# HMMT February 2023

**February 18, 2023**

## Combinatorics Round

1. There are 800 marbles in a bag. Each marble is colored with one of 100 colors, and there are eight marbles of each color. Anna draws one marble at a time from the bag, without replacement, until she gets eight marbles of the same color, and then she immediately stops.

   Suppose Anna has not stopped after drawing 699 marbles. Compute the probability that she stops immediately after drawing the 700th marble.

   *Proposed by: Matthew Cho*

   **Answer:** $\boxed{99/101}$

   **Solution:** In order to not stop after 699 marbles, the last 101 marbles must consist of 2 marbles of one color, and one marble from each other color. Since each of these marbles is equally likely to be the next to be drawn, and we stop after drawing the next marble as long as it's not one of the two of the same color, the desired probability is simply $\frac{99}{101}$.

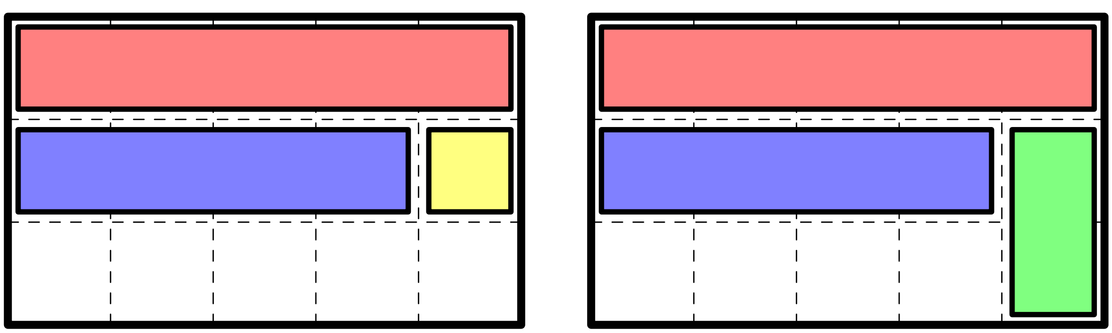
2. Compute the number of ways to tile a $3 \times 5$ rectangle with one $1 \times 1$ tile, one $1 \times 2$ tile, one $1 \times 3$ tile, one $1 \times 4$ tile, and one $1 \times 5$ tile. (The tiles can be rotated, and tilings that differ by rotation or reflection are considered distinct.)

   *Proposed by: Sean Li*

   **Answer:** $\boxed{40}$

   **Solution:** Our strategy is to first place the $1 \times 5$ and the $1 \times 4$ tiles since their size restricts their location. We have three cases:

   - **Case 1: first row.** There are 4 ways to place the $1 \times 4$ tile. There is an empty cell next to the $1 \times 4$ tile, which can either be occupied by the $1 \times 1$ tile or the $1 \times 2$ tile (see diagram). In both cases, there are 2 ways to place the remaining two tiles, so this gives $4 \cdot 2 \cdot 2 = 16$ ways.



   - **Case 2: middle row.** There are 4 ways to place the $1 \times 4$ tile, and the $1 \times 1$ tile must go next to it. There are 2 ways to place the remaining two tiles, so this gives $4 \cdot 2 = 8$ ways.
   - **Case 3: bottom row.** This is the same as Case 1 up to rotation, so there are also 16 ways to place the tiles here.

   In total, we have $16 + 8 + 16 = 40$ ways to place the tiles.

3. Richard starts with the string HHMMMMTT. A *move* consists of replacing an instance of HM with MH, replacing an instance of MT with TM, or replacing an instance of TH with HT. Compute the number of possible strings he can end up with after performing zero or more moves.

*Proposed by: Albert Wang*

**Answer:** $\boxed{70}$

**Solution:** The key claim is that the positions of the Ms fully determines the end configuration. Indeed, since all Hs are initially left of all Ts, the only successful swaps that can occur will involve Ms. So, picking $\binom{8}{4} = 70$ spots for Ms and then filling in the remaining 4 spots with Hs first and then Ts gives all possible arrangements.

It is not hard to show that all of these arrangements are also achievable; just greedily move Ms to their target positions.

4. The cells of a $5 \times 5$ grid are each colored red, white, or blue. Sam starts at the bottom-left cell of the grid and walks to the top-right cell by taking steps one cell either up or to the right. Thus, he passes through 9 cells on his path, including the start and end cells. Compute the number of colorings for which Sam is guaranteed to pass through a total of exactly 3 red cells, exactly 3 white cells, and exactly 3 blue cells no matter which route he takes.

*Proposed by: Sean Li*

**Answer:** $\boxed{1680}$

**Solution:** Let $c_{i,j}$ denote the cell in the $i$-th row from the bottom and the $j$-th column from the left, so Sam starts at $c_{1,1}$ and is traveling to $c_{5,5}$. The key observation (from, say, trying small cases) is that

**Claim.** For $1 \leq i, j < 5$, the cells $c_{i+1,j}$ and $c_{i,j+1}$ must be the same color.

*Proof.* Choose a path $P$ from $c_{1,1}$ to $c_{i,j}$, and a path $Q$ from $c_{i+1,j+1}$ to $c_{5,5}$. Then consider the two paths $P \to c_{i+1,j} \to Q$ and $P \to c_{i,j+1} \to Q$. These both must have 3 cells of each color, but they only differ at cells $c_{i+1,j}$ and $c_{i,j+1}$. So these cells must be the same color. $\square$

Hence, every diagonal $D_k = \{c_{a,b} : a + b = k\}$ must consist of cells of the same color. Moreover, any path that goes from $c_{1,1}$ to $c_{5,5}$ contains exactly one cell in $D_k$ for $k = 2, 3, \ldots, 10$. So we simply need to color the diagonals $D_2, \ldots, D_{10}$ such that there are 3 diagonals of each color. The number of ways to do this is $\binom{9}{3,3,3} = 1680$.

5. Elbert and Yaiza each draw 10 cards from a 20-card deck with cards numbered $1, 2, 3, \ldots, 20$. Then, starting with the player with the card numbered 1, the players take turns placing down the lowest-numbered card from their hand that is greater than every card previously placed. When a player cannot place a card, they lose and the game ends.

Given that Yaiza lost and 5 cards were placed in total, compute the number of ways the cards could have been initially distributed. (The order of cards in a player's hand does not matter.)

*Proposed by: Maxim Li*

**Answer:** $\boxed{324}$

**Solution:** Put each card in order and label them based on if Elbert or Yaiza got them. We will get a string of E's and Y's like $EEYYYE\ldots$, and consider the "blocks" of consecutive letters. It is not hard to see that only the first card of each block is played, and the number of cards played is exactly the number of blocks. Thus, it suffices to count the ways to distribute 10 cards to each player to get exactly 5 blocks.

Note that since Yaiza lost, Elbert must have the last block, and since blocks alternate in player, Elbert also has the first block. Then a card distribution is completely determined by where Yaiza's blocks are relative to Elbert's cards (e.g. one block is between the 4th and 5th card), as well as the number of cards in each block. Since Elbert has 10 cards, there are $\binom{9}{2}$ ways to pick the locations of the blocks, and 9 ways to distribute 10 cards between two blocks. This gives a total answer of $9\binom{9}{2} = 324$.

6. Each cell of a $3 \times 3$ grid is labeled with a digit in the set $\{1, 2, 3, 4, 5\}$. Then, the maximum entry in each row and each column is recorded. Compute the number of labelings for which every digit from 1 to 5 is recorded at least once.

   *Proposed by: Evan Erickson, Luke Robitaille*

   **Answer:** $\boxed{2664}$

   **Solution:** We perform casework by placing the entries from largest to smallest.

   - The grid must have exactly one 5 since an entry equal to 5 will be the maximum in its row and in its column. We can place this in 9 ways.
   - An entry equal to 4 must be in the same row or column as the 5; otherwise, it will be recorded twice, so we only have two records left but 1, 2, and 3 are all unrecorded. Using similar logic, there is at most one 4 in the grid. So there are 4 ways to place the 4.
   - We further split into cases for the 3 entries. Without loss of generality, say the 4 and the 5 are in the same row.
     - If there is a 3 in the same row as the 4 and the 5, then it remains to label a $2 \times 3$ grid with 1s and 2s such that there is exactly one row with all 1s, of which there are $2(2^3 - 1) = 14$ ways to do so.
     - Suppose there is no 3 in the same row as the 4 and the 5. Then there are two remaining empty rows to place a 3.
       There are two possible places we could have a record of 2, the remaining unoccupied row or the remaining unoccupied column. There are 2 ways to pick one of these; without loss of generality, we pick the row. Then the column must be filled with all 1s, and the remaining slots in the row with record 2 can be filled in one of 3 ways (12, 21, or 22). The final empty cell can be filled with a 1, 2, or 3, for a total of 3 ways. Our total here is $2 \cdot 2 \cdot 3 \cdot 5 = 60$ ways.

   Hence, our final answer is $9 \cdot 4 \cdot (14 + 60) = 36 \cdot 74 = 2664$.

7. Svitlana writes the number 147 on a blackboard. Then, at any point, if the number on the blackboard is $n$, she can perform one of the following three operations:

   - if $n$ is even, she can replace $n$ with $\frac{n}{2}$;
   - if $n$ is odd, she can replace $n$ with $\frac{n+255}{2}$; and
   - if $n \geq 64$, she can replace $n$ with $n - 64$.

   Compute the number of possible values that Svitlana can obtain by doing zero or more operations.

   *Proposed by: Jerry Liang, Vidur Jasuja*

   **Answer:** $\boxed{163}$

   **Solution:** The answer is $163 = \sum_{i=0}^{4} \binom{8}{i}$. This is because we can obtain any integer less than $2^8$ with less than or equal to 4 ones in its binary representation. Note that $147 = 2^7 + 2^4 + 2^1 + 2^0$.

   We work in binary. Firstly, no operation can increase the number of ones in $n$'s binary representation. The first two operations cycle the digits of $n$ to the right, and the last operation can change a $11, 10, 01$ at the front of $n$ to $10, 01, 00$, respectively. This provides an upper bound.

   To show we can obtain any of these integers, we'll show that given a number $m_1$ with base 2 sum of digits $k$, we can obtain every number with base 2 sum of digits $k$. Since we can, by cycling, change any 10 to an 01, we can move all of $m_1$'s ones to the end, and then cycle so they're all at the front. From here, we can just perform a series of swaps to obtain any other integer with this same sum of digits. It's also easy to see that we can decrement the sum of digits of $n$, by cycling a 1 to the second digit of the number and then performing the third operation. So this proves the claim.

8. A random permutation $a = (a_1, a_2, \ldots, a_{40})$ of $(1, 2, \ldots, 40)$ is chosen, with all permutations being equally likely. William writes down a $20 \times 20$ grid of numbers $b_{ij}$ such that $b_{ij} = \max(a_i, a_{j+20})$ for all $1 \le i, j \le 20$, but then forgets the original permutation $a$. Compute the probability that, given the values of $b_{ij}$ alone, there are exactly 2 permutations $a$ consistent with the grid.

   *Proposed by: Zixiang Zhou*

   **Answer:** $\boxed{\frac{10}{13}}$

   **Solution:** We can deduce information about $a$ from the grid $b$ by looking at the largest element of it, say $m$. If $m$ fills an entire row, then the value of $a$ corresponding to this row must be equal to $m$. Otherwise, $m$ must fill an entire column, and the value of $a$ corresponding to this column must be equal to $m$. We can then ignore this row/column and continue this reasoning recursively on the remaining part of the grid.

   Near the end, there are two cases. We could have a $1 \times 1$ remaining grid, where there are 2 permutations $a$ consistent with $b$. We could also have a case where one of the dimensions of the remaining grid is 1, the other dimension is at least 2 (say $k$), and the number $k+1$ fills the entire remaining grid. In that case, there are $k!$ ways to arrange the other elements $1, \ldots, k$.

   It follows that there are exactly 2 permutations $a$ consistent with the grid if and only if one of 1 and 2 is assigned to a row and the other is assigned to a column, or they are both assigned to the same type and 3 is assigned to the opposite type. The probability that this does not occur is the probability that $1, 2, 3$ are all assigned to the same type, which happens with probability $\frac{19}{39} \cdot \frac{18}{38} = \frac{18}{2 \cdot 39} = \frac{3}{13}$, so the answer is $1 - \frac{3}{13} = \frac{10}{13}$.

9. There are 100 people standing in a line from left to right. Half of them are randomly chosen to face right (with all $\binom{100}{50}$ possible choices being equally likely), and the others face left. Then, while there is a pair of people who are facing each other and have no one between them, the leftmost such pair leaves the line. Compute the expected number of people remaining once this process terminates.

   *Proposed by: Albert Wang*

   **Answer:** $\boxed{\frac{2^{100}}{\binom{100}{50}} - 1}$

   **Solution:** Notice that the order in which the people leave the line is irrelevant. Give each right-facing person a weight of 1, and each left-facing person a weight of $-1$. We claim the answer for some arrangement of these $2n$ people is $-2$ times the minimum prefix sum. For instance:

$$\texttt{LRRLRLLLRRRRL} \to (-2)(-2) \to 4$$

$$\texttt{RRRLLRLLLRRL} \to (-2)(-1) \to 2$$

   *Proof.* he final configuration is always of the form

$$\underbrace{\texttt{LL} \ldots \texttt{LL}}_{k}\underbrace{\texttt{RR} \ldots \texttt{RR}}_{k}$$

   and the minimum prefix sum is invariant. As the final configuration has minimum prefix sum is $k$, we are done. □

   So, we want to find the expected value of the minimum prefix sum across all such strings of 1s and $-1$s. To find this, we will instead compute the equivalent value

$$\sum_{k=1}^{\infty} \Pr[\text{maximum prefix sum is} \ge k].$$

Consider the $k$th term of this sum, and the corresponding walk from $(0,0)$ to $(2n,0)$ with `L` corresponding to a step of $(1,-1)$ and `R` corresponding to a step of $(1,1)$. Consider the point $P$ at $y=k$ with minimal $x$-coordinate, and reflect the remainder of the walk across $y=k$. This gives a path that ends at $(2n,2k)$. Noting that this is a bijection between walks from $(0,0)$ to $(2n,2k)$ and walks that reach $y=k$, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\Pr[\text{maximum prefix sum is} \geq k] &= \sum_{k=1}^{\infty}\frac{\binom{2n}{n-k}}{\binom{2n}{n}}\\ &= \frac{1}{2}\left[\left(\sum_{k=-\infty}^{\infty}\frac{\binom{2n}{n-k}}{\binom{2n}{n}}\right)-1\right]\\ &= \frac{1}{2}\left(\frac{2^{2n}}{\binom{2n}{n}}-1\right).\end{aligned}$$

Adjusting for the factor of 2 we saved at the beginning, our final answer for $n=50$ is $\frac{2^{100}}{\binom{100}{50}}-1$.

10. Let $x_0 = x_{101} = 0$. The numbers $x_1, x_2, \ldots, x_{100}$ are chosen at random from the interval $[0,1]$ uniformly and independently. Compute the probability that $2x_i \geq x_{i-1}+x_{i+1}$ for all $i = 1, 2, \ldots, 100$.

*Proposed by: Albert Wang*

**Answer:** $\boxed{\frac{1}{100\cdot 100!^2}\binom{200}{99}}$

**Solution:** We solve for general $n$ where $n=100$ in the problem. Notice that the points $(i, A_i)$ must form a convex hull, so there is some unique maximal element $A_i$. Consider the $i-1$ points $A_1, \ldots, A_{i-1}$ left of $i$, and the $i$ *slopes* formed between these points of segments $\overline{A_0A_1}, \ldots, \overline{A_{i-1}A_i}$. Notice that we must choose the $i-1$ points to be decreasing. Ignoring cases where they have some shared $y$-coordinates since this happens with probability 0, we have a $\frac{1}{(i-1)!}$ chance of picking them in ascending order. Now, we order the differences

$$\{A_1-A_0, A_2-A_1, \ldots, A_i-A_{i-1}\}$$

in descending order, obtaining some new list

$$\{d_1, d_2, \ldots, d_i\}$$

and redefining $A_k = \sum_{j=1}^{k} d_j$. Notice that this procedure almost surely maps $(i-1)!i!$ possible sequences of points $A_1, A_2, \ldots, A_{i-1}$ to a valid convex hull, so the chance that the points left of $A_i$ are valid is $\frac{1}{(i-1)!i!}$. Similarly, the chance that the points on the right work is given by $\frac{1}{(n+1-i)!(n-i)!}$. So, for a maximum value at $A_i$ the chance that we get a valid convex hull is $\frac{1}{(i-1)!i!(n+1-i)!(n-i)!}$.

To finish, note that each point is equally likely to be the peak. Our answer is

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(i-1)!i!(n+1-i)!(n-i)!}\\ &= \frac{1}{n\cdot n!^2}\sum_{i=1}^{n}\frac{n!^2}{(i-1)!i!(n+1-i)!(n-i)!}\\ &= \frac{1}{n\cdot n!^2}\sum_{i=1}^{n}\binom{n}{i-1}\binom{n}{n-i} =\\ &= \frac{1}{n\cdot n!^2}\binom{2n}{n-1}\end{aligned}$$

Plugging in $n = 100$ gives the desired answer.