HMMT Spring 2021

March 06, 2021

Geometry Round

1. A circle contains the points $(0, 11)$ and $(0, -11)$ on its circumference and contains all points (x, y) with $x^2 + y^2 < 1$ in its interior. Compute the largest possible radius of the circle.

2. Let X_0 be the interior of a triangle with side lengths 3, 4, and 5. For all positive integers n, define X_n to be the set of points within 1 unit of some point in X_{n-1}. The area of the region outside X_{20} but inside X_{21} can be written as $a\pi + b$, for integers a and b. Compute $100a + b$.

3. Triangle ABC has a right angle at C, and D is the foot of the altitude from C to AB. Points L, M, and N are the midpoints of segments AD, DC, and CA, respectively. If $CL = 7$ and $BM = 12$, compute BN^2.

4. Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 5$, $BC = 9$, $CD = 10$, and $DA = 7$. Lines BC and DA intersect at point E. Let M be the midpoint of CD, and let N be the intersection of the circumcircles of $\triangle BMC$ and $\triangle DMA$ (other than M). If $EN^2 = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

5. Let AEF be a triangle with $EF = 20$ and $AE = AF = 21$. Let B and D be points chosen on segments AE and AF, respectively, such that BD is parallel to EF. Point C is chosen in the interior of triangle AEF such that $ABCD$ is cyclic. If $BC = 3$ and $CD = 4$, then the ratio of areas $\frac{[ABCD]}{[AEF]}$ can be written as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $100a + b$.

6. In triangle ABC, let M be the midpoint of BC, H be the orthocenter, and O be the circumcenter. Let N be the reflection of M over H. Suppose that $OA = ON = 11$ and $OH = 7$. Compute BC^2.

7. Let O and A be two points in the plane with $OA = 30$, and let Γ be a circle with center O and radius r. Suppose that there exist two points B and C on Γ with $\angle ABC = 90^\circ$ and $AB = BC$. Compute the minimum possible value of $\lfloor r \rfloor$.

8. Two circles with radii 71 and 100 are externally tangent. Compute the largest possible area of a right triangle whose vertices are each on at least one of the circles.

9. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AD = BD$. Let M be the midpoint of AB, and let $P \neq C$ be the second intersection of the circumcircle of $\triangle BCD$ and the diagonal AC. Suppose that $BC = 27, CD = 25$, and $AP = 10$. If $MP = \frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

10. Acute triangle ABC has circumcircle Γ. Let M be the midpoint of BC. Points P and Q lie on Γ so that $\angle APM = 90^\circ$ and $Q \neq A$ lies on line AM. Segments PQ and BC intersect at S. Suppose that $BS = 1$, $CS = 3$, $PQ = 8\sqrt{\frac{7}{37}}$, and the radius of Γ is r. If the sum of all possible values of r^2 can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.