HMMT February 2015

Saturday 21 February 2015

Algebra

1. Let Q be a polynomial
$$Q(x) = a_0 + a_1x + \cdots + a_nx^n,$$
where $a_0, \ldots, a_n$ are nonnegative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

2. The fraction $\frac{1}{2015}$ has a unique "(restricted) partial fraction decomposition" of the form
$$\frac{1}{2015} = \frac{a}{5} + \frac{b}{13} + \frac{c}{31},$$
where a, b, c are integers with $0 \le a < 5$ and $0 \le b < 13$. Find $a + b$.

3. Let p be a real number and $c \neq 0$ an integer such that
$$c - 0.1 < x^p \left(\frac{1 - (1+x)^{10}}{1 + (1+x)^{10}} \right) < c + 0.1$$
for all (positive) real numbers x with $0 < x < 10^{-100}$. (The exact value 10^{-100} is not important. You could replace it with any "sufficiently small number".)

   Find the ordered pair (p, c).

4. Compute the number of sequences of integers $(a_1, \ldots, a_{200})$ such that the following conditions hold.

   - $0 \le a_1 < a_2 < \cdots < a_{200} \le 202$.
   - There exists a positive integer N with the following property: for every index $i \in \{1, \ldots, 200\}$ there exists an index $j \in \{1, \ldots, 200\}$ such that $a_i + a_j - N$ is divisible by 203.

5. Let a, b, c be positive real numbers such that $a+b+c = 10$ and $ab+bc+ca = 25$. Let $m = \min\{ab, bc, ca\}$. Find the largest possible value of m.

6. Let a, b, c, d, e be nonnegative integers such that $625a + 250b + 100c + 40d + 16e = 15^3$. What is the maximum possible value of $a + b + c + d + e$?

7. Suppose (a_1, a_2, a_3, a_4) is a 4-term sequence of real numbers satisfying the following two conditions:

   - $a_3 = a_2 + a_1$ and $a_4 = a_3 + a_2$;
   - there exist real numbers a, b, c such that
   $$an^2 + bn + c = \cos(a_n)$$
   for all $n \in \{1, 2, 3, 4\}$.

   Compute the maximum possible value of
   $$\cos(a_1) - \cos(a_4)$$
   over all such sequences (a_1, a_2, a_3, a_4).

8. Find the number of ordered pairs of integers $(a, b) \in \{1, 2, \ldots, 35\}^2$ (not necessarily distinct) such that $ax + b$ is a "quadratic residue modulo $x^2 + 1$ and 35", i.e. there exists a polynomial $f(x)$ with integer coefficients such that either of the following **equivalent** conditions holds:

   - there exist polynomials P, Q with integer coefficients such that $f(x)^2 - (ax + b) = (x^2 + 1)P(x) + 35Q(x)$;

- or more conceptually, the remainder when (the polynomial) $f(x)^2 - (ax+b)$ is divided by (the polynomial) x^2+1 is a polynomial with (integer) coefficients all divisible by 35.

9. Let $N = 30^{2015}$. Find the number of ordered 4-tuples of integers $(A, B, C, D) \in \{1, 2, \ldots, N\}^4$ (not necessarily distinct) such that for every integer n, $An^3 + Bn^2 + 2Cn + D$ is divisible by N.

10. Find all ordered 4-tuples of integers (a, b, c, d) (not necessarily distinct) satisfying the following system of equations:

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= c - b - 2 \\ 2ab &= a - d - 32 \\ 2ac &= 28 - a - d \\ 2ad &= b + c + 31. \end{aligned}$$