14^{th} Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

1. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

2. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

3. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

4. Let H be a regular hexagon of side length x. Call a hexagon in the same plane a "distortion" of H if and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

5. Let $a \star b = ab + a + b$ for all integers a and b. Evaluate $1 \star (2 \star (3 \star (4 \star \ldots (99 \star 100) \ldots)))$.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

7. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

8. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

9. For all real numbers x, let
$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}.$$
Evaluate $(f(f(\ldots(f(2011))\ldots)))^{2011}$, where f is applied 2010 times.

10. Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD, respectively, so that the area of square $ABCD$ equals the area of triangle AEF. If EF intersects BC at X and $BX = 6$, determine DF.

11. Let $f(x) = x^2 + 6x + c$ for all real numbers x, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

12. Let $ABCDEF$ be a convex equilateral hexagon such that lines BC, AD, and EF are parallel. Let H be the orthocenter of triangle ABD. If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.

13. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \le P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

14. Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

15. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \le i \le 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \le |r_2|$ for any function f satisfying these properties, find A.

16. Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP + PQ + CQ$, where P and Q may be any points lying along rays AB and AD, respectively. Determine the maximum value of m.

17. Let $z = \cos\frac{2\pi}{2011} + i\sin\frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \ldots \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3)\ldots P(z^{2010})$.

18. Collinear points A, B, and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x-axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $AB/BC = 2$. If $D = (a, a)$, the circumcircle of triangle ADC intersects $y = x$ again at E, and ray AE intersects $y = 2x$ at F, evaluate AE/EF.

19. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2$; $b_0 = 2$, and $a_{n+1} = a_n\sqrt{1 + a_n^2 + b_n^2} - b_n$; $b_{n+1} = b_n\sqrt{1 + a_n^2 + b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4.

20. Let ω_1 and ω_2 be two circles that intersect at points A and B. Let line l be tangent to ω_1 at P and to ω_2 at Q so that A is closer to PQ than B. Let points R and S lie along rays PA and QA, respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and Q. Let O be the circumcenter of triangle ASR, and let C and D be the midpoints of major arcs AP and AQ, respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.