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There are nine sets of four problems each. The point values of the problems in each set, in order, are 3, 4, 5, 7, 9, 12, 15, 20, 25. Good luck!

Also, a heads up: the problems in the last set (problems 33 to 36) of this guts round will **NOT** give possible partial credits as per tradition, but the set is nevertheless different from a normal guts set. **In light of this change, you may want to reserve more time for the last set than usual.**

1. [**3**] Find the sum of all real solutions to $x^2 + \cos x = 2019$.

2. [**3**] There are 100 people in a room with ages $1, 2, \ldots, 100$. A pair of people is called *cute* if each of them is at least seven years older than half the age of the other person in the pair. At most how many pairwise disjoint cute pairs can be formed in this room?

3. [**3**] Let $S(x)$ denote the sum of the digits of a positive integer x. Find the maximum possible value of $S(x + 2019) - S(x)$.

4. [**3**] Tessa has a figure created by adding a semicircle of radius 1 on each side of an equilateral triangle with side length 2, with semicircles oriented outwards. She then marks two points on the boundary of the figure. What is the greatest possible distance between the two points?

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5. [**4**] Call a positive integer n *weird* if n does not divide $(n-2)!$. Determine the number of weird numbers between 2 and 100 inclusive.

6. [**4**] The pairwise products ab, bc, cd, and da of positive integers a, b, c, and d are $64, 88, 120$, and 165 in some order. Find $a + b + c + d$.

7. [**4**] For any real number α, define

$$\operatorname{sign}(\alpha) = \begin{cases} +1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases}$$

How many triples $(x, y, z) \in \mathbb{R}^3$ satisfy the following system of equations

$$\begin{aligned} x &= 2018 - 2019 \cdot \operatorname{sign}(y + z), \\ y &= 2018 - 2019 \cdot \operatorname{sign}(z + x), \\ z &= 2018 - 2019 \cdot \operatorname{sign}(x + y)? \end{aligned}$$

8. [**4**] A regular hexagon $PROFIT$ has area 1. Every minute, greedy George places the largest possible equilateral triangle that does not overlap with other already-placed triangles in the hexagon, with ties broken arbitrarily. How many triangles would George need to cover at least 90% of the hexagon's area?

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9. [**5**] Define $P = \{\mathrm{S}, \mathrm{T}\}$ and let $\mathcal{P}$ be the set of all proper subsets of P. (A *proper subset* is a subset that is not the set itself.) How many ordered pairs $(\mathcal{S}, \mathcal{T})$ of proper subsets of $\mathcal{P}$ are there such that

   (a) $\mathcal{S}$ is not a proper subset of $\mathcal{T}$ and $\mathcal{T}$ is not a proper subset of $\mathcal{S}$; and

   (b) for any sets $S \in \mathcal{S}$ and $T \in \mathcal{T}$, S is not a proper subset of T and T is not a proper subset of S?

10. [**5**] Let

$$A = (1 + 2\sqrt{2} + 3\sqrt{3} + 6\sqrt{6})(2 + 6\sqrt{2} + \sqrt{3} + 3\sqrt{6})(3 + \sqrt{2} + 6\sqrt{3} + 2\sqrt{6})(6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6}),$$

$$B = (1 + 3\sqrt{2} + 2\sqrt{3} + 6\sqrt{6})(2 + \sqrt{2} + 6\sqrt{3} + 3\sqrt{6})(3 + 6\sqrt{2} + \sqrt{3} + 2\sqrt{6})(6 + 2\sqrt{2} + 3\sqrt{3} + \sqrt{6}).$$

Compute the value of A/B.

11. [**5**] In the Year 0 of Cambridge there is one squirrel and one rabbit. Both animals multiply in numbers quickly. In particular, if there are m squirrels and n rabbits in Year k, then there will be $2m + 2019$ squirrels and $4n - 2$ rabbits in Year $k + 1$. What is the first year in which there will be strictly more rabbits than squirrels?

12. [**5**] Bob is coloring lattice points in the coordinate plane. Find the number of ways Bob can color five points in $\{(x, y) \mid 1 \leq x, y \leq 5\}$ blue such that the distance between any two blue points is *not* an integer.

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13. [**7**] Reimu has 2019 coins $C_0, C_1, \ldots, C_{2018}$, one of which is fake, though they look identical to each other (so each of them is equally likely to be fake). She has a machine that takes any two coins and picks one that is not fake. If both coins are not fake, the machine picks one uniformly at random. For each $i = 1, 2, \ldots, 1009$, she puts C_0 and C_i into the machine once, and machine picks C_i. What is the probability that C_0 is fake?

14. [**7**] Let ABC be a triangle where $AB = 9, BC = 10, CA = 17$. Let Ω be its circumcircle, and let A_1, B_1, C_1 be the diametrically opposite points from A, B, C, respectively, on Ω. Find the area of the convex hexagon with the vertices A, B, C, A_1, B_1, C_1.

15. [**7**] Five people are at a party. Each pair of them are *friends*, *enemies*, or *frenemies* (which is equivalent to being both *friends* and *enemies*). It is known that given any three people A, B, C:

   - If A and B are friends and B and C are friends, then A and C are friends;
   - If A and B are enemies and B and C are enemies, then A and C are friends;
   - If A and B are friends and B and C are enemies, then A and C are enemies.

   How many possible relationship configurations are there among the five people?

16. [**7**] Let $\mathbb{R}$ be the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers x and y, we have

$$f(x^2) + f(y^2) = f(x + y)^2 - 2xy.$$

Let $S = \sum_{n=-2019}^{2019} f(n)$. Determine the number of possible values of S.

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17. [**9**] Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Let A_1, A_2 be points on side BC, B_1, B_2 be points on side CA, and C_1, C_2 be points on side AB. Suppose that there exists a point P such that PA_1A_2, PB_1B_2, and PC_1C_2 are congruent equilateral triangles. Find the area of convex hexagon $A_1A_2B_1B_2C_1C_2$.

18. [**9**] 2019 points are chosen independently and uniformly at random on the interval $[0, 1]$. Tairitsu picks 1000 of them randomly and colors them black, leaving the remaining ones white. Hikari then computes the sum of the positions of the leftmost white point and the rightmost black point. What is the probability that this sum is at most 1?

19. [**9**] Complex numbers a, b, c form an equilateral triangle with side length 18 in the complex plane. If $|a + b + c| = 36$, find $|bc + ca + ab|$.

20. [**9**] On floor 0 of a weird-looking building, you enter an elevator that only has one button. You press the button twice and end up on floor 1. Thereafter, every time you press the button, you go up by one floor with probability $\frac{X}{Y}$, where X is your current floor, and Y is the total number of times you have pressed the button thus far (not including the current one); otherwise, the elevator does nothing. Between the third and the 100^{th} press inclusive, what is the expected number of pairs of consecutive presses that both take you up a floor?

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21. [**12**] A regular hexagon $ABCDEF$ has side length 1 and center O. Parabolas $P_1, P_2, \ldots, P_6$ are constructed with common focus O and directrices AB, BC, CD, DE, EF, FA respectively. Let χ be the set of all distinct points on the plane that lie on at least two of the six parabolas. Compute

$$\sum_{X \in \chi} |OX|.$$

(Recall that the focus is the point and the directrix is the line such that the parabola is the locus of points that are equidistant from the focus and the directrix.)

22. [**12**] Determine the number of subsets S of $\{1, 2, \ldots, 1000\}$ that satisfy the following conditions:
    - S has 19 elements, and
    - the sum of the elements in any non-empty subset of S is not divisible by 20.

23. [**12**] Find the smallest positive integer n such that

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n \text{ 2's}} > \underbrace{((\cdots((100!)!)!\cdots)!)!}_{100 \text{ factorials}}.$$

24. [**12**] Let S be the set of all positive factors of 6000. What is the probability of a random quadruple $(a, b, c, d) \in S^4$ satisfies

$$\operatorname{lcm}(\gcd(a, b), \gcd(c, d)) = \gcd(\operatorname{lcm}(a, b), \operatorname{lcm}(c, d))?$$

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25. [**15**] A 5 by 5 grid of unit squares is partitioned into 5 pairwise incongruent rectangles with sides lying on the gridlines. Find the maximum possible value of the product of their areas.

26. [**15**] Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let I_A, I_B, I_C be the A, B, C excenters of this triangle, and let O be the circumcenter of the triangle. Let $\gamma_A, \gamma_B, \gamma_C$ be the corresponding excircles and ω be the circumcircle. X is one of the intersections between γ_A and ω. Likewise, Y is an intersection of γ_B and ω, and Z is an intersection of γ_C and ω. Compute

$$\cos \angle OXI_A + \cos \angle OYI_B + \cos \angle OZI_C.$$

27. [**15**] Consider the eighth-sphere $\{(x, y, z) \mid x, y, z \geq 0,\ x^2 + y^2 + z^2 = 1\}$. What is the area of its projection onto the plane $x + y + z = 1$?

28. [**15**] How many positive integers $2 \leq a \leq 101$ have the property that there exists a positive integer N for which the last two digits in the decimal representation of a^{2^n} is the same for all $n \geq N$?

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29. [**20**] Yannick picks a number N randomly from the set of positive integers such that the probability that n is selected is 2^{-n} for each positive integer n. He then puts N identical slips of paper numbered 1 through N into a hat and gives the hat to Annie. Annie does not know the value of N, but she draws one of the slips uniformly at random and discovers that it is the number 2. What is the expected value of N given Annie's information?

30. [**20**] Three points are chosen inside a unit cube uniformly and independently at random. What is the probability that there exists a cube with side length $\frac{1}{2}$ and edges parallel to those of the unit cube that contains all three points?

31. [**20**] Let ABC be a triangle with $AB = 6$, $AC = 7$, $BC = 8$. Let I be the incenter of ABC. Points Z and Y lie on the interior of segments AB and AC respectively such that YZ is tangent to the incircle. Given point P such that

$$\angle ZPC = \angle YPB = 90^\circ,$$

find the length of IP.

32. [**20**] For positive integers a and b such that a is coprime to b, define $\text{ord}_b(a)$ as the least positive integer k such that $b \mid a^k - 1$, and define $\varphi(a)$ to be the number of positive integers less than or equal to a which are coprime to a. Find the least positive integer n such that

$$\text{ord}_n(m) < \frac{\varphi(n)}{10}$$

for all positive integers m coprime to n.

Warning: The next (and final) set contains problems in different formats.

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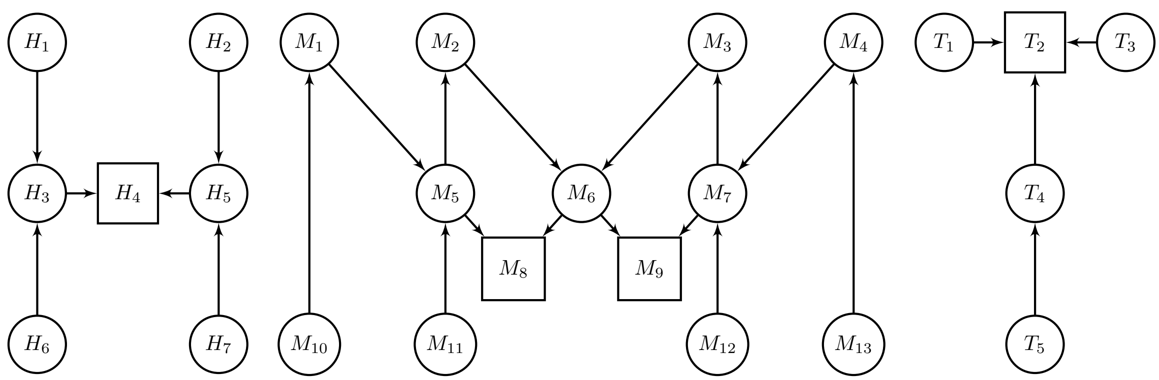
Welcome to **Ultra Relay**! This is an event where teams of twenty-five students race to solve a set of twenty-five problems and ultimately find the answers to the four "anchor" problems. Many of the problems will depend on answers to other problems, hence the team members will need to relay the answers from one person/problem to the next, along the network shown below. (The anchor problems are indicated by a box.) **Only the answers to the four anchor problems will be graded.**



What? You don't have twenty-five people on your team, you say? That's unfortunate. Anyway, here are all the problems. (You may work on them together.) Have fun!

For ease of reference, the label of each problem also denotes the answer to that problem.

H_1. Let $r = H_1$ be the answer to this problem. Given that r is a nonzero real number, what is the value of $r^4 + 4r^3 + 6r^2 + 4r$?

H_2. Given two distinct points A, B and line ℓ that is not perpendicular to AB, what is the maximum possible number of points P on ℓ such that ABP is an isosceles triangle?

H_3. Let $A = H_1, B = H_6 + 1$. A real number x is chosen randomly and uniformly in the interval $[A, B]$. Find the probability that $x^2 > x^3 > x$.

H_4. Let $A = \lceil 1/H_3 \rceil, B = \lceil H_5/2 \rceil$. How many ways are there to partition the set $\{1, 2, \ldots, A+B\}$ into two sets U and V with size A and B respectively such that the probability that a number chosen from U uniformly at random is greater than a number chosen from V uniformly at random is exactly $\frac{1}{2}$?

H_5. Let $A = H_2, B = H_7$. Two circles with radii A and B respectively are given in the plane. If the length of their common external tangent is twice the length of their common internal tangent (where both tangents are considered as segments with endpoints being the points of tangency), find the distance between the two centers.

H_6. How many ways are there to arrange the numbers $21, 22, 33, 35$ in a row such that any two adjacent numbers are relatively prime?

H_7. How many pairs of integers (x, y) are there such that $|x^2 - 2y^2| \leq 1$ and $|3x - 4y| \leq 1$?

M_1. Let $S = M_{10}$. Determine the number of ordered triples (a, b, c) of nonnegative integers such that $a + 2b + 4c = S$.

M_2. Let $S = \lfloor M_5 \rfloor$. Two integers m and n are chosen between 1 and S inclusive uniformly and independently at random. What is the probability that $m^n = n^m$?

M_3. Let $S = \lceil M_7 \rceil$. In right triangle ABC, $\angle C = 90^\circ$, $AC = 27, BC = 36$. A circle with radius S is tangent to both AC and BC and intersects AB at X and Y. Find the length of XY.

M_4. Let $S = M_{13} + 5$. Compute the product of all positive divisors of S.

M_5. Let $A = \sqrt{M_1}, B = \lceil M_{11} \rceil$. Given complex numbers x and y such that $x + \frac{1}{y} = A$, $\frac{1}{x} + y = B$, compute the value of $xy + \frac{1}{xy}$.

M_6. Let $A = \lfloor 1/M_2 \rfloor, B = \lfloor M_3^2/100 \rfloor$. Let P and Q both be quadratic polynomials. Given that the real roots of $P(Q(x)) = 0$ are $0, A, B, C$ in some order, find the sum of all possible values of C.

M_7. Let $A = \lceil \log_2 M_4 \rceil, B = M_{12} + 1$. A 5-term sequence of positive reals satisfy that the first three terms and the last three terms both form an arithmetic sequence and the middle three terms form a geometric sequence. If the first term is A and the fifth term is B, determine the third term of the sequence.

M_8. Let $A = \lfloor M_5^2 \rfloor, B = \lfloor M_6^2 \rfloor$. A regular A-gon, a regular B-gon, and a circle are given in the plane. What is the greatest possible number of regions that these shapes divide the plane into?

M_9. Let A and B be the unit digits of $\lceil 7M_6 \rceil$ and $\lfloor 6M_7 \rfloor$ respectively. When all the positive integers not containing digit A or B are written in increasing order, what is the 2019^{th} number in the list?

M_{10}. What is the smallest positive integer with remainder $2, 3, 4$ when divided by $3, 5, 7$ respectively?

M_{11}. An equiangular hexagon has side lengths $1, 2, 3, 4, 5, 6$ in some order. Find the nonnegative difference between the largest and the smallest possible area of this hexagon.

M_{12}. Determine the second smallest positive integer n such that $n^3 + n^2 + n + 1$ is a perfect square.

M_{13}. Given that A, B are nonzero base-10 digits such that $A \cdot \overline{AB} + B = \overline{BB}$, find $\overline{AB}$.

T_1. Let S, P, A, C, E be (not necessarily distinct) decimal digits where $E \neq 0$. Given that $N = \sqrt{\overline{ESCAPE}}$ is a positive integer, find the minimum possible value of N.

T_2. Let $X = \lfloor T_1/8 \rfloor, Y = T_3 - 1, Z = T_4 - 2$. A point P lies inside the triangle ABC such that $PA = X, PB = Y, PC = Z$. Find the largest possible area of the triangle.

T_3. How many ways can one tile a 2×8 board with 1×1 and 2×2 tiles? Rotations and reflections of the same configuration are considered distinct.

T_4. Let $S = T_5$. Given real numbers a, b, c such that $a^2 + b^2 + c^2 + (a + b + c)^2 = S$, find the maximum possible value of $(a + b)(b + c)(c + a)$.

T_5. A regular tetrahedron has volume 8. What is the volume of the set of all the points in the space (*not necessarily inside the tetrahedron*) that are closer to the center of the tetrahedron than any of the four vertices?

33. [**25**] Determine the value of H_4.

34. [**25**] Determine the value of M_8.

35. [**25**] Determine the value of M_9.

36. [**25**] Determine the value of T_2.