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1. **[5]** A circle of area 1 is cut by two distinct chords. Compute the maximum possible area of the smallest resulting piece.

2. **[5]** Compute the smallest integer $n > 72$ that has the same set of prime divisors as 72.

3. **[5]** The graphs of the lines

$$y = x + 2, \quad y = 3x + 4, \quad y = 5x + 6, \quad y = 7x + 8, \quad y = 9x + 10, \quad y = 11x + 12$$

are drawn. These six lines divide the plane into several regions. Compute the number of regions the plane is divided into.

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4. **[6]** The number 17^6 when written out in base 10 contains 8 distinct digits from $1, 2, \ldots, 9$, with no repeated digits or zeroes. Compute the missing nonzero digit.

5. **[6]** Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 20$, $CD = 24$, and area 880. Compute the area of the triangle formed by the midpoints of AB, AC, and BD.

6. **[6]** The vertices of a cube are labeled with the integers 1 through 8, with each used exactly once. Let s be the maximum sum of the labels of two edge-adjacent vertices. Compute the minimum possible value of s over all such labelings.

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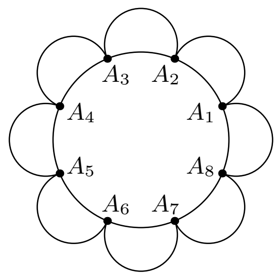
7. **[7]** Let $\mathcal{P}$ be a regular 10-gon in the coordinate plane. Mark computes the number of distinct x-coordinates that vertices of $\mathcal{P}$ take. Across all possible placements of $\mathcal{P}$ in the plane, compute the sum of all possible answers Mark could get.

8. **[7]** Derek is bored in math class and is drawing a flower. He first draws 8 points $A_1, A_2, \ldots, A_8$ equally spaced around an enormous circle. He then draws 8 arcs outside the circle where the ith arc for $i = 1, 2, \ldots, 8$ has endpoints A_i, A_{i+1} with $A_9 = A_1$, such that all of the arcs have radius 1 and any two consecutive arcs are tangent. Compute the perimeter of Derek's 8-petaled flower (not including the central circle).



9. **[7]** Compute the remainder when

$$1\,002\,003\,004\,005\,006\,007\,008\,009$$

is divided by 13.

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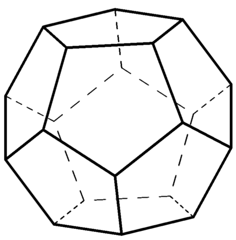
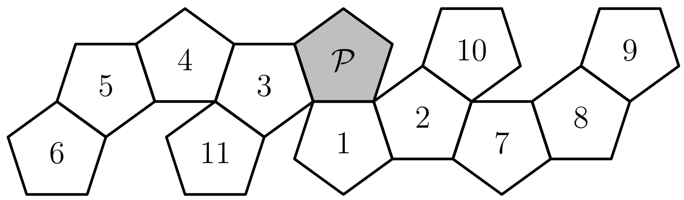
10. **[8]** Compute the largest prime factor of $3^{12} + 3^9 + 3^5 + 1$.

11. **[8]** A four-digit integer in base 10 is *friendly* if its digits are four consecutive digits in any order. A four-digit integer is *shy* if there exist two adjacent digits in its representation that differ by 1. Compute the number of four-digit integers that are both friendly and shy.

12. **[8]** A dodecahedron is a polyhedron shown on the left below. One of its nets is shown on the right. Compute the label of the face opposite to $\mathcal{P}$.





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13. **[9]** Let f and g be two quadratic polynomials with real coefficients such that the equation $f(g(x)) = 0$ has four distinct real solutions: 112, 131, 146, and a. Compute the sum of all possible values of a.

14. **[9]** Let $ABCD$ be a trapezoid with $AB \parallel CD$. Point X is placed on segment $\overline{BC}$ such that $\angle BAX = \angle XDC$. Given that $AB = 5$, $BX = 3$, $CX = 4$, and $CD = 12$, compute AX.

15. **[9]** Compute the sum of the three smallest positive integers n for which

$$\frac{1 + 2 + 3 + \cdots + (2024n - 1) + 2024n}{1 + 2 + 3 + \cdots + (4n - 1) + 4n}$$

is an integer.

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16. **[10]** Compute
$$\frac{2+3+\cdots+100}{1}+\frac{3+4+\cdots+100}{1+2}+\cdots+\frac{100}{1+2+\cdots+99}.$$

17. **[10]** Compute the number of ways to shade in some subset of the 16 cells in a 4×4 grid such that each of the 25 vertices of the grid is a corner of at least one shaded cell.

18. **[10]** Let $ABCD$ be a rectangle whose vertices are labeled in counterclockwise order with $AB = 32$ and $AD = 60$. Rectangle $AB'C'D'$ is constructed by rotating $ABCD$ counterclockwise about A by $60°$. Given that lines BB' and DD' intersect at point X, compute CX.

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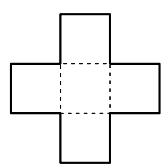
19. **[11]** An equilateral triangle is inscribed in a circle ω. A chord of ω is cut by the perimeter of the triangle into three segments of lengths 55, 121, and 55 in that order. Compute the sum of all possible side lengths of the triangle.

20. **[11]** There exists a unique line tangent to the graph of $y = x^4 - 20x^3 + 24x^2 - 20x + 25$ at two distinct points. Compute the product of the x-coordinates of the two tangency points.

21. **[11]** Two points are chosen independently and uniformly at random from the interior of the X-pentomino shown below. Compute the probability that the line segment between these two points lies entirely within the X-pentomino.



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22. **[12]** Suppose that a and b are positive integers such that $\gcd(a^3 - b^3, (a-b)^3)$ is not divisible by any perfect square except 1. Given that $1 \le a - b \le 50$, compute the number of possible values of $a - b$ across all such a, b.

23. **[12]** Consider a quarter-circle with center O, arc $\widehat{AB}$, and radius 2. Draw a semicircle with diameter $\overline{OA}$ lying inside the quarter-circle. Points P and Q lie on the semicircle and segment $\overline{OB}$, respectively, such that line PQ is tangent to the semicircle. As P and Q vary, compute the maximum possible area of triangle BQP.

24. **[12]** Let $f(x) = x^2 + 6x + 6$. Compute the greatest real number x such that $f(f(f(f(f(f(x)))))) = 0$.

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25. **[13]** Let ABC be an equilateral triangle. A regular hexagon $PXQYRZ$ of side length 2 is placed so that P, Q, and R lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. If points A, X, and Y are collinear, compute BC.

26. **[13]** A right rectangular prism of silly powder has dimensions $20 \times 24 \times 25$. Jerry the wizard applies 10 bouts of highdroxylation to the box, each of which increases one dimension of the silly powder by 1 and decreases a different dimension of the silly powder by 1, with every possible choice of dimensions equally likely to be chosen and independent of all previous choices. Compute the expected volume of the silly powder after Jerry's routine.

27. **[13]** For any positive integer n, let $f(n)$ be the number of ordered triples (a, b, c) of positive integers such that

    - $\max(a, b, c)$ divides n and
    - $\gcd(a, b, c) = 1$.

    Compute $f(1) + f(2) + \cdots + f(100)$.

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28. **[15]** The graph of the equation $\tan(x+y) = \tan(x) + 2\tan(y)$, with its pointwise holes filled in, partitions the coordinate plane into congruent regions. Compute the perimeter of one of these regions.

29. **[15]** Let ABC be a triangle such that $AB = 3$, $AC = 4$, and $\angle BAC = 75^\circ$. Square $BCDE$ is constructed outside triangle ABC. Compute $AD^2 + AE^2$.

30. **[15]** Compute the number of ways to shade exactly 4 distinct cells of a 4×4 grid such that no two shaded cells share one or more vertices.

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31. **[17]** Positive integers a, b, and c have the property that $\operatorname{lcm}(a, b)$, $\operatorname{lcm}(b, c)$, and $\operatorname{lcm}(c, a)$ end in 4, 6, and 7, respectively, when written in base 10. Compute the minimum possible value of $a + b + c$.

32. **[17]** Let ABC be an acute triangle and D be the foot of altitude from A to $\overline{BC}$. Let X and Y be points on the segment $\overline{BC}$ such that $\angle BAX = \angle YAC$, $BX = 2$, $XY = 6$, and $YC = 3$. Given that $AD = 12$, compute BD.

33. **[17]** A grid is called *groovy* if each cell of the grid is labeled with the smallest positive integer that does not appear below it in the same column or to the left of it in the same row. Compute the sum of the entries of a groovy 14×14 grid whose bottom left entry is 1.

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34. **[20]** The largest known prime number as of October 2024 is $2^{136\,279\,841} - 1$. It happens to be an example of a prime number of the form $2x^2 - 1$. Estimate the number of positive integers $x \leq 10^6$ such that $2x^2 - 1$ is prime.

    Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20.99 \max\left(0, 1 - \frac{|E-A|}{A}\right)^{2.5} \right\rfloor$ points.

35. **[20]** There are 1024 players, ranked from 1 (most skilled) to 1024 (least skilled), participating in a single elimination tournament. In each of the 10 rounds, the remaining players are paired uniformly at random. In each match, the player with a lower rank always wins, and the loser is eliminated from the tournament.

    For each positive integer $n \in [1, 1024]$, let $f(n)$ be the expected number of rounds that the participant with rank n participates in. Estimate the minimum positive integer N such that $f(N) < 2$.

    Submit a positive integer E. If the correct answer is A, you will receive $\max\left(0, 20 - \left\lfloor \frac{|E-A|}{2} \right\rfloor\right)$ points.

36. **[20]** Estimate the value of

$$\frac{20! \cdot 40! \cdot 40!}{100!} \cdot \sum_{i=0}^{40} \sum_{j=0}^{40} \frac{(i+j+18)!}{i!j!18!}.$$

    Submit a positive real number E either in decimal or in a fraction of two positive integers written in decimal (such as $\frac{2024}{2025}$). If the correct answer is A, your will receive $\max(20 - \lfloor 5000 \cdot |E-A| \rfloor, 0)$ points.