HMMT February 2018

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Geometry

1. Triangle GRT has $GR = 5$, $RT = 12$, and $GT = 13$. The perpendicular bisector of GT intersects the extension of GR at O. Find TO.

2. Points A, B, C, D are chosen in the plane such that segments AB, BC, CD, DA have lengths 2, 7, 5, 12, respectively. Let m be the minimum possible value of the length of segment AC and let M be the maximum possible value of the length of segment AC. What is the ordered pair (m, M)?

3. How many noncongruent triangles are there with one side of length 20, one side of length 17, and one 60° angle?

4. A paper equilateral triangle of side length 2 on a table has vertices labeled A, B, C. Let M be the point on the sheet of paper halfway between A and C. Over time, point M is lifted upwards, folding the triangle along segment BM, while A, B, and C remain on the table. This continues until A and C touch. Find the maximum volume of tetrahedron $ABCM$ at any time during this process.

5. In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

6. Let ABC be an equilateral triangle of side length 1. For a real number $0 < x < 0.5$, let A_1 and A_2 be the points on side BC such that $A_1B = A_2C = x$, and let $T_A = \triangle AA_1A_2$. Construct triangles $T_B = \triangle BB_1B_2$ and $T_C = \triangle CC_1C_2$ similarly.

   There exist positive rational numbers b, c such that the region of points inside all three triangles T_A, T_B, T_C is a hexagon with area
   $$\frac{8x^2 - bx + c}{(2-x)(x+1)} \cdot \frac{\sqrt{3}}{4}.$$
   Find (b, c).

7. Triangle ABC has sidelengths $AB = 14$, $AC = 13$, and $BC = 15$. Point D is chosen in the interior of $\overline{AB}$ and point E is selected uniformly at random from $\overline{AD}$. Point F is then defined to be the intersection point of the perpendicular to $\overline{AB}$ at E and the union of segments $\overline{AC}$ and $\overline{BC}$. Suppose that D is chosen such that the expected value of the length of $\overline{EF}$ is maximized. Find AD.

8. Let ABC be an equilateral triangle with side length 8. Let X be on side AB so that $AX = 5$ and Y be on side AC so that $AY = 3$. Let Z be on side BC so that AZ, BY, CX are concurrent. Let ZX, ZY intersect the circumcircle of AXY again at P, Q respectively. Let XQ and YP intersect at K. Compute $KX \cdot KQ$.

9. Po picks 100 points $P_1, P_2, \ldots, P_{100}$ on a circle independently and uniformly at random. He then draws the line segments connecting P_1P_2, $P_2P_3, \ldots, P_{100}P_1$. When all of the line segments are drawn, the circle is divided into a number of regions. Find the expected number of regions that have all sides bounded by straight lines.

10. Let ABC be a triangle such that $AB = 6, BC = 5, AC = 7$. Let the tangents to the circumcircle of ABC at B and C meet at X. Let Z be a point on the circumcircle of ABC. Let Y be the foot of the perpendicular from X to CZ. Let K be the intersection of the circumcircle of BCY with line AB. Given that Y is on the interior of segment CZ and $YZ = 3CY$, compute AK.