Notes about 3D C-PML equations for SPECFEM3D

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September 22, 2012

Adjacent to 3DCPML, we assume the motion is governed by 3D elastic wave equation. In differential form it is given by:

$$
\begin{aligned}
\varepsilon_{xx} &= \partial_x u_x \\
\varepsilon_{yy} &= \partial_y u_y \\
\varepsilon_{zz} &= \partial_z u_z \\
\varepsilon_{xy} &= \frac{1}{2}(\partial_x u_y + \partial_y u_x)\ ,\ \varepsilon_{yx} = \varepsilon_{xy} \\
\varepsilon_{xz} &= \frac{1}{2}(\partial_x u_z + \partial_z u_x)\ ,\ \varepsilon_{zx} = \varepsilon_{xz} \\
\varepsilon_{yz} &= \frac{1}{2}(\partial_y u_z + \partial_z u_y)\ ,\ \varepsilon_{zy} = \varepsilon_{yz}
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \\
\sigma_{yy} &= \lambda\,\varepsilon_{xx} + (\lambda + 2\mu)\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \\
\sigma_{zz} &= \lambda\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + (\lambda + 2\mu)\,\varepsilon_{zz} \\
\sigma_{xy} &= 2\mu\,\varepsilon_{xy}\ ,\ \sigma_{yx} = \sigma_{xy} \\
\sigma_{xz} &= 2\mu\,\varepsilon_{xz}\ ,\ \sigma_{zx} = \sigma_{xz} \\
\sigma_{yz} &= 2\mu\,\varepsilon_{yz}\ ,\ \sigma_{yz} = \sigma_{zy}
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
\rho\,\frac{\partial^2 u_x}{\partial t^2} &= \partial_x \sigma_{xx} + \partial_y \sigma_{xy} + \partial_z \sigma_{xz} + f_x \\
\rho\,\frac{\partial^2 u_y}{\partial t^2} &= \partial_x \sigma_{yx} + \partial_y \sigma_{yy} + \partial_z \sigma_{yz} + f_y \\
\rho\,\frac{\partial^2 u_z}{\partial t^2} &= \partial_x \sigma_{zx} + \partial_y \sigma_{zy} + \partial_z \sigma_{zz} + f_z
\end{aligned}
\tag{3}
$$

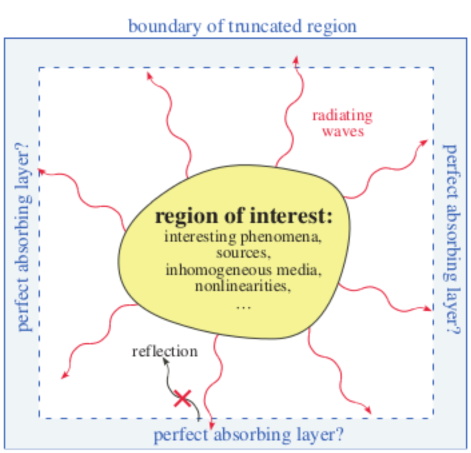

Figure 1. Schematic of a typical wave-equation problem, in which there is some finite region of interest where sources, inhomogeneous media, nonlinearities and etc are being investigated, from which some radiative waves escape to infinity. The perfect absorbing layer, which was used to truncated out computational region, is placed adjacent to the edges of the computational region

To obtain the CPML form, we firstly perform a Fourier transform on above equation, then replace each spatial derivative with that same derivative multiplied by $1/s$ to make the plane wave entering into CPML without reflection and decaying exponentially along the x, y, z or in all directions. Thus,in differential form in 3D the frequency-domain equation govened the motion in CPML gives:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{s_x}\partial_x u_x \\
\varepsilon_{yy} &= \frac{1}{s_y}\partial_y u_y \\
\varepsilon_{zz} &= \frac{1}{s_z}\partial_z u_z \\
\varepsilon_{xy} &= \frac{1}{2}(\frac{1}{s_x}\partial_x u_y + \frac{1}{s_y}\partial_y u_x)\ ,\ \varepsilon_{yx} = \varepsilon_{xy} \\
\varepsilon_{xz} &= \frac{1}{2}(\frac{1}{s_x}\partial_x u_z + \frac{1}{s_z}\partial_z u_x)\ ,\ \varepsilon_{zx} = \varepsilon_{xz} \\
\varepsilon_{yz} &= \frac{1}{2}(\frac{1}{s_y}\partial_y u_z + \frac{1}{s_z}\partial_z u_y)\ ,\ \varepsilon_{zy} = \varepsilon_{yz}
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \\
\sigma_{yy} &= \lambda\,\varepsilon_{xx} + (\lambda + 2\mu)\,\varepsilon_{yy} + \lambda\,\varepsilon_{zz} \\
\sigma_{zz} &= \lambda\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} + (\lambda + 2\mu)\,\varepsilon_{zz} \\
\sigma_{xy} &= 2\mu\,\varepsilon_{xy}\ ,\ \sigma_{yx} = \sigma_{xy} \\
\sigma_{xz} &= 2\mu\,\varepsilon_{xz}\ ,\ \sigma_{zx} = \sigma_{xz} \\
\sigma_{yz} &= 2\mu\,\varepsilon_{yz}\ ,\ \sigma_{yz} = \sigma_{zy}
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
-\rho\,\omega^2 u_x &= \frac{1}{s_x}\partial_x\sigma_{xx} + \frac{1}{s_y}\partial_y\sigma_{xy} + \frac{1}{s_z}\partial_z\sigma_{xz} + f_x \\
-\rho\,\omega^2 u_y &= \frac{1}{s_x}\partial_x\sigma_{yx} + \frac{1}{s_y}\partial_y\sigma_{yy} + \frac{1}{s_z}\partial_z\sigma_{yz} + f_y \\
-\rho\,\omega^2 u_z &= \frac{1}{s_x}\partial_x\sigma_{zx} + \frac{1}{s_y}\partial_y\sigma_{zy} + \frac{1}{s_z}\partial_z\sigma_{zz} + f_z
\end{aligned} \tag{6}$$

where $s_x = \kappa_x + \frac{d_x}{\alpha_x + \mathrm{i}\omega}$, $s_y = \kappa_y + \frac{d_y}{\alpha_y + \mathrm{i}\omega}$ and $s_z = \kappa_z + \frac{d_z}{\alpha_z + \mathrm{i}\omega}$. $\kappa_x, d_x, \alpha_x; \kappa_y, d_y, \alpha_y; \kappa_z, d_z, \alpha_z$ valued following the way used in paper Festa & Vilotte (2005).
In the frequency domain we define:

$$\partial_{\tilde{x}} = \frac{1}{s_x}\,\partial_x, \tag{7}$$

and we know from Komatitsch & Martin (2007) that ∂_x is then transformed in the time domain into:

$$\partial_{\tilde{x}} = \frac{1}{\kappa_x}\partial_x - \frac{d_x}{\kappa_x^2}H(t)e^{-(d_x/\kappa_x + \alpha_x)t} * \partial_x\,. \tag{8}$$

This means that $\frac{1}{s_x}$ is transformed by inverse Fourier transform into $\frac{1}{\kappa_x} - \frac{d_x}{\kappa_x^2}H(t)e^{-(d_x/\kappa_x+\alpha_x)t}*$.

Here we extended the approach used in Matzen (2011) to get an stable CPML implementation for spectral element method: Let us multiply the set of equation (6) by $s_x s_y s_z$, This will results in a more complex mass matrix on the left-hand side. And let us get rid of the source term because there is no source in the PML layers, we have:

$$\begin{aligned}
-\rho s_x s_y s_z \omega^2 u_x &= s_y s_z \partial_x \sigma_{xx} + s_x s_z \partial_y \sigma_{xy} + s_x s_y \partial_z \sigma_{xz} \\
-\rho s_x s_y s_z \omega^2 u_y &= s_y s_z \partial_x \sigma_{yx} + s_x s_z \partial_y \sigma_{yy} + s_x s_y \partial_z \sigma_{yz} \\
-\rho s_x s_y s_z \omega^2 u_z &= s_y s_z \partial_x \sigma_{zx} + s_x s_z \partial_y \sigma_{zy} + s_x s_y \partial_z \sigma_{zz}
\end{aligned} \tag{9}$$

since $s_y s_z$ does not depend on x , $s_x s_z$ does not depend on y and $s_x s_y$ does not depend on z:

$$\begin{aligned}
-\rho s_x s_y s_z \omega^2 u_x &= \partial_x (s_y s_z \sigma_{xx}) + \partial_y (s_x s_z \sigma_{xy}) + \partial_z (s_x s_y \sigma_{xz}) \\
-\rho s_x s_y s_z \omega^2 u_y &= \partial_x (s_y s_z \sigma_{yx}) + \partial_y (s_x s_z \sigma_{yy}) + \partial_z (s_x s_y \sigma_{yz}) \\
-\rho s_x s_y s_z \omega^2 u_z &= \partial_x (s_y s_z \sigma_{zx}) + \partial_y (s_x s_z \sigma_{zy}) + \partial_z (s_x s_y \sigma_{zz})
\end{aligned} \tag{10}$$

By inserting the definition of the stress tensor we get:

$$\begin{aligned}
-\rho s_x s_y s_z \omega^2 u_x &= \partial_x (s_y s_z ((\lambda + 2\mu)\, \varepsilon_{xx} + \lambda\, \varepsilon_{yy} + \lambda\, \varepsilon_{zz})) + \partial_y (s_x s_z (2\mu\, \varepsilon_{xy})) + \partial_z (s_x s_y (2\mu\, \varepsilon_{xz})) \\
-\rho s_x s_y s_z \omega^2 u_y &= \partial_x (s_y s_z (2\mu\, \varepsilon_{xy})) + \partial_y (s_x s_z (\lambda\, \varepsilon_{xx} + (\lambda + 2\mu)\, \varepsilon_{yy} + \lambda\, \varepsilon_{zz})) + \partial_z (s_x s_y (2\mu\, \varepsilon_{yz})) \\
-\rho s_x s_y s_z \omega^2 u_z &= \partial_x (s_y s_z (2\mu\, \varepsilon_{xz})) + \partial_y (s_x s_z (2\mu\, \varepsilon_{yz})) + \partial_z (s_x s_y (\lambda\, \varepsilon_{xx} + \lambda\, \varepsilon_{yy} + (\lambda + 2\mu)\, \varepsilon_{zz}))
\end{aligned} \tag{11}$$

Then, inserting the definition of the strain tensor we get:

$$\begin{aligned}
-\rho s_x s_y s_z \omega^2 u_x &= \partial_x (s_y s_z ((\lambda + 2\mu) \frac{1}{s_x} \partial_x u_x + \lambda \frac{1}{s_y} \partial_y u_y + \lambda \frac{1}{s_z} \partial_z u_z)) + \partial_y (s_x s_z (\mu (\frac{1}{s_x} \partial_x u_y + \frac{1}{s_y} \partial_y u_x))) \\
&\quad + \partial_z (s_x s_y (\mu (\frac{1}{s_x} \partial_x u_z + \frac{1}{s_z} \partial_z u_x)) \\
-\rho s_x s_y s_z \omega^2 u_y &= \partial_x (s_y s_z (\mu (\frac{1}{s_x} \partial_x u_y + \frac{1}{s_y} \partial_y u_x))) + \partial_y (s_x s_z (\lambda \frac{1}{s_x} \partial_x u_x + (\lambda + 2\mu) \frac{1}{s_y} \partial_y u_y + \lambda \frac{1}{s_z} \partial_z u_z)) \\
&\quad + \partial_z (s_x s_y (\mu (\frac{1}{s_y} \partial_y u_z + \frac{1}{s_z} \partial_z u_y))) \\
-\rho s_x s_y s_z \omega^2 u_z &= \partial_x (s_y s_z (\mu (\frac{1}{s_x} \partial_x u_z + \frac{1}{s_z} \partial_z u_x))) + \partial_y (s_x s_z (\mu (\frac{1}{s_y} \partial_y u_z + \frac{1}{s_z} \partial_z u_y))) \\
&\quad + \partial_z (s_x s_y (\lambda \frac{1}{s_x} \partial_x u_x + \lambda \frac{1}{s_y} \partial_y u_y + (\lambda + 2\mu)(\frac{1}{s_z} \partial_z u_z)))
\end{aligned} \tag{12}$$

Grouping the s_x,s_y and s_z factors we get:

$$\begin{aligned}
-\rho s_x s_y s_z \omega^2 u_x &= \partial_x ((\lambda + 2\mu) \frac{s_y s_z}{s_x} \partial_x u_x + \lambda s_z \partial_y u_y + \lambda s_y \partial_z u_z) + \partial_y (\mu s_z \partial_x u_y + \mu \frac{s_x s_z}{s_y} \partial_y u_x) \\
&\quad + \partial_z (\mu s_y \partial_x u_z + \mu \frac{s_x s_y}{s_z} \partial_z u_x) \\
-\rho s_x s_y s_z \omega^2 u_y &= \partial_x (\mu \frac{s_y s_z}{s_x} \partial_x u_y + \mu s_z \partial_y u_x) + \partial_y (\lambda s_z \partial_x u_x + (\lambda + 2\mu) \frac{s_x s_z}{s_y} \partial_y u_y + \lambda s_x \partial_z u_z) \\
&\quad + \partial_z (\mu s_x \partial_y u_z + \mu \frac{s_x s_y}{s_z} \partial_z u_y) \\
-\rho s_x s_y s_z \omega^2 u_z &= \partial_x (\mu \frac{s_y s_z}{s_x} \partial_x u_z + \mu s_y \partial_z u_x) + \partial_y (\mu \frac{s_x s_z}{s_y} \partial_y u_z + \mu s_x \partial_z u_y) \\
&\quad + \partial_z (\lambda s_y \partial_x u_x + \lambda s_x \partial_y u_y + (\lambda + 2\mu) \frac{s_x s_y}{s_z} \partial_z u_z)
\end{aligned} \tag{13}$$

Let us note that the above equation, if interpreted in terms of an equivalent stress tensor, implies that such a stress tensor is not symmetric any more.

With CPML, in variational form, in 3D with a test vector $\mathbf{w} = (w_x, w_y, w_z)$ this gives (let us get rid of the source term because there is no source in the PML layers):

$$\begin{aligned}
\int_\Omega -\rho s_x s_y s_z \omega^2 u_x &= \int_\Omega \Bigg(w_x \partial_x ((\lambda+2\mu)\frac{s_y s_z}{s_x}\partial_x u_x + \lambda s_z \partial_y u_y + \lambda s_y \partial_z u_z) + w_x \partial_y (\mu s_z \partial_x u_y + \mu \frac{s_x s_z}{s_y}\partial_y u_x) \\
&\quad + w_x \partial_z (\mu s_y \partial_x u_z + \mu \frac{s_x s_y}{s_z}\partial_z u_x) \Bigg) \\
\int_\Omega -\rho s_x s_y s_z \omega^2 u_y &= \int_\Omega \Bigg(w_y \partial_x (\mu \frac{s_y s_z}{s_x}\partial_x u_y + \mu s_z \partial_y u_x) + w_y \partial_y (\lambda s_z \partial_x u_x + (\lambda+2\mu)\frac{s_x s_z}{s_y}\partial_y u_y + \lambda s_x \partial_z u_z) \\
&\quad + w_y \partial_z (\mu s_x \partial_y u_z + \mu \frac{s_x s_y}{s_z}\partial_z u_y) \Bigg) \\
\int_\Omega -\rho s_x s_y s_z \omega^2 u_z &= \int_\Omega \Bigg(w_z \partial_x (\mu \frac{s_y s_z}{s_x}\partial_x u_z + \mu s_y \partial_z u_x) + w_z \partial_y (\mu \frac{s_x s_z}{s_y}\partial_y u_z + \mu s_x \partial_z u_y) \\
&\quad + w_z \partial_z (\lambda s_y \partial_x u_x + \lambda s_x \partial_y u_y + (\lambda+2\mu)\frac{s_x s_y}{s_z}\partial_z u_z) \Bigg) \qquad (14)
\end{aligned}$$

Integrating by parts (and ignoring the boundary integral term because of the Dirichlet conditions that we will apply later on that boundary, because its contribution would be erased by the Dirichlet condition) we get

$$\begin{aligned}
\int_\Omega -\rho s_x s_y s_z \omega^2 u_x &= \int_\Omega \Bigg(((\lambda+2\mu)\frac{s_y s_z}{s_x}\partial_x u_x + \lambda s_z \partial_y u_y + \lambda s_y \partial_z u_z)\partial_x w_x + (\mu s_z \partial_x u_y + \mu \frac{s_x s_z}{s_y}\partial_y u_x)\partial_y w_x \\
&\quad + (\mu s_y \partial_x u_z + \mu \frac{s_x s_y}{s_z}\partial_z u_x)\partial_z w_x \Bigg) \\
\int_\Omega -\rho s_x s_y s_z \omega^2 u_y &= \int_\Omega \Bigg((\mu \frac{s_y s_z}{s_x}\partial_x u_y + \mu s_z \partial_y u_x)\partial_x w_y + (\lambda s_z \partial_x u_x + (\lambda+2\mu)\frac{s_x s_z}{s_y}\partial_y u_y + \lambda s_x \partial_z u_z)\partial_y w_y \\
&\quad + (\mu s_x \partial_y u_z + \mu \frac{s_x s_y}{s_z}\partial_z u_y)\partial_z w_y \Bigg) \\
\int_\Omega -\rho s_x s_y s_z \omega^2 u_z &= \int_\Omega \Bigg((\mu \frac{s_y s_z}{s_x}\partial_x u_z + \mu s_y \partial_z u_x)\partial_x w_z + (\mu \frac{s_x s_z}{s_y}\partial_y u_z + \mu s_x \partial_z u_y)\partial_y w_z \\
&\quad + (\lambda s_y \partial_x u_x + \lambda s_x \partial_y u_y + (\lambda+2\mu)\frac{s_x s_y}{s_z}\partial_z u_z)\partial_z w_z \Bigg) \qquad (15)
\end{aligned}$$

Inverse FT of $-\omega^2 s_x s_y s_z$ computed using MAPLE for left-hand side:

$$A_0\delta''(t) + A_1\delta'(t) + A_2\delta(t) + A_3 \mathrm{e}^{-\alpha\, t}\, H(t) + A_4 \mathrm{e}^{-\alpha\, t}\, t\, H(t) + A_5 \mathrm{e}^{-\alpha\, t}\, t^2\, H(t)$$

with

$$A_0 = \kappa_x\kappa_y\kappa_z\,,\; A_1 = d_x\kappa_y\kappa_z + d_y\kappa_x\kappa_z + d_z\kappa_x\kappa_y$$

$$A_2 = d_x d_y \kappa_z + d_x d_z \kappa_y + d_y d_z \kappa_x - \alpha(d_x\kappa_y\kappa_z + d_y\kappa_x\kappa_z + d_z\kappa_x\kappa_y)$$

$$A_3 = d_x d_y d_z - 2\alpha(d_x d_y \kappa_z + d_x d_z \kappa_y + d_y d_z \kappa_x) + \alpha^2(d_x\kappa_y\kappa_z + d_y\kappa_x\kappa_z + d_z\kappa_x\kappa_y)$$

$$A_4 = -2\alpha d_x d_y d_z + \alpha^2(d_x d_y \kappa_z + d_x d_z \kappa_y + d_y d_z \kappa_x), A_5 = \frac{1}{2}\alpha^2 d_x d_y d_z$$

Here in order to solve the singularity problem arised using definition of α_x, α_y and α_z, we set

$$\alpha_x = \alpha_y = \alpha_z = const = \alpha$$

where α_x, α_y or α_z is nonzero

Inverse FT of $\frac{s_y s_z}{s_x}$ computed using MAPLE for right-hand side:

a): $d_x \neq 0$

$$A_6\delta(t) + A_7 \mathrm{e}^{-\alpha\, t}\, H(t) + A_8 \mathrm{e}^{-t(\alpha + \frac{d_x}{\kappa_x})}\, H(t)$$

with

$$A_6 = \frac{\kappa_y \kappa_z}{\kappa_x},\, A_7 = \frac{d_y d_z}{d_x},\, A_8 = \frac{(d_y\kappa_x - d_x\kappa_y)(d_x\kappa_z - d_z\kappa_x)}{d_x\kappa_x^2}$$

b): $d_x = 0$

$$A_6\delta(t) + A_7\mathrm{e}^{-\alpha\, t}\, H(t) + A_8\mathrm{e}^{-\alpha\, t}\, t\, H(t)$$

with

$$A_6 = \frac{\kappa_y \kappa_z}{\kappa_x},\, A_7 = \frac{d_z\kappa_y + d_y\kappa_z}{\kappa_x},\, A_8 = \frac{d_y d_z}{\kappa_x}$$

Inverse FT of $\frac{s_x s_z}{s_y}$ computed using MAPLE for right-hand side:

a): $d_y \neq 0$

$$A_9\delta(t) + A_{10}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\, H(t)$$

with

$$A_9 = \frac{\kappa_x \kappa_z}{\kappa_y},\, A_{10} = \frac{d_x d_z}{d_y},\, A_{11} = \frac{(d_x\kappa_y - d_y\kappa_x)(d_y\kappa_z - d_z\kappa_y)}{d_y\kappa_y^2}$$

b): $d_y = 0$

$$A_9\delta(t) + A_{10}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{11}\mathrm{e}^{-\alpha\, t}\, t\, H(t)$$

with

$$A_9 = \frac{\kappa_x \kappa_z}{\kappa_y},\, A_{10} = \frac{d_z\kappa_x + d_x\kappa_z}{\kappa_y},\, A_{11} = \frac{d_x d_z}{\kappa_y}$$

Inverse FT of $\frac{s_x s_y}{s_z}$ computed using MAPLE for right-hand side:

a): $d_z \neq 0$

$$A_{12}\delta(t) + A_{13}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\, H(t)$$

with

$$A_{12} = \frac{\kappa_x \kappa_y}{\kappa_z},\, A_{13} = \frac{d_x d_y}{d_z},\, A_{14} = \frac{(d_x\kappa_z - d_z\kappa_x)(d_z\kappa_y - d_y\kappa_z)}{d_z\kappa_z^2}$$

b): $d_z = 0$

$$A_{12}\delta(t) + A_{13}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{14}\mathrm{e}^{-\alpha\, t}\, t\, H(t)$$

with

$$A_{12} = \frac{\kappa_x \kappa_y}{\kappa_z},\, A_{13} = \frac{d_y\kappa_x + d_x\kappa_y}{\kappa_z},\, A_{14} = \frac{d_x d_y}{\kappa_z}$$

Inverse FT of s_x computed using MAPLE for right-hand side:

$$A_{15}\delta(t) + A_{16}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{15} = \kappa_x,\, A_{16} = d_x$$

Inverse FT of s_y computed using MAPLE for right-hand side:

$$A_{17}\delta(t) + A_{18}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{17} = \kappa_y,\, A_{18} = d_y$$

Inverse FT of s_z computed using MAPLE for right-hand side:

$$A_{19}\delta(t) + A_{20}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{19} = \kappa_z,\, A_{20} = d_z$$

Thus going back to the time domain, in which a product in the Fourier transform domain becomes a convolution in the time domain, we get

$$
\begin{aligned}
&\int_\Omega \rho\Big(A_0\delta''(t)+A_1\delta'(t)+A_2\delta(t)+A_3\mathrm{e}^{-\alpha\, t}\,H(t)+A_4\mathrm{e}^{-\alpha\, t}\,t\,H(t)+A_5\mathrm{e}^{-\alpha\, t}\,t^2\,H(t)\Big)*u_x\\
&=\int_\Omega\Bigg(\Big((\lambda+2\mu)(A_6\delta(t)+A_7\mathrm{e}^{-\alpha\, t}\,H(t)+A_8\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\,H(t))*\partial_x u_x+\lambda(A_{19}\delta(t)+A_{20}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\,H(t))*\partial_y u_y\\
&\qquad+\lambda(A_{17}\delta(t)+A_{18}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\,H(t))*\partial_z u_z\Big)\partial_x w_x\\
&\quad+\Big(\mu(A_{19}\delta(t)+A_{20}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\,H(t))*\partial_x u_y+\mu(A_9\delta(t)+A_{10}\mathrm{e}^{-\alpha\, t}\,H(t)+A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\,H(t))*\partial_y u_x\Big)\partial_y w_x\\
&\quad+\Big(\mu(A_{17}\delta(t)+A_{18}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\,H(t))*\partial_x u_z+\mu(A_{12}\delta(t)+A_{13}\mathrm{e}^{-\alpha\, t}\,H(t)+A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\,H(t))*\partial_z u_x\Big)\partial_z w_x\Bigg)\\
&\int_\Omega \rho\Big(A_0\delta''(t)+A_1\delta'(t)+A_2\delta(t)+A_3\mathrm{e}^{-\alpha\, t}\,H(t)+A_4\mathrm{e}^{-\alpha\, t}\,t\,H(t)+A_5\mathrm{e}^{-\alpha\, t}\,t^2\,H(t)\Big)*u_y\\
&=\int_\Omega\Bigg(\Big(\mu(A_6\delta(t)+A_7\mathrm{e}^{-\alpha\, t}\,H(t)+A_8\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\,H(t))*\partial_x u_y+\mu(A_{19}\delta(t)+A_{20}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\,H(t))*\partial_y u_x\Big)\partial_x w_y\\
&\quad+\Big(\lambda(A_{19}\delta(t)+A_{20}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\,H(t))*\partial_x u_x+(\lambda+2\mu)(A_9\delta(t)+A_{10}\mathrm{e}^{-\alpha\, t}\,H(t)+A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\,H(t))*\partial_y u_y\\
&\qquad+\lambda(A_{15}\delta(t)+A_{16}\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\,H(t))*\partial_z u_z\Big)\partial_y w_y\\
&\quad+\Big(\mu(A_{15}\delta(t)+A_{16}\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\,H(t))*\partial_y u_z+\mu(A_{12}\delta(t)+A_{13}\mathrm{e}^{-\alpha\, t}\,H(t)+A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\,H(t))*\partial_z u_y\Big)\partial_z w_y\Bigg)\\
&\int_\Omega \rho\Big(A_0\delta''(t)+A_1\delta'(t)+A_2\delta(t)+A_3\mathrm{e}^{-\alpha\, t}\,H(t)+A_4\mathrm{e}^{-\alpha\, t}\,t\,H(t)+A_5\mathrm{e}^{-\alpha\, t}\,t^2\,H(t)\Big)*u_z\\
&=\int_\Omega\Bigg(\Big(\mu(A_6\delta(t)+A_7\mathrm{e}^{-\alpha\, t}\,H(t)+A_8\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\,H(t))*\partial_x u_z+\mu(A_{17}\delta(t)+A_{18}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\,H(t))*\partial_z u_x\Big)\partial_x w_z\\
&\quad+\Big(\mu(A_9\delta(t)+A_{10}\mathrm{e}^{-\alpha\, t}\,H(t)+A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\,H(t))*\partial_y u_z+\mu(A_{15}\delta(t)+A_{16}\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\,H(t))*\partial_z u_y\Big)\partial_y w_z\\
&\quad+\Big(\lambda(A_{17}\delta(t)+A_{18}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\,H(t))*\partial_x u_x+\lambda(A_{15}\delta(t)+A_{16}\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\,H(t))*\partial_y u_y\\
&\qquad+(\lambda+2\mu)(A_{12}\delta(t)+A_{13}\mathrm{e}^{-\alpha\, t}\,H(t)+A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\,H(t))*\partial_z u_z\Big)\partial_z w_z\Bigg)
\end{aligned}
\tag{16}
$$

Take the first equation of set (16) as example,Remembering that $u*\delta'(t)=u'*\delta(t)$ and $u*\delta''(t)=u''*\delta(t)$, we expanding the left hand side of the equation

$$\int_\Omega \rho A_0\frac{\partial^2 u_x}{\partial t^2}w_x+\int_\Omega \rho A_1\frac{\partial u_x}{\partial t}w_x+\int_\Omega \rho A_2 u_x w_x+\int_\Omega \rho(A_3\mathrm{e}^{-\alpha\, t}\,H(t))*u_x+\int_\Omega \rho(A_4\mathrm{e}^{-\alpha\, t}\,t\,H(t))*u_x+\int_\Omega \rho(A_5\mathrm{e}^{-\alpha\, t}\,t^2\,H(t))*u_x$$

Note that, following Hughes (1987) equation (9.1.12) page 491, and as also mentioned by Matzen (2011), the term $\rho\,A_1$ is a damping matrix C related to velocity and thus it adds a $\rho\,\frac{\Delta t}{2}\,A_1$ term to the mass matrix in the explicit Newmark scheme ($\alpha=\frac{1}{2}$, $\beta=0$).

Eq. 16 is the equation set of corner PML where $\kappa_x,d_x,\alpha_x;\kappa_y,d_y,\alpha_y;and\kappa_z,d_z,\alpha_z$ are not zero everywhere. But in the edge PML like XYPML, XZPML and YZPML or surface PML like XPML, YPML and ZPML, it is not the same. In that case, we have:
a):XYPML
κ_x,d_x and κ_y,d_y defined as usual; $\alpha_x=\alpha_y=\alpha$ but $\kappa_z=1,d_z=0,\alpha_z=0$.
In this case inverse FT of $-\omega^2 s_x s_y s_z$ is:

$$A_0\delta''(t)+A_1\delta'(t)+A_2\delta(t)+A_3\mathrm{e}^{-\alpha\, t}\,H(t)+A_4\mathrm{e}^{-\alpha\, t}\,t\,H(t)+A_5\mathrm{e}^{-\alpha\, t}\,t^2\,H(t)$$

with

$$A_0=\kappa_x\kappa_y\,,\;A_1=d_x\kappa_y+d_y\kappa_x\,,\;A_2=d_xd_y-\alpha(d_y\kappa_x+d_x\kappa_y)$$

$$A_3=-2\alpha d_xd_y+\alpha^2(d_x\kappa_y+d_y\kappa_x),A_4=\alpha^2d_xd_y,A_5=0$$

Inverse FT of $\frac{s_ys_z}{s_x}$ computed using MAPLE for right-hand side:

$$A_6\delta(t) + A_7\mathrm{e}^{-\alpha\, t}\, H(t) + A_8\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\, H(t)$$

with

$$A_6 = \frac{\kappa_y}{\kappa_x},\; A_7 = 0,\; A_8 = \frac{d_y\kappa_x - d_x\kappa_y}{\kappa_x^2}$$

Inverse FT of $\frac{s_x s_z}{s_y}$ computed using MAPLE for right-hand side:

$$A_9\delta(t) + A_{10}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\, H(t)$$

with

$$A_9 = \frac{\kappa_x}{\kappa_y},\; A_{10} = 0,\; A_{11} = \frac{(d_x\kappa_y - d_y\kappa_x)}{\kappa_y^2}$$

Inverse FT of $\frac{s_x s_y}{s_z}$ computed using MAPLE for right-hand side:

$$A_{12}\delta(t) + A_{13}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{14}\mathrm{e}^{-\alpha\, t} tH(t)$$

with

$$A_{12} = \kappa_x\kappa_y,\; A_{13} = d_x\kappa_y + d_y\kappa_x,\; A_{14} = d_x d_y$$

Inverse FT of s_x computed using MAPLE for right-hand side:

$$A_{15}\delta(t) + A_{16}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{15} = \kappa_x,\; A_{16} = d_x$$

Inverse FT of s_y computed using MAPLE for right-hand side:

$$A_{17}\delta(t) + A_{18}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{17} = \kappa_y,\; A_{18} = d_y$$

Inverse FT of s_z computed using MAPLE for right-hand side:

$$A_{19}\delta(t) + A_{20}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{19} = 1,\; A_{20} = 0$$

b):XZPML
κ_x, d_x and κ_z, d_z defined as usual; $\alpha_x = \alpha_z = \alpha$ but $\kappa_y = 1, d_y = 0, \alpha_y = 0$
In this case inverse FT of $-\omega^2 s_x s_y s_z$ is:

$$A_0\delta''(t) + A_1\delta'(t) + A_2\delta(t) + A_3\mathrm{e}^{-\alpha\, t}\, H(t) + A_4\mathrm{e}^{-\alpha\, t}\, t\, H(t) + A_5\mathrm{e}^{-\alpha\, t}\, t^2\, H(t)$$

with

$$A_0 = \kappa_x\kappa_z\; ,\; A_1 = d_x\kappa_z + d_z\kappa_x\; ,\; A_2 = d_x d_z - \alpha(d_z\kappa_x + d_x\kappa_z)$$

$$A_3 = -2\alpha d_x d_z + \alpha^2(d_x\kappa_z + d_z\kappa_x), A_4 = \alpha^2 d_x d_z, A_5 = 0$$

Inverse FT of $\frac{s_y s_z}{s_x}$ computed using MAPLE for right-hand side:

$$A_6\delta(t) + A_7\mathrm{e}^{-\alpha\, t}\, H(t) + A_8\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\, H(t)$$

with

$$A_6 = \frac{\kappa_z}{\kappa_x},\; A_7 = 0,\; A_8 = \frac{d_z\kappa_x - d_x\kappa_z}{\kappa_x^2}$$

Inverse FT of $\frac{s_x s_z}{s_y}$ computed using MAPLE for right-hand side:

$$A_9\delta(t) + A_{10}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{11}\mathrm{e}^{-\alpha\, t}\, tH(t)$$

with

$$A_9 = \kappa_x \kappa_z,\; A_{10} = d_x \kappa_z + d_z \kappa_x,\; A_{11} = d_x d_z$$

Inverse FT of $\frac{s_x s_y}{s_z}$ computed using MAPLE for right-hand side:

$$A_{12}\delta(t) + A_{13}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\, H(t)$$

with

$$A_{12} = \frac{\kappa_x}{\kappa_z},\; A_{13} = 0,\; A_{14} = \frac{d_x \kappa_z - d_z \kappa_x}{\kappa_z^2}$$

Inverse FT of s_x computed using MAPLE for right-hand side:

$$A_{15}\delta(t) + A_{16}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{15} = \kappa_x,\; A_{16} = d_x$$

Inverse FT of s_y computed using MAPLE for right-hand side:

$$A_{17}\delta(t) + A_{18}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{17} = 1,\; A_{18} = 0$$

Inverse FT of s_z computed using MAPLE for right-hand side:

$$A_{19}\delta(t) + A_{20}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{19} = \kappa_z,\; A_{20} = d_z$$

c):YZPML
κ_y, d_y and κ_z, d_z defined as usual; $\alpha_y = \alpha_z = \alpha$ but $\kappa_x = 1, d_x = 0, \alpha_x = 0$
In this case inverse FT of $-\omega^2 s_x s_y s_z$ is:

$$A_0\delta''(t) + A_1\delta'(t) + A_2\delta(t) + A_3\mathrm{e}^{-\alpha\, t}\, H(t) + A_4\mathrm{e}^{-\alpha\, t}\, t\, H(t) + A_5\mathrm{e}^{-\alpha\, t}\, t^2\, H(t)$$

with

$$A_0 = \kappa_y \kappa_z\;,\; A_1 = d_y \kappa_z + d_z \kappa_y\;,\; A_2 = d_y d_z - \alpha(d_z \kappa_y + d_y \kappa_z)$$

$$A_3 = -2\alpha d_y d_z + \alpha^2 (d_y \kappa_z + d_z \kappa_y), A_4 = \alpha^2 d_y d_z, A_5 = 0$$

Inverse FT of $\frac{s_y s_z}{s_x}$ computed using MAPLE for right-hand side:

$$A_6\delta(t) + A_7\mathrm{e}^{-\alpha\, t}\, H(t) + A_8\mathrm{e}^{-\alpha\, t}\, tH(t)$$

with

$$A_6 = \kappa_y \kappa_z,\; A_7 = d_y \kappa_z + \kappa_y d_z,\; A_8 = d_y d_z$$

Inverse FT of $\frac{s_x s_z}{s_y}$ computed using MAPLE for right-hand side:

$$A_9\delta(t) + A_{10}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\, H(t)$$

with

$$A_9 = \frac{\kappa_z}{\kappa_y},\; A_{10} = 0,\; A_{11} = \frac{d_z \kappa_y - d_y \kappa_z}{\kappa_y^2}$$

Inverse FT of $\frac{s_x s_y}{s_z}$ computed using MAPLE for right-hand side:

$$A_{12}\delta(t) + A_{13}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\, H(t)$$

with

$$A_{12} = \frac{\kappa_y}{\kappa_z},\; A_{13} = 0,\; A_{14} = \frac{d_y \kappa_z - d_z \kappa_y}{\kappa_z^2}$$

Inverse FT of s_x computed using MAPLE for right-hand side:

$$A_{15}\delta(t) + A_{16}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{15} = 1,\ A_{16} = 0$$

Inverse FT of s_y computed using MAPLE for right-hand side:

$$A_{17}\delta(t) + A_{18}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{17} = \kappa_y,\ A_{18} = d_y$$

Inverse FT of s_z computed using MAPLE for right-hand side:

$$A_{19}\delta(t) + A_{20}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{19} = \kappa_z,\ A_{20} = d_z$$

d):XPML

κ_x, d_x defined as usual; $\alpha_x = \alpha$ but $\kappa_y = 1, d_y = 0, \alpha_y = 0;\ \kappa_z = 1, d_z = 0, \alpha_z = 0$

In this case inverse FT of $-\omega^2 s_x s_y s_z$ is:

$$A_0\delta''(t) + A_1\delta'(t) + A_2\delta(t) + A_3\mathrm{e}^{-\alpha\, t}\, H(t) + A_4\mathrm{e}^{-\alpha\, t}\, t\, H(t) + A_5\mathrm{e}^{-\alpha\, t}\, t^2\, H(t)$$

with

$$A_0 = \kappa_x\ ,\ A_1 = d_x\ ,\ A_2 = -\alpha d_x$$

$$A_3 = \alpha^2 d_x, A_4 = 0, A_5 = 0$$

Inverse FT of $\frac{s_y s_z}{s_x}$ computed using MAPLE for right-hand side:

$$A_6\delta(t) + A_7\mathrm{e}^{-\alpha\, t}\, H(t) + A_8\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\, H(t)$$

with

$$A_6 = \frac{1}{\kappa_x},\ A_7 = 0,\ A_8 = -\frac{d_x}{\kappa_x^2}$$

Inverse FT of $\frac{s_x s_z}{s_y}$ computed using MAPLE for right-hand side:

$$A_9\delta(t) + A_{10}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\, H(t)$$

with

$$A_9 = \kappa_x,\ A_{10} = d_x,\ A_{11} = 0$$

Inverse FT of $\frac{s_x s_y}{s_z}$ computed using MAPLE for right-hand side:

$$A_{12}\delta(t) + A_{13}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\, H(t)$$

with

$$A_{12} = \kappa_x,\ A_{13} = d_x,\ A_{14} = 0$$

Inverse FT of s_x computed using MAPLE for right-hand side:

$$A_{15}\delta(t) + A_{16}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{15} = \kappa_x,\ A_{16} = d_x$$

Inverse FT of s_y computed using MAPLE for right-hand side:

$$A_{17}\delta(t) + A_{18}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{17} = 1,\ A_{18} = 0$$

Inverse FT of s_z computed using MAPLE for right-hand side:

$$A_{19}\delta(t) + A_{20}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{19} = 1,\ A_{20} = 0$$

e):YPML
κ_y, d_y defined as usual; $\alpha_y = \alpha$ but $\kappa_x = 1, d_x = 0, \alpha_x = 0;\ \kappa_z = 1, d_z = 0, \alpha_z = 0$
In this case inverse FT of $-\omega^2 s_x s_y s_z$ is:

$$A_0\delta''(t) + A_1\delta'(t) + A_2\delta(t) + A_3\mathrm{e}^{-\alpha\, t}\, H(t) + A_4\mathrm{e}^{-\alpha\, t}\, t\, H(t) + A_5\mathrm{e}^{-\alpha\, t}\, t^2\, H(t)$$

with

$$A_0 = \kappa_y\,,\ A_1 = d_y\,,\ A_2 = -\alpha d_y$$

$$A_3 = \alpha^2 d_y, A_4 = 0, A_5 = 0$$

Inverse FT of $\frac{s_y s_z}{s_x}$ computed using MAPLE for right-hand side:

$$A_6\delta(t) + A_7\mathrm{e}^{-\alpha\, t}\, H(t) + A_8\mathrm{e}^{-t(\alpha+\frac{d_x}{\kappa_x})}\, H(t)$$

with

$$A_6 = \kappa_y,\ A_7 = d_y,\ A_8 = 0$$

Inverse FT of $\frac{s_x s_z}{s_y}$ computed using MAPLE for right-hand side:

$$A_9\delta(t) + A_{10}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{11}\mathrm{e}^{-t(\alpha+\frac{d_y}{\kappa_y})}\, H(t)$$

with

$$A_9 = \frac{1}{\kappa_y},\ A_{10} = 0,\ A_{11} = -\frac{d_y}{\kappa_y^2}$$

Inverse FT of $\frac{s_x s_y}{s_z}$ computed using MAPLE for right-hand side:

$$A_{12}\delta(t) + A_{13}\mathrm{e}^{-\alpha\, t}\, H(t) + A_{14}\mathrm{e}^{-t(\alpha+\frac{d_z}{\kappa_z})}\, H(t)$$

with

$$A_{12} = \kappa_y,\ A_{13} = d_y,\ A_{14} = 0$$

Inverse FT of s_x computed using MAPLE for right-hand side:

$$A_{15}\delta(t) + A_{16}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{15} = 1,\ A_{16} = 0$$

Inverse FT of s_y computed using MAPLE for right-hand side:

$$A_{17}\delta(t) + A_{18}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{17} = \kappa_y,\ A_{18} = d_y$$

Inverse FT of s_z computed using MAPLE for right-hand side:

$$A_{19}\delta(t) + A_{20}\mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{19} = 1,\ A_{20} = 0$$

f):ZPML
κ_z, d_z defined as usual; $\alpha_z = \alpha$ but $\kappa_x = 1, d_x = 0, \alpha_x = 0;\ \kappa_y = 1, d_y = 0, \alpha_y = 0$
In this case inverse FT of $-\omega^2 s_x s_y s_z$ is:

$$A_0\delta''(t) + A_1\delta'(t) + A_2\delta(t) + A_3\mathrm{e}^{-\alpha\, t}\, H(t) + A_4\mathrm{e}^{-\alpha\, t}\, t\, H(t) + A_5\mathrm{e}^{-\alpha\, t}\, t^2\, H(t)$$

with

$$A_0 = \kappa_z\,,\ A_1 = d_z\,,\ A_2 = -\alpha d_z$$

$$A_3 = \alpha^2 d_z, A_4 = 0, A_5 = 0$$

Inverse FT of $\frac{s_y s_z}{s_x}$ computed using MAPLE for right-hand side:

$$A_6\delta(t) + A_7 \mathrm{e}^{-\alpha\, t}\, H(t) + A_8 \mathrm{e}^{-t(\alpha + \frac{d_x}{\kappa_x})}\, H(t)$$

with

$$A_6 = \kappa_z,\ A_7 = d_z,\ A_8 = 0$$

Inverse FT of $\frac{s_x s_z}{s_y}$ computed using MAPLE for right-hand side:

$$A_9\delta(t) + A_{10} \mathrm{e}^{-\alpha\, t}\, H(t) + A_{11} \mathrm{e}^{-t(\alpha + \frac{d_y}{\kappa_y})}\, H(t)$$

with

$$A_9 = \kappa_z,\ A_{10} = d_z,\ A_{11} = 0$$

Inverse FT of $\frac{s_x s_y}{s_z}$ computed using MAPLE for right-hand side:

$$A_{12}\delta(t) + A_{13} \mathrm{e}^{-\alpha\, t}\, H(t) + A_{14} \mathrm{e}^{-t(\alpha + \frac{d_z}{\kappa_z})}\, H(t)$$

with

$$A_{12} = \frac{1}{\kappa_z},\ A_{13} = 0,\ A_{14} = -\frac{d_z}{\kappa_z^2}$$

Inverse FT of s_x computed using MAPLE for right-hand side:

$$A_{15}\delta(t) + A_{16} \mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{15} = 1,\ A_{16} = 0$$

Inverse FT of s_y computed using MAPLE for right-hand side:

$$A_{17}\delta(t) + A_{18} \mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{17} = 1,\ A_{18} = 0$$

Inverse FT of s_z computed using MAPLE for right-hand side:

$$A_{19}\delta(t) + A_{20} \mathrm{e}^{-\alpha\, t}\, H(t)$$

with

$$A_{19} = \kappa_y,\ A_{20} = d_z$$

The time domain equation of CPML in edge-PML and surface-PML can be derived directly following the above way used to get the time domain equation of CPML in corner-PML

1 TIME SCHEME FOR THE CONVOLUTION TERMS

The last step is to find a recursive way of computing all the convolutions. Equations for first-order convolution performed using the first-order-accurate recursive approach of Luebbers & Hunsberger (1992) (but they are unstable in practice):

$$\partial_{\tilde{x}} = \frac{1}{\kappa_x}\partial_x + \psi_x \tag{17}$$

with

$$\psi_x^n = b_x \psi_x^{n-1} + a_x \left(\partial_x\right)^{n-\frac{1}{2}}\ . \tag{18}$$

Equations for second-order convolution: To get a stable time scheme we thus use the second-order-accurate recursive approach developed recently in in Wang and Texeira 2006, which is that for a mathematical convolution

$$f(t) = a\mathrm{e}^{-bt} H(t) * u(t) = a\psi_(t)$$

from Wang and Texeira 2006 we know that we can compute the discrete form using

$$\psi^n = \mathrm{e}^{-b\Delta t}\psi^{n-1} + (\frac{1}{b}(1-\mathrm{e}^{-b\Delta t/2}))u^n + (\frac{1}{b}(1-\mathrm{e}^{-b\Delta t/2})\mathrm{e}^{-b\Delta t/2})u^{n-1} \quad (19)$$

It can also used to compute mathematical convolutions as following

$$g(t) = [a\mathrm{e}^{-bt}tH(t)] * u(t)$$

which gives

$$g(t) = \int_{-\infty}^{t} a\mathrm{e}^{-b(t-\tau)}(t-\tau)H(t-\tau)u(\tau)d\tau$$

according to linearity of convolution we have

$$g(t) = at\int_{-\infty}^{t} \mathrm{e}^{-b(t-\tau)}H(t-\tau)u(\tau)d\tau - a\int_{-\infty}^{t} \mathrm{e}^{-b(t-\tau)}H(t-\tau)\tau u(\tau)d\tau$$

or

$$g(t) = at[\mathrm{e}^{-bt}H(t) * u(t)] - a[\mathrm{e}^{-bt}H(t) * (tu(t))]$$

Then the same discrete scheme (19) can be used to compute the first and second square brackets term on the right hand. Also we can used to compute mathematical convolutions as following (this kind of term only appears in the 3D case)

$$g(t) = [a\mathrm{e}^{-bt}t^2H(t)] * u(t)$$

which gives

$$g(t) = \int_{-\infty}^{t} a\mathrm{e}^{-b(t-\tau)}(t-\tau)^2H(t-\tau)u(\tau)d\tau$$

according to linearity of convolution we have

$$g(t) = at^2\int_{-\infty}^{t} \mathrm{e}^{-b(t-\tau)}H(t-\tau)u(\tau)d\tau - 2at\int_{-\infty}^{t} \mathrm{e}^{-b(t-\tau)}\tau H(t-\tau)u(\tau)d\tau + \int_{-\infty}^{t} a\mathrm{e}^{-b(t-\tau)}\tau^2H(t-\tau)u(\tau)d\tau$$

or

$$g(t) = at^2[\mathrm{e}^{-bt}H(t) * u(t)] - 2at[\mathrm{e}^{-bt}H(t) * (tu(t))] + [\mathrm{e}^{-bt}H(t) * (t^2u(t))]$$

Then the same discrete scheme (19) can be used to compute the first, second and third square brackets term on the right hand.

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