HMMT November 2024

November 09, 2024

General Round

1. Six consecutive positive integers are written on slips of paper. The slips are then handed out to Ethan, Jacob, and Karthik, such that each of them receives two slips. The product of Ethan's numbers is 20, and the product of Jacob's numbers is 24. Compute the product of Karthik's numbers.

2. Let $RANDOM$ be a regular hexagon with side length 1. Points I and T lie on segments $\overline{RA}$ and $\overline{DO}$, respectively, such that $MI = MT$ and $\angle TMI = 90^\circ$. Compute the area of triangle MIT.

3. Suppose that a, b, and c are *distinct* positive integers such that $a^b b^c = a^c$. Across all possible values of a, b, and c, compute the minimum value of $a + b + c$.

4. Compute the number of ways to pick a 3-element subset of
$$\{10^1 + 1, 10^2 + 1, 10^3 + 1, 10^4 + 1, 10^5 + 1, 10^6 + 1, 10^7 + 1\}$$
such that the product of the 3 numbers in the subset has no digits besides 0 and 1 when written in base 10.

5. Let f be a function on nonnegative integers such that $f(0) = 0$ and
$$f(3n + 2) = f(3n + 1) = f(3n) + 1 = 3f(n) + 1$$
for all integers $n \geq 0$. Compute the sum of all nonnegative integers m such that $f(m) = 13$.

6. A positive integer n is *stacked* if $2n$ has the same number of digits as n and the digits of $2n$ are multiples of the corresponding digits of n. For example, 1203 is stacked because $2 \times 1203 = 2406$, and $2, 4, 0, 6$ are multiples of $1, 2, 0, 3$, respectively. Compute the number of stacked integers less than 1000.

7. Let triangle ABC have $AB = 5, BC = 8$, and $\angle ABC = 60^\circ$. A circle ω tangent to segments $\overline{AB}$ and $\overline{BC}$ intersects segment $\overline{CA}$ at points X and Y such that points C, Y, X, and A lie along $\overline{CA}$ in this order. If ω is tangent to $\overline{AB}$ at point Z and $ZY \parallel BC$, compute the radius of ω.

8. Let
$$f(x) = \Bigg|\bigg|\cdots\bigg|\Big|\big||x| - 1\big| - 2\Big| - 3\bigg| - \cdots\bigg| - 10\Bigg|.$$
Compute $f(1) + f(2) + \cdots + f(54) + f(55)$.

9. Let $ABCDEF$ be a regular hexagon with center O and side length 1. Point X is placed in the interior of the hexagon such that $\angle BXC = \angle AXE = 90^\circ$. Compute all possible values of OX.

10. Let $S = \{1, 2, 3, \ldots, 64\}$. Compute the number of ways to partition S into 16 arithmetic sequences such that each arithmetic sequence has length 4 and common difference 1, 4, or 16.