3^{rd} Annual Harvard-MIT November Tournament

Sunday 7 November 2010

General Test

1. [**2**] Jacob flips five coins, exactly three of which land heads. What is the probability that the first two are both heads?

2. [**3**] How many sequences $a_1, a_2, \ldots, a_8$ of zeroes and ones have $a_1a_2 + a_2a_3 + \cdots + a_7a_8 = 5$?

3. [**3**] Triangle ABC has $AB = 5$, $BC = 7$, and $CA = 8$. New lines not containing but parallel to AB, BC, and CA are drawn tangent to the incircle of ABC. What is the area of the hexagon formed by the sides of the original triangle and the newly drawn lines?

4. [**4**] An ant starts at the point $(1,0)$. Each minute, it walks from its current position to one of the four adjacent lattice points until it reaches a point (x,y) with $|x| + |y| \geq 2$. What is the probability that the ant ends at the point $(1,1)$?

5. [**5**] A polynomial P is of the form $\pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1$. Given that $P(2) = 27$, what is $P(3)$?

6. [**5**] What is the sum of the positive solutions to $2x^2 - x\lfloor x \rfloor = 5$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x?

7. [**6**] What is the remainder when $(1+x)^{2010}$ is divided by $1 + x + x^2$?

8. [**7**] Two circles with radius one are drawn in the coordinate plane, one with center $(0,1)$ and the other with center $(2,y)$, for some real number y between 0 and 1. A third circle is drawn so as to be tangent to both of the other two circles as well as the x axis. What is the smallest possible radius for this third circle?

9. [**7**] What is the sum of all numbers between 0 and 511 inclusive that have an even number of 1s when written in binary?

10. [**8**] You are given two diameters AB and CD of circle Ω with radius 1. A circle is drawn in one of the smaller sectors formed such that it is tangent to AB at E, tangent to CD at F, and tangent to Ω at P. Lines PE and PF intersect Ω again at X and Y. What is the length of XY, given that $AC = \frac{2}{3}$?