# HMMT February 2024

**February 17, 2024**

## Team Round

1. **[20]** Let $a_1, a_2, a_3, \ldots, a_{100}$ be integers such that

$$\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_{100}^2}{a_1 + a_2 + a_3 + \cdots + a_{100}} = 100.$$

Determine, with proof, the maximum possible value of $a_1$.

*Proposed by: Nilay Mishra*

**Answer:** $\boxed{550}$

**Solution 1:** We can rearrange the equation as follows:

$$a_1^2 + a_2^2 + \cdots + a_{99}^2 + a_{100}^2 = 100(a_1 + a_2 + \cdots + a_{99} + a_{100})$$

$$(a_1^2 - 100a_1) + (a_2^2 - 100a_2) + (a_3^2 - 100a_3) + \cdots + (a_{100}^2 - 100a_{100}) = 0$$

$$(a_1 - 50)^2 + (a_2 - 50)^2 + (a_3 - 50)^2 + \cdots + (a_{100} - 50)^2 = 100 \cdot 50^2 = 500^2.$$

Thus $(a_1 - 50)^2 \leq 500^2$ and so $a_1 \leq 550$. Equality holds when $a_1 = 550$ and $a_i = 50$ for all $i > 1$. Therefore, the maximum possible value of $a_1$ is $\boxed{550}$.

**Solution 2:** Let $k = \frac{1}{99}(a_2 + a_3 + \cdots + a_{100})$. Note, that by the Cauchy-Schwarz inequality, we have:

$$99 \cdot (a_2^2 + a_3^2 + \cdots + a_{100}^2) \geq (a_2 + a_3 + \cdots + a_{100})^2 = (99k)^2$$

and so:

$$100 = \frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_{100}^2}{a_1 + a_2 + a_3 + \cdots + a_{100}} \geq \frac{a_1^2 + 99k^2}{a_1 + 99k}$$

Now consider the function:

$$f_t(x) = \frac{99x^2 + t^2}{99x + t}$$

for all reals $t$. Note that for $t = a_1$, we have that $f_t(k) \leq 100$ for some real $k$. Furthermore, because $a_1^2 + a_2^2 + \cdots + a_{100}^2 > 0$, we have $a_1 + a_2 + \cdots + a_{100} > 0$, and so $f_t(x) > 0$ for all valid $x$.

We assume $a_1 > 0$, since we are trying to maximize this element and some positive solution exists (i.e. $(100, 100, \ldots, 100)$). Hence, the minimum point of $f_t(x)$ over $x > 0$ needs to be less than 100. To compute this, set the derivative to be zero as follows:

$$f_t'(x) = 1 - \frac{100t^2}{(t + 99x)^2} = 0$$

which has solutions at $x = \frac{t}{11}$ and $x = -\frac{t}{9}$. Taking a second derivative shows that the point $(\frac{t}{11}, \frac{2t}{11})$ is the only valid minimum point in the first quadrant. Hence, we must have $\frac{2t}{11} \leq 100$ and so $a_1 = t \leq \boxed{550}$, as desired. Equality holds at $(550, 50, \ldots, 50)$.

2. **[25]** Nine distinct positive integers summing to 74 are put into a $3 \times 3$ grid. Simultaneously, the number in each cell is replaced with the sum of the numbers in its adjacent cells. (Two cells are adjacent if they share an edge.) After this, exactly four of the numbers in the grid are 23. Determine, with proof, all possible numbers that could have been originally in the center of the grid.

*Proposed by: Rishabh Das*

**Answer:** $\boxed{18}$

**Solution:** Suppose the initial grid is of the format shown below:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

After the transformation, we end with

$$\begin{bmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & h_n & i_n \end{bmatrix} = \begin{bmatrix} b+d & a+c+e & b+f \\ a+e+g & b+d+f+h & c+e+i \\ d+h & g+e+i & f+h \end{bmatrix}$$

Since $d \neq f$, $a_n = b + d \neq b + f = c_n$. By symmetry, no two corners on the same side of the grid may both be 23 after the transformation.

Since $c \neq g$, $b_n = a + c + e \neq a + e + g = d_n$. By symmetry, no two central-edge squares sharing a corner may both be 23 after the transformation.

Assume for the sake of contradiction that $e_n = 23$. Because $a_n, c_n, g_n, i_n < e_n$, none of $a_n$, $c_n$, $g_n$, $i_n$ can be equal to 23. Thus, 3 of $b_n, d_n, f_n, h_n$ must be 23. WLOG assume $b_n = d_n = f_n = 23$. Thus is a contradiction however, as $b_n \neq d_n$. Thus, $e_n \neq 23$.

This leaves the case with two corners diametrically opposite and two central edge squares diametrically opposite being 23. WLOG assume $a_n = b_n = h_n = i_n = 23$.

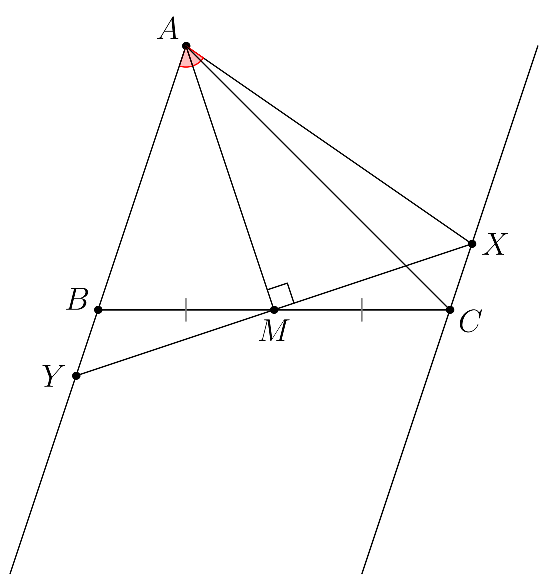
Thus, $92 = 4 \cdot 23 = a_n + b_n + h_n + i_n = (b + d) + (a + c + e) + (e + g + i) + (f + h) = (a + b + c + d + e + f + g + h + i) + e$. Since $a + b + c + d + e + f + g + h + i = 74$, this means that $e = 92 - 74 = 18$.

One possible example of 18 working is $\begin{bmatrix} 4 & 16 & 2 \\ 6 & 18 & 7 \\ 1 & 17 & 3 \end{bmatrix}$. Thus the only possible value for the center is 18.

3. **[25]** Let $ABC$ be a scalene triangle and $M$ be the midpoint of $BC$. Let $X$ be the point such that $CX \parallel AB$ and $\angle AMX = 90°$. Prove that $AM$ bisects $\angle BAX$.

*Proposed by: Pitchayut Saengrungkongka*

**Solution:**



Let $Y$ be the intersection of lines $AB$ and $XM$. Since $BY \parallel CX$, we have $\angle YBM = \angle XCM$. Furthermore, we have $BM = CM$, since $M$ is the midpoint of $BC$. Thus,

$$\triangle BMY \cong \triangle CMX.$$

Thus, $MY = MX$. Combined with the condition $AM \perp XY$, we get that $AYX$ is an isosceles triangle with median $AM$. Therefore, $AM$ bisects $\angle YAX$ which is the same as $\angle BAX$ and we are done.

4. **[30]** Each lattice point with nonnegative coordinates is labeled with a nonnegative integer in such a way that the point $(0,0)$ is labeled by 0, and for every $x, y \geq 0$, the set of numbers labeled on the points $(x,y)$, $(x,y+1)$, and $(x+1,y)$ is $\{n, n+1, n+2\}$ for some nonnegative integer $n$. Determine, with proof, all possible labels for the point $(2000, 2024)$.

*Proposed by: Nithid Anchaleenukoon*

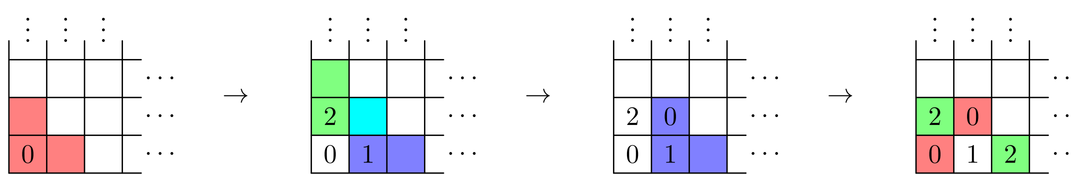
**Answer:** $\boxed{0, 3, 6, 9, \ldots, 6048}$

**Solution:** We claim the answer is all multiples of 3 from 0 to $2000 + 2 \cdot 2024 = 6048$. First, we prove no other values are possible. Let $\ell(x,y)$ denote the label of cell $(x,y)$.

**The label is divisible by 3.**

Observe that for any $x$ and $y$, $\ell(x,y)$, $\ell(x,y+1)$, and $\ell(x+1,y)$ are all distinct mod 3. Thus, for any $a$ and $b$, $\ell(a+1,b+1)$ cannot match $\ell(a+1,b)$ or $\ell(a,b+1)$ mod 3, so it must be equivalent to $\ell(a,b)$ modulo 3.

Since $\ell(a,b+1), \ell(a,b+2), \ell(a+1,b+1)$ are all distinct mod 3, and $\ell(a+1,b+1)$ and $\ell(a,b)$ are equivalent mod 3, then $\ell(a,b), \ell(a,b+1), \ell(a,b+2)$ are all distinct mod 3, and thus similarly $\ell(a,b+1), \ell(a,b+2), \ell(a,b+3)$ are all distinct mod 3, which means that $\ell(a,b+3)$ must be neither $\ell(a,b+1)$ or $\ell(a,b+2)$ mod 3, and thus must be equal to $\ell(a,b)$ mod 3.



These together imply that

$$\ell(w,x) \equiv \ell(y,z) \bmod 3 \iff w - x \equiv y - z \bmod 3.$$

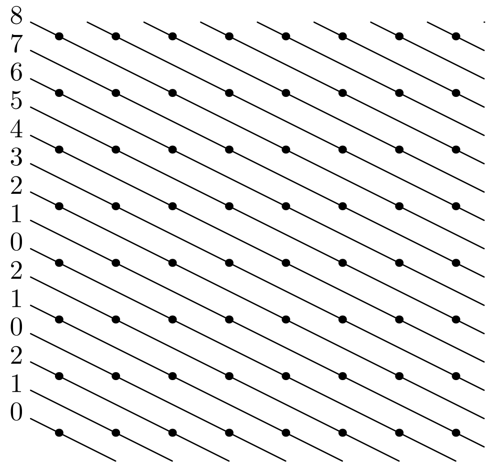
It follows that $\ell(2000, 2024)$ must be equivalent to $\ell(0,0)$ mod 3, which is a multiple of 3.

**The label is at most 6048.**

Note that since $\ell(x+1,y)$, $\ell(x,y+1)$, and $\ell(x,y)$ are 3 consecutive numbers, $\ell(x+1,y) - \ell(x,y)$ and $\ell(x,y+1) - \ell(x,y)$ are both $\leq 2$. Moreover, since $\ell(x+1,y+1) \leq \ell(x,y) + 4$, since it is also the same mod 3, it must be at most $\ell(x,y) + 3$. Thus, $\ell(2000,2000) \leq \ell(0,0) + 3 \cdot 2000$, and $\ell(2000,2024) \leq \ell(2000,2000) + 2 \cdot 24$, so $\ell(2000,2024) \leq 6048$.

**Construction.**

Consider lines $\ell_n$ of the form $x + 2y = n$ (so $(2000, 2024)$ lies on $\ell_{6048}$). Then any three points of the form $(x,y)$, $(x,y+1)$, and $(x+1,y)$ lie on three consecutive lines $\ell_n$, $\ell_{n+1}$, $\ell_{n+2}$ in some order. Thus, for any $k$ which is a multiple of 3, if we label every point on line $\ell_i$ with $\max(i \bmod 3, i-k)$, any three consecutive lines $\ell_n$, $\ell_{n+1}$, $\ell_{n+2}$ will either be labelled 0, 1, and 2 in some order, or $n-k$, $n-k+1$, $n-k+2$, both of which consist of three consecutive numbers. Below is an example with $k = 6$.

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 2 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 1 | 2 | 0 | 1 | 2 | 3 |
| 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 |

Any such labelling is valid, and letting $k$ range from 0 to 6048, we see $(2000, 2024)$ can take any label of the form $6048 - k$, which spans all such multiples of 3.

Hence the possible labels are precisely the multiples of 3 from 0 to 6048.

5. **[40]** Determine, with proof, whether there exist positive integers $x$ and $y$ such that $x + y$, $x^2 + y^2$, and $x^3 + y^3$ are all perfect squares.

*Proposed by: Rishabh Das*

**Answer:** $\boxed{\text{Yes}}$

**Solution:** Take $(x, y) = (184, 345)$. Then $x + y = 23^2$, $x^2 + y^2 = 391^2$, and $x^3 + y^3 = 6877^2$.

*Remark.* We need $x + y, x^2 + y^2, x^2 - xy + y^2$ to be perfect squares. We will find $a, b$ such that $a^2 + b^2, a^2 - ab + b^2$ are perfect squares, and then let $x = a(a + b)$ and $y = b(a + b)$. Experimenting with small Pythagorean triples gives $a = 8, b = 15$ as a solution.

*Remark.* The smallest solution we know of not of the form $(184k^2, 345k^2)$ is

$$(147\,916\,017\,521\,041, 184\,783\,370\,001\,360).$$

6. **[45]** Let $\mathbb{Q}$ be the set of rational numbers. Given a rational number $a \neq 0$, find, with proof, all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying the equation

$$f(f(x) + ay) = af(y) + x$$

for all $x, y \in \mathbb{Q}$.

*Proposed by: Yichen Xiao*

**Answer:** $\boxed{\begin{cases} f(x) = x \\ f(x) = -x \\ f(x) = x + c \text{ for all rational numbers } c \text{ iff } a = 2 \end{cases}}$

**Solution:** Let $P(x, y)$ denote the functional equation. From $P(x, 0)$, we have $f(f(x)) = x + af(0)$. Thus, the tripling trick gives $f(x + af(0)) = f(f(f(x))) = f(x) + af(0)$.

Now, here is the main idea: $P(f(x), y)$ gives

$$\begin{aligned} f\big(f(f(x)) + ay\big) &= af(y) + f(x) \\ f(x + af(0) + ay) &= f(x) + af(y) \\ f(x + ay) &= f(x) + af(y) - af(0). \end{aligned}$$

In particular, plugging in $x = 0$ into this equation gives $f(ay) = af(y) + (1 - a)f(0)$, so inserting it back to the same equation gives

$$f(x + ay) = f(x) + f(ay) - f(0),$$

for all rational numbers $x, y$. In particular, the function $g(x) = f(x) - f(0)$ is additive, so $f$ is linear.

Let $f(x) = bx + c$. By substituting it in, we have $P(x, y)$ iff

$$\begin{aligned} f(ay + bx + c) &= a(by + c) + x \\ aby + b^2x + bc + c &= aby + ac + x \\ (b^2 - 1)x + (b + 1 - a)c &= 0. \end{aligned}$$

Since $x$ is arbitrary, we can state that $b^2 - 1 = 0$ and $(b + 1 - a)c = 0$, thus $b = \pm 1$. As $a \neq 0$, we know $b + 1 - a = 0$ only if $b = 1$ and $a = 2$. When $a \neq 2$ or $b \neq 1$, we know the only solutions are $b = \pm 1, c = 0$, while for $a = 2, b = 1$, the equation is automatically satisfied, so the final answer is

$$\begin{cases} f(x) = x \\ f(x) = -x \\ f(x) = x + c \text{ for all rational number } c \text{ iff } a = 2 \end{cases}$$

**Solution 2:** We will only prove that $f$ is linear. Then, proceed as in the end of Solution 1.

We know $f(f(x)) = af(0)+x$, so as $af(0)+x$ can take any rational number when $x$ takes every rational number, the range of $f$ is $\mathbb{Q}$, and so $f$ is surjective. If $f(x_1) = f(x_2)$, we have $x_1 = f(f(x_1)) - af(0) = f(f(x_2)) - af(0) = x_2$, so $x_1 = x_2$, implying $f$ being injective. Thus, $f$ is bijective.

From $P(x, 0)$, we still get $f(f(x)) = x + af(0)$.

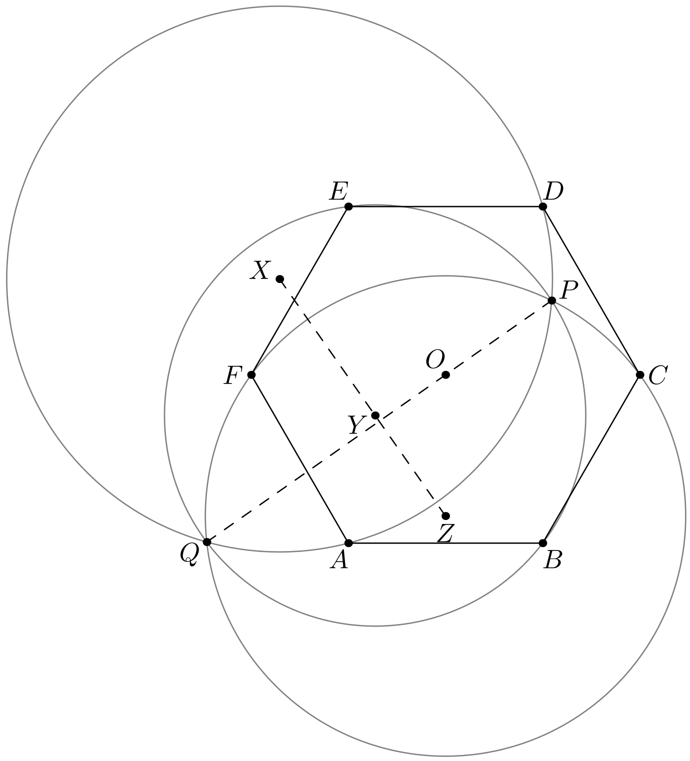
Thus, from $P(f^{-1}(0)), y/a)$, we can get $f(y) = af(y/a) + f^{-1}(0)$. Plugging again $P(f(x), y/a)$, we have $f(f(f(x)) + y) = f(x + y + af(0)) = af(y/a) + f(x) = f(x) + f(y) - f^{-1}(x)$.

Thus, we know $f(x + y) = f(x - af(0)) + f(y) - f^{-1}(0) = f(x) + f(y) - f(0)$. Hence, the function $g(x) = f(x) - f(0)$ is additive, so $g(x) = kx$ for some rational number $k$. Thus, $f$ is a linear function, and we can proceed as in above solution.

7. **[50]** Let $ABCDEF$ be a regular hexagon with $P$ as a point in its interior. Prove that of the three values $\tan\angle APD$, $\tan\angle BPE$, and $\tan\angle CPF$, two of them sum to the third one.

   *Proposed by: Albert Wang*

   **Solution 1:**




WLOG let the side length of the hexagon be 1. Let $O$ be the center of the hexagon. Consider drawing in the circles $(APD)$, $(BPE)$, and $(CPF)$. Note that $O$ lies on the radical axis of all three circles, since $AO \cdot OD = BO \cdot OE = CO \cdot OF$. Since $P$ also lies on the radical axis, all three circles are coaxial.

Let $X$, $Y$, and $Z$ be the centers of $(APD)$, $(BPE)$, and $(CPF)$, respectively. Since the circles are coaxial, $X$, $Y$, and $Z$ are collinear. WLOG $Y$ lies on segment $XZ$. Note that $XO \perp AD, YO \perp BE \implies \angle XOY = 60°$. Similarly, we have $\angle YOZ = 60°$. Now, inverting at $O$ and using van Schooten's Theorem gives that $1/OY = 1/OX + 1/OZ$.

Furthermore, we have

$$\angle APD = 180° - \frac{1}{2}\angle AXD = 180° - \angle AXO \implies \tan \angle APD = -\tan \angle AXO = -\frac{AO}{OX} = -\frac{1}{OX}.$$

Similarly, we have $\tan \angle BPE = -\frac{1}{OY}$ and $\tan \angle CPF = -\frac{1}{OZ}$. Therefore, two of these tangent values sum to the third, as desired.

**Solution 2:**

Firstly, note that $AD$, $BE$, and $CF$ are diameters of the circle $(ABCDEF)$, so the angles $\angle APD$, $\angle BPE$, $\angle CPF$ are all obtuse. Therefore, the desired tangents are well-defined.

WLOG let the side length of the hexagon be 1. Let $O$ be the center of the hexagon, and let $OP = x$. Finally, let $\angle AOP = \theta$.

Now, by Law of Cosines, we have $AP = \sqrt{x^2 + 1 - 2x\cos\theta}$ and $DP = \sqrt{x^2 + 1 + 2x\cos\theta}$. Now, by Law of Cosines again we have

$$\cos \angle APD = \frac{2(x^2+1) - 4}{2\sqrt{(x^2+1)^2 - 4x^2\cos^2\theta}} = \frac{x^2 - 1}{\sqrt{(1-x^2)^2 + 4x^2\sin^2\theta}}$$

$$\implies \tan \angle APD = \frac{2x|\sin\theta|}{x^2 - 1}.$$

Similarly, $\angle(BE, OP)$ and $\angle(CF, OP)$ are $\theta+60^\circ$ and $\theta+120^\circ$, respectively (here we are using directed angles). Therefore, the desired three values are

$$\{\tan\angle APD, \tan\angle BPE, \tan\angle CPF\} = \left\{\frac{2x|\sin\theta|}{x^2-1}, \frac{2x|\sin(\theta+60^\circ)|}{x^2-1}, \frac{2x|\sin(\theta+120^\circ)|}{x^2-1}\right\}.$$

We can scale down the tangent values by $\frac{2x}{x^2-1}$ to get $\{|\sin\theta|, |\sin(\theta+60^\circ)|, |\sin(\theta+120^\circ)|\}$. Now, consider an equilateral triangle with vertices at the third roots of unity rotated by $\theta$ degrees counter-clockwise. The three values represent the distances from the three vertices to the real axis. Since the centroid of this triangle is the origin (lying on the real axis), two of these quantities must sum to the third, as desired.

**Solution 3:** We will show either the three sum to 0 or two of them sum to the third one; since they're all negative, the former case is actually impossible.

Let $(ABCDEF)$ be the unit circle, with $a = 1$, $b = \omega$, and so on, where $\omega = e^{\pi i/3}$. Then $\angle APD$ is the argument of

$$\frac{1-p}{\omega^3 - p} = \frac{(1-p)(\omega^3 - \overline{p})}{(\omega^3 - p)(\omega^3 - \overline{p})} = \frac{\omega^3 - \overline{p} - \omega^3 p + |p|^2}{1 + |p|^2 - \omega^3 p - \omega^3 \overline{p}}.$$

Then $\tan\angle APD$ is the imaginary part divided by the real part of this, which is

$$-\frac{1}{i}\cdot\frac{\overline{p}-p}{|p|^2-1} = c\cdot \operatorname{dist}(P, AD)$$

for some constant $c$. (Note that $\tan APD$ might actually be the negative of this, depending on direction; this is why we added the remark at the beginning about them possibly summing to 0.)

Similarly, $\tan\angle BPE = c\cdot\operatorname{dist}(P, BE)$ and $\tan\angle CPF = c\cdot\operatorname{dist}(P, CF)$. It suffices to show that two $\operatorname{dist}(P, AD)$, $\operatorname{dist}(P, BE)$, and $\operatorname{dist}(P, CF)$ sum to the third. However, this is easy; without loss of generality let $P$ be inside $OAB$, and let the hexagon have side length 1. Then

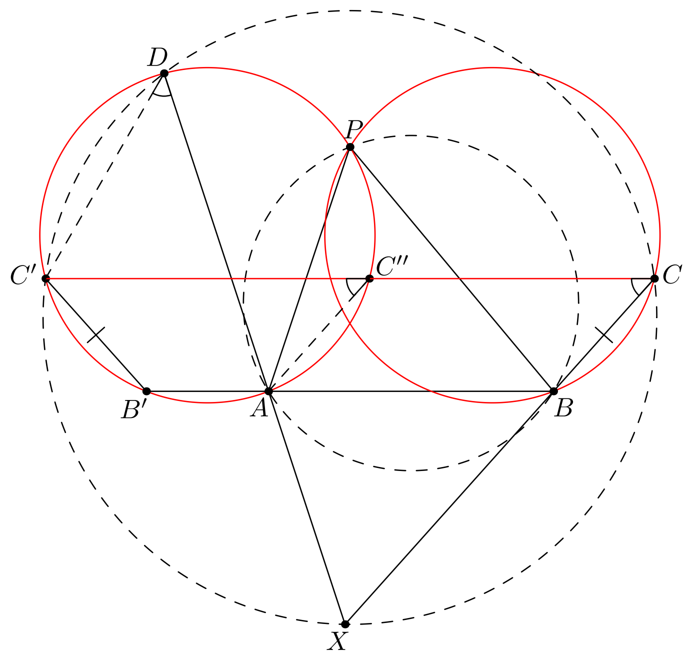
$$\operatorname{dist}(P, AD) + \operatorname{dist}(P, BE) = \frac{\sqrt{3}}{2} - \operatorname{dist}(P, AB) = \operatorname{dist}(P, CF),$$

as desired.

8. **[50]** Let $P$ be a point in the interior of quadrilateral $ABCD$ such that the circumcircles of triangles $PDA$, $PAB$, and $PBC$ are pairwise distinct but congruent. Let the lines $AD$ and $BC$ meet at $X$. If $O$ is the circumcenter of triangle $XCD$, prove that $OP \perp AB$.

*Proposed by: Pitchayut Saengrungkongka*

**Solution 1:**




Because the circles have equal radii, $\angle PDA = \angle ABP$, so if $(PDA)$ intersects line $AB$ again at a point $B'$, then we have $\angle PB'B = \angle PBB'$, which means $PB = PB'$, similarly for the second intersection of $(PCB)$ with $AB$, $A'$; thus, $(PDA)$ and $(PCB)$ are congruent mirror images across the $P$-altitude, as they are $(PAB')$ and $(PBA')$, respectively.

Consider $C'$, the reflection of $C$ across the $P$-altitude. We want to prove that $C'$ lies on $(XCD)$, as then the circumcenter of $(XCD)$ will lie on the perpendicular bisector of $CC'$. Because of our earlier observation, $C', D, P, A$ are concyclic.

We present two approaches to finishing the angle chase from here:

- Add point $C''$, the intersection of $CC'$ with $(XDA)$. Because $AB$ is parallel to the line between the centers, and so is $C''C$, then $ABCC''$ is a parallelogram; thus,

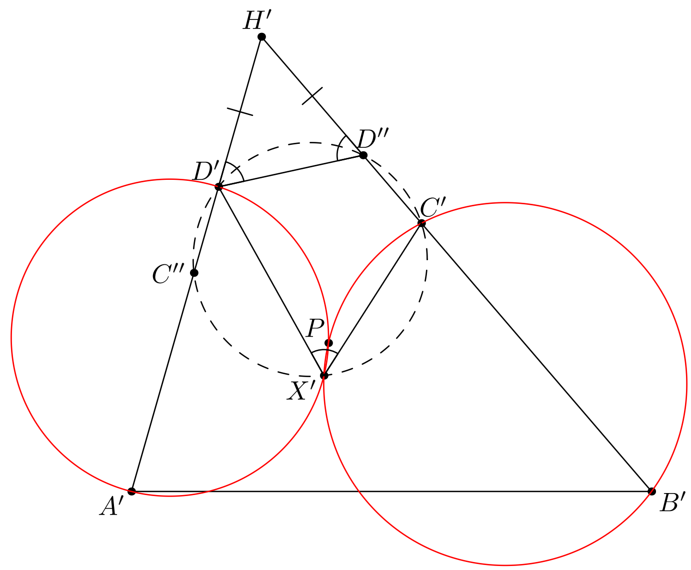
$$\angle C'DA = \angle C'C''A = \angle C'CB = \angle C'CX.$$

- Add point $B'$, the reflection of $B$ over the $P$-altitude. Note that $B'$ lies on $(XDA)$; in particular, $C'CBB'$ is an isosceles trapezoid, as $C'B'$ is the reflection of $CB$ over the $P$-altitude of $\triangle PAB$. Thus,

$$\angle C'DA = 180 - \angle C'B'A = 180 - \angle C'B'B = \angle C'CB = \angle C'CX.$$

*Remark.* It is possible to do the last angle chasing without adding any additional points (beyond $C'$). However, the details are much messier.

**Solution 2:**




Invert about $P$. Because the circles $(PDA), (PAB), (PBC)$ all have equal radii and pass through $P$, the resulting lines $D'A', A'B', B'C'$ are equal distances away from $P$; letting $H'$ be the intersection of lines $D'A'$ and $B'C'$, it follows that $P$ is an incenter or excenter of $\triangle A'B'H'$. Also, in the original diagram, the $P$-altitude of $\triangle PAB$ includes the second intersection of the circles $(PDA)$ and $(PCB)$ (as in the first solution); thus this $P$-altitude inverts to line $PH'$. Finally, $X'$ is the intersection of $(PD'A')$ and $(PB'C')$.

Note that the center of $(XCD)$ lies on the $P$-altitude of $PAB$ iff the inverse of the center of $(X'C'D')$ does. Thus we want to show that the center of $(X'C'D')$ lies on $H'P$. Let $D''$ be the second intersection of $X'C'D'$ with $B'H'$. Then

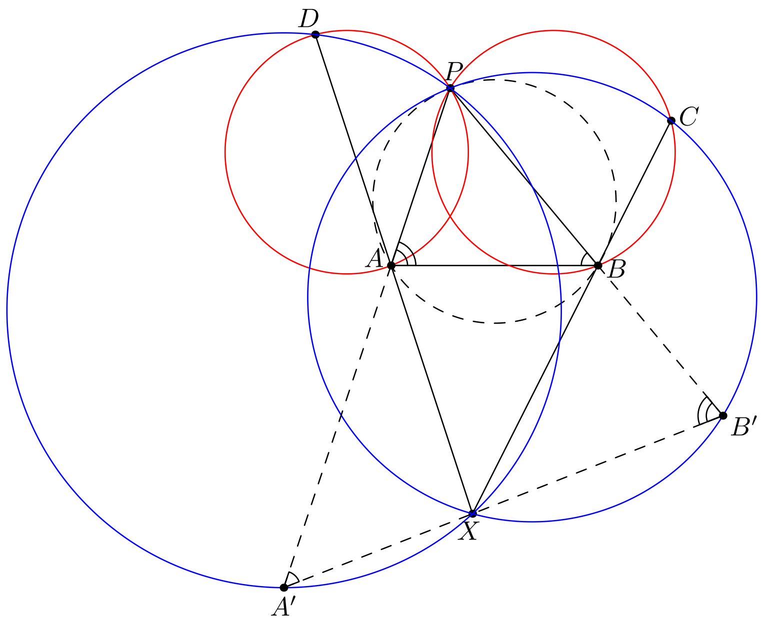
$$\begin{aligned}
\measuredangle(D'D'', H'P) &= \measuredangle D'D''H' + \measuredangle D''H'P \\
&= \measuredangle D'X'C' + \measuredangle B'H'P \\
&= \measuredangle D'X'P + \measuredangle PX'C' + \measuredangle PH'A' \\
&= \measuredangle D'A'P + \measuredangle PB'C' + \measuredangle PH'A' \\
&= \measuredangle PA'B' + \measuredangle PB'H' + \measuredangle PH'A' \\
&= 90^\circ
\end{aligned}$$

(where the last step follows from the fact that $P$ is an incenter or excenter of $\triangle A'B'H'$), and

$$\begin{aligned}
\measuredangle H'D'D'' &= 180^\circ - \measuredangle D''HD' - \measuredangle D'D''H \\
&= 180^\circ - \measuredangle D''H'P - (\measuredangle D''H'P + \measuredangle D'D''H') \\
&= 90^\circ - \measuredangle D''H'P \\
&= \measuredangle D'D''H',
\end{aligned}$$

so $\triangle H'D'D''$ is isosceles, and thus $H'P$ is the perpendicular bisector of $D'D''$. Thus the center of $(X'C'D'D'')$ lies on $H'P$, which means we're done.

**Solution 3:**




Let $A'$ be the other intersection of line $PA$ with $(PDX)$ and $B'$ be the other intersection of $PB$ with $(PCX)$. Consider circles $(PA'B')$ and $(XCD)$. Note that $A$ and $B$ have equal power with respect to both circles, because of $(PDA'X)$ and $(PCB'X)$. Thus, $AB$ is the radical axis of the two cricles. However,

$$\measuredangle PA'X = \measuredangle PDX = \measuredangle PDA = \measuredangle ABP$$

and

$$\measuredangle XB'P = \measuredangle XCP = \measuredangle BCP = \measuredangle PAB,$$

where the last step follows from the fact that the circles have equal radii. Because $\measuredangle B'PA' = \measuredangle BPA$, it follows that $A', X, B'$ are collinear, and in fact $\triangle PAB \sim \triangle PB'A'$. In particular, this means that the $P$-altitude of $\triangle PAB$ passes through the circumcenter of $\triangle PB'A'$, as the circumcenter and orthocenter are isogonal conjugates. Thus, as the circumcenter of $PA'B'$ lies on the $P$-altitude, and the line between the centers of $(PA'B')$ and $(XCD)$ must be perpendicular to their radical axis $AB$, then the circumcenter of $(XCD)$ must lie on the $P$-altitude as well, completing the proof.

9. **[55]** On each cell of a $200 \times 200$ grid, we place a car, which faces in one of the four cardinal directions. In a move, one chooses a car that does not have a car immediately in front of it, and slides it one cell forward. If a move would cause a car to exit the grid, the car is removed instead. The cars are placed so that there exists a sequence of moves that eventually removes all the cars from the grid. Across all such starting configurations, determine the maximum possible number of moves to do so.

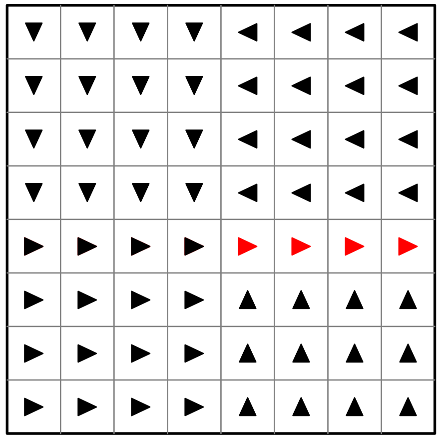
*Proposed by: Ethan Zhou*

**Answer:** $\boxed{6014950}$

**Solution:**

Let $n = 100$. The answer is $\frac{1}{2}n(12n^2 + 3n - 1) = 6014950$.

A construction for an $8 \times 8$ grid instead (so $n = 4$):



Label the rows and columns from 1 to $2n$, and let $(r, c)$ denote the cell at row $r$, column $c$. The cars can be cleared in the following order:

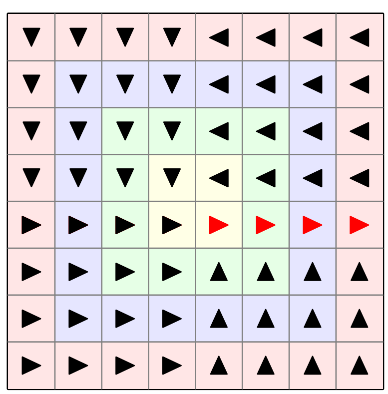
- Remove all cars in row $n$.
- For each row $k = n-1, \ldots, 1$, move the $n$ upward-facing cars in row $k$ once, then remove all remaining cars in row $k$.
- Now all cars in the upper-left quarter of the grid can be removed, then those in the upper-right, then those in the lower-right.

Moreover, this starting configuration indeed requires

$$4 \cdot \frac{n^2(3n+1)}{2} - \frac{n(n+1)}{2} = \frac{1}{2}n(12n^2 + 3n - 1)$$

moves to clear.

Now we show this is the best possible. Take some starting configuration for which it is possible for all cars to leave. For each car $c$, let $d(c)$ denote the number of moves $c$ makes before it exits. Partition the grid into concentric square "rings" $S_1, \ldots, S_n$, such that $S_1$ consists of all cells on the border of the grid, ..., $S_n$ consists of the four central cells:



Since all cars can be removed, each $S_k$ contains some car $c$ which points away from the ring, so that $d(c) = k$. Now fix some ring $S_k$. Then:

- If car $c$ is at a corner of $S_k$, we have $d(c) \leq 2n + 1 - k$.

- Each car $c$ on the bottom edge of $S_k$, say at $(x, k)$ for $k < x < 2n+1-k$, can be paired with the opposing car $c'$ at $(x, 2n+1-k)$. As $c, c'$ cannot point toward each other, we have

$$d(c) + d(c') \leq (2n+1-k) + \max\{x, 2n+1-x\}.$$

Likewise, we can pair each car $c$ at $(k, x)$ with the opposing car $c'$ at $(2n+1-k, x)$, getting the same bound.

- If $d(c) = k$, then pairing it with the opposing car $c'$ gives $d(c) + d(c') \leq 2n+1$. Note that this is less than the previous bound, by at least

$$\max\{x, 2n+1-x\} - k \geq n+1-k > 0.$$

Summing the contributions $d(c)$ from the four corners, each pair among the non-corner cars, and a pair involving an outward-facing car gives

$$\sum_{c \in S_k} d(c) \leq 4(2n+1-k) + 4\left(\sum_{x=k+1}^{n} [(2n+1-k) + (2n+1-x)]\right) - (n+1-k).$$

One can verify that this evaluates to $\frac{1}{2}n(12n^2+3n-1)$; alternatively, note that equality holds in our construction, so summing over all $1 \leq k \leq n$ must yield the desired tight upper bound.

10. **[60]** Across all polynomials $P$ such that $P(n)$ is an integer for all integers $n$, determine, with proof, all possible values of $P(i)$, where $i^2 = -1$.

*Proposed by: Pitchayut Saengrungkongka*

**Answer:** $\boxed{a + bi \text{ works if and only if } a, b \in \mathbb{Q} \text{ and } \nu_p(a), \nu_p(b) \geq 0 \text{ for all } p \equiv 1 \pmod 4.}$

**Solution:** We claim the answer is every complex number $a + bi$ where $a$ and $b$ are rationals whose simplified denominators are not multiples of any prime congruent to 1 modulo 4. The proof consists of two main steps: proving that powers of $p \equiv 1 \bmod 4$ can't appear in the denominator, and showing all possible values are attainable. We show three different methods of the former part.

**Impossibility via elementary number theory**

We first show that no other values are possible. Indeed, it is well known that any polynomial that maps $\mathbb{Z}$ into itself must be of the form $P(n) = \sum_{k=0}^{m} a_k \binom{n}{k}$ for integers $a_k$ and where we treat the binomials as formal polynomials. This may be proved via finite differences.

It is therefore sufficient to show that for any $k$, $\binom{i}{k}$ can be simplified to a fraction of the form $\frac{a+bi}{c}$, where $a, b, c$ are integers and $c$ is not divisible by any prime that is 1 modulo 4. We have that

$$\binom{i}{k} = \frac{i \cdot (i-1) \cdot \ \dots \ \cdot (i-(k-1))}{k \cdot (k-1) \cdot \ \dots \ \cdot 1}.$$

Pick any prime $p$ that is 1 modulo 4. Since $p$ is 1 modulo 4, there exist distinct residue classes $x, y$ modulo $p$ so that $x^2 \equiv y^2 \equiv -1 \bmod p$. We will show that for every integer, $r \leq k$ divisible by $p$ in the denominator, we can pair it with a disjoint set, $\{r_x, r_y\}$, of two positive integers less than $k$ in these two residue classes so that $(i - r_x)(i - r_y)$ has real and complex parts divisible by the highest power of $p$ dividing $r$. Thus any factor of $p$ that is 1 modulo 4 in the denominator, exists in the numerator as well, which suffices. Start with $u = 1$ and repeat the following process for increasing $u$ until there is nothing left to do.

For every positive integer $j \leq k$ such that $p^u | j$, there exist unique $j'_x, j'_y \in [j - p^u, j)$ satisfying $j'_x \equiv x \bmod p$, $j'_y \equiv y \bmod p$ and $p^u | {j'_x}^2 + 1, {j'_y}^2 + 1$. So pair $j$ with the set $\{j'_x, j'_y\}$, and pair its old partners if any to the old partners of $j'_x$ and $j'_y$.

The important feature of this assignment process is that we always have at step $u$, $p^{\min(u,v_p(r))}|r_x + r_y, r_x^2+1, r_y^2+1$. Thus at the end of the process, $(i-r_x)(i-r_y) = \frac{1}{2}((r_x+r_y)^2 - (r_x^2+1) - (r_y^2+1)) - (r_x+r_y)i$ has real and complex parts divisible by $p^{v_p(r)}$ as claimed.

**Impossibility via Gaussian Integers**

We work in the ring of Gaussian Integers $\mathbb{Z}[i] = \{a+bi : a, b \in \mathbb{Z}\}$, which is sitting inside number field $\mathbb{Q}[i] = \{a+bi : a, b \in \mathbb{Q}\}$. It's well-known that $\mathbb{Z}[i]$ is a unique factorization domain. For any Gaussian prime $\pi$, let $\nu_{pi}(z)$ denote the exponent of $\pi$ in the factorization of $z$.

Let $p \equiv 1 \pmod 4$ be a prime. It's well known that $p$ splits into two Gaussian primes, $p = \pi\overline{\pi}$. Note that it suffices to show

$$\nu_\pi(i(i-1)(i-2)\dots(i-k+1)) \geq \nu_\pi(k!) = \left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \dots \quad (*)$$

because the similar statement for $\overline{\pi}$ will follow. The key claim is the following:

**Claim.** For any integer $t \geq 1$ and $n$, at least one of numbers $i-n-1, i-n-2, \dots, i-n-p^t$ is divisible by $\pi^t$.

*Proof.* First, we show that there exists integer $r$ such that $\pi^t \mid i-r$. To that end, by Hensel's lemma, there exists an integer $s$ for which $p^t \mid s^2+1$. Thus,

$$\pi^t\overline{\pi}^t \mid (s-i)(s+i).$$

However, $\gcd(s-i, s+i) \mid 2$, so $\pi^t$ must divide either $s-i$ or $s+i$. In particular, $r = s$ or $r = -s$ work.

To complete the problem, pick the unique $x \in \{1, 2, \dots, p^t\}$ such that $n+x \equiv r \pmod p)^t$, so $i-(n+x) \equiv i-r \pmod{p^t}$ and hence divisible by $\pi^t$. □

Using the claim repeatedly, we find that among numbers $i, i-1, i-2, \dots, i-k+1$,

- at least $\left\lfloor \frac{k}{p} \right\rfloor$ are divisible by $\pi$ (this is by selecting $\left\lfloor \frac{k}{p} \right\rfloor$ disjoint contiguous block of size $p$),
- at least $\left\lfloor \frac{k}{p^2} \right\rfloor$ are divisible by $\pi^2$,
- at least $\left\lfloor \frac{k}{p^3} \right\rfloor$ are divisible by $\pi^3$,
- $\vdots$

Using these altogether suffices to prove $(*)$.

**Impossibility via *p*-adics**

Let $P(i) = a+bi$, and note that $P(-i) = a-bi$. Fix some $p \equiv 1 \bmod 4$, and consider the $p$-adic integers $\mathbb{Z}_p$ lying in $\mathbb{Q}_p$. Note that $x^2+1 = 0$ has a solution in $\mathbb{Z}_p$, and hence $\pm i \in \mathbb{Z}_p$. Now take a sequence of integers $m_k$ converging to $i$ in $\mathbb{Z}_p$, so $-m_k$ converges to $-i$. Then since polynomials are continuous, $P(m_k)$ converges to $P(i)$ and $P(-m_k)$ converges to $P(-i)$, so $P(m_k) + P(-m_k)$ and $P(m_k) - P(-m_k)$ converge to $2x$ and $2yi$ respectively. Finally, since $P(m_k)$ and $P(-m_k)$ are integers, they have nonnegative $p$-adic valuation, and so by continuity, $2x$ and $2y$ have nonegative $p$-adic valuation. Thus, when written as simplified fractions, $a$ and $b$ cannot have any powers of $p$ in their denominator, as desired.

**Construction**

We now show that all of the claimed values are possible. The set of polynomials, $P$, taking $\mathbb{Z}$ to itself is closed under addition and multiplication, and therefore so is the set of possible values of $P(i)$. It clearly contains $\mathbb{Z}[i]$ by taking linear polynomials. Thus it suffices to show that $p^{-1}$ is attainable for every prime $p$ that is not 1 modulo 4. $2^{-1}$ is achieved by taking $P(x) = \frac{1}{4}x(x-1)(x-2)(x-3)+3$, so we may focus our attention only on the case where $p$ is 3 modulo 4. It is then further sufficient to show that some $(a+bi)p^{-1}$ is attainable for $a, b \in \mathbb{Z}$ not both divisible by $p$ because then $(a-bi)\cdot(a+bi)p^{-1} = (a^2+b^2)p^{-1}$ is also obtainable and cannot be an integer since $-1$ is not a quadratic residue modulo $p$, so Bézout's Theorem shows that $p^{-1}$ is attainable. Now with this goal in mind, observe that

$$\left|\binom{i}{p}\right| = \frac{1^2 \cdot (1^2+1^2) \cdot \ \ldots \ \cdot (1^2+(p-1)^2)}{p \cdot (2p-1) \cdot \ \ldots \ \cdot 1}.$$

has denominator divisible by $p$, but numerator not divisible by $p$ since again, $-1$ is not a quadratic residue modulo $p$. Hence we can find some integer $m$ so that $m\binom{i}{p} = (a+bi)p^{-1}$ where $a$ and $b$ that aren't both divisible by $p$ as desired.