2^{nd} Annual Harvard-MIT November Tournament

Saturday 7 November 2009

General Test

1. [**2**] Evaluate the sum:

$$11^2 - 1^2 + 12^2 - 2^2 + 13^2 - 3^2 + \ldots + 20^2 - 10^2.$$

2. [**3**] Given that $a+b+c=5$ and that $1 \leq a,b,c \leq 2$, what is the minimum possible value of $\frac{1}{a+b} + \frac{1}{b+c}$?

3. [**3**] What is the period of the function $f(x) = \cos(\cos(x))$?

4. [**4**] How many subsets A of $\{1,2,3,4,5,6,7,8,9,10\}$ have the property that no two elements of A sum to 11?

5. [**5**] A polyhedron has faces that are all either triangles or squares. No two square faces share an edge, and no two triangular faces share an edge. What is the ratio of the number of triangular faces to the number of square faces?

6. [**5**] Find the maximum value of $x+y$, given that $x^2 + y^2 - 3y - 1 = 0$.

7. [**6**] There are 15 stones placed in a line. In how many ways can you mark 5 of these stones so that there are an odd number of stones between any two of the stones you marked?

8. [**7**] Let $\triangle ABC$ be an equilateral triangle with height 13, and let O be its center. Point X is chosen at random from all points inside $\triangle ABC$. Given that the circle of radius 1 centered at X lies entirely inside $\triangle ABC$, what is the probability that this circle contains O?

9. [**7**] A set of points is *convex* if the points are the vertices of a convex polygon (that is, a non-self-intersecting polygon with all angles less than or equal to $180°$). Let S be the set of points (x,y) such that x and y are integers and $1 \leq x,y \leq 26$. Find the number of ways to choose a convex subset of S that contains exactly 98 points.

10. [**8**] Compute

$$\prod_{n=0}^{\infty}\left(1 - \left(\frac{1}{2}\right)^{3^n} + \left(\frac{1}{4}\right)^{3^n}\right).$$