$A(x) = 2 + x \qquad B(x) = 1 + x$

$n=4 \rightarrow$ חזקה קרובה ל 2.

$w_4^0 = 1 \qquad w_4^1 = i \qquad w_4^2 = -1 \qquad w_4^3 = -i$

$FFT\left((2,1)\right)$

$w_n \leftarrow e^{\frac{2\pi i}{n}}$

$w \leftarrow 1$

$a^0 \leftarrow (2)$

$a^1 \leftarrow (1)$

$y^0 \leftarrow FFT((2)) \leftarrow 2$

$y^1 \leftarrow FFT((1)) \leftarrow 1$

$y_0 = 2 + (1)\cdot 1 = \{A^0(x) = 2,\ A^1(x) = 1,\ A(x) = A(x) = A^0(x^2) + x A^1(x^2)\} = 3$

$y_3 = 2 - 1\cdot 1 = 1$

$y_1 = 2 + i\cdot 1 = 2+i$

$y_4 = 2 - i\cdot 1 = 2-i$

$2 - 2i - i + (-1)$

$A(x)$	$(w^0, 3)\ (w^1, 2+i)\ (w^2, 1)\ (w^3, 2-i)$
$B(x)$	$(w^0, 2)\ (w^1, 1+i)\ (w^2, 0)\ (w^3, 1-i)$
$C(x)$	$(w^0, 6)\ (w^1, 1+3i)\ (w^2, 0)\ (w^3, 1-3i)$

$FFT((1,1))$

$a^0 \leftarrow 1$

$a^1 \leftarrow 1$

$y^0 \leftarrow FFT(1)$

$y^1 \leftarrow FFT(1)$

$\{A(x) = A^0(x) + xA^1(x),\ A^0(x) = 1,\ A^1(x) = 1\}$

$y_0 = 1 + 1\cdot 1 = 2$

$y_3 = 1 - 1\cdot 1 = 0$

$y_1 = 1 + i\cdot 1 = 1+i$

$y_4 = 1 - i\cdot 1 = 1-i$

$c^0 = \frac{1}{4}[6 + 1+3i + 0 + 1-3i] = 2$

$c^1 = \frac{1}{4}\cdot[6\cdot 1 + (1+3i)\cdot(-i) + 0 + (1-3i)i] = \frac{1}{4}[6 - i + 3 + i + 3] = \frac{12}{4} = 3$

$c^2 = \frac{1}{4}[6 + (1+3i)\cdot w_4^{-2} + 0 + (1-3i)w_4^{-6}] = \frac{1}{4}[6 - 1 - 3i + 0 - 1 + 3i] = 1$

$c = (2, 3, 1) \rightarrow C(x) = 2 + 3x + x^2$

$FFT((2))$

return 2

$FFT((1))$

return 1

$$P(x) = \sum_{j=0}^{n-1} c_j x^j \qquad q(x) = \sum_{i=0}^{n-1} c_i x^i \qquad \omega = e^{2\pi i/n} = [\omega^0, \omega^1, \dots, \omega^{n-1}]$$

split P into P_e, P_o and q into q_e, q_o

$$P(x) = P_e(x^2) + xP_o(x^2) \qquad P(-x) = P_e(x^2) - xP_o(x^2)$$

?The Fast Fourier Transform (FFT): Most Ingenious Algorith

```
def FFT(P):
    # P - [p_0, p_1, ..., p_{n-1}] coeff representation
    n = len(P) # n is a power of 2
    if n == 1:
        return P
    ω = e^(2πi/n)
    P_e, P_o = [p_0, p_2, ..., p_{n-2}], [p_1, p_3, ..., p_{n-1}]
    y_e, y_o = FFT(P_e), FFT(P_o)
    y = [0] * n
    for j in range(n/2):
        y[j] = y_e[j] + ω^j y_o[j]
        y[j + n/2] = y_e[j] - ω^j y_o[j]
    return y
```

$$\boxed{\text{FFT} \quad P(x) : [p_0, p_1, \dots, p_{n-1}] \quad \omega = e^{\frac{2\pi i}{n}} : [\omega^0, \omega^1, \dots, \omega^{n-1}]}$$

$$\boxed{n = 1 \Rightarrow P(1)}$$

$$\boxed{\text{FFT} \quad P_e(x^2) : [p_0, p_2, \dots, p_{n-2}] \quad [\omega^0, \omega^2, \dots, \omega^{n-2}]}$$

$$y_e = [P_e(\omega^0), P_e(\omega^2), \dots, P_e(\omega^{n-2})]$$

$$\boxed{\text{FFT} \quad P_o(x^2) : [p_1, p_3, \dots, p_{n-1}] \quad [\omega^0, \omega^2, \dots, \omega^{n-2}]}$$

$$y_o = [P_o(\omega^0), P_o(\omega^2), \dots, P_o(\omega^{n-2})]$$

$$\boxed{\begin{gathered} P(\omega^j) = y_e[j] + \omega^j y_o[j]) \\ P(\omega^{j+n/2}) = y_e[j] - \omega^j y_o[j]) \\ j \in \{0, 1, \dots (n/2 - 1)\} \end{gathered}}$$

$$\boxed{y = [P(\omega^0), P(\omega^1), \dots, P(\omega^{n-1})]}$$

$$x_k = \frac{1}{n} \sum_{j=0}^{n-1} X_j \, \omega_n^{-jk}$$

- x_k are the coefficients we want to recover,
- X_j are the DFT values (the pointwise multiplication result),
- $\omega_n = e^{-2\pi i/n}$ is the primitive nth root of unity (inverse of ω used in FFT),
- n is the number of points (in our case, 4).

Let's go through the **Inverse FFT (IFFT)** calculation in detail, starting from the pointwise multiplication results. The goal of the inverse FFT is to take the frequency-domain representation we obtained (after pointwise multiplication) and return it to the time-domain, or coefficient form, of the polynomial $C(x)$.

Setup for Inverse FFT

After the pointwise multiplication, we have:

$$C' = [6, 1+3i, 0, 1-3i]$$

Since we zero-padded to length 4, we're working with a vector length $n = 4$, and we'll use the 4th roots of unity in the inverse direction to perform the inverse FFT.

The inverse FFT formula for a sequence $C' = [C'(0), C'(1), C'(2), C'(3)]$ is:

$$C(k) = \frac{1}{n} \sum_{j=0}^{n-1} C'(j) \cdot \omega_n^{-jk}$$

where:

- $n = 4$,
- ω_n^{-1} is the inverse root of unity, specifically $e^{-2\pi i/4} = i$,
- We need to divide by $n = 4$ at the end to normalize.

The 4th roots of unity (for $n = 4$) in the inverse FFT are:

- $\omega_0^{-1} = 1$
- $\omega_1^{-1} = -i$
- $\omega_2^{-1} = -1$
- $\omega_3^{-1} = i$

Let's calculate each $C(k)$ for $k = 0, 1, 2, 3$.

Calculation of Each $C(k)$

Step 1: Calculate $C(0)$

$$C(0) = \frac{1}{4}\left(C'(0)\cdot\omega_0^{-1} + C'(1)\cdot\omega_0^{-1} + C'(2)\cdot\omega_0^{-1} + C'(3)\cdot\omega_0^{-1}\right)$$

Since $\omega_0^{-1} = 1$, this becomes:

$$C(0) = \frac{1}{4}(6 + (1+3i) + 0 + (1-3i)) = \frac{1}{4}(6+1+3i+1-3i)$$

$$C(0) = \frac{1}{4}\cdot 8 = 2$$

Step 2: Calculate $C(1)$

$$C(1) = \frac{1}{4}\left(C'(0)\cdot\omega_0^{-1} + C'(1)\cdot\omega_1^{-1} + C'(2)\cdot\omega_2^{-1} + C'(3)\cdot\omega_3^{-1}\right)$$

Substitute the roots of unity values:

$$C(1) = \frac{1}{4}\left(6 + (-i)(1+3i) + (-1)\cdot 0 + i(1-3i)\right)$$

Calculating each term:

1. $(-i)(1+3i) = -i + 3 = 3 - i$
2. $i(1-3i) = i - 3 = -3 + i$

So,

$$C(1) = \frac{1}{4}\left(6 + (3-i) + 0 + (-3+i)\right)$$

Simplify:

$$C(1) = \frac{1}{4}\cdot 12 = 3$$

Step 3: Calculate $C(2)$

$$C(2) = \frac{1}{4}\left(C'(0)\cdot\omega_0^{-2} + C'(1)\cdot\omega_1^{-2} + C'(2)\cdot\omega_2^{-2} + C'(3)\cdot\omega_3^{-2}\right)$$

Since $\omega_2^{-2} = -1$, this simplifies to:

$$C(2) = \frac{1}{4}(6 - (1+3i) + 0 - (1-3i))$$

Simplify further:

1. $-(1+3i) = -1 - 3i$
2. $-(1-3i) = -1 + 3i$

So:

$$C(2) = \frac{1}{4}(6 - 1 - 3i - 1 + 3i) = \frac{1}{4}\cdot 4 = 1$$

Step 4: Calculate $C(3)$

$$C(3) = \frac{1}{4}\left(C'(0)\cdot\omega_0^{-3} + C'(1)\cdot\omega_1^{-3} + C'(2)\cdot\omega_2^{-3} + C'(3)\cdot\omega_3^{-3}\right)$$

Substitute:

$$C(3) = \frac{1}{4}\left(6 + i(1+3i) + (-1)\cdot 0 - i(1-3i)\right)$$

Calculating:

1. $i(1+3i) = i + 3 = 3 + i$
2. $-i(1-3i) = -i + 3 = 3 - i$

Then,

$$C(3) = \frac{1}{4}\left(6 + (3+i) + 0 + (3-i)\right) = \frac{1}{4}\cdot 12 = 3$$

Final Result

Thus, the coefficients of the polynomial $C(x) = 2 + 3x + x^2$ are:

$$[2, 3, 1]$$

where:

- $C(0) = 2$ (constant term),
- $C(1) = 3$ (coefficient of x),
- $C(2) = 1$ (coefficient of x^2).

Final Result

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- $C(0) = 2$ (constant term),
- $C(1) = 3$ (coefficient of x),
- $C(2) = 1$ (coefficient of x^2).

The Inverse FFT has given us the coefficients in the time domain, confirming that:

$$C(x) = 2 + 3x + x^2$$