11^{th} Annual Harvard-MIT Mathematics Tournament

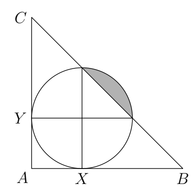
Saturday 23 February 2008

Individual Round: Geometry Test

1. [**3**] How many different values can $\angle ABC$ take, where A,B,C are distinct vertices of a cube?

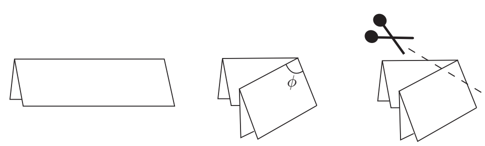
2. [**3**] Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

3. [**4**] Let ABC be a triangle with $\angle BAC = 90°$. A circle is tangent to the sides AB and AC at X and Y respectively, such that the points on the circle diametrically opposite X and Y both lie on the side BC. Given that $AB = 6$, find the area of the portion of the circle that lies outside the triangle.



4. [**4**] In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

5. [**5**] A piece of paper is folded in half. A second fold is made at an angle ϕ ($0° < \phi < 90°$) to the first, and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O, and also that $\angle XOY = 144°$, where X and Y are the the vertices of the hole adjacent to O. Find the value(s) of ϕ (in degrees).

6. [**5**] Let ABC be a triangle with $\angle A = 45°$. Let P be a point on side BC with $PB = 3$ and $PC = 5$. Let O be the circumcenter of ABC. Determine the length OP.

7. [**6**] Let C_1 and C_2 be externally tangent circles with radius 2 and 3, respectively. Let C_3 be a circle internally tangent to both C_1 and C_2 at points A and B, respectively. The tangents to C_3 at A and B meet at T, and $TA = 4$. Determine the radius of C_3.

8. [**6**] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ, visits all three sides of ABC, and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

9. [**7**] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

10. [**7**] Let ABC be a triangle with $BC = 2007$, $CA = 2008$, $AB = 2009$. Let ω be an excircle of ABC that touches the line segment BC at D, and touches extensions of lines AC and AB at E and F, respectively (so that C lies on segment AE and B lies on segment AF). Let O be the center of ω. Let ℓ be the line through O perpendicular to AD. Let ℓ meet line EF at G. Compute the length DG.