EPITA

Mathematics

Midterm exam S3

November 2022

Duration: 3 hours

Name:
First name:
Class:
MARK:
The marking system is for a mark from 0 to 40. It will be divided by 2, to get a mark from 0 to 20.
Instructions:
 Documents and pocket calculators are not allowed. Write your answers on the stapled sheets provided for answering. No other sheet will be corrected. Please, do not use lead pencils for answering.

Exercise 1 (6 points)

1.	Find the nature of the series whose general term is: $u_n = \frac{\sin\left(\frac{1}{n}\right)}{n^2}$. Justify carefully.
2.	Find the nature of the series whose general term is: $u_n = \frac{n^2 e^{-\sqrt{n}}}{2^{2n}}$. Justify carefully.
3.	Find the nature of the series whose general term is: $u_n = (-1)^n \frac{n}{e^n}$. Justify carefully.

Exercise 2 (6 points)

Let $a \in \mathbb{R}$ such that a > 0 and consider the sequence (u_n) defined for all $n \ge 2$ by: $u_n = \frac{(-1)^n}{\sqrt{n^a + (-1)^n}}$.

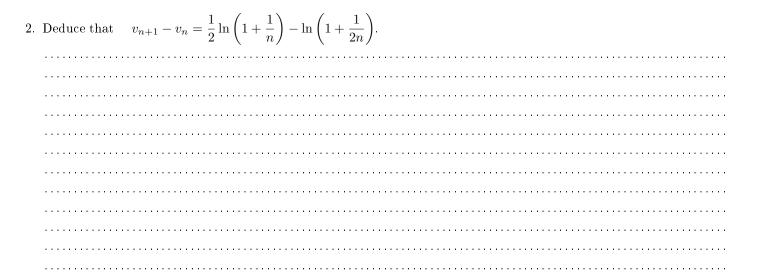
The purpose of the exercise is to study the nature of $\sum u_n$.

1.	Find $c \in \mathbb{R}$ such that $u_n = \frac{(-1)^n}{n^{a/2}} + \frac{c}{n^{3a/2}} + o\left(\frac{1}{n^{3a/2}}\right)$.
2.	Using the previous question, discuss the nature of $\sum u_n$ depending on the value of a .

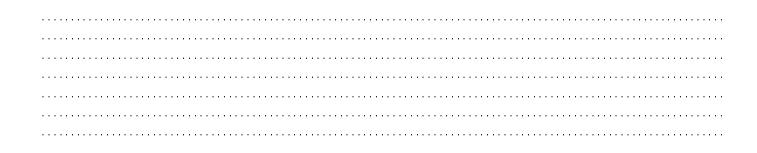
Exercise 3 (8 points)

The purpose of this exercise is to study the sequence (u_n) defined for all $n \in \mathbb{N}^*$ by: $u_n = \frac{2^n n!}{1 \times 3 \times 5 \times \cdots \times (2n-1)}$. In that purpose, consider the auxiliary sequence (v_n) defined by: $v_n = \ln(u_n) - \frac{1}{2}\ln(n)$.

Let $n \in$	№. Comp	oute $\frac{u_{n+1}}{u_n}$.			



3.	3. Find $a \in \mathbb{R}$ such that v_{n+1}	$1 - v_n \sim \frac{a}{n^2}.$	
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4. Show that (v_n) converges. Let ℓ denote its limit.

5	5. Using the previous question, show that there exists $k \in \mathbb{R}$ such that $u_n \sim k \sqrt{n}$. Express k as a function of ℓ .
Ex	ercise 4: Leibniz's rule (5 points)
Let ((u_n) be a numerical sequence with an alternating sign.
1	. Write the statement of the Leibniz's theorem about $\sum u_n$.
	N.B.: prove only the convergence of the series $\sum u_n$. It is not required to prove the upper bound of the remainders.

ercise 5: probabil	ities (5 points)
$egin{array}{ll} \mathbf{ercise} & \mathbf{5:} & \mathbf{probabil} \ X_1, X_2 & \mathbf{and} & X_3 & \mathbf{be} & \mathbf{three} & \mathbf{in} \end{array}$	
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X_1, X_2 and X_3 be three in \mathbb{R}^2 . What are the generating \mathbb{R}^2 . Consider the random value.	dependent random variables, taking their values in $\{1,3\}$, such that for all $i \in [1,3]$, $P(X_i=1)=\frac{1}{3}$ and $P(X_i=3)=\frac{2}{3}$ functions G_{X_i} of these variables?

Ex	ercise 6: power series (10 points)
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$$xf'(x) = \sum_{n=0}^{+\infty} b_n x^n$$
 and $f''(x) = \sum_{n=0}^{+\infty} c_n x^n$

3. By injecting these expressions of xf'(x) and f''(x) into the equation f''(x) + xf'(x) + f(x) = 0, write this equation as:

$$\forall x \in]-R, R[, \quad \sum_{n=0}^{+\infty} d_n x^n = 0$$

	$\forall x \in]-R, R[, \sum_{n=0}^{+\infty} d_n x^n = 0$
	where the coefficients (d_n) depend on the sequence (a_n) .
4.	Keep in mind that the condition $\sum_{n=0}^{+\infty} d_n x^n = 0$ implies that all the d_n equal zero. Using this property, show that
	$a_2=-rac{1}{2}, \qquad a_3=0 \qquad \text{and, as a general rule:} \qquad \forall \in \mathbb{N}, a_{n+2}=-rac{a_n}{n+2}$
5.	What is the value of a_n when n is odd?
6.	Find the value of a_n when n is even.
	<u>Hint:</u> you can set $n = 2k$ ($k \in \mathbb{N}$). To start with, express a_{2k} as a function of $a_{2(k-1)}$, then as a function of $a_{2(k-2)}$, and so on, until you get an expression of a_{2k} as a function of a_0 .

8.	(Bonus) Check that the final expression that you got at previous question is a solution of (C) on the whole set \mathbb{R} .
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