HMMT February 2020

February 15, 2020

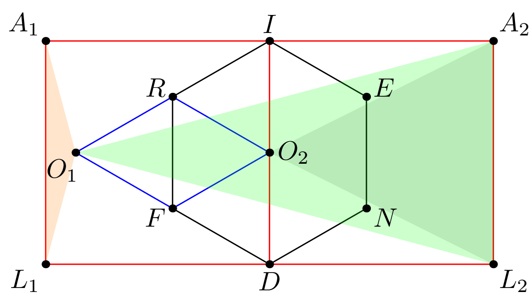
Geometry

1. Let $DIAL$, FOR, and $FRIEND$ be regular polygons in the plane. If $ID = 1$, find the product of all possible areas of OLA.

   Proposed by: Andrew Gu

   Answer: $\boxed{\frac{1}{32}}$

   Solution: Focusing on $FRIEND$ and FOR first, observe that either DIO is an equilateral triangle or O is the midpoint of ID. Next, OLA is always an isosceles triangle with base $LA = 1$. The possible distances of O from LA are 1 and $1 \pm \frac{\sqrt{3}}{2}$ as the distance from O to ID in the equilateral triangle case is $\frac{\sqrt{3}}{2}$.



   The three possibilities are shown in the diagram as shaded triangles $\triangle O_1L_1A_1$, $\triangle O_2L_2A_2$, and $\triangle O_1L_2A_2$.

   The product of all possible areas is thus

$$\frac{1 \cdot \left(1 - \frac{\sqrt{3}}{2}\right) \cdot \left(1 + \frac{\sqrt{3}}{2}\right)}{2^3} = \frac{1}{2^5} = \frac{1}{32}.$$

2. Let ABC be a triangle with $AB = 5$, $AC = 8$, and $\angle BAC = 60°$. Let $UVWXYZ$ be a regular hexagon that is inscribed inside ABC such that U and V lie on side BA, W and X lie on side AC, and Z lies on side CB. What is the side length of hexagon $UVWXYZ$?

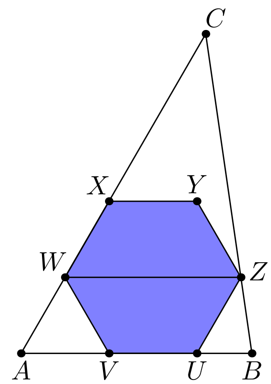
   Proposed by: Ryan Kim

   Answer: $\boxed{\frac{40}{21}}$

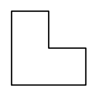
   Solution: Let the side length of $UVWXYZ$ be s. We have $WZ = 2s$ and $WZ \parallel AB$ by properties of regular hexagons. Thus, triangles WCZ and ACB are similar. AWV is an equilateral triangle, so we have $AW = s$. Thus, using similar triangles, we have

$$\frac{WC}{WZ} = \frac{AC}{AB} \implies \frac{8-s}{2s} = \frac{8}{5},$$

   so $5(8-s) = 8(2s) \implies s = \frac{40}{21}$.



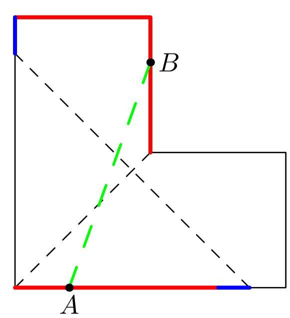
3. Consider the L-shaped tromino below with 3 attached unit squares. It is cut into exactly two pieces of equal area by a line segment whose endpoints lie on the perimeter of the tromino. What is the longest possible length of the line segment?



Proposed by: James Lin

Answer: $\boxed{\frac{5}{2}}$

Solution: Let the line segment have endpoints A and B. Without loss of generality, let A lie below the lines $x + y = \sqrt{3}$ (as this will cause B to be above the line $x + y = \sqrt{3}$) and $y = x$ (we can reflect about $y = x$ to get the rest of the cases):



Now, note that as A ranges from $(0, 0)$ to $(1.5, 0)$, B will range from $(1, 1)$ to $(1, 2)$ to $(0, 2)$, as indicated by the red line segments. Note that these line segments are contained in a rectangle bounded by $x = 0$, $y = 0$, $x = 1.5$, and $y = 2$, and so the longest line segment in this case has length $\sqrt{2^2 + 1.5^2} = \frac{5}{2}$.

As for the rest of the cases, as $A = (x, 0)$ ranges from $(1.5, 0)$ to $(\sqrt{3}, 0)$, B will be the point $(0, \frac{3}{x})$, so it suffices to maximize $\sqrt{x^2 + \frac{9}{x^2}}$ given $1.5 \le x \le \sqrt{3}$. Note that the further away x^2 is from 3, the larger $x^2 + \frac{9}{x^2}$ gets, and so the maximum is achieved when $x = 1.5$, which gives us the same length as before.

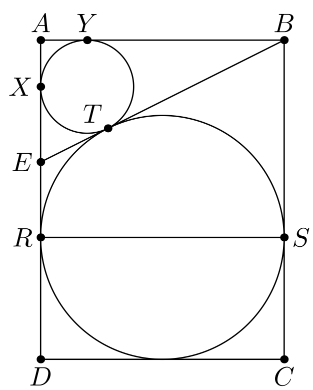
Thus, the maximum length is $\frac{5}{2}$.

4. Let $ABCD$ be a rectangle and E be a point on segment AD. We are given that quadrilateral $BCDE$ has an inscribed circle ω_1 that is tangent to BE at T. If the incircle ω_2 of ABE is also tangent to BE at T, then find the ratio of the radius of ω_1 to the radius of ω_2.

   Proposed by: James Lin

   Answer: $\boxed{\frac{3+\sqrt{5}}{2}}$

   Solution: Let ω_1 be tangent to AD, BC at R, S and ω_2 be tangent to AD, AB at X, Y. Let $AX = AY = r$, $EX = ET = ER = a$, $BY = BT = BS = b$. Then noting that $RS \parallel CD$, we see that $ABSR$ is a rectangle, so $r + 2a = b$. Therefore $AE = a + r$, $AB = b + r = 2(a + r)$, and so $BE = (a+r)\sqrt{5}$. On the other hand, $BE = b + a = r + 3a$. This implies that $a = \frac{1+\sqrt{5}}{2}r$. The desired ratio is then $\frac{RS}{2AY} = \frac{AB}{2r} = \frac{a+r}{r} = \frac{3+\sqrt{5}}{2}$.

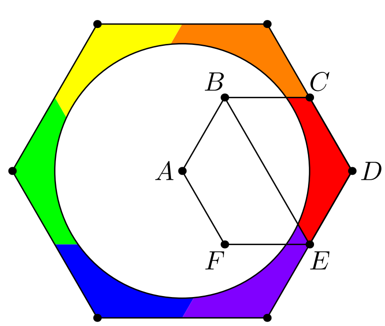


5. Let $ABCDEF$ be a regular hexagon with side length 2. A circle with radius 3 and center at A is drawn. Find the area inside quadrilateral $BCDE$ but outside the circle.

   Proposed by: Carl Joshua Quines

   Answer: $\boxed{4\sqrt{3} - \frac{3}{2}\pi}$

   Solution: Rotate the region 6 times about A to form a bigger hexagon with a circular hole. The larger hexagon has side length 4 and area $24\sqrt{3}$, so the area of the region is $\frac{1}{6}(24\sqrt{3} - 9\pi) = 4\sqrt{3} - \frac{3}{2}\pi$.

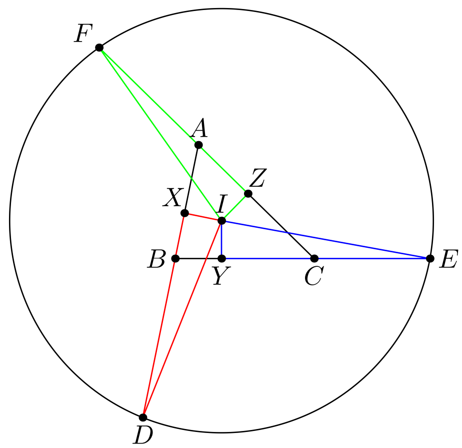


6. Let ABC be a triangle with $AB = 5$, $BC = 6$, $CA = 7$. Let D be a point on ray AB beyond B such that $BD = 7$, E be a point on ray BC beyond C such that $CE = 5$, and F be a point on ray CA beyond A such that $AF = 6$. Compute the area of the circumcircle of DEF.

   Proposed by: James Lin

   Answer: $\boxed{\frac{251}{3}\pi}$

   Solution 1: Let I be the incenter of ABC. We claim that I is the circumcenter of DEF.

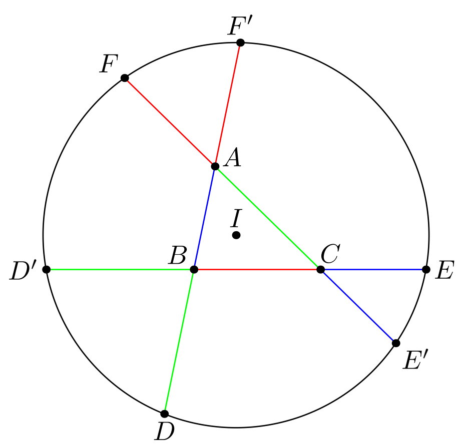


To prove this, let the incircle touch AB, BC, and AC at X, Y, and Z, respectively. Noting that $XB = BY = 2$, $YC = CZ = 4$, and $ZA = AX = 3$, we see that $XD = YE = ZF = 9$. Thus, since $IX = IY = IZ = r$ (where r is the inradius) and $\angle IXD = \angle IYE = \angle IZF = 90°$, we have three congruent right triangles, and so $ID = IE = IF$, as desired.

Let $s = \frac{5+6+7}{2} = 9$ be the semiperimeter. By Heron's formula, $[ABC] = \sqrt{9(9-5)(9-6)(9-7)} = 6\sqrt{6}$, so $r = \frac{[ABC]}{s} = \frac{2\sqrt{6}}{3}$. Then the area of the circumcircle of DEF is

$$ID^2\pi = (IX^2 + XD^2)\pi = (r^2 + s^2)\pi = \frac{251}{3}\pi.$$

Solution 2: Let D' be a point on ray CB beyond B such that $BD' = 7$, and similarly define E', F'. Noting that $DA = E'A$ and $AF = AF'$, we see that $DE'F'F$ is cyclic by power of a point. Similarly, $EF'D'D$ and $FD'E'E$ are cyclic. Now, note that the radical axes for the three circles circumscribing these quadrilaterals are the sides of ABC, which are not concurrent. Therefore, $DD'FF'EE'$ is cyclic. We can deduce that the circumcenter of this circle is I in two ways: either by calculating that the midpoint of $D'E$ coincides with the foot from I to BC, or by noticing that the perpendicular bisector of FF' is AI. The area can then be calculated the same way as the previous solution.



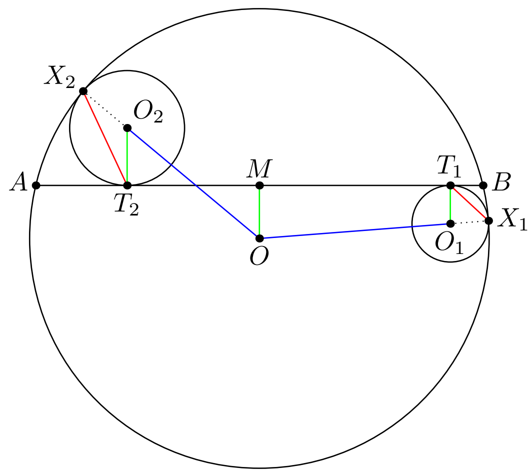
Remark. The circumcircle of DEF is the *Conway circle* of ABC.

7. Let Γ be a circle, and ω_1 and ω_2 be two non-intersecting circles inside Γ that are internally tangent to Γ at X_1 and X_2, respectively. Let one of the common internal tangents of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2, respectively, while intersecting Γ at two points A and B. Given that $2X_1T_1 = X_2T_2$ and that ω_1, ω_2, and Γ have radii 2, 3, and 12, respectively, compute the length of AB.

Proposed by: James Lin

Answer: $\boxed{\frac{96\sqrt{10}}{13}}$

Solution 1: Let ω_1, ω_2, Γ have centers O_1, O_2, O and radii r_1, r_2, R respectively. Let d be the distance from O to AB (signed so that it is positive if O and O_1 are on the same side of AB).



Note that

$$\begin{aligned} OO_i &= R - r_i, \\ \cos\angle T_1O_1O = \frac{O_1T_1 - OM}{OO_1} &= \frac{r_1 - d}{R - r_1}, \\ \cos\angle T_2O_2O = \frac{O_2T_2 + OM}{OO_1} &= \frac{r_2 + d}{R - r_2}. \end{aligned}$$

Then

$$\begin{aligned} X_1T_1 &= r_1\sqrt{2 - 2\cos\angle X_1O_1T_1} \\ &= r_i\sqrt{2 + 2\cos\angle T_1O_1O} \\ &= r_1\sqrt{2 + 2\frac{r_1 - d}{R - r_1}} \\ &= r_1\sqrt{2\frac{R - d}{R - r_1}}. \end{aligned}$$

Likewise,

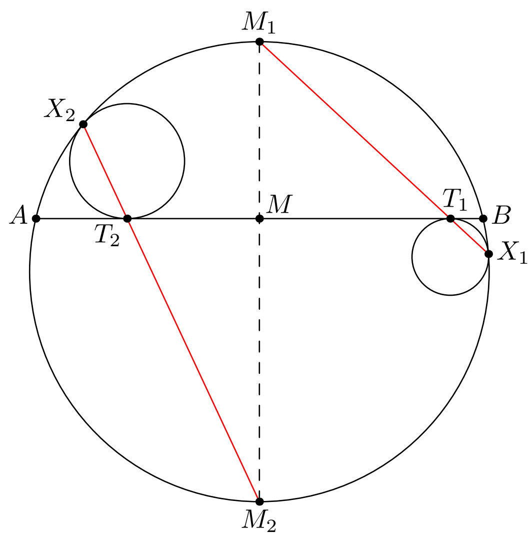
$$X_2T_2 = r_2\sqrt{2\frac{R + d}{R - r_2}}.$$

From $2X_1T_1 = X_2T_2$ we have

$$8r_1^2\left(\frac{R - d}{R - r_1}\right) = 4X_1T_1^2 = X_2T_2^2 = 2r_2^2\left(\frac{R + d}{R - r_2}\right).$$

Plugging in $r_1 = 2$, $r_2 = 3$, $R = 12$ and solving yields $d = \frac{36}{13}$. Hence $AB = 2\sqrt{R^2 - d^2} = \frac{96\sqrt{10}}{13}$.

Solution 2: We borrow the notation from the previous solution. Let X_1T_1 and X_2T_2 intersect Γ again at M_1 and M_2. Note that, if we orient AB to be horizontal, then the circles ω_1 and ω_2 are on opposite sides of AB. In addition, for $i \in \{1, 2\}$ there exist homotheties centered at X_i with ratio $\frac{R}{r_i}$ which send ω_i to Γ. Since T_1 and T_2 are points of tangencies and thus top/bottom points, we see that M_1 and M_2 are the top and bottom points of Γ, and so M_1M_2 is a diameter perpendicular to AB.



Now, note that through power of a point and the aforementioned homotheties,

$$P(M_1, \omega_1) = M_1T_1 \cdot M_1X_1 = X_1T_1^2 \left(\frac{R}{r_1}\right)\left(\frac{R}{r_1} - 1\right) = 30X_1T_1^2,$$

and similarly $P(M_2, \omega_2) = 12X_2T_2^2$. (Here P is the power of a point with respect to a circle). Then

$$\frac{P(M_1, \omega_1)}{P(M_2, \omega_2)} = \frac{30X_1T_1^2}{12X_2T_2^2} = \frac{30}{12(2)^2} = \frac{5}{8}.$$

Let M be the midpoint of AB, and suppose $M_1M = R + d$ (here d may be negative). Noting that M_1 and M_2 are arc bisectors, we have $\angle AX_1M_1 = \angle T_1AM_1$, so $\triangle M_1AT_1 \sim \triangle M_1X_1A$, meaning that $M_1A^2 = M_1T_1 \cdot M_1X_1 = P(M_1, \omega_1)$. Similarly, $\triangle M_2AT_2 \sim \triangle M_2X_2A$, so $M_2A^2 = P(M_2, \omega_2)$. Therefore,

$$\frac{P(M_1, \omega_1)}{P(M_2, \omega_2)} = \frac{M_1A^2}{M_2A^2} = \frac{(R^2 - d^2) + (R + d)^2}{(R^2 - d^2) + (R - d)^2} = \frac{2R^2 + 2Rd}{2R^2 - 2Rd} = \frac{R + d}{R - d} = \frac{5}{8},$$

giving $d = -\frac{3}{13}R$. Finally, we compute $AB = 2R\sqrt{1 - \left(\frac{3}{13}\right)^2} = \frac{8R\sqrt{10}}{13} = \frac{96\sqrt{10}}{13}$.

8. Let ABC be an acute triangle with circumcircle Γ. Let the internal angle bisector of $\angle BAC$ intersect BC and Γ at E and N, respectively. Let A' be the antipode of A on Γ and let V be the point where AA' intersects BC. Given that $EV = 6$, $VA' = 7$, and $A'N = 9$, compute the radius of Γ.

   Proposed by: James Lin

   Answer: $\boxed{\frac{15}{2}}$

   Solution 1: Let H_a be the foot of the altitude from A to BC. Since AE bisects $\angle H_aAV$, by the angle bisector theorem $\frac{AH_a}{H_aE} = \frac{AV}{VE}$. Note that $\triangle AH_aE \sim \triangle ANA'$ are similar right triangles, so $\frac{AN}{NA'} = \frac{AH_a}{H_aE}$.

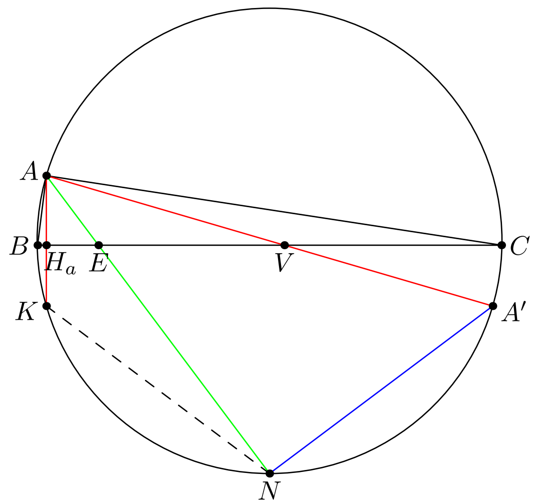
Let R be the radius of Γ. We know that $AA' = 2R$, so $AN = \sqrt{AA'^2 - NA'^2} = \sqrt{4R^2 - 81}$ and $AV = AA' - VA' = 2R - 7$. Therefore

$$\frac{\sqrt{4R^2-81}}{9} = \frac{AN}{NA'} = \frac{AH_a}{H_aE} = \frac{AV}{VE} = \frac{2R-7}{6}.$$

The resulting quadratic equation is

$$0 = 9(2R-7)^2 - 4(4R^2-81) = 20R^2 - 252R + 765 = (2R-15)(10R-51).$$

We are given that ABC is acute so $VA' < R$. Therefore $R = \frac{15}{2}$.



Solution 2: Let Ψ denote inversion about A with radius $\sqrt{AB \cdot AC}$ composed with reflection about AE. Note that Ψ swaps the pairs $\{B, C\}$, $\{E, N\}$, and $\{H_a, A'\}$. Let $K = \Psi(V)$, which is also the second intersection of AH_a with Γ. Since AE bisects $\angle KAA'$, we have $NK = NA' = 9$. By the inversion distance formula,

$$NK = \frac{AB \cdot AC \cdot VE}{AE \cdot AV} = \frac{AE \cdot AN \cdot VE}{AE \cdot AV} = \frac{AN \cdot VE}{AV}.$$

This leads to the same equation as the previous solution.

9. Circles $\omega_a, \omega_b, \omega_c$ have centers A, B, C, respectively and are pairwise externally tangent at points D, E, F (with $D \in BC, E \in CA, F \in AB$). Lines BE and CF meet at T. Given that ω_a has radius 341, there exists a line ℓ tangent to all three circles, and there exists a circle of radius 49 tangent to all three circles, compute the distance from T to ℓ.

   Proposed by: Andrew Gu

   Answer: $\boxed{294}$

   Solution 1: We will use the following notation: let ω be the circle of radius 49 tangent to each of ω_a, ω_b, ω_c. Let ω_a, ω_b, ω_c have radii r_a, r_b, r_c respectively. Let γ be the incircle of ABC, with center I and radius r. Note that DEF is the intouch triangle of ABC and γ is orthogonal to ω_a, ω_b, ω_c (i.e. ID, IE, IF are the common internal tangents). Since AD, BE, CF are concurrent at T, we have $K = AB \cap DE$ satisfies $(A, B; F, K) = -1$, so K is the external center of homothety of ω_a and ω_b. In particular, K lies on ℓ. Similarly, $BC \cap EF$ also lies on ℓ, so ℓ is the polar of T to γ. Hence $IT \perp \ell$ so if L is the foot from I to ℓ, we have $IT \cdot IL = r^2$.

An inversion about γ preserves ω_a, ω_b, ω_c and sends ℓ to the circle with diameter IT. Since inversion preserves tangency, the circle with diameter IT must be ω. Therefore $IT = 98$ by the condition of the problem statement. Letting a, b, c be the radii of ω_a, ω_b, ω_c respectively and invoking Heron's formula as well as $A = rs$ for triangle ABC, we see that γ has radius

$$r = \sqrt{\frac{r_a r_b r_c}{r_a + r_b + r_c}}.$$

We will compute this quantity using Descartes' theorem. Note that there are two circles tangent to ω_a, ω_b, ω_c, one with radius $IT/2$ and one with radius ∞. By Descartes' circle theorem, we have (where $k_a := 1/a$ is the curvature)

$$k_a + k_b + k_c + 2\sqrt{k_a k_b + k_b k_c + k_a k_c} = \frac{1}{IT/2}$$

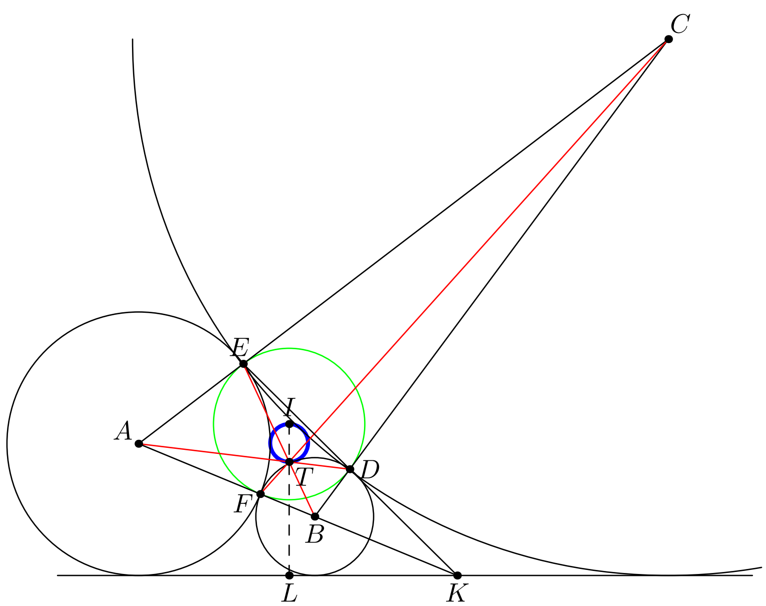
and

$$k_a + k_b + k_c - 2\sqrt{k_a k_b + k_b k_c + k_c k_a} = 0,$$

which implies

$$\sqrt{\frac{r_a + r_b + r_c}{r_a r_b r_c}} = \sqrt{k_a k_b + k_b k_c + k_c k_a} = \frac{1}{2IT}.$$

Therefore $r = 2IT$, which means $IL = \frac{r^2}{IT} = 4IT$ and $TL = 3IT = 294$.



Solution 2: Using the same notation as the previous solution, note that the point T can be expressed with un-normalized barycentric coordinates

$$\left(\frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c}\right)$$

with respect to ABC because T is the Gergonne point of triangle ABC. The distance from T to ℓ can be expressed as a weighted average of the distances from each of the points A, B, C, which is

$$\frac{1/r_a}{1/r_a + 1/r_b + 1/r_c} \cdot r_a + \frac{1/r_b}{1/r_a + 1/r_b + 1/r_c} \cdot r_b + \frac{1/r_c}{1/r_a + 1/r_b + 1/r_c} \cdot r_c = \frac{3}{1/r_a + 1/r_b + 1/r_c}.$$

Note that there are two circles tangent to ω_a, ω_b, ω_c, one with radius 49 and one with radius ∞. By Descartes' circle theorem, we have (where $k_a := 1/r_a$ is the curvature)

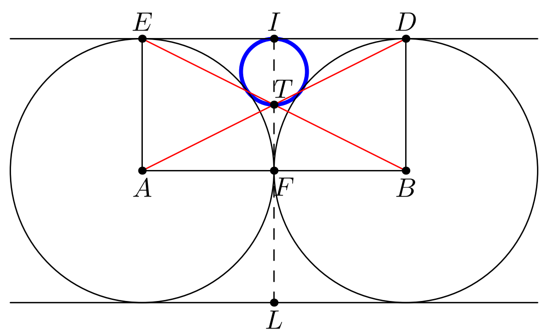
$$k_a + k_b + k_c + 2\sqrt{k_ak_b + k_bk_c + k_ak_c} = \frac{1}{49}$$

and

$$k_a + k_b + k_c - 2\sqrt{k_ak_b + k_bk_c + k_ck_a} = 0,$$

so $k_a + k_b + k_c = 1/98$. The distance from T to ℓ is then $3 \cdot 98 = 294$.

Solution 3: As in the first solution, we deduce that IT is a diameter of ω, with T being the point on ω closest to ℓ. By Steiner's porism, we can hold ω and ℓ fixed while making ω_c into a line parallel to ℓ, resulting in the following figure:



Let R be the common radius of ω_a and ω_b and r be the radius of ω. Notice that $ABDE$ is a rectangle with center T, so $R = AE = 2IT = 4r$. The distance from T to ℓ is $IL - IT = 2R - 2r = 6r = 294$.

10. Let Γ be a circle of radius 1 centered at O. A circle Ω is said to be *friendly* if there exist distinct circles $\omega_1, \omega_2, \ldots, \omega_{2020}$, such that for all $1 \le i \le 2020$, ω_i is tangent to Γ, Ω, and ω_{i+1}. (Here, $\omega_{2021} = \omega_1$.) For each point P in the plane, let $f(P)$ denote the sum of the areas of all friendly circles centered at P. If A and B are points such that $OA = \frac{1}{2}$ and $OB = \frac{1}{3}$, determine $f(A) - f(B)$.

Proposed by: Michael Ren

Answer: $\boxed{\frac{1000}{9}\pi}$

Solution: Let P satisfy $OP = x$. (For now, we focus on $f(P)$ and ignore the A and B from the problem statement.) The key idea is that if we invert at some point along OP such that the images of Γ and Ω are concentric, then ω_i still exist. Suppose that this inversion fixes Γ and takes Ω to Ω' of radius r (and X to X' in general). If the inversion is centered at a point Q along ray OP such that $OQ = d$, then the radius of inversion is $\sqrt{d^2-1}$. Let the diameter of Ω meet OQ at A and B with A closer to Q than B. Then, $(AB; PP_\infty) = -1$ inverts to $(A'B'; P'Q) = -1$, where P_∞ is the point at infinity along line OP, so P' is the inverse of Q in Ω'. We can compute $OP' = \frac{r^2}{d}$ so $P'Q = d - \frac{r^2}{d}$ and $PQ = \frac{d^2-1}{d - \frac{r^2}{d}}$. Thus, we get the equation $\frac{d^2-1}{d-\frac{r^2}{d}} + x = d$, which rearranges to $\frac{1-r^2}{d^2-r^2}d = x$, or $d^2 - x^{-1}(1-r^2)d - r^2 = 0$. Now, we note that the radius of Ω is

$$\frac{1}{2}AB = \frac{1}{2}\left(\frac{d^2-1}{d-r} - \frac{d^2-1}{d+r}\right) = \frac{r(d^2-1)}{d^2-r^2} = r\left(1 + \frac{r^2-1}{d^2-r^2}\right) = r\left(1 - \frac{x}{d}\right).$$

The quadratic formula gives us that $d = \frac{(1-r^2) \pm \sqrt{r^4-(2-4x^2)r^2+1}}{2x}$, so $\frac{x}{d} = -\frac{1-r^2 \pm \sqrt{r^4-(2-4x^2)r^2+1}}{2r^2}$, which means that the radius of Ω is

$$\frac{r^2+1 \pm \sqrt{r^4-(2-4x^2)r^2+1}}{2r} = \frac{r + \frac{1}{r} \pm \sqrt{r^2 + \frac{1}{r^2} - 2 + 4x^2}}{2}.$$

Note that if r gives a valid chain of 2020 circles, so will $\frac{1}{r}$ by homothety/inversion. Thus, we can think of each pair of $r, \frac{1}{r}$ as giving rise to two possible values of the radius of Ω, which are $\frac{r+\frac{1}{r}\pm\sqrt{r^2+\frac{1}{r^2}-1}}{2}$. This means that the pairs have the same sum of radii as the circles centered at O, and the product of the radii is $1-x^2$. (A simpler way to see this is to note that inversion at P with radius $\sqrt{1-x^2}$ swaps the two circles.) From this, it follows that the difference between the sum of the areas for each pair is $2\pi\left(\frac{1}{2^2}-\frac{1}{3^2}\right)=\frac{5}{18}\pi$. There are $\frac{\varphi(2020)}{2}=400$ such pairs, which can be explicitly computed as $\frac{1-\sin\frac{\pi k}{2020}}{1+\sin\frac{\pi k}{2020}}, \frac{1+\sin\frac{\pi k}{2020}}{1-\sin\frac{\pi k}{2020}}$ for positive integers $k<1010$ relatively prime to 2020. Thus, the answer is $\frac{1000}{9}\pi$.