February 2017

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Combinatorics

1. Kelvin the Frog is going to roll three fair ten-sided dice with faces labelled $0, 1, 2, \ldots, 9$. First he rolls two dice, and finds the sum of the two rolls. Then he rolls the third die. What is the probability that the sum of the first two rolls equals the third roll?

2. How many ways are there to insert +'s between the digits of 111111111111111 (fifteen 1's) so that the result will be a multiple of 30?

3. There are 2017 jars in a row on a table, initially empty. Each day, a nice man picks ten consecutive jars and deposits one coin in each of the ten jars. Later, Kelvin the Frog comes back to see that N of the jars all contain the same positive integer number of coins (i.e. there is an integer $d > 0$ such that N of the jars have exactly d coins). What is the maximum possible value of N?

4. Sam spends his days walking around the following 2×2 grid of squares.

1	2
4	3

   Say that two squares are adjacent if they share a side. He starts at the square labeled 1 and every second walks to an adjacent square. How many paths can Sam take so that the sum of the numbers on every square he visits in his path is equal to 20 (not counting the square he started on)?

5. Kelvin the Frog likes numbers whose digits strictly decrease, but numbers that violate this condition in at most one place are good enough. In other words, if d_i denotes the ith digit, then $d_i \leq d_{i+1}$ for at most one value of i. For example, Kelvin likes the numbers 43210, 132, and 3, but not the numbers 1337 and 123. How many 5-digit numbers does Kelvin like?

6. Emily starts with an empty bucket. Every second, she either adds a stone to the bucket or removes a stone from the bucket, each with probability $\frac{1}{2}$. If she wants to remove a stone from the bucket and the bucket is currently empty, she merely does nothing for that second (still with probability $\frac{1}{2}$). What is the probability that after 2017 seconds her bucket contains exactly 1337 stones?

7. There are 2017 frogs and 2017 toads in a room. Each frog is friends with exactly 2 distinct toads. Let N be the number of ways to pair every frog with a toad who is its friend, so that no toad is paired with more than one frog. Let D be the number of distinct possible values of N, and let S be the sum of all possible values of N. Find the ordered pair (D, S).

8. Kelvin and 15 other frogs are in a meeting, for a total of 16 frogs. During the meeting, each pair of distinct frogs becomes friends with probability $\frac{1}{2}$. Kelvin thinks the situation after the meeting is *cool* if for each of the 16 frogs, the number of friends they made during the meeting is a multiple of 4. Say that the probability of the situation being cool can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime. Find a.

9. Let m be a positive integer, and let T denote the set of all subsets of $\{1, 2, \ldots, m\}$. Call a subset S of T δ-*good* if for all $s_1, s_2 \in S$, $s_1 \neq s_2$, $|\Delta(s_1, s_2)| \geq \delta m$, where Δ denotes symmetric difference (the symmetric difference of two sets is the set of elements that is in exactly one of the two sets). Find the largest possible integer s such that there exists an integer m and a $\frac{1024}{2047}$-good set of size s.

10. Compute the number of possible words $w = w_1 w_2 \ldots w_{100}$ satisfying:
    - w has exactly 50 A's and 50 B's (and no other letters).
    - For $i = 1, 2, \ldots, 100$, the number of A's among $w_1, w_2, \ldots, w_i$ is at most the number of B's among $w_1, w_2, \ldots, w_i$.
    - For all $i = 44, 45, \ldots, 57$, if w_i is an B, then w_{i+1} must be an B.