HMMT November 2019

November 9, 2019

Guts Round

1. **[5]** A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

   A polynomial is called *teeny* if it has degree at most 1 and integer coefficients between -7 and 7, inclusive.

   How many nonzero tricky teeny polynomials are there?

   Proposed by: Carl Schildkraut

   Answer: $\boxed{2}$

   If a degree 0 polynomial has 4 as a root, then it must be the constant zero polynomial. Thus, we will only consider polynomials of degree 1.

   If P has degree 1, integer coefficients, and 4 as a root, then it must be of the form $P(x) = a(x-4) = ax - 4a$ for some nonzero integer a. Since all integer coefficients are between -7 and 7, inclusive, we require $-7 \le 4a \le 7$, which gives us $a = -1, 1$. Note that for both values, the coefficient of x is also between -7 and 7, so there are 2 tricky teeny polynomials.

2. **[5]** You are trying to cross a 6 foot wide river. You can jump at most 4 feet, but you have one stone you can throw into the river; after it is placed, you may jump to that stone and, if possible, from there to the other side of the river. However, you are not very accurate and the stone ends up landing uniformly at random in the river. What is the probability that you can get across?

   Proposed by: Carl Schildkraut and Milan Haiman

   Answer: $\boxed{\frac{1}{3}}$

   To be able to cross, the stone must land between 2 and 4 feet from the river bank you are standing on. Therefore the probability is $\frac{2}{6} = \frac{1}{3}$.

3. **[5]** For how many positive integers a does the polynomial

   $$x^2 - ax + a$$

   have an integer root?

   Proposed by: Krit Boonsiriseth

   Answer: $\boxed{1}$

   Let r, s be the roots of $x^2 - ax + a = 0$. By Vieta's, we have $r + s = rs = a$. Note that if one root is an integer, then both roots must be integers, as they sum to an integer a. Then,

   $$rs - (r+s) + 1 = 1 \implies (r-1)(s-1) = 1.$$

   Because we require r, s to be both integers, we have $r - 1 = s - 1 = \pm 1$, which yields $r = s = 0, 2$. If $r = 0$ or $s = 0$, then $a = 0$, but we want a to be a positive integer. Therefore, our only possibility is when $r = s = 2$, which yields $a = 4$, so there is exactly 1 value of a (namely, $a = 4$) such that $x^2 - ax - a$ has an integer root.

4. **[6]** In 2019, a team, including professor Andrew Sutherland of MIT, found three cubes of integers which sum to 42:

   $$42 = (-8053873881207597\_)^3 + (80435758145817515)^3 + (12602123297335631)^3$$

   One of the digits, labeled by an underscore, is missing. What is that digit?

   Proposed by: Andrew Gu

   Answer: $\boxed{4}$

   Let the missing digit be x. Then, taking the equation modulo 10, we see that $2 \equiv -x^3 + 5^3 + 1^3$. This simplifies to $x^3 \equiv 4 \pmod{10}$, which gives a unique solution of $x = 4$.

5. **[6]** A point P is chosen uniformly at random inside a square of side length 2. If P_1, P_2, P_3, and P_4 are the reflections of P over each of the four sides of the square, find the expected value of the area of quadrilateral $P_1P_2P_3P_4$.

   Proposed by: Carl Schildkraut

   Answer: $\boxed{8}$

   Let $ABCD$ denote the square defined in the problem. We see that if P_1 is the reflection of P over $\overline{AB}$, then the area of P_1AB is the same as the area of PAB. Furthermore, if P_4 is the reflection of P over $\overline{DA}$, P_1, A, and P_4 are collinear, as A is the midpoint of $\overline{P_1P_4}$. Repeating this argument for the other points gives us that the desired area is

   $$[P_1AB]+[P_2BC]+[P_3CD]+[P_4DA]+[ABCD] = [PAB]+[PBC]+[PCD]+[PDA]+[ABCD] = 2[ABCD] = 8.$$

6. **[6]** Compute the sum of all positive integers $n < 2048$ such that n has an even number of 1's in its binary representation.

   Proposed by: Milan Haiman

   Answer: $\boxed{1048064}$

   Note that the positive integers less than 2047 are those with at most 11 binary digits. Consider the contribution from any one of those digits. If we set that digit to 1, then the remaining 10 digits can be set in $2^9 = 512$ ways so that the number of 1's is even. Therefore the answer is

   $$512(2^0 + \cdots + 2^{10}) = 512 \cdot 2047 = 1048064.$$

7. **[7]** Let S be the set of all nondegenerate triangles formed from the vertices of a regular octagon with side length 1. Find the ratio of the largest area of any triangle in S to the smallest area of any triangle in S.

   Proposed by: Carl Schildkraut

   Answer: $\boxed{3+2\sqrt{2}}$

   By a smoothing argument, the largest triangle is that where the sides span 3, 3, and 2 sides of the octagon respectively (i.e. it has angles $45°$, $67.5°$, and $67.5°$), and the smallest triangle is that formed by three adjacent vertices of the octagon. Scaling so that the circumradius of the octagon is 1, our answer is

   $$\frac{\sin(90°) + 2\sin(135°)}{2\sin(45°) - \sin(90°)} = \frac{1+\sqrt{2}}{\sqrt{2}-1} = 3 + 2\sqrt{2},$$

   where the numerator is derived from splitting the large triangle by the circumradii, and the denominator is derived from adding the areas of the two triangles formed by the circumradii, then subtracting the area not in the small triangle.

8. **[7]** There are 36 students at the Multiples Obfuscation Program, including a singleton, a pair of identical twins, a set of identical triplets, a set of identical quadruplets, and so on, up to a set of identical octuplets. Two students look the same if and only if they are from the same identical multiple. Nithya the teaching assistant encounters a random student in the morning and a random student in the afternoon (both chosen uniformly and independently), and the two look the same. What is the probability that they are actually the same person?

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{3}{17}}$

   Let X and Y be the students Nithya encounters during the day. The number of pairs (X, Y) for which X and Y look the same is $1 \cdot 1 + 2 \cdot 2 + \ldots + 8 \cdot 8 = 204$, and these pairs include all the ones in which X and Y are identical. As X and Y are chosen uniformly and independently, all 204 pairs are equally likely to be chosen, thus the problem reduces to choosing one of the 36 pairs in 204, the probability for which is $\frac{3}{17}$.

9. **[7]** Let p be a real number between 0 and 1. Jocelin has a coin that lands heads with probability p and tails with probability $1-p$; she also has a number written on a blackboard. Each minute, she flips the coin, and if it lands heads, she replaces the number x on the blackboard with $3x+1$; if it lands tails she replaces it with $x/2$. Given that there are constants a, b such that the expected value of the value written on the blackboard after t minutes can be written as $at+b$ for all positive integers t, compute p.

   Proposed by: Carl Schildkraut

   Answer: $\boxed{\frac{1}{5}}$

   If the blackboard has the value x written on it, then the expected value of the value after one flip is

   $$f(x) = p(3x-1) + (1-p)x/2.$$

   Because this expression is linear, we can say the same even if we only know the blackboard's initial expected value is x. Therefore, if the blackboard value is x_0 at time 0, then after t minutes, the expected blackboard value is $f^t(x_0)$. We are given that $x_0, f(x_0), f^2(x_0), \ldots$ is an arithmetic sequence, so for there to be a constant difference, we must have $f(x) = x + c$ for some c.

   This only occurs when $3p + (1-p)/2 = 1$, so $p = 1/5$.

10. **[8]** Let $ABCD$ be a square of side length 5, and let E be the midpoint of side AB. Let P and Q be the feet of perpendiculars from B and D to CE, respectively, and let R be the foot of the perpendicular from A to DQ. The segments CE, BP, DQ, and AR partition $ABCD$ into five regions. What is the median of the areas of these five regions?

   Proposed by: Carl Schildkraut

   Answer: $\boxed{5}$

   We have $DQ \perp CE$ and $AR \perp DQ$, so $AR || CE$. Thus, we can show that $\triangle ARD \cong \triangle DQC \cong \triangle CPB$, so the median of the areas of the five regions is equal to the area of one of the three triangles listed above.

   Now, note that $\triangle EBC \sim \triangle BPC$, so $\frac{BP}{BC} = \frac{EB}{EC} = \frac{1}{\sqrt{5}}$. This means that $BP = \sqrt{5}$, so $CP = 2\sqrt{5}$. Therefore, the area of $\triangle BPC$, the median area, is 5.

11. **[8]** Let a, b, c, d be real numbers such that

   $$\min(20x+19, 19x+20) = (ax+b) - |cx+d|$$

   for all real numbers x. Find $ab + cd$.

   Proposed by: Yuan Yao

   Answer: $\boxed{380}$

   In general, $\min(p,q) = \frac{p+q}{2} - \left|\frac{p-q}{2}\right|$. Letting $p = 20x+19$ and $q = 19x+20$ gives $a = b = 19.5$ and $c = d = \pm 0.5$. Then the answer is $19.5^2 - 0.5^2 = 19 \cdot 20 = 380$.

12. **[8]** Four players stand at distinct vertices of a square. They each independently choose a vertex of the square (which might be the vertex they are standing on). Then, they each, at the same time, begin running in a straight line to their chosen vertex at 10mph, stopping when they reach the vertex. If at any time two players, whether moving or not, occupy the same space (whether a vertex or a point inside the square), they collide and fall over. How many different ways are there for the players to choose vertices to go to so that none of them fall over?

   Proposed by: Carl Schildkraut

   Answer: $\boxed{11}$

   Observe that no two players can choose the same vertex, and no two players can choose each others vertices. Thus, if two players choose their own vertices, then the remaining two also must choose their own vertices (because they cant choose each others vertices), thus all 4 players must choose their own vertices. There is 1 way to choose the vertices in this case.

Name the players top left, top right, bottom left, and bottom right, based on their initial positions. Assume exactly one player (without loss of generality, say the top left) chooses their own vertex. Then, the remaining 3 players have to form a triangle (recall no two player can choose each others vertices). There are 4 ways to choose which player chooses their own vertex, and 2 ways to choose which direction the players move in the triangle, thus there are 8 ways to choose the vertices in this case.

Lastly, assume no one chooses their own vertex. We will first prove that no player can choose the vertex across them. Assume the contrary, without loss of generality, let the top left player chooses the bottom right vertex. Then, neither of the bottom left and the top right players can choose the others vertex, because they would meet the top left player at the center of the square. As they cant choose bottom right (it is chosen by the top left player), and cant choose their own vertex (by assumption), they both have to choose the top left vertex, which is an immediate contradiction.

Now, the top left player has to choose either the top right vertex or the bottom left. Without loss of generality, let the player choose the top right vertex. Then, the top right player has to choose the bottom right vertex (as they can neither go across nor back to top left), the bottom right player has to choose the bottom left vertex, and the bottom left player has to choose the top left vertex, and all the choices are determined by the first players choice. There are 2 ways to choose where the first player goes, thus there are 2 ways to choose the vertices in this case.

In total, there are $1 + 8 + 2 = 11$ ways to choose the vertices.

13. **[9]** In $\triangle ABC$, the incircle centered at I touches sides AB and BC at X and Y, respectively. Additionally, the area of quadrilateral $BXIY$ is $\frac{2}{5}$ of the area of ABC. Let p be the smallest possible perimeter of a $\triangle ABC$ that meets these conditions and has integer side lengths. Find the smallest possible area of such a triangle with perimeter p.

    Proposed by: Joey Heerens

    Answer: $\boxed{2\sqrt{5}}$

    Note that $\angle BXI = \angle BYI = 90$, which means that AB and BC are tangent to the incircle of ABC at X and Y respectively. So $BX = BY = \frac{AB+BC-AC}{2}$, which means that $\frac{2}{5} = \frac{[BXIY]}{[ABC]} = \frac{AB+BC-AC}{AB+BC+AC}$. The smallest perimeter is achieved when $AB = AC = 3$ and $BC = 4$. The area of this triangle ABC is $2\sqrt{5}$.

14. **[9]** Compute the sum of all positive integers n for which

    $$9\sqrt{n} + 4\sqrt{n+2} - 3\sqrt{n+16}$$

    is an integer.

    Proposed by: Milan Haiman

    Answer: $\boxed{18}$

    For the expression to be an integer at least one of n and $n+2$ must be a perfect square. We also note that at most one of n and $n+2$ can be a square, so exactly one of them is a square.

    Case 1: n is a perfect square. By our previous observation, it must be that $4\sqrt{n+2} = 3\sqrt{n+16} \Rightarrow n = 16$.

    Case 2: $n+2$ is a perfect square. By our previous observation, it must be that $9\sqrt{n} = 3\sqrt{n+16} \Rightarrow n = 2$.

    Consequently, the answer is $16 + 2 = 18$.

15. **[9]** Let a, b, c be positive integers such that

    $$\frac{a}{77} + \frac{b}{91} + \frac{c}{143} = 1.$$

What is the smallest possible value of $a+b+c$?

Proposed by: James Lin

Answer: $\boxed{79}$

We need $13a+11b+7c=1001$, which implies $13(a+b+c-77)=2b+6c$. Then $2b+6c$ must be divisible by both 2 and 13, so it is minimized at 26 (e.g. with $b=10, c=1$). This gives $a+b+c=79$.

16. **[10]** Equilateral $\triangle ABC$ has side length 6. Let ω be the circle through A and B such that CA and CB are both tangent to ω. A point D on ω satisfies $CD=4$. Let E be the intersection of line CD with segment AB. What is the length of segment DE?

Proposed by: Benjamin Qi

Answer: $\boxed{\dfrac{20}{13}}$

Let F be the second intersection of line CD with ω. By power of a point, we have $CF=9$, so $DF=5$. This means that $\frac{[ADB]}{[AFB]}=\frac{DE}{EF}=\frac{DE}{5-DE}$. Now, note that triangle CAD is similar to triangle CFA, so $\frac{FA}{AD}=\frac{CA}{CD}=\frac{3}{2}$. Likewise, $\frac{FB}{BD}=\frac{CB}{CD}=\frac{3}{2}$. Also, note that $\angle ADB=180-\angle DAB-\angle DBA=180-\angle CAB=120$, and $\angle AFB=180-\angle ADB=60$. This means that $\frac{[ADB]}{[AFB]}=\frac{AD\cdot BD\cdot\sin 120}{FA\cdot FB\cdot\sin 60}=\frac{4}{9}$. Therefore, we have that $\frac{DE}{5-DE}=\frac{4}{9}$. Solving yields $DE=\frac{20}{13}$.

17. **[10]** Kelvin the frog lives in a pond with an infinite number of lily pads, numbered 0, 1, 2, 3, and so forth. Kelvin starts on lily pad 0 and jumps from pad to pad in the following manner: when on lily pad i, he will jump to lily pad $(i+k)$ with probability $\frac{1}{2^k}$ for $k>0$. What is the probability that Kelvin lands on lily pad 2019 at some point in his journey?

Proposed by: Nikhil Reddy

Answer: $\boxed{\dfrac{1}{2}}$

Suppose we combine all of the lily pads with numbers greater than 2019 into one lily pad labeled ∞. Also, let Kelvin stop once he reaches one of these lily pads.

Now at every leap, Kelvin has an equal chance of landing on 2019 as landing on ∞. Furthermore, Kelvin is guaranteed to reach 2019 or ∞ within 2020 leaps. Therefore the chance of landing on 2019 is the same as missing it, so our answer is just $\frac{1}{2}$.

18. **[10]** The polynomial x^3-3x^2+1 has three real roots r_1, r_2, and r_3. Compute

$$\sqrt[3]{3r_1-2}+\sqrt[3]{3r_2-2}+\sqrt[3]{3r_3-2}.$$

Proposed by: Milan Haiman

Answer: $\boxed{0}$

Let r be a root of the given polynomial. Then

$$r^3-3r^2+1=0 \implies r^3-3r^2+3r-1=3r-2 \implies r-1=\sqrt[3]{3r-2}.$$

Now by Vieta the desired value is $r_1+r_2+r_3-3=3-3=0$.

19. **[11]** Let ABC be a triangle with $AB=5$, $BC=8$, $CA=11$. The incircle ω and A-excircle[1] Γ are centered at I_1 and I_2, respectively, and are tangent to BC at D_1 and D_2, respectively. Find the ratio of the area of $\triangle AI_1D_1$ to the area of $\triangle AI_2D_2$.

Proposed by: Carl Schildkraut

Answer: $\boxed{\dfrac{1}{9}}$

Let D_1' and D_2' be the points diametrically opposite D_1 and D_2 on the incircle and A-excircle, respectively. As I_x is the midpoint of D_x and D_x', we have

$$\frac{[AI_1D_1]}{[AI_2D_2]} = \frac{[AD_1D_1']}{[AD_2D_2']}.$$

Now, $\triangle AD_1D_1'$ and $\triangle AD_2D_2'$ are homothetic with ratio $\frac{r}{r_A} = \frac{s-a}{s}$, where r is the inradius, r_A is the A-exradius, and s is the semiperimeter. Our answer is thus

$$\left(\frac{s-a}{s}\right)^2 = \left(\frac{4}{12}\right) = \frac{1}{9}.$$

20. **[11]** Consider an equilateral triangle T of side length 12. Matthew cuts T into N smaller equilateral triangles, each of which has side length 1, 3, or 8. Compute the minimum possible value of N.

    Proposed by: Matthew Cho

    Answer: $\boxed{16}$

    Matthew can cut T into 16 equilateral triangles with side length 3. If he instead included a triangle of side 8, then let him include a triangles of side length 3. He must include $12^2 - 8^2 - 3^2a = 80 - 9a$ triangles of side length 1. Thus $a \le 8$, giving that he includes at least

    $$(80 - 9a) + (a) + 1 = 81 - 8a \ge 17$$

    total triangles, so 16 is minimal.

21. **[11]** A positive integer n is *infallible* if it is possible to select n vertices of a regular 100-gon so that they form a convex, non-self-intersecting n-gon having all equal angles. Find the sum of all infallible integers n between 3 and 100, inclusive.

    Proposed by: Benjamin Qi

    Answer: $\boxed{262}$

    Suppose $A_1A_2\dots A_n$ is an equiangular n-gon formed from the vertices of a regular 100-gon. Note that the angle $\angle A_1A_2A_3$ is determined only by the number of vertices of the 100-gon between A_1 and A_3. Thus in order for $A_1A_2\dots A_n$ to be equiangular, we require exactly that $A_1, A_3, \dots$ are equally spaced and $A_2, A_4, \dots$ are equally spaced. If n is odd, then all the vertices must be equally spaced, meaning $n \mid 100$. If n is even, we only need to be able to make a regular $\left(\frac{n}{2}\right)$-gon from the vertices of a 100-gon, which we can do if $n \mid 200$. Thus the possible values of n are $4, 5, 8, 10, 20, 25, 40, 50,$ and 100, for a total of 262.

22. **[12]** Let $f(n)$ be the number of distinct digits of n when written in base 10. Compute the sum of $f(n)$ as n ranges over all positive 2019-digit integers.

    Proposed by: Milan Haiman

    Answer: $\boxed{9\left(10^{2019} - 9^{2019}\right)}$

    Write

    $$f(n) = f_0(n) + \cdots + f_9(n),$$

    where $f_d(n) = 1$ if n contains the digit d and 0 otherwise. The sum of $f_d(n)$ over all 2019-digit positive integers n is just the number of 2019-digit positive integers that contain the digit d. For $1 \le d \le 9$,

    $$\sum_n f_d(n) = 9 \cdot 10^{2018} - 8 \cdot 9^{2018}.$$

    Also,

    $$\sum_n f_0(n) = 9 \cdot 10^{2018} - 9^{2019}.$$

Summing over all possible values of d, we compute

$$\sum_n f(n) = \sum_{d=0}^{9} \sum_n f_d(n) = 9(9 \cdot 10^{2018} - 8 \cdot 9^{2018}) + 9 \cdot 10^{2018} - 9^{2019} = 9\left(10^{2019} - 9^{2019}\right).$$

23. **[12]** For a positive integer n, let, $\tau(n)$ be the number of positive integer divisors of n. How many integers $1 \le n \le 50$ are there such that $\tau(\tau(n))$ is odd?

Proposed by: Kevin Liu

Answer: $\boxed{17}$

Note that $\tau(n)$ is odd if and only if n is a perfect square. Thus, it suffices to find the number of integers n in the given range such that $\tau(n) = k^2$ for some positive integer k.

If $k = 1$, then we obtain $n = 1$ as our only solution. If $k = 2$, we see that n is either in the form pq or p^3, where p and q are distinct primes. The first subcase gives $8 + 4 + 1 = 13$ solutions, while the second subcase gives 2 solutions. $k = 3$ implies that n is a perfect square, and it is easy to see that only $6^2 = 36$ works. Finally, $k \ge 4$ implies that n is greater than 50, so we've exhausted all possible cases. Our final answer is $1 + 13 + 2 + 1 = 17$.

24. **[12]** Let P be a point inside regular pentagon $ABCDE$ such that $\angle PAB = 48^\circ$ and $\angle PDC = 42^\circ$. Find $\angle BPC$, in degrees.

Proposed by: Dylan Liu

Answer: $\boxed{84^\circ}$

Since a regular pentagon has interior angles 108°, we can compute $\angle PDE = 66^\circ$, $\angle PAE = 60^\circ$, and $\angle APD = 360^\circ - \angle AED - \angle PDE - \angle PAE = 126^\circ$. Now observe that drawing PE divides quadrilateral $PAED$ into equilateral triangle PAE and isosceles triangle PED, where $\angle DPE = \angle EDP = 66^\circ$. That is, we get $PA = PE = s$, where s is the side length of the pentagon.

Now triangles PAB and PED are congruent (with angles $48^\circ - 66^\circ - 66^\circ$), so $PD = PB$ and $\angle PDC = \angle PBC = 42^\circ$. This means that triangles PDC and PBC are congruent (side-angle-side), so $\angle BPC = \angle DPC$.

Finally, we compute $\angle BPC + \angle DPC = 2\angle BPC = 360^\circ - \angle APB - \angle EPA - \angle DPE = 168^\circ$, meaning $\angle BPC = 84^\circ$.

25. **[13]** In acute $\triangle ABC$ with centroid G, $AB = 22$ and $AC = 19$. Let E and F be the feet of the altitudes from B and C to AC and AB respectively. Let G' be the reflection of G over BC. If E, F, G, and G' lie on a circle, compute BC.

Proposed by: Milan Haiman

Answer: $\boxed{13}$

Note that B, C, E, F lie on a circle. Moreover, since BC bisects GG', the center of the circle that goes through E, F, G, G' must lie on BC. Therefore, B, C, E, F, G, G' lie on a circle. Specifically, the center of this circle is M, the midpoint of BC, as $ME = MF$ because M is the center of the circumcircle of $BCEF$. So we have $GM = \frac{BC}{2}$, which gives $AM = \frac{3BC}{2}$. Then, by Apollonius's theorem, we have $AB^2 + AC^2 = 2(AM^2 + BM^2)$. Thus $845 = 5BC^2$ and $BC = 13$.

26. **[13]** Dan is walking down the left side of a street in New York City and must cross to the right side at one of 10 crosswalks he will pass. Each time he arrives at a crosswalk, however, he must wait t seconds, where t is selected uniformly at random from the real interval $[0, 60]$ (t can be different at different crosswalks). Because the wait time is conveniently displayed on the signal across the street, Dan employs the following strategy: if the wait time when he arrives at the crosswalk is no more than k seconds, he crosses. Otherwise, he immediately moves on to the next crosswalk. If he arrives at the last crosswalk and has not crossed yet, then he crosses regardless of the wait time. Find the value of k which minimizes his expected wait time.

Answer: $\boxed{60\left(1-\left(\frac{1}{10}\right)^{\frac{1}{9}}\right)}$

With probability $\left(1-\frac{k}{60}\right)^9$, Dan reaches the last crosswalk without crossing at any previous site, in which case the expected value of his wait time is 30 seconds. Otherwise, with probability $1-\left(1-\frac{k}{60}\right)^9$, Dan crosses at an earlier crosswalk, in which case the expected value of his wait time is $\frac{k}{2}$. We want to find the k that minimizes

$$30\left(1-\frac{k}{60}\right)^9+\frac{k}{2}\left(1-\left(1-\frac{k}{60}\right)^9\right)=30-\left(30-\frac{k}{2}\right)\left(1-\left(1-\frac{k}{60}\right)^9\right)$$

Letting $a=1-\frac{k}{60}$, we can use weighted AM-GM:

$$9^{\frac{1}{10}}\left(a\left(1-a^9\right)\right)^{\frac{9}{10}}=\left(9a^9\right)^{\frac{1}{10}}\left(1-a^9\right)^{\frac{9}{10}}\leq\frac{9}{10}$$

where equality occurs when $9a^9=1-a^9$, or $a=\left(\frac{1}{10}\right)^{\frac{1}{9}}$, meaning that $k=60\left(1-\left(\frac{1}{10}\right)^{\frac{1}{9}}\right)$. Because our original expression can be written as

$$30-30a(1-a^9),$$

the minimum occurs at the same value, $k=60\left(1-\left(\frac{1}{10}\right)^{\frac{1}{9}}\right)$.

27. **[13]** For a given positive integer n, we define $\varphi(n)$ to be the number of positive integers less than or equal to n which share no common prime factors with n. Find all positive integers n for which

$$\varphi(2019n)=\varphi(n^2).$$

Proposed by: Carl Schildkraut

Answer: $\boxed{1346, 2016, 2019}$

Let $p_1,p_2,\ldots,p_k$ be the prime divisors of n. Then it is known that $\varphi(n)=n\cdot\frac{p_1-1}{p_1}\ldots\frac{p_k-1}{p_k}$. As n^2 and n has the same set of prime divisors, it also holds that $\varphi(n^2)=n^2\cdot\frac{p_1-1}{p_1}\ldots\frac{p_k-1}{p_k}$. We will examine the equality in four cases.

- $\gcd(n,2019)=1$ In this case, $2019\cdot n$ has also 3 and 673 as prime divisors, thus $\varphi(2019\cdot n)=2019\cdot n\cdot\frac{p_1-1}{p_1}\ldots\frac{p_k-1}{p_k}\cdot\frac{2}{3}\cdot\frac{672}{673}$, and the equality implies $n=1342$, however $\gcd(1342,3)\neq 1$, contradiction. Thus, there is no answer in this case.
- $\gcd(n,2019)=3$ In this case, $2019\cdot n$ has also 673 as a prime divisor, thus $\varphi(2019\cdot n)=2019\cdot n\cdot\frac{p_1-1}{p_1}\ldots\frac{p_k-1}{p_k}\cdot\frac{672}{673}$, and the equality implies $n=2016$, which satisfies the equation. Thus, the only answer in this case is $n=2016$.
- $\gcd(n,2019)=673$ In this case, $2019\cdot n$ has also 3 as a prime divisor, thus $\varphi(2019\cdot n)=2019\cdot n\cdot\frac{p_1-1}{p_1}\ldots\frac{p_k-1}{p_k}\cdot\frac{2}{3}$, and the equality implies $n=1346$, which satisfies the equation. Thus, the only answer in this case is $n=1346$.
- $\gcd(n,2019)=2019$ In this case, $2019\cdot n$ has the same set of prime divisors, thus $\varphi(2019\cdot n)=2019\cdot n\cdot\frac{p_1-1}{p_1}\ldots\frac{p_k-1}{p_k}$, and the equality implies $n=2019$, which satisfies the equation. Thus, the only answer in this case is $n=2019$.

Thus, all the answers are $n=1346, 2016, 2019$.

28. **[15]** A palindrome is a string that does not change when its characters are written in reverse order. Let S be a 40-digit string consisting only of 0's and 1's, chosen uniformly at random out of all such strings. Let E be the expected number of nonempty contiguous substrings of S which are palindromes. Compute the value of $\lfloor E\rfloor$.

Proposed by: Benjamin Qi

Answer: $\boxed{113}$

Note that S has $41-n$ contiguous substrings of length n, so we see that the expected number of palindromic substrings of length n is just $(41-n)\cdot 2^{-\lfloor n/2\rfloor}$. By linearity of expectation, E is just the sum of this over all n from 1 to 40. However, it is much easier to just compute

$$\sum_{n=1}^{\infty}(41-n)\cdot 2^{-\lfloor n/2\rfloor}.$$

The only difference here is that we have added some insignificant negative terms in the cases where $n>41$, so E is in fact slightly greater than this value (in fact, the difference between E and this sum is $\frac{7}{1048576}$). To make our infinite sum easier to compute, we can remove the floor function by pairing up consecutive terms. Then our sum becomes

$$40+\sum_{n=1}^{\infty}\frac{81-4n}{2^n},$$

which is just $40+81-8=113$. E is only slightly larger than this value, so our final answer is $\lfloor E\rfloor=\boxed{113}$.

29. **[15]** In isosceles $\triangle ABC$, $AB=AC$ and P is a point on side BC. If $\angle BAP=2\angle CAP$, $BP=\sqrt{3}$, and $CP=1$, compute AP.

Proposed by: Milan Haiman

Answer: $\boxed{\sqrt{2}}$

Let $\angle CAP=\alpha$, By the Law of Sines, $\frac{\sqrt{3}}{\sin 2\alpha}=\frac{1}{\sin\alpha}$ which rearranges to $\cos\alpha=\frac{\sqrt{3}}{2}\Rightarrow\alpha=\frac{\pi}{6}$. This implies that $\angle BAC=\frac{\pi}{2}$. By the Pythagorean Theorem, $2AB^2=(\sqrt{3}+1)^2$, so $AB^2=2+\sqrt{3}$. Applying Stewart's Theorem, it follows that $AP^2=\frac{(\sqrt{3}+1)(2+\sqrt{3})}{\sqrt{3}+1}-\sqrt{3}\Rightarrow AP=\sqrt{2}$.

30. **[15]** A function $f:\mathbb{Z}\to\mathbb{Z}$ satisfies: $f(0)=0$ and

$$\left|f((n+1)2^k)-f(n2^k)\right|\le 1$$

for all integers $k\ge 0$ and n. What is the maximum possible value of $f(2019)$?

Proposed by: Krit Boonsiriseth

Answer: $\boxed{4}$

Consider a graph on $\mathbb{Z}$ with an edge between $(n+1)2^k$ and $n2^k$ for all integers $k\ge 0$ and n. Each vertex m is given the value $f(m)$. The inequality $|f((n+1)2^k)-f(n2^k)|\le 1$ means that any two adjacent vertices of this graph must have values which differ by at most 1. Then it follows that for all m,

$$f(m)\le\text{number of edges in shortest path from 0 to } m$$

because if we follow a path from 0 to m, along each edge the value increases by at most 1. Conversely, if we define $f(m)$ to be the number of edges in the shortest path between 0 and m, then this is a valid function because for any two adjacent vertices, the lengths of their respective shortest paths to 0 differ by at most 1. Hence it suffices to compute the distance from 0 to 2019 in the graph.

There exists a path with 4 edges, given by

$$0\to 2048\to 2016\to 2018\to 2019.$$

Suppose there existed a path with three edges. In each step, the number changes by a power of 2, so we have $2019=\pm 2^{k_1}\pm 2^{k_2}\pm 2^{k_3}$ for some nonnegative integers k_1,k_2,k_3 and choice of signs. Since 2019 is odd, we must have 2^0 somewhere. Then we have $\pm 2^{k_1}\pm 2^{k_2}\in\{2018,2020\}$. Without loss of generality assume that $k_1 \geq k_2$. Then we can write this as $\pm 2^{k_2}(2^{k_1 k_2} \pm 1) \in \{2018, 2020\}$. It is easy to check that $k_1 = k_2$ is impossible, so the factorization $2^{k_2}(2^{k_1 k_2} \pm 1)$ is a product of a power of two and an odd number. Now compute $2018 = 2 \times 1009$ and $2020 = 4 \times 505$. Neither of the odd parts are of the form $2^{k_1 - k_2} \pm 1$, so there is no path of three steps.

We conclude that the maximum value of $f(2019)$ is 4.

31. **[17]** James is standing at the point $(0, 1)$ on the coordinate plane and wants to eat a hamburger. For each integer $n \geq 0$, the point $(n, 0)$ has a hamburger with n patties. There is also a wall at $y = 2.1$ which James cannot cross. In each move, James can go either up, right, or down 1 unit as long as he does not cross the wall or visit a point he has already visited.

    Every second, James chooses a valid move uniformly at random, until he reaches a point with a hamburger. Then he eats the hamburger and stops moving. Find the expected number of patties that James eats on his burger.

    Proposed by: Joey Heerens

    Answer: $\boxed{\dfrac{7}{3}}$

    Note that we desire to compute the number of times James moves to the right before moving down to the line $y = 0$. Note also that we can describe James's current state based on whether his y-coordinate is 0 or 1 and whether or not the other vertically adjacent point has been visited. Let $E(1, N)$ be the expected number of times James will go right before stopping if he starts at a point with y-coordinate 1 and the other available point with the same x-coordinate has not been visited. Define $E(1, Y), E(2, N)$, and $E(2, Y)$ similarly. Then we can construct equations relating the four variables:

    $$E(1, N) = \frac{1}{3}E(2, Y) + \frac{1}{3}(E(1, N) + 1),$$

    as James can either go up, right, or down with probability $1/3$ each if he starts in the state $(1, N)$. Similarly, we have

    $$E(2, N) = \frac{1}{2}E(1, Y) + \frac{1}{2}(E(2, N) + 1), E(1, Y) = \frac{1}{2}(E(1, N) + 1),$$

    and $E(2, Y) = E(2, N) + 1$. Solving these equations, we get $E(1, N) = \frac{7}{3}$, which is our answer, as James starts in that state having gone left 0 times.

32. **[17]** A sequence of real numbers $a_0, a_1, \ldots, a_9$ with $a_0 = 0$, $a_1 = 1$, and $a_2 > 0$ satisfies

    $$a_{n+2}a_n a_{n-1} = a_{n+2} + a_n + a_{n-1}$$

    for all $1 \leq n \leq 7$, but cannot be extended to a_{10}. In other words, no values of $a_{10} \in \mathbb{R}$ satisfy

    $$a_{10}a_8 a_7 = a_{10} + a_8 + a_7.$$

    Compute the smallest possible value of a_2.

    Proposed by: Dylan Liu

    Answer: $\boxed{\sqrt{2} - 1}$

    Say $a_2 = a$. Then using the recursion equation, we have $a_3 = -1$, $a_4 = \frac{a+1}{a-1}$, $a_5 = \frac{-a+1}{a+1}$, $a_6 = -\frac{1}{a}$, $a_7 = -\frac{2a}{a^2-1}$, and $a_8 = 1$.

    Now we have $a_{10}a_8 a_7 = a_{10} + a_8 + a_7$. No value of a_{10} can satisfy this equation iff $a_8 a_7 = 1$ and $a_8 + a_7 \neq 0$. Since a_8 is 1, we want $1 = a_7 = -\frac{2a}{a^2-1}$, which gives $a^2 + 2a - 1 = 0$. The only positive root of this equation is $\sqrt{2} - 1$.

    This problem can also be solved by a tangent substitution. Write $a_n = \tan \alpha_n$. The given condition becomes

    $$\alpha_{n+2} + \alpha_n + \alpha_{n-1} = 0.$$

We are given $\alpha_0 = 0$, $\alpha_1 = \pi/4$, and $\alpha_2 \in (0, \pi/2)$. Using this, we can recursively compute $\alpha_3, \alpha_4, \ldots$ in terms of α_2 until we get to $\alpha_{10} = \frac{3\pi}{4} - 2\alpha_2$. For a_{10} not to exist, we need $\alpha_{10} \equiv \pi/2 \bmod \pi$. The only possible value of $\alpha_2 \in (0, \pi/2)$ is $\alpha_2 = \pi/8$, which gives $a_2 = \tan \pi/8 = \sqrt{2} - 1$.

33. **[17]** A circle Γ with center O has radius 1. Consider pairs (A, B) of points so that A is inside the circle and B is on its boundary. The circumcircle Ω of OAB intersects Γ again at $C \neq B$, and line AC intersects Γ again at $X \neq C$. The pair (A, B) is called *techy* if line OX is tangent to Ω. Find the area of the region of points A so that there exists a B for which (A, B) is techy.

Proposed by: Carl Schildkraut and Milan Haiman

Answer: $\boxed{\frac{3\pi}{4}}$

We claim that (A, B) is techy if and only if $OA = AB$.

Note that OX is tangent to the circle (OBC) if and only if OX is perpendicular to the angle bisector of $\angle BOC$, since $OB = OC$. Thus (A, B) is techy if and only if OX is parallel to BC. Now since $OC = OX$,

$$OX \parallel BC \iff \angle BCA = \angle OXA \iff \angle BCA = \angle ACO \iff OA = AB.$$

From the claim, the desired region of points A is an annulus between the circles centered at O with radii $\frac{1}{2}$ and 1. So the answer is $\frac{3\pi}{4}$.

34. **[20]** A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

A polynomial is called *k-tiny* if it has degree at most 7 and integer coefficients between $-k$ and k, inclusive.

A polynomial is called *nearly tricky* if it is the sum of a tricky polynomial and a 1-tiny polynomial.

Let N be the number of nearly tricky 7-tiny polynomials. Estimate N.

An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^4 \right\rfloor$ points.

Proposed by: Carl Schildkraut

Answer: $\boxed{64912347}$

A tricky 7-tiny polynomial takes the form

$$(c_6 x^6 + \ldots + c_1 x + c_0)(x - 4).$$

For each fixed value of k, $c_k - 4c_{k+1}$ should lie in $[-7, 7]$, so if we fix c_k, there are around $15/4$ ways of choosing c_{k+1}. Therefore if we pick $c_0, \ldots, c_6$ in this order, there should be around $(15/4)^7$ tricky 7-tiny polynomials.

A 1-tiny polynomial takes the form $\varepsilon_6 x^7 + \cdots + \varepsilon_1 x + \varepsilon_0$ with $\varepsilon_i \in \{-1, 0, +1\}$, so there are 3^8 1-tiny polynomials.

A nearly tricky 7-tiny polynomial P takes the form $Q+T$ where Q is roughly a tricky 7-tiny polynomial, and T is 1-tiny. Furthermore, there is a unique decomposition $Q + T$ because $T(4) = P(4)$ and each integer n can be written in the form $\sum \varepsilon_k 4^k$ in at most one way. Therefore the number of nearly tricky 7-tiny is around $(15/4)^7 \cdot 3^8 \approx 68420920$, which is worth 16 points.

The exact answer can be found by setting up recurrences. Let $t(d, \ell)$ be the number of polynomials of degree at most i of the form

$$(\ell x^{d-1} + c_{d-2} x^{d-2} + \cdots + c_0)(x - 4) + (\varepsilon_{d-1} x^{d-1} + \cdots + \varepsilon_1 x + \varepsilon_0).$$

which has integer coefficients between -7 and 7 except the leading term ℓx^d. It follows that $t(0, 0) = 1, t(0, k) = 0$ for all $k \neq 0$, and $t(d + 1, \ell)$ can be computed as follows: for each value of c_{d-1}, there are

$t(d, c_{d-1})$ ways to pick $c_{d-2}, \ldots, c_0, \varepsilon_{d-1}, \ldots, \varepsilon_0$, and exactly $w(c_{d-1} - 4\ell)$ ways of picking ε_d, where $w(k) = \min(9 - |k|, 3)$ for $|k| \leq 8$ and 0 otherwise. Therefore setting $c = c_{d-1} - 4\ell$ we have

$$t(d+1, \ell) = \sum_{c=-8}^{8} t(d, c+4\ell) w(c).$$

The number of nearly tricky 7-tiny polynomials is simply $t(8, 0)$, which can be computed to be 64912347 using the following C code.

```
int w(int a){
    if(a < -9 || a > 9) return 0;
    else if(a == -8 || a == 8) return 1;
    else if(a == -7 || a == 7) return 2;
    else return 3;
}

int main()
{
    int m=8,n=7,r=4,d,l,c,c4l;
    int mid = 2 + n/r;
    int b = 2*mid+1;
    long int t[500][500];
    for(l=0; l<b; l++){
        t[0][l] = (l == mid) ? 1 : 0;
    }
    for(d=0; d<m+1; d++){
        for(l=0; l<b; l++){
            t[d+1][l] = 0;
            for(c=-8; c<9; c++){
                c4l = c + 4*(l-mid) + mid;
                t[d+1][l] += (c4l >= 0 && c4l <= 2*mid) ? t[d][c4l]*w(c) : 0;
            }
        }
    }
    printf("%ld",t[8][mid]);
}
```

35. **[20]** You are trying to cross a 400 foot wide river. You can jump at most 4 feet, but you have many stones you can throw into the river. You will stop throwing stones and cross the river once you have placed enough stones to be able to do so. You can throw straight, but you can't judge distance very well, so each stone ends up being placed uniformly at random along the width of the river. Estimate the expected number N of stones you must throw before you can get across the river.

An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^3 \right\rfloor$ points.

Proposed by: Carl Schildkraut and Milan Haiman

Answer: $\boxed{712.811}$

If we divide the river into 100 4-foot sections, then to be able to cross we need to get at least one stone into each section. On average, this takes

$$\frac{100}{100} + \frac{100}{99} + \cdots + \frac{100}{1} \approx 100 \ln 100$$

stone throws (it takes $\frac{100}{100-k}$ moves on average to get a stone into a new section if k sections already have a stone). So the answer is at least $100 \ln 100 \approx 450$.

On the other hand, if we divide the river into 200 2-foot sections, then once we have a stone in each section we are guaranteed to be able to cross. By a similar argument, we obtain that the answer is at most $200 \ln 200 \approx 1050$.

Estimates near these bounds earn about 5 to 7 points. An estimate in between can earn close to 20 points.

To compute the answer (almost) exactly, we use the following argument.

Scale the problem so the river is of size 1, and the jumps are of size 0.01. Suppose that after n throws, the stones thrown are located at positions $0 < x_1 < x_2 < \cdots < x_n < 1$. Let $x_0 = 0, x_{n+1} = 1, r = 0.01$. Define $P(n)$ to be the probability that you still cannot cross the river after n throws. In other words, there exists i such that $x_{i+1} - x_i > r$. Then our answer is $\sum_{n=0}^{\infty} P(n)$.

By PIE we can write

$$P(n) = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+1}{i} \max(1 - ir, 0)^n.$$

based on which intervals $x_{i+1} - x_i$ have length greater than r. Now we switch the order of summation:

$$\sum_{n=0}^{\infty} P(n) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+1}{i} \max(1 - ir, 0)^n = \sum_{i=1}^{\infty} (-1)^{i-1} \sum_{n=0}^{\infty} \binom{n+1}{i} \max(1 - ir, 0)^n.$$

Let $x = \max(1 - ir, 0)$. Then

$$\sum_{n=0}^{\infty} \binom{n+1}{i} x^n = x^{i-1} \sum_{j=0}^{\infty} \binom{i+j}{i} x^j = \frac{x^{i-1}}{(1-x)^{i+1}}.$$

Thus, our answer is

$$\sum_{i=1}^{\lfloor 1/r \rfloor} (-1)^{i-1} \frac{(1 - ir)^{i-1}}{(ir)^{i+1}} \approx 712.811,$$

where the last approximation uses the C++ code below.

```
#include <bits/stdc++.h>

using namespace std;

typedef long double ld;

int main() {
ld sum = 0, r = 0.01;
for (int i = 1; ; ++i) {
ld x = 1-r*i; if (x <= 0) break;
ld ex = pow(x/(1-x),i-1)/(1-x)/(1-x);
if (i&1) sum += ex;
else sum -= ex;
}
cout << fixed << setprecision(8) << sum;
}
```

36. **[20]** Let N be the number of sequences of positive integers $(a_1, a_2, a_3, \ldots, a_{15})$ for which the polynomials

$$x^2 - a_i x + a_{i+1}$$

each have an integer root for every $1 \le i \le 15$, setting $a_{16} = a_1$. Estimate N.

An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^2 \right\rfloor$ points.

Proposed by: Krit Boonsiriseth

Answer: $\boxed{1409}$

We note that $a_{i+1} = x(a_i - x)$ for some positive integer x, so $a_{i+1} \geq a_i - 1$. So, the only way a_i can decrease is decreasing by 1.

As it cannot decrease that quickly, we will make the assumption that if $a_i \geq 10$, $a_{i+1} = a_i - 1$, as otherwise it will increase at least above 16 at which point it will take many moves to go back down below 10. Write that $a \to b$ if b is a possible value of a_{i+1} given $a = a_i$. We have

$$5 \to 6,\ 6 \to 5, 8, 9,\ 7 \to 6,\ 8 \to 7,\ 9 \to 8,$$

and in addition by going to 10 and above, 7 can go to 9 in 2 or 4 steps, 8 can in $4, 7, 8$ steps, and 9 can in $6, 10, 12$ steps. We see from this that the vast majority of sequences should pass through 8. By looking at cycles from 8, we can determine exactly when a sequence can start at 8 and return to 8 (there is one way in 3 steps, two in 4 steps, etc.), and from there we can generate a list of types of sequences by when 8s occur. By dividing by the number of 8s and multiplying by 15, we can get the number of sequences that include 8, which gives us an estimate of 1235, giving us 15 points. As we note that this is a lower estimate, we may round up slightly to get better results.

To find the exact answer, we will first show that no element larger than 32 can occur in the sequence. Reorder the sequence to make a_1 maximal; we have

$$a_{i+1} \geq a_i - 1 \implies a_{15} \geq a_1 - 14.$$

Also, since $a_1 > a_{15}$, $a_1 \geq 2a_{15} - 4$, giving

$$a_1 - 14 \leq \frac{a_1 + 4}{2} \implies a_1 \leq 32.$$

We then construct the following Python code:

```
def p36(max_val,length):
L=[[i] for i in range(1,max_val+1)]
for j in range(length):
newL=[]
for k in L:
poss=[x*(k[-1]-x) for x in range(1,k[-1]//2+1)]
for t in poss:
if 1<=t<=max_val:
newL.append(k+[t])
L=newL
return len(L)

print(p36(32,15))
```

This gives the exact answer of 1409.