HMMT February 2015

Saturday 21 February 2015

Team

For any geometry problem (e.g. Problem 2, 4, or 5), your team's solution should include diagrams as appropriate (sufficiently large, in-scale, clearly labeled, etc.). Failure to meet any of these requirements will result in a 2-point automatic deduction. Thanks in advance for your effort and understanding, and we hope you enjoy the problems!

1. [**5**] The complex numbers x, y, z satisfy

$$\begin{aligned} xyz &= -4 \\ (x+1)(y+1)(z+1) &= 7 \\ (x+2)(y+2)(z+2) &= -3. \end{aligned}$$

   Find, with proof, the value of $(x+3)(y+3)(z+3)$.

2. [**10**] Let P be a (non-self-intersecting) polygon in the plane. Let $C_1, \ldots, C_n$ be circles in the plane whose interiors cover the interior of P. For $1 \le i \le n$, let r_i be the radius of C_i. Prove that there is a single circle of radius $r_1 + \cdots + r_n$ whose interior covers the interior of P.

3. [**15**] Let $z = a+bi$ be a complex number with integer real and imaginary parts $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$ (i.e. z is a Gaussian integer). If p is an odd prime number, show that the real part of $z^p - z$ is an integer divisible by p.

4. [**15**] (Convex) quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle Ω; the diagonals of $ABCD$ meet at X. Suppose $AD < AB$, the circumcircle of triangle BCX intersects segment AB at a point $Y \neq B$, and **ray** $\overrightarrow{CY}$ meets Ω again at a point $Z \neq C$. Prove that ray $\overrightarrow{DY}$ bisects angle ZDB.

   (We have only included the conditions $AD < AB$ and that Z lies on **ray** $\overrightarrow{CY}$ for everyone's convenience. With slight modifications, the problem holds in general. But we will only grade your team's solution in this special case.)

5. [**20**] For a convex quadrilateral P, let D denote the sum of the lengths of its diagonals and let S denote its perimeter. Determine, with proof, all possible values of $\frac{S}{D}$.

6. [**30**] \$indy has \$100 in pennies (worth \$0.01 each), nickels (worth \$0.05 each), dimes (worth \$0.10 each), and quarters (worth \$0.25 each). Prove that she can split her coins into two piles, each with total value exactly \$50.

7. [**35**] Let $f : [0,1] \to \mathbb{C}$ be a *nonconstant* complex-valued function on the real interval $[0,1]$. Prove that there exists $\epsilon > 0$ (possibly depending on f) such that for any polynomial P with complex coefficients, there exists a complex number z with $|z| \le 1$ such that $|f(|z|) - P(z)| \ge \epsilon$.

8. [**40**] Let π be a permutation of $\{1, 2, \ldots, 2015\}$. With proof, determine the maximum possible number of ordered pairs $(i,j) \in \{1, 2, \ldots, 2015\}^2$ with $i < j$ such that $\pi(i) \cdot \pi(j) > i \cdot j$.

9. [**40**] Let $z = e^{\frac{2\pi i}{101}}$ and let $\omega = e^{\frac{2\pi i}{10}}$. Prove that

$$\prod_{a=0}^{9} \prod_{b=0}^{100} \prod_{c=0}^{100} (\omega^a + z^b + z^c)$$

   is an integer and find (with proof) its remainder upon division by 101.

10. [**40**] The sequences of real numbers $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ satisfy $a_{n+1} = (a_{n-1} - 1)(b_n + 1)$ and $b_{n+1} = a_n b_{n-1} - 1$ for $n \ge 2$, with $a_1 = a_2 = 2015$ and $b_1 = b_2 = 2013$. Evaluate, with proof, the infinite sum

$$\sum_{n=1}^{\infty} b_n \left(\frac{1}{a_{n+1}} - \frac{1}{a_{n+3}} \right).$$