HMT 2014

THE HARVARD-MIT MATHEMATICS TOURNAMENT

# COMBINATORICS

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This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Answers should be simplified as much as is reasonably possible and must be exact unless otherwise specified. Rational numbers should be written in lowest terms, although denominators of irrationals need not be rationalized. An $n$th root should be simplified so that the radicand is not divisible by the $n$th power of any prime.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to the Science Center Lobby during lunchtime.

Enjoy!

# HMMT 2014

**Saturday 22 February 2014**

## Combinatorics

1. There are 100 students who want to sign up for the class Introduction to Acting. There are three class sections for Introduction to Acting, each of which will fit exactly 20 students. The 100 students, including Alex and Zhu, are put in a lottery, and 60 of them are randomly selected to fill up the classes. What is the probability that Alex and Zhu end up getting into the same section for the class?

2. There are 10 people who want to choose a committee of 5 people among them. They do this by first electing a set of 1, 2, 3, or 4 committee leaders, who then choose among the remaining people to complete the 5-person committee. In how many ways can the committee be formed, assuming that people are distinguishable? (Two committees that have the same members but different sets of leaders are considered to be distinct.)

3. Bob writes a random string of 5 letters, where each letter is either $A$, $B$, $C$, or $D$. The letter in each position is independently chosen, and each of the letters $A, B, C, D$ is chosen with equal probability. Given that there are at least two $A$'s in the string, find the probability that there are at least three $A$'s in the string.

4. Find the number of triples of sets $(A, B, C)$ such that:

   (a) $A, B, C \subseteq \{1, 2, 3, \ldots, 8\}$.

   (b) $|A \cap B| = |B \cap C| = |C \cap A| = 2$.

   (c) $|A| = |B| = |C| = 4$.

   Here, $|S|$ denotes the number of elements in the set $S$.

5. Eli, Joy, Paul, and Sam want to form a company; the company will have 16 shares to split among the 4 people. The following constraints are imposed:

   - Every person must get a positive integer number of shares, and all 16 shares must be given out.
   - No one person can have more shares than the other three people combined.

   Assuming that shares are indistinguishable, but people are distinguishable, in how many ways can the shares be given out?

6. We have a calculator with two buttons that displays an integer $x$. Pressing the first button replaces $x$ by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces $x$ by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to the real number $y$.)

7. Six distinguishable players are participating in a tennis tournament. Each player plays one match of tennis against every other player. There are no ties in this tournament—each tennis match results in a win for one player and a loss for the other. Suppose that whenever $A$ and $B$ are players in the tournament such that $A$ wins strictly more matches than $B$ over the course of the tournament, it is also true that $A$ wins the match against $B$ in the tournament. In how many ways could the tournament have gone?

8. The integers $1, 2, \ldots, 64$ are written in the squares of a $8 \times 8$ chess board, such that for each $1 \leq i < 64$, the numbers $i$ and $i + 1$ are in squares that share an edge. What is the largest possible sum that can appear along one of the diagonals?

9. There is a heads up coin on every integer of the number line. Lucky is initially standing on the zero point of the number line facing in the positive direction. Lucky performs the following procedure: he looks at the coin (or lack thereof) underneath him, and then,

- If the coin is heads up, Lucky flips it to tails up, turns around, and steps forward a distance of one unit.
- If the coin is tails up, Lucky picks up the coin and steps forward a distance of one unit facing the same direction.
- If there is no coin, Lucky places a coin heads up underneath him and steps forward a distance of one unit facing the same direction.

He repeats this procedure until there are 20 coins anywhere that are tails up. How many times has Lucky performed the procedure when the process stops?

10. An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_1), \ldots, (x_k, y_k)$ of points in $\mathbb{R}^2$ such that $(a, b) = (x_1, y_1)$, $(c, d) = (x_k, y_k)$, and for each $1 \leq i < k$ we have that either $(x_{i+1}, y_{i+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$. Two up-right paths are said to intersect if they share any point.

    Find the number of pairs $(A, B)$ where $A$ is an up-right path from $(0, 0)$ to $(4, 4)$, $B$ is an up-right path from $(2, 0)$ to $(6, 4)$, and $A$ and $B$ do not intersect.

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**Combinatorics**

**PUT LABEL HERE**

Name ______________________________ Team ID# __________

Organization ______________________ Team ______________________

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________
7. ____________________
8. ____________________
9. ____________________
10. ____________________

Score: ____________