15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Guts

1. [**2**] Square $ABCD$ has side length 2, and X is a point outside the square such that $AX = XB = \sqrt{2}$. What is the length of the longest diagonal of pentagon $AXBCD$?

2. [**2**] Let $a_0, a_1, a_2, \ldots$ denote the sequence of real numbers such that $a_0 = 2$ and $a_{n+1} = \frac{a_n}{1+a_n}$ for $n \geq 0$. Compute a_{2012}.

3. [**2**] Suppose x and y are real numbers such that $-1 < x < y < 1$. Let G be the sum of the geometric series whose first term is x and whose ratio is y, and let G' be the sum of the geometric series whose first term is y and ratio is x. If $G = G'$, find $x + y$.

4. [**2**] Luna has an infinite supply of red, blue, orange, and green socks. She wants to arrange 2012 socks in a line such that no red sock is adjacent to a blue sock and no orange sock is adjacent to a green sock. How many ways can she do this?

. . . .

5. [**3**] Mr. Canada chooses a positive real a uniformly at random from $(0, 1]$, chooses a positive real b uniformly at random from $(0, 1]$, and then sets $c = a/(a + b)$. What is the probability that c lies between $1/4$ and $3/4$?

6. [**3**] Let rectangle $ABCD$ have lengths $AB = 20$ and $BC = 12$. Extend ray BC to Z such that $CZ = 18$. Let E be the point in the interior of $ABCD$ such that the perpendicular distance from E to $\overline{AB}$ is 6 and the perpendicular distance from E to $\overline{AD}$ is 6. Let line EZ intersect AB at X and CD at Y. Find the area of quadrilateral $AXYD$.

7. [**3**] M is an 8×8 matrix. For $1 \leq i \leq 8$, all entries in row i are at least i, and all entries on column i are at least i. What is the minimum possible sum of the entries of M?

8. [**3**] Amy and Ben need to eat 1000 total carrots and 1000 total muffins. The muffins can not be eaten until all the carrots are eaten. Furthermore, Amy can not eat a muffin within 5 minutes of eating a carrot and neither can Ben. If Amy eats 40 carrots per minute and 70 muffins per minute and Ben eats 60 carrots per minute and 30 muffins per minute, what is the minimum number of minutes it will take them to finish the food?

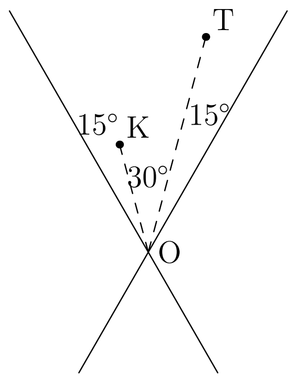
. . . .

. . . .

9. [**5**] Given $\triangle ABC$ with $AB < AC$, the altitude AD, angle bisector AE, and median AF are drawn from A, with D, E, F all lying on $\overline{BC}$. If $\measuredangle BAD = 2\measuredangle DAE = 2\measuredangle EAF = \measuredangle FAC$, what are all possible values of $\measuredangle ACB$?

10. [**5**] Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Compute $P(3)$.

11. [**5**] Knot is on an epic quest to save the land of Hyruler from the evil Gammadorf. To do this, he must collect the two pieces of the Lineforce, then go to the Temple of Lime. As shown on the figure, Knot starts on point K, and must travel to point T, where $OK = 2$ and $OT = 4$. However, he must first reach both solid lines in the figure below to collect the pieces of the Lineforce. What is the minimal distance Knot must travel to do so?



12. [**5**] Knot is ready to face Gammadorf in a card game. In this game, there is a deck with twenty cards numbered from 1 to 20. Each player starts with a five card hand drawn from this deck. In each round, Gammadorf plays a card in his hand, then Knot plays a card in his hand. Whoever played a card with greater value gets a point. At the end of five rounds, the player with the most points wins. If Gammadorf starts with a hand of $1, 5, 10, 15, 20$, how many five-card hands of the fifteen remaining cards can Knot draw which always let Knot win (assuming he plays optimally)?

. . . .

. . . .

13. [**7**] Niffy's favorite number is a positive integer, and Stebbysaurus is trying to guess what it is. Niffy tells her that when expressed in decimal without any leading zeros, her favorite number satisfies the following:

    - Adding 1 to the number results in an integer divisible by 210.
    - The sum of the digits of the number is twice its number of digits.
    - The number has no more than 12 digits.
    - The number alternates in even and odd digits.

    Given this information, what are all possible values of Niffy's favorite number?

14. [**7**] Let triangle ABC have $AB = 5$, $BC = 6$, and $AC = 7$, with circumcenter O. Extend ray AB to point D such that $BD = 5$, and extend ray BC to point E such that $OD = OE$. Find CE.

15. [**7**] Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two distinct real polynomials such that the x-coordinate of the vertex of f is a root of g, the x-coordinate of the vertex of g is a root of f and both f and g have the same minimum value. If the graphs of the two polynomials intersect at the point $(2012, -2012)$, what is the value of $a + c$?

16. [**7**] Let A, B, C, and D be points randomly selected independently and uniformly within the unit square. What is the probability that the six lines $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$ all have positive slope?

. . . .

15^{TH} ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012 — GUTS ROUND

17. [**11**] Mark and William are playing a game. Two walls are placed 1 meter apart, with Mark and William each starting an orb at one of the walls. Simultaneously, they release their orbs directly toward the other. Both orbs are enchanted such that, upon colliding with each other, they instantly reverse direction and go at double their previous speed. Furthermore, Mark has enchanted his orb so that when it collides with a wall it instantly reverses direction and goes at double its previous speed (William's reverses direction at the same speed). Initially, Mark's orb is moving at $\frac{1}{1000}$ meters/s, and William's orb is moving at 1 meter/s. Mark wins when his orb passes the halfway point between the two walls. How fast, in meters/s, is his orb going when this first happens?

18. [**11**] Let x and y be positive real numbers such that $x^2 + y^2 = 1$ and $(3x - 4x^3)(3y - 4y^3) = -\frac{1}{2}$. Compute $x + y$.

19. [**11**] Given that P is a real polynomial of degree at most 2012 such that $P(n) = 2^n$ for $n = 1, 2, \ldots, 2012$, what choice(s) of $P(0)$ produce the minimal possible value of $P(0)^2 + P(2013)^2$?

20. [**11**] Let n be the maximum number of bishops that can be placed on the squares of a 6×6 chessboard such that no two bishops are attacking each other. Let k be the number of ways to put n bishops on an 6×6 chessboard such that no two bishops are attacking each other. Find $n + k$. (Two bishops are considered to be attacking each other if they lie on the same diagonal. Equivalently, if we label the squares with coordinates (x, y), with $1 \le x, y \le 6$, then the bishops on (a, b) and (c, d) are attacking each other if and only if $|a - c| = |b - d|$.)

. . . .

. . . .

21. [**13**] Let N be a three-digit integer such that the difference between any two positive integer factors of N is divisible by 3. Let $d(N)$ denote the number of positive integers which divide N. Find the maximum possible value of $N \cdot d(N)$.

22. [**13**] For each positive integer n, there is a circle around the origin with radius n. Rainbow Dash starts off somewhere on the plane, but not on a circle. She takes off in some direction in a straight path. She moves $\frac{\sqrt{5}}{5}$ units before crossing a circle, then $\sqrt{5}$ units, then $\frac{3\sqrt{5}}{5}$ units. What distance will she travel before she crosses another circle?

23. [**13**] Points X and Y are inside a unit square. The score of a vertex of the square is the minimum distance from that vertex to X or Y. What is the minimum possible sum of the scores of the vertices of the square?

24. [**13**] Franklin has four bags, numbered 1 through 4. Initially, the first bag contains fifteen balls, numbered 1 through 15, and the other bags are empty. Franklin randomly pulls a pair of balls out of the first bag, throws away the ball with the lower number, and moves the ball with the higher number into the second bag. He does this until there is only one ball left in the first bag. He then repeats this process in the second and third bag until there is exactly one ball in each bag. What is the probability that ball 14 is in one of the bags at the end?

. . . .

15$^{\text{TH}}$ ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012 — GUTS ROUND

25. [**17**] FemtoPravis is walking on an 8×8 chessboard that wraps around at its edges (so squares on the left edge of the chessboard are adjacent to squares on the right edge, and similarly for the top and bottom edges). Each femtosecond, FemtoPravis moves in one of the four diagonal directions uniformly at random. After 2012 femtoseconds, what is the probability that FemtoPravis is at his original location?

26. [**17**] Suppose ABC is a triangle with circumcenter O and orthocenter H such that A, B, C, O, and H are all on distinct points with integer coordinates. What is the second smallest possible value of the circumradius of ABC?

27. [**17**] Let S be the set $\{1, 2, \ldots, 2012\}$. A perfectutation is a bijective function h from S to itself such that there exists an $a \in S$ such that $h(a) \neq a$, and that for any pair of integers $a \in S$ and $b \in S$ such that $h(a) \neq a$, $h(b) \neq b$, there exists a positive integer k such that $h^k(a) = b$. Let n be the number of ordered pairs of perfectutations (f, g) such that $f(g(i)) = g(f(i))$ for all $i \in S$, but $f \neq g$. Find the remainder when n is divided by 2011.

28. [**17**] Alice is sitting in a teacup ride with infinitely many layers of spinning disks. The largest disk has radius 5. Each succeeding disk has its center attached to a point on the circumference of the previous disk and has a radius equal to $2/3$ of the previous disk. Each disk spins around its center (relative to the disk it is attached to) at a rate of $\pi/6$ radians per second. Initially, at $t = 0$, the centers of the disks are aligned on a single line, going outward. Alice is sitting at the limit point of all these disks. After 12 seconds, what is the length of the trajectory that Alice has traced out?

. . . .

. . . .

29. [**19**] Consider the cube whose vertices are the eight points (x, y, z) for which each of x, y, and z is either 0 or 1. How many ways are there to color its vertices black or white such that, for any vertex, if all of its neighbors are the same color then it is also that color? Two vertices are neighbors if they are the two endpoints of some edge of the cube.

30. [**19**] You have a twig of length 1. You repeatedly do the following: select two points on the twig independently and uniformly at random, make cuts on these two points, and keep only the largest piece. After 2012 repetitions, what is the expected length of the remaining piece?

31. [**19**] Let S_7 denote all the permutations of $1, 2, \ldots, 7$. For any $\pi \in S_7$, let $f(\pi)$ be the smallest positive integer i such that $\pi(1), \pi(2), \ldots, \pi(i)$ is a permutation of $1, 2, \ldots, i$. Compute $\sum_{\pi \in S_7} f(\pi)$.

32. [**19**] Let S be a set of size 3. How many collections T of subsets of S have the property that for any two subsets $U \in T$ and $V \in T$, both $U \cap V$ and $U \cup V$ are in T?

. . . .

15^{TH} ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012 — GUTS ROUND

33. [**23**] Compute the decimal expansion of $\sqrt{\pi}$. Your score will be $\min(23, k)$, where k is the number of consecutive correct digits immediately following the decimal point in your answer.

For each of the remaining three problems, it is difficult to obtain an exact answer. Instead, give an interval $[L, U]$, where L and U are positive real numbers *written in decimal.* If $[L, U]$ contains the answer to the problem, you will receive credit based off of how close L and U are. Otherwise, you will receive no credit.

34. [**23**] Let Q be the product of the sizes of all the non-empty subsets of $\{1, 2, \ldots, 2012\}$, and let $M = \log_2(\log_2(Q))$. Give lower and upper bounds L and U for M. If $0 < L \leq M \leq U$, then your score will be $\min(23, \left\lfloor \frac{23}{3(U-L)} \right\rfloor)$. Otherwise, your score will be 0.

35. [**23**] Let N be the number of distinct roots of $\prod_{k=1}^{2012}(x^k - 1)$. Give lower and upper bounds L and U on N. If $0 < L \leq N \leq U$, then your score will be $\left\lfloor \frac{23}{(U/L)^{1.7}} \right\rfloor$. Otherwise, your score will be 0.

36. [**23**] Maria is hopping up a flight of stairs with 100 steps. At every hop, she advances some integer number of steps. Each hop she makes has fewer steps. However, the positive difference between the length of consecutive hops decreases. Let P be the number of distinct ways she can hop up the stairs. Find lower and upper bounds L and U for P. If $0 < L \leq P \leq U$, your score will be $\left\lfloor \frac{23}{\sqrt{U/L}} \right\rfloor$. Otherwise, your score will be 0.

. . . .

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________________ Team ____________________________ Team ID# __________

1. **[2]** ____________________

2. **[2]** ____________________

3. **[2]** ____________________

4. **[2]** ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________________ Team ____________________________ Team ID# __________

5. **[3]** ____________________

6. **[3]** ____________________

7. **[3]** ____________________

8. **[3]** ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________________ Team ____________________________ Team ID# __________

9. **[5]** ____________________

10. **[5]** ____________________

11. **[5]** ____________________

12. **[5]** ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________________ Team ____________________________ Team ID# __________

13. **[7]** ____________________

14. **[7]** ____________________

15. **[7]** ____________________

16. **[7]** ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________________ Team ____________________________ Team ID# __________

17. **[11]** ____________________

18. **[11]** ____________________

19. **[11]** ____________________

20. **[11]** ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

21. [**13**] ____________________

22. [**13**] ____________________

23. [**13**] ____________________

24. [**13**] ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

25. [**17**] ____________________

26. [**17**] ____________________

27. [**17**] ____________________

28. [**17**] ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

29. [**19**] ____________________

30. [**19**] ____________________

31. [**19**] ____________________

32. [**19**] ____________________

15TH ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

33. [**23**] ____________________

34. [**23**] ____________________

35. [**23**] ____________________

36. [**23**] ____________________