15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Team A

1. [**20**] Let ABC be a triangle. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D. Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC, respectively. Prove that $BE = CF$.

2. [**20**]

   (a) For what positive integers n do there exist functions $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that for all $1 \le i \le n$, we have that exactly one of $f(g(i)) = i$ and $g(f(i)) = i$ holds?

   (b) What if f, g must be permutations?

3. [**20**] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player somewhere on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

   (a) Suppose that a token can legally move to any orthogonally adjacent square. Show that Bob has a winning strategy for this game.

   (b) Suppose instead that a token can legally move to any square which shares a vertex with the square it is currently on. Show that Alice has a winning strategy for this game.

4. [**20**] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC. Line l is drawn through M so that it is perpendicular to AM, and intersects line AB at point X and line AC at point Y. Prove that $\angle BAC = 90^\circ$ if and only if quadrilateral $XBYC$ is cyclic.

5. [**20**] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza: mushrooms, olives, green peppers, cheese, and pepperoni. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes 5 pizzas, each with some subset of the five toppings, then for each of those Arno tells her if that pizza has any toppings he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she can then make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?

6. [**30**] It has recently been discovered that the right triangle with vertices (0,0), (0,2012), and (2012,0) is a giant pond home to many frogs. Frogs have the special ability that, if they are at a lattice point (x, y), they can hop to any of the three lattice points $(x+1, y+1)$, $(x-2, y+1)$, $(x+1, y-2)$, assuming the given lattice point lies in or on the boundary of the triangle.

   Frog Jeff starts at the corner $(0, 0)$, while Frog Kenny starts at the corner $(0, 2012)$. Show that the set of points Jeff can reach is equal in size to the set of points that Kenny can reach.

7. [**30**] Five points are chosen on a sphere of radius 1. What is the maximum possible volume of their convex hull?

8. [**30**] For integer $m, n \ge 1$, let $A(m, n)$ denote the number of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that $f(j) - f(i) \ge j - i$ for all $1 \le i < j \le n$, and let $B(m, n)$ denote the number of functions $g : \{0, 1, \ldots, 2n + m\} \to \{0, 1, \ldots, m\}$ such that $g(0) = 0$, $g(2n + m) = m$, and $|g(i) - g(i - 1)| = 1$ for all $1 \le i \le 2n + m$. Prove that $A(m, n) = B(m, n)$.

For the remaining problems, let $\varphi(k)$ denote the number of positive integers less than or equal to k that are relatively prime to k.

9. [**40**] For any positive integer n, let $N = \phi(1) + \phi(2) + \ldots + \phi(n)$. Show that there exists a sequence

$$a_1, a_2, \ldots a_N$$

containing exactly $\phi(k)$ instances of k for all positive integers $k \leq n$ such that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + ... + \frac{1}{a_N a_1} = 1.$$

10. [**40**] For positive odd integer n, let $f(n)$ denote the number of matrices A satisfying the following conditions:

- A is $n \times n$.
- Each row and column contains each of $1, 2, \ldots, n$ exactly once in some order.
- $A^T = A$. (That is, the element in row i and column j is equal to the one in row j and column i, for all $1 \leq i, j \leq n$.)

Prove that $f(n) \geq \frac{n!(n-1)!}{\varphi(n)}$, where φ is the totient function.