HMMT February 2020

February 15, 2020

Guts Round

1. **[4]** Submit an integer x as your answer to this problem. The number of points you receive will be $\max(0, 8 - |8x - 100|)$. (Non-integer answers will be given 0 points.)

   Proposed by: Andrew Gu, Andrew Lin

   Answer: $\boxed{12 \text{ or } 13}$

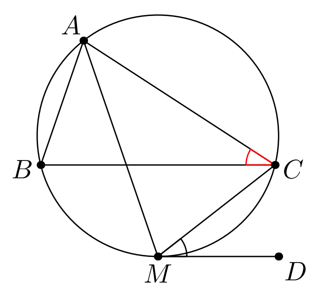
   Solution: We want to minimize $|8x-100|$, so x should equal either the floor or the ceiling of $\frac{100}{8} = 12.5$. Note that no other answers receive any points, while both 12 and 13 receive 4 points.

2. **[4]** Let ABC be a triangle and ω be its circumcircle. The point M is the midpoint of arc BC not containing A on ω and D is chosen so that DM is tangent to ω and is on the same side of AM as C. It is given that $AM = AC$ and $\angle DMC = 38^\circ$. Find the measure of angle $\angle ACB$.

   Proposed by: Joseph Heerens

   Answer: $\boxed{33^\circ}$

   Solution: By inscribed angles, we know that $\angle BAC = 38^\circ \cdot 2 = 76^\circ$ which means that $\angle C = 104^\circ - \angle B$. Since $AM = AC$, we have $\angle ACM = \angle AMC = 90^\circ - \frac{\angle MAC}{2} = 71^\circ$. Once again by inscribed angles, this means that $\angle B = 71^\circ$ which gives $\angle C = 33^\circ$.



3. **[4]** Let ABC be a triangle and D, E, and F be the midpoints of sides BC, CA, and AB respectively. What is the maximum number of circles which pass through at least 3 of these 6 points?

   Proposed by: Andrew Gu

   Answer: $\boxed{17}$

   Solution: All $\binom{6}{3} = 20$ triples of points can produce distinct circles aside from the case where the three points are collinear (BDC, CEA, AFB).

4. **[4]** Compute the value of $\sqrt{105^3 - 104^3}$, given that it is a positive integer.

   Proposed by: Andrew Gu

   Answer: $\boxed{181}$

   Solution 1: First compute $105^3 - 104^3 = 105^2 + 105 \cdot 104 + 104^2 = 3 \cdot 105 \cdot 104 + 1 = 32761$. Note that $180^2 = 32400$, so $181^2 = 180^2 + 2 \cdot 180 + 1 = 32761$ as desired.

Solution 2: We have $105^3 - 104^3 = 105^2 + 105 \cdot 104 + 104^2$. Thus

$$104\sqrt{3} < \sqrt{105^3 - 104^3} < 105\sqrt{3}.$$

Now estimating gives the bounds $180 < 104\sqrt{3}$ and $105\sqrt{3} < 182$. So the answer is 181.

5. **[5]** Alice, Bob, and Charlie roll a 4, 5, and 6-sided die, respectively. What is the probability that a number comes up exactly twice out of the three rolls?

   Proposed by: Andrew Lin

   Answer: $\boxed{\frac{13}{30}}$

   Solution 1: There are $4 \cdot 5 \cdot 6 = 120$ different ways that the dice can come up. The common number can be any of 1, 2, 3, 4, or 5: there are $3+4+5 = 12$ ways for it to be each of 1, 2, 3, or 4, because we pick one of the three people's rolls to disagree, and there are 3, 4, and 5 ways that roll can come up (for Alice, Bob, and Charlie respectively). Finally, there are 4 ways for Bob and Charlie to both roll a 5 and Alice to roll any number. Thus there are 52 different ways to satisfy the problem condition, and our answer is $\frac{52}{120} = \frac{13}{30}$.

   Solution 2: If Bob rolls the same as Alice, Charlie must roll a different number. Otherwise Charlie must roll the same as either Alice or Bob. So the answer is

$$\frac{1}{5} \cdot \frac{5}{6} + \frac{4}{5} \cdot \frac{2}{6} = \frac{13}{30}.$$

   Solution 3: By complementary counting, the answer is

$$1 - \frac{1}{5} \cdot \frac{1}{6} - \frac{4}{5} \cdot \frac{4}{6} = \frac{13}{30}.$$

   The first and second products correspond to rolling the same number three times and rolling three distinct numbers, respectively.

6. **[5]** Two sides of a regular n-gon are extended to meet at a 28° angle. What is the smallest possible value for n?

   Proposed by: James Lin

   Answer: $\boxed{45}$

   Solution: We note that if we inscribe the n-gon in a circle, then according to the inscribed angle theorem, the angle between two sides is $\frac{1}{2}$ times some $x - y$, where x and y are integer multiples of the arc measure of one side of the n-gon. Thus, the angle is equal to $\frac{1}{2}$ times an integer multiple of $\frac{360}{n}$, so $\frac{1}{2} \cdot k \cdot \frac{360}{n} = 28$ for some integer k. Simplifying gives $7n = 45k$, and since all k are clearly attainable, the smallest possible value of n is 45.

7. **[5]** Ana and Banana are rolling a standard six-sided die. Ana rolls the die twice, obtaining a_1 and a_2, then Banana rolls the die twice, obtaining b_1 and b_2. After Ana's two rolls but before Banana's two rolls, they compute the probability p that $a_1 b_1 + a_2 b_2$ will be a multiple of 6. What is the probability that $p = \frac{1}{6}$?

   Proposed by: James Lin

   Answer: $\boxed{\frac{2}{3}}$

Solution: If either a_1 or a_2 is relatively prime to 6, then $p = \frac{1}{6}$. If one of them is a multiple of 2 but not 6, while the other is a multiple of 3 but not 6, we also have $p = \frac{1}{6}$. In other words, $p = \frac{1}{6}$ if $\gcd(a_1, a_2)$ is coprime to 6, and otherwise $p \neq \frac{1}{6}$. The probability that $p = \frac{1}{6}$ is $\frac{(3^2-1)(2^2-1)}{6^2} = \frac{2}{3}$ where $\frac{q^2-1}{q^2}$ corresponds to the probability that at least one of a_1 and a_2 is not divisible by q for $q = 2, 3$.

8. **[5]** Tessa picks three real numbers x, y, z and computes the values of the eight expressions of the form $\pm x \pm y \pm z$. She notices that the eight values are all distinct, so she writes the expressions down in increasing order. For example, if $x = 2$, $y = 3$, $z = 4$, then the order she writes them down is

$$-x-y-z,\ +x-y-z,\ -x+y-z,\ -x-y+z,\ +x+y-z,\ +x-y+z,\ -x+y+z,\ +x+y+z.$$

How many possible orders are there?

Proposed by: Yuan Yao

Answer: $\boxed{96}$

Solution: There are $2^3 = 8$ ways to choose the sign for each of x, y, and z. Furthermore, we can order $|x|$, $|y|$, and $|z|$ in $3! = 6$ different ways. Now assume without loss of generality that $0 < x < y < z$. Then there are only two possible orders depending on the sign of $x + y - z$:

$$-x-y-z,\ +x-y-z,\ -x+y-z,\ -x-y+z,\ x+y-z,\ x-y+z,\ -x+y+z,\ x+y+z$$

$$-x-y-z,\ +x-y-z,\ -x+y-z,\ x+y-z,\ -x-y+z,\ x-y+z,\ -x+y+z,\ x+y+z$$

Thus, the answer is $8 \cdot 6 \cdot 2 = 96$.

9. **[6]** Let $P(x)$ be the monic polynomial with rational coefficients of minimal degree such that $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \ldots, \frac{1}{\sqrt{1000}}$ are roots of P. What is the sum of the coefficients of P?

Proposed by: Carl Joshua Quines

Answer: $\boxed{\frac{1}{16000}}$

Solution: For irrational $\frac{1}{\sqrt{r}}$, $-\frac{1}{\sqrt{r}}$ must also be a root of P. Therefore

$$P(x) = \frac{\left(x^2 - \frac{1}{2}\right)\left(x^2 - \frac{1}{3}\right)\cdots\left(x^2 - \frac{1}{1000}\right)}{\left(x + \frac{1}{2}\right)\left(x + \frac{1}{3}\right)\cdots\left(x + \frac{1}{31}\right)}.$$

We get the sum of the coefficients of P by setting $x = 1$, so we use telescoping to get

$$P(1) = \frac{\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{999}{1000}}{\frac{3}{2} \cdot \frac{4}{3} \cdots \frac{32}{31}} = \frac{1}{16000}.$$

10. **[6]** Jarris is a weighted tetrahedral die with faces F_1, F_2, F_3, F_4. He tosses himself onto a table, so that the probability he lands on a given face is proportional to the area of that face (i.e. the probability he lands on face F_i is $\frac{[F_i]}{[F_1]+[F_2]+[F_3]+[F_4]}$ where $[K]$ is the area of K). Let k be the maximum distance any part of Jarris is from the table after he rolls himself. Given that Jarris has an inscribed sphere of radius 3 and circumscribed sphere of radius 10, find the minimum possible value of the expected value of k.

Proposed by: James Lin

Answer: $\boxed{12}$

Solution: Since the maximum distance to the table is just the height, the expected value is equal to $\frac{\sum_{i=1}^{4} h_i[F_i]}{\sum_{i=1}^{4} [F_i]}$. Let V be the volume of Jarris. Recall that $V = \frac{1}{3}h_i[F_i]$ for any i, but also $V = \frac{r}{3}\left(\sum_{i=1}^{4}[F_i]\right)$ where r is the inradius (by decomposing into four tetrahedra with a vertex at the incenter). Therefore

$$\frac{\sum\limits_{i=1}^{4} h_i[F_i]}{\sum\limits_{i=1}^{4} [F_i]} = \frac{12V}{3V/r} = 4r = 12.$$

11. **[6]** Find the number of ordered pairs of positive integers (x, y) with $x, y \leq 2020$ such that $3x^2 + 10xy + 3y^2$ is the power of some prime.

Proposed by: James Lin

Answer: $\boxed{29}$

Solution: We can factor as $(3x + y)(x + 3y)$. If $x \geq y$, we need $\frac{3x+y}{x+3y} \in \{1, 2\}$ to be an integer. So we get the case where $x = y$, in which we need both to be a power of 2, or the case $x = 5y$, in which case we need y to be a power of 2. This gives us $11+9+9 = 29$ solutions, where we account for $y = 5x$ as well.

12. **[6]** An 11×11 grid is labeled with consecutive rows $0, 1, 2, \ldots, 10$ and columns $0, 1, 2, \ldots, 10$ so that it is filled with integers from 1 to 2^{10}, inclusive, and the sum of all of the numbers in row n and in column n are both divisible by 2^n. Find the number of possible distinct grids.

Proposed by: Joseph Heerens

Answer: $\boxed{2^{1100}}$

Solution: We begin by filling the 10 by 10 grid formed by rows and columns 1 through 10 with any values, which we can do in $(2^{10})^{100} = 2^{1000}$ ways. Then in column 0, there is at most 1 way to fill in the square in row 10, 2 ways for the square in row 9, down to 2^{10} ways in row 0. Similarly, there is 1 way to fill in the square in row 0 and column 10, 2 ways to fill in the square in row 0 and column 9, etc. Overall, the number of ways to fill out the squares in row or column 0 is $2^1 \cdot 2^2 \cdot 2^3 \cdots 2^9 \cdot 2^{10} \cdot 2^9 \cdot 2^8 \cdots 2^1 = 2^{100}$, so the number of possible distinct grids $2^{1000} \cdot 2^{100} = 2^{1100}$.

13. **[8]** Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 1$, and $CA = 4\sqrt{3}$. The angle trisectors of C intersect $\overline{AB}$ at D and E, and lines $\overline{AC}$ and $\overline{BC}$ intersect the circumcircle of $\triangle CDE$ again at X and Y, respectively. Find the length of XY.

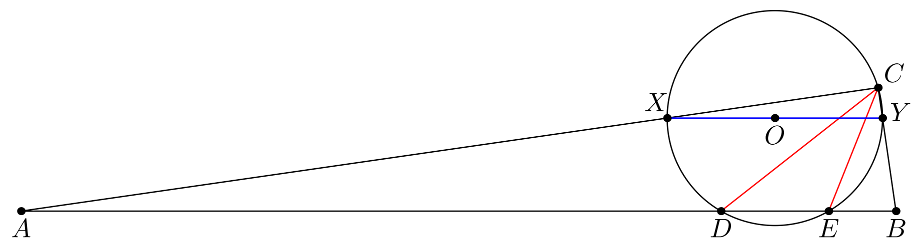
Proposed by: Hahn Lheem

Answer: $\boxed{\frac{112}{65}}$

Solution: Let O be the cirumcenter of $\triangle CDE$. Observe that $\triangle ABC \sim \triangle XYC$. Moreover, $\triangle ABC$ is a right triangle because $1^2 + (4\sqrt{3})^2 = 7^2$, so the length XY is just equal to $2r$, where r is the radius of the circumcircle of $\triangle CDE$. Since D and E are on the angle trisectors of angle C, we see that $\triangle ODE$, $\triangle XDO$, and $\triangle YEO$ are equilateral. The length of the altitude from C to AB is $\frac{4\sqrt{3}}{7}$. The distance from C to XY is $\frac{XY}{AB} \cdot \frac{4\sqrt{3}}{7} = \frac{2r}{7} \cdot \frac{4\sqrt{3}}{7}$, while the distance between lines XY and AB is $\frac{r\sqrt{3}}{2}$. Hence we have

$$\frac{4\sqrt{3}}{7} = \frac{2r}{7} \cdot \frac{4\sqrt{3}}{7} + \frac{r\sqrt{3}}{2}.$$

Solving for r gives that $r = \frac{56}{65}$, so $XY = \frac{112}{65}$.



14. **[8]** Let $\varphi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Let S be the set of positive integers n such that $\frac{2n}{\varphi(n)}$ is an integer. Compute the sum

$$\sum_{n \in S} \frac{1}{n}.$$

Proposed by: Joseph Heerens, James Lin

Answer: $\boxed{\frac{10}{3}}$

Solution: Let T_n be the set of prime factors of n. Then

$$\frac{2n}{\phi(n)} = 2 \prod_{p \in T} \frac{p}{p-1}.$$

We can check that this is an integer for the following possible sets:

$$\varnothing, \{2\}, \{3\}, \{2,3\}, \{2,5\}, \{2,3,7\}.$$

For each set T, the sum of the reciprocals of the positive integers having that set of prime factors is

$$\prod_{p \in T} \left(\sum_{m=1}^{\infty} \frac{1}{p^m} \right) = \prod_{p \in T} \frac{1}{p-1}.$$

Therefore the desired sum is

$$1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{12} = \frac{10}{3}.$$

15. **[8]** You have six blocks in a row, labeled 1 through 6, each with weight 1. Call two blocks $x \leq y$ connected when, for all $x \leq z \leq y$, block z has not been removed. While there is still at least one block remaining, you choose a remaining block uniformly at random and remove it. The cost of this operation is the sum of the weights of the blocks that are connected to the block being removed, including itself. Compute the expected total cost of removing all the blocks.

Proposed by: Benjamin Qi

Answer: $\boxed{\frac{163}{10}}$

Solution: Note that the total cost is the total number of ordered pairs (x, y) with $1 \leq x, y \leq 6$ such that x and y are connected right before x gets removed.

The probability that blocks x and y are connected just before block x is removed is simply $\frac{1}{|x-y|+1}$, since all of the $|x-y|+1$ relevant blocks are equally likely to be removed first. Summing over $1 \le x, y \le 6$, combining terms with the same value of $|x-y|$, we get

$$\frac{2}{6} + \frac{4}{5} + \frac{6}{4} + \frac{8}{3} + \frac{10}{2} + 6 = \frac{163}{10}.$$

16. **[8]** Determine all triplets of real numbers (x, y, z) satisfying the system of equations

$$\begin{aligned} x^2y + y^2z &= 1040 \\ x^2z + z^2y &= 260 \\ (x-y)(y-z)(z-x) &= -540. \end{aligned}$$

Proposed by: Krit Boonsiriseth

Answer: $\boxed{(16,4,1),\ (1,16,4)}$

Solution: Call the three equations $(1), (2), (3)$. $(1)/(2)$ gives $y = 4z$. $(3) + (1) - (2)$ gives

$$(y^2 - z^2)x = 15z^2x = 240,$$

so $z^2x = 16$. Therefore

$$\begin{aligned} z(x+2z)^2 &= x^2z + z^2y + 4z^2x = \frac{81}{5} \\ z(x-2z)^2 &= x^2z + z^2y - 4z^2x = \frac{49}{5} \end{aligned}$$

so $\left|\frac{x+2z}{x-2z}\right| = \frac{9}{7}$. Thus either $x = 16z$ or $x = \frac{z}{4}$.

If $x = 16z$, then (1) becomes $1024z^3 + 16z^3 = 1040$, so $(x, y, z) = (16, 4, 1)$.

If $x = \frac{z}{4}$, then (1) becomes $\frac{1}{4}z^3 + 16z^3 = 1040$, so $(x, y, z) = (1, 16, 4)$.

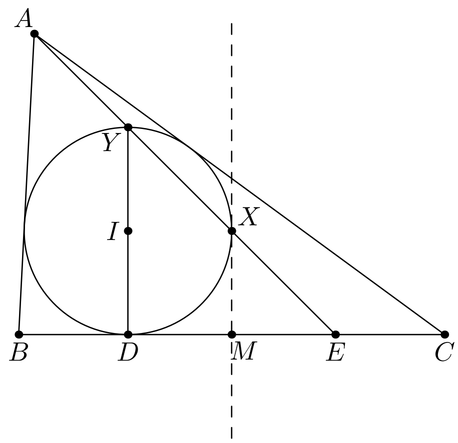
17. **[10]** Let ABC be a triangle with incircle tangent to the perpendicular bisector of BC. If $BC = AE = 20$, where E is the point where the A-excircle touches BC, then compute the area of $\triangle ABC$.

Proposed by: Tristan Shin

Answer: $\boxed{100\sqrt{2}}$

Solution: Let the incircle and BC touch at D, the incircle and perpendicular bisector touch at X, Y be the point opposite D on the incircle, and M be the midpoint of BC. Recall that A, Y, and E are collinear by homothety at A. Additionally, we have $MD = MX = ME$ so $\angle DXY = \angle DXE = 90°$. Therefore E, X, and Y are collinear. Since $MX \perp BC$, we have $\angle AEB = 45°$. The area of ABC is

$$\frac{1}{2}BC \cdot AE \cdot \sin\angle AEB = 100\sqrt{2}.$$



18. **[10]** A *vertex-induced* subgraph is a subset of the vertices of a graph together with any edges whose endpoints are both in this subset.

    An undirected graph contains 10 nodes and m edges, with no loops or multiple edges. What is the minimum possible value of m such that this graph must contain a nonempty vertex-induced subgraph where all vertices have degree at least 5?

    Proposed by: Benjamin Qi

    Answer: $\boxed{31}$

    Solution: Suppose that we want to find the vertex-induced subgraph of maximum size where each vertex has degree at least 5. To do so, we start with the entire graph and repeatedly remove any vertex with degree less than 5.

    If there are vertices left after this process terminates, then the subgraph induced by these vertices must have all degrees at least 5. Conversely, if there is a vertex-induced subgraph where all degrees are at least 5, then none of these vertices can be removed during the removing process. Thus, there are vertices remaining after this process if and only if such a vertex-induced subgraph exists.

    If the process ends with an empty graph, the largest possible number of edges are removed when the first 5 removed vertices all have 4 edges at the time of removal, and the last 5 vertices are all connected to each other, resulting in $5\times 4+4+3+2+1+0 = 30$ removed edges. The answer is $30+1 = 31$.

19. **[10]** The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exist unique positive integers $n_1, n_2, n_3, n_4, n_5, n_6$ such that

$$\sum_{i_1=0}^{100}\sum_{i_2=0}^{100}\sum_{i_3=0}^{100}\sum_{i_4=0}^{100}\sum_{i_5=0}^{100} F_{i_1+i_2+i_3+i_4+i_5} = F_{n_1} - 5F_{n_2} + 10F_{n_3} - 10F_{n_4} + 5F_{n_5} - F_{n_6}.$$

    Find $n_1 + n_2 + n_3 + n_4 + n_5 + n_6$.

    Proposed by: Andrew Gu

    Answer: $\boxed{1545}$

    Solution: We make use of the identity

$$\sum_{i=0}^{\ell} F_i = F_{\ell+2} - 1,$$

(easily proven by induction) which implies

$$\sum_{i=k}^{\ell} F_i = F_{\ell+2} - F_{k+1}.$$

Applying this several times yields

$$\begin{aligned}
&\sum_{i_1=0}^{100}\sum_{i_2=0}^{100}\sum_{i_3=0}^{100}\sum_{i_4=0}^{100}\sum_{i_5=0}^{100} F_{i_1+i_2+i_3+i_4+i_5} \\
=&\sum_{i_1=0}^{100}\sum_{i_2=0}^{100}\sum_{i_3=0}^{100}\sum_{i_4=0}^{100}(F_{i_1+i_2+i_3+i_4+102} - F_{i_1+i_2+i_3+i_4+1}) \\
=&\sum_{i_1=0}^{100}\sum_{i_2=0}^{100}\sum_{i_3=0}^{100}(F_{i_1+i_2+i_3+204} - 2F_{i_1+i_2+i_3+103} + F_{i_1+i_2+i_3+2}) \\
=&\sum_{i_1=0}^{100}\sum_{i_2=0}^{100}(F_{i_1+i_2+306} - 3F_{i_1+i_2+205} + 3F_{i_1+i_2+104} - F_{i_1+i_2+3}) \\
=&\sum_{i_1=0}^{100}(F_{i_1+408} - 4F_{i_1+307} + 6F_{i_1+206} - 4F_{i_1+105} + F_{i_1+4}) \\
=&F_{510} - 5F_{409} + 10F_{308} - 10F_{207} + 5F_{106} - F_5.
\end{aligned}$$

This representation is unique because the Fibonacci terms grow exponentially quickly, so e.g. the F_{510} term dominates, forcing $n_1 = 510$ and similarly for the other terms. The final answer is

$$510 + 409 + 308 + 207 + 106 + 5 = 1545.$$

20. **[10]** There exist several solutions to the equation

$$1 + \frac{\sin x}{\sin 4x} = \frac{\sin 3x}{\sin 2x},$$

where x is expressed in degrees and $0^\circ < x < 180^\circ$. Find the sum of all such solutions.

Proposed by: Benjamin Qi

Answer: $\boxed{320^\circ}$

Solution: We first apply sum-to-product and product-to-sum:

$$\frac{\sin 4x + \sin x}{\sin 4x} = \frac{\sin 3x}{\sin 2x}$$

$$2\sin(2.5x)\cos(1.5x)\sin(2x) = \sin(4x)\sin(3x)$$

Factoring out $\sin(2x) = 0$,

$$\sin(2.5x)\cos(1.5x) = \cos(2x)\sin(3x)$$

Factoring out $\cos(1.5x) = 0$ (which gives us 60° as a solution),

$$\sin(2.5x) = 2\cos(2x)\sin(1.5x)$$

$$\sin(2.5x) = \sin(3.5x) - \sin(0.5x)$$

Convert into complex numbers, we get

$$(x^{3.5} - x^{-3.5}) - (x^{0.5} - x^{-0.5}) = (x^{2.5} - x^{-2.5})$$

$$x^7 - x^6 - x^4 + x^3 + x - 1 = 0$$

$$(x-1)(x^6 - x^3 + 1) = 0$$

We recognize the latter expression as $\frac{x^9+1}{x^3+1}$, giving us

$$x = 0°, 20°, 100°, 140°, 220°, 260°, 340°.$$

The sum of the solutions is

$$20° + 60° + 100° + 140° = 320°.$$

21. **[12]** We call a positive integer t *good* if there is a sequence $a_0, a_1, \ldots$ of positive integers satisfying $a_0 = 15, a_1 = t$, and

$$a_{n-1}a_{n+1} = (a_n - 1)(a_n + 1)$$

for all positive integers n. Find the sum of all good numbers.

Proposed by: Krit Boonsiriseth

Answer: $\boxed{296}$

Solution: By the condition of the problem statement, we have

$$a_n^2 - a_{n-1}a_{n+1} = 1 = a_{n-1}^2 - a_{n-2}a_n.$$

This is equivalent to

$$\frac{a_{n-2} + a_n}{a_{n-1}} = \frac{a_{n-1} + a_{n+1}}{a_n}.$$

Let $k = \frac{a_0+a_2}{a_1}$. Then we have

$$\frac{a_{n-1} + a_{n+1}}{a_n} = \frac{a_{n-2} + a_n}{a_{n-1}} = \frac{a_{n-3} + a_{n-1}}{a_{n-2}} = \cdots = \frac{a_0 + a_2}{a_1} = k.$$

Therefore we have $a_{n+1} = ka_n - a_{n-1}$ for all $n \geq 1$. We know that k is a positive rational number because a_0, a_1, and a_2 are all positive integers. We claim that k must be an integer. Suppose that $k = \frac{p}{q}$ with $\gcd(p, q) = 1$. Since $ka_n = a_{n-1} + a_{n+1}$ is always an integer for $n \geq 1$, we must have $q \mid a_n$ for all $n \geq 1$. This contradicts $a_2^2 - a_1a_3 = 1$. Conversely, if k is an integer, inductively all a_i are integers.

Now we compute $a_2 = \frac{t^2-1}{15}$, so $k = \frac{t^2+224}{15t}$ is an integer. Therefore $15k - t = \frac{224}{t}$ is an integer. Combining with the condition that a_2 is an integer limits the possible values of t to 1, 4, 14, 16, 56, 224. The values $t < 15$ all lead to $a_n = 0$ for some n whereas $t > 15$ leads to a good sequence. The sum of the solutions is

$$16 + 56 + 224 = 296.$$

22. **[12]** Let A be a set of integers such that for each integer m, there exists an integer $a \in A$ and positive integer n such that $a^n \equiv m \pmod{100}$. What is the smallest possible value of $|A|$?

Proposed by: Daniel Zhu

Answer: $\boxed{41}$

Solution: Work in $R = \mathbb{Z}/100\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$.

Call an element $r \in R$ type (s, t) if $s = \nu_2(r) \leq 2$ and $t = \nu_5(r) \leq 2$. Also, define an element $r \in R$ to be *coprime* if it is of type $(0, 0)$, *powerful* if it is of types $(0, 2)$, $(2, 0)$, or $(2, 2)$, and *marginal* otherwise.

Then, note that if if $r \in R$ is marginal, then any power of r is powerful. Therefore all marginal elements must be in A.

We claim that all powerful elements are the cube of some marginal element. To show this take a powerful element r. In modulo 4 or 25, if r is a unit, then since 3 is coprime to both the sizes of $(\mathbb{Z}/4\mathbb{Z})^\times$ and $(\mathbb{Z}/25\mathbb{Z})^\times$, it is the cube of some element. Otherwise, if r is zero then it is the cube of 2 or 5, respectively (since this case happens at least once this means that the constructed cube root is marginal).

We now claim that 4 additional elements are needed to generate the coprime elements. To see this, note that $R^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/20\mathbb{Z}$ since there are primitive roots mod 4 and 25. Under this isomorphism, one can show that $(1, 1)$, $(1, 2)$, $(1, 4)$, and $(0, 1)$ generate anything, and that no element in $R^\times$ has more than one of these as a multiple.

To wrap up, note that there are $100 - (20 + 1)(2 + 1) = 37$ marginal elements, so 41 elements are needed in total.

23. **[12]** A function $f\colon A \to A$ is called *idempotent* if $f(f(x)) = f(x)$ for all $x \in A$. Let I_n be the number of idempotent functions from $\{1, 2, \ldots, n\}$ to itself. Compute

$$\sum_{n=1}^{\infty} \frac{I_n}{n!}.$$

Proposed by: Carl Schildkraut

Answer: $\boxed{e^e - 1}$

Solution: Let $A_{k,n}$ denote the number of idempotent functions on a set of size n with k fixed points. We have the formula

$$A_{k,n} = \binom{n}{k} k^{n-k}$$

for $1 \leq k \leq n$ because there are $\binom{n}{k}$ ways to choose the fixed points and all $n - k$ remaining elements must map to fixed points, which can happen in k^{n-k} ways. Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{I_n}{n!} &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{A_{k,n}}{n!} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{k^{n-k}}{k!(n-k)!} \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} \frac{k^{n-k}}{(n-k)!} \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{k^n}{n!} \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} e^k \\
&= e^e - 1.
\end{aligned}$$

24. **[12]** In $\triangle ABC$, ω is the circumcircle, I is the incenter and I_A is the A-excenter. Let M be the midpoint of arc $\widehat{BAC}$ on ω, and suppose that X, Y are the projections of I onto MI_A and I_A onto MI, respectively. If $\triangle XYI_A$ is an equilateral triangle with side length 1, compute the area of $\triangle ABC$.

Proposed by: Michael Diao

Answer: $\boxed{\frac{\sqrt{6}}{7}}$

Solution 1: Using Fact 5, we know that II_A intersects the circle (ABC) at M_A, which is the center of (II_ABCXY). Let R be the radius of the latter circle. We have $R = \frac{1}{\sqrt{3}}$.

We have $\angle AIM = \angle YII_A = \angle YIX = \frac{\pi}{3}$. Also, $\angle II_AM = \angle IMI_A$ by calculating the angles from the equilateral triangle. Using 90-60-30 triangles, we have:

$$AI = \frac{1}{2}MI = \frac{1}{2}II_A = R$$

$$AM = \frac{\sqrt{3}}{2}MI = \sqrt{3}R$$

$$MM_A^2 = AM^2 + AM_A^2 = 7R^2$$

Now, let J and N be the feet of the altitudes from A and B respectively on MM_A. Note that as M is an arc midpoint of BC, N is actually the midpoint of BC.

$$M_AJ = \frac{AM_A^2}{MM_A} = \frac{4}{\sqrt{7}}R$$

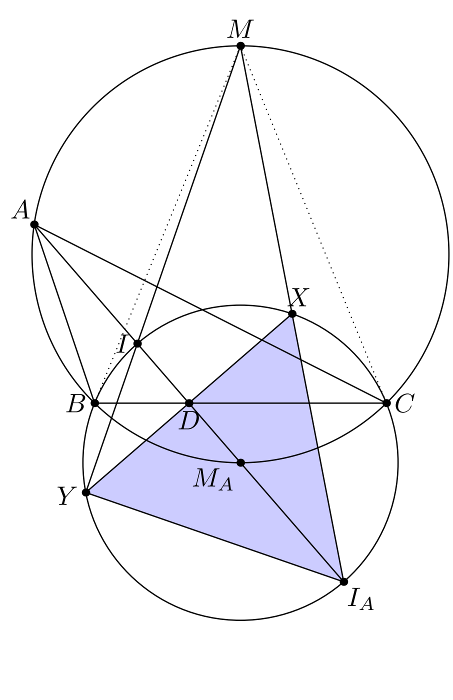
$$M_AN = \frac{BM_A^2}{MM_A} = \frac{1}{\sqrt{7}}R$$

Thus $JN = \frac{3}{\sqrt{7}}R$. Also, we have,

$$BN^2 = M_AN \cdot MN = \frac{6}{7}R^2$$

Now, $[ABC] = \frac{1}{2}JN \cdot BC = JN \cdot BN = \frac{3\sqrt{6}}{7}R^2 = \frac{\sqrt{6}}{7}$.

Solution 2: By Fact 5, we construct the diagram first with $\triangle XYI_A$ as the reference triangle.



Let M_A be the circumcenter of $\triangle XYI_A$ and let Ω be the circumcircle, which has circumradius $R = \frac{1}{\sqrt{3}}$. Then by Fact 5, M_A is the midpoint of minor arc $\widehat{BC}$, and $B, C \in \Omega$. Now we show the following result:

Claim. $b + c = 2a$.

Proof. Letting D be the intersection of II_A with BC, we have $M_AD = \frac{1}{2}R$. Then by the Shooting Lemma,

$$M_AA \cdot M_AD = R^2 \implies M_AA = 2R.$$

On the other hand, by Ptolemy,

$$M_AB \cdot AC + M_AC \cdot AB = M_AA \cdot BC \implies R \cdot (b + c) = 2R \cdot a,$$

whence the result follows. □

By the triangle area formula, we have

$$[ABC] = sr = \frac{1}{2}(a + b + c)r = \frac{3}{2}ar.$$

Therefore, we are left to compute a and r.

Since M_A is the antipode of M on (ABC) we get MB, MC are tangent to Ω; in particular, MBM_AC is a kite with $MB = MC$ and $M_AB = M_AC$. Then by Power of a Point, we get

$$MB^2 = MC^2 = MI \cdot MY = 2R \cdot 3R = 2,$$

where $MI = II_A = 2R$ and $IY = R$.

Since $ID = DM_A$, we know that r is the length of the projection from M_A to BC. Finally, given our above information, we can compute

$$r = \frac{BM_A^2}{M_AM} = \frac{BM_A^2}{\sqrt{BM_A^2 + MB^2}} = \frac{1}{\sqrt{21}}$$
$$a = 2r\frac{BM_A}{MB} = \frac{2\sqrt{2}}{7}.$$

The area of ABC is then

$$\frac{3}{2}\left(\frac{2\sqrt{2}}{\sqrt{7}}\right)\left(\frac{1}{\sqrt{21}}\right) = \frac{\sqrt{6}}{7}.$$

25. **[15]** Let S be the set of 3^4 points in four-dimensional space where each coordinate is in $\{-1, 0, 1\}$. Let N be the number of sequences of points $P_1, P_2, \ldots, P_{2020}$ in S such that $P_iP_{i+1} = 2$ for all $1 \le i \le 2020$ and $P_1 = (0, 0, 0, 0)$. (Here $P_{2021} = P_1$.) Find the largest integer n such that 2^n divides N.

Proposed by: James Lin

Answer: $\boxed{4041}$

Solution: From $(0, 0, 0, 0)$ we have to go to $(\pm 1, \pm 1, \pm 1, \pm 1)$, and from $(1, 1, 1, 1)$ (or any of the other similar points), we have to go to $(0, 0, 0, 0)$ or $(-1, 1, 1, 1)$ and its cyclic shifts. If a_i is the number of ways to go from $(1, 1, 1, 1)$ to point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ in i steps, then we need to find $\nu_2(16a_{2018})$. To find a recurrence relation for a_i, note that to get to some point in $(\pm 1, \pm 1, \pm 1, \pm 1)$, we must either come from a previous point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ or the point $(0, 0, 0, 0)$. In order to go to one point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ through $(0, 0, 0, 0)$ from the point $(\pm 1, \pm 1, \pm 1, \pm 1)$, we have one way of going to the origin and 16 ways to pick which point we go to after the origin.

Additionally, if the previous point we visit is another point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ then we have 4 possible directions to go in. Therefore the recurrence relation for a_i is $a_i = 4a_{i-1} + 16a_{i-2}$. Solving the linear recurrence yields

$$a_i = \frac{1}{\sqrt{5}}(2+2\sqrt{5})^i - \frac{1}{\sqrt{5}}(2-2\sqrt{5})^i = 4^i F_{i+1},$$

so it suffices to find $\nu_2(F_{2019})$. We have $F_n \equiv 0, 1, 1, 2, 3, 1 \pmod 4$ for $n \equiv 0, 1, 2, 3, 4, 5 \pmod 6$, so $\nu_2(F_{2019}) = 1$, and the answer is $4 + 2 \cdot 2018 + 1 = 4041$.

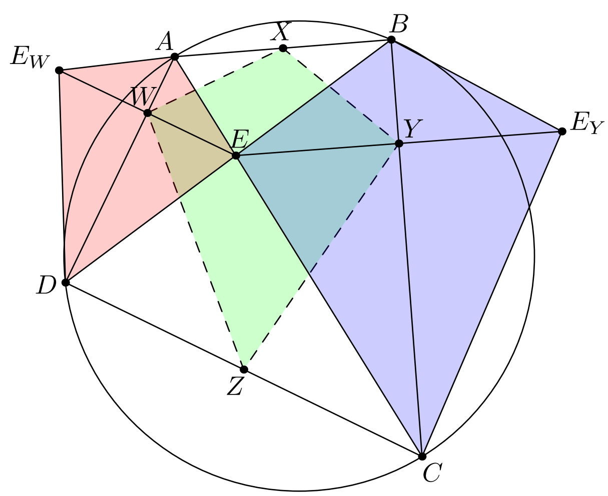
26. **[15]** Let $ABCD$ be a cyclic quadrilateral, and let segments AC and BD intersect at E. Let W and Y be the feet of the altitudes from E to sides DA and BC, respectively, and let X and Z be the midpoints of sides AB and CD, respectively. Given that the area of AED is 9, the area of BEC is 25, and $\angle EBC - \angle ECB = 30^\circ$, then compute the area of $WXYZ$.

Proposed by: James Lin

Answer: $\boxed{17 + \frac{15}{2}\sqrt{3}}$

Solution: Reflect E across DA to E_W, and across BC to E_Y. As $ABCD$ is cyclic, $\triangle AED$ and $\triangle BEC$ are similar. Thus $E_W AED$ and $EBE_Y C$ are similar too.

Now since W is the midpoint of $E_W E$, X is the midpoint of AB, Y is the midpoint of EE_Y, and Z is the midpoint of DC, we have that $WXYZ$ is similar to $E_W AED$ and $EBE_Y C$.



From the given conditions, we have $EW : EY = 3 : 5$ and $\angle WEY = 150^\circ$. Suppose $EW = 3x$ and $EY = 5x$. Then by the law of cosines, we have $WY = \sqrt{34 + 15\sqrt{3}}x$.

Thus, $E_W E : WY = 6 : \sqrt{34 + 15\sqrt{3}}$. So by the similarity ratio,

$$[WXYZ] = [E_W AED]\left(\frac{\sqrt{34+15\sqrt{3}}}{6}\right)^2 = 2 \cdot 9 \cdot \left(\frac{34 + 15\sqrt{3}}{36}\right) = 17 + \frac{15}{2}\sqrt{3}.$$

27. **[15]** Let $\{a_i\}_{i \geq 0}$ be a sequence of real numbers defined by

$$a_{n+1} = a_n^2 - \frac{1}{2^{2020 \cdot 2^n - 1}}$$

for $n \geq 0$. Determine the largest value for a_0 such that $\{a_i\}_{i\geq 0}$ is bounded.

Proposed by: Joshua Lee

Answer: $\boxed{1+\frac{1}{2^{2020}}}$

Solution: Let $a_0 = \frac{1}{\sqrt{2}^{2020}}\left(t+\frac{1}{t}\right)$, with $t \geq 1$. (If $a_0 < \frac{1}{\sqrt{2}^{2018}}$ then no real t exists, but we ignore these values because a_0 is smaller.) Then, we can prove by induction that

$$a_n = \frac{1}{\sqrt{2}^{2020\cdot 2^n}}\left(t^{2^n}+\frac{1}{t^{2^n}}\right).$$

For this to be bounded, it is easy to see that we just need

$$\frac{t^{2^n}}{\sqrt{2}^{2020\cdot 2^n}} = \left(\frac{t}{\sqrt{2}^{2020}}\right)^{2^n}$$

to be bounded, since the second term approaches 0. We see that this is is equivalent to $t \leq 2^{2020/2}$, which means

$$a_0 \leq \frac{1}{\sqrt{2}^{2020}}\left(\sqrt{2}^{2020}+\left(\frac{1}{\sqrt{2}}\right)^{2020}\right) = 1+\frac{1}{2^{2020}}.$$

28. **[15]** Let $\triangle ABC$ be a triangle inscribed in a unit circle with center O. Let I be the incenter of $\triangle ABC$, and let D be the intersection of BC and the angle bisector of $\angle BAC$. Suppose that the circumcircle of $\triangle ADO$ intersects BC again at a point E such that E lies on IO. If $\cos A = \frac{12}{13}$, find the area of $\triangle ABC$.

Proposed by: Michael Diao

Answer: $\boxed{\frac{15}{169}}$

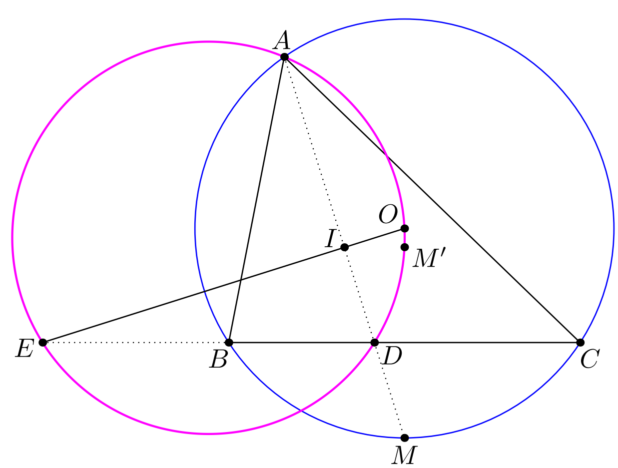
Solution: Consider the following lemma:

Lemma. *$AD \perp EO$.*

Proof. By the Shooting Lemma, the reflection of the midpoint M of arc BC not containing A over BC lies on (ADO). Hence

$$\measuredangle ADE + \measuredangle DEO = \measuredangle MDC + \measuredangle DM'O = \measuredangle MDC + \measuredangle M'MD = 90^\circ.$$

This is enough to imply $AD \perp EO$. $\square$



Thus I is the foot from O onto AD. Now

$$AI^2 + IO^2 = AO^2.$$

By Euler's formula,

$$\left(\frac{r}{\sin\frac{A}{2}}\right)^2 + R^2 - 2Rr = R^2.$$

Hence

$$r = 2R\sin^2\frac{A}{2}.$$

Then

$$s = a + \frac{r}{\tan\frac{A}{2}} = a + R\sin A = 3R\sin A$$

and

$$[ABC] = rs = \left(2R\sin^2\frac{A}{2}\right)(3R\sin A).$$

Since $R = 1$, we get

$$[ABC] = 3(1-\cos A)\sin A.$$

Plugging in $\sin A = \frac{5}{13}$ and $\cos A = \frac{12}{13}$, we get

$$[ABC] = 3\cdot\frac{1}{13}\cdot\frac{5}{13} = \frac{15}{169}.$$

Remark. On the contest, this problem stated that $\triangle ABC$ is an acute triangle and that $\sin A = \frac{5}{13}$, rather than $\cos A = \frac{12}{13}$. This is erroneous because there is no acute triangle satisfying the conditions of the problem statement (the given diagram is not accurate). We apologize for the mistake.

29. **[18]** Let $ABCD$ be a tetrahedron such that its circumscribed sphere of radius R and its inscribed sphere of radius r are concentric. Given that $AB = AC = 1 \leq BC$ and $R = 4r$, find BC^2.

Proposed by: Michael Ren

Answer: $\boxed{1+\sqrt{\frac{7}{15}}}$

Solution: Let O be the common center of the two spheres. Projecting O onto each face of the tetrahedron will divide it into three isosceles triangles. Unfolding the tetrahedron into its net, the reflection of any of these triangles about a side of the tetrahedron will coincide with another one of these triangles. Using this property, we can see that each of the faces is broken up into the same three triangles. It follows that the tetrahedron is isosceles, i.e. $AB = CD$, $AC = BD$, and $AD = BC$.

Let P be the projection of O onto ABC and $x = BC$. By the Pythagorean Theorem on triangle POA, P has distance $\sqrt{R^2 - r^2} = r\sqrt{15}$ from A, B, and C. Using the area-circumcenter formula, we compute

$$[ABC] = \frac{AB\cdot AC\cdot BC}{4PA} = \frac{x}{4r\sqrt{15}}.$$

However, by breaking up the volume of the tetrahedron into the four tetrahedra $OABC$, $OABD$, $OACD$, $OBCD$, we can write $[ABC] = \frac{V}{\frac{4}{3}r}$, where $V = [ABCD]$. Comparing these two expressions for $[ABC]$, we get $x = 3\sqrt{15}V$.

Using the formula for the volume of an isosceles tetrahedron (or some manual calculations), we can compute $V = x^2\sqrt{\frac{1}{72}(2-x^2)}$. Substituting into the previous equation (and taking the solution which is ≥ 1), we eventually get $x^2 = 1 + \sqrt{\frac{7}{15}}$.

30. **[18]** Let $S = \{(x, y) \mid x > 0,\ y > 0,\ x + y < 200, \text{ and } x, y \in \mathbb{Z}\}$. Find the number of parabolas $\mathcal{P}$ with vertex V that satisfy the following conditions:

- $\mathcal{P}$ goes through both $(100, 100)$ and at least one point in S,
- V has integer coordinates, and
- $\mathcal{P}$ is tangent to the line $x + y = 0$ at V.

Proposed by: James Lin

Answer: $\boxed{264}$

Solution: We perform the linear transformation $(x, y) \to (x - y, x + y)$, which has the reverse transformation $(a, b) \to \left(\frac{a+b}{2}, \frac{b-a}{2}\right)$. Then the equivalent problem has a parabola has a vertical axis of symmetry, goes through $A = (0, 200)$, a point $B = (u, v)$ in

$$S' = \{(x, y) \mid x + y > 0,\ x > y,\ y < 200,\ x, y \in \mathbb{Z}, \text{ and } x \equiv y \bmod 2\},$$

and a new vertex $W = (w, 0)$ on $y = 0$ with w even. Then $\left(1 - \frac{u}{w}\right)^2 = \frac{v}{200}$. The only way the RHS can be the square of a rational number is if $\frac{u}{w} = \frac{v'}{10}$ where $v = 2(10 - v')^2$. Since v is even, we can find conditions so that u, w are both even:

$$\begin{aligned} v' \in \{1, 3, 7, 9\} &\implies (2v') \mid u, 20 \mid w \\ v' \in \{2, 4, 6, 8\} &\implies v' \mid u, 10 \mid w \\ v' = 5 &\implies 2 \mid u, 4 \mid w \end{aligned}$$

It follows that any parabola that goes through $v' \in \{3, 7, 9\}$ has a point with $v' = 1$, and any parabola that goes through $v' \in \{4, 6, 8\}$ has a point with $v' = 2$. We then count the following parabolas:

- The number of parabolas going through $(2k, 162)$, where k is a nonzero integer with $|2k| < 162$.
- The number of parabolas going through $(2k, 128)$ not already counted, where k is a nonzero integer with $|2k| < 128$. (Note that this passes through $(k, 162)$.)
- The number of parabolas going through $(2k, 50)$ not already counted, where k is a nonzero integer with $|2k| < 50$. (Note that this passes through $\left(\frac{2k}{5}, 162\right)$, and any overlap must have been counted in the first case.)

The number of solutions is then

$$2\left(80 + \frac{1}{2} \cdot 64 + \frac{4}{5} \cdot 25\right) = 264.$$

31. **[18]** Anastasia is taking a walk in the plane, starting from $(1, 0)$. Each second, if she is at (x, y), she moves to one of the points $(x - 1, y)$, $(x + 1, y)$, $(x, y - 1)$, and $(x, y + 1)$, each with $\frac{1}{4}$ probability. She stops as soon as she hits a point of the form (k, k). What is the probability that k is divisible by 3 when she stops?

Proposed by: Michael Ren

Answer: $\boxed{\frac{3-\sqrt{3}}{3} \text{ or } 1 - \frac{1}{\sqrt{3}}}$

Solution: The key idea is to consider $(a + b, a - b)$, where (a, b) is where Anastasia walks on. Then, the first and second coordinates are independent random walks starting at 1, and we want to find the probability that the first is divisible by 3 when the second reaches 0 for the first time. Let C_n be the nth Catalan number. The probability that the second random walk first reaches 0 after $2n - 1$ steps is

$\frac{C_{n-1}}{2^{2n-1}}$, and the probability that the first is divisible by 3 after $2n-1$ steps is $\frac{1}{2^{2n-1}} \sum_{i \equiv n \bmod 3} \binom{2n-1}{i}$ (by letting i be the number of -1 steps). We then need to compute

$$\sum_{n=1}^{\infty} \left(\frac{C_{n-1}}{4^{2n-1}} \sum_{i \equiv n \bmod 3} \binom{2n-1}{i} \right).$$

By a standard root of unity filter,

$$\sum_{i \equiv n \bmod 3} \binom{2n-1}{i} = \frac{4^n+2}{6}.$$

Letting

$$P(x) = \frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} C_n x^n$$

be the generating function for the Catalan numbers, we find that the answer is

$$\frac{1}{6} P\left(\frac{1}{4}\right) + \frac{1}{12} P\left(\frac{1}{16}\right) = \frac{1}{3} + \frac{1}{12} \cdot \frac{2}{1+\sqrt{\frac{3}{4}}} = \frac{3-\sqrt{3}}{3}.$$

32. **[18]** Find the smallest real constant α such that for all positive integers n and real numbers $0 = y_0 < y_1 < \cdots < y_n$, the following inequality holds:

$$\alpha \sum_{k=1}^{n} \frac{(k+1)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \sum_{k=1}^{n} \frac{k^2+3k+3}{y_k}.$$

Proposed by: Andrew Gu

Answer: $\boxed{\frac{16\sqrt{2}}{9}}$

Solution: We first prove the following lemma:

Lemma. For positive reals a, b, c, d, the inequality

$$\frac{a^{3/2}}{c^{1/2}} + \frac{b^{3/2}}{d^{1/2}} \geq \frac{(a+b)^{3/2}}{(c+d)^{1/2}}$$

holds.

Proof. Apply Hölder's inequality in the form

$$\left(\frac{a^{3/2}}{c^{1/2}} + \frac{b^{3/2}}{d^{1/2}} \right)^2 (c+d) \geq (a+b)^3.$$ □

For $k \geq 2$, applying the lemma to $a = (k-1)^2, b = 8k+8, c = y_{k-1}^2, d = y_k^2 - y_{k-1}^2$ yields

$$\frac{(k-1)^3}{y_{k-1}} + \frac{(8k+8)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \frac{(k+3)^3}{y_k}.$$

We also have the equality

$$\frac{(8 \cdot 1 + 8)^{3/2}}{\sqrt{y_1^2 - y_0^2}} = \frac{(1+3)^3}{y_1}.$$

Summing the inequality from $k = 2$ to $k = n$ with the equality yields

$$\sum_{k=1}^{n} \frac{(8k+8)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \sum_{k=1}^{n} \frac{(k+3)^3 - k^3}{y_k} + \frac{n^3}{y_n} \geq \sum_{k=1}^{n} \frac{9(k^2+3k+3)}{y_k}.$$

Hence the inequality holds for $\alpha = \frac{16\sqrt{2}}{9}$. In the reverse direction, this is sharp when $y_n = n(n+1)(n+2)(n+3)$ (so that $y_{k-1} = \frac{k-1}{k+3} y_k$ for $k = 2, \ldots, n$) and $n \to \infty$.

33. **[22]** Estimate

$$N = \prod_{n=1}^{\infty} n^{n^{-1.25}}.$$

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N) \rfloor$ points.

Proposed by: Sujay Kazi

Answer: $\boxed{\approx 8282580}$

Solution: We approximate

$$\ln N = \sum_{n=1}^{\infty} \frac{\ln n}{n^{5/4}}$$

with an integral as

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x^{5/4}} dx &= \left(-4x^{-1/4} \ln x - 16x^{-1/4} \right) \Big|_1^{\infty} \\ &= 16. \end{aligned}$$

Therefore e^{16} is a good approximation. We can estimate e^{16} by repeated squaring:

$$\begin{aligned} e &\approx 2.72 \\ e^2 &\approx 7.4 \\ e^4 &\approx 55 \\ e^8 &\approx 3000 \\ e^{16} &\approx 9000000. \end{aligned}$$

The true value of e^{16} is around 8886111, which is reasonably close to the value of N. Both e^{16} and 9000000 would be worth 20 points.

34. **[22]** For odd primes p, let $f(p)$ denote the smallest positive integer a for which there does not exist an integer n satisfying $p \mid n^2 - a$. Estimate N, the sum of $f(p)^2$ over the first 10^5 odd primes p.

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N)^3 \rfloor$ points.

Proposed by: Michael Ren

Answer: $\boxed{2266067}$

Solution: Note that the smallest quadratic nonresidue a is always a prime, because if $a = bc$ with $b, c > 1$ then one of b and c is also a quadratic nonresidue. We apply the following heuristic: if p_1, p_2, $\ldots$ are the primes in increasing order, then given a "uniform random prime" q, the values of $\left(\frac{p_1}{q}\right), \left(\frac{p_2}{q}\right), \ldots$ are independent and are 1 with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$.

Of course, there is no such thing as a uniform random prime. More rigorously, for any n, the joint distributions of $\left(\frac{p_1}{q}\right), \ldots, \left(\frac{p_n}{q}\right)$ where q is a uniform random prime less than N converges in distribution to n independent coin flips between 1 and -1 as $N \to \infty$. For ease of explanation, we won't adopt this more formal view, but it is possible to make the following argument rigorous by looking at primes $q < N$ and sending $N \to \infty$. Given any n, the residue of q mod n is uniform over the $\varphi(n)$ residues mod n that are relatively prime to n. By quadratic reciprocity, conditioned on either $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$, exactly half of the nonzero residues mod p_n satisfy $\left(\frac{p_n}{q}\right) = 1$ and exactly half satisfy $\left(\frac{p_n}{q}\right) = -1$ for odd p_n (the case of $p_n = 2$ is slightly different and one must look mod 8, but the result is the same). The residue of q mod 8, p_2, p_3, ..., p_n are independent as these are pairwise relatively prime, yielding our heuristic.

Thus, we may model our problem of finding the smallest quadratic nonresidue with the following process: independent fair coins are flipped for each prime, and we take the smallest prime that flipped heads. We can estimate the expected value of $f(p)^2$ as $\sum_{n=1}^{\infty} \frac{p_n^2}{2^n}$. Looking at the first few terms gives us

$$\frac{2^2}{2} + \frac{3^2}{4} + \frac{5^2}{8} + \frac{7^2}{16} + \frac{11^2}{32} + \frac{13^2}{64} + \frac{17^2}{128} + \frac{19^2}{256} + \frac{23^2}{512} + \frac{29^2}{1024} \approx 22.$$

The terms after this decay rapidly, so a good approximation is $E = 22 \cdot 10^5$, good enough for 20 points. The more inaccurate $E = 20 \cdot 10^5$ earns 15 points.

This Python code computes the exact answer:

```
def smallest_nqr(p):
    for a in range(1,p):
        if pow(a,(p-1)//2,p)==p-1:
            return a

import sympy
print(sum([smallest_nqr(p)**2 for p in sympy.ntheory.primerange(3,sympy.prime(10**5+2))]))
```

Remark. In 1961, Erdős showed that as $N \to \infty$, the average value of $f(p)$ over odd primes $p < N$ will converge to $\sum_{n=1}^{\infty} \frac{p_n}{2^n} \approx 3.675$.

35. **[22]** A collection $\mathcal{S}$ of 10000 points is formed by picking each point uniformly at random inside a circle of radius 1. Let N be the expected number of points of $\mathcal{S}$ which are vertices of the convex hull of the $\mathcal{S}$. (The convex hull is the smallest convex polygon containing every point of $\mathcal{S}$.) Estimate N.

An estimate of $E > 0$ will earn $\max(\lfloor 22 - |E - N| \rfloor, 0)$ points.

Proposed by: Shengtong Zhang

Answer: $\boxed{\approx 72.8}$

Solution: Here is C++ code by Benjamin Qi to estimate the answer via simulation. It is known that the expected number of vertices of the convex hull of n points chosen uniformly at random inside a circle is $O(n^{1/3})$. See "On the Expected Complexity of Random Convex Hulls" by Har-Peled.

36. **[22]** A *snake of length k* is an animal which occupies an ordered k-tuple $(s_1, \ldots, s_k)$ of cells in a $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. If the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \ldots, s_{k-1})$ instead.

Initially, a snake of length 4 is in the grid $\{1, 2, \ldots, 30\}^2$ occupying the positions $(1,1), (1,2), (1,3), (1,4)$ with $(1,1)$ as its head. The snake repeatedly makes a move uniformly at random among moves it can legally make. Estimate N, the expected number of moves the snake makes before it has no legal moves remaining.

An estimate of $E > 0$ will earn $\lfloor 22 \min(N/E, E/N)^4 \rfloor$ points.

Proposed by: Andrew Gu

Answer: $\boxed{\approx 4571.8706930}$

Solution: Let $n = 30$. The snake can get stuck in only 8 positions, while the total number of positions is about $n^2 \times 4 \times 3 \times 3 = 36n^2$. We can estimate the answer as $\frac{36n^2}{8} = 4050$, which is good enough for 13 points.

Let's try to compute the answer as precisely as possible. For each head position (a, b) and tail orientation $c \in [0, 36)$, let $x = 36(na + b) + c$ be an integer denoting the current state of the snake. Let E_x by the expected number of moves the snake makes if it starts at state x. If from state x the snake can transition to any of states $y_1, y_2, \ldots, y_k$, then add an equation of the form

$$E_x - \frac{1}{k} \sum_{i=1}^{k} E_{y_i} = 1.$$

Otherwise, if there are no transitions out of state x then set $E_x = 0$.

It suffices to solve a system of $36n^2$ linear equations for $E_0, E_1, \ldots, E_{36n^2-1}$. Then the answer will equal E_i, where i corresponds to the state described in the problem statement. Naively, using Gaussian elimination would require about $(36n^2)^3 \approx 3.4 \cdot 10^{13}$ operations, which is too slow. Also, it will require too much memory to store $(36n^2)^2$ real numbers at once.

We can use the observation that initially, the maximum difference between any two indices within the same equation is at most $\approx 72n$, so Gaussian elimination only needs to perform approximately $(36n^2) \cdot (72n)^2 \approx 1.5 \cdot 10^{11}$ operations. Furthermore, we'll only need to store $\approx (36n^2) \cdot (72n)$ real numbers at a time. Benjamin Qi's solution ends up finishing in less than two minutes for $n = 30$ (C++ code).

Here are some more examples of problems in which Gaussian elimination can be sped up by exploiting some sort of special structure:

- `https://codeforces.com/contest/963/problem/E`
- `https://codeforces.com/gym/102501/problem/E`