HMMT February 2019

February 16, 2019

Combinatorics

1. How many distinct permutations of the letters of the word REDDER are there that do not contain a palindromic substring of length at least two? (A *substring* is a contiguous block of letters that is part of the string. A string is *palindromic* if it is the same when read backwards.)

2. Your math friend Steven rolls five fair icosahedral dice (each of which is labelled $1, 2, \dots, 20$ on its sides). He conceals the results but tells you that at least half of the rolls are 20. Suspicious, you examine the first two dice and find that they show 20 and 19 in that order. Assuming that Steven is truthful, what is the probability that all three remaining concealed dice show 20?

3. Reimu and Sanae play a game using 4 fair coins. Initially both sides of each coin are white. Starting with Reimu, they take turns to color one of the white sides either red or green. After all sides are colored, the 4 coins are tossed. If there are more red sides showing up, then Reimu wins, and if there are more green sides showing up, then Sanae wins. However, if there is an equal number of red sides and green sides, then *neither* of them wins. Given that both of them play optimally to maximize the probability of winning, what is the probability that Reimu wins?

4. Yannick is playing a game with 100 rounds, starting with 1 coin. During each round, there is a $n\%$ chance that he gains an extra coin, where n is the number of coins he has at the beginning of the round. What is the expected number of coins he will have at the end of the game?

5. Contessa is taking a random lattice walk in the plane, starting at $(1, 1)$. (In a random lattice walk, one moves up, down, left, or right 1 unit with equal probability at each step.) If she lands on a point of the form $(6m, 6n)$ for $m, n \in \mathbb{Z}$, she ascends to heaven, but if she lands on a point of the form $(6m+3, 6n+3)$ for $m, n \in \mathbb{Z}$, she descends to hell. What is the probability that she ascends to heaven?

6. A point P lies at the center of square $ABCD$. A sequence of points $\{P_n\}$ is determined by $P_0 = P$, and given point P_i, point P_{i+1} is obtained by reflecting P_i over one of the four lines AB, BC, CD, DA, chosen uniformly at random and independently for each i. What is the probability that $P_8 = P$?

7. In an election for the Peer Pressure High School student council president, there are 2019 voters and two candidates Alice and Celia (who are voters themselves). At the beginning, Alice and Celia both vote for themselves, and Alice's boyfriend Bob votes for Alice as well. Then one by one, each of the remaining 2016 voters votes for a candidate randomly, with probabilities proportional to the current number of the respective candidate's votes. For example, the first undecided voter David has a $\frac{2}{3}$ probability of voting for Alice and a $\frac{1}{3}$ probability of voting for Celia.

   What is the probability that Alice wins the election (by having more votes than Celia)?

8. For a positive integer N, we color the positive divisors of N (including 1 and N) with four colors. A coloring is called *multichromatic* if whenever a, b and $\gcd(a, b)$ are pairwise distinct divisors of N, then they have pairwise distinct colors. What is the maximum possible number of multichromatic colorings a positive integer can have if it is not the power of any prime?

9. How many ways can one fill a 3×3 square grid with nonnegative integers such that no *nonzero* integer appears more than once in the same row or column and the sum of the numbers in every row and column equals 7?

10. Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position (a_1, a_2, a_3, a_4) to some position (x_1, x_2, x_3, x_4) with integer coordinates satisfying

    $$(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2+(x_4-a_4)^2 = 4 \quad \text{and} \quad |(x_1+x_2+x_3+x_4)-(a_1+a_2+a_3+a_4)| = 2.$$

    In how many ways can Fred reach $(10, 10, 10, 10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk?