15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Team B

For problems 1-5, only a short answer is required.

1. [**10**] Triangle ABC has $AB = 5$, $BC = 3\sqrt{2}$, and $AC = 1$. If the altitude from B to AC and the angle bisector of angle A intersect at at D, what is BD?

2. [**10**] You are given two line segments of length 2^n for each integer $0 \leq n \leq 10$. How many distinct nondegenerate triangles can you form with three of the segments? Two triangles are considered distinct if they aren't congruent.

3. [**10**] Mac is trying to fill 2012 barrels with apple cider. He starts with 0 energy. Every minute, he may rest, gaining 1 energy, or if he has n energy, he may expend k energy ($1 \leq k \leq n$) to fill up to $n(k+1)$ barrels with cider. What is the minimal number of minutes he needs to fill all the barrels?

4. [**10**] A restaurant has some number of seats, arranged in a line. Its customers are in parties arranged in a queue. To seat its customers, the restaurant takes the next party in the queue and attempts to see all of the party's member(s) in a contiguous block of unoccupied seats. If one or more such blocks exist, then the restaurant places the party in an arbitrarily selected block; otherwise, the party leaves.

   Suppose the queue has parties of sizes $6, 4, 2, 5, 3, 1$ from front to back, and all seats are initially empty. What is the minimal number of seats the restaurant needs to guarantee that it will seat all of these customers?

5. [**10**] Steph and Jeff each start with the number 4, and Travis is flipping coins. Every time he flips a heads, Steph replaces her number x with $2x - 1$, and Jeff replaces his number y with $y + 8$. Every time he flips a tails, Steph replaces her number x with $\frac{x+1}{2}$, and Jeff replaces his number y with $y - 3$. After some (positive) number of coin flips, Steph and Jeff miraculously end up with the same number below 2012. How many times was a coin flipped?

For problems 6-10, justification is required.

6. [**20**] Let ABC be a triangle. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D. Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC, respectively. Prove that $BE = CF$.

7. [**20**]

   (a) For what positive integers n do there exist functions $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that for all $1 \leq i \leq n$, we have that exactly one of $f(g(i)) = i$ and $g(f(i)) = i$ holds?

   (b) What if f, g must be permutations?

8. [**20**] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player somewhere on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

   (a) Suppose that a token can legally move to any orthogonally adjacent square. Show that Bob has a winning strategy for this game.

   (b) Suppose instead that a token can legally move to any square which shares a vertex with the square it is currently on. Show that Alice has a winning strategy for this game.

9. [**20**] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC. Line l is drawn through M so that it is perpendicular to AM, and intersects line AB at point X and line AC at point Y. Prove that $\angle BAC = 90^\circ$ if and only if quadrilateral $XBYC$ is cyclic.

10. [**20**] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza: mushrooms, olives, green peppers, cheese, and pepperoni. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes 5 pizzas, each with some subset of the five toppings, then for each of those Arno tells her if that pizza has any toppings he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she can then make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?