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1. [**5**] If $f(x) = x/(x+1)$, what is $f(f(f(f(2009))))$?

2. [**5**] A knight begins on the lower-left square of a standard chessboard. How many squares could the knight end up at after exactly 2009 legal knight's moves? (A knight's move is 2 squares either horizontally or vertically, followed by 1 square in a direction perpendicular to the first.)

3. [**5**] Consider a square, inside which is inscribed a circle, inside which is inscribed a square, inside which is inscribed a circle, and so on, with the outermost square having side length 1. Find the difference between the sum of the areas of the squares and the sum of the areas of the circles.

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4. [**6**] A cube has side length 1. Find the product of the lengths of the diagonals of this cube (a diagonal is a line between two vertices that is not an edge).

5. [**6**] Tanks has a pile of 5 blue cards and 5 red cards. Every morning, he takes a card and throws it down a well. What is the probability that the first card he throws down and the last card he throws down are the same color?

6. [**6**] Find the last two digits of 1032^{1032}. Express your answer as a two-digit number.

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7. [**7**] A *computer program* is a function that takes in 4 bits, where each bit is either a 0 or a 1, and outputs TRUE or FALSE. How many computer programs are there?

8. [**7**] The angles of a convex n-sided polygon form an arithmetic progression whose common difference (in degrees) is a non-zero integer. Find the largest possible value of n for which this is possible. (A polygon is convex if its interior angles are all less than $180°$.)

9. [**7**] Daniel wrote all the positive integers from 1 to n inclusive on a piece of paper. After careful observation, he realized that the sum of all the digits that he wrote was exactly 10,000. Find n.

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10. [**8**] Admiral Ackbar needs to send a 5-character message through hyperspace to the Rebels. Each character is a lowercase letter, and the same letter may appear more than once in a message. When the message is beamed through hyperspace, the characters come out in a random order. Ackbar chooses his message so that the Rebels have at least a $\frac{1}{2}$ chance of getting the same message he sent. How many distinct messages could he send?

11. [**8**] Lily and Sarah are playing a game. They each choose a real number at random between -1 and 1. They then add the squares of their numbers together. If the result is greater than or equal to 1, Lily wins, and if the result is less than 1, Sarah wins. What is the probability that Sarah wins?

12. [**8**] Let ω be a circle of radius 1 centered at O. Let B be a point on ω, and let l be the line tangent to ω at B. Let A be on l such that $\angle AOB = 60°$. Let C be the foot of the perpendicular from B to OA. Find the length of line segment OC.

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13. [**8**] 8 students are practicing for a math contest, and they divide into pairs to take a practice test. In how many ways can they be split up?

14. [**8**] Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial whose roots are all negative integers. If $a + b + c + d = 2009$, find d.

15. [**8**] The curves $x^2 + y^2 = 36$ and $y = x^2 - 7$ intersect at four points. Find the sum of the squares of the x-coordinates of these points.

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16. [**9**] Pick a random digit in the decimal expansion of $\frac{1}{99999}$. What is the probability that it is 0?

17. [**9**] A circle passes through the points $(2, 0)$ and $(4, 0)$ and is tangent to the line $y = x$. Find the sum of all possible values for the y-coordinate of the center of the circle.

18. [**9**] Let f be a function that takes in a triple of integers and outputs a real number. Suppose that f satisfies the equations

$$\begin{aligned} f(a,b,c) &= \frac{f(a+1,b,c)+f(a-1,b,c)}{2} \\ f(a,b,c) &= \frac{f(a,b+1,c)+f(a,b-1,c)}{2} \\ f(a,b,c) &= \frac{f(a,b,c+1)+f(a,b,c-1)}{2} \end{aligned}$$

for all integers a, b, c. What is the minimum number of triples at which we need to evaluate f in order to know its value everywhere?

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19. [**11**] You are trapped in ancient Japan, and a giant enemy crab is approaching! You must defeat it by cutting off its two claws and six legs and attacking its weak point for massive damage. You cannot cut off any of its claws until you cut off at least three of its legs, and you cannot attack its weak point until you have cut off all of its claws and legs. In how many ways can you defeat the giant enemy crab? (Note that the legs are distinguishable, as are the claws.)

20. [**11**] Consider an equilateral triangle and a square both inscribed in a unit circle such that one side of the square is parallel to one side of the triangle. Compute the area of the convex heptagon formed by the vertices of both the triangle and the square.

21. [**11**] Let $f(x) = x^2 + 2x + 1$. Let $g(x) = f(f(\cdots f(x)))$, where there are 2009 fs in the expression for $g(x)$. Then $g(x)$ can be written as

$$g(x) = x^{2^{2009}} + a_{2^{2009}-1}x^{2^{2009}-1} + \cdots + a_1x + a_0,$$

where the a_i are constants. Compute $a_{2^{2009}-1}$.

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22. [**12**] Five cards labeled A, B, C, D, and E are placed consecutively in a row. How many ways can they be re-arranged so that no card is moved more than one position away from where it started? (Not moving the cards at all counts as a valid re-arrangement.)

23. [**12**] Let $a_0, a_1, \ldots$ be a sequence such that $a_0 = 3$, $a_1 = 2$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$. Find

$$\sum_{n=0}^{8} \frac{a_n}{a_{n+1}a_{n+2}}.$$

24. [**12**] Penta chooses 5 of the vertices of a unit cube. What is the maximum possible volume of the figure whose vertices are the 5 chosen points?

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25. [**14**] Find all solutions to $x^4 + 2x^3 + 2x^2 + 2x + 1 = 0$ (including non-real solutions).

26. [**14**] In how many ways can the positive integers from 1 to 100 be arranged in a circle such that the sum of every two integers placed opposite each other is the same? (Arrangements that are rotations of each other count as the same.) Express your answer in the form $a! \cdot b^c$.

27. [**14**] $ABCD$ is a regular tetrahedron of volume 1. Maria glues regular tetrahedra $A'BCD$, $AB'CD$, $ABC'D$, and $ABCD'$ to the faces of $ABCD$. What is the volume of the tetrahedron $A'B'C'D'$?

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28. [**17**] Six men and their wives are sitting at a round table with 12 seats. These men and women are very jealous — no man will allow his wife to sit next to any man except for himself, and no woman will allow her husband to sit next to any woman except for herself. In how many distinct ways can these 12 people be seated such that these conditions are satisfied? (Rotations of a valid seating are considered distinct.)

29. [**17**] For how many integer values of b does there exist a polynomial function with integer coefficients such that $f(2) = 2010$ and $f(b) = 8$?

30. [**17**] Regular hexagon $ABCDEF$ has side length 2. A laser beam is fired inside the hexagon from point A and hits $\overline{BC}$ at point G. The laser then reflects off $\overline{BC}$ and hits the midpoint of $\overline{DE}$. Find BG.

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31. [**20**] There are two buildings facing each other, each 5 stories high. How many ways can Kevin string ziplines between the buildings so that:

   (a) each zipline starts and ends in the middle of a floor.

   (b) ziplines can go up, stay flat, or go down, but can't touch each other (this includes touching at their endpoints).

   Note that you can't string a zipline between two floors of the same building.

32. [**20**] A circle ω_1 of radius 15 intersects a circle ω_2 of radius 13 at points P and Q. Point A is on line PQ such that P is between A and Q. R and S are the points of tangency from A to ω_1 and ω_2, respectively, such that the line AS does not intersect ω_1 and the line AR does not intersect ω_2. If $PQ = 24$ and $\angle RAS$ has a measure of 90°, compute the length of AR.

33. [**20**] Compute

$$\sum_{n=2009}^{\infty} \frac{1}{\binom{n}{2009}}.$$

   Note that $\binom{n}{k}$ is defined as $\frac{n!}{k!(n-k)!}$.

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34. [**25**] How many hits does "3.1415" get on Google? Quotes are for clarity only, and not part of the search phrase. Also note that Google does not search substrings, so a webpage with 3.14159 on it will not match 3.1415. If A is your answer, and S is the correct answer, then you will get $\max(25 - |\ln(A) - \ln(S)|, 0)$ points, rounded to the nearest integer.

35. [**25**] Call an integer $n > 1$ *radical* if $2^n - 1$ is prime. What is the 20th smallest radical number? If A is your answer, and S is the correct answer, you will get $\max\left(25\left(1 - \frac{|A-S|}{S}\right), 0\right)$ points, rounded to the nearest integer.

36. [**25**] Write down a pair of integers (a, b), where $-100000 < a < b < 100000$. You will get $\max(25, k)$ points, where k is the number of other teams' pairs that you interleave. (Two pairs (a, b) and (c, d) of integers *interleave* each other if $a < c < b < d$ or $c < a < d < b$.)

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