HMMT February 2019

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Algebra and Number Theory

1. What is the smallest positive integer that cannot be written as the sum of two nonnegative palindromic integers? (An integer is *palindromic* if the sequence of decimal digits are the same when read backwards.)

   Proposed by: Yuan Yao

   Answer: $\boxed{21}$

   We need to first prove that every positive integer N less than 21 can be written as sum of two nonnegative palindromic integers. If N is in the interval $[1,9]$, then it can be written as $0+N$. If N is in the interval $[10,18]$, it can be written as $9+(N-9)$. In addition, 19 and 20 can be written as $11+8$ and $11+9$, respectively.

   Second, we need to show that 21 cannot be expressed in such a way. Lets suppose $21 = a+b$ with $a \le b$. It follows that b has to be at least 11. Since $b \le 21$, the only way for b to be palindromic is that $b = 11$. However, this leads to $a = 21-b = 10$, which is not a palindrome. Therefore, 21 is the smallest number that satisfy the problem condition.

2. Let $N = 2^{\left(2^2\right)}$ and x be a real number such that $N^{\left(N^N\right)} = 2^{(2^x)}$. Find x.

   Proposed by: Yuan Yao

   Answer: $\boxed{66}$

   We compute

   $$N^{\left(N^N\right)} = 16^{16^{16}} = 2^{4\cdot 2^{4\cdot 2^4}} = 2^{2^{2^6+2}} = 2^{2^{66}},$$

   so $x = 66$.

3. Let x and y be positive real numbers. Define $a = 1+\frac{x}{y}$ and $b = 1+\frac{y}{x}$. If $a^2+b^2 = 15$, compute a^3+b^3.

   Proposed by: Michael Tang

   Answer: $\boxed{50}$

   Note that $a-1 = \frac{x}{y}$ and $b-1 = \frac{y}{x}$ are reciprocals. That is,

   $$(a-1)(b-1) = 1 \implies ab-a-b+1 = 1 \implies ab = a+b.$$

   Let $t = ab = a+b$. Then we can write

   $$a^2+b^2 = (a+b)^2 - 2ab = t^2 - 2t,$$

   so $t^2-2t = 15$, which factors as $(t-5)(t+3) = 0$. Since $a, b > 0$, we must have $t = 5$. Then, we compute

   $$a^3+b^3 = (a+b)^3 - 3ab(a+b) = 5^3 - 3\cdot 5^2 = 50.$$

4. Let $\mathbb{N}$ be the set of positive integers, and let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

   - $f(1) = 1$;
   - for $n \in \mathbb{N}$, $f(2n) = 2f(n)$ and $f(2n+1) = 2f(n)-1$.

   Determine the sum of all positive integer solutions to $f(x) = 19$ that do not exceed 2019.

   Proposed by: Yuan Yao

   Answer: $\boxed{1889}$

   For $n = 2^{a_0}+2^{a_1}+\cdots+2^{a_k}$ where $a_0 > a_1 > \cdots > a_k$, we can show that $f(n) = 2^{a_0}-2^{a_1}-\cdots-2^{a_k} = 2^{a_0+1}-n$ by induction: the base case $f(1) = 1$ clearly holds; for the inductive step, when n is even we note that $f(n) = 2f(\frac{n}{2}) = 2(2^{a_0} - \frac{n}{2}) = 2^{a_0+1} - n$ as desired, and when n is odd we also have $f(n) = 2f(\frac{n-1}{2}) - 1 = 2(2^{a_0} - \frac{n-1}{2}) - 1 = 2^{a_0+1} - n$, again as desired.

Since $19 = f(n) \le 2^{a_0} \le n$, we have $a_0 \ge 5$ and $n = 2^{a_0+1} - 19 \le 2019$ gives $a_0 \le 9$. So the answer is $\sum_{a=5}^{9} (2^{a+1} - 19) = (2^{11} - 2^6) - 19 \cdot 5 = 1889$.

5. Let $a_1, a_2, \dots$ be an arithmetic sequence and $b_1, b_2, \dots$ be a geometric sequence. Suppose that $a_1b_1 = 20$, $a_2b_2 = 19$, and $a_3b_3 = 14$. Find the greatest possible value of a_4b_4.

Proposed by: Michael Tang

Answer: $\boxed{\frac{37}{4}}$

We present two solutions: the first more algebraic and computational, the second more conceptual.

Solution 1. Let $\{a_n\}$ have common difference d and $\{b_n\}$ have common ratio d; for brevity, let $a_1 = a$ and $b_1 = b$. Then we have the equations $ab = 20$, $(a+d)br = 19$, and $(a+2d)br^2 = 14$, and we want to maximize $(a+3d)br^3$.

The equation $(a+d)br = 19$ expands as $abr + dbr = 19$, or $20r + bdr = 19$ since $ab = 20$. Similarly, $(20 + 2bd)r^2 = 14$, or $10r^2 + bdr^2 = 7$. Multiplying the first equation by r and subtracting the second, we get

$$10r^2 = 19r - 7 \implies (5r-7)(2r-1) = 0,$$

so either $r = \frac{7}{5}$ or $r = \frac{1}{2}$.

For each value of r, we have $bd = \frac{19-20r}{r} = \frac{19}{r} - 20$, so

$$(a+3d)br^3 = (20+3bd)r^3 = \left(\frac{57}{r} - 40\right) r^3 = r^2(57 - 40r).$$

The greater value of this expression is $\frac{37}{4}$, achieved when $r = \frac{1}{2}$.

Solution 2. The key is to find a (linear) recurrence relation that the sequence $c_n = a_nb_n$ satisfies. Some knowledge of theory helps here: c_n is of the form $snr^n + tr^n$ for some constants r, s, t, so $\{c_n\}$ satisfies a linear recurrence relation with characteristic polynomial $(x-r)^2 = x^2 - 2rx + r^2$. That is,

$$c_n = 2rc_{n-1} - r^2c_{n-2}$$

for some constant r.

Taking $n = 3$, we get $14 = 2r \cdot 19 - r^2 \cdot 20$, which factors as $(5r-7)(2r-1) = 0$, so either $r = \frac{7}{5}$ or $r = \frac{1}{2}$. Then

$$c_4 = 2rc_3 - r^2c_2 = 28r - 19r^2.$$

This expression is maximized at $r = \frac{14}{19}$, and strictly decreases on either side. Since $\frac{1}{2}$ is closer to $\frac{14}{19}$ than $\frac{7}{5}$, we should choose $r = \frac{1}{2}$, giving the answer $c_4 = 14 - \frac{19}{4} = \frac{37}{4}$.

6. For positive reals p and q, define the *remainder* when p is divided by q as the smallest nonnegative real r such that $\frac{p-r}{q}$ is an integer. For an ordered pair (a, b) of positive integers, let r_1 and r_2 be the remainder when $a\sqrt{2} + b\sqrt{3}$ is divided by $\sqrt{2}$ and $\sqrt{3}$ respectively. Find the number of pairs (a, b) such that $a, b \le 20$ and $r_1 + r_2 = \sqrt{2}$.

Proposed by: Yuan Yao

Answer: $\boxed{16}$

The remainder when we divide $a\sqrt{2} + b\sqrt{3}$ by $\sqrt{2}$ is defined to be the smallest non-negative real r_1 such that $\frac{a\sqrt{2}+b\sqrt{3}-r_1}{\sqrt{2}}$ is integral. As $\frac{x}{\sqrt{2}}$ is integral iff x is an integral multiple of $\sqrt{2}$, it follows that $r_1 = b\sqrt{3} - c\sqrt{2}$, for some integer c. Furthermore given any real r such that $\frac{a\sqrt{2}+b\sqrt{3}-r}{\sqrt{2}}$ is integral, we may add or subtract $\sqrt{2}$ to r and the fraction remains an integer. Thus, the smallest non-negative real r_1 such that the fraction is an integer must satisfy $0 \le r_1 < \sqrt{2}$.

Similarly, we find $r_2 = a\sqrt{2} - d\sqrt{3}$ for some integer d and $0 \le r_2 < \sqrt{3}$. Since $r_1 + r_2 = \sqrt{2}$, then

$$(a-c)\sqrt{2} + (b-d)\sqrt{3} = \sqrt{2} \iff a - c = 1 \text{ and } b - d = 0.$$

Finally, substituting in $c = a - 1$ and $d = b$ plugging back into our bounds for r_1 and r_2, we get

$$\begin{cases} 0 \le b\sqrt{3} - (a-1)\sqrt{2} < \sqrt{2} \\ 0 \le a\sqrt{2} - b\sqrt{3} < \sqrt{3} \end{cases}$$

or

$$\begin{cases} (a-1)\sqrt{2} \le b\sqrt{3} \\ b\sqrt{3} < a\sqrt{2} \\ b\sqrt{3} \le a\sqrt{2} \\ a\sqrt{2} < (b+1)\sqrt{3} \end{cases}$$

Note that $b\sqrt{3} < a\sqrt{2} \implies b\sqrt{3} \le a\sqrt{2}$ and

$$(a-1)\sqrt{2} \le b\sqrt{3} \implies a\sqrt{2} \le b\sqrt{3} + \sqrt{2} < b\sqrt{3} + \sqrt{3} = (b+1)\sqrt{3}$$

so the last two inequalities are redundant. We are left with

$$(a-1)\sqrt{2} \le b\sqrt{3} < a\sqrt{2}.$$

Since the non-negative number line is partitioned by intervals of the form $\big[(a-1)\sqrt{2}, a\sqrt{2}\big)$ for positive integers a, for any positive integer b, we can find a positive integer a that satisfies the inequalities. As clearly $a > b$, it remains to find the number of b such that $a \le 20$. This is bounded by

$$b\sqrt{3} < a\sqrt{2} \le 20\sqrt{2} \iff b < \frac{20\sqrt{2}}{\sqrt{3}} \implies b \le 16$$

so there are 16 values of b and thus 16 ordered pairs of positive integers (a, b) that satisfy the problem.

7. Find the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} \frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

Proposed by: Andrew Gu

Answer: $\boxed{\frac{1}{54}}$

Let S denote the given sum. By summing over all six permutations of the variables a, b, c we obtain

$$\begin{aligned} 6S &= \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} \frac{3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 6abc}{4^{a+b+c}(a+b)(b+c)(c+a)} \\ &= \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} \frac{3}{4^{a+b+c}} \\ &= 3\left(\sum_{a=1}^{\infty}\frac{1}{4^a}\right)\left(\sum_{b=1}^{\infty}\frac{1}{4^b}\right)\left(\sum_{c=1}^{\infty}\frac{1}{4^c}\right) \\ &= 3\left(\frac{1}{3}\right)^3 \\ &= \frac{1}{9}. \end{aligned}$$

Hence $S = \frac{1}{54}$.

8. There is a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) > 0$ and such that

$$\sum_{d|n} f(d) f\left(\frac{n}{d}\right) = 1$$

for all $n \geq 1$. What is $f(2018^{2019})$?

Proposed by: Ashwin Sah

Answer: $\boxed{\frac{\binom{4038}{2019}^2}{2^{8076}} \text{ OR } \frac{(4038!)^2}{(2019!)^4 \cdot 2^{8076}}}$

Fix any prime p, and let $a_n = f(p^n)$ for $n \geq 0$. Notice that using the relation for p^n, we obtain

$$\sum_{i=0}^{n} a_i a_{n-i} = 1,$$

which means that if we let $g(x) = \sum_{n\geq 0} a_n x^n$, then $g(x)^2 = 1 + x + x^2 + \cdots = \frac{1}{1-x}$ as a generating function. Thus $g(x) = (1-x)^{-\frac{1}{2}}$, and this is well-known to have generating function with coefficients $a_n = \frac{\binom{2n}{n}}{4^n}$. One way to see this is using the Taylor series and then reorganizing terms; it is also intimately related to the generating function for the Catalan numbers. In particular, a_n is independent of our choice of p.

Now if we define $f_0(n) = \prod_{p|n} a_{v_p(n)}$, then we see that $f = f_0$ on the prime powers.

If we define the Dirichlet convolution of two functions $\chi_1, \chi_2 : \mathbb{N} \to \mathbb{R}$ as χ_3 such that

$$\chi_3(n) = \sum_{d|n} \chi_1(d) \chi_2\left(\frac{n}{d}\right),$$

then it is well-known that multiplicative functions ($\chi(m)\chi(n) = \chi(mn)$ if $\gcd(m,n)$, so e.g. $\phi(n)$, the Euler totient function) convolve to a multiplicative function.

In particular, f_0 is a multiplicative function by definition (it is equivalent to only define it at prime powers then multiply), so the convolution of f_0 with itself is multiplicative. By definition of a_n, the convolution of f_0 with itself equals 1 at all prime powers. Thus by multiplicativity, it equals the constant function 1 everywhere.

Two final things to note: $f_0(1) = a_0 = 1 > 0$, and f satisfying the conditions in the problem statement is indeed unique (proceed by induction on n that $f(n)$ is determined uniquely and that the resulting algorithm for computing f gives a well-defined function). Therefore f_0, satisfying those same conditions, must equal f.

At last, we have

$$f(p^{2019}) = f_0(p^{2019}) = \frac{\binom{4038}{2019}}{4^{2019}}$$

so

$$f(2018^{2019}) = f(2^{2019}) f(1009^{2019}) = \frac{\binom{4038}{2019}^2}{4^{4038}} = \frac{\binom{4038}{2019}^2}{2^{8076}}.$$

9. Tessa the hyper-ant has a 2019-dimensional hypercube. For a real number k, she calls a placement of nonzero real numbers on the 2^{2019} vertices of the hypercube *k-harmonic* if for any vertex, the sum of all 2019 numbers that are edge-adjacent to this vertex is equal to k times the number on this vertex. Let S be the set of all possible values of k such that there exists a k-harmonic placement. Find $\sum_{k\in S} |k|$.

Proposed by: Yuan Yao

Answer: $\boxed{2040200}$

By adding up all the equations on each vertex, we get $2019S = kS$ where S is the sum of all entries, so $k = 2019$ unless $S = 0$. In the latter case, by adding up all the equations on a half of the cube, we get

$2018S - S = kS$ where S is the sum of all entries on that half of the cube, so $k = 2017$ unless $S = 0$. In the latter case (the sum of all entries of *any* half is zero), by adding up all the equations on a half of the half-cube, we get $2017S - 2S = kS$, so $k = 2015$ unless $S = 0$. We continue this chain of casework until we get that the sum of every two vertices connected by unit segments is zero, in which case we have $k = -2019$. This means that k can take any odd value between -2019 and 2019 inclusive, so the sum of absolute values is $2 \cdot 1010^2 = 2040200$.

To achieve these values, suppose that the vertices of the hypercube are $\{0,1\}^{2019}$ and that the label of $(x_1, x_2, \ldots, x_{2019})$ is $a_1^{x_1} a_2^{x_2} \ldots a_{2019}^{x_{2019}}$ for constants $a_1, a_2, \ldots, a_{2019} \in \{-1, 1\}$, then it is not difficult to see that this labeling is $(a_1 + a_2 + \cdots + a_{2019})$-harmonic for any choice of a_i's, so this can achieve all odd values between -2019 and 2019 inclusive.

10. The sequence of integers $\{a_i\}_{i=0}^{\infty}$ satisfies $a_0 = 3, a_1 = 4$, and

$$a_{n+2} = a_{n+1}a_n + \left\lceil \sqrt{a_{n+1}^2 - 1}\sqrt{a_n^2 - 1} \right\rceil$$

for $n \geq 0$. Evaluate the sum

$$\sum_{n=0}^{\infty} \left(\frac{a_{n+3}}{a_{n+2}} - \frac{a_{n+2}}{a_n} + \frac{a_{n+1}}{a_{n+3}} - \frac{a_n}{a_{n+1}} \right).$$

Proposed by: Ernest Chiu

Answer: $\boxed{\frac{14}{69}}$

The key idea is to note that $a_{n+1}a_n + \sqrt{a_{n+1}^2 - 1}\sqrt{a_n^2 - 1}$ is the larger zero of the quadratic

$$f_n(x) = x^2 - (2a_{n+1}a_n)x + a_n^2 + a_{n+1}^2 - 1.$$

Since a_{n+2} is the smallest integer greater than or equal to this root, it follows that $a_n^2 + a_{n+1}^2 + a_{n+2}^2 - 2a_n a_{n+1} a_{n+2} - 1$ is some small nonnegative integer. For these particular initial conditions ($a_0 = 3, a_1 = 4, a_2 = 12 + \lceil\sqrt{120}\rceil = 23$), this integer is $(3^2+4^2)+(23^2-2\cdot3\cdot4\cdot23)-1 = 25-23-1 = 1$.

We now use induction to prove both

$$a_{n+3} = 2a_{n+2}a_{n+1} - a_n \text{ and } a_n^2 + a_{n+1}^2 + a_{n+2}^2 - 2a_n a_{n+1} a_{n+2} - 1 = 1$$

for $n \geq 0$. The base case is not difficult to check: $a_3 = 4 \cdot 23 + \lceil\sqrt{7920}\rceil = 181 = 2 \cdot 4 \cdot 23 - 3$, and the other equation has been checked above. Since the quadratic equation $f_{n+1}(x) = 1$ has a solution a_n by induction hypothesis. Then, using Vieta's theorem, $2a_{n+1}a_{n+2} - a_n$ is also a solution. Then, the two roots of $f_{n+1}(x) = 0$ must be in strictly between a_n and $2a_{n+1}a_{n+2} - a_n$, so we have that $a_{n+3} \leq 2a_{n+1}a_{n+2} - a_n$ since a_{n+3} is the ceiling of the larger root. In fact, since $a_{n+1}a_{n+2}$ is much larger than a_n for $n \geq 1$ (it is not difficult to see that a_n grows faster than exponential), meaning that the two roots of f_{n+1} are more than 1 away from the minimum, and $f(2a_{n+1}a_{n+2} - a_n) = 1$, we have $f(2a_{n+1}a_{n+2} - a_n - 1) < 0$, which mean that we must have $a_{n+3} = 2a_{n+1}a_{n+2} - a_n$, which simultaneously proves both statements due to Vieta jumping.

To finish, note that the above recurrence gives

$$\frac{a_{n+3}}{a_{n+2}} - \frac{a_{n+2}}{a_n} = 2a_{n+1} - \frac{a_n}{a_{n+2}} - 2a_{n+1} + \frac{a_{n-1}}{a_n} = -\frac{a_n}{a_{n+2}} + \frac{a_{n-1}}{a_n},$$

which telescopes with the other two terms. (Convergence can be shown since the ratio of adjacent terms is bounded above by 1/3. In fact it goes to zero rapidly.) The only leftover terms after telescoping are $\frac{a_3}{a_2} - \frac{a_2}{a_0} = -\frac{a_0}{a_2} + \frac{a_{-1}}{a_0} = -\frac{3}{23} + \frac{1}{3} = \frac{14}{69}$, giving the answer. (Here, we use the backwards recurrence $a_{n-1} = 2a_n a_{n+1} - a_{n+2}$ to find $a_{-1} = 2 \cdot 3 \cdot 4 - 23 = 1$.)