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Combinatorics

1. Kelvin the Frog is going to roll three fair ten-sided dice with faces labelled $0, 1, 2, \ldots, 9$. First he rolls two dice, and finds the sum of the two rolls. Then he rolls the third die. What is the probability that the sum of the first two rolls equals the third roll?

   Proposed by: Yang Liu

   Answer: $\boxed{\frac{11}{200}}$

   First, there are $10^3 = 1000$ triples (a, b, c). Now, we should count how many of these triples satisfy $a + b = c$. If $c = 0$, we get 1 triple $(0, 0, 0)$. If $c = 1$, we get two triples $(1, 0, 1)$ and $(0, 1, 1)$. Continuing, this gives that the total number of triples is $1 + 2 + \cdots + 10 = 55$. Therefore, our final answer is $\frac{55}{1000} = \frac{11}{200}$.

2. How many ways are there to insert +'s between the digits of 111111111111111 (fifteen 1's) so that the result will be a multiple of 30?

   Proposed by: Yang Liu

   Answer: $\boxed{2002}$

   Note that because there are 15 1's, no matter how we insert +'s, the result will always be a multiple of 3. Therefore, it suffices to consider adding +'s to get a multiple of 10. By looking at the units digit, we need the number of summands to be a multiple of 10. Because there are only 15 digits in our number, we have to have exactly 10 summands. Therefore, we need to insert 9 +'s in 14 possible positions, giving an answer of $\binom{14}{9} = 2002$.

3. There are 2017 jars in a row on a table, initially empty. Each day, a nice man picks ten consecutive jars and deposits one coin in each of the ten jars. Later, Kelvin the Frog comes back to see that N of the jars all contain the same positive integer number of coins (i.e. there is an integer $d > 0$ such that N of the jars have exactly d coins). What is the maximum possible value of N?

   Proposed by: Kevin Sun

   Answer: $\boxed{2014}$

   Label the jars $1, 2, \ldots, 2017$. I claim that the answer is 2014. To show this, we need both a construction and an upper bound. For the construction, for $1 \leq i \leq 201$, put a coin in the jars $10i + 1, 10i + 2, \ldots, 10i + 10$. After this, each of the jars $1, 2, \ldots, 2010$ has exactly one coin. Now, put a coin in each of the jars $2008, 2009, \ldots, 2017$. Now, the jars $1, 2, \ldots, 2007, 2011, 2012, \ldots, 2017$ all have exactly one coin. This gives a construction for $N = 2014$ (where $d = 1$).

   Now, we show that this is optimal. Let $c_1, c_2, \ldots, c_{2017}$ denote the number of coins in each of the jars. For $1 \leq j \leq 10$, define

   $$s_j = c_j + c_{j+10} + c_{j+20} + \ldots .$$

   Note that throughout the process, $s_1 = s_2 = \cdots = s_j$. It is also easy to check that the sums $s_1, s_2, \ldots, s_7$ each involve 202 jars, while the sums s_8, s_9, s_{10} each involve 201 jars.

   Call a jar *good* if it has exactly d coins. If there are at least 2015 good jars, then one can check that it is forced that at least one of $s_1, s_2, \ldots, s_7$ only involves good jars, and similarly, at least one of s_8, s_9, s_{10} only involves good jars. But this would mean that $202d = 201d$ as all s_i are equal, contradiction.

4. Sam spends his days walking around the following 2×2 grid of squares.

1	2
4	3

Say that two squares are adjacent if they share a side. He starts at the square labeled 1 and every second walks to an adjacent square. How many paths can Sam take so that the sum of the numbers on every square he visits in his path is equal to 20 (not counting the square he started on)?

Proposed by: Sam Korsky

Answer: $\boxed{167}$

Note that on the first step, Sam can either step on 2 or 4. On the second step, Sam can either step on 1 or 3, regardless of whether he is on 2 or 4. Now, for example, say that Sam takes 8 steps. His total sum will be $2+1+2+1+2+1+2+1+2a$, where a is the number of times that he decides to step on the larger number of his two choices. Solving gives $a = 4$. As he took 8 steps, this gives him $\binom{8}{4} = 70$ ways in this case.

We can follow a similar approach by doing casework on the number of steps he takes. I will simply list them out here for brevity. For 8 steps, we get $\binom{8}{4} = 70$. For 9 steps, we get $\binom{9}{3} = 84$. For 12 steps, we get a contribution on $\binom{12}{1} = 12$. For 13 steps, we get a contribution of $\binom{13}{0} = 1$. Therefore, the final answer is $70 + 84 + 12 + 1 = 167$.

5. Kelvin the Frog likes numbers whose digits strictly decrease, but numbers that violate this condition in at most one place are good enough. In other words, if d_i denotes the ith digit, then $d_i \le d_{i+1}$ for at most one value of i. For example, Kelvin likes the numbers 43210, 132, and 3, but not the numbers 1337 and 123. How many 5-digit numbers does Kelvin like?

Proposed by: Alexander Katz

Answer: $\boxed{14034}$

Suppose first that no digit violates the constraint; i.e. the digits are in strictly decreasing order. There are $\binom{10}{5}$ ways to choose the digits of the number, and each set of digits can be arranged in exactly one way, so there are $\binom{10}{5}$ such numbers.

We now perform casework on which digit violates the constraint. If it is the final digit, the first four digits must be arranged in decreasing order, which there are $\binom{10}{4}$ ways to do. The final digit can then be any digit, but we have overcounted the ones in which the number is in fully decreasing order (this can happen, for example, if the first 4 digits we chose were 5, 4, 3, and 2 – a last digit of 1 was already counted in the first case). Therefore, there are $\binom{10}{4}\binom{10}{1} - 252$ new numbers in this case.

If the offending digit is second from the right, the first 3 digits must be decreasing, as must the last 2 digits. There are $\binom{10}{3}\binom{10}{2}$ ways to do this. As before, we overcount the case where the second digit from the right is not actually an offender, so we again overcount the case where all 5 digits decrease. Hence there are $\binom{10}{3}\binom{10}{2} - 252$ new numbers in this case.

The case where the third digit is the offender is identical to the previous case, so there are another $\binom{10}{3}\binom{10}{2} - 252$ numbers to account for. The final case is when the second digit is the offending digit, in which case there are $\binom{10}{4}$ ways to choose the final 4 digits, but only 9 ways to choose the opening digit (as 0 cannot be a leading digit). Accounting for the usual overcounting, our final answer is

$$252 + \left[\binom{10}{4}\binom{10}{1} - 252\right] + 2\left[\binom{10}{3}\binom{10}{2} - 252\right] + \left[\binom{10}{4} \cdot 9 - 252\right]$$

which is easily calculated as $\boxed{14034}$.

6. Emily starts with an empty bucket. Every second, she either adds a stone to the bucket or removes a stone from the bucket, each with probability $\frac{1}{2}$. If she wants to remove a stone from the bucket and the bucket is currently empty, she merely does nothing for that second (still with probability $\frac{1}{2}$). What is the probability that after 2017 seconds her bucket contains exactly 1337 stones?

Proposed by: Sam Korsky

Answer: $\boxed{\frac{\binom{2017}{340}}{2^{2017}}}$

Replace 2017 with n and 1337 with k and denote the general answer by $f(n,k)$. I claim that $f(n,k) = \frac{\binom{n}{\lfloor \frac{n-k}{2} \rfloor}}{2^n}$. We proceed by induction on n.

The claim is obviously true for $n = 0$ since $f(0,0) = 1$. Moreover, we have that $f(n,0) = \frac{1}{2}f(n-1,0) + \frac{1}{2}f(n-1,1)$ and $f(n,k) = \frac{1}{2}f(n-1,k-1) + \frac{1}{2}f(n-1,k+1)$ for $k > 0$ so the inductive step is immediate by Pascal's identity. This concludes the proof.

7. There are 2017 frogs and 2017 toads in a room. Each frog is friends with exactly 2 distinct toads. Let N be the number of ways to pair every frog with a toad who is its friend, so that no toad is paired with more than one frog. Let D be the number of distinct possible values of N, and let S be the sum of all possible values of N. Find the ordered pair (D,S).

   Proposed by: Yang Liu

   Answer: $\boxed{(1009, 2^{1009} - 2)}$

   I claim that N can equal 0 or 2^i for $1 \le i \le 1008$. We prove this now. Note that the average number of friends a toad has is also 2. If there is a toad with 0 friends, then clearly $N = 0$. If a toad has 1 friend, then it must be paired with its only friend, so we have reduced to a smaller case. Otherwise, all toads and frogs have exactly degree 2, so the graph is a union of cycles. Each cycle can be paired off in exactly two ways. The number of cycles can range anywhere from 1 to 1008, and this completes the proof.

   To construct all $N = 2^1, 2^2, \ldots, 2^{1008}$, we can simply let our graph be a union of i cycles, which would have 2^i matchings. Clearly we can choose any $i = 1, 2, \ldots, 1008$.

   Therefore, $D = 1009$ and $S = 2^1 + 2^2 + \cdots + 2^{1008} = 2^{1009} - 2$.

8. Kelvin and 15 other frogs are in a meeting, for a total of 16 frogs. During the meeting, each pair of distinct frogs becomes friends with probability $\frac{1}{2}$. Kelvin thinks the situation after the meeting is *cool* if for each of the 16 frogs, the number of friends they made during the meeting is a multiple of 4. Say that the probability of the situation being cool can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime. Find a.

   Proposed by: Yang Liu

   Consider the multivariate polynomial

   $$\prod_{1 \le i < j \le 16} (1 + x_i x_j)$$

   We're going to filter this by summing over all 4^{16} 16-tuples $(x_1, x_2, \ldots, x_{16})$ such that $x_j = \pm 1, \pm i$. Most of these evaluate to 0 because $i^2 = (-i)^2 = -1$, and $1 \cdot -1 = -1$. If you do this filtering, you get the following 4 cases:

   Case 1: Neither of i or $-i$ appears. Then the only cases we get are when all the x_j are 1, or they're all -1. Total is 2^{121}. ($120 = \binom{16}{2}$.)

   Case 2: i appears, but $-i$ does not. Then all the remaining x_j must be all 1 or all -1. This contributes a sum of $(1+i)^{15} \cdot 2^{105} + (1-i)^{15} \cdot 2^{105} = 2^{113}$. i can be at any position, so we get $16 \cdot 2^{113}$.

Case 3: $-i$ appears, but i does not. Same contribution as above. $16 \cdot 2^{113}$.

Case 4: Both i and $-i$ appear. Then all the rest of the x_j must be all 1 or all -1. This contributes a sum of $2 \cdot (1 + i(-i)) \cdot (1+i)^{14} \cdot (1-i)^{14} \cdot 2^{91} = 2^{107}$. i and $-i$ can appear in $16 \cdot 15$ places, so we get $240 \cdot 2^{107}$.

So the final answer is this divided a factor for our filter. ($4^{16} = 2^{32}$.) So our final answer is $\frac{2^{89}+16\cdot 2^{82}+240\cdot 2^{75}}{2^{120}} = \frac{1167}{2^{41}}$.

Therefore, the answer is 1167.

9. Let m be a positive integer, and let T denote the set of all subsets of $\{1, 2, \ldots, m\}$. Call a subset S of T δ-*good* if for all $s_1, s_2 \in S$, $s_1 \neq s_2$, $|\Delta(s_1, s_2)| \geq \delta m$, where Δ denotes symmetric difference (the symmetric difference of two sets is the set of elements that is in exactly one of the two sets). Find the largest possible integer s such that there exists an integer m and a $\frac{1024}{2047}$-good set of size s.

   Proposed by: Yang Liu

   Answer: $\boxed{2048}$

   Let $n = |S|$. Let the sets in S be $s_1, s_2, \ldots, s_n$. We bound the sum $\sum_{1 \leq i < j \leq n} |\Delta(s_i, s_j)|$ in two ways. On one hand, by the condition we have the obvious bound

$$\sum_{1 \leq i < j \leq n} |\Delta(s_i, s_j)| \geq \binom{n}{2} \delta m.$$

   On the other hand, for $1 \leq i \leq m$, let $t_i = |\{1 \leq j \leq n : i \in s_j\}|$. Then it is clear that

$$\sum_{1 \leq i < j \leq n} |\Delta(s_i, s_j)| = \sum_{k=1}^{m} t_k(n - t_k) \leq \frac{n^2}{4} m$$

   by AM-GM. Therefore, we get the bound

$$\binom{n}{2} \delta m \leq \frac{n^2}{4} m \Rightarrow n \leq \frac{2\delta}{2\delta - 1} = 2048.$$

   To give a construction with $n = 2048$, take $m = 2047$. For the rest of this construction, we will be interpreting the integers $1, 2, \ldots, m$ as 11-digit integers in binary. Given this interpretation, define a *dot product* $x \odot y$ of two positive integers $0 \leq x, y \leq m$ the following way. If $x = (x_1 x_2 \ldots x_{11})_2, y = (y_1 y_2 \ldots y_{11})_2$ in binary, then

$$x \odot y = \sum x_i y_i \pmod 2.$$

   Now we can define the sets $s_1, s_2, \ldots, s_{2048}$. Define

$$s_i = \{1 \leq j \leq m : (i-1) \odot j = 1.\}.$$

   A computation shows that this construction works.

   Some notes: here is the motivation behind the construction. We are treating the integers $0, 1, \ldots, m$ as the vector space $V = \mathbb{F}_2^{11}$, and the sets s_i correspond to linear functionals $f_i : V \to \mathbb{F}_2$. In particular, the function $f_i : V \to \mathbb{F}_2$ is simply defined as $f_i(x) = (i-1) \odot x$, which one can easily check to be linear. This construction corresponds to Hadamard matrices of size 2^{11}.

10. Compute the number of possible words $w = w_1 w_2 \ldots w_{100}$ satisfying:

- w has exactly 50 A's and 50 B's (and no other letters).
- For $i = 1, 2, \ldots, 100$, the number of A's among $w_1, w_2, \ldots, w_i$ is at most the number of B's among $w_1, w_2, \ldots, w_i$.
- For all $i = 44, 45, \ldots, 57$, if w_i is an B, then w_{i+1} must be an B.

Proposed by: Allen Liu

Call the last property in the problem statement $P(i, j)$ where in the statement $i = 44$, $j = 57$. We show that the number of words satisfying the first two conditions and $P(m, m+k)$ is the same independent of m (assuming k is fixed). It suffices to show that the number of words satisfying $P(m-1, m+k-1)$ is the same as the number of words satisfying $P(m, m+k)$. Construct a bijection as follows: for a word satisfying $P(m-1, m+k-1)$, if it satisfies $P(m, m+k)$, leave it as is. Otherwise, the character in position $m+k$ must be B and the character in position $m+k+1$ must be A. In this case, move these two characters to positions $m-1$,m (shifting all other characters back). It is not difficult to verify that this is indeed a bijection.

Thus, the problem is now equivalent to computing the number of words satisfying the first two conditions and $P(1, 14)$. However, this condition simply means that the first 15 characters must be B. Now we are essentially counting the number of paths from $(15, 0)$ to $(50, 50)$ that don't go above $y = x$. There is a bijection between paths from $(15, 0)$ to $(50, 50)$ that do cross $y = x$ and paths from $(15, 0)$ to $(49, 51)$ (using the standard reflection argument). Thus the answer is $\binom{85}{35} - \binom{85}{34}$