HMMT February 2020

February 15, 2020

Team Round

1. **[20]** Let n be a positive integer. Define a sequence by $a_0 = 1$, $a_{2i+1} = a_i$, and $a_{2i+2} = a_i + a_{i+1}$ for each $i \geq 0$. Determine, with proof, the value of $a_0 + a_1 + a_2 + \cdots + a_{2^n-1}$.

2. **[25]** Let n be a fixed positive integer. An n-staircase is a polyomino with $\frac{n(n+1)}{2}$ cells arranged in the shape of a staircase, with arbitrary size. Here are two examples of 5-staircases:

   Prove that an n-staircase can be dissected into strictly smaller n-staircases.

3. **[25]** Let ABC be a triangle inscribed in a circle ω and ℓ be the tangent to ω at A. The line through B parallel to AC meets ℓ at P, and the line through C parallel to AB meets ℓ at Q. The circumcircles of ABP and ACQ meet at $S \neq A$. Show that AS bisects BC.

4. **[35]** Alan draws a convex 2020-gon $\mathcal{A} = A_1A_2 \cdots A_{2020}$ with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \ldots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells Jason the angles $\theta_1, \theta_2, \ldots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

5. **[40]** Let a_0, b_0, c_0, a, b, c be integers such that $\gcd(a_0, b_0, c_0) = \gcd(a, b, c) = 1$. Prove that there exists a positive integer n and integers $a_1, a_2, \ldots, a_n = a, b_1, b_2, \ldots, b_n = b, c_1, c_2, \ldots, c_n = c$ such that for all $1 \leq i \leq n$, $a_{i-1}a_i + b_{i-1}b_i + c_{i-1}c_i = 1$.

6. **[40]** Let $n > 1$ be a positive integer and S be a collection of $\frac{1}{2}\binom{2n}{n}$ distinct n-element subsets of $\{1, 2, \ldots, 2n\}$. Show that there exists $A, B \in S$ such that $|A \cap B| \leq 1$.

7. **[50]** Positive real numbers x and y satisfy

$$\left|\left|\cdots\left|\left||x| - y\right| - x\right|\cdots - y\right| - x\right| = \left|\left|\cdots\left|\left||y| - x\right| - y\right|\cdots - x\right| - y\right|$$

   where there are 2019 absolute value signs $|\cdot|$ on each side. Determine, with proof, all possible values of $\frac{x}{y}$.

8. **[50]** Let ABC be a scalene triangle with angle bisectors AD, BE, and CF so that D, E, and F lie on segments BC, CA, and AB respectively. Let M and N be the midpoints of BC and EF respectively. Prove that line AN and the line through M parallel to AD intersect on the circumcircle of ABC if and only if $DE = DF$.

9. **[55]** Let $p > 5$ be a prime number. Show that there exists a prime number $q < p$ and a positive integer n such that p divides $n^2 - q$.

10. **[60]** Let n be a fixed positive integer, and choose n positive integers $a_1, \ldots, a_n$. Given a permutation π on the first n positive integers, let $S_\pi = \{i \mid \frac{a_i}{\pi(i)} \text{ is an integer}\}$. Let N denote the number of distinct sets S_π as π ranges over all such permutations. Determine, in terms of n, the maximum value of N over all possible values of $a_1, \ldots, a_n$.