Diffraction at Ultrasonic Waves

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1 Introduction

The term *ultrasound* refers to mechanical oscillations, whose frequencies vary between 16 kHz, the upper audio limit of the human ear, and 10^{10} Hz. A human ear can only hear sound in the range of approx. 16 to 16,000 Hz. Oscillations in the region below 16 Hz are referred to as infra sound (*e.g.* earthquakes) and oscillations with frequencies above 10^{10} Hz are referred to as hyper-sound. This upper frequency-limitation for the ultrasound is determined by the atomic configuration of matter. The wavelength of the ultrasound in this region is in the order of a small multiple of the lattice constant a ($\Lambda \sim 10^3 a$); in the next region of hyper-sound, in which all the thermal movements of atoms and molecules occur, strong quantum-mechanical effects have to be taken into account (phonon [1] theory). Below these very high frequencies of 10^{10} Hz the material can be treated as a continuum *i.e.* the laws of classical acoustics apply to it, which originally only dealt with the problem of the sound within the hearing range.

There are few physical applications for ultrasound, the most importantly the determination of elasticity constants from measurements of the speed of sound. From measurements of the speed of sound and absorbability, the structural nature of the material can be determined within the scope of microscopic theories. Today the applications of ultrasound have grown out of the narrow area of physics; to be mentioned above all is the sonar (found mainly on ships), the non destructive testing of material and medical diagnostics in the human body. An ultrasound microscope is also under development nowadays. Apart from these passive applications there is also a number of active applications where the oscillation energy is used for performing work processes *e.g.* cleaning (ultrasonic baths), welding plastics and treatment of ceramic materials.

2 Production of Ultrasound

The oscillating systems used for the production of ultrasound waves should be suitable for working with high frequencies. This means that all oscillating systems with spring and mass separated, which are used for the production of sound in the hearing range, can not be used to produce ultrasound since we cannot increase their eigen frequencies above a certain value. Instead,

[1] sound particles

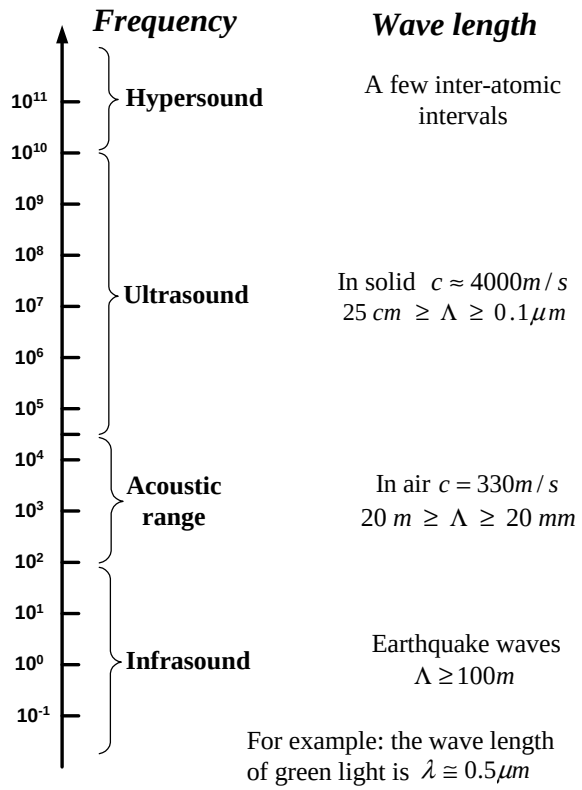

Figure 1: Sonic regions and typical wave length

in the ultrasound range, continua which are able to oscillate are used *e.g.* cavities filled with gas or liquid and solid bodies in the form of plates or bars. In these systems the elasticity of the material plays the role of the spring, and the density together with the geometric properties play the role of the mass *i.e.* the spring and mass are distributed continuously over the oscillator.

One can simulate ultrasonic oscillation with frequencies up to about 100 kHz with purely mechanic oscillators. Examples of *mechanical oscillators* are gas and liquid filled whistles, which work on the same principle as a set of wind instruments. For the production of non-sinusoidal oscillations the hole-siren can be used.

Of far grater importance than mechanical oscillators are the *electromechanical oscillators*. As the name implies electric energy is converted into mechanical oscillation energy. In this group we can find

- Piezoelectric converters.
- Magnetostrictive converters.
- Electrodynamic converters.
- Electrostatic converters.

For the production of higher frequencies magnetostrictive and piezoelectric converters are the most important.

Magnetostrictive converters operate according to the Magnetostriction effect. If a bar from a Ferromagnetic material, mostly from Ni, is magnetized, it's length l will vary slightly by Δl, because the magnetic moment is aligned in the direction of the field and thus affects the deformation of the lattice in the crystal. For a N-bar the relatively big change $\frac{\Delta l}{l}$ amounts to $-2.5 \cdot 10^{-5}$ in a magnetic field of 1 Tesla ($= \frac{1}{4\pi} \cdot 10^5$ m/A). Applying outside forces to an already magnetized N-bar will change the magnetization of the bar, which in turn will cause a voltage surge in the bar. This voltage surge can be measured when an induction coil is wrapped around the bar. Ultrasound can thus be produced and measured using these converters and generators. Magnetostrictive oscillators are mostly suitable to the production of intense sound levels, up to 200 kHz.

The piezoelectric converter is today's most frequently used sonic generator and detector. Compared to the previously discussed technologies, far higher frequencies can be achieved (in the MHz range).

The piezoelectric (or pressure-electric) effect was discovered in 1880 by the Curie brothers. With some crystals, when subjected to pressure or tensile stress in special crystallographic directions, electrical charges are realest on certain crystal surfaces. The produced charges are proportional to the pressure or the stretch applied. The sign of the charges changes if for example a compression alters into a dilation.

The reversed piezoelectric effect was detected soon after in 1881. The same group of crystals , when put between two electrodes with a potential difference, reacts by deformation. The direct piezoelectric effect is used for detec-

tion of ultrasound waves and the reversed piezoelectric effect (Electrostriction) is used for their production. When placing an alternating voltage on two condenser plates, between which the crystal is located, the crystal will oscillate according to the frequency of the alternating voltage. The length variation of the crystal is proportional to the piezoelectric module d and the electrical tension put on it. Because piezoelectric crystals are always anisotropic, d is a tensor and δl also depends on the direction of the applied electric field relative to the crystal axes. Putting a field in parallel to a main axis we obtain

$$\Delta l = d_{jj} U_1$$

(for quartz at low frequencies $d_{11} = 2.3 \cdot 10^{-12} m/V$).

For the receptor, the situation is analogous. The amplitude of the produced change of pressure (by the sonic wave) is proportional to the tension on the condenser plates. However, the proportionality constants are different in this case. The received voltage U_2 is

$$U_2 = h_{ii} \cdot \Delta l$$

where h is the deformation constant (for quartz with low frequencies $h_{11} = 4.9 \cdot 10^9 V/m$). For the same length variation, the voltages U_1 and U_2 thus differ. The proportion $\frac{U_2}{U_1}$ for identical Δl is described by the square of the coupling coefficient k

$$k_{ii} = \sqrt{d_{ii} \cdot h_{ii}}$$

In general the coupling coefficient at low frequencies is smaller than 1. However, with resonance support and low damping the coupling coefficient can become nearly 1.

All crystals that show the piezoelectric effect have several similar qualities, they isolate well and have one or more polar axes. A 180°turn of a polar axis does not result in the same state. The piezoelectric effect now appears in the directions of the polar axes. Crystals on which the piezoelectric effect can be observed are for example: lithium sulfate, tourmaline, zinc blende, Seignette-salt and tartaric acid.

In a quartz crystal, which we want to study as an example, we have three polar axes. Quartz has the chemical formula SiO_2 and it forms hexagonal crystals. Every Si-atom has four positive elementary charges and every O-atom two negative elementary charges. Figure 2(a) shows a structural cell of quartz.

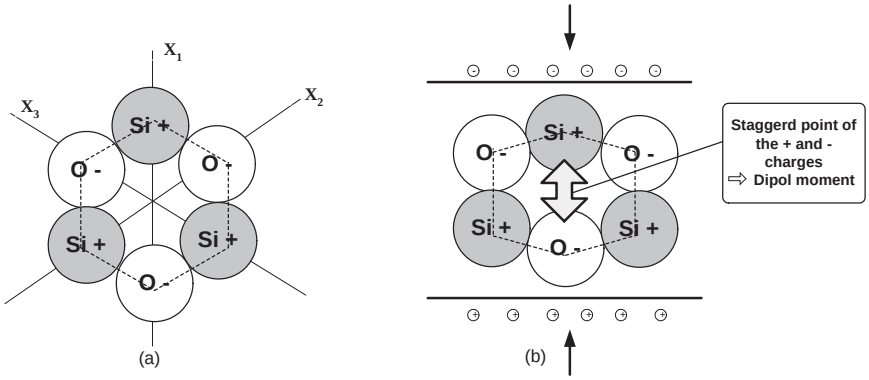

Figure 2: A structural cell of quartz

X_1, X_2 and X_3 are the polar axes. If we now apply, for instance, pressure in direction of the X_1 axis , we reach the situation depicted in figure 2(b) where we see the resulting shift of the atoms and charges on the surface. Qualitatively the same shift can be achieved, if a thrust is applied perpendicularly to the X_1 axis (transverse piezoelectric effect).

Apart from this group of piezoelectric single crystals there is also a series of ferroelectric substances. With theses substances the dipole moment is not only produced by applying pressure or tension, but the electric dipoles already exist within the crystal unit cell, similar to the magnetic moments exist in *e.g.* Fe. For this reason ferroelectric materials have a very high dielectric constant. Applying an electric field at a high temperature aligns these electrical dipole moments in the same way that a magnetic field aligns the magnetic dipole moments in Fe. When a ferromagnetic material in an electric field is cooled down below a certain temperature, known as the Currie-temperature, the aligned electric dipole moments "freeze". The outcome is a permanent macroscopic electric dipole. This modification remains to a large extent, as long as the temperature of the sample does not rise above the Curie-temperature. Above the curie temperature the polarization disappears irreversibly, *i.e.* the sample must be polarized again using an electric field.

In contrast to the example of a quartz piezo-crystal, ferromagnetic sound-converters are not necessarily single crystals. Poly-crystaline materials are sufficient, which can be produced in an inexpensive way and in various forms

(plates, tubes, hollow spheres) by means of sintering. Examples for these materials are: lead zirconate titanate (PZT), barium titanate, lead meta niobium and lithium niobium.

In this experiment a PZT oscillator is used. PZT is a mixture of $PbZrO_3$ and $PbTiO_3$. The piezoelectric constant depends on the mixing proportion of the two materials and can vary within certain limits. The Curie-temperature for the PZT is about 250°C.

3 Propagation of Sound Waves

If pressure or tension[2] is applied to a mechanical continuum *e.g.* a bar of length l and a cross section F it changes its shape (by Δl and ΔF).

For a small perturbation Δl from the rest position we can use Hooke's law to describe the deformation $\frac{\Delta l}{l}$

$$\frac{\Delta l}{l} = \frac{1}{E} \cdot \frac{K}{F}$$

where E is the elasticity module, K the applied force and $\frac{K}{F}$ is the tension. For liquids or gases the compressibility $\mathcal{H}$ is used instead of $\frac{1}{E}$. In systems with static tension the isotermic compressibility is dominating, while in high-frequency dynamic-tension systems the adiabatic compressibility is dominating, since there is not enough time for the heat exchange between the system and its surroundings.

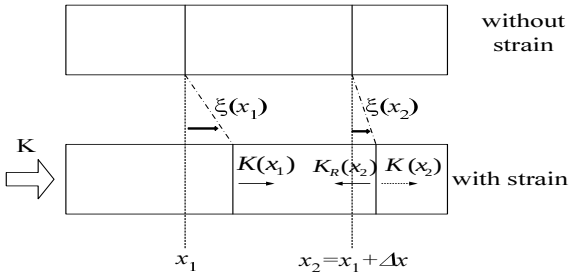

Figure 3:

[2] in the direction of the length

Let's now look at the dynamic equilibrium of a mass element $\rho\,\mathrm{d}F\mathrm{d}x$ in a one dimentional bar or liquid, $\xi(x,t)$ represents a deflection at location x at time t.

A force $K(x_1)$ applied the location x_1 therefore causes a length variation of $\xi(x_1) - \xi(x_2)$ in the volume element according to

$$\frac{\xi(x_1+\Delta x)-\xi(x_1)}{\Delta x} = \frac{\partial \xi}{\partial x}\bigg|_{x=x_1} = \frac{1}{E}\cdot\frac{K(x_1)}{\mathrm{d}F} \tag{1}$$

In an equilibrium, a static (= time independent) force $K(x_1)$ must be balanced by a reaction force $K_R(x_2) = -K(x_2)$ *i.e.* $K(x_1) - K(x_2) = 0$. On the other hand, with dynamic forces we must take the inertia of the volume element $T = \rho\,\mathrm{d}F\mathrm{d}x\frac{\partial^2\xi}{\partial t^2}$ into consideration.

$$K(x_1) - K(x_2) = \frac{\partial K}{\partial x}\mathrm{d}x = \rho\frac{\partial^2\xi}{\partial t^2}\mathrm{d}F\mathrm{d}x \tag{2}$$

By differentiating (1) and placing the result in (2) we obtain

$$\frac{\partial^2\xi}{\partial t^2} - c_s^2\frac{\partial^2\xi}{\partial x^2} = 0, \qquad c_s = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{1}{\mathcal{H}_{ad}\cdot\rho}} \tag{3}$$

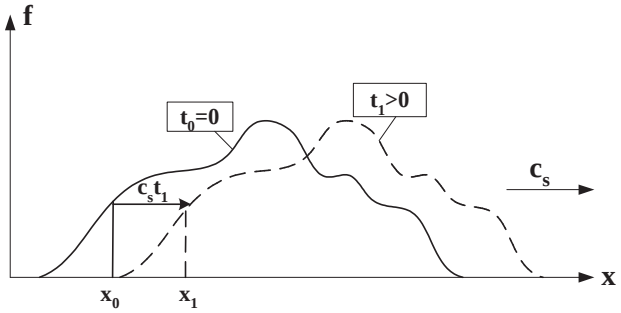

Figure 4: $f(x_1,t_1) = f(x_0,t_0) = f(x_1 - c_s t_1)$

This wave equation is valid for sound waves with small amplitudes $\xi(x,t)$. The most general solution of this equation has the form

$$\xi(x,t) = f(x - c_s t) + g(x + c_s t) \tag{4}$$

Here f and g represent elastic deviations of an arbitrary form, which propagate at the speed of sound c_s. The wave f propagates in the direction of the positive x axis while g propagates along the direction of the negative x axis , see figure 4.

According to our assumptions wave equation (3) applies to longitudinal waves ($\xi \parallel c_s$) in liquids or gas. Transverse waves ($\xi \perp c_s$) on the other hand, cannot be carried forward by liquids or gas. The equation applies also to expansion waves in thin bars ($\Lambda \gg \emptyset =$ diameter of the bar). These expansion waves (Fig.5) are formed due to the lateral contraction during the length variation of the bar.

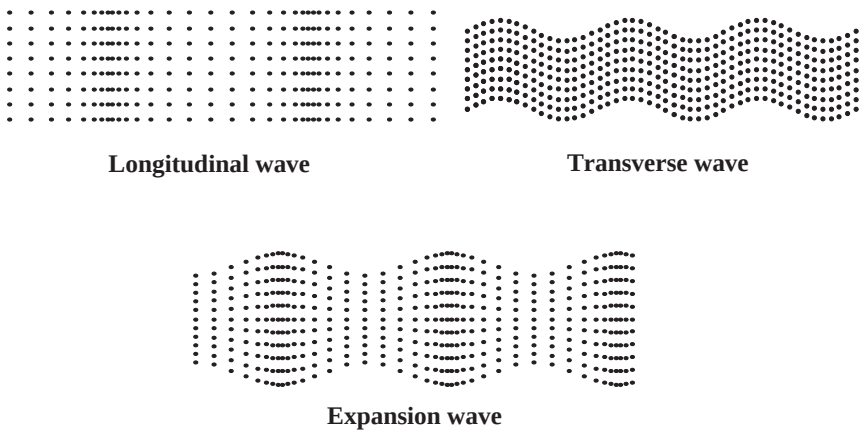

Figure 5: Different kinds of waves

Expansion waves are actually a mixture (linear combination) of longitudinal and transversal waves. When using an infinitely extended medium, these lateral contractions cannot be formed. In this case more complicated elastodynamics equations are required instead of the simple wave equation. Then, the speed of light depends on the lateral contraction coefficient for both longitudinal and transverse waves .The wave velocity, in media with geometrical dimensions of the order of a wavelength, depend on these dimensions or to the wave length Λ respectively.

3.1 Solution of the Wave Equation for Periodical Excitation

Since we try to excite the piezoelectric converter using sines oscillations of frequency Ω, we expect the liquid in which the crystal is placed to have also periodical waves. Hence we make the ansatz

$$\xi(x,t) = \alpha_1 e^{i(\Omega t - Kx)} + \alpha_2 e^{i(\Omega t + Kx)} \tag{5}$$

where $K = \frac{\Omega}{c_s} = \frac{2\pi}{\Lambda}$=wave number.

α_1 is the amplitude of the outgoing wave from the converter and α_2 is the amplitude of the wave reflected from l (see figure 6). $\frac{\alpha_1}{\alpha_2}$ is the reflection factor.

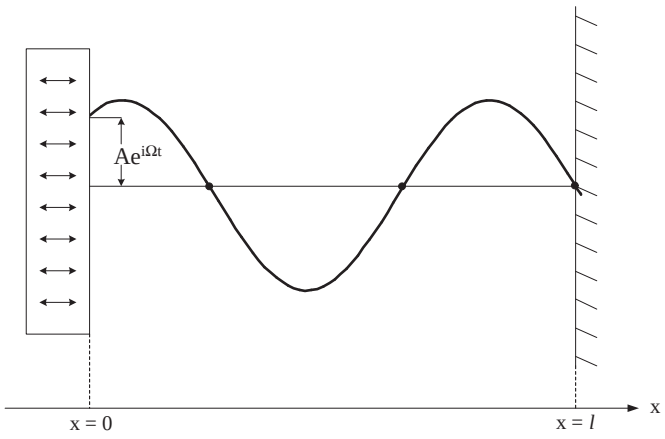

Figure 6: standing wave

The source of the piezoelectric excitation is located at $x = 0$, thus

$$\xi(0,t) = Ae^{i\Omega t} \tag{6}$$

We will assume total reflection from the barrier at location l, this would be the case for an infinitely stiff barrier.

$$\text{hence} \qquad \xi(l,t) = 0 \tag{7}$$

From the two boundary conditions above and equation (5) we obtain a standing wave

$$\xi(x,t) = A \cdot \frac{\sin(K(l-x))}{\sin(Kl)} \cdot e^{i\Omega t} \tag{8}$$

If we would have no reflection at all, *i.e.* perfect transmission or absorption at $x = l$, we would get $\alpha_2 = 0$ and would have only traveling waves. The reality in our experiment is somewhere in the between. Consequently we should formulate the boundary conditions differently. If $\xi_1(x,t)$ is a wave in a adjacent medium, we require that at $x = l$ the deflections and forces are equal.

$$\text{so} \qquad \xi(l,t) = \xi_1(l,t), \qquad E\frac{\partial \xi}{\partial x} = E_1\frac{\partial \xi_1}{\partial x} \tag{9}$$

4 Diffraction of Light at Ultrasound Waves

In 1932 Debye and Sears discovered in the USA and Lucas and Biquard discovered in France that transparent media diffract light when an ultrasound wave is sent trough them. This effect is a consequence of a periodical variation of the refractive index, which in turn is a consequence of a local periodical pressure change caused by the ultrasound wave. Figure 7 shows the experimental setup which allows the observation of diffraction of light by an ultrasonic wave.

A thin slit, lit up by the lamp La serves as a source of light. The lens L_1 is placed in the distance of its focal length from the gap and thus produces a broad beam of parallel light. The light then penetrates a transparent medium (gas, liquid or solid) in which an ultrasonic wave transducer Q, located perpendicularly to the direction of the incidence light beam, produces elastic waves. For experiments with liquids or gases a container with plane-parallel glass walls is required. The second lens L_2 projects a real image S' of the gap S on a screen. If the ultrasound wave is excited, several orders of the spectrum of the lamp La can be seen on both sides of S'. By introducing a filter F into the path of the light beam we produce monochromatic light and obtain only one interference strip of each order near S'.

To understand this phenomenon we must assume that the local periodical

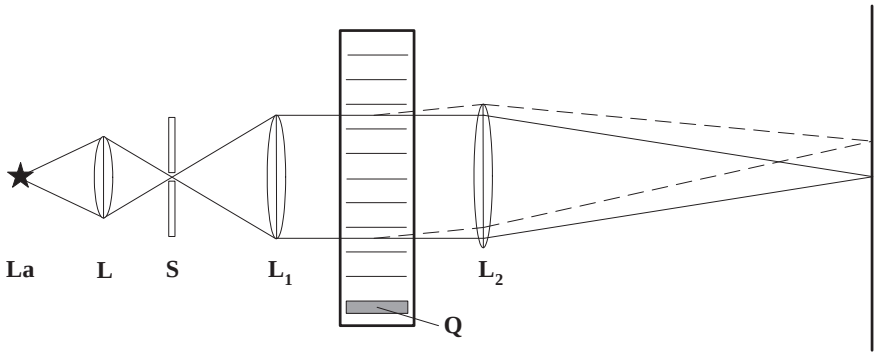

Figure 7: The experimental setup which allowed the observation of diffraction of light by an ultrasonic wave.

pressure changes of the elastic wave create local changes in the refraction index of the medium. The surfaces of equal phase $S(x, y, z)$ of the light wave are then no longer plains ($S(x, y, z) = k_0 x$ for light propagation in the x-direction.) but they become a sine function with the same period as the ultrasonic wave. This arises from the fact that in an area with a higher refractive index n the light wave travel slower than in an area with a lower refractive index ($c = \frac{c_0}{n}$).

When we describe an elastic wave by the local density $\rho(y, t)$ instead of by the deflection $\xi(y, t)$ we can easily convince our selfs that we obtain

$$\rho(y, t) = \rho_0 + \rho_0 \cdot \frac{\partial \xi(y, t)}{\partial y} \tag{10}$$

For the refractive index $n(y, t)$ we expect accordingly

$$n(y, t) = n_0 + \Delta n(y, t) \tag{11}$$

The relation between the refractive index, the dielectric constant ε and the density ρ is described by the well-known Clausius-Mossotti equation from the theory of the dielectric constants.

$$\frac{1}{\rho} \cdot \frac{\varepsilon - 1}{\varepsilon + 2} = \frac{1}{3\varepsilon_0} \cdot \frac{N_L}{M} \cdot \alpha = \text{const.} \tag{12}$$

Where N_L is the Loschmidt number, M the molecular weight and α the polarisability of the atoms or molecules. The polarisability can be regarded as independent of density. At optical frequencies in a non-magnetic medium ($\mu = 1$) we have $\varepsilon = n^2$. We therefore obtain from equation (12)

$$\Delta\rho = \frac{3\rho\Delta\varepsilon}{(\varepsilon+2)(\varepsilon-1)} \tag{13}$$

combining (10) and (11) with $\Delta\varepsilon = 2n \cdot \Delta n$ we find

$$\Delta n(y,t) = \frac{(n_0^2+2)(n_0^2-1)}{6n_0} \cdot \frac{\partial\xi(y,t)}{\partial y} \tag{14}$$

The light wave $E(x,y,z,t)$ (or the field vectors D,H and B) obeys a wave equation in the same way the ultrasonic wave does. The wave equations for light can be obtained from Maxwell's equations,

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{15}$$

$$\nabla \times H = \frac{\partial D}{\partial t} \tag{16}$$

$$\nabla \cdot D = 0 \tag{17}$$

$$B = \mu_0 \mu H \tag{18}$$

$$D = \varepsilon_0 \varepsilon E \tag{19}$$

The exact wave equation for an inhomogeneous medium $\varepsilon = \varepsilon(r,t)$ is very complicated. For a homogeneous medium $\varepsilon = const.$ we can simply obtain the wave equation by deriving equation (16) by time and by using also equations (18) and (15).

$$\frac{\partial^2 \vec{D}}{\partial t^2} = \nabla \times \frac{\partial \vec{H}}{\partial t} = \nabla \times \left(\frac{1}{\mu_0 \mu} \cdot \frac{\partial \vec{B}}{\partial t} \right) =$$
$$-\frac{1}{\mu_0 \mu} \nabla \times \left(\nabla \times \vec{E} \right) = -\frac{1}{\mu \mu_0} \left[\Delta \vec{E} - \nabla (\nabla \cdot \vec{E}) \right]$$

$$\Rightarrow \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{1}{\varepsilon \varepsilon_0 \mu \mu_0} \cdot \Delta \vec{E} = 0. \tag{20}$$

Analogously to the sound wave t term $(\varepsilon \varepsilon_0 \mu \mu_0)^{-1/2}$ is equivalent to the propagation speed of the phase c. Thus in vacuum $c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ and in a medium $c = \frac{c_0}{n}$ $(n = \sqrt{\varepsilon \mu})$. The abbreviation $\vec{E}$ stands for $\{E_1, E_2, E_3\}$.

The separation anzatz $\vec{E} = \vec{u}(x, y, z) e^{i\omega t}$ leads to a time independent wave equation.

$$\Delta \vec{u} + k_0^2 n^2 \vec{u} = 0 \tag{21}$$

Where $k_o = \frac{\omega}{c_0} = \frac{2\pi}{\lambda_0}$. Simple solutions of this equation are, *e.g.* plain waves for $n = \text{const.}$

$$\vec{E}(x, y, z, t) = \vec{A} e^{i\omega t - kx} \qquad k = k_0 n$$

where the modulus of $\vec{A}$ corresponds to the amplitude and the direction to the polarization of the light beam.

From analogy to the plain wave we obtain

$$u_j = A_j(x, y, z) e^{i k_0 S(x,y,z)} \tag{22}$$

From placing this result in the time independent wave equation (21) we see, that in the extreme case of geometrical optics *i.e.* $k_0 \to \infty$, the equation is satisfied if

$$\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 = n^2 \tag{23}$$

$$\frac{1}{A_j}\nabla A_j \cdot \nabla S = -\frac{1}{2} \cdot \Delta S \tag{24}$$

$S(x, y, z)$ is called the *eikonal*. Surfaces that obey the equation $S(x, y, z) =$ const. are constant phase surfaces of the light wave and grad(S) then indicate the direction of the light wave.

Under the condition that the temporal and local changes of the refractive index $n(y, t)$ are small compared to the temporal and local changes of the source of light, *i.e.* $\Omega_{\text{sound}} \ll \omega_{\text{light}}$ and $\Lambda_{\text{sound}} \gg \lambda_{\text{light}}$ we can use an approximation of a slowly varying function $n(y, t)$ instead of the constant refractive index

Exact solutions of the eikonal equation (23) for our problem $n = n_0 + \Delta n(y, t)$ might still be very difficult to calculate. However, the equation tells us how one can obtain an image of the surfaces of constant phase using a simple graphical method. If we have one known surface of a constant phase (*e.g.* boundary condition) we can use it to construct a group of "parallel surfaces" with constant infinitesimal gaps $|\nabla S| = n(y)$. The construction corresponds exactly to Huygen's principle where each point of the wave field is a starting point of a spherical wave. Draw such planes, or curves of constant phase for an ultrasound wave, for a fixed point in time, *i.e.* $n = n_0 + \Delta n_0 \sin(K\Lambda)$ with $\Delta n_0 < n_0$.

5 Intensity distribution of the interference pattern

To calculate the interference pattern on the screen we need to apply Huygen's principle once more. We consider every point on the exit plain $x = a$ after the ultrasound field as a starting point of a spherical wave with an amplitude and phase of the light wave in this point, see figure 8. Since the exact solution of the eikonal S in the domain of the ultrasonic wave is very difficult, we would like to use an approximation in which the light rays pass in parallel through the ultrasound field and are modulated only in the local and temporal phase. This corresponds to a linear approximation of the eikonal and amplitude.

$$S = n(y, t) \cdot x \quad \text{and} \quad A = A_0 \tag{25}$$

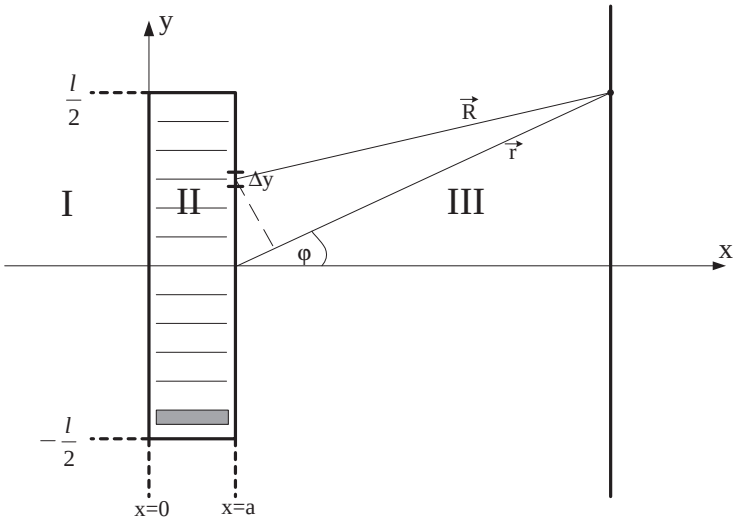

Figure 8: Diffraction

In this approximation all the polarization directions are identical, so we can abandon the vector notation.

In region I we have parallel light so

$$E_I = A_0 e^{i(\omega t - k_0 x)} \tag{26}$$

in region II, from our assumption

$$E_{II} = A_0 e^{i(\omega t - k_0 n(y,t) \cdot x)} \tag{27}$$

we consider the wave in region III as a superposition of spherical waves ΔE_{III}

$$\Delta E_{III} = E_{II} \cdot \frac{1}{R} e^{i(\omega t - k_0 R)} \cdot \Delta y \tag{28}$$

For a big R or observations around small angles φ we have

$$R \cong r - y \sin(\varphi) \tag{29}$$

We assume that the ultrasonic wave propagates in the y-direction, thus

$$n(y,t) = n_0 + \Delta n_0 \sin(\Omega t - Ky) \tag{30}$$

Together with (27), (28) and (29) we get

$$E_{III} = \frac{A_0}{r} e^{i(\omega t - k_0 \cdot n_0 \cdot a - k_0 \cdot r)} \int_{-\frac{l}{2}}^{+\frac{l}{2}} e^{-i(k_0 \cdot \Delta n_0 \cdot a \sin(\Omega t - Ky) - k_0 \cdot y \sin(\varphi)} \mathrm{d}y$$

We can exactly expand the $e^{i \cdot \sin}$−function in the integrand into a Fourier series. We know

$$e^{i \cdot a \sin(\alpha)} = \sum_{\nu=-\infty}^{\nu=+\infty} J_\nu(a) e^{i\nu\alpha}. \tag{31}$$

J_ν are the Bessel functions of integer order. After exchange between the integral and the sum, the integral turns into

$$\sum_{\nu=-\infty}^{\nu=+\infty} \int_{-\frac{l}{2}}^{\frac{l}{2}} (-1)^\nu J_\nu(k_0 \Delta n a) e^{i\nu\Omega t} e^{i(k_0 \sin(\varphi) - \nu K)y} \mathrm{d}y$$

The integral is now simple to calculate and we obtain the following for the light wave of region III

$$\begin{aligned} E_{III} \quad = \quad & \frac{A_0 l}{r} e^{-i(k_o n_0 a + kr)} \sum_{\nu=-\infty}^{\nu=+\infty} (-1)^\nu J_\nu(k_0 \Delta n_0 a) \cdot \\ & \frac{\sin[\frac{l}{2}(k_0 \sin(\varphi) - \nu K)]}{\frac{l}{2}(k_0 \sin(\varphi) - \nu K)} e^{i(\omega + \nu\Omega)t} \end{aligned} \tag{32}$$

The intensity is proportional to the temporal average of the square of the amplitude or in complex notation

$$I = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau (E \cdot E^*) \mathrm{d}t \tag{33}$$

Calculating the last equation with E_{III}, we obtain a double sum of the form

$$I \cong \lim_{\tau\to\infty} \sum_{\nu\mu} F_\nu F_\mu \cdot \frac{1}{\tau} \int_0^\tau e^{i(\nu-\mu)\Omega t} \mathrm{d}t = \sum_{\nu\mu} F_\nu F_\mu \delta_{\nu\mu}$$

The intensity is accordingly

$$I = \frac{A_0^2 l^2}{r^2} \sum_{\nu=-\infty}^{+\infty} J_\nu^2(k_0 \Delta n_0 a) \cdot \left[\frac{\sin[\frac{l}{2}(k_0 \sin(\varphi) - \nu K)]}{\frac{l}{2}(k_0 \sin(\varphi) - \nu K)} \right]^2 \tag{34}$$

The function $\frac{\sin^2 x}{x^2}$ (see figure 9) is maximal for $x = 0$ and drops rapidly from each side of $x = 0$. Thus we have strong intensity (interference fringes) around the angles φ_ν in our wave field ($r \gg l$).

$$k_0 \sin \varphi_\nu - \nu K = 0 \tag{35}$$

For small angles we obtain

$$\varphi_\nu \cong \nu \frac{k}{k_0} = \nu \frac{\lambda_0}{\Lambda} \tag{36}$$

Where ν indicates the order of the diffraction spectrum.

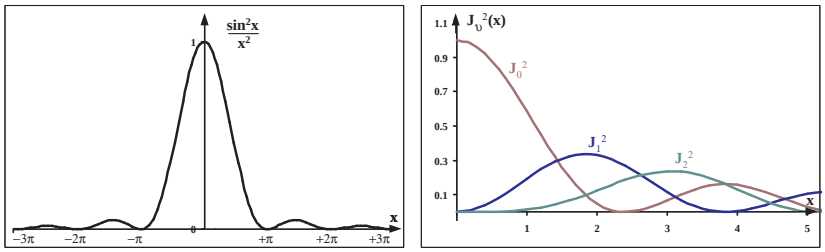

Figure 9:

We find that the distance of the interference fringes from each other is the same as with a diffraction grating with a lattice constant of the length of the ultrasonic wave.

The intensity of the interference fringes of the ν-th order is proportional to the square of the Bessel function J_ν

$$I_\nu \sim J_\nu^2(k_0 \Delta n_0 a) \tag{37}$$

An appropriate choice of light frequency (k_0), width a or intensity (Δn_0) of the ultrasound wave can thus cause the extenuation of specific orders

Next we apply the Bessel functions sum rule

$$\sum_{\nu=-\infty}^{+\infty} J_\nu^2(x) = 1$$

this means that the sum of intensities over all orders is a constant, which corresponds to the intensity of the incident light.

Another special feature resulting from the movement of our phase grating can be explained by wave equation (32) for E_{III}. The light observed in the ν-th order does not have the original frequency ω but is shifted by ν-times the frequency of the ultrasonic wave, see figure 10.

$$\omega_\nu = \omega + \nu\Omega$$

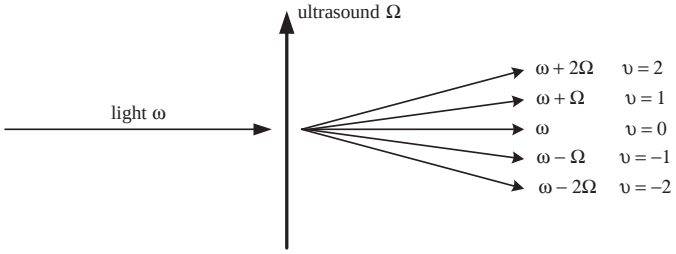

Figure 10:

We can explain this frequency shift with the Doppler effect or with quantum mechanics in an easy manner. In quantum mechanic the sound- and light-(electromagnetic-) fields are quantized, *i.e.* the energy of a field of frequency ω can only change between discrete values ($\hbar\omega$). A sonic-field of frequency

Ω can thus be described through quasi-particles (the so-called phonons) which has an energy of $E = \hbar\Omega$ per particle. The discrete energy change implies now that we can either produce or annihilate (emit or absorb) an entire particle with an energy of $\hbar\Omega$ in the sonic-field. The same explanation applies for the electromagnetic field as well. In the later case the particles are called photons and have the energy $\hbar\omega$. The momentum $\vec{p}$ of these mass-less particles is

$$\vec{p} = \hbar\vec{k} \tag{38}$$

If we look at the diffraction of light from the ultrasonic wave as scattering of photons by phonons, the particles must fulfill the energy and momentum conservation laws. Here we should pay attention, though, that for each photon only whole numbers (ν) of phonons can be absorbed (+) or emitted (-).

$$\vec{k_1} = \vec{k_0} + \nu\vec{K} \qquad \text{momentum conservation} \tag{39}$$

$$\omega_1 = \omega_0 + \nu\Omega \qquad \text{energy conservation} \tag{40}$$

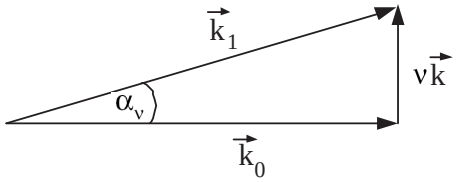

Figure 11:

From the momentum conservation law we attain

$$|k_1|^2 + |k_0|^2 - 2|k_0||k_1|\cos\alpha_\nu = \nu^2 K^2 \tag{41}$$

$|k_1| = \frac{\omega_1}{c_0} n = \frac{2\pi}{\lambda_0} n$ =wave number of the incident light in a liquid with refractive index n. (λ_0 und c_0 are values in a vacuum)

Here $\nu\Omega \ll \omega$, which means $|k_1| \cong |k_0|$ and thus from equation (41) we obtain

$$2\sin\frac{\alpha_\nu}{2} = \frac{K}{k_1} \qquad \alpha_\nu \cong \nu\frac{\lambda_0}{n\Lambda}$$

The angle α_ν is the angle between incident and scattered photons in the liquid. For perpendicular incidence of the photons on the liquid we observe the scattered photons in air, $n_L = 1$, at the angle φ_ν (refraction law).

$$\frac{\sin\varphi_\nu}{\sin\alpha_\nu} \cong \frac{\varphi_\nu}{\alpha_\nu} \cong n$$

$$\text{so} \qquad \varphi_\nu = \nu\frac{\lambda_0}{\Lambda}$$

In the ν-th order of the diffraction spectrum we see now the photons that absorbed ν phonons, or emitted ν phonons for $\nu < 0$ respectively; correspondingly, due to energy conservation, the frequency in the ν-th order is shifted by $\nu\Omega$.

6 Apparatus

6.1 Diffraction from Sonic-Waves

Today we use red He-Ne laser with a wavelength of 6328 Å as a light source. Since the laser beam is narrow and can be amid to arrive perpendicularly to the cuvette the slit S as well as lenses L,L_1 are not required (see figure 7). The laser width can thus be referred to as S. The laser beam coming out of the container should then be focused on the screen using lens L_2 till a sharp image is obtained.

6.2 The Striae method

In contrast to diffraction, in which the slit S is mapped on the screen, it is possible to detect the standing waves in the Striae method directly from their image on the screen through the following mechanism; The lens L_4 focuses the laser S on a bean stop B (figure 12), so in the absence of ultrasound waves the screen remains completely dark. Now after generating a standing ultrasound wave in the cuvette, the diffracted light can pass the shutter and be seen on the screen.

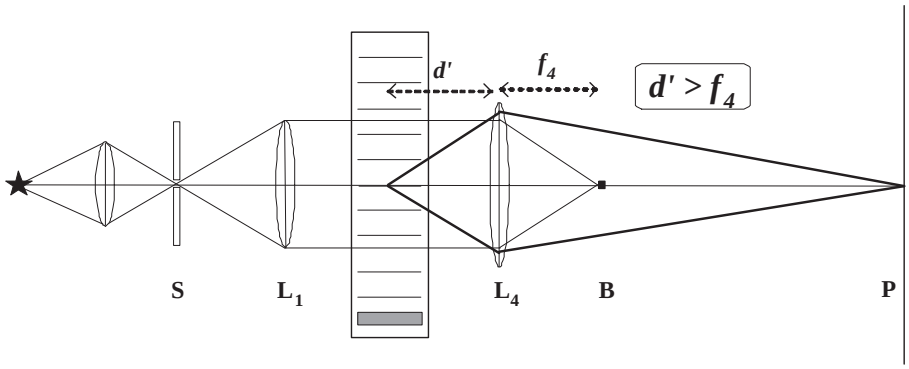

Figure 12: Striae method

7 Problems

7.1 Experimental

a) Photograph the diffraction pattern of different ultrasonic frequencies. Calculate the wave length of the elastic wave from the distance of the interference stripes and determine the sound propagation velocity in xylene.

b) Photograph the ultrasonic wave using the striae method and calculate the sound propagation velocity.

7.2 General Problems

1. Calculate the ultrasonic wave $\xi(x,t)$ in liquid for partial reflection and show, with the help of Eq. (9), that the reflection factor for the amplitude on the boundary layer between two media, 1 and 2, is given by

$$R = \frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}$$

   (where ρ is density and c is the sound velocity) .

2. How big is the maximal change (=amplitude) Δn_0 of the refraction index of a traveling sine wave if the ultrasonic converter emits a power-density of W

$$\begin{aligned} n_{xylene} &= 1.497 \\ \rho &= 0.86 \frac{\text{gr}}{\text{cm}^3} \\ W &= 1 \frac{\text{Watt}}{\text{cm}^2} \end{aligned}$$

Hint: calculate the power-density of the sonic wave as a function of the amplitude A_0

$$\xi(y,t) = A_0 \sin(\Omega t - Ky)$$

3. Use Huygen's principle to draw surfaces of constant phase of the light wave as it passes through the liquid (at a certain time, $t = \text{const}$).

4. Which frequencies can be seen in the different orders of diffraction for standing waves?
   Hint: Treat the standing waves as two counter propagating traveling waves and apply the summarized results from figure 10. Assume that the diffraction occurs first at one of the counter propagating waves and then after at the other one.

5. An appropriate choice of light frequency or ultrasound intensity can, in the case of traveling waves, cause a loss of the zeroth order from the screen. Is this also possible for standing waves?