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1. **[5]** A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

   A polynomial is called *teeny* if it has degree at most 1 and integer coefficients between -7 and 7, inclusive.

   How many nonzero tricky teeny polynomials are there?

2. **[5]** You are trying to cross a 6 foot wide river. You can jump at most 4 feet, but you have one stone you can throw into the river; after it is placed, you may jump to that stone and, if possible, from there to the other side of the river. However, you are not very accurate and the stone ends up landing uniformly at random in the river. What is the probability that you can get across?

3. **[5]** For how many positive integers a does the polynomial

$$x^2 - ax + a$$

   have an integer root?

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4. **[6]** In 2019, a team, including professor Andrew Sutherland of MIT, found three cubes of integers which sum to 42:

$$42 = (-8053873881207597\_)^3 + (80435758145817515)^3 + (12602123297335631)^3$$

   One of the digits, labeled by an underscore, is missing. What is that digit?

5. **[6]** A point P is chosen uniformly at random inside a square of side length 2. If P_1, P_2, P_3, and P_4 are the reflections of P over each of the four sides of the square, find the expected value of the area of quadrilateral $P_1P_2P_3P_4$.

6. **[6]** Compute the sum of all positive integers $n < 2048$ such that n has an even number of 1's in its binary representation.

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7. **[7]** Let S be the set of all nondegenerate triangles formed from the vertices of a regular octagon with side length 1. Find the ratio of the largest area of any triangle in S to the smallest area of any triangle in S.

8. **[7]** There are 36 students at the Multiples Obfuscation Program, including a singleton, a pair of identical twins, a set of identical triplets, a set of identical quadruplets, and so on, up to a set of identical octuplets. Two students look the same if and only if they are from the same identical multiple. Nithya the teaching assistant encounters a random student in the morning and a random student in the afternoon (both chosen uniformly and independently), and the two look the same. What is the probability that they are actually the same person?

9. **[7]** Let p be a real number between 0 and 1. Jocelin has a coin that lands heads with probability p and tails with probability $1-p$; she also has a number written on a blackboard. Each minute, she flips the coin, and if it lands heads, she replaces the number x on the blackboard with $3x+1$; if it lands tails she replaces it with $x/2$. Given that there are constants a, b such that the expected value of the value written on the blackboard after t minutes can be written as $at+b$ for all positive integers t, compute p.

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10. **[8]** Let $ABCD$ be a square of side length 5, and let E be the midpoint of side AB. Let P and Q be the feet of perpendiculars from B and D to CE, respectively, and let R be the foot of the perpendicular from A to DQ. The segments CE, BP, DQ, and AR partition $ABCD$ into five regions. What is the median of the areas of these five regions?

11. **[8]** Let a, b, c, d be real numbers such that

$$\min(20x+19, 19x+20) = (ax+b) - |cx+d|$$

for all real numbers x. Find $ab+cd$.

12. **[8]** Four players stand at distinct vertices of a square. They each independently choose a vertex of the square (which might be the vertex they are standing on). Then, they each, at the same time, begin running in a straight line to their chosen vertex at 10mph, stopping when they reach the vertex. If at any time two players, whether moving or not, occupy the same space (whether a vertex or a point inside the square), they collide and fall over. How many different ways are there for the players to choose vertices to go to so that none of them fall over?

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13. **[9]** In $\triangle ABC$, the incircle centered at I touches sides AB and BC at X and Y, respectively. Additionally, the area of quadrilateral $BXIY$ is $\frac{2}{5}$ of the area of ABC. Let p be the smallest possible perimeter of a $\triangle ABC$ that meets these conditions and has integer side lengths. Find the smallest possible area of such a triangle with perimeter p.

14. **[9]** Compute the sum of all positive integers n for which

$$9\sqrt{n}+4\sqrt{n+2}-3\sqrt{n+16}$$

is an integer.

15. **[9]** Let a, b, c be positive integers such that

$$\frac{a}{77}+\frac{b}{91}+\frac{c}{143}=1.$$

What is the smallest possible value of $a+b+c$?

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16. **[10]** Equilateral $\triangle ABC$ has side length 6. Let ω be the circle through A and B such that CA and CB are both tangent to ω. A point D on ω satisfies $CD=4$. Let E be the intersection of line CD with segment AB. What is the length of segment DE?

17. **[10]** Kelvin the frog lives in a pond with an infinite number of lily pads, numbered 0, 1, 2, 3, and so forth. Kelvin starts on lily pad 0 and jumps from pad to pad in the following manner: when on lily pad i, he will jump to lily pad $(i+k)$ with probability $\frac{1}{2^k}$ for $k>0$. What is the probability that Kelvin lands on lily pad 2019 at some point in his journey?

18. **[10]** The polynomial x^3-3x^2+1 has three real roots r_1, r_2, and r_3. Compute

$$\sqrt[3]{3r_1-2}+\sqrt[3]{3r_2-2}+\sqrt[3]{3r_3-2}.$$

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19. **[11]** Let ABC be a triangle with $AB = 5$, $BC = 8$, $CA = 11$. The incircle ω and A-excircle[1] Γ are centered at I_1 and I_2, respectively, and are tangent to BC at D_1 and D_2, respectively. Find the ratio of the area of $\triangle AI_1D_1$ to the area of $\triangle AI_2D_2$.

20. **[11]** Consider an equilateral triangle T of side length 12. Matthew cuts T into N smaller equilateral triangles, each of which has side length 1, 3, or 8. Compute the minimum possible value of N.

21. **[11]** A positive integer n is *infallible* if it is possible to select n vertices of a regular 100-gon so that they form a convex, non-self-intersecting n-gon having all equal angles. Find the sum of all infallible integers n between 3 and 100, inclusive.

[1] The A-*excircle* of triangle ABC is the unique circle lying outside the triangle that is tangent to segment BC and the extensions of sides AB and AC.

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22. **[12]** Let $f(n)$ be the number of distinct digits of n when written in base 10. Compute the sum of $f(n)$ as n ranges over all positive 2019-digit integers.

23. **[12]** For a positive integer n, let, $\tau(n)$ be the number of positive integer divisors of n. How many integers $1 \leq n \leq 50$ are there such that $\tau(\tau(n))$ is odd?

24. **[12]** Let P be a point inside regular pentagon $ABCDE$ such that $\angle PAB = 48^\circ$ and $\angle PDC = 42^\circ$. Find $\angle BPC$, in degrees.

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25. **[13]** In acute $\triangle ABC$ with centroid G, $AB = 22$ and $AC = 19$. Let E and F be the feet of the altitudes from B and C to AC and AB respectively. Let G' be the reflection of G over BC. If E, F, G, and G' lie on a circle, compute BC.

26. **[13]** Dan is walking down the left side of a street in New York City and must cross to the right side at one of 10 crosswalks he will pass. Each time he arrives at a crosswalk, however, he must wait t seconds, where t is selected uniformly at random from the real interval $[0, 60]$ (t can be different at different crosswalks). Because the wait time is conveniently displayed on the signal across the street, Dan employs the following strategy: if the wait time when he arrives at the crosswalk is no more than k seconds, he crosses. Otherwise, he immediately moves on to the next crosswalk. If he arrives at the last crosswalk and has not crossed yet, then he crosses regardless of the wait time. Find the value of k which minimizes his expected wait time.

27. **[13]** For a given positive integer n, we define $\varphi(n)$ to be the number of positive integers less than or equal to n which share no common prime factors with n. Find all positive integers n for which

$$\varphi(2019n) = \varphi(n^2).$$

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28. **[15]** A palindrome is a string that does not change when its characters are written in reverse order. Let S be a 40-digit string consisting only of 0's and 1's, chosen uniformly at random out of all such strings. Let E be the expected number of nonempty contiguous substrings of S which are palindromes. Compute the value of $\lfloor E \rfloor$.

29. **[15]** In isosceles $\triangle ABC$, $AB = AC$ and P is a point on side BC. If $\angle BAP = 2\angle CAP$, $BP = \sqrt{3}$, and $CP = 1$, compute AP.

30. **[15]** A function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies: $f(0) = 0$ and

$$\left| f((n+1)2^k) - f(n2^k) \right| \leq 1$$

for all integers $k \geq 0$ and n. What is the maximum possible value of $f(2019)$?

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31. **[17]** James is standing at the point $(0, 1)$ on the coordinate plane and wants to eat a hamburger. For each integer $n \geq 0$, the point $(n, 0)$ has a hamburger with n patties. There is also a wall at $y = 2.1$ which James cannot cross. In each move, James can go either up, right, or down 1 unit as long as he does not cross the wall or visit a point he has already visited.

    Every second, James chooses a valid move uniformly at random, until he reaches a point with a hamburger. Then he eats the hamburger and stops moving. Find the expected number of patties that James eats on his burger.

32. **[17]** A sequence of real numbers $a_0, a_1, \ldots, a_9$ with $a_0 = 0$, $a_1 = 1$, and $a_2 > 0$ satisfies

$$a_{n+2}a_na_{n-1} = a_{n+2} + a_n + a_{n-1}$$

for all $1 \leq n \leq 7$, but cannot be extended to a_{10}. In other words, no values of $a_{10} \in \mathbb{R}$ satisfy

$$a_{10}a_8a_7 = a_{10} + a_8 + a_7.$$

Compute the smallest possible value of a_2.

33. **[17]** A circle Γ with center O has radius 1. Consider pairs (A, B) of points so that A is inside the circle and B is on its boundary. The circumcircle Ω of OAB intersects Γ again at $C \neq B$, and line AC intersects Γ again at $X \neq C$. The pair (A, B) is called *techy* if line OX is tangent to Ω. Find the area of the region of points A so that there exists a B for which (A, B) is techy.

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34. **[20]** A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

    A polynomial is called *k-tiny* if it has degree at most 7 and integer coefficients between $-k$ and k, inclusive.

    A polynomial is called *nearly tricky* if it is the sum of a tricky polynomial and a 1-tiny polynomial.

    Let N be the number of nearly tricky 7-tiny polynomials. Estimate N.

    An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^4 \right\rfloor$ points.

35. **[20]** You are trying to cross a 400 foot wide river. You can jump at most 4 feet, but you have many stones you can throw into the river. You will stop throwing stones and cross the river once you have placed enough stones to be able to do so. You can throw straight, but you can't judge distance very well, so each stone ends up being placed uniformly at random along the width of the river. Estimate the expected number N of stones you must throw before you can get across the river.

    An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^3 \right\rfloor$ points.

36. **[20]** Let N be the number of sequences of positive integers $(a_1, a_2, a_3, \ldots, a_{15})$ for which the polynomials

$$x^2 - a_ix + a_{i+1}$$

each have an integer root for every $1 \leq i \leq 15$, setting $a_{16} = a_1$. Estimate N.

An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^2 \right\rfloor$ points.