HMMT February 2022

February 19, 2022

Guts Round

1. **[5]** A regular 2022-gon has perimeter 6.28. To the nearest positive integer, compute the area of the 2022-gon.

   Proposed by: Akash Das

   Answer: $\boxed{3}$

   Solution: Note that the area of a regular 2022-gon is approximately equal to the area of its circumcircle, and the perimeter of a regular 2022-gon approximately equals the perimeter of its circumcircle. Since the perimeter is $6.28 \approx 2\pi$, the circumradius $R \approx 1$, so the area of the 2022-gon is approximately $R^2\pi \approx \pi \approx 3$.

2. **[5]** Three distinct vertices are randomly selected among the five vertices of a regular pentagon. Let p be the probability that the triangle formed by the chosen vertices is acute. Compute $10p$.

   Proposed by: Sheldon Kieren Tan

   Answer: $\boxed{5}$

   Solution: The only way for the three vertices to form an acute triangle is if they consist of two adjacent vertices and the vertex opposite their side. Since there are 5 ways to choose this and $\binom{5}{3} = 10$ ways to choose the three vertices, we have $p = \frac{5}{10} = \frac{1}{2}$.

3. **[5]** Herbert rolls 6 fair standard dice and computes the product of all of his rolls. If the probability that the product is prime can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

   Proposed by: Akash Das

   Answer: $\boxed{2692}$

   Solution: The only way this can happen is if 5 of the dice roll 1 and the last die rolls a prime number (2, 3, or 5). There are 6 ways to choose the die that rolls the prime, and 3 ways to choose the prime. Thus, the probability is $\frac{3\cdot 6}{6^6} = \frac{1}{2592}$.

4. **[5]** For a real number x, let $[x]$ be x rounded to the nearest integer and $\langle x\rangle$ be x rounded to the nearest tenth. Real numbers a and b satisfy $\langle a\rangle + [b] = 98.6$ and $[a] + \langle b\rangle = 99.3$. Compute the minimum possible value of $[10(a+b)]$.

   (Here, any number equally between two integers or tenths of integers, respectively, is rounded up. For example, $[-4.5] = -4$ and $\langle 4.35\rangle = 4.4$.)

   Proposed by: Sean Li

   Answer: $\boxed{988}$

   Solution: Without loss of generality, let a and b have the same integer part or integer parts that differ by at most 1, as we can always repeatedly subtract 1 from the larger number and add 1 to the smaller to get another solution.

   Next, we note that the decimal part of a must round to .6 and the decimal part of b must round to .3. We note that $(a, b) = (49.55, 49.25)$ is a solution and is clearly minimal in fractional parts, giving us $[10(a+b)] = 988$.

5. **[6]** Compute the remainder when

$$1000200040008001600320064012802560512102420484096 8192$$

is divided by 100020004000800160032.

Proposed by: Ankit Bisain

Answer: $\boxed{40968192}$

Solution: Let X_k denote 2^k except with leading zeroes added to make it four digits long. Let $\overline{abc\cdots}$ denote the number obtained upon concatenating $a, b, c, \ldots$. We have

$$2^6 \cdot \overline{X_0 X_1 \ldots X_5} = \overline{X_6 X_7 \ldots X_{11}}.$$

Therefore, $\overline{X_0 X_1 \ldots X_5}$ divides $\overline{X_0 X_1 \ldots X_{11}}$, meaning the remainder when $\overline{X_0 X_1 \ldots X_{13}}$ is divided by $\overline{X_0 X_1 \ldots X_5}$ is

$$\overline{X_{12} X_{13}} = 40968192.$$

6. **[6]** Regular polygons $ICAO$, $VENTI$, and $ALBEDO$ lie on a plane. Given that $IN = 1$, compute the number of possible values of ON.

Proposed by: Sean Li

Answer: $\boxed{2}$

Solution: First, place $ALBEDO$. We then note that $ICAO$ has two orientations, both of which have I on EO. Next, we note that for any given orientation of $ICAO$, the two orientations of $VENTI$ have N symmetric to line EI. Thus, for any given orientation of $ICAO$, we have that ON is the same in both orientations of $VENTI$, which gives a total of 2 possible values for ON.

7. **[6]** A jar contains 8 red balls and 2 blue balls. Every minute, a ball is randomly removed. The probability that there exists a time during this process where there are more blue balls than red balls in the jar can be expressed as $\frac{a}{b}$ for relatively prime integers a and b. Compute $100a + b$.

Proposed by: Brian Liu

Answer: $\boxed{209}$

Solution: One can show that the condition in the problem is satisfied if and only the last ball drown is blue (which happens with probability $\frac{1}{5}$), or the blue balls are drawn second-to-last and third-to-last (which happens with probability $\frac{1}{\binom{10}{2}} = \frac{1}{45}$). Thus the total probability is $\frac{10}{45} = \frac{2}{9}$.

8. **[6]** For any positive integer n, let $\tau(n)$ denote the number of positive divisors of n. If n is a positive integer such that $\frac{\tau(n^2)}{\tau(n)} = 3$, compute $\frac{\tau(n^7)}{\tau(n)}$.

Proposed by: Sean Li

Answer: $\boxed{29}$

Solution: Let the prime factorization of n be $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Then, the problem condition is equivalent to

$$\prod_{i=1}^{k} \frac{2e_i + 1}{e_i + 1} = 3.$$

Note that since $\frac{2x+1}{x+1} \geq 1.5$ for $x \geq 1$, and $1.5^3 > 3$, we have $k \leq 2$. Also, $k = 1$ implies $2e_1 + 1 = 3(e_1 + 1)$, which implies e_1 is negative. Thus, we must have $k = 2$. Then, our equation becomes

$$(2e_1 + 1)(2e_2 + 1) = 3(e_1 + 1)(e_2 + 1),$$

which simplifies to $(e_1 - 1)(e_2 - 1) = 3$. This gives us $e_1 = 2$ and $e_2 = 4$. Thus, we have $n = p^2q^4$ for primes p and q, so $\frac{\tau(n^7)}{\tau(n)} = \frac{\tau(p^{14}q^{28})}{\tau(p^2q^4)} = \frac{15\cdot 29}{3\cdot 5} = 29$.

9. **[7]** An E-shape is a geometric figure in the two-dimensional plane consisting of three rays pointing in the same direction, along with a line segment such that

   - the endpoints of the rays all lie on the segment,
   - the segment is perpendicular to all three rays,
   - both endpoints of the segment are endpoints of rays.

   Suppose two E-shapes intersect each other N times in the plane for some positive integer N. Compute the maximum possible value of N.

   Proposed by: Daniel Xianzhe Hong

   Answer: $\boxed{11}$

   Solution: Define a C-shape to be an E-shape without the middle ray. Then, an E-shape consists of a ray and a C-shape. Two C-shapes can intersect at most 6 times, a C-shape and a ray can intersect at most 2 times, and two rays can intersect at most 1 time. Thus, the number of intersections of two E-shapes is at most $6 + 2 + 2 + 1 = 11$.

10. **[7]** A positive integer n is loose it has six positive divisors and satisfies the property that any two positive divisors $a < b$ of n satisfy $b \geq 2a$. Compute the sum of all loose positive integers less than 100.

    Proposed by: Daniel Zhu

    Answer: $\boxed{512}$

    Solution: Note that the condition in the problem implies that for any divisor d of n, if d is odd then all other divisors of n cannot lie in the interval $[\lceil \frac{d}{2} \rceil, 2d-1]$. If d is even, then all other divisors cannot lie in the interval $[\frac{d}{2}+1, 2d-1]$.

    We first find that n must be of the form p^5 or p^2q for primes p and q. If $n = p^5$, the only solution is when $p = 2$ and $n = 32$.

    Otherwise, $n = p^2q$. Since $100 > n > p^2$, so $p \leq 7$. Now we can do casework on p. When $p = 2$, we find that q cannot lie in $[2, 3]$ or $[3, 7]$, so we must have $q \geq 11$. All such values for q work, giving solutions $n = 44, 52, 68, 76, 92$. When $p = 3$, we find that q cannot lie in $[2, 5]$ or $[5, 17]$, so we must have that $q \geq 19$, so there are no solutions in this case. When $p = 5$ or $p = 7$, the only solution occurs when $q = 2$ (since otherwise $n > 100$). This gives us the solutions $n = 50$ and $n = 98$. Adding these values of n gives 512.

11. **[7]** A regular dodecagon $P_1P_2\cdots P_{12}$ is inscribed in a unit circle with center O. Let X be the intersection of P_1P_5 and OP_2, and let Y be the intersection of P_1P_5 and OP_4. Let A be the area of the region bounded by XY, XP_2, YP_4, and minor arc $\widehat{P_2P_4}$. Compute $\lfloor 120A \rfloor$.

    Proposed by: Gabriel Wu

    Answer: $\boxed{45}$

    Solution: The area of sector OP_2P_4 is one sixth the area of the circle because its angle is $60°$. The desired area is just that of the sector subtracted by the area of equilateral triangle OXY.

Note that the altitude of this triangle is the distance from O to P_1P_5, which is $\frac{1}{2}$. Thus, the side length of the triangle is $\frac{\sqrt{3}}{3}$, implying that the area is $\frac{\sqrt{3}}{12}$. Thus, we find that $A = \frac{\pi}{6} - \frac{\sqrt{3}}{12}$. Thus, $120A = 20\pi - 10\sqrt{3} \approx 62.8 - 17.3$, which has floor 45.

12. **[7]** A unit square $ABCD$ and a circle Γ have the following property: if P is a point in the plane not contained in the interior of Γ, then $\min(\angle APB, \angle BPC, \angle CPD, \angle DPA) \le 60^\circ$. The minimum possible area of Γ can be expressed as $\frac{a\pi}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

    Proposed by: Daniel Zhu

    Answer: $\boxed{106}$

    Solution: Note that the condition for Γ in the problem is equivalent to the following condition: if $\min(\angle APB, \angle BPC, \angle CPD, \angle DPA) > 60^\circ$, then P is contained in the interior of Γ. Let X_1, X_2, X_3, and X_4 be the four points in $ABCD$ such that ABX_1, BCX_2, CDX_3, and DAX_4 are all equilateral triangles. Now, let $\Omega_1, \Omega_2, \Omega_3$, and Ω_4 be the respective circumcircles of these triangles, and let the centers of these circles be O_1, O_2, O_3, and O_4. Note that the set of points P such that $\angle APB, \angle BPC, \angle CPD, \angle DPA > 60^\circ$ is the intersection of $\Omega_1, \Omega_2, \Omega_3$, and Ω_4. We want to find the area of the minimum circle containing this intersection.

    Let Γ_1 and Γ_2 intersect at B and B'. Define C', D' and A' similarly. It is not hard to see that the circumcircle of square $A'B'C'D'$ is the desired circle. Now observe that $\angle AB'D' = \angle AB'D = 60^\circ$. Similarly, $\angle AD'B' = 60^\circ$, so $AB'D'$ is equilateral. Its height is the distance from A to $B'D'$, which is $\frac{1}{\sqrt{2}}$, so its side length is $\frac{\sqrt{6}}{3}$. This is also the diameter of the desired circle, so its area is $\frac{\pi}{4} \cdot \frac{6}{9} = \frac{\pi}{6}$.

13. **[9]** Let z_1, z_2, z_3, z_4 be the solutions to the equation $x^4+3x^3+3x^2+3x+1 = 0$. Then $|z_1|+|z_2|+|z_3|+|z_4|$ can be written as $\frac{a+b\sqrt{c}}{d}$, where c is a square-free positive integer, and a, b, d are positive integers with $\gcd(a, b, d) = 1$. Compute $1000a + 100b + 10c + d$.

    Proposed by: Daniel Zhu

    Answer: $\boxed{7152}$

    Solution: Note that $x = 0$ is clearly not a solution, so we can divide the equation by x^2 to get $\left(x^2 + 2 + \frac{1}{x^2}\right)+3\left(x + \frac{1}{x}\right)+1 = 0$. Letting $y = x+\frac{1}{x}$, we get that $y^2+3y+1 = 0$, so $y = x+\frac{1}{x} = \frac{-3\pm\sqrt{5}}{2}$. Since $\frac{-3+\sqrt{5}}{2}$ has absolute value less than 2, the associated x are on the unit circle, and thus the two solutions for x in this case each have magnitude 1. For $\frac{-3-\sqrt{5}}{2}$, the roots are negative reals that are reciprocals of each other. Thus, the sum of their absolute values is the absolute value of their sum, which is $\frac{3+\sqrt{5}}{2}$. Thus, the sum of the magnitudes of the four solutions are $1 + 1 + \frac{3+\sqrt{5}}{2} = \frac{7+\sqrt{5}}{2}$.

14. **[9]** The area of the largest regular hexagon that can fit inside of a rectangle with side lengths 20 and 22 can be expressed as $a\sqrt{b} - c$, for positive integers a, b, and c, where b is squarefree. Compute $100a + 10b + c$.

    Proposed by: Akash Das

    Answer: $\boxed{134610}$

    Solution: Let s be the sidelength of the hexagon. We can view this problem as finding the maximal rectangle of with sides s and $s\sqrt{3}$ that can fit inside this rectangle. Let $ABCD$ be a rectangle with $AB = 20$ and $BC = 22$ and let $XYZW$ be an inscribed rectangle with X on AB and Y on BC with $XY = s$ and $YZ = s\sqrt{3}$. Let $BX = a$ and $BY = b$. Then, by similar triangles, we have $AX = b\sqrt{3}$ and $CY = a\sqrt{3}$. Thus, we have $a + b\sqrt{3} = 20$ and $a\sqrt{3} + b = 22$. Solving gives us $a = 11\sqrt{3} - 10$ and $b = 10\sqrt{3}-11$, so $s^2 = a^2+b^2 = 884-440\sqrt{3}$. Thus, the area of the hexagon is $\frac{s^2 \cdot 3\sqrt{3}}{2} = 1326\sqrt{3}-1980$.

15. **[9]** Let N be the number of triples of positive integers (a, b, c) satisfying

$$a \leq b \leq c, \qquad \gcd(a, b, c) = 1, \qquad abc = 6^{2020}.$$

Compute the remainder when N is divided by 1000.

Proposed by: Benjamin Wu

Answer: $\boxed{602}$

Solution: Let $n = 2020$. If we let $a = 2^{p_1} \cdot 3^{q_1}, b = 2^{p_2} \cdot 3^{q_2}, c = 2^{p_3} \cdot 3^{q_3}$, then the number of ordered triples (a, b, c) that satisfy the second and third conditions is the number of nonnegative solutions to $p_1 + p_2 + p_3 = n$ and $q_1 + q_2 + q_3 = n$, where at least one of p_1, p_2, p_3 is zero and at least one of q_1, q_2, q_3 is zero (otherwise, $\gcd(a, b, c) > 1$). By complementary counting, the number is

$$\left(\binom{n+2}{2} - \binom{n-1}{2}\right)^2 = 9n^2.$$

Let ℓ be the number of unordered triples (a, b, c) with a, b, c distinct, and m the number of unordered triples (a, b, c) with two numbers equal. Since it is impossible for $a = b = c$, we have $9n^2 = 6\ell + 3m$.

We now count m. Without loss of generality, assume $a = b$. For the factors of 2, we have two choices: either assign 2^{2020} to c or assign 2^{1010} to both a and b. We have a similar two choices for the factors of 3. Therefore $m = 4$.

Our final answer is

$$N = m + n = \frac{6\ell + 3m + 3m}{6} = \frac{9 \cdot 2020^2 + 12}{6} = 2 + 6 \cdot 1010^2 \equiv 602 \pmod{1000}.$$

16. **[9]** Let ABC be an acute triangle with A-excircle Γ. Let the line through A perpendicular to BC intersect BC at D and intersect Γ at E and F. Suppose that $AD = DE = EF$. If the maximum value of $\sin B$ can be expressed as $\frac{\sqrt{a}+\sqrt{b}}{c}$ for positive integers a, b, and c, compute the minimum possible value of $a + b + c$.

Proposed by: Akash Das

Answer: $\boxed{705}$

Solution: First note that we can assume $AB < AC$. Suppose Γ is tangent to BC at T. Let $AD = DE = EF = x$. Then, by Power of a Point, we have $DT^2 = DE \cdot DF = x \cdot 2x = 2x^2 \implies DT = x\sqrt{2}$. Note that $CT = s - b$, and since the length of the tangent from A to Γ is s, we have $s^2 = AE \cdot AF = 6x^2$, so $CT = x\sqrt{6} - b$. Since $BC = BD + DT + TC$, we have $BD = BC - x\sqrt{2} - (x\sqrt{6} - b) = a + b - x(\sqrt{2} + \sqrt{6})$. Since $a + b = 2s - c = 2x\sqrt{6} - c$, we have $BD = x(\sqrt{6} - \sqrt{2}) - c$. Now, by Pythagorean Theorem, we have $c^2 = AB^2 = AD^2 + BD^2 = x^2 + [x(\sqrt{6} - \sqrt{2}) - c]^2$. Simplifying gives $x^2(9 - 4\sqrt{3}) = xc(2\sqrt{6} - 2\sqrt{2})$. This yields

$$\frac{x}{c} = \frac{2\sqrt{6} - 2\sqrt{2}}{9 - 4\sqrt{3}} = \frac{6\sqrt{2} + 10\sqrt{6}}{33} = \frac{\sqrt{72} + \sqrt{600}}{33}.$$

17. **[11]** Compute the number of positive real numbers x that satisfy

$$\left(3 \cdot 2^{\lfloor \log_2 x \rfloor} - x\right)^{16} = 2022x^{13}.$$

Proposed by: Daniel Zhu

Answer: $\boxed{9}$

Solution: Let $f(x) = 3 \cdot 2^{\lfloor \log_2 x \rfloor} - x$. Note that for each integer i, if $x \in [2^i, 2^{i+1})$, then $f(x) = 3 \cdot 2^i - x$. This is a line segment from $(2^i, 2^{i+1})$ to $(2^{i+1}, 2^i)$, including the first endpoint but not the second.

Now consider the function $f(x)^{16}/x^{13}$. This consists of segments of decreasing functions connecting $(2^i, 2^{3i+16})$ and $(2^{i+1}, 2^{3i-13})$. Note that for each $-1 \le i \le 7$, we have that $2^{3i-13} \le 2^{10} < 2022 < 2^{11} \le 2^{3i+16}$. This gives us 9 solutions in total.

18. **[11]** Compute the number of permutations π of the set $\{1, 2, \ldots, 10\}$ so that for all (not necessarily distinct) $m, n \in \{1, 2, \ldots, 10\}$ where $m + n$ is prime, $\pi(m) + \pi(n)$ is prime.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{4}$

Solution: Since π sends pairs (m, n) with $m + n$ prime to pairs (m', n') with $m' + n'$ prime, and there are only finitely many such pairs, we conclude that if $m + n$ is composite, then so is $\pi(m) + \pi(n)$. Also note that $2\pi(1) = \pi(1) + \pi(1)$ is prime because $2 = 1 + 1$ is prime. Thus, $\pi(1) = 1$. Now, since $1 + 2, 1 + 4, 1 + 6$, and $1 + 10$ are all prime, we know that $\pi(2), \pi(4), \pi(6)$, and $\pi(10)$ are all even. Additionally, since $8 + 2, 8 + 6, 8 + 6$, and $8 + 10$ are all composite, it is not hard to see that $\pi(8)$ must also be even. Therefore π preserves parity.

Now, draw a bipartite graph between the odd and even numbers where we have an edge between a and b if and only if $a + b$ composite. We now only need to compute automorphisms of this graph that fix 1. Note that the edges are precisely $1 - 8 - 7 - 2$, $3 - 6 - 9$, and $4 - 5 - 10$. Since 1 is a fixed point of π, we know that π fixes $1, 8, 7$, and 2. Additionally, $\pi(6) = 6$ and $\pi(5) = 5$. We can swap 3 and 9, as well as 4 and 10. Thus, there are $2 \cdot 2 = 4$ possible permutations.

19. **[11]** In right triangle ABC, a point D is on hypotenuse AC such that $BD \perp AC$. Let ω be a circle with center O, passing through C and D and tangent to line AB at a point other than B. Point X is chosen on BC such that $AX \perp BO$. If $AB = 2$ and $BC = 5$, then BX can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

Proposed by: Akash Das

Answer: $\boxed{8041}$

Solution: Note that since $AD \cdot AC = AB^2$, we have the tangency point of ω and AB is B', the reflection of B across A. Let Y be the second intersection of ω and BC. Note that by power of point, we have $BY \cdot BC = BB'^2 = 4AB^2 \implies BY = \frac{4AB^2}{BC}$. Note that AX is the radical axis of ω and the degenerate circle at B, so we have $XB^2 = XY \cdot XC$, so

$$BX^2 = (BC - BX)(BY - BX) = BX^2 - BX(BC + BY) + BC \cdot BY.$$

This gives us

$$BX = \frac{BC \cdot BY}{BC + BY} = \frac{4AB^2 \cdot BC}{4AB^2 + BC^2} = \frac{80}{41}.$$

20. **[11]** Let π be a uniformly random permutation of the set $\{1, 2, \ldots, 100\}$. The probability that $\pi^{20}(20) = 20$ and $\pi^{21}(21) = 21$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$. (Here, π^k means π iterated k times.)

Proposed by: Sean Li

Answer: $\boxed{1025}$

Solution: We look at the cycles formed by π. Let $\text{ord}_\pi(n)$ denote the smallest m such that $\pi^m(n) = n$. In particular, the condition implies that $\text{ord}_\pi(20) \mid 20$ and $\text{ord}_\pi(21) \mid 21$.

Claim 1. 20 and 21 cannot be in the same cycle.

Proof. If 20 and 21 were in the same cycle, then $x = \text{ord}_\pi(20) = \text{ord}_\pi(21)$ for some x. Then $x > 1$ since the cycle contains both 20 and 21, but $x \mid 20, x \mid 21$ implies $x = 1$, a contradiction.

Claim 2. The probability that $a = \text{ord}_\pi(20), b = \text{ord}_\pi(21)$ for some fixed a, b such that $a + b \le 100$ is $\frac{1}{99 \cdot 100}$.

Proof. We can just count these permutations. We first choose $a - 1$ elements of $[100] \setminus \{20, 21\}$ to be in the cycle of 20, then we similarly choose $b - 1$ to be in the cycle of 21. We then have $(a-1)!$ ways to reorder within the cycle of 20, $(b-1)!$ ways to reorder within the cycle of 21, and $(100 - a - b)!$ ways to permute the remaining elements. The total number of ways is just

$$\frac{98!}{(a-1)!(b-1)!(100-a-b)!} \cdot (a-1)!(b-1)!(100-a-b)! = 98!,$$

so the probability this happens is just $\frac{98!}{100!} = \frac{1}{9900}$.

Now, since $\text{ord}_\pi(20) \mid 20$ and $\text{ord}_\pi(21) \mid 21$, we have 6 possible values for $\text{ord}_\pi(20)$ and 4 for $\text{ord}_\pi(21)$, so in total we have a $\frac{6 \cdot 4}{9900} = \frac{2}{825}$ probability that the condition is satisfied.

21. **[12]** In the Cartesian plane, let $A = (0,0)$, $B = (200, 100)$, and $C = (30, 330)$. Compute the number of ordered pairs (x, y) of integers so that $(x + \frac{1}{2}, y + \frac{1}{2})$ is in the interior of triangle ABC.

Proposed by: Ankit Bisain

Answer: $\boxed{31480}$

Solution: We use Pick's Theorem, which states that in a lattice polygon with I lattice points in its interior and B lattice points on its boundary, the area is $I + B/2 - 1$. Also, call a point *center* if it is of the form $(x + \frac{1}{2}, y + \frac{1}{2})$ for integers x and y.

The key observation is the following – suppose we draw in the center points, rotate 45° degrees about the origin and scale up by $\sqrt{2}$. Then, the area of the triangle goes to $2K$, and the set of old lattice points and center points becomes a lattice. Hence, we can also apply Pick's theorem to this new lattice.

Let the area of the original triangle be K, let I_1 and B_1 be the number of interior lattice points and boundary lattice points, respectively. Let I_c and B_c be the number of interior and boundary points that are center points in the original triangle. Finally, let I_2 and B_2 be the number of interior and boundary points that are either lattice points or center points in the new triangle. By Pick's Theorem on both lattices,

$$\begin{aligned} K &= I_1 + B_1/2 - 1 \\ 2K &= I_2 + B_2/2 - 1 \\ \implies (I_2 - I_1) &= K - \frac{B_1 - B_2}{2} \\ \implies I_c &= K - \frac{B_c}{2}. \end{aligned}$$

One can compute that the area is 31500. The number of center points that lie on on AB, BC, and CA are 0, 10, and 30, respectively. Thus, the final answer is $31500 - \frac{0+10+30}{2} = 31480$.

22. **[12]** The function $f(x)$ is of the form $ax^2 + bx + c$ for some integers a, b, and c. Given that

$$\begin{aligned} &\{f(177\,883), f(348\,710), f(796\,921), f(858\,522)\} \\ &\qquad = \{1\,324\,754\,875\,645, 1\,782\,225\,466\,694, 1\,984\,194\,627\,862, 4\,388\,794\,883\,485\}, \end{aligned}$$

compute a.

Proposed by: Daniel Zhu

Answer: $\boxed{23}$

Solution: We first match the outputs to the inputs. To start, we observe that since $a \geq 0$ (since the answer to the problem is nonnegative), we must either have $f(858522) \approx 4.39 \cdot 10^{12}$ or $f(177883) \approx 4.39 \cdot 10^{12}$. However, since 858522 is relatively close to 796921, the first case is unrealistic, meaning that the second case must be true.

Now, looking mod 2, we find that $f(796921) \approx 1.32 \cdot 10^{12}$. Additionally, we find that mod 5, $f(1) \equiv f(3) \equiv 0 \pmod 5$, so $f(x) \equiv a(x-1)(x-3) \pmod 5$. Modulo 5, we now have $\{3a, 4a\} = \{f(0), f(2)\} = \{2, 4\}$, so it follows that $a \equiv 3 \pmod 5$, $f(349710) \approx 1.78 \cdot 10^{12}$ and $f(858522) \approx 1.98 \cdot 10^{12}$.

There are several ways to finish from here. One (somewhat tedious) method is to use mod 9, which tells us that $f(7) = 7$, $f(5) = 8$, $f(3) = 4$, which tells you that $a \equiv 5 \pmod 9$ (take a finite difference). This tells you that $a \equiv 23 \pmod{45}$, and $a \geq 68$ can be ruled out for being too large.

Another method is to work with the numbers themselves. One way to do this is to note that for quadratic polynomials,

$$f'\left(\frac{x+y}{2}\right) = \frac{f(y) - f(x)}{y - x}.$$

Using this for $\{177883, 348710\}$ and $\{796921, 858522\}$, we find that $f'(260000) \approx -1500000$ and $f'(830000) \approx 1000000$. Thus f' (which we know must be linear with slope $2a$) has slope just less than 50.

Either way, we find that $a = 23$. The actual polynomial is $8529708870514 - 27370172x + 23x^2$.

23. **[12]** Let $ABCD$ be an isosceles trapezoid such that $AB = 17$, $BC = DA = 25$, and $CD = 31$. Points P and Q are selected on sides AD and BC, respectively, such that $AP = CQ$ and $PQ = 25$. Suppose that the circle with diameter PQ intersects the sides AB and CD at four points which are vertices of a convex quadrilateral. Compute the area of this quadrilateral.

Proposed by: Fedir Yudin

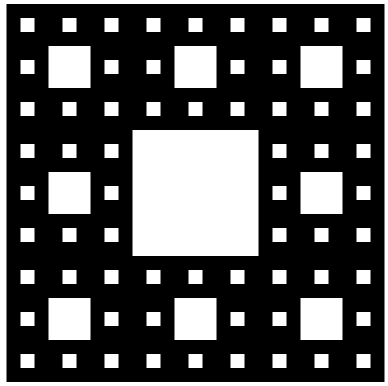
Answer: $\boxed{168}$

Solution:

Let the midpoint of PQ be M; note that M lies on the midline of $ABCD$. Let $B'C'$ be a translate of BC (parallel to AB and CD) so that M is the midpoint of B' and C'. Since $MB' = MC' = 25/2 = MP = MQ$, B' and C' are one of the four intersections of the circle with diameter PQ and the sides AB and CD. We may also define A' and D' similarly and get that they are also among the four points.

It follows that the desired quadrilateral is $B'D'C'A'$, which is a rectangle with height equal to the height of $ABCD$ (which is 24), and width equal to $\frac{1}{2}(31 - 17) = 7$. Thus the area is $24 \cdot 7 = 168$.

24. **[12]** Let S_0 be a unit square in the Cartesian plane with horizontal and vertical sides. For any $n > 0$, the shape S_n is formed by adjoining 9 copies of S_{n-1} in a 3×3 grid, and then removing the center copy. For example, S_3 is shown below:



Let a_n be the expected value of $|x - x'| + |y - y'|$, where (x, y) and (x', y') are two points chosen randomly within S_n. There exist relatively prime positive integers a and b such that

$$\lim_{n\to\infty} \frac{a_n}{3^n} = \frac{a}{b}.$$

Compute $100a + b$.

Proposed by: Gabriel Wu

Answer: $\boxed{1217}$

Solution: By symmetry, we only need to consider the x-distance, then we can multiply our answer by 2. Let this quantity be $g(n) = a_n/2$.

Divide the nth iteration fractal into three meta-columns of equal width. Then the probability that a random point is in the first, second, and third meta-columns is $\frac{3}{8}, \frac{2}{8}$, and $\frac{3}{8}$, respectively. If the two points end up in neighboring meta columns, the expected value of their x-distance is simply the width of a meta-column, which is 3^{n-1}. If they end up in opposite meta-columns (the left and right ones), it is twice this amount, which is $2 \cdot 3^{n-1}$. Finally, if the two points lie in the same meta-column, which happens with probability $(\frac{3}{8})^2 + (\frac{2}{8})^2 + (\frac{3}{8})^2 = \frac{11}{32}$, the expected x-distance is just $g(n-1)$. Thus, we have

$$g(n) = 3^{n-1}\left(2 \cdot \frac{3}{8} \cdot \frac{2}{8} + 2 \cdot \frac{2}{8} \cdot \frac{3}{8}\right) + \left(2 \cdot 3^{n-1}\right)\left(2 \cdot \frac{3}{8} \cdot \frac{3}{8}\right) + \frac{11}{32}g(n-1) = \frac{15}{16} \cdot 3^{n-1} + \frac{11}{32}g(n-1).$$

As n grows, say this is asymptotic to $g(n) = 3^n C$. For some constant C. Then we can write $3^n C = \frac{15}{16} \cdot 3^{n-1} + \frac{11}{32} \cdot 3^{n-1}C \implies C = \frac{6}{17}$. Our final answer is twice this, which is $\frac{12}{17}$.

25. **[14]** Let ABC be an acute scalene triangle with circumcenter O and centroid G. Given that AGO is a right triangle, $AO = 9$, and $BC = 15$, let S be the sum of all possible values for the area of triangle AGO. Compute S^2.

Proposed by: Akash Das

Answer: $\boxed{288}$

Solution: Note that we know that O, H, and G are collinear and that $HG = 2OG$. Thus, let $OG = x$ and $HG = 2x$. We also have $\sin A = \frac{BC}{2R} = \frac{5}{6}$, so $\cos A = \frac{\sqrt{11}}{6}$. Then, if $AG \perp OG$, then we have $x^2 + AG^2 = OG^2 + AG^2 = AO^2 = 81$ and $HG^2 + AG^2 = 4x^2 + AG^2 = AH^2 = (2R\cos A)^2 = 99$. Solving gives us $x = \sqrt{6}$ and $AG = 5\sqrt{3}$. Thus, the area of AGO in this case is $\frac{1}{2} \cdot \sqrt{6} \cdot 5\sqrt{3} = \frac{5\sqrt{3}}{2}$. If we have $AO \perp OG$, then we have $99 = AH^2 = AO^2 + OH^2 = 81 + 9x^2$. This gives us $x = \sqrt{2}$. In this case, we have the area of AGO is $\frac{1}{2} \cdot \sqrt{2} \cdot 9 = \frac{9\sqrt{2}}{2}$. Adding up the two areas gives us $S = 12\sqrt{2}$. Squaring gives $S^2 = 288$.

26. **[14]** Diana is playing a card game against a computer. She starts with a deck consisting of a single card labeled 0.9. Each turn, Diana draws a random card from her deck, while the computer generates a card with a random real number drawn uniformly from the interval $[0, 1]$. If the number on Diana's card is larger, she keeps her current card and also adds the computer's card to her deck. Otherwise, the computer takes Diana's card. After k turns, Diana's deck is empty. Compute the expected value of k.

Proposed by: Gabriel Wu

Answer: $\boxed{100}$

Solution: By linearity of expectation, we can treat the number of turns each card contributes to the total independently. Let $f(x)$ be the expected number of turns a card of value x contributes (we want $f(0.9)$). If we have a card of value x, we lose it after 1 turn with probability $1 - x$. If we don't lose it after the first turn, which happens with probability x, then given this, the expected number of turns this card contributes is $f(x) + \frac{1}{x}\int_0^x f(t)dt$. Thus, we can write the equation

$$f(x) = 1 + xf(x) + \int_0^x f(t)dt.$$

Differentiating both sides gives us

$$f'(x) = xf'(x) + f(x) + f(x) \implies \frac{f'(x)}{f(x)} = \frac{2}{1-x}.$$

Integrating gives us $\ln f(x) = -2\ln(1-x) + C \implies f(x) = \frac{e^C}{(1-x)^2}$. Since $f(0) = 1$, we know that $C = 0$, so $f(x) = (1-x)^{-2}$. Thus, we have $f(0.9) = (1-0.9)^{-2} = 100$.

27. **[14]** In three-dimensional space, let S be the region of points (x, y, z) satisfying $-1 \le z \le 1$. Let $S_1, S_2, \ldots, S_{2022}$ be 2022 independent random rotations of S about the origin $(0, 0, 0)$. The expected volume of the region $S_1 \cap S_2 \cap \cdots \cap S_{2022}$ can be expressed as $\frac{a\pi}{b}$, for relatively prime positive integers a and b. Compute $100a + b$.

Proposed by: Daniel Zhu

Answer: $\boxed{271619}$

Solution: Consider a point P of distance r from the origin. The distance from the origin of a random projection of P onto a line is uniform from 0 to r. Therefore, if $r < 1$ then the probability of P being in all the sets is 1, while for $r \ge 1$ it is r^{-2022}. Therefore the volume is

$$\frac{4\pi}{3} + 4\pi\int_1^\infty r^2 r^{-2022}\,dr = 4\pi\left(\frac{1}{3} + \frac{1}{2019}\right) = \frac{2696\pi}{2019}.$$

28. **[14]** Compute the nearest integer to

$$100\sum_{n=1}^{\infty} 3^n \sin^3\left(\frac{\pi}{3^n}\right).$$

Proposed by: Akash Das

Answer: $\boxed{236}$

Solution: Note that we have

$$\sin 3x = 3\sin x - 4\sin^3 x \implies \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x),$$

which implies that

$$\frac{\sin^3 x}{3x} = \frac{1}{4}\left(\frac{\sin x}{x} - \frac{\sin 3x}{3x}\right).$$

Substituting $x = \frac{\pi}{3^n}$ and simplifying gives us

$$3^n \sin^3 \frac{\pi}{3^n} = \frac{3\pi}{4}\left(\frac{\sin \frac{\pi}{3^n}}{\frac{\pi}{3^n}} - \frac{\sin \frac{\pi}{3^{n-1}}}{\frac{\pi}{3^{n-1}}}\right).$$

Summing this from $n = 1$ to N and telescoping gives us

$$\sum_{n=1}^{N} 3^n \sin^3\left(\frac{\pi}{3^n}\right) = \frac{3\pi}{4}\left(\frac{\sin \frac{\pi}{3^N}}{\frac{\pi}{3^N}} - \frac{\sin \pi}{\pi}\right).$$

Taking $N \to \infty$ gives us an answer of

$$100 \cdot \frac{3\pi}{4} \approx 236.$$

29. **[16]** Let $a \neq b$ be positive real numbers and m, n be positive integers. An $m+n$-gon P has the property that m sides have length a and n sides have length b. Further suppose that P can be inscribed in a circle of radius $a + b$. Compute the number of ordered pairs (m, n), with $m, n \leq 100$, for which such a polygon P exists for some distinct values of a and b.

Proposed by: Daniel Zhu

Answer: $\boxed{940}$

Solution: Letting $x = \frac{a}{a+b}$, we have to solve

$$m \arcsin \frac{x}{2} + n \arcsin \frac{1-x}{2} = \pi.$$

This is convex in x, so if it is to have a solution, we must find that the LHS exceeds π at one of the endpoints. Thus $\max(m, n) \geq 7$. If $\min(m, n) \leq 5$ we can find a solution by by the intermediate value theorem. Also if $\min(m, n) \geq 7$ then

$$m \arcsin \frac{x}{2} + n \arcsin \frac{1-x}{2} \geq 14 \arcsin(1/4) > \pi.$$

The inequality $\arcsin(1/4) > \frac{\pi}{14}$ can be verified by noting that

$$\sin \frac{\pi}{14} < \frac{\pi}{14} < \frac{3.5}{14} = \frac{1}{4}.$$

The final case is when $\min(m, n) = 6$. We claim that this doesn't actually work. If we assume that $n = 6$, we may compute the derivative at 0 to be

$$\frac{m}{2} - 6 \cdot \frac{1}{\sqrt{3}} = \frac{m - \sqrt{48}}{2} > 0,$$

so no solution exists.

30. **[16]** Let $(x_1, y_1), \ldots, (x_k, y_k)$ be the distinct real solutions to the equation

$$(x^2 + y^2)^6 = (x^2 - y^2)^4 = (2x^3 - 6xy^2)^3.$$

Then $\sum_{i=1}^{k}(x_i+y_i)$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a+b$.

Proposed by: Akash Das

Answer: $\boxed{516}$

Solution: Using polar coordinates, we can transform the problem to finding the intersections between $r = \cos 2\theta$ and $r = 2\cos 3\theta$. Drawing this out gives us a four-leaf clover and a large 3-leaf clover, which intersect at 7 points (one point being the origin). Note that since this graph is symmetric about the x-axis, we are only interested in finding the x-coordinates, which is $r\cos\theta = \cos 2\theta \cos\theta = 2\cos^3\theta - \cos\theta$.

Now note that all points of intersection satisfy

$$\cos 2\theta = 2\cos 3\theta \iff 8\cos^3\theta - 2\cos^2\theta - 6\cos\theta + 1 = 0.$$

Now, we want to compute the sum of $2\cos^3\theta - \cos\theta$ over all values of $\cos\theta$ that satisfy the above cubic. In other words, if the solutions for $\cos\theta$ to the above cubic are $a, b,$ and c, we want $2\sum_{\text{cyc}} 2a^3 - a$, since each value for $\cos\theta$ generates two solutions (symmetric about the x-axis).

This is

$$\sum_{\text{cyc}} 4a^3 - 2a = \sum_{\text{cyc}} a^2 + a - \frac{1}{2},$$

where we have used the fact that $a^3 = a^2 + 3a - \frac{1}{2}$. By Vieta's formulas, $a+b+c = \frac{1}{4}$, while

$$a^2+b^2+c^2 = \left(\frac{1}{4}\right)^2 + 2\cdot\frac{3}{4} = \frac{25}{16}.$$

Thus the final answer is $\frac{5}{16}$.

31. **[16]** For a point $P = (x, y)$ in the Cartesian plane, let $f(P) = (x^2 - y^2, 2xy - y^2)$. If S is the set of all P so that the sequence $P, f(P), f(f(P)), f(f(f(P))), \ldots$ approaches $(0,0)$, then the area of S can be expressed as $\pi\sqrt{r}$ for some positive real number r. Compute $\lfloor 100r \rfloor$.

Proposed by: Daniel Zhu

Answer: $\boxed{133}$

Solution: For a point $P = (x, y)$, let $z(P) = x + y\omega$, where ω is a nontrivial third root of unity. Then

$$\begin{aligned} z(f(P)) = (x^2-y^2) + (2xy - y^2)\omega = x^2 + 2xy\omega + y^2(-1-\omega) \\ = x^2 + 2xy\omega + y^2\omega^2 = (x+y\omega)^2 = z(P)^2. \end{aligned}$$

Applying this recursively gives us $z(f^n(P)) = z(f^{n-1}(P))^2 = z(f^{n-2}(P))^4 = \cdots = z(P)^{2^n}$. Thus the condition $f^n(P) \to (0,0)$ is equivalent to $|z(P)| < 1$. The region of such points is the preimage of the unit disk (area π) upon the "shear" sending $(0,1)$ to $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. This shear multiplies areas by a factor of $\frac{\sqrt{3}}{2}$, so the original area was $\frac{2\pi}{\sqrt{3}} = \pi\sqrt{\frac{4}{3}}$.

32. **[16]** An ant starts at the point $(0,0)$ in the Cartesian plane. In the first minute, the ant faces towards $(1,0)$ and walks one unit. Each subsequent minute, the ant chooses an angle θ uniformly at random in the interval $[-90^\circ, 90^\circ]$, and then turns an angle of θ clockwise (negative values of θ correspond to counterclockwise rotations). Then, the ant walks one unit. After n minutes, the ant's distance from $(0,0)$ is d_n. Let the expected value of d_n^2 be a_n. Compute the closest integer to

$$10\lim_{n\to\infty}\frac{a_n}{n}.$$

Proposed by: Carl Schildkraut

Answer: $\boxed{45}$

Solution: Let α_k be a random variable that represents the turn made after step k, choosing α_k uniformly at random on the complex plane among the arc of the unit circle containing 1 from $-i$ to i. It is well known that $\mathbb{E}[\alpha_k] = \frac{2}{\pi}$. We have that

$$a_n = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\prod_{k=1}^{i-1} \alpha_k \prod_{k=1}^{j-1} \alpha_k^{-1}\right].$$

Separating the sum based on $|i-j|$,

$$a_n = n + 2\sum_{t=1}^{n-1}(n-t)\mathbb{E}\left[\alpha_1 \cdots \alpha_t\right] = n + 2\sum_{t=1}^{n-1}(n-t)\left(\frac{2}{\pi}\right)^t.$$

Since terms with large t get very small, we can write

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} 1 + 2\sum_{t=1}^{\infty}\left(1-\frac{t}{n}\right)\left(\frac{2}{\pi}\right)^t = 1 + 2\sum_{t=1}^{\infty}\lim_{n\to\infty}\left(1-\frac{t}{n}\right)\left(\frac{2}{\pi}\right)^t = 1 + 2\sum_{t=1}^{\infty}\left(\frac{2}{\pi}\right)^t.$$

This gives that

$$x = \frac{\pi+2}{\pi-2}.$$

To estimate this quantity we use the approximation $\pi \approx 22/7$, which gives us

$$x \approx \frac{22+14}{22-14} = 4.5.$$

33. **[20]** In last year's HMMT Spring competition, 557 students submitted at least one answer to each of the three individual tests. Let S be the set of these students, and let P be the set containing the 30 problems on the individual tests. Estimate A, the number of subsets $R \subseteq P$ for which some student in S answered the questions in R correctly but no others. An estimate of E earns $\max(0, \lfloor 20 - \frac{2}{3}|A-E|\rfloor)$ points.

Proposed by: Daniel Zhu

Answer: $\boxed{450}$

34. **[20]** Estimate A, the number of unordered triples of integers (a, b, c) so that there exists a nondegenerate triangle with side lengths a, b, and c fitting inside a 100×100 square. An estimate of E earns $\max(0, \lfloor 20 - |A-E|/1000\rfloor)$ points.

Proposed by: Daniel Zhu

Answer: $\boxed{194162}$

Solution: Let's first count the number of such triangles with perimeter equal to p. By Stars and Bars, there are $\binom{p}{2} \approx \frac{p^2}{2}$ ordered triples of positive integers that sum to p. Additionally, note that only about a quarter of them satisfy the triangle inequality, we have only $\frac{p^2}{8}$ possible triples. Dividing by 3! gives us approximately $\frac{p^2}{48}$ nondegenerate triangles with perimeter p. Summing this from $p = 1$ to n gives us approximately $\frac{n^3}{144}$ triangles with perimeter at most n.

Now, note that there are two "extremes" for our triangles. One extreme is a triangle that is very close to a line. In that case, we have that the maximum perimeter is $200\sqrt{2} \approx 283$. In the other extreme, we have a triangle that is very close to an equilateral triangle, in which case we have the maximum perimeter is $3 \cdot \frac{100}{\cos 15^\circ} \approx 311$. Thus, as a compromise between these extremes, we can plug in $n = 300$ to get a value of $\frac{300^3}{144} = 187500$, which would have earned 13 points.

35. **[20]** A random permutation of $\{1, 2, \ldots, 100\}$ is given. It is then sorted to obtain the sequence $(1, 2, \ldots, 100)$ as follows: at each step, two of the numbers which are not in their correct positions are selected at random, and the two numbers are swapped. If s is the expected number of steps (i.e. swaps) required to obtain the sequence $(1, 2, \cdots, 100)$, then estimate $A = \lfloor s \rfloor$. An estimate of E earns $\max(0, \lfloor 20 - \frac{1}{2}|A - E| \rfloor)$ points.

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{2427}$

Solution: Let $f(n)$ be the expected number of steps if there are n elements out of order. Let's consider one of these permutations and suppose that a and b are random elements that are out of order. The probability that swapping a and b sends a to the proper place is $\frac{1}{n-1}$, and the probability that it sends b to the proper place is $\frac{1}{n-1}$. Thus we can approximate

$$f(n) \approx 1 + \frac{2}{n-1} f(n-1) + \frac{n-3}{n-1} f(n).$$

(The chance that both get sent to the right place decreases the overall probability that the number of fixed points increases, but also decreases the expected number of moves after the swap. These effects largely cancel out.)

As a result, we conclude that

$$f(n) \approx f(n-1) + \frac{n-1}{2},$$

and since $f(0) = 0$ we have $f(n) \approx \frac{n(n-1)}{4}$. At the beginning, the expected number of elements that are in the right place is 1, so the answer is approximately $f(99) \approx \frac{99 \cdot 98}{4} \approx 2425$. This is good enough for 19 points.

36. **[20]** For a cubic polynomial $P(x)$ with complex roots z_1, z_2, z_3, let

$$M(P) = \frac{\max(|z_1 - z_2|, |z_1 - z_3|, |z_2 - z_3|)}{\min(|z_1 - z_2|, |z_1 - z_3|, |z_2 - z_3|)}.$$

Over all polynomials $P(x) = x^3 + ax^2 + bx + c$, where a, b, c are nonnegative integers at most 100 and $P(x)$ has no repeated roots, the twentieth largest possible value of $M(P)$ is m. Estimate $A = \lfloor m \rfloor$. An estimate of E earns $\max(0, \lfloor 20 - 20|3\ln(A/E)|^{1/2} \rfloor)$ points.

Proposed by: Daniel Zhu

Answer: $\boxed{8097}$

Solution: Consider fixing a and b. Then, we know that $P'(x) = 3x^2 + 2ax + b$, which has a root at approximately $r \approx -b/2a$, which is rather small compared to 100. Then $P(r) \approx -b^2/4a$. Assuming that this is greater than about -100, then the value of c that produces the roots that are closest together is the closest integer to $-P(r)$ (the chance when this creates a double root is pretty rare. Let $-P(r) = c + s$, so that we can now assume that s is uniformly distributed in $(-1/2, 1/2)$. One can show that the difference between these roots is about $2\sqrt{|s|/a}$. Since these roots are rather small, by Vieta's formulas the other root is near $-a$, so $M(P)$ is about $\frac{1}{2}\sqrt{a^3/|s|}$.

It's clear from this discussion that a needs to be reasonably large for $M(P)$ to be large. Thus the condition $P(r) > -100$ is satisfied close to all the time—we will henceforth ignore it.

Set some L and let's consider the expected number of P so that $M(P) > L$. Then, for a given a, b, we need $|s| < a^3/(2L)^2$. Summing over all a and b, we find the probability is $2 \cdot 100 \cdot 100^4/4 \cdot 1/(2L)^2$. Setting this equal to 20 gives us

$$L^2 = \frac{10^{10}}{160} \implies L \approx 7900.$$

This is good enough for 14 points.