HMMT February 2020

February 15, 2020

Algebra and Number Theory

1. Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

   Proposed by: James Lin

   Answer: $\boxed{-4038}$

   Solution 1: Plugging in $x = r$ gives $r^2 = 2020$. This means $P(1) = 2 - r^2 - 2020 = -4038$.

   Solution 2: Vieta's formulas give the following equations:

$$\begin{aligned} r + s + t &= -1 \\ rs + st + tr &= -r^2 \\ rst &= 2020. \end{aligned}$$

   The second equation is $(r+t)(r+s) = 0$. Without loss of generality, let $r+t = 0$. Then $s = r+s+t = -1$. Finally $r^2 = rst = 2020$, so $P(1) = -4038$.

2. Find the unique pair of positive integers (a, b) with $a < b$ for which

$$\frac{2020 - a}{a} \cdot \frac{2020 - b}{b} = 2.$$

   Proposed by: James Lin

   Answer: $\boxed{(505, 1212)}$

   Solution 1: If either a or b is larger than 2020, then both must be for the product to be positive. However, the resulting product would be less than 1, so this case is impossible. Now, we see that $(\frac{2020-a}{a}, \frac{2020-b}{b})$ must be in the form $(\frac{x}{y}, \frac{2y}{x})$, in some order, for relatively prime positive integers x and y. Then $\frac{2020}{a} = \frac{x+y}{y}$ and $\frac{2020}{b} = \frac{x+2y}{x}$, so $x + y$ and $x + 2y$ are relatively prime factors of 2020. Since $x + y < x + 2y < 2(x + y)$, the only possibility is $x + y = 4$, $x + 2y = 5$. Thus, $(x, y) = (3, 1)$, and $(\frac{2020-a}{a}, \frac{2020-b}{b}) = (3, \frac{2}{3})$ because $a < b$. Solving gives $(a, b) = (505, 1212)$.

   Solution 2: We rearrange to find that $(a+2020)(b+2020) = 2 \cdot 2020^2$. Note that $a+2020$ and $b+2020$ are both less than $4040 < 101^2$, so they must both be divisible by 101. Hence, we divide out a factor of 101^2 and solve the equivalent problem of $(a' + 20)(b' + 20) = 800$, where $a' = \frac{a}{101}$ and $b' = \frac{b}{101}$. Because each factor must be less than $\frac{4040}{101} = 40$, we get that $(a' + 20, b' + 20) = (25, 32)$, which yields $(a, b) = (505, 1212)$.

3. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

   Proposed by: James Lin

   Answer: $\boxed{2^{32}}$

   Solution: Let $y = \log_a x$ so $\log_a \log_a y = \log_{a^2} \log_{a^2} \frac{1}{2}y$. Setting $z = \log_a y$, we find $\log_a z = \log_{a^2}(\frac{1}{2}z - \frac{1}{16})$, or $z^2 - \frac{1}{2}z + \frac{1}{16} = 0$. Thus, we have $z = \frac{1}{4}$, so we can backsolve to get $y = 4$ and $x = 2^{32}$.

4. For positive integers n and k, let $\mho(n,k)$ be the number of distinct prime divisors of n that are at least k. For example, $\mho(90,3)=2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mho(n,k)}{3^{n+k-7}}.$$

Proposed by: Daniel Zhu

Answer: $\boxed{167}$

Solution: A prime p is counted in $\mho(n,k)$ if $p \mid n$ and $k \le p$. Thus, for a given prime p, the total contribution from p in the sum is

$$3^7\sum_{m=1}^{\infty}\sum_{k=1}^{p}\frac{1}{3^{pm+k}} = 3^7\sum_{i\ge p+1}\frac{1}{3^i} = \frac{3^{7-p}}{2}.$$

Therefore, if we consider $p \in \{2,3,5,7,\ldots\}$ we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\mho(n,k)}{3^{n+k-7}} = \frac{3^5}{2}+\frac{3^4}{2}+\frac{3^2}{2}+\frac{3^0}{2}+\varepsilon = 167+\varepsilon,$$

where $\varepsilon < \sum_{i=11}^{\infty}\frac{3^{7-i}}{2} = \frac{1}{108} \ll \frac{1}{2}$. The closest integer to the sum is 167.

5. A positive integer N is *piquant* if there exists a positive integer m such that if n_i denotes the number of digits in m^i (in base 10), then $n_1+n_2+\cdots+n_{10}=N$. Let p_M denote the fraction of the first M positive integers that are piquant. Find $\lim\limits_{M\to\infty} p_M$.

Proposed by: James Lin

Answer: $\boxed{\frac{32}{55}}$

Solution: For notation, let $n_i(m)$ denote the number of digits of m^i and $N(m)=n_1(m)+n_2(m)+\cdots+n_{10}(m)$. Observe that $n_i(10m)=n_i(m)+i$ so $N(10m)=N(m)+55$. We will determine, for $k\to\infty$, how many of the integers from $N(10^k)$ to $N(10^{k+1})-1$, inclusive, are piquant.

Increment m by 1 from 10^k to 10^{k+1}. The number of digits of m^i increases by one if $m^i < 10^h \le (m+1)^i$, or $m < 10^{\frac{h}{i}} \le m+1$ for some integer h. This means that, as we increment m by 1, the sum $n_1+n_2+\cdots+n_{10}$ increases when m "jumps over" $10^{\frac{h}{i}}$ for $i\le 10$. Furthermore, when m is big enough, all "jumps" are distinguishable, i.e. there does not exist two $\frac{h_1}{i_1}\ne\frac{h_2}{i_2}$ such that $m<10^{h_1/i_1}<10^{h_2/i_2}\le m+1$.

Thus, for large k, the number of times $n_1(m)+n_2(m)+\cdots+n_{10}(m)$ increases as m increments by 1 from 10^k to 10^{k+1} is the number of different $10^{\frac{h}{i}}$ in the range $(10^k,10^{k+1}]$. If we take the fractional part of the exponent, this is equivalent to the number of distinct fractions $0<\frac{j}{i}\le 1$ where $1\le i\le 10$. The number of such fractions with denominator i is $\varphi(i)$, so the total number of such fractions is $\varphi(1)+\varphi(2)+\cdots+\varphi(10)=32$.

We have shown that for sufficiently large k, $N(10^{k+1})-N(10^k)=55$ and exactly 32 integers in the range $[N(10^k),N(10^{k+1}))$ are piquant. This implies that $\lim_{M\to\infty}p_M=\frac{32}{55}$.

6. A polynomial $P(x)$ is a *base-n polynomial* if it is of the form $a_dx^d+a_{d-1}x^{d-1}+\cdots+a_1x+a_0$, where each a_i is an integer between 0 and $n-1$ inclusive and $a_d>0$. Find the largest positive integer n such that for any real number c, there exists at most one base-n polynomial $P(x)$ for which $P(\sqrt{2}+\sqrt{3})=c$.

Proposed by: James Lin

Answer: $\boxed{9}$

Solution: It is equivalent to determine the largest n such that we cannot find two distinct base-n polynomials P_1 and P_2 such that $P_1(\sqrt{2}+\sqrt{3}) = P_2(\sqrt{2}+\sqrt{3})$. The difference of two base-n polynomials is a polynomial with integer coefficients whose absolute values are less than n, and all such polynomials are the difference of two base-n polynomials. We compute the minimal polynomial of $x = \sqrt{2}+\sqrt{3}$ first: since $x^2 = 5+2\sqrt{6}$, we have $(x^2-5)^2 = 24$ so $x^4 - 10x^2 + 1 = 0$. Therefore $\sqrt{2}+\sqrt{3}$ is a root of $(x^2+1)(x^4-10x^2+1) = x^6 - 9x^4 - 9x^2 + 1$. The coefficients of this polynomial have magnitude at most 9, so $n < 10$.

In the other direction, observe that $(\sqrt{2}+\sqrt{3})^k$ is of the form $a + b\sqrt{6}$ for integers a and b if k is even, and $a\sqrt{2}+b\sqrt{3}$ if k is odd. As no integer linear combination of the first expression can equal the second, we can treat these cases separately. Suppose $Q(x) = c_d x^{2d} + c_{d-1}x^{2d-2} + \cdots + c_0$ is an even polynomial with $|c_i| < 9$ for all i and $c_d \neq 0$. Let $y = (\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6}$ and observe that $y > 9$. Then

$$\begin{aligned}|c_d y^d| &\geq y^d \\ &> \frac{8}{y-1}(y^d - 1) \\ &= 8y^{d-1} + 8y^{d-2} + \cdots + 8y + 8 \\ &\geq |c_{d-1}y^{d-1} + c_{d-2}y^{d-2} + \cdots + c_0|.\end{aligned}$$

Therefore $Q(\sqrt{2}+\sqrt{3}) = c_d y^d + c_{d-1}y^{d-1} + \cdots + c_0 \neq 0$, so no two distinct base-9 polynomials coincide at $x = \sqrt{2}+\sqrt{3}$.

The same logic applies for the odd polynomial case after dividing out a factor of $\sqrt{2}+\sqrt{3}$, so $n = 9$ works.

7. Find the sum of all positive integers n for which

$$\frac{15 \cdot n!^2 + 1}{2n-3}$$

is an integer.

Proposed by: Andrew Gu

Answer: $\boxed{90}$

Solution: It is clear that $n = 1$ and $n = 2$ work so assume that $n > 2$. If $2n-3$ is composite then its smallest prime factor is at most $\frac{2n-3}{2} < n$ so will be coprime to $15 \cdot n!^2 + 1$. Therefore assume that $2n - 3 = p$ is prime. We can rewrite the numerator as

$$(-1)^n \cdot 15 \cdot \left(1 \cdot 2 \cdots \frac{p+3}{2}\right) \cdot \left(\frac{p-3}{2} \cdot \frac{p-1}{2} \cdots (p-1)\right) + 1 \pmod{p}.$$

By Wilson's Theorem, $(p-1)! \equiv -1 \pmod{p}$, so the expression simplifies to

$$(-1)^{n+1} \cdot 15 \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} \cdot \frac{p+1}{2} \cdot \frac{p+3}{2} + 1 \equiv (-1)^{n+1} \cdot \frac{135}{16} + 1 \pmod{p}.$$

If $p \equiv 3 \pmod{4}$, then we have

$$\frac{135+16}{16} \equiv \frac{151}{16} \equiv 0 \pmod{p}.$$

If $p \equiv 1 \pmod{4}$, then we have

$$\frac{135-16}{16} \equiv \frac{119}{16} \equiv 0 \pmod{p}.$$

So p must be a prime divisor of 151 or 119, which means that $p \in \{7, 17, 151\}$. All of these numbers work aside from 7 (because $7 \equiv 3 \pmod 4$) and the corresponding values of n are 10 and 77. The sum of the solutions is then $1 + 2 + 10 + 77 = 90$.

8. Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

Proposed by: Milan Haiman

Answer: $\boxed{2021 - \binom{4040}{2020}}$

Solution 1: Since $P(0) = 0$, we see that P has no constant term. Let $Q(x) = \frac{P(x^2)-x}{x}$ be a polynomial with degree at most 4039. From the given values of P, we see that $Q(k) = 0$ and $Q(-k) = -2$ for $k = 1, 2, 3, \ldots, 2020$.

Now, consider the polynomial $R(x) = Q(x+1) - Q(x)$, which has degree at most 4038. Then R has roots $-2020, -2019, \ldots, -2, 1, 2, \ldots, 2019$, so

$$R(x) = a(x+2020)\cdots(x+2)(x-1)\cdots(x-2019)$$

for some real number a. Using $R(0) + R(-1) = Q(1) - Q(-1) = 2$ yields $a = -\frac{1}{2020!2019!}$, so

$$Q(2021) = R(2020) + Q(2020) = -\frac{4040\cdots2022\cdot2019\cdots1}{2020!2019!} + 0 = -\frac{1}{2021}\binom{4040}{2020}.$$

It follows that $P(2021^2) = 2021Q(2021) + 2021 = 2021 - \binom{4040}{2020}$.

Solution 2: By Lagrange interpolation,

$$P(x) = \sum_{k=0}^{2020} k \prod_{\substack{0\le j\le 2020\\ j\ne k}} \frac{(x-j^2)}{(k^2-j^2)} = \sum_{k=0}^{2020} \frac{2(-1)^k k}{(2020-k)!(2020+k)!(x-k^2)} \prod_{j=0}^{2020}(x-j^2).$$

Therefore, by applying Pascal's identity multiple times, we get that

$$\begin{aligned}
P(2021^2) &= \sum_{k=0}^{2020} \frac{4042!(-1)^k k}{(2021-k)!(2021+k)!} \\
&= \sum_{k=0}^{2020} \binom{4042}{2021-k}(-1)^k k \\
&= 2021 - \left(\sum_{k=0}^{2021}(-1)^{k+1}k\left(\binom{4041}{2021-k} + \binom{4041}{2020-k}\right)\right) \\
&= 2021 - \left(\sum_{k=1}^{2021}(-1)^{k+1}\binom{4041}{2021-k}\right) \\
&= 2021 - \left(\sum_{k=1}^{2021}(-1)^{k+1}\left(\binom{4040}{2021-k} + \binom{4040}{2020-k}\right)\right) \\
&= 2021 - \binom{4040}{2020}.
\end{aligned}$$

9. Let $P(x) = x^{2020} + x + 2$, which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

   Proposed by: Milan Haiman

   Answer: $\boxed{2020 \cdot 2^{2019}}$

   Solution: Let $P(x)$ have degree $n = 2020$ with roots $r_1, \dots, r_n$. Let $R(x) = \prod_i (x - r_i^2)$. Then

$$\prod_i r_i^n P\left(\frac{x}{r_i}\right) = \prod_i \prod_j (x - r_i r_j) = Q(x)^2 R(x).$$

Using $R(x^2) = (-1)^n P(x)P(-x)$ and Vieta, we obtain

$$P(x)P(-x)Q(x^2)^2 = P(0)^n \prod_i P\left(\frac{x^2}{r_i}\right).$$

Plugging in $x = \alpha$, we use the facts that $P(\alpha) = 4$, $P(-\alpha) = 4 - 2\alpha$, and also

$$P\left(\frac{\alpha^2}{r_i}\right) = \frac{\alpha^{4040}}{r_i^{2020}} + \frac{\alpha^2}{r_i} + 2 = -\frac{(\alpha-2)^2}{r_i + 2} + \frac{\alpha^2}{r_i} + 2 = \frac{2(-\alpha - r_i)^2}{r_i(r_i+2)}.$$

This will give us

$$P(\alpha)P(-\alpha)Q(\alpha^2)^2 = 2^n \prod_i \frac{2(-\alpha - r_i)^2}{r_i(r_i+2)} = 2^n \cdot \frac{2^n P(-\alpha)^2}{P(0)P(-2)}.$$

Therefore,

$$\begin{aligned} Q(\alpha^2)^2 &= \frac{4^n P(-\alpha)}{P(0)P(-2)P(\alpha)} \\ &= \frac{4^n(4 - 2\alpha)}{2 \cdot 2^{2020} \cdot 4} \\ &= \frac{2^{4041}(2-\alpha)}{2^{2023}} \\ &= 2^{2018}(2 - \alpha). \end{aligned}$$

We can check that $P(x) - 4$ has no double roots (e.g. by checking that it shares no roots with its derivative), which means that all possible α are distinct. Therefore, adding over all α gives $2020 \cdot 2^{2019}$, because the sum of the roots of $P(x) - 4$ is 0.

10. We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in $\mathbb{F}_{101}$ (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_{101}$ for each nonnegative integer k. For example, $(x+3)(100x+5) = 100x^2 + 2x + 15$ in $\mathbb{F}_{101}[x]$ because the corresponding coefficients are equal modulo 101.

    We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

$$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x)$$

in $\mathbb{F}_{101}[x]$. Find the number of lucky polynomials.

Proposed by: Michael Ren

Answer: $\boxed{101^{954}}$

Solution 1: Let $p = 101$, $m = 1001$, and work in the ring $R := \mathbb{F}_p[x]/(x^m - 1)$. We want to find the number of elements a of this ring that are of the form $x^p - x$. We first solve this question for a field extension $\mathbb{F}_{p^d}$ of $\mathbb{F}_p$. Note that $(x+n)^p - (x+n) = x^p - x$ for any $n \in \mathbb{F}_p$, and the polynomial $t^p - t = b$ has at most p solutions in $\mathbb{F}_{p^d}$ for any $b \in \mathbb{F}_{p^d}$. Combining these implies that $t^p - t = b$ always has either p or 0 solutions in $\mathbb{F}_{p^d}$, so there are p^{d-1} elements of $\mathbb{F}_{p^d}$ expressible in the form $x^p - x$. Now, note that we may factor R into a product of field extensions of $\mathbb{F}_p$, each corresponding to an irreducible factor of $x^m - 1$ in $\mathbb{F}_p$, as the polynomial $x^m - 1$ has no double roots in $\mathbb{F}_p$ as $p \nmid m$. By the Chinese Remainder Theorem, we may multiply the number of lucky polynomials for each of the field extensions to find the final answer. A field extension of degree d will yield p^{d-1} lucky polynomials. Thus, the final answer is p^{m-q}, where q is the number of fields in the factorization of R into fields.

To do determine q, we first factor

$$x^m - 1 = \prod_{k|m} \Phi_k(x)$$

in $\mathbb{Z}[x]$ where $\Phi_k(x)$ are the cyclotomic polynomials. Then we compute the number of irreducible divisors of the cyclotomic polynomial $\Phi_k(x)$ in $\mathbb{F}_p[x]$. We claim that this is equal to $\frac{\varphi(k)}{\operatorname{ord}_k(p)}$. Indeed, note that given a root ω of Φ_k in the algebraic closure of $\mathbb{F}_p$, the roots of its minimal polynomial are $\omega, \omega^p, \omega^{p^2}, \ldots$, and this will cycle after the numerator repeats modulo k, from which it follows that the degree of the minimal polynomial of ω is $\operatorname{ord}_k(p)$. Thus, $\Phi_k(x)$ factors into $\frac{\varphi(k)}{\operatorname{ord}_k(p)}$ irreducible polynomials.

It remains to compute orders. We have that

$$\begin{aligned}
\operatorname{ord}_7(101) &= \operatorname{ord}_7(3) = 6,\\
\operatorname{ord}_{11}(101) &= \operatorname{ord}_{11}(2) = 10,\\
\operatorname{ord}_{13}(101) &= \operatorname{ord}_{13}(10) = 6.
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{ord}_{77}(101) &= 30,\\
\operatorname{ord}_{91}(101) &= 6,\\
\operatorname{ord}_{143}(101) &= 30,\\
\operatorname{ord}_{1001}(101) &= 30,\\
\operatorname{ord}_{1}(101) &= 1.
\end{aligned}$$

The number of factors of $x^{1001} - 1$ in $\mathbb{F}_{101}[x]$ is thus

$$\frac{1}{1} + \frac{6}{6} + \frac{10}{10} + \frac{12}{6} + \frac{60}{30} + \frac{72}{6} + \frac{120}{30} + \frac{720}{30} = 1 + 1 + 1 + 2 + 2 + 12 + 4 + 24 = 47,$$

so the total number of lucky polynomials is $101^{1001-47} = 101^{954}$.

Solution 2: As in the previous solution, we work in the ring $R = \mathbb{F}_p/(x^m - 1)$, which we can treat as the set of polynomials in $\mathbb{F}_p[x]$ with degree less than m. The problem is asking us for the number of elements of the map $\alpha\colon h \mapsto h^p - h$ in R. Note that this map is linear because $(a+b)^p = a^p + b^p$ in any field where $p = 0$ (which R is an example of). Hence it suffices to determine the size of the kernel of α.

We can directly compute that if

$$h(x) = a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1 x + a_0,$$

then

$$h(x)^p = a_{m-1}x^{p(m-1)} + a_{m-2}x^{p(m-2)} + \cdots + a_1 x^p + a_0,$$

where exponents are taken modulo m. Therefore h is in the kernel if and only if $a_k = a_{pk}$ for all k where indices are taken modulo m. Letting q denote the number of orbits of the map $x \mapsto px$ in $\mathbb{Z}/m\mathbb{Z}$, the size of the kernel is then p^q so the size of the image is p^{m-q}. It remains to compute q, which will end up being the same computation as in the previous solution.