Matching Markets

Economics and Computation

Matching Markets

Think of the setting of assigning dorm rooms to students, houses to buyers, etc. Each person wants one of each item. Each item can be given to at most one person.

Matching Markets

Matching markets are used in a number of real-world settings, including:

- assigning medical students to residency positions in hospitals to continue their training
- assigning students to high schools
- assigning cadaver kidneys or live kidney donations to patients with kidney disease

Matching Markets

- Two-Sided Matching Markets
  - Stable Matching – See Intro to GT slides
- Assignment Problems
  - The House Allocation Problem
  - Prices and the Market-Clearing property

The House Allocation Problem

Let $N = \{1, \ldots, n\}$ denote the set of agents, and G denote the set of items. To keep things simple, we assume that $G = \{1, \ldots, n\}$, so that there are as many items as there are agents. Items are assumed to be indivisible. Let $x = (x_1, \ldots, x_n)$ denote a *feasible assignment*, such that each agent is assigned to one item and no item is assigned more than once. Item $x_i \in G$ is assigned to agent i.

The House Allocation Problem

An agent's preference for assignments is assumed to depend only on the item that it is assigned. Given this, we define preference orders on assignments through preferences on items. Let $\succ_i$ denote a *strict preference order on items*, so that $j \succ_i j'$ for items j and j' means that agent i strictly prefers j over j'. This implies a weak preference order $\succeq_i$ on assignments. In particular, we write $x \sim_i x'$ for two assignments x and x' for which $x_i = x'_i$. Otherwise, the preference of agent i for x over x' depends on the preference order $\succ_i$ applied to item x_i and item x'_i. We write $x \succeq_i x'$ to denote that $x_i = x'_i$, or $x_i \succ_i x'_i$.

Pareto Optimal

In the house allocation problem, the design goal is to implement a *Pareto optimal* assignment.

An assignment x is *Pareto optimal* if there is no assignment x' such that $x' \succ_i x$ for some agent i, and $x' \succeq_k x$ for all agents $k \in N$.

The House Allocation Problem

Let π denote a *priority order* on agents so that $\pi(k) \in N$ for $k \in \{1, \ldots, n\}$ is the agent with priority k and $\pi(k) \neq \pi(k')$ for all $k \neq k'$. Consider the following simple mechanism:

Definition 12.18 (Serial dictatorship)**.** *The* serial dictatorship *(SD) mechanism with priority order π works as follows:*

- *Each agent i makes a simultaneous report about their preference order $\hat{\succ}_i$.*
- *In each step $k \in \{1, 2, \ldots, n\}$, assign agent $i = \pi(k)$ to their most preferred item amongst those items not already assigned, based on preference order $\hat{\succ}_i$.*

The House Allocation Problem

Theorem 12.19. *The serial dictatorship mechanism is strategy-proof and Pareto optimal fo the house allocation problem.*

Proof. The SD mechanism is strategy-proof because agent i with priority k, i.e., $\pi(i) = k$ cannot affect the item allocated to other agents before their turn, and by reporting the true preference order the agent will receive the most preferred item of those still available.

To see that the outcome x is Pareto optimal, we can try to construct x' that Pareto dominates x. We need $x'_{\pi(1)} = x_{\pi(1)}$, because the top priority agent obtains their most preferred item in x. Continuing, we need $x'_{\pi(2)} = x_{\pi(2)}$, because the second priority agent $\pi(2)$ obtains their most preferred item in x across the remaining items. After n steps we require $x' = x$. $\square$

Strict preferences on items are important in proving this result. Without this, some agent might be indifferent between two items, say items 1 and 2, and choose 1, when the next agent in priority has a strict preference for item 1 over 2.

The House Allocation Problem

The SD mechanism also provides a building block for the following randomized mechanism:

Definition 12.20 (Random serial dictatorship). *The* random serial dictatorship *(RSD) mechanism for the house allocation problem runs the SD mechanism on a priority order sampled uniformly at random from all possible orders.*

Theorem 12.21. *The random serial dictatorship mechanism is strategy-proof and Pareto optimal for the house allocation problem.*

In particular, RSD is strategy-proof whatever the realized coin flips internal to the mechanism.

The House Allocation Problem

Another possible design goal is *ex ante Pareto optimality*, which requires that there is no distribution over assignments that is weakly improving for all and strictly improving for at least one agent. The RSD mechanism does not satisfy this property. This is developed in Exercise 12.5, along with the idea of a mechanism that is ex ante Pareto optimal but not strategy-proof.

By adopting a random priority order, RSD provides a form of fairness that is not satisfied by the SD mechanism— it is *anonymous* in the sense that, fixing the reported preferences, the distribution on assignments is invariant to changing the identities of agents. In fact, RSD is conjectured to be the unique mechanism that is anonymous, Pareto optimal, and strategy-proof. See the chapter notes.

The Housing Markets Problem

We now suppose each agent comes to the market with an item. Let x^0 denote the initial assignment. Agent i owns item $x_i^0 \in G$, and each item is owned by some agent. We assume that the initial ownership is known to the mechanism. We require individual-rationality (IR), so that $x_i \succeq_i x_i^0$ for every agent i, where x is the selected assignment. Every agent must be at least as happy with the outcome as with the item brought to the marketplace.

The Housing Markets Problem

A useful property of a matching mechanism for the housing markets problem is that the mechanism produces a *core outcome*. This is a consideration that comes from supposing that participants may be able to break away from the market and trade amongst themselves. The core property requires that no group of participants can trade the items that they own in the initial assignment in a way that makes them all just as happy and some strictly happier than the outcome of the matching mechanism.

The Housing Markets Problem

Let $X(L)$ denote the set of assignments that a set L of agents can obtain by trading the items that they own in the initial assignment amongst themselves. Say that a set of agents form a *blocking coalition* for an assignment if they can do better by trading amongst themselves:

Definition 12.22 (Blocking coalition). *A set $L \subseteq N$ of agents form a* blocking coalition *for assignment x if there is some assignment $x' \in X(L)$ such that $x'_i \succ_i x_i$ for at least one agent $i \in L$ and there is no agent $k \in L$ for whom $x_k \succ_k x'_k$.*

In words, for L to be a blocking coalition, all agents in L must be just as happy, and at least one agent must be strictly happier.

The Housing Markets Problem

Definition 12.23 (Core assignment)**.** *An assignment x is in the core if and only if there is no blocking coalition.*

If a mechanism produces a core outcome then no group of agents could have done better by breaking away and trading amongst themselves. The special case considers a coalition that comprises just a single agent, and shows that a core assignment also satisfies IR.

Core assignments are a strict subset of the Pareto optimal assignments. Certainly, a core assignment is also Pareto optimal because the coalition of all agents would otherwise form a blocking coalition for the assignment. But a Pareto optimal outcome need not be in the core (see Exercise 12.4).

The Housing Markets Problem

A cycle on a directed graph is a sequence of $k > 1$ edges, $i^{(1)} \to i^{(2)} \to i^{(k)} \to i^{(1)}$, involving k vertices. The *top-trading cycles (TTC) mechanism* operates on a directed graph, with agents as vertices, and edges representing agent preferences.

The Housing Markets Problem

Definition 12.24 (Top-trading cycles mechanism)**.** *The* top-trading cycles *mechanism proceeds as follows:*

- *Each agent i makes a simultaneous claim about their preference order, $\hat{\succ}_i$, on items.*
- *Round 1: Form a directed graph with agents as vertices, and a directed edge from each agent i to the agent who owns their most preferred item, based on report $\hat{\succ}_i$. Perform the trades on cycles and remove all agents that trade, along with the items that they owned.*
- *Round k for $k > 1$: form a directed graph with the remaining agents, with a directed edge from each agent i who is still present to the agent who owns their most preferred item and is still present. Perform the trades on cycles and remove all agents that trade, along with the items that they owned.*

Repeat until there are no agents left.

The Housing Markets Problem

Example 12.25. *Suppose there are nine agents, and assume that each agent $i \in \{1, \ldots, 9\}$ owns item i. Consider the following preference profile:*

$$\begin{array}{llllll} \succ_1: & 2 \cdots & \succ_4: & 5 \cdots & \succ_7: & 4\ 5\ 6\ 3\ 8 \cdots \\ \succ_2: & 1 \cdots & \succ_5: & 3 \cdots & \succ_8: & 7 \cdots \\ \succ_3: & 4 \cdots & \succ_6: & 4\ 3\ 6 \cdots & \succ_9: & 6\ 4\ 7\ 3\ 9 \cdots \end{array}$$

We only specify as many of the preferences as required for the example. Figure 12.27 shows the directed graph that is formed in each round of the TTC procedure. In each round, each remaining agent "points at" (i.e., has an out-edge to) the agent with the most preferred item of those remaining in the market.

In the first round there are two cycles, for agents 1 and 2 and agents 3, 4 and 5. The direction of trade is against the direction of the cycle. Agents 1 and 2 swap their items. Item 4 goes to agent 3, item 5 to agent 4, and item 3 to agent 5. These agents are removed along with their items.

In the second round, the edges are updated so that each remaining agent points to the most preferred remaining item; i.e., agent 7 now points to 6 because 4 and 5 have left the market. There is a single cycle, and agent 6 exits the market without trading. In the third round, agent 8 now points at 7 and agent 7 at 8. Agents 7 and 8 swap. In the final round, there is a single cycle and agent 9 leaves the market without trading. The final assignment is:

agent	1	2	3	4	5	6	7	8	9
item	2	1	4	5	3	6	8	7	9

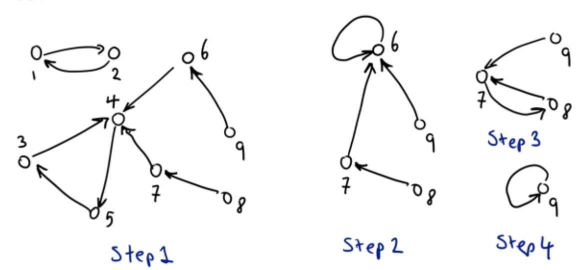

Figure 12.27.: The four rounds of the Top-Trading Cycles (TTC) mechanism on Example 12.25.

The Housing Markets Problem

Lemma 12.26. *The directed graph formed in each round of the TTC mechanism includes at least one cycle, and when there are multiple cycles in a single round no vertex participates in more than one cycle.*

Proof. Exercise 12.4 verifies that there must be at least one cycle in a directed graph with one out-edge from each vertex (perhaps a self-loop, with an agent pointing to itself). To see that all cycles are vertex-disjoint, suppose otherwise, and that there are two cycles that both include agent i. Because every agent has exactly one out-edge, there is a unique directed path from agent i, following out-edges, and there cannot be two directed cycles that involve agent i. $\square$

The Housing Markets Problem

Theorem 12.28. *The TTC mechanism is strategy-proof.*

Proof. Let N_k denote the set of agents that trade in round k under truthful reports. Suppose when agent i is truthful, that i trades in round t, and that following a misreport, agent i trades in round s. Given that i trades in round s, the same trades $N_1, \ldots, N_{s-1}$ still occur, irrespective of i's behavior before round s. The base case for $k = 1$ is immediate: the directed graph is the same other than possibly the out-edge from i, all the cycles that form when i is truthful continue to form whatever the report of i, and no additional cycles form. Given the inductive hypothesis, the argument for round $k > 1$ ($< s$) is unchanged, and we have trades N_k. We proceed by case analysis.

Case $s \geq t$: the misreport is not useful, because by being truthful i gets the best item from the set of items available at round t, and there are only fewer items available in future rounds.

Case $s < t$: suppose agent i gets item x, which means there is a directed path from item x to agent i. When i is truthful, i will point to another item in round s (and does not get it), but the directed path from x to i still exists. Moreover, as long as i continues to participate, the directed path from x to i remains in place, since all people and items on this path remain in the market. Therefore, at round t, item x is still available. By being truthful, i gets the best item from the set of items available at round t, which is at least as good as item x.

$\square$

The Housing Markets Problem

Theorem 12.29. *The TTC mechanism is core-selecting, strategy-proof, Pareto optimal and individual-rational. The core assignment is also unique.*

Proof. Let N_k denote the agents who trade in round k. Let x_{ttc} denote the outcome of TTC.

For the core property, suppose for contradiction that $L \subseteq N$ are blocking, with L able to trade amongst themselves and achieve an assignment x_L that they collectively prefer to x_{ttc} (with $x_L \neq x_{\text{ttc}}$). Because the agents in N_1 receive their most preferred items in x_{ttc}, then any $i \in L \cap N_1$ must receive the same item in x_L as x_{ttc}, else L would not be blocking. In particular, for any such i, all agents on the cycle in TTC where i trades must be part of L. But now, any $j \in L \cap N_2$ must get the same item in x_L as in x_{ttc}, since they obtain the most preferred item amongst those not traded in N_1 in x_{ttc}, and trades involving $L \cap N_1$ are left undisturbed in x_L. In particular, for any such j, all agents on the cycle in TTC where j trades must be part of L. This continues, and we conclude that L must include all agents, and assignment x_L must be the same as x_{ttc}, and a contradiction.

We now prove that the core assignment is unique; specifically, if assignment x_c is a core assignment then it must be assignment x_{ttc}. The agents in set N_1 in TTC trade amongst themselves, and each obtains their most preferred item. Any core assignment must assign these agents as in TTC, otherwise N_1 would be a blocking coalition. The agents in set N_2 trade amongst themselves in x_{ttc}, and each obtains the most preferred item amongst the one or more items not owned by set N_1. Suppose x_c assigns the agents in N_1 in the same way as in TTC, but not the agents in N_2. Then agents in N_2 would form a blocking coalition to x_c. Continuing through all rounds, this proves that the assignment x_c must be equal to x_{ttc} because otherwise there would be a blocking coalition.

Pareto optimality is a property of the core assignment. Without Pareto optimality, the coalition of all agents would be blocking. IR is a property of a core assignment (consider blocking by a single agent). $\square$

The Housing Markets Problem

Practical Considerations and Discussion:

A variation on the TTC mechanism that also needs to handle school priorities for students is used for school choice in some US school districts, including San Francisco, and had been used in the past in New Orleans (see Exercise 12.3 for a development for school choice).

In comparison with the student-proposing DA, which is more popular and is used in Boston, New York and New Orleans, TTC can provide better welfare properties for students. A

drawback is that the TTC mechanism is more complicated to describe, and that it does not ensure stability (intepreted here as providing that students will not envy the placement of another student, given school priorities).

The theoretical properties of TTC are sensitive to small changes in the problem definition. For a sense of the difficulties, there can be no core assignments for settings with more than one kind of item, tables and chairs, for example, and agents with preferences on pairs of items. Indifferences in agent preferences on different assignments can also lead to difficulties. See the chapter notes for additional discussion.

Prices and the Market-Clearing property

See Paul G. Spirakis Slides.

COMP323 – Introduction to Computational Game Theory

Matching Markets

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Outline

Markets

- We think about markets as a prime example of network-structured interaction between many agents.
- When we consider markets creating opportunities for interaction among buyers and sellers, there is an implicit network encoding the access between these buyers and sellers.
- Matching markets form a class of network models of interactions among market participants.

Matching markets

Matching markets have a long history of study in economics, operations research, and other areas because they embody, in a very clean and stylized way, a number of basic principles:

- the way in which people may have different preferences for different kinds of goods,
- the way in which prices can decentralize the allocation of goods to people, and
- the way in which such prices can in fact lead to allocations that are socially optimal.

Allocations of goods based on preferences

- We begin with a setting in which goods will be allocated to people based on preferences.
- These preferences will be expressed in network form, but there is no explicit buying, selling, or price-setting.
- This first simple setting will be a crucial component of the more complex ones that follow.

Bipartite matching problem

The model we start with is called the bipartite matching problem, and we can motivate it via the following scenario.

- Suppose that the administrators of a college dormitory are assigning rooms to returning students for a new academic year.
- Each room is designed for a single student, and each student is asked to list several acceptable options for the room they'd like to get.
- Students can have different preferences over rooms; some people might want larger rooms, quieter rooms, sunnier rooms, and so forth and so the lists provided by the students may overlap in complex ways.

Bipartite matching problem

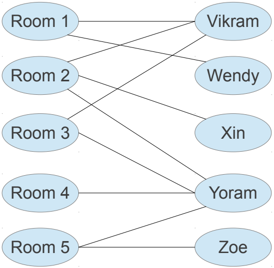
We can model the lists provided by the students using a bipartite graph as follows:

> There is a node for each student, a node for each room, and an edge connecting a student to a room if the student has listed the room as an acceptable option.

- Recall that, in a bipartite graph, the nodes are divided into two categories, and each edge connects a node in one category to a node in the other category.
- In this case, the two categories are students and rooms.
- Bipartite graphs are useful for modeling situations in which individuals or objects of one type are being assigned or matched up with individuals or objects of another type.

Bipartite matching problem

Example



An example with five students and five rooms indicating, for instance, that the student named Vikram has listed each of Rooms 1, 2, and 3 as acceptable options, while the student named Wendy only listed Room 1.

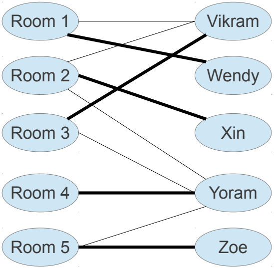
Perfect matchings

- The task that the college dorm administrators are trying to solve is to assign each student a room that they'd be happy to accept.
- This task has a natural interpretation in terms of the bipartite graph we have just drawn: since the edges represent acceptable options for students, we want to assign a distinct room to each student, so that each student is assigned a room to which he or she is connected by an edge.
- Such an assignment is a perfect matching in the bipartite graph, i.e., a choice of edges so that each node is the endpoint of exactly one of the chosen edges.

Perfect matchings

Example

The following figure shows such an assignment, with darkened edges indicating who gets which room.



Constricted sets

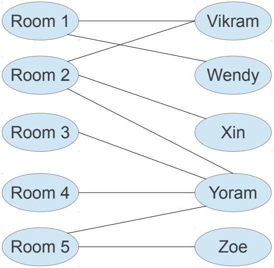
- If a bipartite graph has a perfect matching, it's easy to demonstrate this: you just indicate the edges that form the perfect matching.
- But what if a bipartite graph has no perfect matching? What could you show someone to convince them that there isn't one?

Constricted sets provide a clean way to demonstrate that no perfect matching exists:

- Let $G = (V, E)$ be a bipartite graph with parts A and B ($V = A \cup B$).
- For any $S \subseteq B$ we say that $v \in A$ is a neighbor of S if there exists $s \in S$ such that $(v, s) \in E$.
- The neighbor set of S, denoted $N(S)$, is the set of all neighbors of S.
- A set $S \subset B$ is constricted is S is strictly larger than $N(S)$; that is, $|S| > |N(S)|$.

Constricted sets

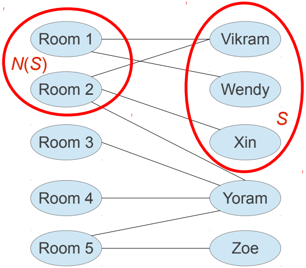
Example



The figure shows a bipartite graph that contains no perfect matching.

Constricted sets

Example

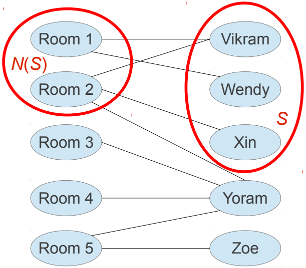


A succinct reason why there is no perfect matching in this graph:

- The set consisting of Vikram, Wendy, and Xin, has collectively provided only two options for acceptable rooms.
- With 3 people and only 2 acceptable rooms, there is clearly no way to construct a perfect matching.

Constricted sets

Example



The set of three students in this example is called a constricted set, since their edges to the other side of the bipartite graph "constrict" the formation of a perfect matching.

The Matching Theorem

- Any time there is a constricted set S in a bipartite graph, it immediately shows that there can be no perfect matching: each node in S would have to be matched to a different node in $N(S)$, but there are more nodes in S than there are in $N(S)$, so this is not possible.
- So it's easy to see that constricted sets form one kind of obstacle to the presence of perfect matchings.
- The Matching Theorem asserts that a perfect matching exists if and only if no constricted set exists.

Theorem (Matching Theorem)

If a bipartite graph with equal numbers of nodes on each part has no perfect matching, then it must contain a constricted set.

The Matching Theorem

- The details of the proof were given in a previous lecture.
- What the theorem says is that the simple notion of a constricted set is in fact the only obstacle to having a perfect matching.
- One way to think about the Matching Theorem, using our example of students and rooms, is as follows:
    - After the students submit their lists of acceptable rooms, its easy for the dormitory administrators to explain to the students what happened, regardless of the outcome.
    - Either they can announce the perfect matching giving the assignment of students to rooms, or they can explain that no assignment is possible by indicating a set of students who collectively gave too small a set of acceptable options.

The latter case is a constricted set.

Valuations

The problem of bipartite matching illustrates some aspects of a market in a very simple form:

- individuals express preferences in the form of acceptable options;
- a perfect matching then solves the problem of allocating objects to individuals according to these preferences; and
- if there is no perfect matching, it is because of a "constriction" in the system that blocks it.

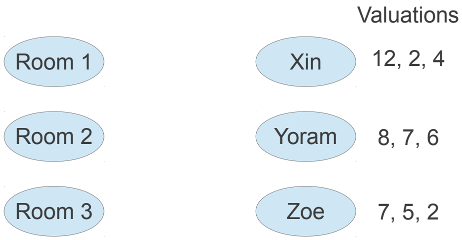
We now extend this model to introduce an additional feature:

- Rather than expressing preferences simply as binary "acceptable-or-not" choices, we allow each individual to express how much they would like each object, in numerical form.
- Thus, each individual has a valuation for each object.

Valuations

Example

In our example of students and rooms, suppose that rather than specifying a list of acceptable rooms, each student provides a valuation (numerical score) for each room, indicating how happy they would be with it.



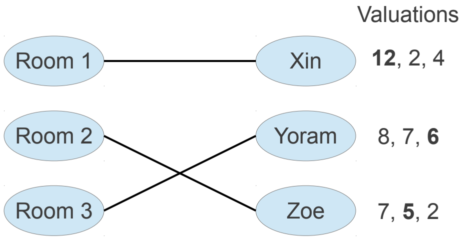
E.g., Yoram's valuations for Rooms 1, 2, and 3 are 8, 7, and 6 respectively.
Note: students may disagree on which rooms are better, and by how much.

Quality of an assignment

Using valuations, we can evaluate the quality of an assignment of objects to individuals:

The quality of an assignment is the sum of each individual's valuation for what they are assigned.

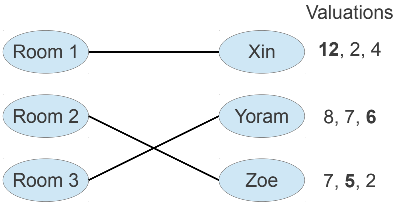
For example, the quality of the following assignment



is $12 + 6 + 5 = 23$.

Optimal assignments

- If the dorm administrators had accurate data on each student's valuations, then a reasonable way to assign rooms would be to choose the assignment of maximum possible quality.
- We will refer to this as the optimal assignment, since it maximizes the total happiness of everyone for what they get.
- While the optimal assignment maximizes total happiness, it does not necessarily give everyone their favorite item:

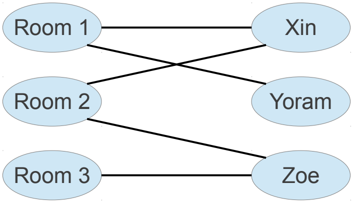


The assignment in our example is optimal, however all the students think Room 1 is the best, but it can only go to one of them.

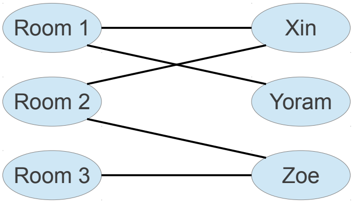
Optimal assignments and bipartite matching

- In a very concrete sense, the problem of finding an optimal assignment also forms a natural generalization of the bipartite matching problem.
- Specifically, it contains the bipartite matching problem as a special case.

To see why this holds, suppose that, in our example, there are an equal number of students and rooms, and each student simply submits a list of acceptable rooms without providing a numerical valuation.
This gives a bipartite graph:



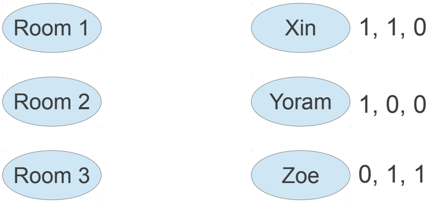
Optimal assignments and bipartite matching



We would like to know if this bipartite graph contains a perfect matching, and we can express precisely this question in the language of valuations and optimal assignments as follows:

- we give each student a valuation of 1 for each room they included on their acceptable list, and
- a valuation of 0 for each room they omitted from their list.

Optimal assignments and bipartite matching



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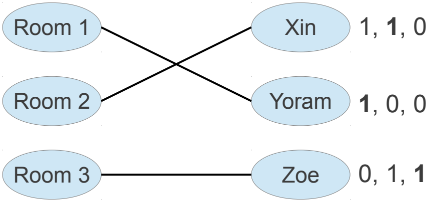
Optimal assignments and bipartite matching



- There is a perfect matching iff we can find an assignment that gives each student a room she values at 1 rather than 0, i.e., iff the optimal assignment has a total valuation equal to the number of students.
- This shows how the problem of bipartite matching is implicit in the broader problem of finding an optimal assignment.
- But is there a comparably natural way to find or characterize the optimal assignment for a given set of valuations?
- We will next describe a way to determine an optimal assignment, in the context of a broader market interpretation of this problem.

Prices and valuations

- Thus far, we have been using the metaphor of a central administrator who determines a perfect matching, or an optimal assignment, by collecting data from everyone and then performing a centralized computation.
- A more standard picture of a market involves much less central coordination, with individuals making decisions based on prices and their own valuations.
- We will see that if we replace the role of the central administrator by a particular scheme for pricing items, then allowing individuals to follow their own self-interest based on valuations and prices can still produce optimal assignments.

Setting

- Suppose that we have a collection of sellers, each with a house for sale, and an equal-sized collection of buyers, each of whom wants a house.
- Each buyer has a valuation for each house.
- Two different buyers may have very different valuations for the same houses.
- The valuation that a buyer j has for the house held by seller i will be denoted v_{ij}.
- We assume that each valuation is a non-negative integer number $(0, 1, 2, \ldots)$ and that sellers have a valuation of 0 for each house; they care only about receiving payment from buyers.

Prices and payoffs

- Each seller i puts her house up for sale, offering to sell it for a price $p_i > 0$.
- If a buyer j buys the house from seller i at this price, the buyer's payoff is her valuation for this house, minus the amount of money she had to pay: $v_{ij} - p_i$.
- Given a set of prices, if buyer j wants to maximize her payoff, she will buy from the seller i for which this quantity $v_{ij} - p_i$ is maximized.

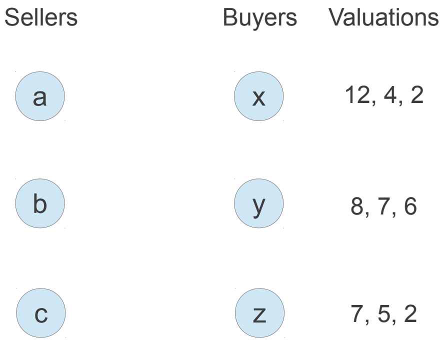
Prices and payoffs

- If $v_{ij} - p_i$ is maximized in a tie between several sellers, then the buyer can maximize her payoff by choosing any one of them.
- If her payoff $v_{ij} - p_i$ is negative for every choice of seller i, then the buyer would prefer not to buy any house: we assume she can obtain a payoff of 0 by simply not transacting.
- We will call the seller or sellers that maximize the payoff for buyer j the preferred sellers of buyer j, provided the payoff from these sellers is not negative.
- We say that buyer j has no preferred seller if the payoffs $v_{ij} - p_i$ are negative for all choices of i.

Prices and payoffs

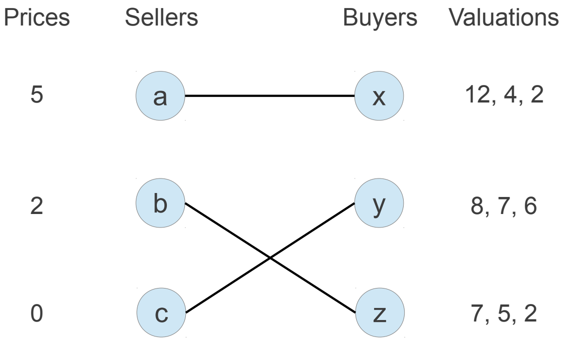
Examples

- Consider the following example of 3 sellers (a, b, and c) and 3 buyers (x, y, and z).
- For each buyer node, the valuations for the houses of the respective sellers appear in the list next to the node.



Prices and payoffs

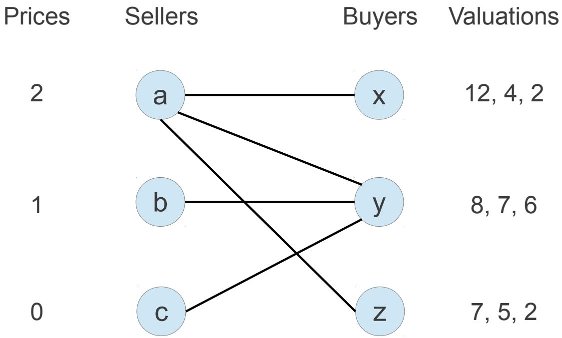
Examples



- Given the prices, each buyer creates a link to her preferred seller.
- For example, buyer x would receive a payoff of $12 - 5 = 7$ if she buys from a, a payoff of $4 - 2 = 2$ if she buys from b, and $2 - 0 = 2$ if she buys from c. This is why a is her unique preferred seller.

Prices and payoffs

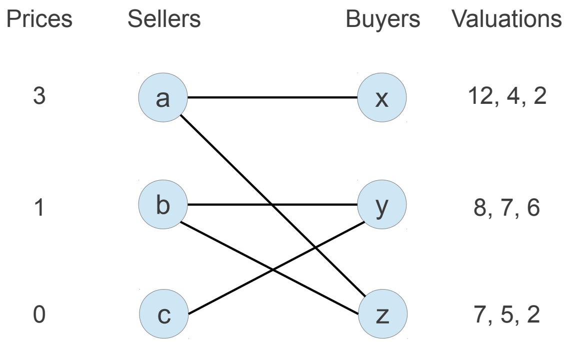
Examples



- The preferred-seller graph for prices 2, 1, and 0.

Prices and payoffs

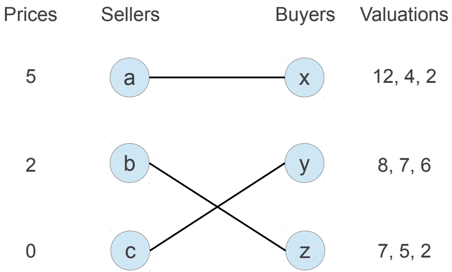
Examples



- The preferred-seller graph for prices 3, 1, and 0.

Market-clearing prices

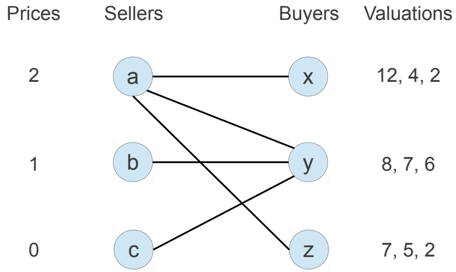
Examples



- This setting has the the nice property that if each buyer claims the house she likes best, each buyer ends up with a different house: somehow the prices have perfectly resolved the contention for houses.
- This happens despite the fact that each of the three buyers value the house of seller a the highest; it is the high price of 5 that dissuades buyers y and z from pursuing this house.
- Such set of prices is called market-clearing.

Market-clearing prices

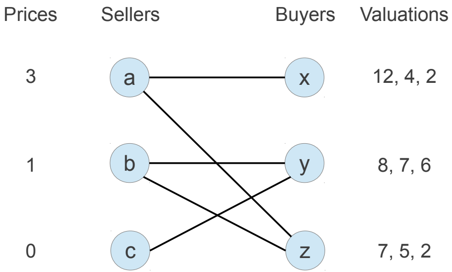
Examples



- Here the prices are not market-clearing, since buyers x and z both want the house offered by seller a, so in this case, when each buyer pursues the house that maximizes their payoff, the contention for houses is not resolved.
- Notice that although each of a, b, and c is a preferred seller for y, since they all give y equal payoffs, this does not help with the contention between x and z.

Market-clearing prices

Examples



- Here, if the buyers coordinate so that each chooses the appropriate preferred seller, then each buyer gets a different house (this requires that y take c's house and z take b's house.)
- This set of prices is market-clearing as well, even though a bit of coordination is required due to ties in the maximum payoffs.
- Ties like this may be inevitable: e.g., if all valuations are the same, then no choice of prices will break this symmetry.

Market-clearing prices

Formulation

Given the possibility of ties, we will think about market-clearing prices more generally as follows.

- For a set of prices, we define the preferred-seller graph on buyers and sellers by simply constructing an edge between each buyer and her preferred seller or sellers.
- There will be no edge out of a buyer if she has no preferred seller.
- Now we simply say:

A set of prices is market-clearing if the resulting preferred-seller graph has a perfect matching.

Properties of market-clearing prices

Optimality

- What is the relationship between market-clearing prices and social welfare?
- Just because market-clearing prices resolve the contention among buyers, causing them to get different houses, does this mean that the total valuation of the resulting assignment will be good?
- In fact, market-clearing prices (for this buyer-seller matching problem) always provide socially optimal outcomes:

Theorem (Optimality of Market-Clearing Prices)

For any set of market-clearing prices, a perfect matching in the resulting preferred-seller graph has the maximum total valuation of any assignment of sellers to buyers.

Properties of market-clearing prices

Proof of optimality

The proof of optimality of market-clearing prices is as follows.

- Consider a set of market-clearing prices, and let M be a perfect matching in the preferred-seller graph.
- Consider the total payoff of this matching, defined simply as the sum of each buyer's payoff for what she gets.
- Since each buyer is grabbing a house that maximizes her payoff individually, M has the maximum total payoff of any assignment of houses to buyers.
- Now how does total payoff relate to total valuation, which is what we're hoping that M maximizes?
- If buyer j chooses house i, then her valuation is v_{ij} and her payoff is $v_{ij} - p_i$.
- Thus, the total payoff to all buyers is simply the total valuation, minus the sum of all prices.

Properties of market-clearing prices

Proof of optimality

Total Payoff of M = Total Valuation of M − Sum of all prices.

- But the sum of all prices is something that does not depend on which matching we choose: it is just the sum of everything the sellers are asking for, regardless of how they get paired up with buyers.
- So a matching M that maximizes the total payoff is also one that maximizes the total valuation, and this completes the proof.

Properties of market-clearing prices

Optimality: a different view

- Suppose that instead of thinking about the total valuation of the matching, we think about the total of the payoffs received by all participants in the market.
- For a buyer, her payoff is defined as above: it is her valuation for the house she gets minus the price she pays.
- A seller's payoff is simply the amount of money she receives in payment for his house.
- Therefore, in any matching, the total of the payoffs to all the sellers is simply equal to the sum of the prices (since they all get paid, and it doesnt matter which buyer pays which seller).
- We just argued that the total of the payoffs to all the buyers is equal to the total valuation of the matching M, minus the sum of all prices.

Properties of market-clearing prices

Optimality: a different view

- Therefore, the total of the payoffs to both the sellers and the buyers is exactly equal to the total valuation of the matching M.
- The point is that the prices detract from the total buyer payoff by exactly the amount that they contribute to the total seller payoff, so sum of the prices cancels out from this calculation.
- Therefore, to maximize the total payoffs to all participants, we want prices and a matching that lead to the maximum total valuation, and this is achieved by using market-clearing prices and a perfect matching in the resulting preferred-seller graph.

We can summarize this as follows.

Theorem (Optimality of Market-Clearing Prices (equivalent version))

A set of market-clearing prices, and a perfect matching in the resulting preferred-seller graph, produces the maximum possible sum of payoffs to all sellers and buyers.

Properties of market-clearing prices

Existence

- Market-clearing prices feel too good to be true.
- We saw that such prices can be achieved in one very small example; but something much more general is true:

Theorem (Existence of Market-Clearing Prices)

For any set of buyer valuations, there exists a set of market-clearing prices.

- This is far from obvious, and we will turn shortly to a method for constructing market-clearing prices that, in the process, proves they always exist.

Existence of market-clearing prices

- We are going to show existence of market-clearing prices by taking an arbitrary set of buyer valuations, and describing a procedure that arrives at market-clearing prices.
- The procedure will in fact be a general kind of auction, taking into account the fact that there are
  - multiple things being auctioned, and
  - multiple buyers with different valuations.
- This particular auction procedure was described by (Demange, Gale, and Sotomayor, 1986), but it's equivalent to a construction of market-clearing prices discovered by (Egerváry, 1916).

Construction of market-clearing prices

Heres how the auction works:

- Initially all sellers set their prices to 0.
- Buyers react by choosing their preferred seller(s), and we look at the resulting preferred-seller graph.
- If this graph has a perfect matching we are done.
- Otherwise (and this is the key point) there is a constricted set of buyers S.
- Consider the set of neighbors $N(S)$, which is a set of sellers.
- The buyers in S only want what the sellers in $N(S)$ have to sell, but there are fewer sellers in $N(S)$ than there are buyers in S.

Construction of market-clearing prices

Heres how the auction works (continued):

- So the sellers in $N(S)$ are in "high demand", i.e., too many buyers are interested in them.
- They respond by each raising their prices by one unit, and the auction then continues.
- One more ingredient is a reduction operation on the prices: if we ever reach a point where all prices are strictly greater than 0, then we reduce the prices by subtracting the smallest price, p, from each one. This drops the lowest price to 0, and shifts all other prices by the same relative amount.

Construction of market-clearing prices

A general round of the auction looks like what we have described:

1. At the start of each round, there is a current set of prices, with the smallest one equal to 0.
2. We construct the preferred-seller graph and check whether there is a perfect matching.
3. If there is, we are done: the current prices are market-clearing.
4. If not, we find a constricted set of buyers S and their neighbors $N(S)$.
5. Each seller in $N(S)$ (simultaneously) raises her price by one unit.
6. If necessary, we reduce the prices: the same amount is subtracted from each price so that the smallest price becomes zero.
7. We now begin the next round of the auction, using these new prices.

Construction of market-clearing prices

Example

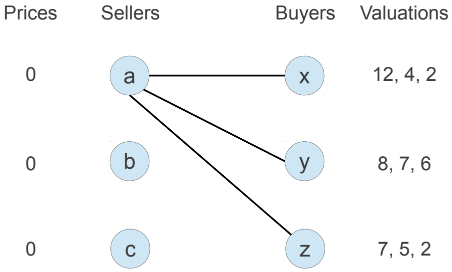
Let us see what happens when we apply the auction procedure to the following example.

Sellers	Buyers	Valuations
a	x	12, 4, 2
b	y	8, 7, 6
c	z	7, 5, 2

Construction of market-clearing prices

Example

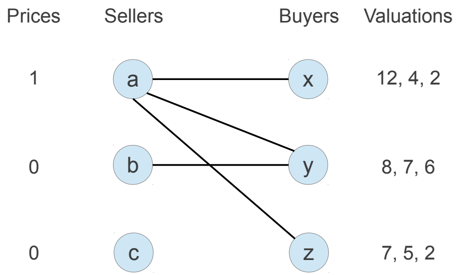
In the first round, all prices start at 0. The set of all buyers forms a constricted set S, with $N(S) = \{\text{a}\}$. So a raises her price by one unit and the auction continues to the second round.



Construction of market-clearing prices

Example

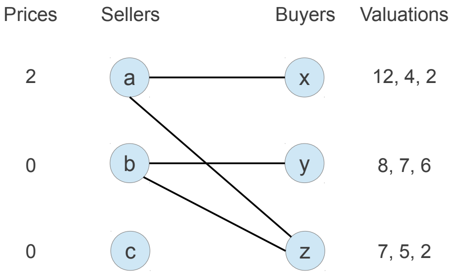
In the second round, the set of buyers consisting of x and z forms a constricted set S, with $N(S)$ again equal to the seller a. Seller a again raises his price by one unit and the auction continues to the third round.



Construction of market-clearing prices

Example

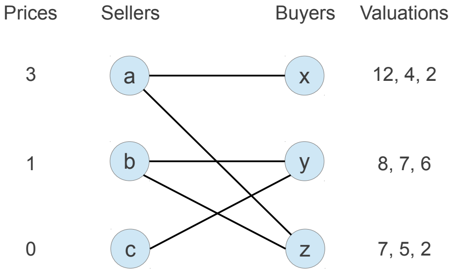
In the third round, the set of all buyers forms a constricted set S, with $N(S)$ equal to the set of two sellers a and b. So a and b simultaneously raise their prices by one unit each, and the auction continues to the fourth round.



Construction of market-clearing prices

Example

In the fourth round, when we build the preferred-seller graph, we find it contains a perfect matching. Hence, the current prices are market-clearing and the auction comes to an end.



Construction of market-clearing prices

Remarks

The previous example illustrates two aspects of this auction that should be emphasized.

- In any round where the set of "over-demanded" sellers $N(S)$ consists of more than one individual, all the sellers in this set raise their prices simultaneously.
- While the auction procedure described produces the market-clearing prices shown in the last step, there can be other market-clearing prices for the same set of buyer valuations (this is because, in some round, we could alternately have identified a different constricted set S). This just means that there can be multiple options for how to run the auction procedure in certain rounds, with any of these options leading to market-clearing prices when the auction comes to an end.

Showing that the auction terminates

- It is not immediately clear why the auction must always come to an end.
- Consider the sequence of steps the auction follows in our example: prices change, different constricted sets form at different points in time, and eventually the auction stops with a set of market-clearing prices.
- But why should this happen in general? Why couldn't there be a set of valuations that cause the prices to constantly shift around so that some set of buyers is always constricted, and the auction never stops?

Showing that the auction terminates

- The only way the auction procedure we described can come to end is if it reaches a set of market-clearing prices; otherwise, the rounds continue.
- So if we can show that the auction must come to an end for any set of buyer valuations, i.e., that the rounds cannot go on forever, then we have shown that market-clearing prices always exist.
- We will show that the auction must always come to an end by a potential function argument: we will identify a certain kind of "potential energy" which is draining out of the auction as it runs; since the auction starts with only a bounded supply of this potential energy, it must eventually run out.

The potential of the auction

Here is how we define this notion of potential precisely.

- For any current set of prices, define the potential of a buyer to be the maximum payoff she can currently get from any seller.
- This is the buyer's potential payoff: the buyer will actually get this payoff if the current prices are market-clearing prices.
- We also define the potential of a seller to be the current price she is charging.
- This is the seller's potential payoff: the seller will actually get this payoff if the current prices are market-clearing prices.
- Finally, we define the potential of the auction to be the sum of the potential of all participants, both buyers and sellers.

The potential as the auction runs

How does the potential of the auction behave as we run it?

- It begins with all sellers having potential 0, and each buyer having a potential equal to her maximum valuation for any house, so the potential of the auction at the start is some $P_0 \geq 0$.
- At the start of each round of the auction, everyone has potential at least 0:
    - The sellers always have potential at least 0 since the prices are always at least 0.
    - Because of the price-reduction step in every round, the lowest price is always 0, and therefore each buyer is always doing at least as well as the option of buying a 0-cost item, which gives a payoff of at least 0. (This also means that each buyer has at least one preferred seller at the start of each round.)
- Finally, since the potentials of the sellers and buyers are all at least 0 at the start of each round, so is the potential of the auction.

The potential as the auction runs

- The potential only changes when the prices change.
- Note that the reduction of prices, as defined above, does not change the potential energy of the auction: if we subtract p from each price, then the potential of each seller drops by p, but the potential of each buyer goes up by p, and it all cancels out.
- When the sellers in $N(S)$ all raise their prices by one unit, each of these sellers' potentials goes up by one unit.
- But the potential of each buyer in S goes down by one unit, since all their preferred houses just got more expensive.
- Since S has strictly more nodes than $N(S)$ does, this means that the potential of the auction goes down by at least one unit more than it goes up, so it strictly decreases by at least one unit.

The potential as the auction runs

So, the potential of the auction

- starts at some fixed value $P_0 \geq 0$;
- decreases by at least one unit in each step that the auction runs; and
- can't drop below 0.

This means that

The auction must come to an end within P_0 steps, and when it comes to an end, we have our market-clearing prices.

Matching markets and single-item auctions

We have seen sealed-bid single-item auctions, and we have just seen a more complex type of auction based on bipartite graphs.

- How do these different kinds of auctions relate to each other?
- We will show a very natural way to view the single-item auction (both the outcome and the procedure itself) as a special case of the bipartite graph auction.

Single-item auctions as bipartite graph auctions

- Suppose we have a set of n buyers and a single seller auctioning an item, and let buyer j have valuation v_j for the item.
- To map this to our model based on perfect matchings, we need an equal number of buyers and sellers, but this is easily dealt with:
  - create $n-1$ "fake" additional sellers, and
  - give buyer j a valuation of 0 for the item offered by each of these fake sellers.

  With the real seller labeled 1, this means we have $v_{1j} = v_j$ and $v_{ij} = 0$ for larger values of i.
- Now we have a genuine instance of our bipartite graph model:
  - from a perfect matching of buyers to sellers, we can see which buyer ends up paired with the real seller (this is the buyer who gets the item), and
  - from a set of market-clearing prices, we will see what the real item sells for.

Single-item auctions as bipartite graph auctions

The price-raising procedure to produce market-clearing prices has a natural meaning here as well:

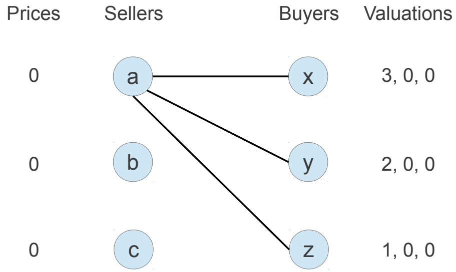
- Initially, all buyers will identify the real seller as their preferred seller.
- The first constricted set S we find is the set of all buyers, and $N(S)$ is the single real seller.
- The seller raises his price by one unit, and this continues as long as at least two buyers have the real seller as their unique preferred seller: they form a constricted set S with $N(S)$ equal to the real seller, and this seller raises his price by a unit.
- The prices of the fake items remain fixed at 0 throughout the auction.
- When all but one buyer has identified other sellers as preferred sellers, the graph has a perfect matching. This happens at precisely the moment that the buyer with the second-highest valuation drops out.

Thus the buyer with the highest valuation gets the item, and pays the second-highest valuation.

Single-item auctions as bipartite graph auctions

An example

A single-item auction can be represented by the bipartite graph model: the item is represented by one seller node, and then there are additional seller nodes for which all buyers have 0 valuation.

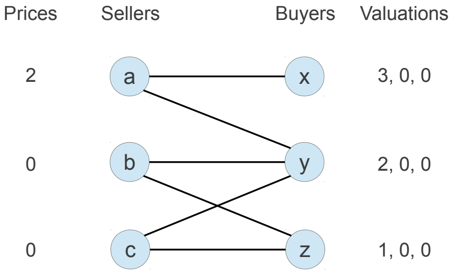


(a) The start of the bipartite graph auction.

Single-item auctions as bipartite graph auctions

An example

A single-item auction can be represented by the bipartite graph model: the item is represented by one seller node, and then there are additional seller nodes for which all buyers have 0 valuation.



(b) The end of the auction, when x gets the item at the valuation of y.

Trading with intermediaries

We will now study a network model for trade based on three fundamental principles:

1. Individual buyers and sellers often trade through intermediaries;
2. not all buyers and sellers have access to the same intermediaries; and
3. not all buyers and sellers trade at the same price.

Rather, the prices that each buyer and seller commands are determined in part by the range of alternatives that their respective network positions provide.

Trading with intermediaries

An example

An example of a market with intermediaries is the market for agricultural goods between local producers and consumers in a developing country:

- In many cases there are middlemen, or traders, who buy from farmers and then resell to consumers.
- Given the often poor transportation networks, the perishability of the products and limited access to capital by farmers, individual farmers can sell only to a limited number of intermediaries.
- Similarly, consumers can buy from only a limited number of intermediaries.
- A developing country may have many such partially overlapping local markets existing alongside modern, more global markets.

A network model of trade

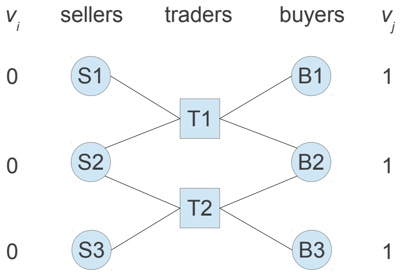
We now describe a simple, yet general enough, model of trade on a network.

- We assume there is a single type of good that comes in indivisible units.
- Each seller i initially holds one unit of the good which she values at v_i; she is willing to sell it at any price that is at least v_i.
- Each buyer j values one copy of the good at v_j, and will try to obtain a copy of the good if she can do it by paying no more than v_j.
- No individual wants more than one copy of the good, so additional copies are valued at 0.
- All buyers, sellers, and traders are assumed to know these valuations.

Network representation

We represent a trading network using the following conventions:

- Sellers are represented by circles on the left.
- Buyers are represented by circles on the right.
- Traders are represented by squares in the middle.
- The value that each seller and buyer places on a copy of the good is written next to the respective node that represents them.



Trading networks vs matching markets

Beyond the fact that we now have intermediaries, there are a few other differences between this model and the model of matching markets:

- We are assuming that buyers have the same valuation for all copies of a good, whereas in matching markets we allowed buyers to have different valuations for the goods offered by different sellers.
- The network here is fixed and externally imposed by constraints such as geography (in agricultural markets) or eligibility to participate (in different financial markets). In matching markets, we began with fixed graphs such as this, but then focused the core of the analysis on preferred-seller graphs that were determined not by external forces but by the preferences of buyers with respect to an evolving set of prices.

Prices and the Flow of Goods

Bid and ask prices

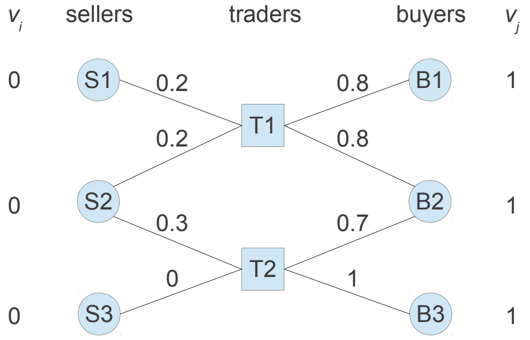
The flow of goods from sellers to buyers is determined by a game in which traders first set prices, and then sellers and buyers react to these prices.

- Each trader t offers a bid price to each seller i that he is connected to; we will denote this bid price by b_{ti}
- This bid price is an offer by t to buy i's copy of the good at a value of b_{ti}.
- Similarly, each trader t offers an ask price to each buyer j that he is connected to.
- This ask price, denoted a_{tj}, is an offer by t to sell a copy of the good to buyer j at a value of a_{tj}

Prices and the Flow of Goods

Bid and ask prices: Example

Each trader posts bid prices to the sellers he is connected to, and ask prices to the buyers he is connected to:



Prices and the Flow of Goods

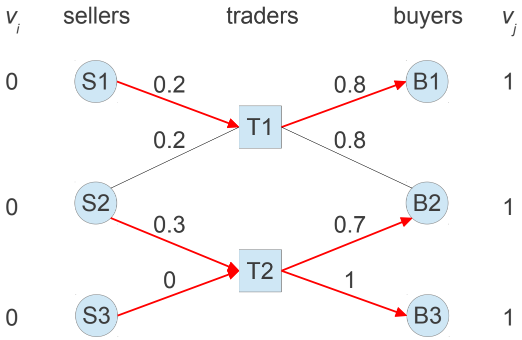
Flow of goods from sellers to buyers

- Once traders announce prices, each seller and buyer chooses at most one trader to deal with.
- Each seller sells his copy of the good to the trader he selects, or keeps his copy of the good if he chooses not to sell it.
- Each buyer purchases a copy of the good from the trader she selects, or receives no copy of the good if she does not select a trader.
- This determines a flow of goods from sellers, through traders, to buyers.

Prices and the Flow of Goods

Flow of goods from sellers to buyers: Example

The bid and ask prices determine a flow of goods, as sellers and buyers each choose the offer that is most favorable to them.

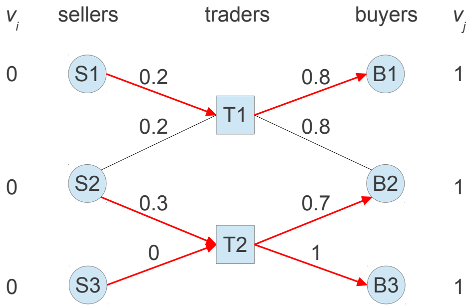


Prices and the Flow of Goods

- Each seller has only one copy of the good, and each buyer only wants one copy, so at most one copy of the good moves along any edge in the network.
- There is no limit on the number of copies of the good that can pass through a single trader node.
- A trader can only sell as many goods to buyers as he receives from sellers.
- There will be incentives for a trader not to produce bid and ask prices that cause more buyers than sellers to accept his offers.
- There will also be incentives for a trader not to be caught in the reverse difficulty, with more sellers than buyers accepting his offers

Prices and the Flow of Goods

In our example, seller S3 accepts the bid even though it is equal to his value, and buyer B3 accepts the ask even though it is equal to his value.



Our assumption in this model is that when a seller or buyer is indifferent between accepting or rejecting, then we (as the modelers) can choose either alternative as the outcome that actually happens.

Strategies

Recall that specifying a game requires a description of the strategies and the payoffs.

We have already discussed the strategies:

- a trader's strategy is a choice of bid and ask prices to propose to each neighboring seller and buyer;
- a seller or buyer's strategy is a choice of a neighboring trader to deal with, or the decision not to take part in a transaction.

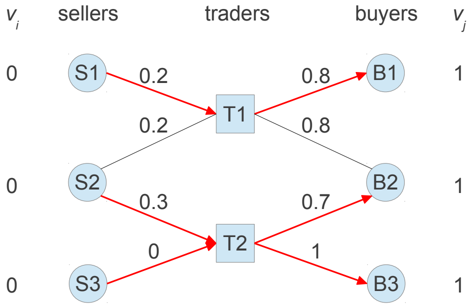
Payoffs

The payoffs follow naturally from the discussion thus far.

- A trader's payoff is the profit he makes from all his transactions: it is the sum of the ask prices of his accepted offers to buyers, minus the sum of the bid prices of his accepted offers to sellers.
- For a seller i, the payoff from selecting trader t is b_{ti} , while the payoff from selecting no trader is v_i. In the former case, the seller receives b_{ti} units of money, while in the latter he keeps his copy of the good, which he values at v_i. (We will consider only cases in which all the seller v_i's are 0.)
- For each buyer j, the payoff from selecting trader t is $v_j - a_{tj}$, while the payoff from selecting no trader is 0. In the former case, the buyer receives the good but gives up a_{tj} units of money.

Payoffs

Example



- The payoff to T1 is $0.8 - 0.2 = 0.6$ and the payoff to T2 is $0.7 + 1 - 0.3 - 0 = 1.4$.
- The payoffs to the three sellers are 0.2, 0.3, and 0 respectively.
- The payoffs to the three buyers are $1 - 0.8 = 0.2$, $1 - 0.7 = 0.3$, and $1 - 1 = 0$ respectively.

Remarks

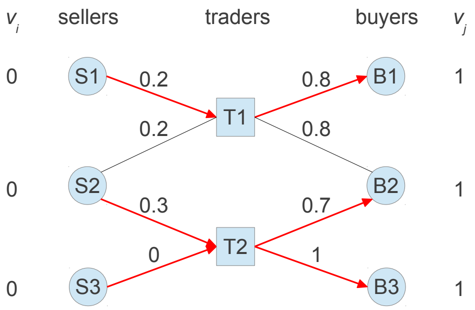
The game defined here has a feature which forms a contrast with other games we have discussed earlier.

- In earlier games, all players moved (i.e., executed their chosen strategies) simultaneously.
- In this game the moves happen in two stages:
  1. In the first stage, all the traders simultaneously choose bid and ask prices.
  2. In the second stage, all the sellers and buyers then simultaneously choose traders to deal with.

  However stage 2 is extremely simple: the best response for each seller and buyer is always simply to choose the trader with the best offer.
- Still, we will have to take the two-stage structure into account when we consider the equilibria for this game.

Best responses and equilibrium

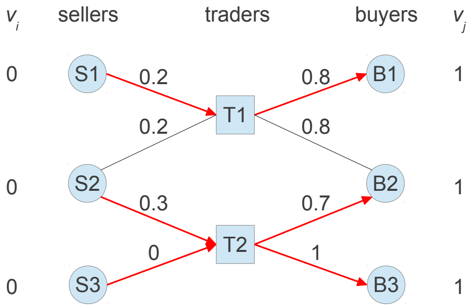
Illustration



We go back to our example in order to motivate the equilibrium concept we will use for this game.

Best responses and equilibrium

Illustration

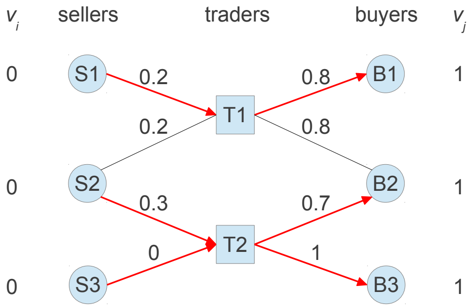


T_1 is making several bad decisions:

- Because of the offers he is making to S2 and B2, he is losing out on this deal to the lower trader T2.
- If he were to raise his bid to S2 to 0.4, and lower his ask to B2 to 0.6, then he would take the trade away from T2: S2 and B2 would both choose him, and would make a profit of 0.2.

Best responses and equilibrium

Illustration

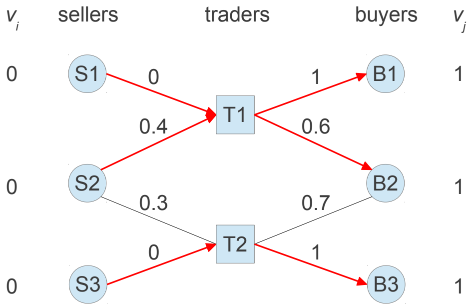


- Also, there is no reason for T1 not to lower his bid to S1, and raise his ask to B1.
- Even with worse offers, S1 and B1 will still want to deal with T1, since they have no other options aside from choosing not to transact.

Best responses and equilibrium

Illustration

Taking these point into account, these are the results of a deviation by T1:



- T1's payoff has now increased to $1 + 0.6 \quad 0 \quad 0.4 = 1.2$.
- S1 and B1 are now indifferent between performing the transaction or not, and as discussed earlier, we give ourselves (as the modelers) the power to break ties in determining the equilibrium for such situations.

Best responses and equilibrium

This discussion motivates the equilibrium concept we will use for this game, which is a generalization of Nash equilibrium.

- The equilibrium will be based on a set of strategies such that each player is choosing a best response to what all the other players are doing.
- However, the definition also needs to take the two-stage structure of the game into account.

The problem faced by the buyers and sellers in the second stage:

- It is a standard game among the buyers and sellers, and each of them chooses a strategy that is a best response to what all other players are doing.

Best responses and equilibrium

The problem faced by the traders in the first stage:

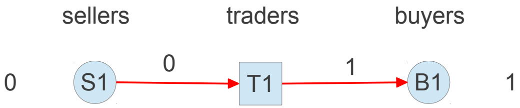
- Each trader chooses a strategy (bid and prices) that is a best response both to the strategies the sellers and buyers will use (what bids and asks they will accept) and the strategies the other traders use (what bids and asks they post).
- So everyone is employing a best response just as in any Nash equilibrium.
- The one difference here is that since the sellers and buyers move second they are required to choose optimally given whatever prices the traders have posted, and the traders know this.
- This equilibrium is called a subgame perfect Nash equilibrium, but we will simply refer to it as an equilibrium.

In order to analyze equilibria in trading networks, we will begin with simple network structures and build up to our example.

Analysis of equilibria in trading networks

Monopoly

Buyers and sellers are subject to monopoly when they have access to only a single trader:

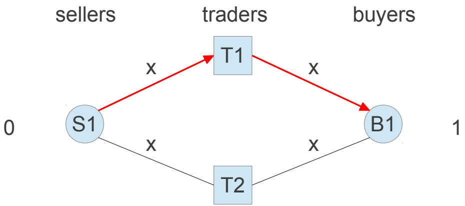


- The trader is in a monopoly position relative to both the seller and the buyer (there is only one trader available to each of them).
- An equilibrium is for the trader to set a bid of 0 to the seller and an ask of 1 to the buyer; they will accept these prices and the good will flow from the seller to the trader and then on to the buyer.
- This is the only equilibrium, since with any other bid and ask between 0 and 1, the trader could slightly lower the bid or raise the ask, thereby performing the transaction at a higher profit.

Analysis of equilibria in trading networks

Perfect competition

Now we consider a basic example showing perfect competition between two traders:

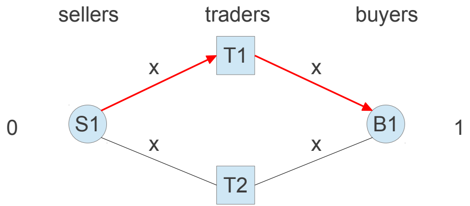


- There is competition between T1 and T2 to buy the copy of the good from S1 and sell it to B1.
- Suppose T1 is performing the trade and making a positive profit: his bid is b and his ask is $a > b$.
- Since T2 is not performing the trade, he has a payoff of zero.

Analysis of equilibria in trading networks

Perfect competition

Now we consider a basic example showing perfect competition between two traders:

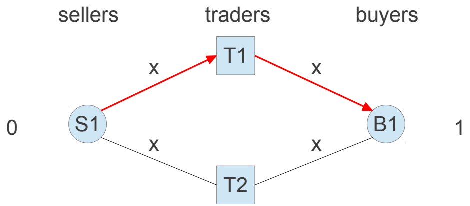


- But then T2's strategy is not a best response to T1: T2 could offer a bid slightly above b and an ask slightly below a, thereby taking the trade away from T1 and receiving a positive payoff.
- So whichever trader is performing the trade at equilibrium must have a payoff of 0: he must be offering the same value x as his bid and ask.

Analysis of equilibria in trading networks

Perfect competition

Now we consider a basic example showing perfect competition between two traders:

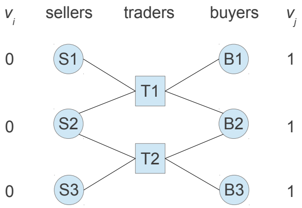


- So the equilibrium occurs at a common bid and ask of x.
- The value of x has to be between 0 and 1: otherwise either the seller wants to sell but the buyer wouldn't want to buy, or conversely the buyer wants to buy but the seller wouldn't want to sell.

Analysis of equilibria in trading networks

Illustration

Let us now compute the equilibria in our example:

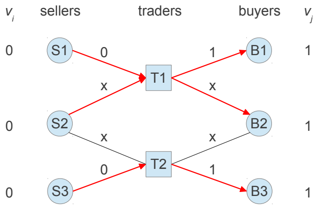


- Sellers S1 and S3, and buyers B1 and B3, are monopolized by their respective traders, and so in any equilibrium these traders will drive the bids and asks all the way to 0 and 1 respectively.
- Seller S2 and buyer B2 benefit from perfect competition between the two traders. The trader performing the transaction must have bid and ask values equal to the same number x, and given this, the other trader must also have bid and ask values equal to x.

Analysis of equilibria in trading networks

Illustration

Thus the equilibria for the trading network are:



Such reasoning is useful in analyzing more complex networks.

- A seller or buyer connected to only a single trade will receive zero payoff in any equilibrium.
- When two traders both connect the same seller and buyer, then neither can make a positive profit in conveying a good from this seller to this buyer.

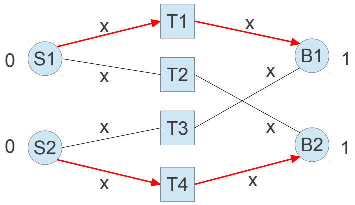
Analysis of equilibria in trading networks

Implicit perfect competition

- In our examples so far, when a trader makes no profit from a transaction, it is always because there is another trader who can precisely replicate the transaction, i.e., a trader who is connected to the same seller and buyer.
- However, it turns out that traders can make zero profit for reasons based more on the global structure of the network, rather than on direct competition with any one trader.

Analysis of equilibria in trading networks

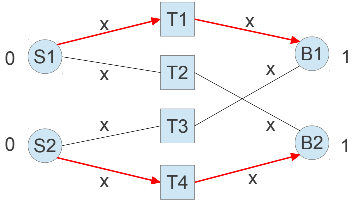
Implicit perfect competition: Example



- In this trading network there is no direct competition for any one “trade route” from a seller to a buyer.
- However, in any equilibrium, all bid and ask prices take on some common value x between 0 and 1, and the goods flow from the sellers to the buyers.
- So all traders again make zero profit.

Analysis of equilibria in trading networks

Implicit perfect competition: Example



- This is an equilibrium: check that each trader is using a best response to all the other traders' strategies.
- We can also verify that in every equilibrium, all bid and ask prices are the same value x: this can be done by checking alternatives in which some trader posts a bid that is less than the corresponding ask, and identifying a deviation that arises.

Further reading

- David Easley and Jon Kleinberg: Networks, Crowds, and Markets: Reasoning About a Highly Connected World, **Chapters 10, 11**. Cambridge University Press, 2010.
- Gabrielle Demange, David Gale, and Marilda Sotomayor: Multi-item auctions. Journal of Political Economy, **94**(4):863–872, 1986.
- László Lovász and Michael Plummer: Matching Theory. North-Holland, 1986.