Midterm exam 1

Duration : three hours
Documents and calculators not allowed

Exercise 1 (2 points)

1. Determine the Taylor expansion around 0 at the order 3 of $\cos(x)^{\sin(x)}$.
2. Determine $\lim_{x\to 0} \dfrac{\ln\big(1+\sin(x)\big) - \sin\big(\ln(1+x)\big)}{x^2\sin(x^2)}$.

Exercise 2 (4,5 points)

1. Using the d'Alembert rule, determine the nature of $\sum \dfrac{2n}{n+2^n}$.
2. Using the d'Alembert rule, determine the nature of $\sum \dfrac{1+n^2}{n!}$.
3. Determine the nature of $\sum \dfrac{\sin\big(\sqrt{n}+1\big)}{n^2}$.
4. Let $\alpha \in \mathbb{R}$. Determine, with precise and detailed arguments, the nature of $\sum \dfrac{(-1)^n}{n^\alpha}$.

Exercise 3 (8 points)

For all $n \in \mathbb{N}^*$, we set

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

1. a. What is the nature of the series $\sum \dfrac{1}{\sqrt{n}}$?

   b. What is the limit of a_n when n tends to $+\infty$?

2. We consider the sequence $(u_n)_{n\geqslant 2}$ defined for all $n \geqslant 2$ by

$$u_n = \frac{(-1)^n}{(-1)^n + a_n}$$

   a. Show that for all $n \geqslant 2$,

$$u_n = \frac{(-1)^n}{a_n} - \frac{1}{a_n^2} + o\left(\frac{1}{a_n^2}\right)$$

   b. Show that the sequence $\left(\dfrac{1}{a_n}\right)_{n\in\mathbb{N}^*}$ is decreasing and converges to 0.

   c. Deduce the nature of the series $\sum \dfrac{(-1)^n}{a_n}$.

   d. Show by induction that, for all $n \in \mathbb{N}^*$,

$$2\sqrt{n+1} - 2 \leqslant a_n \leqslant 2\sqrt{n} - 1$$

   N.B. : you can use the fact, without proving it, that for all $n \in \mathbb{N}^*$, $2\sqrt{n+2} - 2\sqrt{n+1} \leqslant \dfrac{1}{\sqrt{n+1}}$ and $2\sqrt{n+1} - 2\sqrt{n} \geqslant \dfrac{1}{\sqrt{n+1}}$.

e. Deduce the limit of $\dfrac{a_n}{2\sqrt{n}}$ when n tends to $+\infty$. Give an equivalent of a_n in $+\infty$.

f. Deduce the nature of the series $\sum\left(-\dfrac{1}{a_n^2}+o\left(\dfrac{1}{a_n^2}\right)\right)$.

3. Determine the nature of the series $\sum u_n$.

Exercise 4 (4,5 points)

We consider the sequence $(u_n)_{n\geqslant 2}$ such that

$$u_n=\ln\left(\frac{\sqrt{n}+(-1)^n}{\sqrt{n+1}}\right)$$

For $n\geqslant 2$, we set

$$v_n=\frac{\sqrt{n}}{\sqrt{n+1}}$$

1. Show that
$$u_n=\ln(v_n)+\ln\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$$

2. Determine $(\alpha,\beta)\in\mathbb{R}^2$ such that
$$v_n=1-\frac{\alpha}{n}+\frac{\beta}{n^2}+o\left(\frac{1}{n^2}\right)$$

3. Determine $\gamma\in\mathbb{R}$ such that
$$\ln(v_n)=-\frac{\alpha}{n}+\frac{\gamma}{n^2}+o\left(\frac{1}{n^2}\right)$$

4. Show that
$$u_n=\frac{(-1)^n}{\sqrt{n}}-\frac{1}{n}+\frac{(-1)^n}{3n\sqrt{n}}+o\left(\frac{1}{n\sqrt{n}}\right)$$

5. Deduce the nature of $\sum u_n$.
N.B. : your redaction at this last question is expected to be particularly precise and rigorous.

Exercise 5 (2 points)

1. Let (u_n) be a real sequence. Show that : $\sum(u_{n+1}-u_n)$ convergent $\Longleftrightarrow$ (u_n) convergent.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence with strictly positive terms and $u_0\in\mathbb{R}_+^*$. We define $(u_n)_{n\in\mathbb{N}}$ by
$$\forall n\in\mathbb{N},\quad u_{n+1}=u_n+\frac{a_n}{u_n}$$
Show that : (u_n) convergent $\Longleftrightarrow$ $\sum a_n$ convergent.

N.B. : you may use the fact (without proving it) that (u_n) is strictly positive ans strictly increasing.