Planar Emulators for Monge Matrices

Hsien-Chih Chang* Tim Ophelders†

Abstract

We constructively show that any cyclic Monge distance matrix can be represented as the graph distances between vertices on the outer face of a planar graph. The structure of the planar graph depends only on the number of rows of the matrix, and the weight of each edge is a fixed linear combination of constantly many matrix entries. We also show that the size of our constructed graph is worst-case optimal among all planar graphs.

1 Introduction

Monge property, named after the 18th century mathematician Gaspard Monge, roughly say that the sum of shortest-path distances between two crossing pairs of points (x, y) and (z, w) is at least the sum of the ones between corresponding non-crossing pairs (x, z) and (y, w). The original motivation is to study the optimal transport of masses in the plane [30, 39]. As a simple consequence of the Jordan curve theorem, Monge property has been tremendously helpful in designing efficient algorithms for *planar* optimization problems—whether the input is a planar graph or geometric objects lying in the plane [12, 24, 25, 38, 42]. Most famously, Monge property is central to the design of the *SMAWK algorithm* [2] for row-minimum queries in totally monotone matrices and the *Monge heap* data structure [27] for speeding up various optimization algorithms on planar and surface graphs [27, 29, 32, 34, 40, 51]. In some problems where Monge property is evident, it is not clear whether the problem has an obvious connection to planar metrics. Examples are fast dynamic programming using quadrangle inequalities [6, 28], as well as string problems such as the edit distance and longest common subsequence [45, 49]. (See Burkard *et al.* [11, 12], Park [42], and the citations within for additional applications of the Monge properties.) A characterization of matrices satisfying the Monge property is known to exist [7, 10, 44], but the following fundamental question relating planar metric to Monge property remains unanswered: *Given a metric between a finite number of points satisfying some Monge property, is the metric planar?*

We answer this question affirmatively. We show that given any distance matrix satisfying the (cyclic) Monge property, one can construct an edge-weighted planar graph realizing entries of the matrix *exactly* as graph distances between some subset of vertices (called *terminals*). In other words, we construct a *planar emulator* for any (cyclic) Monge matrix with zero diagonals. Moreover, the construction is optimal in size and takes time linear in the size of the distance matrix. In fact, each edge in the graph along with its weight is determined by a constant number of entries in the matrix. Such property is of independent interest and might be useful in designing efficient algorithms under various computation models.

1.1 Related work

Sketching graph distances. Emulators—arbitrary graphs that preserve distances between terminals in the input graph—are known to exist in general [8, 9, 18]. But without additional assumptions on the input graph there is a linear lower-bound on the size of the emulator (with respect to the size of the input graph) when the number of terminals is a polynomial $\Theta(n^\alpha)$ for some range of α strictly less than 1 [18].[1] Chang, Gawrychowski, Mozes, and Weimann [14] constructed the first sub-linear size emulator for any undirected unweighted planar graph: given any k-terminal planar graph with n vertices, an emulator of size $\tilde{O}(\min\{k^2, (kn)^{1/2}\})$ can be constructed in $\tilde{O}(n)$ time, which is optimal up to logarithmic factors.

A related structure, called a *spanner*, which preserves the distances approximately up to additive or multiplicative errors, is relatively well-understood for general graphs [9, 31, 43, 48, 50]. Spanners with stronger guarantees exist for geometrically/topologically constrained graphs [4, 13, 23, 37]. Similarly, *distance oracles* that answer distance queries exactly or approximately are known to exist for planar and surface graphs [1, 5, 15, 27, 35, 36, 41, 46, 47]. (See Ahmed *et al.* [3] for a recent survey on distance sketching.)

Circular planar graphs. One of the central problems in the theory of circular planar graphs considers the following problem: Given measures of effective resistances between all pairs of terminals, can we reconstruct a planar resistor network realizing the measures where the terminals lie on the boundary? Colin de Verdière *et al.* [16, 17] and Curtis *et al.* [20, 21] showed that the reconstruction problem can be solved precisely when the effective resistance matrix is *totally non-negative*. The problem sounds similar to ours

*Duke University, USA.
†Michigan State University, USA.

[1] Interestingly, when the number of terminals is barely sublinear (say $n/2^{\Theta(\log^* n)}$) in an undirected unweighted graph, there is a strictly sublinear-size emulator [8].

in spirit; in fact, when looking closer, the planar emulator problem is equivalent to their reconstruction problem in the $(\min,+)$-semiring instead of the standard $(+,\times)$-ring. The techniques involved in proving their theorem rely crucially on the fact that the weights are over a $(+,\times)$-ring and therefore do not apply to our problem.

1.2 Preliminaries

Monge properties. A matrix M satisfies the *Monge property* if for any two rows $i < i'$ and two columns $j < j'$, one has

$$M[i,j]+M[i',j'] \le M[i',j]+M[i,j'].$$

Matrix M satisfies the *anti-Monge property* if the sign of the above inequality flipped. We often reorder the terms in the inequality to emphasize the monotonicity on the entry differences:

$$M[i',j']-M[i,j'] \le M[i',j]-M[i,j].$$

For the purpose of this paper we only consider *distance matrices*, where the diagonal entries are all zeros, the entries are symmetric and satisfy the triangle inequality. A distance matrix M is *cyclic Monge*[2] if for any four indices i, i', j, j' in cyclic order (that is, $i \le i' \le j \le j'$ after some cyclic reordering of $[i,i',j,j']$), one has

$$M[i,j']+M[i',j] \le M[i,j]+M[i',j'].$$

(Notice the inequality sign flipped comparing to the standard Monge property.) Let M be a cyclic Monge distance matrix and let A and B be two disjoint sub-intervals of the index set of M. Then the submatrix of M between A and B must be an (anti-)Monge matrix.

Planar emulators. Consider an undirected planar graph G with edge weights and let ∂G be the vertices on the boundary of the outer face of G. We consider the distance matrix M between vertices in ∂G: for any pair of vertices i and j in ∂G, we set $M[i,j]$ to be the distance between i and j in G.

It is not immediately clear that any cyclic Monge distance matrix M comes as a distance matrix generated from some planar graph G. A *planar emulator* for a distance matrix M is a graph G whose vertex set $V(G)$ contains the indices of M (and possibly others), and the graph distance $d_G(u,v)$ between any pair of vertices u and v in G is equal to $M[u,v]$. Planarity and the Jordan curve theorem ensures that any distance matrix M of a planar emulator must satisfy the cyclic Monge property. Our main result shows that the converse is also true: *any cyclic Monge distance matrix admits a planar emulator*.

In Section 2 we describe the construction and prove its correctness. We show that the size of the construction is optimal in Section 3, and conclude the paper in Section 4.

[2] This is known as the *Kalmanson matrix* [22,33], which is slightly more restricted than a *triangular Monge matrix* [12] or the *convex quadrangle inequality* [26].

2 Constructing a planar emulator

The goal of this section is to construct planar emulators for arbitrary cyclic Monge distance matrices.

Theorem 1 *Given any $n \times n$ cyclic Monge distance matrix M, there is a planar emulator for M with $\binom{n}{2}$ edges.*

For any given positive integer n, we define a planar graph G^n as follows (see Figure 1). Let the vertices of G^n be the set $\{v_{i,j}\}$, where i ranges in $[1:n]$ and j ranges in $[1:\min\{i,n-i+1\}]$. Define *terminal p_i* to be $v_{i,\min\{i,n-i+1\}}$. The edges of G^n consist of *horizontal* edges and *vertical* edges. A *horizontal edge $e^{\leftrightarrow}_{i,j}$* lies between each $v_{i,j}$ and $v_{i+1,j}$ where j ranges in $[1:\lfloor n/2\rfloor]$ and i ranges in $[j:n-j]$. A *vertical edge $e^{\updownarrow}_{i,j}$* lies between each $v_{i,j}$ and $v_{i,j+1}$ where j ranges in $[1:\min\{i,n+1-i\}-1]$ and i ranges in $[2:n-1]$.

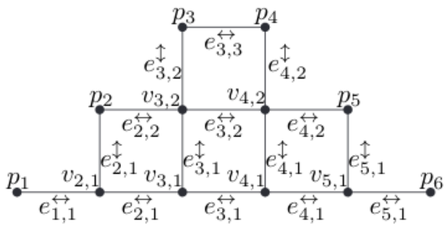


Figure 1: Graph G^6.

Consider a cyclic Monge distance matrix M and for brevity denote $M_{i,j} := M[i,j]$. We define the graph G^n_M as an edge-weighted copy of G^n, where the weight of a horizontal edge $e^{\leftrightarrow}_{i,j}$ is

$$\omega(e^{\leftrightarrow}_{i,j}) := \frac{1}{2}\left(M_{i+1,j}-M_{i,j}+M_{i,n-j+1}-M_{i+1,n-j+1}\right),$$

and the weight of a vertical edge $e^{\updownarrow}_{i,j}$ is

$$\omega(e^{\updownarrow}_{i,j}) := \frac{1}{2}\big(M_{i,j}-M_{i,j+1}+M_{i,n-j+1}-M_{i,n-j}+ M_{j+1,n-j}-M_{j,n-j+1}\big).$$

(See Figure 2.) Henceforth, we will refer to the edge-weighted graph G^n_M as the *canonical realization* of M.

For the rest of the section, we show that $G := G^n_M$ is a planar emulator of M. For this, it suffices to show that $d_G(p_i,p_j) = M[i,j]$ for all pairs of terminals p_i and p_j. First, we derive some properties of G using the fact that M is a cyclic Monge matrix.

Lemma 2 *If M is a cyclic Monge matrix, then all edge weights of G^n_M are non-negative.*

Proof. An edge of G^n_M is either horizontal or vertical. For any horizontal edge $e^{\leftrightarrow}_{i,j}$, the cyclic Monge property states that $M_{i,j}+M_{i+1,n-j+1} \le M_{i+1,j}+M_{i,n-j+1}$, and therefore $2\omega(e^{\leftrightarrow}_{i,j}) = M_{i+1,j}-M_{i,j}+M_{i,n-j+1}-M_{i+1,n-j+1} \ge 0$.

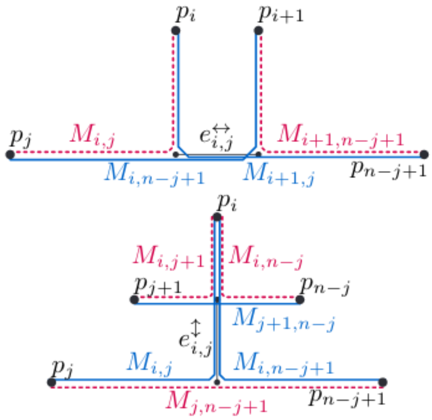


Figure 2: Values used to assign weights to $e^{\leftrightarrow}_{i,j}$ and $e^{\updownarrow}_{i,j}$.

For any vertical edge $e^{\updownarrow}_{i,j}$, the cyclic Monge property states that (1) $M_{i,j+1} + M_{j,n-j} \leq M_{i,j} + M_{j+1,n-j}$ and (2) $M_{i,n-j} + M_{j,n-j+1} \leq M_{j,n-j} + M_{i,n-j+1}$. Combining (1) and (2) gives $2\omega(e^{\updownarrow}_{i,j}) = M_{i,j} - M_{i,j+1} + M_{i,n-j+1} - M_{i,n-j} + M_{j+1,n-j} - M_{j,n-j+1} \geq 0$. □

It follows that the minimum-weight path from p_i to p_j in G is simple.

Next, we show that there is at least one path from p_i to p_j achieving the cost $M[i,j]$. For $i \leq i'$, the path of horizontal edges between $v_{i,j}$ and $v_{i',j}$ in G has weight

$$\begin{aligned}\sum_{x\in[i:i'-1]} \omega(e^{\leftrightarrow}_{x,j}) &= \frac{1}{2} \sum_{x\in[i:i'-1]} \big(M_{x+1,j} - M_{x,j} + M_{x,n-j+1} \\ &\qquad - M_{x+1,n-j+1}\big) \\ &= \frac{1}{2}\big(M_{i',j} - M_{i,j} + M_{i,n-j+1} - M_{i',n-j+1}\big),\end{aligned}$$

and for $j \leq j'$, the path of vertical edges between $v_{i,j}$ and $v_{i,j'}$ has weight

$$\begin{aligned}\sum_{y\in[j:j'-1]} \omega(e^{\updownarrow}_{i,y}) &= \frac{1}{2} \sum_{y\in[j:j'-1]} \big(M_{i,y} - M_{i,y+1} + M_{i,n-y+1} \\ &\qquad - M_{i,n-y} + M_{y+1,n-y} - M_{y,n-y+1}\big) \\ &= \frac{1}{2}\big(M_{i,j} - M_{i,j'} + M_{i,n-j+1} - M_{i,n-j'+1} \\ &\qquad + M_{j',n-j'+1} - M_{j,n-j+1}\big).\end{aligned}$$

Consider two terminals p_i and p_j and assume that $\min\{i, n-i+1\} \geq \min\{j, n-j+1\}$. Let $\pi_{j,i}$ be the unique L-shaped (simple) path from p_j to p_i that consists of a path $\pi^{\leftrightarrow}_{j,i}$ of horizontal edges followed by a path $\pi^{\updownarrow}_{j,i}$ of vertical edges (both paths might possibly be empty). When $\min\{i, n-i+1\} > \min\{j, n-j+1\}$ we define $\pi_{j,i} := \pi_{i,j}$.

Lemma 3 *Let M be a cyclic Monge distance matrix. The weight of $\pi_{j,i}$ in G^n_M is $M_{i,j}$.*

Proof. We assume that $j \leq \lceil n/2 \rceil$ (the other case is symmetric). The vertex at the end of $\pi^{\leftrightarrow}_{j,i}$ (and at the start of $\pi^{\updownarrow}_{j,i}$) is $v_{i,j}$. Let $i' := \min\{i, n-i+1\}$, then the weight of $\pi_{j,i}$ is

$$\begin{aligned}\omega(\pi_{j,i}) &= \sum_{x\in[j:i-1]} \omega(e^{\leftrightarrow}_{x,j}) + \sum_{y\in[j:i'-1]} \omega(e^{\updownarrow}_{i,y}) \\ &= \frac{1}{2}\Big((M_{i,j} - M_{j,j} + M_{j,n-j+1} - M_{i,n-j+1}) + \\ &\qquad (M_{i,j} - M_{i,i'} + M_{i,n-j+1} - M_{i,n-i'+1} + \\ &\qquad M_{i',n-i'+1} - M_{j,n-j+1})\Big) \\ &= \frac{1}{2}(M_{i,j} + M_{i,j} - M_{i,i'} - M_{i,n-i'+1} + M_{i',n-i'+1}),\end{aligned}$$

where either $M_{i,i'} = 0$ and $M_{i,n-i'+1} = M_{i',n-i'+1}$, or $M_{i,n-i'+1} = 0$ and $M_{i,i'} = M_{i',n-i'+1}$; so $\omega(\pi_{j,i}) = M_{i,j}$. □

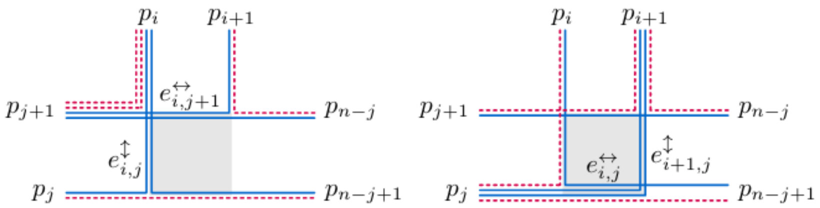
By Lemma 3 we have $d_G(p_i, p_j) \leq M_{i,j}$, so it remains to show that $d_G(p_i, p_j) \geq M_{i,j}$. Define the y-coordinate of a horizontal edge $e^{\leftrightarrow}_{i,j}$ as j, and the x-coordinate of a vertical edge $e^{\updownarrow}_{i,j}$ as i. We next show that G contains a minimum-weight path from p_i to p_j whose horizontal edges all have the same y-coordinate. It follows that there is a minimum-weight path consisting of at most one subpath of horizontal edges.

Lemma 4 *Let M be a cyclic Monge distance matrix. For any pair of terminals p and p', G^n_M has a minimum-weight path from p to p' whose horizontal edges all have the same y-coordinate.*

Proof. For a path π, let $\sigma(\pi)$ be the sum of y-coordinates of its horizontal edges. Let α be a minimum-weight path from p to p' that minimizes $\sigma(\alpha)$ (over all minimum-weight paths from p to p'). We claim that all horizontal edges of α have the same y-coordinate. Suppose not, then α contains a two-edge subpath consisting of a vertical edge $e^{\updownarrow}_{i,j}$ and a horizontal edge $e^{\leftrightarrow}_{i,j+1}$ or $e^{\leftrightarrow}_{i-1,j+1}$. We consider only the case where the subpath has edges $e^{\updownarrow}_{i,j}$ and $e^{\leftrightarrow}_{i,j+1}$ (the other case is symmetric). Consider the path β obtained from α by replacing this subpath by $e^{\leftrightarrow}_{i,j}$ and $e^{\updownarrow}_{i+1,j}$. Then $\sigma(\beta) < \sigma(\alpha)$, so by assumption β cannot be a minimum-weight path. However, Figure 3 shows that the weight of β is at most that of α, contradicting that α is a minimum-weight path that minimizes σ. □

Finally, we show that there is a minimum-weight path for which additionally, its vertical edges all have the same x-coordinate. Together with the fact that all edge weights are non-negative (Lemma 2), it follows that $\pi_{j,i}$ is a minimum-weight path between p_j and p_i.

Lemma 5 *Let M be a cyclic Monge distance matrix. For any pair of terminals p and p', G^n_M has a minimum-weight path from p to p' whose horizontal edges all have the same y-coordinate, and whose vertical edges all have the same x-coordinate.*



$$2\omega(e^{\updownarrow}_{i,j}) + 2\omega(e^{\leftrightarrow}_{i,j+1}) \geq 2\omega(e^{\leftrightarrow}_{i,j}) + 2\omega(e^{\updownarrow}_{i+1,j})$$

Figure 3: The sum of weights of $e^{\leftrightarrow}_{i,j}$ and $e^{\updownarrow}_{i+1,j}$ is at most that of $e^{\updownarrow}_{i,j}$ and $e^{\leftrightarrow}_{i,j+1}$.

$$2\omega(e^{\updownarrow}_{i,j}) + 2\omega(e^{\updownarrow}_{i',j}) + 2\sum_{x=i}^{i'-1} \omega(e^{\leftrightarrow}_{x,j}) \geq 2\sum_{x=i}^{i'-1} \omega(e^{\leftrightarrow}_{x,j+1})$$

Figure 4: The weight of the horizontal path from $v_{i,j+1}$ to $v_{i',j+1}$ is at most the total weight of $e^{\updownarrow}_{i,j}$, $e^{\updownarrow}_{i',j}$, and the horizontal path from $v_{i,j}$ to $v_{i',j}$.

Proof. By Lemma 4, there is a minimum-weight path from p to p' whose horizontal edges all have the same y-coordinate, and without loss of generality assume that this y-coordinate is maximal over all such paths. Because all edges have nonnegative weights by Lemma 2, we may assume that this path consists of a path of vertical edges (with decreasing y-coordinates), followed by a path of horizontal edges whose x-coordinates are increasing or decreasing, and finally a path of vertical edges with increasing y-coordinates. Suppose that the subpath of horizontal edges is surrounded by vertical edges $e^{\updownarrow}_{i,j}$ and $e^{\updownarrow}_{i',j}$ with $i < i'$ (the case $i > i'$ is symmetric). Let α be the path consisting of $e^{\updownarrow}_{i,j}$, the edges $e^{\leftrightarrow}_{x,j}$ for $i \leq x < i'$, and $e^{\updownarrow}_{i',j}$; let β be the path of edges $e^{\leftrightarrow}_{x,j+1}$ for $i \leq x < i'$. Apply cyclic Monge property twice, one can show that $2M_{i',j}+2M_{j+1,n-j}-M_{i',j+1}+2M_{i,n-j+1}-2M_{j,n-j+1}-M_{i,n-j} \geq M_{i',j+1} + M_{i,n-j}$, which implies that the weight of β is at most that of α, so replacing α by β yields a shortest path whose horizontal edges all have the same y-coordinate, but one bigger than that of the horizontal edges of α, which is a contradiction. (See Figure 4.) ☐

As an immediate corollary of Lemmas 2, 3, and 5, every $n \times n$ cyclic Monge distance matrix has a planar emulator of size $\binom{n}{2}$, proving Theorem 1.

3 Lower bound on the size of planar emulators

In this section we show that some Monge distance matrices requires $\binom{n}{2}$ edges in any of its planar emulator. A similar result by Cossarini [19] says that any planar emulator of some *cyclic* Monge matrix requires $\binom{n}{2}$ edges. Therefore, our canonical realization is worst-case optimal in size.

Theorem 6 *Some $n \times n$ Monge distance matrices have no planar emulator with fewer than $\binom{n}{2}$ edges.*

Proof. Let M be a Monge distance matrix. The vector $(M_{i,j})_{i<j} \in \mathbb{R}^{\binom{n}{2}}$ completely determines M since $M_{i,i} = 0$ and $M_{i,j} = M_{j,i}$ as d is a graph metric on the canonical realization of M. The set of such vectors over all Monge distance matrices yields a convex polytope $\mathscr{P}$, as it is bounded only by the hyperplanes arising from the linear inequalities of the triangle inequality and cyclic Monge property. We show that $\mathscr{P}$ is $\binom{n}{2}$-dimensional.

For this, we define a family of $\binom{n}{2}$ sets $(E_e)_{e\in E(G)}$ of edges indexed by the edges of G^n_M. For each horizontal edge $e^{\leftrightarrow}_{i,j}$, let $E_{e^{\leftrightarrow}_{i,j}} := \{e^{\leftrightarrow}_{i,j'} \mid j' \leq j\}$. For each vertical edge $e^{\updownarrow}_{i,j}$, let $E_{e^{\updownarrow}_{i,j}} := \{e^{\updownarrow}_{i,j}\} \cup E_{e^{\leftrightarrow}_{i,j}} \cup E_{e^{\leftrightarrow}_{i+1,j}}$. For each edge e, define the weight function ω_e as the characteristic function of E_e; in other words, let $\omega_e : E \to \{0,1\}$, with $\omega_e(e') = 1$ if $e' \in E_e$, and $\omega_e(e') = 0$ otherwise. We show that the $\binom{n}{2}$ weight functions $(\omega_e)_{e\in E(G)}$ are linearly independent. For each horizontal edge $e^{\leftrightarrow}_{i,1}$, $\omega_{e^{\leftrightarrow}_{i,1}}$ sets only the weight of edge $e^{\leftrightarrow}_{i,1}$ to one, and all other edges to zero. Similarly, for each horizontal edge $e^{\leftrightarrow}_{i,j}$ with $j > 1$, $e \mapsto \omega_{e^{\leftrightarrow}_{i,j}}(e) - \omega_{e^{\leftrightarrow}_{i,j-1}}(e)$ sets only the weight of edge $e^{\leftrightarrow}_{i,j}$ to one. Finally, for each vertical edge $e^{\updownarrow}_{i,j}$, $e \mapsto \omega_{e^{\updownarrow}_{i,j}}(e) - \omega_{e^{\leftrightarrow}_{i,j}}(e) - \omega_{e^{\leftrightarrow}_{i+1,j}}(e)$ sets only the weight of edge $e^{\updownarrow}_{i,j}$ to one. Since each of the $\binom{n}{2}$ edges can be set to weight one while all other edges are set to zero, the defined weight functions are linearly independent, and moreover, any weight function can be obtained as a linear combination of $(\omega_e)_{e\in E(G)}$.

Since the polytope $\mathscr{P}$ is $\binom{n}{2}$-dimensional, there exists a Monge distance matrix whose entries are in general position: there is no indexed family S of fewer than $\binom{n}{2}$ real numbers such that each of the $\binom{n}{2}$ distances can be written as the sum of a subset of S. Since the length of each shortest path in a nonnegatively edge-weighted graph is the sum of a subset of its edge-weights, there is a Monge distance matrix that does not have a planar emulator with fewer than $\binom{n}{2}$ edges. □

The argument of Theorem 6 relies on the fact that the set of distances can be chosen to lie in general position. We present a different, but slightly weaker lower bound for the more general setting where the weights are integers up to $\lceil n/2\rceil$. A Monge matrix M is *unit-Monge* if for all i and j,

$$M[i+1,j]-M[i,j]\in\{-1,0,1\},\text{ and}$$
$$M[i,j]-M[i,j+1]\in\{-1,0,1\}.$$

Theorem 7 *Some $n\times n$ unit-Monge distance matrices have no planar emulator with fewer than $n^2/8+n/2$ edges.*

Proof. Let M be a distance matrix defined as follows. Consider a rectangular grid graph with vertex set $\{0,\dots,w\}\times\{0,\dots,h\}$ and edges between vertices at distance 1, so that vertex (x,y) has (unit-weight) edges to $(x\pm 1,y)$ and $(x,y\pm 1)$. For all y and k, we have $d((0,y),(w,y\pm k)) = w+k$, and symmetrically $d((x,0),(x\pm k,h)) = h+k$ for all x and k. Let M be the distance matrix from the set of vertices $\{(x,0)\}\cup\{(0,y)\}$ to the set of vertices $\{(x,h)\}\cup\{(w,y)\}$; distance matrix M must be unit-Monge.

Consider an arbitrary planar emulator G of M. Let d_G denote the shortest-path metric on G. For vertices i,j,k,ℓ in clockwise-order along the outer face, we have $d_G(i,\ell)+d_G(j,k)\le d_G(i,k)+d_G(j,\ell)$. On the other hand, for any pair of points p and q where p is on a shortest path from i to ℓ and q on a shortest path from j to k, we have $d_G(i,\ell)+d_G(j,k)+2d_G(p,q)\ge d_G(i,k)+d_G(j,\ell)$.

Denote by $\pi_y^{\leftrightarrow}$ a shortest path in G between $(0,y)$ and (w,y), and by $\pi_x^{\updownarrow}$ a shortest path in G between $(x,0)$ and (x,h). We will show that the paths $\pi_x^{\updownarrow}$ are disjoint and have h edges each. Recall that $d_G(i,\ell)+d_G(j,k)+2d_G(p,q)\ge d_G(i,k)+d_G(j,\ell)$, so

$$\begin{aligned}
&\|\pi_y^{\leftrightarrow}\|+\|\pi_{y+k}^{\leftrightarrow}\|+2d_G(\pi_y^{\leftrightarrow},\pi_{y+k}^{\leftrightarrow})\\
&=2w+2d_G(\pi_y^{\leftrightarrow},\pi_{y+k}^{\leftrightarrow})\\
&\ge d_G((0,y),(w,y+k))+d_G((0,y+k),(w,y))\\
&=2(w+k),
\end{aligned}$$

and thus any pair of points $p\in\pi_y^{\leftrightarrow}$ and $q\in\pi_{y+k}^{\leftrightarrow}$ on distinct paths have distance at least $k\ge 1$, so different such paths are vertex-disjoint. Any path $\pi_x^{\updownarrow}$ must cross all the (vertex-disjoint) paths $\pi_0^{\leftrightarrow},\dots\pi_h^{\leftrightarrow}$, and thus have at least h edges (not shared with any path $\pi_y^{\leftrightarrow}$) of length at least 1. Therefore, the paths $\pi_x^{\updownarrow}$ and $\pi_y^{\leftrightarrow}$ (over all x and by symmetric argument y) contain at least $(w+1)h+(h+1)w$ edges. We have $n=2(w+h)$; by taking $w=h=n/4$, this yields a lower bound of

$$2(n/4+1)(n/4)=n^2/8+n/2$$

edges for any planar emulator of M. □

We remark that the argument of Theorem 7 depends only on distances between opposite sides of the grid, and can be made to depend only on the linearly many distances $d((0,y),(w,y+k))$ and $d((x,0),(x+k,h))$ with $k\in\{-1,0,1\}$.

Cossarini [19] proved that any planar emulator for some $n\times n$ *cyclic* unit-Monge matrix must have at least $\binom{n}{2}$ edges. Our result, while slightly weaker in comparison, applies to general unit-Monge matrices, which can be viewed as the *directed* version of the problem.

4 Discussion

In this paper we have shown that any cyclic Monge distance matrix admits a quadratic-size planar emulator. Our construction is universal in the sense that the underlying graph does not depend on the entries of the matrix. And there are metrics for which each edge must be used by some shortest path. We also showed that already for planar emulators of unit-Monge distance matrices (which can be represented in linear space), $\Omega(n^2)$ edges are sometimes necessary.

The cyclic-Monge distance matrices considered in this paper are closely connected to the set of intrinsic metrics of topological disks. In particular, a given metric on points in a circle can be realized as a metric intrinsic to a topological disk bounded by that circle if and only if the metric is a cyclic-Monge distance matrix. We conclude with an open problem.

- Under what conditions do surfaces other than the disk (such as the Möbius strip, or a torus with holes) realize a given metric between points on their boundary? Do such surfaces also have a universal emulator, and if so, one with at most $\binom{n}{2}$ edges?

References

[1] A. Abboud, P. Gawrychowski, S. Mozes, and O. Weimann. Near-optimal compression for the planar graph metric. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 530–549, 2018.

[2] A. Aggarwal, M. M. Klawe, S. Moran, P. Shor, and R. Wilber. Geometric applications of a matrix-searching algorithm. *Algorithmica*, 2:195–208, Nov. 1987.

[3] R. Ahmed, G. Bodwin, F. D. Sahneh, K. Hamm, M. J. L. Jebelli, S. Kobourov, and R. Spence. Graph spanners: A tutorial review. Sept. 2019.

[4] I. Althöfer, G. Das, D. Dobkin, D. Joseph, and J. Soares. On sparse spanners of weighted graphs. *Discrete & Computational Geometry*, 9(1):81–100, Jan. 1993.

[5] S. Arikati, D. Z. Chen, L. P. Chew, G. Das, M. Smid, and C. D. Zaroliagis. Planar spanners and approximate shortest path queries among obstacles in the plane. In *Algorithms — ESA '96*, pages 514–528, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

[6] W. Bein, M. J. Golin, L. L. Larmore, and Y. Zhang. The Knuth-Yao quadrangle-inequality speedup is a consequence of total monotonicity. *ACM Transactions on Algorithms*, 6(1):1–22, Dec. 2009.

[7] W. W. Bein and P. K. Pathak. A characterization of the Monge property and its connection to statistics. *Demonstratio Mathematica*, 29(2):451–457, Apr. 1996.

[8] G. Bodwin. Linear Size Distance Preservers. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 600–615. Society for Industrial and Applied Mathematics, Jan. 2017.

[9] G. Bodwin and V. V. Williams. Better distance preservers and additive spanners. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 855–872. Society for Industrial and Applied Mathematics, Jan. 2016.

[10] V. Y. Burdyuk and V. Trofimov. Generalization of results of Gilmore and Gomory on solution of traveling salesman problem. *Engineering Cybernetics*, 14(3):12–18, 1976.

[11] R. E. Burkard. Monge properties, discrete convexity and applications. *European Journal of Operational Research*, 176(1):1–14, Jan. 2007.

[12] R. E. Burkard, B. Klinz, and R. Rudolf. Perspectives of Monge properties in optimization. *Discrete Applied Mathematics*, 70(2):95–161, Sept. 1996.

[13] N. Catusse, V. Chepoi, and Y. Vaxès. Planar hop spanners for unit disk graphs. In C. Scheideler, editor, *Algorithms for Sensor Systems*, volume 6451, pages 16–30. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[14] H.-C. Chang, P. Gawrychowski, S. Mozes, and O. Weimann. Near-optimal distance emulator for planar graphs. In *26th Annual European Symposium on Algorithms (ESA 2018)*, volume 112 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:17, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[15] V. Cohen-Addad, S. Dahlgaard, and C. Wulff-Nilsen. Fast and compact exact distance oracle for planar graphs. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 962–973, Berkeley, CA, Oct. 2017.

[16] Y. Colin de Verdière, I. Gitler, and D. Vertigan. Réseaux électriques planaires II. *Commentarii Mathematici Helvetici*, 71(1):144–167, Dec. 1996.

[17] Y. D. Colin de Verdière. Réseaux électriques planaires I. *Commentarii Mathematici Helvetici*, 69:351–374, Dec. 1994.

[18] D. Coppersmith and M. Elkin. Sparse sourcewise and pairwise distance preservers. *SIAM Journal on Discrete Mathematics*, 20(2):463–501, 2006.

[19] M. Cossarini. *Discrete Surfaces with Length and Area and Minimal Fillings of the Circle*. Ph.D. Dissertation, Instituto Nacional de Matematica Pura e Aplicada, Sept. 2018.

[20] E. Curtis, E. Mooers, and J. Morrow. Finding the conductors in circular networks from boundary measurements. *ESAIM: Mathematical Modelling and Numerical Analysis*, 28(7):781–814, 1994.

[21] E. B. Curtis, D. Ingerman, and J. A. Morrow. Circular planar graphs and resistor networks. *Linear Algebra and its Applications*, 283(1):115–150, Nov. 1998.

[22] V. G. Deĭneko, J. A. Van der Veen, R. Rudolf, and G. J. Woeginger. Three easy special cases of the euclidean travelling salesman problem. *RAIRO - Operations Research*, 31(4):343–362, 1997.

[23] F. F. Dragan, F. V. Fomin, and P. A. Golovach. Spanners in sparse graphs. *Journal of Computer and System Sciences*, 77(6):1108–1119, Nov. 2011.

[24] T. Dudás and R. Rudolf. Spanning trees and shortest paths in Monge graphs. *Computing*, 60(2):109–119, June 1998.

[25] D. Eppstein. Sequence comparison with mixed convex and concave costs. *Journal of Algorithms*, 11(1):85–101, Mar. 1990.

[26] D. Eppstein, Z. Galil, R. Giancarlo, and G. F. Italiano. Sparse dynamic programming II: Convex and concave cost functions. *Journal of the ACM*, 39(3):546–567, July 1992.

[27] J. Fakcharoenphol and S. Rao. Planar graphs, negative weight edges, shortest paths, and near linear time. *Journal of Computer and System Sciences*, 72(5):868–889, Aug. 2006.

[28] Z. Galil and K. Park. Dynamic programming with convexity, concavity and sparsity. *Theoretical Computer Science*, 92(1):49–76, Jan. 1992.

[29] P. Gawrychowski, S. Mozes, and O. Weimann. Submatrix maximum queries in Monge and partial Monge matrices are equivalent to predecessor search. *ACM Transactions on Algorithms*, 16(2):16:1–16:24, Apr. 2020.

[30] A. J. Hoffman. On simple linear programming problems. In *Proceedings of Symposia in Pure Mathematics*, volume 7, pages 317–327. AMS, 1963.

[31] S.-E. Huang and S. Pettie. Lower Bounds on Sparse Spanners, Emulators, and Diameter-reducing shortcuts. In *16th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2018)*, pages 26:1–26:12, 2018.

[32] G. F. Italiano, Y. Nussbaum, P. Sankowski, and C. Wulff-Nilsen. Improved algorithms for min cut and max flow in undirected planar graphs. In *Proceedings of the 43rd Annual ACM Symposium on Theory of Computing (STOC '11)*, pages 313–322, San Jose, California, USA, 2011.

[33] K. Kalmanson. Edgeconvex Circuits and the Traveling Salesman Problem. *Canadian Journal of Mathematics*, 27(5):1000–1010, Oct. 1975.

[34] H. Kaplan, S. Mozes, Y. Nussbaum, and M. Sharir. Submatrix maximum queries in Monge matrices and partial Monge matrices, and their applications. *ACM Transactions on Algorithms*, 13(2):26:1–26:42, Mar. 2017.

[35] K.-i. Kawarabayashi, P. N. Klein, and C. Sommer. Linear-space approximate distance oracles for planar, bounded-genus and minor-free graphs. In *Proceedings of the 38th International Colloquim Conference on Automata, Languages and Programming*, pages 135–146, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.

[36] P. N. Klein. Multiple-source shortest paths in planar graphs. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 146–155, 2005.

[37] P. N. Klein. A subset spanner for planar graphs, with application to subset TSP. In *Proceedings of the Thirty-Eighth Annual ACM Symposium on Theory of Computing*, STOC '06, pages 749–756, New York, NY, USA, 2006. ACM.

[38] L. L. Larmore and B. Schieber. On-line dynamic programming with applications to the prediction of RNA secondary structure. *Journal of Algorithms*, 12(3):490–515, Sept. 1991.

[39] G. Monge. *Mémoire sur la Théorie des Déblais et des Remblais*. De l'Imprimerie Royale, 1781.

[40] S. Mozes, C. Nikolaev, Y. Nussbaum, and O. Weimann. Minimum cut of directed planar graphs in $O(n \log\log n)$ time. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 477–494, New Orleans, Louisiana, Jan. 2018.

[41] S. Mozes and C. Sommer. Exact Distance Oracles for Planar Graphs. In Y. Rabani, editor, *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 209–222, Philadelphia, PA, Jan. 2012. Society for Industrial and Applied Mathematics.

[42] J. K. Park. *The Monge Array: An Abstraction and Its Applications*. Ph.D. dissertation, MIT, 1991.

[43] D. Peleg and A. A. Schäffer. Graph spanners. *Journal of Graph Theory*, 13(1):99–116, 1989.

[44] R. Rudolf and G. J. Woeginger. The cone of Monge matrices: Extremal rays and applications. *ZOR – Methods and Models of Operations Research*, 42(2):161–168, June 1995.

[45] L. M. Russo. Monge properties of sequence alignment. *Theoretical Computer Science*, 423:30–49, Mar. 2012.

[46] C. Sommer. Shortest-path queries in static networks. *ACM Computing Surveys*, 46(4):1–31, Mar. 2014.

[47] M. Thorup. Compact oracles for reachability and approximate distances in planar digraphs. *J. ACM*, 51(6):993–1024, Nov. 2004.

[48] M. Thorup and U. Zwick. Spanners and emulators with sublinear distance errors. In *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithm*, pages 802–809, Miami, Florida, 2006. ACM Press.

[49] A. Tiskin. Semi-local string comparison: Algorithmic techniques and applications. Nov. 2013.

[50] D. Woodruff. Lower bounds for additive spanners, emulators, and more. In *Proceedings of 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*, pages 389–398, 2006.

[51] J. Łącki and P. Sankowski. Min-cuts and shortest cycles in planar graphs in $O(n \log\log n)$ time. In *Algorithms – ESA 2011*, volume 6942, pages 155–166. Springer Berlin Heidelberg, 2011.