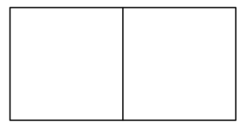
HMMT February 2015

Saturday 21 February 2015

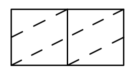
Geometry

1. Let R be the rectangle in the Cartesian plane with vertices at $(0,0),(2,0),(2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown.



Pro selects a point P uniformly at random in the interior of R. Find the probability that the line through P with slope $\frac{1}{2}$ will pass through both unit squares.

Answer: $\boxed{\frac{3}{4}}$



Precisely the middle two (of four) regions satisfiy the problem conditions, and it's easy to compute the (fraction of) areas as $\frac{3}{4}$.

2. Let ABC be a triangle with orthocenter H; suppose that $AB = 13$, $BC = 14$, $CA = 15$. Let G_A be the centroid of triangle HBC, and define G_B, G_C similarly. Determine the area of triangle $G_AG_BG_C$.

Answer: $\boxed{28/3}$ Let D, E, F be the midpoints of BC, CA, and AB, respectively. Then $G_AG_BG_C$ is the DEF about H with a ratio of $\frac{2}{3}$, and DEF is the dilation of ABC about H with a ratio of $-\frac{1}{2}$, so $G_AG_BG_C$ is the dilation of ABC about H with ratio $-\frac{1}{3}$. Thus $[G_AG_BG_C] = \frac{[ABC]}{9}$. By Heron's formula, the area of ABC is $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, so the area of $G_AG_BG_C$ is $[ABC]/9 = 84/9 = 28/3$.

3. Let $ABCD$ be a quadrilateral with $\angle BAD = \angle ABC = 90°$, and suppose $AB = BC = 1$, $AD = 2$. The circumcircle of ABC meets $\overline{AD}$ and $\overline{BD}$ at points E and F, respectively. If lines AF and CD meet at K, compute EK.

Answer: $\boxed{\frac{\sqrt{2}}{2}}$ Assign coordinates such that B is the origin, A is $(0,1)$, and C is $(1,0)$. Clearly, E is the point $(1,1)$. Since the circumcenter of ABC is $(\frac{1}{2},\frac{1}{2})$, the equation of the circumcircle of ABC is $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2 = \frac{1}{2}$. Since line BD is given by $x = 2y$, we find that F is at $(\frac{6}{5},\frac{3}{5})$. The intersection of AF with CD is therefore at $(\frac{3}{2},\frac{1}{2})$, so K is the midpoint of CD. As a result, $EK = \frac{\sqrt{2}}{2}$.

This is in fact a special case of APMO 2013, Problem 5, when the quadrilateral is a square.

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 3, BC = 2, CD = 2, DA = 4$. Let lines perpendicular to $\overline{BC}$ from B and C meet $\overline{AD}$ at B' and C', respectively. Let lines perpendicular to $\overline{AD}$ from A and D meet $\overline{BC}$ at A' and D', respectively. Compute the ratio $\frac{[BCC'B']}{[DAA'D']}$, where $[\varpi]$ denotes the area of figure ϖ.

Answer: $\boxed{\frac{37}{76}}$ To get a handle on the heights CB', etc. perpendicular to BC and AD, let $X = BC \cap AD$, which lies on ray $\overrightarrow{BC}$ and $\overrightarrow{AD}$ since $\widehat{AB} > \widehat{CD}$ (as chords $AB > CD$).

By similar triangles we have equality of ratios $XC : XD : 2 = (XD+4) : (XC+2) : 3$, so we have a system of linear equations: $3XC = 2XD + 8$ and $3XD = 2XC + 4$, so $9XC = 6XD + 24 = 4XC + 32$ gives $XC = \frac{32}{5}$ and $XD = \frac{84/5}{3} = \frac{28}{5}$.

It's easy to compute the trapezoid area ratio $\frac{[BC'B'C]}{[AA'D'D]} = \frac{BC(CB'+BC')}{AD(AD'+DA')} = \frac{BC}{AD}\frac{XC+XB}{XA+XD}$ (where we have similar right triangles due to the common angle at X). This is just $\frac{BC}{AD}\frac{XC+BC/2}{XD+AD/2} = \frac{2}{4}\frac{32/5+1}{28/5+2} = \frac{37}{76}$.

5. Let I be the set of points (x, y) in the Cartesian plane such that

$$x > \left(\frac{y^4}{9} + 2015\right)^{1/4}$$

Let $f(r)$ denote the area of the intersection of I and the disk $x^2 + y^2 \le r^2$ of radius $r > 0$ centered at the origin $(0, 0)$. Determine the minimum possible real number L such that $f(r) < Lr^2$ for all $r > 0$.

Answer: $\boxed{\frac{\pi}{3}}$ Let $B(P, r)$ be the (closed) disc centered at P with radius r. Note that for all $(x, y) \in I$, $x > 0$, and $x > \left(\frac{y^4}{9} + 2015\right)^{1/4} > \frac{|y|}{\sqrt{3}}$. Let $I' = \{(x, y) : x\sqrt{3} > |y|\}$. Then $I \subseteq I'$ and the intersection of I' with $B((0, 0), r)$ is $\frac{\pi}{3}r^2$, so the $f(r) =$ the area of $I \cap B((0, 0), r)$ is also less than $\frac{\pi}{3}r^2$. Thus $L = \frac{\pi}{3}$ works.

On the other hand, if $x > \frac{|y|}{\sqrt{3}} + 7$, then $x > \frac{|y|}{\sqrt{3}} + 7 > \left(\left(\frac{|y|}{9}\right)^4 + 7^4\right)^{1/4} > \left(\frac{y^4}{9} + 2015\right)^{1/4}$, which means that if $I'' = \{(x, y) : (x-7)\sqrt{3} > |y|\}$, then $I'' \subseteq I'$. However, for $r > 7$, the area of $I'' \cap B((7, 0), r-7)$ is $\frac{\pi}{3}(r-7)^2$, and $I'' \subseteq I$, $B((7, 0), r-7) \subseteq B((0, 0), r)$, which means that $f(r) > \frac{\pi}{3}(r-7)^2$ for all $r > 7$, from which it is not hard to see that $L = \frac{\pi}{3}$ is the minimum possible L.

Remark: The lines $y = \pm\sqrt{3}x$ are actually asymptotes for the graph of $9x^4 - y^4 = 2015$. The bulk of the problem generalizes to the curve $|\sqrt{3}x|^\alpha - |y|^\alpha = C$ (for a positive real $\alpha > 0$ and any real C); the case $\alpha = 0$ is the most familiar case of a hyperbola.

6. In triangle ABC, $AB = 2$, $AC = 1 + \sqrt{5}$, and $\angle CAB = 54°$. Suppose D lies on the extension of AC through C such that $CD = \sqrt{5} - 1$. If M is the midpoint of BD, determine the measure of $\angle ACM$, in degrees.

Answer: $\boxed{63}$ Let E be the midpoint of $\overline{AD}$. $EC = \sqrt{5} + 1 - \sqrt{5} = 1$, and $EM = 1$ by similar triangles ($ABD \sim EMD$). $\triangle ECM$ is isosceles, with $m\angle CEM = 54°$. Thus $m\angle ACM = m\angle ECM = \frac{180-54}{2} = 63°$.

7. Let $ABCDE$ be a square pyramid of height $\frac{1}{2}$ with square base $ABCD$ of side length $AB = 12$ (so E is the vertex of the pyramid, and the foot of the altitude from E to $ABCD$ is the center of square $ABCD$). The faces ADE and CDE meet at an acute angle of measure α (so that $0° < \alpha < 90°$). Find $\tan\alpha$.

Answer: $\boxed{\frac{17}{144}}$ Let X be the projection of A onto DE. Let $b = AB = 12$.

The key fact in this computation is that if Y is the projection of A onto face CDE, then the projection of Y onto line DE coincides with the projection of A onto line DE (i.e. X as defined above). We compute $AY = \frac{b}{\sqrt{b^2+1}}$ by looking at the angle formed by the faces and the square base (via $1/2$-$b/2$-$\sqrt{b^2+1}/2$ right triangle). Now we compute $AX = 2[AED]/ED = \frac{b\sqrt{b^2+1}/2}{\sqrt{2b^2+1}/2}$.

But $\alpha = \angle AXY$, so from $(b^2+1)^2 - (\sqrt{2b^2+1})^2 = (b^2)^2$, it easily follows that $\tan\alpha = \frac{\sqrt{2b^2+1}}{b^2} = \frac{17}{144}$.

8. Let S be the set of **discs** D contained completely in the set $\{(x, y) : y < 0\}$ (the region below the x-axis) and centered (at some point) on the curve $y = x^2 - \frac{3}{4}$. What is the area of the union of the elements of S?

Answer: $\boxed{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}$ **Solution 1.** An arbitrary point (x_0, y_0) is contained in S if and only if there exists some (x, y) on the curve $(x, x^2 - \frac{3}{4})$ such that $(x - x_0)^2 + (y - y_0)^2 < y^2$, since the radius of the circle is at most the distance from (x, y) to the x-axis. Some manipulation yields $x^2 - 2y_0(x^2 - \frac{3}{4}) - 2xx_0 + x_0^2 + y_0^2 < 0$.

Observe that $(x_0, y_0) \in S$ if and only if the optimal choice for x that minimizes the expression satisfies the inequality. The minimum is achieved for $x = \frac{x_0}{1-2y_0}$. After substituting and simplifying, we obtain $y_0\left(\frac{-x_0^2}{1-2y_0} + x_0^2 + y_0^2 + \frac{3}{2}y_0\right) < 0$. Since $y_0 < 0$ and $1 - 2y_0 > 0$, we find that we need $-2x_0^2 - 2y_0^2 + \frac{3}{2} - 2y_0 > 0 \iff 1 > x_0 + (y_0 + \frac{1}{2})^2$.

S is therefore the intersection of the lower half-plane and a circle centered at $(0, -\frac{1}{2})$ of radius 1. This is a circle of sector angle $4\pi/3$ and an isosceles triangle with vertex angle $2\pi/3$. The sum of these areas is $\frac{2\pi}{3} + \frac{\sqrt{3}}{4}$.

Solution 2. Let $O = (0, -\frac{1}{2})$ and $\ell = \{y = -1\}$ be the focus and directrix of the given parabola. Let ℓ' denote the x-axis. Note that a point P' is in S iff there exists a point P on the parabola in the lower half-plane for which $d(P, P') < d(P, \ell')$. However, for all such P, $d(P, \ell') = 1 - d(P, \ell) = 1 - d(P, O)$, which means that P' is in S iff there exists a P on the parabola for which $d(P', P) + d(P, O) < 1$. It is not hard to see that this is precisely the intersection of the unit circle centered at O and the lower half-plane, so now we can proceed as in Solution 1.

9. Let $ABCD$ be a regular tetrahedron with side length 1. Let X be the point in triangle BCD such that $[XBC] = 2[XBD] = 4[XCD]$, where $[\varpi]$ denotes the area of figure ϖ. Let Y lie on segment AX such that $2AY = YX$. Let M be the midpoint of BD. Let Z be a point on segment AM such that the lines YZ and BC intersect at some point. Find $\frac{AZ}{ZM}$.

Answer: $\boxed{\frac{4}{7}}$ We apply three-dimensional barycentric coordinates with reference tetrahedron $ABCD$. The given conditions imply that

$$\begin{aligned} X &= (0 : 1 : 2 : 4) \\ Y &= (14 : 1 : 2 : 4) \\ M &= (0 : 1 : 0 : 1) \\ Z &= (t : 1 : 0 : 1) \end{aligned}$$

for some real number t. Normalizing, we obtain $Y = \left(\frac{14}{21}, \frac{1}{21}, \frac{2}{21}, \frac{4}{21}\right)$ and $Z = \left(\frac{t}{t+2}, \frac{1}{t+2}, 0, \frac{1}{t+2}\right)$. If YZ intersects line BC then there exist parameters $\alpha + \beta = 1$ such that $\alpha Y + \beta Z$ has zero A and D coordinates, meaning

$$\begin{aligned} \frac{14}{21}\alpha + \frac{t}{t+2}\beta &= 0 \\ \frac{4}{21}\alpha + \frac{1}{t+2}\beta &= 0 \\ \alpha + \beta &= 1. \end{aligned}$$

Adding twice the second equation to the first gives $\frac{22}{21}\alpha + \beta = 0$, so $\alpha = -22$, $\beta = 21$, and thus $t = \frac{7}{2}$. It follows that $Z = (7 : 2 : 0 : 2)$, and $\frac{AZ}{ZM} = \frac{2+2}{7} = \frac{4}{7}$.

10. Let $\mathcal{G}$ be the set of all points (x, y) in the Cartesian plane such that $0 \le y \le 8$ and

$$(x - 3)^2 + 31 = (y - 4)^2 + 8\sqrt{y(8 - y)}.$$

There exists a unique line ℓ of **negative slope** tangent to $\mathcal{G}$ and passing through the point $(0, 4)$. Suppose ℓ is tangent to $\mathcal{G}$ at a **unique** point P. Find the coordinates (α, β) of P.

Answer: $\boxed{\left(\frac{12}{5}, \frac{8}{5}\right)}$ Let G be $\mathcal{G}$ restricted to the strip of plane $0 \le y \le 4$ (we only care about this region since ℓ has **negative slope** going down from $(0, 4)$). By completing the square, the original equation rearranges to $(x - 3)^2 + (\sqrt{y(8 - y)} - 4)^2 = 1$. One could finish the problem in a completely standard way via the single-variable parameterization $(x, \sqrt{y(8 - y)}) = (3 + \cos t, 4 + \sin t)$ on the appropriate interval of t—just take derivatives with respect to t to find slopes (the computations would probably not be too bad)—but we will present a slightly cleaner solution.

Consider the bijective plane transformation $\Phi : (x, y) \mapsto (x, \sqrt{y(8 - y)})$, with inverse $\Phi^{-1} : (x, y) \mapsto (x, 4 - \sqrt{16 - y^2})$. In general, Φ maps curves as follows: $\Phi(\{(x, y) : f(x, y) = c\}) = \{\Phi(x, y) : f(x, y) = c\} = \{(x', y') : f(\Phi^{-1}(x', y')) = c\}$.

Our line ℓ has the form $y - 4 = -mx$ for some $m > 0$. We have $\Phi(G) = \{(x - 3)^2 + (y - 4)^2 = 1 : 0 \le y \le 4\}$ and $\Phi(\{4 - y = mx : 0 \le y \le 4\}) = \{\sqrt{16 - y^2} = mx : 0 \le y \le 4\}$. Since ℓ is unique, m must also be. But it's easy to see that $m = 1$ gives a tangency point, so if our original tangency point was (u, v), then our new tangency point is $(u, \sqrt{v(8-v)}) = \frac{4}{5}(3, 4) = \left(\frac{12}{5}, \frac{16}{5}\right)$, and so $(u, v) = \left(\frac{12}{5}, \frac{8}{5}\right)$.

Remark. To see what $\mathcal{G}$ looks like, see Wolfram Alpha using the plotting/graphing commands.