1 HW 2 in Electronic Commerce Models, due date 25/12/2024

1.1 Question: Optimal Revenue Auction

Each player can be one of two possible types:

- **Type A**: The cumulative distribution function (CDF) of the valuation $v \in [0, 5]$ is given by

$$F_A(v) = \frac{v^2}{25}.$$

- **Type B**: The CDF of the valuation $v \in [0, 5]$ is given by

$$F_B(v) = \frac{1 - \exp(-v)}{1 - \exp(-5)},$$

.

Tasks:

1. **Virtual Value Function:** For each type:
   - Compute the virtual value function.
   - Verify that the virtual value function is monotone non-decreasing.
   - Compute the value of v where the virtual value function equals zero for each type.
2. **Optimal Revenue Auction Design:** Design the optimal revenue-maximizing auction and compute the expected revenue in the following scenarios:
   - **Case 1:** One item, a single player of type A.
   - **Case 2:** One item, a single player of type B.
   - **Case 3:** One item, a single player of type A and a single player of type B.
   - **Case 4:** One item, two players of type A.
   - **Case 5:** One item, two players of type B.
   - **Case 6:** Two items, one player of type A and one player of type B.

1.2 Question: Optimal Auction Design for n Players and Two Items

Consider designing an **auction** for n players competing for two identical items. Each player i has a private valuation v_i, independently and identically drawn from a uniform distribution over $[0, 1]$.

- For $n = 4$, design an auction that maximizes the **Social Welfare**. Provide an explicit description of the allocation rule and payment rule.
- For $n = 4$, design an auction that maximizes the **Revenue**. Provide an explicit description of the allocation rule and payment rule.
- Compute the interim allocation function $x^*(v)$ in both cases.
- Use the function interim allocation function $x^*(v)$ to compute the revenue of each mechanism
- (Bonus) For $n = 3$, Using payment identity to compute BNE for the first-price auction

1.3 Question: Optimal Revenue Auction with Restrictions

Consider four players, A, B, C, and D, whose valuations are independently and uniformly distributed over $[0, 1]$.
Rules:

- You cannot sell to both players A and B simultaneously.
- You cannot sell to both players D and C simultaneously.
- All other combinations of sales are feasible.

Tasks:

1. Design an optimal revenue-maximizing auction under these constraints.
2. Write an explicit description of the payment rule for each possible allocation.
3. Compute the expected revenue of the auction.

1.4 Question: Prior-Free Analysis

1. **DOP Auction Example:** Construct an example of bids in the DOP auction where the item is sold at different prices to two different bidders.
   - Explain why this outcome might be undesirable.
2. **Profit Extractor Proof:** Prove that the profit extractor is strategy-proof for a fixed reward R.
3. **Expected Revenue - Random Price:** Compute the expected revenue from selling an item whose value is uniformly distributed over $[0, 1]$, where the price is sampled from a uniform $[0, 1]$ distribution.
4. **Expected Revenue - Mean Price:** Compute the expected revenue from selling an item whose value is uniformly distributed over $[0, 1]$, where the price is set to the mean of two sampled values from a uniform $[0, 1]$ distribution.

Density Function of the Mean:

The density function of the mean $M = \frac{X_1+X_2}{2}$, where X_1 and X_2 are independent random variables uniformly distributed over $[0, 1]$, is given by:

$$f_M(m) = \begin{cases} 4m, & 0 \leq m \leq 0.5, \\ 4(1-m), & 0.5 < m \leq 1. \end{cases}$$