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HMMT February 2022, February 19, 2022 — GUTS ROUND

Organization __________________ Team __________________ Team ID# ____________

1. **[5]** A regular 2022-gon has perimeter 6.28. To the nearest positive integer, compute the area of the 2022-gon.

2. **[5]** Three distinct vertices are randomly selected among the five vertices of a regular pentagon. Let p be the probability that the triangle formed by the chosen vertices is acute. Compute $10p$.

3. **[5]** Herbert rolls 6 fair standard dice and computes the product of all of his rolls. If the probability that the product is prime can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a+b$.

4. **[5]** For a real number x, let $[x]$ be x rounded to the nearest integer and $\langle x\rangle$ be x rounded to the nearest tenth. Real numbers a and b satisfy $\langle a\rangle + [b] = 98.6$ and $[a] + \langle b\rangle = 99.3$. Compute the minimum possible value of $[10(a+b)]$.

   (Here, any number equally between two integers or tenths of integers, respectively, is rounded up. For example, $[-4.5] = -4$ and $\langle 4.35\rangle = 4.4$.)

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HMMT February 2022, February 19, 2022 — GUTS ROUND

Organization __________________ Team __________________ Team ID# ____________

5. **[6]** Compute the remainder when

$$10002000400080016003200640128025605121024204840968192$$

   is divided by 100020004000800160032.

6. **[6]** Regular polygons $ICAO$, $VENTI$, and $ALBEDO$ lie on a plane. Given that $IN = 1$, compute the number of possible values of ON.

7. **[6]** A jar contains 8 red balls and 2 blue balls. Every minute, a ball is randomly removed. The probability that there exists a time during this process where there are more blue balls than red balls in the jar can be expressed as $\frac{a}{b}$ for relatively prime integers a and b. Compute $100a+b$.

8. **[6]** For any positive integer n, let $\tau(n)$ denote the number of positive divisors of n. If n is a positive integer such that $\frac{\tau(n^2)}{\tau(n)} = 3$, compute $\frac{\tau(n^7)}{\tau(n)}$.

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Organization ____________________ Team ____________________ Team ID# ____________

9. **[7]** An E-shape is a geometric figure in the two-dimensional plane consisting of three rays pointing in the same direction, along with a line segment such that

   - the endpoints of the rays all lie on the segment,
   - the segment is perpendicular to all three rays,
   - both endpoints of the segment are endpoints of rays.

   Suppose two E-shapes intersect each other N times in the plane for some positive integer N. Compute the maximum possible value of N.

10. **[7]** A positive integer n is loose it has six positive divisors and satisfies the property that any two positive divisors $a < b$ of n satisfy $b \geq 2a$. Compute the sum of all loose positive integers less than 100.

11. **[7]** A regular dodecagon $P_1P_2 \cdots P_{12}$ is inscribed in a unit circle with center O. Let X be the intersection of P_1P_5 and OP_2, and let Y be the intersection of P_1P_5 and OP_4. Let A be the area of the region bounded by XY, XP_2, YP_4, and minor arc $\widehat{P_2P_4}$. Compute $\lfloor 120A \rfloor$.

12. **[7]** A unit square $ABCD$ and a circle Γ have the following property: if P is a point in the plane not contained in the interior of Γ, then $\min(\angle APB, \angle BPC, \angle CPD, \angle DPA) \leq 60^\circ$. The minimum possible area of Γ can be expressed as $\frac{a\pi}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

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HMMT February 2022, February 19, 2022 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# ____________

13. **[9]** Let z_1, z_2, z_3, z_4 be the solutions to the equation $x^4+3x^3+3x^2+3x+1=0$. Then $|z_1|+|z_2|+|z_3|+|z_4|$ can be written as $\frac{a+b\sqrt{c}}{d}$, where c is a squarefree positive integer, and a, b, d are positive integers with $\gcd(a, b, d) = 1$. Compute $1000a + 100b + 10c + d$.

14. **[9]** The area of the largest regular hexagon that can fit inside of a rectangle with side lengths 20 and 22 can be expressed as $a\sqrt{b}-c$, for positive integers a, b, and c, where b is squarefree. Compute $100a+10b+c$.

15. **[9]** Let N be the number of triples of positive integers (a, b, c) satisfying

$$a \leq b \leq c, \qquad \gcd(a, b, c) = 1, \qquad abc = 6^{2020}.$$

   Compute the remainder when N is divided by 1000.

16. **[9]** Let ABC be an acute triangle with A-excircle Γ. Let the line through A perpendicular to BC intersect BC at D and intersect Γ at E and F. Suppose that $AD = DE = EF$. If the maximum value of $\sin B$ can be expressed as $\frac{\sqrt{a}+\sqrt{b}}{c}$ for positive integers a, b, and c, compute the minimum possible value of $a+b+c$.

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Organization ____________________ Team ____________________ Team ID# ____________

17. **[11]** Compute the number of positive real numbers x that satisfy

$$\left(3 \cdot 2^{\lfloor \log_2 x \rfloor} - x\right)^{16} = 2022x^{13}.$$

18. **[11]** Compute the number of permutations π of the set $\{1, 2, \ldots, 10\}$ so that for all (not necessarily distinct) $m, n \in \{1, 2, \ldots, 10\}$ where $m + n$ is prime, $\pi(m) + \pi(n)$ is prime.

19. **[11]** In right triangle ABC, a point D is on hypotenuse AC such that $BD \perp AC$. Let ω be a circle with center O, passing through C and D and tangent to line AB at a point other than B. Point X is chosen on BC such that $AX \perp BO$. If $AB = 2$ and $BC = 5$, then BX can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

20. **[11]** Let π be a uniformly random permutation of the set $\{1, 2, \ldots, 100\}$. The probability that $\pi^{20}(20) = 20$ and $\pi^{21}(21) = 21$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$. (Here, π^k means π iterated k times.)

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Organization ____________________ Team ____________________ Team ID# ____________

21. **[12]** In the Cartesian plane, let $A = (0, 0)$, $B = (200, 100)$, and $C = (30, 330)$. Compute the number of ordered pairs (x, y) of integers so that $(x + \frac{1}{2}, y + \frac{1}{2})$ is in the interior of triangle ABC.

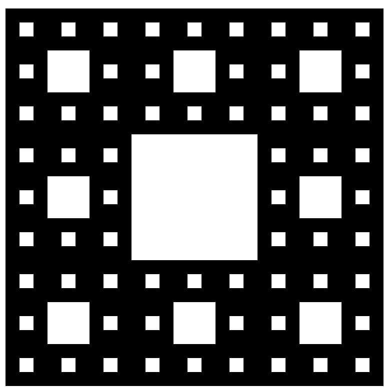
22. **[12]** The function $f(x)$ is of the form $ax^2 + bx + c$ for some integers a, b, and c. Given that

$$\{f(177\,883), f(348\,710), f(796\,921), f(858\,522)\}$$
$$= \{1\,324\,754\,875\,645, 1\,782\,225\,466\,694, 1\,984\,194\,627\,862, 4\,388\,794\,883\,485\},$$

compute a.

23. **[12]** Let $ABCD$ be an isosceles trapezoid such that $AB = 17$, $BC = DA = 25$, and $CD = 31$. Points P and Q are selected on sides AD and BC, respectively, such that $AP = CQ$ and $PQ = 25$. Suppose that the circle with diameter PQ intersects the sides AB and CD at four points which are vertices of a convex quadrilateral. Compute the area of this quadrilateral.

24. **[12]** Let S_0 be a unit square in the Cartesian plane with horizontal and vertical sides. For any $n > 0$, the shape S_n is formed by adjoining 9 copies of S_{n-1} in a 3×3 grid, and then removing the center copy. For example, S_3 is shown below:



Let a_n be the expected value of $|x - x'| + |y - y'|$, where (x, y) and (x', y') are two points chosen randomly within S_n. There exist relatively prime positive integers a and b such that

$$\lim_{n \to \infty} \frac{a_n}{3^n} = \frac{a}{b}.$$

Compute $100a + b$.

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Organization ____________________ Team ____________________ Team ID# ____________

25. **[14]** Let ABC be an acute scalene triangle with circumcenter O and centroid G. Given that AGO is a right triangle, $AO = 9$, and $BC = 15$, let S be the sum of all possible values for the area of triangle AGO. Compute S^2.

26. **[14]** Diana is playing a card game against a computer. She starts with a deck consisting of a single card labeled 0.9. Each turn, Diana draws a random card from her deck, while the computer generates a card with a random real number drawn uniformly from the interval $[0, 1]$. If the number on Diana's card is larger, she keeps her current card and also adds the computer's card to her deck. Otherwise, the computer takes Diana's card. After k turns, Diana's deck is empty. Compute the expected value of k.

27. **[14]** In three-dimensional space, let S be the region of points (x, y, z) satisfying $-1 \leq z \leq 1$. Let $S_1, S_2, \ldots, S_{2022}$ be 2022 independent random rotations of S about the origin $(0, 0, 0)$. The expected volume of the region $S_1 \cap S_2 \cap \cdots \cap S_{2022}$ can be expressed as $\frac{a\pi}{b}$, for relatively prime positive integers a and b. Compute $100a + b$.

28. **[14]** Compute the nearest integer to

$$100 \sum_{n=1}^{\infty} 3^n \sin^3\left(\frac{\pi}{3^n}\right).$$

Organization ________________________ Team ________________________ Team ID# ______________

29. **[16]** Let $a \neq b$ be positive real numbers and m, n be positive integers. An $m+n$-gon P has the property that m sides have length a and n sides have length b. Further suppose that P can be inscribed in a circle of radius $a+b$. Compute the number of ordered pairs (m, n), with $m, n \leq 100$, for which such a polygon P exists for some distinct values of a and b.

30. **[16]** Let $(x_1, y_1), \ldots, (x_k, y_k)$ be the distinct real solutions to the equation

$$(x^2 + y^2)^6 = (x^2 - y^2)^4 = (2x^3 - 6xy^2)^3.$$

Then $\sum_{i=1}^{k}(x_i + y_i)$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

31. **[16]** For a point $P = (x, y)$ in the Cartesian plane, let $f(P) = (x^2 - y^2, 2xy - y^2)$. If S is the set of all P so that the sequence $P, f(P), f(f(P)), f(f(f(P))), \ldots$ approaches $(0, 0)$, then the area of S can be expressed as $\pi\sqrt{r}$ for some positive real number r. Compute $\lfloor 100r \rfloor$.

32. **[16]** An ant starts at the point $(0, 0)$ in the Cartesian plane. In the first minute, the ant faces towards $(1, 0)$ and walks one unit. Each subsequent minute, the ant chooses an angle θ uniformly at random in the interval $[-90^\circ, 90^\circ]$, and then turns an angle of θ clockwise (negative values of θ correspond to counterclockwise rotations). Then, the ant walks one unit. After n minutes, the ant's distance from $(0, 0)$ is d_n. Let the expected value of d_n^2 be a_n. Compute the closest integer to

$$10 \lim_{n \to \infty} \frac{a_n}{n}.$$

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HMMT February 2022, February 19, 2022 — GUTS ROUND

Organization ________________________ Team ________________________ Team ID# ________________

33. **[20]** In last year's HMMT Spring competition, 557 students submitted at least one answer to each of the three individual tests. Let S be the set of these students, and let P be the set containing the 30 problems on the individual tests. Estimate A, the number of subsets $R \subseteq P$ for which some student in S answered the questions in R correctly but no others. An estimate of E earns $\max(0, \lfloor 20 - \frac{2}{3}|A - E| \rfloor)$ points.

34. **[20]** Estimate A, the number of unordered triples of integers (a, b, c) so that there exists a nondegenerate triangle with side lengths a, b, and c fitting inside a 100×100 square. An estimate of E earns $\max(0, \lfloor 20 - |A - E|/1000 \rfloor)$ points.

35. **[20]** A random permutation of $\{1, 2, \ldots, 100\}$ is given. It is then sorted to obtain the sequence $(1, 2, \ldots, 100)$ as follows: at each step, two of the numbers which are not in their correct positions are selected at random, and the two numbers are swapped. If s is the expected number of steps (i.e. swaps) required to obtain the sequence $(1, 2, \cdots, 100)$, then estimate $A = \lfloor s \rfloor$. An estimate of E earns $\max(0, \lfloor 20 - \frac{1}{2}|A - E| \rfloor)$ points.

36. **[20]** For a cubic polynomial $P(x)$ with complex roots z_1, z_2, z_3, let

$$M(P) = \frac{\max(|z_1 - z_2|, |z_1 - z_3|, |z_2 - z_3|)}{\min(|z_1 - z_2|, |z_1 - z_3|, |z_2 - z_3|)}.$$

Over all polynomials $P(x) = x^3 + ax^2 + bx + c$, where a, b, c are nonnegative integers at most 100 and $P(x)$ has no repeated roots, the twentieth largest possible value of $M(P)$ is m. Estimate $A = \lfloor m \rfloor$. An estimate of E earns $\max(0, \lfloor 20 - 20|3\ln(A/E)|^{1/2} \rfloor)$ points.