HMMT November 2024

November 09, 2024

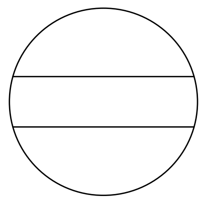
Guts Round

1. **[5]** A circle of area 1 is cut by two distinct chords. Compute the maximum possible area of the smallest resulting piece.

   Proposed by: Derek Liu, Luke Robitaille

   Answer: $\boxed{\frac{1}{3}}$

   Solution: At least 3 pieces are formed, so one of them has area at most $\frac{1}{3}$. This can be achieved with two parallel chords:



2. **[5]** Compute the smallest integer $n > 72$ that has the same set of prime divisors as 72.

   Proposed by: Carlos Rodriguez

   Answer: $\boxed{96}$

   Solution: The prime divisors of 72 are 2 and 3, and we note that $72 = 2^3 \cdot 3^2$. Since we need at least one factor of 2 and one factor of 3, we just need to check the multiples of 6 from 72 onwards.

   $13 \cdot 6$, $14 \cdot 6$, and $15 \cdot 6$ have prime factors of 13, 7, and 5 respectively; thus we get $\boxed{96} = 2^5 \cdot 3 = 16 \cdot 6$ as the smallest number that satisfies the condition given in the problem.

3. **[5]** The graphs of the lines

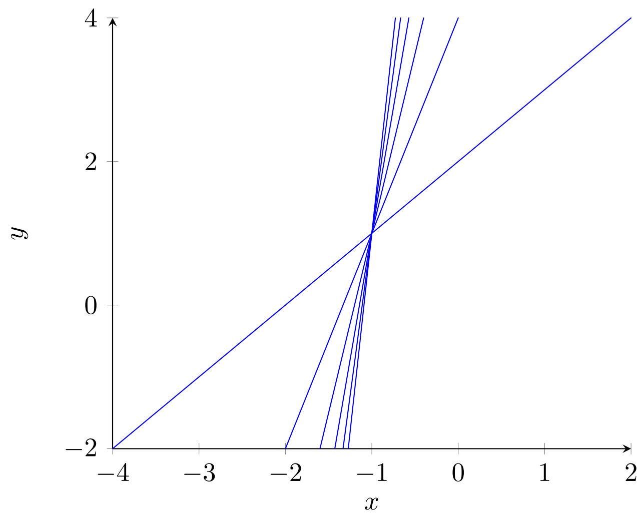
$$y = x + 2, \quad y = 3x + 4, \quad y = 5x + 6, \quad y = 7x + 8, \quad y = 9x + 10, \quad y = 11x + 12$$

   are drawn. These six lines divide the plane into several regions. Compute the number of regions the plane is divided into.

   Proposed by: Arul Kolla

   Answer: $\boxed{12}$

   Solution:



All lines are of the form $y = xk + (k + 1)$. Note that all lines pass through the point $(-1, 1)$, since $1 = (-1)k + (k + 1)$ for all k.

Thus all lines pass through a single point, $(-1, 1)$. The first line divides the plane into two parts, and each subsequent line divides two of the current regions into two more parts. Thus altogether the six lines divide the plane into $\boxed{12}$ parts.

4. **[6]** The number 17^6 when written out in base 10 contains 8 distinct digits from $1, 2, \ldots, 9$, with no repeated digits or zeroes. Compute the missing nonzero digit.

   Proposed by: Karthik Venkata Vedula

   Answer: $\boxed{8}$

   Solution: Observe that

$$17^6 \equiv (-1)^6 = 1 \bmod 9.$$

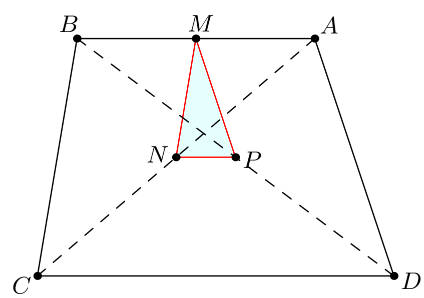
   If x is the missing digit, then the digits of 17^6 sum to $(1+2+\cdots+9)-x = 45-x$, so $45-x \equiv 1 \bmod 9$. We conclude $x = \boxed{8}$.

5. **[6]** Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 20$, $CD = 24$, and area 880. Compute the area of the triangle formed by the midpoints of AB, AC, and BD.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{20}$

   Solution:



We first compute the height of the trapezoid. If h is the height, then the area is

$$880 = \frac{1}{2}h(20 + 24),$$

so $h = 40$. Now, let M, N, P be the midpoints of AB, AC, and BD. Notice that PN is parallel to AB. Thus, the altitude from M to NP has length $\frac{h}{2} = 20$.

To compute NP, let X be the midpoint of BC. Since XN is a midsegment of $\triangle CAB$, we have $XN = \frac{AB}{2} = 10$. Since XP is a midsegment of $\triangle BCD$, we have $XP = \frac{CD}{2} = 12$. Hence, $NP = XP - XN = 2$.

Thus, the area of triangle MNP is $\frac{1}{2} \cdot 2 \cdot 20 = \boxed{20}$.

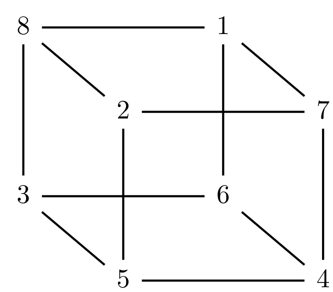
6. **[6]** The vertices of a cube are labeled with the integers 1 through 8, with each used exactly once. Let s be the maximum sum of the labels of two edge-adjacent vertices. Compute the minimum possible value of s over all such labelings.

Proposed by: Derek Liu

Answer: $\boxed{11}$

Solution: The answer must be at least 11, because the label 8 is adjacent to three vertices, one of which has label at least 3.

To show 11 is achievable, note that the following labelling achieves $s = 11$:



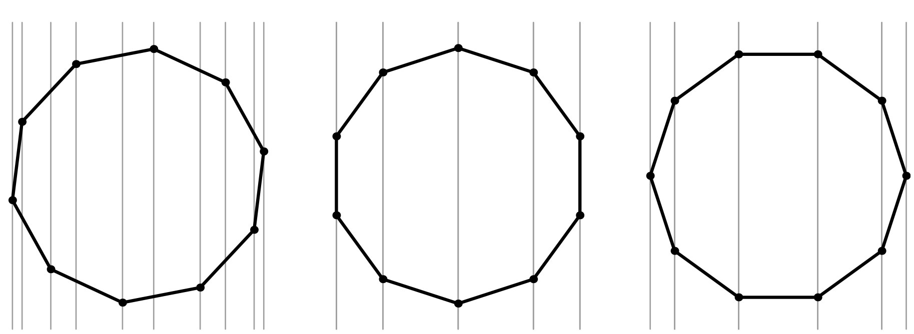
Thus the answer is $\boxed{11}$.

7. **[7]** Let $\mathcal{P}$ be a regular 10-gon in the coordinate plane. Mark computes the number of distinct x-coordinates that vertices of $\mathcal{P}$ take. Across all possible placements of $\mathcal{P}$ in the plane, compute the sum of all possible answers Mark could get.

Proposed by: Srinivas Arun

Answer: $\boxed{21}$

Solution:



10 distinct coordinates 5 distinct coordinates 6 distinct coordinates

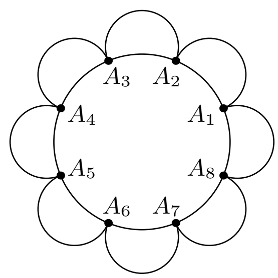
Let $\mathcal{P}$ have vertices $P_1P_2\dots P_{10}$. If no two vertices have the same x-coordinate, then Mark gets 10.

Otherwise, two vertices P_i and P_j have the same x-coordinate. Then P_k and P_{i+j-k} also have the same x-coordinate (indices taken modulo 10), as $P_iP_j \parallel P_kP_{i+j-k}$.

If $i+j$ is odd, the ten vertices of $\mathcal{P}$ pair up into 5 pairs of the form (P_k, P_{i+j-k}), so Mark gets 5. If $i+j$ is even, then the vertices $P_{\frac{i+j}{2}}$ and $P_{\frac{i+j}{2}+5}$ do not pair up, and the remaining 8 vertices form 4 pairs, so Mark gets 6.

Thus, the answer is $10+5+6=\boxed{21}$.

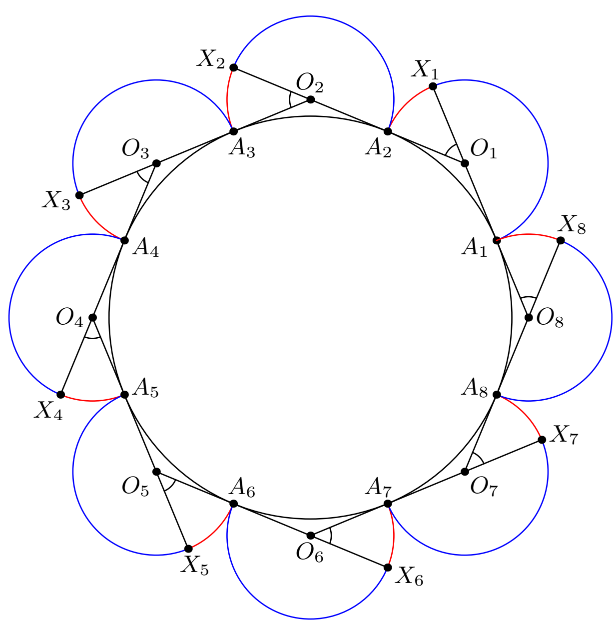
8. **[7]** Derek is bored in math class and is drawing a flower. He first draws 8 points $A_1, A_2, \dots, A_8$ equally spaced around an enormous circle. He then draws 8 arcs outside the circle where the ith arc for $i = 1, 2, \dots, 8$ has endpoints A_i, A_{i+1} with $A_9 = A_1$, such that all of the arcs have radius 1 and any two consecutive arcs are tangent. Compute the perimeter of Derek's 8-petaled flower.



Proposed by: David Dong, Evan Chang, Henrick Rabinovitz, Jackson Dryg, Krishna Pothapragada, Srinivas Arun

Answer: $\boxed{10\pi}$

Solution:



Draw the centers $O_1, \dots, O_8$ of the arcs, and connect these centers to form a regular octagon as shown. For each $i = 1, \dots, 8$, extend line A_iO_i to hit each arc again at X_i.

The blue arcs (arc A_iX_i for each i) are semicircles with total length 8π. Each red arc (arc X_iA_{i+1} for each i) has central angle equal to an exterior angle of the octagon. These exterior angles total 360 degrees, so the red arcs have total length 2π. Hence the perimeter is $\boxed{10\pi}$.

9. **[7]** Compute the remainder when

$$1\,002\,003\,004\,005\,006\,007\,008\,009$$

is divided by 13.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{5}$

Solution: Note that $13 \mid 1001$. Thus we can repeatedly subtract any multiple of 1001 from this number without changing the remainder. In particular, we can repeatedly subtract multiples of 1001 from the left to the right, as follows.

$$\begin{aligned} 1\,002\,003\,004\,005\,006\,007\,008\,009 &\longrightarrow 1\,003\,004\,005\,006\,007\,008\,009 \\ &\longrightarrow 2\,004\,005\,006\,007\,008\,009 \\ &\longrightarrow 2\,005\,006\,007\,008\,009 \\ &\longrightarrow 3\,006\,007\,008\,009 \\ &\longrightarrow 3\,007\,008\,009 \\ &\longrightarrow 4\,008\,009 \\ &\longrightarrow 4\,009 \\ &\longrightarrow \boxed{5}. \end{aligned}$$

10. **[8]** Compute the largest prime factor of $3^{12} + 3^9 + 3^5 + 1$.

Proposed by: Derek Liu

Answer: $\boxed{41}$

Solution: Observe

$$(3^4 + 1)^3 = 3^{12} + 3 \cdot 3^8 + 3 \cdot 3^4 + 1 = 3^{12} + 3^9 + 3^5 + 1,$$

so the answer is the largest prime factor of $3^4 + 1 = 82$, which is $\boxed{41}$.

11. **[8]** A four-digit integer in base 10 is *friendly* if its digits are four consecutive digits in any order. A four-digit integer is *shy* if there exist two adjacent digits in its representation that differ by 1. Compute the number of four-digit integers that are both friendly and shy.

Proposed by: Marin Hristov Hristov

Answer: $\boxed{148}$

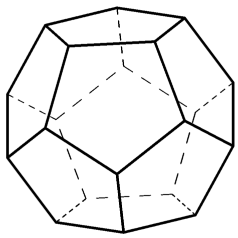
Solution: There are 24 friendly numbers with digits $d_1 = a$, $d_2 = a + 1$, $d_3 = a + 2$, $d_4 = a + 3$, for any $1 \le a \le 6$, and 18 with $a = 0$. Out of these, only the numbers

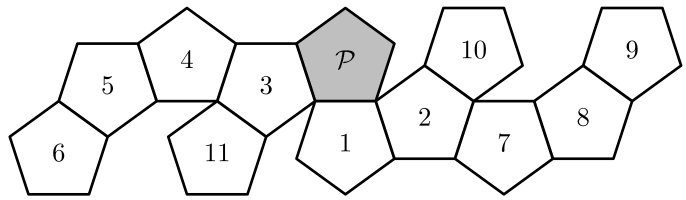
$$\overline{d_2d_4d_1d_3}, \quad \text{and} \quad \overline{d_3d_1d_4d_2}$$

are not shy, none of which has a leading digit zero. Therefore, the answer is:

$$6 \cdot 24 + 18 - 7 \cdot 2 = \boxed{148}.$$

12. **[8]** A dodecahedron is a polyhedron shown on the left below. One of its nets is shown on the right. Compute the label of the face opposite to $\mathcal{P}$.





Proposed by: Arul Kolla

Answer: $\boxed{5}$

Solution: The face opposite to $\mathcal{P}$ must be three faces away from $\mathcal{P}$. It's immediately clear that $\mathcal{P}$ is adjacent to 1 and 3. Also, $\mathcal{P}$, 2 and 10 share a vertex. Furthermore, the faces around 10 are $\mathcal{P}$, 2, 7, 8 and 9 in that order, so $\mathcal{P}$ is adjacent to 9. We conclude the neighbors of $\mathcal{P}$ are faces 3, 1, 2, 10, and 9. These cannot be opposite to $\mathcal{P}$.

Faces 7 and 8 are adjacent to 10, while faces 4 and 11 are adjacent to 3, so they are all two faces away from $\mathcal{P}$ and cannot be its opposite. Furthermore, as $\mathcal{P}$ and 9 are adjacent, by symmetry, 6 and 1 are also adjacent, so 6 also cannot be opposite to $\mathcal{P}$.

That leaves $\boxed{5}$ as the only possible opposite of $\mathcal{P}$.

13. **[9]** Let f and g be two quadratic polynomials with real coefficients such that the equation $f(g(x)) = 0$ has four distinct real solutions: 112, 131, 146, and a. Compute the sum of all possible values of a.

Proposed by: Derek Liu

Answer: $\boxed{389}$

Solution: The key observation is the following.

Claim 1. If a, b, c, d are roots of $f(g(x))$, then one can permute them so that $a + b = c + d$.

Proof. Let v be the point for which $g(v)$ is the local minimum or maximum. Note that if $g(x) = g(y)$, then x and y are symmetric around v, or $x + y = 2v$. Moreover, if a, b, c, d roots of $f(g(x))$, then we can permute them so that $g(a)$ and $g(b)$ are equal to one root of f and $g(c)$ and $g(d)$ are equal to another root of f. This means that $a + b = c + d = 2v$. $\square$

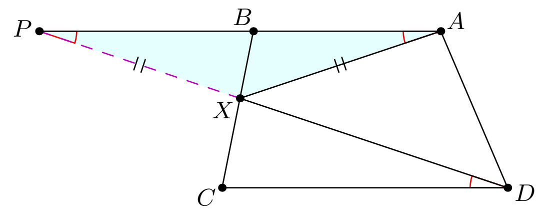
In the case of our problem, if three roots are r, s, and t, then the fourth can be $r + s - t$, $r + t - s$, or $s+t-r$, with sum $r+s+t$. Using the given values, we get that the answer is $112+131+146 = \boxed{389}$. Observe that these are all possible; indeed, if we let $g(x) = x^2 - (r + s)x$, then $g(r) = g(s)$ and $g(t) = g(r + s - t)$. Now let $f(x) = (x - g(r))(x - g(t))$; then $f(g(x))$ has roots r, s, t, and $r + s - t$. The other two values are similarly achievable.

14. **[9]** Let $ABCD$ be a trapezoid with $AB \parallel CD$. Point X is placed on segment $\overline{BC}$ such that $\angle BAX = \angle XDC$. Given that $AB = 5$, $BX = 3$, $CX = 4$, and $CD = 12$, compute AX.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{3\sqrt{6} = \sqrt{54}.}$

Solution:



Let $P = DX \cap AB$. Then, from the angle condition, we get that

$$\angle XAP = \angle XAB = \angle XDC = \angle XPA,$$

so $\triangle XAP$ is isosceles. Moreover, $\triangle XCD$ and $\triangle XBP$ are similar, so $BP = CD \cdot \frac{XB}{XC} = 9$. Thus, if M is the midpoint of AP, then Pythagorean theorem on $\triangle XMP$ gives $XM = \sqrt{3^2 - 2^2} = \sqrt{5}$. Finally, Pythagorean theorem on $\triangle XMA$ gives $AX = \sqrt{7^2 + 5} = \sqrt{54} = 3\sqrt{6}$.

15. **[9]** Compute the sum of the three smallest positive integers n for which

$$\frac{1 + 2 + 3 + \cdots + (2024n - 1) + 2024n}{1 + 2 + 3 + \cdots + (4n - 1) + 4n}$$

is an integer.

Proposed by: David Wei

Answer: $\boxed{89}$

Solution: We simplify the expression as follows:

$$\begin{aligned}
\frac{(2024n)(2024n+1)/2}{(4n)(4n+1)/2} &= \frac{506 \cdot (2024n+1)}{4n+1} \\
&= \frac{506 \cdot (506 \cdot (4n+1) - 505)}{4n+1} \\
&= 506^2 - \frac{506 \cdot 505}{4n+1} \\
&= 506^2 - \frac{2 \cdot 5 \cdot 11 \cdot 23 \cdot 101}{4n+1}.
\end{aligned}$$

Thus, the expression is an integer if and only if $4n + 1$ divides $5 \cdot 11 \cdot 23 \cdot 101$. The smallest divisors of $5 \cdot 11 \cdot 23 \cdot 101$ are

$$1, 5, 11, 23, 55, 101, 253.$$

Since $4n + 1 > 1$ and is 1 modulo 4, the three smallest values it can take are 5, 101, and 253. Hence, the three smallest values of n are 1, 25, and 63, giving the answer of $1 + 25 + 63 = \boxed{89}$.

16. **[10]** Compute

$$\frac{2 + 3 + \cdots + 100}{1} + \frac{3 + 4 + \cdots + 100}{1 + 2} + \cdots + \frac{100}{1 + 2 + \cdots + 99}.$$

Proposed by: Carlos Rodriguez

Answer: $\boxed{9900}$

Solution: Let A denote the sum. We have

$$\begin{aligned} A + 99 &= (1 + 2 + \cdots + 100)\left(\tfrac{1}{1} + \tfrac{1}{1+2} + \cdots + \tfrac{1}{1+2+\cdots+99}\right) \\ &= 5050 \sum_{k=1}^{99} \frac{2}{k(k+1)} \\ &= 10100 \cdot \sum_{k=1}^{99} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 10100\left(1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} - \cdots + \tfrac{1}{99} - \tfrac{1}{100}\right) \\ &= 10100\left(1 - \tfrac{1}{100}\right) = 9999, \end{aligned}$$

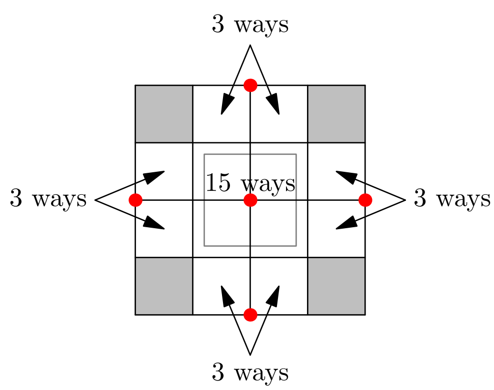
so the answer is $A = \boxed{9900}$

17. **[10]** Compute the number of ways to shade in some subset of the 16 cells in a 4×4 grid such that each of the 25 vertices of the grid is a corner of at least one shaded cell.

Proposed by: Linus Yifeng Tang

Answer: $\boxed{1215}$

Solution:



Observe that every corner cell must be shaded, as they are the only cells incident to the four corners of the grid. Furthermore, for each side of the grid, the midpoint of that side is incident to exactly two cells; at least one must be shaded. Finally, at least one of the four central cells must be shaded to hit the central vertex of the grid.

We claim these conditions are also sufficient for a valid coloring. Let us give the points of the grid coordinates from $(0, 0)$ to $(4, 4)$. Then, the corner cells cover every vertex except for those of the form $(2, x)$ or $(x, 2)$ for $0 \le x \le 4$. Whichever cell covers $(2, 0)$ must also cover $(2, 1)$, and likewise the cells covering $(0, 2)$, $(2, 4)$, and $(4, 2)$ cover $(1, 2)$, $(2, 3)$, and $(3, 2)$ respectively. This leaves the center $(2, 2)$, which is also covered by assumption.

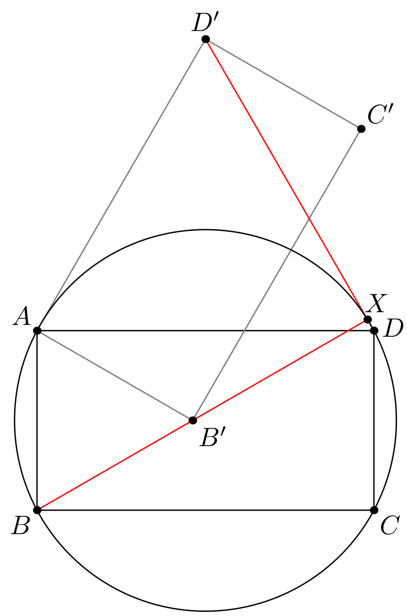
Observe that for each side of the grid, of the two cells incident to its midpoint, there are 3 ways to color at least one of them. Of the 4 central cells, there are $2^4 - 1 = 15$ ways to color at least one. Thus, the number of colorings is $3^4 \cdot 15 = \boxed{1215}$.

18. **[10]** Let $ABCD$ be a rectangle whose vertices are labeled in counterclockwise order with $AB = 32$ and $AD = 60$. Rectangle $AB'C'D'$ is constructed by rotating $ABCD$ counterclockwise about A by $60°$. Given that lines BB' and DD' intersect at point X, compute CX.

Proposed by: David Wei

Answer: $\boxed{34}$

Solution:



The key claim is the following.

Claim 1. $\angle BXD = 90^\circ$.

Proof. We see that $\angle ABB' = \angle AD'D = 60^\circ$ and $\angle BAD' = 90^\circ + \angle DAD' = 150^\circ$, from which we get $\angle BXD' = \angle BXD = 90^\circ$. Note that the fact that BB' and DD' are perpendicular is true regardless of how much we rotate the rectangle. □

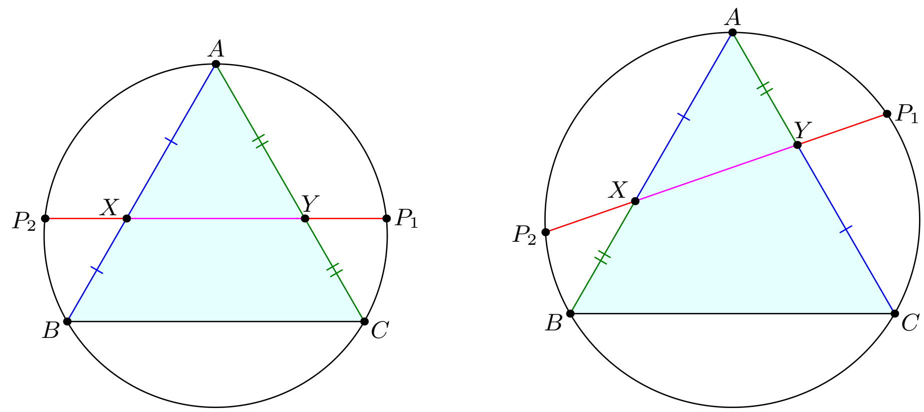
Now since $\angle BAD = \angle BXD$, we establish that X lies on the circumcircle of $ABCD$, which has diameter $\sqrt{32^2 + 60^2} = 68$. Moreover, we have $\angle CDX = 90^\circ + \angle ADD' = 150^\circ$, so we discover that $CX = 34$ by applying the extended law of sines to $\triangle CDX$.

19. **[11]** An equilateral triangle is inscribed in a circle ω. A chord of ω is cut by the perimeter of the triangle into three segments of lengths 55, 121, and 55 in that order. Compute the sum of all possible side lengths of the triangle.

Proposed by: Karthik Venkata Vedula

Answer: $\boxed{410}$

Solution:



Note that the chord splits two of the sides into segments of lengths a, b and c, d, where segments of length a and c is incident to the same vertex of the equilateral triangle. Moreover, $a + b = c + d$ (as the triangle is equilateral) and $ab = cd = 55 \cdot 176$ by Power of a Point. This means that $\{a, b\} = \{c, d\}$.

This means that we have two cases.

- **Case 1: the chord is parallel to the third side.** We must have $a = c = 121$ and by power of point, $b = d = (55 \cdot 176)/121 = 80$, so the side length is $121 + 80 = 201$.
- **Case 2: the chord is not parallel to the third side.** In that case, we have that $a = d$ and $c = b$. Thus, by the Law of Cosines, we have

  $$a^2 + b^2 - ab = 121^2.$$

  Moreover, $ab = 55 \cdot 176$ by power of point. Thus,

  $$a + b = \sqrt{121^2 + 3 \cdot 55 \cdot 176} = 11\sqrt{121 + 3 \cdot 5 \cdot 16} = 209,$$

  so the side length is 209.

This means that the answer is $201 + 209 = 410$.

(One can check that the two triangles indeed exist, as we can solve for a, b, c, d and see that they are positive real.)

20. **[11]** There exists a unique line tangent to the graph of $y = x^4 - 20x^3 + 24x^2 - 20x + 25$ at two distinct points. Compute the product of the x-coordinates of the two tangency points.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{-38}$

Solution: If $f(x)$ is tangent to the x-axis at $(c, 0)$, then $f(x)$ will be divisible by $(x - c)^2$. Thus, if $f(x)$ is tangent at the x-axis at c_1 and c_2, then $f(x) = P(x)(x - c_1)^2(x - c_2)^2$ for some polynomial $P(x)$. By adding $mx + b$, we see that $f(x)$ is tangent to $y = mx + b$ at x-coordinates c_1 and c_2 if and only if

$$f(x) = P(x)(x - c_1)^2(x - c_2)^2 + mx + b$$

for some polynomial $P(x)$.

In the case of our problem, $f(x) = x^4 - 20x^3 + 24x^2 - 20x + 5$, we have by comparing the leading coefficient that $P(x) = 1$. Thus,

$$x^4 - 20x^2 + 24x^2 - 20x + 25 = (x - c_1)^2(x - c_2)^2 + mx + b.$$

By Vieta's formulas,

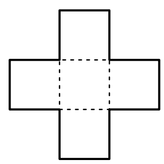
$$2(c_1 + c_2) = 20$$
$$c_1^2 + c_2^2 + 4c_1c_2 = 24.$$

Hence,

$$c_1^2 + 2c_1c_2 + c_2^2 = 10^2 = 100 \implies c_1c_2 = \tfrac{24-100}{2} = \boxed{-38},$$

which is the answer.

21. **[11]** Two points are chosen independently and uniformly at random from the interior of the X-pentomino shown below. Compute the probability that the line segment between these two points lies entirely within the X-pentomino.



Proposed by: Benjamin Shimabukuro

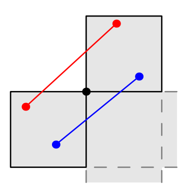
Answer: $\boxed{\frac{21}{25} = 0.84}$

Solution: Here are the cases in which the line segment is guaranteed to lie entirely within the X-pentomino:

- At least one point lies in center square.
- The two points lie in the same square.
- The two points are in opposite edge squares (north and south, or east and west).

Thus, the only case left to consider is the one where one point is in the north square and the other point is in the west square (and the other 3 analogous cases).

By symmetry, the segment will lie outside the X-pentomino half of the time in this case, as illustrated by the red and blue segments in the following diagram.



The probability that two points are in non-opposite edge squares is $\frac{4}{5} \cdot \frac{2}{5} = \frac{8}{25}$.

Half the time, the segment will like outside the X-pentomino. So, the line segment will lie outside the X-pentomino with probability $\frac{1}{2} \cdot \frac{8}{25} = \frac{4}{25}$. Thus, the probability that the line segment lies within the X-pentomino is $1 - \frac{4}{25} = \boxed{\frac{21}{25}}$.

22. **[12]** Suppose that a and b are positive integers such that $\gcd(a^3 - b^3, (a-b)^3)$ is not divisible by any perfect square except 1. Given that $1 \le a - b \le 50$, compute the number of possible values of $a - b$ across all such a, b.

Proposed by: Srinivas Arun

Answer: $\boxed{23}$

Solution: Recall that a positive integer is *squarefree* if it is not divisible by any perfect square except 1. We characterize $a - b$ that works.

Claim 1. Let a and b be positive integers. Then, $\gcd(a^3 - b^3, (a-b)^3)$ is squarefree if and only if $\gcd(a, b) = 1$, $a - b$ is squarefree, and $a - b$ is not divisible by 3.

Proof. If $\gcd(a, b) = d > 1$, then g is divisible by d^3, hence not squarefree. Thus, we now restrict our attention to the case $\gcd(a, b) = 1$. In that case, we factor out $a - b$ from the gcd and simplify it as follows:

$$\begin{aligned} \gcd(a^3 - b^3, (a-b)^3) &= (a-b)\ \gcd(a^2 + ab + b^2, (a-b)^2) \\ &= (a-b)\ \gcd\left((a-b)^2 + 3ab, (a-b)^2\right) \\ &= (a-b)\ \gcd((a-b)^2, 3ab). \end{aligned}$$

Moreover, since $\gcd(a, b) = 1$, we have that $\gcd(a-b, a) = \gcd(a-b, b) = 1$, and so $\gcd((a-b)^2, ab) = 1$, which implies that $\gcd((a-b)^2, 3ab)$ is either 1 or 3.

Thus, each of the following is equivalent to the next.

- $\gcd(a^3 - b^3, (a-b)^3)$ is squarefree.
- $(a-b)\gcd((a-b)^2, 3ab)$ is squarefree.
- $a - b$ is squarefree and at least one of $a - b$ and $\gcd((a-b)^2, 3ab)$ is not divisible by 3.
- $a - b$ is squarefree and $a - b$ is not divisible by 3. $\square$

The claim implies that $c = a - b$ is possible only if c is squarefree and not divisible by 3. For any such c, we may construct (a, b) that works by taking $a = c + 1$ and $b = 1$. Hence, the answer is simply the number of squarefree integers up to 50 that are not divisible by 3.

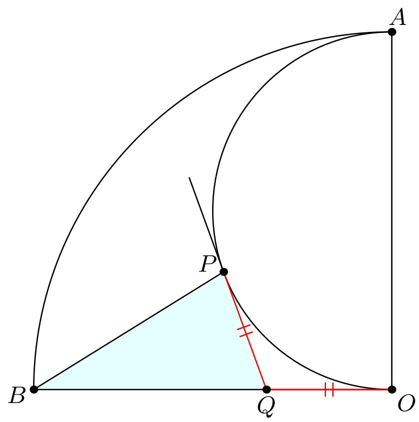
There are 16 multiples of 3 from 1 to 50. Among the remaining numbers, 4, 8, 16, 20, 28, 32, 40, 44, 25, 50, and 49 are not squarefree. This leaves $50 - 16 - 11 = \boxed{23}$ possible values of $a - b$.

23. **[12]** Consider a quarter-circle with center O, arc $\widehat{AB}$, and radius 2. Draw a semicircle with diameter $\overline{OA}$ lying inside the quarter-circle. Points P and Q lie on the semicircle and segment $\overline{OB}$, respectively, such that line PQ is tangent to the semicircle. As P and Q vary, compute the maximum possible area of triangle BQP.

Proposed by: Daeho Jacob Lee

Answer: $\boxed{\frac{1}{2} = 0.5}$

Solution:



Note that we can bound the area of $\triangle BQR$ by

$$\begin{aligned}[BQP] &= \tfrac{1}{2}BQ \cdot QP \sin \angle BQP \\ &\le \tfrac{1}{2}BQ \cdot QP \\ &= \tfrac{1}{2}BQ(2 - BQ) \\ &\le \boxed{\tfrac{1}{2}}.\end{aligned}$$

The maximum occurs when Q is the midpoint of segment $\overline{OB}$.

24. **[12]** Let $f(x) = x^2 + 6x + 6$. Compute the greatest real number x such that $f(f(f(f(f(f(x)))))) = 0$.

Proposed by: Arul Kolla

Answer: $\boxed{\sqrt[64]{3} - 3}$

Solution: Observe that $f(x) = (x+3)^2 - 3$. Now, we claim that

Claim 1. $f^k(x) = (x+3)^{2^k} - 3$ for all positive integers k.

Proof. We use induction. The base case $k = 1$ is clear. To show the inductive step, note that $f^k(x) = (x+3)^{2^k} - 3$ implies

$$f^{k+1}(x) = f(f^k(x)) = f((x+3)^{2^k} - 3) = \left(((x+3)^{2^k} - 3) + 3\right)^2 - 3 = (x+3)^{2^{k+1}} + 3. \qquad \square$$

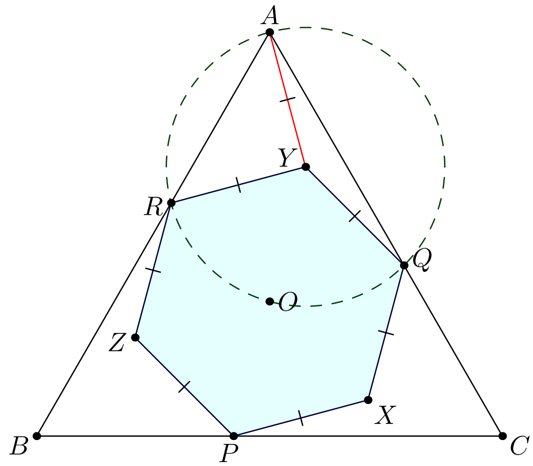
Thus, if r is a real root of f^6, then $(r+3)^{64} = 3$, so $r + 3 = \pm\sqrt[64]{3}$, and hence $r = \pm\sqrt[64]{3} - 3$. The largest value of r is thus $\boxed{\sqrt[64]{3} - 3}$.

25. **[13]** Let ABC be an equilateral triangle. A regular hexagon $PXQYRZ$ of side length 2 is placed so that P, Q, and R lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. If points A, X, and Y are collinear, compute BC.

Proposed by: Rishabh Das

Answer: $\boxed{\sqrt{6} + 3\sqrt{2} = \sqrt{6} + \sqrt{18}}$

Solution:



Notice that $\angle QAR = 60^\circ$, and $\triangle YAR$ is isosceles with base angle 120°. This implies that Y is the circumcenter of $\triangle AQR$. Thus, $YA = YR = YQ = 2$. We have $\angle AYR = 90^\circ$, so $AR = 2\sqrt{2}$. Moreover, $\angle AYQ = 150^\circ$, so $\angle YAQ = 15^\circ$, which implies that $AQ = 4\sin 75^\circ = \sqrt{6}+\sqrt{2}$. By symmetry, we also get that $BR = AQ = \sqrt{6}+\sqrt{2}$. Hence, the answer is $AR + BR = \boxed{\sqrt{6}+3\sqrt{2}}$.

26. **[13]** A right rectangular prism of silly powder has dimensions $20 \times 24 \times 25$. Jerry the wizard applies 10 bouts of highdroxylation to the box, each of which increases one dimension of the silly powder by 1 and decreases a different dimension of the silly powder by 1, with every possible choice of dimensions equally likely to be chosen and independent of all previous choices. Compute the expected volume of the silly powder after Jerry's routine.

Proposed by: Aaron Guo

Answer: $\boxed{11770}$

Solution: Consider the expected change in volume by one bout of highdroxylation. Let the initial dimensions of the silly powder be a, b, and c.

Should a increase to $a+1$ and b decrease to $b-1$, the volume of the silly powder will change from abc to

$$(a+1)(b-1)c = abc + bc - ac - c.$$

If a decreases to $a-1$ and b increases to $b+1$ instead, the new volume is

$$(a-1)(b+1)c = abc - bc + ac - c.$$

Thus, the average change in volume between these two cases is $-c$. By symmetry, the expected change in volume from one bout is $\delta = -\frac{a+b+c}{3}$.

Initially, the dimensions sum to 69, and this sum can never change. Hence, for each bout of hydroxylation, the expected change in volume is $\delta = -\frac{69}{3} = -23$. By linearity of expectation, the expected volume after 10 bouts is $20 \cdot 24 \cdot 25 - 10 \cdot 23 = \boxed{11770}$.

27. **[13]** For any positive integer n, let $f(n)$ be the number of ordered triples (a, b, c) of positive integers such that

- $\max(a, b, c)$ divides n and
- $\gcd(a, b, c) = 1$.

Compute $f(1) + f(2) + \cdots + f(100)$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{1000000}$

Solution: We will show that $\sum_{m=1}^{n} f(m) = n^3$. Indeed, consider the map

$$g : \{1, \ldots, n\}^3 \to \{1, \ldots, n\}^4$$
$$g(a, b, c) = \left(\tfrac{a}{\gcd(a,b,c)}, \tfrac{b}{\gcd(a,b,c)}, \tfrac{c}{\gcd(a,b,c)}, \max(a, b, c)\right).$$

We claim that g is a bijection between $\{1, \ldots, n\}^3$ and tuples (a, b, c, m) that satisfy $m \le n$, $\max(a, b, c)|m$, and $\gcd(a, b, c) = 1$. Note that $g(a, b, c)$ always satisfies these properties because $\gcd\left(\tfrac{a}{\gcd(a,b,c)}, \tfrac{b}{\gcd(a,b,c)}, \tfrac{c}{\gcd(a,b,c)}\right) = \tfrac{\gcd(a,b,c)}{\gcd(a,b,c)} = 1$ and

$$\max\left(\tfrac{a}{\gcd(a,b,c)}, \tfrac{b}{\gcd(a,b,c)}, \tfrac{c}{\gcd(a,b,c)}\right) = \tfrac{\max(a,b,c)}{\gcd(a,b,c)}$$

divides $\max(a, b, c)$. It remains to show g is both injective and surjective.

Injectivity. Suppose that $g(a_1, b_1, c_1) = g(a_2, b_2, c_2)$. Then,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{\gcd(a_1, b_1, c_1)}{\gcd(a_2, b_2, c_2)}.$$

Without loss of generality, assume $a_1 \ge b_1 \ge c_1$, which implies $a_2 \ge b_2 \ge c_2$. Then,

$$a_1 = \max(a_1, b_1, c_1) = \max(a_2, b_2, c_2) = a_2,$$

so $b_1 = b_2$ and $c_1 = c_2$ as well. Thus, g is injective.

Surjectivity. Now suppose we have a tuple $(a, b, c, m) \in \{1, \ldots, n\}^4$ satisfying $\max(a, b, c)|m$ and $\gcd(a, b, c) = 1$. Let $d = \frac{m}{\max(a,b,c)}$. Then, $\gcd(da, db, dc) = d$ and $\max(da, db, dc) = d \cdot \max(a, b, c) = m \le n$, so

$$g(da, db, dc) = \left(\tfrac{da}{d}, \tfrac{db}{d}, \tfrac{dc}{d}, \max(da, db, dc)\right) = (a, b, c, m).$$

Hence, g is surjective.

We conclude g is a bijection between $\{1, \ldots, n\}^3$ and tuples (a, b, c, m) satisfying $\max(a, b, c)|m$ and $\gcd(a, b, c) = 1$. We have $f(m)$ such tuples for each m by definition, so

$$\sum_{m=1}^{n} f(m) = \left|\{1, \ldots, n\}^3\right| = n^3.$$

Thus, $f(1) + \cdots + f(100) = 100^3 = \boxed{1000000}$.

28. **[15]** The graph of the equation $\tan(x + y) = \tan(x) + 2\tan(y)$, with its pointwise holes filled in, partitions the coordinate plane into congruent regions. Compute the perimeter of one of these regions.

Proposed by: Karthik Venkata Vedula

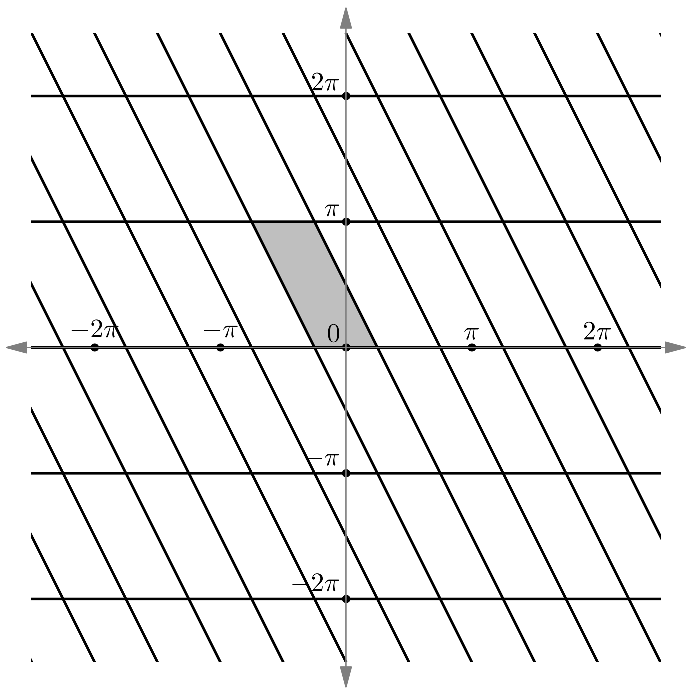
Answer: $\boxed{\pi(\sqrt{5} + 1)}$

Solution: We manipulate the given equation as follows:

$$\begin{aligned}
\tan(x + y) &= \tan x + 2\tan y \\
\frac{\tan x + \tan y}{1 - \tan x \tan y} &= \tan x + 2\tan y \\
\tan x + \tan y &= \big(\tan x + 2\tan y\big) - \tan x \tan y\big(\tan x + 2\tan y\big) \\
\tan x \tan y\big(\tan x + 2\tan y\big) &= \tan y \\
\tan x \tan y \tan(x + y) &= \tan y \\
\tan y(\tan x \tan(x + y) - 1) &= 0
\end{aligned}$$

Thus, the graph of $\tan(x + y) = \tan x + 2\tan y$ is the union of

- the graph of $\tan y = 0$, which is equivalent to $y = n\pi$ for some $n \in \mathbb{Z}$; and
- the graph of $\tan(x+y) = \cot x$, which is equivalent to $2x + y = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$.



Each of the above graphs is a disjoint union of equally spaced parallel lines. Thus, the entire graph partitions the plane into congruent parallelograms. To compute the perimeter, we need to pick two adjacent lines from each bullet point.

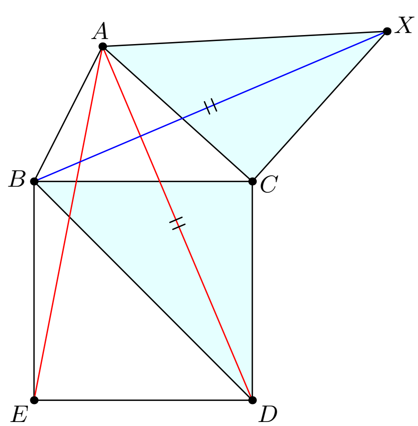
We pick $y = 0$, $y = \pi$, and $2x + y = \pm\pi/2$. This is a parallelogram with vertices $(\pm\pi/4, 0)$, $(-3\pi/4, \pi)$, and $(-\pi/4, \pi)$. This is a parallelogram with side lengths $\pi/2$ and $\pi\sqrt{5}/2$, so the perimeter is $\pi(\sqrt{5}+1)$.

29. **[15]** Let ABC be a triangle such that $AB = 3$, $AC = 4$, and $\angle BAC = 75°$. Square $BCDE$ is constructed outside triangle ABC. Compute $AD^2 + AE^2$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{75 + 24\sqrt{2} = 75 + \sqrt{1152}}$

Solution:



Construct point X such that $\triangle CBD \overset{+}{\sim} \triangle CXA$. Then, $\triangle CBX \cong \triangle CAD$. Thus, $AD = BX$. We have $AB = 3$, $AX = 4\sqrt{2}$, and $\angle BAX = 120^\circ$, so law of cosine gives

$$AD^2 = BX^2 = 3^2 + \left(4\sqrt{2}\right)^2 - 2 \cdot 3 \cdot (4\sqrt{2}) \cdot \cos 120^\circ = 41 + 12\sqrt{2}.$$

Similarly, we may compute

$$AE^2 = 4^2 + \left(3\sqrt{2}\right)^2 - 2 \cdot 4 \cdot (3\sqrt{2}) \cdot \cos 120^\circ = 34 + 12\sqrt{2},$$

so the answer is $75 + 24\sqrt{2}$.

30. **[15]** Compute the number of ways to shade exactly 4 distinct cells of a 4×4 grid such that no two shaded cells share one or more vertices.

Proposed by: Jacob Paltrowitz

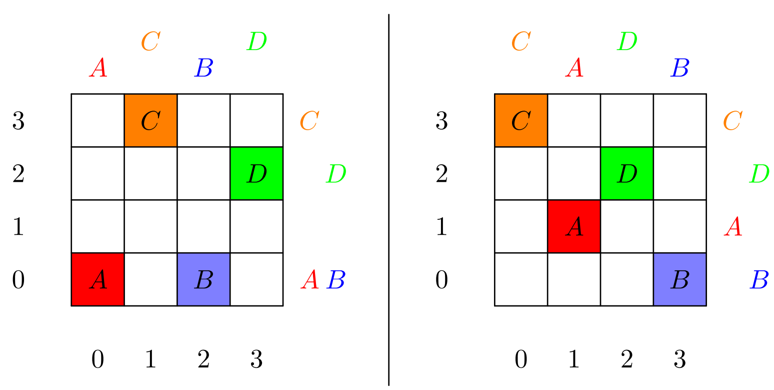
Answer: $\boxed{79}$

Solution: Assign coordinates to the cells of the grid so that the bottom-left, bottom-right, and top-right corners are $(0,0)$, $(3,0)$, and $(3,3)$ respectively.

Observe that for each quadrant of the grid, all four cells of that quadrant share a vertex. Thus, any valid coloring must have exactly one shaded cell in each quadrant. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ and $D = (d_1, d_2)$ denote the shaded cells in the bottom-left, bottom-right, top-left, and top-right quadrants, respectively, so that $0 \le a_1, a_2, b_2, c_1 \le 1$ and $2 \le b_1, c_2, d_1, d_2 \le 3$.

Observe that A and B share a vertex if and only if $|b_1 - a_1| \le 1$ and $|b_2 - a_2| \le 1$. The latter is always true, and the former holds precisely when $a_1 = 1$ and $b_1 = 2$. We conclude that in a valid coloring, (a_1, b_1) must be one of $(0,2)$, $(0,3)$, or $(1,3)$. We can similarly deduce the same holds for (c_1, d_1), (a_2, c_2), and (b_2, d_2).

Suppose the coordinates are chosen according to those constraints. Then, we are guaranteed the pairs of cells (A,B), (C,D), (A,C), and (B,D) do not share any vertices. The only way we get an invalid coloring is if A and D share a vertex, or B and C share a vertex.



Suppose A and D share a vertex. Then we must have $a_1 = a_2 = 1$ and $d_1 = d_2 = 2$, which implies $b_1 = c_2 = 3$ and $b_2 = c_1 = 0$. Thus, there is exactly one way to choose coordinates in the manner above so that A and D share a vertex (as depicted in the figure on the right). Likewise, there is exactly one way for B and C to share a vertex.

There are $3^4 = 81$ ways to choose the coordinates, so the answer is $81 - 2 = \boxed{79}$.

31. **[17]** Positive integers a, b, and c have the property that $\operatorname{lcm}(a,b)$, $\operatorname{lcm}(b,c)$, and $\operatorname{lcm}(c,a)$ end in 4, 6, and 7, respectively, when written in base 10. Compute the minimum possible value of $a+b+c$.

Proposed by: Derek Liu

Answer: $\boxed{28}$

Solution: Note that $a+b+c=28$ is achieved when $(a,b,c)=(19,6,3)$. To show we cannot do better, first observe we would need $a+c<27$ and $\operatorname{lcm}(a,c)\le ac\le 13\cdot 13=169$, which is only possible when $\operatorname{lcm}(a,c)$ is 7, 17, 57, 77, or 117. We do casework on each value:

- $\mathbf{lcm(a,c)=7}$. Then a and c are both 1 or 7, so $\operatorname{lcm}(a,b)$ and $\operatorname{lcm}(b,c)$ are both b or $7b$. It is impossible for one of b and $7b$ to end in 4 and the other to end in 6.
- $\mathbf{lcm(a,c)=17}$. The same argument as above proves this case is also impossible.
- $\mathbf{lcm(a,c)=77}$. Since the divisors of 77 only end in 1 and 7, the same argument as above rules out this case.
- $\mathbf{lcm(a,c)=57}$. As $a+c<27$, we must have $\{a,c\}=\{3,19\}$. Then b is even and less than 6, so it's easy to verify there are no solutions here.
- $\mathbf{lcm(a,c)=117}$. As $a+c<27$, we must have $\{a,c\}=\{9,13\}$. Then b is even and less than 6, so it's easy to verify there are no solutions here.

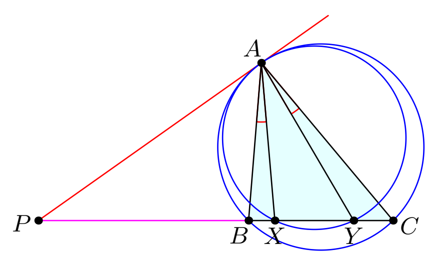
This rules out all cases, so $\boxed{28}$ is optimal.

32. **[17]** Let ABC be an acute triangle and D be the foot of altitude from A to $\overline{BC}$. Let X and Y be points on the segment $\overline{BC}$ such that $\angle BAX=\angle YAC$, $BX=2$, $XY=6$, and $YC=3$. Given that $AD=12$, compute BD.

Proposed by: Sarunyu Thongjarast

Answer: $\boxed{12\sqrt{2}-16=\sqrt{288}-16}$

Solution 1:



Let the line tangent to $\odot(ABC)$ at A intersect line BC at P.

Claim 1. Line PA is tangent to $\odot(AXY)$.

Proof. Note that
$$\angle PAX=\angle PAB+\angle BAX=\angle PCA+\angle CAY=\angle PYA.$$
This proves the desired tangency. $\square$

Now, by power of point, we have,
$$\begin{aligned}PB\cdot PC&=PX\cdot PY=PA^2\\ PB(PB+11)&=(PB+2)(PB+8)\\ PB&=16\\ PA^2&=PB\cdot PC=16\cdot 27\\ PA&=12\sqrt{3}\end{aligned}$$

By Pythagorean theorem, $AD^2 + DP^2 = PA^2$. Therefore, $PD^2 = PA^2 - AD^2 = 432 - 144 = 288$, so $PD = 12\sqrt{2}$. Finally, $BD = PD - PB = 12\sqrt{2} - 16$.

Solution 2: Since $\angle BAX = \angle CAY$, by Steiner ratio theorem, we get that

$$\frac{AB^2}{AC^2} = \frac{BX}{CX} \cdot \frac{BY}{CY} = \frac{2}{9} \cdot \frac{8}{3} = \frac{16}{27}.$$

Thus, if $BD = x$, then by Pythagorean theorem, we get that $AB^2 = 144 + x^2$ and $AC^2 = 144 + (11 - x)^2$. Combining with the displayed equations gives

$$\begin{aligned} 27(144 + x^2) &= 16(144 + (11 - x)^2) \\ 27(144 + x^2) &= 16(144 + x^2) + 16(-22x + 121) \\ 11(144 + x^2) &= 16 \cdot 11 \cdot (-2x + 11) \\ x^2 + 144 + 16(2x - 11) &= 0 \\ x^2 + 32x - 32 &= 0 \\ x &= \boxed{12\sqrt{2} - 16}, \end{aligned}$$

(where we only take the positive solution since $x > 0$).

Solution 3: Suppose that $BD = c$. From the length conditions, we get that

$$\tan\angle BAD = \tfrac{c}{12}, \quad \tan\angle DAX = \tfrac{2-c}{12}, \quad \tan\angle DAC = \tfrac{11-c}{12}, \quad \tan\angle DAY = \tfrac{8-c}{12}$$

Thus, using tangent addition formula, we get that

$$\begin{aligned} \tan\angle BAX &= \tan(\angle BAD + \angle DAX) = \frac{\frac{c}{12} + \frac{2-c}{12}}{1 - \frac{c}{12} \cdot \frac{2-c}{12}} = \frac{\frac{2}{12}}{\frac{144-2c+c^2}{144}} = \frac{24}{144 - 2c + c^2} \\ \tan\angle YAC &= \tan(\angle DAC - \angle DAY) = \frac{\frac{11-c}{12} - \frac{8-c}{12}}{1 + \frac{11-c}{12} \cdot \frac{8-c}{12}} = \frac{\frac{3}{12}}{\frac{c^2-19c+232}{144}} = \frac{36}{c^2 - 19c + 232} \end{aligned}$$

Hence, the condition $\angle BAX = \angle CAY$ translates to

$$\begin{aligned} \frac{24}{144 - 2c + c^2} &= \frac{36}{c^2 - 19c + 232} \\ 36(c^2 - 2c + 144) - 24(c^2 - 19c + 232) &= 0 \\ 3(c^2 - 2c + 144) - 2(c^2 - 19c + 232) &= 0 \\ c^2 + 32c - 32 &= 0 \\ c &= -16 \pm 12\sqrt{2} \end{aligned}$$

Since c is positive, the answer is $BD = \boxed{12\sqrt{2} - 16}$.

33. **[17]** A grid is called *groovy* if each cell of the grid is labeled with the smallest positive integer that does not appear below it in the same column or to the left of it in the same row. Compute the sum of the entries of a groovy 14×14 grid whose bottom left entry is 1.

Proposed by: Jacob Paltrowitz

Answer: $\boxed{1638}$

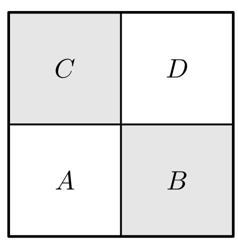
Solution: The following diagram is the entire 16×16 groovy grid computed out. However, one will not need to write out every single entry to obtain the answer.

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2
14	13	16	15	10	9	12	11	6	5	8	7	2	1	4	3
13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11
5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

We prove the following key claim.

Claim 1. In the $2^n \times 2^n$ groovy grid, each row and column is a permutation of the numbers from 1 to 2^n.

Proof. We use induction on n. The base case $n = 0$ is clear. Now, assume that we know this for a $2^n \times 2^n$ groovy grid, and we will prove it for a $2^{n+1} \times 2^{n+1}$ grid. To that end, we divide the $2^{n+1} \times 2^{n+1}$ grid into four subgrids of size $2^n \times 2^n$.



The subgrid labeled A is the groovy grid of size $2^n \times 2^n$, so by induction, each row and column is a permutation of $\{1, \ldots, 2^n\}$. Thus, the bottom left corner of the subgrid labeled B is $2^n + 1$, and so the subgrid labeled B is the groovy grid where each entry is added by 2^n. Hence, by induction hypothesis, each row and column of the subgrid labeled B is a permutation of $\{2^n + 1, 2^n + 2, \ldots, 2^n + 2^n\}$. The same argument applies for the subgrid labeled C.

Finally, the subgrid labeled D has enough numbers from $1, 2, \ldots, 2^n$ and does not need any number greater than 2^n. The bottom left corner is 1. Hence, it must be a groovy grid. Thus, the induction hypothesis applies, and each row and column of the subgrid labeled D is a permutation of $\{1, 2, \ldots, 2^n\}$. By considering all subgrids together, we find that each row and column of the entire grid is a permutation of $\{1, 2, \ldots, 2^{n+1}\}$. $\square$

In particular, we have that every row and column of the 16×16 grid is a permutation of $\{1, 2, \ldots, 16\}$. To compute the sum of entries of the 14×14 grid, we can take out 2 rows and 2 columns and add back the top right 2×2 which we know entries 1, 2, 2, 1 by following the proof of the claim. This gives the final answer of

$$16 \cdot \frac{16 \cdot 17}{2} - 2 \cdot \frac{16 \cdot 17}{2} - 2 \cdot \frac{16 \cdot 17}{2} + (1 + 2 + 2 + 1) = \boxed{1638}.$$

Remark. In fact, one can show that the entry at row x and column y is $((x-1) \oplus (y-1)) + 1$, where $\oplus$ denotes bitwise XOR.

34. **[20]** The largest known prime number as of October 2024 is $2^{136\,279\,841} - 1$. It happens to be an example of a prime number of the form $2x^2 - 1$. Estimate the number of positive integers $x \le 10^6$ such that $2x^2 - 1$ is prime.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20.99 \max\left(0, 1 - \frac{|E-A|}{A}\right)^{2.5} \right\rfloor$ points.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{141444}$

Solution: If $x \le 10^6$, then $2x^2 - 1 < 2 \cdot 10^{12}$. The density of prime numbers up to $2 \cdot 10^{12}$ is roughly

$$\frac{1}{\ln(2 \cdot 10^{12}) - 1} \approx 0.0366.$$

However, $2x^2 - 1$ can never be divisible by 2, 3, or 5. Only $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = \frac{4}{15}$ of numbers are not divisible by 2, 3, or 5, including all of the primes, so the density of primes among such numbers is a factor of $\frac{15}{4}$ higher. Among the 10^6 possible values of x, we get an estimate of

$$\frac{15}{4} \cdot 0.0366 \cdot 10^6 = 137250$$

primes, which is close enough for 19 points.

To refine this estimate further, observe that the values of $2x^2 - 1$ are not uniformly distributed from 1 to $2 \cdot 10^{12}$. Their average is very close to $\frac{2}{3} \cdot 10^{12}$, so as a crude estimate, we can take the density of prime numbers up to $\frac{4}{3} \cdot 10^{12}$ instead:

$$\frac{1}{\ln\left(\frac{4}{3} \cdot 10^{12}\right) - 1} \approx 0.03715.$$

This gives us an estimate of

$$\frac{15}{4} \cdot 0.03715 \cdot 10^6 \approx 139313,$$

which earns all 20 points.

Using `sage`, one can easily obtain the exact answer by the following code.

```
cnt = 0
for x in range(1,10^6+1):
    if is_prime(2*x^2-1):
        cnt += 1
print(cnt)
```

35. **[20]** There are 1024 players, ranked from 1 (most skilled) to 1024 (least skilled), participating in a single elimination tournament. In each of the 10 rounds, the remaining players are paired uniformly at random. In each match, the player with a lower rank always wins, and the loser is eliminated from the tournament.

For each positive integer $n \in [1, 1024]$, let $f(n)$ be the expected number of rounds that the participant with rank n participates in. Estimate the minimum positive integer N such that $f(N) < 2$.

Submit a positive integer E. If the correct answer is A, you will receive $\max\left(0, 20 - \left\lfloor \frac{|E-A|}{2} \right\rfloor\right)$ points.

Proposed by: Aaron Guo

Answer: $\boxed{350}$

Solution: The probability that the rank N player passes round i (where $i = 0$ is implied as 1) is

$$\frac{\binom{1024-2^i}{N-1}}{\binom{1023}{N-1}}.$$

Summing this from $i = 0$ to 9 (each represents the expectation of advancing one round) we must find the minimum N for which the sum goes below 2. Let r indicate $N - 1$.

The first term is 1, the second simplifies to $\frac{1023-r}{1023}$, and round i's contribution is

$$\prod_{j=0}^{2^i-2} \frac{1023-r-j}{1023-j}.$$

At this point, a program may output $N = 350$.

However, one may notice that for larger i, this term vanishes; in fact, estimating this product with k^{2^i-1} where $k = \frac{1023-r}{1023}$.

The trailing terms in the product will be a bit smaller than k, but vanishes. If we do find the estimate for k where $\sum_{i=0}^{9} k^{2^i-1} = 2$, k will be a bit to small, the true k being something slightly greater.

The main concern is how many great of an upper limit for i we choose to approximate with for k. If we stop at $i = 2$, for instance, we have $1 + k + k^3 = 2$, where we might find $k = \frac{2}{3}$ a just estimate, $\frac{2}{3} + \frac{8}{27}$ being just shy of 1. We may produce/cursory check 0.67 or 0.68 as an estimate for k (from this equation, around 0.683). This produces $N = 339$ or $N = 328$.

Estimating the real solution to $1 + k + k^3 + k^7 = 2$ produces very close to the maximum number of points. Given our estimate for k_1, just above $\frac{2}{3}$, we can estimate that $k^7 \approx \frac{128}{2187} \approx 0.06$.

Then, we estimate the solution to $k + k^3 = 1 - 0.06$, for which we might notice that if $k = k_1 - \delta$, then by Newton's method, we can approximate k by

$$\delta \cdot (1 + 3k^2) \approx \delta \cdot \left(1 + 3 \cdot \frac{4}{9}\right) = 0.06,$$

making δ about $\frac{0.06 \cdot 3}{7} \approx 0.026$. If this is subtracted from $k_1 = 0.683$, then $k = 0.657$, producing $N = 351$.

36. **[20]** Estimate the value of

$$\frac{20! \cdot 40! \cdot 40!}{100!} \cdot \sum_{i=0}^{40} \sum_{j=0}^{40} \frac{(i+j+18)!}{i!j!18!}.$$

Submit a positive real number E either in decimal or in a fraction of two positive integers written in decimal (such as $\frac{2024}{2025}$). If the correct answer is A, your will receive $\max(20 - \lfloor 5000 \cdot |E - A| \rfloor, 0)$ points.

Proposed by: Kevin Zhao

Answer: $\boxed{0.1085859}$

Solution: Note that

$$\sum_{i=0}^{40} \sum_{j=0}^{40} \frac{(i+j+18)!}{i!j!18!} = \sum_{i=0}^{40} \sum_{j=0}^{40} \frac{(98-i-j)!}{(40-i)!(40-j)!18!} = \sum_{i=0}^{40} \sum_{j=0}^{40} \binom{98-i-j}{40-i, 40-j, 18}.$$

The multinomial coefficient $\binom{98-i-j}{40-i,40-j,18}$ counts the number of ways to arrange $40-i$ red balls, $40-j$ blue balls, and 18 green balls in a line. Hence, it also counts the number of ways to arrange 40 red, 40 blue, and 20 green balls such that the first $i+1$ balls are i reds followed by a green, and the last $j+1$ balls are j blues preceded by a green. Thus, the summation over all possible i and j counts the number of ways to arrange 40 red, 40 blue, and 20 green balls such that all balls before the first green ball are red and all balls after the last green ball are blue. Since there are $\frac{100!}{20!\cdot 40!\cdot 40!}$ total permutations of these balls, the desired answer is the probability that a random permutation satisfies the above properties.

Since there are 40 blue and 20 green balls, the probability the first non-red ball is green is $\frac{20}{60}=\frac{1}{3}$. Similarly, the probability the the last non-blue ball is green is $\frac{1}{3}$. If we assume these probabilities are independent, we get an answer of $\frac{1}{9}$, which is accurate enough for 8 points.

To get a better estimate, given that the first non-red ball is green, we can find the expected number of red balls that appeared before it. Indeed, this is equivalent to the expected number of red balls before the first non-red ball. There are 60 non-red balls which separate out 61 intervals for the red balls to be in; any given ball is equally likely to be in each interval, so the expected number of red balls in the first interval (i.e., before the first non-red ball) is $\frac{40}{61}$.

Thus, once we assume the first non-red ball is green, we expect there to be $40-\frac{40}{61}$ red balls left and 19 green balls. Then, the probability that the last non-blue ball is green is about $\frac{19}{59-\frac{40}{61}}=\frac{1159}{3559}$. Our final estimate is then $\frac{1}{3}\cdot\frac{1159}{3559}=\frac{1159}{10677}\approx 0.10855$, which is accurate enough for the full 20 points.