# $14^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

**Saturday 12 February 2011**

## Algebra & Calculus Individual Test

1. Let $a$, $b$, and $c$ be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

   **Answer:** $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc, b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

2. Josh takes a walk on a rectangular grid of $n$ rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

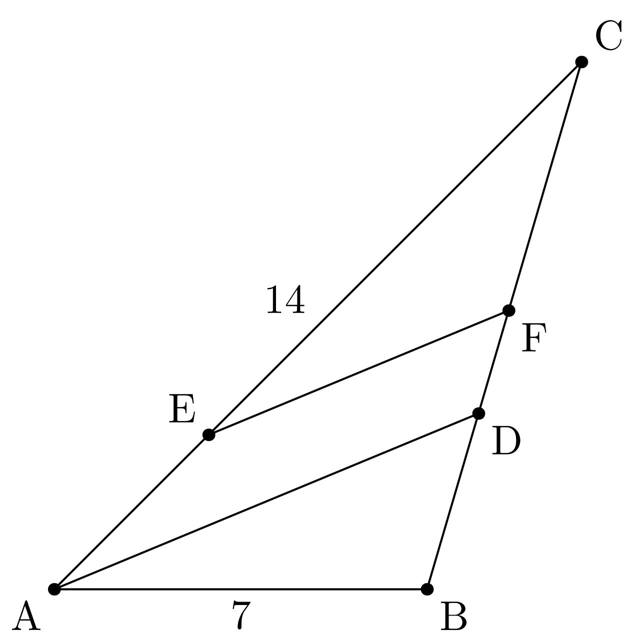
   **Answer:** $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row $k$ to the center square of row $k+1$. So there are $2^{n-1}$ ways to get to the center square of row $n$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f(1) = 1$, and $|f'(x)| \leq 2$ for all real numbers $x$. If $a$ and $b$ are real numbers such that the set of possible values of $\int_0^1 f(x)\,dx$ is the open interval $(a, b)$, determine $b - a$.

   **Answer:** $\boxed{\frac{3}{4}}$ Draw lines of slope $\pm 2$ passing through $(0, 0)$ and $(1, 1)$. These form a parallelogram with vertices $(0, 0), (.75, 1.5), (1, 1), (.25, -.5)$. By the mean value theorem, no point of $(x, f(x))$ lies outside this parallelogram, but we can construct functions arbitrarily close to the top or the bottom of the parallelogram while satisfying the condition of the problem. So $(b - a)$ is the area of this parallelogram, which is $\frac{3}{4}$.

4. Let $ABC$ be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line $BC$ at $D$. If there exist points $E$ and $F$ on sides $AC$ and $BC$, respectively, such that lines $AD$ and $EF$ are parallel and divide triangle $ABC$ into three parts of equal area, determine the number of possible integral values for $BC$.

   **Answer:** $\boxed{13}$



Note that such $E, F$ exist if and only if
$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$
([] denotes area.) Since $AD$ is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,
$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$
Hence (1) is equivalent to $AC = 2AB = 14$. Then $BC$ can be any length $d$ such that the triangle inequalities are satisfied:
$$d + 7 > 14$$
$$7 + 14 > d$$
Hence $7 < d < 21$ and there are 13 possible integral values for $BC$.

5. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

**Answer:** $\boxed{\frac{5}{11}}$ For $1 \le k \le 6$, let $x_k$ be the probability that the current player, say $A$, will win when the number on the tally at the beginning of his turn is $k$ modulo 7. The probability that the total is $l$ modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod 7$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that $A$ will win if he leaves $l$ on the board after his turn is $1 - x_l$. Hence for $1 \le k \le 6$,
$$x_k = \frac{1}{6} \sum_{1 \le l \le 6,\, l \ne k} (1 - x_l) + \frac{1}{6}.$$
Letting $s = \sum_{l=1}^{6} x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \cdots = x_6$, and $6x_k = s$ for every $k$. Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

6. Let $a \star b = ab + a + b$ for all integers $a$ and $b$. Evaluate $1 \star (2 \star (3 \star (4 \star \ldots (99 \star 100) \ldots)))$.

**Answer:** $\boxed{101! - 1}$ We will first show that $\star$ is both commutative and associative.

- Commutativity: $a \star b = ab + a + b = b \star a$
- Associativity: $a \star (b \star c) = a(bc + b + c) + a + bc + b + c = abc + ab + ac + bc + a + b + c$ and $(a \star b) \star c = (ab + a + b)c + ab + a + b + c = abc + ab + ac + bc + a + b + c$. So $a \star (b \star c) = (a \star b) \star c$.

So we need only calculate $((\ldots(1 \star 2) \star 3) \star 4) \ldots \star 100)$. We will prove by induction that
$$((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n + 1)! - 1.$$

- Base case ($n = 2$): $(1 \star 2) = 2 + 1 + 2 = 5 = 3! - 1$
- Inductive step:
  Suppose that
  $$(((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n + 1)! - 1.$$
  Then,
  $$\begin{aligned} ((((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) \star (n + 1)) &= ((n + 1)! - 1) \star (n + 1) \\ &= (n + 1)!(n + 1) - (n + 1) + (n + 1)! - 1 + (n + 1) \\ &= (n + 2)! - 1 \end{aligned}$$

Hence, $((\ldots(1\star 2)\star 3)\star 4)\ldots\star n) = (n+1)! - 1$ for all $n$. For $n = 100$, this results to $101! - 1$.

7. Let $f : [0,1) \to \mathbb{R}$ be a function that satisfies the following condition: if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} = .a_1a_2a_3\ldots$$

is the decimal expansion of $x$ and there does not exist a positive integer $k$ such that $a_n = 9$ for all $n \geq k$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^{2n}}.$$

Determine $f'(\frac{1}{3})$.

**Answer:** $\boxed{0}$ Note that $\frac{1}{3} = \sum_{n=1}^{\infty} \frac{3}{10^n}$.

Clearly $f$ is an increasing function. Also for any integer $n \geq 1$, we see from decimal expansions that $f(\frac{1}{3} \pm \frac{1}{10^n}) - f(\frac{1}{3}) = \pm\frac{1}{10^{2n}}$.

Consider $h$ such that $10^{-n-1} \leq |h| < 10^{-n}$. The two properties of $f$ outlined above show that $|f(\frac{1}{3} + h) - f(\frac{1}{3})| < \frac{1}{10^{2n}}$. And from $|\frac{1}{h}| \leq 10^{n+1}$, we get $\left|\frac{f(\frac{1}{3}+h)-f(\frac{1}{3})}{h}\right| < \frac{1}{10^{n-1}}$. Taking $n \to \infty$ gives $h \to 0$ and $f'(\frac{1}{3}) = \lim_{n\to\infty} \frac{1}{10^{n-1}} = 0$.

8. Find all integers $x$ such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

**Answer:** $\boxed{-3, 2, 5}$ Let $f(x) = 2x^2 + x - 6 = (2x-3)(x+2)$. Suppose a positive integer $a$ divides both $2x - 3$ and $x + 2$. Then $a$ must also divide $2(x+2) - (2x-3) = 7$. Hence, $a$ can either be 1 or 7. As a result, $2x - 3 = 7^n$ or $-7^n$ for some positive integer $n$, or either $x + 2$ or $2x - 3$ is $\pm 1$. We consider the following cases:

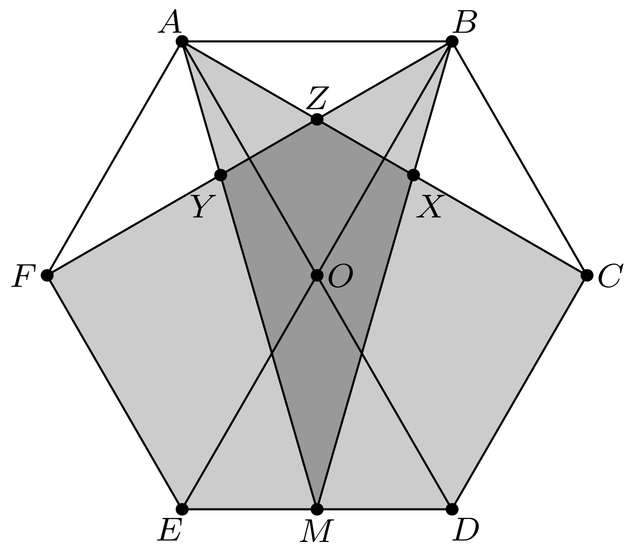
- $(2x-3) = 1$. Then $x = 2$, which yields $f(x) = 4$, a prime power.
- $(2x-3) = -1$. Then $x = 1$, which yields $f(x) = -3$, not a prime power.
- $(x+2) = 1$). Then $x = -1$, which yields $f(x) = -5$ not a prime power.
- $(x+2) = -1$. Then $x = -3$, which yields $f(x) = 9$, a prime power.
- $(2x-3) = 7$. Then $x = 5$, which yields $f(x) = 49$, a prime power.
- $(2x-3) = -7$. Then $x = -2$, which yields $f(x) = 0$, not a prime power.
- $(2x-3) = \pm 7^n$, for $n \geq 2$. Then, since $x + 2 = \dfrac{(2x-3)+7}{2}$, we have that $x + 2$ is divisible by 7 but not by 49. Hence $x + 2 = \pm 7$, yielding $x = 5, -9$. The former has already been considered, while the latter yields $f(x) = 147$.

So $x$ can be either -3, 2 or 5.

(Note: In the official solutions packet we did not list the answer -3. This oversight was quickly noticed on the day of the test, and only the answer $-3, 2, 5$ was marked as correct.

9. Let $ABCDEF$ be a regular hexagon of area 1. Let $M$ be the midpoint of $DE$. Let $X$ be the intersection of $AC$ and $BM$, let $Y$ be the intersection of $BF$ and $AM$, and let $Z$ be the intersection of $AC$ and $BF$. If $[P]$ denotes the area of polygon $P$ for any polygon $P$ in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

**Answer:** $\boxed{0}$




Let $O$ be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60° = \sin 120°$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

10. For all real numbers $x$, let

$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}.$$

Evaluate $(f(f(\ldots(f(2011))\ldots)))^{2011}$, where $f$ is applied 2010 times.

**Answer:** $\boxed{2011^{2011}}$ Direct calculation shows that $f(f(x)) = \dfrac{\sqrt[2011]{1-x^{2011}}}{-x}$ and $f(f(f(x))) = x$. Hence $(f(f(\ldots(f(x))\ldots))) = x$, where $f$ is applied 2010 times. So $(f(f(\ldots(f(2011))\ldots)))^{2011} = 2011^{2011}$.

11. Evaluate $\displaystyle\int_1^\infty \left(\frac{\ln x}{x}\right)^{2011} dx$.

**Answer:** $\boxed{\frac{2011!}{2010^{2012}}}$ By the chain rule, $\dfrac{d}{dx}(\ln x)^n = \dfrac{n \ln^{n-1} x}{x}$.

We calculate the definite integral using integration by parts:

$$\int_{x=1}^{\infty} \frac{(\ln x)^n}{x^{2011}} dx = \left[\frac{(\ln x)^n}{-2010x^{2010}}\right]_{x=1}^{x=\infty} - \int_{x=1}^{\infty} \frac{n(\ln x)^{n-1}}{-2010x^{2011}} dx$$

But $\ln(1) = 0$, and $\lim_{x\to\infty} \frac{(\ln x)^n}{x^{2010}} = 0$ for all $n > 0$. So

$$\int_{x=1}^{\infty} \frac{(\ln x)^n}{x^{2011}} dx = \int_{x=1}^{\infty} \frac{n(\ln x)^{n-1}}{2010x^{2011}} dx$$

It follows that

$$\int_{x=1}^{\infty} \frac{(\ln x)^n}{x^{2011}} dx = \frac{n!}{2010^n} \int_{x=1}^{\infty} \frac{1}{x^{2011}} dx = \frac{n!}{2010^{n+1}}$$

So the answer is $\frac{2011!}{2010^{2012}}$.

12. Let $f(x) = x^2 + 6x + c$ for all real numbers $x$, where $c$ is some real number. For what values of $c$ does $f(f(x))$ have exactly 3 distinct real roots?

**Answer:** $\boxed{\frac{11-\sqrt{13}}{2}}$ Suppose $f$ has only one distinct root $r_1$. Then, if $x_1$ is a root of $f(f(x))$, it must be the case that $f(x_1) = r_1$. As a result, $f(f(x))$ would have at most two roots, thus not satisfying the problem condition. Hence $f$ has two distinct roots. Let them be $r_1 \neq r_2$.

Since $f(f(x))$ has just three distinct roots, either $f(x) = r_1$ or $f(x) = r_2$ has one distinct root. Assume without loss of generality that $r_1$ has one distinct root. Then $f(x) = x^2 + 6x + c = r_1$ has one root, so that $x^2 + 6x + c - r_1$ is a square polynomial. Therefore, $c - r_1 = 9$, so that $r_1 = c - 9$. So $c - 9$ is a root of $f$. So $(c-9)^2 + 6(c-9) + c = 0$, yielding $c^2 - 11c + 27 = 0$, or $(c - \frac{11}{2})^2 = \frac{13}{2}$. This results to $c = \frac{11 \pm \sqrt{13}}{2}$.

If $c = \frac{11-\sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11-\sqrt{13}}{2} = (x + \frac{7+\sqrt{13}}{2})(x + \frac{5-\sqrt{13}}{2})$. We know $f(x) = \frac{-7-\sqrt{13}}{2}$ has a double root, -3. Now $\frac{-5+\sqrt{13}}{2} > \frac{-7-\sqrt{13}}{2}$ so the second root is above the vertex of the parabola, and is hit twice.

If $c = \frac{11+\sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11+\sqrt{13}}{2} = (x + \frac{7-\sqrt{13}}{2})(x + \frac{5+\sqrt{13}}{2})$. We know $f(x) = \frac{-7+\sqrt{13}}{2}$ has a double root, -3, and this is the value of $f$ at the vertex of the parabola, so it is its minimum value. Since $\frac{-5-\sqrt{13}}{2} < \frac{-7+\sqrt{13}}{2}$, $f(x) = \frac{-5-\sqrt{13}}{2}$ has no solutions. So in this case, $f$ has only one real root.

So the answer is $c = \frac{11-\sqrt{13}}{2}$.

Note: In the solutions packet we had both roots listed as the correct answer. We noticed this oversight on the day of the test and awarded points only for the correct answer.

13. Sarah and Hagar play a game of darts. Let $O_0$ be a circle of radius 1. On the $n$th turn, the player whose turn it is throws a dart and hits a point $p_n$ randomly selected from the points of $O_{n-1}$. The player then draws the largest circle that is centered at $p_n$ and contained in $O_{n-1}$, and calls this circle $O_n$. The player then colors every point that is inside $O_{n-1}$ but not inside $O_n$ her color. Sarah goes first, and the two players alternate turns. Play continues indefinitely. If Sarah's color is red, and Hagar's color is blue, what is the expected value of the area of the set of points colored red?

**Answer:** $\boxed{\frac{6\pi}{7}}$ Let $f(r)$ be the average area colored red on a dartboard of radius $r$ if Sarah plays first. Then $f(r)$ is proportional to $r^2$. Let $f(r) = (\pi x)r^2$ for some constant $x$. We want to find $f(1) = \pi x$.

In the first throw, if Sarah's dart hits a point with distance $r$ from the center of $O_0$, the radius of $O_1$ will be $1-r$. The expected value of the area colored red will be $(\pi - \pi(1-r)^2) + (\pi(1-r)^2 - f(1-r)) = \pi - f(1-r)$. The value of $f(1)$ is the average value of $\pi - f(1-r)$ over all points in $O_0$. Using polar coordinates, we get

$$f(1) = \frac{\int_0^{2\pi}\int_0^1 (\pi - f(1-r)) r\,dr\,d\theta}{\int_0^{2\pi}\int_0^1 r\,dr\,d\theta}$$

$$\pi x = \frac{\int_0^1 (\pi - \pi x(1-r)^2) r\,dr}{\int_0^1 r\,dr}$$

$$\frac{\pi x}{2} = \int_0^1 \pi r - \pi x r(1-r)^2 dr$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \pi x(\frac{1}{2} - \frac{2}{3} + \frac{1}{4})$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \frac{\pi x}{12}$$

$$\pi x = \frac{6\pi}{7}$$

14. How many polynomials $P$ with integer coefficients and degree at most 5 satisfy $0 \leq P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

    **Answer:** $\boxed{86400000}$ For each nonnegative integer $i$, let $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$. (Define $x^{\underline{0}} = 1$.)

    Lemma: Each polynomial with integer coefficients $f$ can be uniquely written in the form

    $$f(x) = a_n x^{\underline{n}} + \ldots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}, a_n \neq 0.$$

    Proof: Induct on the degree. The base case (degree 0) is clear. If $f$ has degree $m$ with leading coefficient $c$, then by matching leading coefficients we must have $m = n$ and $a_n = c$. By the induction hypothesis, $f(x) - cx^{\underline{n}}$ can be uniquely written as $a_{n-1}x^{\underline{n-1}}(x) + \ldots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}$.

    There are 120 possible choices for $a_0$, namely any integer in $[0, 120)$. Once $a_0, \ldots, a_{i-1}$ have been chosen so $0 \leq P(0), \ldots, P(i-1) < 120$, for some $0 \leq i \leq 5$, then we have

    $$P(i) = a_i i! + a_{i-1} i^{\underline{i-1}} + \cdots + a_0$$

    so by choosing $a_i$ we can make $P(i)$ any number congruent to $a_{i-1}i^{\underline{i-1}} + \cdots + a_0$ modulo $i!$. Thus there are $\frac{120}{i!}$ choices for $a_i$. Note the choice of $a_i$ does not affect the value of $P(0), \ldots, P(i-1)$. Thus all polynomials we obtain in this way are valid. The answer is

    $$\prod_{i=0}^{5} \frac{120}{i!} = 86400000.$$

    Note: Their is also a solution involving finite differences that is basically equivalent to this solution. One proves that for $i = 0, 1, 2, 3, 4, 5$ there are $\frac{5!}{i!}$ ways to pick the $i$th finite difference at the point 0.

15. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(f(x)) = 1$ for all $x \in [0, 1]$. Determine the set of possible values of $\displaystyle\int_0^1 f(x)\, dx$.

    **Answer:** $\boxed{(\frac{3}{4}, 1]}$ Since the maximum value of $f$ is 1, $\int_0^1 f(x)dx \leq 1$.

    By our condition $f(f(x)) = 1$, $f$ is 1 at any point within the range of $f$. Clearly, 1 is in the range of $f$, so $f(1) = 1$. Now $f(x)$ is continuous on a closed interval so it attains a minimum value $c$. Since $c$ is in the range of $f$, $f(c) = 1$.

    If $c = 1$, $f(x) = 1$ for all $x$ and $\int_0^1 f(x)dx = 1$.

    Now assume $c < 1$. By the intermediate value theorem, since $f$ is continuous it attains all values between $c$ and 1. So for all $x \geq c$, $f(x) = 1$. Therefore,

    $$\int_0^1 f(x)dx = \int_0^c f(x)dx + (1-c).$$

    Since $f(x) \geq c$, $\int_0^c f(x)dx > c^2$, and the equality is strict because $f$ is continuous and thus cannot be $c$ for all $x < c$ and 1 at $c$. So

    $$\int_0^1 f(x)dx > c^2 + (1-c) = (c - \frac{1}{2})^2 + \frac{3}{4} \geq \frac{3}{4}.$$

    Therefore $\frac{3}{4} < \int_0^1 f(x)dx \leq 1$, and it is easy to show that every value in this interval can be reached.

16. Let $f(x) = x^2 - r_2x + r_3$ for all real numbers $x$, where $r_2$ and $r_3$ are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers $n$ by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \le i \le 2011$; (ii) there exists a positive integer $j$ such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If $A$ is the greatest number such that $A \le |r_2|$ for any function $f$ satisfying these properties, find $A$.

**Answer:** $\boxed{2}$ Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some $x$, namely, for $x = g_{2i+1}, 0 \le i \le 2011$. Since $f(x) - x$ is positive for $x$ of large absolute value, the graph of $f(x) - x$ crosses the $x$-axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \ne b$ be the number such that $f(c) = f(b) = b$. Note that $b$ is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, $c$ exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest $j$ such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \ge b$ then $f(g_i) \ge g_i \ge b$ so by induction, $g_{i+1} \ge g_i$ for all $i \ge j$. Hence $j > 1$; in fact $j \ge 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of $f$ is less than $c$. The minimum value is the value of $f$ evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned} f\left(\frac{b+a-1}{2}\right) &< c \\ \left(\frac{b+a-1}{2} - a\right)\left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a - 1 \\ \frac{1-(b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\ \frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\ 4 &< (b-a-1)^2. \end{aligned}$$

Then either $b - a - 1 < -2$ or $b - a - 1 > 2$, but $b > a$, so the latter must hold and $(b - a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b - a)^2$ (the square of the difference of the two roots) and $(r_2 + 1)^2 - 4r_3$ (from the coefficients), so $(r_2 + 1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \ge 0$ as follows. Let $N \ge 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x + 2 + \varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define $G_i$ recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following.

(i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \le i \le N-1$. First note $G_{2N} = -\sqrt{4 + 2\varepsilon} > -\sqrt{4 + 2\varepsilon + \varepsilon^2} = -2 - \varepsilon$. Let $l$ be the negative solution to $h(x) = x$. Note that $-2 - \varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \ge -2 - \varepsilon$, and it sends $(-2 - \varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2 - \varepsilon, l)$. It follows that the $G_i$, $0 \le i \le 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \le i \le N - 1$ by backwards induction on $i$, so the desired inequalities follow.

(ii) $G_i$ is increasing for $i \ge 2N + 1$. Indeed, if $x \ge 2 + \varepsilon$, then $x^2 - x = x(x - 1) > 2 + \varepsilon$ so $h(x) > x$. Hence $2 + \varepsilon = G_{2N+1} < G_{2N+2} < \cdots$.

(iii) $G_i$ is unbounded. This follows since $h(x) - x = x(x - 2) - 2 - \varepsilon$ is increasing for $x > 2 + \varepsilon$, so $G_i$ increases faster and faster for $i \ge 2N + 1$.

Now define $f(x) = h(x + G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^{\infty}$ satisfies (i), (ii), and (iii), so does $g_i$.

We claim that we can make $G_0$ arbitrarily close to $-1$ by choosing $N$ large enough and $\varepsilon$ small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing $N$ large corresponds to taking $G_0$ to be a larger iterate of $2 + \varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to $x$ and $\varepsilon$, it suffices to take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\cdots g}_{n}(2)\cdots) \to -1 \text{ as } n \to \infty.$$

But note that for $0 \le \theta \le \frac{\pi}{2}$,

$$g(-2\cos\theta) = -\sqrt{2 - 2\cos\theta} = -2\sin\left(\frac{\theta}{2}\right) = 2\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Hence by induction, $g^{(n)}(-2\cos\theta) = -2\cos\left(\frac{\pi}{2} - \frac{\pi}{4} + \cdots + (-1)^n\left(\theta - \frac{\pi}{2^n}\right)\right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2\cos 0)$ converges to $-2\cos(\frac{\pi}{2} - \frac{\pi}{4} + \cdots) = -2\cos(\frac{\pi}{3}) = -1$, as needed.

17. Let $f : (0,1) \to (0,1)$ be a differentiable function with a continuous derivative such that for every positive integer $n$ and odd positive integer $a < 2^n$, there exists an odd positive integer $b < 2^n$ such that $f\left(\frac{a}{2^n}\right) = \frac{b}{2^n}$. Determine the set of possible values of $f'\left(\frac{1}{2}\right)$.

    **Answer:** $\boxed{\{-1, 1\}}$ The key step is to notice that for such a function $f$, $f'(x) \neq 0$ for any $x$.

    Assume, for sake of contradiction that there exists $0 < y < 1$ such that $f'(y) = 0$. Since $f'$ is a continuous function, there is some small interval $(c,d)$ containing $y$ such that $|f'(x)| \le \frac{1}{2}$ for all $x \in (c,d)$. Now there exists some $n, a$ such that $\frac{a}{2^n}, \frac{a+1}{2^n}$ are both in the interval $(c,d)$. From the definition, $\dfrac{f(\frac{a+1}{2^n}) - f(\frac{a}{2^n})}{\frac{a+1}{2^n} - \frac{a}{2^n}} = 2^n(\dfrac{b'}{2^n} - \dfrac{b}{2^n}) = b' - b$ where $b, b'$ are integers; one is odd, and one is even. So $b' - b$ is an odd integer. Since $f$ is differentiable, by the mean value theorem there exists a point where $f' = b' - b$. But this point is in the interval $(c,d)$, and $|b' - b| > \frac{1}{2}$. This contradicts the assumption that $|f'(x)| \le \frac{1}{2}$ for all $x \in (c,d)$.

    Since $f'(x) \neq 0$, and $f'$ is a continuous function, $f'$ is either always positive or always negative. So $f$ is either increasing or decreasing. $f(\frac{1}{2}) = \frac{1}{2}$ always. If $f$ is increasing, it follows that $f(\frac{1}{4}) = \frac{1}{4}$, $f(\frac{3}{4}) = \frac{3}{4}$, and we can show by induction that indeed $f(\frac{a}{2^n}) = \frac{a}{2^n}$ for all integers $a, n$. Since numbers of this form are dense in the interval $(0,1)$, and $f$ is a continuous function, $f(x) = x$ for all $x$.

    It can be similarly shown that if $f$ is decreasing $f(x) = 1 - x$ for all $x$. So the only possible values of $f'(\frac{1}{2})$ are $-1, 1$.

    Query: if the condition that the derivative is continuous were omitted, would the same result still hold?

18. Let $z = \cos\dfrac{2\pi}{2011} + i\sin\dfrac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \ldots \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers $x$. Evaluate $P(z)P(z^2)P(z^3)\ldots P(z^{2010})$.

**Answer:** $\boxed{2011^{2009} \cdot (1005^{2011} - 1004^{2011})}$ Multiply $P(x)$ by $x - 1$ to get

$$P(x)(x-1) = x^{2009} + 2x^{2008} + \ldots + 2009x - \frac{2009 \cdot 2010}{2},$$

or,

$$P(x)(x-1) + 2010 \cdot 1005 = x^{2009} + 2x^{2008} + \ldots + 2009x + 2010.$$

Multiplying by $x - 1$ once again:

$$\begin{aligned}(x-1)(P(x)(x-1) + \frac{2010 \cdot 2011}{2}) &= x^{2010} + x^{2009} + \ldots + x - 2010,\\ &= (x^{2010} + x^{2009} + \ldots + x + 1) - 2011.\end{aligned}$$

Hence,

$$P(x) = \frac{\frac{(x^{2010} + x^{2009} + \ldots + x + 1) - 2011}{x-1} - 2011 \cdot 1005}{x-1}$$

Note that $x^{2010}+x^{2009}+\ldots+x+1$ has $z, z^2, \ldots, z^{2010}$ as roots, so they vanish at those points. Plugging those 2010 powers of $z$ into the last equation, and multiplying them together, we obtain

$$\prod_{i=1}^{2010} P(z^i) = \frac{(-2011) \cdot 1005 \cdot (x - \frac{1004}{1005})}{(x-1)^2}.$$

Note that $(x-z)(x-z^2)\ldots(x-z^{2010}) = x^{2010} + x^{2009} + \ldots + 1$. Using this, the product turns out to be $2011^{2009} \cdot (1005^{2011} - 1004^{2011})$.

19. Let

$$F(x) = \frac{1}{(2-x-x^5)^{2011}},$$

and note that $F$ may be expanded as a power series so that $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Find an ordered pair of positive real numbers $(c, d)$ such that $\lim_{n\to\infty} \frac{a_n}{n^d} = c$.

**Answer:** $\boxed{(\frac{1}{6^{2011}2010!}, 2010)}$ First notice that all the roots of $2-x-x^5$ that are not 1 lie strictly outside the unit circle. As such, we may write $2-x-x^5$ as $2(1-x)(1-r_1x)(1-r_2x)(1-r_3x)(1-r_4x)$ where $|r_i| < 1$, and let $\frac{1}{(2-x-x^5)} = \frac{b_0}{(1-x)} + \frac{b_1}{(1-r_1x)} + \ldots + \frac{b_4}{(1-r_4x)}$. We calculate $b_0$ as $\lim_{x\to 1} \frac{(1-x)}{(2-x-x^5)} = \lim_{x\to 1} \frac{(-1)}{(-1-5x^4)} = \frac{1}{6}$.

Now raise the equation above to the 2011th power.

$$\frac{1}{(2-x-x^5)^{2011}} = \left(\frac{1/6}{(1-x)} + \frac{b_1}{(1-r_1x)} + \ldots + \frac{b_4}{(1-r_4x)}\right)^{2011}$$

Expand the right hand side using multinomial expansion and then apply partial fractions. The result will be a sum of the terms $(1-x)^{-k}$ and $(1-r_ix)^{-k}$, where $k \le 2011$.

Since $|r_i| < 1$, the power series of $(1-r_ix)^{-k}$ will have exponentially decaying coefficients, so we only need to consider the $(1-x)^{-k}$ terms. The coefficient of $x^n$ in the power series of $(1-x)^{-k}$ is $\binom{n+k-1}{k-1}$, which is a $(k-1)$th degree polynomial in variable $n$. So when we sum up all coefficients, only the power series of $(1-x)^{-2011}$ will have impact on the leading term $n^{2010}$.

The coefficient of the $(1-x)^{-2011}$ term in the multinomial expansion is $(\frac{1}{6})^{2011}$. The coefficient of the $x^n$ term in the power series of $(1-x)^{-2011}$ is $\binom{n+2010}{2010} = \frac{1}{2010!}n^{2010} + \ldots$. Therefore, $(c, d) = (\frac{1}{6^{2011}2010!}, 2010)$.

20. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2$; $b_0 = 2$, and $a_{n+1} = a_n\sqrt{1+a_n^2+b_n^2} - b_n$; $b_{n+1} = b_n\sqrt{1+a_n^2+b_n^2} + a_n$. Find the ternary (base 3) representation of $a_4$ and $b_4$.

**Answer:** $\boxed{1000001100111222 \text{ and } 2211100110000012}$
Note first that $\sqrt{1+a_n^2+b_n^2} = 3^{2^n}$. The proof is by induction; the base case follows trivially from what is given. For the inductive step, note that $1+a_{n+1}^2+b_{n+1}^2 = 1+a_n^2(1+a_n^2+b_n^2)+b_n^2-2a_nb_n\sqrt{1+a_n^2+b_n^2}+b_n^2(1+a_n^2+b_n^2)+a_n^2+2a_nb_n\sqrt{1+a_n^2+b_n^2} = 1+(a_n^2+b_n^2)(1+a_n^2+b_n^2)+a_n^2+b_n^2 = (1+a_n^2+b_n^2)^2$. Invoking the inductive hypothesis, we see that $\sqrt{1+a_{n+1}^2+b_{n+1}^2} = (3^{2^n})^2 = 3^{2^{n+1}}$, as desired.

The quickest way to finish from here is to consider a sequence of complex numbers $\{z_n\}$ defined by $z_n = a_n + b_ni$ for all nonnegative integers $n$. It should be clear that $z_0 = 2+2i$ and $z_{n+1} = z_n(3^{2^n}+i)$. Therefore, $z_4 = (2+2i)(3^{2^0}+i)(3^{2^1}+i)(3^{2^2}+i)(3^{2^3}+i)$. This product is difficult to evaluate in the decimal number system, but in ternary the calculation is a cinch! To speed things up, we will use *balanced ternary*[1], in which the three digits allowed are $-1, 0$, and 1 rather than $0, 1$, and 2. Let $x + yi = (3^{2^0} + i)(3^{2^1} + i)(3^{2^2} + i)(3^{2^3} + i)$, and consider the balanced ternary representation of $x$ and $y$. For all $0 \leq j \leq 15$, let $x_j$ denote the digit in the $3^j$ place of $x$, let $y_j$ denote the digit in the $3^j$ place of $y$, and let $b(j)$ denote the number of ones in the binary representation of $j$. It should be clear that $x_j = -1$ if $b(j) \equiv 2 \pmod 4$, $x_j = 0$ if $b(j) \equiv 1 \pmod 2$, and $x_j = 1$ if $b(j) \equiv 0 \pmod 4$. Similarly, $y_j = -1$ if $b(j) \equiv 1 \pmod 4$, $y_j = 0$ if $b(j) \equiv 0 \pmod 2$, and $y_j = 1$ if $b(j) \equiv 3 \pmod 4$. Converting to ordinary ternary representation, we see that $x = 221211221122001_3$ and $y = 110022202212120_3$. It remains to note that $a_4 = 2x - 2y$ and $b_4 = 2x + 2y$ and perform the requisite arithmetic to arrive at the answer above.

[1] http://en.wikipedia.org/wiki/Balanced_ternary