HMMT February 2025

February 15, 2025

Guts Round

1. **[5]** Call a 9-digit number a *cassowary* if it uses each of the digits 1 through 9 exactly once. Compute the number of cassowaries that are prime.

   Proposed by: Rishabh Das, Jacob Paltrowitz

   Answer: $\boxed{0}$

   Solution: Every cassowary is divisible by 3, as the sum of its digits is $1+2+\cdots+9=45$. Since all such numbers are divisible by 3 and and greater than 3, none of them are prime. So, there are $\boxed{0}$ prime cassowaries.

2. **[5]** Compute

$$\frac{20+\frac{1}{25-\frac{1}{20}}}{25+\frac{1}{20-\frac{1}{25}}}.$$

   Proposed by: Rishabh Das

   Answer: $\boxed{\frac{4}{5}}$

   Solution: We can use the fact that

$$x+\frac{1}{y-\frac{1}{x}}=x+\frac{x}{xy-1}=\frac{x^2y}{xy-1}.$$

   Letting $x=20$, $y=25$ and vice versa in the above expression, we get

$$\frac{x+\frac{1}{y-\frac{1}{x}}}{y+\frac{1}{x-\frac{1}{y}}}=\frac{x^2y}{xy^2}=\frac{x}{y}=\boxed{\frac{4}{5}}.$$

3. **[5]** Jacob rolls two fair six-sided dice. If the outcomes of these dice rolls are the same, he rolls a third fair six-sided die. Compute the probability that the sum of outcomes of all the dice he rolls is even.

   Proposed by: Rishabh Das

   Answer: $\boxed{\frac{5}{12}}$

   Solution: There's a $\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$ probability that he rolls an even number without getting doubles: whatever the first roll is, there is a $\frac{1}{2}$ chance that the second roll is of opposite parity, and we subtract the $\frac{1}{6}$ chance that the second roll is the same.

   There's a $\frac{1}{6}\cdot\frac{1}{2}=\frac{1}{12}$ probability that he gets doubles and then rolls an even number.

   Summing $\frac{1}{3}$ and $\frac{1}{12}$ gets us $\boxed{\frac{5}{12}}$.

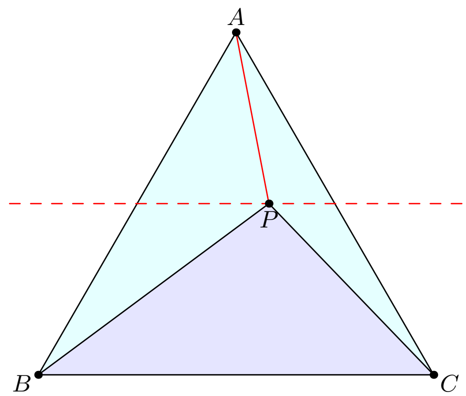
4. **[5]** Let $\triangle ABC$ be an equilateral triangle with side length 4. Across all points P inside triangle $\triangle ABC$ satisfying $[PAB]+[PAC]=[PBC]$, compute the minimum possible length of PA.

   (Here, $[XYZ]$ denotes the area of triangle $\triangle XYZ$.)

   Proposed by: Isabella Zhu

   Answer: $\boxed{\sqrt{3}}$

   Solution:



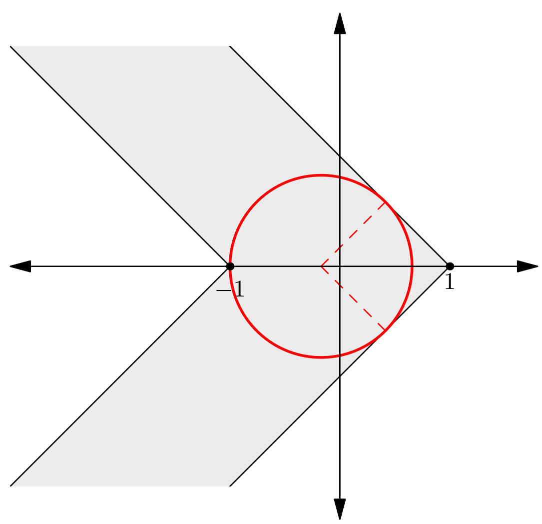
The area condition implies $[ABC] = 2[PBC]$. Hence, P lies on the A-midline of $\triangle ABC$. Therefore, the minimum possible value of PA is the distance from A to this midline. This is achieved by taking P to be the foot of the perpendicular from A to the A-midline. This distance is half the altitude of ABC, which has side length 4, so the answer is $\frac{1}{2}(2\sqrt{3}) = \boxed{\sqrt{3}}$.

5. **[6]** Compute the largest possible radius of a circle contained in the region defined by $|x + |y|| \leq 1$ in the coordinate plane.

   Proposed by: Rishabh Das

   Answer: $\boxed{2\sqrt{2} - 2 = 2(\sqrt{2} - 1)}$

   Solution:



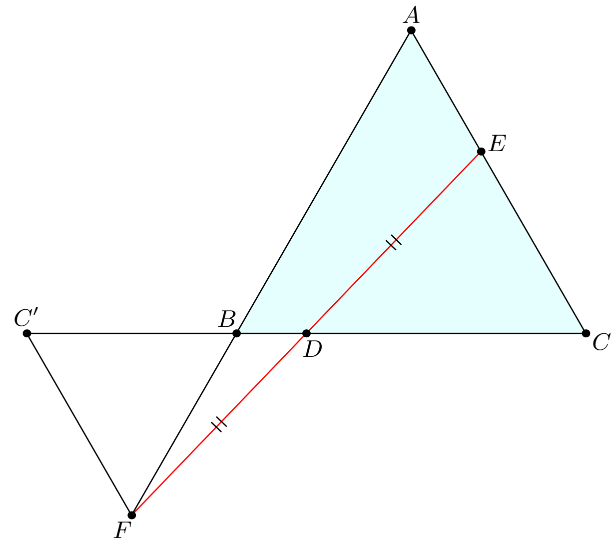
After drawing the graph, it's clear that the circle should pass through $(-1, 0)$ and be tangent to $y = x - 1$ and $y = -x + 1$. Letting the radius of this circle be r, we have $r\sqrt{2} + r = 2$, so $\boxed{r = 2\sqrt{2} - 2}$.

6. **[6]** Let $\triangle ABC$ be an equilateral triangle. Point D lies on segment $\overline{BC}$ such that $BD = 1$ and $DC = 4$. Points E and F lie on rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$, respectively, such that D is the midpoint of $\overline{EF}$. Compute EF.

   Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{2\sqrt{13}}$

Solution:



Let C' be the reflection of C over D. Then, $\overline{EC} \parallel \overline{C'F}$ since $ECFC'$ is a parallelogram. Thus, BFC' is an equilateral triangle, so $BF = BC' = 3$ and $\angle FBD = 120°$. By Law of Cosines, we get $DF = \sqrt{3^2 + 3 \cdot 1 + 1^2} = \sqrt{13}$ and $EF = \boxed{2\sqrt{13}}$.

7. **[6]** The number

$$\frac{9^9 - 8^8}{1001}$$

is an integer. Compute the sum of its prime factors.

Proposed by: Derek Liu

Answer: $\boxed{231}$

Solution: Observe

$$\begin{aligned} 9^9 - 8^8 &= 27^6 - 16^6 \\ &= (27^2 - 16^2)(27^4 + 27^2 \cdot 16^2 + 16^4) \\ &= (27 - 16)(27 + 16)(27^2 - 27 \cdot 16 + 16^2)(27^2 + 27 \cdot 16 + 16^2) \\ &= 11 \cdot 43 \cdot 553 \cdot 1417. \end{aligned}$$

The remaining factorizations are motivated by the fact that $1001 = 7 \cdot 11 \cdot 13$. We see that $553 = 7 \cdot 79$ and $1417 = 13 \cdot 109$, so the answer is $43 + 79 + 109 = \boxed{231}$.

8. **[6]** A *checkerboard* is a rectangular grid of cells colored black and white such that the top-left corner is black and no two cells of the same color share an edge. Two checkerboards are *distinct* if and only if they have a different number of rows or columns. For example, a 20×25 checkerboard and a 25×20 checkerboard are considered distinct.

Compute the number of distinct checkerboards that have exactly 41 black cells.

Proposed by: Albert Wang

Answer: $\boxed{9}$

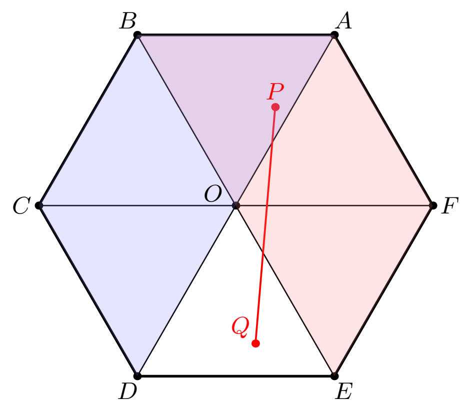
Solution: Since there is a black corner on the checkerboard, the number of white squares is at most the number of black squares. So, the board either has 40 or 41 white squares. Therefore, we want to compute the number of ordered pairs (r, c) with a product of 81 or 82. Since $81 = 3^4$ has 5 divisors and $82 = 41 \cdot 2$ has 4 divisors, there are $\boxed{9}$ checkerboards with exactly 41 black cells.

9. **[7]** Let P and Q be points selected uniformly and independently at random inside a regular hexagon $ABCDEF$. Compute the probability that segment $\overline{PQ}$ is entirely contained in at least one of the quadrilaterals $ABCD$, $BCDE$, $CDEF$, $DEFA$, $EFAB$, or $FABC$.

Proposed by: Isabella Zhu

Answer: $\boxed{\frac{5}{6}}$

Solution:



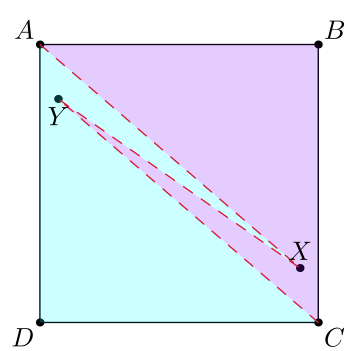
Let O be the center of the hexagon. Without loss of generality, assume P is in $\triangle ABO$. Then, segment PQ is entirely contained in one of the given quadrilaterals if and only if Q is not in $\triangle DEO$. The probability that Q is in $\triangle DEO$ is $\frac{[DEO]}{[ABCDEF]} = \frac{1}{6}$, so the answer is $\boxed{\frac{5}{6}}$.

10. **[7]** A square of side length 1 is dissected into two congruent pentagons. Compute the least upper bound of the perimeter of one of these pentagons.

Proposed by: Isabella Zhu

Answer: $\boxed{2 + 3\sqrt{2}}$

Solution:



Let P_1 and P_2 be the two congruent pentagons. Let $p(P)$ denote the perimeter of polygon P.

We give an upper bound for $p(P_1)+p(P_2)$. Note that since a square has four sides, at least four sides of P_1 and P_2 combined lie on the sides of the square. These sides have total length at most 4, the perimeter of $ABCD$.

Each of the remaining sides has length at most $\sqrt{2}$, since the longest possible length of a segment inside $ABCD$ is $\sqrt{2}$. There are at most 6 remaining sides, so

$$p(P_1)+p(P_2) \le 4+6\sqrt{2}.$$

Since P_1 and P_2 are congruent, this implies

$$p(P_1)=p(P_2) \le \boxed{2+3\sqrt{2}}.$$

This least upper bound can be achieved by placing X close to C and Y close to A, as seen in the diagram.

11. **[7]** Let $f(n)=n^2+100$. Compute the remainder when $\underbrace{f(f(\cdots f(f(}_{2025\ f\text{'s}}1))\cdots))$ is divided by 10^4.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{3101}$

Solution: We claim that $f^k(n) \equiv 1+100(2^n-1) \bmod 10^4$. We can see this by induction, as

$$\begin{aligned} f(1+100(2^n-1)) &= (1+100(2^n-1))^2+100 \\ &\equiv 1+200(2^n-1)+100 \pmod{10^4} \\ &\equiv 1+100(2^{n+1}-1) \pmod{10^4}. \end{aligned}$$

Thus, it suffices to compute $2^{2025} \bmod 100$. We note that $2^{2025} \equiv 0 \pmod 4$ and by Euler's Totient theorem, $2^{2025} \equiv 2^5 \pmod{25}$, so $2^{2025} \equiv 32 \pmod{100}$. Hence, we can compute

$$f^{2025}(1) \equiv 1+100(31) \equiv \boxed{3101} \bmod 10^4.$$

12. **[7]** Holden has a collection of polygons. He writes down a list containing the measure of each interior angle of each of his polygons. He writes down the list $30°$, $50°$, $60°$, $70°$, $90°$, $100°$, $120°$, $160°$, and $x°$, in some order. Compute x.

Proposed by: Rishabh Das

Answer: $\boxed{220}$

Solution: We work in degrees. The sum of all 9 angles is $680+x$. The sum of the angles in a polygon with n sides is $180(n-2) \equiv 180n \bmod 360$. Since there are 9 angles, the polygons have a total of 9 sides, so the sum of the 9 angles must be $9 \cdot 180 \equiv 180 \bmod 360$. Thus $680+x \equiv 180 \bmod 360$, so $x \equiv 220 \bmod 360$. Since $0<x<360$, we know $x=\boxed{220}$.

13. **[9]** A number is *upwards* if its digits in base 10 are nondecreasing when read from left to right. Compute the number of positive integers less than 10^6 that are both upwards and multiples of 11.

Proposed by: Srinivas Arun

Answer: $\boxed{219}$

Solution: For a number $\underline{d_5d_4d_3d_2d_1d_0}$ (allowing leading 0s) to be upwards and a multiple of 11, we must have

$$d_5 \leq d_4 \leq d_3 \leq d_2 \leq d_1 \leq d_0,$$
$$d_0 - d_1 + d_2 - d_3 + d_4 - d_5 \equiv 0 \pmod{11}.$$

Note that $d_0 - d_1$, $d_2 - d_3$, and $d_4 - d_5$ are all nonnegative. Thus,

$$\begin{aligned}
0 &\leq (d_0 - d_1) + (d_2 - d_3) + (d_4 - d_5) \\
&\leq (d_0 - d_1) + (d_1 - d_2) + (d_2 - d_3) + (d_3 - d_4) + (d_4 - d_5) \\
&= d_0 - d_5 \\
&\leq 9.
\end{aligned}$$

Therefore,

$$(d_0 - d_1) + (d_2 - d_3) + (d_4 - d_5) = 0,$$

which can only occur when $d_0 = d_1$, $d_2 = d_3$, and $d_4 = d_5$, i.e. the number is of the form $\underline{aabbcc}$. We can easily verify that all numbers of the form $\underline{aabbcc}$ for digits $a \leq b \leq c$ satisfy our conditions, so we simply have to count them.

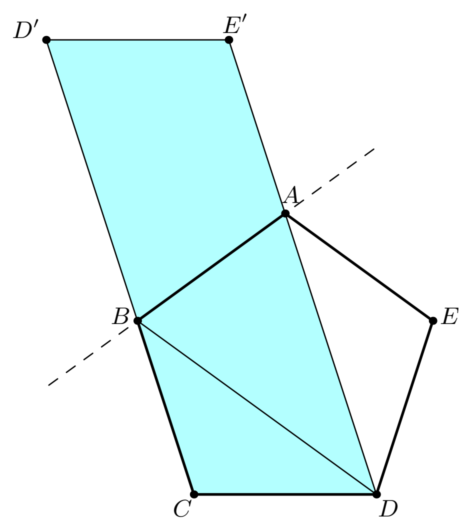
There are $\binom{12}{3} = 220$ such triples of digits (a, b, c). However, one of these triples is $(0, 0, 0)$, which corresponds to the number 0. Thus our answer is $220 - 1 = \boxed{219}$.

14. **[9]** A parallelogram P can be folded over a straight line so that the resulting shape is a regular pentagon with side length 1. Compute the perimeter of P.

Proposed by: Arul Kolla

Answer: $\boxed{5 + \sqrt{5}}$

Solution:



In regular pentagon $ABCDE$ (labeled clockwise), reflect $ABDE$ across AB to obtain $ABD'E'$. Then, $CDE'D'$ is one such parallelogram P. The length of CD' is

$$CB + BD = 1 + 2\cos\angle CBD = 1 + 2\cos(\pi/5) = 1 + \tfrac{\sqrt{5}+1}{2} = \tfrac{\sqrt{5}+3}{2}.$$

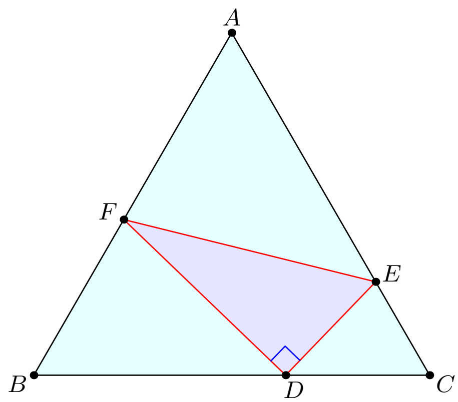
Hence, the perimeter of the desired parallelogram is $2\left(1 + \frac{\sqrt{5}+3}{2}\right) = \boxed{5 + \sqrt{5}}$.

15. **[9]** Right triangle $\triangle DEF$ with $\angle D = 90^\circ$ and $\angle F = 30^\circ$ is inscribed in equilateral triangle $\triangle ABC$ such that D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Given that $BD = 7$ and $DC = 4$, compute DE.

Proposed by: Pitchayut Saengrungkongka

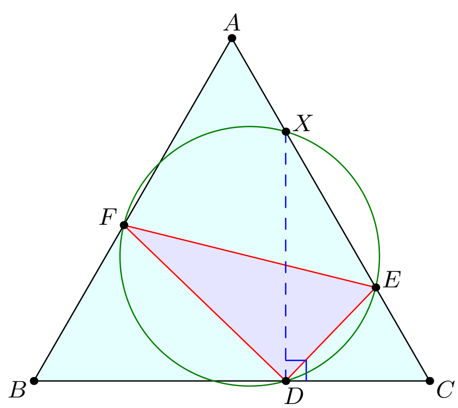
Answer: $\boxed{\sqrt{13}}$

Solution 1:



From $\angle E = 60^\circ$, we get that $\angle AEF = 120^\circ - \angle CED = \angle CDE$. Therefore, $\triangle AEF \sim \triangle CDE$. Since $EF : DE = 2 : 1$, the ratio of similarity must be $2 : 1$, so $AE = 2CD = 8$. Recall ABC has side length $7 + 4 = 11$, so $EC = 11 - 8 = 3$. Law of Cosines on $\triangle CDE$ gives $DE^2 = \sqrt{3^2 + 4^2 - 3 \cdot 4} = \boxed{\sqrt{13}}$.

Solution 2:



Let $\odot(DEF)$ meet AC again at point X. Then, $\angle FXA = 180^\circ - \angle FXE = \angle FDE = 90^\circ$ and $\angle XDC = 180^\circ - \angle DCX - \angle DXC = 120^\circ - \angle DXE = 120^\circ - \angle DFE = 90^\circ$. It follows that $CX = 2CD = 8$, so $AX = 11 - CX = 3$, and $AF = 2AX = 6$. Thus, Law of Cosines on $\triangle AEF$ gives $EF = \sqrt{8^2 + 6^2 - 8 \cdot 6} = 2\sqrt{13}$, implying that $DE = \boxed{\sqrt{13}}$.

16. **[9]** The *Cantor set* is defined as the set of real numbers x such that $0 \leq x < 1$ and the digit 1 does not appear in the base-3 expansion of x. Two numbers are uniformly and independently selected at random from the Cantor set. Compute the expected value of their absolute difference.

(Formally, one can pick a number x uniformly at random from the Cantor set by first picking a real number y uniformly at random from the interval $[0, 1)$, writing it out in binary, reading its digits as if they were in base-3, and setting x to 2 times the result.)

Proposed by: Derek Liu

Answer: $\boxed{\frac{2}{5}}$

Solution: Let d be the expected value of the absolute difference. Observe that the Cantor set is made up of two smaller copies of itself, each scaled down by a factor of 3. There is a $\frac{1}{2}$ chance that the two selected numbers are in the same copy, in which case the expected value of their absolute difference is $\frac{1}{3}d$. Otherwise, we can write them as $\frac{2+x}{3}$ and $\frac{y}{3}$ for independently and uniformly randomly selected x and y in the Cantor set. Their difference is $\frac{2+(x-y)}{3}$, which by symmetry has expected value $\frac{2}{3}$. Thus

$$d = \frac{1}{2} \cdot \frac{1}{3}d + \frac{1}{2} \cdot \frac{2}{3} \implies d = \boxed{\frac{2}{5}}.$$

17. **[11]** Let f be a quadratic polynomial with real coefficients, and let g_1, g_2, g_3, $\ldots$ be a geometric progression of real numbers. Define $a_n = f(n) + g_n$. Given that a_1, a_2, a_3, a_4, and a_5 are equal to 1, 2, 3, 14, and 16, respectively, compute $\frac{g_2}{g_1}$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{-\frac{19}{10}}$

Solution: We will use the method of finite differences. Define $b_n = a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n$. Since f is quadratic, the third finite difference of f is zero. So, $b_n = g_{n+3} - 3g_{n+2} + 3g_{n+1} - g_n$. Letting the common ratio of the geometric sequence be r, we get that $b_n = (r^3 - 3r^2 + 3r - 1)g_n$. So, b_n is a constant multiple of g_n. Thus the ratio $\frac{g_2}{g_1} = \frac{b_2}{b_1}$. Computing $b_1 = 14 - 3 \cdot 3 + 3 \cdot 2 - 1 = 10$ and $b_2 = 16 - 3 \cdot 14 + 3 \cdot 3 - 3 \cdot 2 = -19$, we get

$$\frac{g_2}{g_1} = \frac{b_2}{b_1} = \boxed{-\frac{19}{10}}.$$

18. **[11]** Let $f : \{1, 2, 3, \ldots, 9\} \to \{1, 2, 3, \ldots, 9\}$ be a permutation chosen uniformly at random from the 9! possible permutations. Compute the expected value of $\underbrace{f(f(\cdots f(f(}_{2025\ f\text{'s}}1))\cdots))$.

Proposed by: Linus Yifeng Tang

Answer: $\boxed{\frac{7}{2}}$

Solution: We first compute the probability that $f(1) = 1$. Note that $f(1) = 1$ if and only if 1 is part of a cycle whose length divides 2025.

We claim that for any given k, the probability that 1 is in a cycle of length k is $\frac{1}{9}$. Indeed, the probability that $f(1) \neq 1$ is $\frac{8}{9}$. Given this, there are 8 possible values remaining for $f(f(1))$, so the probability that $f(f(1)) \neq 1$ is $\frac{7}{8}$, and so on. Finally, there are $10 - k$ possible values remaining for $f^k(1)$, so the probability that $f^k(1) = 1$ given all previous assumptions is $\frac{1}{10-k}$. Thus, the probability that 1 is in a cycle of length k is

$$\frac{8}{9} \cdot \frac{7}{8} \cdots \frac{10-k}{11-k} \cdot \frac{1}{10-k} = \frac{1}{9}.$$

Hence, the probability that $f^{2025}(1) = 1$ is the probability that 1 is in a cycle of length 1, 3, 5, or 9, which is $\frac{4}{9}$.

If $f(1) \neq 1$, then $f(1)$ is equally likely to be any of 2 through 9 by symmetry, averaging 5.5.

Therefore, the expected value of $f(1)$ is

$$\frac{4}{9} \cdot 1 + \frac{5}{9} \cdot 5.5 = \boxed{\frac{7}{2}}.$$

19. **[11]** A subset S of $\{1, 2, 3, \ldots, 2025\}$ is called *balanced* if for all elements a and b both in S, there exists an element c in S such that 2025 divides $a + b - 2c$. Compute the number of *nonempty* balanced subsets.

Proposed by: Jacob Paltrowitz

Answer: $\boxed{3751}$

Solution: We work mod 2025, so the condition becomes that for any a, $b \in S$, we have $\frac{a+b}{2} \in S$.

First, we prove that S must be an arithmetic sequence. Observe that if S is balanced, then so is the shift $S + k = \{s + k \mid s \in S\}$ for all k, so we can assume $0 \in S$. Let s be an element of S such that $d = \gcd(s, 2025)$ is minimal. Observe that for any $t \in S$, we have $\frac{t}{2} = \frac{0+t}{2} \in S$. Thus, $\frac{s}{2^n}$ is in S for all n. Because $\gcd(2, 2025) = 1$, this implies $2^n s \bmod 2025 \in S$ for all n. We prove the following claim.

Claim 1. For all positive integers m, we have $ms \in S$.

Proof. We proceed with induction on the number of 1's in the binary representation of m.

Base Case: m has one 1 in binary, so m is a power of 2. Then $ms \in S$ as noted above.

Induction Step: Assume the claim holds for all m which have k 1's in their binary representations. Suppose m has $k+1$ 1's in its binary representation. Let 2^n be the largest power of 2 which is at most m. Then $m - 2^n$ has k 1's in its binary representation, so $2(m - 2^n)$ does as well. By the induction hypothesis, $2(m - 2^n)s \in S$. Also, $2^{n+1}s \in S$, so $ms = \frac{2^{n+1}s + 2(m-2^n)s}{2} \in S$, as desired. $\square$

It follows that every multiple of s is in S. The multiples of s are precisely the multiples of d, so S contains every multiple of d. Now assume for sake of contradiction that S contains some element t which is not a multiple of d, so we can write $t = cd + r$ such that $0 < r < d$. Then $2t \in S$, so $r = \frac{2t+(-2c)d}{2} \in S$. But then $\gcd(r, 2025) \leq r < d$, contradicting the minimality of d. Thus S is precisely the multiples of d.

It can be verified that for any $d \mid 2025$, the set of multiples of d is balanced. Indeed, as d is odd, for any ad, $bd \in S$, their average $\frac{(a+b)d}{2}$ is a multiple of d and hence also in S. Thus, any shift of such a set is also balanced; as seen above, these classify all balanced sets. For each $d \mid 2025$, there are d choices for S, so the answer is $\sum_{d|2025} d = \frac{3^5-1}{2} \cdot \frac{5^3-1}{4} = \boxed{3751}$.

20. **[11]** Compute the 100th smallest positive multiple of 7 whose digits in base 10 are all strictly less than 3.

Proposed by: Srinivas Arun

Answer: $\boxed{221221}$

Solution:

We construct an order-preserving bijection between positive multiples of 7 in base 10 whose digits are all less than 3 and positive multiples of 7. For any multiple of 7 in base 10 with digits all less than 3, interpret it as a base 3 number and convert it to a base 10 decimal, which will be a multiple of 7. For example, 1022 would map to $1022_3 = 35_{10}$.

We first show that this is a valid mapping. Let $\underline{a_n \ldots a_1 a_0}$ be an arbitrary multiple of 7 (in base 10) whose digits are all less than 3. This has a natural interpretation in base 3, and converting this interpretation to base 10 gives

$$\underline{a_n \ldots a_1 a_0}_3 = a_n \cdot 3^n + \cdots + a_1 \cdot 3^1 + a_0 \cdot 3^0 \equiv a_n \cdot 10^n + \cdots + a_1 \cdot 10^1 + a_0 \cdot 10^0 = \underline{a_n \ldots a_1 a_0} \pmod 7,$$

which is a multiple of 7.

This mapping is an injection because base conversion is an injection, and this mapping is also a surjection because for any multiple of 7 in base 10, we can convert it to base 3 and interpret this number as a base 10 decimal. (Reversing the above steps, we see the resulting decimal number is a multiple of 7.) Furthermore, for two positive multiples of 7 whose digits are less than 3, the larger one will have a larger base 3 representation, so this mapping is order-preserving.

Thus, this mapping is a order-preserving bijection between multiples of 7 in base 10 with digits less than 3 and multiples of 7 in base 3. Therefore, the answer is the preimage of the 100th smallest positive multiple of 7, which is equal to $700_{10} = \boxed{221221}_3$.

21. **[12]** Compute the unique 5-digit positive integer $\underline{abcde}$ such that $a \neq 0$, $c \neq 0$, and

$$\underline{abcde} = (\underline{ab} + \underline{cde})^2.$$

Proposed by: Rishabh Das

Answer: $\boxed{88209}$

Solution: Let $\underline{ab} = X$ and $\underline{cde} = Y$. The original equation is equivalent to $1000X + Y = (X+Y)^2$. Taking this modulo 999, we get $(X+Y)^2 \equiv X + Y \pmod{999}$. Therefore, $27 \cdot 37 = 999$ divides $(X+Y)(X+Y-1)$. Since $\gcd(X+Y, X+Y-1) = 1$, each of 27 and 37 can divide at most one of $X+Y$ and $X+Y-1$. Therefore, $X+Y$ is either 0 or 1 modulo 27, as well as 0 or 1 modulo 37.

By the Chinese Remainder Theorem, the four possible residues for $X+Y$ modulo 999 are 0, 1, 297, and 703. Since $(X+Y)^2 = \underline{abcde}$ must be a 5-digit integer, we know $100 \leq X+Y \leq 316$, so the only possible value for $X+Y$ is 297. Thus, the answer is $(X+Y)^2 = 297^2 = \boxed{88209}$.

22. **[12]** Let a, b, and c be real numbers such that $a^2(b+c) = 1$, $b^2(c+a) = 2$, and $c^2(a+b) = 5$. Given that there are three possible values for abc, compute the minimum possible value of abc.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{-5-\sqrt{5}}{2}}$

Solution: Let $x = abc$. Multiplying all equations together and simplifying gives

$$\begin{aligned}
(abc)^2(a+b)(b+c)(c+a) &= 10, \\
(abc)^2\Big(a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc\Big) &= 10, \\
x^2(1+2+5+2x) &= 10, \\
x^2(x+4) &= 5.
\end{aligned}$$

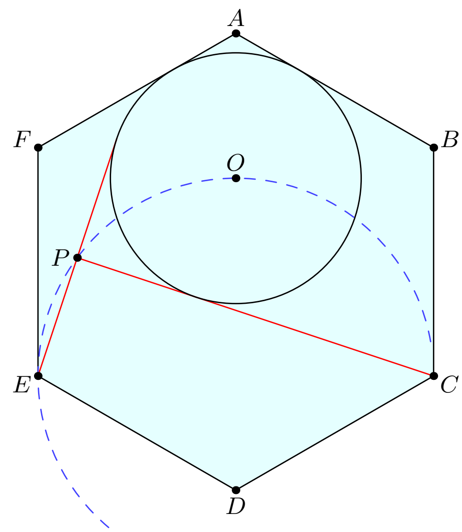
The resulting cubic factors as $(x-1)(x^2+5x+5) = 0$. Therefore, the smallest possible value of abc is $\frac{-5-\sqrt{5^2-4\cdot 5}}{2} = \boxed{\frac{-5-\sqrt{5}}{2}}$.

23. **[12]** Regular hexagon $ABCDEF$ has side length 2. Circle ω lies inside the hexagon and is tangent to segments $\overline{AB}$ and $\overline{AF}$. There exist two perpendicular lines tangent to ω that pass through C and E, respectively. Given that these two lines do not intersect on line AD, compute the radius of ω.

Proposed by: Karthik Venkata Vedula

Answer: $\boxed{\frac{3\sqrt{3}-3}{2} = \frac{3}{2}(\sqrt{3}-1)}$

Solution 1:



Let O be the center of ω, and let the two tangent lines intersect at P. Note that O lies on the external angle bisector of $\angle CPE$ because the tangents are symmetric about line PO. Additionally, O lies on the perpendicular bisector of CE by symmetry. By Fact 5, $COPE$ is cyclic and $\angle COE = 90°$. To finish, observe that $\angle COD = 45°$. Dropping the altitude CH down to AD gives $OH = CH = \sqrt{3}$. So, $AO = AH - OH = 3 - \sqrt{3}$. The desired answer is then $\frac{\sqrt{3}}{2} \cdot AO = \boxed{\frac{3\sqrt{3}-3}{2}}$.

Solution 2: Another way to get $\angle COE = 90°$ is as follows.

Let ω meet the tangents from C and E at Q and R, respectively. Observe $OC = OE$ (as O lies on the perpendicular bisector of CE) and $OQ = OR$, so $\triangle OCQ \stackrel{+}{\cong} \triangle OER$. Then $\angle COE = \angle QOR = 90°$.

24. **[12]** For any integer x, let

$$f(x) = 100! \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{100}}{100!}\right).$$

A positive integer a is chosen such that $f(a) - 20$ is divisible by 101^2. Compute the remainder when $f(a + 101)$ is divided by 101^2.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{1939}$

Solution 1: By the binomial theorem,

$$(a + 101)^n \equiv a^n + \binom{n}{1} a^{n-1} 101 = a^n + 101na^{n-1} \pmod{101^2}.$$

Using this gives (all congruences are modulo 101^2)

$$\begin{aligned} f(a+101) &= 100! \sum_{n=0}^{100} \frac{(a+101)^n}{n!} \\ &\equiv 100! \sum_{n=0}^{100} \left(\frac{a^n}{n!} + \frac{101 n a^{n-1}}{n!} \right) \\ &\equiv f(a) + 100! \cdot 101 \sum_{n=1}^{100} \frac{a^{n-1}}{(n-1)!} \\ &\equiv f(a) + 101 f(a) - 100! \cdot 101 \frac{a^{100}}{100!} \\ &\equiv f(a) + 101(f(a)-1) \\ &\equiv 20 + 101(20-1) = \boxed{1939} \pmod{101^2}. \end{aligned}$$

Solution 2: The above solution can be viewed as a consequence of **Hensel's lemma** as follows. Because 101 is prime, for any integer x not divisible by 101, we have that

$$f'(x) = 100! \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{99}}{99!} \right) = f(x) - x^{100} \equiv f(x) - 1 \pmod{101}.$$

Clearly $101 \nmid a$. Hence, by Hensel's lemma, we get that

$$f(a+101) \equiv f(a) + 101 f'(a) \equiv 20 + 101 \cdot 19 \equiv \boxed{1939} \pmod{101^2}.$$

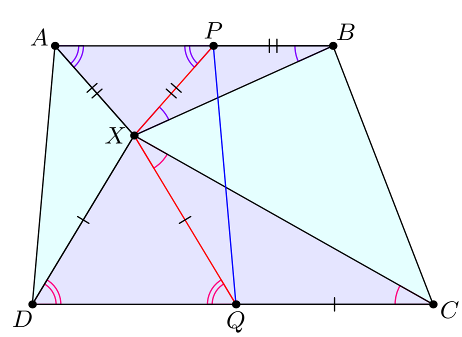
Remark. All possible a's are $a \equiv 1012, 6670$, and $9885 \pmod{101^2}$. They all lead to the same answer.

25. **[14]** Let $ABCD$ be a trapezoid such that $AB \parallel CD$, $AD = 13$, $BC = 15$, $AB = 20$, and $CD = 34$. Point X lies inside the trapezoid such that $\angle XAB = 2\angle XBA$ and $\angle XDC = 2\angle XCD$. Compute $XD - XA$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{4}$

Solution:



Construct point P on AB such that $XA = XP$ and point Q on CD such that $XD = XQ$. The angle condition gives $QC = XQ = XD$ and $PB = XP = XA$. Moreover, $ADQP$ is an isosceles trapezoid.

Let S be the projection of A onto CD, and let T be on CD such that $AT \parallel BC$. Then ADT is a 13-14-15 triangle, so $DS = 5$. Therefore, $QD - PA = 10$. Finally, we get

$$XD - XA = QC - PB = (34 - QD) - (20 - PA) = 14 - 10 = \boxed{4}.$$

26. **[14]** Isabella has a bag with 20 blue diamonds and 25 purple diamonds. She repeats the following process 44 times: she removes a diamond from the bag uniformly at random, then puts one blue diamond and one purple diamond into the bag. Compute the expected number of blue diamonds in the bag after all 44 repetitions.

Proposed by: Henrick Rabinovitz, Srinivas Arun

Answer: $\boxed{\frac{173}{4}}$

Solution: Let $a = 20$ and $b = 25$ be the initial numbers of blue and purple diamonds, respectively, and let $c = 44$ be the number of times Isabella performs the operation. Suppose that at some point, the bag contains x blue diamonds and y purple diamonds, for $x+y = z$ total diamonds. After one step, the bag will have $z+1$ diamonds. The expected change in the number of blue diamonds in this step is $(-x/z)+1 = y/z$, and likewise this quantity for purple diamonds is x/z. Thus, the expected change in the difference between the number of blue and purple diamonds is $(y-x)/z$. Since this difference was initially $x - y$, the expected value of this difference is multiplied by $(z-1)/z$ at each step (regardless of $x - y$). Since z starts at $a + b$ and ends at $a + b + c$, the expected difference after c operations is

$$(a-b) \cdot \prod_{z=a+b}^{a+b+c-1} \frac{z-1}{z} = \frac{(a+b-1)(a-b)}{(a+b+c-1)},$$

and as the total number of diamonds is $a + b + c$, the expected number of blue diamonds at the end is

$$\frac{a+b+c}{2} + \frac{(a+b-1)(a-b)}{2(a+b+c-1)}.$$

Plugging in $a = 20$, $b = 25$, and $c = 44$ gives us the answer, $\boxed{\frac{173}{4}}$.

27. **[14]** Compute the number of ordered pairs (m, n) of *odd* positive integers both less than 80 such that

$$\gcd(4^m + 2^m + 1, 4^n + 2^n + 1) > 1.$$

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{820}$

Solution: First, we characterize all ordered pairs of general (not necessarily odd) positive integers (m, n) such that $\gcd(4^m + 2^m + 1, 4^n + 2^n + 1) > 1$. We claim that (m, n) works if and only if either

- m and n are both even, or
- $\nu_3(m) = \nu_3(n)$.

Proof of necessity. Suppose that p is a prime that divides both $4^m + 2^m + 1$ and $4^n + 2^n + 1$. If $p = 3$, m and n must both be even. Henceforth assume that $p \neq 3$. Let d be the order of 2 modulo p. Then, as $p \mid 2^{3m} - 1$, we find that $d \mid 3m$. Furthermore, we cannot have $p \mid 2^m - 1$. Otherwise, by lifting the exponent (and $p \neq 3$),

$$\nu_p(2^{3m} - 1) = \nu_p(2^m - 1) + \nu_p(3) = \nu_p(2^m - 1) \implies \nu_p(4^m + 2^m + 1) = 0.$$

The previous two results imply $\nu_3(d) = \nu_3(m) + 1$. Similarly, $\nu_3(d) = \nu_3(n) + 1$, so $\nu_3(m) = \nu_3(n)$.

Proof of sufficiency. If m and n are both even, then 3 divides both $4^m + 2^m + 1$ and $4^n + 2^n + 1$. If $\nu_3(m) = \nu_3(n) = k$, then we claim that $4^{3^k} + 2^{3^k} + 1$ divides both $4^m + 2^m + 1$ and $4^n + 2^n + 1$. Indeed, note that as polynomials, $x^2 + x + 1$ divides $x^{2\ell} + x^\ell + 1$ when $3 \nmid \ell$. Plugging in $x = 2^{3^k}$ with $\ell = m/3^k$ and $\ell = n/3^k$ yields the desired result.

Answer Extraction. We count pairs of odd integers (m, n) less than 80 with $\nu_3(m) = \nu_3(n)$. Of all integers in the set $\{1, 3, 5, \ldots, 79\}$, there are 27, 9, 3, and 1 of them that have ν_3 equal to 0, 1, 2, and 3, respectively. Hence, the answer is $27^2 + 9^2 + 3^2 + 1^2 = \boxed{820}$.

28. **[14]** Let f be a function from nonnegative integers to nonnegative integers such that $f(0) = 0$ and

$$f(m) = f\left(\left\lfloor \frac{m}{2} \right\rfloor\right) + \left\lceil \frac{m}{2} \right\rceil^2$$

for all positive integers m. Compute

$$\frac{f(1)}{1 \cdot 2} + \frac{f(2)}{2 \cdot 3} + \frac{f(3)}{3 \cdot 4} + \cdots + \frac{f(31)}{31 \cdot 32}.$$

(Here, $\lfloor z \rfloor$ is the greatest integer less than or equal to z, and $\lceil z \rceil$ is the least positive integer greater than or equal to z.)

Proposed by: Carlos Rodriguez

Answer: $\boxed{\frac{341}{32}}$

Solution 1: For all positive integers n, let $\omega(n) = f(n) - f(n-1)$. We claim that $\omega(n)$ is the largest odd divisor of n for all $n > 0$. Indeed, for all positive integers k, we have

$$\omega(2k) = f(2k) - f(2k-1) = f(k) + k^2 - (f(k-1) + k^2) = f(k) - f(k-1) = \omega(k)$$

and

$$\omega(2k+1) = f(2k+1) - f(2k) = f(k) + (k+1)^2 - (f(k) + k^2) = 2k + 1.$$

Inducting on the positive integers implies that $\omega(n)$ is indeed the largest odd divisor of n.

We can now rewrite the sum as

$$\sum_{n=1}^{31} \frac{f(n)}{n(n+1)} = \sum_{n=1}^{31} \left(\frac{f(n)}{n} - \frac{f(n)}{n+1} \right) = \left(\sum_{n=1}^{30} \frac{f(n) - f(n-1)}{n} \right) - \frac{f(31)}{32}.$$

Note that by using the original recursive definition, we can compute

$$f(31) = 16^2 + 8^2 + 4^2 + 2^2 + 1^2 = 341.$$

Moreover, we also see that $\frac{f(n)-f(n-1)}{n} = \frac{\omega(n)}{n} = 2^{-\nu_2(n)}$, where $2^{\nu_2(n)}$ is the largest power of 2 dividing n. Thus, our desired sum is

$$\left(\sum_{n=1}^{31} 2^{-\nu_2(n)} \right) - \frac{341}{32} = 16 \cdot 2^{-0} + 8 \cdot 2^{-1} + 4 \cdot 2^{-2} + 2 \cdot 2^{-3} + 1 \cdot 2^{-4} - \frac{341}{32} = \boxed{\frac{341}{32}}.$$

Solution 2: From the original recursion, for all positive integers n, we have

$$\frac{f(2n)}{2n(2n+1)} = \frac{f(n) + n^2}{2n(2n+1)} = \frac{f(n)}{2n(2n+1)} + \frac{n}{2(2n+1)}$$

and

$$\frac{f(2n+1)}{(2n+1)(2n+2)} = \frac{f(n)+(n+1)^2}{(2n+1)(2n+2)} = \frac{f(n)}{(2n+1)(2n+2)} + \frac{n+1}{2(2n+1)}.$$

Adding these two equations gives

$$\frac{f(2n)}{2n(2n+1)} + \frac{f(2n+1)}{(2n+1)(2n+2)} = \frac{f(n)}{2n(n+1)} + \frac{1}{2}.$$

Thus, if $T(n) = \sum_{k=1}^{n} \frac{f(k)}{k(k+1)}$, we have

$$\begin{aligned} T(2n+1) &= \sum_{k=1}^{2n+1} \frac{f(k)}{k(k+1)} \\ &= \frac{f(1)}{1(2)} + \sum_{m=1}^{n} \left(\frac{f(2m)}{2m(2m+1)} + \frac{f(2m+1)}{(2m+1)(2m+2)} \right) \\ &= \frac{1}{2} + \sum_{m=1}^{n} \left(\frac{f(m)}{2m(m+1)} + \frac{1}{2} \right) \\ &= \frac{n+1}{2} + \frac{1}{2} T(n). \end{aligned}$$

Starting from $T(1) = 1$, we can compute $T(3) = \frac{5}{4}$, $T(7) = \frac{21}{8}$, $T(15) = \frac{85}{16}$, and finally, $T(31) = \boxed{\frac{341}{32}}$.

29. **[16]** Points A and B lie on circle ω with center O. Let X be a point inside ω. Suppose that $XO = 2\sqrt{2}$, $XA = 1$, $XB = 3$, and $\angle AXB = 90^\circ$. Points Y and Z are on ω such that $Y \neq A$ and triangles $\triangle AXB$ and $\triangle YXZ$ are similar with the same orientation. Compute XY.

Proposed by: Ethan Liu

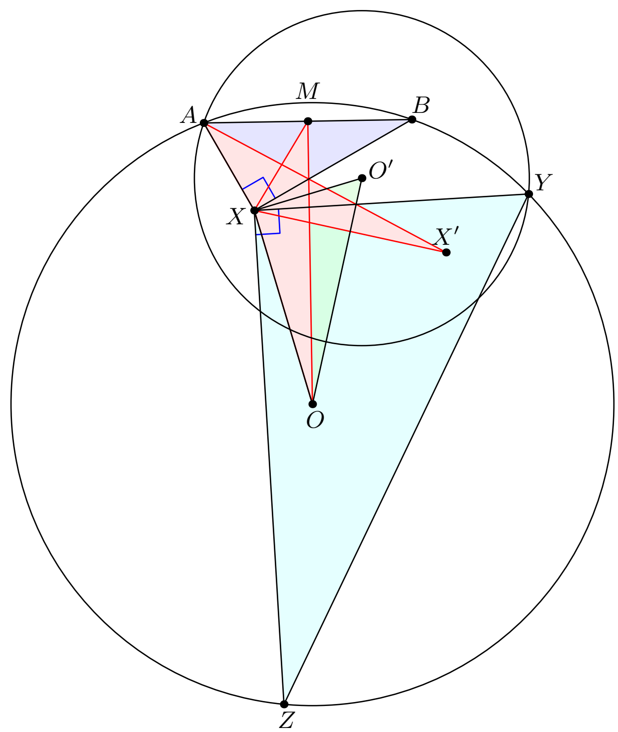
Answer: $\boxed{\frac{11}{5}}$

Solution 1: Consider a rotation about X by 90° followed by a homothety with ratio $\frac{1}{3}$ that sends B to A. This sends ω to ω' with radius $\frac{1}{3}$ of the radius of ω and center O'. Since A is the image of B under this rotation, we know A lies on both circles; the same argument shows Y must lie on both circles. Thus, Y is the reflection of A over OO'. In particular, this means that $XY = AX'$, where X' is the reflection of X over OO'.

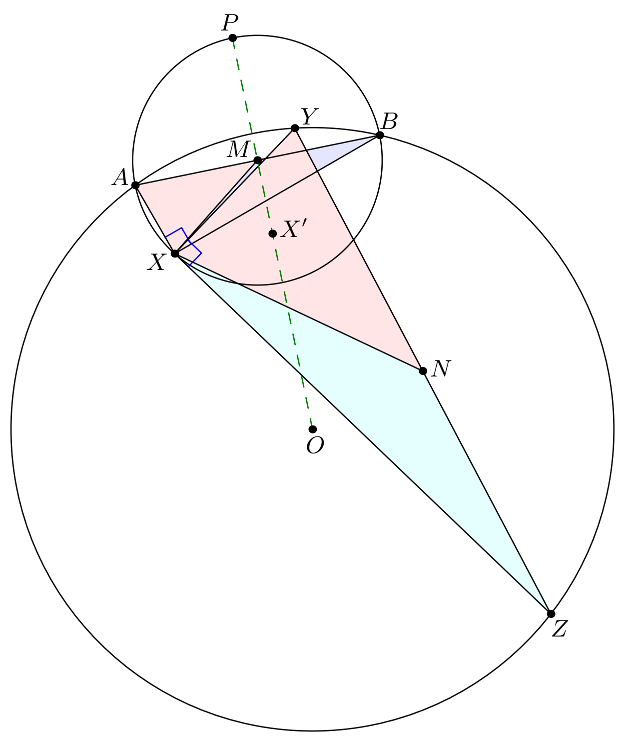
Let M be the midpoint of AB. Note that because $\triangle OXO' \sim \triangle BXA$, we also have $\triangle XOX' \sim \triangle XMA$, as they are both isosceles and $\angle XOX' = 2\angle XOO' = 2\angle XBA = \angle XMA$. This implies that $\triangle XOM \sim \triangle XX'A$. Thus, we know that $AX' = OM \cdot \frac{XA}{XM} = \frac{2}{\sqrt{10}} OM$. It remains to compute OM; noting that the distance between O and the foot from X to AB is $\frac{2}{5}\sqrt{10}$, and that the altitude of $\triangle AXY$ has length $\frac{3}{10}\sqrt{10}$, we get that the distance from O to AB is

$$\frac{3}{10}\sqrt{10} + \sqrt{\left(2\sqrt{2}\right)^2 - \left(\frac{2}{5}\sqrt{10}\right)^2} = \frac{11}{10}\sqrt{10}$$

by the Pythagorean theorem, which means that $XY = AX' = \boxed{\frac{11}{5}}$.



Solution 2:



Let M be the midpoint of AB. We will find MO first.

Let the internal bisector of $\angle AXB$ intersect $\odot(AXB)$ at P. From Ptolemy, $XP = 2\sqrt{2} = XO$. Let X' be the foot of altitude from X to MO. Observe that O is the reflection of P across XX'. By the area of $\triangle AXB$, we have $X'M = \frac{3}{\sqrt{10}}$. Therefore,

$$MO = OX' + X'M = X'P + X'M = 2X'M + MP = \frac{11\sqrt{10}}{10}.$$

By the spiral similarity $\triangle XAB \mapsto \triangle XYZ$, we have that $AY \perp BZ$ and $\angle AOB + \angle YOZ = 90°$. Therefore, $\triangle XAM \sim \triangle XYN$ where N is the midpoint of YZ. Thus, $YZ = \frac{11\sqrt{10}}{5}$ and $XY = \boxed{\frac{11}{5}}$.

Note: if $\triangle XAB \,\bar{\sim}\, \triangle XYZ$, just consider another $\triangle XY'Z'$ caused by reflecting $\triangle XYZ$ across XO.

30. **[16]** Let a, b, and c be real numbers satisfying the system of equations

$$\begin{aligned} a\sqrt{1+b^2} + b\sqrt{1+a^2} &= \tfrac{3}{4}, \\ b\sqrt{1+c^2} + c\sqrt{1+b^2} &= \tfrac{5}{12}, \text{ and} \\ c\sqrt{1+a^2} + a\sqrt{1+c^2} &= \tfrac{21}{20}. \end{aligned}$$

Compute a.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{7}{2\sqrt{30}}}$

Solution 1: Recall that the functions $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^{-x}+e^{-x}}{2}$ satisfy the relation

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \sinh(x)\sqrt{1+\sinh(y)^2} + \sinh(y)\sqrt{1+\sinh(x)^2}.$$

Since sinh is surjective, we can perform the substitution $a = \sinh(x)$, $b = \sinh(y)$, and $c = \sinh(z)$, which turns the equations into

$$\begin{aligned} \sinh(x+y) &= \frac{2 - \frac{1}{2}}{2}, \\ \sinh(y+z) &= \frac{\frac{3}{2} - \frac{2}{3}}{2}, \\ \sinh(z+x) &= \frac{\frac{5}{2} - \frac{2}{5}}{2}. \end{aligned}$$

Thus, $x + y = \log(2)$, $y + z = \log(3/2)$, and $z + x = \log(5/2)$. Solving these equations gives $x = \log(\sqrt{10/3})$, so $a = \frac{1}{2}\left(\sqrt{\frac{10}{3}} - \sqrt{\frac{3}{10}}\right) = \boxed{\frac{7}{2\sqrt{30}}}$.

Solution 2: We can find positive real numbers x, y, and z such that $a = \frac{x^2-1}{2x}$, $b = \frac{y^2-1}{2y}$, and $c = \frac{z^2-1}{2z}$. Then, the first equation becomes

$$\frac{x^2-1}{2x} \cdot \frac{y^2+1}{2y} + \frac{y^2-1}{2y} \cdot \frac{x^2+1}{2x} = \frac{3}{4},$$

which simplifies to

$$xy - \frac{1}{xy} = \frac{3}{2},$$

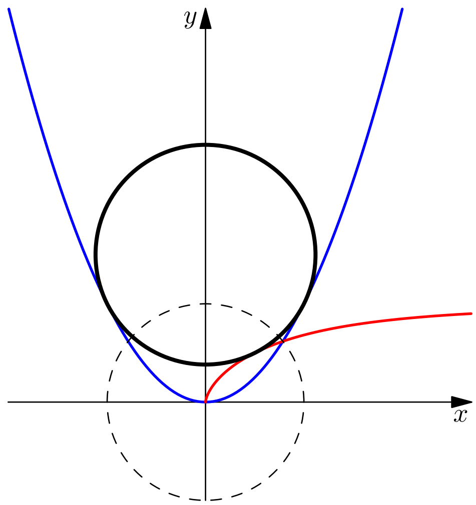
from which it follows that $xy = 2$. Similarly, $yz - \frac{1}{yz} = \frac{5}{6}$ and $zx - \frac{1}{zx} = \frac{21}{10}$, so $yz = \frac{3}{2}$ and $zx = \frac{5}{2}$. Thus $x = \sqrt{(2 \cdot \frac{5}{2})/(\frac{3}{2})} = \sqrt{\frac{10}{3}}$, and $a = \frac{(10/3)-1}{2\sqrt{10/3}} = \boxed{\frac{7}{2\sqrt{30}}}$.

31. **[16]** There exists a unique circle that is both tangent to the parabola $y = x^2$ at two points and tangent to the curve $x = \sqrt{\frac{y^3}{1-y}}$. Compute the radius of this circle.

Proposed by: Karthik Venkata Vedula

Answer: $\boxed{\frac{\sqrt{5}}{2}}$

Solution:



We can square both sides of the second curve to get $x^2 = \frac{y^3}{1-y}$, which further rearranges to

$$\frac{x^2}{(x^2+y^2)^2} = \frac{y}{x^2+y^2}.$$

This relation implies that curves $y = x^2$ and $x^2 = \frac{y^3}{1-y}$ map to each other under inversion about the unit circle $x^2 + y^2 = 1$. Therefore, the unique circle we seek must be invariant under inversion about $x^2 + y^2 = 1$.

Since the circle is tangent to $y = x^2$, we know that the circle is of the form

$$x^2 + (y - y_0)^2 = r^2.$$

We know that the length of the tangent from $(0, 0)$ to this circle is 1. Since the distance from $(0, 0)$ to the center of the circle is y_0, using the Pythagorean Theorem gives $r^2 + 1 = y_0^2$. Because the parabola $y = x^2$ is tangent to this circle at two distinct points, the equation $x^2 + (x^2 - y_0)^2 = r^2 = y_0^2 - 1$ must have two double roots. Therefore,

$$x^2 + (x^2 - y_0)^2 - (y_0^2 - 1) = x^4 - (2y_0 - 1)x^2 + 1$$

must be a perfect square, so $2y_0 - 1 = 2 \implies y_0 = \frac{3}{2}$. This means $r^2 = y_0^2 - 1 = \frac{5}{4}$, so the radius of the circle is $\boxed{\frac{\sqrt{5}}{2}}$.

32. **[16]** In the coordinate plane, a closed lattice loop of length $2n$ is a sequence of lattice points P_0, P_1, P_2, $\ldots$, P_{2n} such that P_0 and P_{2n} are both the origin and $P_iP_{i+1} = 1$ for each i. A closed lattice loop of length 2026 is chosen uniformly at random from all such loops. Let k be the maximum integer such that the line ℓ with equation $x + y = k$ passes through at least one point of the loop. Compute the expected number of indices i such that $0 \le i \le 2025$ and P_i lies on ℓ.

(A lattice point is a point with integer coordinates.)

Proposed by: Carlos Rodriguez, Jordan Lefkowitz

Answer: $\boxed{\frac{1013}{507}}$

Solution: We claim that if 2026 is replaced with $2n$, the answer is $\frac{2n}{n+1}$.

Write the path as a sequence of U, D, L, and R moves. The possible sequences that can result are precisely those with an equal number of U's and D's, and an equal number of R's and L's. We first project this sequence onto a single dimension by converting each U and R to a 1, and each D and L to a -1. The resulting sequence will have an equal number of 1's and -1's.

We claim that every such sequence of n 1's and n -1's corresponds to the same number of closed lattice loops. Indeed, given such a sequence, a corresponding lattice loop can be made by replacing all 1's with U's and R's, and all -1's with D's and L's, so that there are an equal number of U's and D's. Nothing in this replacement process is order-dependent, so the number of ways to create this loop does not depend on the initial sequence.

We can see that each of the 1's move the path towards ℓ and the -1's move the path away. Hence, we can think of the path as a one-dimensional walk, starting and ending in the same place, where the points on ℓ correspond exactly to the maxima of this one-dimensional walk.

Using Catalan numbers, the probability that any given point is at the maxima is

$$\frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1},$$

and thus by linearity, the expected number of $i \in [0, 2025]$ such that P_i is on ℓ is $\frac{2n}{n+1}$. Plugging in $n = 1013$ gives a final answer of $\boxed{\frac{1013}{507}}$.

33. **[20]** Estimate the total number of pages that teams submitted to the Team Round this year. (All pages associated to at least one problem number count as submitted pages, even blank cover sheets for a problem.)

Submit a positive integer E. If the correct answer is A, you will receive $\max\left(0, \left\lceil 20\left(1 - \left(\frac{|E-A|}{100}\right)^{2/3}\right)\right\rceil\right)$ points.

Proposed by: Derek Liu

Answer: $\boxed{1003}$

Solution: Including individual teams, 106 teams registered this year, of which 101 teams submitted a nonzero number of pages to the Team Round. A surprisingly accurate estimate of 1000, which scores 19 points, can be obtained by simply assuming 100 teams competed and each team submitted an average of one page per problem. (Not every team submits all of their cover pages, which mitigates the effect of multiple-page submissions.)

34. **[20]** On the perimeter of a unit circle, 12 points are chosen uniformly and independently at random. Estimate the expected value of the area of the convex 12-gon formed by these points.

Submit a positive number E written in decimal. If the correct answer is A, you will receive $\operatorname{round}\left(20e^{-15|E-A|}\right)$ points.

Proposed by: Isabella Zhu

Answer: $\boxed{\frac{467775}{2\pi^9} - \frac{155925}{2\pi^7} + \frac{10395}{\pi^5} - \frac{1485}{2\pi^3} + \frac{33}{\pi} \approx 2.55915198}$

Solution: We compute the exact answer as given above. Let $n = 12$. and $\theta_1, \theta_2, \dots \theta_n$ be uniformly randomly generated such that $\theta_1 + \cdots + \theta_n = 2\pi$. We are trying to estimate

$$\mathbb{E}\left[\frac{1}{2}\sum_{i=1}^{n}\sin(\theta_i)\right] = \frac{1}{2}\sum_{i=1}^{n}\mathbb{E}[\sin(\theta_i)].$$

We do this by computing the marginal distribution of θ_i, which is proportional to the area of the $n-2$ dimensional cross section

$$\sum_{j\neq i}\theta_j = 2\pi - \theta_i.$$

The volume of this cross section is proportional to $(2\pi - \theta_i)^{n-2}$. Thus, the marginal probability distribution of θ_i can be written as

$$p(\theta_i) = C(2\pi - \theta_i)^{n-2}$$

for some constant C. We can solve for C because we know $\int_0^{2\pi} p(\theta_i)\, d\theta_i = 1$. The result is that

$$p(\theta_i) = \frac{n-1}{2\pi}\left(1 - \frac{\theta_i}{2\pi}\right)^{n-2}.$$

Thus, the quantity that we seek to estimate is

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{2}\sum_{i=1}^{n}\sin\theta_i\right] &= \frac{n}{2}\,\mathbb{E}[\sin\theta_i] \\
&= n\int_0^{2\pi} p(\theta_i)\,\sin\theta_i\, d\theta_i \\
&= \frac{n(n-1)}{4\pi}\int_0^{2\pi}\left(1 - \frac{\theta}{2\pi}\right)^{n-2}\sin\theta\, d\theta \\
&= \frac{33}{\pi}\int_0^{2\pi}\left(1 - \frac{\theta}{2\pi}\right)^{10}\sin\theta\, d\theta.
\end{aligned}$$

We can integrate this using tabular integration by parts.

Differentiate	Integrate	Sign
$\left(1 - \frac{\theta}{2\pi}\right)^{10}$	$\sin\theta$	
$-\frac{5}{\pi}\left(1 - \frac{\theta}{2\pi}\right)^{9}$	$\cos\theta$	$+$
$\frac{45}{2\pi^2}\left(1 - \frac{\theta}{2\pi}\right)^{8}$	$-\sin\theta$	$-$
$-\frac{90}{\pi^3}\left(1 - \frac{\theta}{2\pi}\right)^{7}$	$-\cos\theta$	$+$
$\frac{315}{\pi^4}\left(1 - \frac{\theta}{2\pi}\right)^{6}$	$\sin\theta$	$-$
$-\frac{945}{\pi^5}\left(1 - \frac{\theta}{2\pi}\right)^{5}$	$\cos\theta$	$+$
$\frac{4725}{2\pi^6}\left(1 - \frac{\theta}{2\pi}\right)^{4}$	$-\sin\theta$	$-$
$-\frac{4725}{\pi^7}\left(1 - \frac{\theta}{2\pi}\right)^{3}$	$-\cos\theta$	$+$
$\frac{14175}{2\pi^8}\left(1 - \frac{\theta}{2\pi}\right)^{2}$	$\sin\theta$	$-$
$-\frac{14175}{2\pi^9}\left(1 - \frac{\theta}{2\pi}\right)^{1}$	$\cos\theta$	$+$
$\frac{14175}{4\pi^{10}}$	$-\sin\theta$	$-$
	$\cos\theta$	$+$

To compute the product, note that any term of the form $C\left(1 - \frac{\theta}{2\pi}\right)^n \sin\theta$ vanishes when taking definite integral from 0 to 2π. Therefore, the desired value is

$$\begin{aligned}
&\tfrac{33}{\pi}\Big(\left(1-\tfrac{\theta}{2\pi}\right)^{10}-\tfrac{45}{2\pi^2}\left(1-\tfrac{\theta}{2\pi}\right)^{8}+\tfrac{315}{\pi^4}\left(1-\tfrac{\theta}{2\pi}\right)^{6}-\tfrac{4725}{2\pi^6}\left(1-\tfrac{\theta}{2\pi}\right)^{4}\\
&\qquad\qquad\qquad+\tfrac{14175}{2\pi^8}\left(1-\tfrac{\theta}{2\pi}\right)^{2}+\tfrac{14175}{4\pi^{10}}\Big)\cos\theta\Big|_0^{2\pi}\\
&=\frac{33}{\pi}\left(1-\frac{45}{2\pi^2}+\frac{315}{\pi^4}-\frac{4725}{2\pi^6}+\frac{14175}{2\pi^8}\right)\\
&=\boxed{\frac{33}{\pi}-\frac{1485}{2\pi^3}+\frac{10395}{\pi^5}-\frac{155925}{\pi^7}+\frac{467775}{2\pi^9}}
\end{aligned}$$

35. **[20]** Call an 8-digit number a *flamingo* if it uses each of the digits 2 through 9 exactly once. Estimate the number of flamingos that are prime.

Submit a positive integer E. If the correct answer is A, you will receive round $\left(20\cdot\min\left(\frac{A}{E},\frac{E}{A}\right)^{21}\right)$ points.

Proposed by: Rishabh Das

Answer: $\boxed{3098}$

Solution: There are $8! = 40320$ flamingos which have average $m = 61111110.5$. No flamingo is divisible by 3, as they all have digit sum 44. Assuming each flamingo has a $\frac{3}{2}\cdot\frac{1}{\ln m}$ probability of being prime, we get an estimate of 3373, which scores 3 points.

We can do a lot better by noting that $\frac{3}{8}$ of flamingos aren't divisible by 2 or 5 (the ones ending in 3, 7, or 9), as compared to $\frac{4}{10}$ of numbers in general. Adjusting our old estimate by a factor of $\frac{3}{8}\cdot\frac{10}{4}$, we get a new estimate of 3162, which scores 12 points.

We can still make some minor improvements. Since the digit sum of a flamingo is 44, a flamingo is divisible by 11 if and only if the alternating digit sums are both 22 (as they can't be as low as 11). There are 8 four-element subsets of $\{2,3,\dots,9\}$ which have sum 22, and 70 four-element subsets in all, so $\frac{8}{70}$ of flamingos are divisible by 11. Adjusting our estimate by a factor of $\frac{62}{70}\cdot\frac{11}{10}$, we get a new estimate of 3081, which scores 17 points.

Finally, the estimate of $\frac{1}{\ln m}$ isn't the most accurate. We can do better with the logarithmic integral; the number of primes from 23456789 to 98765432 is approximately $\int_{23456789}^{98765433}\frac{\mathrm{d}t}{\ln t}$, which can be approximated (e.g., using the asymptotic series). Its true value is around 4220000. There are 75308644 numbers from 23456789 to 98765432, respectively. Using these values to estimate prime density instead, we arrive at a final estimate of

$$\frac{4220000}{75308644}\cdot 8!\cdot\frac{3}{2}\cdot\frac{3}{8}\cdot\frac{10}{4}\cdot\frac{62}{70}\cdot\frac{11}{10}\approx 3096,$$

which is accurate enough for all 20 points.

36. **[20]** Ethan initially writes some numbers on a blackboard, each of which is either a 3 or a 5. He then repeatedly picks two numbers and replaces them with their sum, difference, product, or quotient (if the divisor is nonzero). Let $f(n)$ denote the minimum number of numbers Ethan must initially write for him to be able to eventually write the number n. For example, $f(2025)\le 6$ because Ethan could start with 3, 3, 3, 3, 5, and 5 on the board, then repeatedly multiply two numbers at a time to eventually get 2025.

Submit a comma-separated ordered 8-tuple of integers corresponding to the values of $f(164)$, $f(187)$, $f(191)$, $f(224)$, $f(255)$, $f(286)$, $f(374)$, and $f(479)$, in that order, or an X for any value you wish to leave blank. For instance, if you think $f(164)=9$ and $f(224)=8$, you should submit "9, X, X, 8, X, X, X, X". You will earn $\left\lfloor 0.6^W \cdot \frac{(C+1)^2}{4} \right\rfloor$ points, where C is the number of correct answers you submit and W is the number of incorrect (non-blank) answers.

Proposed by: Derek Liu

Answer: $\boxed{6, 6, 6, 5, 5, 6, 6, 7}$

Solution: The following expressions represent optimal ways for Ethan to make each of the 8 given numbers.

$$\begin{aligned}
164 &= 3(5(5+5)+3)+5 \\
187 &= 3(3+5)(3+5)-5 \\
191 &= 5 \cdot 5(3+5) - 3 \cdot 3 \\
224 &= (3+5)(5 \cdot 5+3) \\
255 &= 5 \cdot 5(5+5)+5 \\
286 &= (3+5+5)(5 \cdot 5-3) \\
374 &= 3 \cdot 5 \cdot 5 \cdot 5 - 3/3 = 3 \cdot 5 \cdot 5 \cdot 5 - 5/5 \\
479 &= (5 \cdot 5-3)(5 \cdot 5-3)-5 = (3-5 \cdot 5)(3-5 \cdot 5)-5
\end{aligned}$$

It can be checked by code that these are optimal.