HMMT February 2018

February 10, 2018

Geometry

1. Triangle GRT has $GR = 5$, $RT = 12$, and $GT = 13$. The perpendicular bisector of GT intersects the extension of GR at O. Find TO.

   Proposed by: Henrik Boecken

   Answer: $\boxed{\frac{169}{10}}$

   First, note that $TO = GO$ as O lies on the perpendicular bisector of GT. Then if M is the midpoint of GT, we have that $\triangle GRT \sim \triangle GMO$, so we can compute $TO = GO = GM \cdot \dfrac{GT}{GR} = \dfrac{13}{2} \cdot \dfrac{13}{5} = \dfrac{169}{10}$.

2. Points A, B, C, D are chosen in the plane such that segments AB, BC, CD, DA have lengths 2, 7, 5, 12, respectively. Let m be the minimum possible value of the length of segment AC and let M be the maximum possible value of the length of segment AC. What is the ordered pair (m, M)?

   Proposed by: Kevin Sun

   Answer: $\boxed{(7, 9)}$

   By the triangle inequality on triangle ACD, $AC + CD \geq AD$, or $AC \geq 7$. The minimum of 7 can be achieved when A, C, D lie on a line in that order. By the triangle inequality on triangle ABC, $AB + BC \geq AC$, or $AC \leq 9$. The maximum of 9 can be achieved when A, B, C lie on a line in that order. This gives the answer $(7, 9)$.

3. How many noncongruent triangles are there with one side of length 20, one side of length 17, and one 60° angle?

   Proposed by: Dai Yang

   Answer: $\boxed{2}$

   There are 3 possible vertices that can have an angle of 60°, we will name them. Call the vertex where the sides of length 20 and 17 meet α, denote the vertex where 17 doesn't meet 20 by β, and the final vertex, which meets 20 but not 17, we denote by γ.

   The law of cosines states that if we have a triangle, then we have the equation $c^2 = a^2 + b^2 - 2ab\cos C$ where C is the angle between a and b. But $\cos 60^\circ = \frac{1}{2}$ so this becomes $c^2 = a^2 + b^2 - ab$. We then try satisfying this equation for the 3 possible vertices and find that, for α the equation reads $c^2 = 400 + 289 - 340 = 349$ so that $c = \sqrt{349}$. For β we find that $400 = 289 + b^2 - 17b$ or rather $b^2 - 17b - 111 = 0$ this is a quadratic, solving we find that it has two roots $b = \frac{17 \pm \sqrt{289+444}}{2}$, but since $\sqrt{733} > 17$ only one of these roots is positive. We can also see that this isn't congruent to the other triangle we had, as for both the triangles the shortest side has length 17, and so if they were congruent the lengths of all sides would need to be equal, but $18 < \sqrt{349} < 19$ and since $23^2 < 733$ clearly $\frac{17 \pm \sqrt{289+444}}{2} > \frac{17+23}{2} = 20$ and so the triangles aren't congruent. If we try applying the law of cosines to γ however, we get the equation $289 = a^2 + 400 - 20a$ which we can rewrite as $a^2 - 20a + 111 = 0$ which has no real solutions, as the discriminant $400 - 4 * 111 = -44$ is negative. Thus, γ cannot be 60°, and there are exactly two non congruent triangles with side lengths 20 and 17 with an angle being 60°.

4. A paper equilateral triangle of side length 2 on a table has vertices labeled A, B, C. Let M be the point on the sheet of paper halfway between A and C. Over time, point M is lifted upwards, folding the triangle along segment BM, while A, B, and C remain on the table. This continues until A and C touch. Find the maximum volume of tetrahedron $ABCM$ at any time during this process.

   Proposed by: Dhruv Rohatgi

   Answer: $\boxed{\frac{\sqrt{3}}{6}}$

View triangle ABM as a base of this tetrahedron. Then relative to triangle ABM, triangle CBM rotates around segment BM on a hinge. Therefore the volume is maximized when C is farthest from triangle ABM, which is when triangles ABM and CBM are perpendicular. The volume in this case can be calculated using the formula for the volume of a tetrahedron as $\frac{1}{6} \cdot 1 \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{6}$.

5. In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

   Proposed by: Yuan Yao

   Answer: $\boxed{\frac{8\sqrt{2}}{9}}$

   Since AE bisects $\angle RAM$, we have $RE = EM$, and E, A lie on different sides of RM. Since AM is a diameter, $\angle ARM = 90°$. If the midpoint of RM is N, then from $[RAM] = [REM]$ and $\angle ARM = 90°$, we find $AR = NE$. Note that O, the center of ω, N, and E are collinear, and by similarity of triangles NOM and RAM, $ON = \frac{1}{2}AR = \frac{1}{2}NE$. Therefore, $ON = \frac{1}{3}$ and $NE = \frac{2}{3}$. By the Pythagorean theorem on triangle RAM, $RM = \frac{4\sqrt{2}}{3}$, Therefore, the area of $MARE$ is $2 \cdot \frac{1}{2} \cdot \frac{4\sqrt{2}}{3} \cdot \frac{2}{3} = \frac{8\sqrt{2}}{9}$.

6. Let ABC be an equilateral triangle of side length 1. For a real number $0 < x < 0.5$, let A_1 and A_2 be the points on side BC such that $A_1B = A_2C = x$, and let $T_A = \triangle AA_1A_2$. Construct triangles $T_B = \triangle BB_1B_2$ and $T_C = \triangle CC_1C_2$ similarly.

   There exist positive rational numbers b, c such that the region of points inside all three triangles T_A, T_B, T_C is a hexagon with area

   $$\frac{8x^2 - bx + c}{(2-x)(x+1)} \cdot \frac{\sqrt{3}}{4}.$$

   Find (b, c).

   Proposed by: Kevin Sun

   Answer: $\boxed{(8, 2)}$

   Solution 1:

   Notice that the given expression is defined and continuous not only on $0 < x < 0.5$, but also on $0 \le x \le 0.5$. Let $f(x)$ be the function representing the area of the (possibly degenerate) hexagon for $x \in [0, 0.5]$. Since $f(x)$ is equal to the given expression over $(0, 0.5)$, we can conclude that $f(0)$ and $f(0.5)$ will also be equal to the expression when $x = 0$ and $x = 0.5$ respectively. (In other words, $f(x)$ is equal to the expression over $[0, 0.5]$.) In each of the cases, we can compute easily that $f(0) = \frac{\sqrt{3}}{4}$ and $f(0.5) = 0$, so by plugging them in, we get $\frac{c}{2 \cdot 1} = 1$ and $\frac{2 - b/2 + c}{(3/2) \cdot (3/2)} = 0$, which gives $b = 8$ and $c = 2$.

   Solution 2:

   Let $P = AA_1 \cap CC_2, Q = AA_2 \cap BB_1, R = BB_1 \cap CC_2$. These three points are the points on the hexagon farthest away from A. For reasons of symmetry, the area of the hexagon (call it H for convenience) is:

   $$[H] = [ABC] - 3[BPRQC].$$

   Also, $[BPC] = [BQC]$ by symmetry, so:

   $$[BPRQC] = [BPC] + [BQC] - [BRC]$$

   $$[BPRQC] = 2[BPC] - [BRC].$$

   From this, one can see that all we need to do is calculate the A-level of the points P and R in barycentric coordinates. Ultimately, the A-level of P is $\frac{x}{x+1}$, and the A-level of R is $\frac{x}{2-x}$. From this, straightforward calculation shows that:

   $$[H] = \frac{8x^2 - 8x + 2}{(2-x)(x+1)} \cdot \frac{\sqrt{3}}{4},$$

   thus giving us the answer $(b, c) = (8, 2)$.

7. Triangle ABC has sidelengths $AB = 14$, $AC = 13$, and $BC = 15$. Point D is chosen in the interior of $\overline{AB}$ and point E is selected uniformly at random from $\overline{AD}$. Point F is then defined to be the intersection point of the perpendicular to $\overline{AB}$ at E and the union of segments $\overline{AC}$ and $\overline{BC}$. Suppose that D is chosen such that the expected value of the length of $\overline{EF}$ is maximized. Find AD.

   Proposed by: Gabriel Mintzer

   Answer: $\boxed{\sqrt{70}}$

   Let G be the intersection of the altitude to $\overline{AB}$ at point D with $\overline{AC} \cup \overline{BC}$. We first note that the maximal expected value is obtained when $DG = \frac{[ADGC]}{AD}$, where $[P]$ denotes the area of polygon P. Note that if DG were not equal to this value, we could move D either closer or further from A and increase the value of the fraction, which is the expected value of EF. We note that this equality can only occur if D is on the side of the altitude to $\overline{AB}$ nearest point B. Multiplying both sides of this equation by AD yields $AD \cdot DG = [ADGC]$, which can be interpreted as meaning that the area of the rectangle with base $\overline{AD}$ and height $\overline{DG}$ must have area equal to that of quadrilateral $ADGC$. We can now solve this problem with algebra.

   Let $x = BD$. We first compute the area of the rectangle with base $\overline{AD}$ and height $\overline{DG}$. We have that $AD = AB - BD = 14 - x$. By decomposing the $13-14-15$ triangle into a $5-12-13$ triangle and a $9-12-15$ triangle, and using a similarity argument, we find that $DG = \frac{4}{3}x$. Thus, the area of this rectangle is $\frac{4}{3}x(14-x) = \frac{56}{3}x - \frac{4}{3}x^2$.

   We next compute the area of quadrilateral $ADGC$. We note that $[ADGC] = [ABC] - [BDG]$. We have that $[ABC] = \frac{1}{2}(12)(14) = 84$. We have $BD = x$ and $DG = \frac{4}{3}x$, so $[BDG] = \frac{1}{2}(x)\left(\frac{4}{3}x\right) = \frac{2}{3}x^2$. Therefore, we have $[ADGC] = [ABC] - [BDG] = 84 - \frac{2}{3}x^2$.

   Equating these two areas, we have

   $$\frac{56}{3}x - \frac{4}{3}x^2 = 84 - \frac{2}{3}x^2,$$

   or, simplifying,

   $$x^2 - 28x + 126 = 0.$$

   Solving yields $x = 14 \pm \sqrt{70}$, but $14 + \sqrt{70}$ exceeds AB, so we discard it as an extraneous root. Thus, $BD = 14 - \sqrt{70}$ and

   $$AD = AB - BD = 14 - (14 - \sqrt{70}) = \boxed{\sqrt{70}}.$$

   Remark: if the altitude to point C meets $\overline{AB}$ at point H, then the general answer to this problem is $\sqrt{AH \cdot AB}$. This result can be derived by considering the effects of dilation in the $\overline{AB}$ direction and dilation in the $\overline{CH}$ direction then performing dilations such that $\angle C$ is right and carrying out the calculation described above while considering congruent triangles.

8. Let ABC be an equilateral triangle with side length 8. Let X be on side AB so that $AX = 5$ and Y be on side AC so that $AY = 3$. Let Z be on side BC so that AZ, BY, CX are concurrent. Let ZX, ZY intersect the circumcircle of AXY again at P, Q respectively. Let XQ and YP intersect at K. Compute $KX \cdot KQ$.

   Proposed by: Allen Liu

   Answer: $\boxed{304}$

   Let BY and CX meet at O. O is on the circumcircle of AXY, since $\triangle AXC \cong \triangle CYB$.

We claim that KA and KO are tangent to the circumcircle of AXY. Let XY and BC meet at L. Then, $LBZC$ is harmonic. A perspectivity at X gives $AYOP$ is harmonic. Similarly, a perspectivity at Y gives $AXOQ$ is harmonic. Thus, K is the pole of chord AO.

Now we compute. Denote r as the radius and θ as $\angle AXO$. Then,

$$r = \frac{XY}{\sqrt{3}} = \frac{\sqrt{5^2+3^2-3\cdot 5}}{\sqrt{3}} = \sqrt{\frac{19}{3}};$$

$$\sin\theta = \sin 60^\circ \cdot \frac{AC}{XC} = \frac{\sqrt{3}}{2} \cdot \frac{8}{\sqrt{5^2+8^2-5\cdot 8}} = \frac{4}{7}\sqrt{3};$$

$$KX \cdot KQ = KA^2 = (r \cdot \tan\theta)^2 = (\sqrt{\frac{19}{3}} \cdot 4\sqrt{3})^2 = 304.$$

9. Po picks 100 points $P_1, P_2, \ldots, P_{100}$ on a circle independently and uniformly at random. He then draws the line segments connecting $P_1P_2,\ P_2P_3, \ldots, P_{100}P_1$. When all of the line segments are drawn, the circle is divided into a number of regions. Find the expected number of regions that have all sides bounded by straight lines.

   Proposed by: Allen Liu

   Answer: $\boxed{\frac{4853}{3}}$

   If the 100 segments do not intersect on the interior, then the circle will be cut into 101 regions. By Euler's formula, each additional intersection cuts two edges into two each, and adds one more vertex, so since $V - E + F$ is constant, there will be one more region as well. It then suffices to compute the expected number of intersections, where two segments that share a vertex are not counted as intersections.

   We use linearity of expectation to compute this value. It suffices to compute the expected number of segments that each segment intersects. Consider one such segment P_1P_2. It cannot possibly intersect a segment that shares an endpoint, so that leaves 97 possible other segments. Again, by linearity of expection, it suffices to compute the probability that P_1P_2 intersects P_iP_{i+1}. However, since each of the points was chosen uniformly at random, this is equal to the probability that AC intersects BD, where A, B, C, D are chosen uniformly at random from the circle. Since this probability is $1/3$, each segment intersects with $\frac{97}{3}$ segments on average.

   Now, we can sum over all segments and divide by two to get $(100 \cdot \frac{97}{3})/2 = \frac{4850}{3}$ intersections, since each intersection is counted twice. Accounting for the fact that there are 101 regions to begin with, and exactly 100 of them have an arc on the boundary, we get $\frac{4850}{3} + 101 - 100 = \frac{4853}{3}$ as the answer.

10. Let ABC be a triangle such that $AB = 6, BC = 5, AC = 7$. Let the tangents to the circumcircle of ABC at B and C meet at X. Let Z be a point on the circumcircle of ABC. Let Y be the foot of the perpendicular from X to CZ. Let K be the intersection of the circumcircle of BCY with line AB. Given that Y is on the interior of segment CZ and $YZ = 3CY$, compute AK.

    Proposed by: Allen Liu

    Answer: $\boxed{\frac{147}{10}}$

    Let ω_1 denote the circumcircle of ABC and ω_2 denote the circle centered at X through B and C. Let ω_2 intersect AB, AC again at B', C'. The (signed) power of Y with respect to ω_1 is $-CY \cdot YZ$. The power of Y with respect to ω_2 is $XY^2 - CX^2 = -CY^2$. Thus the ratio of the powers of Y with respect to the two circles is $3:1$. The circumcircle of BCY passes through the intersection points of ω_1 and ω_2 (B and C) and thus contains exactly the set of points such that the ratio of their powers with respect to ω_1 and ω_2 is $3:1$ (this fact can be verified in a variety of ways). We conclude that K must be the point on line AB such that $\frac{KB'}{KA} = 3$. It now suffices to compute AB'. Note $AB' = \frac{AC \cdot B'C'}{BC}$ by similar triangles. Also an angle chase gives that $B'XC'$ are collinear. We compute

    $$BX = \frac{BC}{2\cos A} = \frac{5}{2 \cdot \frac{5}{7}} = \frac{7}{2}$$

and thus $B'C' = 7$ so $AB' = \frac{49}{5}$ and $AK = \frac{3}{2}AB' = \frac{147}{10}$.