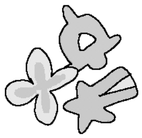
HMMT November 2019

November 9, 2019

Theme Round

Illustrations by Hanna Yang



1. For breakfast, Mihir always eats a bowl of Lucky Charms cereal, which consists of oat pieces and marshmallow pieces. He defines the *luckiness* of a bowl of cereal to be the ratio of the number of marshmallow pieces to the total number of pieces. One day, Mihir notices that his breakfast cereal has exactly 90 oat pieces and 9 marshmallow pieces, and exclaims, "This is such an unlucky bowl!" How many marshmallow pieces does Mihir need to add to his bowl to double its luckiness?

Proposed by: Krit Boonsiriseth

Answer: $\boxed{11}$

Let x be the number of marshmallows to add. We are given that

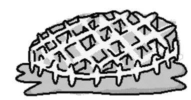
$$2 \cdot \frac{9}{99} = \frac{9+x}{99+x}.$$

Rearanging this gives

$$2(99+x) = 11(9+x).$$

Thus $9x = 99$ and $x = 11$.

2. Sandy likes to eat waffles for breakfast. To make them, she centers a circle of waffle batter of radius 3cm at the origin of the coordinate plane and her waffle iron imprints non-overlapping unit-square holes **centered** at each lattice point. How many of these holes are contained entirely within the area of the waffle?

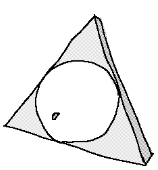


Proposed by: Carl Schildkraut

Answer: $\boxed{21}$

First, note that each divet must have its sides parallel to the coordinate axes; if the divet centered at the lattice point (a, b) does not have this orientation, then it contains the point $(a + 1/2, b)$ in its interior, so it necessarily overlaps with the divet centered at $(a + 1, b)$.

If we restrict our attention to one quadrant, we see geometrically that the divets centered at $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, and $(2, 1)$ are completely contained in the waffle, and no others are. We can make this more rigorous by considering the set of points (x, y) such that $x^2 + y^2 < 9$. We count 1 divet centered at the origin, 8 divets centered on the axes that are not centered at the origin, and 12 divets not centered on the axes, for a total of 21 divets.



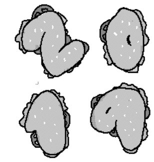
3. For breakfast, Milan is eating a piece of toast shaped like an equilateral triangle. On the piece of toast rests a single sesame seed that is one inch away from one side, two inches away from another side, and four inches away from the third side. He places a circular piece of cheese on top of the toast that is tangent to each side of the triangle. What is the area of this piece of cheese?

Proposed by: Carl Schildkraut

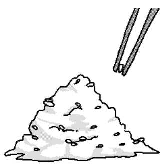
Answer: $\boxed{\frac{49\pi}{9}}$

Suppose the toast has side length s. If we draw the three line segments from the sesame seed to the three vertices of the triangle, we partition the triangle into three smaller triangles, with areas $\frac{s}{2}$, s, and $2s$, so the entire piece of toast has area $\frac{7s}{2}$. Suppose the cheese has radius r. We similarly see that the toast has area $\frac{3rs}{2}$. Equating these, we see that $r = \frac{7}{3}$, so the area of the cheese is $\pi\left(\frac{7}{3}\right)^2 = \frac{49\pi}{9}$.

4. To celebrate 2019, Faraz gets four sandwiches shaped in the digits 2, 0, 1, and 9 at lunch. However, the four digits get reordered (but not flipped or rotated) on his plate and he notices that they form a 4-digit multiple of 7. What is the greatest possible number that could have been formed?



Proposed by: Milan Haiman
Answer: $\boxed{1092}$
Note that 2 and 9 are equivalent mod 7. So we will replace the 9 with a 2 for now. Since 7 is a divisor of 21, a four digit multiple of 7 consisting of 2, 0, 1, and 2 cannot have a 2 followed by a 1 (otherwise we could subtract a multiple of 21 to obtain a number of the form $2 \cdot 10^k$). Thus our number either starts with a 1 or has a 0 followed by a 1. We can check that 2201 and 2012 are not divisible by 7. Thus our number starts with a 1. Checking 1220, 1202, and 1022 gives that 1022 is the only possibility. So the answer is 1092.



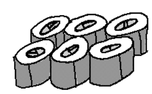
5. Alison is eating 2401 grains of rice for lunch. She eats the rice in a very peculiar manner: every step, if she has only one grain of rice remaining, she eats it. Otherwise, she finds the smallest positive integer $d > 1$ for which she can group the rice into equal groups of size d with none left over. She then groups the rice into groups of size d, eats one grain from each group, and puts the rice back into a single pile. How many steps does it take her to finish all her rice?

Proposed by: Carl Schilkraut
Answer: $\boxed{17}$
Note that $2401 = 7^4$. Also, note that the operation is equivalent to replacing n grains of rice with $n \cdot \frac{p-1}{p}$ grains of rice, where p is the smallest prime factor of n.

Now, suppose that at some moment Alison has 7^k grains of rice. After each of the next four steps, she will have $6 \cdot 7^{k-1}, 3 \cdot 7^{k-1}, 2 \cdot 7^{k-1}$, and 7^{k-1} grains of rice, respectively. Thus, it takes her 4 steps to decrease the number of grains of rice by a factor of 7 given that she starts at a power of 7.

Thus, it will take $4 \cdot 4 = 16$ steps to reduce everything to $7^0 = 1$ grain of rice, after which it will take one step to eat it. Thus, it takes a total of 17 steps for Alison to eat all of the rice.

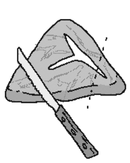
6. Wendy eats sushi for lunch. She wants to eat six pieces of sushi arranged in a 2×3 rectangular grid, but sushi is sticky, and Wendy can only eat a piece if it is adjacent to (not counting diagonally) at most two other pieces. In how many orders can Wendy eat the six pieces of sushi, assuming that the pieces of sushi are distinguishable?



Proposed by: Milan Haiman
Answer: $\boxed{360}$
Call the sushi pieces A, B, C in the top row and D, E, F in the bottom row of the grid. Note that Wendy must first eat either A, C, D, or F. Due to the symmetry of the grid, all of these choices are equivalent. Without loss of generality, suppose Wendy eats piece A.

Now, note that Wendy cannot eat piece E, but can eat all other pieces. If Wendy eats piece B, D, or F, then in the resulting configuration, all pieces of sushi are adjacent to at most 2 pieces, so she will have 4! ways to eat the sushi. Thus, the total number of possibilities in this case is $4 \cdot 3 \cdot 4! = 288$.

If Wendy eats A and then C, then Wendy will only have 3 choices for her next piece of sushi, after which she will have 3! ways to eat the remaining 3 pieces of sushi. Thus, the total number of possibilities in this case is $4 \cdot 1 \cdot 3 \cdot 3! = 72$.
Thus, the total number of ways for Wendy to eat the sushi is $288 + 72 = 360$.



7. Carl only eats food in the shape of equilateral pentagons. Unfortunately, for dinner he receives a piece of steak in the shape of an equilateral triangle. So that he can eat it, he cuts off two corners with straight cuts to form an equilateral pentagon. The set of possible perimeters of the pentagon he obtains is exactly the interval $[a, b)$, where a and b are positive real numbers. Compute $\frac{a}{b}$.

Proposed by: Krit Boonsiriseth and Milan Haiman

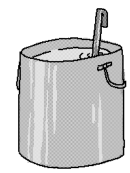
Answer: $\boxed{4\sqrt{3}-6}$

Assume that the triangle has side length 1. We will show the pentagon side length x is in $[2\sqrt{3}-3, \frac{1}{2})$. Call the triangle ABC and let corners B, C be cut. Choose P on AB, Q, R on BC, and S on AC such that $APQRS$ is equilateral. If $x \geq \frac{1}{2}$ then Q is to the right of R, causing self-intersection. Also the distance from P to BC is at most x, so

$$x = PQ \geq PB \sin 60^\circ = (1-x) \cdot \frac{\sqrt{3}}{2}.$$

Solving gives $(2+\sqrt{3})x \geq \sqrt{3}$, or $x \geq \sqrt{3}(2-\sqrt{3}) = 2\sqrt{3}-3$. Finally, these are attainable if we choose P such that $AP = x$, then Q such that $PQ = x$, and so on. Therefore $\frac{a}{b} = 4\sqrt{3}-6$.

8. Omkar, Krit$_1$, Krit$_2$, and Krit$_3$ are sharing $x > 0$ pints of soup for dinner. Omkar always takes 1 pint of soup (unless the amount left is less than one pint, in which case he simply takes all the remaining soup). Krit$_1$ always takes $\frac{1}{6}$ of what is left, Krit$_2$ always takes $\frac{1}{5}$ of what is left, and Krit$_3$ always takes $\frac{1}{4}$ of what is left. They take soup in the order of Omkar, Krit$_1$, Krit$_2$, Krit$_3$, and then cycle through this order until no soup remains. Find all x for which everyone gets the same amount of soup.



Proposed by: Krit Boonsiriseth and Milan Haiman

Answer: $\boxed{\dfrac{49}{3}}$

The main observation is that if $x > 1$ pints of soup are left, then in one round, Omkar gets 1 and each Krit$_n$ gets $\frac{x-1}{6}$, with $\frac{x-1}{2}$ soup left. Thus it is evident that each Krit$_n$ gets the same amount of soup, which means it suffices to find x for which Omkar gets $\frac{x}{4}$.

Omkar gets 1 for each cycle and then all the remaining soup when there is less than one pint remaining. The amount of soup becomes (after each cycle)

$$x \to \frac{x-1}{2} \to \frac{x-3}{4} \to \cdots \to \frac{x+1}{2^n} - 1,$$

so if n is the number of cycles, then Omkar's soup is $n + \frac{x+1}{2^n} - 1$. Setting this equal to $\frac{x}{4}$, we obtain

$$x = \frac{n + 1/2^n - 1}{1/4 - 1/2^n} = \frac{(n-1)2^n + 1}{2^{n-2} - 1}.$$

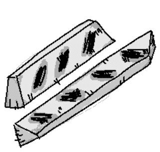
This immediately implies $n > 2$. On the other hand, we necessarily have $0 \leq \frac{x+1}{2^n} - 1 \leq 1$, so $2^n \leq x+1 \leq 2^{n+1}$. But

$$x + 1 = \frac{(n-1)2^n + 2^{n-2}}{2^{n-2} - 1} \leq \frac{(n-1)2^n + 2^n}{2^{n-3}} = 8n,$$

So $2^n \leq 8n \implies n \leq 5$. Testing $n = 3, 4, 5$:

- For $n = 3$ we get $x = 17$ which is greater than 2^4.
- For $n = 4$ we get $x = \frac{49}{3}$ which works.
- For $n = 5$ we get $x = \frac{129}{7}$ which is less than 2^5.

We see that only $n = 4$ and $x = \frac{49}{3}$ works.



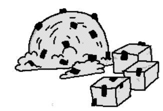
9. For dinner, Priya is eating grilled pineapple spears. Each spear is in the shape of the quadrilateral $PINE$, with $PI = 6$cm, $IN = 15$cm, $NE = 6$cm, $EP = 25$cm, and $\angle NEP + \angle EPI = 60^\circ$. What is the area of each spear, in cm^2?

Proposed by: Milan Haiman

Answer: $\boxed{\dfrac{100\sqrt{3}}{3}}$

We consider a configuration composed of 2 more quadrilaterals congruent to $PINE$. Let them be $P'I'N'E'$, with $E' = P$ and $N' = I$, and $P''I''N''E''$ with $P'' = E, E'' = P', N'' = I'$, and $I'' = N$. Notice that this forms an equilateral triangle of side length 25 since $\angle PP'P'' = \angle PP''P' = \angle P'PP'' = 60°$. Also, we see that the inner triangle $NN'N''$ forms an equilateral triangle of side length 15 since all the side lengths are equal. So the area inside the big equilateral triangle and outside the small one is $\frac{625\sqrt{3}}{4} - \frac{225\sqrt{3}}{4} = 100\sqrt{3}$. Since there are two other congruent quadrilaterals to $PINE$, we have that the area of one of them is $\frac{100\sqrt{3}}{3}$.

10. For dessert, Melinda eats a spherical scoop of ice cream with diameter 2 inches. She prefers to eat her ice cream in cube-like shapes, however. She has a special machine which, given a sphere placed in space, cuts it through the planes $x = n$, $y = n$, and $z = n$ for every integer n (not necessarily positive). Melinda centers the scoop of ice cream uniformly at random inside the cube $0 \le x, y, z \le 1$, and then cuts it into pieces using her machine. What is the expected number of pieces she cuts the ice cream into?



Proposed by: Carl Schildkraut

Answer: $\boxed{7 + \dfrac{13\pi}{3}}$

Note that if we consider the division of $\mathbb{R}^3$ into unit cubes by the given planes, we only need to compute the sum of the probabilities that the ice cream scoop intersects each cube. There are three types of cubes that can be intersected:

- The cube $0 \le x, y, z \le 1$ in which the center lies, as well as the 6 face-adjacent cubes are always intersected, for a total of 7.
- The cubes edge-adjacent to the center cube are intersected if the center of the ice cream lies within 1 unit of the connecting edge, which happens with probability $\frac{\pi}{4}$. There are 12 such cubes, for a total of 3π.
- The cubes corner-adjacent to the center cube are intersected if the center of the ice cream lies within 1 unit of the connecting corner, which happens with probability $\frac{\pi}{6}$. There are 8 such cubes, for a total of $\frac{4\pi}{3}$.

Adding these all up gives our answer of $7 + \frac{13\pi}{3}$.

An alternate solution is possible:

We compute the number of regions into which a convex region S in $\mathbb{R}^3$ is divided by planes: Let a be the number of planes intersecting S. Let b be the number of lines (intersections of two planes) passing through S. Let c be the number of points (intersections of three planes) lying inside S. Then S is divided into $a + b + c + 1$ regions. Then the computation for the problem is fairly straight forward. Note that the only planes, lines, and points that can intersect the ice cream scoop I are the faces, edges, and vertices of the cube $0 \le x, y, z \le 1$. The computation is essentially the same as in the first solution. The scoop intersects each of the 6 faces with probability 1, each of the 12 edges with probability $\frac{\pi}{4}$, and each of the 8 vertices with probability $\frac{\pi}{6}$, for a total expected number of regions $1 + 6 + 3\pi + \frac{4\pi}{3} = 7 + \frac{13\pi}{3}$.