Recursion Tree

Lecture by

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Recursion tree

- The outcome of this session will be the
  - Understand the concept of “Recursion tree”
  - Apply recursion tree method to solve recurrence relations.

Motivation of the topic

The motivation of the topic is to:

- Learn how to solve recurrence relations.

Recursive tree

- Recursion tree method is one of the methods to solve recurrences.

- A recursion tree is a tree where each node represents the cost of a certain recursive sub-problem.

- sum up the numbers in each node to get the cost of the entire algorithm.

Steps to Solve Recurrence Relations Using Recursion Tree Method

Step- 1:

- Draw a recursion tree based on the given recurrence relation.

Step- 2:

Determine-

- Cost of each level
- Total number of levels in the recursion tree
- Number of nodes in the last level
- Cost of the last level

Step- 3:

- Add cost of all the levels of the recursion tree and simplify the expression so obtained in terms of asymptotic notation.

Example

In general, we consider the second term in recurrence as root.

Consider the recurrence relation

$$T(n) = 3T(n/4) + cn^2 \quad \text{for some constant } c.$$

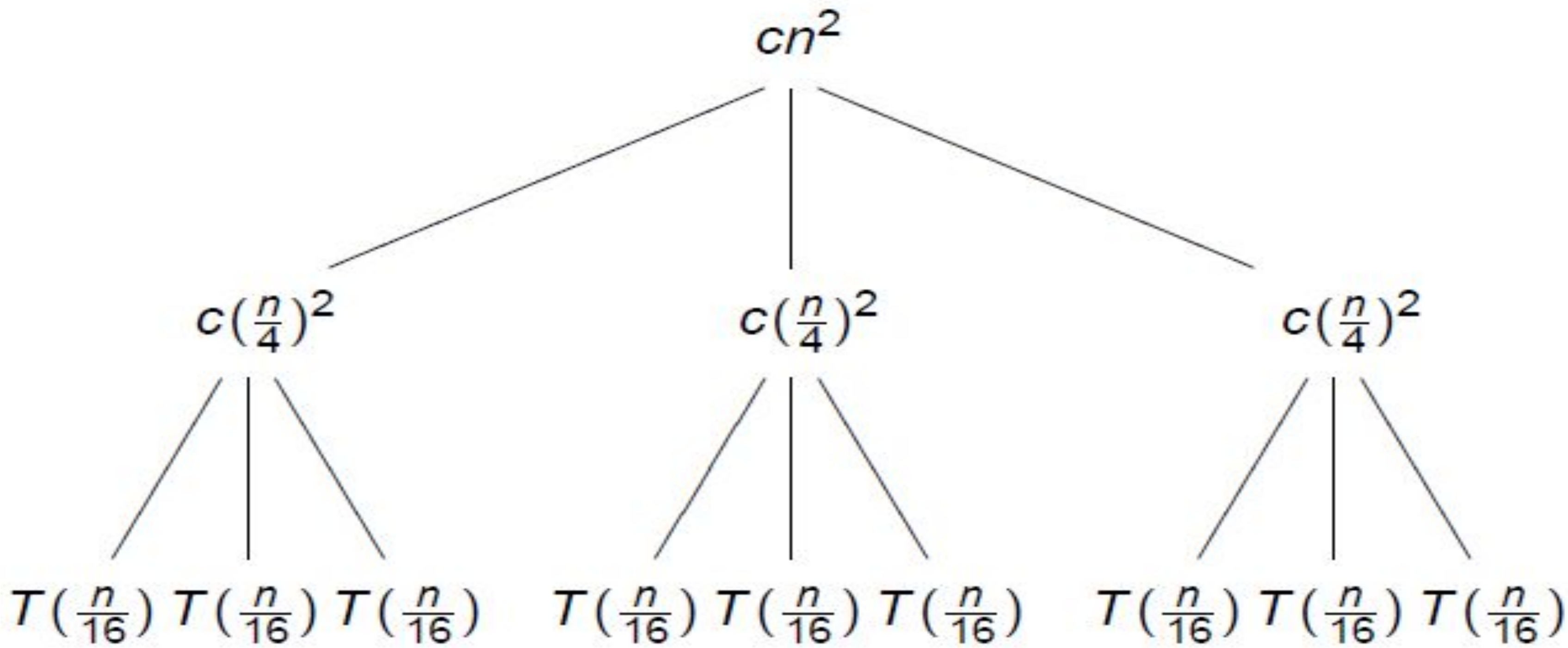
We assume that n is an exact power of 4.

In the recursion-tree method we expand $T(n)$ into a tree:

$$\begin{tikzcd}
T(n) \arrow[r] & & cn^2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
& T(\frac{n}{4}) & T(\frac{n}{4}) & T(\frac{n}{4})
\end{tikzcd}$$

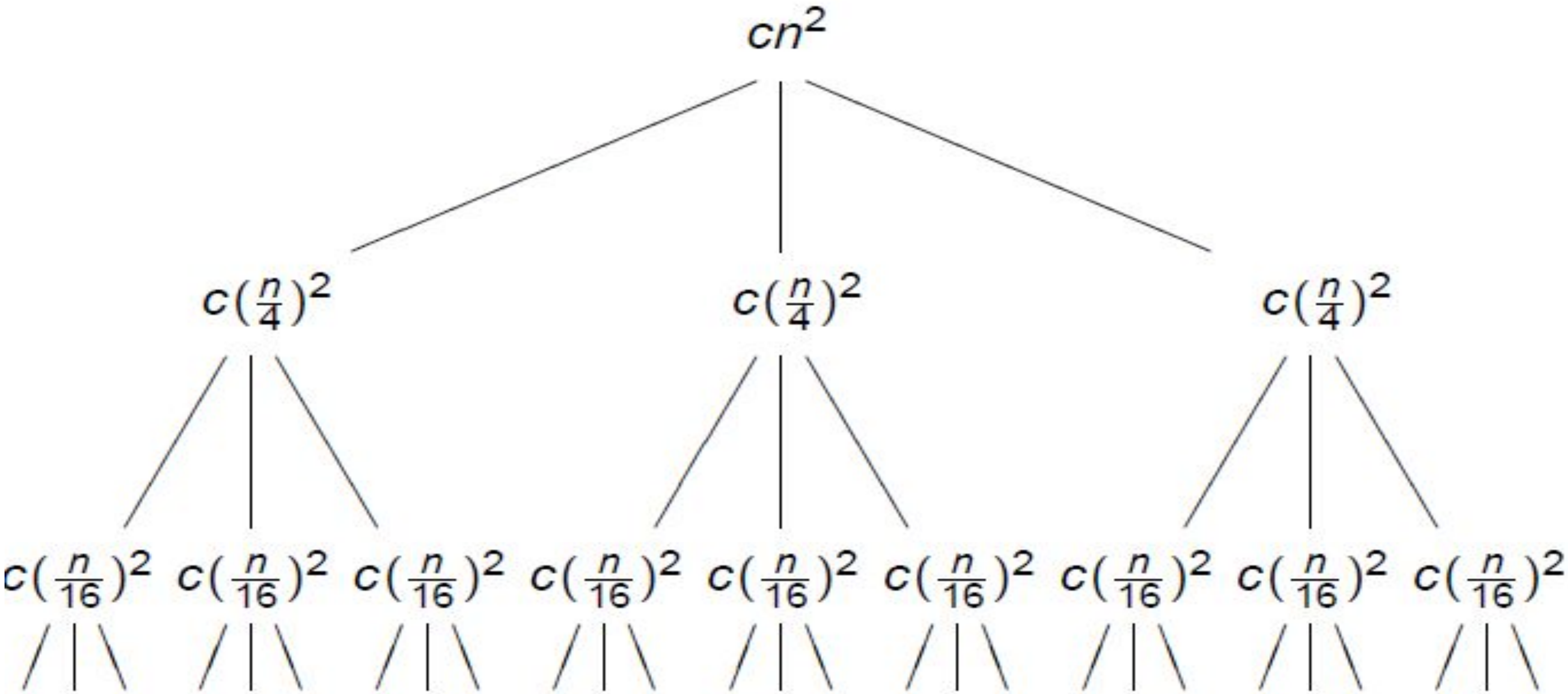
Expand T(n/4):

Applying $T(n) = 3T(n/4) + cn^2$ to $T(n/4)$ leads to
$T(n/4) = 3T(n/16) + c(n/4)^2$, expanding the leaves:



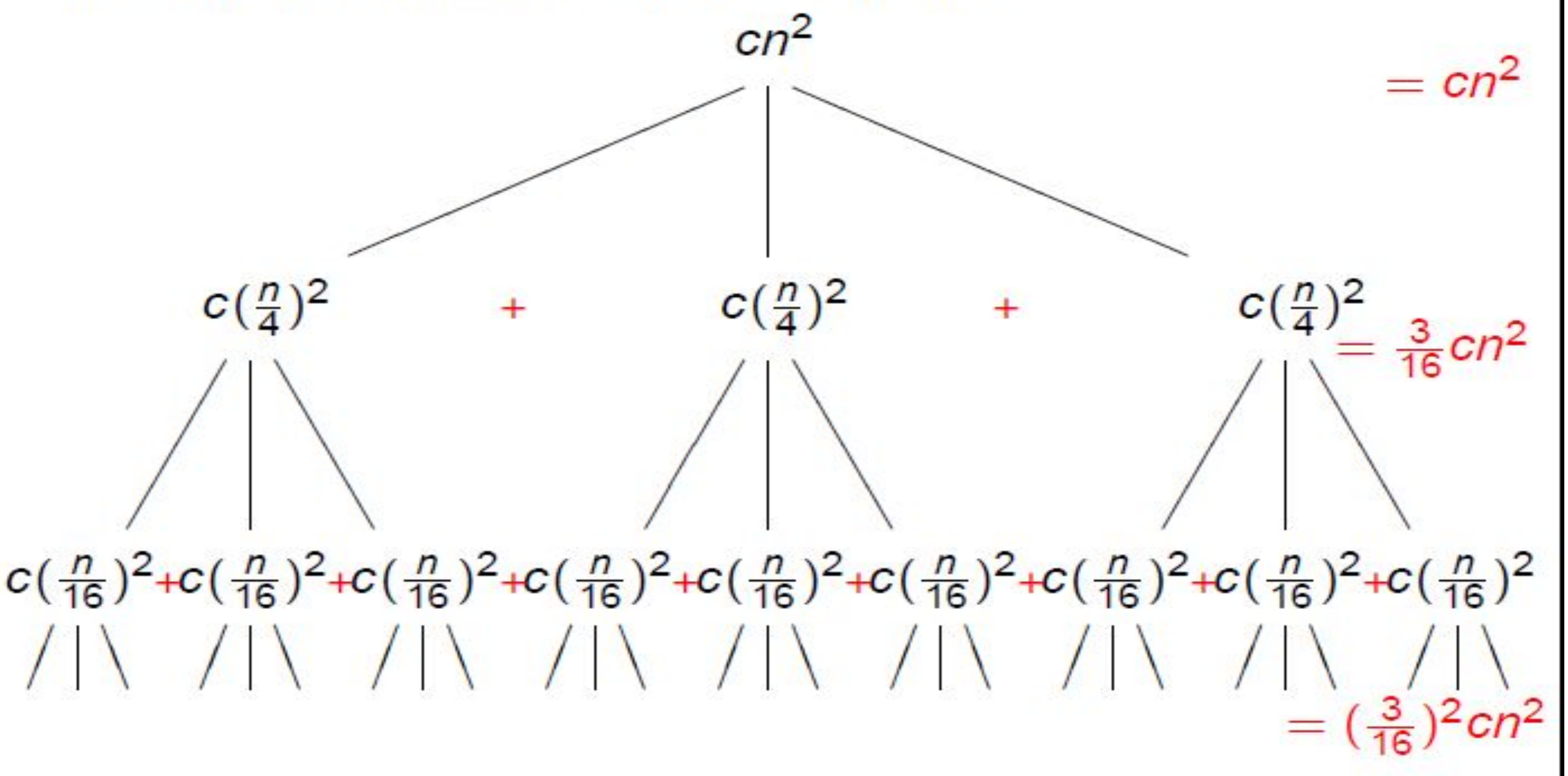
Expand T(n/16)

Applying $T(n) = 3T(n/4) + cn^2$ to $T(n/16)$ leads to
$T(n/16) = 3T(n/64) + c(n/16)^2$, expanding the leaves:



Summing the cost at each level

We sum the cost at each level of the tree:



Adding up the costs

$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots \\
&= cn^2 \left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \cdots\right)
\end{aligned}
$$

The $\cdots$ disappear if $n = 16$,
or the tree has depth at least 2 if $n \geq 16 = 4^2$.

For $n = 4^k$, $k = \log_4(n)$, we have:

$$
T(n) = cn^2 \sum_{i=0}^{\log_4(n)} \left(\frac{3}{16}\right)^i .
$$

Applying the geometric sum

Applying

$$S_n = \sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

to

$$T(n) = cn^2 \sum_{i=0}^{\log_4(n)} \left(\frac{3}{16}\right)^i$$

with $r = \frac{3}{16}$ leads to

$$T(n) = cn^2 \frac{\left(\frac{3}{16}\right)^{\log_4(n)+1} - 1}{\frac{3}{16} - 1}.$$

Polishing the result we get

Instead of $T(n) \leq dn^2$ for some constant d, we have

$$T(n) = cn^2 \frac{\left(\frac{3}{16}\right)^{\log_4(n)+1} - 1}{\frac{3}{16} - 1}.$$

Recall

$$T(n) = cn^2 \sum_{i=0}^{\log_4(n)} \left(\frac{3}{16}\right)^i.$$

To remove the $\log_4(n)$ factor, we consider

$$\begin{aligned} T(n) &\leq cn^2 \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i \\ &= cn^2 \frac{-1}{\frac{3}{16} - 1} \leq dn^2, \text{ for some constant } d. \end{aligned}$$

Verify the guess by applying substitution method

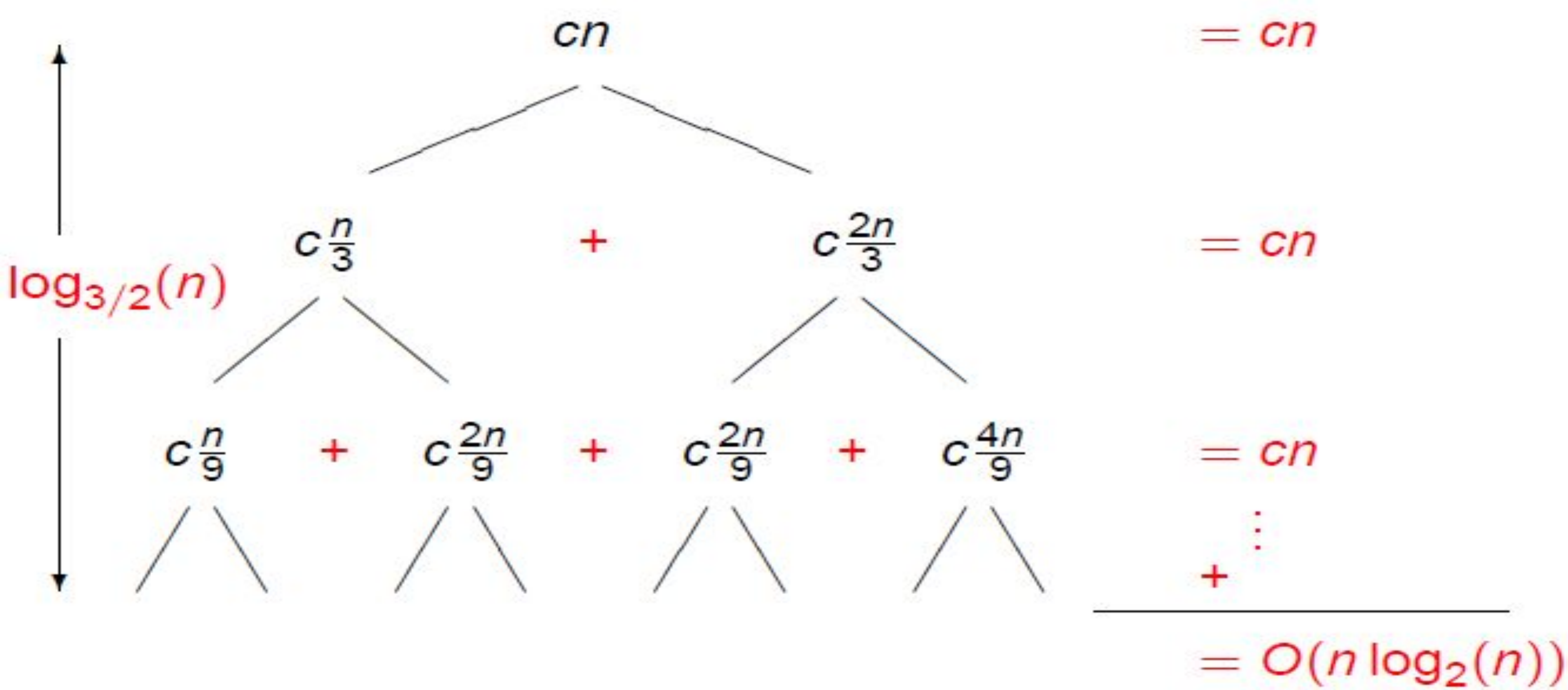
Let us see if $T(n) \leq dn^2$ is good for $T(n) = 3T(n/4) + cn^2$.

Applying the substitution method:

$$
\begin{aligned}
T(n) &= 3T(n/4) + cn^2 \\
&\leq 3d\left(\frac{n}{4}\right)^2 + cn^2 \\
&= \left(\frac{3}{16}d + c\right)n^2 \\
&= \frac{3}{16}\left(d + \frac{16}{3}c\right)n^2 \\
&\leq \frac{3}{16}(2d)n^2, \quad \text{if } d \geq \frac{16}{3}c \\
&\leq dn^2
\end{aligned}
$$

Lets see another example

Consider $T(n) = T(n/3) + T(2n/3) + cn$.



- When we add the values across the levels of the recursion trees, we get a value of n for every level.

- The longest path from the root to leaf is

$$n \longrightarrow \frac{2}{3}n \longrightarrow \left(\frac{2}{3}\right) n \longrightarrow \ldots 1$$

Since $\left(\frac{2}{3}\right)$ n=1 when i=$\log_{\frac{3}{2}}$ n.

Thus the height of the tree is $\log_{\frac{3}{2}}$ n.

$$T(n) = n + n + n + \ldots\ldots + \log_{\frac{3}{2}} n \text{ times.} = \boldsymbol{\theta(n \log n)}$$

Lets practice:

- Consider $T(n) = 3T(n/2) + n$. *Use a recursion tree to* derive a guess for an asymptotic upper bound for $T(n)$ and verify the guess with the substitution method.

- $T(n) = T(n/2) + n^2$.

- $T(n) = 2T(n - 1) + 1$.


Thank you