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1. [**5**] Let x and y be complex numbers such that $x+y=\sqrt{20}$ and $x^2+y^2=15$. Compute $|x-y|$.

2. [**5**] Sherry is waiting for a train. Every minute, there is a 75% chance that a train will arrive. However, she is engrossed in her game of sudoku, so even if a train arrives she has a 75% chance of not noticing it (and hence missing the train). What is the probability that Sherry catches the train in the next five minutes?

3. [**5**] Let $PROBLEMZ$ be a regular octagon inscribed in a circle of unit radius. Diagonals MR, OZ meet at I. Compute LI.

4. [**5**] Consider a three-person game involving the following three types of fair six-sided dice.

   - Dice of type A have faces labelled 2, 2, 4, 4, 9, 9.
   - Dice of type B have faces labelled 1, 1, 6, 6, 8, 8.
   - Dice of type C have faces labelled 3, 3, 5, 5, 7, 7.

   All three players simultaneously choose a die (more than one person can choose the same type of die, and the players don't know one another's choices) and roll it. Then the score of a player P is the number of players whose roll is less than P's roll (and hence is either 0, 1, or 2). Assuming all three players play optimally, what is the expected score of a particular player?

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5. [**6**] Patrick and Anderson are having a snowball fight. Patrick throws a snowball at Anderson which is shaped like a sphere with a radius of 10 centimeters. Anderson catches the snowball and uses the snow from the snowball to construct snowballs with radii of 4 centimeters. Given that the total volume of the snowballs that Anderson constructs cannot exceed the volume of the snowball that Patrick threw, how many snowballs can Anderson construct?

6. [**6**] Consider a $2\times n$ grid of points and a path consisting of $2n-1$ straight line segments connecting all these $2n$ points, starting from the bottom left corner and ending at the upper right corner. Such a path is called *efficient* if each point is only passed through once and no two line segments intersect. How many efficient paths are there when $n=2016$?

7. [**6**] A contest has six problems worth seven points each. On any given problem, a contestant can score either 0, 1, or 7 points. How many possible total scores can a contestant achieve over all six problems?

8. [**6**] For each positive integer n and non-negative integer k, define $W(n,k)$ recursively by

$$W(n,k)=\begin{cases} n^n & k=0 \\ W(W(n,k-1),k-1) & k>0. \end{cases}$$

   Find the last three digits in the decimal representation of $W(555,2)$.

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9. [**7**] Victor has a drawer with two red socks, two green socks, two blue socks, two magenta socks, two lavender socks, two neon socks, two mauve socks, two wisteria socks, and 2000 copper socks, for a total of 2016 socks. He repeatedly draws two socks at a time from the drawer at random, and stops if the socks are of the same color. However, Victor is red-green colorblind, so he also stops if he sees a red and green sock.

   What is the probability that Victor stops with two socks of the same color? Assume Victor returns both socks to the drawer at each step.

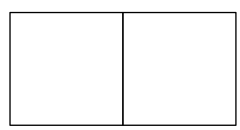
10. [**7**] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let O be the circumcenter of ABC. Find the distance between the circumcenters of triangles AOB and AOC.

11. [**7**] Define $\phi^!(n)$ as the product of all positive integers less than or equal to n and relatively prime to n. Compute the remainder when

$$\sum_{\substack{2\le n\le 50\\ \gcd(n,50)=1}} \phi^!(n)$$

   is divided by 50.

12. [**7**] Let R be the rectangle in the Cartesian plane with vertices at $(0,0), (2,0), (2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.



   Compute the number of ways to choose one or more of the seven edges such that the resulting figure is traceable without lifting a pencil. (Rotations and reflections are considered distinct.)

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13. [**9**] A right triangle has side lengths a, b, and $\sqrt{2016}$ in some order, where a and b are positive integers. Determine the smallest possible perimeter of the triangle.

14. [**9**] Let ABC be a triangle such that $AB = 13$, $BC = 14$, $CA = 15$ and let E, F be the feet of the altitudes from B and C, respectively. Let the circumcircle of triangle AEF be ω. We draw three lines, tangent to the circumcircle of triangle AEF at A, E, and F. Compute the area of the triangle these three lines determine.

15. [**9**] Compute $\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right)$.

16. [**9**] Determine the number of integers $2 \le n \le 2016$ such that $n^n - 1$ is divisible by 2, 3, 5, 7.

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17. [**11**] Compute the sum of all integers $1 \le a \le 10$ with the following property: there exist integers p and q such that p, q, $p^2 + a$ and $q^2 + a$ are all distinct prime numbers.

18. [**11**] Alice and Bob play a game on a circle with 8 marked points. Alice places an apple beneath one of the points, then picks five of the other seven points and reveals that none of them are hiding the apple. Bob then drops a bomb on any of the points, and destroys the apple if he drops the bomb either on the point containing the apple or on an adjacent point. Bob wins if he destroys the apple, and Alice wins if he fails. If both players play optimally, what is the probability that Bob destroys the apple?

19. [**11**] Let

$$A = \lim_{n\to\infty} \sum_{i=0}^{2016} (-1)^i \cdot \frac{\binom{n}{i}\binom{n}{i+2}}{\binom{n}{i+1}^2}$$

Find the largest integer less than or equal to $\frac{1}{A}$.

The following decimal approximation might be useful: $0.6931 < \ln(2) < 0.6932$, where ln denotes the natural logarithm function.

20. [**11**] Let ABC be a triangle with $AB = 13$, $AC = 14$, and $BC = 15$. Let G be the point on AC such that the reflection of BG over the angle bisector of $\angle B$ passes through the midpoint of AC. Let Y be the midpoint of GC and X be a point on segment AG such that $\frac{AX}{XG} = 3$. Construct F and H on AB and BC, respectively, such that $FX \parallel BG \parallel HY$. If AH and CF concur at Z and W is on AC such that $WZ \parallel BG$, find WZ.

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21. [**12**] Tim starts with a number n, then repeatedly flips a fair coin. If it lands heads he subtracts 1 from his number and if it lands tails he subtracts 2. Let E_n be the expected number of flips Tim does before his number is zero or negative. Find the pair (a, b) such that

$$\lim_{n\to\infty} (E_n - an - b) = 0.$$

22. [**12**] On the Cartesian plane $\mathbb{R}^2$, a circle is said to be *nice* if its center is at the origin $(0, 0)$ and it passes through at least one lattice point (i.e. a point with integer coordinates). Define the points $A = (20, 15)$ and $B = (20, 16)$. How many nice circles intersect the open segment AB?

For reference, the numbers 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691 are the only prime numbers between 600 and 700.

23. [**12**] Let $t = 2016$ and $p = \ln 2$. Evaluate in closed form the sum

$$\sum_{k=1}^{\infty} \left(1 - \sum_{n=0}^{k-1} \frac{e^{-t}t^n}{n!}\right) (1-p)^{k-1} p.$$

24. [**12**] Let ΔA_1B_1C be a triangle with $\angle A_1B_1C = 90^\circ$ and $\frac{CA_1}{CB_1} = \sqrt{5} + 2$. For any $i \ge 2$, define A_i to be the point on the line A_1C such that $A_iB_{i-1} \perp A_1C$ and define B_i to be the point on the line B_1C such that $A_iB_i \perp B_1C$. Let Γ_1 be the incircle of ΔA_1B_1C and for $i \ge 2$, Γ_i be the circle tangent to Γ_{i-1}, A_1C, B_1C which is smaller than Γ_{i-1}.

How many integers k are there such that the line A_1B_{2016} intersects Γ_k?

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25. [**14**] A particular coin can land on heads (H), on tails (T), or in the middle (M), each with probability $\frac{1}{3}$. Find the expected number of flips necessary to observe the contiguous sequence HMMTHMMT...HMMT, where the sequence HMMT is repeated 2016 times.

26. [**14**] For positive integers a, b, $a \uparrow\uparrow b$ is defined as follows: $a \uparrow\uparrow 1 = a$, and $a \uparrow\uparrow b = a^{a \uparrow\uparrow (b-1)}$ if $b > 1$.

    Find the smallest positive integer n for which there exists a positive integer a such that $a \uparrow\uparrow 6 \not\equiv a \uparrow\uparrow 7 \mod n$.

27. [**14**] Find the smallest possible area of an ellipse passing through $(2,0)$, $(0,3)$, $(0,7)$, and $(6,0)$.

28. [**14**] Among citizens of Cambridge there exist 8 different types of blood antigens. In a crowded lecture hall are 256 students, each of whom has a blood type corresponding to a distinct subset of the antigens; the remaining of the antigens are foreign to them.

    Quito the Mosquito flies around the lecture hall, picks a subset of the students uniformly at random, and bites the chosen students in a random order. After biting a student, Quito stores a bit of any antigens that student had. A student bitten while Quito had k blood antigen foreign to him/her will suffer for k hours. What is the expected total suffering of all 256 students, in hours?

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29. [**16**] Katherine has a piece of string that is 2016 millimeters long. She cuts the string at a location chosen uniformly at random, and takes the left half. She continues this process until the remaining string is less than one millimeter long. What is the expected number of cuts that she makes?

30. [**16**] Determine the number of triples $0 \le k, m, n \le 100$ of integers such that

$$2^m n - 2^n m = 2^k.$$

31. [**16**] For a positive integer n, denote by $\tau(n)$ the number of positive integer divisors of n, and denote by $\phi(n)$ the number of positive integers that are less than or equal to n and relatively prime to n. Call a positive integer n *good* if $\varphi(n) + 4\tau(n) = n$. For example, the number 44 is good because $\varphi(44) + 4\tau(44) = 44$.

    Find the sum of all good positive integers n.

32. [**16**] How many equilateral hexagons of side length $\sqrt{13}$ have one vertex at $(0,0)$ and the other five vertices at lattice points?

    (A lattice point is a point whose Cartesian coordinates are both integers. A hexagon may be concave but not self-intersecting.)

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33. [**20**] **(Lucas Numbers)** The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for every $n \geq 0$. There are N integers $1 \leq n \leq 2016$ such that L_n contains the digit 1. Estimate N.

    An estimate of E earns $\lfloor 20 - 2|N - E| \rfloor$ or 0 points, whichever is greater.

34. [**20**] **(Caos)** A cao [sic] has 6 legs, 3 on each side. A walking pattern for the cao is defined as an ordered sequence of raising and lowering each of the legs exactly once (altogether 12 actions), starting and ending with all legs on the ground. The pattern is safe if at any point, he has at least 3 legs on the ground and not all three legs are on the same side. Estimate N, the number of safe patterns.

    An estimate of $E > 0$ earns $\left\lfloor 20 \min(N/E, E/N)^4 \right\rfloor$ points.

35. [**20**] **(Maximal Determinant)** In a 17×17 matrix M, all entries are ± 1. The maximum possible value of $|\det M|$ is N. Estimate N.

    An estimate of $E > 0$ earns $\left\lfloor 20 \min(N/E, E/N)^2 \right\rfloor$ points.

36. [**20**] **(Self-Isogonal Cubics)** Let ABC be a triangle with $AB = 2$, $AC = 3$, $BC = 4$. The *isogonal conjugate* of a point P, denoted P^*, is the point obtained by intersecting the reflection of lines PA, PB, PC across the angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively.

    Given a point Q, let $\mathfrak{K}(Q)$ denote the unique cubic plane curve which passes through all points P such that line PP^* contains Q. Consider:

    (a) the M'Cay cubic $\mathfrak{K}(O)$, where O is the circumcenter of $\triangle ABC$,

    (b) the Thomson cubic $\mathfrak{K}(G)$, where G is the centroid of $\triangle ABC$,

    (c) the Napoleon-Feurerbach cubic $\mathfrak{K}(N)$, where N is the nine-point center of $\triangle ABC$,

    (d) the Darboux cubic $\mathfrak{K}(L)$, where L is the de Longchamps point (the reflection of the orthocenter across point O),

    (e) the Neuberg cubic $\mathfrak{K}(X_{30})$, where X_{30} is the point at infinity along line OG,

    (f) the nine-point circle of $\triangle ABC$,

    (g) the incircle of $\triangle ABC$, and

    (h) the circumcircle of $\triangle ABC$.

    Estimate N, the number of points lying on at least two of these eight curves. An estimate of E earns $\left\lfloor 20 \cdot 2^{-|N-E|/6} \right\rfloor$ points.