HMMT February 2018

February 10, 2018

Team Round

1. [**20**] In an $n \times n$ square array of 1×1 cells, at least one cell is colored pink. Show that you can always divide the square into rectangles along cell borders such that each rectangle contains exactly one pink cell.

   Proposed by: Kevin Sun

   We claim that the statement is true for arbitrary rectangles. We proceed by induction on the number of marked cells. Our base case is $k = 1$ marked cell, in which case the original rectangle works.

   To prove it for k marked cells, we split the rectangle into two smaller rectangles, both of which contains at least one marked cell. By induction, we can divide the two smaller rectangles into rectangles with exactly one marked cell. Combining these two sets of rectangles gives a way to divide our original rectangle into rectangles with exactly one marked cell, completing the induction.

2. [**25**] Is the number
$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2018}\right)$$
   greater than, less than, or equal to 50?

   Proposed by: Henrik Boecken

   Call the expression S. Note that
$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2018}\right) < \left(1+\frac{1}{1}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\cdots\left(1+\frac{1}{2017}\right)$$
   Multiplying these two products together, we get
$$\begin{aligned}&\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{2018}\right)\\ &= \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{2019}{2018}\\ &= 2019\end{aligned}$$
   This shows that
$$S^2 < 2019 \implies S < \sqrt{2019} < 50$$
   as desired.

3. [**30**] Michelle has a word with 2^n letters, where a word can consist of letters from any alphabet. Michelle performs a *switcheroo* on the word as follows: for each $k = 0, 1, \ldots, n-1$, she switches the first 2^k letters of the word with the next 2^k letters of the word. For example, for $n = 3$, Michelle changes
$$ABCDEFGH \to BACDEFGH \to CDBAEFGH \to EFGHCDBA$$
   in one switcheroo.

   In terms of n, what is the minimum positive integer m such that after Michelle performs the switcheroo operation m times on any word of length 2^n, she will receive her original word?

   Proposed by: Mehtaab Sawhney

   Let $m(n)$ denote the number of switcheroos needed to take a word of length 2^n back to itself. Consider a word of length 2^n for some $n > 1$. After 2 switcheroos, one has separately performed a switcheroo on the first half of the word and on the second half of the word, while returning the (jumbled) first half of the word to the beginning and the (jumbled) second half of the word to the end.

After $2 \cdot m(n-1)$ switcheroos, one has performed a switcheroo on each half of the word $m(n-1)$ times while returning the halves to their proper order. Therefore, the word is in its proper order. However, it is never in its proper order before this, either because the second half precedes the first half (i.e. after an odd number of switcheroos) or because the halves are still jumbled (because each half has had fewer than $m(n-1)$ switcheroos performed on it).

It follows that $m(n) = 2m(n-1)$ for all $n > 1$. We can easily see that $m(1) = 2$, and a straightforward proof by induction shows that $m = 2^n$.

4. [**30**] In acute triangle ABC, let D, E, and F be the feet of the altitudes from A, B, and C respectively, and let L, M, and N be the midpoints of BC, CA, and AB, respectively. Lines DE and NL intersect at X, lines DF and LM intersect at Y, and lines XY and BC intersect at Z. Find $\frac{ZB}{ZC}$ in terms of AB, AC, and BC.

   Proposed by: Faraz Masroor

   Because $NL||AC$ we have triangles DXL and DEC are similar. From angle chasing, we also have that triangle DEC is similar to triangle ABC. We have $\angle XNA = 180° - \angle XNB = 180° - \angle LNB = 180 - CAB = \angle LMA$. In addition, we have $\frac{NX}{NA} = \frac{XD \cdot XE}{XL \cdot NA} = \frac{AB}{BC}\frac{XE}{LC}\frac{NM}{NA} = \frac{AB}{BC}\frac{ED}{DC}\frac{BC}{AB} = \frac{ED}{DC} = \frac{AB}{AC} = \frac{ML}{MA}$. These two statements mean that triangles ANX and AML are similar, and $\angle XAB = \angle XAN = \angle LAM = \angle LAC$. Similarly, $\angle XAY = \angle LAC$, making A, X, and Y collinear, with $\angle YAB = \angle XAB = \angle LAC$; ie. line AXY is a symmedian of triangle ABC.

   Then $\frac{ZB}{ZC} = \frac{AB}{AC}\frac{\sin\angle ZAB}{\sin\angle ZAC} = \frac{AB}{AC}\frac{\sin\angle LAC}{\sin\angle LAB}$, by the ratio lemma. But using the ratio lemma, $1 = \frac{LB}{LC} = \frac{AB}{AC}\frac{\sin\angle LAB}{\sin\angle LAC}$, so $\frac{\sin\angle LAC}{\sin\angle LAB} = \frac{AB}{AC}$, so $\frac{ZB}{ZC} = \frac{AB^2}{AC^2}$.

5. [**30**] Is it possible for the projection of the set of points (x, y, z) with $0 \leq x, y, z \leq 1$ onto some two-dimensional plane to be a simple convex pentagon?

   Proposed by: Yuan Yao

   It is not possible. Consider P, the projection of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ onto the plane. Since for any point (x, y, z) in the cube, $(1-x, 1-y, 1-z)$ is also in the cube, and the midpoint of their projections will be the projection of their midpoint, which is P, the projection of the cube onto this plane will be a centrally symmetric region around P, and thus cannot be a pentagon.

6. [**35**] Let $n \geq 2$ be a positive integer. A subset of positive integers S is said to be *comprehensive* if for every integer $0 \leq x < n$, there is a subset of S whose sum has remainder x when divided by n. Note that the empty set has sum 0. Show that if a set S is comprehensive, then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

   Proposed by: Allen Liu

   We will show that if $|S| \geq n$, we can remove one element from S and still have a comprehensive set. Doing this repeatedly will always allow us to find a comprehensive subset of size at most $n-1$.

   Write $S = \{s_1, s_2, \ldots, s_k\}$ for some $k \geq n$. Now start with the empty set and add in the elements s_i in order. During this process, we will keep track of all possible remainders of sums of any subset.

   If T is the set of current remainders at any time, and we add an element s_i, the set of remainders will be $T \cup \{t + s_i | t \in T\}$. In particular, the set of remainders only depends on the previous set of remainders and the element we add in.

   At the beginning of our process, the set of possible remainders is $\{0\}$ for the empty set. Since we assumed that S is comprehensive, the final set is $\{0, 1, \ldots, n\}$. The number of elements changes from 1 to $n-1$. However, since we added $k \geq n$ elements, at least one element did not change the size of our remainder set. This implies that adding this element did not contribute to making any new remainders and S is still comprehensive without this element, proving our claim.

7. [**50**] Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. We randomly choose a function $f : [n] \to [n]$, out of the n^n possible functions. We also choose an integer a uniformly at random from $[n]$. Find the probability that there exist positive integers $b, c \geq 1$ such that $f^b(1) = a$ and $f^c(a) = 1$. ($f^k(x)$ denotes the result of applying f to x k times).

   Proposed by: Allen Liu

   Answer: $\boxed{\frac{1}{n}}$

   Given a function f, define $N(f)$ to be the number of numbers that are in the same cycle as 1 (including 1 itself), if there is one, and zero if there is no such cycle. The problem is equivalent to finding $\mathbb{E}(N(f))/n$. Note that

$$P(N(f) = k) = \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \cdots \cdot \frac{n-k+1}{n} \cdot \frac{1}{n},$$

   and it suffices to compute $\sum_{k=1}^{n} P_k$ where $P_k = \frac{k}{n} P(N(f) = k)$. Observe that

$$\begin{aligned}
P_n &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \cdots \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n}\right) \cdot \frac{n}{n} && \cdot \frac{1}{n} \\
P_{n-1} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \cdots \cdot \frac{3}{n} \cdot \frac{2}{n}\right) \cdot \frac{n-1}{n} && \cdot \frac{1}{n} \\
\Rightarrow P_n + P_{n-1} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \cdots \cdot \frac{3}{n} \cdot \frac{2}{n}\right) && \cdot \frac{1}{n} \\
P_{n-2} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \cdots \cdot \frac{3}{n}\right) \cdot \frac{n-2}{n} && \cdot \frac{1}{n} \\
\Rightarrow P_n + P_{n-1} + P_{n-2} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \cdots \cdot \frac{3}{n}\right) && \cdot \frac{1}{n} \\
&\cdots \\
\Rightarrow \sum_{k=1}^{n} P_k &= 1 && \cdot \frac{1}{n}
\end{aligned}$$

   Therefore the answer is $\frac{1}{n}$.

8. [**60**] Allen plays a game on a tree with $2n$ vertices, each of whose vertices can be red or blue. Initially, all of the vertices of the tree are colored red. In one move, Allen is allowed to take two vertices of the same color which are connected by an edge and change both of them to the opposite color. He wins if at any time, all of the vertices of the tree are colored blue.

   (a) (**20**) Show that Allen can win if and only if the vertices can be split up into two groups V_1 and V_2 of size n, such that each edge in the tree has one endpoint in V_1 and one endpoint in V_2.

   (b) (**40**) Let $V_1 = \{a_1, \ldots, a_n\}$ and $V_2 = \{b_1, \ldots, b_n\}$ from part (a). Let M be the minimum over all permutations σ of $\{1, \ldots, n\}$ of the quantity

$$\sum_{i=1}^{n} d(a_i, b_{\sigma(i)}),$$

   where $d(v, w)$ denotes the number of edges along the shortest path between vertices v and w in the tree.

   Show that if Allen can win, then the minimum number of moves that it can take for Allen to win is equal to M.

   (A *graph* consists of a set of vertices and some edges between distinct pairs of vertices. It is *connected* if every pair of vertices are connected by some path of one or more edges. A *tree* is a graph which is connected, in which the number of edges is one less than the number of vertices.)

Proposed by: Kevin Sun

Part (a): First we show that if we can't split the vertices in the desired way then Allen cannot win. To do so, observe that there is a unique way to split the vertices into two groups so that all edges cross between the two groups, since trees are bipartite. Each of Allen's moves either adds one more blue vertex to each group or removes a blue vertex from each group, so the difference in the number of blue vertices between the two groups is invariant. Both groups initially have no blue vertices, so Allen can only possibly arrive at a state where both groups have all blue vertices if both groups were initially of equal size.

To show that Allen can win when the splitting is possible, we induct on n. It's trivial for $n = 1$. A tree on $2n$ vertices has $2n - 1$ edges, so the total degree for each of the two n-vertex groups is $2n - 1$. Therefore, at least one vertex in each group is a leaf, say L_1 in one group and L_2 in the other. Allen can then flip all the vertices from L_1 to L_2 since there is a path of even length (in terms of number of vertices) from one to the other, including flipping L_1 and L_2, and then flip back all of the middle vertices in the path, with the end result being that L_1 and L_2 are blue while all other vertices in the tree remain red. This lets us remove L_1 and L_2 from consideration and apply the inductive hypothesis on the remaining graph.

Part (b): For each edge e in the graph, we can determine a minimum number of times Allen must perform an operation on the endpoints of that edge (henceforth, "flip this edge"), as follows. Observe that removing e disconnects the graph into two components. Each component is still a tree which can be split into two groups as before, though not necessarily with an equal number of vertices. In particular, suppose that the two groups differ by d vertices. Flipping any edge other than e does not change the difference in number of blue vertices between the two groups of the component. So e must flip at least d times if Allen is to win. Let m_e be the minimum number of times that edge e must flip, as determined in this way, and let M' be the total sum of all of the m_e. And let T' be the original tree with every edge duplicated m_e times (or removed if $m_e = 0$).

Observe that any permutation σ that gives us n paths must have total length at least M' for the same reasons as in the above paragraph, so $M' \leq M$.

In fact, we will show that $M' = M$. To show that $M' = M$, it suffices to show that $M' \geq M$, which we will show by finding a partitioning of T' into n odd-length paths for which every vertex is the endpoint of exactly one path. This will show that M' is the sum of path lengths for some permutation, meaning it is at least the sum of path lengths for the optimal permutation. We do this by strong induction. The base case is trivial, and if any m_e is 0 then we're immediately done because we can split the T' into two smaller trees. Otherwise we repeat the argument from part (a) where we use a path between two leaves, since each group of n vertices must contain a leaf, thereby reducing the number of vertices in our tree by 2 and allowing us to use our inductive hypothesis.

It now suffices to show that Allen can win the game in M' moves, but this is exactly the same induction as in the previous paragraph. If any m_e is 0 then we're immediately done by splitting T' into two smaller trees, otherwise we use the path between the two leaves to use our inductive hypothesis.

9. [**60**] Evan has a simple graph with v vertices and e edges. Show that he can delete at least $\frac{e-v+1}{2}$ edges so that each vertex still has at least half of its original degree.

Proposed by: Allen Liu

Fix v. We use strong induction on the number of edges e. If $e \leq v - 1$, the result trivially holds by removing 0 edges. Now take $e > v - 1$ and assume the result has been shown for all smaller values of e. Consider a graph G with v vertices and e edges.

Suppose G contains a cycle C of even length $2k$, where vertices (but not edges) may be repeated in the cycle. Let G' be the subgraph of G with the edges of C removed. Then G' has v vertices and $e - 2k$ edges. By the inductive hypothesis, it is possible to remove $\frac{e-2k-v+1}{2}$ edges from G' so that each vertex still has at least half its original degree. In the original graph G, remove these same edges, and also remove every other edge of C (so, if the vertices of C are $v_1, \cdots, v_{2k}$ in order, we remove the edges between v_{2i-1} and v_{2i} for $1 \leq i \leq k$). In total, we have removed $\frac{e-2k-v+1}{2} + k = \frac{e-v+1}{2}$ edges. Furthermore, all vertices in G still have at least half their original degrees, as desired.

The remaining case to consider is if G has no cycles of even length. Then no two cycles in G can have any vertices or edges in common. Suppose the contrary; then two odd cycles overlap, so their union is connected and has an even number of edges. This union has an Eulerian tour, which is a cycle with an even number of edges, contradicting our assumption.

The number of edges in G is at most $v + c - 1$, where c is the number of cycles. So, we must remove at least $\frac{e-v+1}{2} = \frac{c}{2}$ edges from G. But we can remove c edges from G, one from each cycle. No vertex has its degree decreased by more than 1, and each vertex whose degree is decreased is in a cycle and so has degree at least 2. Therefore each vertex still has at least half of its original degree, and we have removed at least $\frac{e-v+1}{2}$ edges, as desired.

Thus our claim holds for a graph with e edges, and thus by induction holds for any number of edges, as needed.

10. [**60**] Let n and m be positive integers which are at most 10^{10}. Let R be the rectangle with corners at $(0,0), (n,0), (n,m), (0,m)$ in the coordinate plane. A simple non-self-intersecting quadrilateral with vertices at integer coordinates is called *far-reaching* if each of its vertices lie on or inside R, but each side of R contains at least one vertex of the quadrilateral. Show that there is a far-reaching quadrilateral with area at most 10^6.

    (A side of a rectangle includes the two endpoints.)

    Proposed by: Kevin Sun

    Let $g = \gcd(n, m)$, with $n = g \cdot a$ and $m = g \cdot b$. Note that the number of points on the diagonal of R connecting $(0,0)$ and (n,m) is $g + 1$. We construct two far-reaching quadrilaterals and show that at least one of them has small area.

    For our first quadrilateral, let (x_1, y_1) and (x_2, y_2) be the points with the shortest nonzero distances to the diagonal between $(0,0)$ and (n,m) which lie above and below the diagonal, respectively. Now consider the quadrilateral with vertices $(0,0)$, (x_1, y_1), (n,m), (x_2, y_2). Note that the only lattice points which can lie on or inside this quadrilateral are (x_1, y_1), (x_2, y_2), and points on the diagonal of R, as otherwise we could find a closer point to the diagonal than (x_1, y_1) and (x_2, y_2). Thus by Pick's Theorem, the area of this quadrilateral is at most g.

    For our second quadrilateral, we will take as our vertices the points $(0,0)$, $(n-1, m)$, (a,b), and $(n, m-1)$. This is a concave quadrilateral which can be split into two triangle of areas $\frac{a}{2}$ and $\frac{b}{2}$, so the area of this quadrilateral is at most $\frac{1}{2}(a+b)$.

    We have therefore shown that there is always a far-reaching quadrilateral with area at most $\min(g, \frac{1}{2}(a+b))$. Since $g \cdot \frac{1}{2}(a+b) = \frac{1}{2}(n+m) \leq 10^{10}$, we have that $\min(g, \frac{1}{2}(a+b)) \leq 10^5$, so we can always find a far-reaching quadrilateral with area at most 10^5 as desired.