12^{th} Annual Harvard-MIT Mathematics Tournament

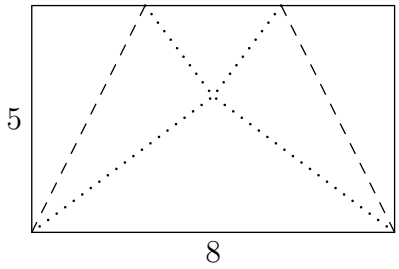
Saturday 21 February 2009

Individual Round: General Test, Part 1

1. [**2**] If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

2. [**2**] Suppose N is a 6-digit number having base-10 representation $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{e}\,\underline{f}$. If N is 6/7 of the number having base-10 representation $\underline{d}\,\underline{e}\,\underline{f}\,\underline{a}\,\underline{b}\,\underline{c}$, find N.

3. [**3**] A rectangular piece of paper with side lengths 5 by 8 is folded along the dashed lines shown below, so that the folded flaps just touch at the corners as shown by the dotted lines. Find the area of the resulting trapezoid.



4. [**3**] If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

5. [**4**] Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

6. [**4**] The corner of a unit cube is chopped off such that the cut runs through the three vertices adjacent to the vertex of the chosen corner. What is the height of the cube when the freshly-cut face is placed on a table?

7. [**5**] Let $s(n)$ denote the number of 1's in the binary representation of n. Compute

$$\frac{1}{255} \sum_{0 \le n < 16} 2^n (-1)^{s(n)}.$$

8. [**5**] Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

9. [**6**] How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1, 2, 3, 4, 5\}$?

10. [**6**] A *kite* is a quadrilateral whose diagonals are perpendicular. Let kite $ABCD$ be such that $\angle B = \angle D = 90^\circ$. Let M and N be the points of tangency of the incircle of $ABCD$ to AB and BC respectively. Let ω be the circle centered at C and tangent to AB and AD. Construct another kite $AB'C'D'$ that is similar to $ABCD$ and whose incircle is ω. Let N' be the point of tangency of $B'C'$ to ω. If $MN' \parallel AC$, then what is the ratio of $AB : BC$?