11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Algebra Test

1. [**3**] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

   Answer: $\boxed{\frac{1}{6}}$ We have $2x^2y^2 = (x^2+y^2)^2 - (x^4+y^4) = \frac{1}{18}$, so $xy = \frac{1}{6}$.

2. [**3**] Let $f(n)$ be the number of times you have to hit the $\sqrt{\ }$ key on a calculator to get a number less than 2 starting from n. For instance, $f(2) = 1, f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?

   Answer: $\boxed{242}$ This is $[2^1, 2^2) \cup [2^4, 2^8) \cup [2^{16}, 2^{32}) \ldots$, and $2^8 < 2008 < 2^{16}$ so we have exactly the first two intervals.

3. [**4**] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x.

   Answer: $\boxed{1, 2}$ Let $f(x) = x^2 + 2ax + 3a$. Note that $f(-3/2) = 9/4$, so the graph of f is a parabola that goes through $(-3/2, 9/4)$. Then, the condition that $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution means that the parabola has exactly one point in the strip $-1 \leq y \leq 1$, which is possible if and only if the parabola is tangent to $y = 1$. That is, $x^2 + 2ax + 3a = 2$ has exactly one solution. Then, the discriminant $\Delta = 4a^2 - 4(3a-2) = 4a^2 - 12a + 8$ must be zero. Solving the equation yields $a = 1, 2$.

4. [**4**] The function f satisfies

$$f(x) + f(2x+y) + 5xy = f(3x-y) + 2x^2 + 1$$

   for all real numbers x, y. Determine the value of $f(10)$.

   Answer: $\boxed{-49}$ Setting $x = 10$ and $y = 5$ gives $f(10) + f(25) + 250 = f(25) + 200 + 1$, from which we get $f(10) = -49$.

   Remark: By setting $y = \frac{x}{2}$, we see that the function is $f(x) = -\frac{1}{2}x^2 + 1$, and it can be checked that this function indeed satisfies the given equation.

5. [**5**] Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$, and the roots of g are the squares of the roots of f. Find $g(9)$.

   Answer: $\boxed{899}$ Let a, b, c be the zeros of f. Then $f(x) = (x-a)(x-b)(x-c)$. Then, the roots of g are a^2, b^2, c^2, so $g(x) = k(x-a^2)(x-b^2)(x-c^2)$ for some constant k. Since $abc = -f(0) = -1$, we have $k = ka^2b^2c^2 = -g(0) = 1$. Thus,

$$g(x^2) = (x^2-a^2)(x^2-b^2)(x^2-c^2) = (x-a)(x-b)(x-c)(x+a)(x+b)(x+c) = -f(x)f(-x).$$

   Setting $x = 3$ gives $g(9) = -f(3)f(-3) = -(31)(-29) = 899$.

6. [**5**] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

   Answer: $\boxed{8}$ The only real roots of unity are 1 and -1. If ζ is a complex root of unity that is also a root of the equation $z^2 + az + b$, then its conjugate $\bar{\zeta}$ must also be a root. In this case, $|a| = |\zeta + \bar{\zeta}| \leq |\zeta| + |\bar{\zeta}| = 2$ and $b = \zeta\bar{\zeta} = 1$. So we only need to check the quadratics $z^2+2z+1, z^2+z+1, z^2+1, z^2-z+1, z^2-2z+1$. We find 8 roots of unity: ± 1, $\pm i$, $\frac{1}{2}(\pm 1 \pm \sqrt{3}i)$.

7. [**5**] Compute $\displaystyle\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$.

Answer: $\boxed{\frac{4}{9}}$ We change the order of summation:

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{k}{2^{n+k}} = \sum_{k=1}^{\infty}\frac{k}{2^k}\sum_{n=k+1}^{\infty}\frac{1}{2^n} = \sum_{k=1}^{\infty}\frac{k}{4^k} = \frac{4}{9}.$$

(The last two steps involve the summation of an infinite geometric series, and what is sometimes called an infinite arithmetico-geometric series. These summations are quite standard, and thus we omit the details here.)

8. [**6**] Compute $\arctan(\tan 65^\circ - 2\tan 40^\circ)$. (Express your answer in degrees as an angle between 0° and 180°.)

Answer: $\boxed{25^\circ}$ **First Solution:** We have

$$\tan 65^\circ - 2\tan 40^\circ = \cot 25^\circ - 2\cot 50^\circ = \cot 25^\circ - \frac{\cot^2 25^\circ - 1}{\cot 25^\circ} = \frac{1}{\cot 25^\circ} = \tan 25^\circ.$$

Therefore, the answer is 25°.

Second Solution: We have

$$\tan 65^\circ - 2\tan 40^\circ = \frac{1+\tan 20^\circ}{1-\tan 20^\circ} - \frac{4\tan 20^\circ}{1-\tan^2 20^\circ} = \frac{(1-\tan 20^\circ)^2}{(1-\tan 20^\circ)(1+\tan 20^\circ)} = \tan(45^\circ - 20^\circ) = \tan 25^\circ.$$

Again, the answer is 25°.

9. [**7**] Let S be the set of points (a,b) with $0 \le a, b \le 1$ such that the equation

$$x^4 + ax^3 - bx^2 + ax + 1 = 0$$

has at least one real root. Determine the area of the graph of S.

Answer: $\boxed{\frac{1}{4}}$ After dividing the equation by x^2, we can rearrange it as

$$\left(x+\frac{1}{x}\right)^2 + a\left(x+\frac{1}{x}\right) - b - 2 = 0$$

Let $y = x+\frac{1}{x}$. We can check that the range of $x+\frac{1}{x}$ as x varies over the nonzero reals is $(-\infty,-2]\cup[2,\infty)$. Thus, the following equation needs to have a real root:

$$y^2 + ay - b - 2 = 0.$$

Its discriminant, $a^2+4(b+2)$, is always positive since $a, b \ge 0$. Then, the maximum absolute value of the two roots is

$$\frac{a+\sqrt{a^2+4(b+2)}}{2}.$$

We need this value to be at least 2. This is equivalent to

$$\sqrt{a^2+4(b+2)} \ge 4 - a.$$

We can square both sides and simplify to obtain

$$2a \ge 2 - b$$

This equation defines the region inside $[0,1]\times[0,1]$ that is occupied by S, from which we deduce that the desired area is $1/4$.

10. [**8**] Evaluate the infinite sum

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$

Answer: $\boxed{\sqrt{5}}$ **First Solution:** Note that

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{n! \cdot n!} = \frac{(2n)(2n-2)(2n-4)\cdots(2)}{n!} \cdot \frac{(2n-1)(2n-3)(2n-5)\cdots(1)}{n!} \\
&= 2^n \cdot \frac{(-2)^n}{n!} \left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-n+1\right) \\
&= (-4)^n \binom{-\frac{1}{2}}{n}.
\end{aligned}$$

Then, by the binomial theorem, for any real x with $|x| < \frac{1}{4}$, we have

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4x)^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

Therefore,

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1}{5}\right)^n = \frac{1}{\sqrt{1-\frac{4}{5}}} = \sqrt{5}.$$

Second Solution: Consider the generating function

$$f(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

It has formal integral given by

$$g(x) = I(f(x)) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1} = \sum_{n=0}^{\infty} C_n x^{n+1} = x \sum_{n=0}^{\infty} C_n x^n,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. Let $h(x) = \sum_{n=0}^{\infty} C_n x^n$; it suffices to compute this generating function. Note that

$$1 + xh(x)^2 = 1 + x \sum_{i,j \geq 0} C_i C_j x^{i+j} = 1 + x \sum_{k \geq 0} \left(\sum_{i=0}^{k} C_i C_{k-i} \right) x^k = 1 + \sum_{k \geq 0} C_{k+1} x^{k+1} = h(x),$$

where we've used the recurrence relation for the Catalan numbers. We now solve for $h(x)$ with the quadratic equation to obtain

$$h(x) = \frac{1/x \pm \sqrt{1/x^2 - 4/x}}{2} = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Note that we must choose the $-$ sign in the $\pm$, since the $+$ would lead to a leading term of $\frac{1}{x}$ for h (by expanding $\sqrt{1-4x}$ into a power series). Therefore, we see that

$$f(x) = D(g(x)) = D(xh(x)) = D\left(\frac{1-\sqrt{1-4x}}{2}\right) = \frac{1}{\sqrt{1-4x}}$$

and our answer is hence $f(1/5) = \sqrt{5}$.