11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Team Round: B Division

Tropical Mathematics [95]

For real numbers x and y, let us consider the two operations $\oplus$ and $\odot$ defined by

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \odot y = x + y.$$

We also include ∞ in our set, and it satisfies $x \oplus \infty = x$ and $x \odot \infty = \infty$ for all x. When unspecified, $\odot$ precedes $\oplus$ in the order of operations.

1. [**10**] (Distributive law) Prove that $(x \oplus y) \odot z = x \odot z \oplus y \odot z$ for all $x, y, z \in \mathbb{R} \cup \{\infty\}$.

2. [**10**] (Freshman's Dream) Let z^n denote $z \odot z \odot z \odot \cdots \odot z$ with z appearing n times. Prove that $(x \oplus y)^n = x^n \oplus y^n$ for all $x, y \in \mathbb{R} \cup \{\infty\}$ and positive integer n.

3. [**35**] By a *tropical polynomial* we mean a function of the form

$$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0,$$

where exponentiation is as defined in the previous problem.

Let p be a tropical polynomial. Prove that

$$p\left(\frac{x+y}{2}\right) \geq \frac{p(x) + p(y)}{2}$$

for all $x, y \in \mathbb{R} \cup \{\infty\}$. (This means that all tropical polynomials are concave.)

4. [**40**] (Fundamental Theorem of Algebra) Let p be a tropical polynomial:

$$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0, \quad a_n \neq \infty$$

Prove that we can find $r_1, r_2, \ldots, r_n \in \mathbb{R} \cup \{\infty\}$ so that

$$p(x) = a_n \odot (x \oplus r_1) \odot (x \oplus r_2) \odot \cdots \odot (x \oplus r_n)$$

for all x.

Juggling [125]

A *juggling sequence* of *length* n is a sequence $j(\cdot)$ of n nonnegative integers, usually written as a string

$$j(0)j(1)\ldots j(n-1)$$

such that the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(t) = t + j(\overline{t})$$

is a permutation of the integers. Here $\bar{t}$ denotes the remainder of t when divided by n. In this case, we say that f is the corresponding *juggling pattern*.

For a juggling pattern f (or its corresponding juggling sequence), we say that it has b *balls* if the permutation induces b infinite orbits on the set of integers. Equivalently, b is the maximum number such that we can find a set of b integers $\{t_1, t_2, \ldots, t_b\}$ so that the sets $\{t_i, f(t_i), f(f(t_i)), f(f(f(t_i))), \ldots\}$ are all infinite and mutually disjoint (i.e. non-overlapping) for $i = 1, 2, \ldots, b$. (This definition will become clear in a second.)

Now is probably a good time to pause and think about what all this has to do with juggling. Imagine that we are juggling a number of balls, and at time t, we toss a ball from our hand up to a height $j(\bar{t})$. This ball stays up in the air for $j(\bar{t})$ units of time, so that it comes back to our hand at time $f(t) = t + j(\bar{t})$. Then, the juggling pattern presents a simplified model of how balls are juggled (for instance, we ignore information such as which hand we use to toss the ball). A throw height of 0 (i.e., $j(\bar{t}) = 0$ and $f(t) = t$) represents that no thrown takes place at time t, which could correspond to an empty hand. Then, b is simply the minimum number of balls needed to carry out the juggling.

The following graphical representation may be helpful to you. On a horizontal line, an curve is drawn from t to $f(t)$. For instance, the following diagram depicts the juggling sequence 441 (or the juggling sequences 414 and 144). Then b is simply the number of contiguous "paths" drawn, which is 3 in this case.

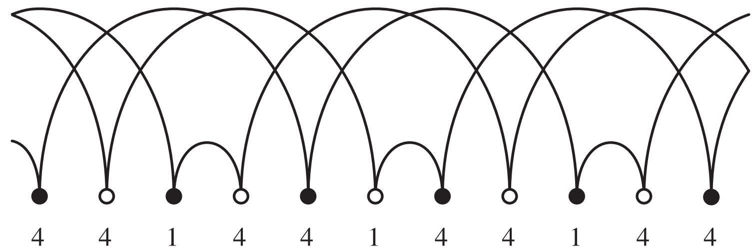


Figure 1: Juggling diagram of 441.

5. [**10**] Prove that 572 is not a juggling sequence.

6. [**40**] Suppose that $j(0)j(1)\cdots j(n-1)$ is a valid juggling sequence. For $i = 0, 1, \ldots, n-1$, Let a_i denote the remainder of $j(i) + i$ when divided by n. Prove that $(a_0, a_1, \ldots, a_{n-1})$ is a permutation of $(0, 1, \ldots, n-1)$.

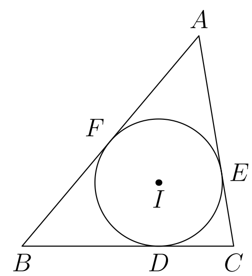
7. [**30**] Determine the number of juggling sequences of length n with exactly 1 ball.

8. [**40**] Prove that the number of balls b in a juggling sequence $j(0)j(1)\cdots j(n-1)$ is simply the average
$$b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n}.$$

9. [**5**] Show that the converse of the previous statement is false by providing a non-juggling sequence $j(0)j(1)j(2)$ of length 3 where the average $\frac{1}{3}(j(0) + j(1) + j(2))$ is an integer. Show that your example works.

Incircles [180]

In the following problems, ABC is a triangle with incenter I. Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB, respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. [**15**] Let a, b, c denote the side lengths of BC, CA, AB. Find the lengths of AE, BF, CD in terms of a, b, c.

11. [**15**] Show that lines AD, BE, CF pass through a common point.

12. [**35**] Show that the incenter of triangle AEF lies on the incircle of ABC.

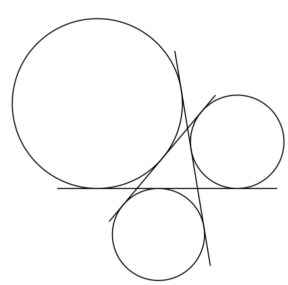
13. [**35**] Let A_1, B_1, C_1 be the incenters of triangle AEF, BDF, CDE, respectively. Show that A_1D, B_1E, C_1F all pass through the orthocenter of $A_1B_1C_1$.

14. [**40**] Let X be the point on side BC such that $BX = CD$. Show that the excircle ABC opposite of vertex A touches segment BC at X.

15. [**40**] Let X be as in the previous problem. Let T be the point diametrically opposite to D on on the incircle of ABC . Show that A, T, X are collinear.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC, tangent to all three sides.
- The *excircles* of a triangle ABC are the three circles on the exterior the triangle but tangent to all three lines AB, BC, CA.



- The *orthocenter* of a triangle is the point of concurrency of the three altitudes.
- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB, respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$