14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

1. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

2. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f(1) = 1$, and $|f'(x)| \le 2$ for all real numbers x. If a and b are real numbers such that the set of possible values of $\int_0^1 f(x)\,dx$ is the open interval (a, b), determine $b - a$.

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

5. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

7. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

8. Let $f : [0, 1) \to \mathbb{R}$ be a function that satisfies the following condition: if
$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} = .a_1a_2a_3\ldots$$
is the decimal expansion of x and there does not exist a positive integer k such that $a_n = 9$ for all $n \ge k$, then
$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^{2n}}.$$
Determine $f'\left(\frac{1}{3}\right)$.

9. The integers from 1 to n are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.

10. Evaluate $\displaystyle\int_1^{\infty} \left(\frac{\ln x}{x}\right)^{2011} dx$.

11. Mike and Harry play a game on an 8×8 board. For some positive integer k, Mike chooses k squares and writes an M in each of them. Harry then chooses $k + 1$ squares and writes an H in each of them. After Harry is done, Mike wins if there is a sequence of letters forming "HMM" or "MMH," when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of k for which Mike has a winning strategy.

12. Sarah and Hagar play a game of darts. Let O_0 be a circle of radius 1. On the nth turn, the player whose turn it is throws a dart and hits a point p_n randomly selected from the points of O_{n-1}. The player then draws the largest circle that is centered at p_n and contained in O_{n-1}, and calls this circle O_n. The player then colors every point that is inside O_{n-1} but not inside O_n her color. Sarah goes first, and the two players alternate turns. Play continues indefinitely. If Sarah's color is red, and Hagar's color is blue, what is the expected value of the area of the set of points colored red?

13. The ordered pairs $(2011, 2), (2010, 3), (2009, 4), \ldots$, $(1008, 1005)$, $(1007, 1006)$ are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs (x_i, y_i) and (x_j, y_j), with (x_i, y_i) left of (x_j, y_j), erases them, and writes $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j}\right)$ in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs (x, y) could appear on the blackboard after the process has come to a conclusion?

14. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(f(x)) = 1$ for all $x \in [0, 1]$. Determine the set of possible values of $\displaystyle\int_0^1 f(x)\, dx$.

15. Let $A = \{1, 2, \ldots, 2011\}$. Find the number of functions f from A to A that satisfy $f(n) \leq n$ for all n in A and attain exactly 2010 distinct values.

16. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A.

17. Let $f : (0, 1) \to (0, 1)$ be a differentiable function with a continuous derivative such that for every positive integer n and odd positive integer $a < 2^n$, there exists an odd positive integer $b < 2^n$ such that $f\left(\frac{a}{2^n}\right) = \frac{b}{2^n}$. Determine the set of possible values of $f'\left(\frac{1}{2}\right)$.

18. Let n be an odd positive integer, and suppose that n people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander's left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other $n - 1$ members vote randomly.

19. Let
$$F(x) = \frac{1}{(2 - x - x^5)^{2011}},$$
and note that F may be expanded as a power series so that $F(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Find an ordered pair of positive real numbers (c, d) such that $\displaystyle\lim_{n\to\infty} \frac{a_n}{n^d} = c$.

20. Alice and Bob play a game in which two thousand and eleven 2011×2011 grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers $1, 2, \ldots, 2011^2$ so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?