HMMT November 2014

Saturday 15 November 2014

Team Round

1. [**3**] What is the smallest positive integer n which cannot be written in any of the following forms?

   - $n = 1 + 2 + \cdots + k$ for a positive integer k.
   - $n = p^k$ for a prime number p and integer k.
   - $n = p + 1$ for a prime number p.

   Answer: $\boxed{22}$ Consider $1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19$ are in the form p^k. So we are left with $6, 10, 12, 14, 15, 18, 20, 21, 22, ...$

   Next, $6, 12, 14, 18, 20$ are in the form $p + 1$, so we are left with $10, 15, 21, 22, ...$

   Finally, $10, 15, 21$ are in the form $n = 1 + 2 + \cdots + k$, so we are left with $22, ...$

   Since $22 = 2 \cdot 11$ is not a prime power, $22 - 1 = 21$ is not prime, and $1 + 2 + \cdots + 6 = 21 < 22 < 28 = 1 + 2 + \cdots + 7$, 22 is the smallest number not in the three forms, as desired.

2. [**5**] Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

   Answer: $\boxed{23}$ Consider that $f(x) = x^2 + 6x + 7 = (x + 3)^2 - 2$. So $f(x) \geq -2$ for real numbers x. Also, f is increasing on the interval $[-3, \infty)$. Therefore

   $$f(f(x)) \geq f(-2) = -1,$$

   $$f(f(f(x))) \geq f(-1) = 2,$$

   and

   $$f(f(f(f(x)))) \geq f(2) = 23.$$

   Thus, the minimum value of $f(f(f(f(x))))$ is 23 and equality is obtained when $x = -3$.

3. [**5**] The side lengths of a triangle are distinct positive integers. One of the side lengths is a multiple of 42, and another is a multiple of 72. What is the minimum possible length of the third side?

   Answer: $\boxed{7}$ Suppose that two of the side lengths are $42a$ and $72b$, for some positive integers a and b. Let c be the third side length. We know that $42a$ is not equal to $72b$, since the side lengths are distinct. Also, $6 | 42a - 72b$. Therefore, by the triangle inequality, we get $c > |42a - 72b| \geq 6$ and thus $c \geq 7$. Hence, the minimum length of the third side is 7 and equality is obtained when $a = 7$ and $b = 4$.

4. [**3**] How many ways are there to color the vertices of a triangle red, green, blue, or yellow such that no two vertices have the same color? Rotations and reflections are considered distinct.

   Answer: $\boxed{24}$ There are 4 ways to color the first vertex, then 3 ways to color the second vertex to be distinct from the first, and finally 2 ways to color the third vertex to be distinct from the earlier two vertices. Multiplying gives 24 ways.

5. [**5**] Let A, B, C, D, E be five points on a circle; some segments are drawn between the points so that each of the $\binom{5}{2} = 10$ pairs of points is connected by either zero or one segments. Determine the number of sets of segments that can be drawn such that:

   - It is possible to travel from any of the five points to any other of the five points along drawn segments.
   - It is possible to divide the five points into two nonempty sets S and T such that each segment has one endpoint in S and the other endpoint in T.

Answer: $\boxed{195}$ First we show that we can divide the five points into sets S and T according to the second condition in only one way. Assume that we can divide the five points into $S \cup T$ and $S' \cup T'$. Then, let $A = S' \cap S, B = S' \cap T, C = T' \cap S$, and $D = T' \cap T$. Since S, T and S', T' partition the set of five points, A, B, C, D also partition the set of five points.

Now, according to the second condition, there can only be segments between S and T and between S' and T'. Therefore, the only possible segments are between points in A and D, or between points in B and C. Since, according to the first condition, the points are all connected via segments, it must be that $A = D = \varnothing$ or $B = C = \varnothing$. If $A = D = \varnothing$, then it follows that $S' = T$ and $T' = S$. Otherwise, if $B = C = \varnothing$, then $S' = S$ and $T' = T$. In either case, S, T and S', T' are the same partition of the five points, as desired.

We now determine the possible sets of segments with regard to the sets S and T.

Case 1: the two sets contain 4 points and 1 point. Then, there are $\binom{5}{1} = 5$ ways to partition the points in this manner. Moreover, the 1 point (in its own set) must be connected to each of the other 4 points, and these are the only possible segments. Therefore, there is only 1 possible set of segments, which, combining with the 5 ways of choosing the sets, gives 5 possible sets of segments.

Case 2: the two sets contain 3 points and 2 points. Then, there are $\binom{5}{2} = 10$ ways to partition the points in this manner. Let S be the set containing 3 points and T the set containing 2 points. We consider the possible degrees of the points in T.

- If both points have degree 3, then each point must connect to all points in S, and the five points are connected via segments. So the number of possible sets of segments is 1.
- If the points have degree 3 and 2. Then, we can swap the points in 2 ways, and, for the point with degree 2, we can choose the elements of S it connects to in $\binom{3}{2} = 3$ ways. In each case, the five points are guaranteed to be connected via segments. Hence 6 ways.
- If the points have degree 3 and 1. Similarly, we can swap the points in 2 ways and connect the point with degree 1 to the elements of S in $\binom{3}{1} = 3$ ways. Since all five points are connected in all cases, we have 6 ways.
- If both points have degree 2. Then, in order for the five points to be connected, the two points must connect to a common element of S. Call this common element A. Then, for the other two elements of S, each one must be connected to exactly one element of T. We can choose A in 3 ways, and swap the correspondence between the other two elements of S with the elements of T in 2 ways. Hence 6 ways.
- If the points have degree 2 and 1. Then, in order to cover S, the point with degree 2 must connect to 2 points in S, and the point with degree 1 to the remaining point in S. But then, the five points will not be connected via segments, an impossibility.
- If both points have degree 1. Then, similar to the previous case, it is impossible to cover all the 3 points in S with only 2 segments, a contradiction.

Combining the subcases, we have $1+6+6+6 = 19$ possible sets of segments with regard to a partition. With 10 possible partitions, we have a total of $19 \cdot 10 = 190$ possible sets of segments.

Finally, combining this number with the 5 possibilities from case 1, we have a total of $5 + 190 = 195$ possibilities, as desired.

6. [**6**] Find the number of strictly increasing sequences of nonnegative integers with the following properties:

   - The first term is 0 and the last term is 12. In particular, the sequence has at least two terms.
   - Among any two consecutive terms, exactly one of them is even.

Answer: $\boxed{144}$ For a natural number n, let A_n be a set containing all sequences which satisfy the problem conditions but which 12 is replaced by n. Also, let a_n be the size of A_n.

We first consider a_1 and a_2. We get $a_1 = 1$, as the only sequence satisfying the problem conditions is $0, 1$. We also get $a_2 = 1$, as the only possible sequence is $0, 1, 2$.

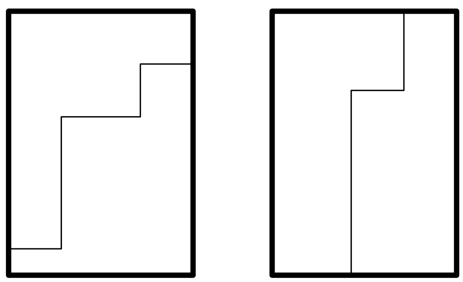
Next, we show that $a_{n+2} = a_{n+1} + a_n$ for all natural number n. We consider the second-to-last terms of each sequence in A_{n+2}.

Case 1. The second-to-last term is $n+1$. When we leave out the last term, the remaining sequence will still satisfy the problem conditions, and hence is in A_{n+1}. Conversely, for a sequence in A_{n+1}, we could add $n+2$ at the end of that sequence, and since $n+1$ and $n+2$ have different parities, the resulting sequence will be in A_{n+2}. Therefore, there is a one-to-one correspondence between the sequences in this case and the sequences in A_{n+1}. So the number of sequences in this case is a_{n+1}.

Case 2. The second-to-last term is less than or equal n. But n and $n+2$ have the same parity, so the second-to-last term cannot exceed $n-1$. When we substitute the last term $(n+2)$ with n, the resulting sequence will satisfy the problem conditions and will be in A_n. Conversely, for a sequence in A_n, we could substitute its last term n, with $n+2$. As n and $n+2$ have the same parity, the resulting sequence will be in A_n. Hence, in this case, the number of sequences is a_n.

Now, since $a_{n+2} = a_{n+1} + a_n$ for all natural numbers n, we can recursively compute that the number of all possible sequences having their last terms as 12 is $a_{12} = 144$. Note that the resulting sequence (a_n) is none other than the Fibonacci numbers.

7. [**7**] Sammy has a wooden board, shaped as a rectangle with length 2^{2014} and height 3^{2014}. The board is divided into a grid of unit squares. A termite starts at either the left or bottom edge of the rectangle, and walks along the gridlines by moving either to the right or upwards, until it reaches an edge opposite the one from which the termite started. Depicted below are two possible paths of the termite.



The termite's path dissects the board into two parts. Sammy is surprised to find that he can still arrange the pieces to form a new rectangle not congruent to the original rectangle. This rectangle has perimeter P. How many possible values of P are there?

Answer: $\boxed{4}$ Let R be the original rectangle and R' the new rectangle which is different from R. We see that the perimeter of R' depends on the possibilities for the side lengths of R'.

We will prove that the dividing line must have the following characterization: starting from the lower left corner of R, walk to the right by distance a, then walk up by distance b, for some positive number a and b, and repeat the two steps until one reaches the upper right corner of R, with the condition that the last step is a walk to the right. (The directions stated here depends on the orientation of R, but we can always orient R so as to fit the description.) Let there be $n+1$ walks to the right and n walks to the top, then we have that this division would rearrange a rectangle of dimension $(n+1)a \times nb$ into a rectangle of dimension $na \times (n+1)b$.

Let us first assume the above. Now, according to the problem, it suffices to find n, a, b such that $(n+1)a = 2^{2014}, nb = 3^{2014}$ or $(n+1)a = 3^{2014}, nb = 2^{2014}$. This means that $n+1$ and n are a power of 3 and a power of 2, whose exponents do not exceed 2014. This corresponds to finding nonnegative integers $k, l \leq 2014$ such that $|2^k - 3^l| = 1$. The only possible pairs of $(2^k, 3^l)$ are $(2, 1), (2, 3), (3, 4)$ and $(8, 9)$. So there are 4 possible configurations of R'.

Now, we prove our claim. For completeness, we will actually prove the claim more generally for any cut, not just ones that move right and up (hence the length of the solution which follows, but only the above two paragraphs are relevant for the purposes of finding the answer).

First we show that the dividing boundary between the two pieces must meet the boundary of R at two points, each being on opposite sides of R as the other. To see why, consider that otherwise, there would be two consecutive sides of R which belong to the same piece. Then, the smallest rectangle containing such a configuration must have each side being as large as each of the two sides, and thus it is R. Since this piece is also part of R', R' must contain R, but their areas are equal, so $R' = R$, a contradiction.

Now, let the dividing boundary go from the top side to the bottom side of R, and call the right piece "piece 1" and the left piece "piece 2." We orient R' in such a way that piece 1 is fixed and piece 2 is moved from the original position in some way to create R'. We will show that piece 2 must be moved by translation by some vector v, t_v. Otherwise, piece 2 is affected by t_v as a well as a rotation by $90°$ or $180°$. We show that these cases are impossible.

First, consider the case where there is a $90°$ rotation. Let the distance from the top side to the bottom side of R be x. Then, the two pieces are contained between a pair of horizontal lines which are of distance x apart from one another. If piece 2 is rotated by $90°$, then these horizontal lines become a pair of *vertical* lines which are of distance x apart from one another. So R' is contained within a union of regions between a pair of horizontal lines and a pair of vertical lines.

Now, we show that R' must be contained within only *one* of these regions. Consider if there exists points (x_1, y_1) and (x_2, y_2) in R' such that (x_1, y_1) is not in the horizontal region (so that y_1 is out of range) and (x_2, y_2) is not in the vertical region (so that x_2 is out of range). Then, it follows that (x_2, y_1) is also in the rectangle R'. But (x_2, y_1) cannot be contained in either region, since both of its x and y coordinates are out of range, a contradiction.

So let us assume, without loss of generality, that R' is contained in the vertical region (the one which contains piece 2). Then, the horizontal side of R' cannot have length greater than x, the width of the region. However, piece 2 is contained in the region, and its width is *exactly* x. Therefore, the width of R' must be exactly x, rendering it to be the same shape as R, a contradiction.

Next, we show that the case with a $180°$ rotation is also impossible. We modify our considerations from the previous case by considering a *half*-region of the region between a pair of horizontal lines (which are still are of distance x apart), which we define as a part of the region on the right or on the left of a certain vertical line. Then, piece 2 is contained within a certain half-region going to the *right* and piece 1 is contained within a certain half-region going to the *left*. Now, in R', since piece 2 is rotated by $180°$, we would have both half-regions going to the left, and R' is contained within a union of them.

Now, consider the "end" of each half-region (the part of the boundary that is vertical). The ends of both half-regions must be contained in R', since they are part of piece 1 and piece 2. Consider a vector that maps the end of one half-region to the other. If the vector is horizontal, then the union of the regions have vertical distance x. Similarly to the previous case, we deduce that the vertical side of R must be of length no more than x, and so must be exactly x, but then $R' = R$, a contradiction.

Now, if the vector has both nonzero horizontal and vertical components, then the parallelogram generated by the locus of the end of a half-region being translated by the vector to the end of the other half-region must be contained within R' (since R' is convex). However, the parallelogram is not contained within the union of the two half-regions, a contradiction.

Finally, if the vector is vertical, then the two half-regions must be on top of one another, and so will have no region in common. Then, since R' is a rectangle, the intersection of R' with each half-region will also be a rectangle. So pieces 1 and 2 must be rectangles. But then a rotation of $180°$ would map piece 2 to itself. So this reduces to a case of pure translation.

We now consider the translation t_v by a vector v on piece 2. Since R' must contain the ends of the half-regions (which retain their original orientations), the vertical side of R' must be at least of length x. But $R' \neq R$, so the vertical side of R' has length strictly greater than x. This implies that the horizontal side of R' must be strictly shorter than that of R, since they have equal area. However, the horizontal side of R' is at least as long as the horizontal distance between the ends of the two half-regions, so the ends of the two half-regions must have moved closer to one another horizontally.

This implies that the vector v has a positive x component. Also, v cannot be entirely horizontal, because there is no more space for piece 2 to move into. So v has a nonzero y component. Without loss of generality, let us assume that v has a positive y component.

Before we continue further, let us label the vertices of R as A, B, C, D, going in counter-clockwise direction, with the left side of R being AB and the right side of R being CD. So AB is in piece 2 and CD is in piece 1. Call the translated piece 2 that is part of the rectangle R' piece $2'$, with the corresponding points A' and B'.

Now, consider the half-regions of piece 1 and piece 2. They are half-regions with the end AB going to the right and the one with the end CD going to the left. So, in R', the half-regions are with end $A'B'$ going to the right and with end CD going to the left, and R' is contained within the union of these two. Now, there cannot be a point in R' that is to the left of $A'B'$, since the smallest rectangle containing that point and A' would not be contained in the union of the two half-regions. Similarly, there cannot be a point in R' that is to the right of CD for the same reason. These restrictions imply that R' must be contained within the union of the two half-regions that lie horizontally between $A'B'$ and CD. However, since the smallest rectangle containing A' and C is precisely this region, R' must be this region. Let $A'B'$ intersect BC at L and let CD intersect the line passing through A' which is parallel to AD at M. We have $R' = A'LCM$.

Now, consider the segments BL and LB'. We know that BL is a boundary of piece 2. Also, since LB' is a boundary of R' and it is below piece $2'$, it cannot be a boundary of piece $2'$. Therefore, it must be a boundary of piece 1. Since LB' is not a boundary of R but is a boundary of piece 1, it must be part of the dividing boundary between piece 1 and 2, and so must also be a boundary of piece 2.

We now prove that: from a sequence of $B, B', B'', ...$, each being translated from the preceding one by v, one of them must eventually lie on AD. Also, if $L, L', L'', ...$ is also a sequence of L being successively translated by v, then, using B_i and L_i to designate the ith term of each sequence: B_iL_i and L_iB_{i+1} must be part of the boundary of piece 2 for all $i \leq N-1$, where B_N is the last point, the one that lies on AD.

We have already proved the assertion for $i = 1$. We now set out to prove, by induction, that B_iL_i and L_iB_{i+1} must be part of the boundary of piece 2 for all i such that B_{i+1} is still within R or on the edge of R.

Consider, by induction hypothesis, that $B_{i-1}L_{i-1}$ and $L_{i-1}B_i$ are parts of the boundary of piece 2. Then, by mapping, B_iL_i and L_iB_{i+1} must be parts of the boundary of piece $2'$. Since the dividing boundary of R go from top to bottom, L_iB_{i+1} cannot be on the rightmost edge of R, which is also the rightmost edge of R'. This means that L_iB_{i+1} is not a boundary of R'. So we have that B_iL_i and L_iB_{i+1} are not boundaries of R' but are boundaries of piece $2'$, so they must be boundaries of piece 1. Now, since they are not boundaries of R either but are boundaries of piece 1, they must be boundaries of piece 2, as desired.

Now we are left to show that one of B_i must lie exactly on AD. Let B_i be the last term of the sequence that is contained within R or its boundary. Then, by the previous result, $B_{i-1}L_{i-1}$ and $L_{i-1}B_i$ are boundaries of piece 2. Then, by mapping, B_iL_i is a boundary of piece $2'$. Since B_iL_i is on or below AD, it cannot be on the boundary of R', but since it is a boundary of piece $2'$, it must also a boundary of piece 1. Now, if B_i is not on AD, then B_iL_i will not be a boundary of R, but since it is a boundary of piece 1, it must also be a boundary of piece 2. By mapping, this implies that $B_{i+1}L_{i+1}$ must be a boundary of piece $2'$. However, $B_{i+1}L_{i+1}$ is not contained within R or its boundary, and so cannot be on piece 1's boundary. Therefore, since $B_{i+1}L_{i+1}$ is a boundary of piece $2'$ but not of piece 1, it must be a boundary of R'. This means that $B_{i+1}L_{i+1}$ is on the upper edge of R'. Mapping back, we get that B_iL_i must be on the upper edge of R, a contradiction to the assumption that B_i is not on AD (the upper edge of R). So B_i is on AD, as desired.

Let B_N be the term of the sequence that is on AD. We have now shown that

$$L_1B_2, B_2L_2, \ldots, B_{N-1}L_{N-1}, L_{N-1}B_N$$

completely defines the dividing line between piece 1 and piece 2. Moreover, $B_1 = B$, defining the starting point of the dividing line. We now add the final description: $L_N = D$. To see why, note that

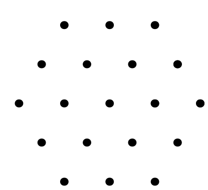
L_N must be on the upper edge of R, as it is in the same horizontal level as B_N. However, since L_{N-1} is one of furthest points to the right of piece 2, by mapping, L_N must be one of the furthest points to the right of piece $2'$, and so must be on the rightmost edge of piece $2'$, which is the rightmost edge of R' and of R. Therefore, L_N is on the upper edge and the rightmost edge of R, and so it must be D, as desired.

8. [**3**] Let $\mathcal{H}$ be a regular hexagon with side length one. Peter picks a point P uniformly and at random within $\mathcal{H}$, then draws the largest circle with center P that is contained in $\mathcal{H}$. What is this probability that the radius of this circle is less than $\frac{1}{2}$?

Answer: $\boxed{\frac{2\sqrt{3}-1}{3}}$ We first cut the regular hexagon $\mathcal{H}$ by segments connecting its center to each vertex into six different equilateral triangles with side lengths 1. Therefore, each point inside $\mathcal{H}$ is contained in some equilateral triangle. We first see that for each point inside an equilateral triangle, the radius of the largest circle with center P which is contained in $\mathcal{H}$ equals the shortest distance from P to the nearest side of the hexagon, which is also a side of the triangle in which it is contained.

Consider that the height of each triangle is $\frac{\sqrt{3}}{2}$. Therefore, the region inside the triangle containing all points with distance more than $\frac{1}{2}$ to the side of the hexagon is an equilateral triangle with a height of $\frac{\sqrt{3}-1}{2}$. Consequently, the area inside the triangle containing all points with distance less than $\frac{1}{2}$ to the side of the hexagon has area $\frac{\sqrt{3}}{4}\left(1-\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)^2\right) = \frac{\sqrt{3}}{4}\cdot\left(\frac{2\sqrt{3}-1}{3}\right)$. This is of the ratio $\frac{2\sqrt{3}-1}{3}$ to the area of the triangle, which is $\frac{\sqrt{3}}{4}$. Since all triangles are identical and the point P is picked uniformly within $\mathcal{H}$, the probability that the radius of the largest circle with center P which is contained in $\mathcal{H}$ is less than $\frac{1}{2}$ is $\frac{2\sqrt{3}-1}{3}$, as desired.

9. [**5**] How many lines pass through exactly two points in the following hexagonal grid?



Answer: $\boxed{60}$ *First solution.* From a total of 19 points, there are $\binom{19}{2} = 171$ ways to choose two points. We consider lines that pass through more than 2 points.

- There are $6+6+3 = 15$ lines that pass through exactly three points. These are: the six sides of the largest hexagon, three lines through the center (perpendicular to the sides of the largest hexagon), and the other six lines perpendiculars to the sides of the largest hexagon.
- There are 6 lines that pass through exactly four points. (They are parallel to the sides of the largest hexagon.)
- There are 3 lines that pass through exactly five points. (They all pass through the center.)

For each $n = 3, 4, 5$, a line that passes through n points will be counted $\binom{n}{2}$ times, and so the corresponding amount will have to be subtracted. Hence the answer is

$$171 - \binom{3}{2}\cdot 15 - \binom{4}{2}\cdot 6 - \binom{5}{2}\cdot 3 = 171 - 45 - 36 - 30 = 60.$$

Second solution. We divide the points into 4 groups as follows.

- Group 1 consists of the center point.
- Group 2 consists of the 6 points surrounding the center.
- Group 3 consists of the 6 vertices of the largest hexagon.

- Group 4 consists of the 6 midpoints of the sides of the largest hexagon.

We wish to count the number of lines that pass through exactly 2 points. Consider: all lines connecting points in group 1 and 2, 1 and 3, and 1 and 4 pass through more than 2 points. So it is sufficient to restrict our attention to group 2, 3 and 4.

- For lines connecting group 2 and 2, the only possibilities are those that the two endpoints are 120 degrees apart with respect to the center, so 6 possibilities.
- For lines connecting group 3 and 3, it is impossible.
- For lines connecting group 4 and 4, the two endpoints must be 60 degrees apart with respect to the center, so 6 possibilities.
- For lines connecting group 3 and 2. For each point in group 3, the only possible points in group 2 are those that are 120 degrees apart from the point in group 3. So $2 \cdot 6 = 12$ possibilities.
- For lines connecting group 4 and 2, the endpoints must be 150 degrees apart with respect to the center, so $2 \cdot 6 = 12$ possibilities.
- For lines connecting group 4 and 3. For each point in group 4, any point in group 3 works except those that are on the side on the largest hexagon of which the point in group 4 is the midpoint. Hence $4 \cdot 6 = 24$ possibilities.

Therefore, the number of lines passing through 2 points is $6 + 6 + 12 + 12 + 24 = 60$, as desired.

10. [**8**] Let $ABCDEF$ be a convex hexagon with the following properties.

   (a) $\overline{AC}$ and $\overline{AE}$ trisect $\angle BAF$.

   (b) $\overline{BE} \parallel \overline{CD}$ and $\overline{CF} \parallel \overline{DE}$.

   (c) $AB = 2AC = 4AE = 8AF$.

   Suppose that quadrilaterals $ACDE$ and $ADEF$ have area 2014 and 1400, respectively. Find the area of quadrilateral $ABCD$.

   Answer: $\boxed{7295}$ From conditions (a) and (c), we know that triangles AFE, AEC and ACB are similar to one another, each being twice as large as the preceding one in each dimension. Let $\overline{AE} \cap \overline{FC} = P$ and $\overline{AC} \cap \overline{EB} = Q$. Then, since the quadrilaterals $AFEC$ and $AECB$ are similar to one another, we have $AP : PE = AQ : QC$. Therefore, $\overline{PQ} \parallel \overline{EC}$.

   Let $\overline{PC} \cap \overline{QE} = T$. We know by condition (b) that $\overline{BE} \parallel \overline{CD}$ and $\overline{CF} \parallel \overline{DE}$. Therefore, triangles PQT and ECD have their three sides parallel to one another, and so must be similar. From this we deduce that the three lines joining the corresponding vertices of the two triangles must meet at a point, i.e., that PE, TD, QC are concurrent. Since PE and QC intersect at A, the points A, T, D are collinear. Now, because $TCDE$ is a parallelogram, $\overline{TD}$ bisects $\overline{EC}$. Therefore, since A, T, D are collinear, $\overline{AD}$ also bisects $\overline{EC}$. So the triangles ADE and ACD have equal area.

   Now, since the area of quadrilateral $ACDE$ is 2014, the area of triangle ADE is $2014/2 = 1007$. And since the area of quadrilateral $ADEF$ is 1400, the area of triangle AFE is $1400 - 1007 = 393$. Therefore, the area of quadrilateral $ABCD$ is $16 \cdot 393 + 1007 = 7295$, as desired.