HMMT February 2016

February 20, 2016

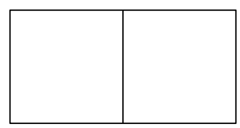
Combinatorics

1. For positive integers n, let S_n be the set of integers x such that n distinct lines, no three concurrent, can divide a plane into x regions (for example, $S_2 = \{3, 4\}$, because the plane is divided into 3 regions if the two lines are parallel, and 4 regions otherwise). What is the minimum i such that S_i contains at least 4 elements?

2. Starting with an empty string, we create a string by repeatedly appending one of the letters H, M, T with probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, respectively, until the letter M appears twice consecutively. What is the expected value of the length of the resulting string?

3. Find the number of ordered pairs of integers (a, b) such that a, b are divisors of 720 but ab is not.

4. Let R be the rectangle in the Cartesian plane with vertices at $(0, 0), (2, 0), (2, 1)$, and $(0, 1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.



How many subsets of these seven edges form a connected figure?

5. Let a, b, c, d, e, f be integers selected from the set $\{1, 2, \ldots, 100\}$, uniformly and at random with replacement. Set
$$M = a + 2b + 4c + 8d + 16e + 32f.$$
What is the expected value of the remainder when M is divided by 64?

6. Define the sequence $a_1, a_2 \ldots$ as follows: $a_1 = 1$ and for every $n \geq 2$,
$$a_n = \begin{cases} n-2 & \text{if } a_{n-1} = 0 \\ a_{n-1} - 1 & \text{if } a_{n-1} \neq 0 \end{cases}$$
A non-negative integer d is said to be *jet-lagged* if there are non-negative integers r, s and a positive integer n such that $d = r + s$ and that $a_{n+r} = a_n + s$. How many integers in $\{1, 2, \ldots, 2016\}$ are jet-lagged?

7. Kelvin the Frog has a pair of standard fair 8-sided dice (each labelled from 1 to 8). Alex the sketchy Kat also has a pair of fair 8-sided dice, but whose faces are labelled differently (the integers on each Alex's dice need not be distinct). To Alex's dismay, when both Kelvin and Alex roll their dice, the probability that they get any given sum is equal!

   Suppose that Alex's two dice have a and b total dots on them, respectively. Assuming that $a \neq b$, find all possible values of $\min\{a, b\}$.

8. Let X be the collection of all functions $f : \{0, 1, \ldots, 2016\} \to \{0, 1, \ldots, 2016\}$. Compute the number of functions $f \in X$ such that
$$\max_{g \in X} \left(\min_{0 \leq i \leq 2016} \big(\max(f(i), g(i)) \big) - \max_{0 \leq i \leq 2016} \big(\min(f(i), g(i)) \big) \right) = 2015.$$

9. Let $V = \{1, \ldots, 8\}$. How many permutations $\sigma : V \to V$ are automorphisms of some tree?

   (A *graph* consists of a some set of vertices and some edges between pairs of distinct vertices. It is *connected* if every two vertices in it are connected by some path of one or more edges. A *tree* G on V is a connected graph with vertex set V and exactly $|V| - 1$ edges, and an *automorphism* of G is a permutation $\sigma : V \to V$ such that vertices $i, j \in V$ are connected by an edge if and only if $\sigma(i)$ and $\sigma(j)$ are.)

10. Kristoff is planning to transport a number of indivisible ice blocks with positive integer weights from the north mountain to Arendelle. He knows that when he reaches Arendelle, Princess Anna and Queen Elsa will name an ordered pair (p, q) of nonnegative integers satisfying $p+q \leq 2016$. Kristoff must then give Princess Anna *exactly* p kilograms of ice. Afterward, he must give Queen Elsa *exactly* q kilograms of ice.

    What is the minimum number of blocks of ice Kristoff must carry to guarantee that he can always meet Anna and Elsa's demands, regardless of which p and q are chosen?