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1. **[5]** Compute the sum of all integers n such that $n^2 - 3000$ is a perfect square.

2. **[5]** Jerry and Neil have a 3-sided die that rolls the numbers 1, 2, and 3, each with probability $\frac{1}{3}$. Jerry rolls first, then Neil rolls the die repeatedly until his number is at least as large as Jerry's. Compute the probability that Neil's final number is 3.

3. **[5]** Compute the number of even positive integers $n \le 2024$ such that $1, 2, \ldots, n$ can be split into $\frac{n}{2}$ pairs, and the sum of the numbers in each pair is a multiple of 3.

4. **[5]** Equilateral triangles ABF and BCG are constructed outside regular pentagon $ABCDE$. Compute $\angle FEG$.

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5. **[6]** Let a, b, and c be real numbers such that

$$\begin{aligned} a+b+c &= 100, \\ ab+bc+ca &= 20, \text{ and} \\ (a+b)(a+c) &= 24. \end{aligned}$$

Compute all possible values of bc.

6. **[6]** In triangle ABC, points M and N are the midpoints of AB and AC, respectively, and points P and Q trisect BC. Given that A, M, N, P, and Q lie on a circle and $BC = 1$, compute the area of triangle ABC.

7. **[6]** Positive integers a, b, and c have the property that a^b, b^c, and c^a end in 4, 2, and 9, respectively. Compute the minimum possible value of $a+b+c$.

8. **[6]** Three points, A, B, and C, are selected independently and uniformly at random from the interior of a unit square. Compute the expected value of $\angle ABC$.

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9. **[7]** Compute the sum of all positive integers n such that $n^2 - 3000$ is a perfect square.

10. **[7]** Alice, Bob, and Charlie are playing a game with 6 cards numbered 1 through 6. Each player is dealt 2 cards uniformly at random. On each player's turn, they play one of their cards, and the winner is the person who plays the median of the three cards played. Charlie goes last, so Alice and Bob decide to tell their cards to each other, trying to prevent him from winning whenever possible. Compute the probability that Charlie wins regardless.

11. **[7]** Let $ABCD$ be a rectangle such that $AB = 20$ and $AD = 24$. Point P lies inside $ABCD$ such that triangles PAC and PBD have areas 20 and 24, respectively. Compute all possible areas of triangle PAB.

12. **[7]** Compute the number of quadruples (a, b, c, d) of positive integers satisfying

$$12a + 21b + 28c + 84d = 2024.$$

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13. **[9]** Mark has a cursed six-sided die that never rolls the same number twice in a row, and all other outcomes are equally likely. Compute the expected number of rolls it takes for Mark to roll every number at least once.

14. **[9]** Compute the smallest positive integer such that, no matter how you rearrange its digits (in base ten), the resulting number is a multiple of 63.

15. **[9]** Let $a \star b = ab - 2$. Compute the remainder when $(((579 \star 569) \star 559) \star \cdots \star 19) \star 9$ is divided by 100.

16. **[9]** Let ABC be an acute isosceles triangle with orthocenter H. Let M and N be the midpoints of sides $\overline{AB}$ and $\overline{AC}$, respectively. The circumcircle of triangle MHN intersects line BC at two points X and Y. Given $XY = AB = AC = 2$, compute BC^2.

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17. **[11]** The numbers $1, 2, \ldots, 20$ are put into a hat. Claire draws two numbers from the hat uniformly at random, $a < b$, and then puts them back into the hat. Then, William draws two numbers from the hat uniformly at random, $c < d$.

    Let N denote the number of integers n that satisfy exactly one of $a \le n \le b$ and $c \le n \le d$. Compute the probability N is even.

18. **[11]** An ordered pair (a, b) of positive integers is called *spicy* if $\gcd(a + b, ab + 1) = 1$. Compute the probability that both $(99, n)$ and $(101, n)$ are spicy when n is chosen from $\{1, 2, \ldots, 2024!\}$ uniformly at random.

19. **[11]** Let $A_1A_2 \ldots A_{19}$ be a regular nonadecagon. Lines A_1A_5 and A_3A_4 meet at X. Compute $\angle A_7XA_5$.

20. **[11]** Compute $\sqrt[4]{5508^3 + 5625^3 + 5742^3}$, given that it is an integer.

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21. **[12]** Kelvin the frog currently sits at $(0, 0)$ in the coordinate plane. If Kelvin is at (x, y), either he can walk to any of $(x, y + 1)$, $(x + 1, y)$, or $(x + 1, y + 1)$, or he can jump to any of $(x, y + 2)$, $(x + 2, y)$ or $(x + 1, y + 1)$. Walking and jumping from (x, y) to $(x + 1, y + 1)$ are considered distinct actions. Compute the number of ways Kelvin can reach $(6, 8)$.

22. **[12]** Let $x < y$ be positive real numbers such that

$$\sqrt{x} + \sqrt{y} = 4 \quad \text{and} \quad \sqrt{x+2} + \sqrt{y+2} = 5.$$

    Compute x.

23. **[12]** Let ℓ and m be two non-coplanar lines in space, and let P_1 be a point on ℓ. Let P_2 be the point on m closest to P_1, P_3 be the point on ℓ closest to P_2, P_4 be the point on m closest to P_3, and P_5 be the point on ℓ closest to P_4. Given that $P_1P_2 = 5$, $P_2P_3 = 3$, and $P_3P_4 = 2$, compute P_4P_5.

24. **[12]** A circle is tangent to both branches of the hyperbola $x^2 - 20y^2 = 24$ as well as the x-axis. Compute the area of this circle.

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25. **[14]** Point P is inside a square $ABCD$ such that $\angle APB = 135^\circ$, $PC = 12$, and $PD = 15$. Compute the area of this square.

26. **[14]** It can be shown that there exists a unique polynomial P in two variables such that for all positive integers m and n,

$$P(m,n) = \sum_{i=1}^{m} \sum_{j=1}^{n} (i+j)^7.$$

Compute $P(3,-3)$.

27. **[14]** A deck of 100 cards is labeled $1, 2, \ldots, 100$ from top to bottom. The top two cards are drawn; one of them is discarded at random, and the other is inserted back at the bottom of the deck. This process is repeated until only one card remains in the deck. Compute the expected value of the label of the remaining card.

28. **[14]** Given that the 32-digit integer

$$64\,312\,311\,692\,944\,269\,609\,355\,712\,372\,657$$

is the product of 6 consecutive primes, compute the sum of these 6 primes.

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29. **[16]** For each prime p, a polynomial $P(x)$ with rational coefficients is called *p-good* if and only if there exist three integers a, b, and c such that $0 \le a < b < c < \frac{p}{3}$ and p divides all the numerators of $P(a)$, $P(b)$, and $P(c)$, when written in simplest form. Compute the number of ordered pairs (r,s) of rational numbers such that the polynomial $x^3 + 10x^2 + rx + s$ is p-good for infinitely many primes p.

30. **[16]** Let ABC be an equilateral triangle with side length 1. Points D, E, F lie inside triangle ABC such that A, E, F are collinear, B, F, D are collinear, C, D, E are collinear, and triangle DEF is equilateral. Suppose that there exists a unique equilateral triangle XYZ with X on side $\overline{BC}$, Y on side $\overline{AB}$, and Z on side $\overline{AC}$ such that D lies on side $\overline{XZ}$, E lies on side $\overline{YZ}$, and F lies on side $\overline{XY}$. Compute AZ.

31. **[16]** Ash and Gary independently come up with their own lineups of 15 fire, grass, and water monsters. Then, the first monster of both lineups will fight, with fire beating grass, grass beating water, and water beating fire. The defeated monster is then substituted with the next one from their team's lineup; if there is a draw, both monsters get defeated.

    Gary completes his lineup randomly, with each monster being equally likely to be any of the three types. Without seeing Gary's lineup, Ash chooses a lineup that maximizes the probability p that his monsters are the last ones standing. Compute p.

32. **[16]** Over all pairs of complex numbers (x,y) satisfying the equations

$$x + 2y^2 = x^4 \quad \text{and} \quad y + 2x^2 = y^4,$$

compute the minimum possible real part of x.

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33. **[20]** Let p denote the proportion of teams, out of all participating teams, who submitted a negative response to problem 5 of the Team round (e.g. "there are no such integers"). Estimate $P = \lfloor 10000p \rfloor$. An estimate of E earns $\max(0, \lfloor 20 - |P - E|/20 \rfloor)$ points.

    If you have forgotten, problem 5 of the Team round was the following: "Determine, with proof, whether there exist positive integers x and y such that $x + y$, $x^2 + y^2$, and $x^3 + y^3$ are all perfect squares."

34. **[20]** Estimate the number of positive integers $n \le 10^6$ such that $n^2 + 1$ has a prime factor greater than n.

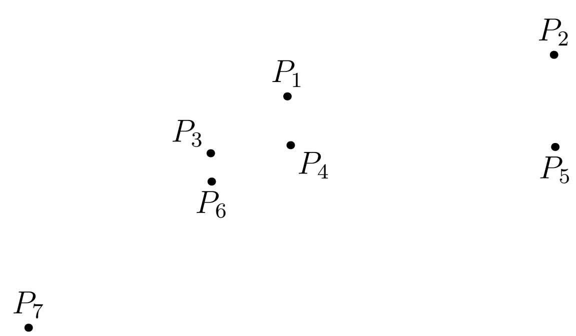
    Submit a positive integer E. If the correct answer is A, you will receive $\max\left(0, \left\lfloor 20 \cdot \min\left(\frac{E}{A}, \frac{10^6-E}{10^6-A}\right)^5 + 0.5\right\rfloor\right)$ points.

35. **[20]** Barry picks infinitely many points inside a unit circle, each independently and uniformly at random, $P_1, P_2, \ldots$. Compute the expected value of N, where N is the smallest integer such that P_{N+1} is inside the convex hull formed by the points $P_1, P_2, \ldots, P_N$.

    Submit a positive real number E. If the correct answer is A, you will receive $\lfloor 100 \cdot \max(0.2099 - |E - A|, 0) \rfloor$ points.

36. **[20]** Let ABC be a triangle. The following diagram contains points $P_1, P_2, \ldots, P_7$, which are the following triangle centers of triangle ABC in some order:

    - the incenter I;
    - the circumcenter O;
    - the orthocenter H;
    - the symmedian point L, which is the intersections of the reflections of B-median and C-median across angle bisectors of $\angle ABC$ and $\angle ACB$, respectively;
    - the Gergonne point G, which is the intersection of lines from B and C to the tangency points of the incircle with $\overline{AC}$ and $\overline{AB}$, respectively;
    - the Nagel point N, which is the intersection of line from B to the tangency point between B-excircle and $\overline{AC}$, and line from C to the tangency point between C-excircle and $\overline{AB}$; and
    - the Kosnita point K, which is the intersection of lines from B and C to the circumcenters of triangles AOC and AOB, respectively.



Note that the triangle ABC is not shown. Compute which triangle centers $\{I, O, H, L, G, N, K\}$ corresponds to P_k for $k \in \{1, 2, 3, 4, 5, 6, 7\}$.

Your answer should be a seven-character string containing I, O, H, L, G, N, K, or X for blank. For instance, if you think $P_2 = H$ and $P_6 = L$, you would answer $XHXXXLX$. If you attempt to identify $n > 0$ points and get them **all** correct, then you will receive $\lceil (n-1)^{5/3} \rceil$ points. Otherwise, you will receive 0 points.