1^{st} Annual Harvard-MIT November Tournament

Saturday 8 November 2008

Individual Round

1. [**2**] Find the minimum of $x^2 - 2x$ over all real numbers x.

   Answer: $\boxed{\text{-1}}$ Write $x^2 - 2x = x^2 - 2x + 1 - 1 = (x-1)^2 - 1$. Since $(x-1)^2 \geq 0$, it is clear that the minimum is -1.

   Alternate method: The graph of $y = x^2 - 2x$ is a parabola that opens up. Therefore, the minimum occurs at its vertex, which is at $\frac{-b}{2a} = \frac{-(-2)}{2} = 1$. But $1^2 - 2 \cdot 1 = -1$, so the minimum is -1.

2. [**3**] What is the units digit of 7^{2009}?

   Answer: $\boxed{7}$ Note that the units digits of $7^1, 7^2, 7^3, 7^4, 7^5, 7^6, \ldots$ follows the pattern $7, 9, 3, 1, 7, 9, 3, 1, \ldots$. The 2009th term in this sequence should be 7.

   Alternate method: Note that the units digit of 7^4 is equal to 1, so the units digit of $(7^4)^{502}$ is also 1. But $(7^4)^{502} = 7^{2008}$, so the units digit of 7^{2008} is 1, and therefore the units digit of 7^{2009} is 7.

3. [**3**] How many diagonals does a regular undecagon (11-sided polygon) have?

   Answer: $\boxed{44}$ There are 8 diagonals coming from the first vertex, 8 more from the next, 7 from the next, 6 from the next, 5 from the next, etc., and 1 from the last, for $8+8+7+6+5+4+3+2+1 = 44$ total.

   Third method: Each vertex has 8 diagonals touching it. There are 11 vertices. Since each diagonal touches two vertices, this counts every diagonal twice, so there are $\frac{8 \cdot 11}{2} = 44$ diagonals.

4. [**4**] How many numbers between 1 and $1,000,000$ are perfect squares but not perfect cubes?

   Answer: $\boxed{990}$ $1000000 = 1000^2 = 10^6$. A number is both a perfect square and a perfect cube if and only if it is exactly a perfect sixth power. So, the answer is the number of perfect squares, minus the number of perfect sixth powers, which is $1000 - 10 = 990$.

5. [**5**] Joe has a triangle with area $\sqrt{3}$. What's the smallest perimeter it could have?

   Answer: $\boxed{6}$ The minimum occurs for an equilateral triangle. The area of an equilateral triangle with side-length s is $\frac{\sqrt{3}}{4}s^2$, so if the area is $\sqrt{3}$ then $s = \sqrt{\sqrt{3}\frac{4}{\sqrt{3}}} = 2$. Multiplying by 3 to get the perimeter yields the answer.

6. [**5**] We say "s grows to r" if there exists some integer $n > 0$ such that $s^n = r$. Call a real number r "sparse" if there are only finitely many real numbers s that grow to r. Find all real numbers that are sparse.

   Answer: $\boxed{\text{-1,0,1}}$ For any number x, other than these 3, x, $\sqrt[3]{x}$, $\sqrt[5]{x}$, $\sqrt[7]{x}, \ldots$ provide infinitely many possible values of s, so these are the only possible sparse numbers. On the other hand, -1 is the only possible value of s for $r = -1$, 0 is the only value for $r = 0$, and -1 and 1 are the only values for $r = 1$. Therefore, -1, 0, and 1 are all sparse.

7. [**6**] Find all ordered pairs (x, y) such that

$$(x - 2y)^2 + (y - 1)^2 = 0.$$

   Answer: $\boxed{\text{(2,1)}}$ The square of a real number is always at least 0, so to have equality we must have $(x - 2y)^2 = 0$ and $(y-1)^2 = 0$. Then $y = 1$ and $x = 2y = 2$.

8. [**7**] How many integers between 2 and 100 inclusive *cannot* be written as $m \cdot n$, where m and n have no common factors and neither m nor n is equal to 1? Note that there are 25 primes less than 100.

Answer: $\boxed{35}$ A number cannot be written in the given form if and only if it is a power of a prime. We can see this by considering the prime factorization. Suppose that $k = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, with $p_1, \ldots, p_n$ primes. Then we can write $m = p_1^{e_1}$ and $n = p_2^{e_2} \cdots p_n^{e_n}$. So, we want to find the powers of primes that are less than or equal to 100. There are 25 primes, as given in the problem statement. The squares of primes are $2^2, 3^2, 5^2, 7^2$. The cubes of primes are $2^3, 3^3$. The fourth powers of primes are $2^4, 3^4$. The fifth powers of primes are 2^5, The sixth powers of primes are 2^6. There are no seventh or higher powers of primes between 2 and 100. This adds 10 non-primes to the list, so that in total there are $10 + 25 = 35$ such integers.

9. [**7**] Find the product of all real x for which

$$2^{3x+1} - 17 \cdot 2^{2x} + 2^{x+3} = 0.$$

Answer: $\boxed{\text{-3}}$ We can re-write the equation as $2^x(2\cdot(2^x)^2 - 17\cdot(2^x) + 8) = 0$, or $2\cdot(2^x)^2 - 17\cdot(2^x) + 8 = 0$. Make the substitution $y = 2^x$. Then we have $2y^2 - 17y + 8 = 0$, which has solutions (by the quadratic formula) $y = \frac{17 \pm \sqrt{289-64}}{4} = \frac{17 \pm 15}{4} = 8, \frac{1}{2}$, so $2^x = 8, \frac{1}{2}$ and $x = 3, -1$. The product of these numbers is -3.

10. [**8**] Find the largest positive integer n such that $n^3 + 4n^2 - 15n - 18$ is the cube of an integer.

Answer: $\boxed{19}$ Note that the next cube after n^3 is $(n+1)^3 = n^3 + 3n^2 + 3n + 1$. After that, it is $(n+2)^3 = n^3 + 6n^2 + 12n + 8$. $n^3 + 6n^3 + 12n + 8$ is definitely bigger than $n^3 + 4n^2 - 15n - 18$, so the largest cube that $n^3 + 4n^2 - 15n - 18$ could be is $(n+1)^3$. On the other hand, for $n \geq 4$, $n^3 + 4n^2 - 15n - 18$ is larger than $(n-2)^3 = n^3 - 6n^2 + 12n - 8$ (as $4n^2 - 15n - 18 > -6n^2 + 12n - 8 \iff 10n^2 - 27n - 10 > 0$, which is true for $n \geq 4$).

So, we will check for all solutions to $n^3 + 4n^2 - 15n - 18 = (n-1)^3, n^3, (n+1)^3$. The first case yields

$$n^3 + 4n^2 - 15n - 18 = n^3 - 3n^2 + 3n - 1 \iff 7n^2 - 18n - 17 = 0$$

which has no integer solutions. The second case yields

$$n^3 + 4n^2 - 15n - 18 = n^3 \iff 4n^2 - 15n - 18 = 0$$

which also has no integer solutions. The final case yields

$$n^3 + 4n^2 - 15n - 18 = n^3 + 3n^2 + 3n + 1 \iff n^2 - 18n - 19 = 0$$

which has integer solutions $n = -1, 19$. So, the largest possible n is 19.

Remark: The easiest way to see that the first two polynomials have no integer solutions is using the *Rational Root Theorem*, which states that the rational solutions of a polynomial $ax^n + \ldots + b$ are all of the form $\pm\frac{b'}{a'}$, where b' divides b and a' divides a.