# HMMT February 2016

**February 20, 2016**

## Algebra

1. Let $z$ be a complex number such that $|z| = 1$ and $|z - 1.45| = 1.05$. Compute the real part of $z$.

2. For which integers $n \in \{1, 2, \ldots, 15\}$ is $n^n + 1$ a prime number?

3. Let $A$ denote the set of all integers $n$ such that $1 \leq n \leq 10000$, and moreover the sum of the decimal digits of $n$ is 2. Find the sum of the squares of the elements of $A$.

4. Determine the remainder when
$$\sum_{i=0}^{2015} \left\lfloor \frac{2^i}{25} \right\rfloor$$
is divided by 100, where $\lfloor x \rfloor$ denotes the largest integer not greater than $x$.

5. An infinite sequence of real numbers $a_1, a_2, \ldots$ satisfies the recurrence
$$a_{n+3} = a_{n+2} - 2a_{n+1} + a_n$$
for every positive integer $n$. Given that $a_1 = a_3 = 1$ and $a_{98} = a_{99}$, compute $a_1 + a_2 + \cdots + a_{100}$.

6. Call a positive integer $N \geq 2$ "special" if for every $k$ such that $2 \leq k \leq N$, $N$ can be expressed as a sum of $k$ positive integers that are relatively prime to $N$ (although not necessarily relatively prime to each other). How many special integers are there less than 100?

7. Determine the smallest positive integer $n \geq 3$ for which
$$A \equiv 2^{10n} \pmod{2^{170}}$$
where $A$ denotes the result when the numbers $2^{10}$, $2^{20}$, $\ldots$, $2^{10n}$ are written in decimal notation and concatenated (for example, if $n = 2$ we have $A = 10241048576$).

8. Define $\phi^!(n)$ as the product of all positive integers less than or equal to $n$ and relatively prime to $n$. Compute the number of integers $2 \leq n \leq 50$ such that $n$ divides $\phi^!(n) + 1$.

9. For any positive integer $n$, $S_n$ be the set of all permutations of $\{1, 2, 3, \ldots, n\}$. For each permutation $\pi \in S_n$, let $f(\pi)$ be the number of ordered pairs $(j, k)$ for which $\pi(j) > \pi(k)$ and $1 \leq j < k \leq n$. Further define $g(\pi)$ to be the number of positive integers $k \leq n$ such that $\pi(k) \equiv k \pm 1 \pmod{n}$. Compute
$$\sum_{\pi \in S_{999}} (-1)^{f(\pi) + g(\pi)}.$$

10. Let $a, b$ and $c$ be positive real numbers such that
$$\begin{aligned} a^2 + ab + b^2 &= 9 \\ b^2 + bc + c^2 &= 52 \\ c^2 + ca + a^2 &= 49. \end{aligned}$$
Compute the value of $\dfrac{49b^2 - 33bc + 9c^2}{a^2}$.