1 HW 1 in Electronic Commerce Models, due date 5/12/2024

1.1 Question 1: VCG Mechanism for Lowest-Cost Path

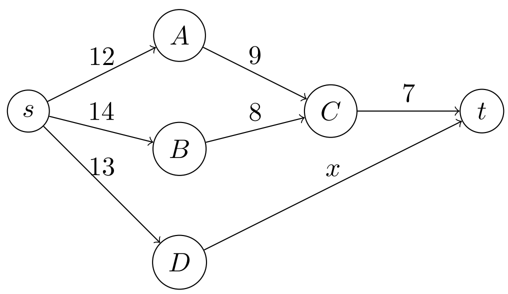
Consider a graph $G = (V, E)$, where $V = \{s, A, B, C, D, t\}$ and $E \subseteq V \times V$. Each edge $e \in E$ has a private cost c_e controlled by an agent:

$$c_{(s,A)} = 12, \quad c_{(s,B)} = 14, \quad c_{(A,C)} = 9, \quad c_{(B,C)} = 8,$$

$$c_{(s,D)} = 13, , \quad c_{(C,t)} = 7, \quad c_{(D,t)} = x.$$

The goal is to find the **lowest-cost path** from s to t using the **VCG mechanism**.

1. For each $x \in \{10, 20\}$ (the private cost of edge (D, t)), determine the VCG output (selected path) and calculate the payment for each edge.
2. Generalize the VCG mechanism for a graph $G = (V, E)$ with any number of vertices and edges. Explain how the payments and selection process would work.



1.2 Question 2: VCG Mechanism for Two Projects

A community is deciding between two potential projects: **Project A** and **Project B**. Due to location constraints, at most one project can be built. Each of the n players has a private valuation v_i^A for Project A and v_i^B for Project B, which are drawn independently from known distributions. The cost to build each project is fixed at C_A and C_B, respectively. The goal is to select the project that maximizes social welfare while ensuring truthful reporting of valuations. It is also possible that no project will be built if neither project's total social welfare exceeds its cost.

1. Write the expression for the total social welfare for Project A and for Project B. Define the condition under which a project should be selected.
2. Consider the following example:
   - Four players have valuations:

     $$v_1^A = 20, \quad v_2^A = 30, \quad v_3^A = 10, \quad v_4^A = 5,$$

     $$v_1^B = 25, \quad v_2^B = 20, \quad v_3^B = 12, \quad v_4^B = x.$$

     The costs are $C_A = 50$ and $C_B = 45$.
   - For each $x \in \{5, 15\}$, determine which project is selected under the VCG mechanism and compute the payments for each player.
3. Write the simplest possible expression for the payments of the players in the VCG mechanism (Hint: divide into cases).

1.3 Question 3: Public Goods Provision and Bayesian Nash Equilibrium

Five players are considering contributing to a community project. Each player has a private valuation v_i for the project, drawn independently from a uniform distribution between $[0, 30]$. The project will only be funded if at least three players contribute, and each player must pay a contribution cost of 10 if they choose to participate.

Consider a threshold policy $\theta(t^*)$, where a player contributes if and only if $v_i \geq t^*$.

1. Assume all players follow the strategy $\theta(12)$:
   (a) What is the probability that the project will be built?
   (b) Given that player i contributes, what is the probability that the project will be built?
   (c) Is $\theta(12)$ a Bayesian Nash Equilibrium? Explain your reasoning.
2. Compute all possible value of t^* such that if all players follow $\theta(t^*)$, it constitutes a Bayesian Nash Equilibrium.
3. Compute the expected social welfare in this Bayesian Nash Equilibrium.
4. (Bonus) Design a VCG mechanism for this problem and compute the expected social welfare under the mechanism.

1.4 Question 4: Bayesian Nash Equilibrium in an All-Pay Auction

Consider an **all-pay auction** with n players competing for a single item. Each player i has a private valuation v_i, independently drawn from a uniform distribution over $[0, 1]$. In this auction:

- Each player submits a bid b_i, which they must pay regardless of whether they win or lose.
- The player with the highest bid wins the item and receives utility $u_i = v_i - b_i$.
- Players who do not win receive utility $u_i = -b_i$.

1. For $n = 2$, if all players bid $\beta(v) = \frac{v}{2}$, is it a Bayesian Nash Equilibrium (BNE)? Provide a full explanation.
2. For $n = 2$, if all players bid $\beta(v) = \frac{v^2}{2}$, is it a Bayesian Nash Equilibrium (BNE)? Provide a full explanation.
3. For a general number of players n, compute the coefficient α such that if all players bid $\beta(v) = \alpha \cdot v^n$, it forms a Bayesian Nash Equilibrium. Provide a full explanation.