HMMT February 2018

February 10, 2018

Team Round

1. [**20**] In an $n \times n$ square array of 1×1 cells, at least one cell is colored pink. Show that you can always divide the square into rectangles along cell borders such that each rectangle contains exactly one pink cell.

2. [**25**] Is the number
$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2018}\right)$$
greater than, less than, or equal to 50?

3. [**30**] Michelle has a word with 2^n letters, where a word can consist of letters from any alphabet. Michelle performs a *switcheroo* on the word as follows: for each $k = 0, 1, \ldots, n-1$, she switches the first 2^k letters of the word with the next 2^k letters of the word. For example, for $n = 3$, Michelle changes
$$ABCDEFGH \to BACDEFGH \to CDBAEFGH \to EFGHCDBA$$
in one switcheroo.

   In terms of n, what is the minimum positive integer m such that after Michelle performs the switcheroo operation m times on any word of length 2^n, she will receive her original word?

4. [**30**] In acute triangle ABC, let D, E, and F be the feet of the altitudes from A, B, and C respectively, and let L, M, and N be the midpoints of BC, CA, and AB, respectively. Lines DE and NL intersect at X, lines DF and LM intersect at Y, and lines XY and BC intersect at Z. Find $\frac{ZB}{ZC}$ in terms of AB, AC, and BC.

5. [**30**] Is it possible for the projection of the set of points (x, y, z) with $0 \leq x, y, z \leq 1$ onto some two-dimensional plane to be a simple convex pentagon?

6. [**35**] Let $n \geq 2$ be a positive integer. A subset of positive integers S is said to be *comprehensive* if for every integer $0 \leq x < n$, there is a subset of S whose sum has remainder x when divided by n. Note that the empty set has sum 0. Show that if a set S is comprehensive, then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

7. [**50**] Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. We randomly choose a function $f : [n] \to [n]$, out of the n^n possible functions. We also choose an integer a uniformly at random from $[n]$. Find the probability that there exist positive integers $b, c \geq 1$ such that $f^b(1) = a$ and $f^c(a) = 1$. ($f^k(x)$ denotes the result of applying f to x k times).

8. [**60**] Allen plays a game on a tree with $2n$ vertices, each of whose vertices can be red or blue. Initially, all of the vertices of the tree are colored red. In one move, Allen is allowed to take two vertices of the same color which are connected by an edge and change both of them to the opposite color. He wins if at any time, all of the vertices of the tree are colored blue.

   (a) (**20**) Show that Allen can win if and only if the vertices can be split up into two groups V_1 and V_2 of size n, such that each edge in the tree has one endpoint in V_1 and one endpoint in V_2.

   (b) (**40**) Let $V_1 = \{a_1, \ldots, a_n\}$ and $V_2 = \{b_1, \ldots, b_n\}$ from part (a). Let M be the minimum over all permutations σ of $\{1, \ldots, n\}$ of the quantity
$$\sum_{i=1}^{n} d(a_i, b_{\sigma(i)}),$$
   where $d(v, w)$ denotes the number of edges along the shortest path between vertices v and w in the tree.

   Show that if Allen can win, then the minimum number of moves that it can take for Allen to win is equal to M.

(A *graph* consists of a set of vertices and some edges between distinct pairs of vertices. It is *connected* if every pair of vertices are connected by some path of one or more edges. A *tree* is a graph which is connected, in which the number of edges is one less than the number of vertices.)

9. [**60**] Evan has a simple graph with v vertices and e edges. Show that he can delete at least $\frac{e-v+1}{2}$ edges so that each vertex still has at least half of its original degree.

10. [**60**] Let n and m be positive integers which are at most 10^{10}. Let R be the rectangle with corners at $(0,0),(n,0),(n,m),(0,m)$ in the coordinate plane. A simple non-self-intersecting quadrilateral with vertices at integer coordinates is called *far-reaching* if each of its vertices lie on or inside R, but each side of R contains at least one vertex of the quadrilateral. Show that there is a far-reaching quadrilateral with area at most 10^6.

    (A side of a rectangle includes the two endpoints.)