HMMT 2014

Saturday 22 February 2014

Geometry

1. Let O_1 and O_2 be concentric circles with radii 4 and 6, respectively. A chord AB is drawn in O_1 with length 2. Extend AB to intersect O_2 in points C and D. Find CD.

   Answer: $\boxed{2\sqrt{21}}$ Let O be the common center of O_1 and O_2, and let M be the midpoint of AB. Then $OM \perp AB$, so by the Pythagorean Theorem, $OM = \sqrt{4^2 - 1^2} = \sqrt{15}$. Thus $CD = 2CM = 2\sqrt{6^2 - 15} = 2\sqrt{21}$.

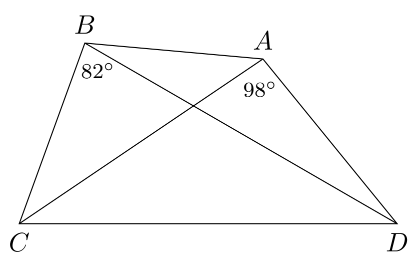
2. Point P and line ℓ are such that the distance from P to ℓ is 12. Given that T is a point on ℓ such that $PT = 13$, find the radius of the circle passing through P and tangent to ℓ at T.

   Answer: $\boxed{169/24}$ Let O be the center of the given circle, Q be the foot of the altitude from P to ℓ, and M be the midpoint of PT. Then since $OM \perp PT$ and $\angle OTP = \angle TPQ$, $\Delta OMP \sim \Delta TQP$. Thus the $OP = TP \cdot \frac{PM}{PQ} = 13 \cdot \frac{13/2}{12} = \frac{169}{24}$

3. ABC is a triangle such that $BC = 10$, $CA = 12$. Let M be the midpoint of side AC. Given that BM is parallel to the external bisector of $\angle A$, find area of triangle ABC. (Lines AB and AC form two angles, one of which is $\angle BAC$. The *external bisector* of $\angle A$ is the line that bisects the other angle.)

   Answer: $\boxed{8\sqrt{14}}$ Since BM is parallel to the external bisector of $\angle A = \angle BAM$, it is perpendicular to the angle bisector of $\angle BAM$. Thus $BA = BM = \frac{1}{2}BC = 6$. By Heron's formula, the area of ΔABC is therefore $\sqrt{(14)(8)(4)(2)} = 8\sqrt{14}$.

4. In quadrilateral $ABCD$, $\angle DAC = 98°, \angle DBC = 82°, \angle BCD = 70°$, and $BC = AD$. Find $\angle ACD$.



   Answer: $\boxed{28}$ Let B' be the reflection of B across CD. Note that $AD = BC$, and $\angle DAC + \angle CB'D = 180°$, so $ACB'D$ is a cyclic trapezoid. Thus, $ACB'D$ is an isosceles trapezoid, so $\angle ACB' = 98°$. Note that $\angle DCB' = \angle BCD = 70°$, so $\angle ACD = \angle ACB' - \angle DCB' = 98° - 70° = 28°$.

5. Let $\mathcal{C}$ be a circle in the xy plane with radius 1 and center $(0,0,0)$, and let P be a point in space with coordinates $(3,4,8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base $\mathcal{C}$ and vertex P.

   Answer: $\boxed{3 - \sqrt{5}}$ Consider the plane passing through P that is perpendicular to the plane of the circle. The intersection of the plane with the cone and sphere is a cross section consisting of a circle inscribed in a triangle with a vertex P. By symmetry, this circle is a great circle of the sphere, and hence has the same radius. The other two vertices of the triangle are the points of intersection between the plane and the unit circle, so the other two vertices are $(\frac{3}{5}, \frac{4}{5}, 0), (-\frac{3}{5}, -\frac{4}{5}, 0)$.

   Using the formula $A = rs$ and using the distance formula to find the side lengths, we find that $r = \frac{2A}{2s} = \frac{2*8}{2+10+4\sqrt{5}} = 3 - \sqrt{5}$.

6. In quadrilateral $ABCD$, we have $AB = 5, BC = 6, CD = 5, DA = 4$, and $\angle ABC = 90°$. Let AC and BD meet at E. Compute $\frac{BE}{ED}$.

   Answer: $\boxed{\sqrt{3}}$ We find that $AC = \sqrt{61}$, and applying the law of cosines to triangle ACD tells us that $\angle ADC = 120$. Then $\frac{BE}{ED}$ is the ratio of the areas of triangles ABC and ADC, which is $\frac{(5)(6)}{(4)(5)\frac{\sqrt{3}}{2}} = \sqrt{3}$.

7. Triangle ABC has sides $AB = 14, BC = 13$, and $CA = 15$. It is inscribed in circle Γ, which has center O. Let M be the midpoint of AB, let B' be the point on Γ diametrically opposite B, and let X be the intersection of AO and MB'. Find the length of AX.

   Answer: $\boxed{65/12}$ Since $B'B$ is a diameter, $\angle B'AB = 90°$, so $B'A \parallel OM$, so $\frac{OM}{B'A} = \frac{BM}{BA} = \frac{1}{2}$. Thus $\frac{AX}{XO} = \frac{B'A}{OM} = 2$, so $AX = \frac{2}{3}R$, where $R = \frac{abc}{4A} = \frac{(13)(14)(15)}{4(84)} = \frac{65}{8}$ is the circumradius of ABC. Putting it all together gives $AX = \frac{65}{12}$.

8. Let ABC be a triangle with sides $AB = 6$, $BC = 10$, and $CA = 8$. Let M and N be the midpoints of BA and BC, respectively. Choose the point Y on ray CM so that the circumcircle of triangle AMY is tangent to AN. Find the area of triangle NAY.

   Answer: $\boxed{600/73}$ Let $G = AN \cap CM$ be the centroid of ABC. Then $GA = \frac{2}{3}GN = \frac{10}{3}$ and $GM = \frac{1}{3}CM = \frac{1}{3}\sqrt{8^2+3^2} = \frac{\sqrt{73}}{3}$. By power of a point, $(GM)(GY) = GA^2$, so $GY = \frac{GA^2}{GY} = \frac{(10/3)^2}{\frac{\sqrt{73}}{3}} = \frac{100}{3\sqrt{73}}$. Thus

$$\begin{aligned}[NAY] &= [GAM] \cdot \frac{[GAY]}{[GAM]} \cdot \frac{[NAY]}{[GAY]} \\ &= \frac{1}{6}[ABC] \cdot \frac{GY}{GM} \cdot \frac{NA}{GA} \\ &= 4 \cdot \frac{100}{73} \cdot \frac{3}{2} = \frac{600}{73}\end{aligned}$$

9. Two circles are said to be *orthogonal* if they intersect in two points, and their tangents at either point of intersection are perpendicular. Two circles ω_1 and ω_2 with radii 10 and 13, respectively, are externally tangent at point P. Another circle ω_3 with radius $2\sqrt{2}$ passes through P and is orthogonal to both ω_1 and ω_2. A fourth circle ω_4, orthogonal to ω_3, is externally tangent to ω_1 and ω_2. Compute the radius of ω_4.

   Answer: $\boxed{\frac{92}{61}}$ Let ω_i have center O_i and radius r_i. Since ω_3 is orthogonal to $\omega_1, \omega_2, \omega_4$, it has equal power r_3^2 to each of them. Thus O_3 is the radical center of $\omega_1, \omega_2, \omega_4$, which is equidistant to the three sides of $\triangle O_1O_2O_4$ and therefore its incenter.

   For distinct $i, j \in \{1, 2, 4\}$, $\omega_i \cap \omega_j$ lies on the circles with diameters O_3O_i and O_3O_j, and hence ω_3 itself. It follows that ω_3 is the incircle of $\triangle O_1O_2O_4$, so $8 = r_3^2 = \frac{r_1r_2r_4}{r_1+r_2+r_4} = \frac{130r_4}{23+r_4} \implies r_4 = \frac{92}{61}$.

   Comment: The condition $P \in \omega_3$ is unnecessary.

10. Let ABC be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Let Γ be the circumcircle of ABC, let O be its circumcenter, and let M be the midpoint of minor arc $\widehat{BC}$. Circle ω_1 is internally tangent to Γ at A, and circle ω_2, centered at M, is externally tangent to ω_1 at a point T. Ray AT meets segment BC at point S, such that $BS - CS = 4/15$. Find the radius of ω_2.

    Answer: $\boxed{1235/108}$ Let N be the midpoint of BC. Notice that $BS - CS = \frac{4}{15}$ means that $NS = \frac{2}{15}$. let lines MN and AS meet at P, and let D be the foot of the altitude from A to BC. Then $BD = 5$ and $AD = 12$, so $DN = 2$ and $DS = \frac{32}{15}$. Thus $NP = AD\frac{SN}{SD} = 12\frac{2/15}{32/15} = \frac{3}{4}$. Now $OB = R = \frac{abc}{4A} = \frac{(13)(14)(15)}{4(84)} = \frac{65}{8}$, so $ON = \sqrt{OB^2 - BN^2} = \sqrt{\left(\frac{65}{8}\right)^2 - 7^2} = \frac{33}{8}$. Thus $OP = \frac{27}{8}$ and $PM = OM - OP = \frac{19}{4}$. By Monge's theorem, the exsimilicenter of ω_1 and Γ (which is A), the insimilicenter of ω_1 and ω_2 (which is T), and the insimilicenter of ω_2 and Γ (call this P') are collinear. But notice that this means $P' = OM \cap AT = P$. From this we get

$$\frac{\text{radius of } \omega_2}{R} = \frac{MP}{OP} = \frac{38}{27}.$$

    Thus the radius of ω_2 is $\frac{65}{8} \cdot \frac{38}{27} = \frac{1235}{108}$.