4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

General Test

1. **[3]** Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x + xy = 4$.

2. **[3]** Let ABC be a triangle, and let D, E, and F be the midpoints of sides BC, CA, and AB, respectively. Let the angle bisectors of $\angle FDE$ and $\angle FBD$ meet at P. Given that $\angle BAC = 37^\circ$ and $\angle CBA = 85^\circ$, determine the degree measure of $\angle BPD$.

3. **[4]** Alberto, Bernardo, and Carlos are collectively listening to three different songs. Each is simultaneously listening to exactly two songs, and each song is being listened to by exactly two people. In how many ways can this occur?

4. **[4]** Determine the remainder when

$$2^{\frac{1\cdot 2}{2}} + 2^{\frac{2\cdot 3}{2}} + \cdots + 2^{\frac{2011\cdot 2012}{2}}$$

is divided by 7.

5. **[5]** Find all real values of x for which

$$\frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} = \frac{1}{4}.$$

6. **[5]** Five people of heights 65, 66, 67, 68, and 69 inches stand facing forwards in a line. How many orders are there for them to line up, if no person can stand immediately before or after someone who is exactly 1 inch taller or exactly 1 inch shorter than himself?

7. **[5]** Determine the number of angles θ between 0 and 2π, other than integer multiples of $\pi/2$, such that the quantities $\sin\theta$, $\cos\theta$, and $\tan\theta$ form a geometric sequence in some order.

8. **[6]** Find the number of integers x such that the following three conditions all hold:
   - x is a multiple of 5
   - $121 < x < 1331$
   - When x is written as an integer in base 11 with no leading 0s (i.e. no 0s at the very left), its rightmost digit is strictly greater than its leftmost digit.

9. **[7]** Let P and Q be points on line l with $PQ = 12$. Two circles, ω and Ω, are both tangent to l at P and are externally tangent to each other. A line through Q intersects ω at A and B, with A closer to Q than B, such that $AB = 10$. Similarly, another line through Q intersects Ω at C and D, with C closer to Q than D, such that $CD = 7$. Find the ratio AD/BC.

10. **[8]** Let $r_1, r_2, \ldots, r_7$ be the distinct complex roots of the polynomial $P(x) = x^7 - 7$. Let

$$K = \prod_{1 \le i < j \le 7} (r_i + r_j),$$

that is, the product of all numbers of the form $r_i + r_j$, where i and j are integers for which $1 \le i < j \le 7$. Determine the value of K^2.