HMMT February 2016

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Algebra

1. Let z be a complex number such that $|z| = 1$ and $|z - 1.45| = 1.05$. Compute the real part of z.

   Proposed by: Evan Chen

   Answer: $\boxed{\frac{20}{29}}$

   From the problem, let A denote the point z on the unit circle, B denote the point 1.45 on the real axis, and O the origin. Let AH be the height of the triangle OAH and H lies on the segment OB. The real part of z is OH. Now we have $OA = 1, OB = 1.45$, and $AB = 1.05$. Thus

$$OH = OA \cos\angle AOB = \cos\angle AOB = \frac{1^2 + 1.45^2 - 1.05^2}{2 \cdot 1 \cdot 1.45} = \frac{20}{29}.$$

2. For which integers $n \in \{1, 2, \ldots, 15\}$ is $n^n + 1$ a prime number?

   Proposed by: Evan Chen

   Answer: $\boxed{1,2,4}$

   $n = 1$ works. If n has an odd prime factor, you can factor, and this is simulated also by $n = 8$:

$$a^{2k+1} + 1 = (a + 1)(\sum_{i=0}^{2k}(-a)^i)$$

   with both parts larger than one when $a > 1$ and $k > 0$. So it remains to check 2 and 4, which work. Thus the answers are 1,2,4.

3. Let A denote the set of all integers n such that $1 \leq n \leq 10000$, and moreover the sum of the decimal digits of n is 2. Find the sum of the squares of the elements of A.

   Proposed by: Evan Chen

   Answer: $\boxed{7294927}$

   From the given conditions, we want to calculate

$$\sum_{i=0}^{3}\sum_{j=i}^{3}(10^i + 10^j)^2.$$

   By observing the formula, we notice that each term is an exponent of 10. 10^6 shows up 7 times, 10^5 shows up 2 times, 10^4 shows up 9 times, 10^3 shows up 4 times, 10^2 shows up 9 times, 10 shows 2 times, 1 shows up 7 times. Thus the answer is 7294927.

4. Determine the remainder when

$$\sum_{i=0}^{2015}\left\lfloor \frac{2^i}{25} \right\rfloor$$

   is divided by 100, where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

   Proposed by: Alexander Katz

   Answer: $\boxed{14}$

   Let r_i denote the remainder when 2^i is divided by 25. Note that because $2^{\phi(25)} \equiv 2^{20} \equiv 1 \pmod{25}$, r is periodic with length 20. In addition, we find that 20 is the order of 2 mod 25. Since 2^i is never a multiple of 5, all possible integers from 1 to 24 are represented by $r_1, r_2, ..., r_{20}$ with the exceptions of 5, 10 ,15, and 20. Hence, $\sum_{i=1}^{20} r_i = \sum_{i=1}^{24} i - (5 + 10 + 15 + 20) = 250$.

We also have

$$\begin{aligned}
\sum_{i=0}^{2015}\left\lfloor\frac{2^i}{25}\right\rfloor &= \sum_{i=0}^{2015}\frac{2^i-r_i}{25}\\
&= \sum_{i=0}^{2015}\frac{2^i}{25}-\sum_{i=0}^{2015}\frac{r_i}{25}\\
&= \frac{2^{2016}-1}{25}-\sum_{i=0}^{1999}\frac{r_i}{25}-\sum_{i=0}^{15}\frac{r_i}{25}\\
&= \frac{2^{2016}-1}{25}-100\left(\frac{250}{25}\right)-\sum_{i=0}^{15}\frac{r_i}{25}\\
&\equiv \frac{2^{2016}-1}{25}-\sum_{i=0}^{15}\frac{r_i}{25} \pmod{100}
\end{aligned}$$

We can calculate $\sum_{i=0}^{15} r_i = 185$, so

$$\sum_{i=0}^{2015}\left\lfloor\frac{2^i}{25}\right\rfloor \equiv \frac{2^{2016}-186}{25} \pmod{100}$$

Now $2^{\phi(625)} \equiv 2^{500} \equiv 1 \pmod{625}$, so $2^{2016} \equiv 2^{16} \equiv 536 \pmod{625}$. Hence $2^{2016}-186 \equiv 350 \pmod{625}$, and $2^{2016}-186 \equiv 2 \pmod{4}$. This implies that $2^{2016}-186 \equiv 350 \pmod{2500}$, and so $\frac{2^{2016}-186}{25} \equiv \boxed{14} \pmod{100}$.

5. An infinite sequence of real numbers $a_1, a_2, \ldots$ satisfies the recurrence

$$a_{n+3} = a_{n+2} - 2a_{n+1} + a_n$$

for every positive integer n. Given that $a_1 = a_3 = 1$ and $a_{98} = a_{99}$, compute $a_1 + a_2 + \cdots + a_{100}$.

Proposed by: Evan Chen

Answer: $\boxed{3}$

A quick telescope gives that $a_1 + \cdots + a_n = 2a_1 + a_3 + a_{n-1} - a_{n-2}$ for all $n \geq 3$:

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= a_1 + a_2 + a_3 + \sum_{k=1}^{n-3}(a_k - 2a_{k+1} + 2a_{k+2})\\
&= a_1 + a_2 + a_3 + \sum_{k=1}^{n-3} a_k - 2\sum_{k=2}^{n-2} a_k + \sum_{k=3}^{n-1} a_k\\
&= 2a_1 + a_3 - a_{n-2} + a_{n-1}.
\end{aligned}$$

Putting $n = 100$ gives the answer.

One actual value of a_2 which yields the sequence is $a_2 = \frac{742745601954}{597303450449}$.

6. Call a positive integer $N \geq 2$ "special" if for every k such that $2 \leq k \leq N$, N can be expressed as a sum of k positive integers that are relatively prime to N (although not necessarily relatively prime to each other). How many special integers are there less than 100?

Proposed by: Casey Fu

Answer: $\boxed{50}$

We claim that all odd numbers are special, and the only special even number is 2. For any even $N > 2$, the numbers relatively prime to N must be odd. When we consider $k = 3$, we see that N can't be expressed as a sum of 3 odd numbers.

Now suppose that N is odd, and we look at the binary decomposition of N, so write $N = 2^{a_1} + 2^{a_2} + ... + 2^{a_j}$ as a sum of distinct powers of 2. Note that all these numbers only have factors of 2 and are therefore relatively prime to N. We see that $j < \log_2 N + 1$.

We claim that for any $k \geq j$, we can write N as a sum of k powers of 2. Suppose that we have N written as $N = 2^{a_1} + 2^{a_2} + ... + 2^{a_k}$. Suppose we have at least one of these powers of 2 even, say 2^{a_1}. We can then write $N = 2^{a_1-1} + 2^{a_1-1} + 2^{a_2} + ... + 2^{a_k}$, which is $k+1$ powers of 2. The only way this process cannot be carried out is if we write N as a sum of ones, which corresponds to $k = N$. Therefore, this gives us all $k > \log_2 N$.

Now we consider the case $k = 2$. Let 2^a be the largest power of 2 such that $2^a < N$. We can write $N = 2^a + (N - 2^a)$. Note that since 2^a and N are relatively prime, so are $N - 2^a$ and N. Note that $a < \log_2 N$. Now similar to the previous argument, we can write 2^a as a sum of k powers of 2 for $1 < k < 2^a$, and since $2^a > \frac{N}{2}$, we can achieve all k such that $2 \leq k < \frac{N}{2} + 1$.

Putting these together, we see that since $\frac{N}{2} + 1 > \log_2 N$ for $N \geq 3$, we can achieve all k from 2 through N, where N is odd.

7. Determine the smallest positive integer $n \geq 3$ for which

$$A \equiv 2^{10n} \pmod{2^{170}}$$

where A denotes the result when the numbers 2^{10}, 2^{20}, $\ldots$, 2^{10n} are written in decimal notation and concatenated (for example, if $n = 2$ we have $A = 10241048576$).

Proposed by: Evan Chen

Answer: $\boxed{14}$

Note that

$$2^{10n} = 1024^n = 1.024^n \times 10^{3n}.$$

So 2^{10n} has roughly $3n+1$ digits for relatively small n's. (Actually we have that for $0 < x < 1$,

$$(1+x)^2 = 1 + 2x + x^2 < 1 + 3x.$$

Therefore, $1.024^2 < 1.03^2 < 1.09$, $1.09^2 < 1.27$, $1.27^2 < 1.81 < 2$, and $2^2 = 4$, so $1.024^{16} < 4$. Thus the conclusion holds for $n \leq 16$.)

For any positive integer $n \leq 16$,

$$A = \sum_{i=1}^{n} 2^{10i} \times 10^{\sum_{j=i+1}^{n}(3j+1)}.$$

Let

$$A_i = 2^{10i} \times 10^{\sum_{j=i+1}^{n}(3j+1)}$$

for $1 \leq i \leq n$, then we know that

$$A - 2^{10n} = \sum_{i=1}^{n-1} A_i$$

and

$$A_i = 2^{10i + \sum_{j=i+1}^{n}(3j+1)} \times 5^{\sum_{j=i+1}^{n}(3j+1)} = 2^{u_i} \times 5^{v_i}$$

where $u_i = 10i + \sum_{j=i+1}^{n}(3j+1)$, $v_i = \sum_{j=i+1}^{n}(3j+1)$. We have that

$$u_i - u_{i-1} = 10 - (3i+1) = 3(3-i).$$

Thus, for $1 \leq i \leq n-1$, u_i is minimized when $i = 1$ or $i = n-1$, with $u_1 = \frac{3n^2+5n+12}{2}$ and $u_{n-1} = 13n - 9$. When $n = 5$,

$$A - 2^{10n} = A_1 + A_2 + A_3 + A_4 = 2^{10} \times 10^{46} + 2^{20} \times 10^{39} + 2^{30} \times 10^{29} + 2^{40} \times 10^{16}$$

is at most divisible by 2^{57} instead of 2^{170}. For all other n's, we have that $u_1 \neq u_{n-1}$, so we should have that both $170 \leq u_1$ and $170 \leq u_{n-1}$. Therefore, since $170 \leq u_{n-1}$, we have that $14 \leq n$. We can see that $u_1 > 170$ and $14 < 16$ in this case. Therefore, the minimum of n is $\boxed{14}$.

8. Define $\phi^!(n)$ as the product of all positive integers less than or equal to n and relatively prime to n. Compute the number of integers $2 \le n \le 50$ such that n divides $\phi^!(n) + 1$.

   Proposed by: Alexander Katz

   Answer: $\boxed{30}$

   Note that, if k is relatively prime to n, there exists a unique $0 < k^{-1} < n$ such that $kk^{-1} \equiv 1 \pmod{n}$. Hence, if $k^2 \not\equiv 1 \pmod{n}$, we can pair k with its inverse to get a product of 1.

   If $k^2 \equiv 1 \pmod{n}$, then $(n-k)^2 \equiv 1 \pmod{n}$ as well, and $k(n-k) \equiv -k^2 \equiv -1 \pmod{n}$. Hence these k can be paired up as well, giving products of -1. When $n \neq 2$, there is no k such that $k^2 \equiv 1 \pmod{n}$ and $k \equiv n - k \pmod{n}$, so the total product $\pmod{n}$ is $(-1)^{\frac{m}{2}}$, where m is the number of k such that $k^2 \equiv 1 \pmod{n}$.

   For prime p and positive integer i, the number of solutions to $k^2 \equiv 1 \pmod{p^i}$ is 2 if p is odd, 4 if $p = 2$ and $i \ge 3$, and 2 if $p = i = 2$. So, by Chinese remainder theorem, if we want the product to be -1, we need $n = p^k, 2p^k$, or 4. We can also manually check the $n = 2$ case to work.

   Counting the number of integers in the allowed range that are of one of these forms (or, easier, doing complementary counting), we get an answer of 30.

   (Note that this complicated argument basically reduces to wanting a primitive root.)

9. For any positive integer n, S_n be the set of all permutations of $\{1, 2, 3, \ldots, n\}$. For each permutation $\pi \in S_n$, let $f(\pi)$ be the number of ordered pairs (j, k) for which $\pi(j) > \pi(k)$ and $1 \le j < k \le n$. Further define $g(\pi)$ to be the number of positive integers $k \le n$ such that $\pi(k) \equiv k \pm 1 \pmod{n}$. Compute

$$\sum_{\pi \in S_{999}} (-1)^{f(\pi)+g(\pi)}.$$

   Proposed by: Ritesh Ragavender

   Answer: $\boxed{995 \times 2^{998}}$

   Define an $n \times n$ matrix $A_n(x)$ with entries $a_{i,j} = x$ if $i \equiv j \pm 1 \pmod{n}$ and 1 otherwise. Let $F(x) = \sum_{\pi \in S_n} (-1)^{f(\pi)} x^{g(\pi)}$ (here $(-1)^{f(\pi)}$ gives the sign $\prod \frac{\pi(u)-\pi(v)}{u-v}$ of the permutation π). Note by construction that $F(x) = \det(A_n(x))$.

   We find that the eigenvalues of $A_n(x)$ are $2x + n - 2$ (eigenvector of all ones) and $(x-1)(\omega_j + \omega_j^{-1})$, where $\omega_j = e^{\frac{2\pi j i}{n}}$, for $1 \le j \le n-1$. Since the determinant is the product of the eigenvalues,

$$F(x) = (2x + n - 2)2^{n-1}(x-1)^{n-1} \prod_{k=1}^{n-1} \cos\left(\frac{2\pi k}{n}\right).$$

   Evaluate the product and plug in $x = -1$ to finish. (As an aside, this approach also tells us that the sum is 0 whenever n is a multiple of 4.)

10. Let a, b and c be positive real numbers such that

$$\begin{aligned} a^2 + ab + b^2 &= 9 \\ b^2 + bc + c^2 &= 52 \\ c^2 + ca + a^2 &= 49. \end{aligned}$$

   Compute the value of $\dfrac{49b^2 - 33bc + 9c^2}{a^2}$.

   Proposed by: Alexander Katz

   Answer: $\boxed{52}$

   Consider a triangle ABC with Fermat point P such that $AP = a, BP = b, CP = c$. Then

$$AB^2 = AP^2 + BP^2 - 2AP \cdot BP \cos(120^\circ)$$

by the Law of Cosines, which becomes

$$AB^2 = a^2 + ab + b^2$$

and hence $AB = 3$. Similarly, $BC = \sqrt{52}$ and $AC = 7$.

Furthermore, we have

$$\begin{aligned} BC^2 = 52 &= AB^2 + BC^2 - 2AB \cdot BC \cos\angle BAC \\ &= 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cos\angle BAC \\ &= 58 - 42 \cos\angle BAC \end{aligned}$$

And so $\cos\angle BAC = \frac{1}{7}$.

Invert about A with arbitrary radius r. Let B', P', C' be the images of B, P, C respectively. Since $\angle APB = \angle AB'P' = 120°$ and $\angle APC = \angle AC'P' = 120°$, we note that $\angle B'P'C' = 120° - \angle BAC$, and so

$$\begin{aligned} \cos\angle B'P'C' &= \cos(120° - \angle BAC) \\ &= \cos 120° \cos\angle BAC - \sin 120° \sin\angle BAC \\ &= -\frac{1}{2}\left(\frac{1}{7}\right) + \frac{\sqrt{3}}{2}\left(\frac{4\sqrt{3}}{7}\right) \\ &= \frac{11}{14} \end{aligned}$$

Furthermore, using the well-known result

$$B'C' = \frac{r^2 BC}{AB \cdot AC}$$

for an inversion about A, we have

$$\begin{aligned} B'P' &= \frac{BPr^2}{AB \cdot AP} \\ &= \frac{br^2}{a \cdot 3} \\ &= \frac{br^2}{3a} \end{aligned}$$

and similarly $P'C' = \frac{cr^2}{7a}$, $B'C' = \frac{r^2\sqrt{52}}{21}$. Applying the Law of Cosines to $B'P'C'$ gives us

$$\begin{aligned} B'C'^2 &= B'P'^2 + P'C'^2 - 2B'P' \cdot P'C' \cos(120° - \angle BAC) \\ \implies \frac{52r^4}{21^2} &= \frac{b^2r^4}{9a^2} + \frac{c^2r^4}{49a^2} - \frac{11bcr^4}{147a^2} \\ \implies \frac{52}{21^2} &= \frac{b^2}{9a^2} + \frac{c^2}{49a^2} - \frac{11bc}{147a^2} \\ \implies \frac{52}{21^2} &= \frac{49b^2 - 33bc + 9c^2}{21^2a^2} \end{aligned}$$

and so $\frac{49b^2-33bc+9c^2}{a^2} = \boxed{52}$.

Motivation: the desired sum looks suspiciously like the result of some Law of Cosines, so we should try building a triangle with sides $\frac{7b}{a}$ *and* $\frac{3c}{a}$. *Getting the* $-\frac{33bc}{a}$ *term is then a matter of setting* $\cos\theta = \frac{11}{14}$. *Now there are two possible leaps: noticing that* $\cos\theta = \cos(120 - \angle BAC)$, *or realizing that it's pretty difficult to contrive a side of* $\frac{7b}{a}$ *– but it's much easier to contrive a side of* $\frac{b}{3a}$. *Either way leads to the natural inversion idea, and the rest is a matter of computation.*