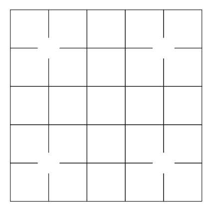
11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Team Round: A Division

Lattice Walks [90]

1. [**20**] Determine the number of ways of walking from $(0,0)$ to $(5,5)$ using only up and right unit steps such that the path does not pass through any of the following points: $(1,1),(1,4),(4,1),(4,4)$.



2. [**20**] Let $n > 2$ be a positive integer. Prove that there are $\frac{1}{2}(n-2)(n+1)$ ways to walk from $(0,0)$ to $(n,2)$ using only up and right unit steps such that the walk never visits the line $y = x$ after it leaves the origin.

3. [**20**] Let $n > 4$ be a positive integer. Determine the number of ways to walk from $(0,0)$ to $(n,2)$ using only up and right unit steps such that the path does not meet the lines $y = x$ or $y = x - n + 2$ except at the start and at the end.

4. [**30**] Let $n > 6$ be a positive integer. Determine the number of ways to walk from $(0,0)$ to $(n,3)$ using only up and right unit steps such that the path does not meet the lines $y = x$ or $y = x - n + 3$ except at the start and at the end.

Lattice and Centroids [130]

A d-dimensional *lattice point* is a point of the form $(x_1, x_2, \ldots, x_d)$ where $x_1, x_2, \ldots, x_d$ are all integers. For a set of d-dimensional points, their *centroid* is the point found by taking the coordinate-wise average of the given set of points.

Let $f(n,d)$ denote the minimal number f such that any set of f lattice points in the d-dimensional Euclidean space contains a subset of size n whose centroid is also a lattice point.

5. [**10**] Let S be a set of 5 points in the 2-dimensional lattice. Show that we can always choose a pair of points in S whose midpoint is also a lattice point.

6. [**10**] Construct a set of 2^d d-dimensional lattice points so that for any two chosen points A, B, the line segment AB does not pass through any other lattice point.

7. [**35**] Show that for positive integers n and d,

$$(n-1)2^d + 1 \le f(n,d) \le (n-1)n^d + 1.$$

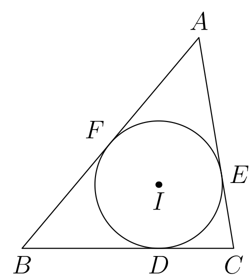
8. [**40**] Show that for positive integers n_1, n_2 and d,

$$f(n_1 n_2, d) \leq f(n_1, d) + n_1 \left(f(n_2, d) - 1 \right).$$

9. [**35**] Determine, with proof, a simple closed-form expression for $f(2^a, d)$.

Incircles [180]

In the following problems, ABC is a triangle with incenter I. Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB, respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. On the circumcircle of ABC, let A' be the midpoint of arc BC (not containing A).
    (a) [**10**] Show that A, I, A' are collinear.
    (b) [**20**] Show that A' is the circumcenter of BIC.
11. [**30**] Let lines BI and EF meet at K. Show that I, K, E, C, D are concyclic.
12. [**40**] Let K be as in the previous problem. Let M be the midpoint of BC and N the midpoint of AC. Show that K lies on line MN.
13. [**40**] Let M be the midpoint of BC, and T diametrically opposite to D on the incircle of ABC. Show that DT, AM, EF are concurrent.
14. [**40**] Let P be a point inside the incircle of ABC. Let lines DP, EP, FP meet the incircle again at D', E', F'. Show that AD', BE', CF' are concurrent.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point. A set of points are *concyclic* if they lie on a common circle.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC, tangent to all three sides.

- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB, respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

- *"Trig" Ceva* states that given ABC a triangle, and points X, Y, Z inside the triangle, the lines AX, BY, CZ are concurrent if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = 1.$$