Minimum-Norm Zero-Forcing Equalization for Fat MIMO Channels via Convex Optimization

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1 Introduction

Zero-forcing (ZF) equalization is a classical technique in multi-input multi-output (MIMO) communication systems, where the goal is to eliminate inter-stream interference by inverting the channel response. While the ZF solution is unique in the square, full-rank case, practical systems often operate in regimes where the effective channel matrix is rectangular and fat, leading to an underdetermined system with infinitely many feasible solutions.

In such scenarios, it is natural to select the equalizer that minimizes the transmitted or processed energy, which corresponds to a minimum-norm solution. This choice is not only physically meaningful, but also leads to improved numerical stability and reduced sensitivity to noise amplification.

In this work, we derive the minimum-norm ZF equalizer for a fat, full-row-rank MIMO channel using a constrained convex optimization approach. Rather than introducing the Moore–Penrose pseudoinverse a priori, the solution is obtained directly from first principles via Lagrangian optimization. This makes explicit the role of rank conditions, matrix dimensions, and adjoint operations in determining the structure of the optimal equalizer.

2 Derivation

We know from previous points that $\bar{\mathbf{T}}$, the block Toeplitz matrix associated with the channel response, must be full rank fat in the sense that $\bar{\mathbf{T}} \in \mathbb{C}^{m \times n}$ with $m < n$ or square in order for the system to be either underdetermined or perfectly determined and allow ZF equalization. We also know that each of the received streams can be optimized separately so that we can write

$$\sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\mathrm{ZF}*}\bar{\mathbf{T}} = \bar{\mathbf{d}}_{\mathbf{j}}^{*}$$

as the condition that we need our system to satisfy. In the square case, in which we already assumed full rank, $\bar{\mathbf{T}}$ is invertible and therefore the equation turns into

$$\bar{\mathbf{w}}_{\mathbf{j}}^{\mathrm{ZF}*} = \frac{1}{\sqrt{G}}\bar{\mathbf{d}}_{\mathbf{j}}^{*}\bar{\mathbf{T}}^{-1}$$

when $\bar{\mathbf{T}}$ in turn is fat, we have infinite solutions to our equation, and it is often desirable to choose the minimum-norm one, as this is usually associated with smaller energy consumption in real systems. This turns our problem into a convex optimization problem where we need

$$\begin{aligned} &\min ||\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}}||^2 \\ \text{s.t. } &\sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{T}} = \bar{\mathbf{d}}_{\mathbf{j}}^* \end{aligned}$$

In order to solve this we make use of lagrangian multipliers, but we must first turn the condition into a homogeneous equation and so

$$\sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{T}} = \bar{\mathbf{d}}_{\mathbf{j}}^* \implies \sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{T}} - \bar{\mathbf{d}}_{\mathbf{j}}^* = 0$$

We also express the euclidean norm as $||\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}}||^2 = \bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}}$ so that we can express our lagrangian as

$$\mathcal{L}(\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}}, \lambda) = \bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}} + 2\mathbb{R}\left\{\lambda^*(\sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{T}} - \bar{\mathbf{d}}_{\mathbf{j}}^*)\right\}$$

were we took the real part of the multiplier in order to ensure that the expression is real. This leads to

$$\mathcal{L}(\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}}, \lambda) = \bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}} + \lambda^*(\sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{T}} - \bar{\mathbf{d}}_{\mathbf{j}}^*) + (\sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{T}} - \bar{\mathbf{d}}_{\mathbf{j}}^*)^*\lambda$$

we can now safely take

$$\frac{\partial\mathcal{L}(\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}}, \lambda)}{\partial\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}} = \bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}} + \sqrt{G}\lambda^*\bar{\mathbf{T}} = 0 \implies \bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}} = -\sqrt{G}\lambda^*\bar{\mathbf{T}}$$

To plug back into our equation we remember we must take the conjugate so that

$$\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*} = \left(-\sqrt{G}\lambda^*\bar{\mathbf{T}}\right)^* = -\sqrt{G}\bar{\mathbf{T}}^*\lambda$$

so that we can do

$$\sqrt{G}\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*}\bar{\mathbf{T}} = \bar{\mathbf{d}}_{\mathbf{j}}^* \implies -G\bar{\mathbf{T}}^*\lambda\bar{\mathbf{T}} = \bar{\mathbf{d}}_{\mathbf{j}}^*$$

as $\bar{\mathbf{T}}$ is fat $\bar{\mathbf{T}}^*\bar{\mathbf{T}}$ is singular and therefore not invertible so we must multiply on the right by $\bar{\mathbf{T}}^*(\bar{\mathbf{T}}\bar{\mathbf{T}}^*)^{-1}$

$$\begin{aligned} -G\bar{\mathbf{T}}^*\lambda\bar{\mathbf{T}}\bar{\mathbf{T}}^*(\bar{\mathbf{T}}\bar{\mathbf{T}}^*)^{-1} &= \bar{\mathbf{d}}_{\mathbf{j}}^*\bar{\mathbf{T}}^*(\bar{\mathbf{T}}\bar{\mathbf{T}}^*)^{-1} \\ -G\bar{\mathbf{T}}^*\lambda &= \bar{\mathbf{d}}_{\mathbf{j}}^*\bar{\mathbf{T}}^*(\bar{\mathbf{T}}\bar{\mathbf{T}}^*)^{-1} \\ -\sqrt{G}\bar{\mathbf{T}}^*\lambda &= \frac{1}{\sqrt{G}}\bar{\mathbf{d}}_{\mathbf{j}}^*\bar{\mathbf{T}}^*(\bar{\mathbf{T}}\bar{\mathbf{T}}^*)^{-1} \end{aligned}$$

which we can plug into the previous expression for $\bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}}$ to obtain

$$\begin{aligned} \bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}*} &= -\sqrt{G}\bar{\mathbf{T}}^*\lambda = \frac{1}{\sqrt{G}}\bar{\mathbf{d}}_{\mathbf{j}}^*\bar{\mathbf{T}}^*(\bar{\mathbf{T}}\bar{\mathbf{T}}^*)^{-1} \\ &\implies \bar{\mathbf{w}}_{\mathbf{j}}^{\text{ZF}} = \frac{1}{\sqrt{G}}(\bar{\mathbf{T}}\bar{\mathbf{T}}^*)^{-1}\bar{\mathbf{T}}\bar{\mathbf{d}}_{\mathbf{j}} \end{aligned}$$

Which leads to a Moore-Penrose pseudoinverse (indicated with †) as

$$\bar{\mathbf{w}}_{\mathbf{j}}^{\mathrm{ZF}} = \frac{1}{\sqrt{G}}(\bar{\mathbf{T}}^{*})^{\dagger}\bar{\mathbf{d}}_{\mathbf{j}}$$

3 Conclusions

In this document, we derived the minimum-norm zero-forcing equalizer for a fat MIMO channel matrix by formulating the problem as a constrained convex optimization task. By minimizing the Euclidean norm of the equalizer subject to the ZF constraint, we showed that the optimal solution necessarily involves the inverse of the Gram matrix $\bar{\mathbf{T}}\bar{\mathbf{T}}^{*}$, which is well-defined under the full row-rank assumption.

The resulting expression coincides with the Moore–Penrose pseudoinverse applied to the desired response, thereby recovering a well-known result through an explicit variational derivation. This approach clarifies why the pseudoinverse arises naturally in underdetermined ZF problems and highlights the importance of dimensional consistency and adjoint ordering in complex-valued optimization.

Beyond its immediate application to MIMO equalization, this derivation provides a useful template for treating related minimum-norm problems in signal processing and communications, including extensions to regularized (MMSE) formulations and numerically robust implementations.