HMMT February 2020

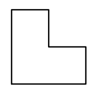
February 15, 2020

Geometry

1. Let $DIAL$, FOR, and $FRIEND$ be regular polygons in the plane. If $ID = 1$, find the product of all possible areas of OLA.

2. Let ABC be a triangle with $AB = 5$, $AC = 8$, and $\angle BAC = 60^\circ$. Let $UVWXYZ$ be a regular hexagon that is inscribed inside ABC such that U and V lie on side BA, W and X lie on side AC, and Z lies on side CB. What is the side length of hexagon $UVWXYZ$?

3. Consider the L-shaped tromino below with 3 attached unit squares. It is cut into exactly two pieces of equal area by a line segment whose endpoints lie on the perimeter of the tromino. What is the longest possible length of the line segment?



4. Let $ABCD$ be a rectangle and E be a point on segment AD. We are given that quadrilateral $BCDE$ has an inscribed circle ω_1 that is tangent to BE at T. If the incircle ω_2 of ABE is also tangent to BE at T, then find the ratio of the radius of ω_1 to the radius of ω_2.

5. Let $ABCDEF$ be a regular hexagon with side length 2. A circle with radius 3 and center at A is drawn. Find the area inside quadrilateral $BCDE$ but outside the circle.

6. Let ABC be a triangle with $AB = 5$, $BC = 6$, $CA = 7$. Let D be a point on ray AB beyond B such that $BD = 7$, E be a point on ray BC beyond C such that $CE = 5$, and F be a point on ray CA beyond A such that $AF = 6$. Compute the area of the circumcircle of DEF.

7. Let Γ be a circle, and ω_1 and ω_2 be two non-intersecting circles inside Γ that are internally tangent to Γ at X_1 and X_2, respectively. Let one of the common internal tangents of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2, respectively, while intersecting Γ at two points A and B. Given that $2X_1T_1 = X_2T_2$ and that ω_1, ω_2, and Γ have radii 2, 3, and 12, respectively, compute the length of AB.

8. Let ABC be an acute triangle with circumcircle Γ. Let the internal angle bisector of $\angle BAC$ intersect BC and Γ at E and N, respectively. Let A' be the antipode of A on Γ and let V be the point where AA' intersects BC. Given that $EV = 6$, $VA' = 7$, and $A'N = 9$, compute the radius of Γ.

9. Circles $\omega_a, \omega_b, \omega_c$ have centers A, B, C, respectively and are pairwise externally tangent at points D, E, F (with $D \in BC, E \in CA, F \in AB$). Lines BE and CF meet at T. Given that ω_a has radius 341, there exists a line ℓ tangent to all three circles, and there exists a circle of radius 49 tangent to all three circles, compute the distance from T to ℓ.

10. Let Γ be a circle of radius 1 centered at O. A circle Ω is said to be *friendly* if there exist distinct circles $\omega_1, \omega_2, \ldots, \omega_{2020}$, such that for all $1 \leq i \leq 2020$, ω_i is tangent to Γ, Ω, and ω_{i+1}. (Here, $\omega_{2021} = \omega_1$.) For each point P in the plane, let $f(P)$ denote the sum of the areas of all friendly circles centered at P. If A and B are points such that $OA = \frac{1}{2}$ and $OB = \frac{1}{3}$, determine $f(A) - f(B)$.