LOFO 2020

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1A. Prove that $(\neg P \vee Q) \Leftrightarrow (P \Rightarrow Q)$ is a tautology.

p	q	$\neg p$	$\neg p \vee q$	$p \Rightarrow q$	φ
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	0	1	1	1

1B. Prove that $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ is a tautology.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	φ
0	0	1	1	0	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	1	0	0	1
1	1	0	0	1	0	0	1

2A. Is the following Hilbert system complete with regards to $\mathcal{F}_{\{\perp,\neg,\vee,\Rightarrow\}}$?

$$\frac{A \Rightarrow B \qquad A}{B}\ [\textit{Modus Ponens}]$$

$$\frac{}{\perp \Rightarrow A}\ [\perp] \qquad \frac{}{A \Rightarrow B \Rightarrow A}\ [\Rightarrow_1] \qquad \frac{}{(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C}\ [\Rightarrow_2]$$

$$\frac{}{(A \Rightarrow \perp) \Rightarrow \neg A}\ [\neg_1] \qquad \frac{}{A \Rightarrow \neg A \Rightarrow \perp}\ [\neg_2]$$

No, because no rule can handle the $\vee$ symbol.

2B. Is the following Hilbert system complete with regards to $\mathcal{F}_{\{\bot,\wedge,\vee,\Rightarrow\}}$?

$$\dfrac{A \Rightarrow B \qquad A}{B}\ [\textit{Modus Ponens}]$$

$$\dfrac{}{A \Rightarrow A \vee B}\ [\vee_1] \qquad \dfrac{}{B \Rightarrow A \vee B}\ [\vee_2] \qquad \dfrac{}{A \vee B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C}\ [\vee_3]$$

$$\dfrac{}{A \wedge B \Rightarrow A}\ [\wedge_1] \qquad \dfrac{}{A \wedge B \Rightarrow B}\ [\wedge_2] \qquad \dfrac{}{A \Rightarrow B \Rightarrow A \wedge B}\ [\wedge_3]$$

$$\dfrac{}{A \Rightarrow B \Rightarrow A}\ [\Rightarrow_1] \qquad \dfrac{}{(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C}\ [\Rightarrow_2]$$

No, because no rule can handle the $\bot$ symbol.

3A. Prove that $\{Q \wedge P, R\} \vdash_{\mathcal{N}} P \wedge (R \wedge Q)$. This can be done using a proof of depth 4.

$$\dfrac{\dfrac{Q \wedge P}{P}\ \wedge\text{Elim2} \qquad \dfrac{R \qquad \dfrac{Q \wedge P}{Q}\ \wedge\text{Elim1}}{R \wedge Q}\ \wedge\text{Intro}}{P \wedge (R \wedge Q)}\ \wedge\text{Intro}$$

3B. Prove that $\{P \vee Q\} \vdash_{\mathcal{N}} P \vee (Q \vee R)$. This can be done using a proof of depth 4.

$$\dfrac{P \vee Q \qquad \dfrac{[P]}{P \vee (Q \vee R)}\ \vee\text{Intro1} \qquad \dfrac{\dfrac{[Q]}{Q \vee R}\ \vee\text{Intro1}}{P \vee (Q \vee R)}\ \vee\text{Intro2}}{P \vee (Q \vee R)}\ \vee\text{Elim}$$

4A. Prove that $\{\neg P \Rightarrow Q\} \vdash_{\mathcal{N}} P \vee Q$ by filling the blanks of the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg P \Rightarrow Q \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P}\,{}^{1}}{P \vee Q}\ [\vee_I^l] \qquad \dfrac{}{\neg(P \vee Q)}\,{}^{2}}{\bot}\ [\neg_E]}{\neg P}\ [\neg_I]^1}{Q}\ [\Rightarrow_E]}{P \vee Q}\ [\vee_I^r] \qquad \dfrac{}{\neg(P \vee Q)}\,{}^{2}}{\dfrac{\bot}{\neg\neg(P \vee Q)}\ [\neg_I]^2}\ [\neg_E]}{P \vee Q}\ [\neg\neg]$$

4B. Prove that $\vdash_{\mathcal{N}} ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ by filling the blanks of the following tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \Rightarrow Q) \Rightarrow P}^{3} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{P}^{1} \quad \dfrac{}{\neg P}^{2}}{\bot}[\neg_E]}{Q}[\bot_E]}{P \Rightarrow Q}[\Rightarrow_I]^1}{P}[\Rightarrow_E] \quad \dfrac{}{\neg P}^{2}}{\bot}[\neg_E]}{\neg\neg P}[\neg_I]^2}{P}[\neg\neg]}{((P \Rightarrow Q) \Rightarrow P) \Rightarrow P}[\Rightarrow_I]^3
$$

5A. Define a term Square $\in \Lambda$ such that for any natural integer n:

$$\text{Square}\ \underline{n} \rightarrow_\beta^* \underline{n^2}$$

Then guess its type.

Square $= \lambda u \cdot \text{Mult}\ uu$ is of type $((\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma)) \rightarrow ((\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma))$.

5B. Define a term Double $\in \Lambda$ such that for any natural integer n:

$$\text{Double}\ \underline{n} \rightarrow_\beta^* \underline{2 \times n}$$

Then guess its type.

Double $= \lambda u \cdot \text{Plus}\ uu$ is of type $((\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma)) \rightarrow ((\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma))$.

6. Prove that $\Theta = (\lambda xy \cdot y(xxy))(\lambda uv \cdot v(uuv))$ is a fixed-point combinator.

$$
\begin{array}{rcl}
\Theta A & = & \theta\theta A \\
 & = & ((\lambda xy \cdot y(xxy)\theta)A \\
 & \rightarrow_\beta & (\lambda y \cdot y(\theta\theta y))A \\
 & \rightarrow_\beta & A(\theta\theta A) \\
 & = & A\Theta A
\end{array}
$$

Thus $\Theta A \leftrightarrow_\beta^* A(\Theta A)$.

7. Prove that $\vdash KI : \tau \rightarrow \sigma \rightarrow \sigma$.

K is of primary type $\sigma \rightarrow \tau \rightarrow \sigma$, thus also of type $(\sigma \rightarrow \sigma) \rightarrow \tau \rightarrow (\sigma \rightarrow \sigma)$. I is of type $\sigma \rightarrow \sigma$. KI is therefore of type $\tau \rightarrow \sigma \rightarrow \sigma$ by the application rule.

8A. Prove that $\vdash_{\mathcal{NI}} (P \wedge Q) \Rightarrow (Q \wedge P)$.
Then find a term in Λ_{ext} of type $\sigma \times \tau \rightarrow \tau \times \sigma$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{$P \wedge Q$}
\RightLabel{$[\wedge_E^r]$}
\UnaryInfC{Q}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{$P \wedge Q$}
\RightLabel{$[\wedge_E^l]$}
\UnaryInfC{P}
\RightLabel{$[\wedge_I]$}
\BinaryInfC{$Q \wedge P$}
\RightLabel{$[\Rightarrow_I]^1$}
\UnaryInfC{$(P \wedge Q) \Rightarrow (Q \wedge P)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{$x : \sigma \times \tau$}
\UnaryInfC{$\Pi_2(x) : \tau$}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{$x : \sigma \times \tau$}
\UnaryInfC{$\Pi_1(x) : \sigma$}
\BinaryInfC{$\langle \Pi_2(x), \Pi_1(x) \rangle : \tau \times \sigma$}
\RightLabel{^{1}}
\UnaryInfC{$\lambda x \cdot \langle \Pi_2(x), \Pi_1(x) \rangle : \sigma \times \tau \to \tau \times \sigma$}
\end{prooftree}
$$

8B. Prove that $\vdash_{\mathcal{NI}} (P \vee Q) \Rightarrow (Q \vee P)$.
Then find a term in Λ_{ext} of type $\sigma \cup \tau \to \tau \cup \sigma$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{$P \vee Q$}
\AxiomC{}
\RightLabel{^{2}}
\UnaryInfC{P}
\RightLabel{$[\vee_I^r]$}
\UnaryInfC{$Q \vee P$}
\AxiomC{}
\RightLabel{^{2}}
\UnaryInfC{Q}
\RightLabel{$[\vee_I^l]$}
\UnaryInfC{$Q \vee P$}
\RightLabel{$[\vee_E]^2$}
\TrinaryInfC{$Q \vee P$}
\RightLabel{$[\Rightarrow_I]^1$}
\UnaryInfC{$(P \vee Q) \Rightarrow (Q \vee P)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{$x : \sigma \cup \tau$}
\AxiomC{}
\RightLabel{^{2}}
\UnaryInfC{$y : \sigma$}
\UnaryInfC{$K_2(y) : \tau \cup \sigma$}
\AxiomC{}
\RightLabel{^{2}}
\UnaryInfC{$z : \tau$}
\UnaryInfC{$K_1(z) : \tau \cup \sigma$}
\RightLabel{^{2}}
\TrinaryInfC{$\oplus(K_2(y), K_1(z), x) : \tau \cup \sigma$}
\RightLabel{^{1}}
\UnaryInfC{$\lambda x \cdot \oplus(K_2(y), K_1(z), x) : (\sigma \cup \tau) \to (\tau \cup \sigma)$}
\end{prooftree}
$$