12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Guts Round

. . . .

12^{th} HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

1. [**5**] Compute
$$1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \cdots + 19 \cdot 20^2.$$

2. [**5**] Given that $\sin A + \sin B = 1$ and $\cos A + \cos B = 3/2$, what is the value of $\cos(A - B)$?

3. [**5**] Find all pairs of integer solutions (n, m) to
$$2^{3^n} = 3^{2^m} - 1.$$

. . . .

12^{th} HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

4. [**6**] Simplify: $i^0 + i^1 + \cdots + i^{2009}$.

5. [**6**] In how many distinct ways can you color each of the vertices of a tetrahedron either red, blue, or green such that no face has all three vertices the same color? (Two colorings are considered the same if one coloring can be rotated in three dimensions to obtain the other.)

6. [**6**] Let ABC be a right triangle with hypotenuse AC. Let B' be the reflection of point B across AC, and let C' be the reflection of C across AB'. Find the ratio of $[BCB']$ to $[BC'B']$.

. . . .

12^{th} HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

7. [**6**] How many perfect squares divide $2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9$?

8. [**6**] Which is greater, $\log_{2008}(2009)$ or $\log_{2009}(2010)$?

9. [**6**] An icosidodecahedron is a convex polyhedron with 20 triangular faces and 12 pentagonal faces. How many vertices does it have?

. . . .

. . . .

10. [**7**] Let a, b, and c be real numbers. Consider the system of simultaneous equations in x and y:

$$\begin{aligned} ax + by &= c - 1 \\ (a+5)x + (b+3)y &= c + 1 \end{aligned}$$

Determine the value(s) of c, in terms of a, such that the system has a solution for any a and b.

11. [**7**] There are 2008 distinct points on a circle. If you connect two of these points to form a line and then connect another two points (distinct from the first two) to form another line, what is the probability that the two lines intersect inside the circle?

12. [**7**] Bob is writing a sequence of letters of the alphabet, each of which can be either uppercase or lowercase, according to the following two rules:

- If he had just written an **uppercase** letter, he can either write the same letter in **lowercase** after it, or the **next** letter of the alphabet in **uppercase**.
- If he had just written a **lowercase** letter, he can either write the same letter in **uppercase** after it, or the **preceding** letter of the alphabet in **lowercase**.

For instance, one such sequence is $aAaABCDdcbBC$. How many sequences of 32 letters can he write that start at (lowercase) a and end at (lowercase) z? (The alphabet contains 26 letters from a to z, without wrapping around, so that z does not precede a.)

. . . .

13. [**8**] How many ordered quadruples (a, b, c, d) of four distinct numbers chosen from the set $\{1, 2, 3, \ldots, 9\}$ satisfy $b < a$, $b < c$, and $d < c$?

14. [**8**] Compute

$$\sum_{k=1}^{2009} k \left(\left\lfloor \frac{2009}{k} \right\rfloor - \left\lfloor \frac{2008}{k} \right\rfloor \right).$$

Here $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.

15. [**8**] Stan has a stack of 100 blocks and starts with a score of 0, and plays a game in which he iterates the following two-step procedure:

(a) Stan picks a stack of blocks and splits it into 2 smaller stacks each with a positive number of blocks, say a and b. (The order in which the new piles are placed does not matter.)

(b) Stan adds the product of the two piles' sizes, ab, to his score.

The game ends when there are only 1-block stacks left. What is the expected value of Stan's score at the end of the game?

. . . .

. . . .

16. [**9**] A spider is making a web between $n > 1$ distinct leaves which are equally spaced around a circle. He chooses a leaf to start at, and to make the base layer he travels to each leaf one at a time, making straight lines of silk from one leaf to another, such that no two of the lines of silk cross each other and he visits every leaf exactly once. In how many ways can the spider make the base layer of the web? Express your answer in terms of n.

17. [**9**] How many positive integers $n \leq 2009$ have the property that $\lfloor \log_2(n) \rfloor$ is odd?

18. [**9**] Find the positive integer n such that $n^3 + 2n^2 + 9n + 8$ is the cube of an integer.

. . . .

19. [**10**] Shelly writes down a vector $v = (a, b, c, d)$, where $0 < a < b < c < d$ are integers. Let $\sigma(v)$ denote the set of 24 vectors whose coordinates are a, b, c, and d in some order. For instance, $\sigma(v)$ contains (b, c, d, a). Shelly notes that there are 3 vectors in $\sigma(v)$ whose sum is of the form (s, s, s, s) for some s. What is the smallest possible value of d?

20. [**10**] A positive integer is called *jubilant* if the number of 1's in its binary representation is even. For example, $6 = 110_2$ is a jubilant number. What is the 2009th smallest jubilant number?

21. [**10**] Simplify
$$2\cos^2(\ln(2009)i) + i\sin(\ln(4036081)i).$$

. . . .

22. [**10**] A circle having radius r_1 centered at point N is externally tangent to a circle of radius r_2 centered at M. Let l and j be the two common external tangent lines to the two circles. A circle centered at P with radius r_2 is externally tangent to circle N at the point at which l coincides with circle N, and line k is externally tangent to P and N such that points M, N, and P all lie on the same side of k. For what ratio r_1/r_2 are j and k parallel?

23. [**10**] The roots of $z^6 + z^4 + z^2 + 1 = 0$ are the vertices of a convex polygon in the complex plane. Find the sum of the squares of the side lengths of the polygon.

24. [**10**] Compute, in terms of n,
$$\sum_{k=0}^{n} \binom{n-k}{k} 2^k.$$
Note that whenever $s < t$, $\binom{s}{t} = 0$.

. . . .

. . . .

25. [**12**] Four points, A, B, C, and D, are chosen randomly on the circumference of a circle with independent uniform probability. What is the expected number of sides of triangle ABC for which the projection of D onto the line containing the side lies between the two vertices?

26. [**12**] Define the sequence $\{x_i\}_{i\geq 0}$ by $x_0 = 2009$ and $x_n = -\frac{2009}{n}\sum_{k=0}^{n-1} x_k$ for all $n \geq 1$. Compute the value of $\sum_{k=0}^{2009} 2^n x_n$.

27. [**12**] Circle Ω has radius 5. Points A and B lie on Ω such that chord AB has length 6. A unit circle ω is tangent to chord AB at point T. Given that ω is also internally tangent to Ω, find $AT \cdot BT$.

. . . .

12th HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

28. [**15**] The vertices of a regular hexagon are labeled $\cos(\theta), \cos(2\theta), \ldots, \cos(6\theta)$. For every pair of vertices, Benjamin draws a blue line through the vertices if one of these functions can be expressed as a polynomial function of the other (that holds for all real θ), and otherwise Roberta draws a red line through the vertices. In the resulting graph, how many triangles whose vertices lie on the hexagon have at least one red and at least one blue edge?

29. [**15**] The average of a set of distinct primes is 27. What is the largest prime that can be in this set?

30. [**15**] Let f be a polynomial with integer coefficients such that the greatest common divisor of all its coefficients is 1. For any $n \in \mathbb{N}$, $f(n)$ is a multiple of 85. Find the smallest possible degree of f.

. . . .

12th HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

31. [**18**] How many ways are there to win tic-tac-toe in $\mathbb{R}^n$? (That is, how many lines pass through three of the lattice points $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$ with each coordinate a_i in $\{1, 2, 3\}$?) Express your answer in terms of n.

32. [**18**] Circle Ω has radius 13. Circle ω has radius 14 and its center P lies on the boundary of circle Ω. Points A and B lie on Ω such that chord AB has length 24 and is tangent to ω at point T. Find $AT \cdot BT$.

33. [**18**] Let m be a positive integer. Let $d(n)$ denote the number of divisors of n, and define the function

$$F(x) = \sum_{n=1}^{105^m} \frac{d(n)}{n^x}.$$

Define the numbers $a(n)$ to be the positive integers for which

$$F(x)^2 = \sum_{n=1}^{105^{2m}} \frac{a(n)}{n^x}$$

for all real x. Express $a(105^m)$ in terms of m.

. . . .

. . . .

34. [$\leq$ **25**] How many integer lattice points (points of the form (m, n) for integers m and n) lie inside or on the boundary of the disk of radius 2009 centered at the origin?

    If your answer is higher than the correct answer, you will receive 0 points. If your answer is d less than the correct answer, your score on this problem will be the larger of 0 and $25 - \lfloor d/10 \rfloor$.

35. [$\leq$ **25**] **Von Neumann's Poker:** The first step in Von Neumann's game is selecting a random number on $[0, 1]$. To generate this number, Chebby uses the factorial base: the number $0.A_1A_2A_3A_4\ldots$ stands for $\sum_{n=0}^{\infty} \frac{A_n}{(n+1)!}$, where each A_n is an integer between 0 and n, inclusive.

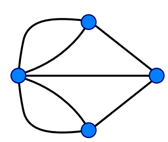
    Chebby has an infinite number of cards labeled $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$. He begins by putting cards $\mathbf{0}$ and $\mathbf{1}$ into a hat and drawing randomly to determine A_1. The card assigned A_1 does not get reused. Chebby then adds in card **2** and draws for A_2, and continues in this manner to determine the random number. At each step, he only draws one card from two in the hat.

    Unfortunately, this method does not result in a uniform distribution. What is the expected value of Chebby's final number?

    Your score on this problem will be the larger of 0 and $\lfloor 25(1 - d) \rfloor$, where d is the positive difference between your answer and the correct answer.

36. [$\leq$ **25**] **Euler's Bridge:** The following figure is the graph of the city of Konigsburg in 1736 - vertices represent sections of the cities, edges are bridges. An *Eulerian path* through the graph is a path which moves from vertex to vertex, crossing each edge exactly once. How many ways could World War II bombers have knocked out some of the bridges of Konigsburg such that the Allied victory parade could trace an Eulerian path through the graph? (The order in which the bridges are destroyed matters.)

    Your score on this problem will be the larger of 0 and $25 - \lfloor d/10 \rfloor$, where d is the positive difference between your answer and the correct answer.



. . . .