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1. **[5]** The formula to convert Celsius to Fahrenheit is

$$F^\circ = 1.8 \cdot C^\circ + 32.$$

In Celsius, it is 10° warmer in New York right now than in Boston. In Fahrenheit, how much warmer is it in New York than in Boston?

2. **[5]** Compute the number of dates in the year 2023 such that when put in MM/DD/YY form, the three numbers are in strictly increasing order.

For example, 06/18/23 is such a date since $6 < 18 < 23$, while today, 11/11/23, is not.

3. **[5]** Let $ABCD$ be a rectangle with $AB = 20$ and $AD = 23$. Let M be the midpoint of CD, and let X be the reflection of M across point A. Compute the area of triangle XBD.

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4. **[6]** The number 5.6 may be expressed uniquely (ignoring order) as a product $\underline{a}.\underline{b} \times \underline{c}.\underline{d}$ for digits a, b, c, d all nonzero. Compute $\underline{a}.\underline{b} + \underline{c}.\underline{d}$.

5. **[6]** Let $ABCDE$ be a convex pentagon such that

$$\begin{aligned} AB + BC + CD + DE + EA &= 64 \text{ and} \\ AC + CE + EB + BD + DA &= 72. \end{aligned}$$

Compute the perimeter of the convex pentagon whose vertices are the midpoints of the sides of $ABCDE$.

6. **[6]** There are five people in a room. They each simultaneously pick two of the other people in the room independently and uniformly at random and point at them. Compute the probability that there exists a group of three people such that each of them is pointing at the other two in the group.

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7. **[7]** Suppose a and b be positive integers not exceeding 100 such that

$$ab = \left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)^2.$$

Compute the largest possible value of $a+b$.

8. **[7]** Six standard fair six-sided dice are rolled and arranged in a row at random. Compute the expected number of dice showing the same number as the sixth die in the row.

9. **[7]** The largest prime factor of $101\,101\,101\,101$ is a four-digit number N. Compute N.

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10. **[8]** A real number x is chosen uniformly at random from the interval $(0, 10)$. Compute the probability that $\sqrt{x}$, $\sqrt{x+7}$, and $\sqrt{10-x}$ are the side lengths of a non-degenerate triangle.

11. **[8]** Let $ABCD$ and $WXYZ$ be two squares that share the same center such that $WX \parallel AB$ and $WX < AB$. Lines CX and AB intersect at P, and lines CZ and AD intersect at Q. If points P, W, and Q are collinear, compute the ratio AB/WX.

12. **[8]** A jar contains 97 marbles that are either red, green, or blue. Neil draws two marbles from the jar without replacement and notes that the probability that they would be the same color is $\frac{5}{12}$. After Neil puts his marbles back, Jerry draws two marbles from the jar with replacement. Compute the probability that the marbles that Jerry draws are the same color.

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13. **[9]** Suppose x, y, and z are real numbers greater than 1 such that

$$\begin{aligned} x^{\log_y z} &= 2, \\ y^{\log_z x} &= 4, \text{ and} \\ z^{\log_x y} &= 8. \end{aligned}$$

Compute $\log_x y$.

14. **[9]** Suppose that point D lies on side BC of triangle ABC such that AD bisects $\angle BAC$, and let ℓ denote the line through A perpendicular to AD. If the distances from B and C to ℓ are 5 and 6, respectively, compute AD.

15. **[9]** Lucas writes two distinct positive integers on a whiteboard. He decreases the smaller number by 20 and increases the larger number by 23, only to discover the product of the two original numbers is equal to the product of the two altered numbers. Compute the minimum possible sum of the original two numbers on the board.

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16. **[10]** Compute the number of tuples $(a_0, a_1, a_2, a_3, a_4, a_5)$ of (not necessarily positive) integers such that $a_i \leq i$ for all $0 \leq i \leq 5$ and

$$a_0 + a_1 + \cdots + a_5 = 6.$$

17. **[10]** Let ABC be an equilateral triangle of side length 15. Let A_b and B_a be points on side AB, A_c and C_a be points on side AC, and B_c and C_b be points on side BC such that $\triangle AA_bA_c$, $\triangle BB_cB_a$, and $\triangle CC_aC_b$ are equilateral triangles with side lengths 3, 4, and 5, respectively. Compute the radius of the circle tangent to segments $\overline{A_bA_c}$, $\overline{B_aB_c}$, and $\overline{C_aC_b}$.

18. **[10]** Over all real numbers x and y such that

$$x^3 = 3x + y \qquad \text{and} \qquad y^3 = 3y + x,$$

compute the sum of all possible values of $x^2 + y^2$.

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19. **[11]** Suppose a, b, and c be real numbers such that

$$\begin{aligned} a^2 - bc &= \quad 14, \\ b^2 - ca &= \quad 14, \text{ and} \\ c^2 - ab &= -3. \end{aligned}$$

Compute $|a + b + c|$.

20. **[11]** Let $ABCD$ be a square of side length 10. Point E is on ray $\overrightarrow{AB}$ such that $AE = 17$, and point F is on ray $\overrightarrow{AD}$ such that $AF = 14$. The line through B parallel to CE and the line through D parallel to CF meet at P. Compute the area of quadrilateral $AEPF$.

21. **[11]** An integer n is chosen uniformly at random from the set $\{1, 2, 3, \ldots, 2023!\}$. Compute the probability that

$$\gcd(n^n + 50, n + 1) = 1.$$

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22. **[12]** There is a 6×6 grid of lights. There is a switch at the top of each column and on the left of each row. A light will only turn on if the switches corresponding to both its column and its row are in the "on" position. Compute the number of different configurations of lights.

23. **[12]** The points $A = (4, \frac{1}{4})$ and $B = (-5, -\frac{1}{5})$ lie on the hyperbola $xy = 1$. The circle with diameter AB intersects this hyperbola again at points X and Y. Compute XY.

24. **[12]** Compute the smallest positive integer k such that 49 divides $\binom{2k}{k}$.

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25. **[13]** A right triangle and a circle are drawn such that the circle is tangent to the legs of the right triangle. The circle cuts the hypotenuse into three segments of lengths 1, 24, and 3, and the segment of length 24 is a chord of the circle. Compute the area of the triangle.

26. **[13]** Compute the smallest multiple of 63 with an odd number of ones in its base two representation.

27. **[13]** Compute the number of ways to color the vertices of a regular heptagon red, green, or blue (with rotations and reflections distinct) such that no isosceles triangle whose vertices are vertices of the heptagon has all three vertices the same color.

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28. **[15]** There is a unique quadruple of positive integers (a, b, c, k) such that c is not a perfect square and $a+\sqrt{b+\sqrt{c}}$ is a root of the polynomial $x^4-20x^3+108x^2-kx+9$. Compute c.

29. **[15]** Let $A_1A_2\dots A_6$ be a regular hexagon with side length $11\sqrt{3}$, and let $B_1B_2\dots B_6$ be another regular hexagon completely inside $A_1A_2\dots A_6$ such that for all $i \in \{1, 2, \dots, 5\}$, A_iA_{i+1} is parallel to B_iB_{i+1}. Suppose that the distance between lines A_1A_2 and B_1B_2 is 7, the distance between lines A_2A_3 and B_2B_3 is 3, and the distance between lines A_3A_4 and B_3B_4 is 8. Compute the side length of $B_1B_2\dots B_6$.

30. **[15]** An HMMT party has m MIT students and h Harvard students for some positive integers m and h, For every pair of people at the party, they are either friends or enemies. If every MIT student has 16 MIT friends and 8 Harvard friends, and every Harvard student has 7 MIT enemies and 10 Harvard enemies, compute how many pairs of friends there are at the party.

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31. **[17]** Let $s(n)$ denote the sum of the digits (in base ten) of a positive integer n. Compute the number of positive integers n at most 10^4 that satisfy

$$s(11n) = 2s(n).$$

32. **[17]** Compute

$$\sum_{\substack{a+b+c=12 \\ a\geq 6,\, b,c\geq 0}} \frac{a!}{b!c!(a-b-c)!},$$

where the sum runs over all triples of nonnegative integers (a, b, c) such that $a + b + c = 12$ and $a \geq 6$.

33. **[17]** Let ω_1 and ω_2 be two non-intersecting circles. Suppose the following three conditions hold:

- The length of a common internal tangent of ω_1 and ω_2 is equal to 19.
- The length of a common external tangent of ω_1 and ω_2 is equal to 37.
- If two points X and Y are selected on ω_1 and ω_2, respectively, uniformly at random, then the expected value of XY^2 is 2023.

Compute the distance between the centers of ω_1 and ω_2.

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34. **[20]** Compute the smallest positive integer that does not appear in any problem statement on any round at HMMT November 2023.

Submit a positive integer A. If the correct answer is C, you will receive $\max(0, 20 - 5|A - C|)$ points.

35. **[20]** Dorothea has a 3×4 grid of dots. She colors each dot red, blue, or dark gray. Compute the number of ways Dorothea can color the grid such that there is no rectangle whose sides are parallel to the grid lines and whose vertices all have the same color.

Submit a positive integer A. If the correct answer is C and your answer is A, you will receive $\left\lfloor 20 \left(\min\left(\frac{A}{C}, \frac{C}{A}\right)\right)^2 \right\rfloor$ points.

36. **[20]** Isabella writes the expression $\sqrt{d}$ for each positive integer d not exceeding $8!$ on the board. Seeing that these expressions might not be worth points on HMMT, Vidur simplifies each expression to the form $a\sqrt{b}$, where a and b are integers such that b is not divisible by the square of a prime number. (For example, $\sqrt{20}$, $\sqrt{16}$, and $\sqrt{6}$ simplify to $2\sqrt{5}$, $4\sqrt{1}$, and $1\sqrt{6}$, respectively.) Compute the sum of $a + b$ across all expressions that Vidur writes.

Submit a positive real number A. If the correct answer is C and your answer is A, you get $\max\left(0, \left\lceil 20\left(1 - |\log(A/C)|^{1/5}\right)\right\rceil\right)$ points.