Decoherence Related Error Rate in Dense Coding

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(Dated: February 14, 2025)

I. INTRODUCTION

Quantum communication protocols, such as dense coding, exploit quantum entanglement to enhance the efficiency of classical communication. However, understanding how decoherence affects these protocols is critical for assessing their real-world viability. This study investigates how various types of decoherence influence the error rate of the dense coding protocol.

A. Dense Coding

Dense Coding is a quantum communication protocol which allows the transmission of 2 classical bits from Alice to Bob by sending a single qubit, which is part of a shared entangled pair by following the steps:

- A source generates a maximally entangled EPR pair which is shared by Alice and Bob. For example it can be prepared in the $|\phi^+\rangle$.
- There are 4 possible combination of bits Alice can wish to send (00,01,10,11) which determine which unitary transformation she performs on her qubit of the pair, respectively $U = \{I, \sigma_x, \sigma_y, \sigma_z\}$. These transformations modify the state of the total EPR pair, following:

$$\begin{aligned} (I \otimes I)|\phi^+\rangle &= |\phi^+\rangle \\ (\sigma_x \otimes I)|\phi^+\rangle &= |\psi^+\rangle \\ (\sigma_y \otimes I)|\phi^+\rangle &= |\phi^-\rangle \\ (\sigma_z \otimes I)|\phi^+\rangle &= |\psi^-\rangle \end{aligned} \tag{1.1}$$

- Alice sends her half of the EPR pair to Bob
- Bob performs the operation $(CNOT(H \otimes I))^{-1} = (H \otimes I)CNOT$ which leads to $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ which can be measured in computational basis to read the 2 classical bits.

B. Decoherence

The word decoherence, used in its broader meaning, denotes any quantum-noise process due to unavoidable coupling of the system to the environment. In this analysis, we will make use of the Kraus representation of decoherence, namely

$$\rho' = \sum_k E_k \rho E_k^\dagger \tag{1.2}$$

where ρ denotes the density matrix of our qubit (or in general of the quantum system of interest) and ρ' the final density matrix after it has undergone the transformation. We are gonna focus on a discrete-time representation of decoherence, as opposed to a more general Markovian model, as we are not interested in deriving an analytical definition of the time-evolution of the density operator of the system and it would lead to useless additional complexity. The main analyzed channels will be: bit-flip, phase-flip, depolarizing and amplitude damping. The Kraus operators were derived from the Lindblad operators, see Appendix A.

II. METRICS

We now introduce the metrics needed to study and analyze this problem.

A. Error Rate

We can define the error rate as the probability of obtaining an output different than the intended one. It is important to note that the actual input of the channel are the unitary operations contained in U as defined in a previous section. To calculate the error rate without loss of generalization we can analyze the case were Alice chooses $I \in U$, and therefore the 2-qubit system is in state $|\phi^+\rangle$. Its density matrix is defined as $\rho = |\phi^+\rangle\langle\phi^+|$, which under decoherence becomes

$$\rho' = \sum_k F_k \rho F_k^\dagger \tag{2.1}$$

were we introduced the Kraus operators acting on only the first of the 2 qubits as

$$F_k = (E_k \otimes I) \tag{2.2}$$

where E_k is the Kraus operator that models the selected type of decoherence on one qubit only. Bob now applies the final operation which leads to

$$\rho'' = ((H \otimes I)CNOT)\rho'((H \otimes I)CNOT)^\dagger \tag{2.3}$$

Which has as diagonal elements the probability of measuring each of the combinations of 2 bits $(p_{00}, p_{01}, p_{10}, p_{11})$. This comes as a result of

$$p_{ij} = Tr(\rho''|ij\rangle\langle ij|) = Tr(\langle ij|\rho''|ij\rangle) = \langle ij|\rho''|ij\rangle \tag{2.4}$$

where $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ represent the computational bases for the subsystems of respectively the first and second qubit, and the last equality comes from the 1×1

dimension of the resulting matrix. The error rate, under the assumptions made above, is now calculated as

$$E(p) = 1 - p_{00} \tag{2.5}$$

where p is the decoherence model parameter.

B. Entanglement of Formation

The entanglement of formation of a mixed state is defined as the minimum average entanglement of an ensemble of pure states that represents the given mixed state. It can give us a good idea of how the entanglement between the 2 qubits evolves as a function of the decoherence parameter, which can be particularly interesting as we start from an EPR pair.

$$E_F(\rho) = H_{bin}\left(\frac{1+\sqrt{1-C^2(\rho)}}{2}\right) \tag{2.6}$$

where $H_{bin}(x)$ is the Shannon entropy of a binary variable with probability distribution $x, 1-x$. C is in this case the concurrence

$$C(\rho) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{2.7}$$

with λ_i the square roots of the eigenvalues of R in descending order, where $R = \rho\tilde{\rho}$ and $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ is the spin-flipped density matrix.

C. Trace Distance

The trace distance is a simple measure of how far our final state is from our starting point

$$D(\rho, \rho') = \frac{1}{2}Tr|\rho - \rho'| \tag{2.8}$$

Namely

$$D(\rho, \rho') = 0 \leftrightarrow \rho = \rho' \tag{2.9}$$
$$D(\rho, \rho') = 1 \leftrightarrow \rho\rho' = 0 \tag{2.10}$$

III. BIT-FLIP

This type of decoherence represents the probability p of a given qubit of being flipped from $|0\rangle$ to $|1\rangle$ or the other way around. Visualized on the Bloch sphere, it appears as a transformation into an ellipsoid symmetrically aligned with the x-axis.
The bit-flip case is usually modelled through the use of the Kraus operators

$$E_0 = \sqrt{1-p}\ I \quad E_1 = p\sigma_x$$

with $p \in [0,1]$ probability of bit-flip in our channel. We extend this operators to our 2-qubit system by applying it only to the first qubit, the one transmitted from Alice to Bob.

$$F_0 = E_0 \otimes I \quad F_1 = E_1 \otimes I \tag{3.1}$$

As mentioned before we analyze just the case $|\phi^+\rangle$ from which we expect $|00\rangle$ as final state. This state has density matrix

$$\rho = |\phi^+\rangle\langle\phi^+| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

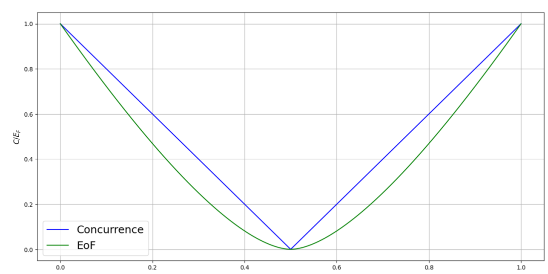
As a consequence of decoherence the state undergoes a unitary transformation

$$\rho'_{BF} = \sum_{i=0}^{1} F_i \rho F_i^\dagger = \frac{1}{2}\begin{pmatrix} 1-p & 0 & 0 & 1-p \\ 0 & p & p & 0 \\ 0 & p & p & 0 \\ 1-p & 0 & 0 & 1-p \end{pmatrix} \tag{3.3}$$

The concurrence, calculated as seen in (2.7) results in

$$C_{BF} = |2p - 1| \tag{3.4}$$

The state becomes separable only for exactly $p = 0.5$,



otherwise it retains some degree of quantum correlation. Concerning trace distance using (2.8) we get

$$D(\rho, \rho'_{BF}) = \frac{1}{2}Tr|\rho - \rho'_{BF}| = p \tag{3.5}$$

Again, quite self-explanatory, as for $p = 0$ the state doesn't undergo any transformation and for $p = 1$ it gets to an orthogonal one.
Concerning the error rate of the protocol, for brevity we use

$$B = (H \otimes I)CNOT \tag{3.6}$$

We have

$$B\rho'_{BF}B^\dagger = diag\{1-p, p, 0, 0\} \tag{3.7}$$

Which already leads us to an interesting result. As we can expect, the probability of correctly measuring both qubits in computational basis decreases with p, although in general not with linear behavior. The interesting part is the fact that no matter p we always measure either 00 or 01 so we can't have errors on the first qubit, while the probability of error on the second one is p. Of course this changes with the unitary operation chosen by Alice so on other states the error probability will fall on the first qubit. This make sense as we are just dealing with a statistical mixture of state $|\phi^+\rangle$ and $|\psi^+\rangle$ as we can write

$$\rho'_{BF} = (1-p)|\phi^+\rangle\langle\phi^+| + p|\psi^+\rangle\langle\psi^+| \tag{3.8}$$

IV. PHASE-FLIP

We are now modeling a finite probability of a qubit's relative phase between its computational basis components being flipped. For example, the state $|+\rangle$ will be mapped to $|-\rangle$, while $|0\rangle$ and $|1\rangle$ remain unchanged. This can also be viewed as a classical mapping of the Bloch sphere onto an ellipsoid, symmetric about the z-axis. This case is pretty similar to the bit-flip case, but the single qubit operators are

$$E_0 = \sqrt{1-p}\; I \quad E_1 = \sqrt{p}\; \sigma_z \tag{4.1}$$

Which gives us a post-decoherence density matrix after expanding our Kraus operators to the 2 qubit Hilbert space

$$\rho'_{PF} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1-2p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-2p & 0 & 0 & 1 \end{pmatrix} \tag{4.2}$$

Using the same method as before, the concurrence results the same, as expected as the qubit is undergoing a similar process with a different axis symmetry

$$C_{PF} = |2p-1| \tag{4.3}$$

The same argument goes for the trace distance which is intuitively identical to the bit-flip case. Reguarding the error rate

$$B\rho'_{PF}B^\dagger = diag\{1-p, 0, p, 0\} \tag{4.4}$$

We, once again, see the similarities with the bit-flip case but different axis simply calls for a change in the probability distribution between the states.

V. DEPOLARIZATION

Depolarization is used to model loss of information to the environment. In the Bloch sphere representation this corresponds to the Bloch vector shortening until it reaches a maximally mixed state. p is in this case a more abstract parameter, representing the amount of information loss ($p = 0$ means no loss, $p = 1$ all the information is lost).
The single qubit Kraus operators used are

$$E_0 = \sqrt{1-p}\, I \;\; E_1 = \sqrt{\frac{p}{3}}\, \sigma_x \;\; E_2 = \sqrt{\frac{p}{3}}\, \sigma_y \;\; E_3 = \sqrt{\frac{p}{3}}\, \sigma_z \tag{5.1}$$

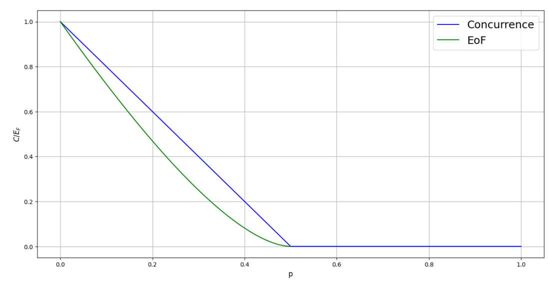
which thanks to our usual expansion on the 2-qubit Hilbert space are mapped into the respective F_i.
This leads us to

$$\rho'_D = \frac{1}{6}\begin{pmatrix} 3-2p & 0 & 0 & 3-4p \\ 0 & 0 & 2p & 0 \\ 0 & 2p & 0 & 0 \\ 3-4p & 0 & 0 & 3-2p \end{pmatrix} \tag{5.2}$$

Using (2.7) we get

$$C_D = max\{0, 2p-1\} \tag{5.3}$$

This result aids some deeper physical insights. At $p = 0.5$



we can see the quantum correlations vanish and the system transitions into a separable state (separability threshold) but, opposed to the bit-flip case (3.4) the information is this time completely lost, the entanglement can't therefore increase, and the state stays separable even for $p > 0.5$.
The error rate calculations lead to an even more interesting result as

$$B\rho'_D B^\dagger = diag\{1-p, \frac{p}{3}, \frac{p}{3}, \frac{p}{3}\} \tag{5.4}$$

For $p = 1$, indicating a complete loss of information about the qubit sent from Alice to Bob, we do not achieve the expected equal probability of guessing each state (0.25 probability of being correct); instead, we face absolute certainty of error. This becomes clearer when comparing (1.1) with (5.1). We see that as $F_i = E_i \otimes I$, these operators transition $|\phi^+\rangle$ to another one of the states Alice can prepare, each with a probability of $\frac{p}{3}$, leading to a statistical mixture. With probability $1-p$, the identity operator is applied, meaning the intended bits will be correctly measured.
Regarding trace distance, $D(\rho, \rho'_D) = p$ which is quite expected: for $p = 1$ we get a mixture of states orthogonal to ρ, while for $p = 0$ the original state is preserved.

VI. AMPLITUDE DAMPING

Amplitude damping aims to model the loss of energy to the environment, with as a consequence the tendency of the system to fall onto its ground state (conventionally $|0\rangle$). The parameter p now models how much of the energy is actually lost. The single-qubit Kraus operators used are

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \quad E_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \tag{6.1}$$

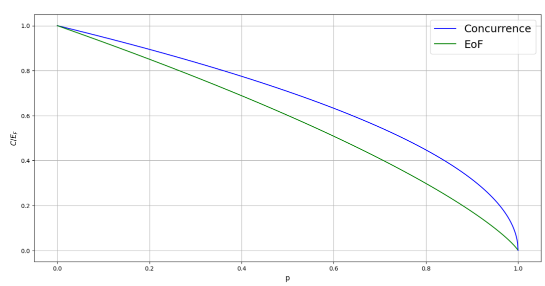
which are then, again, extended to act only of the first qubit of our 2-qubit space.
We can readily see that we are now working on a more asymmetric decoherence, which will lead to interesting results. The evolved density matrix is

$$\rho'_{AD} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \sqrt{1-p} \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-p} & 0 & 0 & 1-p \end{pmatrix} \tag{6.2}$$

The concurrence calculations lead us to

$$C_{AD} = \sqrt{1-p} \tag{6.3}$$

Which at its limits means that for $p = 0$ as we expect



the state is unperturbed and the entanglement is at its maximum, while complete loss of energy for $p = 1$ results in a separable state as we decay to the ground state.
Calculating the final density matrix ρ''_{AD} we have a situation we were yet to encounter as

$$\rho''_{AD} = B\rho'_{AD}B^\dagger = \frac{1}{4}\begin{pmatrix} 2-p+2\sqrt{1-p} & 0 & p & 0 \\ 0 & p & 0 & p \\ p & 0 & 2-p-2\sqrt{1-p} & 0 \\ 0 & p & 0 & p \end{pmatrix} \tag{6.4}$$

The density operator for the final state is not diagonal, there are still present coherences which signal some type of quantum correlation between the qubits is still present for $p \neq 0$. For $p = 0$ instead we, as expected get

$$\rho''_{AD}\Big|_{p=0} = |00\rangle\langle 00| \tag{6.5}$$

This aids a more in depth analysis. We calculate the concurrence using (2.7) for this final density matrix and, fairly enough

$$C_{\rho''_{AD}} = 0 \implies E_F = 0 \tag{6.6}$$

Which concludes that the quantum correlations present are not entanglement-related but take part into the system's quantum discord. Calculations of the quantum discord as

$$D(A;B) = I(A;B) - S(\rho_A) + \min_{\{\Pi_j\}} S(\rho_{AB}|\{\Pi_j\}) \tag{6.7}$$

can become quite complex and are beyond the scope of this brief review.
Returning to the error rate, we get

$$E_{AD}(p) = \frac{2+p+2\sqrt{1-p}}{4} \tag{6.8}$$

Appendix A: Kraus Operators Derivation

In this appendix we simply explain how the Kraus operators were derived from the Lindblad operators for the master equation, in particular for the bit-flip case as the others follow easily. We start from the GKLS master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{1}{2}\sum_\mu \{L^\dagger_\mu L_\mu, \rho\} + \sum_\mu L_\mu \rho L^\dagger_\mu \tag{A1}$$

We are considering a purely decoherence based model so we can assume the absence of an Hamiltonian-related evolution. The Lindblad operator for the bit-flip is

$$L = \sqrt{\gamma}\ \sigma_x \tag{A2}$$

Knowing that the Pauli operators are Hermitian and $\sigma_x^2 = I$

$$\begin{aligned} \dot{\rho} = -\frac{\gamma}{2}(\sigma_x^2\rho + \rho\sigma_x^2) + \gamma\sigma_x\rho\sigma_x \\ \dot{\rho} = -\gamma\rho + \gamma\sigma_x\rho\sigma_x \end{aligned} \tag{A3}$$

We now use the first-order Euler expansion

$$\rho(t+dt) = \rho(t) + \dot{\rho}(t)dt + O(dt^2) \tag{A4}$$

To step into the discrete time realm, from which plugging in (A3) we get

$$\begin{aligned} \rho(t+dt) = \rho(t) + \gamma(\sigma_x\rho\sigma_x - \rho)dt \\ \rho(t+dt) = \gamma dt \sigma_x\rho\sigma_x + (1-\gamma dt)\rho \end{aligned} \tag{A5}$$

We now can recognize the 2 operators

$$E_0 = \sqrt{1-\gamma dt}\ I \quad E_1 = \sqrt{\gamma dt}\ \sigma_x \tag{A6}$$

Substituting $p = \gamma dt$

$$E_0 = \sqrt{1-p}\ I \quad E_1 = \sqrt{p}\ \sigma_x \tag{A7}$$

Which are exactly the operators as written in (3.1).