HMMT February 2019

February 16, 2019

Team Round

1. [**20**] Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and AC intersects XY at point Z. Prove that

$$\frac{AB}{AX}+\frac{AD}{AY}=\frac{AC}{AZ}.$$

Proposed by: Yuan Yao

Solution 1. (Similar Triangles)

Let X' and Y' lie on segments AB and AD respectively such that $ZX' \parallel AD$ and $ZY' \parallel AB$. We note that triangles AXY and $Y'YZ$ are similar, and that triangles $AY'Z$ and ADC are similar. Thus, we have

$$\frac{AC}{AZ}=\frac{AD}{AY'} \text{ and } \frac{AY'}{AY}=\frac{XZ}{XY}.$$

This means that

$$\frac{AD}{AY}=\frac{AD}{AY'}\cdot\frac{AY'}{AY}=\frac{XZ}{XY}\cdot\frac{AC}{AZ},$$

and similarly,

$$\frac{AB}{AX}=\frac{ZY}{XY}\cdot\frac{AC}{AZ}.$$

Therefore we have

$$\frac{AB}{AX}+\frac{AD}{AY}=\left(\frac{XZ}{XY}+\frac{ZY}{XY}\right)\cdot\frac{AC}{AZ}=\frac{AC}{AZ},$$

as desired.

Solution 2. (Affine Transformations)

We recall that affine transformations preserve both parallel lines and ratios between distances of collinear points. It thus suffices to show the desired result when $ABCD$ is a square. This can be done in a variety of ways. For instance, a coordinate bash can be applied by setting A to be the origin. Let the length of the square be 1 and set X and Y as $(a,0)$ and $(0,b)$ respectively, so the line XY has equation $bx+ay=ab$. Then, we note that Z is the point $(\frac{ab}{a+b},\frac{ab}{a+b})$, so

$$\frac{AB}{AX}+\frac{AD}{AY}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}=\frac{AC}{AZ}.$$

2. [**20**] Let $\mathbb{N}=\{1,2,3,\ldots\}$ be the set of all positive integers, and let f be a bijection from $\mathbb{N}$ to $\mathbb{N}$. Must there exist some positive integer n such that $(f(1),f(2),\ldots,f(n))$ is a permutation of $(1,2,\ldots,n)$?

Proposed by: Michael Tang

Answer: $\boxed{\text{No}}$

Consider the bijection f defined by

$$(f(1),f(2),f(3),f(4),\ldots)=(2,4,6,1,8,3,10,5,12,\ldots),$$

which alternates between even and odd numbers after the second entry. (More formally, we define $f(n)=2n$ for $n=1,2$, $f(n)=n+3$ for odd $n\geq 3$ and $f(n)=n-3$ for even $n\geq 4$.) No such n can exist for this f as the largest number among $f(1),f(2),\ldots,f(n)$ is more than n for all n: for $k\geq 2$, the maximum of the first $2k-1$ or $2k$ values is achieved by $f(2k-1)=2k+2$. (Checking $n=1$ and $n=2$ is trivial.)

3. [**25**] For any angle $0 < \theta < \pi/2$, show that

$$0 < \sin\theta + \cos\theta + \tan\theta + \cot\theta - \sec\theta - \csc\theta < 1.$$

Proposed by: Yuan Yao

We use the following geometric construction, which follows from the geometric definition of the trigonometric functions: Let Z be a point on the unit circle in the coordinate plane with origin O. Let X_1, Y_1 be the projections of Z onto the x- and y-axis respectively, and let X_2, Y_2 lie on x- and y-axis respectively such that X_2Y_2 is tangent to the unit circle at Z. Then we have

$$OZ = X_1Y_1 = 1, X_1Z = \sin\theta, Y_1Z = \cos\theta, X_2Z = \tan\theta, Y_2Z = \cot\theta, OX_2 = \sec\theta, OY_2 = \csc\theta.$$

It then suffices to show that $0 < X_2Y_2 - X_1X_2 - Y_1Y_2 < 1 = X_1Y_1$. The left inequality is true because X_1X_2 and Y_1Y_2 are the projections of ZX_2 and ZY_2 onto x- and y-axis respectively. The right inequality is true because $X_1X_2 + X_1Y_1 + Y_1Y_2 > X_2Y_2$ by triangle inequality. Therefore we are done.

4. [**35**] Find all positive integers n for which there do not exist n consecutive composite positive integers less than $n!$.

Proposed by: Brian Reinhart

Answer: $\boxed{1, 2, 3, 4}$

Solution 1. First, note that clearly there are no composite positive integers less than $2!$, and no 3 consecutive composite positive integers less than $3!$. The only composite integers less than $4!$ are

$$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22,$$

and it is easy to see that there are no 4 consecutive composite positive integers among them. Therefore, all $n \leq 4$ works.

Define $M = \operatorname{lcm}(1, 2, \ldots, n+1)$. To see that there are no other such positive integers, we first show that for all $n \geq 5$, $n! > M$. Let $k = \lfloor \log_2(n+1) \rfloor$. Note that $v_2(M) = k$, while

$$v_2((n+1)!) = \sum_{i=1}^{k} \left\lfloor \frac{n+1}{2^i} \right\rfloor \geq \sum_{i=1}^{k} \left(\frac{n+1}{2^i} - 1 \right) = \left(n+1 - \frac{n+1}{2^k} \right) - k \geq (n+1-2) - k = n-k-1.$$

This means that at least $(n-k-1) - k = n-2k-1$ powers of 2 are lost when going from $(n+1)!$ to M. Since $M | (n+1)!$, when $n - 2k - 1 \geq k+1 \iff n \geq 3k+2$, we have

$$M \leq \frac{(n+1)!}{2^{k+1}} \leq \frac{(n+1)!}{2(n+1)} < n!,$$

as desired. Since $n \geq 2^k - 1$, we can rule out all k such that $2^k \geq 3k+3$, which happens when $k \geq 4$ or $n \geq 15$. Moreover, when $k = 3$, we may also rule out all $n \geq 3k+2 = 11$.

We thus need only check values of n between 5 and 10:

$n = 5$: $n! = 120, M = 60$;

$n = 6$: $n! = 720, M = 420$;

$n = 7$: $n! = 5040, M = 840$;

$n \in \{8, 9, 10\}$: $n! \geq 40320, M \leq 27720$.

In all cases, $n! > M$, as desired.

To finish, note that $M-2, M-3, \ldots, M-(n+1)$ are all composite (divisible by $2, 3, \ldots, n+1$ respectively), which gives the desired n consecutive numbers. Therefore, all integers $n \geq 5$ do not satisfy the problem condition, and we are done.

Solution 2. Here is a different way to show that constructions exist for $n \geq 5$. Note that when $n+1$ is not prime, the numbers $n! - 2, n! - 3, \ldots, n! - (n+1)$ are all composite (the first $n-1$ are clearly composite, the last one is composite because $n+1 \mid n!$ and $n! > 2(n+1)$). Otherwise, if $n = p-1$ for prime $p \geq 7$, then the numbers $(n-1)!, (n-1)!-1, (n-1)!-2, \ldots, (n-1)!-(n-1)$ are all composite (the first one and the last $n-2$ are clearly composite since $(n-1)! > 2(n-1)$, the second one is composite since $p \mid (p-2)!-1 = (n-1)!-1$ by Wilson's theorem).

5. [**40**] Find all positive integers n such that the unit segments of an $n \times n$ grid of unit squares can be partitioned into groups of three such that the segments of each group share a common vertex.

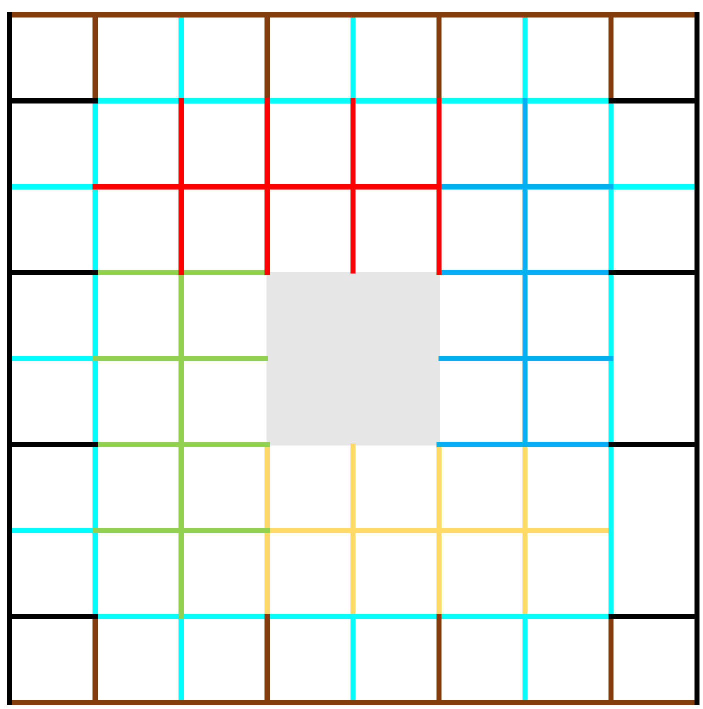
   Proposed by: Yuan Yao

   Answer: $\boxed{n \equiv 0, 2 \pmod 6}$

   We first prove that $n \equiv 0, 2 \pmod 6$ is necessary for there to be such a partitioning. We break this down into proving that n has to be even and that $n \equiv 0, 2 \pmod 3$.

   The only way a segment on a side of the square can be part of such a T-shape is as one of the two consecutive segments along the longer side of the T-shape, so they must come in pairs and therefore, the length of each side has to be even. On the other hand, the total number of segments, which is $2n(n+1)$, has to be a multiple of three as each T-shape consists of three segments, hence either n or $n+1$ is a multiple of 3, implying that $n \equiv 0, 2 \pmod 3$.

   We can then show that these two conditions is sufficient by showing that $n = 2$ and $n = 6$ works and $n = k+6$ works whenever $n = k$ works. The construction for $n = 2$ is simple; just put a T-shape with the longer side on each of the four sides. For $n = 6$ and to go from $n = k$ to $n = k+6$, consider the following diagram:



   There are two main parts – the cycle of stacks of T's in all four orientation (see the red, blue, yellow, and green stacks), and the border (seen here by the cyan, brown, and black T-shapes). The case $n = 6$ can be considered as a special case where the middle square is a single point.

6. [**45**] Scalene triangle ABC satisfies $\angle A = 60°$. Let the circumcenter of ABC be O, the orthocenter be H, and the incenter be I. Let D, T be the points where line BC intersects the internal and external

angle bisectors of $\angle A$, respectively. Choose point X on the circumcircle of $\triangle IHO$ such that $HX \parallel AI$. Prove that $OD \perp TX$.

Proposed by: Wanlin Li

Let I_A denote the A-excenter. Because $\angle A = 60°$, AI is the perpendicular bisector of OH and B, H, O, C all lie on the circle with diameter II_A. We are given that X is on this circle as well, and since $HI = OI$, $XIOI_A$ is also an isosceles trapezoid. But II_A is a diameter, so this means X must be diametrically opposite O on (BOC) and is actually the intersection of the tangents to (ABC) from B and C.

Now $(T, D; B, C) = -1$, so T is on the polar of D with respect to (ABC). BC is the polar of X and D lies on BC, so X must also lie on the polar of D. Therefore TX is the polar of D with respect to (ABC), and $OD \perp TX$ as desired.

7. [**50**] A convex polygon on the plane is called *wide* if the projection of the polygon onto any line in the same plane is a segment with length at least 1. Prove that a circle of radius $\frac{1}{3}$ can be placed completely inside any wide polygon.

Proposed by: Shengtong Zhang

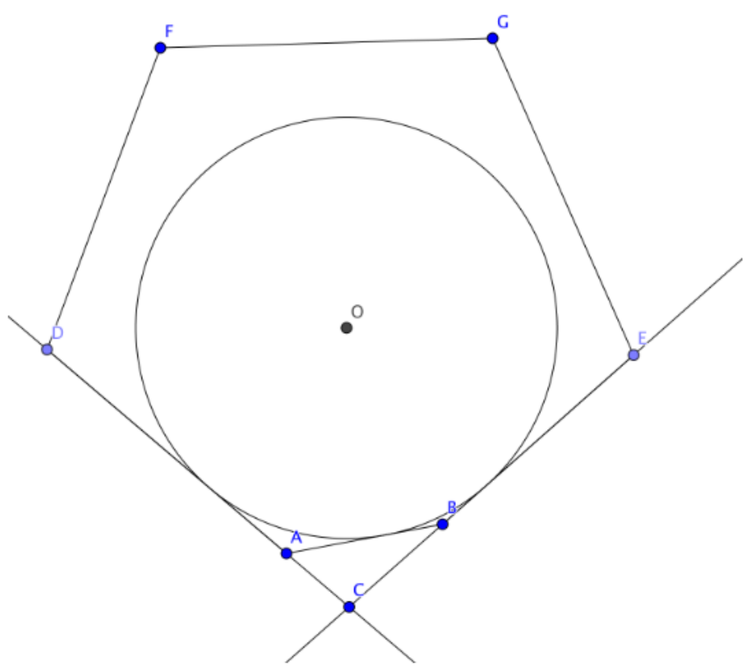
Solution 1.

Lemma. for any polygon including its boundary, there exists a largest circle contained inside it.

Proof. Its easy to see that for any circle inside the polygon, it can be increased in size until it is tangent to at least three sides of the polygon. Then for any three sides of the polygon, there is only one circle tangent to all three, so there are only finitely many possibilities. Therefore there exists a largest one.

Alternatively, one can show that the space of valid (x, y, r) such that the circle with center (x, y) and radius r is compact, e.g. by showing the complement is open and that the complement is open. Then the map $(x, y, r) \to r$ is continuous and therefore has a maximum.

Now, take the largest circle. It clearly must be tangent to three sides. If the circle lies inside the triangle made by the three lines, we can expand the polygon to that triangle and solve it for the triangle instead. Otherwise, we have the following diagram:



Here the circle is an excircle of the triangle ABC made by the lines AB, AD, and BE. (Note that AD and AB dont have to be consecutive sides of the polygon, but the ones in between dont really matter.)

Then since the circle is an excircle, we can consider a homothety at C with power $1+\epsilon$, which sends the circle to a slightly larger circle which does not touch line AB. If this homothety causes the circle to leave the polygon for small enough ϵ, it must be because the circle was initially tangent to another line ℓ, for which it would be an incircle of the triangle made by ℓ and lines AD, BE, bringing us back to the first case.

Thus we can reduce to a case where we have a triangle with each height at least 1, and we want to show the inradius is at least 1/3. Let K be the area of the triangle, so the heights $\frac{2K}{a}, \frac{2K}{b}, \frac{2K}{c}$ are all at least 1. Then the inradius r satisfies

$$r = \frac{K}{s} = \frac{2K}{a+b+c} \geq \frac{2K}{2K+2K+2K} = \frac{1}{3},$$

as desired.

Solution 2. Consider the center of mass G. We will use the notion of *support lines* for convex shapes. (*Support lines* are the lines that touches the shape but does not cut through it.) If a circle centered at G with radius 1/3 cannot be contained inside the polygon, then there exist a point P on the boundary that $GP < 1/3$. Let ℓ_1 be the support line passing through p, ℓ_2 be the line parallel to ℓ_1 and passing through G, and ℓ_3 be the other support line that is parallel to ℓ_1, touching the polygon at P'. Suppose ℓ_2 intersects the polygon at A and B. Extend $P'A$ and $P'B$, intersecting ℓ_1 at A' and B'. Then, if we consider the two parts of the polygon that ℓ_2 divides the polygon into, we have:

- the part of the polygon that contains P' contains the triangle $P'AB$;
- the part of the polygon that contains P is contained in the quadrilateral $AA'B'B$.

Then we conclude that the center of mass G' of the triangle $P'A'B'$ lies between ℓ_2 and ℓ_1, which by assumption is less than $\frac{1}{3}$ away from ℓ_1. However, because the height from P' to ℓ_1 is at least 1, the distance from G' to ℓ_1 is at least $\frac{1}{3}$, so we have a contradiction. Therefore no such P exists and the circle can be placed inside the polygon.

8. [**50**] Can the set of lattice points $\{(x,y)|x,y \in \mathbb{Z}, 1 \leq x,y \leq 252, x \neq y\}$ be colored using 10 distinct colors such that for all $a \neq b, b \neq c$, the colors of (a,b) and (b,c) are distinct?

   Proposed by: Franklyn Wang

   Answer: $\boxed{\text{Yes}}$

   Associate to each number from 1 to 252 a distinct 5-element subset of $S = \{1,2,\ldots,10\}$. Then assign to (a,b) an element of S that is in the subset associated to a but not in that associated to b. It's not difficult to see that this numerical assignment is a valid coloring: the color assigned to (a,b) is not in b, while the color assigned to (b,c) is in b, so they must be distinct.

9. [**55**] Let $p > 2$ be a prime number. $\mathbb{F}_p[x]$ is defined as the set of all polynomials in x with coefficients in $\mathbb{F}_p$ (the integers modulo p with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_p$ for each nonnegative integer k. For example, $(x+2)(2x+3) = 2x^2+2x+1$ in $\mathbb{F}_5[x]$ because the corresponding coefficients are equal modulo 5.

   Let $f, g \in \mathbb{F}_p[x]$. The pair (f,g) is called *compositional* if

   $$f(g(x)) \equiv x^{p^2} - x$$

   in $\mathbb{F}_p[x]$. Find, with proof, the number of compositional pairs (in terms of p).

   Proposed by: Ashwin Sah

   Answer: $\boxed{4p(p-1)}$

   Solution 1. First, notice that $(\deg f)(\deg g) = p^2$ and both polynomials are clearly nonconstant. Therefore there are three possibilities for the ordered pair $(\deg f, \deg g)$, which are $(1,p^2)$, $(p^2,1)$, and (p,p).

In the subsequent parts of the solution, equalities are modulo p. If $f(x) = ax + b, a \neq 0$ is linear, then it is invertible so then g is uniquely determined as $g(x) = f^{-1}(f(g(x))) = \frac{x^{p^2}-x-b}{a}$. Similarly, if $g(x) = cx + d, c \neq 0 \pmod p$ is linear then f is uniquely determined as $f(x) = f(g(g^{-1}(x))) = \left(\frac{x-d}{c}\right)^{p^2} - \left(\frac{x-d}{c}\right)$. In each case there are $p(p-1)$ compositional pairs.

The last case is $\deg f = \deg g = p$. We take the derivative of both sides (we use the formal derivative $D_x f(x) = \sum_{n\geq 1} n f_n x^{n-1}$, which satisfies the usual chain and product rules but can be used on arbitrary polynomials, including those in $\mathbb{F}_p[x]$).

Thus

$$f'(g(x))g'(x) = p^2 x^{p^2-1} - 1 = -1,$$

using that $p = 0$ in $\mathbb{F}_p$. Now $g'(x)$ and $f'(g(x))$ must both be constant polynomials. Since g is nonconstant, this means that $f'(x)$ is also a constant polynomial. We must be careful here, as unlike in $\mathbb{R}$, nonlinear polynomials can have constant derivatives. From the formula of derivative, we see that $h'(x) = 0$ as a polynomial exactly when $h(x)$ is a linear combination of $1, x^p, x^{2p}, \ldots$ (remember that $p = 0$). Thus f', g' both being constant and f, g being of degree p tells us

$$f(x) = ax^p + bx + c, g(x) = dx^p + ex + f$$

where a, b, c, d, e, f are some elements of $\mathbb{F}_p$. Now we must have

$$a(dx^p + ex + f)^p + b(dx^p + ex + f) + c = x^{p^2} - x$$

over $\mathbb{F}_p[x]$. We use the fact that $(x+y)^p = x^p + y^p$ as polynomials in $\mathbb{F}_p$, since the binomial coefficients $\binom{p}{j} \equiv 0 \pmod p$ for $1 \leq j \leq p-1$. This implies $(x+y+z)^p = x^p + y^p + z^p$. Therefore we can expand the previous equation as

$$a(d^p x^{p^2} + e^p x^p + f^p) + b(dx^p + ex + f) + c = x^{p^2} - x.$$

Equating coefficients, we see that

$$\begin{aligned} ad^p &= 1, \\ ae^p + bd &= 0, \\ be &= -1, \\ af^p + bf + c &= 0. \end{aligned}$$

The first and third equations imply that a, d, b, e are nonzero $\pmod p$ and $a = d^{-p}, b = -e^{-1}$. Then $ae^p + bd = 0$ gives

$$d^{-p}e^p - e^{-1}d = 0$$

or $e^{p+1} = d^{p+1}$. Recalling that $e^{p-1} = d^{p-1} = 1$ in $\pmod p$, this tells us $d^2 = e^2$ so $d = \pm e$. Furthermore, any choice of such (d, e) give unique (a, b) which satisfy the first three equations. Finally, once we have determined a, b, d, e, any choice of f gives a unique valid choice of c.

Thus we have $p - 1$ choices for d, two choices for e after choosing d (n.b. for $p = 2$ there is only one choice for e, so the assumption $p > 2$ is used here), and then p choices for f, for a total of $2p(p-1)$ compositional pairs in this case.

Finally, adding the number of compositional pairs from all three cases, we obtain $4p(p-1)$ compositional pairs in total.

Solution 2. The key step is obtaining

$$f(x) = ax^p + bx + c, g(x) = dx^p + ex + f$$

in the case where $\deg f = \deg g = p$. We present an alternative method of obtaining this, with the rest of the solution being the same as the first solution. Let

$$\begin{aligned} f(x) &= f_p x^p + f_{p-1} x^{p-1} + \cdots + f_0 \\ g(x) &= g_p x^p + g_{p-1} x^{p-1} + \cdots + g_0 \end{aligned}$$

where f_p, g_p are nonzero. Like before, we have $g(x)^p = g(x^p)$ in $\mathbb{F}_p[x]$, so

$$x^{p^2} - x = f_p g(x^p) + f_{p-1} g(x)^{p-1} + \cdots + f_0.$$

Consider the maximal $k < p$ for which $f_k \neq 0$. (It is not hard to see that in fact $k \geq 1$, as $f_p g(x^p) + f_0$ cannot be $x^{p^2} - x$.) First assume that $k > 1$. We look at the x^{kp-1} coefficient, which is affected only by the $f_k g(x)^k$ term. By expanding, the coefficient is $k f_k g_p^{k-1} g_{p-1}$. Therefore $g_{p-1} = 0$. Then we look at the x^{kp-2} coefficient, then the x^{kp-3} coefficient, etc. down to the x^{kp-p+1} coefficient to conclude that $g_{p-1} = g_{p-2} = \cdots = g_1 = 0$. However, then the x coefficient of $f(g(x))$ is zero, contradiction.

Therefore we must have $k = 1$, so f is of the form $ax^p + bx + c$. Using the same method as we used when $k > 1$, we get $g_{p-1} = g_{p-2} = \cdots g_2 = 0$, though the x^{kp-p+1} coefficient is now the x coefficient which we want to be nonzero. Hence we do not obtain $g_1 = 0$ anymore and we find that g is of the form $dx^p + ex + f$.

10. [**60**] Prove that for all positive integers n, all complex roots r of the polynomial

$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$

lie on the unit circle (i.e. $|r| = 1$).

Proposed by: Faraz Masroor

Note that neither 0 nor 1 are roots of the polynomial. Consider the function

$$Q(x) = P(x)/x^n = (2n)x^n + (2n)x^{-n} + (2n-1)x^{n-1} + (2n-1)x^{-n+1} + \cdots + (n+1)x^1 + (n+1)x^{-1} + n.$$

All $2n$ of the complex roots of $P(x)$ will be roots of $Q(x)$.
If $|x| = 1$, then $x = e^{i\theta}$, and

$$\begin{aligned}
Q(x) &= (2n)(x^n + x^{-n}) + (2n-1)(x^{n-1} + x^{-n+1}) + \cdots + (n+1)(x + x^{-1}) + n \\
&= (2n)(e^{in\theta} + e^{-in\theta}) + (2n-1)(e^{i(n-1)\theta} + e^{-i(n-1)\theta}) + \cdots + (n+1)(e^{i\theta} + e^{-i\theta}) + n \\
&= (2n)(2\cos(n\theta)) + (2n-1)(2\cos((n-1)\theta)) + \cdots + (n+1)(\cos(\theta)) + n,
\end{aligned}$$

which is real. Thus on the unit circle, we have $Q(x)$ is real, and we want to show it has $2n$ roots there. Rewrite

$$\begin{aligned}
P(x) &= (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + 2n \\
&= (2n)(x^{2n} + x^{2n-1} + \cdots + 1) \\
&\qquad - (x^{2n-1} + 2x^{2n-2} + \cdots + (n-1)x^n + nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x^2 + x) \\
&= 2n\frac{x^{2n+1} - 1}{x - 1} - x(x^{2n-2} + 2x^{2n-3} + \cdots + (n-1)x^n + nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x + 1) \\
&= 2n\frac{x^{2n+1} - 1}{x - 1} - x(x^{n-1} + x^{n-2} + \cdots + x + 1)^2 \\
&= 2n\frac{x^{2n+1} - 1}{x - 1} - x\left(\frac{x^n - 1}{x - 1}\right)^2,
\end{aligned}$$

and thus

$$Q(x) = \frac{2n}{x^n}\frac{x^{2n+1} - 1}{x - 1} - \frac{x}{x^n}\left(\frac{x^n - 1}{x - 1}\right)^2.$$

Consider the roots of unity $r_j = e^{i\frac{2\pi}{2n}j}$, for $j = 0$ to $2n - 1$. There are $2n$ such roots of unity: they all have $r_j^{2n} = 1$, and they alternate between those which satisfy $r_j^n = 1$ or $r_j^n = -1$. At those $x = r_j$, if

$r_j^n = 1$ but $x \neq 1$, then

$$
\begin{aligned}
Q(x) &= \frac{2n}{x^n}\frac{x^{2n+1}-1}{x-1} - \frac{x}{x^n}\left(\frac{x^n-1}{x-1}\right)^2 \\
&= 2n\frac{x^1-1}{x-1} - x\left(\frac{1-1}{x-1}\right)^2 = 2n > 0.
\end{aligned}
$$

At $x = 1$, we can easily see $Q(1) > 0$.
If $r_j^n = -1$, then

$$
\begin{aligned}
Q(x) &= \frac{2n}{x^n}\frac{x^{2n+1}-1}{x-1} - \frac{x}{x^n}\left(\frac{x^n-1}{x-1}\right)^2 \\
&= -2n\frac{x^1-1}{x-1} + x\left(\frac{-1-1}{x-1}\right)^2 \\
&= -2n + \frac{4x}{(x-1)^2} \\
&= -2n + \frac{4}{x-2+1/x} \\
&= -2n + \frac{4}{2\cos(\frac{2\pi}{2n}j)-2} < -2n-4 < 0
\end{aligned}
$$

since the denominator of this second term is strictly negative ($j \neq 0$).

Thus at each of the $2n$-roots of unity, $Q(x)$ alternates in sign, and because $Q(x)$ is real and continuous on the unit circle, it has at least one root between every pair of consecutive roots of unity. Since there are $2n$ of these pairs, and we know that $Q(x)$ has exactly $2n$ roots (by the Fundamental Theorem of Algebra), we have found all of Q's roots, and therefore those of P.