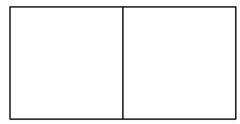
HMMT February 2015

Saturday 21 February 2015

Geometry

1. Let R be the rectangle in the Cartesian plane with vertices at $(0,0),(2,0),(2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown.



Pro selects a point P uniformly at random in the interior of R. Find the probability that the line through P with slope $\frac{1}{2}$ will pass through both unit squares.

2. Let ABC be a triangle with orthocenter H; suppose that $AB = 13$, $BC = 14$, $CA = 15$. Let G_A be the centroid of triangle HBC, and define G_B, G_C similarly. Determine the area of triangle $G_A G_B G_C$.

3. Let $ABCD$ be a quadrilateral with $\angle BAD = \angle ABC = 90°$, and suppose $AB = BC = 1$, $AD = 2$. The circumcircle of ABC meets $\overline{AD}$ and $\overline{BD}$ at points E and F, respectively. If lines AF and CD meet at K, compute EK.

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 3, BC = 2, CD = 2, DA = 4$. Let lines perpendicular to $\overline{BC}$ from B and C meet $\overline{AD}$ at B' and C', respectively. Let lines perpendicular to $\overline{AD}$ from A and D meet $\overline{BC}$ at A' and D', respectively. Compute the ratio $\frac{[BCC'B']}{[DAA'D']}$, where $[\varpi]$ denotes the area of figure ϖ.

5. Let I be the set of points (x, y) in the Cartesian plane such that

$$x > \left(\frac{y^4}{9} + 2015\right)^{1/4}$$

Let $f(r)$ denote the area of the intersection of I and the disk $x^2 + y^2 \le r^2$ of radius $r > 0$ centered at the origin $(0,0)$. Determine the minimum possible real number L such that $f(r) < Lr^2$ for all $r > 0$.

6. In triangle ABC, $AB = 2$, $AC = 1 + \sqrt{5}$, and $\angle CAB = 54°$. Suppose D lies on the extension of AC through C such that $CD = \sqrt{5} - 1$. If M is the midpoint of BD, determine the measure of $\angle ACM$, in degrees.

7. Let $ABCDE$ be a square pyramid of height $\frac{1}{2}$ with square base $ABCD$ of side length $AB = 12$ (so E is the vertex of the pyramid, and the foot of the altitude from E to $ABCD$ is the center of square $ABCD$). The faces ADE and CDE meet at an acute angle of measure α (so that $0° < \alpha < 90°$). Find $\tan\alpha$.

8. Let S be the set of **discs** D contained completely in the set $\{(x, y) : y < 0\}$ (the region below the x-axis) and centered (at some point) on the curve $y = x^2 - \frac{3}{4}$. What is the area of the union of the elements of S?

9. Let $ABCD$ be a regular tetrahedron with side length 1. Let X be the point in triangle BCD such that $[XBC] = 2[XBD] = 4[XCD]$, where $[\varpi]$ denotes the area of figure ϖ. Let Y lie on segment AX such that $2AY = YX$. Let M be the midpoint of BD. Let Z be a point on segment AM such that the lines YZ and BC intersect at some point. Find $\frac{AZ}{ZM}$.

10. Let $\mathcal{G}$ be the set of all points (x, y) in the Cartesian plane such that $0 \le y \le 8$ and

$$(x-3)^2 + 31 = (y-4)^2 + 8\sqrt{y(8-y)}.$$

There exists a unique line ℓ of **negative slope** tangent to $\mathcal{G}$ and passing through the point $(0, 4)$. Suppose ℓ is tangent to $\mathcal{G}$ at a **unique** point P. Find the coordinates (α, β) of P.