HMMT February 2019

February 16, 2019

Algebra and Number Theory

1. What is the smallest positive integer that cannot be written as the sum of two nonnegative palindromic integers? (An integer is *palindromic* if the sequence of decimal digits are the same when read backwards.)

2. Let $N = 2^{\left(2^2\right)}$ and x be a real number such that $N^{\left(N^N\right)} = 2^{(2^x)}$. Find x.

3. Let x and y be positive real numbers. Define $a = 1 + \frac{x}{y}$ and $b = 1 + \frac{y}{x}$. If $a^2 + b^2 = 15$, compute $a^3 + b^3$.

4. Let $\mathbb{N}$ be the set of positive integers, and let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

   - $f(1) = 1$;
   - for $n \in \mathbb{N}$, $f(2n) = 2f(n)$ and $f(2n+1) = 2f(n) - 1$.

   Determine the sum of all positive integer solutions to $f(x) = 19$ that do not exceed 2019.

5. Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1b_1 = 20$, $a_2b_2 = 19$, and $a_3b_3 = 14$. Find the greatest possible value of a_4b_4.

6. For positive reals p and q, define the *remainder* when p is divided by q as the smallest nonnegative real r such that $\frac{p-r}{q}$ is an integer. For an ordered pair (a, b) of positive integers, let r_1 and r_2 be the remainder when $a\sqrt{2} + b\sqrt{3}$ is divided by $\sqrt{2}$ and $\sqrt{3}$ respectively. Find the number of pairs (a, b) such that $a, b \leq 20$ and $r_1 + r_2 = \sqrt{2}$.

7. Find the value of
$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} \frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

8. There is a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) > 0$ and such that
$$\sum_{d|n} f(d) f\left(\frac{n}{d}\right) = 1$$
   for all $n \geq 1$. What is $f(2018^{2019})$?

9. Tessa the hyper-ant has a 2019-dimensional hypercube. For a real number k, she calls a placement of nonzero real numbers on the 2^{2019} vertices of the hypercube *k-harmonic* if for any vertex, the sum of all 2019 numbers that are edge-adjacent to this vertex is equal to k times the number on this vertex. Let S be the set of all possible values of k such that there exists a k-harmonic placement. Find $\sum_{k \in S} |k|$.

10. The sequence of integers $\{a_i\}_{i=0}^{\infty}$ satisfies $a_0 = 3, a_1 = 4$, and
$$a_{n+2} = a_{n+1}a_n + \left\lceil \sqrt{a_{n+1}^2 - 1}\sqrt{a_n^2 - 1} \right\rceil$$
    for $n \geq 0$. Evaluate the sum
$$\sum_{n=0}^{\infty}\left(\frac{a_{n+3}}{a_{n+2}} - \frac{a_{n+2}}{a_n} + \frac{a_{n+1}}{a_{n+3}} - \frac{a_n}{a_{n+1}}\right).$$