HMMT November 2012

Saturday 10 November 2012

Theme Round

1. [**3**] If $4^{4^4} = \sqrt[128]{2^{2^{2^n}}}$, find n.

   Answer: $\boxed{4}$ We rewrite the left hand side as

   $$(2^2)^{4^4} = 2^{2\cdot 4^4} = 2^{2^9},$$

   and the right hand side as

   $$(2^{2^{2^n}})^{\frac{1}{128}} = 2^{2^{(2^n-7)}}.$$

   Equating, we find $2^n - 7 = 9$, yielding $n = 4$.

2. [**4**] If $x^x = 2012^{2012^{2013}}$, find x.

   Answer: $\boxed{2012^{2012}}$ We have

   $$2012^{2012^{2013}} = 2012^{2012\cdot 2012^{2012}} = (2012^{2012})^{2012^{2012}}.$$

   Thus, $x = 2012^{2012}$.

3. [**4**] Find the smallest positive integer n such that $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n} > 3^{3^{3^3}}$. (The notation $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$ is used to denote a power tower with n 2's. For example, $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$ with $n = 4$ would equal $2^{2^{2^2}}$.)

   Answer: $\boxed{6}$ Clearly, $n \geq 5$. When we take $n = 5$, we have

   $$2^{2^{2^{2^2}}} = 2^{2^{16}} < 3^{3^{27}} = 3^{3^{3^3}}.$$

   On the other hand, when $n = 6$, we have

   $$2^{2^{2^{2^{2^2}}}} = 2^{2^{65536}} = 4^{2^{65535}} > 4^{4^{27}} > 3^{3^{27}} = 3^{3^{3^3}}.$$

   Our answer is thus $n = 6$.

4. [**7**] Find the sum of all real solutions for x to the equation $(x^2+2x+3)^{(x^2+2x+3)^{(x^2+2x+3)}} = 2012$.

   Answer: $\boxed{-2}$ When $y = x^2+2x+3$, note that there is a unique real number y such that $y^{y^y} = 2012$ because y^{y^y} is increasing in y. The sum of the real distinct solutions of the equation $x^2+2x+3 = y$ is -2 by Vieta's Formulae as long as $2^2 + 4(y-3) > 0$, which is equivalent to $y > 2$. This is easily seen to be the case; therefore, our answer is -2.

5. [**7**] Given any positive integer, we can write the integer in base 12 and add together the digits of its base 12 representation. We perform this operation on the number $7^{6^{5^{4^{3^{2^1}}}}}$ repeatedly until a single base 12 digit remains. Find this digit.

   Answer: $\boxed{4}$ For a positive integer n, let $s(n)$ be the sum of digits when n is expressed in base 12. We claim that $s(n) \equiv n \pmod{11}$ for all positive integers n. Indeed, if $n = d_k 12^k + d_{k-1}12^{k-1} + \cdots + d_0$ with each d_i an integer between 0 and 11, inclusive, because $12 \equiv 1 \pmod{11}$, reducing modulo 11 gives exactly $s(n)$. Thus, our answer is congruent to $N = 7^{6^{5^{4^{3^{2^1}}}}}$ modulo 11, and furthermore must be a one-digit integer in base 12; these two conditions uniquely determine the answer.

   By Fermat's Little Theorem, $7^{10} \equiv 1 \pmod{11}$, and also observe that $6^{5^{4^{3^{2^1}}}} \equiv 6 \pmod{10}$ because $6 \cong 0 \pmod 2$ and $6 \cong 1 \pmod 5$. Thus, $N \equiv 7^6 \equiv 343^2 \equiv 2^2 \equiv 4 \pmod{11}$, which is our answer. (Additionally, we note that this process of writing the number in base twelve and summing the digits must eventually terminate because the value decreases after each step.)

6. [**3**] A rectangular piece of paper with vertices $ABCD$ is being cut by a pair of scissors. The pair of scissors starts at vertex A, and then cuts along the angle bisector of DAB until it reaches another edge of the paper. One of the two resulting pieces of paper has 4 times the area of the other piece. What is the ratio of the longer side of the original paper to the shorter side?

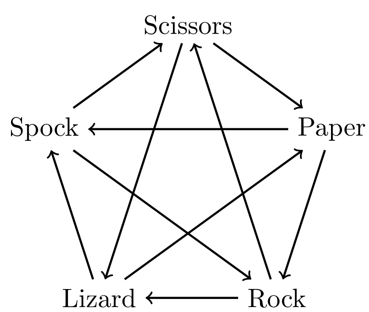
Answer: $\boxed{\frac{5}{2}}$ Without loss of generality, let $AB > AD$, and let $x = AD$, $y = AB$. Let the cut along the angle bisector of $\angle DAB$ meet CD at E. Note that ADE is a 45-45-90 triangle, so $DE = AD = x$, and $EC = y - x$. Now, $[ADE] = \frac{x^2}{2}$, and $[AECB] = x(y - \frac{x}{2}) = 4[ADE]$. Equating and dividing both sides by x, we find that $2x = y - \frac{x}{2}$, so $y/x = \frac{5}{2}$.

7. [**4**] The game of rock-scissors is played just like rock-paper-scissors, except that neither player is allowed to play paper. You play against a poorly-designed computer program that plays rock with 50% probability and scissors with 50% probability. If you play optimally against the computer, find the probability that after 8 games you have won at least 4.

Answer: $\boxed{\frac{163}{256}}$ Since rock will always win against scissors, the optimum strategy is for you to always play rock; then, you win a game if and only if the computer plays scissors. Let p_n be the probability that the computer plays scissors n times; we want $p_0 + p_1 + p_2 + p_3 + p_4$. Note that by symmetry, $p_n = p_{8-n}$ for $n = 0, 1, \ldots, 8$, and because $p_0 + p_1 + \cdots + p_8 = 1$, $p_0 + \cdots + p_3 = p_5 + \cdots + p_8 = (1 - p_4)/2$. Our answer will thus be $(1 + p_4)/2$.

If the computer is to play scissors exactly 4 times, there are $\binom{8}{4}$ ways in which it can do so, compared to 2^8 possible combinations of eight plays. Thus, $p_4 = \binom{8}{4}/2^8 = 35/128$. Our answer is thus $\frac{1+\frac{35}{128}}{2} = \frac{163}{256}$.

8. [**4**] In the game of rock-paper-scissors-lizard-Spock, rock defeats scissors and lizard, paper defeats rock and Spock, scissors defeats paper and lizard, lizard defeats paper and Spock, and Spock defeats rock and scissors, as shown in the below diagram. As before, if two players choose the same move, then there is a draw. If three people each play a game of rock-paper-scissors-lizard-Spock at the same time by choosing one of the five moves at random, what is the probability that one player beats the other two?



Answer: $\boxed{\frac{12}{25}}$ Let the three players be A, B, C. Our answer will simply be the sum of the probability that A beats both B and C, the probability that B beats both C and A, and the probability that C beats A and B, because these events are all mutually exclusive. By symmetry, these three probabilities are the same, so we only need to compute the probability that A beats both B and C. Given A's play, the probability that B's play loses to that of A is $2/5$, and similarly for C. Thus, our answer is $3 \cdot \left(\frac{2}{5}\right) \cdot \left(\frac{2}{5}\right) = \frac{12}{25}$.

9. [**6**] 64 people are in a single elimination rock-paper-scissors tournament, which consists of a 6-round knockout bracket. Each person has a different rock-paper-scissors skill level, and in any game, the person with the higher skill level will always win. For how many players P is it possible that P wins the first four rounds that he plays?

(A *6-round knockout bracket* is a tournament which works as follows:

(a) In the first round, all 64 competitors are paired into 32 groups, and the two people in each group play each other. The winners advance to the second round, and the losers are eliminated.

(b) In the second round, the remaining 32 players are paired into 16 groups. Again, the winner of each group proceeds to the next round, while the loser is eliminated.

(c) Each round proceeds in a similar way, eliminating half of the remaining players. After the sixth round, only one player will not have been eliminated. That player is declared the champion.)

Answer: $\boxed{49}$ Note that a sub-bracket, that is, a subset of games of the tournament that themselves constitute a bracket, is always won by the person with the highest skill level. Therefore, a person wins her first four rounds if and only if she has the highest skill level among the people in her 16-person sub-bracket. This is possible for all but the people with the $16-1=15$ lowest skill levels, so our answer is $64-15=49$.

10. [**8**] In a game of rock-paper-scissors with n people, the following rules are used to determine a champion:

(a) In a round, each person who has not been eliminated randomly chooses one of rock, paper, or scissors to play.

(b) If at least one person plays rock, at least one person plays paper, and at least one person plays scissors, then the round is declared a tie and no one is eliminated. If everyone makes the same move, then the round is also declared a tie.

(c) If exactly two moves are represented, then everyone who made the losing move is eliminated from playing in all further rounds (for example, in a game with 8 people, if 5 people play rock and 3 people play scissors, then the 3 who played scissors are eliminated).

(d) The rounds continue until only one person has not been eliminated. That person is declared the champion and the game ends.

If a game begins with 4 people, what is the expected value of the number of rounds required for a champion to be determined?

Answer: $\boxed{\frac{45}{14}}$ For each positive integer n, let E_n denote the expected number of rounds required to determine a winner among n people. Clearly, $E_1=0$. When $n=2$, on the first move, there is a $\frac{1}{3}$ probability that there is a tie, and a $\frac{2}{3}$ probability that a winner is determined. In the first case, the expected number of additional rounds needed is exactly E_2; in the second, it is E_1. Therefore, we get the relation

$$E_2=\frac{1}{3}(E_2+1)+\frac{2}{3}(E_1+1),$$

from which it follows that $E_2=\frac{3}{2}$.

Next, if $n=3$, with probability $\frac{1}{9}$ there is only one distinct play among the three players, and with probability $\frac{6}{27}=\frac{2}{9}$ all three players make different plays. In both of these cases, no players are eliminated. In all remaining situations, which occur with total probability $\frac{2}{3}$, two players make one play and the third makes a distinct play; with probability $\frac{1}{3}$ two players are eliminated and with probability $\frac{1}{3}$ one player is eliminated. This gives the relation

$$E_3=\frac{1}{3}(E_3+1)+\frac{1}{3}(E_2+1)+\frac{1}{3}(E_1+1),$$

from which we find that $E_3=\frac{9}{4}$.

Finally, suppose $n=4$. With probability $\frac{1}{27}$, all four players make the same play, and with probability $\frac{3\cdot 6\cdot 2}{81}=\frac{4}{9}$, two players make one play, and the other two players make the other two plays; in both cases no players are eliminated, with total probability $\frac{1}{27}+\frac{4}{9}=\frac{13}{27}$ over the two cases. With probability $\frac{6\cdot 4}{81}=\frac{8}{27}$, three players make one play and the fourth makes another; thus, there is a probability of $\frac{4}{27}$ for exactly one player being eliminated and a probability of $\frac{4}{27}$ of three players being eliminated.

Then, there is a remaining probability of $\frac{6\cdot 3}{81} = \frac{2}{9}$, two players make one play and the other two players make another. Similar analysis from before yields

$$E_4 = \frac{13}{27}(E_4+1) + \frac{4}{27}(E_3+1) + \frac{2}{9}(E_2+1) + \frac{4}{27}(E_1+1),$$

so it follows that $E_4 = \frac{45}{14}$.