

# HMMT November 2014

Saturday 15 November 2014

## General Test

1. Two circles  $\omega$  and  $\gamma$  have radii 3 and 4 respectively, and their centers are 10 units apart. Let  $x$  be the shortest possible distance between a point on  $\omega$  and a point on  $\gamma$ , and let  $y$  be the longest possible distance between a point on  $\omega$  and a point on  $\gamma$ . Find the product  $xy$ .
2. Let  $ABC$  be a triangle with  $\angle B = 90^\circ$ . Given that there exists a point  $D$  on  $AC$  such that  $AD = DC$  and  $BD = BC$ , compute the value of the ratio  $\frac{AB}{BC}$ .
3. Compute the greatest common divisor of  $4^8 - 1$  and  $8^{12} - 1$ .

4. In rectangle  $ABCD$  with area 1, point  $M$  is selected on  $\overline{AB}$  and points  $X, Y$  are selected on  $\overline{CD}$  such that  $AX < AY$ . Suppose that  $AM = BM$ . Given that the area of triangle  $MXY$  is  $\frac{1}{2014}$ , compute the area of trapezoid  $AXYB$ .

5. Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed  $2^{100}$  wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move?

(By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

6. Let  $ABC$  be a triangle with  $AB = 5$ ,  $AC = 4$ ,  $BC = 6$ . The angle bisector of  $C$  intersects side  $AB$  at  $X$ . Points  $M$  and  $N$  are drawn on sides  $BC$  and  $AC$ , respectively, such that  $\overline{XM} \parallel \overline{AC}$  and  $\overline{XN} \parallel \overline{BC}$ . Compute the length  $MN$ .

7. Consider the set of 5-tuples of positive integers at most 5. We say the tuple  $(a_1, a_2, a_3, a_4, a_5)$  is *perfect* if for any distinct indices  $i, j, k$ , the three numbers  $a_i, a_j, a_k$  do *not* form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

8. Let  $a, b, c, x$  be reals with  $(a+b)(b+c)(c+a) \neq 0$  that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \quad \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \quad \text{and} \quad \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

Compute  $x$ .

9. For any positive integers  $a$  and  $b$ , define  $a \oplus b$  to be the result when adding  $a$  to  $b$  in binary (base 2), neglecting any carry-overs. For example,  $20 \oplus 14 = 10100_2 \oplus 1110_2 = 11010_2 = 26$ . (The operation  $\oplus$  is called the *exclusive or*.) Compute the sum

$$\sum_{k=0}^{2^{2014}-1} \left( k \oplus \left\lfloor \frac{k}{2} \right\rfloor \right).$$

Here  $\lfloor x \rfloor$  is the greatest integer not exceeding  $x$ .

10. Suppose that  $m$  and  $n$  are integers with  $1 \leq m \leq 49$  and  $n \geq 0$  such that  $m$  divides  $n^{n+1} + 1$ . What is the number of possible values of  $m$ ?