HMMT February 2022

February 19, 2022

Team Round

1. **[20]** Let $(a_1, a_2, \ldots, a_8)$ be a permutation of $(1, 2, \ldots, 8)$. Find, with proof, the maximum possible number of elements of the set

$$\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_8\}$$

that can be perfect squares.

Proposed by: Akash Das

Answer: $\boxed{5}$

Solution: We claim the maximum is 5, achieved by the sequence $(1, 3, 5, 7, 2, 4, 6, 8)$. Now we prove that we cannot do better.

Since $a_1 + a_2 + \ldots + a_8 = 1 + 2 + \ldots + 8 = 36$, then there are at most 6 squares in

$$\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_8\}.$$

Note that if $a_1 + \cdots + a_k = n^2$ and $a_1 + \cdots + a_j = (n+1)^2$, then $a_{k+1} + \ldots + a_j = 2n+1$. Since $2n+1$ is odd, a_m must be odd for some $m \in [k+1, j]$.

Thus, if all of $1, 4, 9, 16, 25$, and 36 are in

$$\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_8\},$$

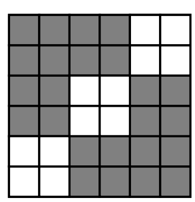
then $a_1 = 1$, and there are five more odd values in $\{a_1, a_2, \ldots, a_8\}$, which is a contradiction because there are only four odd numbers in $\{1, 2, \ldots, 8\}$.

2. **[25]** Find, with proof, the maximum positive integer k for which it is possible to color $6k$ cells of 6×6 grid such that, for any choice of three distinct rows R_1, R_2, R_3 and three distinct columns C_1, C_2, C_3, there exists an uncolored cell c and integers $1 \le i, j \le 3$ so that c lies in R_i and C_j.

Proposed by: Saba Lepsveridze

Answer: $\boxed{4}$

Solution: The answer is $k = 4$. This can be obtained with the following construction:



It now suffices to show that $k = 5$ and $k = 6$ are not attainable. The case $k = 6$ is clear. Assume for sake of contradiction that the $k = 5$ is attainable. Let r_1, r_2, r_3 be the rows of three distinct uncolored cells, and let c_1, c_2, c_3 be the columns of the other three uncolored cells. Then we can choose R_1, R_2, R_3 from $\{1, 2, 3, 4, 5, 6\} \setminus \{r_1, r_2, r_3\}$ and C_1, C_2, C_3 from $\{1, 2, 3, 4, 5, 6\} \setminus \{c_1, c_2, c_3\}$ to obtain a contradiction.

3. **[25]** Let triangle ABC be an acute triangle with circumcircle Γ. Let X and Y be the midpoints of minor arcs $\widehat{AB}$ and $\widehat{AC}$ of Γ, respectively. If line XY is tangent to the incircle of triangle ABC and the radius of Γ is R, find, with proof, the value of XY in terms of R.

   Proposed by: Akash Das

   Answer: $\boxed{\sqrt{3}R}$

   Solution: Note that X and Y are the centers of circles (AIB) and (AIC), respectively, so we have XY perpendicularly bisects AI, where I is the incenter. Since XY is tangent to the incircle, we have AI has length twice the inradius. Thus, we get $\angle A = 60°$. Thus, since $\widehat{XY} = \frac{\widehat{BAC}}{2}$, we have $\widehat{XY}$ is a $120°$ arc. Thus, we have $XY = R\sqrt{3}$.

4. **[30]** Suppose $n \geq 3$ is a positive integer. Let $a_1 < a_2 < \cdots < a_n$ be an increasing sequence of positive real numbers, and let $a_{n+1} = a_1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{a_{k+1}} > \sum_{k=1}^{n} \frac{a_{k+1}}{a_k}.$$

   Proposed by: Akash Das

   Solution 1: We will use induction. The base case is $n = 3$. In this case, we want to show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_1} > \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_1}{a_3}.$$

   Equivalently, we want to show

$$\begin{aligned} a_1^2a_3 + a_2^2a_1 + a_3^2a_2 > a_1^2a_2 + a_2^2a_3 + a_3^2a_1 &\iff a_1^2(a_3 - a_2) + a_3^2(a_2 - a_1) > a_2^2(a_3 - a_1) \\ &\iff (a_3^2 - a_2^2)(a_2 - a_1) > (a_2^2 - a_1^2)(a_3 - a_2) \\ &\iff a_3 + a_2 > a_2 + a_1, \end{aligned}$$

   which is true.

   Now assume the claim is true for $n \geq 3$. Then, we have that

$$\frac{a_1}{a_3} + \frac{a_3}{a_4} + \cdots + \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} > \frac{a_3}{a_1} + \frac{a_4}{a_3} + \cdots + \frac{a_{n+1}}{a_n} + \frac{a_1}{a_{n+1}}.$$

   We also have that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_1} > \frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_3}{a_1}.$$

   Adding the two inequalities and simplifying gives the desired result.

   Solution 2: The points $(a_1, \frac{1}{a_1}), (a_2, \frac{1}{a_2}), (a_3, \frac{1}{a_3}), \ldots, (a_n, \frac{1}{a_n})$ form a counter-clockwise oriented polygon. Thus, we have the area, A, which must be positive, can be calculated by Shoelace theorem:

$$A = \frac{1}{2}\left(\sum_{k=1}^{n} \frac{a_k}{a_{k+1}} - \sum_{k=1}^{n} \frac{a_{k+1}}{a_k}\right).$$

   Since A is positive, we are done.

   Solution 3: For $1 \leq i \leq n-1$, let $r_i = a_{i+1}/a_i$. Then the inequality becomes

$$r_1r_2\cdots r_{n-1} + \frac{1}{r_1} + \cdots + \frac{1}{r_{n-1}} > \frac{1}{r_1r_2\cdots r_{n-1}} + r_1 + \cdots + r_{n-1}.$$

If we let $s_i = \log r_i$ and $f(s) = e^s - e^{-s}$, this is the same as

$$f(s_1 + \cdots + s_{n-1}) > f(s_1) + \cdots + f(s_{n-1}).$$

This follows from the convexity of f and the fact that $f(0) = 0$.

5. **[40]** Let ABC be a triangle with centroid G, and let E and F be points on side BC such that $BE = EF = FC$. Points X and Y lie on lines AB and AC, respectively, so that X, Y, and G are not collinear. If the line through E parallel to XG and the line through F parallel to YG intersect at $P \neq G$, prove that GP passes through the midpoint of XY.

   Proposed by: Eric Shen

   Solution: Let CG intersect AB at N. Then N is the midpoint of AB and it is known that $\frac{CG}{AB} = 2 = \frac{CE}{EB}$, so $EG \parallel AB$. Moreover, since $FE = EB$, we have $[EFG] = [EXG]$. Similarly, $[EFG] = [FYG]$. Now we have $[PXG] = [EXG] = [EFG] = [FYG] = [PYG]$, so PG bisects XY, as desired.

6. **[45]** Let $P(x) = x^4 + ax^3 + bx^2 + x$ be a polynomial with four distinct roots that lie on a circle in the complex plane. Prove that $ab \neq 9$.

   Proposed by: Akash Das

   Answer:

   Solution: If either $a = 0$ the problem statement is clearly true. Thus, assume that $a \neq 0$. Let the roots be $0, z_1, z_2, z_3$, and let the circle through these points be C. Note that we have

$$\frac{3}{z_1 + z_2 + z_3} = -\frac{3}{a},$$
$$\frac{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}}{3} = -\frac{b}{3}.$$

   Note that the map $z \to \frac{1}{z}$ maps C to some line L. Thus, the second equation represents the average of three points on L, which must be a point on L, while the second equation represents the reciprocal of the centroid of z_1, z_2, z_3. Since this centroid doesn't lie on C, we must have its reciprocal doesn't lie on L. Thus, we have

$$-\frac{3}{a} \neq -\frac{b}{3} \implies ab \neq 9.$$

7. **[50]** Find, with proof, all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that

$$f(x)^2 - f(y)f(z) = x(x + y + z)(f(x) + f(y) + f(z))$$

   for all real x, y, z such that $xyz = 1$.

   Proposed by: Akash Das

   Answer: $\boxed{f(x) = 0 \text{ or } f(x) = x^2 - \frac{1}{x}.}$

   Solution 1: The answer is either $f(x) = 0$ for all x or $f(x) = x^2 - \frac{1}{x}$ for all x. These can be checked to work.

   Now, I will prove that these are the only solutions. Let $P(x, y, z)$ be the assertion of the problem statement.

   Lemma 1. $f(x) \in \{0, x^2 - \frac{1}{x}\}$ for all $x \in \mathbb{R} \setminus \{0\}$.

Proof. $P(1,1,1)$ yields $f(1)=0$. Then, $P(x,1,\frac{1}{x})$ and $P(1,x,\frac{1}{x})$ yield

$$f(x)^2 = x(x+\frac{1}{x}+1)(f(x)+f(\frac{1}{x})),$$
$$-f(x)f(\frac{1}{x}) = (x+\frac{1}{x}+1)(f(x)+f(\frac{1}{x})).$$

Thus, we have $f(x)^2 = -xf(x)f(\frac{1}{x})$, so we have $f(x)=0$ or $f(\frac{1}{x}) = -\frac{f(x)}{x}$. Plugging in the latter into the first equation above gives us

$$f(x)^2 = x(x+\frac{1}{x}+1)(f(x)-\frac{f(x)}{x}),$$

which gives us $f(x)=0$ or $f(x)=x^2-\frac{1}{x}$. This proves Lemma 1.

Lemma 2. If $f(t)=0$ for some $t\neq 1$, then we have $f(x)=0$ for all x.

Proof. $P(x,t,\frac{1}{tx})$ and $P(t,x,\frac{1}{tx})$ give us

$$f(x)^2 = x(x+\frac{1}{tx}+t)(f(x)+f(\frac{1}{tx})),$$
$$-f(x)f(\frac{1}{tx}) = t(x+\frac{1}{tx}+t)(f(x)+f(\frac{1}{tx})).$$

Thus we have $tf(x)^2 = -xf(x)f(\frac{1}{tx})$, so $f(x)=0$ or $f(\frac{1}{tx}) = -\frac{t}{x}f(x)$. Plugging in the latter into the first equation gives us

$$f(x)^2 = x(x+\frac{1}{tx}+t)(f(x)-\frac{tf(x)}{x}),$$

which gives us either $f(x)=0$ or $f(x) = x(x+t+\frac{1}{tx})(1-\frac{t}{x}) = x^2-\frac{1}{x}-(t^2-\frac{1}{t})$. Note that since the ladder expression doesn't equal $x^2-\frac{1}{x}$, since $t\neq 1$, we must have that $f(x)=0$. Thus, we have proved lemma 2.

Combining these lemmas finishes the problem.

Solution 2: Suppose $xyz=1$ and $x+y+z\neq 0$ and that x,y,z are not all the same. Then we have

$$f(x)^2 - f(y)f(z) = x(x+y+z)(f(x)+f(y)+f(z)),$$
$$f(y)^2 - f(z)f(x) = y(x+y+z)(f(x)+f(y)+f(z)),$$
$$f(z)^2 - f(x)f(y) = z(x+y+z)(f(x)+f(y)+f(z)).$$

Squaring the first equation and subtracting the second equation times the third gives us: $f(x)F(x,y,z) = (x^2-yz)G(x,y,z)^2$, where $F(x,y,z) = f(x)^3+f(y)^3+f(z)^3-3f(x)f(y)f(z)$ and $G(x,y,z) = (x+y+z)(f(x)+f(y)+f(z))$. If $F(x,y,z)=0$, it is not too hard to see that we get $f(x)=f(y)=f(z)=0$. If not, then we can let $K=\frac{G^2}{F}$ and we substitute $(f(x),f(y),f(z)) = (K(x^2-yz),K(y^2-xz),K(z^2-xy))$ into the first equation to get $K^2x(x^3+y^3+z^3-3xyz) = Kx(x^3+y^3+z^3-3xyz)$. Thus, we have $K=0$ or $K=1$. Thus, we have either $(f(x),f(y),f(z)) = (0,0,0)$ or $(f(x),f(y),f(z)) = (x^2-yz,y^2-zx,z^2-xy)$.

Thus, $f(0.5)=0$ or $f(0.5)=0.5^2-\frac{1}{0.5}$. If the former is true, then for all y and z such that $yz=2$ and $y+z\neq 0.5$, we have $f(y)=0$. However, this gives that $f(y)=0$ for all y. Likewise, if the latter were true, we would have $f(y)=y^2-\frac{1}{y}$ for all y, so we are done.

8. **[50]** Let $P_1P_2\cdots P_n$ be a regular n-gon in the plane and $a_1,\ldots,a_n$ be nonnegative integers. It is possible to draw m circles so that for each $1\le i\le n$, there are exactly a_i circles that contain P_i on their interior. Find, with proof, the minimum possible value of m in terms of the a_i.

Proposed by: Daniel Zhu

Answer: $\boxed{\max\left(a_1,\ldots,a_n,\frac{1}{2}\sum_{i=1}^{n}|a_i-a_{i+1}|\right)}$

Solution: For convenience, we take all indices modulo n. Let $[n]$ be the set $\{1,2,\ldots,n\}$. Also, let $M=\max(a_1,\ldots,a_n)$, $d=\frac{1}{2}\sum_i|a_i-a_{i+1}|$, and $M'=\max(M,d)$. We claim that M' is the answer.

Let Ω be the circumcircle of the polygon.

First let's prove that $m\geq M'$. Obviously $m\geq M$. Also, there must be at least $|a_i-a_{i+1}|$ circles crossing Ω between P_i and P_{i+1}, and a circle can cross Ω at most twice. Thus $m\geq d$.

We will present two ways to arrive at a construction.

Inductive construction. We use induction on $\sum_i a_i$. If all the a_i are zero, then the problem is trivial. Now assume that not all the a_i are zero the idea is that we are going to subtract 1 from a consecutive subset of the a_i so that the value of M' goes down by 1.

There are two cases. First of all, if $a_i=0$ for some i, then we can choose such an i so that $a_{i+1}>0$. Then, let j be the minimal positive integer so that $a_{i+j}=0$. Then subtract 1 from $a_{i+1},\ldots,a_{i+j-1}$. It is clear that d decreases by 1. If $a_{i+j}=a_{i+j+1}=\cdots=a_i=0$, then M also goes down by 1. If not, then $M<d$, so M' goes down by 1 anyway.

The second case is when $a_i>0$ for all i. If all the a_i are the same then we are done by subtracting 1 from everything. If not, we can find i,j with $j>i+1$ so that $a_i=M$, $a_j=M$, and $a_{i+1},a_{i+2},\ldots,a_{j-1}<M$. Then subtract 1 from the complement of $a_j,a_{j+1},\ldots,a_{i-1}$. Then M goes down by 1 and d goes down by 1.

Non-inductive construction. We will prove that if $M\leq d$, then we may choose $m=d$. If $M>d$, then since $d\geq M-\min(a_1,\ldots,a_n)$ we can subtract $M-d$ from every a_i, draw $M-d$ circles containing every point, and apply the below construction.

Let $a_i'=a_i-\min_j(a_j)$, $A_h=\{i\mid a_i'<h,a_{i+1}'\geq h\}$, $B_h=\{i\mid a_i'\geq h,a_{i+1}'<h\}$. Also, let $s_h=|A_h|=|B_h|$. Note that $d=\sum_h s_h$ and that $s_h>0\iff h\leq\max(a_i')$.

For $h\leq\max(a_i')$ and $1\leq j\leq s_h$, define an arrangement of circles $C_h^{(j)}$ as follows: let the elements of A_h and B_h be $a_1,b_1,a_2,b_2,\ldots$ in order. Then for each $i\leq s_h$ add a circle covering the points in the interval $(a_i,b_{i+j}]$. One can show that point P_i is covered by circles j times if $a_i'\geq h$ and $j-1$ times otherwise.

Now, for some choice of j_h for all h, consider taking $\bigcup_h C_h^{(j_h)}$. Then, P_i is covered by circles $\sum_h j_h+a_i-M$ times. If we choose the j_h so that $\sum_h j_h=M$, which can be shown to be possible, we are done.

9. **[55]** Let Γ_1 and Γ_2 be two circles externally tangent to each other at N that are both internally tangent to Γ at points U and V, respectively. A common external tangent of Γ_1 and Γ_2 is tangent to Γ_1 and Γ_2 at P and Q, respectively, and intersects Γ at points X and Y. Let M be the midpoint of the arc $\widehat{XY}$ that does not contain U and V. Let Z be on Γ such $MZ\perp NZ$, and suppose the circumcircles of QVZ and PUZ intersect at $T\neq Z$. Find, with proof, the value of $TU+TV$, in terms of R, r_1, and r_2, the radii of Γ, Γ_1, and Γ_2, respectively.

Proposed by: Akash Das

Answer: $\boxed{\frac{(Rr_1+Rr_2-2r_1r_2)2\sqrt{r_1r_2}}{|r_1-r_2|\sqrt{(R-r_1)(R-r_2)}}}$

Solution: By Archimedes lemma, we have M,Q,V are collinear and M,P,U are collinear as well. Note that inversion at M with radius MX shows that $PQUV$ is cyclic. Thus, we have $MP\cdot MU=MQ\cdot MV$, so M lies on the radical axis of (PUZ) and (QVZ), thus T must lie on the line MZ. Thus, we have $MZ\cdot MT=MQ\cdot MV=MN^2$, which implies triangles MZN and MNT are similar. Thus, we have $NT\perp MN$. However, since the line through O_1 and O_2 passes through N and is perpendicular to MN, we have T lies on line O_1O_2. Additionally, since $MZ\cdot MT=MN^2=MX^2$, inversion at M with radius MX swaps Z and T, and since (MXY) maps to line XY, this means T also lies on XY.

Therefore, T is the intersection of PQ and O_1O_2, and thus by Monge's Theorem, we must have T lies on UV.

Now, to finish, we will consider triangle OUV. Since O_1O_2T is a line that cuts this triangle, by Menelaus, we have

$$\frac{OO_1}{O_1U}\cdot\frac{UT}{VT}\cdot\frac{VO_2}{O_2O}=1.$$

Using the values of the radii, this simplifies to

$$\frac{R-r_1}{r_1}\cdot\frac{UT}{VT}\cdot\frac{r_2}{R-r_2}=1 \implies \frac{UT}{VT}=\frac{r_1(R-r_2)}{r_2(R-r_1)}.$$

Now, note that

$$TU\cdot TV=TP\cdot TQ=\frac{4r_1^2r_2^2}{(r_1-r_2)^2}.$$

Now, let $TU=r_1(R-r_2)k$ and $TU=r_2(R-r_1)k$. Plugging this into the above equation gives

$$r_1r_2(R-r_1)(R-r_2)k^2=\frac{4(r_1r_2)^2}{(r_1-r_2)^2}.$$

Solving gives

$$k=\frac{2\sqrt{r_1r_2}}{|r_1-r_2|\sqrt{(R-r_1)(R-r_2)}}.$$

To finish, note that

$$TU+TV=k(Rr_1+Rr_2-2r_1r_2)=\frac{2(Rr_1+Rr_2-2r_1r_2)\sqrt{r_1r_2}}{|r_1-r_2|\sqrt{(R-r_1)(R-r_2)}}.$$

10. **[60]** On a board the following six vectors are written:

$$(1,0,0),\quad(-1,0,0),\quad(0,1,0),\quad(0,-1,0),\quad(0,0,1),\quad(0,0,-1).$$

Given two vectors v and w on the board, a move consists of erasing v and w and replacing them with $\frac{1}{\sqrt{2}}(v+w)$ and $\frac{1}{\sqrt{2}}(v-w)$. After some number of moves, the sum of the six vectors on the board is u. Find, with proof, the maximum possible length of u.

Proposed by: Daniel Zhu

Answer: $\boxed{2\sqrt{3}}$

Solution: For a construction, note that one can change

$$(1,0,0),(-1,0,0)\to(\sqrt{2},0,0),(0,0,0)\to(1,0,0),(1,0,0)$$

and similarly for $(0,1,0),(0,-1,0)$ and $(0,0,1),(0,0,-1)$. Then $u=(2,2,2)$.

For the bound, argue as follows: let the vectors be $v_1,\ldots,v_6$, $n=(x,y,z)$ be any unit vector, and $S=\sum_i(n\cdot v_i)^2$, where the sum is over all vectors on the board. We claim that S is invariant. Indeed, we have

$$\begin{aligned}\left(n\cdot\frac{1}{\sqrt{2}}(v+w)\right)^2+\left(n\cdot\frac{1}{\sqrt{2}}(v-w)\right)^2&=\left(\frac{n\cdot v+n\cdot w}{\sqrt{2}}\right)^2+\left(\frac{n\cdot v-n\cdot w}{\sqrt{2}}\right)^2\\&=\frac{2(n\cdot v)^2+2(n\cdot w)^2}{2}\\&=(n\cdot v)^2+(n\cdot w)^2.\end{aligned}$$

Also, at the beginning we have $S = 2x^2 + 2y^2 + 2z^2 = 2$. Therefore we must always have $S = 2$. Thus, by the Cauchy-Schwarz inequality we have

$$n \cdot u = \sum n \cdot v_i \leq \sqrt{\sum_i (n \cdot v_i)^2}\sqrt{6} = \sqrt{12} = 2\sqrt{3}.$$

But since n is arbitrary, this implies that $|u| \leq 2\sqrt{3}$; otherwise we could pick $n = u/|u|$ and reach a contradiction.