Exercise 1 (5,5 points)

1. Solve on $I =]0, +\infty[$ the differential equation $(E) xy' + \frac{1}{2}y = -2.$

Homogeneous equation: $\mu + \frac{1}{2}\gamma = 0$ $\gamma = 2e^{-\frac{1}{2}indx} = 2e^{-\frac{1}{2}in(x)} = \frac{1}{2}$

Particular salution: the constant function your - - 4

(onclusion: S= (30, table => 1/2 | k C/R)

2. Solve on $\mathbb R$ the differential equation (E) $2y'' + 8y' + 8y = 3e^{-2x}$

Homogeneous equation: y"+ 4y + 4y = 0

we characteristic equation is X + 4X + 4 = 0

Thus 4/21- (b 11+6-) - 2x

Porticular solution: we search up in the form up=P(N)e-20

1) - CC - 7 (0# 10) - 74 0/ -1 -0\ -24 00 -71 - -

40 ES (= > 2 (P" - 4P + 4P) e 14 + 8 (P' - 2P) e 14 + 8 (P - 2P) e 14 = 3e 2 (= \ e - 2 \left(28" + 0 P' + 0 P \right) = 3 e - 2 u

(=> P! = 3

Let P' = 3 x P" = 3 x2, then 40(2)= 3 12e-21

S= (R -> M × +> (3 x2+k,x+k2) = 20 ,(k,k2) E 122

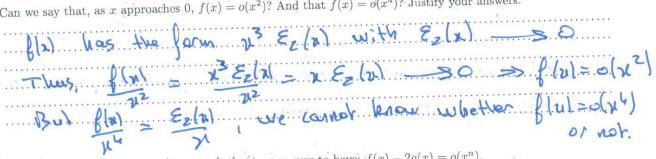
Exercise 2 (5 points)

The questions of the exercise are mutually independent.

1. Let f and g be two functions such that, as x approaches 0:

$$f(x) = o(x^3)$$
 and $g(x) = x^2 \varepsilon(x)$ with $\lim_{x \to 0} \varepsilon(x) = 0$

(a) Can we say that, as x approaches 0, $f(x) = o(x^2)$? And that $f(x) = o(x^4)$? Justify your answers.



(b) Find the greatest natural number n such that we are sure to have: $f(x) - 2g(x) = o(x^n)$.

$$||f(x)|| = 2g(x) = o(x^2) \quad \text{frecause} \quad ||f(x)|| = 2g(x) = x^3 \mathcal{E}_{\mathcal{L}}(x) - 2g(x) = x^2 \mathcal{E}_{\mathcal$$

$$f(x) = 1 + x + x^2 + o(x^3)$$
 and $g(x) = 2x + x^2 - x^3 + o(x^3)$

Find simple equivalents in 0 of: f(x), g(x) and 2xf(x) - g(x).

$$g(x) = 2x + o(x) \Rightarrow g(x) \times 2x$$

$$g(x) = 2x + o(x) \Rightarrow g(x) \times 2x$$

$$2x + o(x) = 2x + 2x^2 + 2x^3 + o(x^4) - 2x - x^2 + x^3 + o(x^3) = x^2 + o(x^2) \times x^2$$

3. Propose a Taylor expansion in 0, at the order 3, of a non-zero function h which would satisfy:

$$h(x) \sim -3x \text{ and } h(x) + 3x \sim 5x^{2}$$
Let $h(x) = -3x + 5x^{2} + 37x^{3} + 5(x^{3})$
Then $h(x) = -3x + 6(x) \Rightarrow h(x) \approx -3x$ and $h(x) + 3x = 5x^{2} + 6(x^{2})$ as $5x^{2}$

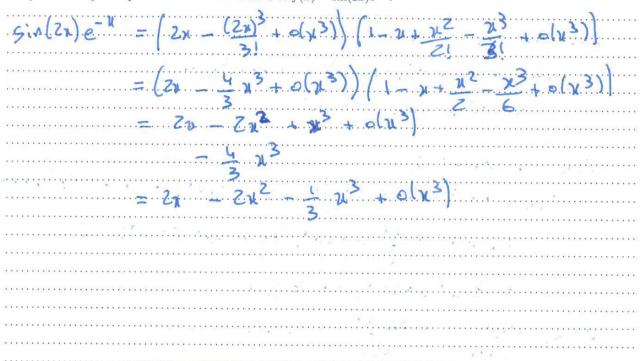
4. Propose a Taylor expansion in 0, at the order 4, of a non-zero function i which would satisfy:

$$i(x) = o(x^3) \text{ and } \lim_{x \to 0} \frac{i(x)}{x^4} = 2$$
Let $i(x) = 2x^4 + o(x^4)$

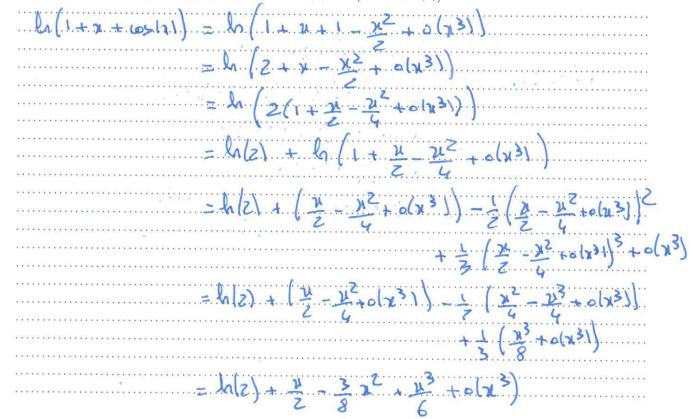
Exercise 3 (5 points)

In your redaction, write explicitly the basic Taylor expansions that you use.

1. Compute the Taylor expansion in 0 at the order 3 of $f(x) = \sin(2x)e^{-x}$.



2. Compute the Taylor expansion in 0 at the order 3 of $g(x) = \ln (1 + x + \cos(x))$.



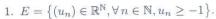
Exercise 4 (5 points)

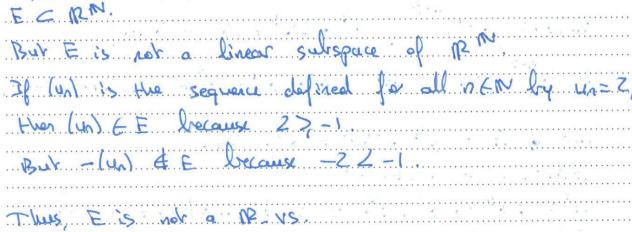
1	Din d	$\lim_{x\to 0}$	$\sqrt{1+2x^2} - \cos(2x^2) - x^2$
1.	Find		$e^{-x} + \sin(x) - 1$

,	Find $\lim_{x\to 0} \frac{1}{e^{-x} + \sin(x) - 1}$
	The denomination is
	$D(u) = (1 - u + 2u^2 + o(u^2)) + (x + o(x^2)) - 1$
	$D u = (1 - u + \frac{u^2}{2} + o(u^2)) + (x + o(x^2)) - 1$ $= \frac{u^2}{2} + o(u^2) + o(x^2)$
	2
	The numerator is
	The numerator is N(x) = 1+ \frac{1}{2}(2x^2) + \frac{1}{2}(2x^2)^2 + o(x^4) - \left(1 - \frac{1}{2}x^2)^2 + o(x^4)\right).
	$= 1 + 2^{2} - 2^{4} + o(2^{4}) - 1 + 22^{4} + o(2^{4}) - 2^{2}$
	Z
	$=\frac{3}{2}\chi^{2}+o(\chi^{2}) N \stackrel{?}{=} \chi^{2}$
	$= \frac{3}{2} x^{4} + o(x^{4}) N = \frac{3}{2} x^{4}$ Thus, Mind N 3/2 x = 3x ² $\frac{1}{2} N(x) = \frac{3}{2} x^{2}$ $\frac{1}{2} N(x) = 0$
	$O(x)$ x^2/z $O(x)$
	Sinu $3x^2 \rightarrow 0$, 0 0 0 0 0 0 0 0 0 0
2.	Find $\lim_{x\to +\infty} \left(x\sin\left(\frac{1}{x}\right)\right)$.
	$\sqrt{2} \sqrt{2} \ln \left(2 \sin \left(\frac{1}{2} \right) \right)$
	/n & (1) = e
	31/11/11
	$= \frac{\chi^2 \ln \left(\pi \left(\frac{1}{2} - \frac{1}{6\chi^2} + o\left(\frac{1}{\chi^2} \right) \right) \right)}{\left(\frac{1}{2} - \frac{1}{6\chi^2} + o\left(\frac{1}{2} \right) \right)}$
	$= e^{\lambda^2 \ln \left(1 - \frac{1}{6\lambda^2} + o\left(\frac{1}{\lambda^2}\right)\right)}$
	$= e^{12} \ln \left(1 - \frac{1}{6\pi^2} + o\left(\frac{1}{3\epsilon}\right)\right)$ $= e^{12} \left(-\frac{1}{6\pi^2} + o\left(\frac{1}{3\epsilon}\right)\right)$ $= e^{12} \left(-\frac{1}{6\pi^2} + o\left(\frac{1}{3\epsilon}\right)\right)$
	$= e^{\chi^2} \ln \left(1 - \frac{1}{6\chi^2} + o(\frac{1}{\chi^2}) \right)$ $= e^{\chi^2} \left(-\frac{1}{6\chi^2} + o(\frac{1}{\chi^2}) \right)$
	$= e^{12} \ln \left(1 - \frac{1}{6\pi^2} + o\left(\frac{1}{3\epsilon}\right)\right)$ $= e^{12} \left(-\frac{1}{6\pi^2} + o\left(\frac{1}{3\epsilon}\right)\right)$ $= e^{12} \left(-\frac{1}{6\pi^2} + o\left(\frac{1}{3\epsilon}\right)\right)$
	$= e^{12} \ln \left(1 - \frac{1}{6\pi^2} + o(\frac{1}{3^2}) \right)$ $= e^{12} \left(-\frac{1}{6\pi^2} + o(\frac{1}{3^2}) \right)$ $= e^{-\frac{1}{6}} + o(1)$ $= e^{-\frac{1}{6}} + o(1)$ $= e^{-\frac{1}{6}} + o(1)$ $= e^{-\frac{1}{6}} + o(1)$
	$= e^{12} \ln \left(1 - \frac{1}{6\pi^2} + o(\frac{1}{3^2}) \right)$ $= e^{12} \left(-\frac{1}{6\pi^2} + o(\frac{1}{3^2}) \right)$ $= e^{-\frac{1}{6}} + o(1)$ $= e^{-\frac{1}{6}} + o(1)$ $= e^{-\frac{1}{6}} + o(1)$ $= e^{-\frac{1}{6}} + o(1)$

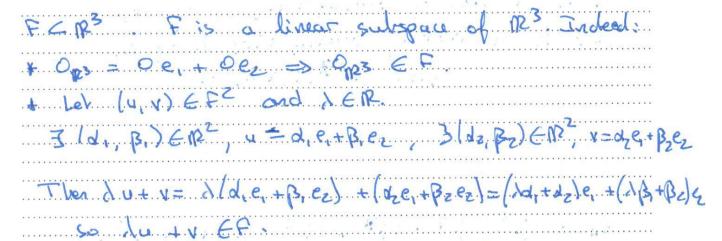
Exercise 5 (6 points)

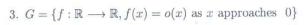
Are the following sets \mathbb{R} -vector spaces? Justify rigorously your answers.

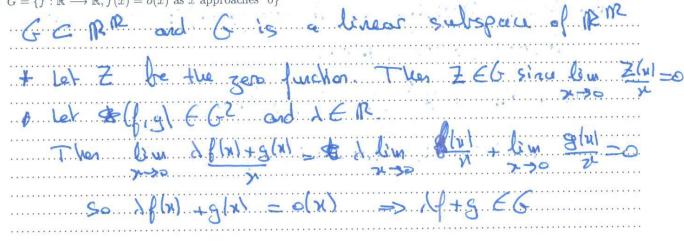






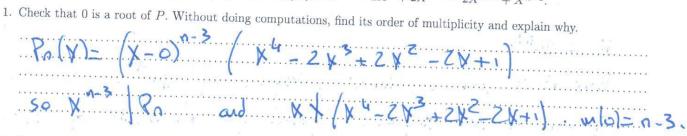




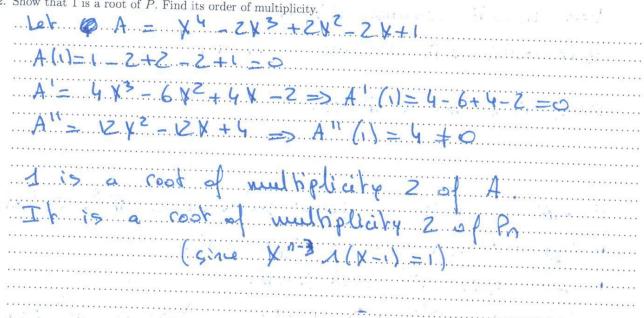


Exercise 6 (4,5 points)

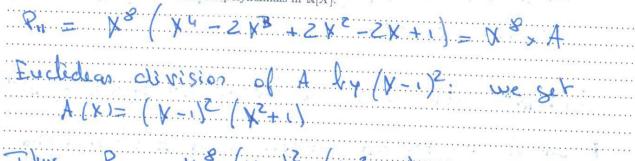
Consider a natural number $n \ge 5$ and the polynomial $P_n(X) = X^{n+1} - 2X^n + 2X^{n-1} - 2X^{n-2} + X^{n-3}$.



2. Show that 1 is a root of P. Find its order of multiplicity.



3. Assume in this question that n = 11. Thus, $P_{11}(X) = X^{12} - 2X^{11} + 2X^{10} - 2X^9 + X^8$. Using the previous questions, factorize P_{11} as a product of irreducible polynomials in $\mathbb{R}[X]$.



Exercise 7 (4 points)

The purpose of the exercise is to find all the polynomials P of degree 3 such that $(X-1)^2|P(X)-1$ and $(X+1)^2|P(X)+1$. Consider a polynomial $P(X)=aX^3+bX^2+cX+d$ with $(a,b,c,d)\in\mathbb{R}^4$ and satisfying the hypothesis:

$$(H)$$
: $(X-1)^2 | P(X) - 1$ and $(X+1)^2 | P(X) + 1$

Let us define the two polynomials: A(X) = P(X) - 1 and B(X) = P(X) + 1.

1.	Write all the	information a	about	A and	B that	can be	deduced	from	the hypothesis	$(H)^{\circ}$
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A(i) = A'(i) = 0 and B(+i) = B'(-i) = 0

2. Deduce the values of P(1), P'(1), P(-1) and P'(-1).

e(i)=1 e(-i)=-1

3. Find all the polynomials P of degree 3 who satisfy (H).

Let P= ax3+ lx2+ cx+d Then P=30x2+2lx+c

3a + 2b + 6 = 0 (P(1)=0)

-a + b - c + d = -1 (P(-1=-1)

3a -2b+c =0 (e'(-1)=0)

 $(\Rightarrow) 0 + b + c + d = 1$ $(\Rightarrow) (+ 1 + 6 = 2)$

3a + 2b + c =0 (£q2)

(=> b=0, d=-b=0, c=-3a (from Eq 3)

We inject in Eq 1: $-2a=1 \Rightarrow a=-\frac{1}{2}$ and $c=\frac{3}{2}$

We get: P= - 18 + 3 X