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Journal of Computer and System Sciences 72 (2006) 868–889

JOURNAL OF
COMPUTER
AND SYSTEM
SCIENCES

www.elsevier.com/locate/jcss

Planar graphs, negative weight edges, shortest paths, and near linear time

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Received 17 May 2002; received in revised form 6 May 2004

Available online 7 February 2006

Abstract

In this paper, we present an $O(n \log^3 n)$ time algorithm for finding shortest paths in an n -node planar graph with real weights. This can be compared to the best previous strongly polynomial time algorithm developed by Lipton, Rose, and Tarjan in 1978 which runs in $O(n^{3/2})$ time, and the best polynomial time algorithm developed by Henzinger, Klein, Subramanian, and Rao in 1994 which runs in $\tilde{O}(n^{4/3})$ time. We also present significantly improved data structures for reporting distances between pairs of nodes and algorithms for updating the data structures when edge weights change.
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Keywords: Planar graphs; Negative edge weights; Shortest paths; Algorithms

1. Introduction

The shortest path problem with real (positive and negative) weights is the problem of finding the shortest distances from a specified source node to all the nodes in the graph. For this paper, we assume that the graph has no negative cycles since the shortest path between two nodes will be undefined in the presence of negative cycles. In general, algorithms for

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¹ Supported by the Fulbright scholarship and the scholarship from the Faculty of Engineering, Kasetsart University, Thailand. Work done while at CS Division, University of California, Berkeley.

the shortest path problem can, however, easily be modified to output a negative cycle, if one exists. This also holds for the algorithms in this paper.

The shortest path problem has long been studied and continues to find applications in diverse areas. The problem has wide application even when the underlying graph is a grid. For example, there are recent image segmentation approaches that use negative cycle detection [1,2]. Some of our other favorite applications for planar graphs include separator algorithms [3], multi-source multi-sink flow algorithms [4], or algorithms for finding minimum weighted cuts [5].

In 1958, Bellman and Ford [6,7] gave an $O(mn)$ algorithm for finding shortest paths on an m -edge, n -node graph with arbitrary real edge weights. Gabow and Tarjan [8] showed that this problem could indeed be solved in $\tilde{O}(\sqrt{nm} \log C)$,² where C denotes the largest absolute weights in the graph. Their algorithm depends on the values of the edge weights. For strongly polynomial algorithms, Bellman–Ford remains the best known.

As for graphs with non-negative edge weights, the problem is much easier. For example, Dijkstra's shortest path algorithm [9] can be implemented in $O(m + n \log n)$ time.

For planar graphs, upon the discovery of planar separator theorems [10], an $O(n^{3/2})$ algorithm was given by Lipton, Rose, and Tarjan [11]. Their algorithm is based on partitioning the graph into pieces and recursively computing distances between the borders of each piece using numerous invocations of Dijkstra's algorithm to build a dense graph. Then they use the Bellman–Ford algorithm on the resulting dense graph to construct a global solution. Their algorithms works not only for planar graphs but for any \sqrt{n} -separable one.³

Combining a similar approach with a weakly polynomial algorithm of Goldberg [12] for general graphs, Henzinger et al. [13] gave an $\tilde{O}(n^{4/3} \log C)$ algorithm for the shortest path problem on planar graphs or any graphs with an $O(\sqrt{n})$ sized separator, where C denotes the largest absolute weights.

In this paper, we present an $O(n \log^3 n)$ time algorithm for finding shortest paths in a planar graph with real weights. We also present algorithms for query and dynamic versions of the shortest path problems.

1.1. The idea

Our approach is similar to the approaches discussed above in that, give a planar graph G , it constructs a rather dense non-planar graph G_D on a subset of nodes and then computes a shortest path tree G_D .

We observe that there exists a shortest path tree in G_D that must obey a non-crossing property in the geometric embedding of the graph inherited from the embedding of the original planar graph G . Using this non-crossing condition, we can compute a shortest path tree of the G_D in time that is nearly linear in the number of nodes in G_D and significantly less than linear in the number of edges. Specifically, we decompose our dense graph G_D into a set of bipartite graphs whose distance matrices obey a non-crossing condition (called the Monge condition; see definitions in Section 2.3).

² The $\tilde{O}(\cdot)$ notation ignores logarithmic factors.

³ Assuming that they are given a recursive decomposition of the graph.

Our algorithm proceeds by combining Dijkstra's and the Bellman–Ford algorithms with methods for searching Monge matrices in sublinear time. We use an on-line method for searching Monge arrays with our version of Dijkstra's algorithm on the dense graph.

We note that our algorithms rely heavily on planarity, whereas some of the previous methods only require that the graphs are $O(\sqrt{n})$ -separable. Recently, Smith [14] suggests that our algorithm works with the same time and space bound as well on graphs of bounded genus, by using the result of Hutchinson and Miller [15] and Djidjev and Venkatesan [16] on finding planarizing separators for bounded genus graphs at the topmost level of the decomposition.

1.2. Our results

We give the following results.

- An $O(n \log^3 n)$ algorithm for finding shortest paths in planar graphs with real weights.
- An algorithm that requires $O(n \log^3 n)$ preprocessing time and answers distance queries between pairs of nodes in time $O(\sqrt{n} \log^2 n)$. The best previous algorithms had an $\Theta(n^2)$ query-preprocessing time product, whereas ours is $\tilde{O}(n^{3/2})$.
- An algorithm that supports distance queries and update operations that change edge weights in amortized $O(n^{2/3} \log^{7/3} n)$ time per operation. This algorithm works for positive edge weights.
- An algorithm that supports distance queries and update operations that change edge weights in amortized $O(n^{4/5} \log^{13/5} n)$ time per operation. This algorithm works for negative edge weights as well.

We also present an on-line Monge searching problem and methods to solve it that may be novel and of independent interest.

1.3. More related work

For planar graphs with positive edge weights, Henzinger et al. [13] gave an $O(n)$ time algorithm to compute single-source shortest paths. Their work improves on work of Fredrickson [17] who had previously given $O(n\sqrt{\log n})$ algorithms for this problem.

For non-negative integer weights, if the distance queries are for ϵ -approximate answers, Thorup [18] gave two algorithms. His first algorithm preprocesses the input in time $O(n\epsilon^{-1} \log^2 n \log \Delta)$ and builds a data structure supporting each query in time $O(\log \log \Delta + \epsilon^{-1} \log n)$, where Δ is an upper bound on the largest finite distance. His second algorithm gives a faster query time of $O(\log \log \Delta + 1/\epsilon)$ while the preprocessing time becomes $O(n \log^3 \log \Delta / \epsilon^2)$.

Recently, Klein [19] gives an $O(n \log n)$ -time algorithm for planar graphs with non-negative edge weights that construct a data structure supporting each distance query in time $O(\log n)$. Using this algorithm, he derives faster algorithms for constructing the data structure for distance query for graphs with non-negative weights, which runs in time $O(n \log^2 n)$, and for the dynamic case with the amortized running time of $O(n^{2/3} \log^{5/3} n)$

per operation. His technique also improve the preprocessing time of Thorup's first algorithm mentioned in the previous paragraph.

Frederickson [20] gave an improved all-pairs shortest path algorithm for planar graphs with small hammock decompositions. Djidjev et al. [21] gave dynamic algorithms whose complexity are linear in the size of the hammock decomposition. This could be quite efficient in certain cases, e.g., when the graph is outerplanar. But for general planar graphs—even grid graphs—their algorithms are no better than those in [11].

Efficient algorithms for searching for minima in Monge arrays have been developed previously. See, for example, [22,23].

A binary searching technique similar to the one we use in the Monge searching problem also appeared in an algorithm for finding shortest paths on a three-dimensional polygon by Mitchell et al. [24].

2. Preliminaries

In this section, we give backgrounds on shortest paths, algorithms for finding them, and Monge arrays. Subsection 2.1 describes two basic algorithms that we use, namely, Dijkstra's algorithm and the Bellman–Ford algorithm. We discuss price functions and reduced costs as tools for speeding up shortest path computations when there are many sources in Subsection 2.2. Finally, in Subsection 2.3, we give an introduction to the Monge property. We begin with the definition of shortest path labellings.

Given a directed graph $G = (V, E)$, and a weight function $w : E \rightarrow R$ on the directed edges, a distance labeling for a source node s is a function $d : V \rightarrow R$ such that $d(v)$ is the minimum over all s to v paths P of P 's length, i.e., $\sum_{e \in P} w(e)$.

2.1. Algorithms

The algorithms we use work through a sequence of *edge relaxations*. They start with a labeling $d(\cdot)$ and choose an edge to relax. The relax operation proceeds for an edge $e = (u, v)$ by setting the distance label $d(v)$ to the minimum of $d(v)$ and $d(u) + d(e)$.

Dijkstra's algorithm (described below) correctly computes a distance labeling when the weights on the edges are non-negative, i.e., $d(e) \geq 0$ for all $e \in E$.

```
Algorithm Dijkstra( $G = (V, E)$ ,  $w$ ,  $s$ )
   $d(v) = \infty$ ,  $\forall v \neq s$ .
   $d(s) = 0$ .
   $S = \{s\}$ .
  while  $S \neq V$  do
     $u = \text{findMin}_d(V \setminus S)$ 
    foreach  $e = (u, v)$ 
       $d(v) = \min(d(v), d(u) + w(e))$  /* This is an edge relaxation */
     $S = S \cup \{u\}$ .
```

For distance functions where $d(e)$ could be less than zero, Bellman and Ford suggested the following algorithm, which is guaranteed to compute a distance labeling if there is no cycle in the graph whose total weight under $d(\cdot)$ is negative. The algorithm is as follows.

```
Algorithm Bellman–Ford ( $G = (V, E)$ ,  $w$ ,  $s$ )
   $d(v) = \infty$ ,  $\forall v \neq s$ .
   $d(s) = 0$ .
   $phase = 0$ .
  while  $phase \leq n$  do
    relax all edges.
     $phase \leftarrow phase + 1$ .
```

2.2. Feasible price functions and relabellings

A *price function* p is a function from V to the set of real numbers. The *reduced cost function* w_p over the edge set induced by the price function p is defined as

$$w_p(u, v) = p(u) + w(u, v) - p(v).$$

It is well known that the reduced cost function preserves negative cycles and also the shortest paths [25].

We say that the price function p is *feasible* if and only if for all edges $e = (u, v)$, $w_p(u, v) \geq 0$. Hence, for any feasible price function p , we can find a distance labeling from any source node using Dijkstra's algorithm on the modified graph with w_p as weights (called the *relabelled graph*). The distance labeling for the original graph can be easily recovered from the relabeled one. We note that a valid set of distance labels for any source node is a feasible price function.

Thus, computing shortest paths from k sources in a graph with negative weight edges can be accomplished with only one application of the Bellman–Ford algorithm and $k - 1$ applications of Dijkstra's algorithm.

2.3. Monge arrays

Given ordered sets A and B and a distance function $d : A \times B \rightarrow \mathcal{R}$ between pairs of element in A and B , we say that d has the *Monge property* if for all $u, v \in A$ and $x, y \in B$, $u \leq v$ and $x \leq y$ implies that

$$d(u, x) + d(v, y) \leq d(v, y) + d(u, x),$$

i.e., the sum of the distances when the pairs do not cross is at most the sum when the pairs cross (see Fig. 1(a)). We can also view the triplet (A, B, d) as a metric on a complete “ordered” bipartite graph. Naturally, we call each element in A and B a node. Nodes in A are *left nodes*, and nodes in B are *right nodes*.

An example of distance functions with the Monge property is the shortest distances between border nodes in planar graphs. Figure 1(b) gives an example. Since any paths from u to x and from v to y always cross at some node w , we can break up the distance on the left-hand side and derive the inequality.

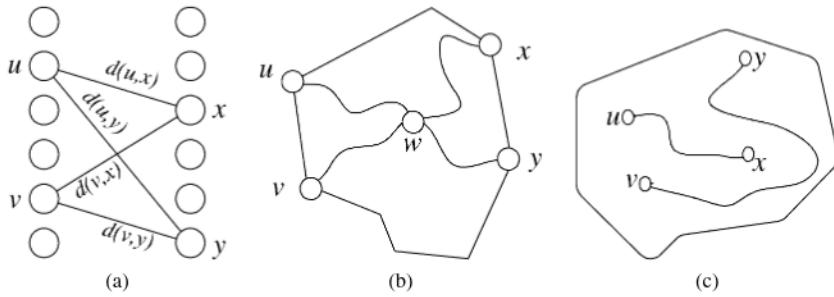


Fig. 1. (a) The pairs of distances, (b) the Monge property for distances in planar graphs, (c) when two paths do not cross.

The Monge property also implies another important property that we use. That is, for $y \in B$, if $u \in A$ is the node that minimizes $d(u, y)$, for any $v \geq u \in A$ and $x \leq y \in B$, $d(v, x) \geq d(u, x)$. Figure 1(b) also demonstrates this fact in the case of distances in planar graphs. To see this, note that the paths $u-y$ and $v-x$ must cross at some node w , whose distance label must be either from u or v . However, it must be the case that the distance from u to w is no worse than the current label of w because u minimizes $d(u, y)$. This implies that $d(u, x) \geq d(v, x)$. The requirement that these two paths cross is very crucial, as this fact is not true in the example in Fig. 1(c). Intuitively, this fact says that shortest paths need not cross. We call this property a *non-crossing* property.

The non-crossing property can be stated in the form that we use as follows. If $u \in A$ minimizes $d(u, y)$ for some $y \in B$, then for all $v \geq u \in A$ and $x \leq y \in B$,

$$d(v, x) \geq d(u, x).$$

To see this, note that from the Monge property, we have $d(u, y) + d(v, x) \geq d(u, x) + d(v, y)$. Since u minimizes $d(u, y)$, $d(u, y) \leq d(v, y)$, and it follows that $d(v, x) \geq d(u, x)$.

Given (A, B, d) with the Monge property, the *Monge matching problem* is to find a parent $p(v) \in A$ for all $v \in B$ such that $d(p(v), v) \leq d(u, v)$ for any $u \in A$. The set of $(p(v), v)$ form the *minimum Monge matching*. The non-crossing property states that it is enough to look for function $p(\cdot)$ that has no “crossing.” Therefore, we can use standard divide-and-conquer techniques to derive an $O(n \log n)$ algorithm for the problem, where $n = |A| + |B|$. The idea is to find parent $u \in A$ which minimizes $d(u, v)$ for the middle right node v first by checking all the left nodes, and then recurse on the top and bottom half of the right nodes. Each right node is to be considered at most twice for each recursive level.

3. The algorithm

We proceed in this section with a description of our algorithm. In Section 3.1, we define our main tool, the dense distance graph, which is an efficiently searchable representation of distances in the planar graph. In Section 3.2, we show how to compute the graph inductively, by relying on some Monge data structures and efficient implementations of

Dijkstra's algorithm and the Bellman–Ford algorithm. In Section 3.3, we show how to use it to compute a shortest path labeling of the graph. In Sections 3.4 to 3.6, we use the dense distance graph as the basis for query and dynamic shortest path algorithms.

3.1. The dense distance graph

A *decomposition* of a graph is a set of subsets P_1, P_2, \dots, P_k (not necessarily disjoint) such that the union of all the sets is V and for all $e = (u, v) \in E$, there is a unique P_i that contains e . A node v is a *border node* of a set P_i if $v \in P_i$ and there exists an edge $e = (v, x)$ where $x \notin P_i$. We refer to the subgraph induced on a subset P_i as a *piece* of the decomposition.

We assume that we are given a recursive decomposition where at each level, a piece with n nodes and r border nodes is divided into two subpieces such that each subpiece has no more than $2n/3$ nodes and at most $2r/3 + c\sqrt{n}$ border nodes, for some constant c . (The recursion stops when a piece contains a single edge.)

In this recursive context, we define a border node of a subpiece to be any border node of the original piece or any new border node introduced by the decomposition of the current piece.

It is convenient to define the *level of a decomposition* in the natural way, with the entire graph being the only piece in the level 0 decomposition, the pieces of the decomposition of the entire graph being the level 1 pieces in the decomposition, and so on. A node is a *level i border node* if it is a border node of a level i piece. Note that a node may be a border node for many levels. Indeed, any level i border node is also a level j border node for all $j > i$.

Given an embedding of the piece, a *hole* is a bounded face where all adjacent nodes are border nodes. For simplicity, we assume inductively that there is a planar embedding of any piece in the recursive decomposition where all the border nodes are on a single face and are circularly ordered. This assumption implies that no piece of the decomposition has a hole. Although this assumption is true for interesting classes of planar graphs, e.g., grid graphs, it is not true in general. In Section 5, we show how to generalize the algorithm to work with a piece with a constant number of holes and describe how one can find a recursive decomposition of that form in $O(n \log n)$ time.

We assume, without loss of generality, that the graph is a bounded-degree graph and, for simplicity, that each piece is connected. If some piece is not connected, the algorithm works with each connected component separately.

For each piece of the decomposition, we recursively compute the all-pairs shortest path distances between all its border nodes along paths that lie entirely inside the piece. These all-pair distances form the edge set of a non-planar graph representing shortest paths between border nodes. Taking the union of these graphs over all the levels, we have the *dense distance graph* of the planar graph.

The *level i dense distance graph* is the subgraph of the dense distance graph on the level i border nodes. We refer to the *level i dense distance graph of piece P* as the subgraph of the level i dense distance graph whose edges correspond to paths that lie in P . Note that in this definition, piece P might not be a level i piece.

This graph underlies previous algorithms for shortest paths in planar graphs. We give a better algorithm to construct and use it.

3.2. Computing the dense distance graph

We assume (recursively) while computing the level i dense distance graph that we have the level $i + 1$ graph and the distances between all the border nodes of each piece in that level. We will show how to find the edges of the level i dense distance graph that correspond to a particular piece P , which has n nodes and r border nodes.

Recall that the level i dense distance graph for P consists of the all-pairs shortest path distances between border nodes within each of its subpieces in the level $i + 1$ dense distance graph.

Also, note that the level $i + 1$ dense distance graph may contain negative edges. By finding a feasible price function using a single Bellman–Ford computation from any source, however, we can find the shortest path distances from any other source using only the Dijkstra computation as stated in Section 2.2.

We proceed by doing a single Bellman–Ford computation in the level $i + 1$ dense distance graph of P from one border node, and then doing $r - 1$ Dijkstra computations on the relabeled graph to compute the shortest path distances from the remaining border nodes.

This, again, is exactly what previous researchers did. Their algorithms, however, used implementations for the Bellman–Ford and Dijkstra’s algorithms which depended linearly on the number of edges that are present in the level $i + 1$ dense distance graph.

Our methods depend near linearly on the number of nodes in the dense distance graph level $i + 1$ of piece P , which is proportional to the square root of the number of its edges.

We assume that P contains n nodes and $r = c\sqrt{n}$ border nodes. By a property of the decomposition, we assume that each of the two subpieces of P contain at most $c'\sqrt{n}$ border nodes. Thus, the level $i + 1$ dense distance graph contains at most $r' = O(\sqrt{n})$ nodes.

3.2.1. The Bellman–Ford step

The Bellman–Ford algorithm that we run proceeds as in Fig. 2.

The total number of border nodes in every subpiece of P is $r' = O(\sqrt{n})$, so the number of edges is $O(n)$. Therefore, if we relax every edge directly as in [11], the running time for each step of edge relaxation would be $O(n)$ for all of P . The total running time for the Bellman–Ford step would then be $O(n^{3/2})$.

However, we will relax the edges in time that is nearly linear in the number of nodes, i.e., in $O(\sqrt{n} \log^2 n)$ time. This gives a running time of $O(n \log^2 n)$ for the Bellman–Ford step.

```
Algorithm Bellman–Ford ( $P, w, s$ )
 $d(s) = 0.$ 
 $d(v) = \infty, \forall v \neq s.$ 
 $phase = 0.$ 
while  $phase \leq r'$  do
    relax all edges.
     $phase \leftarrow phase + 1.$ 
```

Fig. 2. The implementation of the Bellman–Ford algorithm.

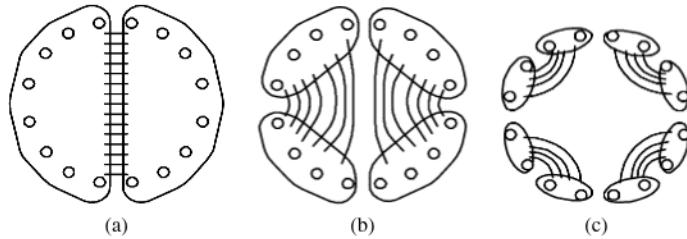


Fig. 3. Partitions of nodes to form $O(\log n)$ Monge arrays: (a) the first and second arrays, (b) the third and forth arrays, and (c) the fifth and sixth arrays.

We accomplish this by maintaining the edges of each subpiece of P in $O(\log n)$ levels of Monge arrays. After the definition of the Monge arrays, we describe how all edges in each Monge array with k nodes can be relaxed in $O(k \log k)$ time, where k is the number of nodes in the data structure.

The first Monge array that we define is formed as follows. Divide the border nodes in some subpiece into two halves, the first (or left) half in the circular order (with an arbitrary starting point) and the second (or right) half. Consider the set of edges in the dense distance graph that go from the left border nodes to the right ones. The edges obey the Monge property, since there is an underlying shortest path tree in which the corresponding paths do not cross.

Using the same left-right partitioning, we can define another Monge array with the direction of edges reversed (i.e., edges in the array go from the right border nodes to the left border nodes).

Successive Monge arrays are constructed by recursively dividing the left and right halves further. Each node will occur in at most $O(\log n)$ data structures, and each edge will occur in one data structure. Figure 3 shows how we partition nodes and edges between them.

We can relax all the edges in a Monge array as follows. The nodes on the left have a label associated with them, and a node v on the right must choose a left node u which minimizes $d(u) + w(u, v)$. We say that u is the *parent* of v and edge (u, v) is the *parent edge* of v . However, because of the planarity of the piece, the parent edges of two right nodes need not cross, and this gives us the Monge property. For this special case, we can use a the Monge matching algorithm, described in Subsection 2.3, to find all the parents in time $O(r \log r)$, where the number of nodes in the Monge array is r .

The total number of nodes in all the data structures is $O(\sqrt{n} \log n)$ for each subpiece of P in the decomposition.

In P we have two subpieces. Therefore, the time for relaxing all of P 's edges is $O(\sqrt{n} \log^2 n)$. The number of phases the Bellman–Ford runs is the number of nodes in the longest path, which is $O(\sqrt{n})$. Thus, the time for the Bellman–Ford step is $O(n \log^2 n)$ on an n -node piece.

3.2.2. The Dijkstra step

After one invocation of Bellman–Ford, we have a shortest path tree from some border node of P . Now, using the relabeling property, we can modify all edge weights so that they

```

Algorithm Dijkstra ( $P, \{P'_i\}, s$ )
   $d(v) \leftarrow \infty, \forall v \neq s.$ 
   $d(s) \leftarrow 0.$ 
   $S \leftarrow \{s\}.$ 
  for all  $P'_i \ni s$  do
    AddToHeap( $H, (d(s), P'_i)$ ).
  while  $H$  is not empty do
     $P'_{\min} \leftarrow \text{ExtractMin}(H).$ 
     $v \leftarrow \text{ExtractMinInSubpiece}(P'_{\min}).$ 
    if  $v \notin S$ ,
      for all  $P'_i \ni v$  do
        ScanInSubpiece( $P'_i, v, d_v$ ).
         $v_{\min} \leftarrow \text{FindMinInSubpiece}(P'_i)$ 
        UpdateHeap( $H, (d(v_{\min}), P'_i)$ ).
    else
       $v_{\min} \leftarrow \text{FindMinInSubpiece}(P'_{\min})$ 
      UpdateHeap( $H, (d(v_{\min}), P'_{\min})$ ).
     $S \leftarrow S \cup \{v\}.$ 

```

Fig. 4. Pseudocode for Dijkstra implementation.

are all positive and the shortest paths remain unchanged. With these modified weights, we repeatedly apply Dijkstra's algorithm to compute all-pairs shortest distances among the border nodes in P .

In order to compute the shortest path distances from each border node s of P , we proceed as in Dijkstra.

While working at level i of the decomposition, we view each subpiece of level $i + 1$ separately. Each subpiece maintains a data structure that allow us to scan a node (relax all edges in the dense distance graph adjacent to that node in that subpiece) and find the minimum labeled node in the subpiece efficiently.

As in Dijkstra's algorithm, we will maintain a set of scanned nodes S and a global heap H for keeping minimum labeled nodes from all subpieces. A node is extracted from the global heap. This node can belong to many subpieces, so we scan it in all the subpieces containing it. After we scanned the node, the minimum labeled node in subpieces might change, so we have to update the entry of these subpieces in the global heap H .

The primary difference between our implementation of Dijkstra's algorithm and the normal one is that in our implementation a node that is already scanned can appear again in the heap. This is because the data structure in each subpiece does not guarantee that a minimum border node after being scanned will never reappear as a minimum node again. The data structure does, however, guarantee that node can reappear at most $O(\log r)$ times.

Let $\{P'_i\}$ denote the set of subpieces in P . The pseudocode in Fig. 4 describes the algorithm for computing the shortest path tree starting at a border node s . It uses the following operations on data structures that are maintained for each of the subpieces.

- $\text{ScanInSubpiece}(P'_i, v, d_v)$: Relax all edges of v in the dense distance graph at level $i + 1$ in piece P'_i conditioned on $d(v) = d_v$.

A sequence of l calls to ScanInSubpiece can be implemented in $O(l \log^2 r)$ time.

- $\text{FindMinInSubpiece}(P'_i)$: Return the border node (which might already be scanned) in piece P'_i whose label is no greater than all unscanned nodes in the piece.
This procedure can be implemented in $O(1)$ time.
- $\text{ExtractMinInSubpiece}(P'_i)$: Return the border node (which might already be scanned) in piece P'_i whose label is no greater than all unscanned nodes in the piece, and attempt to remove it from the heap in the piece. We say ‘attempt’ because the extracted node can be returned as the minimum node again, however no node can be returned by this procedure more than $O(\log r)$ times.
A sequence of l calls to $\text{ExtractMinInSubpiece}$ can be implemented in $O(l \log r)$ time.

We will show how to implement this data structure in Section 4.1. At this point we assume the bounds stated above and use them to bound the running time of the Dijkstra step.

The algorithm also uses heap H which stores references to subpieces. It supports the following operations: $\text{AddToHeap}(H, (k, P'_i))$ which adds subpiece P'_i with key k ; $\text{ExtractMin}(H)$ which returns the subpiece with the minimum key, and delete it from H ; and $\text{UpdateHeap}(H, (k, P'_i))$ which updates P'_i 's key.

We stress that an already scanned node might be returned from FindMinInSubpiece and $\text{ExtractMinInSubpiece}$. The data structure does, however, guarantee that any border nodes will not be returned from $\text{ExtractMinInSubpiece}$ more than $O(\log r)$ times.

When the data structure for each subpiece returns a minimum labeled unscanned node, it is the minimum unscanned node in the subpiece. Also, the algorithm only scans a node which is the minimum over all nodes returned from all the subpieces. Therefore, every time the algorithm scans a node v , v has the minimum distance label over all unscanned nodes.

Thus, our algorithm is a valid implementation of Dijkstra's algorithm in that it only scans the minimum labeled nodes. Thus, it correctly computes a shortest path labeling.

3.2.3. Analysis of the running time of the Dijkstra step

For piece P , there are $O(r) + O(\sqrt{n}) = O(\sqrt{n})$ border nodes in consideration, including those from the level $i + 1$ dense distance graph of P .

The number of calls to ScanInSubpiece is bounded by the number of nodes in the level $i + 1$ dense distance graph. (Because the degree for each node is bounded, the node belongs to a constant number of subpieces.) Since each node is scanned once as in Dijkstra's algorithm, there are $O(r)$ calls to ScanInSubpiece .

$\text{ExtractMinInSubpiece}$ is called at most $O(\log r)$ times on each node in the level $i + 1$ dense distance graph. So the total number of operations is $O(r \log r)$ for a total cost of $O(r \log^2 r)$.

Finally, the number of calls to FindMinInSubpiece is bounded by the number of calls to ScanInSubpiece for a subpiece and to $\text{ExtractMinInSubpiece}$.

The time for each operation on heap H is $O(1)$ because there are a constant number of subpieces.

Therefore, the running time for computing each shortest path tree is $O(r \log^2 r) = O(\sqrt{n} \log^2 n)$, and the total running time for computing all trees is $O(r \sqrt{n} \log^2 n) = O(\sqrt{n} \sqrt{n} \log^2 n) = O(n \log^2 n)$.

3.2.4. The running time for constructing the dense distance graph

For each level of the decomposition, the time for doing the Bellman–Ford step is $O(n \log^2 n)$, and the time for all Dijkstra computations is $O(n \log^2 n)$. Since there are at most $O(\log n)$ levels in the decomposition, the time to construct the whole dense distance graph is $O(n \log^3 n)$.

3.3. Shortest path

To actually solve the shortest path problem for a source s , we use the dense distance graph as follows.

Assume that s lies on the outer face of the embedding; if it does not, one can transform the embedding in linear time. Then, we add s as a “border” node to all the pieces that contain it and compute the dense distance graph on the resulting decomposition. We compute a shortest path labeling for the source s in the level 1 dense distance graph to the border nodes using the Bellman–Ford algorithm above.

We then extend the distances to the internal nodes recursively, again, using the Bellman–Ford algorithm.

The Bellman–Ford computation costs $O(n \log^2 n)$ for each level. Therefore, the running time for computing all the distances is $O(n \log^3 n)$.

3.4. Supporting queries when the graph is static

The dense distance graph and the Dijkstra procedure above can be used to answer shortest path queries between a pair of nodes. In this section we show how to use the dense distance graph to find the shortest distance between any pair of nodes in $O(\sqrt{n} \log^2 n)$ time.

The algorithm for this is very similar to the one for the Dijkstra step in the shortest path algorithm. Suppose the query is for the distance of a pair (u, v) . The shortest (u, v) path can be viewed as a sequence of paths between border nodes of the nested pieces that contain u and v . The lengths of these paths are represented in the dense distance graph as an edge between border nodes and the higher level border nodes or as an edge among border nodes of a piece. Thus, we can perform a Dijkstra computation on this subgraph of the dense distance graph to compute the shortest (u, v) path.

We derive the bound on the number of border nodes in the pieces containing u as follows. Each piece, except the first one, which is G , is a piece in the decomposition of another piece. Hence the number of nodes goes down geometrically. Also, the number of border nodes, which is bounded above by the square root of the number of nodes in the piece, goes down geometrically. Therefore the number of border nodes involved is $O(\sqrt{n})$.

We use the same algorithm as in the Dijkstra step, but now we work with many pieces from many levels of the decomposition. The algorithm in the Dijkstra step continues to work in time $O(k \log^2 n)$ where k is the total number of nodes involved in the Dijkstra step.

Since the total number of nodes in the graph that we are searching is $O(\sqrt{n})$, the running time is bounded by $O(\sqrt{n} \log^2 n)$.

3.5. Dynamic algorithms for graphs with only positive edge weights

A dynamic data structure answers shortest path queries and allows edge cost updates, where the cost of an edge may be decreased or increased. (Edge additions and deletions are not addressed in this paper.)

The query algorithm in the previous subsection works only with the pieces in the recursive decompositions that contain the query pair. It can avoid the other pieces because all the distances in those pieces are reflected in the distances among the border nodes of the pieces containing them.

In the dynamic version, we will use the same algorithm as in the query-only case. Since our query step uses Dijkstra's algorithm, it is crucial that all weights are non-negative. However, some update might introduce a negative edge in the relabeled graph. To simplify the presentation, we first discuss the case that all edges have positive weights in this section. In the following section we extend the idea to the general case.

We do not know how to efficiently maintain an explicit representation of the dense distance graph when an update occurs. But, only the pieces containing an updated edge will not have the correct distances among their border nodes. That is, an edge in the dense distance graph between two border nodes of a region containing an updated edge may have an accurate distance label. However, edges in other pieces in the dense distance graph have the correct label.

We call the pieces that contain updated edges *activated pieces* and call the border nodes of these pieces *activated nodes*. (See Fig. 5(b) for example.)

To properly recompute the distance for piece P that contains an updated edge, we need to consider the distances among all the border nodes of pieces that are contained in P . Thus, we define the *activated graph* to be all the valid edges between border nodes of the pieces containing an updated edge and their sibling pieces. (See Fig. 5(c).)

We answer a query for a pair (u, v) by adding the valid edges of border nodes of pieces containing u and v to the activated graph and running a Dijkstra's computation on the resulting graph. We call this graph the *extended activated graph for (u, v)* . (See Fig. 5(d).)

We proceed by deriving a bound on the number of nodes involved in the computation assuming that we allow a maximum of k updates before rebuilding the entire data structure.

For each update, the number of border nodes on the pieces that need to be in Dijkstra's computation is $O(\sqrt{n})$. Naively, one can bound the total number of activated nodes by $O(k\sqrt{n})$. However, if we consider a top-down process that divides any piece that contains an updated edge, we can show that the total number of activated nodes is $O(\sqrt{nk})$ as follows.

Consider the decomposition tree. There are at most k leaves that are activated. Hence, at most $k - 1$ pieces have both their children activated; call these pieces *branching pieces*. Because the number of nodes goes down geometrically along the tree, we can bound the total number of activated border nodes using the number of border nodes of the branching pieces. The worst case is that all $k - 1$ branching pieces are in the highest level of the decomposition tree, i.e., they form a balanced binary tree. We note that the pieces on the same level partition the graph; thus, the number of border nodes is maximized when they partition the graph evenly. Hence, on level l there are at most $2^l \sqrt{n}/2^l = \sqrt{n}2^l$ border

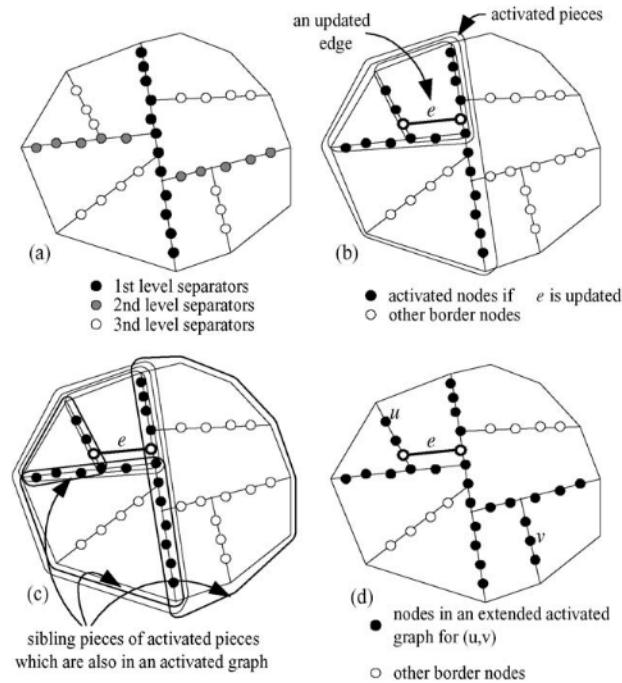


Fig. 5. (a) A graph with a three-level decomposition. (b) Activated nodes for (u, v) . (c) An activated graph when e is updated. (d) An extended activated graph for (u, v) .

nodes. The sum on the last level dominates the total sum; therefore, the number of border nodes is $O(\sqrt{nk})$.

Thus, the Dijkstra computation described in Subsection 3.2.2 will run in time $O(\sqrt{nk} \log^2 n)$.

Therefore, the total cost of a sequence of k updates and queries is $O(k\sqrt{nk} \log^2 n)$ for the queries plus $O(n \log^3 n)$ for (re)building the dense distance graph. By choosing k to be $n^{1/3} \log^{2/3} n$ we get an amortized complexity of $O(n^{2/3} \log^{7/3} n)$ per operation.

3.6. Dynamic algorithms for graphs with negative edge weights

We follow the same strategy in this case, as well. That is, we simply maintain the notion of the activated graph during a sequence of k updates. To answer a query for a pair (u, v) , we compute a distance labeling in the extended activated graph for (u, v) .

Unfortunately, there may be negative edges in the extended activated graph so we cannot just do a Dijkstra computation as above.

We note that if we have a feasible price function over the node set of the extended activated graph, if only one edge $e = (u, v)$ is updated with a negative weight w , we can use one computation of Dijkstra's algorithm to update the price function as follows. We compute the shortest distance labels $d(\cdot)$ of all the nodes starting from v . If $d(u)$ is greater than $-w$, changing the weight of e does not introduce any edge with a negative reduced

cost on the graph with $d(\cdot)$ as a price function; hence, we can update e and update the price function to be $d(\cdot)$. If, however, $d(u)$ is less than $-w$, the shortest path from v to u together with the updated edge (u, v) would form a negative cycle.

Therefore, if we already have k updates, we can compute a feasible price function in the extended activated graph by performing k Dijkstra computations by starting with the original price function on the extended activated graph, and for each update, we update the price function as described above.

After we have the feasible price function for the extended activated graph which includes all the updates, we can proceed as in the previous section.

After k queries and updates, we rebuild the dense distance graph. Thus, the total time for a sequence of k queries and updates is $O(k^2 \sqrt{nk} \log^2 n)$ for recomputing the price function and $O(n \log^3 n)$ for (re)building the dense distance graph. By choosing k to be $n^{1/5} \log^{2/5} n$, we get an amortized complexity of $O(n^{4/5} \log^{13/5} n)$ per operation.

4. Monge searching data structures

Recall that given a complete ordered bipartite graph (A, B, E) with edge weights d satisfying the Monge property, the Monge matching problem is to find a parent $p(v) \in A$ for all $v \in B$ that minimizes $d(p(v), v)$. The non-crossing property ensure that there exists a solution where $p(\cdot)$ do not “cross.” Subsection 2.3 describes a divide-and-conquer algorithm for finding the minimum Monge matching in time $O(n \log n)$, where $n = |A| + |B|$.

In this section, we develop the on-line data structure that underlies the algorithms in Section 4.1. The data structure is extended to handle the non-bipartite case in Section 4.2. We note that the interface of this data structure is rather involved. The data structure was used mainly in the Dijkstra step of the algorithms. Also, the technique for reducing the general case to the bipartite case is used in the edge relaxation step in our implementation of Bellman–Ford.

4.1. On-line bipartite Monge searching

Given ordered sets A and B and a distance function $d : A \times B \rightarrow \mathcal{R}$ between pairs of element in A and B , the Monge property ensures that for all $u, v \in A$ and $x, y \in B$, $u \leq v$ and $x \leq y$ implies that $d(u, x) + d(v, y) \leq d(v, y) + d(u, x)$. This condition on the distance function still holds when an offset distance $D(u)$ on each left node u is given, i.e., the cost for an edge $e = (u, v)$ becomes $D(u) + d(u, v)$.

We now consider an on-line version of this problem in which the offset distances on the left-side nodes are to be specified on-line.

We are given a complete ordered bipartite graph $G = (A, B, E)$ with a distance function d having the Monge property and a not-fully-specified initial distance D for every node in A ; the cost for an edge (u, v) is now $D(u) + d(u, v)$. Initially the distance $D(u) = \infty$ for every $u \in A$, and $D(u)$ will be specified once for each u , over the life of the data structure. We want the data structure to maintain the minimum Monge matching, i.e., the parent $p(v)$ for each $v \in B$.

To make the interface suitable for the application of the data structure, we introduce a growing subset $S \subseteq B$. Initially, $S = \emptyset$. Given S , the *best matched node* is node $v \in B \setminus S$ which minimizes $D(p(v)) + d(p(v), v)$. We only allow the user to (1) query for the best matched node v and (2) add the current best matched node v to S . Basically, the set S denotes the set of nodes in B that have the “correct” matches, i.e., if $v \in S$, v ’s best parent, $p(v)$, would remain the same after this point.

One of the interpretations for a node to have the correct match in the on-line setting is the following. In the context of Dijkstra’s algorithm, the minimum node v in $B \setminus S$ certainly has the correct match (or correct label) when v itself is scanned, because that means its label will never decrease.

When the current minimum node has the correct match, the user must add it to S to be able to query for other best matched nodes, since the data structure only allows queries for the current best matched node outside the correctly matched set.

Without loss of generality, assume that $|A| = |B| = n$; otherwise we can add dummy nodes to the one smaller. Let A be the ordered set $\{a_1, \dots, a_n\}$ and B be the ordered set $\{b_1, \dots, b_n\}$. The data structure maintains the initial distance variables $D(u)$ for all $u \in A$ and the subset $S \subseteq B$, and supports the following operations.

- $\text{ActivateLeft}(u, d_u)$: for $u \in A$, set $D(u) = d_u$.
- $\text{FindMin}()$: return a node $v \in B \setminus S$ such that $v = \arg \min_{v \in B \setminus S} \min_{u \in A} D(u) + d(u, v)$.
- $\text{ExtractMin}()$: let $v = \text{FindMin}()$, set $S \leftarrow S \cup \{v\}$, and return v .

To build this data structure, we will use a range search tree data structure (see, e.g., [26]), which, for an ordered set $Q = \{q_1, \dots, q_l\}$, supports a query of the form $\min_{s \leq i \leq t} q_i$, for any given $1 \leq s, t \leq l$. The interval tree can be implemented using a balanced binary tree, whose leaves are Q and of which each internal node keeps the minimum value over all its children. The time for each query is $O(\log l)$, where l is the size of the ordered set.

For each active left node a_i , we let $s(a_i)$ denote the start index of the interval of the right nodes of which a_i is the best left node. Also, we let $t(a_i)$ denote the index of the last node in a_i ’s interval plus one.

The algorithm maintains the invariant that for every $a_i \in A$ which has the initial distance $D(a_i) \neq \infty$, the node a_i is the best left node for the right nodes $b_{s(a_i)}, b_{s(a_i)+1}, \dots, b_{t(a_i)-1}$, i.e., $D(a_i) + d(a_i, b_j) \leq D(a) + d(a, b_j)$ for all $s(a_i) \leq j < t(a_i)$ and $a \in A$. Note that this implies that the ranges $[s(a_i), t(a_i))$ are non-overlapping. (For example, see Fig. 6(a).)

The data structure maintains:

- for each node in A , whether it is active;
- the left neighbor tree N , an ordered (by index) binary tree for nodes in A which are the best left neighbors for some right node;
- the heap H for the minimum edges (a_i, b_j) of every $a_i \in N$.

Also, for each active node $a_i \in A$, the data structure maintains the range $[s(a_i), t(a_i))$.

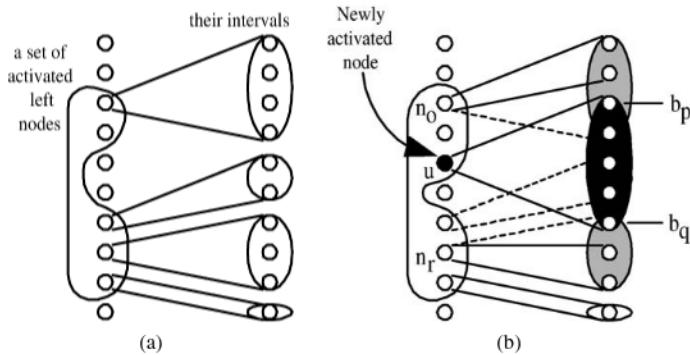


Fig. 6. (a) A bipartite graph with activated nodes and their intervals. (b) Changes after u is activated.

We assume that for each $u \in A$, an interval tree $T(u)$ representation for the values $d(u, b_1), \dots, d(u, b_n)$ is given.⁴

We now describe how each operation is performed.

- **ActivateLeft(u, d_u):** If u is the first node activated, let $s(u) = 1$ and $t(u) = n + 1$, and insert u into N . Otherwise, we will find a set of right nodes $\{b_p, \dots, b_q\}$ of which the node u is now a better parent.

We describe how to find b_p , the top-most right match of u . Suppose that $u = a_i$. Denote the nodes in N as $\{n_1, \dots, n_l\}$, ordered as in A . We find the maximum index j such that $n_j < u$ in A . If there is no such node, we let $p = 1$. Otherwise, we find the activated left node n_o whose interval contains b_p by sequentially comparing, for $k = j, j - 1, \dots, 1$, the distance label of the top right match of n_k , $b_{s(n_k)}$ with the new distance label it will get from u . We continue comparing until the distance from u is no better than the old one. Therefore, we know that $b_p \in \{b_{s(n_o)}, \dots, b_{t(n_o)-1}\}$, and we can binary search for b_p .

We use a similar method to find b_q . Let n_r be the left-hand side node that has b_q in its interval.

We need to modify the data structure for parent change. The internal data structures affected are the tree N , and the heap H . The nodes n_o and n_r that previously had b_p and b_q in their intervals have to shrink their intervals. All other nodes n_{o+1}, \dots, n_{r-1} will have their interval removed, and they are removed from N . We set $s(u) \leftarrow p$ and $t(u) \leftarrow q + 1$.

Finally, for every node whose interval is affected, we have to find its new best right neighbor and update the heap H accordingly. That is, we delete the entries corresponding to n_{o+1}, \dots, n_{r-1} and we modify the entries corresponding to n_o and n_r . Figure 6(b) shows how the data structure is modified after u is activated.

- **FindMin():** We use H to find the minimum edge (a_j, b_i) and return b_i .

⁴ These interval trees must be constructed a priori, e.g., when the distances $d(\cdot, \cdot)$ are computed. The data structure is given these distances as input. This is easy to add to the representation of the dense distance graph.

- ExtractMin(): We use H to find the minimum edge (a_j, b_i) . Then we create two new nodes a'_j and a''_j and put them next to a_j in N such that $a'_j < a_j < a''_j$. We set $s(a'_j) \leftarrow s(a_j)$; $t(a'_j) \leftarrow i$; $s(a''_j) \leftarrow i + 1$; and $t(a''_j) \leftarrow t(a_j)$. Also we set $s(a_j) \leftarrow i$ and $t(a_j) \leftarrow i + 1$, and remove (a_j, b_i) from the heap H . Finally, for a'_j and a''_j , we use a_j 's range search tree to find their best right neighbors and add them to H .

4.1.1. Analysis of the running time

We note that the size of N might be greater than n during the execution of the algorithm because we create some nodes every time ExtractMin is called. However, it is called at most n times; thus, we create no more than $2n = O(n)$ nodes.

We now analyze the running time for each operation.

- ActivateLeft(u, d_u): First, searching for the index j in N takes time $O(\log n)$. To find b_p we do a sequential search and a binary search. Every node in N that we examined during the sequential search is removed except the last one. We charge the cost for the sequential search to the cost for removing and updating these nodes. The cost for the binary search is $O(\log n)$. The search for the lower end costs the same. Next, u has to pick its best right neighbor and add it to H . This can be done in $O(\log n)$ time by querying the range search tree $T(u)$ for the minimum right node of u over the range $\{b_p, \dots, b_q\}$. After the interval is found, some other node in N must update its data structure. At most two nodes have to change their intervals, pick their best right neighbors, and update their entries in H ; this takes $O(\log n)$ time. All other nodes are deleted and will never reappear. Each delete takes time $O(\log n)$ and we charge this to the time the node was inserted to the data structure. Therefore, the operation takes $O(\log n)$ amortized time.
- FindMin(): We can read the top-most item in H in $O(1)$ time.
- ExtractMin(): We can find the current minimum node b_i in $O(1)$ time. It takes $O(\log n)$ time to find the left matched node a_j for b_i . We then do a constant number of operations on N and H which take $O(\log n)$ time. Therefore, ExtractMin runs in time $O(\log n)$.

4.2. Non-bipartite on-line Monge searching

We generalize our data structure to support the case when the graph is not bipartite in this section. We have the graph $G = (V, E)$ with the distance function $d : E \rightarrow \mathcal{R}$. The nodes in V are in a circular order, and the distance function d satisfies the property that

$$d(u, w) + d(v, x) \geq d(u, x) + d(v, w), \quad (1)$$

for every $u, v, w, x \in V$ such that $u \leq v \leq w \leq x$ in V . Notice that the sign of the inequality is reversed because in this case (u, w) crosses (v, x) , contrary to the bipartite case that (u, x) crosses (v, w) .

We note the difficulties in this case. In the bipartite case, the set of nodes which has a particular node as their parent is consecutive, i.e., that particular node owns a single

interval. This is not true in the non-bipartite case. However, we show in this section how to reduce this problem to $O(\log n)$ bipartite problems. The idea is to partition the edges as in Section 3.2.1.

From the graph G , we create $2\lceil \log n \rceil$ bipartite graphs, because for each left–right partition, edges between them can go in two directions. We denote these bipartite graphs as $G_0, G_1, \dots, G_{2\lceil \log n \rceil - 1}$. Under this reduction, each edge belongs to one and only one bipartite graph. We refer to each bipartite graph G_j as a level of G . Let \mathcal{G} denote the set of these $O(\log n)$ bipartite graphs.

The operations that we need from this non-bipartite data structure are the following. We want to be able to set the initial offset distance D as in the bipartite case, and also, we want to find a labeled node which is the minimum one over all the levels. However, the notion of the set S is different now. Suppose that currently a node v on level i is the minimum node. When v is extracted, the level i label of v is definitely correct, but its labels on the other levels can still change. Therefore, we only add the node v to the set S of the level i bipartite graph. This has a drawback: a call to `FindMin` can return v again. However, since each node belongs to at most $O(\log n)$ levels, the node v can reappear at most $O(\log n)$ times.

The data structure for the non-bipartite case consists of $O(\log n)$ data structures for the bipartite case, for all $G_j \in \mathcal{G}$. It maintains a heap H' of minimum nodes over all levels, and initially the distance offset $D(v) = \infty$ for all v in all levels. To make the names of the procedures consistent with the algorithm that constructs the dense distance graph, we call these procedures `ScanInSubpiece`, `FindMinInSubpiece`, and `ExtractMinInSubpiece` instead of `ActivateNode`, `FindMin`, and `ExtractMin`, respectively. We now describe the operations that the data structure supports together with their implementations and running times.

- `ScanInSubpiece(v, d_v)`: let $D(v) = d_v$.

This operation can be done by calling `ActivateLeft(v, d_v)` on every $G_j \in \mathcal{G}$ of which v is in the left-hand side node. On those affected levels, we call `FindMin` and update their entries in the heap H' . This operation can be done in $O(\log^2 n)$ amortized time, because there are $O(\log n)$ levels and each call to `ActivateLeft` costs $O(\log n)$ amortized time. The time for finding the minimum nodes and updating the heap is only $O(\log n \log \log n)$, because the heap H' is of size $O(\log n)$.

- `FindMinInSubpiece()`: find the minimum distance node over all levels.

This can be done in $O(1)$ time by returning the minimum entry in the heap H' .

- `ExtractMinInSubpiece()`: find the minimum distance node over all levels, remove that node from its level, and attempt to add the node to the set S of the data structure.

For this operation, we do as in `FindMinInSubpiece`, but after the minimum node is found, we call `ExtractMin` once on the bipartite data structure of the level to which the minimum node belongs and update that level's entry in H' . The cost for `ExtractMin` is $O(\log n)$ time, and the cost for updating H' is $O(\log \log n)$, because the size of H' is $O(\log n)$. Therefore, this operation can be done in $O(\log n)$ time.

As noted in the discussion above, after $O(\log n)$ attempts to extract a node, it will never be returned as the minimum node again.

5. Dealing with holes

It is not difficult to work with a piece with holes if only a constant number of holes are present. We will describe the algorithm for finding such a decomposition in Subsection 5.1. Here, we continue to discuss how one can modify the algorithm given the required decomposition.

We assume that there are at most h holes on each piece. For each hole, the edges between two border nodes adjacent to it can be handled using the method described in previous sections. Thus, we need h copies of the on-line Monge data structures.

For edges between border nodes adjacent to different holes, we need to modify the data structure. First note that the on-line bipartite Monge data structure in Section 4.1 can be modified to work with edges going between two holes. In this case, we need to consider the nodes in circular order, which can be done by allowing the intervals of left-hand side nodes to wrap around. Therefore, since there are at most $h(h - 1)$ pair of holes, we need another $h(h - 1)$ data structures for edges category.

The running time for data structure operations increases by a factor of at most $O(h^2)$, which is a constant. Therefore, if every piece has $O(1)$ holes, we can implement the algorithm to run in the same time bound.

5.1. Graph decomposition

As mentioned in Section 3, we cannot always have a decomposition that introduces no holes in any pieces (see Fig. 7(a), for example), however it is possible to keep the number of holes in each piece below some constant h .

In this section, we give a procedure that decomposes the graph such that for every constant number of levels of the decomposition, the numbers of nodes and border nodes on each piece go down geometrically while the number of holes remains at most a constant.

We use the simple cycle separator algorithm of Miller [27] to divide each piece into smaller ones. The algorithm takes as an input a planar graph G with constant face size containing n nodes with node weights, and finds a simple cycle separator of size $O(\sqrt{n})$ which separates the graph into two subgraphs, each of which contains at most $2/3$ of the total weight.

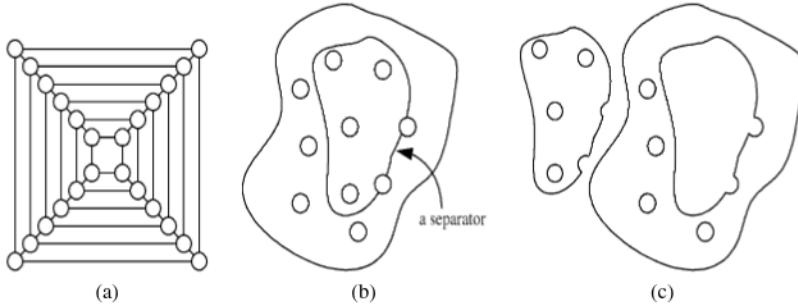


Fig. 7. (a) An example of a graph which cannot be decomposed without introducing a hole. (b) A simple cycle separator dividing the holes between two subpieces. (c) The two subpieces of the decomposition.

We show how to use the simple cycle separator algorithm to divide the graph so that we introduce at most one hole each time. We assume first that the graph is connected. To get the face size to be a constant, we triangulate the graph. The edges added during the triangulation process allow the cycle to jump over some faces, and because of these edges, the resulting subgraphs might contain many connected components. However, at most one hole gets introduced to at most one of the subgraphs.

For the piece with many connected components, we will apply Miller's algorithm on at most one of the components; hence, at most one hole is created. We proceed as follows. We note that if no component has more than $2/3$ of the weights, we can divide the piece, without introducing any separator nodes, into two subpieces each with at most $2/3$ fraction of the total weight. Now assume that there exists a unique component C that has more than $2/3$ of the weight. We run the Miller algorithm only on C , introducing at most one hole. Now, we are left with components each having at most $2/3$ fraction of the weights, so we can divide them without introducing new separator nodes.

With different weight assignments, we can divide the piece so that the subpieces satisfy various constraints. It is a standard application of the algorithm to divide the graph so that the number of nodes or border nodes in the subpieces drops by a factor of $2/3$.

To divide the piece so that the number of holes decreases, we contract the border nodes on each hole into a super node and place uniform weights on the super nodes. After applying the Miller algorithm, each subgraph contains at most a $2/3$ fraction of all super nodes. In other words, it contains at most $2/3$ of the original holes. For any super node on the separator, the hole corresponding to that super node is broken into two parts and the border nodes of that hole become parts of the newly introduced hole. However, we introduce at most one hole during the process. Figures 7(b) and (c) illustrate the way we divide the holes.

We now describe how to obtain a decomposition with the property claimed in the beginning of Section 5. We divide a piece in the way that depends on which level of the decomposition it belongs to. More specifically, at level $3i$ we reduce the number of nodes, at level $3i + 1$ we reduce the number of border nodes, and at level $3i + 2$ we reduce the number of holes.

On every $3i$ th level, the number of nodes decreases by a factor of $2/3$. At each level we introduce at most new $O(\sqrt{n})$ border nodes. The number of nodes in each subpiece decreases geometrically, and the number border nodes in each subpiece with n nodes is at most $O(\sqrt{n})$.

On each level that we decompose a piece, we introduce at most one new hole. Since for every 3 levels, the number of holes drops by a factor of $2/3$, the number of holes remains at most a constant.

The Miller algorithm runs in linear time; thus, the time for decomposing the graph is $O(n \log n)$.

Acknowledgments

We thank Chris Harrelson for his careful reading of this paper. We also greatly thank the anonymous referees for numerous suggestions that help improving the presentation of the paper.

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