HMMO 2020

November 14–21, 2020

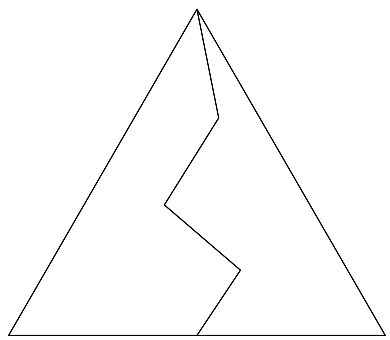
Guts Round

1. **[5]** Two hexagons are attached to form a new polygon P. Compute the minimum number of sides that P can have.

   Proposed by: Daniel Zhu

   Answer: $\boxed{3}$

   Solution: A triangle can be split into two hexagons; in other words, two concave hexagons can be attached to form a triangle, like the following:



2. **[5]** Let a be a positive integer such that $2a$ has units digit 4. What is the sum of the possible units digits of $3a$?

   Proposed by: Daniel Zhu

   Answer: $\boxed{7}$

   Solution: If $2a$ has last digit 4, then the last digit of a is either 2 or 7. In the former case, $3a$ has last digit 6, and in the latter case, $3a$ has last digit 1. This gives a final answer of $6 + 1 = 7$.

3. **[5]** How many six-digit multiples of 27 have only 3, 6, or 9 as their digits?

   Proposed by: Daniel Zhu

   Answer: $\boxed{51}$

   Solution: Divide by 3. We now want to count the number of six-digit multiples of 9 that only have 1, 2, or 3 as their digits. Due to the divisibility rule for 9, we only need to consider when the digit sum is a multiple of 9. Note that $3 \cdot 6 = 18$ is the maximum digit sum.

   If the sum is 18, the only case is 333333.

   Otherwise, the digit sum is 9. The possibilities here, up to ordering of the digits, are 111222 and 111123. The first has $\binom{6}{3} = 20$ cases, while the second has $6 \cdot 5 = 30$. Thus the final answer is $1 + 20 + 30 = 51$.

4. **[6]** Ainsley and Buddy play a game where they repeatedly roll a standard fair six-sided die. Ainsley wins if two multiples of 3 in a row are rolled before a non-multiple of 3 followed by a multiple of 3, and Buddy wins otherwise. If the probability that Ainsley wins is $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

Proposed by: Dora Woodruff

Answer: $\boxed{109}$

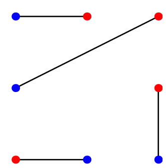
Solution: We let X be the event of a multiple of 3 being rolled and Y be the event of a nonmultiple of 3 being rolled. In order for Ainsley to win, she needs event X to happen consecutively; meanwhile, Buddy just needs Y then X to occur. Thus, if Y occurs in the first two rolls, Buddy will be guaranteed to win, since the next time X happens, it will have been preceded by an X. Thus, the probability of A winning is equivalent to the probability of X happening in each of the first two times, or $(1/3)^2 = 1/9$.

5. **[6]** The points $(0,0), (1,2), (2,1), (2,2)$ in the plane are colored red while the points $(1,0), (2,0), (0,1), (0,2)$ are colored blue. Four segments are drawn such that each one connects a red point to a blue point and each colored point is the endpoint of some segment. The smallest possible sum of the lengths of the segments can be expressed as $a + \sqrt{b}$, where a, b are positive integers. Compute $100a + b$.

   Proposed by: Andrew Gu

   Answer: $\boxed{305}$

   Solution:

   If $(2,2)$ is connected to $(0,1)$ or $(1,0)$, then the other 6 points can be connected with segments of total length 3, which is minimal. This leads to a total length of $3 + \sqrt{5}$.

   On the other hand, if $(2,2)$ is connected to $(0,2)$ or $(0,2)$, then connecting the other points with segments of total length 2 is impossible, so the minimal length is at least $2 + 2 + \sqrt{2} = 4 + \sqrt{2} > 3 + \sqrt{5}$.

6. **[6]** If x, y, z are real numbers such that $xy = 6$, $x - z = 2$, and $x + y + z = 9$, compute $\frac{x}{y} - \frac{z}{x} - \frac{z^2}{xy}$.

   Proposed by: Ragulan Sivakumar

   Answer: $\boxed{2}$

   Solution: Let $k = \frac{x}{y} - \frac{z}{x} - \frac{z^2}{xy} = \frac{x^2 - yz - z^2}{xy}$. We have

   $$k + 1 = \frac{x^2 + xy - yz - z^2}{xy} = \frac{x^2 - xz + xy - yz + zx - z^2}{xy} = \frac{(x+y+z)(x-z)}{xy} = \frac{9 \cdot 2}{6} = 3,$$

   so $k = 2$.

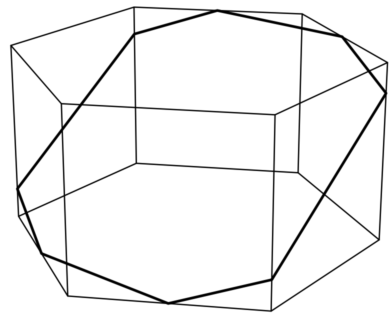
7. **[7]** Compute the maximum number of sides of a polygon that is the cross-section of a regular hexagonal prism.

   Proposed by: Hahn Lheem

   Answer: $\boxed{8}$

   Solution: Note that since there are 8 faces to a regular hexagonal prism and a cross-section may only intersect a face once, the upper bound for our answer is 8.

   Indeed, we can construct a cross-section of the prism with 8 sides. Let $ABCDEF$ and $A'B'C'D'E'F'$ be the two bases of the prism, with A being directly over A'. Choose points P and Q on line segments AB and BC, respectively, and choose points P' and Q' on segments $D'E'$ and $E'F'$, respectively, such that $PQ \parallel P'Q'$. Then, the cross-section of the prism from the plane that goes through P, Q, P', and Q' forms a polygon with 8 sides.



8. **[7]** A small village has n people. During their yearly elections, groups of three people come up to a stage and vote for someone in the village to be the new leader. After every possible group of three people has voted for someone, the person with the most votes wins.

   This year, it turned out that everyone in the village had the exact same number of votes! If $10 \leq n \leq 100$, what is the number of possible values of n?

   Proposed by: Vincent Bian

   Answer: $\boxed{61}$

   Solution: The problem asks for the number of n that divide $\binom{n}{3}$, which happens exactly when $\frac{(n-1)(n-2)}{2\cdot 3}$ is an integer. Regardless of the parity of n, $(n-1)(n-2)$ is always divisible by 2. Also, $(n-1)(n-2)$ is divisible by 3 if and only if n is not a multiple of 3. Of the 91 values from 10 to 100, 30 are divisible by 3, so our answer is 61.

9. **[7]** A fair coin is flipped eight times in a row. Let p be the probability that there is exactly one pair of consecutive flips that are both heads and exactly one pair of consecutive flips that are both tails. If $p = \frac{a}{b}$, where a, b are relatively prime positive integers, compute $100a + b$.

   Proposed by: Yannick Yao

   Answer: $\boxed{1028}$

   Solution: Separate the sequence of coin flips into alternating blocks of heads and tails. Of the blocks of heads, exactly one block has length 2, and all other blocks have length 1. The same statement applies to blocks of tails. Thus, if there are k blocks in total, there are $k-2$ blocks of length 1 and 2 blocks of length 2, leading to $k+2$ coins in total. We conclude that $k = 6$, meaning that there are 3 blocks of heads and 3 blocks of tails.

   The blocks of heads must have lengths $1, 1, 2$ in some order, and likewise for tails. There are $3^2 = 9$ ways to choose these two orders, and 2 ways to assemble these blocks into a sequence, depending on whether the first coin flipped is heads or tails. Thus the final probability is $18/2^8 = 9/128$.

10. **[8]** The number 3003 is the only number known to appear eight times in Pascal's triangle, at positions

$$\binom{3003}{1}, \binom{3003}{3002}, \binom{a}{2}, \binom{a}{a-2}, \binom{15}{b}, \binom{15}{15-b}, \binom{14}{6}, \binom{14}{8}.$$

   Compute $a + b(15-b)$.

   Proposed by: Carl Joshua Quines

   Answer: $\boxed{128}$

Solution: We first solve for a. Note that $3003 = 3 \cdot 7 \cdot 11 \cdot 13$. We have $3003 = \binom{a}{2} = \frac{a(a-1)}{2} \approx \frac{a^2}{2}$. This means we can estimate $a \approx \sqrt{3003 \cdot 2}$, so a is a little less than 80. Furthermore, $11 \mid 2 \cdot 3003 = a(a-1)$, meaning one of a or $a-1$ must be divisible by 11. Thus, either $a = 77$ or $a = 78$. Conveniently, $13 \mid 78$, so we get $a = 78$ and we can verify that $\binom{78}{2} = 3003$.

We solve for $b < 15 - b$ satisfying $\binom{15}{b} = 3003$. Because $\binom{15}{b} = \frac{15!}{b!(15-b)!}$ is divisible by 11, we must have $b \geq 5$. But we're given $\binom{14}{6} = 3003$, so $b < 6$. We conclude that $b = 5$, and it follows that $a + b(15 - b) = 128$.

11. **[8]** Two diameters and one radius are drawn in a circle of radius 1, dividing the circle into 5 sectors. The largest possible area of the smallest sector can be expressed as $\frac{a}{b}\pi$, where a, b are relatively prime positive integers. Compute $100a + b$.

Proposed by: William Qian

Answer: $\boxed{106}$

Solution: Let the two diameters split the circle into four sectors of areas A, B, A, and B, where $A + B = \frac{\pi}{2}$. Without loss of generality, let $A \leq B$.

If our radius cuts into a sector of area A, the area of the smallest sector will be of the form $\min(x, A - x)$. Note that $\min(A - x, x) \leq \frac{A}{2} \leq \frac{\pi}{8}$.

If our radius cuts into a sector of area B, then the area of the smallest sector will be of the form $\min(A, x, B - x) \leq \min(A, \frac{B}{2}) = \min(A, \frac{\pi}{4} - \frac{A}{2})$. This equals A if $A \leq \frac{\pi}{6}$ and it equals $\frac{\pi}{4} - \frac{A}{2}$ if $A \geq \frac{\pi}{6}$. This implies that the area of the smallest sector is maximized when $A = \frac{\pi}{6}$, and we get an area of $\frac{\pi}{6}$.

12. **[8]** In a single-elimination tournament consisting of $2^9 = 512$ teams, there is a strict ordering on the skill levels of the teams, but Joy does not know that ordering. The teams are randomly put into a bracket and they play out the tournament, with the better team always beating the worse team. Joy is then given the results of all 511 matches and must create a list of teams such that she can guarantee that the third-best team is on the list. What is the minimum possible length of Joy's list?

Proposed by: Cory Hixson

Answer: $\boxed{45}$

Solution: The best team must win the tournament. The second-best team has to be one of the 9 teams that the first best team beat; call these teams *marginal*. The third best team must have lost to either the best or the second-best team, so it must either be marginal or have lost to a marginal team. Since there is exactly one marginal team that won k games for each integer $0 \leq k \leq 8$, we can then conclude that there are $1 + 2 + \cdots + 9 = 45$ teams that are either marginal or lost to a marginal team. Moreover, it is not hard to construct a scenario in which the third-best team is any of these 45 teams, so we cannot do better.

13. **[9]** Wendy is playing darts with a circular dartboard of radius 20. Whenever she throws a dart, it lands uniformly at random on the dartboard. At the start of her game, there are 2020 darts placed randomly on the board. Every turn, she takes the dart farthest from the center, and throws it at the board again. What is the expected number of darts she has to throw before all the darts are within 10 units of the center?

Proposed by: Vincent Bian

Answer: $\boxed{6060}$

Solution: Consider an individual dart. There is a $\frac{1}{4}$ probability it is already within 10 units of the center. If not, for every throw there is a $\frac{1}{4}$ probability it is not thrown again. Thus, if E is the expected value of times it is thrown, we find $E = 1 + \frac{3}{4}E \implies E = 4$.

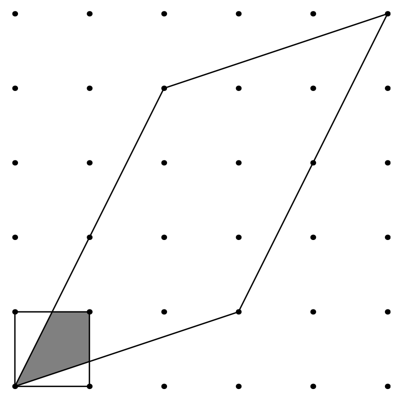
As a result, the expected number of times each dart is thrown is $\frac{3}{4} \cdot 4 = 3$. By linearity of expectation, the answer is $2020 \cdot 3 = 6060$.

14. **[9]** A point (x, y) is selected uniformly at random from the unit square $S = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. If the probability that $(3x + 2y, x + 4y)$ is in S is $\frac{a}{b}$, where a, b are relatively prime positive integers, compute $100a + b$.

Proposed by: Christopher Xu

Answer: $\boxed{820}$

Solution:



Under the transformation $(x, y) \mapsto (3x + 2y, x + 4y)$, S is mapped to a parallelogram with vertices $(0, 0)$, $(3, 1)$, $(5, 5)$, and $(2, 4)$. Using the shoelace formula, the area of this parallelogram is 10.

The intersection of the image parallelogram and S is the quadrilateral with vertices $(0, 0)$, $\left(1, \frac{1}{3}\right)$, $(1, 1)$, and $\left(\frac{1}{2}, 1\right)$. To get this quadrilateral, we take away a right triangle with legs 1 and $\frac{1}{2}$ and a right triangle with legs 1 and $\frac{1}{3}$ from the unit square. So the quadrilateral has area $1 - \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3} = \frac{7}{12}$. Then the fraction of the image parallelogram that lies within S is $\frac{\frac{7}{12}}{10} = \frac{7}{120}$, which is the probability that a point stays in S after the mapping.

15. **[9]** For a real number r, the quadratics $x^2 + (r - 1)x + 6$ and $x^2 + (2r + 1)x + 22$ have a common real root. The sum of the possible values of r can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

Proposed by: Andrew Yao

Answer: $\boxed{405}$

Solution: Let the common root be s. Then,

$$s^2 + (r - 1)s + 6 = s^2 + (2r + 1)s + 22,$$

and $s = -\frac{16}{r+2}$. Substituting this into $s^2 + (r - 1)s + 6 = 0$ yields

$$\frac{256}{(r + 2)^2} - \frac{16(r - 1)}{r + 2} + 6 = 0.$$

After multiplying both sides by $(r + 2)^2$, the equation becomes

$$256 - 16(r - 1)(r + 2) + 6(r + 2)^2 = 0,$$

which simplifies into

$$5r^2 - 4r - 156 = 0.$$

Thus, by Vieta's Formulas, the sum of the possible values of r is $\frac{4}{5}$.

16. **[10]** Three players play tic-tac-toe together. In other words, the three players take turns placing an "A", "B", and "C", respectively, in one of the free spots of a 3×3 grid, and the first player to have three of their label in a row, column, or diagonal wins. How many possible final boards are there where the player who goes third wins the game? (Rotations and reflections are considered different boards, but the order of placement does not matter.)

Proposed by: Andrew Lin

Answer: $\boxed{148}$

Solution: In all winning cases for the third player, every spot in the grid must be filled. There are two ways that player C wins along a diagonal, and six ways that player C wins along a row or column. In the former case, any arrangement of the As and Bs is a valid board, since every other row, column, and diagonal is blocked. So there are $\binom{6}{3} = 20$ different finishing boards each for this case. However, in the latter case, we must make sure players A and B do not complete a row or column of their own, so only $20-2 = 18$ of the finishing boards are valid. The final answer is $2 \cdot 20 + 6 \cdot 18 = 148$.

17. **[10]** Let $\mathbb{N}_{>1}$ denote the set of positive integers greater than 1. Let $f : \mathbb{N}_{>1} \to \mathbb{N}_{>1}$ be a function such that $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}_{>1}$. If $f(101!) = 101!$, compute the number of possible values of $f(2020 \cdot 2021)$.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{66}$

Solution: For a prime p and positive integer n, we let $v_p(n)$ denote the largest nonnegative integer k such that $p^k \mid n$. Note that f is determined by its action on primes. Since $f(101!) = 101!$, by counting prime factors, f must permute the set of prime factors of $101!$; moreover, if p and q are prime factors of $101!$ and $f(p) = q$, we must have $v_p(101!) = v_q(101!)$. This clearly gives $f(2) = 2$, $f(5) = 5$, so it suffices to find the number of possible values for $f(43 \cdot 47 \cdot 101)$. (We can factor $2021 = 45^2 - 2^2 = 43 \cdot 47$.)

There are 4 primes with $v_p(101!) = 2$ (namely, 37, 41, 43, 47), so there are 6 possible values for $f(43 \cdot 47)$. Moreover, there are 11 primes with $v_p(101!) = 1$ (namely, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101). Hence there are 66 possible values altogether.

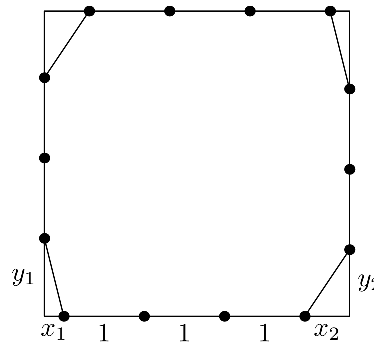
18. **[10]** Suppose Harvard Yard is a 17×17 square. There are 14 dorms located on the perimeter of the Yard. If s is the minimum distance between two dorms, the maximum possible value of s can be expressed as $a - \sqrt{b}$ where a, b are positive integers. Compute $100a + b$.

Proposed by: Hahn Lheem

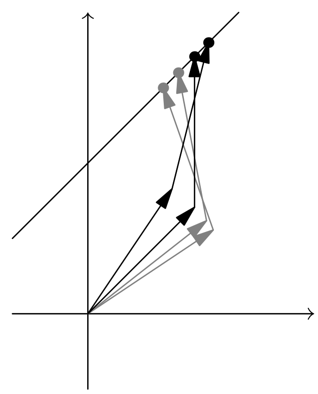
Answer: $\boxed{602}$

Solution: If two neighboring dorms are separated by a distance of more than s, we can move them slightly closer together and adjust the other dorms, increasing s. Therefore, in an optimal arrangement, the dorms form an equilateral 14-gon with side length s.

By scaling, the problem is now equivalent to finding the smallest a such that there exist 14 vertices on the boundary of an $a \times a$ square that form an equilateral 14-gon with side length 1. Such a 14-gon must be centrally symmetric, yielding the following picture:



We know that $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$ and $x_1 + x_2 + 3 = y_1 + y_2 + 2 = a$. Moreover, if these equations are satisfied, then such a 14-gon exists. We now consider the vectors $\vec{v}_1 = (x_1, y_1)$ and $\vec{v}_2 = (x_2, y_2)$. These unit vectors are in the first quadrant and add to $(a-3, a-2)$, which lies on the line $y = x + 1$.



Since $\vec{v}_1$ and $\vec{v}_2$ must lie on the first quadrant, from the above diagram we deduce that the minimum value of a occurs when one of $\vec{v}_1, \vec{v}_2$ is $(0, 1)$, meaning that $(a-3, a-2) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} + 1)$. This means that $a = \frac{\sqrt{2}}{2} + 3$, so the maximum possible value of s is

$$\frac{17}{\frac{\sqrt{2}}{2} + 3} = 17 \cdot \frac{3 - \frac{\sqrt{2}}{2}}{17/2} = 6 - \sqrt{2}.$$

19. **[11]** Three distinct vertices of a regular 2020-gon are chosen uniformly at random. The probability that the triangle they form is isosceles can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

Proposed by: Hahn Lheem

Answer: $\boxed{773}$

Solution: The number of isosceles triangles that share vertices with the 2020-gon is $2020 \cdot 1009$, since there are 2020 ways to choose the apex of the triangle and then 1009 ways to choose the other two vertices. (Since 2020 is not divisible by 3, there are no equilateral triangles, so no triangle is overcounted.)

Therefore, the probability is

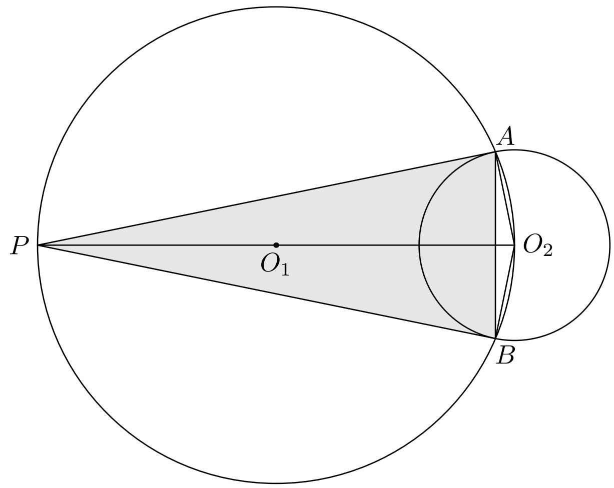
$$\frac{2020 \cdot 1009}{\binom{2020}{3}} = \frac{2020 \cdot 2018/2}{2020 \cdot 2019 \cdot 2018/6} = \frac{3}{2019} = \frac{1}{673}.$$

20. **[11]** Let ω_1 be a circle of radius 5, and let ω_2 be a circle of radius 2 whose center lies on ω_1. Let the two circles intersect at A and B, and let the tangents to ω_2 at A and B intersect at P. If the area of $\triangle ABP$ can be expressed as $\frac{a\sqrt{b}}{c}$, where b is square-free and a, c are relatively prime positive integers, compute $100a + 10b + c$.

Proposed by: Hahn Lheem

Answer: $\boxed{19285}$

Solution:



Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Because

$$\angle O_2AP + \angle O_2BP = 90^\circ + 90^\circ = 180^\circ,$$

quadrilateral O_2APB is cyclic. But O_2, A, and B lie on ω_1, so P lies on ω_1 and O_2P is a diameter of ω_1.

From the Pythagorean theorem on triangle PAO_2, we can calculate $AP = 4\sqrt{6}$, so $\sin \angle AO_2P = \frac{2\sqrt{6}}{5}$ and $\cos \angle AO_2P = \frac{1}{5}$. Because $\triangle AO_2P$ and $\triangle BO_2P$ are congruent, we have

$$\sin \angle APB = \sin 2\angle AO_2P = 2 \sin \angle AO_2P \cos \angle AO_2P = \frac{4\sqrt{6}}{25},$$

implying that

$$[APB] = \frac{PA \cdot PB}{2} \sin \angle APB = \frac{192\sqrt{6}}{25}.$$

21. **[11]** Let $f(n)$ be the number of distinct prime divisors of n less than 6. Compute

$$\sum_{n=1}^{2020} f(n)^2.$$

Proposed by: Milan Haiman

Answer: $\boxed{3431}$

Solution: Define

$$\mathbf{1}_{a|n} = \begin{cases} 1 & a \mid n \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned}
f(n)^2 &= (\mathbf{1}_{2|n} + \mathbf{1}_{3|n} + \mathbf{1}_{5|n})^2 \\
&= \mathbf{1}_{2|n} + \mathbf{1}_{3|n} + \mathbf{1}_{5|n} + 2(\mathbf{1}_{2|n}\mathbf{1}_{3|n} + \mathbf{1}_{2|n}\mathbf{1}_{5|n} + \mathbf{1}_{3|n}\mathbf{1}_{5|n}) \\
&= \mathbf{1}_{2|n} + \mathbf{1}_{3|n} + \mathbf{1}_{5|n} + 2(\mathbf{1}_{6|n} + \mathbf{1}_{10|n} + \mathbf{1}_{15|n}).
\end{aligned}$$

So summing $f(n)^2$ over integers $1 \leq n \leq 2020$ is the same as summing 1 for each time n is divisible by 2, 3, or 5, and additionally summing 2 for each time n is divisible by 6, 10, or 15.

$$\begin{aligned}
\sum_{n=1}^{2020} f(n)^2 &= \left\lfloor \frac{2020}{2} \right\rfloor + \left\lfloor \frac{2020}{3} \right\rfloor + \left\lfloor \frac{2020}{5} \right\rfloor + 2\left(\left\lfloor \frac{2020}{6} \right\rfloor + \left\lfloor \frac{2020}{10} \right\rfloor + \left\lfloor \frac{2020}{15} \right\rfloor \right) \\
&= 1010 + 673 + 404 + 2(336 + 202 + 134) = 3431.
\end{aligned}$$

22. **[12]** In triangle ABC, $AB = 32$, $AC = 35$, and $BC = x$. What is the smallest positive integer x such that $1 + \cos^2 A$, $\cos^2 B$, and $\cos^2 C$ form the sides of a non-degenerate triangle?

Proposed by: Hahn Lheem

Answer: $\boxed{48}$

Solution: By the triangle inequality, we wish $\cos^2 B + \cos^2 C > 1 + \cos^2 A$. The other two inequalities are always satisfied, since $1 + \cos^2 A \geq 1 \geq \cos^2 B, \cos^2 C$. Rewrite the above as

$$2 - \sin^2 B - \sin^2 C > 2 - \sin^2 A,$$

so it is equivalent to $\sin^2 B + \sin^2 C < \sin^2 A$. By the law of sines, $\sin A : \sin B : \sin C = BC : AC : AB$. Therefore,

$$\sin^2 B + \sin^2 C < \sin^2 A \iff CA^2 + AB^2 < x^2.$$

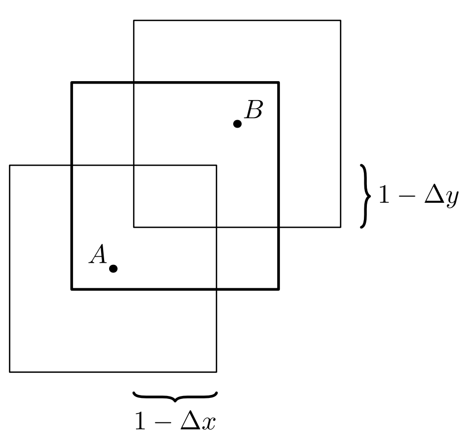
Since $CA^2 + AB^2 = 2249$, the smallest possible value of x such that $x^2 > 2249$ is 48.

23. **[12]** Two points are chosen inside the square $\{(x, y) \mid 0 \leq x, y \leq 1\}$ uniformly at random, and a unit square is drawn centered at each point with edges parallel to the coordinate axes. The expected area of the union of the two squares can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

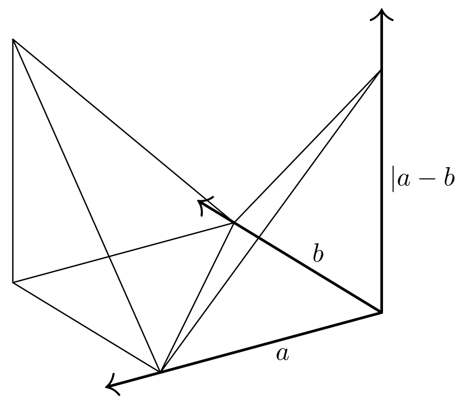
Proposed by: Yannick Yao

Answer: $\boxed{1409}$

Solution:



Let Δx and Δy be the positive differences between the x coordinates and y coordinates of the centers of the squares, respectively. Then, the length of the intersection of the squares along the x dimension is $1 - \Delta x$, and likewise the length along the y dimension is $1 - \Delta y$. In order to find the expectation of Δx and Δy, we can find the volume of the set of points (a, b, c) such that $0 \leq a, b \leq 1$ and $c \leq |a - b|$. This set is composed of the two pyramids of volume $\frac{1}{6}$ shown below:



Since the expected distance between two points on a unit interval is therefore $\frac{1}{3}$, we have that $\mathbb{E}[1 - \Delta x] = \mathbb{E}[1 - \Delta y] = \frac{2}{3}$. The expectation of the product of independent variables equals the product of their expectations, so the expected area of intersection is $\frac{4}{9}$ and the expected area of union is $2 - \frac{4}{9} = \frac{14}{9}$.

24. **[12]** Compute the number of positive integers less than 10! which can be expressed as the sum of at most 4 (not necessarily distinct) factorials.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{648}$

Solution: Since $0! = 1! = 1$, we ignore any possible 0!'s in our sums.

Call a sum of factorials *reduced* if for all positive integers k, the term $k!$ appears at most k times. It is straightforward to show that every positive integer can be written uniquely as a reduced sum of factorials. Moreover, by repeatedly replacing $k+1$ occurrences of $k!$ with $(k+1)!$, every non-reduced sum of factorials is equal to a reduced sum with strictly fewer terms, implying that the aforementioned reduced sum associated to a positive integer n in fact uses the minimum number of factorials necessary.

It suffices to compute the number of nonempty reduced sums involving $\{1!, 2!, \ldots, 9!\}$ with at most 4 terms. By stars and bars, the total number of such sums, ignoring the reduced condition, is $\binom{13}{9} = 714$.

The sums that are not reduced must either contain two copies of 1!, three copies of 2!, or four copies of 3!. Note that at most one of these conditions is true, so we can count them separately. If k terms are fixed, there are $\binom{13-k}{9}$ ways to choose the rest of the terms, meaning that we must subtract $\binom{11}{9} + \binom{10}{9} + \binom{9}{9} = 66$. Our final answer is $714 - 66 = 648$.

25. **[13]** Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers where $a_1 = \sum_{i=0}^{100} i!$ and $a_i + a_{i+1}$ is an odd perfect square for all $i \geq 1$. Compute the smallest possible value of a_{1000}.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{7}$

Solution: Note that $a_1 \equiv 1 + 1 + 2 + 6 \equiv 2 \pmod{8}$. Since $a_1 + a_2$ must be an odd perfect square, we must have $a_1 + a_2 \equiv 1 \pmod{8} \implies a_2 \equiv 7 \pmod{8}$. Similarly, since $a_2 + a_3$ is an odd perfect square, we must have $a_3 \equiv 2 \pmod 8$. We can continue this to get $a_{2k-1} \equiv 2 \pmod 8$ and $a_{2k} \equiv 7 \pmod 8$, so in particular, we have $a_{1000} \equiv 7 \pmod 8$, so $a_{1000} \geq 7$.

Now, note that we can find some large enough odd perfect square t^2 such that $t^2 - a_1 \geq 23$. Let $a_2 = t^2 - a_1$. Since $a_2 \equiv 7 \pmod 8$, we can let $a_2 - 7 = 8k$ for some integer $k \geq 2$. Now, since we have $(2k+1)^2 - (2k-1)^2 = 8k$, if we let $a_3 = (2k-1)^2 - 7$, then

$$a_2 + a_3 = a_2 + ((2k-1)^2 - 7) = (2k-1)^2 + (a_2 - 7) = (2k-1)^2 + 8k = (2k+1)^2,$$

which is an odd perfect square. Now, we can let $a_4 = 7$ and we will get $a_3 + a_4 = (2k-1)^2$. From here, we can let $2 = a_5 = a_7 = a_9 = \cdots$ and $7 = a_4 = a_6 = a_8 = \cdots$, which tells us that the least possible value for a_{1000} is 7.

26. **[13]** Two players play a game where they are each given 10 indistinguishable units that must be distributed across three locations. (Units cannot be split.) At each location, a player wins at that location if the number of units they placed there is at least 2 more than the units of the other player. If both players distribute their units randomly (i.e. there is an equal probability of them distributing their units for any attainable distribution across the 3 locations), the probability that at least one location is won by one of the players can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{1011}$

Solution: By stars and bars, the total number of distributions is $\binom{12}{2}^2 = 66^2$. If no locations are won, either both distributions are identical or the difference between the two is $(1, 0, -1)$, in some order. The first case has 66 possibilities. If the difference is $(1, 0, -1)$, we can construct all such possiblities by choosing nonnegative integers a, b, c that sum to 9, and having the two players choose $(a+1, b, c)$ and $(a, b, c+1)$. This can be done in $\binom{11}{2} = 55$ ways. In total, the second case has $6 \cdot 55 = 5 \cdot 66$ possibilities.

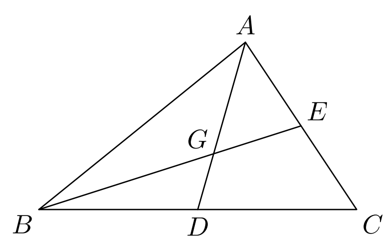
Thus the probability that no locations are won is $\frac{6 \cdot 66}{66^2} = \frac{1}{11}$, meaning that the answer is $\frac{10}{11}$.

27. **[13]** In $\triangle ABC$, D and E are the midpoints of BC and CA, respectively. AD and BE intersect at G. Given that $GECD$ is cyclic, $AB = 41$, and $AC = 31$, compute BC.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{49}$

Solution:



By Power of a Point,

$$\tfrac{2}{3}AD^2 = AD \cdot AG = AE \cdot AC = \tfrac{1}{2} \cdot 31^2$$

so $AD^2 = \frac{3}{4} \cdot 31^2$. The median length formula yields

$$AD^2 = \tfrac{1}{4}(2AB^2 + 2AC^2 - BC^2),$$

whence

$$BC = \sqrt{2AB^2 + 2AC^2 - 4AD^2} = \sqrt{2 \cdot 41^2 + 2 \cdot 31^2 - 3 \cdot 31^2} = 49.$$

28. **[15]** Bernie has 2020 marbles and 2020 bags labeled $B_1, \ldots, B_{2020}$ in which he randomly distributes the marbles (each marble is placed in a random bag independently). If E the expected number of integers $1 \le i \le 2020$ such that B_i has at least i marbles, compute the closest integer to $1000E$.

    Proposed by: Benjamin Kang

    Answer: $\boxed{1000}$

    Solution: Let p_i be the probability that a bag has i marbles. Then, by linearity of expectation, we find

    $$E = (p_1 + p_2 + \cdots) + (p_2 + p_3 + \cdots) + \cdots = p_1 + 2p_2 + 3p_3 + \cdots.$$

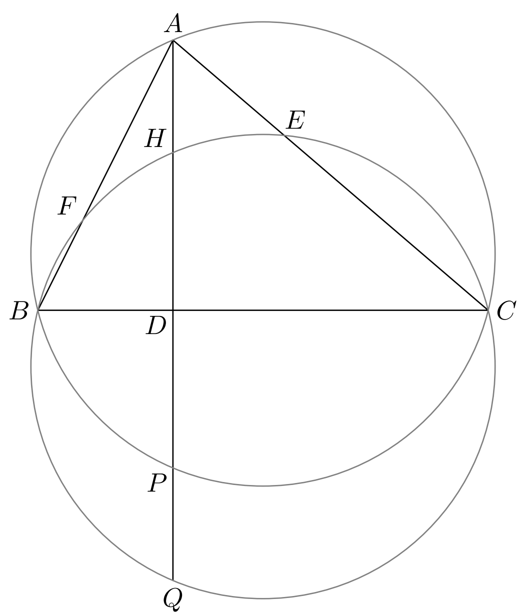
    This is precisely the expected value of the number of marbles in a bag. By symmetry, this is 1.

29. **[15]** In acute triangle ABC, let H be the orthocenter and D the foot of the altitude from A. The circumcircle of triangle BHC intersects AC at $E \ne C$, and AB at $F \ne B$. If $BD = 3$, $CD = 7$, and $\frac{AH}{HD} = \frac{5}{7}$, the area of triangle AEF can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

    Proposed by: Andrew Yao

    Answer: $\boxed{12017}$

    Solution:



Let AH intersect the circumcircle of $\triangle ABC$ again at P, and the circumcircle of $\triangle BHC$ again at Q. Because $\angle BHC = 180 - \angle A = \angle BPC$, P is the reflection of H over D. Thus, we know that $PD = HD$. From power of a point and $AD = \frac{12HD}{7}$,

$$BD \cdot CD = AD \cdot PD = \frac{12HD^2}{7}.$$

From this, $HD = \frac{7}{2}$ and $AH = \frac{5}{2}$. Furthermore, because $\triangle BHC$ is the reflection of $\triangle BPC$ over BC, the circumcircle of $\triangle BHC$ is the reflection of the circumcircle of $\triangle ABC$ over BC. Then, $AQ = 2AD = 12$. Applying Power of a Point,

$$AC \cdot AE = AB \cdot AF = AH \cdot AQ = 30.$$

We can compute $AC = \sqrt{85}$ and $AB = 3\sqrt{5}$, which means that $AE = \frac{6\sqrt{85}}{17}$ and $AF = 2\sqrt{5}$. Also, $[ABC] = \frac{BC \cdot AD}{2} = 30$. Therefore,

$$[AEF] = \frac{AE \cdot AF}{AC \cdot AB} \cdot [ABC] = \frac{4}{17} \cdot 30 = \frac{120}{17}.$$

30. **[15]** Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers that satisfies

$$\sum_{n=k}^{\infty} \binom{n}{k} a_n = \frac{1}{5^k},$$

for all positive integers k. The value of $a_1 - a_2 + a_3 - a_4 + \cdots$ can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

Proposed by: Akash Das

Answer: $\boxed{542}$

Solution: Let $S_k = \frac{1}{5^k}$. In order to get the coefficient of a_2 to be -1, we need to have $S_1 - 3S_3$. This subtraction makes the coefficient of a_3 become -6. Therefore, we need to add $7S_3$ to make the coefficient of a_4 equal to 1. The coefficient of a_4 in $S_1 - 3S_3 + 7S_5$ is 14, so we must subtract $15S_4$. We can continue to pattern to get that we want to compute $S_1 - 3S_2 + 7S_3 - 15S_4 + 31S_5 - \cdots$. To prove that this alternating sum equals $a_1 - a_2 + a_3 - a_4 + \cdots$, it suffices to show

$$\sum_{i=1}^{n} (-(-2)^i + (-1)^i) \binom{n}{i} = (-1)^{i+1}.$$

To see this is true, note that the left hand side equals $-(1-2)^i + (1-1)^i = (-1)^{i+1}$ by binomial expansion. (We may rearrange the sums since the positivity of the a_i's guarantee absolute convergence.) Now, all that is left to do is to compute

$$\sum_{i=1}^{\infty} \frac{(2^i - 1)(-1)^{i-1}}{5^i} = \sum_{i=1}^{\infty} \frac{(-1)^i}{5^i} - \sum_{i=1}^{\infty} \frac{(-2)^i}{5^i} = \frac{\frac{-1}{5}}{1 - \frac{-1}{5}} - \frac{\frac{-2}{5}}{1 - \frac{-2}{5}} = \frac{5}{42}.$$

31. **[17]** For some positive real α, the set S of positive real numbers x with $\{x\} > \alpha x$ consists of the union of several intervals, with total length 20.2. The value of α can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$. (Here, $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.)

Proposed by: Daniel Zhu

Answer: $\boxed{4633}$

Solution: If we note that $x = \{x\} + \lfloor x \rfloor$, then we can rewrite our given inequality as $\{x\} > \frac{\alpha}{1-\alpha} \lfloor x \rfloor$. However, since $\{x\} < 1$, we know that we must have $\frac{\alpha}{1-\alpha} \lfloor x \rfloor < \{x\} < 1$, so each interval is of the form $(n + \frac{\alpha}{1-\alpha} n, n+1)$ for some integer n, which has length $\frac{1-(n+1)\alpha}{1-\alpha}$. If we let k be the smallest integer such that $\frac{1-(k+1)\alpha}{1-\alpha} < 0$, then the total length of all our intervals is the sum

$$\sum_{n=0}^{k-1} \frac{1-(n+1)\alpha}{1-\alpha} = \frac{k - \frac{k(k+1)}{2}\alpha}{1-\alpha}.$$

If we set this to 20.2, we can solve for α to get

$$\alpha = \frac{k - 20.2}{\frac{k(k+1)}{2} - 20.2}.$$

Since we defined k to be the smallest integer such that $1 - (k+1)\alpha < 0$, we know that k is the largest integer such that $k\alpha < 1$. If we plug in our value for α, we get that this is equivalent to

$$\frac{k^2 - 20.2k}{\frac{k(k+1)}{2} - 20.2} < 1 \implies k < 40.4.$$

Thus, we have $k = 40$, and plugging this in for our formula for α gives us

$$\alpha = \frac{40 - 20.2}{\frac{40\cdot 41}{2} - 20.2} = \frac{33}{1333}.$$

32. **[17]** The numbers $1, 2, \ldots, 10$ are written in a circle. There are four people, and each person randomly selects five consecutive integers (e.g. $1, 2, 3, 4, 5$, or $8, 9, 10, 1, 2$). If the probability that there exists some number that was not selected by any of the four people is p, compute $10000p$.

Proposed by: Hahn Lheem

Answer: $\boxed{3690}$

Solution: The unselected numbers must be consecutive. Suppose that $\{1, 2, \ldots, k\}$ are the unselected numbers for some k.

In this case, 1 cannot be selected, so there are 5 possible sets of consecutive numbers the people could have chosen. This leads to 5^4 possibilities. Moreover, 10 must be selected, so we must subtract 4^4 possibilities where neither 1 nor 10 are selected.

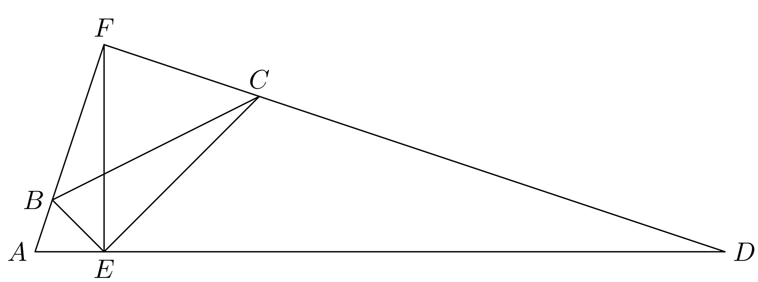
Therefore, accounting for the rotation of the unselected numbers, we find $p = \frac{10(5^4-4^4)}{10^4} = \frac{3690}{10000}$.

33. **[17]** In quadrilateral $ABCD$, there exists a point E on segment AD such that $\frac{AE}{ED} = \frac{1}{9}$ and $\angle BEC$ is a right angle. Additionally, the area of triangle CED is 27 times more than the area of triangle AEB. If $\angle EBC = \angle EAB$, $\angle ECB = \angle EDC$, and $BC = 6$, compute the value of AD^2.

Proposed by: Akash Das

Answer: $\boxed{320}$

Solution:



Extend sides AB and CD to intersect at point F. The angle conditions yield $\triangle BEC \sim \triangle AFD$, so $\angle AFD = 90°$. Therefore, since $\angle BFC$ and $\angle BEC$ are both right angles, quadrilateral $EBFC$ is cyclic and

$$\angle EFC = \angle EBC = 90° - \angle ECB = 90° - \angle EDF,$$

implying that $EF \perp AD$.

Since AFD is a right triangle, we have $(\frac{FA}{FD})^2 = \frac{AE}{ED} = \frac{1}{9}$, so $\frac{FA}{FD} = \frac{1}{3}$. Therefore $\frac{EB}{EC} = \frac{1}{3}$. Since the area of CED is 27 times more than the area of AEB, $ED = 9 \cdot EA$, and $EC = 3 \cdot EB$, we get that $\angle DEC = \angle AEB = 45°$. Since $BECF$ is cyclic, we obtain $\angle FBC = \angle FCB = 45°$, so $FB = FC$.

Since $BC = 6$, we get $FB = FC = 3\sqrt{2}$. From $\triangle EAB \sim \triangle EFC$ we find $AB = \frac{1}{3}FC = \sqrt{2}$, so $FA = 4\sqrt{2}$. Similarly, $FD = 12\sqrt{2}$. It follows that $AD^2 = FA^2 + FD^2 = 320$.

34. **[20]** Let a be the proportion of teams that correctly answered problem 1 on the Guts round. Estimate $A = \lfloor 10000a \rfloor$. An estimate of E earns $\max(0, \lfloor 20 - |A-E|/20 \rfloor)$ points. If you have forgotten, question 1 was the following:

Two hexagons are attached to form a new polygon P. What is the minimum number of sides that P can have?

Proposed by: Daniel Zhu

Answer: $\boxed{2539}$

Solution: 689 teams participated in the guts round. Of these,

- 175 teams submitted 3, the correct answer;
- 196 teams submitted 4;
- 156 teams submitted 10 (the correct answer if the hexagons had to be regular);
- 64 teams submitted 6 (the correct answer if one of the hexagons had to be regular);
- 19 teams submitted 8 (the correct answer if the hexagons had to be convex);
- 17 teams submitted 11;
- 13 teams submitted other incorrect answers;
- 49 teams did not submit an answer.

35. **[20]** Estimate A, the number of times an 8-digit number appears in Pascal's triangle. An estimate of E earns $\max(0, \lfloor 20 - |A-E|/200 \rfloor)$ points.

Proposed by: Daniel Zhu

Answer: $\boxed{180020660}$

Solution: We can obtain a good estimate by only counting terms of the form $\binom{a}{1}$, $\binom{a}{2}$, $\binom{a}{a-1}$, and $\binom{a}{a-2}$. The last two cases are symmetric to the first two, so we will only consider the first two and multiply by 2 at the end.

Since $\binom{a}{1} = a$, there are 90000000 values of a for which $\binom{a}{1}$ has eight digits. Moreover, since $\binom{a}{2} \approx a^2/2$, the values of a for which $\binom{a}{2}$ has eight digits vary from about $\sqrt{2 \cdot 10^7}$ to $\sqrt{2 \cdot 10^8}$, leading to about $10^4\sqrt{2}(1 - 10^{-1/2}) \approx 14000 \cdot 0.69 = 9660$ values for a.

Therefore, these terms yield an estimate of 180019320, good enough for 13 points. Of course, one would expect this to be an underestimate, and even rounding up to 180020000 would give 16 points.

36. **[20]** Let p_i be the ith prime. Let

$$f(x) = \sum_{i=1}^{50} p_i x^{i-1} = 2 + 3x + \cdots + 229x^{49}.$$

If a is the unique positive real number with $f(a) = 100$, estimate $A = \lfloor 100000a \rfloor$. An estimate of E will earn $\max(0, \lfloor 20 - |A-E|/250 \rfloor)$ points.

Proposed by: Daniel Zhu

Answer: $\boxed{83601}$

Solution: Note $f(x)$ is increasing. Since $f(0) = 2$ and $f(1) \approx 50000$, we have $0 < a < 1$.

Since we know that $p_{50} = 229$, we can crudely bound

$$f(x) \lesssim \sum_{i=1}^{\infty} 5ix^{i-1} = \frac{5}{(1-x)^2}.$$

Setting this equal to 100 yields $x = 1 - 20^{-1/2} \approx 0.78$, so this is a good lower bound for a, though just outside the window to receive points.

A better estimate can be obtained by noting that since $p_{25} = 100$, it is more accurate to write

$$f(x) \lesssim \sum_{i=1}^{\infty} 4ix^{i-1} = \frac{4}{(1-x)^2},$$

which yields $a = 0.8$, good enough for 5 points.

However, we can do better. If we know that $a \approx 0.8$, the "most significant terms" will occur at the i where $p_i/p_{i+1} \approx 0.8$. The first few primes are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$, so this transition occurs roughly at $p_8 = 19$. Thus, it is more accurate to approximate $f(x) = \sum_{i=1}^{\infty} \frac{19}{8} ix^{i-1}$, so $a = 1 - \sqrt{19/800} \approx 1 - 40^{-1/2} \approx 0.85$, good enough for 14 points.

Repeating this process again with the new estimate for a reveals that $p_9 = 23$ may have been a better choice, which yield $a = 1 - \sqrt{23/900} \approx 1 - \sqrt{0.0256} = 0.84$. This is good enough for 18 points.