14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

Team Round A

Warm-Up [50]

1. [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. Alice goes first, and the players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $0 < y - x < 1$.

   (a) [**10**] If the first player to write a number greater than or equal to 2010 wins, determine, with proof, who has the winning strategy.

   (b) [**10**] If the first player to write a number greater than or equal to 2010 on her 2011th turn or later wins (if a player writes a number greater than or equal to 2010 on her 2010th turn or earlier, she loses immediately), determine, with proof, who has the winning strategy.

2. [**15**] Rachel and Brian are playing a game in a grid with 1 row of 2011 squares. Initially, there is one white checker in each of the first two squares from the left, and one black checker in the third square from the left. At each stage, Rachel can choose to either run or fight. If Rachel runs, she moves the black checker 1 unit to the right, and Brian moves each of the white checkers one unit to the right. If Rachel chooses to fight, she pushes the checker immediately to the left of the black checker 1 unit to the left, the black checker is moved 1 unit to the right, and Brian places a new white checker in the cell immediately to the left of the black one. The game ends when the black checker reaches the last cell. How many different final configurations are possible?

3. [**15**] Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be a set of positive integers such that $a_1 = 2$ and $a_m = \varphi(a_{m+1})$ for all $1 \leq m \leq n-1$, where, for all positive integers k, $\varphi(k)$ denotes the number of positive integers less than or equal to k that are relatively prime to k. Prove that $a_n \geq 2^{n-1}$.

Complex Numbers [35]

4. [**15**] Let a, b, and c be complex numbers such that $|a| = |b| = |c| = |a+b+c| = 1$. If $|a-b| = |a-c|$ and $b \neq c$, prove that $|a+b||a+c| = 2$.

5. [**20**] Let a and b be positive real numbers. Define two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ for all positive integers n by $(a+bi)^n = a_n + b_n i$. Prove that

$$\frac{|a_{n+1}| + |b_{n+1}|}{|a_n| + |b_n|} \geq \frac{a^2+b^2}{a+b}$$

for all positive integers n.

Coin Flipping [75]

In a one-player game, the player begins with $4m$ fair coins. On each of m turns, the player takes 4 unused coins, flips 3 of them randomly to heads or tails, and then selects whether the 4th one is heads or tails (these four coins are then considered used). After m turns, when the sides of all $4m$ coins have been determined, if half the coins are heads and half are tails, the player wins; otherwise, the player loses.

6. [**10**] Prove that whenever the player must choose the side of a coin, the optimal strategy is to choose heads if more coins have been determined tails than heads and to choose tails if more coins have been determined heads than tails.

7. [**15**] Let T denote the number of coins determined tails and H denote the number of coins determined heads at the end of the game. Let $p_m(k)$ be the probability that $|T-H|=k$ after a game with $4m$ coins, assuming that the player follows the optimal strategy outlined in problem 6. Clearly $p_m(k)=0$ if k is an odd integer, so we need only consider the case when k is an even integer. (By definition, $p_0(0)=1$ and $p_0(k)=0$ for $k \geq 1$). Prove that $p_m(0) \geq p_{m+1}(0)$ for all nonnegative integers m.

8. (a) [**5**] Find a_1, a_2, and a_3, so that the following equation holds for all $m \geq 1$:

$$p_m(0) = a_1 p_{m-1}(0) + a_2 p_{m-1}(2) + a_3 p_{m-1}(4)$$

   (b) [**5**] Find b_1, b_2, b_3, and b_4, so that the following equation holds for all $m \geq 1$:

$$p_m(2) = b_1 p_{m-1}(0) + b_2 p_{m-1}(2) + b_3 p_{m-1}(4) + b_4 p_{m-1}(6)$$

   (c) [**5**] Find c_1, c_2, c_3, and c_4, so that the following equation holds for all $m \geq 1$ and $j \geq 2$:

$$p_m(2j) = c_1 p_{m-1}(2j-2) + c_2 p_{m-1}(2j) + c_3 p_{m-1}(2j+2) + c_4 p_{m-1}(2j+4)$$

9. [**15**] We would now like to examine the behavior of $p_m(k)$ as m becomes arbitrarily large; specifically, we would like to discern whether $\lim_{m\to\infty} p_m(0)$ exists and, if it does, to determine its value. Let $\lim_{m\to\infty} p_m(k) = A_k$.

   (a) [**5**] Prove that $\frac{2}{3} p_m(k) \geq p_m(k+2)$ for all m and k.

   (b) [**10**] Prove that A_0 exists and that $A_0 > 0$. Feel free to assume the result of analysis that a non-increasing sequence of real numbers that is bounded below by a constant c converges to a limit that is greater than or equal to c.

10. [**20**] Once it has been demonstrated that $\lim_{n\to\infty} p_n(0)$ exists and is greater than 0, it follows that $\lim_{n\to\infty} p_n(k)$ exists and is greater than 0 for all even positive integers k and that $\sum_{k=0}^{\infty} A_{2k} = 1$. It also follows that $A_0 = a_1 A_0 + a_2 A_2 + a_3 A_4$, $A_2 = b_1 A_0 + b_2 A_2 + b_3 A_4 + b_4 A_6$, and $A_{2j} = c_1 A_{2j-2} + c_2 A_{2j} + c_3 A_{2j+2} + c_4 A_{2j+4}$ for all positive integers $j \geq 2$, where a_1, a_2, a_3, b_1, b_2, b_3, b_4, and c_1, c_2, c_3, c_4 are the constants you found in problem 8. Assuming these results, determine, with proof, the value of A_0.

Geometry [90]

11. [**20**] Let ABC be a non-isosceles, non-right triangle, let ω be its circumcircle, and let O be its circumcenter. Let M be the midpoint of segment BC. Let the circumcircle of triangle AOM intersect ω again at D. If H is the orthocenter of triangle ABC, prove that $\angle DAH = \angle MAO$.

12. [**70**] Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C, respectively. If A is not a right angle, prove that the circumcenter of triangle AEF lies on the incircle of triangle ABC if and only if the incenter of triangle ABC lies on the circumcircle of triangle AEF.

Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C, respectively. If A is not a right angle, prove that the circumcenter of triangle AEF lies on the incircle of triangle ABC if and only if the incenter of triangle ABC lies on the circumcircle of triangle AEF.

Last Writes [110]

13. [**30**] Given positive integers a and b such that $a > b$, define a sequence of ordered pairs (a_l, b_l) for nonnegative integers l by $a_0 = a$, $b_0 = b$, and $(a_{l+1}, b_{l+1}) = (b_l, a_l \mod b_l)$, where, for all positive integers x and y, $x \mod y$ is defined to be the remainder left by x upon division by y. Define $f(a,b)$ to be the smallest positive integer j such that $b_j = 0$. Given a positive integer n, define $g(n)$ to be $\max_{1 \le k \le n-1} f(n,k)$.

    (a) [**15**] Given a positive integer m, what is the smallest positive integer n_m such that $g(n_m) = m$?

    (b) [**15**] What is the second smallest?

14. [**25**] Given a positive integer n, a sequence of integers $a_1, a_2, \ldots, a_r$, where $0 \le a_i \le k$ for all $1 \le i \le r$, is said to be a "*k-representation*" of n if there exists an integer c such that

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} a_i k^{c-i} = n.$$

    Prove that every positive integer n has a *k-representation*, and that the *k-representation* is unique if and only if 0 does not appear in the base-k representation of $n-1$.

15. [**55**] Denote $\{1, 2, ..., n\}$ by $[n]$, and let S be the set of all permutations of $[n]$. Call a subset T of S *good* if every permutation σ in S may be written as $t_1 t_2$ for elements t_1 and t_2 of T, where the product of two permutations is defined to be their composition. Call a subset of U of S *extremely good* if every permutation σ in S may be written as $s^{-1}us$ for elements s of S and u of U. Let τ be the smallest value of $|T|/|S|$ for all good subsets T, and let υ be the smallest value of $|U|/|S|$ for all extremely good subsets U. Prove that $\sqrt{\upsilon} \ge \tau$.