HMMT November 2024

November 09, 2024

Theme Round

1. Compute the number of ways to fill each of the 12 empty cells in the grid below with one of T, A, L, or C such that each of the four rows, columns, and bolded 2×2 square regions contains each letter exactly once.

T			
	A		
		L	
			C

Proposed by: Arul Kolla, Katelyn Zhou

Answer: $\boxed{2}$

Solution:

T	?		
	A		
		L	
			C

Consider the cell marked with a ?. It may be replaced with either an L or a C, each of which leads to a unique filling of the grid. The answer is $\boxed{2}$, shown below:

T	C	A	L
L	A	C	T
C	T	L	A
A	L	T	C

T	L	C	A
C	A	T	L
A	C	L	T
L	T	A	C

Note that they are symmetric with respect to reflection about the diagonal.

2. Paul is in the desert and has a pile of gypsum crystals. No matter how he divides the pile into two nonempty piles, at least one of the resulting piles has a number of crystals that, when written in base 10, has a sum of digits at least 7. Given that Paul's initial pile has at least two crystals, compute the smallest possible number of crystals in the initial pile.

Proposed by: Albert Wang, Carlos Rodriguez

Answer: $\boxed{49}$

Solution: Denote the digit sum of a positive integer m as $s(m)$.

Let the pile have n gypsum crystals, so that n can be written as $10^{i_1} + 10^{i_2} + \cdots + 10^{i_{s(n)}}$.

First, $s(n)$ cannot be 1 (i.e. n cannot be a power of 10), since otherwise we could split the gypsum pile into two equal parts each with digit sum 5. We also can't have $1 < s(n) \leq 12$, as otherwise $n = 10^{i_1} + 10^{i_2} + \cdots + 10^{i_{s(n)}}$ can be split into two groups of at most six terms each (which have digit sum at most 6). Hence $s(n)$ is at least 13, forcing n to be at least 49.

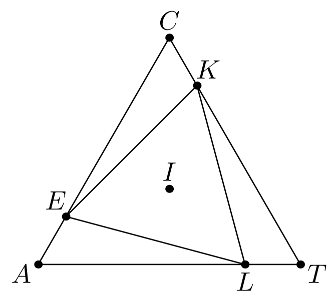
Now we show that $n = 49$ indeed works. Note that for any splitting into two piles of size a, b with $a+b = 49$, we do not carry any digits when adding a and b. Hence we must have $s(a)+s(b) = s(n) = 13$. This forces at least one of $s(a)$, $s(b)$ to be at least 7, so $n = \boxed{49}$ indeed works.

3. Points K, A, L, C, I, T, E are such that triangles CAT and ELK are equilateral, share a center I, and points E, L, K lie on sides $\overline{CA}, \overline{AT}, \overline{TC}$ respectively. If the area of triangle CAT is double the area of triangle ELK and $CI = 2$, compute the minimum possible value of CK.

   Proposed by: Albert Wang, Isaac Zhu

   Answer: $\boxed{\sqrt{3}-1}$

   Solution 1:



First, compute that triangle CAT has side length $2\sqrt{3}$ and area $3\sqrt{3}$. Since triangle ELK has half the area of CAT, the triangles CEK, AEL, TLK must each have $\frac{1}{6}$ the area of CAT. Now by sine area formula, we have

$$\frac{1}{6}\cdot 3\sqrt{3} = [CEK] = \frac{1}{2}CE\cdot CK\cdot \sin(60^\circ) \implies CE\cdot CK = 2$$

Also, note that $CE + CK = CA = 2\sqrt{3}$ by symmetry. So we have the equation

$$CE\cdot(2\sqrt{3} - CE) = 2 \implies CE = \sqrt{3}\pm 1.$$

The answer is $\boxed{\sqrt{3}-1}$.

Solution 2: Let M be the midpoint of CT. Note that KIM is a right triangle. Using the given ratio of areas, we know that $CI^2 = 2\cdot KI^2$, implying that $KI = \sqrt{2}$. Pythagorean theorem on right triangle KIM tells us that $KM = 1$. Since CIM is a $30^\circ - 60^\circ - 90^\circ$ triangle, we know that $CM = \sqrt{3}$. The smallest possible value of $CK = \sqrt{3}\pm 1$ is then $\boxed{\sqrt{3}-1}$.

4. Compute

$$\sum_{i=1}^{4}\sum_{t=1}^{4}\sum_{e=1}^{4}\left\lfloor\frac{ite}{5}\right\rfloor.$$

   Proposed by: Derek Liu

   Answer: $\boxed{168}$

   Solution: Note that 5 never divides ite because 5 is prime. Thus, we can pair up the terms: since $ite \equiv -(5-i)te \bmod 5$,

$$\left\lfloor\frac{ite}{5}\right\rfloor + \left\lfloor\frac{(5-i)te}{5}\right\rfloor = \frac{ite}{5} + \frac{(5-i)te}{5} - 1.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{4}\sum_{t=1}^{4}\sum_{e=1}^{4}\left\lfloor\frac{ite}{5}\right\rfloor &= \left(\sum_{i=1}^{4}\sum_{t=1}^{4}\sum_{e=1}^{4}\frac{ite}{5}\right)-\frac{4^3}{2} \\
&= \frac{1}{5}\left(\sum_{i=1}^{4} i\right)^3-\frac{64}{2} \\
&= \frac{1}{5}(10)^3-32 \\
&= 2\cdot 100-32 \\
&= \boxed{168}.
\end{aligned}$$

5. Alf, the alien from the 1980s TV show, has a big appetite for the mineral apatite. However, he's currently on a diet, so for each integer $k \geq 1$, he can eat exactly k pieces of apatite on day k. Additionally, if he eats apatite on day k, he cannot eat on any of days $k+1$, $k+2$, $\ldots$, $2k-1$. Compute the maximum total number of pieces of apatite Alf could eat over days $1, 2, \ldots, 99, 100$.

   Proposed by: Marin Hristov Hristov

   Answer: $\boxed{197}$

   Solution 1: If Alf doesn't eat on day 100, he could have changed his diet so that he eats on all the same days except the last day is changed to 100. This attains strictly more apatite, and therefore an optimal diet must have Alf eating on day 100.

   Knowing this, Alf must not have eaten anything on days $51, \ldots, 99$. Now, by the same logic, Alf must have eaten on day 50. Continuing the logic recursively gives that Alf must have eaten on days

$$100, 50, 25, 12, 6, 3, 1.$$

   The sum of these numbers is $\boxed{197}$.

   Solution 2: The answer is $\boxed{197}$, achieved by Alf eating on days 1, 3, 6, 12, 25, 50, 100. We show that we could not do better.

   Let $a_1 > a_2 > \cdots > a_k$ be the days that Alf ate apatite. By problem's condition, $a_i \geq 2a_{i+1}$ for all i. Thus, beginning with $a_1 \leq 100$, we deduce that

   - $a_2 \leq \left\lfloor \frac{a_1}{2} \right\rfloor = 50$,
   - $a_3 \leq \left\lfloor \frac{a_2}{2} \right\rfloor = 25$,
   - $a_4 \leq \left\lfloor \frac{a_3}{2} \right\rfloor = 12$,
   - $a_5 \leq \left\lfloor \frac{a_4}{2} \right\rfloor = 6$,
   - $a_6 \leq \left\lfloor \frac{a_5}{2} \right\rfloor = 3$,
   - $a_7 \leq \left\lfloor \frac{a_6}{2} \right\rfloor = 1$,

   and hence $k \leq 7$. Thus, $a_1 + \cdots + a_k \leq 100 + 50 + 25 + 12 + 6 + 3 + 1 = 197$.

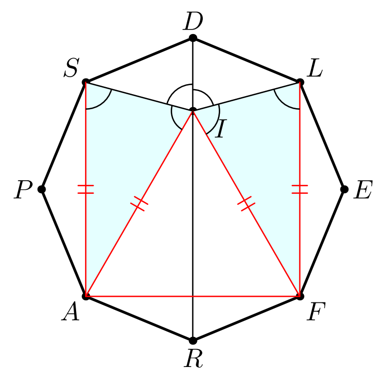
   Remark. The alternative answer 198193, which can be obtained if the contestant read $99, 100$ as the five-digit number 99100, was also accepted.

6. Let $FELDSPAR$ be a regular octagon, and let I be a point in its interior such that $\angle FIL = \angle LID = \angle DIS = \angle SIA$. Compute $\angle IAR$ in degrees.

Proposed by: Derek Liu

Answer: $\boxed{82.5^\circ = \left(\frac{165}{2}\right)^\circ}$

Solution:



Observe that I lies on line DR due to symmetry, so $ID \| FL$. Thus $\angle FLI = \angle LID = \angle FIL$, implying that triangle FIL is isosceles with $FI = FL$. Similarly, $AI = AS$. Since $FLSA$ is a square, $FI = FL = AS = AI = FA$. Therefore, FIA is equilateral, so $\angle AIR = \frac{1}{2}\angle FIA = 30^\circ$ and $\angle IAR = 180^\circ - 30^\circ - \frac{1}{2} \cdot 135^\circ = \boxed{82.5^\circ}$.

7. Jasper and Rose are playing a game. Twenty-six 32-ounce jugs are in a line, labeled Quart A through Quart Z from left to right. All twenty-six jugs are initially full. Jasper and Rose take turns making one of the following two moves:

- Remove a positive integer number of ounces from the leftmost nonempty jug, possibly emptying it
- Remove an *equal* positive integer number of ounces from the two leftmost nonempty jugs, possibly emptying one or both of them. (Attempting to remove more ounces from a jug than it currently contains is not allowed.)

Jasper plays first. A player's score is the number of ounces they take from Quart Z. If both players play to maximize their score, compute the maximum score that Jasper can guarantee.

Proposed by: Derek Liu

Answer: $\boxed{31}$

Solution: Notice that after any sequence of moves, the leftmost nonempty jug has at most as many ounces as the second leftmost nonempty jug.

Alice's strategy for 31 is as follows: as long as at least two jugs are nonempty, remove all but one ounce from the first jug. This will ensure Bob only ever gets to take one ounce from one or two jugs at a time, so on Alice's turn the first jug will always have more than one ounce. Eventually, Bob will be forced to take one ounce from jug Y and at most one from jug Z, leaving at least 31 for Alice.

It remains to show 32 is not attainable. If all but jugs Y and Z are empty on Bob's turn, then Bob can guarantee at least one ounce. Thus, the only way Alice could guarantee all 32 ounces in Quart Z is by making Bob empty jug X without touching jugs Y or Z, so that Alice can then take all 32 from Z in one move. This can only happen if Bob is forced to empty jugs W and X simultaneously; otherwise, Bob would have the option to empty part of Y as well. But then Bob could just empty W only, contradiction.

Thus $\boxed{31}$ is maximal.

8. For all positive integers r and s, let $\text{Top}(r, s)$ denote the top number (i.e., numerator) when $\frac{r}{s}$ is written in simplified form. For instance, $\text{Top}(20, 24) = 5$. Compute the number of ordered pairs of positive integers (a, z) such that $200 \le a \le 300$ and $\text{Top}(a, z) = \text{Top}(z, a - 1)$.

   Proposed by: David Dong, Derek Liu, Henrick Rabinovitz, Jackson Dryg, Krishna Pothapragada, Linus Yifeng Tang, Pitchayut Saengrungkongka, Srinivas Arun

   Answer: $\boxed{38}$

   Solution: In general, $\text{Top}(r, s) = \frac{r}{\gcd(r,s)}$. We characterize all possible (a, z) as follows.

   Claim 1. For any positive integers a and z, we have $\text{Top}(a, z) = \text{Top}(z, a - 1)$ if and only if there exists positive integers d and e such that $e \mid d^2 - 1$, $a = d^2$, and $z = de$.

   Proof. ($\Leftarrow$) From $e \mid d^2 - 1$, we deduce that $\gcd(d, e) = 1$. Thus, $\text{Top}(a, z) = \frac{d^2}{\gcd(d^2,de)} = \frac{d^2}{d} = d$. We also have $\gcd(z, a - 1) = \gcd(de, d^2 - 1) = e$, so $\text{Top}(z, a - 1) = \frac{de}{e} = d$ as well.

   ($\Rightarrow$) Let $d = \gcd(a, z)$ and $e = \gcd(z, a - 1)$. We have that $\gcd(d, e) = 1$ because it divides both a and $a - 1$. The equation implies that $\frac{a}{d} = \frac{z}{e}$, or $\frac{a}{z} = \frac{d}{e}$. The left side has simplified form $\frac{a/d}{z/d}$, and the right side is already simplified. Thus, $a = d^2$ and $z = de$. Finally, $e = \gcd(z, a - 1) \mid a - 1 = d^2 - 1$. $\square$

   The condition that $200 \le a \le 300$ implies $d \in \{15, 16, 17\}$. Once we select d, each divisor e of $d^2 - 1$ yields a solution. Thus, the answer is

   $$\begin{aligned} \tau(15^2 - 1) + \tau(16^2 - 1) + \tau(17^2 - 1) &= \tau(14 \cdot 16) + \tau(15 \cdot 17) + \tau(16 \cdot 18) \\ &= 12 + 8 + 18 = \boxed{38}, \end{aligned}$$

   where $\tau(n)$ is the number of divisors of n.

9. Compute the number of ways to color each cell of an 18×18 square grid either ruby or sapphire such that each contiguous 3×3 subgrid has exactly 1 ruby cell.

   Proposed by: Jacob Paltrowitz

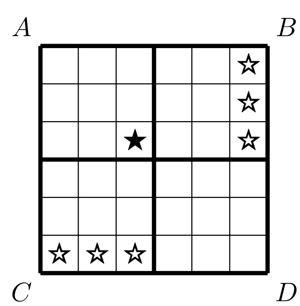
   Answer: $\boxed{4365}$

   Solution: Subdivide the grid into 36 subgrids of size 3×3. Each contains exactly one ruby cell.

   Consider four adjacent subgrids

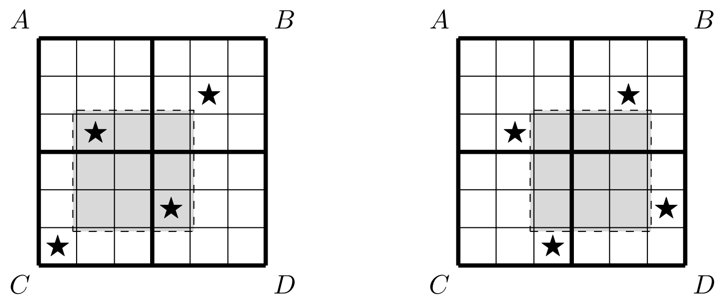
   $$\begin{array}{c|c} A & B \\ \hline C & D \end{array}$$

   and the relative positions of the ruby cells within their respective 3×3 subgrids. Between A and C, we can check that the ruby cells differ by only a horizontal shift. Likewise, between A and B, the ruby cells differ by only a vertical shift.



The black star indicates the ruby in A, and the white stars indicate potential locations for rubies in B, C.

It can further be checked that the shift between A, B is the same as between C, D, and likewise between A, C and B, D. Also, it cannot be the case that both a horizontal and a vertical shift is present, as some 3×3 subgrid will have an invalid number of ruby cells:



Now, in the original grid

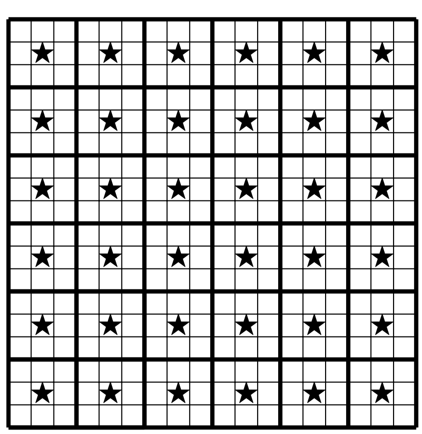
A	B	$\cdots$
C	D	$\cdots$
$\vdots$	$\vdots$	$\ddots$

we conclude that between each of the 5 pairs of adjacent columns of subgrids, there is some constant vertical shift, and that between each of the 5 pairs of adjacent rows of subgrids, there is some constant horizontal shift. However, if there exists both a pair of columns (x_1, x_2) with a nontrivial vertical shift and also a pair of rows (y_1, y_2) with a nontrivial horizontal shift, the four subgrids at their intersection $(x_1, x_2) \times (y_1, y_2)$ would violate our reasoning above. Hence, there are either only vertical shifts, only horizontal shifts, or neither. Equivalently, all rubies are contained within either six equally spaced rows or six equally spaced columns. It can be checked that all such configurations are valid.

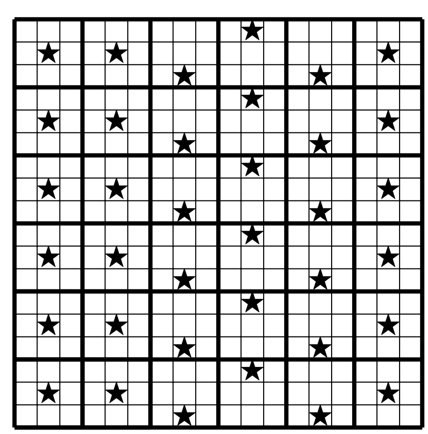
To count the number of ways to place all rubies into six equally spaced rows, we have 3 choices for which rows to choose, and 3 choices within each row for where the rubies are located. This gives 3^7. Symmetrically, there are 3^7 ways to place all rubies into six equally spaced columns. To adjust for overcounting, we note that there are 9 possibilities where all rubies are contained in six rows and six columns. The answer is

$$2 \cdot 3^7 - 9 = \boxed{4365}.$$

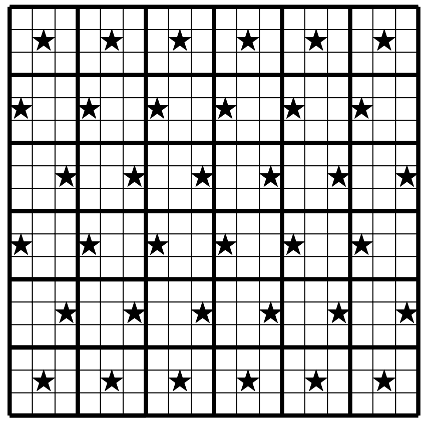
For completeness, examples of each of the three cases (no shifts, horizontal shifts only, vertical shifts only) is shown below.



No shifts

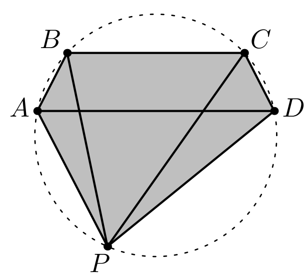


Vertical shifts only



Horizontal shifts only

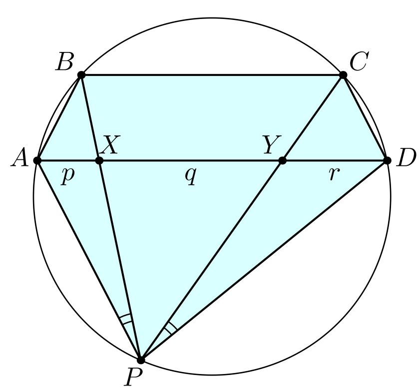
10. Isabella the geologist discovers a diamond deep underground via an X-ray machine. The diamond has the shape of a convex cyclic pentagon $PABCD$ with $AD \parallel BC$. Soon after the discovery, her X-ray breaks, and she only recovers partial information about its dimensions. She knows that $AD = 70$, $BC = 55$, $PA : PD = 3 : 4$, and $PB : PC = 5 : 6$. Compute PB.



Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{25\sqrt{6}.}$

Solution 1:



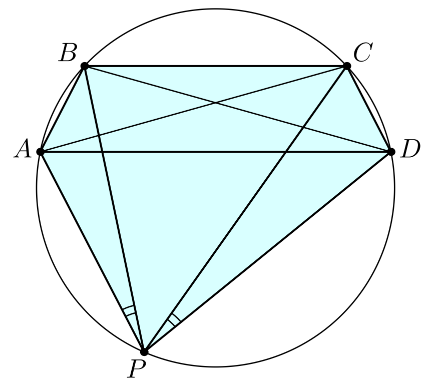
Let $X = PB \cap AD$ and $Y = PC \cap AD$. Let $AX = p$, $XY = q$, and $YD = r$. From $AB \parallel CD$, we get that $AB = CD$, and so $\angle APX = \angle DPY$. Thus, we may apply Steiner ratio theorem on $\triangle PAD$ and $\triangle PXY$ to get that

$$\frac{p(p+q)}{r(q+r)} = \frac{3^2}{4^2}, \quad \frac{p(q+r)}{r(p+q)} = \frac{5^2}{6^2}.$$

Multiplying these two equations gives $p : r = 5 : 8$, and using each individual equations gives $p : q : r = 5 : 22 : 8$. Thus, $p = 10$, $q = 44$, and $r = 16$.

Now, from $XY \parallel BC$, we have $PX : XB = 4 : 1$, so set $PX = 4t$ and $XB = t$. However, $4t^2 = PY \cdot YC = 10 \cdot 60 = 600$. Solving this gives $t = \sqrt{150} = 5\sqrt{6}$, hence $PB = 5t = \boxed{25\sqrt{6}}$.

Solution 2:



Let $AB = CD = a$, $AC = BD = b$, $\frac{AP}{3} = \frac{DP}{4} = x$, and $\frac{BP}{5} = \frac{CP}{6} = y$. Applying Ptolemy's theorem for the quadrilaterals $ABCP$, $BCDP$, and $ABCD$ yields:

$$b \cdot 5y = 55 \cdot 3x + a \cdot 6y \tag{1}$$
$$b \cdot 6y = 55 \cdot 4x + a \cdot 5y \tag{2}$$
$$b^2 = 55 \cdot 70 + a^2 \tag{3}$$

Equating the left-hand sides of (1) and (2) leads to

$$6 \cdot (165x + 6ay) = 5 \cdot (220x + 5ay) \implies 110x = 11ay \implies 10x = ay. \tag{4}$$

Substituting $220x = 22ay$ into (2) implies $27ay = 6by$, or $b = \frac{9}{2}a$. Plugging this into (3), we find $a^2 = 200$, so $a = 10\sqrt{2}$, and therefore $b = 45\sqrt{2}$. Furthermore, $x = y\sqrt{2}$ after replacing a with $10\sqrt{2}$ in (4). We now apply Law of Cosines for $\triangle ABC$ and $\triangle APC$:

$$55^2 + (10\sqrt{2})^2 - 2 \cdot (10\sqrt{2}) \cdot 55 \cos\theta = (45\sqrt{2})^2 \tag{5}$$
$$(3\sqrt{2}y)^2 + (6y)^2 + 2 \cdot (3\sqrt{2}y) \cdot (6y) \cos\theta = (45\sqrt{2})^2 \tag{6}$$

where $\theta = \angle ABC$. Solving (5) yields

$$\cos\theta = \frac{55^2 + (10\sqrt{2})^2 - (45\sqrt{2})^2}{2 \cdot (10\sqrt{2}) \cdot 55} = -\frac{3\sqrt{2}}{8}.$$

Plugging this into (6), we can compute:

$$y^2 = \frac{(45\sqrt{2})^2}{(3\sqrt{2})^2 + 6^2 - 27} = \frac{4050}{27} = 150.$$

Therefore, $BP = 5y = 5\sqrt{150} = \boxed{25\sqrt{6}}$.