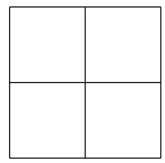
HMMT February 2018, February 10, 2018 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

1. [**4**] A square can be divided into four congruent figures as shown:



If each of the congruent figures has area 1, what is the area of the square?

2. [**4**] John has a 1 liter bottle of pure orange juice. He pours half of the contents of the bottle into a vat, fills the bottle with water, and mixes thoroughly. He then repeats this process 9 more times. Afterwards, he pours the remaining contents of the bottle into the vat. What fraction of the liquid in the vat is now water?

3. [**4**] Allen and Yang want to share the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. How many ways are there to split all ten numbers among Allen and Yang so that each person gets at least one number, and either Allen's numbers or Yang's numbers sum to an even number?

4. [**4**] Find the sum of the digits of $11 \cdot 101 \cdot 111 \cdot 110011$.

HMMT February 2018, February 10, 2018 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

5. [**6**] Randall proposes a new temperature system called *Felsius* temperature with the following conversion between Felsius $^\circ E$, Celsius $^\circ C$, and Fahrenheit $^\circ F$:

$$^\circ E = \frac{7 \times {}^\circ C}{5} + 16 = \frac{7 \times {}^\circ F - 80}{9}.$$

For example, $0\,^\circ C = 16\,^\circ E$. Let x, y, z be real numbers such that $x\,^\circ C = x\,^\circ E$, $y\,^\circ E = y\,^\circ F$, $z\,^\circ C = z\,^\circ F$. Find $x + y + z$.

6. [**6**] A bug is on a corner of a cube. A *healthy* path for the bug is a path along the edges of the cube that starts and ends where the bug is located, uses no edge multiple times, and uses at most two of the edges adjacent to any particular face. Find the number of healthy paths.

7. [**6**] A triple of integers (a, b, c) satisfies $a + bc = 2017$ and $b + ca = 8$. Find all possible values of c.

8. [**6**] Suppose a real number $x > 1$ satisfies

$$\log_2(\log_4 x) + \log_4(\log_{16} x) + \log_{16}(\log_2 x) = 0.$$

Compute

$$\log_2(\log_{16} x) + \log_{16}(\log_4 x) + \log_4(\log_2 x).$$

. . . .

HMMT February 2018, February 10, 2018 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

9. [**7**] In a game, N people are in a room. Each of them simultaneously writes down an integer between 0 and 100 inclusive. A person wins the game if their number is exactly two-thirds of the average of all the numbers written down. There can be multiple winners or no winners in this game. Let m be the maximum possible number such that it is possible to win the game by writing down m. Find the smallest possible value of N for which it is possible to win the game by writing down m in a room of N people.

10. [**7**] Let a positive integer n be called a *cubic square* if there exist positive integers a, b with $n = \gcd(a^2, b^3)$. Count the number of cubic squares between 1 and 100 inclusive.

11. [**7**] FInd the value of

$$\sum_{k=1}^{60}\sum_{n=1}^{k}\frac{n^2}{61-2n}.$$

12. [**7**] $\triangle PNR$ has side lengths $PN = 20$, $NR = 18$, and $PR = 19$. Consider a point A on PN. $\triangle NRA$ is rotated about R to $\triangle N'RA'$ so that R, N', and P lie on the same line and AA' is perpendicular to PR. Find $\frac{PA}{AN}$.

. . . .

HMMT February 2018, February 10, 2018 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

13. [**9**] Suppose $\triangle ABC$ has lengths $AB = 5$, $BC = 8$, and $CA = 7$, and let ω be the circumcircle of $\triangle ABC$. Let X be the second intersection of the external angle bisector of $\angle B$ with ω, and let Y be the foot of the perpendicular from X to BC. Find the length of YC.

14. [**9**] Given that x is a positive real, find the maximum possible value of

$$\sin\left(\tan^{-1}\left(\frac{x}{9}\right) - \tan^{-1}\left(\frac{x}{16}\right)\right).$$

15. [**9**] Michael picks a random subset of the complex numbers $\{1, \omega, \omega^2, \ldots, \omega^{2017}\}$ where ω is a primitive 2018^{th} root of unity and all subsets are equally likely to be chosen. If the sum of the elements in his subset is S, what is the expected value of $|S|^2$? (The sum of the elements of the empty set is 0.)

16. [**9**] Solve for x:

$$x\lfloor x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor\rfloor = 122.$$

. . . .

Organization ______________________________ Team ______________________________ Team ID# ___________

17. [**10**] Compute the value of
$$\frac{\cos 30.5^\circ + \cos 31.5^\circ + ... + \cos 44.5^\circ}{\sin 30.5^\circ + \sin 31.5^\circ + ... + \sin 44.5^\circ}.$$

18. [**10**] Compute the number of integers $n \in \{1, 2, \ldots, 300\}$ such that n is the product of two distinct primes, and is also the length of the longest leg of some nondegenerate right triangle with integer side lengths.

19. [**10**] Suppose there are 100 cookies arranged in a circle, and 53 of them are chocolate chip, with the remainder being oatmeal. Pearl wants to choose a contiguous subsegment of exactly 67 cookies and wants this subsegment to have exactly k chocolate chip cookies. Find the sum of the k for which Pearl is guaranteed to succeed regardless of how the cookies are arranged.

20. [**10**] Triangle $\triangle ABC$ has $AB = 21$, $BC = 55$, and $CA = 56$. There are two points P in the plane of $\triangle ABC$ for which $\angle BAP = \angle CAP$ and $\angle BPC = 90^\circ$. Find the distance between them.

. . . .

HMMT February 2018, February 10, 2018 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

21. [**12**] You are the first lucky player to play in a slightly modified episode of Deal or No Deal! Initially, there are sixteen cases marked 1 through 16. The dollar amounts in the cases are the powers of 2 from $2^1 = 2$ to $2^{16} = 65536$, in some random order. The game has eight turns. In each turn, you choose a case and claim it, without opening it. Afterwards, a random remaining case is opened and revealed to you, then removed from the game.

    At the end of the game, all eight of your cases are revealed and you win all of the money inside them.

    However, the hosts do not realize you have X-ray vision and can see the amount of money inside each case! What is the expected amount of money you will make, given that you play optimally?

22. [**12**] How many graphs are there on 10 vertices labeled $1, 2, \ldots, 10$ such that there are exactly 23 edges and no triangles?

23. [**12**] Kevin starts with the vectors $(1, 0)$ and $(0, 1)$ and at each time step, he replaces one of the vectors with their sum. Find the cotangent of the minimum possible angle between the vectors after 8 time steps.

24. [**12**] Find the largest positive integer n for which there exist n finite sets $X_1, X_2, \ldots, X_n$ with the property that for every $1 \le a < b < c \le n$, the equation
$$|X_a \cup X_b \cup X_c| = \left\lceil \sqrt{abc} \right\rceil$$
holds.

. . . .

Organization ______________________________ Team ______________________________ Team ID# ___________

25. [**15**] Fran writes the numbers $1, 2, 3, \ldots, 20$ on a chalkboard. Then she erases all the numbers by making a series of moves; in each move, she chooses a number n uniformly at random from the set of all numbers still on the chalkboard, and then erases all of the divisors of n that are still on the chalkboard (including n itself). What is the expected number of moves that Fran must make to erase all the numbers?

26. [**15**] Let ABC be a triangle with $\angle A = 18^\circ, \angle B = 36^\circ$. Let M be the midpoint of AB, D a point on ray CM such that $AB = AD$; E a point on ray BC such that $AB = BE$, and F a point on ray AC such that $AB = AF$. Find $\angle FDE$.

27. [**15**] There are 2018 frogs in a pool and there is 1 frog on the shore. In each time-step thereafter, one random frog moves position. If it was in the pool, it jumps to the shore, and vice versa. Find the expected number of time-steps before all frogs are in the pool for the first time.

28. [**15**] Arnold and Kevin are playing a game in which Kevin picks an integer $1 \leq m \leq 1001$, and Arnold is trying to guess it. On each turn, Arnold first pays Kevin 1 dollar in order to guess a number k of Arnold's choice. If $m \geq k$, the game ends and he pays Kevin an additional $m - k$ dollars (possibly zero). Otherwise, Arnold pays Kevin an additional 10 dollars and continues guessing.

    Which number should Arnold guess first to ensure that his worst-case payment is minimized?

. . . .

HMMT February 2018, February 10, 2018 — GUTS ROUND

Organization ______________________________ Team ______________________________ Team ID# ___________

29. [**17**] Let a, b, c be positive integers. All the roots of each of the quadratics

$$ax^2 + bx + c, ax^2 + bx - c, ax^2 - bx + c, ax^2 - bx - c$$

    are integers. Over all triples (a, b, c), find the triple with the third smallest value of $a + b + c$.

30. [**17**] Find the number of *unordered* pairs $\{a, b\}$, where $a, b \in \{0, 1, 2, \ldots, 108\}$ such that 109 divides $a^3 + b^3 - ab$.

31. [**17**] In triangle ABC, $AB = 6$, $BC = 7$ and $CA = 8$. Let D, E, F be the midpoints of sides BC, AC, AB, respectively. Also let O_A, O_B, O_C be the circumcenters of triangles AFD, BDE, and CEF, respectively. Find the area of triangle $O_AO_BO_C$.

32. [**17**] How many 48-tuples of positive integers $(a_1, a_2, \ldots, a_{48})$ between 0 and 100 inclusive have the property that for all $1 \leq i < j \leq 48$, $a_i \notin \{a_j, a_j + 1\}$?

. . . .

Organization ____________________________ Team ____________________________ Team ID# __________

33. [**20**] 679 contestants participated in HMMT February 2017. Let N be the number of these contestants who performed at or above the median score in at least one of the three individual tests. Estimate N.

    An estimate of E earns $\left\lfloor 20 - \frac{|E-N|}{2} \right\rfloor$ or 0 points, whichever is greater.

34. [**20**] The integers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are written on a blackboard. Each day, a teacher chooses one of the integers uniformly at random and decreases it by 1. Let X be the expected value of the number of days which elapse before there are no longer positive integers on the board. Estimate X.

    An estimate of E earns $\lfloor 20 \cdot 2^{-|X-E|/8} \rfloor$ points.

35. [**20**] In a wooden block shaped like a cube, all the vertices and edge midpoints are marked. The cube is cut along *all* possible planes that pass through at least four marked points. Let N be the number of pieces the cube is cut into. Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N) \rfloor$ points.

36. [**20**] In the game of Connect Four, there are seven vertical columns which have spaces for six tokens. These form a 7×6 grid of spaces. Two players White and Black move alternately. A player takes a turn by picking a column which is not already full and dropping a token of their color into the lowest unoccupied space in that column. The game ends when there are four consecutive tokens of the same color in a line, either horizontally, vertically, or diagonally. The player who has four tokens in a row of their color wins.

    Assume two players play this game randomly. Each player, on their turn, picks a random column which is not full and drops a token of their color into that column. This happens until one player wins or all of the columns are filled. Let P be the probability that all of the columns are filled without any player obtaining four tokens in a row of their color. Estimate P.

    An estimate of $E > 0$ earns $\lfloor 20 \min(P/E, E/P) \rfloor$ points.