February 2017
February 18, 2017

Team

1. [**15**] Let $P(x), Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers y, then $P(x) = Q(x)$ for all real numbers x.

2. [**25**] Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when x and y are both positive?

3. [**30**] A polyhedron has $7n$ faces. Show that there exist $n+1$ of the polyhedron's faces that all have the same number of edges.

4. [**35**] Let $w = w_1 w_2 \dots w_n$ be a word. Define a *substring* of w to be a word of the form $w_i w_{i+1} \dots w_{j-1} w_j$, for some pair of positive integers $1 \le i \le j \le n$. Show that w has at most n distinct palindromic substrings.

   For example, *aaaaa* has 5 distinct palindromic substrings, and *abcata* has 5 (a, b, c, t, ata).

5. [**35**] Let ABC be an acute triangle. The altitudes BE and CF intersect at the orthocenter H, and point O denotes the circumcenter. Point P is chosen so that $\angle APH = \angle OPE = 90^\circ$, and point Q is chosen so that $\angle AQH = \angle OQF = 90^\circ$. Lines EP and FQ meet at point T. Prove that points A, T, O are collinear.

6. [**40**] Let r be a positive integer. Show that if a graph G has no cycles of length at most $2r$, then it has at most $|V|^{2016}$ cycles of length exactly $2016r$, where $|V|$ denotes the number of vertices in the graph G.

7. [**45**] Let p be a prime. A *complete residue class modulo* p is a set containing at least one element equivalent to $k \pmod{p}$ for all k.

   (a) (**20**) Show that there exists an n such that the nth row of Pascal's triangle forms a complete residue class modulo p.

   (b) (**25**) Show that there exists an $n \le p^2$ such that the nth row of Pascal's triangle forms a complete residue class modulo p.

8. [**45**] Does there exist an irrational number $\alpha > 1$ such that

$$\lfloor \alpha^n \rfloor \equiv 0 \pmod{2017}$$

for all integers $n \ge 1$?

9. [**65**] Let n be a positive odd integer greater than 2, and consider a regular n-gon $\mathcal{G}$ in the plane centered at the origin. Let a *subpolygon* $\mathcal{G}'$ be a polygon with at least 3 vertices whose vertex set is a subset of that of $\mathcal{G}$. Say $\mathcal{G}'$ is *well-centered* if its centroid is the origin. Also, say $\mathcal{G}'$ is *decomposable* if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.

10. [**65**] Let LBC be a fixed triangle with $LB = LC$, and let A be a variable point on arc LB of its circumcircle. Let I be the incenter of $\triangle ABC$ and $\overline{AK}$ the altitude from A. The circumcircle of $\triangle IKL$ intersects lines KA and BC again at $U \neq K$ and $V \neq K$. Finally, let T be the projection of I onto line UV. Prove that the line through T and the midpoint of $\overline{IK}$ passes through a fixed point as A varies.