HMMO 2020

November 14, 2020

Team Round

1. **[20]** For how many positive integers $n \leq 1000$ does the equation in real numbers

$$x^{\lfloor x \rfloor} = n$$

have a positive solution for x? (For a real number x, $\lfloor x \rfloor$ denotes the largest integer that is not greater than x.)

Proposed by: John Michael Wu

Answer: $\boxed{412}$

Solution: If $\lfloor x \rfloor = 0$, then $x^{\lfloor x \rfloor} = 1 = 1^{\lfloor 1 \rfloor}$, so we can safely ignore this case, as it does not introduce new solutions.

If $\lfloor x \rfloor = k$ for some $k > 0$, $x \mapsto x^{\lfloor x \rfloor} = x^k$ is a continuous and increasing function on the interval $[k, k+1)$. Therefore, the $x^{\lfloor x \rfloor}$ can take on any value in $[k^k, (k+1)^k)$ when $\lfloor x \rfloor = k$. Because $5^4 < 1000 < 5^5$, it suffices to count the number of integers in the interval $[k^k, (k+1)^k)$ for $1 \leq k \leq 4$.

Thus, the number of valid n is simply

$$\sum_{k=1}^{4} (k+1)^k - k^k = (2-1) + (9-4) + (64-27) + (625-256) = 412.$$

2. **[25]** How many ways are there to arrange the numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$ in a circle so that every two adjacent elements are relatively prime? Consider rotations and reflections of the same arrangement to be indistinguishable.

Proposed by: Daniel Zhu

Answer: $\boxed{36}$

Solution: Note that 6 can only be adjacent to 1, 5, and 7, so there are $\binom{3}{2} = 3$ ways to pick its neighbors. Since each of 1, 5, and 7 is relatively prime to every number in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ but itself (and hence can have arbitrary neighbors), without loss of generality suppose we have picked 1 and 5 as neighbors of 6. Observe that fixing the positions of 1, 5, and 6 eliminates the indistinguishability of rotations and reflections.

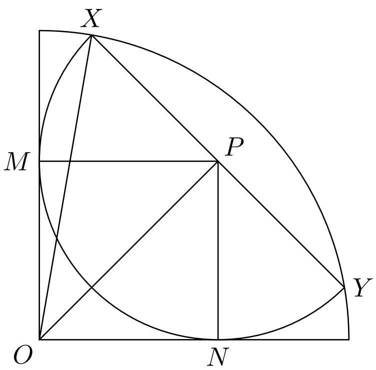
Now, we have to consecutively arrange $\{2, 3, 4, 7, 8\}$ so that no two of 2, 4, and 8 are adjacent. There are $3! \cdot 2! = 12$ ways of doing so, so the final answer is $3 \cdot 12 = 36$.

3. **[30]** Let A be the area of the largest semicircle that can be inscribed in a quarter-circle of radius 1. Compute $\frac{120A}{\pi}$.

Proposed by: Akash Das

Answer: $\boxed{20}$

Solution:



The optimal configuration is when the two ends X and Y of the semicircle lie on the arc of the quarter circle. Let O and P be the centers of the quarter circle and semicircle, respectively. Also, let M and N be the points where the semicircle is tangent to the radii of the quartercircle.

Let r be the radius of the semicircle. Since $PM = PN$, $PMON$ is a square and $OP = \sqrt{2}r$. By the Pythagorean theorem on triangle OPX, $1 = 2r^2 + r^2$, so $r = 1/\sqrt{3}$. The area of the semicircle is therefore $\frac{\pi}{2}\frac{1}{3} = \frac{\pi}{6}$.

4. **[35]** Marisa has two identical cubical dice labeled with the numbers $\{1, 2, 3, 4, 5, 6\}$. However, the two dice are not fair, meaning that they can land on each face with different probability. Marisa rolls the two dice and calculates their sum. Given that the sum is 2 with probability 0.04, and 12 with probability 0.01, the maximum possible probability of the sum being 7 is p. Compute $\lfloor 100p \rfloor$.

   Proposed by: Shengtong Zhang

   Answer: $\boxed{28}$

   Solution: Let p_i be the probability that the dice lands on the number i. The problem gives that $p_1^2 = 0.04, p_6^2 = 0.01$, so we have

$$p_1 = 0.2, \quad p_6 = 0.1, \quad p_2 + p_3 + p_4 + p_5 = 0.7.$$

   We are asked to maximize

$$2(p_1p_6 + p_2p_5 + p_3p_4) = 2(0.02 + p_2p_5 + p_3p_4).$$

   Let $x = p_2 + p_5$ and $y = p_3 + p_4$. Then by AM-GM, $p_2p_5 \le \frac{x^2}{4}$, $p_3p_4 \le \frac{y^2}{4}$. Also,

$$\frac{x^2 + y^2}{4} \le \frac{x^2 + 2xy + y^2}{4} = \frac{(x + y)^2}{4} = \frac{0.7^2}{4} = 0.1225.$$

   Hence,

$$2(p_1p_6 + p_2p_5 + p_3p_4) \le 2(0.02 + 0.1225) = 0.285,$$

   where equality holds if $p_2 = p_5 = 0.35, p_3 = p_4 = 0$.

   Thus, we conclude that $p = 0.285$ and $\lfloor 100p \rfloor = 28$.

5. **[40]** For each positive integer n, let a_n be the smallest nonnegative integer such that there is only one positive integer at most n that is relatively prime to all of $n, n + 1, \ldots, n + a_n$. If $n < 100$, compute the largest possible value of $n - a_n$.

   Proposed by: Hahn Lheem

   Answer: $\boxed{16}$

Solution: Note that 1 is relatively prime to all positive integers. Therefore, the definition of a_n can equivalently be stated as: "a_n is the smallest nonnegative integer such that for all integers x, $2 \leq x \leq n$, x shares a prime factor with at least one of $n, n+1, \ldots n+a_n$."

The condition is equivalent the statement that the integers from n to $n+a_n$ must include multiples of all primes less than n. Therefore, if p is the largest prime satisfying $p < n$, then $n + a_n \geq 2p$.

We now claim that $a_n = 2p - n$ works for all $n > 11$. For all primes q at most $a_n + 1$, it is apparent that $n, n+1, \ldots, n+a_n$ indeed contains a multiple of q. For primes $a_n + 1 < q \leq p$, we then find that $2q \leq n + a_n$. To finish, we claim that $2q \geq n$, which would be implied by $2(a_n + 2) \geq n \iff p \geq 3n/4 - 1$. This is indeed true for all $11 < n < 100$.

We therefore wish to maximize $n - a_n = n - (2p - n) = 2(n - p)$. Therefore, the answer is twice the largest difference between two primes less than 100. This difference is 8 (from 89 to 97), so the answer is 16. Since this is greater than 11, we have not lost anything by ignoring the smaller cases.

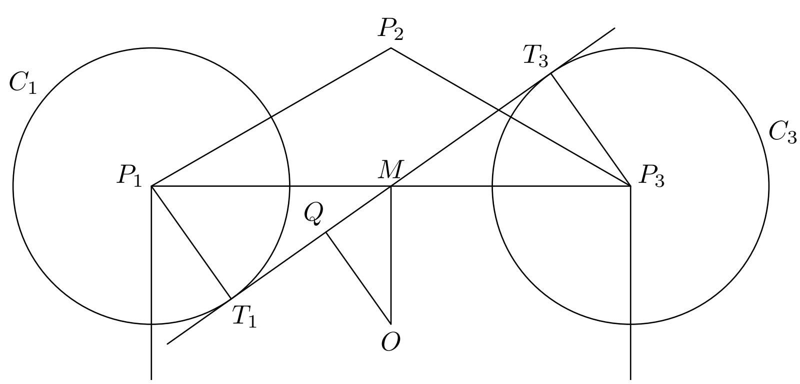
6. **[40]** Regular hexagon $P_1P_2P_3P_4P_5P_6$ has side length 2. For $1 \leq i \leq 6$, let C_i be a unit circle centered at P_i and ℓ_i be one of the internal common tangents of C_i and C_{i+2}, where $C_7 = C_1$ and $C_8 = C_2$. Assume that the lines $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ bound a regular hexagon. The area of this hexagon can be expressed as $\sqrt{\frac{a}{b}}$, where a and b are relatively prime positive integers. Compute $100a + b$.

Proposed by: Daniel Zhu

Answer: $\boxed{1603}$

Solution: The only way for the lines ℓ_i to bound a regular hexagon H is if they are rotationally symmetric around the center O of the original hexagon. (A quick way to see this is to note that the angle between the two internal common tangents of C_i and C_{i+2} cannot be a multiple of $60°$.) Thus all we need to do is to compute h, the distance from the center O to the sides of H, because then we can compute the side length of H as $\frac{2}{\sqrt{3}}h$ and thus its area as

$$6\frac{\sqrt{3}}{4}\left(\frac{2h}{\sqrt{3}}\right)^2 = 2\sqrt{3}h^2.$$



Without loss of generality, let's only consider ℓ_1. Let M be the midpoint of P_1P_3 and let T_1 and T_3 be the tangency points between ℓ_1 and C_1 and C_3, respectively. Without loss of generality, assume T_1 is closer to O than T_3. Finally, let Q be the projection of O onto ℓ_1, so that $h = OQ$.

Now, note that $\angle OMQ = 90° - \angle T_1MP_1 = \angle MP_1T_1$, so $\triangle OMQ \sim \triangle MP_1T_1$. Therefore, since $OM = OP_2/2 = 1$, we find

$$h = OQ = \frac{OQ}{OM} = \frac{T_1M}{MP_1} = \frac{\sqrt{MP_1^2 - P_1T_1^2}}{MP_1} = \sqrt{\frac{2}{3}},$$

since $MP_1 = P_1P_3/2 = \sqrt{3}$. Thus the final area is $\frac{2}{3}2\sqrt{3} = \sqrt{16/3}$.

7. **[45]** Roger the ant is traveling on a coordinate plane, starting at $(0,0)$. Every second, he moves from one lattice point to a different lattice point at distance 1, chosen with equal probability. He will continue to move until he reaches some point P for which he could have reached P more quickly had he taken a different route. For example, if he goes from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(1,2)$ to $(0,2)$, he stops at $(0,2)$ because he could have gone from $(0,0)$ to $(0,1)$ to $(0,2)$ in only 2 seconds. The expected number of steps Roger takes before he stops can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a+b$.

   Proposed by: Carl Schildkraut

   Answer: $\boxed{1103}$

   Solution: Roger is guaranteed to be able to take at least one step. Suppose he takes that step in a direction u. Let e_1 be the expectation of the number of additional steps Roger will be able to take after that first move. Notice that Roger is again guaranteed to be able to make a move, and that three types of steps are possible:

   (1) With probability $\frac{1}{4}$, Roger takes a step in the direction $-u$ and his path ends.

   (2) With probability $\frac{1}{4}$, Roger again takes a step in the direction u, after which he is expected to take another e_1 steps.

   (3) With probability $\frac{1}{2}$, Roger takes a step in a direction w perpendicular to u, after which he is expected to take some other number e_2 of additional steps.

   If Roger makes a move of type (3), he is again guaranteed to be able to take a step. Here are the options:

   (1) With probability $\frac{1}{2}$, Roger takes a step in one of the directions $-u$ and $-w$ and his path ends.

   (2) With probability $\frac{1}{2}$, Roger takes a step in one of the directions u and w, after which he is expected to take an additional e_2 steps.

   Using these rules, we can set up two simple linear equations to solve the problem.

$$e_2 = \frac{1}{2}e_2 + 1 \implies e_2 = 2$$
$$e_1 = \frac{1}{2}e_2 + \frac{1}{4}e_1 + 1 = \frac{1}{4}e_1 + 2 \implies e_1 = \frac{8}{3}$$

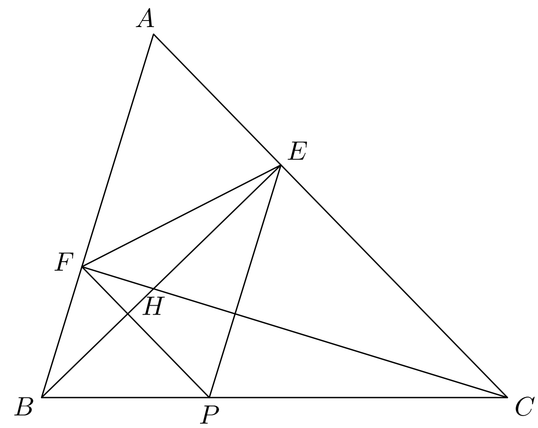
   Since Roger takes one step before his expectation is e_1, the answer is $\frac{11}{3}$.

8. **[50]** Altitudes BE and CF of acute triangle ABC intersect at H. Suppose that the altitudes of triangle EHF concur on line BC. If $AB=3$ and $AC=4$, then $BC^2=\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a+b$.

   Proposed by: Jeffrey Lu

   Answer: $\boxed{33725}$

   Solution:



Let P be the orthocenter of $\triangle EHF$. Then $EH \perp FP$ and $EH \perp AC$, so FP is parallel to AC. Similarly, EP is parallel to AB. Using similar triangles gives

$$1 = \frac{BP}{BC} + \frac{CP}{BC} = \frac{AE}{AC} + \frac{AF}{AB} = \frac{AB\cos A}{AC} + \frac{AC\cos A}{AB},$$

so $\cos A = \frac{12}{25}$. Then by the law of cosines, $BC^2 = 3^2 + 4^2 - 2(3)(4)(\frac{12}{25}) = \frac{337}{25}$.

9. **[55]** Alice and Bob take turns removing balls from a bag containing 10 black balls and 10 white balls, with Alice going first. Alice always removes a black ball if there is one, while Bob removes one of the remaining balls uniformly at random. Once all balls have been removed, the expected number of black balls which Bob has can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

Proposed by: Benjamin Qi

Answer: $\boxed{4519}$

Solution: Suppose a is the number of black balls and b is the number of white balls, and let $E_{a,b}$ denote the expected number of black balls Bob has once all the balls are removed with Alice going first. Then we want to find $E_{10,10}$. It is evident that if $E_{0,b} = 0$. Also, since Bob chooses a black ball with probability $\frac{a-1}{a+b-1}$, if $a > 0$ we have

$$\begin{aligned} E_{a,b} &= \frac{a-1}{a+b-1}\left(E_{a-2,b} + 1\right) + \frac{b}{a+b-1}E_{a-1,b-1} \\ &= \frac{(a-1)(E_{a-2,b}+1) + bE_{a-1,b-1}}{a+b-1} \end{aligned}$$

We claim that $E_{a,b} = \frac{a(a-1)}{2(a+b-1)}$, which will yield an answer of $\frac{45}{19}$. To prove this, we use induction. In the base case of $a = 0$ we find $\frac{a(a-1)}{2(a+b-1)} = 0$, as desired. Also, for $a > 0$ we have that by the inductive hypothesis

$$\begin{aligned} E_{a,b} &= \frac{(a-1)((a-2)(a-3) + 2(a+b-3)) + b(a-1)(a-2)}{2(a+b-1)(a+b-3)} \\ &= \frac{(a-1)(a-2)(a+b-3) + 2(a-1)(a+b-3)}{2(a+b-1)(a+b-3)} \\ &= \frac{a(a-1)}{2(a+b-1)}, \end{aligned}$$

as desired.

10. **[60]** Let x and y be non-negative real numbers that sum to 1. Compute the number of ordered pairs (a, b) with $a, b \in \{0, 1, 2, 3, 4\}$ such that the expression $x^a y^b + y^a x^b$ has maximum value 2^{1-a-b}.

Proposed by: Shengtong Zhang

Answer: $\boxed{17}$

Solution: Let $f(x, y) = x^a y^b + y^a x^b$. Observe that 2^{1-a-b} is merely the value of $f(\frac{1}{2}, \frac{1}{2})$, so this value is always achievable.

We claim (call this result $(\divideontimes)$) that if (a, b) satisfies the condition, so does $(a + 1, b + 1)$. To see this, observe that if $f(x, y) \le 2^{1-a-b}$, then multiplying by the inequality $xy \le \frac{1}{4}$ yields $x^{a+1}y^{b+1} + y^{a+1}x^{b+1} \le 2^{-1-a-b}$, as desired.

For the rest of the solution, without loss of generality we consider the $a \ge b$ case. If $a = b = 0$, then $f(x, y) = 2$, so $(0, 0)$ works. If $a = 1$ and $b = 0$, then $f(x, y) = x + y = 1$, so $(1, 0)$ works. For $a \ge 2$, $(a, 0)$ fails since $f(1, 0) = 1 > 2^{1-a}$.

If $a = 3$ and $b = 1$, $f(x, y) = xy(x^2+y^2) = xy(1-2xy)$, which is maximized at $xy = \frac{1}{4} \iff x = y = \frac{1}{2}$, so $(3, 1)$ works. However, if $a = 4$ and $b = 1$, $f(x, y) = xy(x^3 + y^3) = xy((x + y)^3 - 3xy(x + y)) = xy(1 - 3xy)$, which is maximized at $xy = \frac{1}{6}$. Thus $(4, 1)$ does not work.

From these results and $(\divideontimes)$, we are able to deduce all the pairs that do work (↙ represents those pairs that work by $(\divideontimes)$):

4	✗	✗	↙	↙	↙
3	✗	✓	↙	↙	↙
2	✗	↙	↙	↙	↙
1	✓	↙	↙	✓	✗
0	✓	✓	✗	✗	✗
	0	1	2	3	4