HMMT 2014

THE HARVARD-MIT MATHEMATICS TOURNAMENT

GUTS

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1. [**4**] Compute the prime factorization of 159999.

2. [**4**] Let $x_1, \ldots, x_{100}$ be defined so that for each i, x_i is a (uniformly) random integer between 1 and 6 inclusive. Find the expected number of integers in the set $\{x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_{100}\}$ that are multiples of 6.

3. [**4**] Let $ABCDEF$ be a regular hexagon. Let P be the circle inscribed in $\triangle BDF$. Find the ratio of the area of circle P to the area of rectangle $ABDE$.

4. [**4**] Let D be the set of divisors of 100. Let Z be the set of integers between 1 and 100, inclusive. Mark chooses an element d of D and an element z of Z uniformly at random. What is the probability that d divides z?

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5. [**5**] If four fair six-sided dice are rolled, what is the probability that the lowest number appearing on any die is exactly 3?

6. [**5**] Find all integers n for which $\dfrac{n^3+8}{n^2-4}$ is an integer.

7. [**5**] The Evil League of Evil is plotting to poison the city's water supply. They plan to set out from their headquarters at $(5, 1)$ and put poison in two pipes, one along the line $y = x$ and one along the line $x = 7$. However, they need to get the job done quickly before Captain Hammer catches them. What's the shortest distance they can travel to visit both pipes and then return to their headquarters?

8. [**5**] The numbers $2^0, 2^1, \cdots, 2^{15}, 2^{16} = 65536$ are written on a blackboard. You repeatedly take two numbers on the blackboard, subtract one from the other, erase them both, and write the result of the subtraction on the blackboard. What is the largest possible number that can remain on the blackboard when there is only one number left?

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9. [**6**] Compute the side length of the largest cube contained in the region

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq 25 \text{ and } x \geq 0\}$$

of three-dimensional space.

10. [**6**] Find the number of nonempty sets $\mathcal{F}$ of subsets of the set $\{1, \ldots, 2014\}$ such that:

   (a) For any subsets $S_1, S_2 \in \mathcal{F}$, $S_1 \cap S_2 \in \mathcal{F}$.

   (b) If $S \in \mathcal{F}$, $T \subseteq \{1, \ldots, 2014\}$, and $S \subseteq T$, then $T \in \mathcal{F}$.

11. [**6**] Two fair octahedral dice, each with the numbers 1 through 8 on their faces, are rolled. Let N be the remainder when the product of the numbers showing on the two dice is divided by 8. Find the expected value of N.

12. [**6**] Find a nonzero monic polynomial $P(x)$ with integer coefficients and minimal degree such that $P(1 - \sqrt[3]{2} + \sqrt[3]{4}) = 0$. (A polynomial is called *monic* if its leading coefficient is 1.)

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13. [**8**] An auditorium has two rows of seats, with 50 seats in each row. 100 indistinguishable people sit in the seats one at a time, subject to the condition that each person, except for the first person to sit in each row, must sit to the left or right of an occupied seat, and no two people can sit in the same seat. In how many ways can this process occur?

14. [**8**] Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $\angle D = 90°$. Suppose that there is a point E on CD such that $AE = BE$ and that triangles AED and CEB are similar, but not congruent. Given that $\frac{CD}{AB} = 2014$, find $\frac{BC}{AD}$.

15. [**8**] Given a regular pentagon of area 1, a *pivot line* is a line not passing through any of the pentagon's vertices such that there are 3 vertices of the pentagon on one side of the line and 2 on the other. A *pivot point* is a point inside the pentagon with only finitely many non-pivot lines passing through it. Find the area of the region of pivot points.

16. [**8**] Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

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17. [**11**] Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following conditions:

    (a) $f(1) = 1$.

    (b) $f(a) \leq f(b)$ whenever a and b are positive integers with $a \leq b$.

    (c) $f(2a) = f(a) + 1$ for all positive integers a.

    How many possible values can the 2014-tuple $(f(1), f(2), ..., f(2014))$ take?

18. [**11**] Find the number of ordered quadruples of positive integers (a, b, c, d) such that a, b, c, and d are all (not necessarily distinct) factors of 30 and $abcd > 900$.

19. [**11**] Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at E, the bisectors of $\angle ABC$ and $\angle BCD$ meet at F, the bisectors of $\angle BCD$ and $\angle CDA$ meet at G, and the bisectors of $\angle DAB$ and $\angle ABC$ meet at H. Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle ABH has area 25. Find the area of triangle CDG.

20. [**11**] A deck of 8056 cards has 2014 ranks numbered 1–2014. Each rank has four suits—hearts, diamonds, clubs, and spades. Each card has a rank and a suit, and no two cards have the same rank and the same suit. How many subsets of the set of cards in this deck have cards from an odd number of distinct ranks?

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21. [**14**] Compute the number of ordered quintuples of nonnegative integers $(a_1, a_2, a_3, a_4, a_5)$ such that $0 \leq a_1, a_2, a_3, a_4, a_5 \leq 7$ and 5 divides $2^{a_1} + 2^{a_2} + 2^{a_3} + 2^{a_4} + 2^{a_5}$.

22. [**14**] Let ω be a circle, and let $ABCD$ be a quadrilateral inscribed in ω. Suppose that BD and AC intersect at a point E. The tangent to ω at B meets line AC at a point F, so that C lies between E and F. Given that $AE = 6, EC = 4, BE = 2$, and $BF = 12$, find DA.

23. [**14**] Let $S = \{-100, -99, -98, \ldots, 99, 100\}$. Choose a 50-element subset T of S at random. Find the expected number of elements of the set $\{|x| : x \in T\}$.

24. [**14**] Let $A = \{a_1, a_2, \ldots, a_7\}$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of A is an integer. Find the smallest possible value of the sum of the elements in A.

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25. [**17**] Let ABC be an equilateral triangle of side length 6 inscribed in a circle ω. Let A_1, A_2 be the points (distinct from A) where the lines through A passing through the two trisection points of BC meet ω. Define B_1, B_2, C_1, C_2 similarly. Given that $A_1, A_2, B_1, B_2, C_1, C_2$ appear on ω in that order, find the area of hexagon $A_1A_2B_1B_2C_1C_2$.

26. [**17**] For $1 \leq j \leq 2014$, define
$$b_j = j^{2014} \prod_{i=1, i \neq j}^{2014} (i^{2014} - j^{2014})$$
where the product is over all $i \in \{1, \ldots, 2014\}$ except $i = j$. Evaluate
$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{2014}}.$$

27. [**17**] Suppose that $(a_1, \ldots, a_{20})$ and $(b_1, \ldots, b_{20})$ are two sequences of integers such that the sequence $(a_1, \ldots, a_{20}, b_1, \ldots, b_{20})$ contains each of the numbers $1, \ldots, 40$ exactly once. What is the maximum possible value of the sum
$$\sum_{i=1}^{20} \sum_{j=1}^{20} \min(a_i, b_j)?$$

28. [**17**] Let $f(n)$ and $g(n)$ be polynomials of degree 2014 such that $f(n) + (-1)^n g(n) = 2^n$ for $n = 1, 2, \ldots, 4030$. Find the coefficient of x^{2014} in $g(x)$.

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29. [**20**] Natalie has a copy of the unit interval $[0,1]$ that is colored white. She also has a black marker, and she colors the interval in the following manner: at each step, she selects a value $x \in [0,1]$ uniformly at random, and

    (a) If $x \le \frac{1}{2}$ she colors the interval $[x, x+\frac{1}{2}]$ with her marker.

    (b) If $x > \frac{1}{2}$ she colors the intervals $[x,1]$ and $[0, x-\frac{1}{2}]$ with her marker.

    What is the expected value of the number of steps Natalie will need to color the entire interval black?

30. [**20**] Let ABC be a triangle with circumcenter O, incenter I, $\angle B = 45^\circ$, and $OI \parallel BC$. Find $\cos\angle C$.

31. [**20**] Compute
$$\sum_{k=1}^{1007}\left(\cos\left(\frac{\pi k}{1007}\right)\right)^{2014}.$$

32. [**20**] Find all ordered pairs (a,b) of complex numbers with $a^2+b^2 \neq 0$, $a+\frac{10b}{a^2+b^2}=5$, and $b+\frac{10a}{a^2+b^2}=4$.

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33. [**25**] An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_1), \ldots, (x_k, y_k)$ of points in $\mathbb{R}^2$ such that $(a, b) = (x_1, y_1)$, $(c, d) = (x_k, y_k)$, and for each $1 \le i < k$ we have that either $(x_{i+1}, y_{i+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$.

    Let S be the set of all up-right paths from $(-400, -400)$ to $(400, 400)$. What fraction of the paths in S do not contain any point (x, y) such that $|x|, |y| \le 10$? Express your answer as a decimal number between 0 and 1.

    If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1 - 10|C - A|), 0\} \rceil$.

34. [**25**] Consider a number line, with a lily pad placed at each integer point. A frog is standing at the lily pad at the point 0 on the number line, and wants to reach the lily pad at the point 2014 on the number line. If the frog stands at the point n on the number line, it can jump directly to either point $n+2$ or point $n+3$ on the number line. Each of the lily pads at the points $1, \cdots, 2013$ on the number line has, independently and with probability $1/2$, a snake. Let p be the probability that the frog can make some sequence of jumps to reach the lily pad at the point 2014 on the number line, without ever landing on a lily pad containing a snake. What is $p^{1/2014}$? Express your answer as a decimal number.

    If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1 - 20|C - A|), 0\} \rceil$.

35. [**25**] How many times does the letter "e" occur in all problem statements in this year's HMMT February competition?

    If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1 - |\log_2(C/A)|), 0\} \rceil$.

36. [**25**] We have two concentric circles C_1 and C_2 with radii 1 and 2, respectively. A random chord of C_2 is chosen. What is the probability that it intersects C_1?

    Your answer to this problem must be expressed in the form $\frac{m}{n}$, where m and n are positive integers. If your answer is in this form, your score for this problem will be $\lfloor \frac{25 \cdot X}{Y} \rfloor$, where X is the total number of teams who submit the answer $\frac{m}{n}$ (including your own team), and Y is the total number of teams who submit a valid answer. Otherwise, your score is 0. (Your answer is *not* graded based on correctness, whether your fraction is in lowest terms, whether it is at most 1, etc.)

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5. [**5**] ____________________

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9. [**6**] ____________________

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18. [**11**] ________________

19. [**11**] ________________

20. [**11**] ________________

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21. [**14**] ________________

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24. [**14**] ________________

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29. [**20**] ________________

30. [**20**] ________________

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36. [**25**] ________________