1 HW 2 in Electronic Commerce Models, due date 25/12/2024

1.1 Question: Optimal Revenue Auction

Each player can be one of two possible types:

- **Type A**: The cumulative distribution function (CDF) of the valuation $v \in [0, 5]$ is given by

$$F_A(v) = \frac{v^2}{25}.$$

- **Type B**: The CDF of the valuation $v \in [0, 5]$ is given by

$$F_B(v) = \frac{1 - \exp(-v)}{1 - \exp(-5)},$$

.

Tasks:

1. **Virtual Value Function:** For each type:
   - Compute the virtual value function.
   - Verify that the virtual value function is monotone non-decreasing.
   - Compute the value of v where the virtual value function equals zero for each type.
2. **Optimal Revenue Auction Design:** Design the optimal revenue-maximizing auction and compute the expected revenue in the following scenarios:
   - **Case 1:** One item, a single player of type A.
   - **Case 2:** One item, a single player of type B.
   - **Case 3:** One item, a single player of type A and a single player of type B.
   - **Case 4:** One item, two players of type A.
   - **Case 5:** One item, two players of type B.
   - **Case 6:** Two items, one player of type A and one player of type B (Each player has identical valuations for both items, according to their type, and can receive at most one item.).

1.2 Question: Optimal Auction Design for n Players and Two Items

Consider designing an auction for n players competing for two identical items, where each player can receive at most one item. Each player i has a private valuation v_i, which is independently and identically drawn from a uniform distribution over $[0, 1]$.

- For $n = 4$, design an auction that maximizes the **Social Welfare**. Provide an explicit description of the allocation rule and payment rule.
- For $n = 4$, design an auction that maximizes the **Revenue**. Provide an explicit description of the allocation rule and payment rule.
- Compute the interim allocation function $x^*(v)$ in both cases.
- Use the function interim allocation function $x^*(v)$ to compute the revenue of each mechanism
- (Bonus) For $n = 3$, Using payment identity to compute BNE for the first-price auction

1.3 Question: Optimal Revenue Auction with Restrictions

Consider four players, A, B, C, and D, where each player can receive at most one item and whose valuations for the item are independently and uniformly distributed over $[0, 1]$.

Rules:

- You cannot sell to both players A and B simultaneously.
- You cannot sell to both players D and C simultaneously.
- All other combinations of sales are feasible.

Tasks:

1. Design an optimal revenue-maximizing auction under these constraints.
2. Write an explicit description of the payment rule for each possible allocation.
3. Compute the expected revenue of the auction.

1.4 Question: Prior-Free Analysis

1. **DOP Auction Example:** Provide an example of bids in a DOP auction where two bidders are asked to pay different prices, and discuss why this outcome might be undesirable.

   (Bonus) Prove that if the item is sold to two different bidders in a DOP auction, they will always be asked to pay the same price, assuming the tie-breaking rule selects the minimal price.

2. **Profit Extractor Proof:** Prove that the profit extractor is strategy-proof for a fixed reward R.
3. **Expected Revenue - Random Price:** Compute the expected revenue from selling an item whose value is uniformly distributed over $[0, 1]$, where the price is sampled from a uniform $[0, 1]$ distribution.
4. **Expected Revenue - Mean Price:** Compute the expected revenue from selling an item whose value is uniformly distributed over $[0, 1]$, where the price is set to the mean of two sampled values from a uniform $[0, 1]$ distribution.

Density Function of the Mean:

The density function of the mean $M = \frac{X_1+X_2}{2}$, where X_1 and X_2 are independent random variables uniformly distributed over $[0, 1]$, is given by:

$$f_M(m) = \begin{cases} 4m, & 0 \leq m \leq 0.5, \\ 4(1-m), & 0.5 < m \leq 1. \end{cases}$$

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CDF $= x$ $[0,1]$ items $=2$

$f = 1$

VCG - giving $v = (v_1, \dots v_n)$

choice rule $x(v) = \arg\max(\sum \hat{v}_i)$

payment rule $t_i(\hat{v}) = \left[\sum_{j \neq i} \hat{v}_j(a^{-i})\right] - \left[\sum_{j \neq i} \hat{v}_j(a^*)\right]$

The mechanism choose the two higest clímed values.

Assme we will sort the by thire values, named the accordingly (1 to n)

Assme k itmes to sell

The payment

$$t_i(v) = \begin{cases} \left(\sum_{j \neq i} \hat{v}(a^{-i})\right) - \sum v_j(a^*) \\ 0 & i > k \end{cases}$$

$$t_i(v) = \begin{cases} \left(\sum_{j=1}^{j=k+1} v_j\right) - v_i - \left[\left(\sum_{j=1}^{j=k} v_j\right) - v_i\right] & \\ 0 & i > k \end{cases}$$

$$t_i(v) = \begin{cases} \left(\sum_{j=1}^{j=k+1} v_j\right) - \left[\left(\sum_{j=1}^{j=k+1} v_j\right) - v_{k+1}\right] & i \leq k \\ 0 & i > k \end{cases}$$

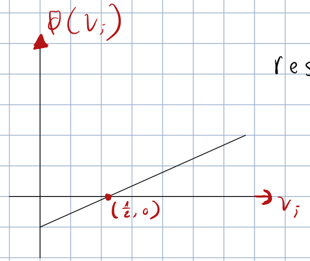
$$t_i(v) = \begin{cases} v_{k+1} & i \leq k \\ 0 & i > k \end{cases}$$

give our case when $n=4$

$$t_i(v) = \begin{cases} v_3 & i \leq 2 \\ 0 & i > 2 \end{cases}$$

1.2) $n=4$, we want to max the revenue, we will need to maximize V.V.

$$\phi(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)} = v_i - \frac{1-v_i}{1} = 2v_i - 1$$



reserved price $\frac{1}{2}$.

allocation take the top 2 players, v_1, v_2

if $v_1 > \frac{1}{2}$ && $v_2 \leq \frac{1}{2} \rightarrow p_1 = \frac{1}{2}$, only v_1 get product

if $v_1 < \frac{1}{2}$ && $v_2 < \frac{1}{2} \rightarrow$ no one get the product

if $v_1 \geq \frac{1}{2}$ && $v_2 \geq \frac{1}{2} \rightarrow p_1 = v_2$ $\quad v_2 = \begin{cases} v_3 & v_3 > \frac{1}{2} \\ \frac{1}{2} & v_3 < \frac{1}{2} \end{cases}$

1.2 The interim allocation function

$$x_i^*(v) = p\left(v_i \geq \max(v)\right) + p\left(\max(v) \geq v_i \geq \max\left(v(\max(v)^{-1})\right)\right)$$

$$p\left(v_i \geq \max(v)\right) = v^n = v^3$$

$$p\left(\max(v) \geq v_i \geq \max\left(v(\max(v)^{-1})\right)\right) = \binom{3}{1} \cdot (1-v) \cdot (v)^3 = 3 \cdot v^2 (1-v)$$

$$x_i^*(v) = v^3 + 3 \cdot v^2 \cdot (1-v)$$

⊗ compute the revenue

$$rev = \int_0^1 p(v)\, x(v) \cdot f(v)\, dv = \int_0^1 12x^2 \cdot (1-x) = 1$$

$$p(v) = v_3 \quad v_3 \sim 12\, x^2 \cdot (1-x)$$

n = 4 revenue max v.v

$$E[r] = \int_{\frac{1}{2}}^1 \phi(v) \cdot x^*(v) \cdot f(v)\, dv = \int_{\frac{1}{2}}^1 (2v-1) \cdot \left[\frac{3v^2(1-v) + v^3}{c}\right] \cdot 1 = 0.225$$

1.3 The mechanism to max $E[rev]$

we will v.v to select the player/s with the higest v.v that hold the condition

$$\phi(v) = 2v - 1 \qquad r = \frac{1}{2}$$

② The payment need to be the minimum value the player can say and still get the allocation

let's say the values are orders $V_1 \to V_4$ $\quad V_1 \geq V_2 \geq V_3 \geq V_4$

A, B is symetric to C, D.

So we have this combintions

$\{\} \to$ no alloctin

$\{A\} \to$ A is the max allocting, and $V_B < \frac{1}{2}$

$\{A, B\} \to$ A+B is the max alloction.

$\{B\}$

The payment is the min value needed to say and get the product.

$\{\} \to 0$ $\qquad u(v) = \begin{cases} 0 & v < \frac{1}{2} \\ v \end{cases}$

$\{A\} \to P_A = \left[\max\left(u(V_C) + u(V_D), \frac{1}{2}\right)\right]$

$\{A, B\} \to P_A = \left[\max\left(u(V_C) + u(V_D), \frac{1}{2}\right)\right] - V_B$

$\{B\} \to \left[\max\left(u(V_C) + u(V_D), \frac{1}{2}\right)\right] - V_A$

8)

$\{\} \to \left(\frac{1}{2}\right)^4 \to \frac{1}{8} = p \longrightarrow$

$\{C\} \to \frac{1}{8}$

$\{D\} \to \frac{1}{8}$

$\{C+D\} = \frac{1}{8}$

$\{A\} \to p\left(V_A > \max\left(u(V_C) + u(V_D), \frac{1}{2}\right)\right) \cdot p\left(B < \frac{1}{2}\right) = \frac{1}{8}$

$\{A, B\} \to \frac{1}{8}$

$\{B\} \to \frac{1}{8}$ $\quad \underbrace{p\left(x > \frac{1}{2}\right)}_{\frac{1}{2}} \cdot \underbrace{p\left(y < \frac{1}{2}\right)}_{\frac{1}{2}} \cdot \underbrace{p(z > x)}_{\frac{1}{2}}$

$$E\{r\} = \left[p\left(\{B\}, \{A\}, \{C\}, \{D\}\right) \cdot \int_{\frac{1}{2}}^{1} x\right] dx + p\left(\{A,B\}, \{C,D\}\right) + \int_{1}^{2} 2 - x \quad dx$$

(1.4) D of Auction A, B so that $V_A > V_B$

$V_A = 0.8 \qquad V_B = 0.6$

$p_A = V_B \qquad p_B = r = 0.4$

this is undesirable because this seems unfair.

(2) In other PDF

(3) $$\int_0^1 \int_p^1 p \cdot dV \, dp = \int_0^1 p(1-p) = \left[p - p^2\right]_0^1 = \frac{1}{6}$$

(4) $$\int_0^1 {}^{4m}_{4(1-m)} \Rightarrow \int \left[\int_0^{\frac{1}{2}} 4x \, dx + \int_{\frac{1}{2}}^1 4(1-x) \, dx \right]$$

$$\int f = \begin{cases} 2x^2 & 0 \le x \le \frac{1}{2} \\ 4x - x^2 & \frac{1}{2} < x < 1 \end{cases}$$

$$E[r] = \int_0^{\frac{1}{2}} x \cdot (4x) \cdot (1-x) + \int_{\frac{1}{2}}^1 x \cdot 4(1-x) \cdot (1-x) \, dx = \frac{5}{24}$$