# HMMT November 2012

**Saturday 10 November 2012**

## Guts Round

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1. [**5**] Find the number of prime numbers less than 30.

2. [**5**] Albert is very hungry, and goes to his favorite burger shop to buy a meal, which consists of a burger, a side, and a drink. Given that there are 5 different types of burgers offered, 3 different types of sides, and 12 different types of drinks, find the number of meals Albert could get.

3. [**5**] Find the area of the region between two concentric circles that have radii 100 and 99.

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4. [**6**] $ABCD$ is a concave quadrilateral such that $\angle CBA = 50^\circ$, $\angle BAD = 80^\circ$, $\angle ADC = 30^\circ$, and $CB = CD$. Find $\angle CBD$.

5. [**6**] $a$ and $b$ are complex numbers such that $2a + 3b = 10$ and $4a^2 + 9b^2 = 20$. Find $ab$.

6. [**6**] Given the following formulas:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

find

$$(1^3 + 3 \cdot 1^2 + 3 \cdot 1) + (2^3 + 3 \cdot 2^2 + 3 \cdot 2) + \cdots + (99^3 + 3 \cdot 99^2 + 3 \cdot 99).$$

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7. [**7**] Consider the sequence given by $a_0 = 1, a_1 = 1 + 3, a_2 = 1 + 3 + 3^2, a_3 = 1 + 3 + 3^2 + 3^3, \ldots$. Find the number of terms among $a_0, a_1, a_2, \ldots, a_{2012}$ that are divisible by 7.

8. [**7**] Three cards are drawn from the top of a shuffled standard 52-card deck. Find the probability that they are all of different suits. (A suit is either spades, clubs, hearts, or diamonds.)

9. [**7**] How many sets consist of distinct composite numbers that add up to 23?

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10. [**8**] Consider the sequence defined by $a_0 = 0, a_1 = 1$, and $a_n = 2013a_{n-1} + 2012a_{n-2} + n$ for $n \geq 2$. Find the remainder when $a_{2012}$ is divided by 2012.

11. [**8**] Find the smallest positive integer $n$ such that the number of zeroes that $n!$ ends with is a positive multiple of 5.

12. [**8**] Three circles of radius 1 are drawn, whose centers form the vertices of an equilateral triangle of side length 1. Find the area of the region common to at least two of the circles.

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13. [**9**] Triangle $ABC$ has a perimeter of 18. $\angle A = 60^\circ$, and $BC = 7$. Find the area of the triangle.

14. [**9**] Mark would like to place the numbers 1 to 7 on a circle such that the sequences along both arcs going from 1 to 7 are increasing. For example, one arc could be 1, 2, 4, 7 and the other could be 1, 3, 5, 6, 7. In how many distinct ways can Mark place the numbers? Two arrangements are distinct if and only if one cannot be rotated to match the other.

15. [**9**] Find the area of the region in the $xy$-plane consisting of all points $(a, b)$ such that the quadratic $ax^2 + 2(a + b - 7)x + 2b = 0$ has fewer than two real solutions for $x$.

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16. [**10**] Yuhang is making a bracelet for his one true love using beads. How many distinct bracelets can be made from 2 red beads, 2 green beads, and 2 blue beads, if two bracelets are distinct if and only if one cannot be made into the other through rotations and reflections?

17. [**10**] Given that $x = \ln 30$, $y = \ln 360$, and $z = \ln 270$, and that there are rational numbers $p, q, r$ such that $\ln 5400 = px + qy + rz$, find the ordered triple $(p, q, r)$.

18. [**10**] Let $\triangle ABC$ be a triangle with $AC = 1$ and $\angle ABC$ obtuse. Let $D$ and $E$ be points on $AC$ such that $\angle DBC = \angle ABE = 90^\circ$. If $AD = DE = EC$, find $AB + AC$.

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19. [**11**] Find the number of triples of nonnegative integers $(x, y, z)$ such that $15x + 21y + 35z = 525$.

20. [**11**] An elementary school teacher is taking a class of 20 students on a field trip. To make sure her students don't get lost, she uses the buddy system—using the complete class roster, she pairs the students into 10 pairs and leaves if no person reports that someone from his or her pair is missing. In particular, the teacher will not notice if both students from a pair go missing. Suppose each student independently gets lost with probability $\frac{1}{10}$. Given that the teacher leaves, what is the probability that no student got lost?

21. [**11**] $ABC$ is a triangle with $AB = 7$, $BC = 10$, and $CA = 13$. Point $D$ lies on segment $BC$ such that $DC = 2BD$. Point $E$ lies on segment $AD$ such that $AE = 4ED$. Point $F$ lies on segment $AC$ such that $FC = 5AF$. Find the area of $\triangle EFC$.

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22. [**12**] Alice generates an infinite decimal by rolling a fair 6-sided die with the numbers 1, 2, 3, 5, 7, and 9 infinitely many times, and appending the resulting numbers after a decimal point. What is the expected value of her number?

23. [**12**] $ABC$ is a triangle with $AB = 4$, $BC = 6$, and $CA = 7$. Let $\Omega$ be the circumcircle of $ABC$; the tangent to $\Omega$ passing through $A$ meets the extension of side $BC$ at $P$. Find the length $PB$.

24. [**12**] 12 children sit around a circle. Find the number of ways we can distribute 4 pieces of candy among them such that each child gets at most one piece of candy and no two adjacent children both get a piece of candy.

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25. [**13**] Triangle $ABC$ satisfies $AB = 8$, $BC = 9$, and $CA = 10$. Evaluate $\dfrac{\sin^2 B + \sin^2 C - \sin^2 A}{\sin B \cdot \sin C}$.

26. [**13**] $x_1, x_2, x_3, \ldots$ is a sequence of real numbers satisfying $x_1 = 1$, $x_2 = 2$, and $x_{n+1} = 2x_n - x_{n-1} + 2^n$ for $n \geq 2$. Find $x_{2012}$.

27. [**13**] Find the number of ordered triples of positive integers $(a, b, c)$ satisfying $a^2 - b^2 + ac - bc = 2012$.

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28. [**15**] Find the set of all possible values that can be attained by the expression $\dfrac{ab + b^2}{a^2 + b^2}$, where $a$ and $b$ are positive real numbers. Express your answer in interval notation.

29. [**15**] Find the sum of the real values of $x$ satisfying $(x + 1)(2x + 1)(3x + 1)(4x + 1) = 16x^4$.

30. [**15**] A monkey forms a string of letters by repeatedly choosing one of the letters $a$, $b$, or $c$ to type at random. Find the probability that he first types the string $aaa$ before he first types the string $abc$.

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31. [**17**] Let $ABCD$ be a regular tetrahedron with $AB = 1$, and let $P$ be the center of face $BCD$. Let $M$ be the midpoint of segment $AP$. Ray $BM$ meets face $ACD$ at $Q$. Determine the area of triangle $QDC$.

32. [**17**] Define $f(n)$ to be the remainder when $n^{n^n}$ is divided by 23 for each positive integer $n$. Find the smallest positive integer $k$ such that $f(n + k) = f(n)$ for all positive integers $n$.

33. [**17**] You are playing pool on a $1 \times 1$ table, and the cue ball is at the bottom left corner of the square. Find the smallest angle larger than $45^\circ$ at which you could shoot the ball such that the ball bounces against walls exactly 2012 times before arriving at a vertex of the square.

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34. [**20**] For a positive integer $n$, let $\tau(n)$ be the number of divisors of $n$. Determine $\sum_{n=1}^{2012} \tau(n)$. Your score will be $\max\left\{0, \left\lfloor 20\left(1 - \frac{|S-k|}{S}\right)^3 \right\rfloor\right\}$, where $k$ is your answer and $S$ is the actual answer.

35. [**20**] A regular 2012-gon has a circumcircle with radius 1. Compute the area of the 2012-gon. Your score will be $\min(20, \lfloor \frac{k^2}{5} \rfloor)$, where $k$ is the number of consecutive correct digits immediately following the decimal point of your answer.

36. [**20**] Let $\pi_1$ and $\pi_2$ be permutations of the numbers from 1 through 7. Call $\pi_1$ *superior to* $\pi_2$ if the sum of all $i$ such that $\pi_1(i) > \pi_2(i)$ exceeds the sum of all $i$ such that $\pi_2(i) > \pi_1(i)$. Write down a permutation of the integers from 1 through 7. Let $N$ be the total number of answers submitted for this problem, and let $n$ be the number of submitted answers your answer is superior to. Your score will be $\lceil 20\frac{n}{N} \rceil$.

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