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HMMT February 2025, February 15, 2025 — GUTS ROUND

Organization __________________________ Team __________________________ Team ID# _______________

1. **[5]** Call a 9-digit number a *cassowary* if it uses each of the digits 1 through 9 exactly once. Compute the number of cassowaries that are prime.

2. **[5]** Compute
$$\frac{20+\frac{1}{25-\frac{1}{20}}}{25+\frac{1}{20-\frac{1}{25}}}.$$

3. **[5]** Jacob rolls two fair six-sided dice. If the outcomes of these dice rolls are the same, he rolls a third fair six-sided die. Compute the probability that the sum of outcomes of all the dice he rolls is even.

4. **[5]** Let $\triangle ABC$ be an equilateral triangle with side length 4. Across all points P inside triangle $\triangle ABC$ satisfying $[PAB]+[PAC]=[PBC]$, compute the minimum possible length of PA.

   (Here, $[XYZ]$ denotes the area of triangle $\triangle XYZ$.)

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HMMT February 2025, February 15, 2025 — GUTS ROUND

Organization __________________________ Team __________________________ Team ID# _______________

5. **[6]** Compute the largest possible radius of a circle contained in the region defined by $|x+|y||\le 1$ in the coordinate plane.

6. **[6]** Let $\triangle ABC$ be an equilateral triangle. Point D lies on segment $\overline{BC}$ such that $BD=1$ and $DC=4$. Points E and F lie on rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$, respectively, such that D is the midpoint of $\overline{EF}$. Compute EF.

7. **[6]** The number
$$\frac{9^9-8^8}{1001}$$
is an integer. Compute the sum of its prime factors.

8. **[6]** A *checkerboard* is a rectangular grid of cells colored black and white such that the top-left corner is black and no two cells of the same color share an edge. Two checkerboards are *distinct* if and only if they have a different number of rows or columns. For example, a 20×25 checkerboard and a 25×20 checkerboard are considered distinct.

   Compute the number of distinct checkerboards that have exactly 41 black cells.

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HMMT February 2025, February 15, 2025 — GUTS ROUND

Organization __________________________ Team __________________________ Team ID# _______________

9. **[7]** Let P and Q be points selected uniformly and independently at random inside a regular hexagon $ABCDEF$. Compute the probability that segment $\overline{PQ}$ is entirely contained in at least one of the quadrilaterals $ABCD$, $BCDE$, $CDEF$, $DEFA$, $EFAB$, or $FABC$.

10. **[7]** A square of side length 1 is dissected into two congruent pentagons. Compute the least upper bound of the perimeter of one of these pentagons.

11. **[7]** Let $f(n) = n^2 + 100$. Compute the remainder when $\underbrace{f(f(\cdots f(f(}_{2025\ f\text{'s}}1))\cdots))$ is divided by 10^4.

12. **[7]** Holden has a collection of polygons. He writes down a list containing the measure of each interior angle of each of his polygons. He writes down the list 30°, 50°, 60°, 70°, 90°, 100°, 120°, 160°, and x°, in some order. Compute x.

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HMMT February 2025, February 15, 2025 — GUTS ROUND

Organization __________________________ Team __________________________ Team ID# _______________

13. **[9]** A number is *upwards* if its digits in base 10 are nondecreasing when read from left to right. Compute the number of positive integers less than 10^6 that are both upwards and multiples of 11.

14. **[9]** A parallelogram P can be folded over a straight line so that the resulting shape is a regular pentagon with side length 1. Compute the perimeter of P.

15. **[9]** Right triangle $\triangle DEF$ with $\angle D = 90^\circ$ and $\angle F = 30^\circ$ is inscribed in equilateral triangle $\triangle ABC$ such that D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Given that $BD = 7$ and $DC = 4$, compute DE.

16. **[9]** The *Cantor set* is defined as the set of real numbers x such that $0 \leq x < 1$ and the digit 1 does not appear in the base-3 expansion of x. Two numbers are uniformly and independently selected at random from the Cantor set. Compute the expected value of their absolute difference.

(Formally, one can pick a number x uniformly at random from the Cantor set by first picking a real number y uniformly at random from the interval $[0, 1)$, writing it out in binary, reading its digits as if they were in base-3, and setting x to 2 times the result.)

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HMMT February 2025, February 15, 2025 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# ____________

17. **[11]** Let f be a quadratic polynomial with real coefficients, and let $g_1, g_2, g_3, \ldots$ be a geometric progression of real numbers. Define $a_n = f(n) + g_n$. Given that a_1, a_2, a_3, a_4, and a_5 are equal to 1, 2, 3, 14, and 16, respectively, compute $\frac{g_2}{g_1}$.

18. **[11]** Let $f : \{1, 2, 3, \ldots, 9\} \to \{1, 2, 3, \ldots, 9\}$ be a permutation chosen uniformly at random from the 9! possible permutations. Compute the expected value of $\underbrace{f(f(\cdots f(f(}_{2025\ f\text{'s}}1))\cdots))$.

19. **[11]** A subset S of $\{1, 2, 3, \ldots, 2025\}$ is called *balanced* if for all elements a and b both in S, there exists an element c in S such that 2025 divides $a + b - 2c$. Compute the number of *nonempty* balanced subsets.

20. **[11]** Compute the 100th smallest positive multiple of 7 whose digits in base 10 are all strictly less than 3.

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HMMT February 2025, February 15, 2025 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# ____________

21. **[12]** Compute the unique 5-digit positive integer $\underline{abcde}$ such that $a \neq 0$, $c \neq 0$, and
$$\underline{abcde} = (\underline{ab} + \underline{cde})^2.$$

22. **[12]** Let a, b, and c be real numbers such that $a^2(b + c) = 1$, $b^2(c + a) = 2$, and $c^2(a + b) = 5$. Given that there are three possible values for abc, compute the minimum possible value of abc.

23. **[12]** Regular hexagon $ABCDEF$ has side length 2. Circle ω lies inside the hexagon and is tangent to segments $\overline{AB}$ and $\overline{AF}$. There exist two perpendicular lines tangent to ω that pass through C and E, respectively. Given that these two lines do not intersect on line AD, compute the radius of ω.

24. **[12]** For any integer x, let
$$f(x) = 100! \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{100}}{100!}\right).$$
A positive integer a is chosen such that $f(a) - 20$ is divisible by 101^2. Compute the remainder when $f(a + 101)$ is divided by 101^2.

25. **[14]** Let $ABCD$ be a trapezoid such that $AB \parallel CD$, $AD = 13$, $BC = 15$, $AB = 20$, and $CD = 34$. Point X lies inside the trapezoid such that $\angle XAB = 2\angle XBA$ and $\angle XDC = 2\angle XCD$. Compute $XD - XA$.

26. **[14]** Isabella has a bag with 20 blue diamonds and 25 purple diamonds. She repeats the following process 44 times: she removes a diamond from the bag uniformly at random, then puts one blue diamond and one purple diamond into the bag. Compute the expected number of blue diamonds in the bag after all 44 repetitions.

27. **[14]** Compute the number of ordered pairs (m, n) of *odd* positive integers both less than 80 such that

$$\gcd(4^m + 2^m + 1, 4^n + 2^n + 1) > 1.$$

28. **[14]** Let f be a function from nonnegative integers to nonnegative integers such that $f(0) = 0$ and

$$f(m) = f\left(\left\lfloor \frac{m}{2} \right\rfloor\right) + \left\lceil \frac{m}{2} \right\rceil^2$$

for all positive integers m. Compute

$$\frac{f(1)}{1 \cdot 2} + \frac{f(2)}{2 \cdot 3} + \frac{f(3)}{3 \cdot 4} + \cdots + \frac{f(31)}{31 \cdot 32}.$$

(Here, $\lfloor z \rfloor$ is the greatest integer less than or equal to z, and $\lceil z \rceil$ is the least positive integer greater than or equal to z.)

Organization ____________________ Team ____________________ Team ID# ____________

29. **[16]** Points A and B lie on circle ω with center O. Let X be a point inside ω. Suppose that $XO = 2\sqrt{2}$, $XA = 1$, $XB = 3$, and $\angle AXB = 90^\circ$. Points Y and Z are on ω such that $Y \neq A$ and triangles $\triangle AXB$ and $\triangle YXZ$ are similar with the same orientation. Compute XY.

30. **[16]** Let a, b, and c be real numbers satisfying the system of equations

$$\begin{aligned} a\sqrt{1+b^2} + b\sqrt{1+a^2} &= \tfrac{3}{4}, \\ b\sqrt{1+c^2} + c\sqrt{1+b^2} &= \tfrac{5}{12}, \text{ and} \\ c\sqrt{1+a^2} + a\sqrt{1+c^2} &= \tfrac{21}{20}. \end{aligned}$$

Compute a.

31. **[16]** There exists a unique circle that is both tangent to the parabola $y = x^2$ at two points and tangent to the curve $x = \sqrt{\frac{y^3}{1-y}}$. Compute the radius of this circle.

32. **[16]** In the coordinate plane, a closed lattice loop of length $2n$ is a sequence of lattice points P_0, P_1, P_2, $\ldots$, P_{2n} such that P_0 and P_{2n} are both the origin and $P_iP_{i+1} = 1$ for each i. A closed lattice loop of length 2026 is chosen uniformly at random from all such loops. Let k be the maximum integer such that the line ℓ with equation $x + y = k$ passes through at least one point of the loop. Compute the expected number of indices i such that $0 \le i \le 2025$ and P_i lies on ℓ.

(A lattice point is a point with integer coordinates.)

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Organization ____________________ Team ____________________ Team ID# __________

33. **[20]** Estimate the total number of pages that teams submitted to the Team Round this year. (All pages associated to at least one problem number count as submitted pages, even blank cover sheets for a problem.)

    Submit a positive integer E. If the correct answer is A, you will receive $\max\left(0, \left\lceil 20\left(1-\left(\frac{|E-A|}{100}\right)^{2/3}\right)\right\rceil\right)$ points.

34. **[20]** On the perimeter of a unit circle, 12 points are chosen uniformly and independently at random. Estimate the expected value of the area of the convex 12-gon formed by these points.

    Submit a positive number E written in decimal. If the correct answer is A, you will receive $\operatorname{round}\left(20e^{-15|E-A|}\right)$ points.

35. **[20]** Call an 8-digit number a *flamingo* if it uses each of the digits 2 through 9 exactly once. Estimate the number of flamingos that are prime.

    Submit a positive integer E. If the correct answer is A, you will receive $\operatorname{round}\left(20 \cdot \min\left(\frac{A}{E}, \frac{E}{A}\right)^{21}\right)$ points.

36. **[20]** Ethan initially writes some numbers on a blackboard, each of which is either a 3 or a 5. He then repeatedly picks two numbers and replaces them with their sum, difference, product, or quotient (if the divisor is nonzero). Let $f(n)$ denote the minimum number of numbers Ethan must initially write for him to be able to eventually write the number n. For example, $f(2025) \leq 6$ because Ethan could start with 3, 3, 3, 3, 5, and 5 on the board, then repeatedly multiply two numbers at a time to eventually get 2025.

    Submit a comma-separated ordered 8-tuple of integers corresponding to the values of $f(164)$, $f(187)$, $f(191)$, $f(224)$, $f(255)$, $f(286)$, $f(374)$, and $f(479)$, in that order, or an X for any value you wish to leave blank. For instance, if you think $f(164) = 9$ and $f(224) = 8$, you should submit "9, X, X, 8, X, X, X, X". You will earn $\left\lfloor 0.6^W \cdot \frac{(C+1)^2}{4} \right\rfloor$ points, where C is the number of correct answers you submit and W is the number of incorrect (non-blank) answers.