HMMT February 2020

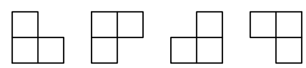
February 15, 2020

Combinatorics

1. How many ways can the vertices of a cube be colored red or blue so that the color of each vertex is the color of the majority of the three vertices adjacent to it?

2. How many positive integers at most 420 leave different remainders when divided by each of 5, 6, and 7?

3. Each unit square of a 4×4 square grid is colored either red, green, or blue. Over all possible colorings of the grid, what is the maximum possible number of L-trominos that contain exactly one square of each color? (L-trominos are made up of three unit squares sharing a corner, as shown below.)



4. Given an 8×8 checkerboard with alternating white and black squares, how many ways are there to choose four black squares and four white squares so that no two of the eight chosen squares are in the same row or column?

5. Let S be a set of intervals defined recursively as follows:

   - Initially, $[1, 1000]$ is the only interval in S.
   - If $l \neq r$ and $[l, r] \in S$, then both $\left[l, \left\lfloor \frac{l+r}{2} \right\rfloor\right], \left[\left\lfloor \frac{l+r}{2} \right\rfloor + 1, r\right] \in S$.

   (Note that S can contain intervals such as $[1, 1]$, which contain a single integer.) An integer i is chosen uniformly at random from the range $[1, 1000]$. What is the expected number of intervals in S which contain i?

6. Alice writes 1001 letters on a blackboard, each one chosen independently and uniformly at random from the set $S = \{a, b, c\}$. A move consists of erasing two distinct letters from the board and replacing them with the third letter in S. What is the probability that Alice can perform a sequence of moves which results in one letter remaining on the blackboard?

7. Anne-Marie has a deck of 16 cards, each with a distinct positive factor of 2002 written on it. She shuffles the deck and begins to draw cards from the deck without replacement. She stops when there exists a nonempty subset of the cards in her hand whose numbers multiply to a perfect square. What is the expected number of cards in her hand when she stops?

8. Let Γ_1 and Γ_2 be concentric circles with radii 1 and 2, respectively. Four points are chosen on the circumference of Γ_2 independently and uniformly at random, and are then connected to form a convex quadrilateral. What is the probability that the perimeter of this quadrilateral intersects Γ_1?

9. Farmer James wishes to cover a circle with circumference 10π with six different types of colored arcs. Each type of arc has radius 5, has length either π or 2π, and is colored either red, green, or blue. He has an unlimited number of each of the six arc types. He wishes to completely cover his circle without overlap, subject to the following conditions:

   - Any two adjacent arcs are of different colors.
   - Any three adjacent arcs where the middle arc has length π are of three different colors.

   Find the number of distinct ways Farmer James can cover his circle. Here, two coverings are equivalent if and only if they are rotations of one another. In particular, two colorings are considered distinct if they are reflections of one another, but not rotations of one another.

10. Max repeatedly throws a fair coin in a hurricane. For each throw, there is a 4% chance that the coin gets blown away. He records the number of heads H and the number of tails T before the coin is lost. (If the coin is blown away on a toss, no result is recorded for that toss.) What is the expected value of $|H - T|$?