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Remedial Assignment 2

17 Irrotational vector

$$\vec{F} = (y^2 \cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$$

The condition for scalar potential is

$$\vec{F} = \vec{\nabla}\phi$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 \cos x + z^3) & (2y\sin x - 4) & (3xz^2) \end{vmatrix}$$

$$\nabla \times \vec{F} = 0$$

Hence, $\vec{F}$ is irrotational

Now

$$(y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\hat{j} + 3xz^2\hat{k}$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \quad - (1)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x \quad - (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \quad - (3)$$

Integrating (1) p.w.r to 'x'

We get $\phi = y^2 \sin x - 4 z^3 x f_1(y,z)$ — (4)

Integrating (2) p.w.r to 'y'

We get $\phi = y^2 \sin x - 4y + f_2(x,z)$ — (5)

Integrating (3) p.w.r to 'z'

We get $\phi = xz^3 + f_3(x,y)$ — (6)

Combining (4), (5) & (6), we get

$\phi = y^2 \sin x + z^3 x - 4y + c$

where c is a constant.

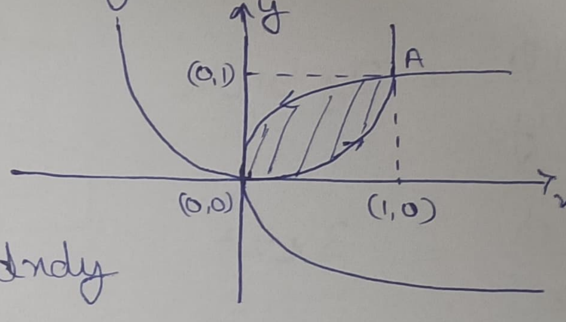
27 $\int_C [(x-y)\,dx + (x+y)\,dy]$, curves $y = x^2$ and $y^2 = x$

By green's theorem in plane,

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \oint_C (u\,dx + v\,dy)$$

Here $u = x - y$, $v = x + y$

$\frac{\partial v}{\partial x} = 1$, $\frac{\partial u}{\partial y} = -1$



$$\text{LHS} = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$= \iint_R (1-(-1))\,dx\,dy$

$= \iint_R 2\,dx\,dy$

$= 2\int_0^1 \int_{y^2}^{\sqrt{y}} dx\,dy$

$= 2\int_0^1 [x]_{y^2}^{\sqrt{y}}\,dy$

$= 2\int_0^1 (\sqrt{y}-y^2)\,dy$

$= 2\left[\frac{y^{3/2}}{3/2} - \frac{y^3}{3}\right]_0^1$

$= 2\left[\frac{2}{3}-\frac{1}{3}\right] = \frac{2}{3}$

RHS $= \oint_C (u\,dx + v\,dy)$

Along $y=x^2$, $dy = 2x\,dx$

Along $y=x^2$, then line integral equation

$\rightarrow \int_0^1 [(x-x^2)dx + (x+x^2)(2x)\,dx]$

$= \int_0^1 (2x^3+x^2+x)\,dx$

$= \int_0^1 \left[2\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{2}+\frac{1}{3}+\frac{1}{2} = \frac{4}{3}$

Along $y^2 = x$, $2y\,dy = dx$

$\therefore$ Along $y^2 = x$, line integral equals

$$= \int_1^0 [(y^2-y)\,dy(2y)\,dy + (y^2+y)\,dy]$$

$$= \int_1^0 (2y^3 - y^2 + y)\,dy$$

$$= \left[2\frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2}\right]_1^0$$

$$= 0 - \left[2\frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2}\right]_1^0$$

$$= \frac{-2}{3}$$

$\therefore$ Required line integral $\Rightarrow \frac{4}{3} + \left(\frac{-2}{3}\right) = \frac{2}{3}$

LHS = RHS, Hence Greene's theorum verified

3) $f(t) = \begin{cases} 1, & 0 < t < a/2 \\ -1, & \frac{a}{2} < t < a \end{cases}$

where $f(t+a) = f(t)$

$\Rightarrow f(t)$ is of period $T = a$

$$= L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

$$= \frac{\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt}{1 - s^{-sa}}$$

$$= \frac{\int_0^{a/2} e^{-st} \cdot 1\, dt + \int_{a/2}^{a} e^{-st} \underbrace{f(t)}_{(-1)}\, dt}{1-e^{-sa}}$$

$$= \frac{1}{[1-e^{-sa}]} \left\{ \left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a} \right\}$$

$$= \frac{1}{1-e^{-sa}} \left[\frac{-e^{-sa/2}}{s} + \frac{e^{0}}{s} + \frac{e^{-sa}}{s} - \frac{e^{-sa/2}}{s} \right]$$

$$= \frac{1}{s(1-e^{-sa})} \left[\frac{-e^{-sa/2}}{\cancel{s}} + \frac{e^{0}}{\cancel{s}} + \frac{e^{-sa}}{\cancel{s}} \right] - \frac{e^{-sa/2}}{\cancel{s}} \Bigg]$$

$$= \frac{1}{s(1-e^{-sa})} \left[1 - 2e^{-sa/2} + e^{-sa} \right]$$

$$= \frac{[1-e^{-sa/2}]^2}{s[1-e^{-sa/2}][1+e^{-sa/2}]} = \frac{1-e^{-sa/2}}{s(1+e^{-sa/2})}$$

$$= \frac{1}{s} \tanh \frac{sa}{4}$$

47 $L\left[t \sin 3t \cos 2t\right]$

Let $f(t) = \sin 3t \cos 2t$

$\therefore F(s) = L\left[f(t)\right] = L(\sin 3t \cos 2t)$

$$= L\left[\frac{\sin(3t+2t) + \sin(3t-2t)}{2}\right]$$

$$= L\left[\frac{\sin 5t + \sin t}{2}\right]$$

$$= \frac{1}{2}\left[L(\sin 5t) + L(\sin t)\right]$$

$$= \frac{1}{2}\left[\frac{5}{12+25} + \frac{1}{s^2+1}\right]$$

Now $t \sin 3t \cos 2t = t f(t)$ is the multiplication of $f(t)$ by t,

$$\therefore L\left[tf(t)\right] = \frac{-\partial F(s)}{\partial s} = \frac{-\partial}{\partial s}\left[\frac{1}{2}\left(\frac{s}{s^2+25} + \frac{1}{s^2+1}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{\partial}{\partial s}\left(\frac{5}{s^2+25}\right) + \frac{\partial}{\partial s}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{-10s}{(s^2+25)^2} - \frac{2s}{(s^2+1)^2}\right]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

5) $y' - 3y = e^{2t}$, $y(0) = 1$

take laplace on both sides

$$L[y' - 3y] = L[e^{2t}]$$

$$L[y'] - 3L[y] = \frac{1}{s-2}$$

$$[s\bar{y}(s) - y(0)] - 3\bar{y}(s) = \frac{1}{s-2}$$

$\because (y(0) = 1)$

$$= s\bar{y}(s) - 1 - 3\bar{y}(s) = \frac{1}{s-2}$$

$$= \bar{y}(s)(-3 + s) = \frac{1}{s-2} + 1$$

$$\bar{y}(s) = \frac{(s-1)}{(s-2)(s-3)}$$

$$= \frac{(s-1)}{(s-2)(s-3)} = \frac{A}{(s-2)} + \frac{B}{(s-3)}$$

$$A(s-3) + B(s-2) = s-1$$

if $s = 3$, $B = 2$

$s = 2$, $A = -1$

$$\Rightarrow \bar{y}(s) = \frac{-1}{s-2} + \frac{2}{s-3}$$

taking inverse laplace transform

$$= -1L^{-1}\left[\frac{1}{s-2}\right] + 2L^{-1}\left[\frac{1}{s-3}\right]$$

$$y(t) = -1e^{2t} + 2e^{3t}$$

$$y(t) = 2e^{3t} - e^{2t}$$