3^{rd} Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Team Round

Polyhedron Hopping

1. [**3**] Travis is hopping around on the vertices of a cube. Each minute he hops from the vertex he's currently on to the other vertex of an edge that he is next to. After four minutes, what is the probability that he is back where he started?

   Answer: $\boxed{\frac{7}{27}}$ Let the cube have vertices all 0 or 1 in the x, y, z, coordinate system. Travis starts at $(0,0,0)$. If after 3 moves he is at $(1,1,1)$ he cannot get back to $(0,0,0)$. From any other vertex he has a $\frac{1}{3}$ chance of getting back on the final move. There is a $\frac{2}{9}$ chance he ends up at $(1,1,1)$, and thus a $\frac{7}{9}$ chance he does not end up there, and thus a $\frac{7}{27}$ chance he ends up at $(0,0,0)$.

2. [**6**] In terms of k, for $k > 0$ how likely is he to be back where he started after $2k$ minutes?

   Answer: $\boxed{\frac{1}{4}+\frac{3}{4}\left(\frac{1}{9}\right)^k}$ Again, Travis starts at $(0,0,0)$. At each step, exactly one of the three coordinates will change. The parity of the sum of the three coordinates will change at each step, so after $2k$ steps, the sum of the coordinates must be even. There are only four possibilites for Travis's position: $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$. Let p_k be the probability that Travis is at $(0,0,0)$ after $2k$ steps. Then $1-p_k$ is the probability that he is on $(1,1,0)$, $(1,0,1)$, or $(0,1,1)$. Suppose we want to compute p_{k+1}. There are two possibilities: we were either at $(0,0,0)$ after $2k$ steps or not. If we were, then there is a $\frac{1}{3}$ probability that we will return (since our $(2k+1)^{\text{th}}$ step can be arbitary, but there is a $\frac{1}{3}$ chance that we will reverse that as our $(2k+2)^{\text{th}}$ step). If we were not at $(0,0,0)$ after our $2k^{\text{th}}$ steps, then two of our coordinates must have been ones. There is a $\frac{2}{3}$ probability that the $(2k+1)^{\text{th}}$ step will change one of those to a zero, and there is a $\frac{1}{3}$ step that that the $(2k+2)^{\text{th}}$ step will change the remaining one. Hence, in this case, there is a $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)=\frac{2}{9}$ probability that Travis ends up $(0,0,0)$ in this case. So we have:

$$p_{k+1} = p_k\left(\frac{1}{3}\right) + (1-p_k)\left(\frac{2}{9}\right)$$
$$p_{k+1} = \frac{1}{9}p_k + \frac{2}{9}$$
$$\left(p_{k+1}-\frac{1}{4}\right) = \frac{1}{9}\left(p_k - \frac{1}{4}\right)$$

   (We get the value $\frac{1}{4}$ either by guessing that the sequence $p_0, p_1, p_2, \ldots$ should converge to $\frac{1}{4}$ or simply by solving the equation $-\frac{1}{9}x + x = \frac{2}{9}$.) This shows that $p_0-\frac{1}{4}, p_1-\frac{1}{4}, \ldots$ is a geometric series with ratio $\frac{1}{9}$. Since $p_0 - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$, we get that $p_k - \frac{1}{4} = \frac{3}{4}\left(\frac{1}{9}\right)^k$, or that $p_k = \frac{1}{4}+\frac{3}{4}\left(\frac{1}{9}\right)^k$.

3. [**3**] While Travis is having fun on cubes, Sherry is hopping in the same manner on an octahedron. An octahedron has six vertices and eight regular triangular faces. After five minutes, how likely is Sherry to be one edge away from where she started?

   Answer: $\boxed{\frac{11}{16}}$ Let the starting vertex be the 'bottom' one. Then there is a 'top' vertex, and 4 'middle' ones. If $p(n)$ is the probability that Sherry is on a middle vertex after n minutes, $p(0) = 0$,

$p(n+1) = (1 - p(n)) + p(n) \cdot \frac{1}{2}$. This recurrence gives us the following equations.

$$p(n+1) = 1 - \frac{p(n)}{2}$$
$$p(0) = 0$$
$$p(1) = 1$$
$$p(2) = \frac{1}{2}$$
$$p(3) = \frac{3}{4}$$
$$p(4) = \frac{5}{8}$$
$$p(5) = \frac{11}{16}$$

4. [**6**] In terms of k, for $k > 0$, how likely is it that after k minutes Sherry is at the vertex opposite the vertex where she started?

Answer: $\boxed{\frac{1}{6} + \frac{1}{3(-2)^k}}$ Take $p(n)$ from the last problem. By examining the last move of Sherry, the probability that she ends up on the original vertex is equal to the probability that she ends up on the top vertex, and both are equal to $\frac{1-p(n)}{2}$ for $n \geq 1$.

From the last problem,

$$p(n+1) = 1 - \frac{p(n)}{2}$$
$$p(n+1) - \frac{2}{3} = -\frac{1}{2}\left(p(n) - \frac{2}{3}\right)$$

and so $p(n) - \frac{2}{3}$ is a geometric series with ratio $-\frac{1}{2}$. Since $p(0) = 0$, we get $p(n) - \frac{2}{3} = -\frac{2}{3}\left(-\frac{1}{2}\right)^n$, or that $p(n) = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n$.

Now, for $k \geq 1$, we have that the probability of ending up on the vertex opposite Sherry's initial vertex after k minutes is $\frac{1-p(k)}{2} = \frac{1}{2} - \frac{1}{2}\left(\frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^k\right) = \frac{1}{6} + \frac{1}{3}\left(-\frac{1}{2}\right)^k = \frac{1}{6} + \frac{1}{3(-2)^k}$.

Circles in Circles

5. [**4**] Circle O has chord AB. A circle is tangent to O at T and tangent to AB at X such that $AX = 2XB$. What is $\frac{AT}{BT}$?

Answer: $\boxed{2}$ Let TX meet circle O again at Y. Since the homethety centered at T takes X to Y also takes AB to the tangent line of circle O passing through Y, we have Y is the midpoint of arc AB. This means that $\angle ATY = \angle YTB$. By the Angle Bisector Theorem, $\frac{AT}{BT} = \frac{AX}{BX} = 2$.

6. [**6**] AB is a diameter of circle O. X is a point on AB such that $AX = 3BX$. Distinct circles ω_1 and ω_2 are tangent to O at T_1 and T_2 and to AB at X. The lines T_1X and T_2X intersect O again at S_1 and S_2. What is the ratio $\frac{T_1T_2}{S_1S_2}$?

Answer: $\boxed{\frac{3}{5}}$ Since the problem only deals with ratios, we can assume that the radius of O is 1. As we have proven in Problem 5, points S_1 and S_2 are midpoints of arc AB. Since AB is a diameter, S_1S_2 is also a diameter, and thus $S_1S_2 = 2$.

Let O_1, O_2, and P denote the center of circles ω_1, ω_2, and O. Since ω_1 is tangent to O, we have $PO_1 + O_1X = 1$. But $O_1X \perp AB$. So $\triangle PO_1X$ is a right triangle, and $O_1X^2 + XP^2 = O_1P^2$. Thus, $O_1X^2 + 1/4 = (1 - O_1X)^2$, which means $O_1X = \frac{3}{8}$ and $O_1P = \frac{5}{8}$.

Since $T_1T_2 \parallel O_1O_2$, we have $T_1T_2 = O_1O_2 \cdot \frac{PT_1}{PO_1} = 2O_1X \cdot \frac{PT_1}{PO_1} = 2\left(\frac{3}{8}\right)\frac{1}{5/8} = \frac{6}{5}$. Thus $\frac{T_1T_2}{S_1S_2} = \frac{6/5}{2} = \frac{3}{5}$.

7. [**7**] ABC is a right triangle with $\angle A = 30°$ and circumcircle O. Circles ω_1, ω_2, and ω_3 lie outside ABC and are tangent to O at T_1, T_2, and T_3 respectively and to AB, BC, and CA at S_1, S_2, and S_3, respectively. Lines T_1S_1, T_2S_2, and T_3S_3 intersect O again at A', B', and C', respectively. What is the ratio of the area of $A'B'C'$ to the area of ABC?

Answer: $\boxed{\frac{\sqrt{3}+1}{2}}$ Let $[PQR]$ denote the area of $\triangle PQR$. The key to this problem is following fact: $[PQR] = \frac{1}{2} PQ \cdot PR \sin \angle QPR$.

Assume that the radius of O is 1. Since $\angle A = 30°$, we have $BC = 1$ and $AB = \sqrt{3}$. So $[ABC] = \frac{\sqrt{3}}{2}$. Let K denote the center of O. Notice that $\angle B'KA' = 90°$, $\angle AKC' = 90°$, and $\angle B'KA = \angle KAB = 30°$. Thus, $\angle B'KC' = \angle B'KA + \angle AKC' = 120°$ and consequently $\angle C'KA' = 150°$.

Therefore, $[A'B'C'] = [A'KB'] + [B'KC'] + [C'KA'] = \frac{1}{2} + \frac{1}{2}\sin 120° + \frac{1}{2}\sin 150° = \frac{3}{4} + \frac{\sqrt{3}}{4}$. This gives the desired result that $[A'B'C'] = \frac{\sqrt{3}+1}{2}[ABC]$.

Linear? What's The Problem?

A function $f(x_1, x_2, \ldots, x_n)$ is said to be linear in each of its variables if it is a polynomial such that no variable appears with power higher than one in any term. For example, $1 + x + xy$ is linear in x and y, but $1 + x^2$ is not. Similarly, $2x + 3yz$ is linear in x, y, and z, but xyz^2 is not.

8. [**4**] A function $f(x, y)$ is linear in x and in y. $f(x, y) = \frac{1}{xy}$ for $x, y \in \{3, 4\}$. What is $f(5, 5)$?

Answer: $\boxed{\frac{1}{36}}$ The main fact that we will use in solving this problem is that $f(x+2, y) - f(x+1, y) = f(x+1, y) - f(x, y)$ whenever f is linear in x and y. Suppose that $f(x, y) = axy + by + cx + d = x(ay + c) + (by + d)$ for some constants a, b, c, and d. Then it is easy to see that

$$f(x+2, y) - f(x+1, y) = (x+2)(ay+c) + (by+d) - (x+1)(ay+c) - (by+d) = ay + c$$
$$f(x+1, y) - f(x, y) = (x+1)(ay+c) + (by+d) - x(ay+c) - (by+d) = ay + c,$$

which implies that $f(x+2, y) - f(x+1, y) = f(x+1, y) - f(x, y)$. In particular, $f(5, y) - f(4, y) = f(4, y) - f(3, y)$, so $f(5, y) = 2f(4, y) - f(3, y)$. Similarly, $f(x, 5) = 2f(x, 4) - f(x, 3)$. Now we see that:

$$\begin{aligned} f(5,5) &= 2f(5,4) - f(5,3) \\ &= 2[2f(4,4) - f(3,4)] - [2f(4,3) - f(3,3)] \\ &= 4f(4,4) - 2f(3,4) - 2f(4,3) + f(3,3) \\ &= \frac{4}{16} - \frac{4}{12} + \frac{1}{9} \\ &= \frac{1}{4} - \frac{1}{3} + \frac{1}{9} \\ &= \frac{1}{9} - \frac{1}{12} \\ &= \frac{1}{36}, \end{aligned}$$

so the answer is $\frac{1}{36}$.

9. [**5**] A function $f(x, y, z)$ is linear in x, y, and z such that $f(x, y, z) = \frac{1}{xyz}$ for $x, y, z \in \{3, 4\}$. What is $f(5, 5, 5)$?

Answer: $\boxed{\frac{1}{216}}$ We use a similar method to the previous problem. Notice that $f(x, y, 5) = 2f(x, y, 4) - f(x, y, 3)$. Let f_2 denote the function from the previous problem and f_3 the function from this problem.

Since $3f_3(x,y,3)$ is linear in x and y, and $3f_3(x,y,3) = \frac{1}{xy}$ for all $x, y \in \{3,4\}$, the previous problem implies that $3f_3(5,5,3) = \frac{1}{36} = f_2(5,5)$. Similarly, $4f_3(5,5,4) = f_2(5,5)$. Now we have

$$\begin{aligned} f_3(5,5,5) &= 2f_3(5,5,4) - f_3(5,5,3) \\ &= \frac{1}{2}f_2(5,5) - \frac{1}{3}f_2(5,5) \\ &= \frac{1}{6}f_2(5,5) \\ &= \frac{1}{6 \cdot 36} \\ &= \frac{1}{216}. \end{aligned}$$

10. [**6**] A function $f(x_1, x_2, \ldots, x_n)$ is linear in each of the x_i and $f(x_1, x_2, \ldots, x_n) = \frac{1}{x_1 x_2 \cdots x_n}$ when $x_i \in \{3,4\}$ for all i. In terms of n, what is $f(5,5,\ldots,5)$?

Answer: $\boxed{\frac{1}{6^n}}$ Let $f_n(x_1, x_2, \ldots, x_n)$ denote the n-variable version of the function. We will prove that $f_n(5,\ldots,5) = \frac{1}{6^n}$ by induction. The base case was done in the two previous problems. Suppose we know that $f_{n-1}(5,5,\ldots,5) = \frac{1}{\cdot 6^{n-1}}$. Let $g(x_1,\ldots,x_{n-1}) = 3f_n(x_1,\ldots,x_{n-1},3)$. We have that g is linear in $x_1,\ldots,x_{n-1}$ and $g(x_1,\ldots,x_{n-1}) = \frac{1}{x_1 \cdots x_{n-1}}$ for all $x_1,\ldots,x_{n-1} \in \{3,4\}$. By the inductive hypothesis, we have $g(5,\ldots,5) = \frac{1}{6^{n-1}} = f_{n-1}(5,\ldots,5)$. Therefore, $f_n(5,\ldots,5,3) = \frac{f_{n-1}(5,\ldots,5)}{3}$. Similarly, $f_n(5,\ldots,5,4) = \frac{f_{n-1}(5,\ldots,5)}{4}$.

$$\begin{aligned} f_n(5,5,\ldots,5,5) &= 2f_n(5,5,\ldots,5,4) - f_n(5,5,\ldots,5,3) \\ &= \frac{1}{2}f_{n-1}(5,5,\ldots,5) - \frac{1}{3}f_{n-1} \\ &= \frac{1}{6}f_{n-1}(5,5,\ldots,5) \\ &= \frac{1}{6 \cdot 6^{n-1}} \\ &= \frac{1}{6^n}, \end{aligned}$$

and this proves our conjecture by induction.