HMMT February 2015

Saturday 21 February 2015

Combinatorics

1. Evan's analog clock displays the time 12:13; the number of seconds is not shown. After 10 seconds elapse, it is still 12:13. What is the expected number of seconds until 12:14?

   Answer: $\boxed{25}$ At first, the time is uniformly distributed between 12:13:00 and 12:13:50. After 10 seconds, the time is uniformly distributed between 12:13:10 and 12:14:00. Thus, it takes on average 25 seconds to reach 12:14(:00).

2. Victor has a drawer with 6 socks of 3 different types: 2 complex socks, 2 synthetic socks, and 2 trigonometric socks. He repeatedly draws 2 socks at a time from the drawer at random, and stops if the socks are of the same type. However, Victor is "synthetic-complex type-blind", so he also stops if he sees a synthetic and a complex sock.

   What is the probability that Victor stops with 2 socks of the same type? Assume Victor returns both socks to the drawer after each step.

   Answer: $\boxed{\frac{3}{7}}$ Let the socks be $C_1, C_2, S_1, S_2, T_1, T_2$, where C, S and T stand for complex, synthetic and trigonometric respectively. The possible stopping points consist of three pairs of socks of the same type plus four different complex-synthetic (C-S) pairs, for a total of 7. So the answer is $\frac{3}{7}$.

3. Starting with the number 0, Casey performs an infinite sequence of moves as follows: he chooses a number from $\{1, 2\}$ at random (each with probability $\frac{1}{2}$) and adds it to the current number. Let p_m be the probability that Casey ever reaches the number m. Find $p_{20} - p_{15}$.

   Answer: $\boxed{\frac{11}{2^{20}}}$ We note that the only way n does not appear in the sequence is if $n-1$ and then $n+1$ appears. Hence, we have $p_0 = 1$, and $p_n = 1 - \frac{1}{2}p_{n-1}$ for $n > 0$. This gives $p_n - \frac{2}{3} = -\frac{1}{2}\left(p_{n-1} - \frac{2}{3}\right)$, so that
$$p_n = \frac{2}{3} + \frac{1}{3} \cdot \left(-\frac{1}{2}\right)^n,$$
   so $p_{20} - p_{15}$ is just
$$\frac{1 - (-2)^5}{3 \cdot 2^{20}} = \frac{11}{2^{20}}.$$

4. Alice Czarina is bored and is playing a game with a pile of rocks. The pile initially contains 2015 rocks. At each round, if the pile has N rocks, she removes k of them, where $1 \le k \le N$, with each possible k having equal probability. Alice Czarina continues until there are no more rocks in the pile. Let p be the probability that the number of rocks left in the pile after each round is a multiple of 5. If p is of the form $5^a \cdot 31^b \cdot \frac{c}{d}$, where a, b are integers and c, d are positive integers relatively prime to $5 \cdot 31$, find $a + b$.

   Answer: $\boxed{-501}$ We claim that $p = \frac{1}{5}\frac{6}{10}\frac{11}{15}\frac{16}{20}\cdots\frac{2006}{2010}\frac{2011}{2015}$. Let p_n be the probability that, starting with n rocks, the number of rocks left after each round is a multiple of 5. Indeed, using recursions we have
$$p_{5k} = \frac{p_{5k-5} + p_{5k-10} + \cdots + p_5 + p_0}{5k}$$
   for $k \ge 1$. For $k \ge 2$ we replace k with $k - 1$, giving us
$$p_{5k-5} = \frac{p_{5k-10} + p_{5k-15} + \cdots + p_5 + p_0}{5k - 5}$$
$$\implies (5k-5)p_{5k-5} = p_{5k-10} + p_{5k-15} + \cdots + p_5 + p_0$$
   Substituting this back into the first equation, we have
$$5kp_{5k} = p_{5k-5} + (p_{5k-10} + p_{5k-15} + \cdots + p_5 + p_0) = p_{5k-5} + (5k-5)p_{5k-5},$$

which gives $p_{5k} = \frac{5k-4}{5k} p_{5k-5}$. Using this equation repeatedly along with the fact that $p_0 = 1$ proves the claim.

Now, the power of 5 in the denominator is $v_5(2015!) = 403 + 80 + 16 + 3 = 502$, and 5 does not divide any term in the numerator. Hence $a = -502$. (The sum counts multiples of 5 plus multiples of 5^2 plus multiples of 5^3 and so on; a multiple of 5^n but not 5^{n+1} is counted exactly n times, as desired.)

Noting that $2015 = 31 \cdot 65$, we found that the numbers divisible by 31 in the numerator are those of the form $31 + 155k$ where $0 \le k \le 12$, including $31^2 = 961$; in the denominator they are of the form $155k$ where $1 \le k \le 13$. Hence $b = (13 + 1) - 13 = 1$ where the extra 1 comes from 31^2 in the numerator. Thus $a + b = -501$.

5. For positive integers x, let $g(x)$ be the number of blocks of consecutive 1's in the binary expansion of x. For example, $g(19) = 2$ because $19 = 10011_2$ has a block of one 1 at the beginning and a block of two 1's at the end, and $g(7) = 1$ because $7 = 111_2$ only has a single block of three 1's. Compute $g(1) + g(2) + g(3) + \cdots + g(256)$.

   Answer: $\boxed{577}$ **Solution 1.** We prove that $g(1)+g(2)+\cdots+g(2^n) = 1+2^{n-2}(n+1)$ for all $n \ge 1$, giving an answer of $1 + 2^6 \cdot 9 = 577$. First note that $g(2^n) = 1$, and that we can view $0, 1, \ldots, 2^n - 1$ as n-digit binary sequences by appending leading zeros as necessary. (Then $g(0) = 0$.)

   Then for $0 \le x \le 2^n - 1$, x and $2^n - x$ are complementary n-digit binary sequences (of 0's and 1's), with x's strings of 1's (0's) corresponding to $2^n - x$'s strings of 0's (resp. 1's). It follows that $g(x) + g(2^n - x)$ is simply 1 more than the number of digit changes in x (or $2^n - x$), i.e. the total number of 01 and 10 occurrences in x. Finally, because exactly half of all n-digit binary sequences have $0, 1$ or $1, 0$ at positions $k, k+1$ (for $1 \le k \le n-1$ fixed), we conclude that the average value of $g(x) + g(2^n - x)$ is $1 + \frac{n-1}{2} = \frac{n+1}{2}$, and thus that the total value of $g(x)$ is $\frac{1}{2} \cdot 2^n \cdot \frac{n+1}{2} = (n+1)2^{n-2}$, as desired.

   Solution 2. We prove that $g(1) + g(2) + \cdots + g(2^n - 1) = 2^{n-2}(n+1)$. Identify each block of 1's with its *rightmost* 1. Then it suffices to count the number of these "rightmost 1's." For each $1 \le k \le n-1$, the kth digit from the left is a rightmost 1 if and only if the k and $k+1$ digits from the left are 1 and 0 respectively. Thus there are 2^{n-2} possible numbers. For the rightmost digit, it is a rightmost 1 if and only if it is a 1, so there are 2^{n-1} possible numbers. Sum up we have: $(n-1)2^{n-2} + 2^{n-1} = 2^{n-2}(n+1)$, as desired.

   Remark. We can also solve this problem using recursion or generating functions.

6. Count the number of functions $f : \mathbb{Z} \to \{\text{'green'}, \text{'blue'}\}$ such that $f(x) = f(x + 22)$ for all integers x and there does **not** exist an integer y with $f(y) = f(y + 2) = $ 'green'.

   Answer: $\boxed{39601}$ It is clear that f is determined by $f(0), \ldots, f(21)$. The colors of the 11 even integers are independent of those of the odd integers because evens and odds are never exactly 2 apart.

   First, we count the number of ways to "color" the even integers. $f(0)$ can either be 'green' or 'blue'. If $f(0)$ is 'green', then $f(2) = f(20) = $ 'blue'. A valid coloring of the 8 other even integers corresponds bijectively to a string of 8 bits such that no two consecutive bits are 1. In general, the number of such length n strings is well known to be F_{n+2} (indexed according to $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$), which can be proven by recursion. Therefore, the number of colorings of even integers in this case is $F_{10} = 55$.

   If $f(0)$ is 'blue', then a valid coloring of the 10 other even integers corresponds bijectively to a string as above, of 10 bits. The number of colorings for this case is $F_{12} = 144$. The total number of colorings of even integers is $55 + 144 = 199$.

   Using the same reasoning for coloring the odd integers, we see that the number of colorings of all of the integers is $199^2 = 39601$.

7. 2015 people sit down at a restaurant. Each person orders a soup with probability $\frac{1}{2}$. Independently, each person orders a salad with probability $\frac{1}{2}$. What is the probability that the number of people who ordered a soup is exactly one more than the number of people who ordered a salad?

Answer: $\boxed{\frac{\binom{4030}{2016}}{2^{4030}} \text{ OR } \frac{\binom{4030}{2014}}{2^{4030}} \text{ OR } \frac{\binom{4032}{2016}-2\binom{4030}{2015}}{2^{4031}}}$ **Solution 1.** Note that total soups = total salads + 1 is equivalent to total soups + total not-salads = 2016. So there are precisely $\binom{2015+2015}{2016}$ possibilities, each occurring with probability $(1/2)^{2015+2015}$. Thus our answer is $\frac{\binom{4030}{2016}}{2^{4030}}$.

Solution 2. To count the number of possibilities, we can directly evaluate the sum $\sum_{i=0}^{2014}\binom{2015}{i}\binom{2015}{i+1}$. One way is to note $\binom{2015}{i+1} = \binom{2015}{2014-i}$, and finish with Vandermonde's identity: $\sum_{i=0}^{2014}\binom{2015}{i}\binom{2015}{2014-i} = \binom{2015+2015}{2014} = \binom{4030}{2014}$ (which also equals $\binom{4030}{2016}$).

(We could have also used $\binom{2015}{i} = \binom{2015}{2015-i}$ to get $\sum_{i=0}^{2014}\binom{2015}{2015-i}\binom{2015}{i+1} = \binom{2015+2015}{2016}$ directly, which is closer in the spirit of the previous solution.)

Solution 3 (sketch). It's also possible to get a handle on $\sum_{i=0}^{2014}\binom{2015}{i}\binom{2015}{i+1}$ by squaring Pascal's identity $\binom{2015}{i}+\binom{2015}{i+1} = \binom{2016}{i+1}$ and summing over $0 \le i \le 2014$. This gives an answer of $\frac{\binom{4032}{2016}-2\binom{4030}{2015}}{2^{4031}}$, which can be simplified by noting $\binom{4032}{2016} = \frac{4032}{2016}\binom{4031}{2015}$, and then applying Pascal's identity.

8. Let S be the set of all 3-digit numbers with all digits in the set $\{1,2,3,4,5,6,7\}$ (so in particular, **all three digits are nonzero**). For how many elements $\overline{abc}$ of S is it true that at least one of the (not necessarily distinct) "digit cycles"
$$\overline{abc}, \overline{bca}, \overline{cab}$$
is divisible by 7? (Here, $\overline{abc}$ denotes the number whose base 10 digits are a, b, and c in that order.)

Answer: $\boxed{127}$ Since the value of each digit is restricted to $\{1,2,\dots,7\}$, there is exactly one digit representative of each residue class modulo 7.

Note that $7 \mid \overline{abc}$ if and only if $100a+10b+c \equiv 0 \pmod 7$ or equivalently $2a+3b+c \equiv 0$. So we want the number of triples of residues (a,b,c) such that at least one of $2a+3b+c \equiv 0$, $2b+3c+a \equiv 0$, $2c+3a+b \equiv 0$ holds.

Let the solution sets of these three equations be S_1, S_2, S_3 respectively, so by PIE and cyclic symmetry we want to find $3|S_1| - 3|S_1 \cap S_2| + |S_1 \cap S_2 \cap S_3|$.

Clearly $|S_1| = 7^2$, since for each of a and b there is a unique c that satisfies the equation.

For $S_1 \cap S_2$, we may eliminate a to get the system $0 \equiv 2(2b+3c)-(3b+c) = b+5c$ (and $a \equiv -2b-3c$), which has 7 solutions (one for each choice of c).

For $S_1 \cap S_2 \cap S_3 \subseteq S_1 \cap S_2$, we have from the previous paragraph that $b \equiv -5c$ and $a \equiv 10c - 3c \equiv 0$. By cyclic symmetry, $b, c \equiv 0$ as well, so there's exactly 1 solution in this case.

Thus the answer is $3 \cdot 7^2 - 3 \cdot 7 + 1 = 127$.

9. Calvin has a bag containing 50 red balls, 50 blue balls, and 30 yellow balls. Given that after pulling out 65 balls at random (without replacement), he has pulled out 5 more red balls than blue balls, what is the probability that the next ball he pulls out is red?

Answer: $\boxed{\frac{9}{26}}$ **Solution 1.** The only information this gives us about the number of yellow balls left is that it is even. A bijection shows that the probability that there are k yellow balls left is equal to the probability that there are $30-k$ yellow balls left (flip the colors of the red and blue balls, and then switch the 65 balls that have been picked with the 65 balls that have not been picked). So the expected number of yellow balls left is 15. Therefore the expected number of red balls left is 22.5. So the answer is $\frac{22.5}{65} = \frac{45}{130} = \frac{9}{26}$.

Solution 2. Let $w(b) = \binom{50}{b}\binom{50}{r=b+5}\binom{30}{60-2b}$ be the number of possibilities in which b blue balls have been drawn (precisely $15 \le b \le 30$ are possible). **For fixed** b, the probability of drawing red next is $\frac{50-r}{50+50+30-65} = \frac{45-b}{65}$. So we want to evaluate
$$\frac{\sum_{b=15}^{30} w(b)\frac{45-b}{65}}{\sum_{b=15}^{30} w(b)}.$$

Note the symmetry of weights:

$$w(45-b) = \binom{50}{45-b}\binom{50}{50-b}\binom{30}{2b-30} = \binom{50}{b+5}\binom{50}{b}\binom{30}{60-2b},$$

so the $\frac{45-b}{65}$ averages out with $\frac{45-(45-b)}{65}$ to give a final answer of $\frac{45/2}{65} = \frac{9}{26}$.

Remark. If one looks closely enough, the two approaches are not so different. The second solution may be more conceptually/symmetrically phrased in terms of the number of yellow balls.

10. A group of friends, numbered $1, 2, 3, \ldots, 16$, take turns picking random numbers. Person 1 picks a number uniformly (at random) in $[0, 1]$, then person 2 picks a number uniformly (at random) in $[0, 2]$, and so on, with person k picking a number uniformly (at random) in $[0, k]$. What is the probability that the 16 numbers picked are strictly increasing?

Answer: $\boxed{\frac{17^{15}}{16!^2}}$ **Solution 1 (intuitive sketch).** If person i picks a_i, this is basically a continuous version of Catalan paths (always $y \le x$) from $(0, 0)$ to $(17, 17)$, with "up-right corners" at the (i, a_i).

A **cyclic shifts argument** shows that "$\frac{1}{17}$" of the increasing sequences $(x_1, \ldots, x_{16})$ in $[0, 17]^{16}$ work (i.e. have $x_i \in [0, i]$ for all i), so contribute volume[1] $\frac{1}{17}\frac{17^{16}}{16!}$. Explicitly, the cyclic shift we're using is

$$T_C : (x_1, \ldots, x_{16}) \mapsto (x_2 - x_1, \ldots, x_{16} - x_1, C - x_1)$$

for $C = 17$ (though it's the same for any $C > 0$), which sends increasing sequences in $[0, C]^{16}$ to increasing sequences in $[0, C]^{16}$. The "$\frac{1}{17}$" essentially follows from the fact that T has period 17, and almost every[2] **T-orbit (of 17 (increasing) sequences) contains exactly 1 working sequence**.[3] But to be more rigorous, we still need some more justification.[4]

The volume contribution of permitted sequences (i.e. $a_i \in [0, i]$ for all i; those under consideration in the first place) $(a_1, \ldots, a_{16}) \in [0, 17]^{16}$ is $16!$, so based on the previous paragraph, our final probability is $\frac{17^{15}}{16!^2}$.

Solution 2. Here we present a discrete version of the previous solution.

To do this, we consider several related events.

Let X be a 16-tuple chosen uniformly and randomly from $[0, 17]^{16}$ (used to define events A, B, C). Let Z be a 16-tuple chosen uniformly and randomly from $\{1, 2, \ldots, 17\}^{16}$ (used to define event D).

- A is the event that X's coordinates are ordered ascending;
- B is the event that X lies in the "box" $[0, 1] \times \cdots \times [0, 16]$;
- C is the event that when X's coordinates are sorted ascending to form Y (e.g. if $X = (1, 3.2, 3, 2, 5, 6, \ldots, 16)$ then $Y = (1, 2, 3, 3.2, 5, 6, \ldots, 16)$), Y lies in the box;
- D is the event that when Z's coordinates are sorted ascending to form W, W lies in the aforementioned box. When Z satisfies this condition, Z is known as a *parking function*.

[1] If we wanted to be completely rigorous, we'd want to also show that the set of such sequences is *measurable*. However, our sets here are simple enough that this is not a major concern.

[2] i.e. "outside a set of measure 0"

[3] Try to visualize this yourself! Compare the "exactly 1 of 17" with the discrete solutions below.

[4] We're using a symmetry/"bijection-of-integrals" argument here (to equate 17 integrals with sum $\frac{17^{16}}{16!}$), so we need to be a little careful with the bijections. In particular, we must make sure the cyclic shift

$$T : (x_1, \ldots, x_{16}) \mapsto (x_2 - x_1, \ldots, x_{16} - x_1, 17 - x_1)$$

is "measure-preserving" on $\mathbb{R}^{16}$. (An example of a non-measure-preserving bijection is $t \mapsto t^2$ on $[0, 1]$.) The standard way to check this is by taking a Jacobian (determinant); more conceptually, since $T^{17} = \mathrm{Id}$, and T is affine (so the Jacobian is constant), we should be able to avoid direct computation by using the chain rule. Alternatively, more "combinatorially", it should also be possible to do so by properly discretizing our space.

We want to find $P(A|B)$ because given that X is in B, X has a uniform distribution in the box, just as in the problem. Now note

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} \frac{P(A)}{P(B)} = P(B|A) \frac{P(A)}{P(B)}.$$

C is invariant with respect to permutations, so $\frac{1}{16!} = P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B)}{P(C)}$. Since $P(A) = \frac{1}{16!}$, we have $P(B|A) = \frac{P(A \cap B)}{P(A)} = P(C)$.

Furthermore, $P(C) = P(D)$ because C only depends on the ceilings of the coordinates. So $P(A|B) = P(C)\frac{P(A)}{P(B)} = P(D)\frac{P(A)}{P(B)}$. (*)

Given a 16-tuple Z from $\{1, 2, \ldots, 17\}^{16}$, let $Z + n$ (for integers n) be the 16-tuple formed by adding n to each coordinate and then reducing modulo 17 so that each coordinate lies in $[1, 17]$.

Key claim (discrete analog of cyclic shifts argument). Exactly one of $Z, Z+1, \ldots, Z+16$ is a parking function.

First, assuming this claim, it easily follows that $P(D) = \frac{1}{17}$. Substituting $P(A) = \frac{1}{16!}$, $P(B) = \frac{16!}{17^{16}}$ into (*) gives $P(A|B) = \frac{17^{15}}{16!^2}$. It now remains to prove the claim.

Proof. Consider the following process.

Begin with 17 parking spots around a circle, labelled 1 to 17 clockwise and all unoccupied. There are 16 cars, 1 to 16, and they park one at a time, from 1 to 16. The ith car tries to park in the spot given by the ith coordinate of Z. If this spot is occupied, that car parks in the closest unoccupied spot in the clockwise direction. Because there are only 16 cars, each car will be able to park, and exactly one spot will be left.

Suppose that number 17 is left. For any integer n $(1 \le n \le 16)$, the n cars that ended up parking in spots 1 through n must have corresponded to coordinates at most n. (If not, then the closest spot in the clockwise direction would have to be before spot 17 and greater than n, a contradiction.) It follows that the nth lowest coordinate is at most n and that when Z is sorted, it lies in the box.

Suppose now that D is true. For any integer n $(1 \le n \le 16)$, the nth lowest coordinate is at most n, so there are (at least) n cars whose corresponding coordinates are at most n. At least one of these cars does not park in spots 1 through $n-1$. Consider the first car to do so. It either parked in spot n, or skipped over it because spot n was occupied. Therefore spot n is occupied at the end. This is true for all n not equal to 17, so spot 17 is left.

It follows that Z is a parking function if and only if spot 17 is left. The same is true for $Z+1$ (assuming that the process uses $Z+1$ instead of Z), etc.

Observe that the process for $Z+1$ is exactly that of Z, rotated by 1 spot clockwise. In particular, its empty spot is one more than that of Z, (where 1 is one more than 17.) It follows that exactly one of $Z, Z+1, \ldots, Z+16$ leaves the spot 17, and that exactly one of these is a parking function. $\square$

Solution 3. Suppose that person i picks a number in the interval $[b_i - 1, b_i]$ where $b_i \le i$. Then we have the condition: $b_1 \le b_2 \le \cdots \le b_{16}$. Let c_i be the number of b_j's such that $b_j = i$. Then, for each admissible sequence $b_1, b_2, \ldots, b_{16}$, there is the probability $\frac{1}{c_1! c_2! \cdots c_{16}!}$ that the problem condition holds, since if c_i numbers are picked uniformly and randomly in the interval $[i-1, i]$, then there is $\frac{1}{c_i!}$ chance of them being in an increasing order. Thus the answer we are looking for is

$$\frac{1}{16!} \sum_{\substack{b_i \le i \\ b_1 \le \cdots \le b_{16}}} \frac{1}{c_1! c_2! \cdots c_{16}!} = \frac{1}{16!^2} \sum_{\substack{b_i \le i \\ b_1 \le \cdots \le b_{16}}} \binom{c_1 + \cdots + c_{16}}{c_1, c_2, \ldots, c_{16}}.$$

Thus it suffices to prove that

$$\sum_{\substack{b_i \le i \\ b_1 \le \cdots \le b_{16}}} \binom{c_1 + \cdots + c_{16}}{c_1, c_2, \ldots, c_{16}} = 17^{15}.$$

The left hand side counts the number of 16-tuple such that the nth smallest entry is less than or equal to n. In other words, this counts the number of parking functions of length 16.[5] Since the number of parking functions of length n is $\frac{1}{n+1} \cdot (n+1)^n = (n+1)^{n-1}$ (as proven for $n = 16$ in the previous solution), we obtain the desired result.

[5] See `http://www-math.mit.edu/~rstan/transparencies/parking.pdf`.