HMMT November 2014

Saturday 15 November 2014

Theme Round

1. Find the probability that the townspeople win if there are initially two townspeople and one goon.

   Answer: $\boxed{\frac{1}{3}}$ The goon is chosen on the first turn with probability $\frac{1}{3}$, and this is necessary and sufficient for the townspeople to win.

2. Find the smallest positive integer n such that, if there are initially $2n$ townspeople and 1 goon, then the probability the townspeople win is greater than 50%.

   Answer: $\boxed{3}$ We instead consider the probability the goon wins. The game clearly must last n days. The probability the goon is not sent to jail on any of these n days is then

$$\frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \dots \cdot \frac{2}{3}.$$

   If $n = 2$ then the probability the goon wins is $\frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15} > \frac{1}{2}$, but when $n = 3$ we have $\frac{6}{7} \cdot \frac{8}{15} = \frac{16}{35} < \frac{1}{2}$, so the answer is $n = 3$.

   Alternatively, the let p_n be the probability that $2n$ townspeople triumph against 1 goon. There is a $\frac{1}{2n+1}$ chance that the goon is jailed during the first morning and the townspeople win. Otherwise, the goon eliminates one townperson during the night. We thus have $2n-2$ townspeople and 1 goon left, so the probability that the town wins is p_{n-1}. We obtain the recursion

$$p_n = \frac{1}{2n+1} + \frac{2n}{2n+1} p_{n-1}.$$

   By the previous question, we have the initial condition $p_1 = \frac{1}{3}$. We find that $p_2 = \frac{7}{15} < \frac{1}{2}$ and $p_3 = \frac{19}{35} > \frac{1}{2}$, yielding $n = 3$ as above.

3. Find the smallest positive integer n such that, if there are initially $n+1$ townspeople and n goons, then the probability the townspeople win is less than 1%.

   Answer: $\boxed{6}$ By a similar inductive argument, the probability for a given n is

$$p_n = \frac{n!}{(2n+1)!!}.$$

   Clearly this is decreasing in n. It is easy to see that

$$p_5 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} = \frac{8}{693} > 0.01$$

   and

$$p_6 = \frac{6}{13} p_5 = \frac{48}{693 \cdot 13} < 0.01.$$

   Hence the answer is $n = 6$. Heuristically, $p_{n+1} \approx \frac{1}{2} p_n$ for each n, so arriving at these estimates for the correct answer of n is not difficult.

4. Suppose there are initially 1001 townspeople and two goons. What is the probability that, when the game ends, there are exactly 1000 people in jail?

   Answer: $\boxed{\frac{3}{1003}}$ By considering the parity of the number of people in jail, we see that this situation arises if and only if the goons win after the 500th night. That means that at this point we must have exactly one townsperson and two goons remaining. In other words, this situation arises if and only if no goon is ever sent to jail. The probability that this occurs is

$$\frac{1001}{1003} \cdot \frac{999}{1001} \cdot \frac{997}{999} \cdots \frac{3}{5} = \frac{3}{1003}.$$

5. Suppose that there are initially eight townspeople and one goon. One of the eight townspeople is named *Jester*. If Jester is sent to jail during some morning, then the game ends immediately in his sole victory. (However, the Jester does not win if he is sent to jail during some night.)

   Find the probability that only the Jester wins.

   Answer: $\boxed{\frac{1}{3}}$ Let a_n denote the answer when there are $2n-1$ regular townies, one Jester, and one goon. It is not hard to see that $a_1 = \frac{1}{3}$. Moreover, we have a recursion

$$a_n = \frac{1}{2n+1} \cdot 1 + \frac{1}{2n+1} \cdot 0 + \frac{2n-1}{2n+1}\left(\frac{1}{2n-1} \cdot 0 + \frac{2n-2}{2n-1} \cdot a_{n-1}\right).$$

   The recursion follows from the following consideration: during the day, there is a $\frac{1}{2n+1}$ chance the Jester is sent to jail and a $\frac{1}{2n+1}$ chance the goon is sent to jail, at which point the game ends. Otherwise, there is a $\frac{1}{2n-1}$ chance that the Jester is selected to be jailed from among the townies during the evening. If none of these events occur, then we arrive at the situation of a_{n-1}.

   Since $a_1 = \frac{1}{3}$, we find that $a_n = \frac{1}{3}$ for all values of n. This gives the answer.

6. Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be pairwise distinct parabolas in the plane. Find the maximum possible number of intersections between two or more of the $\mathcal{P}_i$. In other words, find the maximum number of points that can lie on two or more of the parabolas $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$.

   Answer: $\boxed{12}$ Note that two distinct parabolas intersect in at most 4 points, which is not difficult to see by drawing examples.[1] Given three parabolas, each pair intersects in at most 4 points, for at most $4 \cdot 3 = 12$ points of intersection in total. It is easy to draw an example achieving this maximum, for example, by slanting the parabolas at different angles.

7. Let $\mathcal{P}$ be a parabola with focus F and directrix ℓ. A line through F intersects $\mathcal{P}$ at two points A and B. Let D and C be the feet of the altitudes from A and B onto ℓ, respectively. Given that $AB = 20$ and $CD = 14$, compute the area of $ABCD$.

   Answer: $\boxed{140}$ Observe that $AD + BC = AF + FB = 20$, and that $ABCD$ is a trapezoid with height $BC = 14$. Hence the answer is $\frac{1}{2}(AD + BC)(14) = 140$.

8. Consider the parabola consisting of the points (x, y) in the real plane satisfying

$$(y + x) = (y - x)^2 + 3(y - x) + 3.$$

   Find the minimum possible value of y.

   Answer: $\boxed{-\frac{1}{2}}$ Let $w = y - x$. Adding w to both sides and dividing by two gives

$$y = \frac{w^2 + 4w + 3}{2} = \frac{(w+2)^2 - 1}{2},$$

   which is minimized when $w = -2$. This yields $y = -\frac{1}{2}$.

9. In equilateral triangle ABC with side length 2, let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2, respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2, respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and CB at C_1 and C_2, respectively.

   Find the perimeter of the triangle formed by lines A_1A_2, B_1B_2, C_1C_2.

   Answer: $\boxed{66 - 36\sqrt{3}}$ Since everything is equilateral it's easy to find the side length of the wanted triangle. By symmetry, it's just $AA_1 + 2A_1B_2 = 3AA_1 - AB$. Using the definition of a parabola, $AA_1 = \frac{\sqrt{3}}{2}A_1B$ so some calculation gives a side length of $2(11 - 6\sqrt{3})$, thus the perimeter claimed.

[1] This can be seen rigorously by showing that five points in the plane determine at most one quadratic equation $ax^2 + by^2 + cxy + dx + ey + f = 0$, up to scaling.

10. Let z be a complex number and k a positive integer such that z^k is a positive real number other than 1. Let $f(n)$ denote the real part of the complex number z^n. Assume the parabola $p(n) = an^2 + bn + c$ intersects $f(n)$ four times, at $n = 0, 1, 2, 3$. Assuming the smallest possible value of k, find the largest possible value of a.

Answer: $\boxed{\frac{1}{3}}$ Let $r = |z|, \theta = \arg z$, and $C = \frac{\Re z}{|z|} = \cos\theta = \cos\frac{2\pi j}{k}$ for some j with $\gcd(j,k) = 1$. The condition of the four consecutive points lying on a parabola is equivalent to having the finite difference

$$f(3) - 3f(2) + 3f(1) - f(0) = 0.$$

This implies

$$\begin{aligned} f(3) - f(0) &= 3\left[f(2) - f(1)\right] \\ \iff r^3\cos(3\theta) - 1 &= 3\left(r^2\cos(2\theta) - r\cos(\theta)\right) \\ \iff r^3(4C^3 - 3C) - 1 &= 3(r^2(2C^2 - 1) - rC). \end{aligned}$$

Now we simply test the first few possible values of k.

$k = 1$ implies $C = 1$, which gives $r^3 - 1 = 3(r^2 - r) \implies (r-1)^3 = 0 \implies r = 1$. This is not allowed since $r = 1$ implies a periodic function.

$k = 2$ implies $C = -1$, which gives $-r^3 - 1 = 3r^2 + r \implies (r+1)^3 = 0$, again not allowed since $r > 0$.

$k = 3$ implies $C = -\frac{1}{2}$. This gives $r^3 - 1 = \frac{-3}{2}(r^2 - r) \implies (r-1)(r+\frac{1}{2})(r+2) = 0$. These roots are either negative or 1, again not allowed.

$k = 4$ implies $C = 0$. This gives $-1 = -3r^2 \implies r = \pm\frac{1}{\sqrt{3}}$. $r = \frac{1}{\sqrt{3}}$ is allowed, so this will generate our answer.

Again by finite differences (or by any other method of interpolating with a quadratic), we get $2a = f(0) + f(2) - 2f(1) = \frac{2}{3}$, so $a = \frac{1}{3}$.