HMMT November 2023

November 11, 2023

Team Round

1. **[20]** Let ABC be an equilateral triangle with side length 2 that is inscribed in a circle ω. A chord of ω passes through the midpoints of sides AB and AC. Compute the length of this chord.

2. **[20]** A real number x satisfies $9^x + 3^x = 6$. Compute the value of $16^{1/x} + 4^{1/x}$.

3. **[25]** Two distinct similar rhombi share a diagonal. The smaller rhombus has area 1, and the larger rhombus has area 9. Compute the side length of the larger rhombus.

4. **[30]** There are six empty slots corresponding to the digits of a six-digit number. Claire and William take turns rolling a standard six-sided die, with Claire going first. They alternate with each roll until they have each rolled three times. After a player rolls, they place the number from their die roll into a remaining empty slot of their choice. Claire wins if the resulting six-digit number is divisible by 6, and William wins otherwise. If both players play optimally, compute the probability that Claire wins.

5. **[35]** A complex quartic polynomial Q is *quirky* if it has four distinct roots, one of which is the sum of the other three. There are four complex values of k for which the polynomial $Q(x) = x^4 - kx^3 - x^2 - x - 45$ is quirky. Compute the product of these four values of k.

6. **[45]** The pairwise greatest common divisors of five positive integers are
$$2, 3, 4, 5, 6, 7, 8, p, q, r$$
in some order, for some positive integers p, q, r. Compute the minimum possible value of $p + q + r$.

7. **[45]** Let $ABCD$ be a convex trapezoid such that $\angle BAD = \angle ADC = 90^\circ$, $AB = 20$, $AD = 21$, and $CD = 28$. Point $P \neq A$ is chosen on segment AC such that $\angle BPD = 90^\circ$. Compute AP.

8. **[55]** There are $n \geq 2$ coins, each with a different positive integer value. Call an integer m *sticky* if some subset of these n coins have total value m. We call the entire set of coins a *stick* if all the sticky numbers form a consecutive range of integers. Compute the minimum total value of a stick across all sticks containing a coin of value 100.

9. **[60]** Let r_k denote the remainder when $\binom{127}{k}$ is divided by 8. Compute $r_1 + 2r_2 + 3r_3 + \cdots + 63r_{63}$.

10. **[65]** Compute the number of ways a non-self-intersecting concave quadrilateral can be drawn in the plane such that two of its vertices are $(0, 0)$ and $(1, 0)$, and the other two vertices are two distinct lattice points $(a, b), (c, d)$ with $0 \leq a, c \leq 59$ and $1 \leq b, d \leq 5$.

    (A concave quadrilateral is a quadrilateral with an angle strictly larger than 180°. A lattice point is a point with both coordinates integers.)