ka^2);
kre=sqrt((kz-kh).^2+ka^2);
p = (exp(i*kz) -exp(i*kre))./(1-kz./kR);
```

```
MAT> z=linspace(.2,4,500);
MAT> p = focused_on_axis(z,6.35,1480,10,76.2);
MAT> plot(zr,abs(p))
MAT> xlabel('z/R')
```

Paraxial Approximation

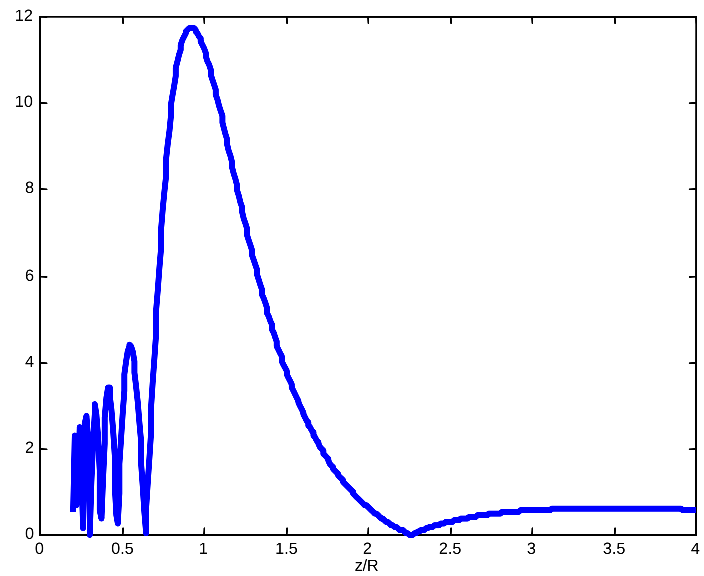
$$r_e \cong z + \frac{a^2 q_0}{2z} \qquad\qquad q_0 = 1 - z/R_0$$

on-axis pressure:

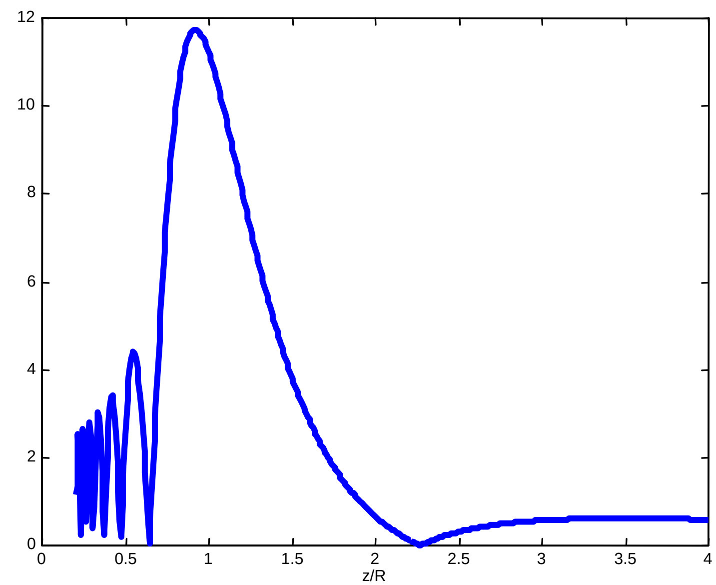
$$p(z,\omega) = \underbrace{\rho c v_0 \exp(ikz)}_{\text{plane wave}} \underbrace{\left\{ \frac{1}{q_0}\left[1 - \exp\left(\frac{ika^2 q_0}{2z}\right)\right]\right\}}_{\substack{\text{diffraction correction} \\ C_1(a,\, z,\, R_0,\, \omega)}}$$

on-axis pressure

exact $\frac{|p|}{\rho c v}$



paraxial $\frac{|p|}{\rho c v_0}$

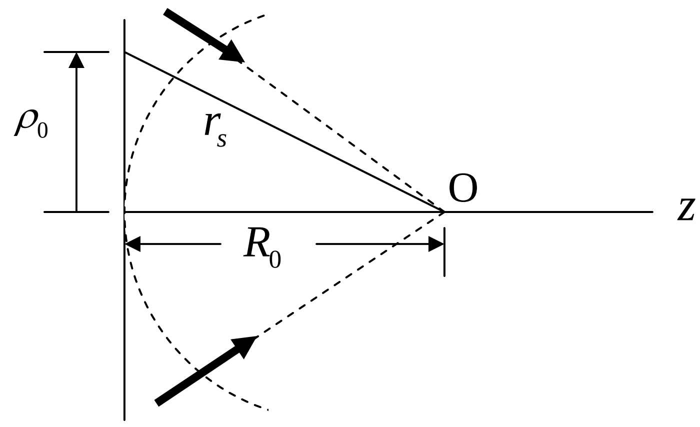


```
function p = par_focused_on_axis(zR, A,c,F,R)
% on axis pressure of a spherically focused probe,paraxial approx.
% as a function of the normalized distance, zR = z/R
%A, radius of the transducer in mm. R , focal length in mm.
%c, the wave speed in m/sec, and F the frequency in MHz
al=1000*A*F/c;
ka=2*pi*al;
zN=(R/A)*(1/al)*(zR);
kz=ka*al*zN;
kR=2000*pi*F*R/c;
qo=1-kz./kR;
p = (1-exp(i*ka*(A/R)*qo./(2*zR)))./qo;
```

```
MAT> z=linspace(.2,4,500);
MAT> p = par_focused_on_axis(z,6.35,1480,10,76.2);
MAT> plot(zr, abs(p))
MAT> xlabel('z/R')
```

Another way to model focusing (in the paraxial approximation)

suppose on a planar aperture we have a spherical wave propagating (generated by a lens, for example)



then on the aperture we have a phase given approximately in the paraxial approximation $(\rho_0 / R_0 << 1)$ by

$$\exp\left(-ik\left[r_s - R_0\right]\right) = \exp\left[-ik\left[\sqrt{\rho_0^2 + R_0^2} - R_0\right]\right]$$

$$\cong \exp\left(-ik\rho_0^2 / 2R_0\right)$$

Thus, suppose we use a Rayleigh-Sommerfeld model for a planar transducer and place this phase (in the paraxial approximation) in the integral:

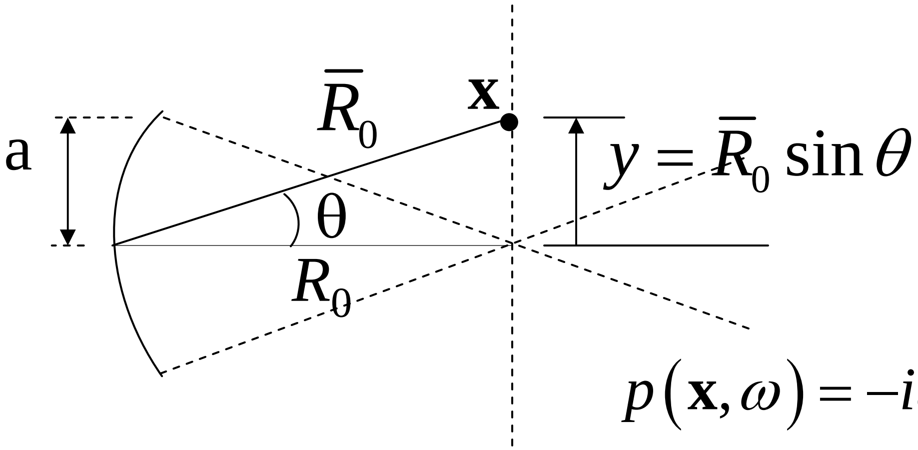
$$p(\mathbf{x},\omega) = \frac{-i\omega\rho\, v_0(\omega)}{2\pi} \iint_S \exp\left(-ik\rho_0^2 / 2R_0\right) \frac{\exp(ikr)}{r} dS$$

Using the paraxial approximation and evaluating this integral exactly for **x** on the transducer axis gives for a circular transducer of radius a:

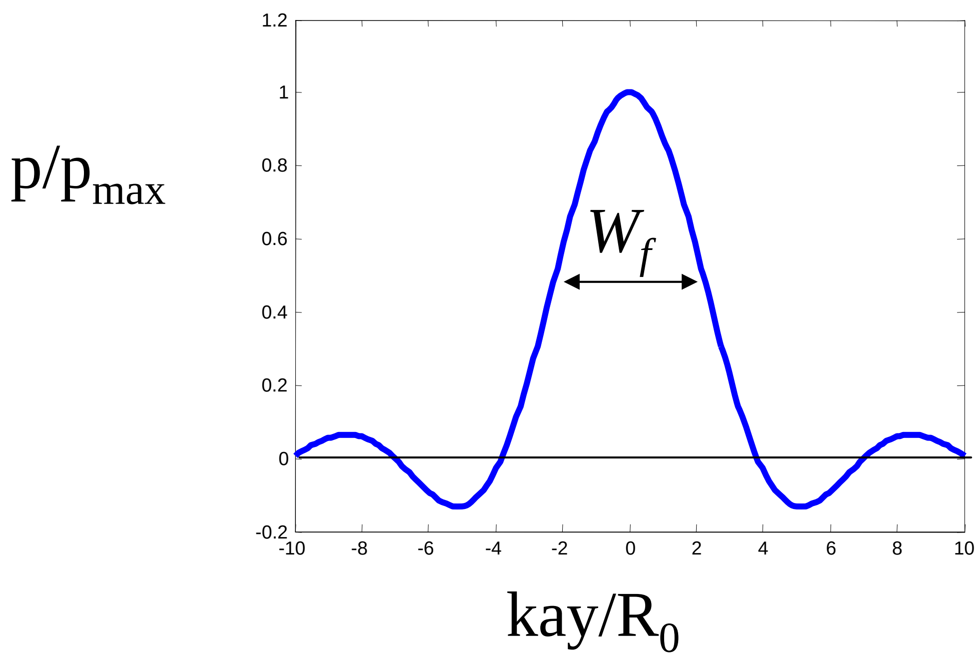
$$p(z,\omega) = \frac{\rho c v_0 \exp(ikz)}{q_0} \left[1 - \exp\left(ika^2 q_0 / 2z\right)\right]$$

Similarly, off-axis values will also represent those from a focused transducer

Wave field in the plane at the geometric focus of a spherically focused transducer



$$p(\mathbf{x},\omega) = -i\omega\rho v_0 a^2 \frac{\exp\left(ik\bar{R}_0\right)}{\bar{R}_0} \frac{J_1\left(kay/\bar{R}_0\right)}{kay/\bar{R}_0}$$

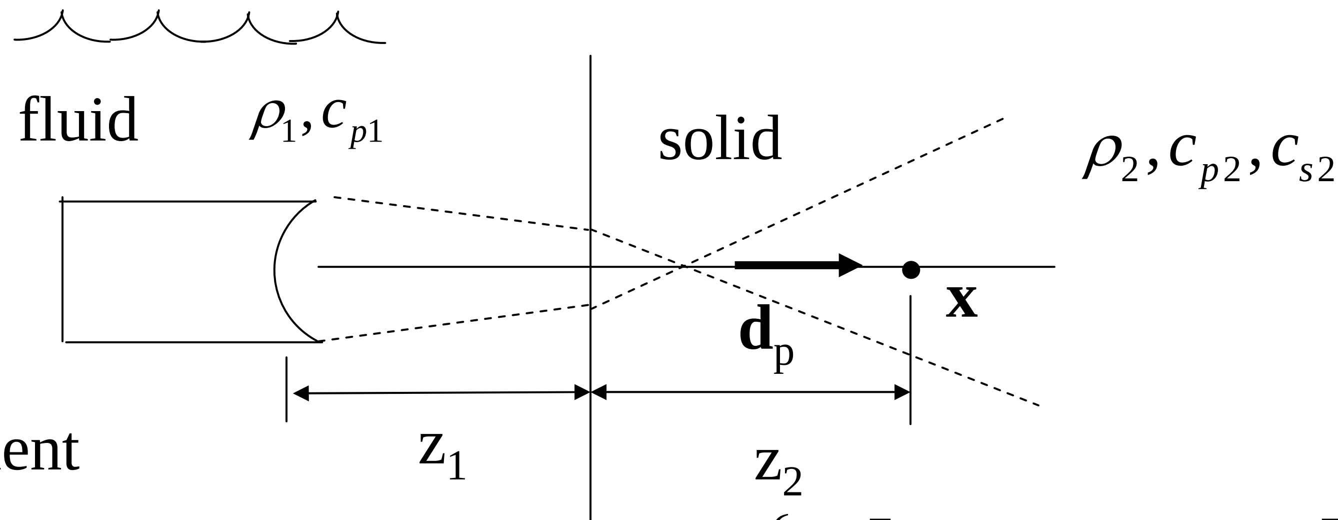


$$W_f\Big|_{6\,dB} = 4.43\frac{R_0}{ka} = 1.41\,\lambda\,F$$

λ … wavelength

$F = R_0 / 2a$ … transducer F number

On-axis response at normal incidence to an interface (paraxial approximation)



displacement

$$\mathbf{u}(\mathbf{x},\omega) = \frac{v_0}{-i\omega} T_{12}^{P;P} \mathbf{d}_p \exp\left(ik_{p1}z_1 + k_{p2}z_2\right) \underbrace{\left\{ \frac{1}{\tilde{q}_0} \left[1 - \exp\left(\frac{ik_{p1}a^2\tilde{q}_0}{2\tilde{z}} \right) \right] \right\}}_{C(a,\omega,\tilde{z})}$$

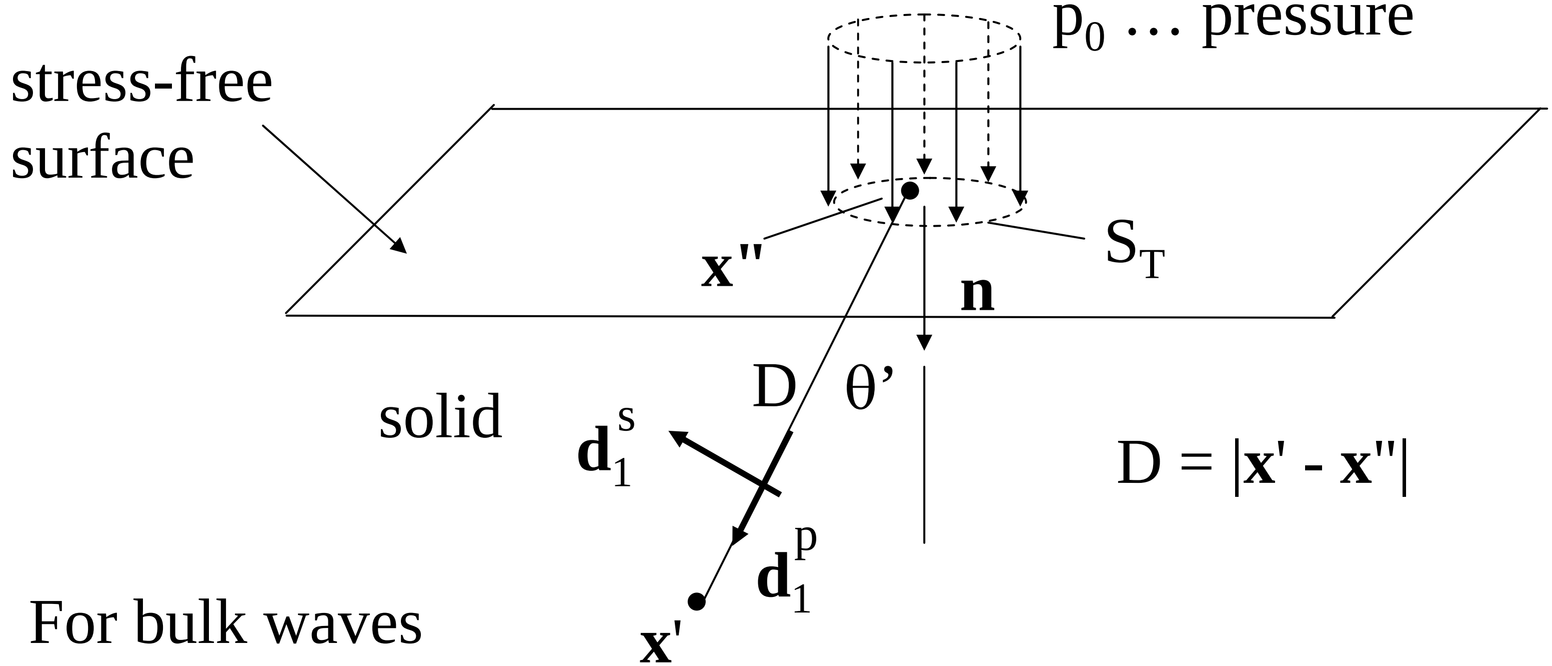
transmission coefficient
(velocity/velocity)

$$\tilde{z} = z_1 + \frac{c_{p2}}{c_{p1}} z_2$$

$$\tilde{q}_0 = 1 - \frac{\tilde{z}}{R_0}$$

same diffraction correction expression as for a fluid

Contact P-wave Transducer Model



For bulk waves

$$\mathbf{u}(\mathbf{x}',\omega) = \frac{p_0}{2\pi\rho_1 c_{s1}^2}\int_{S_T} K_s(\theta')\mathbf{d}_1^s \frac{\exp(ik_{s1}D)}{D}dS(\mathbf{x}'')$$

$$+\frac{p_0}{2\pi\rho_1 c_{p1}^2}\int_{S_T} K_p(\theta')\,\mathbf{d}_1^p \frac{\exp(ik_{p1}D)}{D}dS(\mathbf{x}'')$$

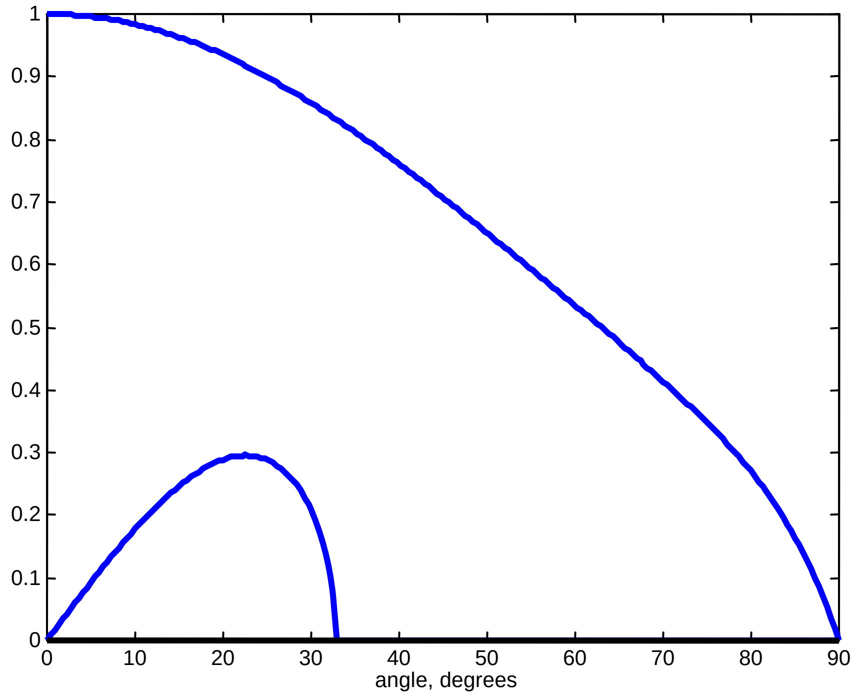
Directivity functions

$$K_p(\theta') = \frac{\cos\theta' \kappa_1^2 \left(\kappa_1^2/2 - \sin^2\theta'\right)}{2G(\sin\theta')}$$

$$K_s(\theta') = \frac{\kappa_1^3 \cos\theta' \sin\theta' \sqrt{1-\kappa_1^2 \sin^2\theta'}}{2G(\sin\theta')}$$

$$G(x) = \left(x^2 - \kappa_1^2/2\right)^2 + x^2\sqrt{1-x^2}\sqrt{\kappa_1^2 - x^2} \qquad \kappa_1 = \frac{c_{p1}}{c_{s1}}$$

K_p , K



```
function [kp,ks] = directivity(ang, cp, cs)
% computes the directivity functions for a p-wave contact
%transducer. ang is angle in degrees, cp, cs are p- and s-wave
%speeds
k = cp/cs;
angr = ang*pi/180;
x = sin(angr);
c =cos(angr);
g=(x.^2 -k^2/2).^2 + x.^2.*sqrt(1 - x.^2).*sqrt(k^2 - x.^2);
kp = c.*(k^2).*(k.^2/2 -x.^2)./(2.*g);
ks = (k*x <1).*c.*(k^3).*x.*sqrt(1 - k^2.*x.^2)./(2.*g);
```

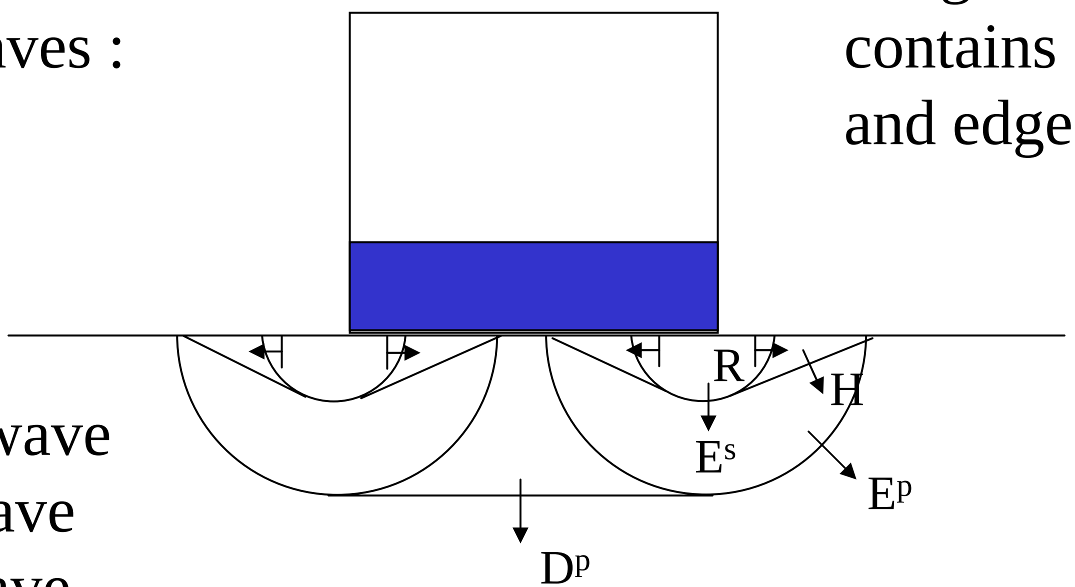
```
MAT> x = linspace(0,90,200);
MAT> [kp,ks] = directivity(x, 5900, 3200);
MAT> plot(x, kp)
MAT> hold on
MAT> plot(x, ks)
MAT> xlabel('angle, degrees')
```

For θ' small $K_p = 1$, $K_s = 0$

$$\mathbf{u}(\mathbf{x}',\omega) = \frac{p_0\mathbf{n}}{2\pi\rho_1 c_{p1}^2}\int_{S_T}\frac{\exp(ik_{p1}D)}{D}dS$$

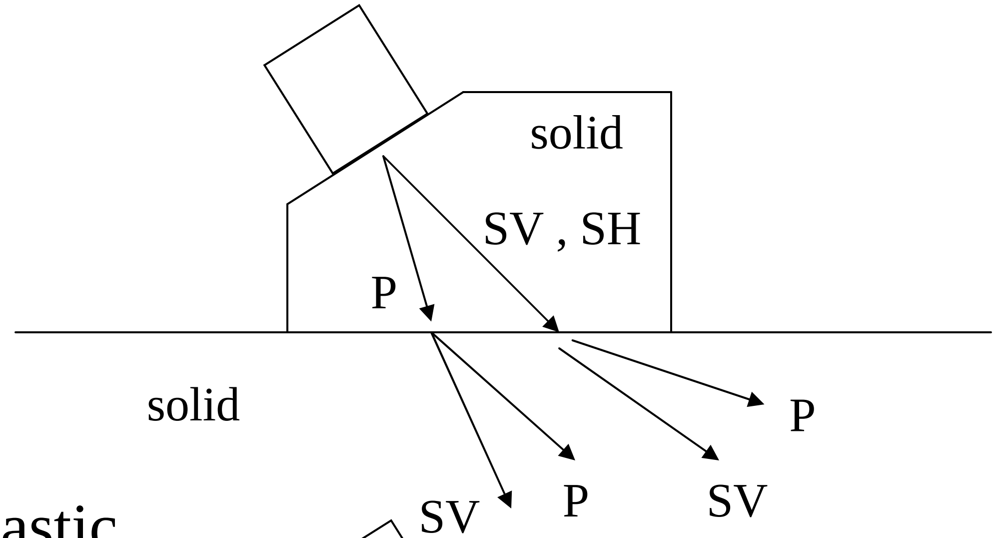
integral contains direct and edge P-waves

Full set of waves :

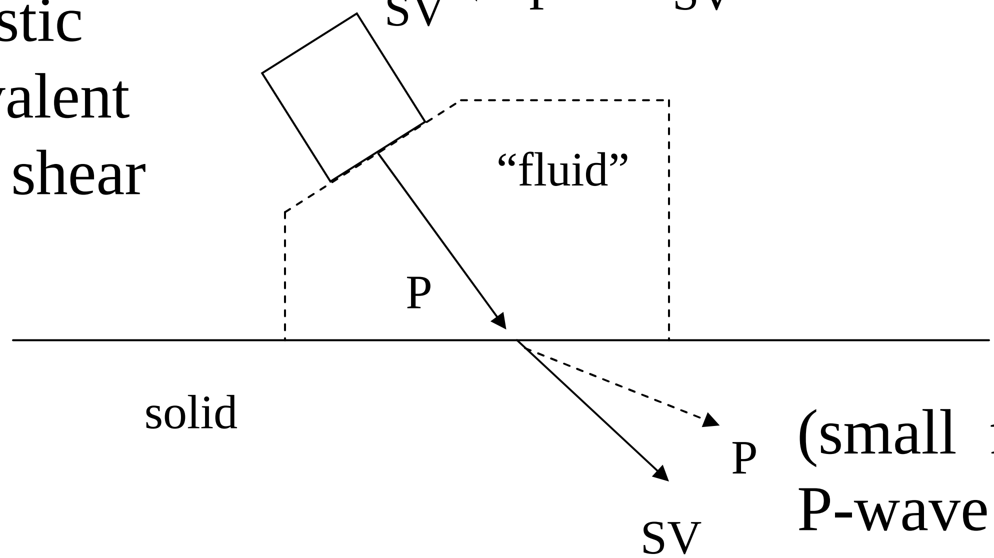


D^p … Direct P-wave
E^p … Edge P-wave
E^s … Edge S-wave
H … Head wave
R … Rayleigh wave

Angle Beam Shear Wave Transducer Model



Can replace elastic wedge by equivalent "fluid" (neglect shear waves)



(small for incident P-wave beyond critical angle)

Ultrasonic Beam Models

Numerically Intense Models

- EFIT - Langenberg
- Finite Elements - Lord
- Boundary Elements - Rizzo
- Edge Elements - Schmerr, Lerch

Surface Integral Models

- Generalized Point Source -Spies
- Rayleigh- Sommerfeld + High Freq. Asymptotics - Schmerr, Lhemery, others

Line Integral Models

- Boundary Diffraction Wave - Schmerr, Lerch

Ultrasonic Beam Models

Other Basis Function Models
Gauss-Hermite Models - Thompson, Gray, Newberry, Minachi, Margetan

Multi- Gaussian Models
Minachi, Spies, Schmerr and Rudolph, Cerveny (Seismology)

A few references – mostly paraxial models

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