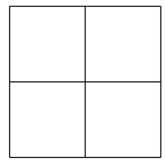
HMMT February 2018

February 10, 2018

Guts Round

1. [**4**] A square can be divided into four congruent figures as shown:



If each of the congruent figures has area 1, what is the area of the square?

Proposed by: Kevin Sun

Answer: $\boxed{4}$

There are four congruent figures with area 1, so the area of the square is 4.

2. [**4**] John has a 1 liter bottle of pure orange juice. He pours half of the contents of the bottle into a vat, fills the bottle with water, and mixes thoroughly. He then repeats this process 9 more times. Afterwards, he pours the remaining contents of the bottle into the vat. What fraction of the liquid in the vat is now water?

Proposed by: Farrell Eldrian Wu

Answer: $\boxed{\frac{5}{6}}$

All the liquid was poured out eventually. 5 liters of water was poured in, and he started with 1 liter of orange juice, so the fraction is $\frac{5}{1+5} = \frac{5}{6}$.

3. [**4**] Allen and Yang want to share the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. How many ways are there to split all ten numbers among Allen and Yang so that each person gets at least one number, and either Allen's numbers or Yang's numbers sum to an even number?

Proposed by: Kevin Sun

Answer: $\boxed{1022}$

Since the sum of all of the numbers is odd, exactly one of Allen's sum and Yang's sum must be odd. Therefore any way of splitting the numbers up where each person receives at least one number is valid, so the answer is $2^{10} - 2 = 1022$.

4. [**4**] Find the sum of the digits of $11 \cdot 101 \cdot 111 \cdot 110011$.

Proposed by: Evan Chen

Answer: $\boxed{48}$

There is no regrouping, so the answer is $2 \cdot 2 \cdot 3 \cdot 4 = 48$. The actual product is 13566666531.

5. [**6**] Randall proposes a new temperature system called *Felsius* temperature with the following conversion between Felsius $^\circ E$, Celsius $^\circ C$, and Fahrenheit $^\circ F$:

$$^\circ E = \frac{7 \times {}^\circ C}{5} + 16 = \frac{7 \times {}^\circ F - 80}{9}.$$

For example, $0\,^\circ C = 16\,^\circ E$. Let x, y, z be real numbers such that $x\,^\circ C = x\,^\circ E$, $y\,^\circ E = y\,^\circ F$, $z\,^\circ C = z\,^\circ F$. Find $x + y + z$.

Proposed by: Yuan Yao

Answer: $\boxed{-120}$

Notice that $(5k)\,^\circ C = (7k + 16)\,^\circ E = (9k + 32)\,^\circ F$, so Felsius is an exact average of Celsius and Fahrenheit at the same temperature. Therefore we conclude that $x = y = z$, and it is not difficult to compute that they are all equal to -40.

6. [**6**] A bug is on a corner of a cube. A *healthy* path for the bug is a path along the edges of the cube that starts and ends where the bug is located, uses no edge multiple times, and uses at most two of the edges adjacent to any particular face. Find the number of healthy paths.

   Proposed by: Dhruv Rohatgi

   Answer: $\boxed{6}$

   There are 6 symmetric ways to choose the first two edges on the path. After these are chosen, all subsequent edges are determined, until the starting corner is reached once again.

7. [**6**] A triple of integers (a, b, c) satisfies $a + bc = 2017$ and $b + ca = 8$. Find all possible values of c.

   Proposed by: Ashwin Sah

   Answer: $\boxed{-6, 0, 2, 8}$

   Add and subtract the two equations to find

$$(b + a)(c + 1) = 8 + 2017,$$

$$(b - a)(c - 1) = 2017 - 8.$$

   We see that c is even and then that every integer c with $c + 1 | 2025, c - 1 | 2009$ works. We factor and solve.

   The full solutions are $(2017, 8, 0), (-667, 1342, 2), (-59, -346, -6), (-31, 256, 8)$.

8. [**6**] Suppose a real number $x > 1$ satisfies

$$\log_2(\log_4 x) + \log_4(\log_{16} x) + \log_{16}(\log_2 x) = 0.$$

   Compute

$$\log_2(\log_{16} x) + \log_{16}(\log_4 x) + \log_4(\log_2 x).$$

   Proposed by: Michael Tang

   Answer: $\boxed{-\frac{1}{4}}$

   Let A and B be these sums, respectively. Then

$$\begin{aligned} B - A &= \log_2\left(\frac{\log_{16} x}{\log_4 x}\right) + \log_4\left(\frac{\log_2 x}{\log_{16} x}\right) + \log_{16}\left(\frac{\log_4 x}{\log_2 x}\right) \\ &= \log_2(\log_{16} 4) + \log_4(\log_2 16) + \log_{16}(\log_4 2) \\ &= \log_2\left(\frac{1}{2}\right) + \log_4 4 + \log_{16}\left(\frac{1}{2}\right) \\ &= (-1) + 1 + \left(-\frac{1}{4}\right) \\ &= -\frac{1}{4}. \end{aligned}$$

   Since $A = 0$, we have the answer $B = -\frac{1}{4}$.

9. [**7**] In a game, N people are in a room. Each of them simultaneously writes down an integer between 0 and 100 inclusive. A person wins the game if their number is exactly two-thirds of the average of all the numbers written down. There can be multiple winners or no winners in this game. Let m be the maximum possible number such that it is possible to win the game by writing down m. Find the smallest possible value of N for which it is possible to win the game by writing down m in a room of N people.

   Proposed by: Kevin Sun

   Answer: $\boxed{34}$

Since the average of the numbers is at most 100, the winning number is an integer which is at most two-thirds of 100, or at most 66. This is achieved in a room with 34 people, in which 33 people pick 100 and one person picks 66, so the average number is 99.

Furthermore, this cannot happen with less than 34 people. If the winning number is 66 and there are N people, the sum of the numbers must be 99. then we must have that $99N \leq 66 + 100(N-1)$, which reduces to $N \geq 34$.

10. [**7**] Let a positive integer n be called a *cubic square* if there exist positive integers a, b with $n = \gcd(a^2, b^3)$. Count the number of cubic squares between 1 and 100 inclusive.

Proposed by: Ashwin Sah

Answer: $\boxed{13}$

This is easily equivalent to $v_p(n) \not\equiv 1, 5 \pmod 6$ for all primes p. We just count: $p \geq 11 \implies v_p(n) = 1$ is clear, so we only look at the prime factorizations with primes from $\{2, 3, 5, 7\}$. This is easy to compute: we obtain 13.

11. [**7**] FInd the value of

$$\sum_{k=1}^{60} \sum_{n=1}^{k} \frac{n^2}{61 - 2n}.$$

Proposed by: Henrik Boecken

Answer: $\boxed{-18910}$

Change the order of summation and simplify the inner sum:

$$\begin{aligned} \sum_{k=1}^{60} \sum_{n=1}^{k} \frac{n^2}{61 - 2n} &= \sum_{n=1}^{60} \sum_{k=n}^{60} \frac{n^2}{61 - 2n} \\ &= \sum_{n=1}^{60} \frac{n^2(61-n)}{61 - 2n} \end{aligned}$$

Then, we rearrange the sum to add the terms corresponding to n and $61 - n$:

$$\begin{aligned} \sum_{n=1}^{60} \frac{n^2(61-n)}{61-2n} &= \sum_{n=1}^{30} \left(\frac{n^2(61-n)}{61-2n} + \frac{(61-n)^2(61-(61-n))}{61-2(61-n)} \right) \\ &= \sum_{n=1}^{30} \frac{n^2(61-n) - n(61-n)^2}{61-2n} \\ &= \sum_{n=1}^{30} \frac{n(61-n)(n-(61-n))}{61-2n} \\ &= \sum_{n=1}^{30} -n(61-n) \\ &= \sum_{n=1}^{30} n^2 - 61n \end{aligned}$$

Finally, using the formulas for the sum of the first k squares and sum of the first k positive integers, we conclude that this last sum is

$$\frac{30(31)(61)}{6} - 61\frac{30(31)}{2} = -18910$$

So, the original sum evaluates to -18910.

12. [**7**] $\triangle PNR$ has side lengths $PN = 20$, $NR = 18$, and $PR = 19$. Consider a point A on PN. $\triangle NRA$ is rotated about R to $\triangle N'RA'$ so that R, N', and P lie on the same line and AA' is perpendicular to PR. Find $\frac{PA}{AN}$.

Proposed by: Henrik Boecken

Answer: $\boxed{\frac{19}{18}}$

Denote the intersection of PR and AA' be D. Note $RA' = RA$, so D, being the altitude of an isosceles triangle, is the midpoint of AA'. Thus,

$$\angle ARD = \angle A'RD = \angle NRA$$

so RA is the angle bisector of PNR through R. By the angle bisector theorem, we have $\frac{PA}{AN} = \frac{PR}{RN} = \frac{19}{18}$

13. [**9**] Suppose $\triangle ABC$ has lengths $AB = 5$, $BC = 8$, and $CA = 7$, and let ω be the circumcircle of $\triangle ABC$. Let X be the second intersection of the external angle bisector of $\angle B$ with ω, and let Y be the foot of the perpendicular from X to BC. Find the length of YC.

Proposed by: Caleb He

Answer: $\boxed{\frac{13}{2}}$

Extend ray $\overrightarrow{AB}$ to a point D, Since BX is an angle bisector, we have $\angle XBC = \angle XBD = 180^\circ - \angle XBA = \angle XCA$, so $XC = XA$ by the inscribed angle theorem. Now, construct a point E on BC so that $CE = AB$. Since $\angle BAX \cong \angle BCX$, we have $\triangle BAX \cong \triangle ECX$ by SAS congruence. Thus, $XB = XE$, so Y bisects segment BE. Since $BE = BC - EC = 8 - 5 = 3$, we have $YC = EC + YE = 5 + \frac{1}{2} \cdot 3 = \frac{13}{2}$.

(Archimedes Broken Chord Thoerem).

14. [**9**] Given that x is a positive real, find the maximum possible value of

$$\sin\left(\tan^{-1}\left(\frac{x}{9}\right) - \tan^{-1}\left(\frac{x}{16}\right)\right).$$

Proposed by: Yuan Yao

Answer: $\boxed{\frac{7}{25}}$

Consider a right triangle AOC with right angle at O, $AO = 16$ and $CO = x$. Moreover, let B be on AO such that $BO = 9$. Then $\tan^{-1}\frac{x}{9} = \angle CBO$ and $\tan^{-1}\frac{x}{16} = \angle CAO$, so their difference is equal to $\angle ACB$. Note that the locus of all possible points C given the value of $\angle ACB$ is part of a circle that passes through A and B, and if we want to maximize this angle then we need to make this circle as small as possible. This happens when OC is tangent to the circumcircle of ABC, so $OC^2 = OA \cdot OB = 144 = 12^2$, thus $x = 12$, and it suffices to compute $\sin(\alpha - \beta)$ where $\sin\alpha = \cos\beta = \frac{4}{5}$ and $\cos\alpha = \sin\beta = \frac{3}{5}$. By angle subtraction formula we get $\sin(\alpha - \beta) = (\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{7}{25}$.

15. [**9**] Michael picks a random subset of the complex numbers $\{1, \omega, \omega^2, \ldots, \omega^{2017}\}$ where ω is a primitive 2018^{th} root of unity and all subsets are equally likely to be chosen. If the sum of the elements in his subset is S, what is the expected value of $|S|^2$? (The sum of the elements of the empty set is 0.)

Proposed by: Nikhil Reddy

Answer: $\boxed{\frac{1009}{2}}$

Consider a and $-a$ of the set of complex numbers. If x is the sum of some subset of the other complex numbers, then expected magnitude squared of the sum including a and $-a$ is

$$\frac{(x+a)(\overline{x+a}) + x\overline{x} + x\overline{x} + (x-a)(\overline{x-a})}{4}$$

$$x\overline{x} + \frac{a\overline{a}}{2}$$

$$x\overline{x}+\frac{1}{2}$$

By repeating this process on the remaining 2016 elements of the set, we can obtain a factor of $\frac{1}{2}$ every time. In total, the answer is

$$\frac{1009}{2}$$

16. [**9**] Solve for x:

$$x\lfloor x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor\rfloor = 122.$$

Proposed by: John Michael Wu

Answer: $\boxed{\frac{122}{41}}$

This problem can be done without needless casework.

(For negative values of x, the left hand side will be negative, so we only need to consider positive values of x.)

The key observation is that for $x \in [2, 3)$, 122 is an extremely large value for the expression. Indeed, we observe that:

$$\begin{aligned}
\lfloor x\rfloor &= 2 & \\
\lfloor x\lfloor x\rfloor\rfloor &\leq 2(3)-1 &= 5\\
\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor &\leq 3(5)-1 &= 14\\
\lfloor x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor\rfloor &\leq 3(14)-1 &= 41\\
x\lfloor x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor\rfloor &< 3(41) &= 123
\end{aligned}$$

So the expression can only be as large as 122 if ALL of those equalities hold (the the fourth line equaling 40 isn't good enough), and $x = \frac{122}{41}$. Note that this value is extremely close to 3. We may check that this value of x indeed works. Note that the expression is strictly increasing in x, so $x = \frac{122}{41}$ is the only value that works.

17. [**10**] Compute the value of

$$\frac{\cos 30.5^\circ + \cos 31.5^\circ + ... + \cos 44.5^\circ}{\sin 30.5^\circ + \sin 31.5^\circ + ... + \sin 44.5^\circ}.$$

Proposed by: Sujay Kazi

Answer: $\boxed{(\sqrt{2}-1)(\sqrt{3}+\sqrt{2}) = 2-\sqrt{2}-\sqrt{3}+\sqrt{6}}$

Consider a 360-sided regular polygon with side length 1, rotated so that its sides are at half-degree inclinations (that is, its sides all have inclinations of $0.5^\circ, 1.5^\circ, 2.5^\circ$, and so on. Go to the bottom point on this polygon and then move clockwise, numbering the sides $1, 2, 3, ..., 360$ as you go. Then, take the section of 15 sides from side 31 to side 45. These sides have inclinations of $30.5^\circ, 31.5^\circ, 32.5^\circ$, and so on, up to 44.5°. Therefore, over this section, the horizontal and vertical displacements are, respectively:

$$H = \cos 30.5^\circ + \cos 31.5^\circ + ... + \cos 44.5^\circ$$

$$V = \sin 30.5^\circ + \sin 31.5^\circ + ... + \sin 44.5^\circ$$

However, we can also see that, letting R be the circumradius of this polygon:

$$H = R(\sin 45^\circ - \sin 30^\circ)$$

$$V = R[(1-\cos 45^\circ) - (1-\cos 30^\circ)]$$

From these, we can easily compute that our desired answer is $\frac{H}{V} = (\sqrt{2}-1)(\sqrt{3}+\sqrt{2}) = 2-\sqrt{2}-\sqrt{3}+\sqrt{6}$.

18. [**10**] Compute the number of integers $n \in \{1, 2, \ldots, 300\}$ such that n is the product of two distinct primes, and is also the length of the longest leg of some nondegenerate right triangle with integer side lengths.

Proposed by:

Answer: $\boxed{13}$

Let $n = p \cdot q$ for primes $p < q$. If n is the second largest side of a right triangle there exist integers c, a such that $a < pq$ and $(pq)^2 = c^2 - a^2 = (c-a)(c+a)$. Since $c - a < c + a$ there are three cases for the values of $c - a, c + a$, and in each case we determine when $a < pq$.

(a) $c - a = 1$ and $c + a = p^2q^2$: Then $a = \frac{p^2q^2-1}{2} > pq$, so there are no solutions.

(b) $c - a = p$ and $c + a = pq^2$: Then $a = \frac{pq^2-p}{2} > pq$.

(c) $c - a = p^2$ and $c + a = q^2$. Then $a = \frac{q^2-p^2}{2}$ which we require to be less than pq. This is equivalent to

$$\begin{aligned} \frac{q^2-p^2}{2} &< pq \\ q^2 &< 2pq + p^2 \\ 2q^2 &< (q+p)^2 \\ \sqrt{2}q &< q + p \\ (\sqrt{2}-1)q &< p < q \end{aligned}$$

So the problem is equivalent to finding the number of distinct prime pairs (p, q) such that $pq < 300$ and $(\sqrt{2} - 1)q < p < q$. There are $\boxed{13}$ such pairs:

$$\{(3,5),(3,7),(5,7),(5,11),(7,11),(7,13),(11,13),(11,17),(11,19),(11,23),(13,17),(13,19),(13,23)\}$$

and $13 \cdot 23 = 299$ which is the biggest such pair.

The most interesting borderline case are $(3,7) : \frac{3}{7} \approx .42 > \sqrt{2}-1$, which leads to the $(20, 21, 29)$ triangle, $(5, 13) : \frac{5}{13} \approx .385 < \sqrt{2} - 1$, which leads to the $(65, 72, 97)$ triangle, and $(7, 17) : \frac{7}{17} \approx .411 < \sqrt{2} - 1$ which leads to the $(119, 120, 169)$ right triangle.

19. [**10**] Suppose there are 100 cookies arranged in a circle, and 53 of them are chocolate chip, with the remainder being oatmeal. Pearl wants to choose a contiguous subsegment of exactly 67 cookies and wants this subsegment to have exactly k chocolate chip cookies. Find the sum of the k for which Pearl is guaranteed to succeed regardless of how the cookies are arranged.

Proposed by: Alexander Wei

Answer: $\boxed{71}$

We claim that the only values of k are 35 and 36.

WLOG assume that the cookies are labelled 0 through 99 around the circle. Consider the following arrangement: cookies 0 through 17, 34 through 50, and 67 through 84 are chocolate chip, and the remaining are oatmeal. (The cookies form six alternating blocks around the circle of length $18, 16, 17, 16, 18, 15$.) Consider the block of 33 cookies that are not chosen. It is not difficult to see that since the sum of the lengths of each two adjacent block is always at least 33 and at most 34, this block of unchosen cookies always contains at least one complete block of cookies of the same type (and no other cookies of this type). So this block contains 17 or 18 or $33 - 16 = 17$ or $33 - 15 = 18$ chocolate chip cookies. Therefore, the block of 67 chosen cookies can only have $53 - 17 = 36$ or $53 - 18 = 35$ chocolate chip cookies.

Now we show that 35 and 36 can always be obtained. Consider all possible ways to choose 67 cookies: cookies 0 through 66, 1 through 67, ..., 99 through 65. It is not difficult to see that the number of chocolate chip cookies in the block changes by at most 1 as we advance from one way to the next.

Moreover, each cookie will be chosen 67 times, so on average there will be $\frac{67\cdot 53}{100} = 35.51$ chocolate chip cookies in each block. Since not all blocks are below average and not all blocks are above average, there must be a point where a block below average transition into a block above average. The difference of these two blocks is at most 1, so one must be 35 and one must be 36.

Therefore, the sum of all possible values of k is $35 + 36 = 71$.

20. [**10**] Triangle $\triangle ABC$ has $AB = 21$, $BC = 55$, and $CA = 56$. There are two points P in the plane of $\triangle ABC$ for which $\angle BAP = \angle CAP$ and $\angle BPC = 90°$. Find the distance between them.

Proposed by: Michael Tang

Answer: $\boxed{\frac{5}{2}\sqrt{409}}$

Let P_1 and P_2 be the two possible points P, with $AP_1 < AP_2$. Both lie on the $\angle A$-bisector and the circle γ with diameter BC. Let D be the point where the $\angle A$-bisector intersects BC, let M be the midpoint of BC, and let X be the foot of the perpendicular from M onto the $\angle A$-bisector.

Since we know the radius of γ, to compute P_1P_2 it suffices to compute MX. By the angle bisector theorem we find $BD = 15$ and $DC = 40$, so Stewart's theorem gives

$$15 \cdot 40 \cdot 55 + 55 \cdot AD^2 = 21^2 \cdot 40 + 56^2 \cdot 15 \implies AD = 24.$$

Then $\cos\angle ADB = \frac{-21^2+15^2+24^2}{2\cdot 15\cdot 24} = \frac{1}{2}$, so $\angle ADB = \angle MDX = 60°$. Since $DM = BM - BD = \frac{55}{2} - 15 = \frac{25}{2}$, we get $MX = DM \sin\angle MDX = \frac{25\sqrt{3}}{4}$. Hence

$$P_1P_2 = 2\sqrt{\left(\frac{55}{2}\right)^2 - \left(\frac{25\sqrt{3}}{4}\right)^2} = \frac{5\sqrt{409}}{2}.$$

21. [**12**] You are the first lucky player to play in a slightly modified episode of Deal or No Deal! Initially, there are sixteen cases marked 1 through 16. The dollar amounts in the cases are the powers of 2 from $2^1 = 2$ to $2^{16} = 65536$, in some random order. The game has eight turns. In each turn, you choose a case and claim it, without opening it. Afterwards, a random remaining case is opened and revealed to you, then removed from the game.

At the end of the game, all eight of your cases are revealed and you win all of the money inside them.

However, the hosts do not realize you have X-ray vision and can see the amount of money inside each case! What is the expected amount of money you will make, given that you play optimally?

Proposed by: Kevin Sun

Answer: $\boxed{\dfrac{7\cdot 2^{18}+4}{15} \text{ (or } \dfrac{1835012}{15}\text{)}}$

Firstly, note that it is always optimal for you to take the case with the largest amount of money. To prove this rigorously, consider a strategy where you don't - then change the first move where you deviate to taking the maximal case. This can only increase your return.

We calculate the probability $f(n, k)$ that, if there are n cases numbered $1, \cdots, n$ in increasing order of value, that you will take case k in the course of play. We claim that $f(n, k) = \frac{k-1}{n-1}$ and prove this by induction on $n/2$ (n always even). The base case $n = 2$ is true because you will always take case 2 and leave case 1. Then, for the general n, you will always take case n (so $f(n, n) = 1$). Afterward, one case at random will be removed. When calculating $f(n, k)$ there is a $\frac{n-1-k}{n-1}$ probability a case numbered greater than k is removed, which inductively gives a probability $f(n-2, k)$. Also, there is a $\frac{k-1}{n-1}$ probability a case numbered less than k is removed, which inductively gives a probability

$f(n-2,k-1)$. We can compute

$$\begin{aligned}
f(n,k) &= \frac{n-1-k}{n-1} f(n-2,k) + \frac{k-1}{n-1} f(n-2,k-1) \\
&= \frac{n-1-k}{n-1} \cdot \frac{k-1}{n-3} + \frac{k-1}{n-1}\frac{k-2}{n-3} \\
&= \frac{k-1}{(n-1)(n-3)}(n-1-k+k-2) \\
&= \frac{k-1}{n-1}
\end{aligned}$$

as desired.

Finally, we must find $\sum_{i=1}^{16} f(16,i)2^{i-1}$. Using standard procedures, we get

$$\begin{aligned}
\sum_{i=1}^{16} f(16,i)2^{i-1} &= \sum_{i=1}^{15} \frac{i}{15} 2^i \\
&= \sum_{i=1}^{15} \frac{i}{15}(2^{17} - 2^{i+1}) \\
&= \frac{1}{15}(15)(2^{17}) - \frac{1}{15}\left(\sum_{i=1}^{15} 2^{i+1}\right) \\
&= 2^{17} - \frac{1}{15}(2^{17} - 4) \\
&= \boxed{\frac{14 \cdot 2^{17} + 4}{15}}
\end{aligned}$$

22. [**12**] How many graphs are there on 10 vertices labeled $1, 2, \ldots, 10$ such that there are exactly 23 edges and no triangles?

Proposed by: Allen Liu

Answer: $\boxed{42840}$

Note that the sum of the degrees of the graph is $23 \cdot 2 = 46$, so at least one vertex has degree 5 or more. We casework on the maximal degree n.

Case 1: $n \geq 7$, then none of the n neighbors can have an edge between each other, for $\binom{n}{2}$ edges unusable, and the vertex with maximal degree cannot connect to the $9-n$ other vertices. Then we have $\binom{n}{2} + 9 - n > \binom{10}{2} - 23 = 22$ when $n \geq 7$, so there cannot be any graph in this case.

Case 2: $n = 6$. WLOG suppose that 1 is connected to $2, 3, 4, 5, 6, 7$, then none of $2, 3, 4, 5, 6, 7$ can connect to each other.

Case 2.1: There is at least one edge between $8, 9, 10$, then each of $2, 3, 4, 5, 6, 7$ can connect to at most two of $8, 9, 10$, for at most $6 \cdot 2 + \binom{3}{2} = 15$ additional edges. Along with the 6 original edges, it is not enough to each 23 edges.

Case 2.2: There are no edges between $8, 9, 10$, then this graph is a bipartite graph between $1, 8, 9, 10$ and $2, 3, 4, 5, 6, 7$. There can be at most $4 \cdot 6 = 24$ edges in this graph, so exactly one edge is removed from this graph. There are $\binom{10}{4} \cdot 24 = 5040$ possible graphs in this case.

Case 3: $n = 5$. WLOG suppose that 1 is connected to $2, 3, 4, 5, 6$, then none of $2, 3, 4, 5, 6$ can connect to each other.

Case 3.1: There is at least one edge between $7, 8, 9, 10$. Then each of $2, 3, 4, 5, 6$ can connect to at most three of $7, 8, 9, 10$, for $5 \cdot 3 = 15$ edges. In this case at least three of $7, 8, 9, 10$ must not be connected to each other, so there can be at most three edges, for $5 + 15 + 3 = 23$ edges at most. However, this requires the three disconnected vertices of $7, 8, 9, 10$ to be connected to all of $2, 3, 4, 5, 6$ and the other vertex of $7, 8, 9, 10$, causing them to have degree 6. We can therefore ignore this case. (The case where $2, 3, 4, 5, 6$ can connect to two or less of $7, 8, 9, 10$ can be easily ruled out.)

Case 3.2: There are no edges between $7, 8, 9, 10$, then this graph is a bipartite graph between $1, 7, 8, 9, 10$ and $2, 3, 4, 5, 6$. This is a $K_{5,5}$ with two edges removed, which accounts for $\binom{10}{5}/2 \cdot \binom{25}{2} = 126 \cdot 300 = 37800$ graphs.

It is not difficult to see that Case 2.2 and Case 3.2 are disjoint (by considering max degree), so there are $5040 + 37800 = 42840$ graphs in total.

23. [**12**] Kevin starts with the vectors $(1, 0)$ and $(0, 1)$ and at each time step, he replaces one of the vectors with their sum. Find the cotangent of the minimum possible angle between the vectors after 8 time steps.

Proposed by: Allen Liu

Answer: $\boxed{987}$

Say that the vectors Kevin has at some step are (a, b) and (c, d). Notice that regardless of which vector he replaces with $(a+c, b+d)$, the area of the triangle with vertices $(0, 0)$, (a, b), and (c, d) is preserved with the new coordinates. We can see this geometrically: the parallelogram with vertices $(0, 0)$, (a, b), (c, d), and $(a + c, b + d)$ can be cut in half by looking at the triangle formed by any 3 of the vertices, which include the original triangle, and both possible triangles that might arise in the next step.

Because the area is preserved, the minimum possible angle then arises when the two vectors, our sides, are as long as possible. This occurs when we alternate which vector is getting replaced for the sum. Given two vectors (a, b) and (c, d), with $\sqrt{a^2 + b^2} > \sqrt{c^2 + d^2}$, we would rather replace (c, d) than (a, b), and $(a + c, b + d)$ has a larger norm than (a, b). Then at the nth step, Kevin has the vectors (F_n, F_{n-1}) and (F_{n+1}, F_n), where $F_0 = 0$ and $F_1 = 1$. The tangent of the angle between them is the tangent of the difference of the angles they make with the x-axis, which is just their slope. We can then compute the cotangent as

$$\left| \frac{1 + \frac{F_{n-1}}{F_n} \cdot \frac{F_n}{F_{n+1}}}{\frac{F_n}{F_{n+1}} - \frac{F_{n-1}}{F_n}} \right| = \left| \frac{F_n(F_{n+1} + F_{n-1})}{F_n^2 - F_{n-1}F_{n+1}} \right|.$$

We can show (by induction) that $F_n(F_{n+1} + F_{n-1}) = F_{2n}$ and $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$. Thus at the 8th step, the cotangent of the angle is $F_{16} = 987$.

24. [**12**] Find the largest positive integer n for which there exist n finite sets $X_1, X_2, \ldots, X_n$ with the property that for every $1 \le a < b < c \le n$, the equation

$$|X_a \cup X_b \cup X_c| = \left\lceil \sqrt{abc} \right\rceil$$

holds.

Proposed by: Pakawut Jiradilok

Answer: $\boxed{4}$

First, we construct an example for $N = 4$. Let X_1, X_2, X_3, X_4 be pairwise disjoint sets such that $X_1 = \varnothing$, $|X_2| = 1$, $|X_3| = 2$, and $|X_4| = 2$. It is straightforward to verify the condition.

We claim that there are no five sets $X_1, X_2, \ldots, X_5$ for which $\#(X_a \cup X_b \cup X_c) = \left\lceil \sqrt{abc} \right\rceil$, for $1 \le a < b < c \le 5$. Note that showing the non-existence of five such sets implies that there are no n sets with the desired property for $n \ge 5$ as well.

Suppose, for sake of contradiction, that there are such $X_1, \ldots, X_5$. Then, note that $|X_1 \cup X_2 \cup X_4| = 3$, $|X_1 \cup X_2 \cup X_5| = 4$, and $|X_2 \cup X_4 \cup X_5| = 7$. Note that

$$|X_1 \cup X_2 \cup X_4| + |X_1 \cup X_2 \cup X_5| = |X_2 \cup X_4 \cup X_5|.$$

For any sets A, B, C, D, we have the following two inequalities:

$$|A \cup B \cup C| + |A \cup B \cup D| \geq |A \cup B \cup C \cup D| \geq |B \cup C \cup D|.$$

For $A = X_1$, $B = X_2$, $C = X_4$, and $D = X_5$ in the situation above, we conclude that the equalities must both hold in both inequalities. The first equality shows that $X_1 \cup X_2 = \varnothing$, and therefore both X_1 and X_2 are empty.

Now observe that $|X_1 \cup X_4 \cup X_5| = 5 \neq 7 = |X_2 \cup X_4 \cup X_5|$. This gives a contradiction.

25. [**15**] Fran writes the numbers $1, 2, 3, \ldots, 20$ on a chalkboard. Then she erases all the numbers by making a series of moves; in each move, she chooses a number n uniformly at random from the set of all numbers still on the chalkboard, and then erases all of the divisors of n that are still on the chalkboard (including n itself). What is the expected number of moves that Fran must make to erase all the numbers?

Proposed by: Michael Tang

Answer: $\boxed{\frac{131}{10}}$

For each n, $1 \leq n \leq 20$, consider the first time that Fran chooses one of the multiples of n. It is in this move that n is erased, and all the multiples of n at most 20 are equally likely to be chosen for this move. Hence this is the only move in which Fran could possibly choose n; since there are $\lfloor 20/n \rfloor$ multiples of n at most 20, this means that the probability that n is ever chosen is $1/\lfloor 20/n \rfloor$. Therefore the expected number of moves is

$$\begin{aligned} E &= \sum_{n=1}^{20} \frac{1}{\lfloor 20/n \rfloor} \\ &= \frac{1}{20} + \frac{1}{10} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + 4\left(\frac{1}{2}\right) + 10\,(1) = \frac{131}{10}. \end{aligned}$$

(This sum is easier to compute than it may seem, if one notes that $1/20 + 1/5 + 1/4 = 1/2$ and $1/6 + 1/3 = 1/2$).

26. [**15**] Let ABC be a triangle with $\angle A = 18°, \angle B = 36°$. Let M be the midpoint of AB, D a point on ray CM such that $AB = AD$; E a point on ray BC such that $AB = BE$, and F a point on ray AC such that $AB = AF$. Find $\angle FDE$.

Proposed by: Faraz Masroor

Answer: $\boxed{27}$

Let $\angle ABD = \angle ADB = x$, and $\angle DAB = 180 - 2x$. In triangle ACD, by the law of sines, $CD = \frac{AD}{\sin \angle ACM} \cdot \sin 198 - 2x$, and by the law of sines in triangle BCD, $CD = \frac{BD}{\sin \angle BCM} \cdot \sin x + 36$. Combining the two, we have $2\cos x = \frac{BD}{AD} = \frac{\sin 198-2x}{\sin x+36} \cdot \frac{\sin \angle BCM}{\sin \angle ACM}$. But by the ratio lemma, $1 = \frac{MB}{MA} = \frac{CB}{CA} \frac{\sin \angle BCM}{\sin \angle ACM}$, meaning that $\frac{\sin \angle BCM}{\sin \angle ACM} = \frac{CA}{CB} = \frac{\sin 36}{\sin 18} = 2\cos 18$. Plugging this in and simplifying, we have $2\cos x = \frac{\sin 198-2x}{\sin x+36} \cdot 2\cos 18 = \frac{\cos 108-2x}{\cos 54-x} \cdot 2\cos 18$, so that $\frac{\cos x}{\cos 18} = \frac{\cos 108-2x}{\cos 54-x}$. We see that $x = 36°$ is a solution to this equation, and by carefully making rough sketches of both functions, we can convince ourselves that this is the only solution where x is between 0 and 90 degrees. Therefore $\angle ABD = \angle ADB = 36$, $\angle DAB = 108$. Simple angle chasing yields $\angle AEB = 72, \angle ECA = 54, \angle EAC = 54, \angle EAB = 72$, making D, A, and E collinear, and so $\angle BDE = 36$. And because $AF = AB = AD$, $\angle FDB = 1/2\angle FAB = 9$, so $\angle FDE = 36 - 9 = 27$.

27. [**15**] There are 2018 frogs in a pool and there is 1 frog on the shore. In each time-step thereafter, one random frog moves position. If it was in the pool, it jumps to the shore, and vice versa. Find the expected number of time-steps before all frogs are in the pool for the first time.

Proposed by: Dhruv Rohatgi

Answer: $\boxed{2^{2018} - 1}$

Consider the general case of n frogs. Let E_i be the expected time for all frogs to enter the pool when i frogs are on the shore and $n-i$ frogs are in the pool. We have $E_0 = 0$, $E_n = 1 + E_{n-1}$, and

$$E_i = \frac{i}{n}E_{i-1} + \frac{n-i}{n}E_{i+1} + 1$$

for $0 < i < n$. Define f_i so that

$$E_i = \frac{f_i}{(n-1)(n-2)\cdots(i)} + E_{i-1}.$$

Then by plugging this equation into the first equation, we can show that

$$f_i = n(n-1)\cdots(i+1) + (n-i)f_{i+1}.$$

Furthermore, we know that $f_n = 1$. Therefore

$$\begin{aligned} f_1 &= \sum_{i=1}^{n} \frac{n!}{i!}\frac{(n-1)!}{(n-i)!} \\ &= (n-1)!\sum_{i=1}^{n}\binom{n}{i} \\ &= (n-1)!(2^n-1). \end{aligned}$$

Therefore

$$E_1 = \frac{(n-1)!(2^n-1)}{(n-1)!} + E_0 = 2^n - 1.$$

Plugging in $n = 2018$ yields $E_1 = 2^{2018} - 1$.

28. [**15**] Arnold and Kevin are playing a game in which Kevin picks an integer $1 \leq m \leq 1001$, and Arnold is trying to guess it. On each turn, Arnold first pays Kevin 1 dollar in order to guess a number k of Arnold's choice. If $m \geq k$, the game ends and he pays Kevin an additional $m-k$ dollars (possibly zero). Otherwise, Arnold pays Kevin an additional 10 dollars and continues guessing.

Which number should Arnold guess first to ensure that his worst-case payment is minimized?

Proposed by:

Answer: $\boxed{859}$

We let $f(n)$ denote the smallest amount we can guarantee to pay at most if Arnold's first choice is n. For each $k < n$, if Arnold's first choice is $k+1$, in both worst case scenarios, he could end up paying either $n-k$ or $11 + f(k)$. It is then clear that $f(n) = \min_{k+1<n} \max\{n-k, 11+f(k)\}$.

Now clearly $f(k)$ is a non-decreasing function of k, and $n-k$ is a strictly decreasing function of k. Therefore if there exists k such that $n-k = 11+f(k)$, we have $f(n) = n-k = 11+f(k)$ with picking $k+1$ as an optimal play (and picking $K+1$ also optimal iff $K \geq k$ and $f(K) = f(k)$.

Now note that $f(k) = k$ for $k \leq 12$ (but $f(13) = 12$ though it's not relevant to the solution). Let $a_1 = 11$. Now recursively define a_i such that $a_i - a_{i-1} = 11 + f(a_{i-1})$. Thus $f(a_i) = a_i - a_{i-1}$ with the optimal move to pick $a_{i-1}+1$.

$a_1 = 11$

$a_2 - 11 = 11 + 11 : a_2 = 33, f(a_2) = 22$

$a_3 - 33 = 11 + f(33) : a_3 = 66, f(a_3) = 33$

It is clear by induction that a_i is 11 times the ith triangular number. 1001 is $11 \times 91 = \frac{14\times13}{2}$, so the optimal strategy is to pick 1 more than $11 \times \frac{12\times13}{2} = 858$. So the answer is 859.

29. [**17**] Let a, b, c be positive integers. All the roots of each of the quadratics

$$ax^2+bx+c, ax^2+bx-c, ax^2-bx+c, ax^2-bx-c$$

are integers. Over all triples (a,b,c), find the triple with the third smallest value of $a+b+c$.

Proposed by: Henrik Boecken

Answer: $\boxed{(1,10,24)}$

The quadratic formula yields that the answers to these four quadratics are $\frac{\pm b\pm\sqrt{b^2\pm 4ac}}{2a}$. Given that all eight of these expressions are integers, we can add or subtract appropriate pairs to get that $\frac{b}{a}$ and $\frac{\sqrt{b^2\pm 4ac}}{a}$ are integers. Let $b'=\frac{b}{a}$ and $c'=\frac{4c}{a}$. We can rewrite the expressions to get that b' and $\sqrt{b'^2\pm c'}$ are positive integers, which also tells us that c' is a positive integer. Let $b'^2+c'=n^2$, $b'^2-c'=m^2$.

Notice that $a+b+c=a(1+b'+\frac{c'}{4})$, so to find the third smallest value of $a+b+c$, we first find small solutions to (b',c'). To do this, we find triples (m,b',n) such that m^2,b'^2,n^2 form an arithmetic sequence. Because odd squares are 1 mod 4 and even squares are 0 mod 4, if any of these three terms is odd, then all three terms must be odd. By dividing these terms by the largest possible power of 2 then applying the same logic, we can extend our result to conclude that $v_2(m)=v_2(b')=v_2(n)$. Thus, we only need to look at (m,b',n) all odd, then multiply them by powers of 2 to get even solutions.

We then plug in $b'=3,5,7,9$, and find that out of these options, only $(n,b',m)=(1,5,7)$ works, giving $(b',c')=(5,24)$, $a+b+c=12a$. Multiplying by 2 yields that $(n,b',m)=(2,10,14)$ also works, giving $(b',c')=(10,96)$, $a+b+c=35a$. For $11\le b\le 17$, we can check that $m=b+2$ fails to give an integer n. For $11\le b\le 17$, $m\neq b+2$, $a+b+c=a(1+b'+\frac{c'}{4})\ge a(1+11+\frac{15^2-11^2}{4})=38a$, the smallest possible value of which is greater than $12a$ with $a=1$, $12a$ with $a=2$, and $35a$ with $a=1$. Thus, it cannot correspond to the solution with the third smallest $a+b+c$. For $b\ge 19$, $a+b+c=a(1+b'+\frac{c'}{4})\ge a(1+19+\frac{21^2+19^2}{4})=40a$, which, similar as before, can't correspond to the solution with the third smallest $a+b+c$.

Thus the smallest solution is $(a,b',c')=(1,5,24),(a,b,c)=(1,5,6)$, the second smallest solution is $(a,b',c')=(2,5,24),(a,b,c)=(2,10,12)$, and the third smallest solution that the problem asks for is $(a,b',c')=(1,10,96),(a,b,c)=(1,10,24)$.

30. [**17**] Find the number of *unordered* pairs $\{a,b\}$, where $a,b\in\{0,1,2,\ldots,108\}$ such that 109 divides a^3+b^3-ab.

Proposed by: Henrik Boecken

Answer: $\boxed{54}$

We start with the equation

$$a^3+b^3\equiv ab \pmod{109}.$$

If either a or b are 0, then we get $a^3\equiv 0$, implying that both are 0. Thus, $\{0,0\}$ is a pair. For the rest of the problem, let's assume that neither a nor b are 0. Multiplying both sides by $a^{-1}b^{-2}$ yields

$$(ab^{-1})^2+a^{-1}b\equiv b^{-1}$$

from which we make the substitution

$$a=xy^{-1}$$
$$b=y^{-1}$$

to get the equation

$$y\equiv x^2+x^{-1}.$$

Plugging this value back into (a,b), we get that all solutions must be of the form

$$(a,b)=((x+x^{-2})^{-1},(x^2+x^{-1})^{-1}),$$

where $1 \le x \le 108$. It now suffices to find all nonzero unordered pairs $\{m, n\}$ of the form $\{x+x^{-2}, x^2+x^{-1}\}$, where $1 \le x \le 108$. There are four values of x for which $x+x^{-2} \equiv x^2+x^{-1}$, and of these values, three of them give $x+x^{-2} \equiv 0$. This is because we can re-arrange the equation at hand to get

$$x^4 - x^3 + x - 1 \equiv 0,$$

which factors into

$$(x-1)(x^3+1) \equiv 0.$$

If $x = 1$, then $\{m, n\} = \{2, 2\}$, and if $x^3 + 1 \equiv 0$ (which has three solutions: $46, 64$ and 108), then

$$\{m, n\} = \{x^{-1}(x^3+1), x^{-2}(x^3+1)\} = \{0, 0\}.$$

Therefore, we keep $x = 1$ and discard $x = 46, 64, 108$. Of the remaining 104 values of x, $m \neq n$, and neither are 0. We have to worry about collisions between distinct values of x. There are two ways a collision can occur: if there exists $x \neq y$ such that

$$(x+x^{-2}, x^2+x^{-1}) = (y+y^{-2}, y^2+y^{-1}),$$

or if there exists $x \neq y$ such that

$$(x+x^{-2}, x^2+x^{-1}) = (y^2+y^{-1}, y+y^{-2}).$$

The first case cannot occur: if $x+x^{-2} \equiv y+y^{-2}$, we have that $x^2+x^{-1} = x(x+x^{-2}) \neq y(x+x^{-2}) = y(y+y^{-2}) = y^2+y^{-1}$. As a consequence of this, the second case only occurs if $y = x^{-1}$. Therefore, the remaining 104 values of x can be partitioned into 52 pairs of (x, x^{-1}), which ends up producing 52 distinct unordered pairs $\{m, n\}$. Adding this to the $x = 1$ case and $\{0, 0\}$, we get a total of

$$52 + 1 + 1 = \boxed{54}$$

unordered pairs.

31. [**17**] In triangle ABC, $AB = 6$, $BC = 7$ and $CA = 8$. Let D, E, F be the midpoints of sides BC, AC, AB, respectively. Also let O_A, O_B, O_C be the circumcenters of triangles AFD, BDE, and CEF, respectively. Find the area of triangle $O_AO_BO_C$.

Proposed by: Henrik Boecken

Let $AB = z, BC = x, CA = y$. Let X, Y, Z, O, N be the circumcenter of AEF, BFD, CDE, ABC, DEF respectively. Note that N is the nine-point center of ABC, and X, Y, Z are the midpoints of OA, OB, OC respectively, and thus XYZ is the image of homothety of ABC with center O and ratio $\frac{1}{2}$, so this triangle has side lengths $\frac{x}{2}, \frac{y}{2}, \frac{z}{2}$. Since NX perpendicularly bisects EF, which is parallel to BC and thus YZ, we see that N is the orthocenter of XYZ. Moreover, O_1 lies on YN and O_1X is perpendicular to XY.

To compute the area of $O_1O_2O_3$, it suffices to compute $[NO_1O_2] + [NO_2O_3] + [NO_3O_1]$. Note that O_1X is parallel to NO_2, and O_2Y is parallel to XN, so $[NO_1O_2] = [NXO_2] = [NXY]$. Similarly the other two triangles have equal area as $[NYZ]$ and $[NZX]$ respectively, so the desired area is simply the area of $[XYZ]$, which is

$$\frac{1}{4}\frac{\sqrt{(x+y+z)(x+y-z)(x-y+z)(-x+y+z)}}{4} = \frac{\sqrt{21 \cdot 9 \cdot 5 \cdot 7}}{16} = \frac{21\sqrt{15}}{16}.$$

32. [**17**] How many 48-tuples of positive integers $(a_1, a_2, \ldots, a_{48})$ between 0 and 100 inclusive have the property that for all $1 \le i < j \le 48$, $a_i \notin \{a_j, a_j + 1\}$?

Proposed by: Mehtaab Sawhney

Answer: $\boxed{54^{48}}$

(With Ashwin Sah) The key idea is write the elements of the sequence in increasing order. These sets are in bijection with solutions to $d_1 + \ldots + d_k = 48$ and $a_1 + \ldots + a_{k+1} = 53$ with $d_i \ge 1$, $a_i \ge 1$ for

$2 \leq I \leq k$, and $a_1, a_{k+1} \geq 0$. Notice that there are $\binom{54}{k}$ solutions to the second equation and then there are $\frac{48!}{d_1! \cdots d_k!}$ solutions for each $\{d_i\}$ set. Then this gives that the answer is

$$\begin{aligned}
&\sum_{1 \leq k \leq 48} \binom{54}{k} \sum_{d_1 + \ldots + d_k = 48} \frac{48!}{\prod_{i=1}^{k} d_i!} \\
&= 48![x^{48}] \sum_{1 \leq k \leq 48} (e^x - 1)^k \binom{54}{k} \\
&= 48![x^{48}] \sum_{0 \leq k \leq 54} (e^x - 1)^k \binom{54}{k} \\
&= 48! \, [x^{48}] \, (e^x)^{54} \\
&= 54^{48}.
\end{aligned}$$

33. [**20**] 679 contestants participated in HMMT February 2017. Let N be the number of these contestants who performed at or above the median score in at least one of the three individual tests. Estimate N.

    An estimate of E earns $\left\lfloor 20 - \frac{|E-N|}{2} \right\rfloor$ or 0 points, whichever is greater.

    Proposed by: Henrik Boecken

    Answer: $\boxed{516}$

    Out of the 679 total contestants at HMMT February 2017, 188 contestants scored at least the median on all three tests, 159 contestants scored at least the median on two tests, and 169 contestants scored at least the median on one test, giving a total of 516 contestants

34. [**20**] The integers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are written on a blackboard. Each day, a teacher chooses one of the integers uniformly at random and decreases it by 1. Let X be the expected value of the number of days which elapse before there are no longer positive integers on the board. Estimate X.

    An estimate of E earns $\lfloor 20 \cdot 2^{-|X-E|/8} \rfloor$ points.

    Proposed by: Mehtaab Sawhney

    Answer: $\boxed{120.75280458176904}$

    29673761980310360290435187545023378925682165679114151937768966599764336431473191952862689256885663426948

35. [**20**] In a wooden block shaped like a cube, all the vertices and edge midpoints are marked. The cube is cut along *all* possible planes that pass through at least four marked points. Let N be the number of pieces the cube is cut into. Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N) \rfloor$ points.

    Proposed by: Yuan Yao

    Answer: $\boxed{15600}$

36. [**20**] In the game of Connect Four, there are seven vertical columns which have spaces for six tokens. These form a 7×6 grid of spaces. Two players White and Black move alternately. A player takes a turn by picking a column which is not already full and dropping a token of their color into the lowest unoccupied space in that column. The game ends when there are four consecutive tokens of the same color in a line, either horizontally, vertically, or diagonally. The player who has four tokens in a row of their color wins.

    Assume two players play this game randomly. Each player, on their turn, picks a random column which is not full and drops a token of their color into that column. This happens until one player wins or all of the columns are filled. Let P be the probability that all of the columns are filled without any player obtaining four tokens in a row of their color. Estimate P.

    An estimate of $E > 0$ earns $\lfloor 20 \min(P/E, E/P) \rfloor$ points.

    Proposed by: Allen Liu

    Answer: $\boxed{0.0025632817}$