HMMO 2020

November 14–21, 2020

Team Round

1. **[20]** For how many positive integers $n \leq 1000$ does the equation in real numbers

$$x^{\lfloor x \rfloor} = n$$

have a positive solution for x? (For a real number x, $\lfloor x \rfloor$ denotes the largest integer that is not greater than x.)

2. **[25]** How many ways are there to arrange the numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$ in a circle so that every two adjacent elements are relatively prime? Consider rotations and reflections of the same arrangement to be indistinguishable.

3. **[30]** Let A be the area of the largest semicircle that can be inscribed in a quarter-circle of radius 1. Compute $\frac{120A}{\pi}$.

4. **[35]** Marisa has two identical cubical dice labeled with the numbers $\{1, 2, 3, 4, 5, 6\}$. However, the two dice are not fair, meaning that they can land on each face with different probability. Marisa rolls the two dice and calculates their sum. Given that the sum is 2 with probability 0.04, and 12 with probability 0.01, the maximum possible probability of the sum being 7 is p. Compute $\lfloor 100p \rfloor$.

5. **[40]** For each positive integer n, let a_n be the smallest nonnegative integer such that there is only one positive integer at most n that is relatively prime to all of $n, n+1, \ldots, n+a_n$. If $n < 100$, compute the largest possible value of $n - a_n$.

6. **[40]** Regular hexagon $P_1P_2P_3P_4P_5P_6$ has side length 2. For $1 \leq i \leq 6$, let C_i be a unit circle centered at P_i and ℓ_i be one of the internal common tangents of C_i and C_{i+2}, where $C_7 = C_1$ and $C_8 = C_2$. Assume that the lines $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ bound a regular hexagon. The area of this hexagon can be expressed as $\sqrt{\frac{a}{b}}$, where a and b are relatively prime positive integers. Compute $100a + b$.

7. **[45]** Roger the ant is traveling on a coordinate plane, starting at $(0, 0)$. Every second, he moves from one lattice point to a different lattice point at distance 1, chosen with equal probability. He will continue to move until he reaches some point P for which he could have reached P more quickly had he taken a different route. For example, if he goes from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(1, 2)$ to $(0, 2)$, he stops at $(0, 2)$ because he could have gone from $(0, 0)$ to $(0, 1)$ to $(0, 2)$ in only 2 seconds. The expected number of steps Roger takes before he stops can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

8. **[50]** Altitudes BE and CF of acute triangle ABC intersect at H. Suppose that the altitudes of triangle EHF concur on line BC. If $AB = 3$ and $AC = 4$, then $BC^2 = \frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

9. **[55]** Alice and Bob take turns removing balls from a bag containing 10 black balls and 10 white balls, with Alice going first. Alice always removes a black ball if there is one, while Bob removes one of the remaining balls uniformly at random. Once all balls have been removed, the expected number of black balls which Bob has can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

10. **[60]** Let x and y be non-negative real numbers that sum to 1. Compute the number of ordered pairs (a, b) with $a, b \in \{0, 1, 2, 3, 4\}$ such that the expression $x^a y^b + y^a x^b$ has maximum value 2^{1-a-b}.