3^{rd} Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Guts Round

1. [**5**] David, Delong, and Justin each showed up to a problem writing session at a random time during the session. If David arrived before Delong, what is the probability that he also arrived before Justin?

   Answer: $\boxed{\frac{2}{3}}$ Let t_1 be the time that David arrives, let t_2 be the time that Delong arrives, and let t_3 be the time that Justin arrives. We can assume that all times are pairwise distinct because the probability of any two being equal is zero. Because the times were originally random and independent before we were given any information, then all orders $t_1 < t_2 < t_3$, $t_1 < t_3 < t_2$, $t_3 < t_1 < t_2$, $t_2 < t_1 < t_3$, $t_2 < t_3 < t_1$, $t_3 < t_2 < t_1$ must all be equally likely. Since we are given that $t_1 < t_2$, then we only have the first three cases to consider, and $t_1 < t_3$ in two cases of these three. Thus, the desired probability is $\frac{2}{3}$.

2. [**5**] A circle of radius 6 is drawn centered at the origin. How many squares of side length 1 and integer coordinate vertices intersect the interior of this circle?

   Answer: $\boxed{132}$ By symmetry, the answer is four times the number of squares in the first quadrant. Let's identify each square by its coordinates at the bottom-left corner, (x, y). When $x = 0$, we can have $y = 0 \ldots 5$, so there are 6 squares. (Letting $y = 6$ is not allowed because that square intersects only the *boundary* of the circle.) When $x = 1$, how many squares are there? The equation of the circle is $y = \sqrt{36 - x^2} = \sqrt{36 - 1^2}$ is between 5 and 6, so we can again have $y = 0 \ldots 5$. Likewise for $x = 2$ and $x = 3$. When $x = 4$ we have $y = \sqrt{20}$ which is between 4 and 5, so there are 5 squares, and when $x = 5$ we have $y = \sqrt{11}$ which is between 3 and 4, so there are 4 squares. Finally, when $x = 6$, we have $y = 0$, and no squares intersect the interior of the circle. This gives $6 + 6 + 6 + 6 + 5 + 4 = 33$. Since this is the number in the first quadrant, we multiply by four to get 132.

3. [**5**] Jacob flipped a fair coin five times. In the first three flips, the coin came up heads exactly twice. In the last three flips, the coin also came up heads exactly twice. What is the probability that the third flip was heads?

   Answer: $\boxed{\frac{4}{5}}$ How many sequences of five flips satisfy the conditions, and have the third flip be heads? We have __H__, so exactly one of the first two flips is heads, and exactly one of the last two flips is heads. This gives $2 \times 2 = 4$ possibilities. How many sequences of five flips satisfy the conditions, and have the third flip be tails? Now we have __T__, so the first two and the last two flips must all be heads. This gives only 1 possibility. So the probability that the third flip was heads is $\frac{4}{(4+1)} = \frac{4}{5}$

4. [**6**] Let x be a real number. Find the maximum value of $2^{x(1-x)}$.

   Answer: $\boxed{\sqrt[4]{2}}$ Consider the function 2^y. This is monotonically increasing, so to maximize 2^y, you simply want to maximize y. Here, $y = x(1 - x) = -x^2 + x$ is a parabola opening downwards. The vertex of the parabola occurs at $x = (-1)/(-2) = 1/2$, so the maximum value of the function is $2^{(1/2)(1/2)} = \sqrt[4]{2}$.

5. [**6**] An icosahedron is a regular polyhedron with twenty faces, all of which are equilateral triangles. If an icosahedron is rotated by θ degrees around an axis that passes through two opposite vertices so that it occupies exactly the same region of space as before, what is the smallest possible positive value of θ?

   Answer: $\boxed{72^\circ}$ Because this polyhedron is regular, all vertices must look the same. Let's consider just one vertex. Each triangle has a vertex angle of 60°, so we must have fewer than 6 triangles; if we had 6, there would be 360° at each vertex and you wouldn't be able to "fold" the polyhedron up (that is, it would be a flat plane). It's easy to see that we need at least 3 triangles at each vertex, and this gives a triangular pyramid with only 4 faces. Having 4 triangles meeting at each vertex gives an octahedron (two square pyramids with the squares glued together) with 8 faces. Therefore, an icosahedron has 5 triangles meeting at each vertex, so rotating by $\frac{360^\circ}{5} = 72^\circ$ gives another identical icosahedron.

Alternate solution: Euler's formula tells us that $V - E + F = 2$, where an icosahedron has V vertices, E edges, and F faces. We're told that $F = 20$. Each triangle has 3 edges, and every edge is common to 2 triangles, so $E = \frac{3(20)}{2} = 30$. Additionally, each triangle has 3 vertices, so if every vertex is common to n triangles, then $V = \frac{3(20)}{n} = \frac{60}{n}$. Plugging this into the formula, we have $\frac{60}{n} - 30 + 20 = 2$, so $\frac{60}{n} = 12$ and $n = 5$. Again this shows that the rotation is $\frac{360°}{5} = 72°$

6. [**6**] How many ordered pairs (S, T) of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are there whose union contains exactly three elements?

Answer: $\boxed{3240}$ Let the three elements in the union be a, b, and c. We know that a can be only in S, only in T, or both, so there are 3 possibilities for placing it. (Recall that $S = \{a\}, T = \{b, c\}$ is different from $S = \{b, c\}, T = \{a\}$ because S and T are an *ordered* pair.) Likewise for b and c. The other 7 elements are in neither S nor T, so there is only 1 possibility for placing them. This gives $3^3 = 27$ ways to pick S and T once you've picked the union. There are $\binom{10}{3} = 120$ ways to pick the elements in the union, so we have $120 \times 27 = 3240$ ways total.

7. [**7**] Let $f(x, y) = x^2 + 2x + y^2 + 4y$. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) be the vertices of a square with side length one and sides parallel to the coordinate axes. What is the minimum value of $f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) + f(x_4, y_4)$?

Answer: $\boxed{-18}$ The square's corners must be at (x, y), $(x+1, y)$, $(x+1, y+1)$, and $(x, y+1)$ for some x and y. So,

$$\begin{aligned}
&f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) + f(x_4, y_4) \\
&\qquad = 2(x^2 + 2x) + 2((x+1)^2 + 2(x+1)) + 2(y^2 + 4y) + 2((y+1)^2 + 4(y+1)) \\
&\qquad = 4x^2 + 12x + 6 + 4y^2 + 20y + 10 \\
&\qquad = (2x+3)^2 - 3 + (2y+5)^2 - 15 \\
&\qquad \geq -18
\end{aligned}$$

This attains its minimum value of -18 when $x = -\frac{3}{2}$ and $y = -\frac{5}{2}$.

8. [**7**] What is the sum of all four-digit numbers that are equal to the cube of the sum of their digits (leading zeros are not allowed)?

Answer: $\boxed{10745}$ We want to find all integers x between 1000 and 9999 that are the cube of the sum of their digits. Of course, our search is only restricted to perfect cubes. The smallest such cube is $10^3 = 1000$ and the largest such cube is $21^3 = 9261$. This means we only have to check 12 different cubes, which is quite doable, but we can reduce the search even further with a little number theory.

Suppose we write our number as $x = 1000a + 100b + 10c + d$, where a, b, c, and d are the decimal digits of x. Then we have

$$(a + b + c + d)^3 \equiv 1000a + 100b + 10c + d \equiv a + b + c + d \pmod 9$$

If we let $k = a + b + c + d$, then k must be a solution to the modular equation $k^3 \equiv k \pmod 9$. A quick check of the values 0 through 8 shows that the only solutions are 0, 1, and 8.

Now, in our search, we only have to check values that are the cube of a number which is either 0, 1, or 8 mod 9.

$$\begin{aligned}
10^3 &= 1000, \text{ but } 1 + 0 + 0 + 0 \neq 10. \\
17^3 &= 4913, \text{ and } 4 + 9 + 1 + 3 = 17. \\
18^3 &= 5832, \text{ and } 5 + 8 + 3 + 2 = 18. \\
19^3 &= 6859, \text{ but } 6 + 8 + 5 + 9 \neq 19.
\end{aligned}$$

So the only solutions are 4913 and 5832, which sum to 10745.

9. [**7**] How many functions $f : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 10\}$ satisfy the property that $f(i) + f(j) = 11$ for all values of i and j such that $i + j = 11$.

   Answer: $\boxed{100000}$ To construct such a function f, we just need to choose a value for $f(x)$ from $\{1, 2, ..., 10\}$ for each $x \in \{1, 2, ..., 10\}$. But the condition that $f(i) + f(j) = 11$ whenever $i + j = 11$ means that

$$\begin{aligned} f(10) &= 11 - f(1). \\ f(9) &= 11 - f(2). \\ &\vdots \\ f(6) &= 11 - f(5). \end{aligned}$$

   This means that once we have chosen $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$, the five remaining values of $f(6)$, $f(7)$, $f(8)$, $f(9)$, and $f(10)$ are already determined. The answer is therefore just the number of ways to choose these first five values. Since there are 10 possibilities for each one, we get that the answer is $10^5 = 100000$.

10. [**8**] What is the smallest integer greater than 10 such that the sum of the digits in its base 17 representation is equal to the sum of the digits in its base 10 representation?

    Answer: $\boxed{153}$ We assume that the answer is at most three digits (in base 10). Then our desired number can be expressed in the form $\overline{abc}_{10} = \overline{def}_{17}$, where a, b, c are digits in base 10, and d, e, f are digits in base 17. These variables then satisfy the equations

$$\begin{aligned} 100a + 10b + c &= 289d + 17e + f, \\ a + b + c &= d + e + f. \end{aligned}$$

    Subtracting the second equation from the first, we obtain $99a + 9b = 288d + 16e$, or $9(11a + b) = 16(18d + e)$. From this equation, we find that $11a + b$ must be divisible by 16, and $18d + e$ must be divisible by 9. To minimize $\overline{abc}$, we find the minimal possible value of a: If $a = 0$, then the only way for $11a + b = b$ to be divisible by 16 is to set $b = 0$; however, this is disallowed by the problem condition, which stipulates that the number must be greater than 10. If we try $a = 1$, then we find that the only possible value of b which lets $11a + b = b + 11$ be divisible by 16 is $b = 5$. Plugging these in and simplifying, we find that we must have $18d + e = 9$. The only possible solution to this is $d = 0, e = 9$. Now to satisfy $a + b + c = d + e + f$, we must have $1 + 5 + c = 0 + 9 + f$, or $c = f + 3$. The minimal possible solution to this is $c = 3, f = 0$. So our answer is $\overline{abc} = 153$, which is also equal to 090_{17}.

11. [**8**] How many nondecreasing sequences $a_1, a_2, \ldots, a_{10}$ are composed entirely of at most three distinct numbers from the set $\{1, 2, \ldots, 9\}$ (so $1, 1, 1, 2, 2, 2, 3, 3, 3, 3$ and $2, 2, 2, 2, 5, 5, 5, 5, 5, 5$ are both allowed)?

    Answer: $\boxed{3357}$ From any sequence $a_1, a_2, \ldots, a_{10}$, construct a sequence $b_1, b_2, \ldots, b_9$, where b_i counts the number of times i occurs in the sequence. There is a correspondence from all possible sequences $b_1, b_2, \ldots, b_9$ with at most 3 nonzero terms which add to 10, since any sequence of $a_1, a_2, \ldots, a_{10}$ will be converted to this form, and from any sequence $b_1, b_2, \ldots, b_9$, we can construct a unique sequence of a-s by listing i b_i times (for $1 \leq i \leq 9$) in nondecreasing order.

    Our goal now is to count the number of possible sequences $b_1, b_2, \ldots, b_9$ meeting our conditions. We casework on the number of nonzero terms in the sequence:
    Case 1: The sequence has exactly one nonzero term.
    Then exactly one of $b_1, b_2, \ldots, b_9$ is equal to 10, and all the rest are equal to 0. This gives us 9 possible sequences in this case.
    Case 2: The sequence has exactly two nonzero terms.
    There are $\binom{9}{2} = 36$ ways to choose the two terms b_i, b_j $(i < j)$ which are nonzero. From here, we have 9 choices for the value of b_i, namely 1 through 9 (since both b_i and b_j must be nonzero), and b_j will be fixed, so this case gives us $36 \cdot 9 = 324$ possible sequences.
    Case 3: The sequence has exactly three nonzero terms.
    There are $\binom{9}{3} = 84$ ways to choose the three terms b_i, b_j, b_k $(i < j < k)$ which are nonzero. Letting

$c_i = b_i - 1, c_j = b_j - 1, c_k = b_k - 1$, we have that c_i, c_j, c_k are nonnegative integers which sum to 7. There are $\binom{9}{2} = 36$ solutions to this equation (consider placing two dividers in the nine spaces between the ten elements), giving $84 \cdot 36 = 3024$ possbilities in this case.

We then have $9 + 324 + 3024 = 3357$ possible sequences.

12. [**8**] An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point $(5, 5)$?

Answer: $\boxed{84}$ We can change the ant's sequence of moves to a sequence $a_1, a_2, \ldots, a_{10}$, with $a_i = 0$ if the i-th step is up, and $a_i = 1$ if the i-th step is right. We define a subsequence of moves $a_i, a_{i+1}, \ldots, a_j$, $(i \leq j)$ as an *up run* if all terms of the subsequence are equal to 0, and a_{i-1} and a_{j+1} either do not exist or are not equal to 0, and define a *right run* similarly. In a sequence of moves, up runs and right runs alternate, so the number of up rights can differ from the number of right runs by at most one.

Now let $f(n)$ denote the number of sequences $a_1, a_2, \ldots, a_n$ where $a_i \in \{1, 2\}$ for $1 \leq i \leq n$, and $a_1 + a_2 + \cdots + a_n = 5$. (In essence, we are splitting the possible 5 up moves into up runs, and we are doing the same with the right moves). We can easily compute that $f(3) = 3, f(4) = 4, f(5) = 1$, and $f(n) = 0$ otherwise.

For each possible pair of numbers of up runs and right runs, we have two choices of which type of run is first. Our answer is then $2(f(3)^2 + f(3)f(4) + f(4)^2 + f(4)f(5) + f(5)^2) = 2(9+12+16+4+1) = 84$.

13. [**8**] How many sequences of ten binary digits are there in which neither two zeroes nor three ones ever appear in a row?

Answer: $\boxed{28}$ Let a_n be the number of binary sequences of length n satisfying the conditions and ending in 0, let b_n be the number ending in 01, and let c_n be the number ending in 11. From the legal sequences of length 2 $01, 11, 10$, we find that $a_2 = b_2 = c_2 = 1$. We now establish a recursion by building sequences of length $n + 1$ from sequences of length n. We can add a 0 to a sequence of length n if and only if it ended with a 1, so $a_{n+1} = b_n + c_n$. We can have a sequence of length $n + 1$ ending with 01 only by adding a 1 to a sequence of length n ending in 0, so $b_{n+1} = a_n$. We can have a sequence of length $n + 1$ ending with 11 only by adding a 1 to a sequence of length n ending in 01, so $c_{n+1} = b_n$. We can now run the recursion:

n	a_n	b_n	c_n
2	1	1	1
3	2	1	1
4	2	2	1
5	3	2	2
6	4	3	2
7	5	4	3
8	7	5	4
9	9	7	5
10	12	9	7

Our answer is then $12 + 9 + 7 = 28$.

14. [**8**] The positive integer i is chosen at random such that the probability of a positive integer k being chosen is $\frac{3}{2}$ times the probability of $k+1$ being chosen. What is the probability that the i^{th} digit after the decimal point of the decimal expansion of $\frac{1}{7}$ is a 2?

Answer: $\boxed{\frac{108}{665}}$ First we note that the probability that n is picked is $\frac{1}{2} \times \left(\frac{2}{3}\right)^n$, because this is the sequence whose terms decrease by a factor of $\frac{2}{3}$ each time and whose sum is 1 (recall that probabilities must sum to 1).

Now note that $\frac{1}{7} = .142857142857\ldots$, meaning that 2 occurs at digits 3, 9, 15, 21, etc. We can then calculate the probability that we ever pick 2 as

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{2} \cdot \left(\frac{2}{3}\right)^{6k+3} &= \frac{4}{27} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{6k} \\ &= \frac{4}{27} \cdot \frac{1}{1-\left(\frac{2}{3}\right)^6} \\ &= \frac{4}{27} \cdot \frac{729}{729-64} \\ &= \frac{4}{27} \cdot \frac{729}{665} \\ &= \frac{108}{665}.\end{aligned}$$

15. [**8**] Distinct points A, B, C, D are given such that triangles ABC and ABD are equilateral and both are of side length 10. Point E lies inside triangle ABC such that $EA = 8$ and $EB = 3$, and point F lies inside triangle ABD such that $FD = 8$ and $FB = 3$. What is the area of quadrilateral $AEFD$?

Answer: $\boxed{\frac{91\sqrt{3}}{4}}$ Since $AEB \cong DFB$, we have $\angle EBA = \angle FBD$. Thus, $\angle EBF = \angle EBA + \angle ABF = \angle FBD + \angle ABF = \angle ABD = 60°$. Since $EB = BF = 3$, this means that EBF is an equilateral triangle of side length 3. Now we have $[AEFD] = [AEBD] - [EBF] - [FBD] = [AEB] + [ABD] - [EBF] - [FBD] = [ABD] - [EBF] = \frac{\sqrt{3}}{4}(10^2 - 3^2) = \frac{91\sqrt{3}}{4}$.

16. [**9**] Triangle ABC is given in the plane. Let AD be the angle bisector of $\angle BAC$; let BE be the altitude from B to AD, and let F be the midpoint of AB. Given that $AB = 28, BC = 33, CA = 37$, what is the length of EF?

Answer: $\boxed{14}$ $\triangle ABE$ is a right triangle, and F is the midpoint of the hypotenuse (and therefore the circumcenter), so $EF = BF = AF = 14$.

17. [**9**] A triangle with side lengths $5, 7, 8$ is inscribed in a circle C. The diameters of C parallel to the sides of lengths 5 and 8 divide C into four sectors. What is the area of either of the two smaller ones?

Answer: $\boxed{\frac{49}{18}\pi}$ Let $\triangle PQR$ have sides $p = 7$, $q = 5$, $r = 8$. Of the four sectors determined by the diameters of C that are parallel to PQ and PR, two have angles equal to P and the other two have angles equal to $\pi - P$. We first find P using the law of cosines: $49 = 25 + 64 - 2(5)(8)\cos P$ implies $\cos P = \frac{1}{2}$ implies $P = \frac{\pi}{3}$. Thus the two smaller sectors will have angle $\frac{\pi}{3}$. Next we find the circumradius of $\triangle PQR$ using the formula $R = \frac{pqr}{4[PQR]}$, where $[PQR]$ is the area of $\triangle PQR$. By Heron's Formula we have $[PQR] = \sqrt{10(5)(3)(2)} = 10\sqrt{3}$; thus $R = \frac{5\cdot7\cdot8}{4(10\sqrt{3})} = \frac{7}{\sqrt{3}}$. The area of a smaller sector is thus $\frac{\pi/3}{2\pi}\left(\pi R^2\right) = \frac{\pi}{6}\left(\frac{7}{\sqrt{3}}\right)^2 = \frac{49}{18}\pi$

18. [**9**] Jeff has a 50 point quiz at 11 am. He wakes up at a random time between 10 am and noon, then arrives at class 15 minutes later. If he arrives on time, he will get a perfect score, but if he arrives more than 30 minutes after the quiz starts, he will get a 0, but otherwise, he loses a point for each minute he's late (he can lose parts of one point if he arrives a nonintegral number of minutes late). What is Jeff's expected score on the quiz?

Answer: $\boxed{\frac{55}{2}}$ If Jeff wakes up between 10:00 and 10:45, he gets 50. If he wakes up between 10:45 and 11:15, and he wakes up k minutes after 10:45, then he gets $50 - k$ points. Finally, if he wakes up between 11:15 and 12:00 he gets 0 points. So he has a $\frac{3}{8}$ probability of 50, a $\frac{3}{8}$ probability of 0, and a $\frac{1}{4}$ probability of a number chosen uniformly between 20 and 50 (for an average of 35). Thus his expected score is $\frac{3}{8} \times 50 + \frac{1}{4} \times 35 = \frac{75+35}{4} = \frac{110}{4} = \frac{55}{2}$.

19. [**11**] How many 8-digit numbers begin with 1, end with 3, and have the property that each successive digit is either one more or two more than the previous digit, considering 0 to be one more than 9?

Answer: $\boxed{21}$ Given an 8-digit number a that satifies the conditions in the problem, let a_i denote the difference between its $(i+1)$th and ith digit. Since $i \in \{1,2\}$ for all $1 \le i \le 7$, we have $7 \le a_1 + a_2 + \cdots + a_7 \le 14$. The difference between the last digit and the first digit of m is $3 - 1 \equiv 2 \pmod{10}$, which means $a_1 + \cdots + a_7 = 12$. Thus, exactly five of the a_is equal to 2 and the remaining two equal to 1. The number of permutations of five 2s and two 1s is $\binom{7}{2} = 21$.

20. [**11**] Given a permutation π of the set $\{1, 2, \ldots, 10\}$, define a rotated cycle as a set of three integers i, j, k such that $i < j < k$ and $\pi(j) < \pi(k) < \pi(i)$. What is the total number of rotated cycles over all permutations π of the set $\{1, 2, \ldots, 10\}$?

    Answer: $\boxed{72576000}$ Let us consider a triple (i, j, k) with $i < j < k$ and determine how many permutations rotate it. There are $\binom{10}{3}$ choices for the values of $\pi(i), \pi(j), \pi(k)$ and the choice of this set of three determines the values of $\pi(i), \pi(j), \pi(k)$. The other 7 values then have 7! ways to be arranged (any permutation of them will work), so exactly $\binom{10}{3}7!$ permutations rotate (i, j, k). Therefore, as there are $\binom{10}{3}$ such triples, the total number of rotated triples is $\binom{10}{3}^2 \cdot 7! = 72576000$.

21. [**11**] George, Jeff, Brian, and Travis decide to play a game of hot potato. They begin by arranging themselves clockwise in a circle in that order. George and Jeff both start with a hot potato. On his turn, a player gives a hot potato (if he has one) to a randomly chosen player among the other three (if a player has two hot potatoes on his turn, he only passes one). If George goes first, and play proceedes clockwise, what is the probability that Travis has a hot potato after each player takes one turn?

    Answer: $\boxed{\frac{5}{27}}$ Notice that Travis can only have the hot potato at the end if he has two potatoes before his turn. A little bit of casework shows that this can only happen when
    Case 1: George gives Travis his potato, while Jeff gives Brian his potato, which in then goes to Travis. The probability of this occuring is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$
    Case 2: George gives Travis his potato, while Jeff gives Travis his potato. The probability of this occuring is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$
    Case 3: George gives Brian his potato, Jeff gives Travis his potato, and then Brian gives Travis his potato. The probability of this occuring is $\frac{1}{27}$
    Because these events are all disjoint, the probability that Travis ends up with the hot potato is $\frac{5}{27}$

22. [**12**] Let $g_1(x) = \frac{1}{3}(1 + x + x^2 + \cdots)$ for all values of x for which the right hand side converges. Let $g_n(x) = g_1(g_{n-1}(x))$ for all integers $n \ge 2$. What is the largest integer r such that $g_r(x)$ is defined for some real number x?

    Answer: $\boxed{5}$ Notice that the series is geometric with ratio x, so it converges if $-1 < x < 1$. Also notice that where $g_1(x)$ is defined, it is equal to $\frac{1}{3(1-x)}$. The image of $g_1(x)$ is then the interval $(\frac{1}{6}, \infty)$. The image of $g_2(x)$ is simply the values of $g_1(x)$ for x in $(\frac{1}{6}, 1)$, which is the interval $(\frac{2}{5}, \infty)$. Similarly, the image of $g_3(x)$ is $(\frac{5}{9}, \infty)$, the image of $g_4(x)$ is $(\frac{3}{4}, \infty)$, and the image of $g_5(x)$ is $(\frac{4}{3}, \infty)$. As this does not intersect the interval $(-1, 1)$, $g_6(x)$ is not defined for any x, so the answer is 5.

23. [**12**] Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers such that for integers $n > 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. How many such sequences $\{a_n\}$ are there such that $a_{2010} \le 2^{2012}$?

    Answer: $\boxed{36 \cdot 2^{2009} + 36}$ Consider the characteristic polynomial for the recurrence $a_{n+2} - 3a_{n+1} + 2a_n = 0$, which is $x^2 - 3x + 2$. The roots are at 2 and 1, so we know that numbers a_i must be of the form $a_i = a2^{i-1} + b$ for integers a and b. Therefore a_{2010} must equal to $a2^{2009} + b$, where a and b are both integers. If the expression is always positive, it is sufficient to say a_1 is positive and a is nonnegative, or $a + b > 0$, and $a \ge 0$.
    For a given value of a, $1 - a \le b \le 2^{2012} - a2^{2009}$, so there are $2^{2012} - a2^{2009} + a$ possible values of b for each a (where the quantity is positive). a can take any value between 0 and 2^3, we sum over all such a in this range, to attain $9 \cdot 2^{2012} - (1+2+3+4+5+6+7+8)2^{2009} + (1+2+3+4+5+6+7+8)$, or $36\left(2^{2009}\right) + 36$, which is our answer.

24. [**12**] Let $P(x)$ be a polynomial of degree at most 3 such that $P(x) = \frac{1}{1+x+x^2}$ for $x = 1, 2, 3, 4$. What is $P(5)$?

Answer: $\boxed{\frac{-3}{91}}$ The forward difference of a polynomial P is $\Delta P(x) = P(x+1) - P(x)$, which is a new polynomial with degree reduced by one. Therefore, if we apply this operation three times we'll get a constant function, and we can work back up to get a value of $P(5)$. Practically, we create the following table of differences:

$$\begin{array}{ccccccc}
\frac{1}{3} & & \frac{1}{7} & & \frac{1}{13} & & \frac{1}{21} \\
& \frac{-4}{21} & & \frac{-6}{91} & & \frac{-8}{273} & \\
& & \frac{34}{273} & & \frac{10}{273} & & \\
& & & \frac{-24}{273} & & &
\end{array}$$

Then extend it to be the following table:

$$\begin{array}{ccccccccc}
\frac{1}{3} & & \frac{1}{7} & & \frac{1}{13} & & \frac{1}{21} & & \frac{-9}{273} \\
& \frac{-4}{21} & & \frac{-6}{91} & & \frac{-8}{273} & & \frac{-22}{273} & \\
& & \frac{34}{273} & & \frac{10}{273} & & \frac{-14}{273} & & \\
& & & \frac{-24}{273} & & \frac{-24}{273} & & &
\end{array}$$

So our answer is $\frac{-9}{273} = \frac{-3}{91}$

25. [**14**] Triangle ABC is given with $AB = 13$, $BC = 14$, $CA = 15$. Let E and F be the feet of the altitudes from B and C, respectively. Let G be the foot of the altitude from A in triangle AFE. Find AG.

Answer: $\boxed{\frac{396}{65}}$ By Heron's formula we have $[ABC] = \sqrt{21(8)(7)(6)} = 84$. Let D be the foot of the altitude from A to BC; then $AD = 2 \cdot \frac{84}{14} = 12$. Notice that because $\angle BFC = \angle BEC$, $BFEC$ is cyclic, so $\angle AFE = 90 - \angle EFC = 90 - \angle EBC = \angle C$. Therefore, we have $\triangle AEF \sim \triangle ABC$, so $\frac{AG}{AD} = \frac{AE}{AB}$; $\frac{1}{2}(BE)(AC) = 84 \implies BE = \frac{56}{5} \implies AE = \sqrt{13^2 - \left(\frac{56}{5}\right)^2} = \sqrt{\frac{65^2 - 56^2}{5^2}} = \frac{33}{5}$. Then $AG = AD \cdot \frac{AE}{AB} = 12 \cdot \frac{33/5}{13} = \frac{396}{65}$.

26. [**14**] w, x, y, z are real numbers such that

$$\begin{aligned}
w + x + y + z &= 5 \\
2w + 4x + 8y + 16z &= 7 \\
3w + 9x + 27y + 81z &= 11 \\
4w + 16x + 64y + 256z &= 1
\end{aligned}$$

What is the value of $5w + 25x + 125y + 625z$?

Answer: $\boxed{-60}$ We note this system of equations is equivalent to evaluating the polynomial (in a) $P(a) = wa + xa^2 + ya^3 + za^4$ at 1, 2, 3, and 4. We know that $P(0) = 0$, $P(1) = 5$, $P(2) = 7$, $P(3) = 11$, $P(4) = 1$. The finite difference of a polynomial f is $f(n+1) - f(n)$, which is a polynomial with degree one less than the degree of f. The second, third, etc finite differences come from applying this operation repeatedly. The fourth finite difference of this polynomial is constant because this is a fourth degree polynomial. Repeatedly applying finite differences, we get

$$\begin{array}{ccccccccc}
0 & & 5 & & 7 & & 11 & & 1 \\
& 5 & & 2 & & 4 & & -10 & \\
& & -3 & & 2 & & -14 & & \\
& & & 5 & & -16 & & & \\
& & & & -21 & & & &
\end{array}$$

and we see that the fourth finite difference is -21. We can extend this table, knowing that the fourth finite difference is always -21, and we find that that $P(5) = -60$. The complete table is

$$\begin{array}{ccccccccccc}
0 & & 5 & & 7 & & 11 & & 1 & & -60 \\
& 5 & & 2 & & 4 & & -10 & & -61 & \\
& & -3 & & 2 & & -14 & & -51 & & \\
& & & 5 & & -16 & & -37 & & & \\
& & & & -21 & & -21 & & & &
\end{array}$$

27. [**14**] Let $f(x) = -x^2 + 10x - 20$. Find the sum of all 2^{2010} solutions to $\underbrace{f(f(\ldots(x)\ldots))}_{2010 fs} = 2$.

Answer: $\boxed{5 \cdot 2^{2010}}$ Define $g(x) = f(f(\ldots(x)\ldots))$. We calculate:

$$f(10-x) = -(10-x)^2 + 10(10-x) - 20 = -100 + 20x - x^2 + 100 - 10x - 20 = -x^2 + 10x - 20 = f(x).$$

This implies that $g(10-x) = g(x)$. So if $g(x) = 2$, then $g(10-x) = 2$. Moreover, we can calculate $f(5) = -25 + 50 - 20 = 5$, so $g(5) = 5 \neq 2$. Thus the possible solutions to $g(x) = 2$ can be grouped into pairs, $(x_1, 10 - x_1), (x_2, 10 - x_2), \ldots$. The sum of the members of each pair is 10, and there are 2^{2009} pairs, so the sum is

$$10 \cdot 2^{2009} = 5 \cdot 2^{2010}.$$

28. [**17**] In the game of set, each card has four attributes, each of which takes on one of three values. A set deck consists of one card for each of the 81 possible four-tuples of attributes. Given a collection of 3 cards, call an attribute *good* for that collection if the three cards either all take on the same value of that attribute or take on all three different values of that attribute. Call a collection of 3 cards *two-good* if exactly two attributes are good for that collection. How many two-good collections of 3 cards are there? The order in which the cards appear does not matter.

Answer: $\boxed{25272}$ In counting the number of sets of 3 cards, we first want to choose which of our two attributes will be good and which of our two attributes will not be good. There are $\binom{4}{2} = 6$ such choices.

Now consider the two attributes which are not good, attribute X and attribute Y. Since these are not good, some value should appear exactly twice. Suppose the value a appears twice and b appears once for attribute X and that the value c appears twice and d appears once for attribute Y. There are three choices for a and then two choices for b; similarly, there are three choices for c and then two choices for d. This gives $3 \cdot 2 \cdot 3 \cdot 2 = 36$ choices of a, b, c, and d.

There are two cases to consider. The first is that there are two cards which both have a and c, while the other card has both b and d. The second case is that only one card has both a and c, while one card has a and d and the other has b and c.

Case 1:

Card 1	*Card 2*	*Card 3*
— Good attribute 1 —		
— Good attribute 2 —		
a	a	b
c	c	d

The three cards need to be distinct. Card 3 is necessarily distinct from Card 1 and Card 2, but we need to ensure that Card 1 and Card 2 are distinct from each other. There are 9 choices for the two good attributes of Card 1, and then 8 choices for the two good attributes of Card 2. But we also want to divide by 2 since we do not care about the order of Card 1 and Card 2. So there are $\frac{9 \cdot 8}{2} = 36$ choices for the good attributes on Card 1 and Card 2. Then, the values of the good attributes of Card 1 and Card 2 uniquely determine the values of the good attributes of Card 3.

Case 2:

Card 1	*Card 2*	*Card 3*
— Good attribute 1 —		
— Good attribute 2 —		
a	a	b
c	d	c

Card 1, Card 2, and Card 3 will all be distinct no matter what the values of the good attributes are, because the values of attributes X and Y are unique to each card. So there are 9 possibilities for the the values of the good attributes on card 1, and then there are 9 more possibilities for the values of the good attribute on Card 2. We do not have to divide by 2 this time, since Card 1 and Card 2 have distinct values in X and Y. So there are $9^2 = 81$ possibilities here.

So our final answer is $6 \cdot 6^2 \cdot (36 + 81) = 25272$.

29. [**17**] In the game of Galactic Dominion, players compete to amass cards, each of which is worth a certain number of points. Say you are playing a version of this game with only two kinds of cards, planet cards and hegemon cards. Each planet card is worth 2010 points, and each hegemon card is worth four points per planet card held. You start with no planet cards and no hegemon cards, and, on each turn, starting at turn one, you take either a planet card or a hegemon card, whichever is worth more points given the hand you currently hold. Define a sequence $\{a_n\}$ for all positive integers n by setting a_n to be 0 if on turn n you take a planet card and 1 if you take a hegemon card. What is the smallest value of N such that the sequence $a_N, a_{N+1}, \ldots$ is necessarily periodic (meaning that there is a positive integer k such that $a_{n+k} = a_n$ for all $n \geq N$)?

    Answer: $\boxed{503}$ If you have P planets and H hegemons, buying a planet gives you $2010 + 4H$ points while buying a hegemon gives you $4P$ points. Thus you buy a hegemon whenever $P - H \geq 502.5$, and you buy a planet whenever $P - H \leq 502.5$. Therefore $a_i = 1$ for $1 \leq i \leq 503$. Starting at $i = 504$ (at which point you have bought 503 planets) you must alternate buying planets and hegemons. The sequence $\{a_i\}_{i \geq 503}$ is periodic with period 2.

30. [**17**] In the game of projective set, each card contains some nonempty subset of six distinguishable dots. A projective set deck consists of one card for each of the 63 possible nonempty subsets of dots. How many collections of five cards have an even number of each dot? The order in which the cards appear does not matter.

    Answer: $\boxed{109368}$ We'll first count sets of cards where the order does matter. Suppose we choose the first four cards. Then there is exactly one card that can make each dot appear twice. However, this card could be empty or it could be one of the cards we've already chosen, so we have to subtract for these two cases. First, there are $63 \cdot 62 \cdot 61 \cdot 60$ ways to choose the first four cards. Let's now count how many ways there are that the fifth card could be empty.

    The fifth card is empty if and only if the first four cards already have an even number of each dot. Suppose we choose the first two cards. There is a possible fourth card if the third card is not either of the first two or the card that completes a set. If that is the case, then the fourth card is unique. This comes to $63 \cdot 62 \cdot 60$ cases.

    Now consider how many ways there are for the fifth card to be a duplicate. This is just the number of ways for three cards to have an even number of each dot, then have two copies of the same card in the other two slots, one of which needs to be the fifth slot. The number of ways for three cards to have an even number of each dot is just the number of ways to choose two cards. Therefore, we'll choose two cards ($63 \cdot 62$ ways), choose the slot in the first four positions for the duplicate card (4 ways), and the duplicate card, which can't be any of the nonduplicated cards, so there are 60 choices. Therefore, there are $63 \cdot 62 \cdot 4 \cdot 60$ ways for the fifth card to be the same as one of the first four.

    This means that the number of five card sets where the order does matter is $63 \cdot 62 \cdot 61 \cdot 60 - 63 \cdot 62 \cdot 60 - 63 \cdot 62 \cdot 4 \cdot 60$, so our final answer is $\frac{63 \cdot 62 \cdot 61 \cdot 60 - 63 \cdot 62 \cdot 60 - 63 \cdot 62 \cdot 4 \cdot 60}{120} = \frac{63 \cdot 62 \cdot (61 - 1 - 4)}{2} = 63 \cdot 31 \cdot 56 = 109368$.

31. [**20**] What is the perimeter of the triangle formed by the points of tangency of the incircle of a 5-7-8 triangle with its sides?

Answer: $\boxed{\frac{9\sqrt{21}}{7}+3}$ Let $\triangle ABC$ be a triangle with sides $a=7$, $b=5$, and $c=8$. Let the incircle of $\triangle ABC$ be tangent to sides BC, CA, and AB at points D, E, and F. By the law of cosines (using the form $\cos(A)=\frac{b^2+c^2-a^2}{2bc}$), we have

$$\begin{aligned}\cos(A) &= \frac{8^2+5^2-7^2}{2(5)(8)} = \frac{1}{2}\\ \cos(B) &= \frac{8^2+7^2-5^2}{2(7)(8)} = \frac{11}{14}\\ \cos(C) &= \frac{5^2+7^2-8^2}{2(7)(5)} = \frac{1}{7}\end{aligned}$$

Now we observe that AEF, BDF, and CDE are all isosceles. Let us call the lengths of the legs of these triangles s, t, and u, respectively. Then we know that $s+t=8$, $t+u=7$, and $u+s=5$, so $s=3$, $t=5$, and $u=2$.

Our final observation is that an isosceles angle with legs of length l and whose non-equal angle is θ has a base of length $l\sqrt{2(1-\cos(\theta))}$. This can be proven using the law of cosines or the Pythagorean theorem.

Using this, we can calculate that

$$\begin{aligned}DE &= 2\sqrt{2(1-\cos(C))}\\ &= 2\sqrt{\frac{12}{7}}\\ EF &= 3\sqrt{2(1-\cos(A))}\\ &= 3\\ FD &= 5\sqrt{2(1-\cos(B))}\\ &= 5\sqrt{\frac{3}{7}},\end{aligned}$$

and then

$$\begin{aligned}DE+EF+FD &= 2\sqrt{\frac{12}{7}}+3+5\sqrt{\frac{3}{7}}\\ &= 3+9\sqrt{\frac{3}{7}}\\ &= 3+9\frac{\sqrt{21}}{7}.\end{aligned}$$

32. [**20**] Let T be the set of numbers of the form 2^a3^b where a and b are integers satisfying $0\le a,b\le 5$. How many subsets S of T have the property that if n is in S then all positive integer divisors of n are in S?

Answer: $\boxed{924}$ Consider the correspondence $(a,b)\leftrightarrow 2^a3^b$ for non-negative integers a and b. So we can view T as the square of lattice points (a,b) where $0\le a,b\le 5$, and subsets of T as subsets of this square.

Notice then that the integer corresponding to (a_1,b_1) is a divisor of the integer corresponding to (a_2,b_2) if and only if $0\le a_1\le a_1$ and $0\le b_1\le b_2$. This means that subsets $S\subset T$ with the desired property, correspond to subsets of the square where if a point is in the set, then so are all points to the left and south of it.

Consider any such subset S. For each $0 \leq x \leq 5$, let S_x be the maximum y value of any point $(x, y) \in S$, or -1 if there is no such point. We claim the values S_x uniquely characterize S. This is because each S_x characterizes the points of the form (x, y) in S. In particular, (x, z) will be in S if and only if $z \leq S_x$. If $(x, z) \in S$ with $z > S_x$, then S_x is not the maximum value, and if $(x, z) \notin S$ with $z \leq S_x$, then S fails to satisfy the desired property.

We now claim that $S_x \geq S_y$ for $x < y$, so the sequence $S_0, \ldots, S_1$ is decreasing. This is because if (y, S_y) is in the set S, then so must be (x, S_y). Conversely, it is easy to see that if $S_0, \ldots, S_1$ is decreasing, then S is a set satisfying the desired property.

We now claim that decreasing sequences $S_0, \ldots, S_5$ are in bijective correspondence with walks going only right and down from $(-1, 5)$ to $(5, -1)$. The sequence $S_0, \ldots, S_5$ simply corresponds to the walk $(-1, 5) \to (-1, S_0) \to (0, S_0) \to (0, S_1) \to (1, S_1) \to \cdots \to (4, S_5) \to (5, S_5) \to (5, -1)$. Geometrically, we are tracing out the outline of the set S.

The number of such walks is simply $\binom{12}{6}$, since we can view it as choosing the 6 of 12 steps at which to move right. Thus the number of subsets S of T with the desired property is $\binom{12}{6} = 924$.

33. [**20**] Convex quadrilateral $BCDE$ lies in the plane. Lines EB and DC intersect at A, with $AB = 2$, $AC = 5$, $AD = 200$, $AE = 500$, and $\cos \angle BAC = \frac{7}{9}$. What is the largest number of nonoverlapping circles that can lie in quadrilateral $BCDE$ such that all of them are tangent to both lines BE and CD?

Answer: $\boxed{5}$ Let $\theta = \angle BAC$, and $\cos\theta = \frac{7}{9}$ implies $\cos\frac{\theta}{2} = \sqrt{\frac{1+\frac{7}{9}}{2}} = \frac{2\sqrt{2}}{3}$; $\sin\frac{\theta}{2} = \frac{1}{3}$; $BC = \sqrt{4 + 25 - 2(2)(5)\frac{7}{9}} = \frac{11}{3}$. Let O_1 be the excircle of $\triangle ABC$ tangent to lines AB and AC, and let r_1 be its radius; let O_1 be tangent to line AB at point P_1. Then $AP_1 = \frac{AB+BC+CA}{2}$ and $\frac{r_1}{AP_1} = \tan\frac{\theta}{2} = \frac{1}{2\sqrt{2}} \implies r_1 = \frac{16}{3\cdot 2\sqrt{2}}$. Let O_n be a circle tangent to O_{n-1} and the lines AB and AC, and let r_n be its radius; let O_n be tangent to line AB at point P_n. Then $\frac{O_nP_n}{AO_n} = \sin\frac{\theta}{2} = \frac{1}{3}$; since $\triangle AP_nO_n \sim \triangle AP_{n-1}O_{n-1}$ and $O_nO_{n-1} = r_n + r_{n-1}$, we have $\frac{1}{3} = \frac{O_nP_n}{AO_n} = \frac{r_n}{AO_{n-1}+O_{n-1}O_n} = \frac{r_n}{3r_{n-1}+r_n+r_{n-1}} \implies r_n = 2r_{n-1} = 2^{n-1}\frac{16}{3\cdot 2\sqrt{2}}$. We want the highest n such that O_n is contained inside $\triangle ADE$. Let the incircle of $\triangle ADE$ be tangent to AD at X; then the inradius of $\triangle ADE$ is $\frac{AX}{\tan\frac{\theta}{2}} = \frac{\frac{500+200-\frac{1100}{3}}{2}}{2\sqrt{2}} = \frac{500}{3\cdot 2\sqrt{2}}$. We want the highest n such that $r_n \leq \frac{500}{3\cdot 2\sqrt{2}}$; thus $2^{n-1} \cdot 16 \leq 500 \implies n = 5$.

34. [**25**] Estimate the sum of all the prime numbers less than $1,000,000$. If the correct answer is X and you write down A, your team will receive $\min\left(\lfloor\frac{25X}{A}\rfloor, \lfloor\frac{25A}{X}\rfloor\right)$ points, where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

Answer: $\boxed{37550402023}$ A decent approximation to the sum of all the primes can be obtained with the following two facts. First, there are approximately $\frac{n}{\ln n}$ primes less than n and second, the n^{th} prime is approximately $n\ln n$. We'll approximate $\ln 1000000$ as 15 (the actual number is 13.8), so there are approximately $\frac{10^6}{15}$ primes. Then we want $\sum_{n=1}^{\frac{10^6}{15}} n\ln n$. If you know calculus, this can be approximated by the integral $\int_1^{\frac{10^6}{15}} x\ln x\,dx$, which has the antiderivative $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$, giving an answer of around $\frac{1}{2}\cdot\frac{10^{12}}{225}\cdot(15 - \ln 15) - \frac{1}{4}\cdot\frac{10^{12}}{225}$. Estimating $\ln 15$ as about 3, this is approximately $\frac{23\cdot 10^{12}}{900} = 2.5\cdot 10^{10} = 25,000,000,000$. The actual answer is 37,550,402,023, so an approximation of this accuracy would get 16 points. We can arrive at the same answer without calculus by approximating $\ln n$ as $\sum_{k=1}^{n}\frac{1}{k}$, so that our sum becomes $\sum_{n=1}^{\frac{10^6}{15}}\sum_{k=1}^{n}\frac{n}{k} = \sum_{k=1}^{\frac{10^6}{15}}\sum_{n=k}^{\frac{10^6}{15}}\frac{n}{k} \approx \sum_{k=1}^{\frac{10^6}{15}}\frac{1}{2}\cdot\frac{\frac{10^{12}}{225}-k^2}{k} = \frac{1}{2}\cdot\frac{10^{12}}{225}\cdot\ln(\frac{10^6}{15}) - \frac{1}{2}\sum_{k=1}^{\frac{10^6}{15}} k \approx \frac{1}{2}\cdot\frac{10^{12}}{225}\cdot\ln(\frac{10^6}{15}) - \frac{1}{4}\frac{10^{12}}{225} \approx 2.5\cdot 10^{10}$.

35. [**25**] A mathematician M' is called a descendent of mathematician M if there is a sequence of mathematicians $M = M_1, M_2, \ldots, M_k = M'$ such that M_i was M_{i+1}'s doctoral advisor for all i. Estimate

the number of descendents that the mathematician who has had the largest number of descendents has had, according to the Mathematical Genealogy Project. Note that the Mathematical Genealogy Project has records dating back to the 1300s. If the correct answer is X and you write down A, your team will receive $\max\left(25 - \lfloor\frac{|X-A|}{100}\rfloor, 0\right)$ points, where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

Answer: $\boxed{82310}$ First let's estimate how many "generations" of mathematicians there have been since 1300. If we suppose that a mathematician gets his PhD around age 30 and becomes a PhD advisor around age 60, then we'll get a generation length of approximately 30 years. However, not all mathematicians will train more than one PhD. Let's say that only 40% of mathematicians train at least 2 PhDs. Then effectively we have only 40% of the generations, or in other words each effective generation takes 75 years. Then we have $\frac{22}{3}$ branching generations. If we assume that all of these only train 2 PhDs, then we get an answer of $2^{\frac{22}{3}} \approx 1625$. But we can ensure that our chain has at least a single person who trained 100 PhDs (this is approximately the largest number of advisees for a single mathematician), allowing us to change one factor of 2 into a factor of 100. That gives us an answer of $1625 \cdot 50 = 81250$, which is very close to the actual value of 82310.

36. [**25**] Paul Erdős was one of the most prolific mathematicians of all time and was renowned for his many collaborations. The Erdős number of a mathematician is defined as follows. Erdős has an Erdős number of 0, a mathematician who has coauthored a paper with Erdős has an Erdős number of 1, a mathematician who has not coauthored a paper with Erdős, but has coauthored a paper with a mathematician with Erdős number 1 has an Erdős number of 2, etc. If no such chain exists between Erdős and another mathematician, that mathematician has an Erdős number of infinity. Of the mathematicians with a finite Erdős number (including those who are no longer alive), what is their average Erdős number according to the Erdős Number Project? If the correct answer is X and you write down A, your team will receive $\max(25 - \lfloor 100|X - A|\rfloor, 0)$ points where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

Answer: $\boxed{4.65}$ We'll suppose that each mathematician collaborates with approximately 20 people (except for Erdős himself, of course). Furthermore, if a mathematician has Erdős number k, then we'd expect him to be the cause of approximately $\frac{1}{2^k}$ of his collaborators' Erdős numbers. This is because as we get to higher Erdős numbers, it is more likely that a collaborator has a lower Erdős number already. Therefore, we'd expect about 10 times as many people to have an Erdős number of 2 than with an Erdős number of 1, then a ratio of 5, 2.5, 1.25, and so on. This tells us that more mathematicians have an Erdős number of 5 than any other number, then 4, then 6, and so on. If we use this approximation, we have a ratio of mathematicians with Erdős number 1, 2, and so on of about $1 : 10 : 50 : 125 : 156 : 97 : 30 : 4 : 0.3$, which gives an average Erdős number of 4.8. This is close to the actual value of 4.65.