# $11^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

**Saturday 23 February 2008**

**Individual Round: Calculus Test**

1. [**3**] Let $f(x) = 1 + x + x^2 + \cdots + x^{100}$. Find $f'(1)$.

   **Answer:** $\boxed{5050}$ Note that $f'(x) = 1 + 2x + 3x^2 + \cdots + 100x^{99}$, so $f'(1) = 1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050$.

2. [**3**] Let $\ell$ be the line through $(0,0)$ and tangent to the curve $y = x^3 + x + 16$. Find the slope of $\ell$.

   **Answer:** $\boxed{13}$ Let the point of tangency be $(t, t^3 + t + 16)$, then the slope of $\ell$ is $(t^3 + t + 16)/t$. On the other hand, since $dy/dx = 3x^2 + 1$, the slope of $\ell$ is $3t^2 + 1$. Therefore,

   $$\frac{t^3 + t + 16}{t} = 3t^2 + 1.$$

   Simplifying, we get $t^3 = 8$, so $t = 2$. It follows that the slope is $3(2)^2 + 1 = 13$.

3. [**4**] Find all $y > 1$ satisfying $\displaystyle\int_1^y x \ln x \, dx = \frac{1}{4}$.

   **Answer:** $\boxed{\sqrt{e}}$ Applying integration by parts with $u = \ln x$ and $v = \frac{1}{2}x^2$, we get

   $$\int_1^y x \ln x \, dx = \frac{1}{2}x^2 \ln x \Big|_1^y - \frac{1}{2}\int_1^y x \, dx = \frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 + \frac{1}{4}.$$

   So $y^2 \ln y = \frac{1}{2}y^2$. Since $y > 1$, we obtain $\ln y = \frac{1}{2}$, and thus $y = \sqrt{e}$.

4. [**4**] Let $a, b$ be constants such that $\displaystyle\lim_{x \to 1} \frac{(\ln(2-x))^2}{x^2 + ax + b} = 1$. Determine the pair $(a, b)$.

   **Answer:** $\boxed{(-2, 1)}$ When $x = 1$, the numerator is 0, so the denominator must be zero as well, so $1 + a + b = 0$. Using l'Hôpital's rule, we must have

   $$1 = \lim_{x \to 1} \frac{(\ln(2-x))^2}{x^2 + ax + b} = \lim_{x \to 1} \frac{2\ln(2-x)}{(x-2)(2x+a)},$$

   and by the same argument we find that $2 + a = 0$. Thus, $a = -2$ and $b = 1$. This is indeed a solution, as can be seen by finishing the computation.

5. [**4**] Let $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$ for all real numbers $x$. Determine $f^{(2008)}(0)$ (i.e., $f$ differentiated 2008 times and then evaluated at $x = 0$).

   **Answer:** $\boxed{\frac{3}{8}}$ We have

   $$\begin{aligned}\sin^6 x + \cos^6 x &= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x) \\ &= 1 - 3\sin^2 x \cos^2 x = 1 - \frac{3}{4}\sin^2 2x = 1 - \frac{3}{4}\left(\frac{1 - \cos 4x}{2}\right) \\ &= \frac{5}{8} + \frac{3}{8}\cos 4x.\end{aligned}$$

   It follows that $f(x) = \frac{5}{8} + \frac{3}{8}\cos x$. Thus $f^{(2008)}(x) = \frac{3}{8}\cos x$. Evaluating at $x = 0$ gives $\frac{3}{8}$.

6. [**5**] Determine the value of $\displaystyle\lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k}^{-1}$.

**Answer:** $\boxed{2}$ Let $S_n$ denote the sum in the limit. For $n \geq 1$, we have $S_n \geq \binom{n}{0}^{-1} + \binom{n}{n}^{-1} = 2$. On the other hand, for $n \geq 3$, we have

$$S_n = \binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \binom{n}{n-1}^{-1} + \binom{n}{n}^{-1} + \sum_{k=2}^{n-2} \binom{n}{k}^{-1} \leq 2 + \frac{2}{n} + (n-3)\binom{n}{2}^{-1}$$

which goes to 2 as $n \to \infty$. Therefore, $S_n \to 2$.

7. [**5**] Find $p$ so that $\lim_{x\to\infty} x^p \left(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x}\right)$ is some non-zero real number.

**Answer:** $\boxed{\frac{5}{3}}$ Make the substitution $t = \frac{1}{x}$. Then the limit equals to

$$\lim_{t\to 0} t^{-p} \left(\sqrt[3]{\frac{1}{t}+1} + \sqrt[3]{\frac{1}{t}-1} - 2\sqrt[3]{\frac{1}{t}}\right) = \lim_{t\to 0} t^{-p-\frac{1}{3}} \left(\sqrt[3]{1+t} + \sqrt[3]{1-t} - 2\right).$$

We need the degree of the first nonzero term in the MacLaurin expansion of $\sqrt[3]{1+t} + \sqrt[3]{1-t} - 2$. We have

$$\sqrt[3]{1+t} = 1 + \frac{1}{3}t - \frac{1}{9}t^2 + o(t^2), \qquad \sqrt[3]{1-t} = 1 - \frac{1}{3}t - \frac{1}{9}t^2 + o(t^2).$$

It follows that $\sqrt[3]{1+t} + \sqrt[3]{1-t} - 2 = -\frac{2}{9}t^2 + o(t^2)$. By consider the degree of the leading term, it follows that $-p - \frac{1}{3} = -2$. So $p = \frac{5}{3}$.

8. [**7**] Let $T = \displaystyle\int_0^{\ln 2} \frac{2e^{3x} + e^{2x} - 1}{e^{3x} + e^{2x} - e^x + 1}\, dx$. Evaluate $e^T$.

**Answer:** $\boxed{\frac{11}{4}}$ Divide the top and bottom by $e^x$ to obtain that

$$T = \int_0^{\ln 2} \frac{2e^{2x} + e^x - e^{-x}}{e^{2x} + e^x - 1 + e^{-x}}\, dx$$

Notice that $2e^{2x} + e^x - e^{-x}$ is the derivative of $e^{2x} + e^x - 1 + e^{-x}$, and so

$$T = \Big[\ln|e^{2x} + e^x - 1 + e^{-x}|\Big]_0^{\ln 2} = \ln\left(4 + 2 - 1 + \frac{1}{2}\right) - \ln 2 = \ln\left(\frac{11}{4}\right)$$

Therefore, $e^T = \frac{11}{4}$.

9. [**7**] Evaluate the limit $\lim\limits_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)} \left(1^1 \cdot 2^2 \cdot \cdots \cdot n^n\right)^{\frac{1}{n^2}}$.

**Answer:** $\boxed{e^{-1/4}}$ Taking the logarithm of the expression inside the limit, we find that it is

$$-\frac{1}{2}\left(1 + \frac{1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k \ln k = \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n} \ln\left(\frac{k}{n}\right).$$

We can recognize this as the as Riemann sum expansion for the integral $\int_0^1 x \ln x\, dx$, and thus the limit of the above sum as $n \to \infty$ equals to the value of this integral. Evaluating this integral using integration by parts, we find that

$$\int_0^1 x \ln x\, dx = \frac{1}{2}x^2 \ln x\Big|_0^1 - \int_0^1 \frac{x}{2}\, dx = -\frac{1}{4}.$$

Therefore, the original limit is $e^{-1/4}$.

10. [**8**] Evaluate the integral $\int_0^1 \ln x \ln(1-x)\ dx$.

**Answer:** $\boxed{2 - \frac{\pi^2}{6}}$ We have the MacLaurin expansion $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$. So

$$\int_0^1 \ln x \ln(1-x)\ dx = -\int_0^1 \ln x \sum_{n=1}^{\infty} \frac{x^n}{n}\ dx = -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln x\ dx.$$

Using integration by parts, we get

$$\int_0^1 x^n \ln x\ dx = \left.\frac{x^{n+1}\ln x}{n+1}\right|_0^1 - \int_0^1 \frac{x^n}{n+1}\ dx = -\frac{1}{(n+1)^2}.$$

(We used the fact that $\lim_{x\to 0} x^n \ln x = 0$ for $n > 0$, which can be proven using l'Hôpital's rule.) Therefore, the original integral equals to

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right).$$

Telescoping the sum and using the well-known identity $\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we see that the above sum is equal to $2 - \frac{\pi^2}{6}$.