15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Guts

1. [**2**] Square $ABCD$ has side length 2, and X is a point outside the square such that $AX = XB = \sqrt{2}$. What is the length of the longest diagonal of pentagon $AXBCD$?

   Answer: $\boxed{\sqrt{10}}$ Since $AX = XB = \sqrt{2}$ and $AB = 2$, we have $\angle AXB = 90°$. Hence, the distance from X to AB is 1 and the distance from X to CD is 3. By inspection, the largest diagonals are thus $BX = CX = \sqrt{3^2 + 1^2}$.

2. [**2**] Let $a_0, a_1, a_2, \ldots$ denote the sequence of real numbers such that $a_0 = 2$ and $a_{n+1} = \frac{a_n}{1+a_n}$ for $n \geq 0$. Compute a_{2012}.

   Answer: $\boxed{\frac{2}{4025}}$ Calculating out the first few terms, note that they follow the pattern $a_n = \frac{2}{2n+1}$. Plugging this back into the recursion shows that it indeed works.

3. [**2**] Suppose x and y are real numbers such that $-1 < x < y < 1$. Let G be the sum of the geometric series whose first term is x and whose ratio is y, and let G' be the sum of the geometric series whose first term is y and ratio is x. If $G = G'$, find $x + y$.

   Answer: $\boxed{1}$ We note that $G = x/(1-y)$ and $G' = y/(1-x)$. Setting them equal gives $x/(1-y) = y/(1-x) \Rightarrow x^2 - x = y^2 - x \Rightarrow (x+y-1)(x-y) = 0$, so we get that $x + y - 1 = 0 \Rightarrow x + y = 1$.

4. [**2**] Luna has an infinite supply of red, blue, orange, and green socks. She wants to arrange 2012 socks in a line such that no red sock is adjacent to a blue sock and no orange sock is adjacent to a green sock. How many ways can she do this?

   Answer: $\boxed{4 \cdot 3^{2011}}$ Luna has 4 choices for the first sock. After that, she has 3 choices for each of 2011 remaining socks for a total of $4 \cdot 3^{2011}$.

5. [**3**] Mr. Canada chooses a positive real a uniformly at random from $(0, 1]$, chooses a positive real b uniformly at random from $(0, 1]$, and then sets $c = a/(a + b)$. What is the probability that c lies between $1/4$ and $3/4$?

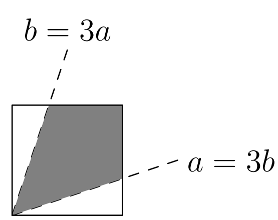
   Answer: $\boxed{2/3}$ From $c \geq 1/4$ we get

   $$\frac{a}{a+b} \geq \frac{1}{4} \Longleftrightarrow b \leq 3a$$

   and similarly $c \leq 3/4$ gives

   $$\frac{a}{a+b} \leq \frac{3}{4} \Longleftrightarrow a \leq 3b.$$

   Choosing a and b randomly from $[0, 1]$ is equivalent to choosing a single point uniformly and randomly from the unit square, with a on the horizontal axis and b on the vertical axis:



   To find the probability that $b \leq 3a$ and $a \leq 3b$, we need to find the area of the shaded region of the square. The area of each of the triangles on the side is $(1/2)(1)(1/3) = 1/6$, and so the area of the shaded region is $1 - 2(1/6) = 2/3$.

6. [**3**] Let rectangle $ABCD$ have lengths $AB = 20$ and $BC = 12$. Extend ray BC to Z such that $CZ = 18$. Let E be the point in the interior of $ABCD$ such that the perpendicular distance from E to $\overline{AB}$ is 6 and the perpendicular distance from E to $\overline{AD}$ is 6. Let line EZ intersect AB at X and CD at Y. Find the area of quadrilateral $AXYD$.

   Answer: $\boxed{72}$ Draw the line parallel to $\overline{AD}$ through E, intersecting $\overline{AB}$ at F and $\overline{CD}$ at G. It is clear that XFE and YGE are congruent, so the area of $AXYD$ is equal to that of $AFGD$. But $AFGD$ is simply a 12 by 6 rectangle, so the answer must be 72. (Note: It is also possible to directly compute the values of AX and DY, then use the formula for the area of a trapezoid.)

7. [**3**] M is an 8×8 matrix. For $1 \leq i \leq 8$, all entries in row i are at least i, and all entries on column i are at least i. What is the minimum possible sum of the entries of M?

   Answer: $\boxed{372}$ Let s_n be the minimum possible sum for an n by n matrix. Then, we note that increasing it by adding row $n+1$ and column $n+1$ gives $2n+1$ additional entries, each of which has minimal size at least $n+1$. Consequently, we obtain $s_{n+1} = s_n + (2n+1)(n+1) = s_n + 2n^2 + 3n + 1$. Since $s_0 = 0$, we get that $s_8 = 2(7^2 + \ldots + 0^2) + 3(7 + \ldots + 0) + 8 = 372$.

8. [**3**] Amy and Ben need to eat 1000 total carrots and 1000 total muffins. The muffins can not be eaten until all the carrots are eaten. Furthermore, Amy can not eat a muffin within 5 minutes of eating a carrot and neither can Ben. If Amy eats 40 carrots per minute and 70 muffins per minute and Ben eats 60 carrots per minute and 30 muffins per minute, what is the minimum number of minutes it will take them to finish the food?

   Answer: $\boxed{23.5 \text{ or } 47/2}$ Amy and Ben will continuously eat carrots, then stop (not necessarily at the same time), and continuously eat muffins until no food is left. Suppose that Amy and Ben finish eating the carrots in T_1 minutes and the muffins T_2 minutes later; we wish to find the minimum value of $T_1 + T_2$. Furthermore, suppose Amy finishes eating the carrots at time a_1, and Ben does so at time b_1, so that $T_1 = \max(a_1, b_1)$.

   First, suppose that $a_1 \leq b_1$, and let $b_1 - a_1 = c$. We have $40(T_1 - c) + 60T_1 = 1000$, so T_1 is minimized when $c = 0$. Also, $30(T_2 - 5) + 70(T_2 - \max(5 - c, 0)) = 1000$. Wee see that $T_1 + T_2$ is minimized when $c = 5$, and $T_1 + T_2 = 23.5$. In a similar way, we see that when $b_1 \leq a_1$, $T_1 + T_2 > 23.5$, so our answer is 23.5.

9. [**5**] Given $\triangle ABC$ with $AB < AC$, the altitude AD, angle bisector AE, and median AF are drawn from A, with D, E, F all lying on $\overline{BC}$. If $\measuredangle BAD = 2\measuredangle DAE = 2\measuredangle EAF = \measuredangle FAC$, what are all possible values of $\measuredangle ACB$?

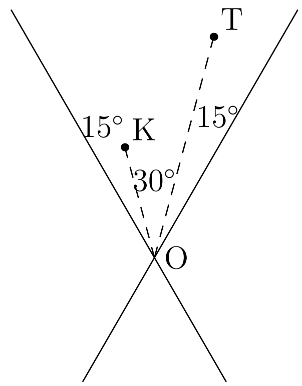
   Answer: $\boxed{30^\circ \text{ or } \pi/6 \text{ radians}}$ Let H and O be the orthocenter and circumcenter of ABC, respctively: it is well-known (and not difficult to check) that $\measuredangle BAH = \measuredangle CAO$. However, note that $\measuredangle BAH = \measuredangle BAD = \measuredangle CAF$, so $\measuredangle CAF = \measuredangle CAO$, that is, O lies on median AF, and since $AB < AC$, it follows that $F = O$. Therefore, $\measuredangle BAC = 90^\circ$.

   Now, we compute $\measuredangle ACB = \measuredangle BAD = \frac{2}{6}\measuredangle BAC = 30^\circ$.

10. [**5**] Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Compute $P(3)$.

    Answer: $\boxed{5}$ Plugging in $x = -1, 1, 2$ results in the trio of equations $1 = P(-1) = P(0) - P(1) + P(2)$, $P(1) = P(0) + P(1) + P(2) \Rightarrow P(1) + P(2) = 0$, and $P(2) = P(0) + 2P(1) + 4P(2)$. Solving these as a system of equations in $P(0), P(1), P(2)$ gives $P(0) = -1, P(1) = -1, P(2) = 1$. Consequently, $P(x) = x^2 - x - 1 \Rightarrow P(3) = 5$.

11. [**5**] Knot is on an epic quest to save the land of Hyruler from the evil Gammadorf. To do this, he must collect the two pieces of the Lineforce, then go to the Temple of Lime. As shown on the figure, Knot starts on point K, and must travel to point T, where $OK = 2$ and $OT = 4$. However, he must first reach both solid lines in the figure below to collect the pieces of the Lineforce. What is the minimal distance Knot must travel to do so?



Answer: $\boxed{2\sqrt{5}}$ Let l_1 and l_2 be the lines as labeled in the above diagram. First, suppose Knot visits l_1 first, at point P_1, then l_2, at point P_2. Let K' be the reflection of K over l_1, and let T' be the reflection of T over l_2. The length of Knot's path is at least

$$KP_1 + P_1P_2 + P_2T = K'P_1 + P_1P_2 + P_2T' \geq K'T'$$

by the Triangle Inequality (This bound can be achieved by taking P_1, P_2 to be the intersections of $K'T'$ with l_1, l_2, respectively.) Also, note that $\measuredangle K'OT' = 90^\circ$, so that $K'T' = 2\sqrt{5}$.

Now, suppose Knot instead visits l_2 first, at point Q_2, then l_1, at point Q_1. Letting K'' be the reflection of K over l_2 and T'' be the reflection of T over l_1, by similar logic to before the length of his path is at least the length of $K''T''$. However, by inspection $K''T'' > K'T'$, so our answer is $2\sqrt{5}$.

12. [**5**] Knot is ready to face Gammadorf in a card game. In this game, there is a deck with twenty cards numbered from 1 to 20. Each player starts with a five card hand drawn from this deck. In each round, Gammadorf plays a card in his hand, then Knot plays a card in his hand. Whoever played a card with greater value gets a point. At the end of five rounds, the player with the most points wins. If Gammadorf starts with a hand of $1, 5, 10, 15, 20$, how many five-card hands of the fifteen remaining cards can Knot draw which always let Knot win (assuming he plays optimally)?

    Answer: $\boxed{2982}$ Knot can only lose if all of his cards are lower than 10; if not he can win by playing the lowest card that beats Gammadorf's card, or if this is not possible, his lowest card, each turn. There are $\binom{7}{5} = 21$ losing hands, so he has $\binom{15}{5} - \binom{7}{5}$ possible winning hands.

13. [**7**] Niffy's favorite number is a positive integer, and Stebbysaurus is trying to guess what it is. Niffy tells her that when expressed in decimal without any leading zeros, her favorite number satisfies the following:

    - Adding 1 to the number results in an integer divisible by 210.
    - The sum of the digits of the number is twice its number of digits.
    - The number has no more than 12 digits.
    - The number alternates in even and odd digits.

    Given this information, what are all possible values of Niffy's favorite number?

    Answer: $\boxed{1010309}$ Note that Niffy's favorite number must end in 9, since adding 1 makes it divisible by 10. Also, the sum of the digits of Niffy's favorite number must be even (because it is equal to twice the number of digits) and congruent to 2 modulo 3 (because adding 1 gives a multiple of 3). Furthermore, the sum of digits can be at most 24, because there at most 12 digits in Niffy's favorite number, and must be at least 9, because the last digit is 9. This gives the possible sums of digits 14 and 20. However, if the sum of the digits of the integer is 20, there are 10 digits, exactly 5 of which are odd, giving an odd sum of digits, which is impossible. Thus, Niffy's favorite number is a 7 digit number with sum of digits 14.

The integers which we seek must be of the form $\overline{ABCDEF9}$, where A, C, E are odd, B, D, F are even, and $A+B+C+D+E+F=5$. Now, note that $\{A, C, E\} = \{1, 1, 1\}$ or $\{1, 1, 3\}$, and these correspond to $\{B, D, F\} = \{0, 0, 2\}$ and $\{0, 0, 0\}$, respectively. It suffices to determine which of these six integers are congruent to $-1 \pmod 7$, and we see that Niffy's favorite number must be 1010309.

14. [**7**] Let triangle ABC have $AB = 5$, $BC = 6$, and $AC = 7$, with circumcenter O. Extend ray AB to point D such that $BD = 5$, and extend ray BC to point E such that $OD = OE$. Find CE.

    Answer: $\boxed{\sqrt{59}-3}$ Because $OD = OE$, D and E have equal power with respect to the circle, so $(EC)(EB) = (DB)(DA) = 50$. Letting $EC = x$, we have $x(x+6) = 50$, and taking the positive root gives $x = \sqrt{59} - 3$.

15. [**7**] Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two distinct real polynomials such that the x-coordinate of the vertex of f is a root of g, the x-coordinate of the vertex of g is a root of f and both f and g have the same minimum value. If the graphs of the two polynomials intersect at the point $(2012, -2012)$, what is the value of $a + c$?

    Answer: $\boxed{-8048}$ It is clear, by symmetry, that 2012 is the equidistant from the vertices of the two quadratics. Then it is clear that reflecting f about the line $x = 2012$ yields g and vice versa. Thus the average of each pair of roots is 2012. Thus the sum of the four roots of f and g is 8048, so $a + c = -8048$.

16. [**7**] Let A, B, C, and D be points randomly selected independently and uniformly within the unit square. What is the probability that the six lines $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$ all have positive slope?

    Answer: $\boxed{\frac{1}{24}}$ Consider the sets of x-coordinates and y-coordinates of the points. In order to make 6 lines of positive slope, we must have smallest x-coordinate must be paired with the smallest y-coordinate, the second smallest together, and so forth. If we fix the order of the x-coordinates, the probability that the corresponding y-coordinates are in the same order is $1/24$.

17. [**11**] Mark and William are playing a game. Two walls are placed 1 meter apart, with Mark and William each starting an orb at one of the walls. Simultaneously, they release their orbs directly toward the other. Both orbs are enchanted such that, upon colliding with each other, they instantly reverse direction and go at double their previous speed. Furthermore, Mark has enchanted his orb so that when it collides with a wall it instantly reverses direction and goes at double its previous speed (William's reverses direction at the same speed). Initially, Mark's orb is moving at $\frac{1}{1000}$ meters/s, and William's orb is moving at 1 meter/s. Mark wins when his orb passes the halfway point between the two walls. How fast, in meters/s, is his orb going when this first happens?

    Answer: $\boxed{2^{17}/125}$ If the two orbs leave their respective walls at the same time, then they will return to their walls at the same time (because colliding affects both their speeds). After returning to the wall n times, Mark's orb will travel at $\frac{4^n}{1000}$ meter/s and William's will travel at 2^n meter/s. Mark wins when his orb is traveling faster at $n = 10$. $\frac{4^{10}}{1000} = \frac{2^{17}}{125}$

18. [**11**] Let x and y be positive real numbers such that $x^2 + y^2 = 1$ and $(3x - 4x^3)(3y - 4y^3) = -\frac{1}{2}$. Compute $x + y$.

    Answer: $\boxed{\frac{\sqrt{6}}{2}}$ Solution 1: Let $x = \cos(\theta)$ and $y = \sin(\theta)$. Then, by the triple angle formulae, we have that $3x - 4x^3 = -\cos(3\theta)$ and $3y - 4y^3 = \sin(3\theta)$, so $-\sin(3\theta)\cos(3\theta) = -\frac{1}{2}$. We can write this as $2\sin(3\theta)\cos(3\theta) = \sin(6\theta) = 1$, so $\theta = \frac{1}{6}\sin^{-1}(1) = \frac{\pi}{12}$. Thus, $x + y = \cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{6}}{2}$.

    Solution 2: Expanding gives $9xy + 16x^3y^3 - 12xy^3 - 12x^3y = 9(xy) + 16(xy)^3 - 12(xy)(x^2+y^2) = -\frac{1}{2}$, and since $x^2 + y^2 = 1$, this is $-3(xy) + 16(xy)^3 = -\frac{1}{2}$, giving $xy = -\frac{1}{2}, \frac{1}{4}$. However, since x and y are positive reals, we must have $xy = \frac{1}{4}$. Then, $x + y = \sqrt{x^2 + y^2 + 2xy} = \sqrt{1 + 2 \cdot \frac{1}{4}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$.

19. [**11**] Given that P is a real polynomial of degree at most 2012 such that $P(n) = 2^n$ for $n = 1, 2, \ldots, 2012$, what choice(s) of $P(0)$ produce the minimal possible value of $P(0)^2 + P(2013)^2$?

Answer: $\boxed{1 - 2^{2012}}$ Define $\Delta^1(n) = P(n+1) - P(n)$ and $\Delta^i(n) = \Delta^{i-1}(n+1) - \Delta^{i-1}(n)$ for $i > 1$. Since $P(n)$ has degree at most 2012, we know that $\Delta^{2012}(n)$ is constant. Computing, we obtain $\Delta^1(0) = 2 - P(0)$ and $\Delta^i(0) = 2^{i-1}$ for $1 < i \leq 2012$. We see that continuing on gives $\Delta^{2012}(0) = \Delta^{2012}(1) = P(0)$ and $\Delta^i(2012 - i) = 2^{2013-i}$ for $1 \leq i \leq 2011$. Then, $P(2013) = P(2012) + \Delta^1(2012) = \ldots = P(2012) + \Delta^1(2011) + \ldots + \Delta^{2012}(0) = P(0) + 2^{2013} - 2$. Now, we want to minimize the value of $P(0)^2 + P(2013)^2 = 2P(0)^2 + 2P(0)(2^{2013} - 2) + (2^{2013} - 2)^2$, but this occurs simply when $P(0) = -\frac{1}{2}(2^{2013} - 2) = 1 - 2^{2012}$.

20. [**11**] Let n be the maximum number of bishops that can be placed on the squares of a 6×6 chessboard such that no two bishops are attacking each other. Let k be the number of ways to put n bishops on an 6×6 chessboard such that no two bishops are attacking each other. Find $n + k$. (Two bishops are considered to be attacking each other if they lie on the same diagonal. Equivalently, if we label the squares with coordinates (x, y), with $1 \leq x, y \leq 6$, then the bishops on (a, b) and (c, d) are attacking each other if and only if $|a - c| = |b - d|$.)

Answer: $\boxed{74}$ Color the square with coordinates (i, j) black if $i + j$ is odd and white otherwise, for all $1 \leq i, j \leq 6$. Looking at the black squares only, we note that there are six distinct diagonals which run upward and to the right, but that two of them consist only of a corner square; we cannot simultaneously place bishops on both of these corner squares. Consequently, we can place at most five bishops on black squares. (This can be achieved by placing bishops on $(1, 2), (1, 4), (6, 1), (6, 3), (6, 5)$.) If there are five bishops on black squares, there must be exactly one bishop on one of the two black corner squares, $(6, 1)$ and $(1, 6)$: suppose without loss of generality that we place a bishop on $(1, 6)$. Then, exactly one of $(3, 6)$ and $(1, 4)$ must also contain a bishop, and there are 2 ways to place two bishops on the four remaining black squares that are not yet under attack. Thus, we have a total of $2 \cdot 2 \cdot 2$ possible placements on black squares.

Similarly, there are at most 5 bishops which can be placed on white squares and 2^3 ways to place them, so that $n = 10$ and $k = 2^6$. Finally, $n + k = 10 + 2^6 = 74$.

21. [**13**] Let N be a three-digit integer such that the difference between any two positive integer factors of N is divisible by 3. Let $d(N)$ denote the number of positive integers which divide N. Find the maximum possible value of $N \cdot d(N)$.

Answer: $\boxed{5586}$ We first note that all the prime factors of n must be 1 modulo 3 (and thus 1 modulo 6). The smallest primes with this property are $7, 13, 19, \ldots$. Since $7^4 = 2401 > 1000$, the number can have at most 3 prime factors (including repeats). Since $7 \cdot 13 \cdot 19 = 1729 > 1000$, the most factors N can have is 6. Consider the number $7^2 \cdot 19 = 931$, which has 6 factors. For this choice of N, $N \cdot d(N) = 5586$. For another N to do better, it must have at least 6 factors, for otherwise, $N \cdot d(N) < 1000 \cdot 5 = 5000$. It is easy to verify that $7^2 \cdot 19$ is the greatest number with 6 prime factors satisfying our conditions, so the answer must be 5586.

22. [**13**] For each positive integer n, there is a circle around the origin with radius n. Rainbow Dash starts off somewhere on the plane, but not on a circle. She takes off in some direction in a straight path. She moves $\frac{\sqrt{5}}{5}$ units before crossing a circle, then $\sqrt{5}$ units, then $\frac{3\sqrt{5}}{5}$ units. What distance will she travel before she crosses another circle?

Answer: $\boxed{\frac{2\sqrt{170} - 9\sqrt{5}}{5}}$ Note that the distance from Rainbow Dash's starting point to the first place in which she hits a circle is irrelevant, except in checking that this distance is small enough that she does not hit another circle beforehand. It will be clear at the end that our configuration does not allow this (by the Triangle Inequality). Let O be the origin, and let Rainbow Dash's first three meeting points be A, B, C so that $AB = \sqrt{5}$ and $BC = \frac{3\sqrt{5}}{5}$.

Consider the lengths of OA, OB, OC. First, note that if $OA = OC = n$ (i.e. A and C lie on the same circle), then we need $OB = n - 1$, but since she only crosses the circle containing B once, it follows that the circle passing through B is tangent to AC, which is impossible since $AB \neq AC$. If $OA = OB = n$, note that $OC = n+1$. Dropping a perpendicular from O to AB, we see that by the Pythagorean Theorem,

$$n^2 - \frac{5}{4} = (n+1)^2 - \frac{121}{20},$$

from which we get that n is not an integer. Similarly, when $OB = OC = n$, we have $OA = n+1$, and n is not an integer.

Therefore, either $OA = n+2, OB = n+1, OC = n$ or $OA = n, OB = n+1, OC = n+2$. In the first case, by Stewart's Theorem,

$$\frac{24\sqrt{5}}{5} + (n+1)^2 \cdot \frac{8\sqrt{5}}{5} = n^2 \cdot \sqrt{5} + (n+2)^2 \cdot \frac{3\sqrt{5}}{5}.$$

This gives a negative value of n, so the configuration is impossible. In the final case, we have, again by Stewart's Theorem,

$$\frac{24\sqrt{5}}{5} + (n+1)^2 \cdot \frac{8\sqrt{5}}{5} = (n+2)^2 \cdot \sqrt{5} + n^2 \cdot \frac{3\sqrt{5}}{5}.$$

Solving gives $n = 3$, so $OA = 3, OB = 4, OC = 5$.

Next, we compute, by the Law of Cosines, $\cos\angle OAB = -\frac{1}{3\sqrt{5}}$, so that $\sin\angle OAB = \frac{2\sqrt{11}}{3\sqrt{5}}$. Let the projection from O to line AC be P; we get that $OP = \frac{2\sqrt{11}}{\sqrt{5}}$. Rainbow Dash will next hit the circle of radius 6 at D. Our answer is now $CD = PD - PC = \frac{2\sqrt{170}}{5} - \frac{9\sqrt{5}}{5}$ by the Pythagorean Theorem.

23. [**13**] Points X and Y are inside a unit square. The score of a vertex of the square is the minimum distance from that vertex to X or Y. What is the minimum possible sum of the scores of the vertices of the square?

Answer: $\boxed{\frac{\sqrt{6}+\sqrt{2}}{2}}$ Let the square be $ABCD$. First, suppose that all four vertices are closer to X than Y. Then, by the triangle inequality, the sum of the scores is $AX + BX + CX + DX \geq AB + CD = 2$. Similarly, suppose exactly two vertices are closer to X than Y. Here, we have two distinct cases: the vertices closer to X are either adjacent or opposite. Again, by the Triangle Inequality, it follows that the sum of the scores of the vertices is at least 2.

On the other hand, suppose that A is closer to X and B, C, D are closer to Y. We wish to compute the minimum value of $AX + BY + CY + DY$, but note that we can make $X = A$ to simply minimize $BY + CY + DY$. We now want Y to be the Fermat point of triangle BCD, so that $\measuredangle BYC = \measuredangle CYD = \measuredangle DYB = 120°$. Note that by symmetry, we must have $\measuredangle BCY = \measuredangle DCY = 45°$, so $\measuredangle CBY = \measuredangle CDY = 15°$.

And now we use the law of sines: $BY = DY = \frac{\sin 45°}{\sin 120°}$ and $CY = \frac{\sin 15°}{\sin 120°}$. Now, we have $BY + CY + DY = \frac{\sqrt{2}+\sqrt{6}}{2}$, which is less than 2, so this is our answer.

24. [**13**] Franklin has four bags, numbered 1 through 4. Initially, the first bag contains fifteen balls, numbered 1 through 15, and the other bags are empty. Franklin randomly pulls a pair of balls out of the first bag, throws away the ball with the lower number, and moves the ball with the higher number into the second bag. He does this until there is only one ball left in the first bag. He then repeats this process in the second and third bag until there is exactly one ball in each bag. What is the probability that ball 14 is in one of the bags at the end?

Answer: $\boxed{\frac{2}{3}}$ Pretend there is a 16th ball numbered 16. This process is equivalent to randomly drawing a tournament bracket for the 16 balls, and playing a tournament where the higher ranked ball always wins. The probability that a ball is left in a bag at the end is the probability that it loses to ball 16. Of the three balls 14, 15, 16, there is a $\frac{1}{3}$ chance 14 plays 15 first, a $\frac{1}{3}$ chance 14 plays 16 first, and a $\frac{1}{3}$ chance 15 plays 16 first. In the first case, 14 does not lose to 16, and instead loses to 15; otherwise 14 loses to 16, and ends up in a bag. So the answer is $\frac{2}{3}$.

25. [**17**] FemtoPravis is walking on an 8×8 chessboard that wraps around at its edges (so squares on the left edge of the chessboard are adjacent to squares on the right edge, and similarly for the top and bottom edges). Each femtosecond, FemtoPravis moves in one of the four diagonal directions uniformly at random. After 2012 femtoseconds, what is the probability that FemtoPravis is at his original location?

Answer: $\boxed{\left(\frac{1+2^{1005}}{2^{1007}}\right)^2}$ We note the probability that he ends up in the same row is equal to the probability that he ends up in the same column by symmetry. Clearly these are independent, so we calculate the probability that he ends up in the same row.

Now we number the rows $0-7$ where 0 and 7 are adjacent. Suppose he starts at row 0. After two more turns, the probability he is in row 2 (or row 6) is $\frac{1}{4}$, and the probability he is in row 0 again is $\frac{1}{2}$.

Let a_n, b_n, c_n and d_n denote the probability he is in row 0,2,4,6 respectively after $2n$ moves.

We have $a_0 = 1$, and for $n \geq 0$ we have the following equations:

$$a_{n+1} = \frac{1}{2}a_n + \frac{1}{4}b_n + \frac{1}{4}d_n$$

$$b_{n+1} = \frac{1}{2}b_n + \frac{1}{4}a_n + \frac{1}{4}c_n$$

$$c_{n+1} = \frac{1}{2}c_n + \frac{1}{4}b_n + \frac{1}{4}d_n$$

$$d_{n+1} = \frac{1}{2}d_n + \frac{1}{4}a_n + \frac{1}{4}c_n$$

From which we get the following equations:

$$a_n + c_n = \frac{1}{2}$$

$$x_n = a_n - c_n = \frac{1}{2}(a_{n-1} - c_{n-1}) = \frac{x_{n-1}}{2}$$

So

$$a_{1006} + c_{1006} = \frac{1}{2}$$

$$x_0 = 1, x_{1006} = \frac{1}{2^{1006}}$$

$$a_{1006} = \frac{1 + 2^{1005}}{2^{1007}}$$

And thus the answer is $\left(\frac{1+2^{1005}}{2^{1007}}\right)^2$.

26. [**17**] Suppose ABC is a triangle with circumcenter O and orthocenter H such that A, B, C, O, and H are all on distinct points with integer coordinates. What is the second smallest possible value of the circumradius of ABC?

Answer: $\boxed{\sqrt{10}}$ Assume without loss of generality that the circumcenter is at the origin. By well known properties of the Euler line, the centroid G is such that O, G, and H are collinear, with G in between O and H, such that $GH = 2GO$. Thus, since $G = \frac{1}{3}(A + B + C)$, and we are assuming O is the origin, we have $H = A + B + C$. This means that as long as A, B, and C are integer points, H will be as well.

However, since H needs to be distinct from A, B, and C, we must have $\triangle ABC$ not be a right triangle, since in right triangles, the orthocenter is the vertex where the right angle is.

Now, if a circle centered at the origin has any integer points, it will have at least four integer points. (If it has a point of the form $(a, 0)$, then it will also have $(-a, 0)$, $(0, a)$, and $(0, -a)$. If it has a point of the form (a, b), with $a, b \neq 0$, it will have each point of the form $(\pm a, \pm b)$.) But in any of these cases where there are only four points, any triangle which can be made from those points is a right triangle.

Thus we need the circumcircle to contain at least eight lattice points. The smallest radius this occurs at is $\sqrt{1^2+2^2}=\sqrt{5}$, which contains the eight points $(\pm 1, \pm 2)$ and $(\pm 2, \pm 1)$. We get at least one valid triangle with this circumradius:

$$A=(-1,2),\ B=(1,2),\ C=(2,1).$$

The next valid circumradius is $\sqrt{1^2+3^2}=\sqrt{10}$ which has the valid triangle

$$A=(-1,3),\ B=(1,3),\ C=(3,1).$$

27. [**17**] Let S be the set $\{1,2,\ldots,2012\}$. A perfectutation is a bijective function h from S to itself such that there exists an $a \in S$ such that $h(a) \neq a$, and that for any pair of integers $a \in S$ and $b \in S$ such that $h(a) \neq a$, $h(b) \neq b$, there exists a positive integer k such that $h^k(a)=b$. Let n be the number of ordered pairs of perfectutations (f,g) such that $f(g(i))=g(f(i))$ for all $i \in S$, but $f \neq g$. Find the remainder when n is divided by 2011.

Answer: $\boxed{2}$ Note that both f and g, when written in cycle notation, must contain exactly one cycle that contains more than 1 element. Assume f has k fixed points, and that the other $2012-k$ elements form a cycle, (of which there are $(2011-k)!$ ways).

Then note that if f fixes a then $f(g(a))=g(f(a))=g(a)$ implies f fixes $g(a)$ So g must send fixed points of f to fixed points of f. It must, therefore send non-fixed points to non-fixed points. This partitions S into two sets, at least one of which must be fixed by g, since g is a perfectutation.

If g fixes all of the the non-fixed points of f, then, since any function commutes with the identity, g fixes some m of the fixed points and cycles the rest in $(k-m-1)!$ ways. So there are $\sum_{m=0}^{k-2}\binom{k}{m}(k-m-1)!$ choices, which is $\sum_{m=0}^{k-2}\frac{k!}{(k-m)m!}$.

If g fixes all of the fixed points of f, then order the non-fixed points of f $a_1, a_2, \ldots, a_{2012-k}$ such that $f(a_i)=a_{i+1}$. If $g(a_i)=a_j$ then $f(g(a_i))=a_{j+1}$ thus $g(a_{i+1})=a_{j+1}$. Therefore the choice of $g(a_1)$ uniquely determines $g(a_i)$ for the rest of the i, and $g(a_m)=a_{m+j-i}$. But g has to be a perfectutation, so g cycles through all the non-fixed points of f, which happens if and only if $j-i$ is relatively prime to $2012-k$. So there are $\phi(2012-k)$ choices.

Therefore for any f there are $\sum_{m=0}^{k-2}\frac{k!}{(k-m)m!}+\phi(2012-k)$ choices of g, but one of them will be $g=f$, which we cannot have by the problem statement. So there are $-1+\sum_{m=0}^{k-2}\frac{k!}{(k-m)m!}+\phi(2012-k)$ options.

Now note that a permutation can not fix all but one element. So $n=\sum_{k=0}^{2010}\binom{2012}{k}(2011-k)!(-1+\sum_{m=0}^{k-2}\frac{k!}{(k-m)m!}+\phi(2012-k))$

Modulo 2011 (which is prime), note that all terms in the summand except the one where $k=1$ vanish. Thus, $n \equiv (2010)!(-1+(-1)) \equiv 2 \pmod{2011}$ by Wilson's Theorem.

28. [**17**] Alice is sitting in a teacup ride with infinitely many layers of spinning disks. The largest disk has radius 5. Each succeeding disk has its center attached to a point on the circumference of the previous disk and has a radius equal to $2/3$ of the previous disk. Each disk spins around its center (relative to the disk it is attached to) at a rate of $\pi/6$ radians per second. Initially, at $t=0$, the centers of the disks are aligned on a single line, going outward. Alice is sitting at the limit point of all these disks. After 12 seconds, what is the length of the trajectory that Alice has traced out?

Answer: $\boxed{18\pi}$ Suppose the center of the largest teacup is at the origin in the complex plane, and let $z=\frac{2}{3}e^{\pi i t/6}$. The center of the second disk is at $5e^{\pi i t/6}$ at time t; that is, $\frac{15}{2}z$. Then the center of the third disk relative to the center of the second disk is at $\frac{15}{2}z^2$, and so on. Summing up a geometric series, we get that Alice's position is

$$\begin{aligned}\frac{15}{2}(z+z^2+z^3+\cdots) &= \frac{15}{2}(1+z^2+z^3+\cdots)-\frac{15}{2}\\ &= \frac{15}{2}\left(\frac{1}{1-z}\right)-\frac{15}{2}.\end{aligned}$$

Now, after 12 seconds, z has made a full circle in the complex plane centered at 0 and of radius $2/3$. Thus $1-z$ is a circle centered at 1 of radius $2/3$.

So $1-z$ traces a circle, and now we need to find the path that $\frac{1}{1-z}$ traces. In the complex plane, taking the reciprocal corresponds to a reflection about the real axis followed by a geometric inversion about the unit circle centered at 0. It is well known that geometric inversion maps circles not passing through the center of the inversion to circles.

Now, the circle traced by $1-z$ contains the points $1-2/3=1/3$, and $1+2/3=5/3$. Therefore the circle $\frac{1}{1-z}$ contains the points 3 and $3/5$, with the center lying halfway between. So the radius of the circle is

$$\frac{1}{2}\left(3-\frac{3}{5}\right)=\frac{6}{5}$$

and so the perimeter is $2\pi(6/5)=12\pi/5$. Scaling by $15/2$ gives an answer of

$$\frac{15}{2}\left(\frac{12\pi}{5}\right)=18\pi.$$

29. [**19**] Consider the cube whose vertices are the eight points (x,y,z) for which each of x, y, and z is either 0 or 1. How many ways are there to color its vertices black or white such that, for any vertex, if all of its neighbors are the same color then it is also that color? Two vertices are neighbors if they are the two endpoints of some edge of the cube.

Answer: $\boxed{118}$ Divide the 8 vertices of the cube into two sets A and B such that each set contains 4 vertices, any two of which are diagonally adjacent across a face of the cube. We do casework based on the number of vertices of each color in set A.

- Case 1: 4 black. Then all the vertices in B must be black, for 1 possible coloring.
- Case 2: 3 black, 1 white. Then there are 4 ways to assign the white vertex. The vertex in B surrounded by the black vertices must also be black. Meanwhile, the three remaining vertices in B may be any configuration except all black, for a total of $4(2^3-1)=28$ possible colorings.
- Case 3: 2 black, 2 white. Then, there are 6 ways to assign the 2 white vertices. The 4 vertices of B cannot all be the same color. Additionally, we cannot have 3 black vertices of B surround a white vertex of A with the other vertex of B white, and vice-versa, so we have a total of $6(2^4-2-4)=60$ possible colorings.
- Case 4: 1 black, 3 white. As in case 2, there are 28 possible colorings.
- Case 5: 5 white. As in case 1, there is 1 possible coloring.

So there is a total of $1+28+60+28+1=118$ possible colorings.

30. [**19**] You have a twig of length 1. You repeatedly do the following: select two points on the twig independently and uniformly at random, make cuts on these two points, and keep only the largest piece. After 2012 repetitions, what is the expected length of the remaining piece?

Answer: $\boxed{(11/18)^{2012}}$ First let $p(x)$ be the probability density of x being the longest length.

Let a_n be the expected length after n cuts. $a_n=\int_0^1 p(x)\cdot(xa_{n-1})dx=a_{n-1}\int_0^1 xp(x)dx=a_1a_{n-1}$.

It follows that $a_n=a_1^n$, so our answer is $(a_1)^{2012}$. We now calculate a_1.

Let $P(z)$ be the probability that the longest section is $\le z$. Clearly $P(z)=0$ for $z\le\frac{1}{3}$.

To simulate making two cuts we pick two random numbers x,y from $[0,1]$, and assume without loss of generality that $x\le y$. Then picking two such points is equivalent to picking a point in the top left triangle half of the unit square. This figure has area $\frac{1}{2}$ so our $P(z)$ will be double the area where $x\le z$, $y\ge 1-z$ and $y-x\le z$. For $\frac{1}{3}\le z\le\frac{1}{2}$ the probability is double the area bounded by $x=z, 1-z=y, y-x=z$. This is $2(\frac{1}{2}(3z-1)^2)=(3z-1)^2$.

For $\frac{1}{2} \leq z \leq 1$ this value is double the hexagon bounded by $x = 0$, $y = 1 - z$, $y = x$, $x = z$, $y = 1$, $y = x + z$. The complement of this set, however, is three triangles of area $\frac{(1-z)^2}{2}$, so $P(z) = 1 - 3(1-z)^2$ for $\frac{1}{2} \leq z \leq 1$.

Now note that $P'(z) = p(z)$. Therefore by integration by parts $a_1 = \int_0^1 zp(z)dz = \int_0^1 zP'(z)dz = [_0^1 zP(z) - \int_0^1 P(z)dz$. This equals

$$1 - \int_{\frac{1}{3}}^{\frac{1}{2}} (3z-1)^2 dz - \int_{\frac{1}{2}}^{1} 1 - 3(1-z)^2 dz$$

$$= 1 - [_{\frac{1}{3}}^{\frac{1}{2}} \frac{(3z-1)^3}{9} - \frac{1}{2} + [_{\frac{1}{2}}^{1} (z-1)^3$$

.

$$= 1 - \frac{1}{72} - \frac{1}{2} + \frac{1}{8} = \frac{1}{2} + \frac{1}{9}$$

$$= \frac{11}{18}$$

So the answer is $(\frac{11}{18})^{2012}$.

31. [**19**] Let S_7 denote all the permutations of $1, 2, \ldots, 7$. For any $\pi \in S_7$, let $f(\pi)$ be the smallest positive integer i such that $\pi(1), \pi(2), \ldots, \pi(i)$ is a permutation of $1, 2, \ldots, i$. Compute $\sum_{\pi \in S_7} f(\pi)$.

Answer: $\boxed{29093}$ Extend the definition of f to apply for any permutation of $1, 2, \ldots, n$, for any positive integer n. For positive integer n, let $g(n)$ denote the number of permutations π of $1, 2, \ldots, n$ such that $f(\pi) = n$. We have $g(1) = 1$. For fixed n, k (with $k \leq n$), the number of permutations π of $1, 2, \ldots, n$ such that $f(\pi) = k$ is $g(k)(n-k)!$. This gives us the recursive formula $g(n) = n! - \sum_{k=1}^{n-1} g(k)(n-k)!$. Using this formula, we find that the first 7 values of g are $1, 1, 3, 13, 71, 461, 3447$. Our sum is then equal to $\sum_{k=1}^{7} k \cdot g(k)(7-k)!$. Using our computed values of g, we get that the sum evaluates to 29093.

32. [**19**] Let S be a set of size 3. How many collections T of subsets of S have the property that for any two subsets $U \in T$ and $V \in T$, both $U \cap V$ and $U \cup V$ are in T?

Answer: $\boxed{74}$ Let us consider the collections T grouped based on the size of the set $X = \bigcup_{U \in T} U$, which we can see also must be in T as long as T contains at least one set. This leads us to count the number of collections on a set of size at most 3 satisfying the desired property with the additional property that the entire set must be in the collection. Let C_n denote that number of such collections on a set of size n. Our answer will then be $1 + \binom{3}{0}C_0 + \binom{3}{1}C_1 + \binom{3}{2}C_2 + \binom{3}{3}C_3$, with the additional 1 coming from the empty collection.

Now for such a collection T on a set of n elements, consider the set $I = \bigcap_{U \in T} U$. Suppose this set has size k. Then removing all these elements from consideration gives us another such collection on a set of size $n - k$, but now containing the empty set. We can see that for each particular choice of I, this gives a bijection to the collections on the set S to the collections on the set $S - I$. This leads us to consider the further restricted collections that must contain both the entire set and the empty set.

It turns out that such restricted collections are a well-studied class of objects called *topological spaces*. Let T_n be the number of topological spaces on n elements. Our argument before shows that $C_n = \sum_{k=0}^{n} \binom{n}{k} T_k$. It is relatively straightforward to see that $T_0 = 1, T_1 = 1$, and $T_2 = 4$. For a set of size 3, there are the following spaces. The number of symmetric versions is shown in parentheses.

- $\emptyset, \{a, b, c\}$ (1)
- $\emptyset, \{a, b\}, \{a, b, c\}$ (3)
- $\emptyset, \{a\}, \{a, b, c\}$ (3)
- $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}$ (6)
- $\emptyset, \{a\}, \{b, c\}, \{a, b, c\}$ (3)

- $\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}$ (3)
- $\emptyset, \{a\}, \{b\}, \{a,b\}.\{a,b,c\}$ (3)
- $\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}$ (6)
- $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$ (1)

which gives $T_3 = 29$. Tracing back our reductions, we have that $C_0 = \binom{0}{0}T_0 = 1, C_1 = \binom{1}{0}T_0 + \binom{1}{1}T_1 = 2, C_2 = \binom{2}{0}T_0 + \binom{2}{1}T_1 + \binom{2}{2}T_2 = 7, C_3 = \binom{3}{0}T_0 + \binom{3}{1}T_1 + \binom{3}{2}T_2 + \binom{3}{3}T_3 = 45$, and then our answer is $1 + \binom{3}{0}C_0 + \binom{3}{1}C_1 + \binom{3}{2}C_2 + \binom{3}{3}C_3 = 1 + 1 + 6 + 21 + 45 = 74$.

33. [**23**] Compute the decimal expansion of $\sqrt{\pi}$. Your score will be $\min(23, k)$, where k is the number of consecutive correct digits immediately following the decimal point in your answer.

Answer: $\boxed{1.77245385090551602729816...}$ For this problem, it is useful to know the following square root algorithm that allows for digit-by-digit extraction of $\sqrt{x}$ and gives one decimal place of $\sqrt{x}$ for each two decimal places of x. We will illustrate how to extract the second digit after the decimal point of $\sqrt{\pi}$, knowing that $\pi = 3.1415\cdots$ and $\sqrt{\pi} = 1.7\cdots$.

Let d be the next decimal digit. Then d should be the largest digit such that $(1.7+0.01d)^2 < \pi$, which in this case we will treat as $(1.7+0.01d)^2 < 3.1415$. Expanding this, we get $2.89+0.034d+0.0001d^2 < 3.1415$, from which we get the value of d to be approximately $\lfloor \frac{3.1415-2.89}{0.034} \rfloor = \lfloor \frac{0.2515}{0.034} \rfloor = 7$, since the $0.0001d^2$ term is negligible. Indeed, 7 is the largest such digit, and so $d = 7$ is the second digit of $\sqrt{\pi}$. Because we are constantly subtracting the square of our extracted answer so far, we can record the difference in a manner similar to long division, which yields a quick method of extracting square roots by hand.

34. [**23**] Let Q be the product of the sizes of all the non-empty subsets of $\{1, 2, \ldots, 2012\}$, and let $M = \log_2(\log_2(Q))$. Give lower and upper bounds L and U for M. If $0 < L \le M \le U$, then your score will be $\min(23, \left\lfloor \frac{23}{3(U-L)} \right\rfloor)$. Otherwise, your score will be 0.

Answer: $\boxed{2015.318180...}$ In this solution, all logarithms will be taken in base 2. It is clear that $\log(Q) = \sum_{k=1}^{2012} \binom{2012}{k} \log(k)$. By paring k with $2012 - k$, we get $\sum_{k=1}^{2011} 0.5 * \log(k(2012-k))\binom{2012}{k} + \log(2012)$, which is between $0.5 * \log(2012) \sum_{k=0}^{2012} \binom{2012}{k}$ and $\log(2012) \sum_{k=0}^{2012} \binom{2012}{k}$; i.e., the answer is between $\log(2012)2^{2011}$ and $\log(2012)2^{2012}$. Thus $\log(\log(Q))$ is between $2011 + \log(\log(2012))$ and $2012 + \log(\log(2012))$. Also $3 < \log(\log(2012)) < 4$. So we get $2014 < M < 2016$.

35. [**23**] Let N be the number of distinct roots of $\prod_{k=1}^{2012}(x^k - 1)$. Give lower and upper bounds L and U on N. If $0 < L \le N \le U$, then your score will be $\left\lfloor \frac{23}{(U/L)^{1.7}} \right\rfloor$. Otherwise, your score will be 0.

Answer: $\boxed{1231288}$ For x to be such a number is equivalent to x being an k^{th} root of unity for some k up to 2012. For each k, there are $\varphi(k)$ primitive k^{th} roots of unity, so the total number of roots is $\sum_{k=1}^{2012} \varphi(k)$.

We will give a good approximation of this number using well known facts about the Möbius function, defined by $\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors} \end{cases}$. It turns out that if $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d})$. Using this fact, since $n = \sum_{d|n} \varphi(d)$, we have that $\varphi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$.

Now we have reduced the problem to estimating $\sum_{k=1}^{2012} \sum_{d|k} \mu(d)\frac{k}{d}$. Let $a = \frac{k}{d}$, so we obtain $\sum_{k=1}^{2012} \sum_{d|k} a\mu(d)$.

We can interchange the order of summation by writing

$$\begin{aligned}\sum_{d=1}^{2012}\sum_{a=1}^{\lfloor\frac{2012}{d}\rfloor} a\mu(d) &\approx \sum_{d=1}^{2012}\mu(d)\frac{1}{2}\left(\lfloor\frac{2012}{d}\rfloor\right)^2\\ &\approx \sum_{d=1}^{2012}\mu(d)\frac{2012^2}{2d^2}\\ &= \frac{2012^2}{2}\sum_{d=1}^{2012}\frac{\mu(d)}{d^2}\\ &\approx \frac{2012^2}{2}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}\end{aligned}$$

The Möbius function also satisfies the property that $\sum_{d|n}\mu(d) = \begin{cases}1 & \text{if } n=1\\ 0 & \text{otherwise}\end{cases}$, which can be seen as a special case of the theorem above (letting $f(n)=1, g(n) = \begin{cases}1 & \text{if } n=1\\ 0 & \text{otherwise}\end{cases}$). We can then see that $\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}\right)\left(\sum_{c=1}^{\infty}\frac{1}{c^2}\right) = \frac{1}{1^2} = 1$, so $\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} = \frac{6}{\pi^2}$. Therefore, we have $\sum_{k=1}^{2012}\varphi(k) \approx \frac{3}{\pi^2}\cdot 2012^2 = 1230488.266...$ 2012 is large enough that all of our approximations are pretty accurate and we should be comfortable perturbing this estimate by a small factor to give bounding values.

36. [**23**] Maria is hopping up a flight of stairs with 100 steps. At every hop, she advances some integer number of steps. Each hop she makes has fewer steps. However, the positive difference between the length of consecutive hops decreases. Let P be the number of distinct ways she can hop up the stairs. Find lower and upper bounds L and U for P. If $0 < L \leq P \leq U$, your score will be $\left\lfloor \frac{23}{\sqrt{U/L}} \right\rfloor$. Otherwise, your score will be 0.

Answer: $\boxed{6922}$ Consider the sequence of hops backwards. It is an increasing sequence where the first finite differences are increasing, so all the second finite differences are all positive integers. Furthermore, given positive integers a, e_0 (representing the initial value and initial first finite difference), and a sequence of positive integers $e_1, e_2, \ldots, e_k$, representing the second finite differences, we obtain a sequence of $k+2$ integers satisfying our constraints where the i-th term is equal to $a+\sum_{j=0}^{i-2}(i-j-1)e_j$. The sum of these values is equal to $(k+2)a+\sum_{j=0}^{k}\frac{(j+1)(j+2)}{2}e_{k-j}$. The number of sequences of length $k+2$ with sum 100 is then the number of positive integer solutions to $(k+2)a+\sum_{j=0}^{k}\frac{(j+1)(j+2)}{2}e_{k-j} = 100$ in $a, e_0, \ldots, e_k$.

We can now approximate a count of the total number of solutions by caseworking over the possible values of k. First, we must consider all sequences of length 1 or 2; it easy to see that there are 1 and 49 of these, respectively. Otherwise, we want to consider the following equations:

$$\begin{aligned} x_1+3x_2+3x_3 &= 100,\\ x_1+4x_2+3x_3+6x_4 &= 100,\\ x_1+5x_2+3x_3+6x_4+10x_5 &= 100,\\ x_1+6x_2+3x_3+6x_4+10x_5+15x_6 &= 100,\\ x_1+7x_2+3x_3+6x_4+10x_5+15x_6+21x_7 &= 100,\\ x_1+8x_2+3x_3+6x_4+10x_5+15x_6+21x_7+28x_8 &= 100.\end{aligned}$$

(Any larger values of k will yield equations with no solutions, as the sum of the coefficients will be greater than 100 in all of these.) We can compute the number of solutions to the last equation easily: there are 8. For the remaining 5 equations, we can use the following observation to estimate the number of solutions. Suppose we wanted to count the number of positive integer solutions to

$\sum_{i=1}^k c_i x_i = n$, where $c_i = 1$. This is equivalent to finding the number of nonnegative integer solutions to $\sum_{i=1}^k c_i x_i = n - \sum_{i=1}^k c_i$, which is also equivalent to finding the number of nonnegative integer solutions to $\sum_{i=2}^k c_i x_i \leq n - \sum_{i=1}^k c_i$. Let $n' = n - \sum_{i=1}^k c_i$. Each x_i can take values from 0 to $\frac{n'}{c_i}$, giving about $\frac{(n')^{k-1}}{\prod_{i=2}^k c_i}$ choices. Of course, not all of these choices work; we try to estimate the probability that one does. We can think of a random selection of x_i as approximated by choosing X_i uniformly from $[0, 1]$, then setting $x_i = X_i \cdot \frac{n'}{c_i}$. The condition would then be $\sum_{i=2}^k X_k \leq 1$. The probability of this occuring is $\frac{1}{(k-1)!}$, so an approximate number of solutions would be $\frac{(n')^{k-1}}{(k-1)! \prod_{i=2}^k c_i}$. We can use these values to get a lower bound of around 4000, and we can also use n instead of n' for a reasonable upper bound of 16000.