HMMT November 2019
November 9, 2019

General Round

1. Dylan has a 100×100 square, and wants to cut it into pieces of area at least 1. Each cut must be a straight line (not a line segment) and must intersect the interior of the square. What is the largest number of cuts he can make?

2. Meghana writes two (not necessarily distinct) primes q and r in base 10 next to each other on a blackboard, resulting in the concatenation of q and r (for example, if $q = 13$ and $r = 5$, the number on the blackboard is now 135). She notices that three more than the resulting number is the square of a prime p. Find all possible values of p.

3. Katie has a fair 2019-sided die with sides labeled $1, 2, \ldots, 2019$. After each roll, she replaces her n-sided die with an $(n+1)$-sided die having the n sides of her previous die and an additional side with the number she just rolled. What is the probability that Katie's 2019th roll is a 2019?

4. In $\triangle ABC$, $AB = 2019$, $BC = 2020$, and $CA = 2021$. Yannick draws three regular n-gons in the plane of $\triangle ABC$ so that each n-gon shares a side with a distinct side of $\triangle ABC$ and no two of the n-gons overlap. What is the maximum possible value of n?

5. Let a, b, c be positive real numbers such that $a \leq b \leq c \leq 2a$. Find the maximum possible value of
$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

6. Find all ordered pairs (a, b) of positive integers such that $2a + 1$ divides $3b - 1$ and $2b + 1$ divides $3a - 1$.

7. In Middle-Earth, nine cities form a 3 by 3 grid. The top left city is the capital of Gondor and the bottom right city is the capital of Mordor. How many ways can the remaining cities be divided among the two nations such that all cities in a country can be reached from its capital via the grid-lines without passing through a city of the other country?

8. Compute the number of ordered pairs of integers (x, y) such that $x^2 + y^2 < 2019$ and
$$x^2 + \min(x, y) = y^2 + \max(x, y).$$

9. Let $ABCD$ be an isosceles trapezoid with $AD = BC = 255$ and $AB = 128$. Let M be the midpoint of CD and let N be the foot of the perpendicular from A to CD. If $\angle MBC = 90°$, compute $\tan \angle NBM$.

10. An *up-right path* between two lattice points P and Q is a path from P to Q that takes steps of 1 unit either up or to the right. A lattice point (x, y) with $0 \leq x, y \leq 5$ is chosen uniformly at random. Compute the expected number of up-right paths from $(0, 0)$ to $(5, 5)$ not passing through (x, y).