14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

Team Round B

Think Carefully! [55]

The problems in this section require only short answers.

1. [**55**] Tom, Dick, and Harry play a game in which they each pick an integer between 1 and 2011. Tom picks a number first and informs Dick and Harry of his choice. Then Dick picks a different number and informs Harry of his choice. Finally, Harry picks a number different from both Tom's and Dick's. After all the picks are complete, an integer is randomly selected between 1 and 2011. The player whose number is closest wins 2 dollars, unless there is a tie, in which case each of the tied players wins 1 dollar. If Tom knows that Dick and Harry will each play optimally and select randomly among equally optimal choices, there are two numbers Tom can pick to maximize his expected profit; what are they?

Complex Numbers [90]

The problems in this section require only short answers.

The *norm* of a complex number $z = a + bi$, denoted by $|z|$, is defined to be $\sqrt{a^2 + b^2}$. In the following problems, it may be helpful to note that the norm is multiplicative and that it obeys the triangle inequality. In other words, please observe that for all complex numbers x and y, $|xy| = |x||y|$ and $|x + y| \leq |x| + |y|$. (You may verify these facts for yourself if you like).

2. [**20**] Let a, b, and c be complex numbers such that $|a| = |b| = |c| = |a + b + c| = 1$. If $|a - b| = |a - c|$ and $b \neq c$, evaluate $|a + b||a + c|$.

3. [**30**] Let x and y be complex numbers such that $|x| = |y| = 1$.

   (a) [**15**] Determine the maximum value of $|1 + x| + |1 + y| - |1 + xy|$.

   (b) [**15**] Determine the maximum value of $|1+x|+|1+xy|+|1+xy^2|+\ldots+|1+xy^{2011}|-1006|1+y|$.

4. [**40**] Let a, b, and c be complex numbers such that $|a| = |b| = |c| = 1$. If

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = 1$$

as well, determine the product of all possible values of $|a + b + c|$.

Warm Up Your Proof Skills! [40]

The problems in this section require complete proofs.

5. (a) [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $0 < y - x < 1$. The first player to write a number greater than or equal to 2010 wins. If Alice goes first, determine, with proof, who has the winning strategy.

   (b) [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $y - x \in (0, 1)$. The first player to write a number greater than or equal to 2010 on her 2011th turn or later wins. If a player writes a number greater than or equal to 2010 on her 2010th turn or before, she loses immediately. If Alice goes first, determine, with proof, who has the winning strategy.

The Euler Totient Function [40]

The problems in this section require complete proofs.

Euler's *totient* function, denoted by φ, is a function whose domain is the set of positive integers. It is especially important in number theory, so it is often discussed on the radio or on national TV. (Just kidding). But what is it, exactly? For all positive integers k, $\varphi(k)$ is defined to be the number of positive integers less than or equal to k that are relatively prime to k. It turns out that φ is what you would call a *multiplicative function*, which means that if a and b are relatively prime positive integers, $\varphi(ab) = \varphi(a)\varphi(b)$. Unfortunately, the proof of this result is highly nontrivial. However, there is much more than that to φ, as you are about to discover!

6. [**10**] Let n be a positive integer such that $n > 2$. Prove that $\varphi(n)$ is even.
7. [**10**] Let n be an even positive integer. Prove that $\varphi(n) \leq \frac{n}{2}$.
8. [**20**] Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be a set of positive integers such that $a_1 = 2$ and $a_m = \varphi(a_{m+1})$ for all $1 \leq m \leq n-1$. Prove that $a_n \geq 2^{n-1}$.

Introduction to the Symmedian [70]

The problems in this section require complete proofs.

If A, B, and C are three points in the plane that do not all lie on the same line, the symmedian from A in triangle ABC is defined to be the reflection of the median from A in triangle ABC about the bisector of angle A. Like the φ function, it turns out that the symmedian satisfies some interesting properties, too. For instance, just like how the medians from A, B, and C all intersect at the centroid of triangle ABC, the symmedians from A, B, and C all intersect at what is called (no surprises here) the symmedian point of triangle ABC. The proof of this fact is not easy, but it is unremarkable. In this section, you will investigate some surprising alternative constructions of the symmedian.

9. [**25**] Let ABC be a non-isosceles, non-right triangle, let ω be its circumcircle, and let O be its circumcenter. Let M be the midpoint of segment BC. Let the tangents to ω at B and C intersect at X. Prove that $\angle OAM = \angle OXA$. (Hint: use SAS similarity).
10. [**15**] Let the circumcircle of triangle AOM intersect ω again at D. Prove that points A, D, and X are collinear.
11. [**10**] Let H be the intersection of the three altitudes of triangle ABC. (This point is usually called the orthocenter). Prove that $\angle DAH = \angle MAO$.
12. [**10**] Prove that line AD is the symmedian from A in triangle ABC by showing that $\angle DAB = \angle MAC$.
13. [**10**] Prove that line AD is also the symmedian from D in triangle DBC.

Last Writes [65]

The problems in this section require complete proofs.

14. [**25**] Rachel and Brian are playing a game in a grid with 1 row of 2011 squares. Initially, there is one white checker in each of the first two squares from the left, and one black checker in the third square from the left. At each stage, Rachel can choose to either run or fight. If Rachel runs, she moves the black checker moves 1 unit to the right, and Brian moves each of the white checkers one unit to the right. If Rachel chooses to fight, she pushes the checker immediately to the left of the black checker 1 unit to the left; the black checker is moved 1 unit to the right, and Brian places a new white checker in the cell immediately to the left of the black one. The game ends when the black checker reaches the last cell. How many different final configurations are possible?
15. [**40**] On Facebook, there is a group of people that satisfies the following two properties: (i) there exists a positive integers k such that any subset of $2k-1$ people in the group contains a subset of k people in the group who are all friends with each other, and (ii) every member of the group has 2011 friends or fewer.
    (a) [**15**] If $k = 2$, determine, with proof, the maximum number of people the group may contain.
    (b) [**25**] If $k = 776$, determine, with proof, the maximum number of people the group may contain.