February 2017

February 18, 2017

Algebra and Number Theory

1. Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \le a_i < 3$ for all $0 \le i \le n$.

   Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

2. Find the value of

$$\sum_{1 \le a < b < c} \frac{1}{2^a 3^b 5^c}$$

   (i.e. the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x)f(y) = f(x - y)$. Find all possible values of $f(2017)$.

4. Find all pairs (a, b) of positive integers such that $a^{2017} + b$ is a multiple of ab.

5. Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c \in \mathbb{N}}} (a + b + c),$$

   where $\mathbb{N}$ denotes the positive integers.

6. A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $\lfloor 2017P(2017) \rfloor$.

7. Determine the largest real number c such that for any 2017 real numbers $x_1, x_2, \ldots, x_{2017}$, the inequality

$$\sum_{i=1}^{2016} x_i(x_i + x_{i+1}) \ge c \cdot x_{2017}^2$$

   holds.

8. Consider all ordered pairs of integers (a, b) such that $1 \le a \le b \le 100$ and

$$\frac{(a + b)(a + b + 1)}{ab}$$

   is an integer.

   Among these pairs, find the one with largest value of b. If multiple pairs have this maximal value of b, choose the one with largest a. For example choose $(3, 85)$ over $(2, 85)$ over $(4, 84)$. Note that your answer should be an ordered pair.

9. The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \ge 2$. Find the smallest positive integer m such that $F_m \equiv 0 \pmod{127}$ and $F_{m+1} \equiv 1 \pmod{127}$.

10. Let $\mathbb{N}$ denote the natural numbers. Compute the number of functions $f : \mathbb{N} \to \{0, 1, \ldots, 16\}$ such that

$$f(x + 17) = f(x) \quad \text{and} \quad f(x^2) \equiv f(x)^2 + 15 \pmod{17}$$

    for all integers $x \ge 1$.