HMMT February 2023

February 18, 2023

Team Round

1. **[30]** For any positive integer a, let $\tau(a)$ be the number of positive divisors of a. Find, with proof, the largest possible value of $4\tau(n) - n$ over all positive integers n.

   Proposed by: Vidur Jasuja

   Answer: $\boxed{12}$

   Solution: Let d be the number of divisors of n less than or equal to $\frac{n}{4}$. Then, $\tau(n) - 3 \leq d \leq \frac{n}{4} \implies 4\tau(n) - n \leq 12$. We claim the answer is 12. This is achieved by $n = 12$.

   Remark. It turns out that $n = 12$ is the only equality case. One can see this by analyzing when exactly $\tau(n) = \frac{n}{4} + 3$.

2. **[30]** Prove that there do not exist pairwise distinct complex numbers a, b, c, and d such that

$$a^3 - bcd = b^3 - cda = c^3 - dab = d^3 - abc.$$

   Proposed by: Rishabh Das

   Solution 1: First suppose none of them are 0. Let the common value of the four expressions be k, and let $abcd = P$. Then for $x \in \{a, b, c, d\}$,

$$x^3 - \frac{P}{x} = k \implies x^4 - kx - P = 0.$$

   However, Vieta's tells us $abcd = -P$, meaning $P = -P$, so $P = 0$, a contradiction.

   Now if $a = 0$, then $-bcd = b^3 = c^3 = d^3$. Then without loss of generality $b = x$, $c = x\omega$, and $d = x\omega^2$. But then $-bcd = -x^3 \neq x^3$, a contradiction.

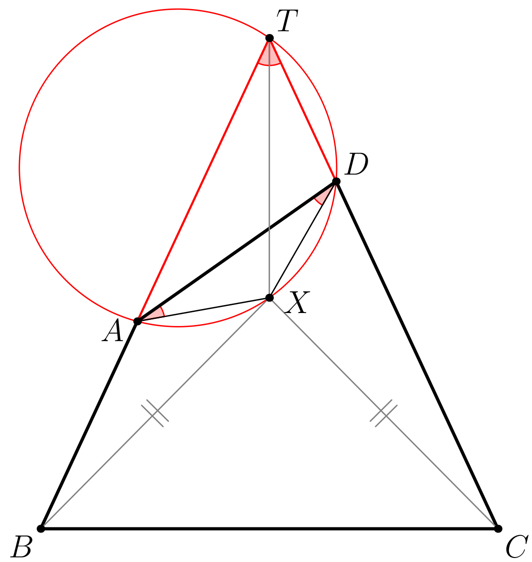
   Thus, there do not exist distinct complex numbers satisfying the equation.

   Solution 2: Subtracting the first two equations and dividing by $a - b$ gives $a^2 + b^2 + ab + cd = 0$. Similarly, $c^2 + d^2 + ab + cd = 0$. So, $a^2 + b^2 = c^2 + d^2$. Similarly, $a^2 + c^2 = b^2 + d^2$. So, $b^2 = c^2$. Similarly, $a^2 = b^2 = c^2 = d^2$. Now by Pigeonhole, two of these 4 must be the same.

3. **[35]** Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle BCD = \theta$ for some angle $\theta < 90^\circ$. Point X lies inside the quadrilateral such that $\angle XAD = \angle XDA = 90^\circ - \theta$. Prove that $BX = XC$.

   Proposed by: Pitchayut Saengrungkongka

   Solution 1:

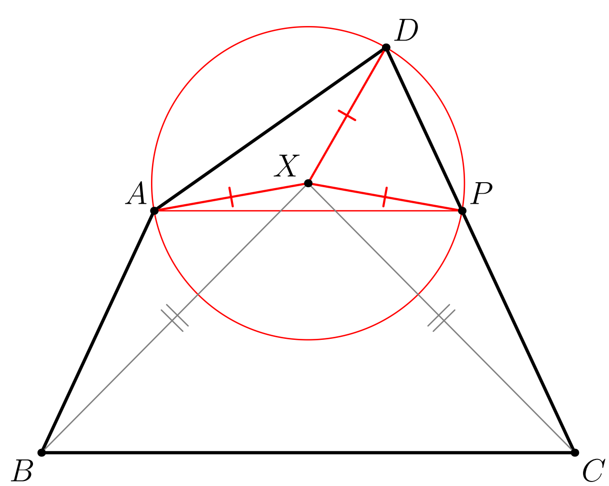


Let lines AB and CD meet at T. Notice that

$$\angle ATD = 180^\circ - \angle ABC - \angle DBC = 180^\circ - 2\theta$$
$$\angle AXD = 180^\circ - 2(90^\circ - \theta) = 2\theta.$$

Therefore, A, T, X, and D are concyclic. In particular, this implies that $\angle XTA = 90^\circ - \theta = \angle XTD$. Thus, XT bisects $\angle BTC$. However, notice that $\angle TBC$ is isosceles, so XT is actually the perpendicular bisector of BC, implying that $BX = XC$.

Solution 2:

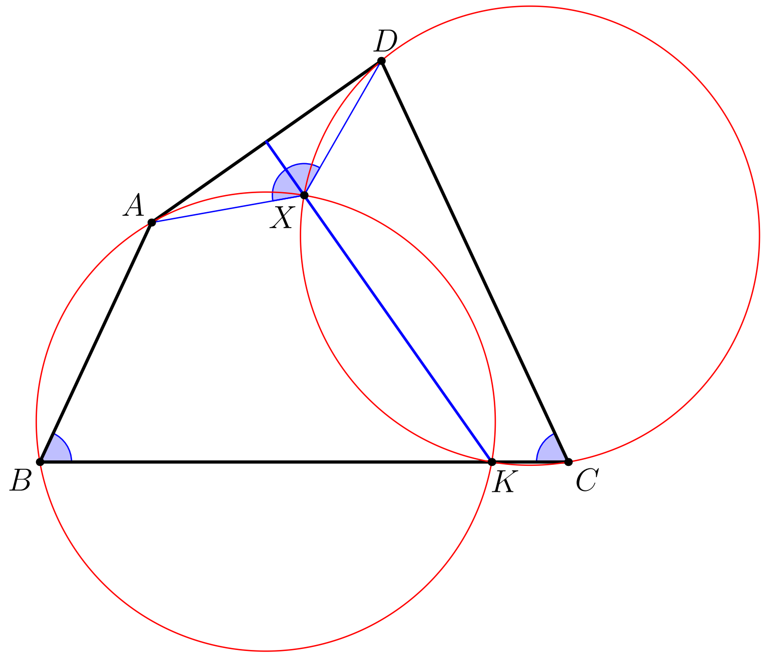


Without loss of generality, let $AB > CD$. Draw the circle γ centered at X and passes through A and D. Let this circle intersects CD again at point $P \neq D$. Then, notice that

$$\angle APD = \frac{\angle AXD}{2} = \theta,$$

implying that $AP \parallel BC$. Combining with $\angle ABC = \angle BCP$, we get that quadrilateral $APCB$ is isosceles trapezoid. Since $X \in \gamma$, we have X lies on the perpendicular bisector of AP, which is the same as the perpendicular bisector of BC, so we are done.

Solution 3:



Let the perpendicular bisector of AD intersects BC at point K. Notice that

$$\angle AXK = 90^\circ + \angle XAD = 180^\circ - \theta = \angle ABK \implies A, B, X, K \text{ are concyclic.}$$
$$\implies \angle XBC = \angle XAK$$

Similarly, we get that $\angle XCB = \angle XDK$. However, since both X and K lies on the perpendicular bisector of AD, implying that $\angle XBC = \angle XCB$.

Solution 4: Fix θ and points A, B, and C. Animate point D along the fixed line through C. Since $\triangle XAD$ as a fixed shape, it follows that X moves linearly along a fixed line. Since we want to show that X lies on the perpendicular bisector of BC, which is fixed, it suffices to prove this for only two locations of D.

- When $AD \parallel BC$, it follows that $ABCD$ is an isosceles trapezoid, implying the result.
- When $D = C$, we notice that
$$\angle AXC = 2\theta = 2\angle ABC,$$
implying that X is the circumcenter of $\triangle ABC$, and the result follows.

4. **[35]** Philena and Nathan are playing a game. First, Nathan secretly chooses an ordered pair (x, y) of positive integers such that $x \leq 20$ and $y \leq 23$. (Philena knows that Nathan's pair must satisfy $x \leq 20$ and $y \leq 23$.) The game then proceeds in rounds; in every round, Philena chooses an ordered pair (a, b) of positive integers and tells it to Nathan; Nathan says YES if $x \leq a$ and $y \leq b$, and NO otherwise. Find, with proof, the smallest positive integer N for which Philena has a strategy that guarantees she can be certain of Nathan's pair after at most N rounds.

Proposed by: Holden Mui, Milan Haiman

Answer: $\boxed{9}$

Solution: It suffices to show the upper bound and lower bound.

Upper bound. Loosen the restriction on y to $y \le 24$. We'll reduce our remaining possibilities by binary search; first, query half the grid to end up with a 10×24 rectangle, and then half of that to go down to 5×24. Similarly, we can use three more queries to reduce to 5×3.

It remains to show that for a 5×3 rectangle, we can finish in 4 queries. First, query the top left 4×2 rectangle. If we are left with the top left 4×2, we can binary search both coordinates with our remaining three queries. Otherwise, we can use another query to be left with either a 4×1 or 1×3 rectangle, and binary searching using our final two queries suffices.

Lower bound. At any step in the game, there will be a set of ordered pairs consistent with all answers to Philena's questions up to that point. When Philena asks another question, each of these possibilities is consistent with only one of YES or NO. Alternatively, this means that one of the answers will leave at least half of the possibilities. Therefore, in the worst case, Nathan's chosen square will always leave at least half of the possibilities. For such a strategy to work in N questions, it must be true that $\frac{460}{2^N} \le 1$, and thus $N \ge 9$.

5. **[40]** Let S be the set of all points in the plane whose coordinates are positive integers less than or equal to 100 (so S has 100^2 elements), and let $\mathcal{L}$ be the set of all lines ℓ such that ℓ passes through at least two points in S. Find, with proof, the largest integer $N \ge 2$ for which it is possible to choose N distinct lines in $\mathcal{L}$ such that every two of the chosen lines are parallel.

Proposed by: Ankit Bisain, Brian Liu, Carl Schildkraut, Luke Robitaille, Maxim Li, William Wang

Answer: $\boxed{4950}$

Solution: Let the lines all have slope $\frac{p}{q}$ where p and q are relatively prime. Without loss of generality, let this slope be positive. Consider the set of points that consists of the point of S with the smallest coordinates on each individual line in the set L. Consider a point (x, y) in this, because there is no other point in S on this line with smaller coordinates, either $x \le q$ or $y \le p$. Additionally, since each line passes through at least two points in S, we need $x + q \le 100$ and $y + p \le 100$.

The shape of this set of points will then be either a rectangle from $(1, 1)$ to $(100 - q, 100 - p)$ with the rectangle from $(q + 1, p + 1)$ to $(100 - q, 100 - p)$ removed, or if $100 - q < q + 1$ or $100 - p < p + 1$, just the initial rectangle. This leads us to two formulas for the number of lines,

$$N = \begin{cases} (100 - p)(100 - q) - (100 - 2p)(100 - 2q) & p, q < 50 \\ (100 - p)(100 - q) & \text{otherwise} \end{cases}$$

In the first case, we need to minimize the quantity

$$\begin{aligned} (100 - p)(100 - q) - (100 - 2p)(100 - 2q) &= 100(p + q) - 3pq \\ &= \frac{10000}{3} - 3\left(q - \frac{100}{3}\right)\left(p - \frac{100}{3}\right), \end{aligned}$$

if one of p, q is above $100/3$ and the other is below it, we would want to maximize how far these two are from $100/3$. The case $(p, q) = (49, 1)$ will be the optimal case since all other combinations will have p, q's closer to $100/3$, this gives us 4853 cases.

In the second case, we need to minimize p and q while keeping at least one above 50 and them relatively prime. From here we need only check $(p, q) = (50, 1)$ since for all other cases, we can reduce either p or q to increase the count. This case gives a maximum of 4950.

6. **[50]** For any odd positive integer n, let $r(n)$ be the odd positive integer such that the binary representation of $r(n)$ is the binary representation of n written backwards. For example, $r(2023) =$

$r(11111100111_2) = 11100111111_2 = 1855$. Determine, with proof, whether there exists a strictly increasing eight-term arithmetic progression $a_1, \ldots, a_8$ of odd positive integers such that $r(a_1), \ldots, r(a_8)$ is an arithmetic progression in that order.

Proposed by: Daniel Zhu

Solution: The main idea is the following claim.

Claim: If a, b, c are in arithmetic progression and have the same number of digits in their binary representations, then $r(a), r(b), r(c)$ cannot be in arithmetic progression in that order.

Proof. Consider the least significant digit that differs in a and b. Then c will have the same value of that digit as a, which will be different from b. Since this becomes the most significant digit in $r(a), r(b), r(c)$, then of course b cannot be between a and c. □

To finish, we just need to show that if there are 8 numbers in arithmetic progression, which we'll write as $a_1, a_1 + d, a_1 + 2d, \ldots, a_1 + 7d$, three of them have the same number of digits. We have a few cases.

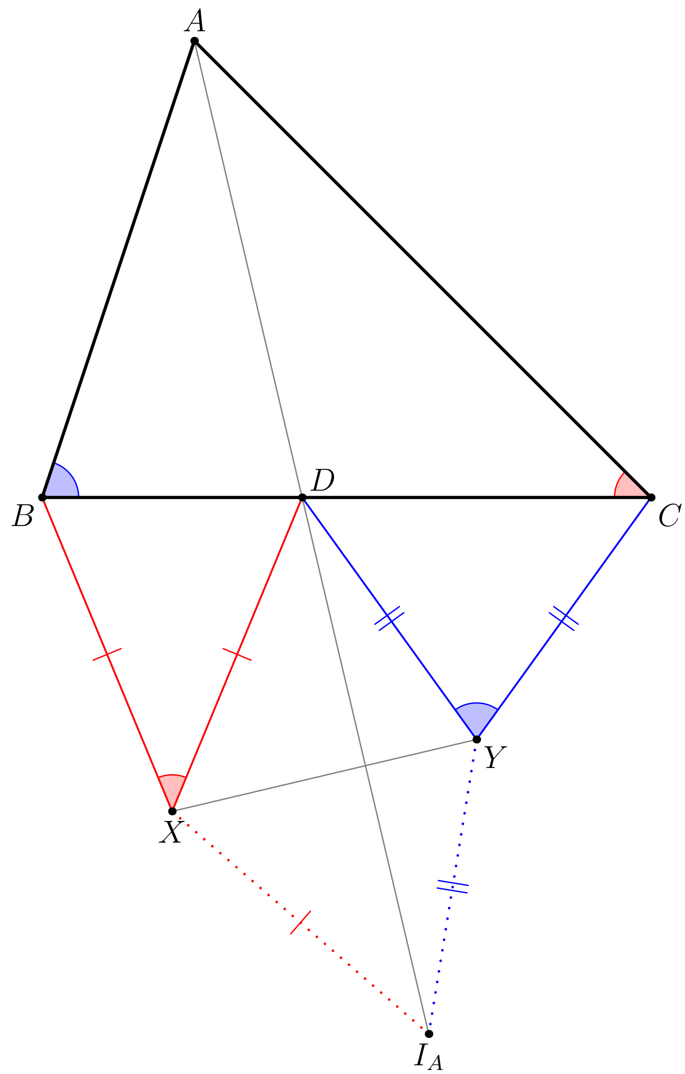
- If $a_1 + 3d < 2^k \leq a_1 + 4d$, then $a_1 + 4d, a_1 + 5d, a_1 + 6d$ will have the same number of digits.
- If $a_1 + 4d < 2^k \leq a_1 + 5d$, then $a_1 + 5d, a_1 + 6d, a_1 + 7d$ will have the same number of digits.
- If neither of these assumptions are true, $a_1 + 3d, a_1 + 4d, a_1 + 5d$ will have the same number of digits.

Having exhausted all cases, we are done.

7. **[55]** Let ABC be a triangle. Point D lies on segment BC such that $\angle BAD = \angle DAC$. Point X lies on the opposite side of line BC as A and satisfies $XB = XD$ and $\angle BXD = \angle ACB$. Analogously, point Y lies on the opposite side of line BC as A and satisfies $YC = YD$ and $\angle CYD = \angle ABC$. Prove that lines XY and AD are perpendicular.

Proposed by: Pitchayut Saengrungkongka

Solution 1:

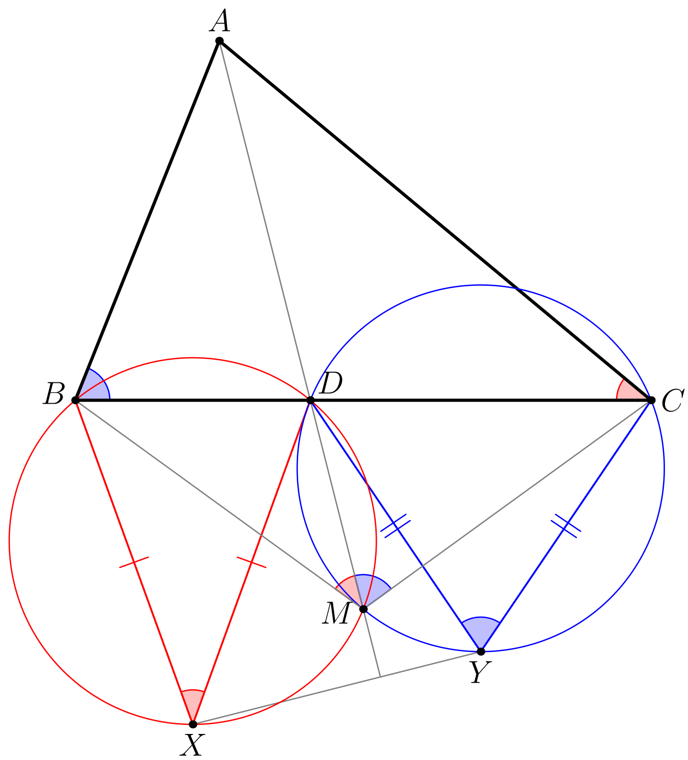


Let I and I_A be the incenter and the A-excenter of $\triangle ABC$. The key observation is that X is the circumcenter of $\triangle BDI_A$. To see why this is true, note that

$$\angle BXD = \angle C = 2\angle ICB = 2\angle II_AB = 2\angle DI_AB.$$

Analogously, Y is the circumcenter of $\triangle CDI_A$. Hence, XY is the perpendicular bisector of DI_A, which is clearly perpendicular to AD.

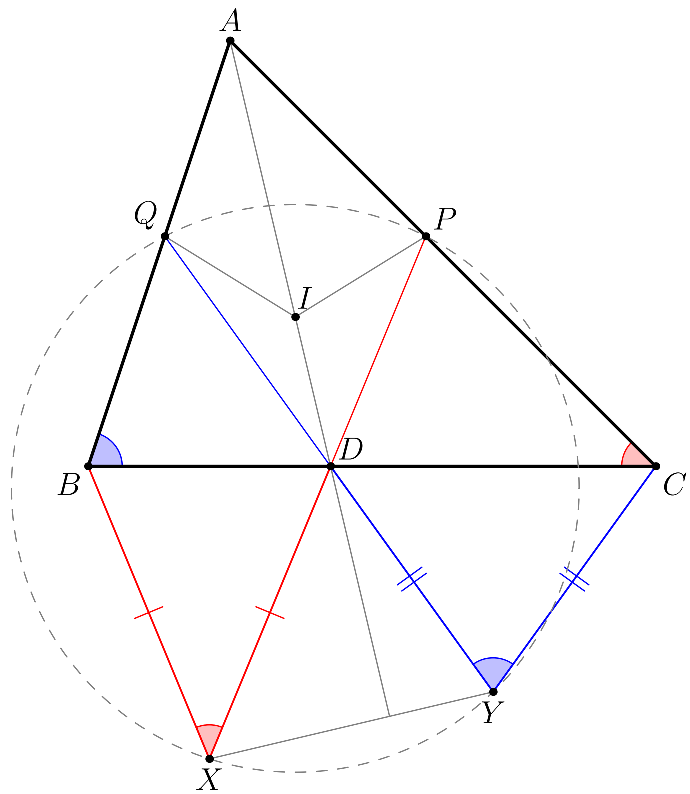
Solution 2:



Denote ω_B and ω_C as the circumcircle of $\triangle BXD$ and $\triangle CYD$. Also, let AD intersects the circumcircle of $\triangle ABC$ at M. Since $\angle BXD = \angle ACB = \angle AMB$, we get that $M \in \omega_B$. Similarly, $M \in \omega_C$. From here, there are two ways to finish.

- Note by Law of Sine that the radius of ω_B and ω_C are $DB/(2\sin\angle MDB)$ and $DB/(2\sin\angle MDC)$, so they are actually equal. Thus, if O_B and O_C are the centers of ω_B and ω_C, then $XO_B = YO_C$. Moreover, XO_B and YO_C are both clearly perpendicular to BC, so XO_BO_CY is parallelogram, implying that $XY \parallel O_BO_C \perp DM$.
- Observe that
$$\angle MXD = \angle MBD = \frac{\angle A}{2} = 90^\circ - \angle XDY,$$
so $XM \perp DY$. Similarly, $YM \perp XD$, so M is the orthocenter of $\triangle DXY$, implying the result.

Solution 3:



Let DX intersects AC at P and DY intersects AB at Q. Observe that $\angle BCP = \angle BXP$, so B, C, P, X are concyclic. This implies that $CD = CP$ and that $DB \cdot DC = DX \cdot DP$.

Similarly, we have $BD = BQ$ and that $DB \cdot DC = DY \cdot DQ$. Thus, we actually have $DX \cdot DP = DY \cdot DQ$, implying that X, Y, P, Q are concyclic.

Now, let I be the incenter of $\triangle ABC$. Since $BD = BQ$, it follows that BI is the perpendicular bisector of DQ, so $ID = IQ$. Similarly, $ID = IP$, so I is actually the circumcenter of $\triangle DPQ$.

We then finish by angle chasing:

$$\angle XDI = 180^\circ - \angle PDI = 90^\circ + \angle DQP = 90^\circ + \angle DXY,$$

implying the result.

8. **[60]** Find, with proof, all nonconstant polynomials $P(x)$ with real coefficients such that, for all nonzero real numbers z with $P(z) \neq 0$ and $P(\frac{1}{z}) \neq 0$, we have

$$\frac{1}{P(z)} + \frac{1}{P(\frac{1}{z})} = z + \frac{1}{z}.$$

Proposed by: Luke Robitaille

Answer: $\boxed{P(x) = \frac{x(x^{4k+2}+1)}{x^2+1} \text{ or } P(x) = \frac{x(1-x^{4k})}{x^2+1}}$

Solution: It is straightforward to plug in and verify the above answers. Hence, we focus on showing that these are all possible solutions. The key claim is the following.

Claim: If $r \neq 0$ is a root of $P(z)$ with multiplicity n, then $1/r$ is also a root of $P(z)$ with multiplicity n.

Proof 1 (Elementary). Let n' be the multiplicity of $1/r$. It suffices to show that $n \le n'$ because we can apply the same assertion on $1/r$ to obtain that $n' \le n$.

To that end, suppose that $(z-r)^n$ divides $P(z)$. From the equation, we have

$$z^N \left[P\left(\frac{1}{z}\right) + P(z) \right] = z^N \left[\left(z + \frac{1}{z}\right) P(z) P\left(\frac{1}{z}\right) \right],$$

where $N \gg \deg P + 1$ to guarantee that both sides are polynomial. Notice that the factor $z^N P(z)$ and the right-hand side is divisible by $(z-r)^n$, so $(z-r)^n$ must also divide $z^N P\left(\frac{1}{z}\right)$. This means that there exists a polynomial $Q(z)$ such that $z^N P\left(\frac{1}{z}\right) = (z-r)^n Q(z)$. Replacing z with $\frac{1}{z}$, we get

$$\frac{P(z)}{z^N} = \left(\frac{1}{z} - r\right)^n Q\left(\frac{1}{z}\right) \implies P(z) = z^{N-n}(1-rz)^n Q\left(\frac{1}{z}\right),$$

implying that $P(z)$ is divisible by $(z-1/r)^n$. □

Proof 2 (Complex Analysis). Here is more advanced proof of the main claim.

View both sides of the equations as meromorphic functions in the complex plane. Then, a root r with multiplicity n of $P(z)$ is a pole of $\frac{1}{P(z)}$ of order n. Since the right-hand side is analytic around r, it follows that the other term $\frac{1}{P(1/z)}$ has a pole at r with order n as well. By replacing z with $1/z$, we find that $\frac{1}{P(z)}$ has a pole at $1/r$ of order n. This finishes the claim. □

The claim implies that there exists an integer k and a constant ϵ such that

$$P(z) = \epsilon z^k P\left(\frac{1}{z}\right).$$

By replacing z with $1/z$, we get that

$$z^k P\left(\frac{1}{z}\right) = \epsilon P(z).$$

Therefore, $\epsilon = \pm 1$. Moreover, using the main equation, we get that

$$\frac{1}{P(z)} + \frac{\epsilon z^k}{P(z)} = z + \frac{1}{z} \implies P(z) = \frac{z(1+\epsilon z^k)}{1+z^2}.$$

This is a polynomial if and only if ($\epsilon = 1$ and $k \equiv 2 \pmod 4$) or ($\epsilon = -1$ and $k \equiv 0 \pmod 4$), so we are done.

9. **[75]** Let ABC be a triangle with $AB < AC$. The incircle of triangle ABC is tangent to side BC at D and intersects the perpendicular bisector of segment BC at distinct points X and Y. Lines AX and AY intersect line BC at P and Q, respectively. Prove that, if $DP \cdot DQ = (AC - AB)^2$, then $AB + AC = 3BC$.

Proposed by: Luke Robitaille

Solution: Let E be the extouch point on BC, let I be the incenter, and D' the reflection of D over I. Note that $DE = AC - AB$, so $DP \cdot DQ = DE^2$. Now let F be the reflection of E across D. The length condition implies $(E, F; P, Q)$ is a harmonic bundle. We also know XY is the perpendicular bisector of DE, so the midpoint M of $D'E$ lies on XY. But then $IM \parallel BC$, so $IM \perp XY$, and M is the midpoint of XY. Since A, D', E are collinear, this means AE bisects XY. Now consider projecting $(E, F; P, Q)$ onto XY. P and Q are taken to X and Y, while E is taken to the midpoint of XY. Thus, F is taken to the point at infinity, so $AF \perp BC$. Now since D is the midpoint of EF, we see that $AF = 2DD'$, or $h_a = 2r$, where h_a is the height from A and r is the inradius. But $\frac{1}{2}ah_a = \frac{a+b+c}{2}r$, so $a = \frac{a+b+c}{2}$, or $3a = b + c$, as desired. □

10. **[90]** One thousand people are in a tennis tournament where each person plays against each other person exactly once, and there are no ties. Prove that it is possible to put all the competitors in a line so that each of the 998 people who are not at an end of the line either defeated both their neighbors or lost to both their neighbors.

Proposed by: Maxim Li

Solution: Take the natural graph theoretic interpretation, where an edge points towards the loser of each pair, and call such a line an alternating path. Consider the longest alternating path, and suppose it doesn't contain everyone. We will show we can make the path longer, which would be a contradiction.

First, assume the path has an odd number of vertices, labeled $v_1, \ldots, v_n$. WLOG v_1 points towards v_2, v_3 points towards v_2, all the way up to v_n which points towards v_{n-1} (otherwise, reverse all the edges). Also WLOG v_1 points towards v_n (if not, label the vertices backwards). Let w be a vertex not in the path. Note that if v_n points towards w, we can make the path longer by adding w to the end. Thus, w must point towards v_n. But now we can take the path $w, v_n, v_1, \ldots, v_{n-1}$, which is longer than before, and so a contradiction.

Now assume the path has an even number of vertices, labeled $v_1, \ldots, v_n$, and WLOG v_1 points towards v_2 again. Then v_{n-1} will point towards v_n. Since 1000 is even, there are at least 2 vertices not in the path, say w_1 and w_2. If either one points towards v_n, we can add it to the end of the path, so v_n must point towards both. Similarly, they must both point towards v_1. But now note that if v_{n-1} points to either, we can make the path $v_1, \ldots, v_{n-1}, w_i, v_n$, which is longer. Thus, w_1, w_2 must both point towards v_{n-1}. Now restrict our attention to $v_1, \ldots, v_{n-1}$. Note that w_1, w_2 both point towards both ends, so we can WLOG assume v_1 points towards v_{n-1}. Also WLOG w_1 points towards w_2. Then we can form the path $w_2, w_1, v_{n-1}, v_1, \ldots, v_{n-2}$, which has $n+1$ vertices. Thus, if the longest alternating path does not contain every vertex, we can make it longer, which is a contradiction, so there must exist an alternating path with all 1000 vertices. $\square$