HMMT 2014

Saturday 22 February 2014

Algebra

1. Given that x and y are nonzero real numbers such that $x + \frac{1}{y} = 10$ and $y + \frac{1}{x} = \frac{5}{12}$, find all possible values of x.

   Answer: $\boxed{4, 6 \text{ OR } 6, 4}$ Let $z = \frac{1}{y}$. Then $x + z = 10$ and $\frac{1}{x} + \frac{1}{z} = \frac{5}{12}$. Since $\frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz} = \frac{10}{xz}$, we have $xz = 24$. Thus, $x(10 - x) = 24$, so $x^2 - 10x + 24 = (x - 6)(x - 4) = 0$, whence $x = 6$ or $x = 4$.

   Alternate solution: Clearing denominators gives $xy + 1 = 10y$ and $yx + 1 = \frac{5}{12}x$, so $x = 24y$. Thus we want to find all real (nonzero) x such that $\frac{x^2}{24} + 1 = \frac{5}{12}x$ (and for any such x, $y = x/24$ will satisfy the original system of equations). This factors as $(x - 4)(x - 6) = 0$, so precisely $x = 4, 6$ work.

2. Find the integer closest to
$$\frac{1}{\sqrt[4]{5^4 + 1} - \sqrt[4]{5^4 - 1}}.$$

   Answer: $\boxed{250}$ Let $x = (5^4 + 1)^{1/4}$ and $y = (5^4 - 1)^{1/4}$. Note that x and y are both approximately 5. We have
$$\begin{aligned}\frac{1}{x-y} &= \frac{(x+y)(x^2+y^2)}{(x-y)(x+y)(x^2+y^2)} = \frac{(x+y)(x^2+y^2)}{x^4-y^4} \\ &= \frac{(x+y)(x^2+y^2)}{2} \approx \frac{(5+5)(5^2+5^2)}{2} = 250.\end{aligned}$$

   Note: To justify the $\approx$, note that $1 = x^4 - 5^4$ implies
$$0 < x - 5 = \frac{1}{(x+5)(x^2+5^2)} < \frac{1}{(5+5)(5^2+5^2)} = \frac{1}{500},$$
   and similarly $1 = 5^4 - y^4$ implies
$$0 < 5 - y = \frac{1}{(5+y)(5^2+y^2)} < \frac{1}{(4+4)(4^2+4^2)} = \frac{1}{256}.$$
   Similarly,
$$0 < x^2 - 5^2 = \frac{1}{x^2+5^2} < \frac{1}{2 \cdot 5^2} = \frac{1}{50}$$
   and
$$0 < 5^2 - y^2 = \frac{1}{5^2+y^2} < \frac{1}{5^2+4.5^2} < \frac{1}{45}.$$
   Now
$$|x + y - 10| = |(x-5) - (5-y)| < \max(|x-5|, |5-y|) < \frac{1}{256},$$
   and similarly $|x^2 + y^2 - 2 \cdot 5^2| < \frac{1}{45}$. It's easy to check that $(10 - 1/256)(50 - 1/45) > 499.5$ and $(10 + 1/256)(50 + 1/45) < 500.5$, so we're done.

3. Let
$$A = \frac{1}{6}\left((\log_2(3))^3 - (\log_2(6))^3 - (\log_2(12))^3 + (\log_2(24))^3\right).$$
   Compute 2^A.

   Answer: $\boxed{72}$ Let $a = \log_2(3)$, so $2^a = 3$ and $A = \frac{1}{6}[a^3 - (a+1)^3 - (a+2)^3 + (a+3)^3]$. But $(x+1)^3 - x^3 = 3x^2 + 3x + 1$, so $A = \frac{1}{6}[3(a+2)^2 + 3(a+2) - 3a^2 - 3a] = \frac{1}{2}[4a + 4 + 2] = 2a + 3$. Thus $2^A = (2^a)^2(2^3) = 9 \cdot 8 = 72$.

4. Let b and c be real numbers, and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$, and that $P(1) \neq P(2)$. Find $P(0)$.

Answer: $\boxed{-\frac{3}{2} \text{ OR } -1.5 \text{ OR } -1\frac{1}{2}}$ Since $P(P(1)) = P(P(2)) = 0$, but $P(1) \neq P(2)$, it follows that $P(1) = 1 + b + c$ and $P(2) = 4 + 2b + c$ are the distinct roots of the polynomial $P(x)$. Thus, $P(x)$ factors:

$$\begin{aligned} P(x) &= x^2 + bx + c \\ &= (x - (1 + b + c))(x - (4 + 2b + c)) \\ &= x^2 - (5 + 3b + 2c)x + (1 + b + c)(4 + 2b + c). \end{aligned}$$

It follows that $-(5 + 3b + 2c) = b$, and that $c = (1 + b + c)(4 + 2b + c)$. From the first equation, we find $2b + c = -5/2$. Plugging in $c = -5/2 - 2b$ into the second equation yields

$$-5/2 - 2b = (1 + (-5/2) - b)(4 + (-5/2)).$$

Solving this equation yields $b = -\frac{1}{2}$, so $c = -5/2 - 2b = -\frac{3}{2}$.

5. Find the sum of all real numbers x such that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$.

Answer: $\boxed{1}$ Rearrange the equation to $x^5 + (1 - x)^5 - 12 = 0$. It's easy to see this has two real roots, and that r is a root if and only if $1 - r$ is a root, so the answer must be 1.

Alternate solution: Note that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 5x(x-1)(x^2-x+1) - 11 = 5u(u+1) - 11$, where $u = x^2 - x$. Of course, $x^2 - x = u$ has real roots if and only if $u \geq -\frac{1}{4}$, and distinct real roots if and only if $u > -\frac{1}{4}$. But the roots of $5u(u+1) - 11$ are $\frac{-5 \pm \sqrt{5^2 + 4(5)(11)}}{2 \cdot 5} = \frac{-5 \pm 7\sqrt{5}}{10}$, one of which is greater than $-\frac{1}{4}$ and the other less than $-\frac{1}{4}$. For the larger root, $x^2 - x = u$ has exactly two distinct real roots, which sum up to 1 by Vieta's.

6. Given that w and z are complex numbers such that $|w + z| = 1$ and $|w^2 + z^2| = 14$, find the smallest possible value of $|w^3 + z^3|$. Here, $|\cdot|$ denotes the absolute value of a complex number, given by $|a + bi| = \sqrt{a^2 + b^2}$ whenever a and b are real numbers.

Answer: $\boxed{\frac{41}{2} \text{ OR } 20.5 \text{ OR } 20\frac{1}{2}}$ We can rewrite $|w^3 + z^3| = |w + z||w^2 - wz + z^2| = |w^2 - wz + z^2| = |\frac{3}{2}(w^2 + z^2) - \frac{1}{2}(w + z)^2|$.

By the triangle inequality, $|\frac{3}{2}(w^2+z^2) - \frac{1}{2}(w+z)^2 + \frac{1}{2}(w+z)^2| \leq |\frac{3}{2}(w^2+z^2) - \frac{1}{2}(w+z)^2| + |\frac{1}{2}(w+z)^2|$. By rearranging and simplifying, we get $|w^3+z^3| = |\frac{3}{2}(w^2+z^2) - \frac{1}{2}(w+z)^2| \geq \frac{3}{2}|w^2+z^2| - \frac{1}{2}|w+z|^2 = \frac{3}{2}(14) - \frac{1}{2} = \frac{41}{2}$.

To achieve $41/2$, it suffices to take w, z satisfying $w + z = 1$ and $w^2 + z^2 = 14$.

7. Find the largest real number c such that

$$\sum_{i=1}^{101} x_i^2 \geq cM^2$$

whenever $x_1, \ldots, x_{101}$ are real numbers such that $x_1 + \cdots + x_{101} = 0$ and M is the median of $x_1, \ldots, x_{101}$.

Answer: $\boxed{\frac{5151}{50} \text{ OR } 103.02 \text{ OR } 103\frac{1}{50}}$ Suppose without loss of generality that $x_1 \leq \cdots \leq x_{101}$ and $M = x_{51} \geq 0$.

Note that $f(t) = t^2$ is a convex function over the reals, so we may "smooth" to the case $x_1 = \cdots = x_50 \leq x_{51} = \cdots = x_{101}$ (the $x_{51} = \cdots$ is why we needed to assume $x_{51} \geq 0$). Indeed, by Jensen's inequality, the map $x_1, x_2, \ldots, x_{50} \to \frac{x_1 + \cdots + x_{50}}{50}, \ldots, \frac{x_1 + \cdots + x_{50}}{50}$ will decrease or fix the LHS, while preserving the ordering condition and the zero-sum condition.

Similarly, we may without loss of generality replace $x_{51}, \ldots, x_{101}$ with their average (which will decrease or fix the LHS, but also either fix or increase the RHS). But this simplified problem has $x_1 = \cdots =$

$x_{50} = -51r$ and $x_{51} = \cdots = x_{101} = 50r$ for some $r \geq 0$, and by homogeneity, C works if and only if $C \leq \frac{50(51)^2+51(50)^2}{50^2} = \frac{51(101)}{50} = \frac{5151}{50}$.

Comment: One may also use the Cauchy-Schwarz inequality or the QM-AM inequality instead of Jensen's inequality.

Comment: For this particular problem, there is another solution using the identity $101\sum x_i^2 - (\sum x_i)^2 = \sum (x_j - x_i)^2$. Indeed, we may set $u = x_{51} - (x_1 + \cdots + x_{50})/50$ and $v = (x_{52} + \cdots + x_{101})/50 - x_{51}$, and use the fact that $(u-v)^2 \leq u^2 + v^2$.

8. Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

Answer: $\boxed{\pm\frac{9}{4} \text{ (OR } \frac{9}{4}, -\frac{9}{4} \text{ OR } -\frac{9}{4}, \frac{9}{4}) \text{ OR } \pm 2\frac{1}{4} \text{ OR } \pm 2.25}$ Any real quartic has an even number of real roots *with multiplicity*, so there exists real r such that $x^4 + kx^3 + x^2 + 4kx + 16$ either takes the form $(x+r)^4$ (clearly impossible) or $(x+r)^2(x^2+ax+b)$ for some real a, b with $a^2 < 4b$. Clearly $r \neq 0$, so $b = \frac{16}{r^2}$ and $4k = 4(k)$ yields $\frac{32}{r} + ar^2 = 4(2r+a) \implies a(r^2-4) = 8\frac{r^2-4}{r}$. Yet $a \neq \frac{8}{r}$ (or else $a^2 = 4b$), so $r^2 = 4$, and $1 = r^2 + 2ra + \frac{16}{r^2} \implies a = \frac{-7}{2r}$. Thus $k = 2r - \frac{7}{2r} = \pm\frac{9}{4}$ (since $r = \pm 2$).

It is easy to check that $k = \frac{9}{4}$ works, since $x^4 + (9/4)x^3 + x^2 + 4(9/4)x + 16 = \frac{1}{4}(x+2)^2(4x^2 - 7x + 16)$. The polynomial given by $k = -\frac{9}{4}$ is just $\frac{1}{4}(-x+2)^2(4x^2+7x+16)$.

Alternate solution: $x^4 + kx^3 + x^2 + 4kx + 16 = (x^2 + \frac{k}{2}x + 4)^2 + (1 - 8 - \frac{k^2}{4})x^2$, so for some $\epsilon \in \{-1, 1\}$, $2x^2 + (k - \epsilon\sqrt{k^2+28})x + 8$ has a single real root and thus takes the form $2(x+r)^2$ (using the same notation as above). But then $(k - \epsilon\sqrt{k^2+28})^2 = 4(2)(8) = 8^2$, so we conclude that $(k \pm 8)^2 = (\epsilon\sqrt{k^2+28})^2$ and $k = \pm(4 - \frac{7}{4}) = \pm\frac{9}{4}$.

9. Given that a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$
$$b^2 + bc + c^2 = -2$$
$$c^2 + ca + a^2 = 1,$$

compute $(ab+bc+ca)^2$. (Here, $i = \sqrt{-1}$.)

Answer: $\boxed{\frac{-11-4i}{3} \text{ OR } -\frac{11+4i}{3}}$ More generally, suppose $a^2 + ab + b^2 = z$, $b^2 + bc + c^2 = x$, $c^2 + ca + a^2 = y$ for some complex numbers a, b, c, x, y, z.

We show that

$$f(a,b,c,x,y,z) = \left(\frac{1}{2}(ab+bc+ca)\sin 120^\circ\right)^2 - \left(\frac{1}{4}\right)^2[(x+y+z)^2 - 2(x^2+y^2+z^2)]$$

holds in general. Plugging in $x = -2$, $y = 1$, $z = 1+i$ will then yield the desired answer,

$$\begin{aligned}(ab+bc+ca)^2 &= \frac{16}{3}\frac{1}{16}[(x+y+z)^2 - 2(x^2+y^2+z^2)] \\ &= \frac{i^2 - 2(4+1+(1+i)^2)}{3} = \frac{-1-2(5+2i)}{3} = \frac{-11-4i}{3}.\end{aligned}$$

Solution 1: Plug in $x = b^2 + bc + c^2$, etc. to get a polynomial g in a, b, c (that agrees with f for every valid choice of a, b, c, x, y, z). It suffices to show that $g(a,b,c) = 0$ for all positive reals a, b, c, as then the polynomial g will be identically 0.

But this is easy: by the law of cosines, we get a geometrical configuration with a point P inside a triangle ABC with $PA = a$, $PB = b$, $PC = c$, $\angle PAB = \angle PBC = \angle PCA = 120^\circ$, $x = BC^2$,

$y = CA^2$, $z = AB^2$. By Heron's formula, we have

$$\begin{aligned}
(\frac{1}{2}(ab+bc+ca)\sin 120^\circ)^2 &= [ABC]^2 \\
&= \frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})\prod_{\text{cyc}}(\sqrt{x}+\sqrt{y}-\sqrt{z})}{2^4} \\
&= \frac{1}{16}[(\sqrt{x}+\sqrt{y})^2 - z][(\sqrt{x}-\sqrt{y})^2 - z] \\
&= \frac{1}{16}[(x-y)^2 + z^2 - 2z(x+y)] \\
&= (\frac{1}{4})^2[(x+y+z)^2 - 2(x^2+y^2+z^2)],
\end{aligned}$$

as desired.

Solution 2: Let $s = a+b+c$. We have $x - y = b^2 + bc - ca - a^2 = (b-a)s$ and cyclic, so $x + as = y + bs = z + cs$ (they all equal $\sum a^2 + \sum bc$). Now add all equations to get

$$x + y + z = 2\sum a^2 + \sum bc = s^2 + \frac{1}{2}\sum (b-c)^2;$$

multiplying both sides by $4s^2$ yields $4s^2(x+y+z) = 4s^4 + 2\sum(z-y)^2$, so $[2s^2 - (x+y+z)]^2 = (x+y+z)^2 - 2\sum(z-y)^2 = 6\sum yz - 3\sum x^2$. But $2s^2 - (x+y+z) = s^2 - \frac{1}{2}\sum(b-c)^2 = (a+b+c)^2 - \frac{1}{2}\sum(b-c)^2 = 3(ab+bc+ca)$, so

$$9(ab+bc+ca)^2 = 6\sum yz - 3\sum x^2 = 3[(x+y+z)^2 - 2(x^2+y^2+z^2)],$$

which easily rearranges to the desired.

Comment: Solution 2 can be done with less cleverness. Let $u = x + as = y + bs = z + cs$, so $a = \frac{u-x}{s}$, etc. Then $s = \sum \frac{u-x}{s}$, or $s^2 = 3u - s(x+y+z)$. But we get another equation in s, u by just plugging in directly to $a^2 + ab + b^2 = z$ (and after everything is in terms of s, we can finish without too much trouble).

10. For an integer n, let $f_9(n)$ denote the number of positive integers $d \le 9$ dividing n. Suppose that m is a positive integer and $b_1, b_2, \dots, b_m$ are real numbers such that $f_9(n) = \sum_{j=1}^{m} b_j f_9(n-j)$ for all $n > m$. Find the smallest possible value of m.

    Answer: $\boxed{28}$ Let $M = 9$. Consider the generating function

$$F(x) = \sum_{n\ge1} f_M(n)x^n = \sum_{d=1}^{M}\sum_{k\ge1} x^{dk} = \sum_{d=1}^{M} \frac{x^d}{1-x^d}.$$

Observe that $f_M(n) = f_M(n + M!)$ for all $n \ge 1$ (in fact, all $n \le 0$ as well). Thus $f_M(n)$ satisfies a degree m linear recurrence if and only if it *eventually* satisfies a degree m linear recurrence. But the latter occurs if and only if $P(x)F(x)$ is a polynomial for some degree m polynomial $P(x)$. (Why?)

Suppose $P(x)F(x) = Q(x)$ is a polynomial for some polynomial P of degree m. We show that $x^s - 1 \mid P(x)$ for $s = 1, 2, \dots, M$, or equivalently that $P(\omega) = 0$ for all primitive sth roots of unity $1 \le s \le M$). Fix a primitive sth root of unity ω, and define a function

$$F_\omega(z) = (1-\omega^{-1}z)\sum_{s\nmid d\le M} \frac{z^d}{1-z^d} + \sum_{s\mid d\le M} \frac{z^d}{1+(\omega^{-1}z)+\cdots+(\omega^{-1}z)^{d-1}}$$

for all z where all denominators are nonzero (in particular, this includes $z = \omega$).

Yet $F_\omega(z) - F(z)(1-\omega^{-1}z) = 0$ for all complex z such that $z^1, z^2, \dots, z^M \ne 1$, so $P(z)F_\omega(z) - Q(z)(1-\omega^{-1}z) = 0$ holds for all such z as well. In particular, the rational function $P(x)F_\omega(x) - Q(x)(1-\omega^{-1}x)$

has infinitely many roots, so must be identically zero *once we clear denominators*. But no denominator vanishes at $x = \omega$, so we may plug in $x = \omega$ to the polynomial identity and then divide out by the original (nonzero) denominators to get $0 = P(\omega)F_\omega(\omega) - Q(\omega)(1 - \omega^{-1}\omega) = P(\omega)F_\omega(\omega)$. However,

$$F_\omega(\omega) = \sum_{s|d \le M} \frac{\omega^d}{1 + (\omega^{-1}\omega) + \cdots + (\omega^{-1}\omega)^{d-1}} = \sum_{s|d \le M} \frac{1}{d}$$

is a positive integer multiple of $1/d$, and therefore nonzero. Thus $P(\omega) = 0$, as desired.

Conversely, if $x^s - 1 \mid P(x)$ for $s = 1, 2, \ldots, M$, then $P(x)$ will clearly suffice. So we just want the degree of the least common multiple of the $x^s - 1$ for $s = 1, 2, \ldots, M$, or just the number of roots of unity of order at most M, which is $\sum_{s=1}^{M} \phi(s) = 1 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 = 28$.

Comment: Only at the beginning do we treat $F(x)$ strictly as a formal power series; later once we get the rational function representation $\sum_{d=1}^{6} \frac{x^d}{1-x^d}$, we can work with polynomial identities in general and don't have to worry about convergence issues for $|x| \ge 1$.