HMMO 2020

November 14–21, 2020

Theme Round

1. Chelsea goes to La Verde's at MIT and buys 100 coconuts, each weighing 4 pounds, and 100 honeydews, each weighing 5 pounds. She wants to distribute them among n bags, so that each bag contains at most 13 pounds of fruit. What is the minimum n for which this is possible?

   Proposed by: Daniel Zhu

   Answer: $\boxed{75}$

   Solution: The answer is $n = 75$, given by 50 bags containing one honeydew and two coconuts (13 pounds), and 25 bags containing two honeydews (10 pounds).

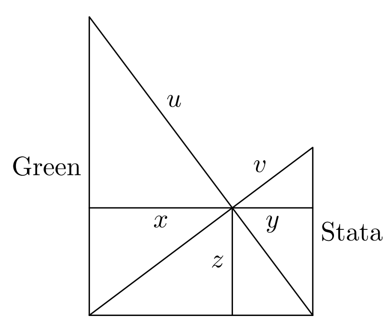
   To show that this is optimal, assign each coconut 1 point and each honeydew 2 points, so that 300 points worth of fruit are bought in total. Then, we claim that each bag can contain at most 4 points of fruit, thus requiring $n \geq 300/4 = 75$. To see this, note that each bag containing greater than 4 points must contain either five coconuts (20 pounds), three coconuts and a honeydew (17 pounds), one coconut and two honeydews (14 pounds), or three honeydews (15 pounds).

2. In the future, MIT has attracted so many students that its buildings have become skyscrapers. Ben and Jerry decide to go ziplining together. Ben starts at the top of the Green Building, and ziplines to the bottom of the Stata Center. After waiting a seconds, Jerry starts at the top of the Stata Center, and ziplines to the bottom of the Green Building. The Green Building is 160 meters tall, the Stata Center is 90 meters tall, and the two buildings are 120 meters apart. Furthermore, both zipline at 10 meters per second. Given that Ben and Jerry meet at the point where the two ziplines cross, compute $100a$.

   Proposed by: Esha Bhatia

   Answer: $\boxed{740}$

   Solution: Define the following lengths:



   Note that due to all the 3-4-5 triangles, we find $\frac{x}{z} = \frac{z}{y} = \frac{4}{3}$, so $120 = x + y = \frac{25}{12}z$. Then,

$$u = \frac{5}{3}x = \frac{20}{9}z = \frac{16}{15}120 = 128,$$

   while

$$v = \frac{5}{4}y = \frac{15}{16}z = \frac{9}{20}120 = 54.$$

   Thus $u - v = 74$, implying that $a = 7.4$.

3. Harvard has recently built a new house for its students consisting of n levels, where the kth level from the top can be modeled as a 1-meter-tall cylinder with radius k meters. Given that the area of all the lateral surfaces (i.e. the surfaces of the external vertical walls) of the building is 35 percent of the total surface area of the building (including the bottom), compute n.

Proposed by: Daniel Zhu

Answer: $\boxed{13}$

Solution: The kth layer contributes a lateral surface area of $2k\pi$, so the total lateral surface area is

$$2(1+2+\cdots+n)\pi = n(n+1)\pi.$$

On the other hand, the vertical surface area is $2n^2\pi$ (No need to sum layers, just look at the building from above and from below). Therefore,

$$n+1 = \frac{7}{20}(3n+1),$$

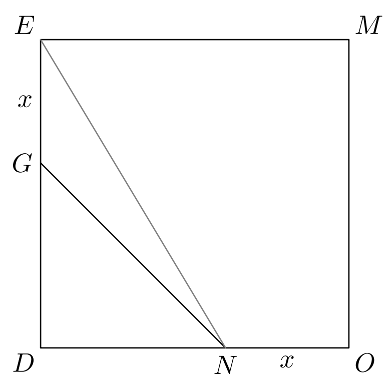
and $n = 13$.

4. Points G and N are chosen on the interiors of sides ED and DO of unit square $DOME$, so that pentagon $GNOME$ has only two distinct side lengths. The sum of all possible areas of quadrilateral $NOME$ can be expressed as $\frac{a-b\sqrt{c}}{d}$, where a, b, c, d are positive integers such that $\gcd(a,b,d) = 1$ and c is square-free (i.e. no perfect square greater than 1 divides c). Compute $1000a + 100b + 10c + d$.

Proposed by: Andrew Lin

Answer: $\boxed{10324}$

Solution:



Since $MO = ME = 1$, but ON and GE are both less than 1, we must have either $ON = NG = GE = x$ (call this case 1) or $ON = GE = x, NG = 1$ (call this case 2).

Either way, the area of $NOME$ (a trapezoid) is $\frac{1+x}{2}$, and triangle NGT is a 45-45-90 triangle. In case 1, we have $1 = ON + NT = x\left(1 + \frac{\sqrt{2}}{2}\right)$, so $x = 2 - \sqrt{2}$ and the area of the trapezoid is $\frac{3-\sqrt{2}}{2}$. In case 2, we have $1 = ON + NT = x + \frac{\sqrt{2}}{2}$, which yields an area of $\frac{4-\sqrt{2}}{4}$ as $x = \frac{2-\sqrt{2}}{2}$. The sum of these two answers is $\frac{10-3\sqrt{2}}{4}$.

5. The classrooms at MIT are each identified with a positive integer (with no leading zeroes). One day, as President Reif walks down the Infinite Corridor, he notices that a digit zero on a room sign has fallen off. Let N be the original number of the room, and let M be the room number as shown on the sign.

The smallest interval containing all possible values of $\frac{M}{N}$ can be expressed as $[\frac{a}{b}, \frac{c}{d})$ where a, b, c, d are positive integers with $\gcd(a, b) = \gcd(c, d) = 1$. Compute $1000a + 100b + 10c + d$.

Proposed by: Andrew Lin

Answer: $\boxed{2031}$

Solution: Let A represent the portion of N to the right of the deleted zero, and B represent the rest of N. For example, if the unique zero in $N = 12034$ is removed, then $A = 34$ and $B = 12000$. Then, $\frac{M}{N} = \frac{A+B/10}{A+B} = 1 - \frac{9}{10}\frac{B}{N}$.

The maximum value for B/N is 1, which is achieved when $A = 0$. Also, if the 0 removed is in the 10^k's place ($k = 2$ in the example above), we find that $A < 10^k$ and $B \geq 10^{k+1}$, meaning that $A/B < 1/10$ and thus $B/N > 10/11$. Also, B/N can get arbitrarily close to $10/11$ via a number like $1099\ldots9$.

Therefore the fraction $\frac{M}{N}$ achieves a minimum at $\frac{1}{10}$ and always stays below $\frac{2}{11}$, though it can get arbitrarily close. The desired interval is then $[\frac{1}{10}, \frac{2}{11})$.

6. The elevator buttons in Harvard's Science Center form a 3×2 grid of identical buttons, and each button lights up when pressed. One day, a student is in the elevator when all the other lights in the elevator malfunction, so that only the buttons which are lit can be seen, but one cannot see which floors they correspond to. Given that at least one of the buttons is lit, how many distinct arrangements can the student observe? (For example, if only one button is lit, then the student will observe the same arrangement regardless of which button it is.)

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{44}$

Solution 1: We first note that there are $2^6 - 1 = 63$ possibilities for lights in total. We now count the number of duplicates we need to subtract by casework on the number of buttons lit. To do this, we do casework on the size of the minimal "bounding box" of the lights:

- If the bounding box is 1×1, the only arrangement up to translation is a solitary light, which can be translated 6 ways. This means we must subtract 5.
- If the bounding box is 2×1, there is 1 arrangement and 4 translations, so we must subtract 3.
- If the bounding box is 1×2, there is 1 arrangement and 3 translations, so we must subtract 2.
- If the bounding box is 3×1, there are 2 arrangements and 2 translations, so we must subtract 2.
- If the bounding box is 2×2, there are 2 arrangements with 2 lights, 4 with 3 lights, and 1 with 4 lights—7 in total. Since there are two translations, we must subtract 7.

The final answer is $63 - 5 - 3 - 2 - 2 - 7 = 44$.

Solution 2: We may also count duplicates by doing casework on buttons lit:

- 1 buttons lit: There are 6 arrangements but all are the same, so we need to subtract 5 duplicates in this case.
- 2 buttons lit: There are 4 indistinguishable ways for the buttons to be vertically adjacent, 3 to be horizontally adjacent, 2 ways for the buttons to be diagonally adjacent for each of 2 directions of diagonals, and 2 for when the lights are in the same vertical line but not adjacent. Since we need to count each of these cases only once, the number of duplicates we need to subtract is 3 (2 vertically adjacent), 2 (2 horizontally adjacent), 2×1 (2 diagonally adjacent), and 1 (2 in same vertical line but not adjacent) for a total of 8 duplicates.
- 3 buttons lit: There are 2 indistinguishable ways for all the buttons in a column to be lit and 2 ways for the buttons to be lit in the shape of an L, given the rotation of the L. Thus, the number of duplicates we need to subtract is 1 (1 column), 1×4 (rotations of L), for a total of 5 duplicates.

- 4 buttons lit: There are 2 indistinguishable ways for the lights to be arranged in a square (and no other duplicates), so we need to subtract 1 duplicate in this case.
- When there are 5 or 6 buttons lit, all of the arrangements of lights are distinct, so we do not subtract any duplicates for these cases.

Thus, the total number of arrangements is $64 - (1 + 5 + 8 + 5 + 1) = 44$.

7. While waiting for their food at a restaurant in Harvard Square, Ana and Banana draw 3 squares $\square_1, \square_2, \square_3$ on one of their napkins. Starting with Ana, they take turns filling in the squares with integers from the set $\{1, 2, 3, 4, 5\}$ such that no integer is used more than once. Ana's goal is to minimize the minimum value M that the polynomial $a_1x^2 + a_2x + a_3$ attains over all real x, where a_1, a_2, a_3 are the integers written in $\square_1, \square_2, \square_3$ respectively. Banana aims to maximize M. Assuming both play optimally, compute the final value of $100a_1 + 10a_2 + a_3$.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{451}$

Solution: Relabel a_1, a_2, a_3 as a, b, c. This is minimized at $x = \frac{-b}{2a}$, so $M = c - \frac{b^2}{4a}$.

If in the end $a = 5$ or $b \in \{1, 2\}$, then $\frac{b^2}{4a} \leq 1$ and $M \geq 0$. The only way for Ana to block this is to set $b = 5$, which will be optimal if we show that it allows Ana to force $M < 0$, which we will now do. At this point, Banana has two choices:

- If Banana fixes a value of a, Ana's best move is to pick $c = 1$, or $c = 2$ if it has not already been used. The latter case yields $M < -1$, while the optimal move in the latter case ($a = 4$) yields $M = 1 - \frac{25}{16} > -1$.
- If Banana fixes a value of c, then if that a value is not 1 Ana can put $a = 1$, yielding $M \leq 4 - \frac{25}{4} < -1$. On the other hand, if Banana fixes $c = 1$ then Ana's best move is to put $a = 2$, yielding $M = 1 - \frac{25}{8} < -1$.

Thus Banana's best move is to set $a = 4$, eliciting a response of $c = 1$. Since $1 - \frac{25}{16} < 0$, this validates our earlier claim that $b = 5$ was the best first move.

8. After viewing the John Harvard statue, a group of tourists decides to estimate the distances of nearby locations on a map by drawing a circle, centered at the statue, of radius $\sqrt{n}$ inches for each integer $2020 \leq n \leq 10000$, so that they draw 7981 circles altogether. Given that, on the map, the Johnston Gate is 10-inch line segment which is entirely contained between the smallest and the largest circles, what is the minimum number of points on this line segment which lie on one of the drawn circles? (The endpoint of a segment is considered to be on the segment.)

Proposed by: Daniel Zhu

Answer: $\boxed{49}$

Solution: Consider a coordinate system on any line ℓ where 0 is placed at the foot from $(0, 0)$ to ℓ. Then, by the Pythagorean theorem, a point (x, y) on ℓ is assigned a coordinate u for which $x^2 + y^2 = u^2 + a$ for some fixed a (dependent only on ℓ). Consider this assignment of coordinates for our segment.

First, suppose that along the line segment u never changes sign; without loss of generality, assume it is positive. Then, if u_0 is the minimum value of u, the length of the interval covered by u^2 is $(u_0 + 10)^2 - u_0^2 = 100 + 20u_0 \geq 100$, meaning that at least 100 points lie on the given circles.

Now suppose that u is positive on a length of k and negative on a length of $10 - k$. Then, it must intersect the circles at least $\lfloor k^2 \rfloor + \lfloor (10 - k)^2 \rfloor$ points, which can be achieved for any k by setting $a = 2020 + \varepsilon$ for very small ε.

To minimize this quantity note that $k^2+(10-k)^2 \geq 50$, so $\lfloor k^2 \rfloor + \lfloor (10-k)^2 \rfloor > k^2+(10-k)^2-2 \geq 48$, proving the bound. For a construction, set $k = 4.99999$.

9. While waiting for their next class on Killian Court, Alesha and Belinda both write the same sequence S on a piece of paper, where S is a 2020-term strictly increasing geometric sequence with an integer common ratio r. Every second, Alesha erases the two smallest terms on her paper and replaces them with their geometric mean, while Belinda erases the two largest terms in her paper and replaces them with their geometric mean. They continue this process until Alesha is left with a single value A and Belinda is left with a single value B. Let r_0 be the minimal value of r such that $\frac{A}{B}$ is an integer. If d is the number of positive factors of r_0, what is the closest integer to $\log_2 d$?

   Proposed by: Hahn Lheem

   Answer: $\boxed{2018}$

   Solution: Because we only care about when the ratio of A to B is an integer, the value of the first term in S does not matter. Let the initial term in S be 1. Then, we can write S as $1, r, r^2, \ldots, r^{2019}$. Because all terms are in terms of r, we can write $A = r^a$ and $B = r^b$. We will now solve for a and b.

   Observe that the geometric mean of two terms r^m and r^n is simply $r^{\frac{m+n}{2}}$, or r raised to the arithmetic mean of m and n. Thus, to solve for a, we can simply consider the sequence $0, 1, 2, \ldots, 2019$, which comes from the exponents of the terms in S, and repeatedly replace the smallest two terms with their arithmetic mean. Likewise, to solve for b, we can consider the same sequence $0, 1, 2, \ldots, 2019$ and repeatedly replace the largest two terms with their arithmetic mean.

   We begin by computing a. If we start with the sequence $0, 1, \ldots, 2019$ and repeatedly take the arithmetic mean of the two smallest terms, the final value will be

$$a = \frac{\frac{\frac{\frac{0+1}{2}+2}{2}+3}{2} + \cdots + 2019}{2} = \sum_{k=1}^{2019} \frac{k}{2^{2020-k}}.$$

   Then, we can compute

$$\begin{aligned} 2a &= \sum_{k=1}^{2019} \frac{k}{2^{2019-k}} \\ \implies a = 2a - a &= \sum_{k=1}^{2019} \frac{k}{2^{2019-k}} - \sum_{k=1}^{2019} \frac{k}{2^{2020-k}} \\ &= \sum_{k=1}^{2019} \frac{k}{2^{2019-k}} - \sum_{k=0}^{2018} \frac{k+1}{2^{2019-k}} \\ &= 2019 - \sum_{j=1}^{2019} \frac{1}{2^j} \\ &= 2019 - \left(1 - \frac{1}{2^{2019}}\right) = 2018 + \frac{1}{2^{2019}}. \end{aligned}$$

   Likewise, or by symmetry, we can find $b = 1 - \frac{1}{2^{2019}}$.

   Since we want $\frac{A}{B} = \frac{r^a}{r^b} = r^{a-b}$ to be a positive integer, and $a - b = \left(2018 + \frac{1}{2^{2019}}\right) - \left(1 - \frac{1}{2^{2019}}\right) = 2017 + \frac{1}{2^{2018}}$, r must be a perfect $(2^{2018})^{\text{th}}$ power. Because $r > 1$, the minimal possible value is $r = 2^{2^{2018}}$. Thus, $d = 2^{2018} + 1$, and so $\log_2 d$ is clearly closest to 2018.

10. Sean enters a classroom in the Memorial Hall and sees a 1 followed by 2020 0's on the blackboard. As he is early for class, he decides to go through the digits from right to left and independently erase the nth digit from the left with probability $\frac{n-1}{n}$. (In particular, the 1 is never erased.) Compute the expected value of the number formed from the remaining digits when viewed as a base-3 number. (For example, if the remaining number on the board is 1000, then its value is 27.)

Proposed by: Vincent Bian

Answer: $\boxed{681751}$

Solution: Suppose Sean instead follows this equivalent procedure: he starts with $M = 10\ldots0$, on the board, as before. Instead of erasing digits, he starts writing a new number on the board. He goes through the digits of M one by one from left to right, and independently copies the nth digit from the left with probability $\frac{1}{n}$. Now, let a_n be the expected value of Sean's new number after he has gone through the first n digits of M. Note that the answer to this problem will be the expected value of a_{2021}, since M has 2021 digits.

Note that $a_1 = 1$, since the probability that Sean copies the first digit is 1.

For $n > 1$, note that a_n is $3a_{n-1}$ with probability $\frac{1}{n}$, and is a_{n-1} with probability $\frac{n-1}{n}$. Thus,

$$\mathbb{E}[a_n] = \frac{1}{n}\mathbb{E}[3a_{n-1}] + \frac{n-1}{n}\mathbb{E}[a_{n-1}] = \frac{n+2}{n}\mathbb{E}[a_{n-1}].$$

Therefore,

$$\mathbb{E}[a_{2021}] = \frac{4}{2} \cdot \frac{5}{3} \cdots \frac{2023}{2021} = \frac{2022 \cdot 2023}{2 \cdot 3} = 337 \cdot 2023 = 681751.$$