HMMT February 2018

February 10, 2018

Algebra and Number Theory

1. For some real number c, the graphs of the equation $y = |x-20| + |x+18|$ and the line $y = x + c$ intersect at exactly one point. What is c?

2. Compute the positive real number x satisfying
$$x^{(2x^6)} = 3.$$

3. There are two prime numbers p so that $5p$ can be expressed in the form $\left\lfloor \frac{n^2}{5} \right\rfloor$ for some positive integer n. What is the sum of these two prime numbers?

4. Distinct prime numbers p, q, r satisfy the equation
$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$
for some positive integer A. What is A?

5. Let $\omega_1, \omega_2, \dots, \omega_{100}$ be the roots of $\frac{x^{101}-1}{x-1}$ (in some order). Consider the set
$$S = \{\omega_1^1, \omega_2^2, \omega_3^3, \dots, \omega_{100}^{100}\}.$$
Let M be the maximum possible number of unique values in S, and let N be the minimum possible number of unique values in S. Find $M - N$.

6. Let α, β, and γ be three real numbers. Suppose that
$$\cos\alpha + \cos\beta + \cos\gamma = 1$$
$$\sin\alpha + \sin\beta + \sin\gamma = 1.$$
Find the smallest possible value of $\cos\alpha$.

7. Rachel has the number 1000 in her hands. When she puts the number x in her left pocket, the number changes to $x + 1$. When she puts the number x in her right pocket, the number changes to x^{-1}. Each minute, she flips a fair coin. If it lands heads, she puts the number into her left pocket, and if it lands tails, she puts it into her right pocket. She then takes the new number out of her pocket. If the expected value of the number in Rachel's hands after eight minutes is E, then compute $\lfloor \frac{E}{10} \rfloor$.

8. For how many pairs of sequences of nonnegative integers $(b_1, b_2, \dots, b_{2018})$ and $(c_1, c_2, \dots, c_{2018})$ does there exist a sequence of nonnegative integers $(a_0, \dots, a_{2018})$ with the following properties:
   - For $0 \le i \le 2018$, $a_i < 2^{2018}$;
   - For $1 \le i \le 2018$, $b_i = a_{i-1} + a_i$ and $c_i = a_{i-1} | a_i$;

   where $|$ denotes the bitwise or operation?

   (The *bitwise or* of two nonnegative integers $x = \cdots x_3x_2x_1x_0$ and $y = \cdots y_3y_2y_1y_0$ expressed in binary is defined as $x|y = \cdots z_3z_2z_1z_0$, where $z_i = 1$ if at least one of x_i and y_i is 1, and 0 otherwise.)

9. Assume the quartic $x^4 - ax^3 + bx^2 - ax + d = 0$ has four real roots $\frac{1}{2} \le x_1, x_2, x_3, x_4 \le 2$. Find the maximum possible value of $\frac{(x_1+x_2)(x_1+x_3)x_4}{(x_4+x_2)(x_4+x_3)x_1}$ (over all valid choices of a, b, d).

10. Let S be a randomly chosen 6-element subset of the set $\{0, 1, 2, \dots, n\}$. Consider the polynomial $P(x) = \sum_{i \in S} x^i$. Let X_n be the probability that $P(x)$ is divisible by some nonconstant polynomial $Q(x)$ of degree at most 3 with integer coefficients satisfying $Q(0) \neq 0$. Find the limit of X_n as n goes to infinity.