3. Foundations of Scalar Diffraction Theory

Introduction to Fourier Optics, Chapter 3, J. Goodman

Maxwell's equations

in the absence of free charge,

$$\vec{\nabla}\cdot\varepsilon\vec{E}=0$$

$$\vec{\nabla}\cdot\mu\vec{H}=0$$

$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}$$

(See Appendix.) ▶

Scalar wave equation

in a linear, isotropic, homogeneous, and nondispersive medium,

$$\nabla^2 u(P,t)-\frac{n^2}{c^2}\frac{\partial^2 u(P,t)}{\partial t^2}=0,$$

where,

$$n=\sqrt{\frac{\varepsilon}{\varepsilon_o}},\quad c=\frac{1}{\sqrt{\mu_o\varepsilon_o}}$$

If the medium is inhomogeneous with $\varepsilon(P)$ that depends on position *P*,

$$\nabla^2\vec{E}+2\vec{\nabla}\left(\vec{E}\cdot\vec{\nabla}\ln\ n\right)-\frac{n^2}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0$$

Helmholtz Equation

For a monochromatic wave,

$$u(P,t) = A(P)\cos\left[2\pi\nu t + \phi(P)\right]$$

$$u(P,t) = \mathrm{Re}\left\{U(P)\exp\left(-j2\pi\nu t\right)\right\}$$ **Complex notation**

The complex function of position (called a *phasor*)

$$U(P) = A(P)\exp\left[-j\phi(P)\right]$$

$$\left(\nabla^2 + k^2\right)U = 0$$

where,

$$\boldsymbol{k = 2\pi n \frac{\nu}{c} = \frac{2\pi}{\lambda}}$$

Helmholtz Equation

Helmholtz equation

Helmholtz Equation $\Delta w + \lambda w = -\Phi(\mathbf{x})$

Many problems related to steady-state oscillations (mechanical, acoustical, thermal, electromagnetic) lead to the two-dimensional Helmholtz equation.

The two-dimensional Helmholtz equation has the following form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \lambda w = -\Phi(x, y) \quad \text{in the Cartesian coordinate system,}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \lambda w = -\Phi(r, \varphi) \quad \text{in the polar coordinate system.}$$

Helmholtz sought to synthesize Maxwell's electromagnetic theory of light with the central force theorem.
To accomplish this, he formulated an electrodynamic theory of action at a distance in which electric and magnetic forces were propagated instantaneously.

Helmholtz, Hermann von (1821-1894)

In 3 dimension,

Cartesian

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho} \right) + \mathbf{e}_3 \frac{1}{\rho} \left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi} \right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi} \right]$$

$$+ \mathbf{e}_2 \left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r A_3) \right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_2) - \frac{\partial A_1}{\partial\theta} \right]$$

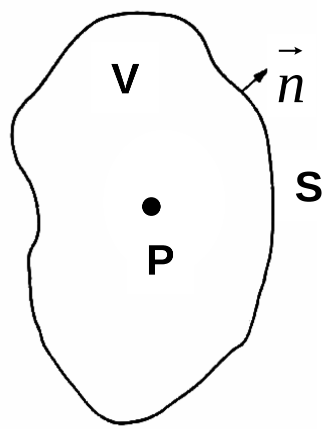
$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

[Note that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$]

Green's Theorem

Let $U(P)$ and $G(P)$ be any two complex-valued functions of position, and let S be a closed surface surrounding a volume V.

If U, G, and their first and second partial derivatives are single-valued and continuous within and on S, then we have



$$\iiint_{v} \left(U\nabla^2 G - G\nabla^2 U \right) d\upsilon = \iint_{s} \left(U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \right) ds$$

Where $\dfrac{\partial}{\partial n}$ signifies a partial derivative in the outward normal direction at each point on S.

Green's Function

Consider an inhomogeneous linear differential equation of,

$$a_2(x)\frac{d^2U}{dx^2}+a_1(x)\frac{dU}{dx}+a_0(x)U=V(x)$$

where *V(x)* is a driving function and *U(x)* satisfies a known set of boundary conditions.

If *G(x)* is the solution when *V(x)* is replaced by the δ*(x-x'),* such that

$$a_2(x)\frac{d^2G(x-x')}{dx^2}+a_1(x)\frac{dG(x-x')}{dx}+a_0(x)G(x-x')=\delta(x-x')$$

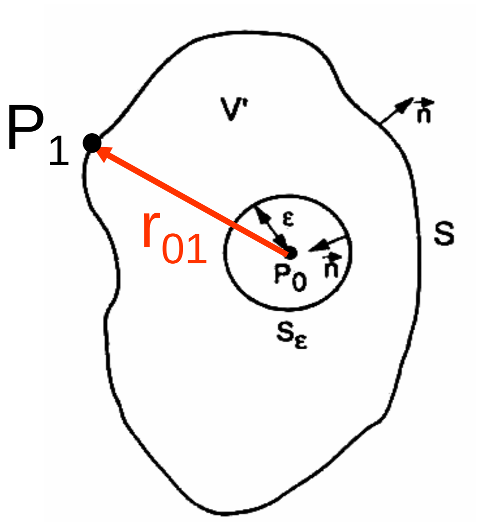
Then, the general solution *U(x)* can be expressed through a convolution integral,

$$U(x)=\int G(x-x')V(x')dx'$$

Since the system is linear shift-invariant.

"Green's function" : impulse response of the system

Kirchhoff's Green's function



V' **: the volume lying between S and S_ε**

S' = S + S_ε

Kirchhoff's Green's function :

A unit amplitude spherical wave expanding about the point P_o (called "free-space Green's function")

$$G(P_1) = \frac{\exp(jkr_{01})}{r_{01}}$$

Within the volume V', the disturbance G satisfies

$$\left(\nabla^2 + k^2\right) G = \mathbf{0}$$

The general Green's function G

$$\nabla^2 G(x, y, z; x_0, y_0, z_0) + k^2 G = -\delta(x - x_0, y - y_0, z - z_0) = -\delta(\underline{r} - \underline{r_0}). \qquad (1.4)$$

δ satisfies the normalization condition

$$\iiint_V \delta(x - x_0, y - y_0, z - z_0)\, dx\, dy\, dz = \iiint_V \delta(\underline{r} - \underline{r_0})\, dv = 1$$

$$\boxed{G = \frac{1}{4\pi} \frac{\exp(-jkr_{01})}{r_{01}},} \qquad (1.7)$$

$$r_{01} = |\underline{r_0} - \underline{r}| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

(1) By direct differentiation, $\nabla^2 G + k^2 G$ is clearly zero everywhere in any homogeneous medium except at $\underline{r} \approx \underline{r_0}$. Therefore, Eq. (1.4) is satisfied within the volume V', which is V minus V_1 (with boundary S_1) of a small sphere with radius r_ε enclosing $\underline{r_0}$ in the limit as r_ε approaches zero.

(2) In order to find out the behavior of G near $\underline{r_0}$, we note that $|G| \to \infty$ as $r_{01} \to 0$. If we perform the volume integration of the left hand side of Eq. (1.4) over the volume V_1,

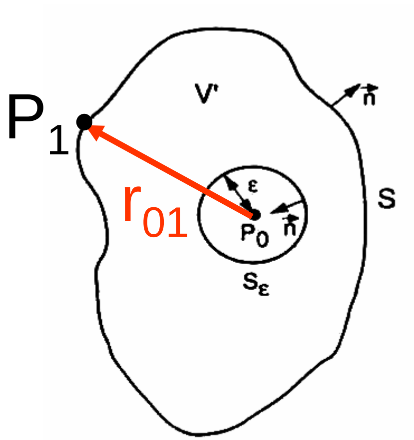
$$\lim_{r_\varepsilon \to 0} \iiint_{V_1} [\nabla \cdot \nabla G + k^2 G]\, dv = \iint_{S_1} \nabla G \cdot \underline{n}\, ds = \lim_{r_\varepsilon \to 0} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left[-\frac{e^{-jkr_\varepsilon}}{4\pi\, r_\varepsilon^2} \right] r_\varepsilon^2 \sin\theta\, d\theta\, d\phi = -1.$$

Thus, using this Green's function, the volume integration of the left hand side of Eq. (1.4) yields the same result as the volume integration of the δ function. In short, the G given in Eq. (1.7) satisfies Eq. (1.4) for any homogeneous medium.

Integral Theorem of Helmholtz and Kirchhoff

Green's Theorem

$$\iiint_{V}\left(U\nabla^2 G - G\nabla^2 U\right)d\upsilon = \iint_{S}\left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)ds$$



V' : the volume lying between S and S_ε

$S' = S + S_\varepsilon$

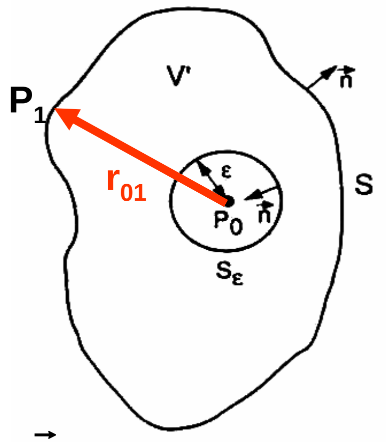
Kirchhoff's Green's function

$$G(P_1) = \frac{\exp\left(jkr_{01}\right)}{r_{01}}$$

$$\iiint_{V'}\left(U\nabla^2 G - G\nabla^2 U\right)\ d\upsilon = -\iiint_{V'}\left(\cancel{Uk^2G - Gk^2U}\right)\ d\upsilon = \mathbf{0} = \iint_{S'}\left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)\ ds$$

$$\iint_{S'}\left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)ds = \mathbf{0} \ \Rightarrow\ -\iint_{S\varepsilon}\left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)ds = \iint_{S}\left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)ds$$

Note that for a general point P_1 on S', we have



$$G(P_1) = \frac{\exp(jkr_{01})}{r_{01}}$$

$$\frac{\partial G(P_1)}{\partial n} = \cos(\vec{n}, \vec{r}_{01})(jk - \frac{1}{r_{01}})\frac{\exp(jkr_{01})}{r_{01}}$$

where $\cos(\vec{n}, \vec{r}_{01})$ *represents the cosine of the angle between* $\vec{n}$ *and* $\vec{r}_{01}$*.*

For a particular case of P_1 on S_ε, $\cos(\vec{n}, \vec{r}_{01}) = -1$ *and*

$$G(P_1) = \frac{\exp(jk\varepsilon)}{\varepsilon} \qquad \frac{\partial G(P_1)}{\partial n} = \frac{\exp(jk\varepsilon)}{\varepsilon}(\frac{1}{\varepsilon} - jk)$$

Letting ε -> 0,

$$\lim_{\varepsilon \to \infty} \iint_{S\varepsilon} \left(U \frac{\partial G}{\partial n} - G \frac{\partial U}{\partial n} \right) ds = \lim_{\varepsilon \to \infty} 4\pi\varepsilon^2 \left[U(P_o) \frac{\exp(jk\varepsilon)}{\varepsilon} \left(\frac{1}{\varepsilon} - jk \right) - \frac{\partial U(P_o)}{\partial n} \frac{\exp(jk\varepsilon)}{\varepsilon} \right] = 4\pi U(P_o)$$

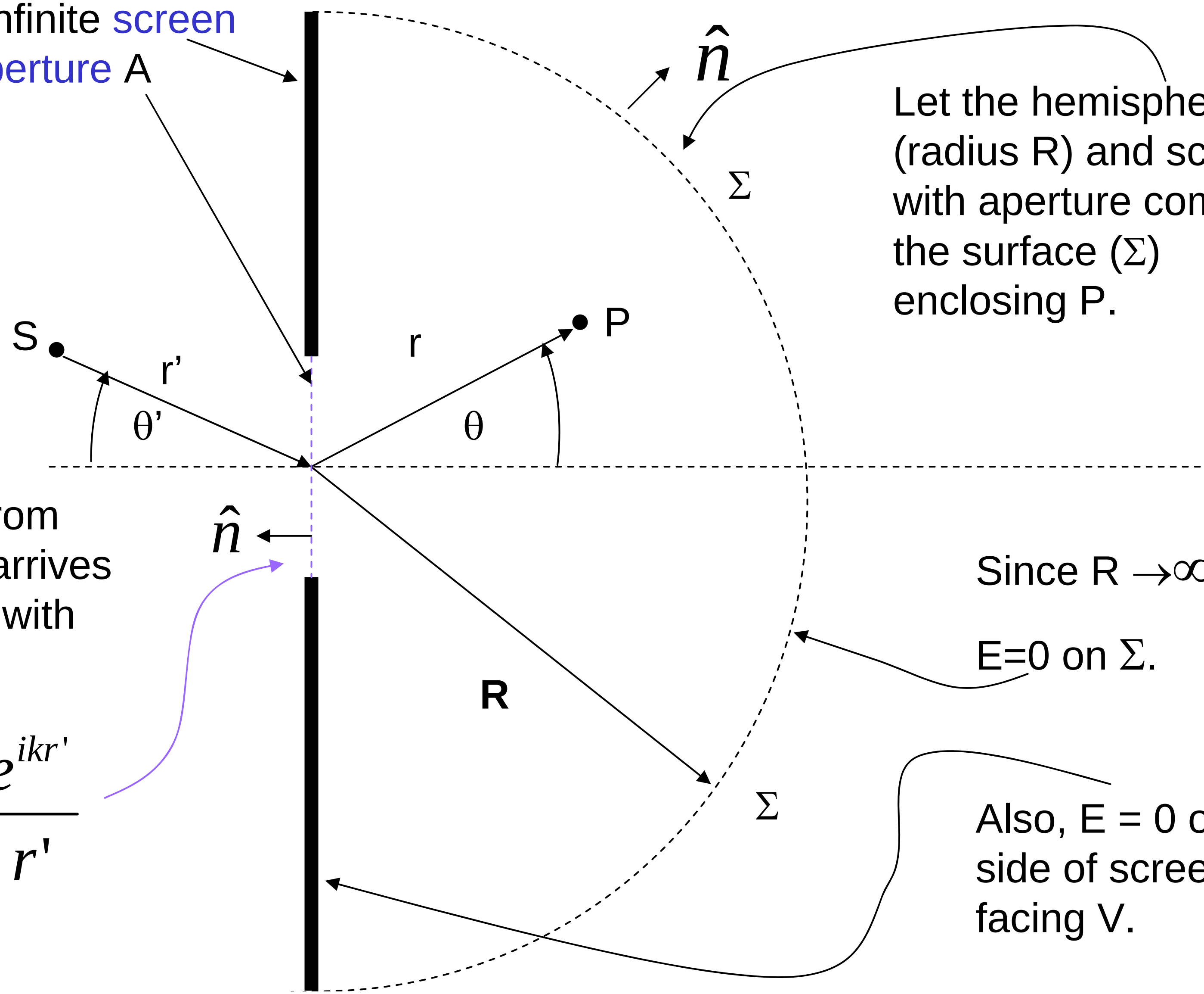
$$U(P_0) = \frac{1}{4\pi} \iint_S \left\{ \frac{\partial U}{\partial n} \left[\frac{\exp(jkr_{01})}{r_{01}} \right] - U \frac{\partial}{\partial n} \left[\frac{\exp(jkr_{01})}{r_{01}} \right] \right\} ds$$

Integral theorem of Helmholz and Kirchhoff

Kirchhoff's Formulation of Diffraction

(A planar screen with a single aperture)

Have infinite screen with aperture A



Let the hemisphere (radius R) and screen with aperture comprise the surface (Σ) enclosing P.

Radiation from source, S, arrives at aperture with amplitude

$$E = E_o \frac{e^{ikr'}}{r'}$$

Since R $\rightarrow\infty$

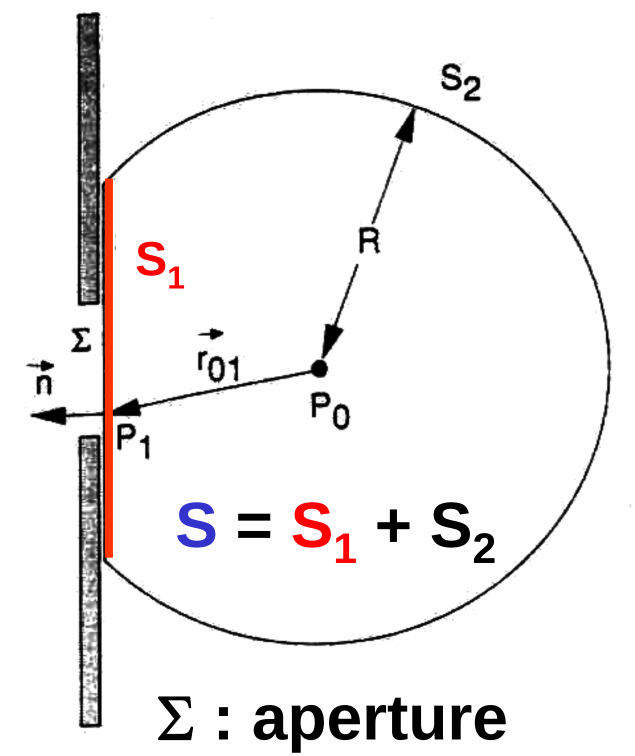
E=0 on Σ.

Also, E = 0 on side of screen facing V.

Kirchhoff's Formulation of Diffraction

From the integral theorem of Helmholz and Kirchhoff

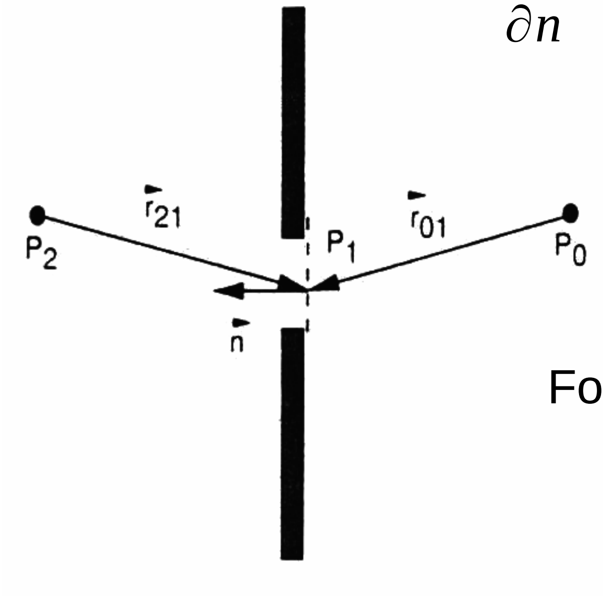
$$U(P_0) = \frac{1}{4\pi} \iint_{S_1+S_2} \left(\frac{\partial U}{\partial n} G - U \frac{\partial G}{\partial n} \right) ds, \quad where \quad G = \frac{\exp(jkr_{01})}{r_{01}}.$$



Kirchhoff boundary conditions

- On Σ, U and $\partial U/\partial n$ are as if there were no screen.
- On S_1 except $\Sigma, U = 0$ and $\partial U/\partial n = 0$.

Fresnel-Kirchhoff's Diffraction Formula (I)



$$\frac{\partial G(P_1)}{\partial n} = \cos(\vec{n}, \vec{r}_{01})\left(jk - \frac{1}{r_{01}}\right)\frac{\exp(jkr_{01})}{r_{01}}$$

$$\approx jk \cos(\vec{n}, \vec{r}_{01})\frac{\exp(jkr_{01})}{r_{01}} \quad \left(\textit{for} \;\; r_{01} >> \lambda \rightarrow k >> \frac{1}{r_{01}} \right)$$

For an illumination consisting of a single point source at P_2 :

$$U(P_1) = \frac{A\exp(jkr_{21})}{r_{21}}$$

$$\boxed{U(P_0) = \frac{A}{j\lambda}\iint_{\Sigma}\frac{\exp[jk(r_{21}+r_{01})]}{r_{21}r_{01}}\left[\frac{\cos(\vec{n},\vec{r}_{01}) - \cos(\vec{n},\vec{r}_{21})}{2}\right]ds}$$

Fresnel-Kirchhoff's Diffraction Formula (II)

$$U(P_0) = \iint_{\Sigma} U'(P_1) \frac{\exp(jkr_{01})}{r_{01}} ds$$

Huygens-Fresnel

$$\psi(P) = A \frac{\exp[-i(\omega t - kr_0)]}{r_0} \iint_S \frac{\exp(iks)}{s} K(\chi)\, dS$$

$$U'(P_1) = \frac{1}{j\lambda} \left[\frac{A \exp(jkr_{21})}{r_{21}} \right] \left[\frac{\cos(\vec{n}, \vec{r}_{01}) - \cos(\vec{n}, \vec{r}_{21})}{2} \right]$$

restricted to the case of an aperture illumination consisting of a single expanding spherical wave.

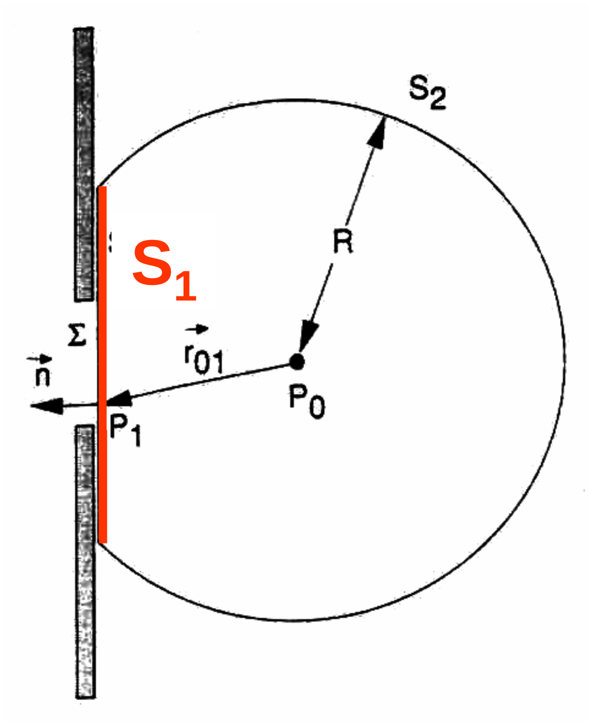
Kirchhoff's boundary conditions are inconsistent! : Potential theory says that "If 2-D potential function and it normal derivative vanish together along any finite curve segment, then the potential function must vanish over the entire plane".

➡ Rayleigh-Sommerfeld theory

- ◆ the scalar theory holds.
- ◆ Both U and G satisfy the homogeneous scalar wave equation.
- ◆The Sommerfeld radiation condition is satisfied.

Sommerfeld radiation condition

$$U(P_0) = \frac{1}{4\pi} \iint_{S2} \left(\frac{\partial U}{\partial n} G - U \frac{\partial G}{\partial n} \right) ds = 0 \quad as \;\; R \to \infty$$



On the surface of S_2, for large R,

$$G = \frac{\exp(jkR)}{R}$$

$$\frac{\partial G}{\partial n} = \left(jk - \frac{1}{R} \right) \frac{\exp(jkR)}{R} \approx jkG$$

$$\iint_{S2} \left[G \frac{\partial U}{\partial n} - (jkG)U \right] ds = \int_{\Omega} G \left(\frac{\partial U}{\partial n} - jkU \right) R^2 d\omega$$

$$\boxed{\therefore \lim_{R \to \infty} R \left(\frac{\partial U}{\partial n} - jkU \right) = 0}$$

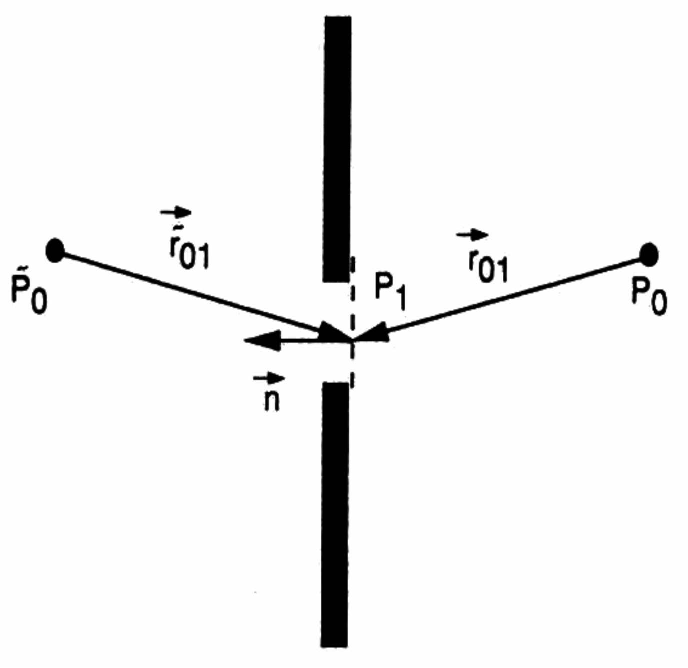
It is satisfied if U vanishes at least as fast as a diverging spherical wave.

The radiation condition is essentially a mathematical statement that there is no incoming wave at very large R.

First Rayleigh-Sommerfeld Solution
(When *U* at Σ is known)

$$U(P_0) = \frac{1}{4\pi}\iint_{S_1}\left(\frac{\partial U}{\partial n}G - U\frac{\partial G}{\partial n}\right)ds$$

Suppose two point sources at P_0 and $\tilde{P}_0$ (mirror image):



$$G_-(P_1) = \frac{\exp(jkr_{01})}{r_{01}} - \frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}$$

$$U_I(P_0) = \frac{-1}{4\pi}\iint_{\Sigma} U\frac{\partial G_-}{\partial n}ds \qquad \frac{\partial G_-(P_1)}{\partial n} = 2\frac{\partial G(P_1)}{\partial n}$$

$$\boxed{U_I(P_0) = \frac{-1}{2\pi}\iint_{\Sigma} U\frac{\partial G}{\partial n}ds}$$

Second Rayleigh-Sommerfeld Solution (When U'_n at Σ is known)

$$G_+(P_1) = \frac{\exp(jkr_{01})}{r_{01}} + \frac{\exp(jk\tilde{r}_{01})}{\tilde{r}_{01}}$$

$$U_{II}(P_0) = \frac{1}{4\pi}\iint_{\Sigma} \frac{\partial U}{\partial n} G_+ ds$$

$$G_+ = 2G$$

$$\boxed{U_{II}(P_0) = \frac{1}{2\pi}\iint_{\Sigma} \frac{\partial U}{\partial n} G ds}$$

Green's function, G_1, for U known on a planar aperture

$$G_1 = \frac{1}{4\pi}\left[\frac{e^{-jkr_{01}}}{r_{01}} - \frac{e^{-jkr_{i1}}}{r_{i1}}\right]$$

Green's function for ∇U known on a planar aperture, G_2

$$G_2 = \frac{1}{4\pi}\left[\frac{e^{-jkr_{01}}}{r_{01}} + \frac{e^{-jkr_{i1}}}{r_{i1}}\right]$$

It is most important to note that, in principle, if we substitute the true U and ∇U into any one of the integrals using G, G_1 or G_2, we should get the same answer.

However, we do not know the true U and ∇U, it is customary to use just the incident U in optics without considering the electromagnetic effects involving the aperture Ω. For example, when we used the incident radiation U as the U in the aperture, we ignored the induced currents near the edge of the aperture. This is an approximation. In this case, we will obtain the same result from the three different Green's functions only in the paraxial approximation, i.e. for $z \gg \lambda$, for an observer located at a relatively small angle from the z axis and for a limited aperture size Ω. In the paraxial approximation, no information concerning the fringe field at small z values or at large angles of observation can be obtained.

Rayleigh-Sommerfeld Diffraction Formula

The 1st and 2nd R_S solutions are,

$$U_I(P_0)=\frac{1}{j\lambda}\iint_{\Sigma}U(P_1)\frac{\exp(jkr_{01})}{r_{01}}\cos(\vec{n},\vec{r}_{01})ds$$

$$U_{II}(P_0)=\frac{1}{2\pi}\iint_{\Sigma}\frac{\partial U(P_1)}{\partial n}\frac{\exp(jkr_{01})}{r_{01}}ds$$

For the case of a spherical wave illumination, $U(P_1)=A\frac{\exp(jkr_{21})}{r_{21}}$

$$U_I(P_0)=\frac{A}{j\lambda}\iint_{\Sigma}\frac{\exp[jk(r_{21}+r_{01})]}{r_{21}r_{01}}\cos(\vec{n},\vec{r}_{01})ds$$

$$U_{II}(P_0)=-\frac{A}{j\lambda}\iint_{\Sigma}\frac{\exp[jk(r_{21}+r_{01})]}{r_{21}r_{01}}\cos(\vec{n},\vec{r}_{21})ds$$

Comparison (I)

F – K :

$$U(P_0) = \frac{1}{4\pi} \iint_{\Sigma} \left(\frac{\partial U}{\partial n} G_K - U \frac{\partial G_K}{\partial n} \right) ds$$

1st R-S :

$$U_1(P_0) = -\frac{1}{2\pi} \iint_{\Sigma} U \frac{\partial G_K}{\partial n} ds$$

2nd R-S :

$$U_{II}(P_0) = \frac{1}{2\pi} \iint_{\Sigma} \frac{\partial U}{\partial n} G_K ds$$

Wow! Surprising!

The Kirchhoff solution is the arithmetic average of the two Rayleigh-Sommerfeld solutions!

Comparison (II)

For a spherical wave illumination,

$$U(P_0) = \frac{A}{j\lambda}\iint_{\Sigma} \frac{\exp\left[jk\left(r_{21}+r_{01}\right)\right]}{r_{21}r_{01}}\psi ds\ , \qquad \psi : \textit{obliquity} \quad \textit{factor}$$

$$\psi = \begin{cases} \frac{1}{2}\left[\cos\left(\vec{n},\vec{r}_{01}\right) - \cos\left(\vec{n},\vec{r}_{21}\right)\right] & \textit{Kirchhoff theory} \\ \cos\left(\vec{n},\vec{r}_{01}\right) & \textit{First Rayleigh-Sommerfeld solution} \\ -\cos\left(\vec{n},\vec{r}_{21}\right) & \textit{Second Rayleigh-Sommerfeld solution} \end{cases}$$

For a normal plane wave illumination, (that means r_{21} → infinite)

$$\psi = \begin{cases} \frac{1}{2}\left[1+\cos\theta\right] & \textit{Kirchhoff theory} \\ \cos\theta & \textit{First Rayleigh-Sommerfeld solution} \\ 1 & \textit{Second Rayleigh-Sommerfeld solution} \end{cases}$$

Again, remember …… After the Huygens-Fresnel principle ……

Fresnel's shortcomings :

He did not mention the existence of backward secondary wavelets, however, there also would be a reverse wave traveling back toward the source. He introduce a quantity of the obliquity factor, but he did little more than conjecture about this kind.

$$\psi(P) = A\frac{\exp[-i(\omega t - kr_0)]}{r_0}\iint\limits_S \frac{\exp(iks)}{s}K(\chi)\,dS$$

Gustav Kirchhoff : Fresnel-Kirhhoff diffraction theory

A more rigorous theory based directly on the solution of the differential wave equation. He, although a contemporary of Maxwell, employed the older elastic-solid theory of light. He found K(χ) = (1 + cosθ)/2. K(0) = 1 in the forward direction, K(π) = 0 with the back wave.

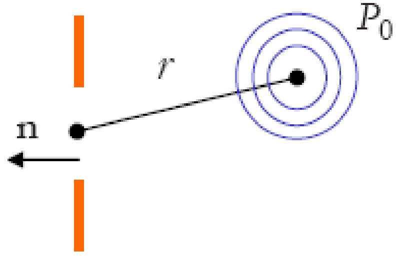
$$\psi(P) = -\left(\frac{ia}{2\lambda}\right)\iint\limits_A \left[\frac{\exp(ikr)}{r}\right]\left[\frac{\exp(iks)}{s}\right][\cos(n,r) - \cos(n,s)]\,dS$$

Arnold Johannes Wilhelm Sommerfeld : Rayleigh-Sommerfeld diffraction theory

A very rigorous solution of partial differential wave equation. The first solution utilizing the electromagnetic theory of light.

$$\psi(P) = -\left(\frac{ia}{\lambda}\right)\iint\limits_A \left[\frac{\exp(ikr)}{r}\right]\left[\frac{\exp(iks)}{s}\right]\cos(n,s)\,dS$$

Again, let's compare the Kirchhof's and Sommerfeld ......

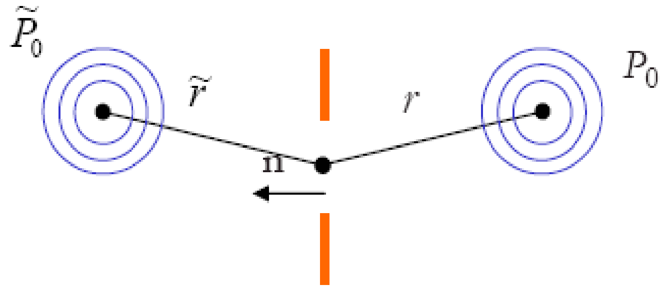


Kirchhoff's choice of $G = \dfrac{e^{ikr}}{r}$ leads to a mathematical inconsistency

(**potential theory**: if U and ∇U vanish on the screen around the aperture, they should vanish over the entire plane S)

Sommerfeld solved the problem by choosing a more complicated G, $G = \dfrac{e^{ikr}}{r} - \dfrac{e^{ik\tilde{r}}}{\tilde{r}}$

$\Rightarrow$ Rayleigh-Sommerfeld diffraction formula



For small angles (paraxial geometry), however, the two choises give essentially the same results

Huygens-Fresnel Principle revised by 1st R – S solution

$$U(P_0)=\frac{1}{j\lambda}\iint_{\Sigma}U(P_1)\frac{\exp(jkr_{01})}{r_{01}}\cos\theta\, ds$$

$$amplitude \quad \propto \quad \frac{1}{\lambda}\propto\nu$$

$$phase: \quad \frac{1}{j}\ \text{(lead the incident phase by } 90^{\circ}\text{)}$$

$$U(P_0)=\iint_{\Sigma}h(P_0,P_1)U(P_1)ds$$

$$h(P_0,P_1)=\frac{1}{j\lambda}\frac{\exp(jkr_{01})}{r_{01}}\cos\theta$$

Each secondary wavelet has a directivity pattern *cos*θ

Generalization to Nonmonochromatic Waves

At the observation point

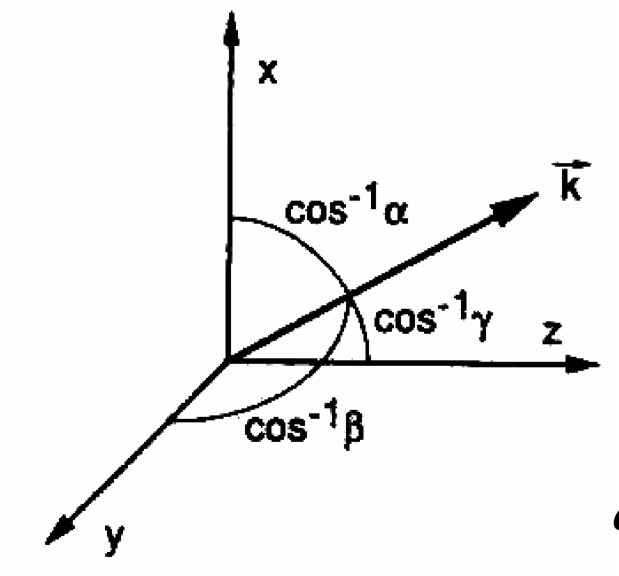
$$u(P_0,t)=\int_{-\infty}^{\infty}U(P_0,\nu)\exp(j2\pi\nu t)d\nu$$

At an aperture point

$$u(P_1,t)=\int_{-\infty}^{\infty}U(P_1,\nu)\exp(j2\pi\nu t)d\nu$$

$$\boxed{u(P_0,t)=\iint_{\Sigma}\frac{\cos(\vec{n},\vec{r}_{01})}{2\pi\upsilon r_{01}}\frac{d}{dt}u\left(P_1,t-\frac{r_{01}}{\upsilon}\right)ds}$$

Angular Spectrum



$$A\left(f_X, f_Y; \mathbf{0}\right) = \int\int_{-\infty}^{\infty} U\left(x, y, \mathbf{0}\right) \exp\left[-j\mathbf{2}\pi\left(f_X x + f_Y y\right)\right] dxdy$$

$$P(x, y, z) = \exp\left(j\vec{k}\cdot\vec{r}\right) = e^{j\frac{2\pi}{\lambda}(\alpha x+\beta y)} e^{j\frac{2\pi}{\lambda}\gamma z}$$

$$where, \vec{k} = \frac{2\pi}{\lambda}(\alpha, \beta, \gamma)$$

$$\alpha = \lambda f_X \quad \beta = \lambda f_Y \quad \gamma = \sqrt{1 - (\lambda f_X)^2 - (\lambda f_Y)^2}$$

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; \mathbf{0}\right) = \int\int_{-\infty}^{\infty} U\left(x, y, \mathbf{0}\right) \exp\left[-j\mathbf{2}\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)\right] dxdy$$

Angular spectrum

Propagation of the Angular Spectrum

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=\int_{-\infty}^{\infty}\int U(x,y,z)\exp\left[-j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)\right]dxdy$$

$$U(x,y,z)=\int_{-\infty}^{\infty}\int A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)\exp\left[j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)\right]d\frac{\alpha}{\lambda}d\frac{\beta}{\lambda}$$

U must satisfy the Helmholtz equation, $\nabla^2 U + k^2 U = 0$

$$\frac{d^2}{dz^2}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)+\left(\frac{2\pi}{\lambda}\right)^2\left[1-\alpha^2-\beta^2\right]A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=0$$

$$\therefore A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)\exp\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,z\right)$$

Propagation of the Angular Spectrum

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)\exp\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,z\right)$$

$\boldsymbol{\alpha}^2+\boldsymbol{\beta}^2<\mathbf{0}$: *each plane-wave component propagates at a different angle*

$\boldsymbol{\alpha}^2+\boldsymbol{\beta}^2>\mathbf{0}$: *evanescent waves*

$$U(x,y,z)=\int\int_{-\infty}^{\infty}A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)\exp\left(j\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,z\right)$$

$$\times\,\mathrm{circ}\left(\sqrt{\alpha^2+\beta^2}\right)\exp\left[j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y\right)\right]d\frac{\alpha}{\lambda}\,d\frac{\beta}{\lambda}$$

Effect of Aperture

$$t_A(x,y) = \frac{U_t(x,y;\mathbf{0})}{U_i(x,y;\mathbf{0})} : amplitude\ transmittance\ \ function\ of\ \ the\ aperture$$

$$A_t\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) = A_i\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) \otimes T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)$$

- For the case of a unit amplitude plane wave incidence,

$$A_i\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) = \delta\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)$$

$$A_t\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) = \delta\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) \otimes T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) = T\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)$$

Propagation as a Linear Spatial Filter

$$A(f_X, f_Y; z) = A(f_X, f_Y; 0)\,\mathrm{circ}\left(\sqrt{(\lambda f_X)^2 + (\lambda f_Y)^2}\right)$$
$$\times \exp\left[j2\pi \frac{z}{\lambda} \sqrt{1 - (\lambda f_X)^2 - (\lambda f_Y)^2} \right]$$

Transfer function of wave propagation phenomenon in free space

$$H(f_X, f_y) = \begin{cases} \exp\left[j2\pi \frac{z}{\lambda} \sqrt{1 - (\lambda f_X)^2 - (\lambda f_Y)^2} \right] & \sqrt{f_X^2 + f_Y^2} \langle \frac{1}{\lambda} \\ 0 & \text{otherwise} \end{cases}$$

BPM (Beam Propagation Method)

$\bar{n}(z) = average[n(x, y, z)]$ on (x,y) plane

$$A_h\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z + \delta z\right) = A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right)\exp\left(j\delta z\sqrt{\bar{n}(z)^2 k_o^2 - (2\pi\frac{\alpha}{\lambda})^2 - (2\pi\frac{\beta}{\lambda})^2}\right)$$

$$U_h(x, y, z + \delta z) = \int\int_{-\infty}^{\infty} A_h\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z + \delta z\right)\exp\left(j\frac{2\pi}{\lambda}(\alpha x + \beta y)\right) d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right)$$

$$U(x, y, z + \delta z) = U_h(x, y, z + \delta z)\exp[- j\delta n(z) k_o \delta z]$$

Invalid for wide angle propagation -> WPM (wave propagation method)

Appendix

From Maxwell's equations to wave equations

– http://www.ee.washington.edu/people/faculty/lin_lih/EE485/ Lih Y. Lin

From Maxwell's equation to Wave equation:

Maxwell's equations in free space

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Necessary condition

$$\mathbf{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + E_z \hat{\mathbf{z}}$$

: Electric field (V/m)

$$\mathbf{H} = H_x \hat{\mathbf{x}} + H_y \hat{\mathbf{y}} + H_z \hat{\mathbf{z}}$$

: Magnetic field (A/m)

Wave equation

$$\nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} = 0$$

$$u = E_{x,y,z} \text{ or } H_{x,y,z}$$

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 (m/s)$$

: Speed of light in free space

$\varepsilon_0 = (1/36\pi) \times 10^{-9} (F/m)$: Electric permittivity

$\mu_0 = 4\pi \times 10^{-7} (H/m)$: Magnetic permeability

Maxwell's Equations in a Medium

Assume a non-magnetic medium with no free electric charges or currents

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t},\ \mathbf{D}\text{: Electric displacement}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},\ \mathbf{B}\text{: Magnetic flux density}$$

$$\nabla \cdot \mathbf{D} = 0$$

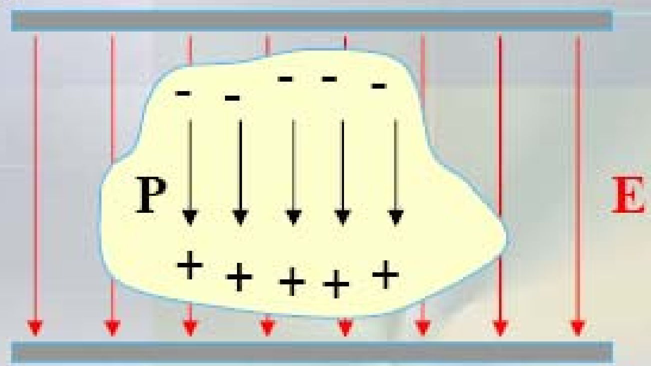
$$\nabla \cdot \mathbf{B} = 0$$

Physical meaning of the electric displacement:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

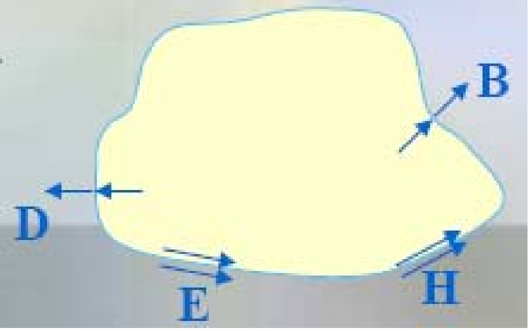
($\nabla \cdot \mathbf{D} = \rho$ if the medium has a charge density ρ)

$$\mathbf{B} = \mu_0 \mathbf{H}$$



Boundary conditions:

- Tangential components of **E** and **H** are continuous.
- Normal components of **D** and **B** are continuous.



Power flow per unit area:

$$\mathbf{S} = \mathrm{Re}\{\mathbf{E}\} \times \mathrm{Re}\{\mathbf{H}\}\ (\mathrm{W/m^2})\text{: Poynting vector}$$

Linear, Nondispersive, Homogeneous, and Isotropic Media

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$$

χ: Electric susceptibility

$$\Rightarrow$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\varepsilon = \varepsilon_0 (1 + \chi)$$

$\varepsilon / \varepsilon_0$: Dielectric constant

Maxwell's equations:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \qquad\qquad \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = 0 \qquad\qquad \nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad\qquad \nabla \cdot \mathbf{H} = 0$$

Identical to Maxwell's equations in free space with ε replaced by ε_0.

	In free space		In a medium
Wave equation:	$\nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} = 0$	$\Rightarrow$	$\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$
Speed of light:	$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$	$\Rightarrow$	$c = \frac{1}{\sqrt{\varepsilon \mu_0}} = \frac{c_0}{n}$ $n = \sqrt{\varepsilon / \varepsilon_0} = \sqrt{1 + \chi}$: Refractive index

Monochromatic Electromagnetic Waves

Let's relate harmonic waves to electromagnetic waves

$$\psi(\mathbf{r},t) \equiv \psi(\mathbf{r})\exp(-i\omega t) \quad \Longleftrightarrow \quad \begin{aligned} \mathbf{E} &= \mathbf{E}(\mathbf{r})e^{-i\omega t} \\ \mathbf{H} &= \mathbf{H}(\mathbf{r})e^{-i\omega t} \end{aligned}$$

(E-M wave represented by complex numbers)

Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{H} &= \varepsilon \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \times \mathbf{E} &= -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{H} &= 0 \end{aligned} \quad \Longrightarrow \quad \boxed{\begin{aligned} \nabla \times \mathbf{H}(\mathbf{r}) &= -i\omega\varepsilon\mathbf{E}(\mathbf{r}) \\ \nabla \times \mathbf{E}(\mathbf{r}) &= i\omega\mu_0\mathbf{H}(\mathbf{r}) \\ \nabla \cdot \mathbf{E}(\mathbf{r}) &= 0 \\ \nabla \cdot \mathbf{H}(\mathbf{r}) &= 0 \end{aligned}}$$

Helmholtz equation:

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = 0 \quad \Longleftrightarrow \quad \boxed{\begin{aligned} &(\nabla^2 + k^2)u(\mathbf{r}) = 0 \\ &u(\mathbf{r}) = E_{x,y,z}(\mathbf{r}) \text{ or } H_{x,y,z}(\mathbf{r}) \\ &k = \omega\sqrt{\varepsilon\mu_0} = nk_0 \end{aligned}}$$

Optical intensity:

$$I = \langle |\mathbf{S}| \rangle = \left| \mathrm{Re}\left\{ \frac{1}{2}\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^* \right\} \right|$$

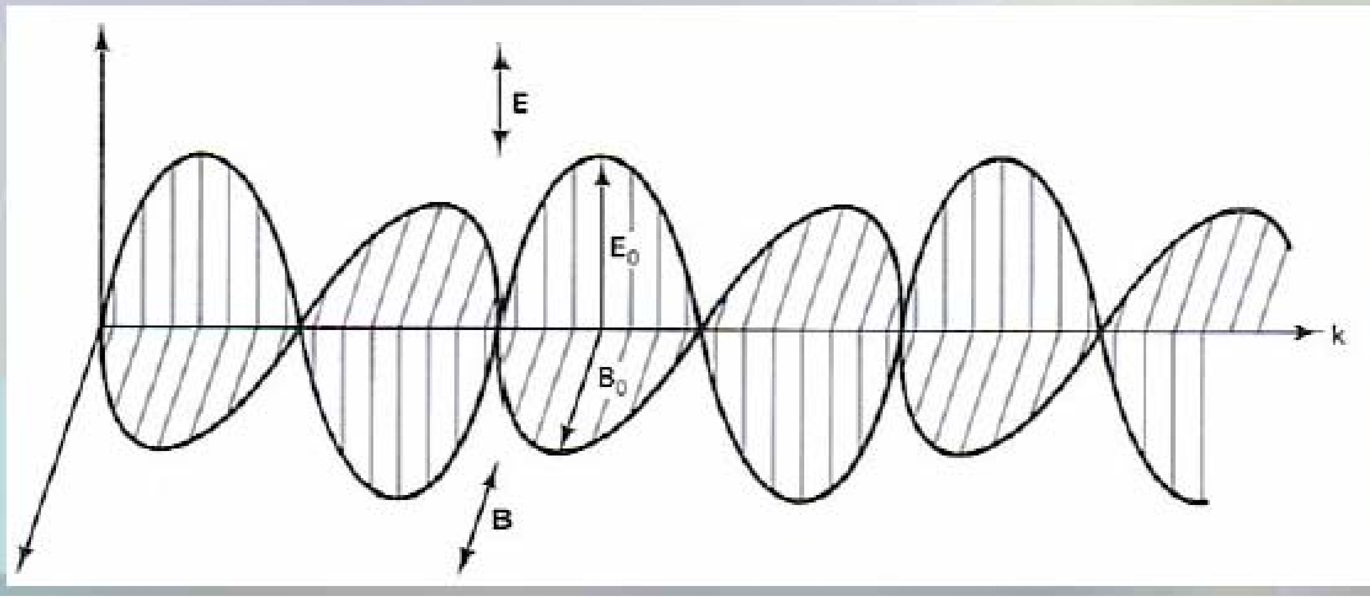
Plane Electromagnetic Wave (I)

$$\psi = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad \Rightarrow \quad \begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \equiv \mathbf{E}(\mathbf{r})e^{-i\omega t} \\ \mathbf{H}(\mathbf{r},t) &= \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \equiv \mathbf{H}(\mathbf{r})e^{-i\omega t} \end{aligned}$$

Substituting into Maxwell's equations:

$$\begin{aligned} \nabla\times\mathbf{H}(\mathbf{r}) &= -i\omega\varepsilon\mathbf{E}(\mathbf{r}) \\ \nabla\times\mathbf{E}(\mathbf{r}) &= i\omega\mu_0\mathbf{H}(\mathbf{r}) \end{aligned} \longrightarrow \begin{aligned} \mathbf{k}\times\mathbf{H}_0 &= -\omega\varepsilon\mathbf{E}_0 \\ \mathbf{k}\times\mathbf{E}_0 &= \omega\mu_0\mathbf{H}_0 \end{aligned}$$

→ E, H, and k are mutually orthogonal — Transverse electromagnetic (TEM) wave.



Relationship between the amplitude of the electric field and the magnetic field:

$$\frac{E_0}{H_0} = \eta$$

$$\eta = \frac{\eta_0}{n} \text{ : Impedance of the medium}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi = 377\Omega \text{ : Impedance of free space}$$

Optical intensity:

$$I = \langle |\mathbf{S}| \rangle = \left| \mathrm{Re}\left\{ \frac{1}{2} \mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})^* \right\} \right| = \frac{1}{2} \left| E_0 H_0^* \right| = \frac{\left| E_0 \right|^2}{2\eta}$$

Absorption

Represented by a complex susceptibility, therefore complex dielectric constant:

$$\chi = \chi' + i\chi'' \rightarrow \qquad \frac{\varepsilon}{\varepsilon_0} = 1 + \chi = 1 + \chi' + i\chi''$$

The Helmholtz equation $(\nabla^2 + k^2)u(\mathbf{r}) = 0$ remains applicable.

$$k = \sqrt{\frac{\varepsilon}{\varepsilon_0}} k_0 = \sqrt{1 + \chi' + i\chi''}\, k_0$$

$$k \equiv \beta + i\frac{1}{2}\alpha$$

$$\exp(ikz) = \exp(-1/2\alpha z)\exp(i\beta z)$$

$$\text{Intensity } I \propto \exp(-\alpha z)$$

α: Absorption coefficient (attenuation coefficient)

$\beta \equiv nk_0$: Propagation constant

n: Effective refractive index

Absorption Bands of Optical Materials

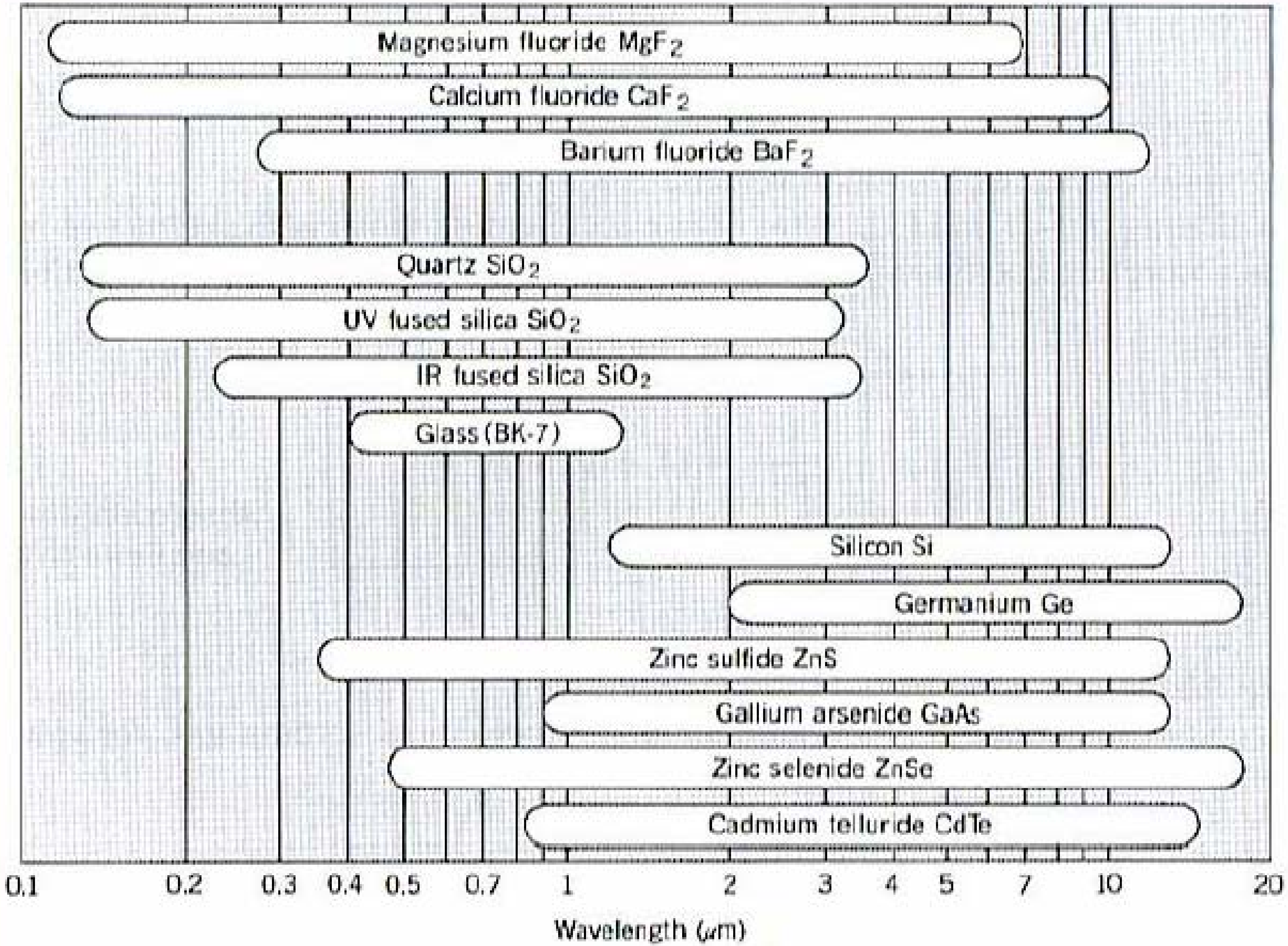


Figure 5.5-1 The spectral bands within which selected optical materials transmit light.