1 HW 3 in Electronic Commerce Models, due date 10/2/2025

1.1 Question: The GSP auction

Assume for this question that the ad quality is the same for all bidders.

1. Verify the claim in Example 10.10 that the bid profile is a Nash equilibrium.
2. Verify the claim in Example 10.10 that the bid profile is an envy-free Nash equilibrium.
3. Considering Example 10.10, verify that there are no useful deviations from truthful bidding.
4. Property (GSP1): Prove that the bid profile in any envy-free Nash equilibrium of the GSP auction is value ordered.
5. Give an example (different from Example 10.10) of a Nash equilibrium of the GSP auction that is not value ordered.

1.2 Question: Power of additional bidders

In all the above sub-questions, the auction sells at most two items (at most one per player), and the players' values are independently and uniformly distributed over $[0, 1]$.

1. Compute the expected revenue of an auction for 4 players that maximizes social welfare.
2. Compute the expected revenue of an auction for 2 players that maximizes revenue.
3. Prove that if bidder values are IID and drawn from a regular distribution, the expected revenue in an auction that maximizes social welfare on $n + 2$ bidders is at least the expected revenue of the revenue-optimal auction on n bidders.

1.3 Question: Single-minded combinatorial auction

1. Suppose there are two items, A and B, and four bidders, with target bundles and values $(A, \$10)$, $(AB, \$19)$, $(B, \$8)$, and $(AB, \$6)$ for bidders 1, 2, 3, and 4 respectively.
   (a) What is the efficient allocation?
   (b) Determine the outcome of the single-minded CA (Definition 8.2) for each choice of score function σ_{val}, σ_{density}, and σ_{sqrt}. Which allocations are optimal?
   (c) Confirm for the σ_{density} score function that bidder 2 does not have a useful deviation.
2. Suppose a greedy algorithm is comparing two bids. One bid is for \$1000 on 100 items, and one bid is for \$100 on 10 items. Provide some intuition for why σ_{sqrt} might provide an allocation with a better total value than σ_{density}.
3. Design a strategy-proof mechanism for single-minded CA that provides a $\min(d, \sqrt{m})$-approximation for the efficient allocation, where the size of any target bundle is at most d and there are m items. (Hint: Use a scoring rule based on the maximum target bundle, and prove that an algorithm that alternates between these scoring rules is strategy-proof.)

1.4 The RSD(Random serial dictatorship) mechanism

1. Prove that the RSD mechanism is equivalent (in the sense of generating the same probability distribution on assignments for every preference profile) to the TTC mechanism when the initial assignment x^0 in the housing markets problem is selected uniformly at random.

2. Consider the following example, where the items are a, b, c and d:

$$\begin{array}{lcccc} \succ_1: & a & b & c & d \\ \succ_2: & a & b & c & d \\ \succ_3: & b & a & d & c \\ \succ_4: & b & a & d & c \end{array}$$

   Use either analysis or simulation to determine the probability that each agent is assigned each item by RSD. [Hint: for each item, the sum of probabilities across agents should sum to 1. For each agent, the sum of probabilities across items should sum to 1. The distribution can be expressed as a 4×4 table of probabilities.]

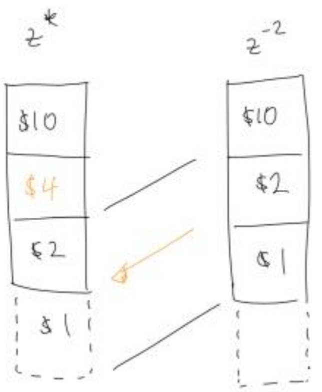

Figure 10.8.: The presence of bid 2 at price \$4 pushes bid 3 down one position relative to z^{-2}, and prevents bid 4 from being allocated.

Example 10.10. *See Figure 10.8. There are three positions (position effects* 0.2, 0.18, 0.1*) and four bidders (per-click values* $10, 4, 2$ *and* 1*, all with the same quality). If truthful, positions* $1, 2, 3$ *go to bidders* 1, 2 *and* 3*, respectively. For bidder* 1, $p_{\text{vcg},1}(b) = (1/pos_1)((pos_1 - pos_2)b_2 + (pos_2 - pos_3)b_3 + (pos_3 - pos_4)b_4) = (1/0.2)((0.2 - 0.18)4 + (0.18 - 0.1)2 + (0.1 - 0)1) = 0.34/0.2 = 17/10$. *For bidder* 2, $p_{\text{vcg},2}(b) = (1/pos_2)((pos_2 - pos_3)b_3 + (pos_3 - pos_4)b_4) = (1/0.18)((0.18 - 0.1)2 + (0.1 - 0)1) = 0.26/0.18 = 13/9$. *For bidder* 3, $p_{\text{vcg},3}(b) = (1/pos_3)(pos_3 - pos_4)b_4 = (1/0.1)(0.1 - 0) = 0.1/0.1 = 1$, *and equal to* b_4.

position	position effect	value-per-click	bid-per-click	VCG price	GSP price
1	0.2	10	10	17/10	4
2	0.18	4	4	13/9	2
3	0.1	2	2	1	1
–	0	1	1	0	0

Table 10.12.: Comparing VCG and GSP prices (all bids have the same quality).

Example 10.10 (continuing from p. 245). *Table 10.16 provides a Nash equilibrium for Example 10.10. The bids are not value ordered in this equilibrium. Figure 10.15 illustrates the best response analysis for the bidder with value \$10 in position 2. This bidder can either deviate upwards, and take position 1 at price \$4, or deviate downwards, and take, for example, position 3 at price \$1. For an upward deviation, we have* $0.18(10-2) = 1.44 \geq 0.2(10-4) = 1.2$, *where the price would be \$4 because it needs to out bid the bid currently in position 1, and see that this is not useful. For a downward deviation, we have* $0.18(10-2) = 1.44 \geq 0.1(10-1) = 0.9$, *and see that this is not useful.*

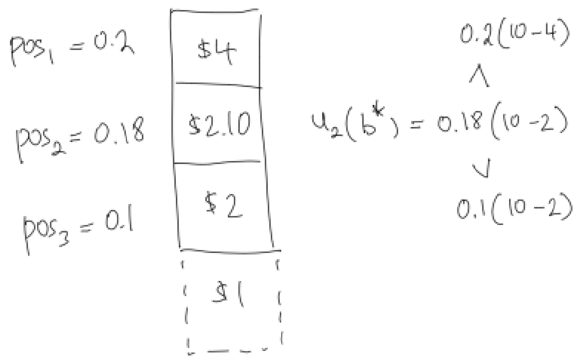

Figure 10.15.: Best response analysis for the bidder in position 2, with value \$10, in a Nash equilibrium of the GSP auction. The bids are \$4, \$2.1, \$2 and \$1. **xx should be** $0.1(10-1)$ **not** $0.1(10-2)$ **xx**

position	position effect	value-per-click	bid-per-click	GSP price
1	0.2	4	4	2.1
2	0.18	10	2.1	2
3	0.1	2	2	1
–	0	1	1	0

Table 10.16.: A Nash equilibrium of the GSP auction.

position	position effect	value-per-click	bid-per-click	GSP price
1	0.2	10	$b_1^* = 10$	17/10
2	0.18	4	$b_2^* = 17/10$	13/9
3	0.1	2	$b_3^* = 13/9$	1
–	0	1	$b_4^* = 1$	0

Table 10.20.: An envy-free Nash equilibrium of the GSP auction.

Example 10.10 (continuing from p. 247). *Suppose, for example, that advertisers play the Nash equilibrium in Table 10.16, and receive feedback about prices \$2.1, \$2, and \$1, for positions 1, 2, and 3, respectively. This is not envy-free, because the bidder with value \$10 would envy the bidder in position 1, preferring position 1 at price \$2.1 to position 2 at price \$2, since* $0.2(10-2.1) = 1.58 > 0.18(10-2) = 1.44$.

Example 10.10 (continuing from p. 249). *Table 10.20 provides an envy-free Nash equilibrium for Example 10.10. For example, bidder* 2 *does not envy bidder* 1 *because* $0.18(4-13/9) \geq 0.2(4-17/10)$*. To confirm this is a Nash equilibrium, we can check for example that bidder* 2 *does not have a useful deviation to a bid between* 1 *and* $13/9$ *to take position* 3*, because* $0.18(4-13/9) \geq 0.1(4-1)$.

Example 10.10 (continued). *Consider bidder* 2 *in Table 10.20. Any bid* $b_2 \in (13/9, 10)$ *is a best response. But how to select a bid in this range? Why not bid \$9.99, which increases the price for bidder 1 and may drive that bidder out of future auctions. A possible concern is that bidder 1 could retaliate, and bid \$9.98, leaving bidder 2 to pay \$9.98 for position 1. The balanced bidding condition requires that bid* b_2 *satisfies*

$$0.18(4-13/9) = 0.2(4-b_2), \tag{10.16}$$

and $b_2 = 17/10$*. This is the highest bid that leaves bidder 2 indifferent between remaining in position 2 and having bidder 1 retaliate.*

Example 10.10 (continuing from p. 250). *We construct the bid profile in Table 10.20 by looking for a value-ordered profile that satisfies balanced bidding. First, we set* $b_4 = v_4 = 1$*. For bid 3, we want* $0.1(2-1) = 0.1 = 0.18(2-b_3)$*, and* $b_3 = 13/9$ *(so that bidder 3 is indifferent between position 3 at \$1 and position 2 at* b_3*). For bid 2, we want* $0.18(4-13/9) = 0.2(4-b_2)$*, and* $b_2 = 17/10$*. For bid 1 we can choose any bid larger than* $17/10$*, for example the truthful bid* $b_1 = v_1 = 10$*. We can check that bid profile* $(10, 17/10, 13/9, 1)$ *is an envy-free Nash equilibrium.*