HMMO 2020

November 14–21, 2020

General Round

1. In the Cartesian plane, a line segment with midpoint $(2020, 11)$ has one endpoint at $(a, 0)$ and the other endpoint on the line $y = x$. Compute a.

   Proposed by: Lingyi Qiu

   Answer: $\boxed{4018}$

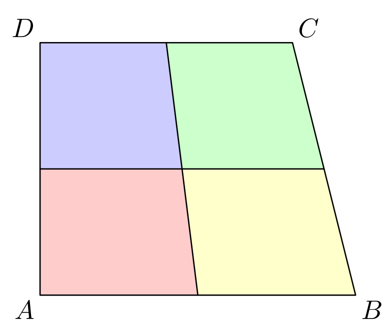
   Solution: Let the other endpoint be (t, t). The midpoint of $(a, 0)$ and (t, t) is $\left(\frac{a+t}{2}, \frac{t}{2}\right)$. So, we know that $\frac{a+t}{2} = 2020$ and $\frac{t}{2} = 11$. The second equation yields $t = 22$. Substituting this into the first yields $a = 2 \cdot 2020 - 22 = 4018$.

2. Let T be a trapezoid with two right angles and side lengths 4, 4, 5, and $\sqrt{17}$. Two line segments are drawn, connecting the midpoints of opposite sides of T and dividing T into 4 regions. If the difference between the areas of the largest and smallest of these regions is d, compute $240d$.

   Proposed by: Shengtong Zhang

   Answer: $\boxed{120}$

   Solution:



   By checking all the possibilities, one can show that T has height 4 and base lengths 4 and 5. Orient T so that the shorter base is on the top.

   Then, the length of the cut parallel to the bases is $\frac{4+5}{2} = \frac{9}{2}$. Thus, the top two pieces are trapezoids with height 2 and base lengths 2 and $\frac{9}{4}$, while the bottom two pieces are trapezoids with height 2 and base lengths $\frac{9}{4}$ and $\frac{5}{2}$. Thus, using the area formula for a trapezoid, the difference between the largest and smallest areas is

$$d = \frac{\left(\frac{5}{2} + \frac{9}{4} - \frac{9}{4} - 2\right) \cdot 2}{2} = \frac{1}{2}.$$

3. Jody has 6 distinguishable balls and 6 distinguishable sticks, all of the same length. How many ways are there to use the sticks to connect the balls so that two disjoint non-interlocking triangles are formed? Consider rotations and reflections of the same arrangement to be indistinguishable.

   Proposed by: Daniel Zhu

   Answer: $\boxed{7200}$

   Solution 1: For two disjoint triangles to be formed, three of the balls must be connected into a triangle by three of the sticks, and the three remaining balls must be connected by the three remaining sticks.

There are $\binom{6}{3}$ ways to pick the 3 balls for the first triangle. Note that once we choose the 3 balls for the first triangle, the remaining 3 balls must form the vertices of the second triangle.

Now that we have determined the vertices of each triangle, we can assign the 6 sticks to the 6 total edges in the two triangles. Because any ordering of the 6 sticks works, there are $6! = 720$ total ways to assign the sticks as edges.

Finally, because the order of the two triangles doesn't matter (i.e. our initial choice of 3 balls could have been used for the second triangle), we must divide by 2 to correct for overcounting. Hence the final answer is $\binom{6}{3} \cdot 6!/2 = 7200$.

Solution 2: First, we ignore all the symmetries in the problem. There are then 6! ways to arrange the balls and 6! ways to arrange the sticks. However, each triangle can be rotated or reflected, so we have overcounted by a factor of 6^2. Moreover, the triangles can be swapped, so we must also divide by 2. Thus the answer is

$$\frac{(6!)^2}{6^2 \cdot 2} = \frac{120^2}{2} = 7200.$$

4. Nine fair coins are flipped independently and placed in the cells of a 3 by 3 square grid. Let p be the probability that no row has all its coins showing heads and no column has all its coins showing tails. If $p = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

   Proposed by: Daniel Zhu

   Answer: $\boxed{8956}$

   Solution: Consider the probability of the complement. It is impossible for some row to have all heads and some column to have tails, since every row intersects every column. Let q be the probability that some row has all heads. By symmetry, q is also the probability that some column has all tails. We can then conclude that $p = 1 - 2q$.

   The probability that a given row does not have all heads is $\frac{7}{8}$. So, the probability that none of the three rows have all heads is $\left(\frac{7}{8}\right)^3$, implying that $q = 1 - \frac{343}{512} = \frac{169}{512}$. Thus $p = 1 - \frac{169}{256} = \frac{87}{256}$.

5. Compute the sum of all positive integers $a \leq 26$ for which there exist integers b and c such that $a + 23b + 15c - 2$ and $2a + 5b + 14c - 8$ are both multiples of 26.

   Proposed by: David Vulakh

   Answer: $\boxed{31}$

   Solution: Assume b and c exist. Considering the two values modulo 13, we find

$$\begin{cases} a + 10b + 2c \equiv 2 \pmod{13} \\ 2a + 5b + c \equiv 8 \pmod{13}. \end{cases}$$

   Subtracting twice the second equation from the first, we get $-3a \equiv -14 \pmod{13}$. So, we have $a \equiv 9 \pmod{13}$. Therefore we must either have $a = 9$ or $a = 22$.

   Moreover, both $a = 9$ and $a = 22$ yield solutions with $b = 0$ and $c = 3, 16$, depending on the value of a. Thus the answer is $9 + 22 = 31$.

6. A sphere is centered at a point with integer coordinates and passes through the three points $(2, 0, 0)$, $(0, 4, 0)$, $(0, 0, 6)$, but not the origin $(0, 0, 0)$. If r is the smallest possible radius of the sphere, compute r^2.

Proposed by: James Lin

Answer: $\boxed{51}$

Solution: Let (x, y, z) be the center of the sphere. By the given condition, we have

$$(x-2)^2 + y^2 + z^2 = x^2 + (y-4)^2 + z^2 = x^2 + y^2 + (z-6)^2.$$

Subtracting $x^2 + y^2 + z^2$ yields

$$x^2 - (x-2)^2 = y^2 - (y-4)^2 = z^2 - (z-6)^2,$$

or

$$4(x-1) = 8(y-2) = 12(z-3).$$

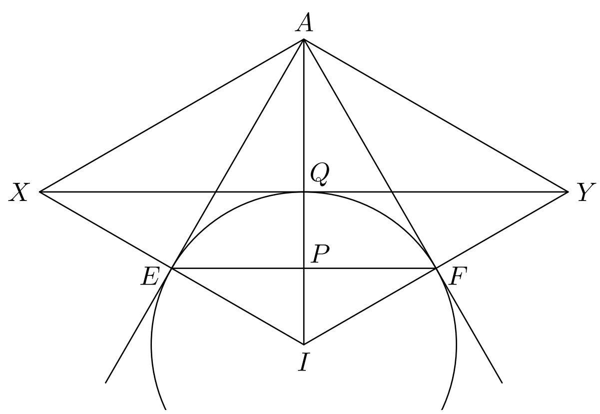
Therefore $(x-1, y-2, z-3)$ must be $(6t, 3t, 2t)$ for some integer t. Checking small values of t yields that the possibilities for (x, y, z) closest to $(2, 0, 0)$ are $(-5, -1, 1)$, $(1, 2, 3)$, and $(7, 5, 5)$. The second yields a sphere that passes through the origin and is thus forbidden. The other two yield $r^2 = 51$ and $r^2 = 75$, so 51 is the answer.

7. In triangle ABC with $AB = 8$ and $AC = 10$, the incenter I is reflected across side AB to point X and across side AC to point Y. Given that segment XY bisects AI, compute BC^2. (The incenter I is the center of the inscribed circle of triangle ABC.)

   Proposed by: Carl Schildkraut

   Answer: $\boxed{84}$

   Solution 1:



Let E, F be the tangency points of the incircle to sides AC, AB, respectively. Due to symmetry around line AI, $AXIY$ is a rhombus. Therefore

$$\angle XAI = 2\angle EAI = 2(90^\circ - \angle EIA) = 180^\circ - 2\angle XAI,$$

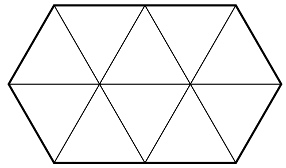
which implies that $60^\circ = \angle XAI = 2\angle EAI = \angle BAC$. By the law of cosines,

$$BC^2 = 8^2 + 10^2 - 2 \cdot 8 \cdot 10 \cdot \cos 60^\circ = 84.$$

Solution 2: Define points as above and additionally let P and Q be the intersections of AI with EF and XY, respectively. Since $IX = 2IE$ and $IY = 2IF$, $\triangle IEF \sim \triangle IXY$ with ratio 2, implying that $IP = \frac{1}{2}IQ = \frac{1}{4}IA$.

Let $\theta = \angle EAI = \angle IEP$. Then $\frac{IP}{IA} = \frac{IP}{IE}\frac{IE}{IA} = \sin^2\theta$, implying that $\sin\theta = 1/2$ and $\theta = 30^\circ$. From here, proceed as in solution 1.

8. A bar of chocolate is made of 10 distinguishable triangles as shown below:



How many ways are there to divide the bar, along the edges of the triangles, into two or more contiguous pieces?

Proposed by: Steven Noah Raphael

Answer: $\boxed{1689}$

Solution: Every way to divide the bar can be described as a nonempty set of edges to break, with the condition that every endpoint of a broken edge is either on the boundary of the bar or connects to another broken edge.

Let the center edge have endpoints X and Y. We do casework on whether the center edge is broken.

If the center edge is broken, then we just need some other edge connecting to X to be broken, and some other edge connecting to Y to be broken. We have 2^5 choices for the edges connecting to X, of which 1 fails. Similarly, we have $2^5 - 1$ valid choices for the edges connecting to Y. This yields $(2^5 - 1)^2 = 961$ possibilities.

If the center edge is not broken, then the only forbidden arrangements are those with exactly one broken edge at X or those with exactly one broken edge at Y. Looking at just the edges connecting to X, we have 5 cases with exactly one broken edge. Thus, there are $2^5 - 5 = 27$ ways to break the edges connecting to X. Similarly there are 27 valid choices for the edges connecting to Y. This yields $27^2 - 1 = 728$ cases, once we subtract the situation where no edges are broken.

The final answer is $961 + 728 = 1689$.

9. In the Cartesian plane, a perfectly reflective semicircular room is bounded by the upper half of the unit circle centered at $(0, 0)$ and the line segment from $(-1, 0)$ to $(1, 0)$. David stands at the point $(-1, 0)$ and shines a flashlight into the room at an angle of $46°$ above the horizontal. How many times does the light beam reflect off the walls before coming back to David at $(-1, 0)$ for the first time?

Proposed by: Kevin Tong

Answer: $\boxed{65}$

Solution: Note that when the beam reflects off the x-axis, we can reflect the entire room across the x-axis instead. Therefore, the number of times the beam reflects off a circular wall in our semicircular room is equal to the number of times the beam reflects off a circular wall in a room bounded by the unit circle centered at $(0, 0)$. Furthermore, the number of times the beam reflects off the x-axis wall in our semicircular room is equal to the number of times the beam crosses the x-axis in the room bounded by the unit circle. We will count each of these separately.

We first find the number of times the beam reflects off a circular wall. Note that the path of the beam is made up of a series of chords of equal length within the unit circle, each chord connecting the points from two consecutive reflections. Through simple angle chasing, we find that the angle subtended by each chord is $180 - 2 \cdot 46 = 88°$. Therefore, the nth point of reflection in the unit circle is $(-\cos(88n), \sin(88n))$. The beam returns to $(-1, 0)$ when

$$88n \equiv 0 \pmod{360} \iff 11n \equiv 0 \pmod{45} \to n = 45$$

but since we're looking for the number of time the beam is reflected before it comes back to David, we only count $45 - 1 = 44$ of these reflections.

Next, we consider the number of times the beam is reflected off the x-axis. This is simply the number of times the beam crosses the x-axis in the unit circle room before returning to David, which happens every 180° around the circle. Thus, we have $\frac{88\cdot 45}{180} - 1 = 21$ reflections off the x-axis, where we subtract 1 to remove the instance when the beam returns to $(-1, 0)$. Thus, the total number of reflections is $44 + 21 = 65$.

10. A sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies

$$a_{n+1} = n\left\lfloor \frac{a_n}{n} \right\rfloor + 1$$

for all positive integers n. If $a_{30} = 30$, how many possible values can a_1 take? (For a real number x, $\lfloor x \rfloor$ denotes the largest integer that is not greater than x.)

Proposed by: Carl Joshua Quines

Answer: $\boxed{274}$

Solution: It is straightforward to show that if $a_1 = 1$, then $a_n = n$ for all n. Since a_{n+1} is an increasing function in a_n, it follows that the set of possible a_1 is of the form $\{1, 2, \ldots, m\}$ for some m, which will be the answer to the problem.

Consider the sequence $b_n = a_{n+1} - 1$, which has the recurrence

$$b_{n+1} = n\left\lfloor \frac{b_n + 1}{n} \right\rfloor.$$

It has the property that b_n is divisible by n. Rearranging the recurrence, we see that

$$\frac{b_{n+1}}{n+1} \le \frac{b_n + 1}{n+1} < \frac{b_{n+1}}{n+1} + 1,$$

and as the b_i are integers, we get $b_{n+1} - 1 \le b_n < b_{n+1} + n$. For $n \ge 2$, this means that the largest possible value of b_n (call this b_n^*) is the smallest multiple of n which is at least b_{n+1}. Also, since $b_1 = b_0 + 1$, we find $b_0^* = b_1^* - 1$, meaning that the largest value for a_1 is b_1^*, and thus the answer is b_1^*.

We have now derived a procedure for deriving b_1^* from $b_{29}^* = 29$. To speed up the computation, let $c_n = b_n^*/n$. Then, since

$$b_n^* = n\left\lceil \frac{b_{n+1}^*}{n} \right\rceil,$$

we find

$$c_n = \left\lceil \frac{n+1}{n} c_{n+1} \right\rceil = c_{n+1} + \left\lceil \frac{c_{n+1}}{n} \right\rceil.$$

We now start from $c_{29} = 1$ and wish to find c_1.

Applying the recurrence, we find $c_{28} = 2$, $c_{27} = 3$, and so on until we reach $c_{15} = 15$. Then, $\lceil c_{n+1}/n \rceil$ becomes greater than 1 and we find $c_{14} = 17$, $c_{13} = 19$, and so on until $c_{11} = 23$. The rest can be done manually, with $c_{10} = 26$, $c_9 = 29$, $c_8 = 33$, $c_7 = 38$, $c_6 = 45$, $c_5 = 54$, $c_4 = 68$, $c_3 = 91$, $c_2 = 137$, and $c_1 = 274$. The last few steps may be easier to perform by converting back into the b_n^*.