HMMT February 2016

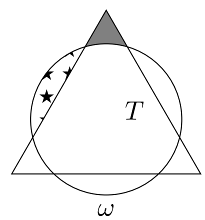
February 20, 2016

Geometry

1. Dodecagon $QWARTZSPHINX$ has all side lengths equal to 2, is not self-intersecting (in particular, the twelve vertices are all distinct), and moreover each interior angle is either 90° or 270°. What are all possible values of the area of $\triangle SIX$?

2. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let H be the orthocenter of ABC. Find the distance between the circumcenters of triangles AHB and AHC.

3. In the below picture, T is an equilateral triangle with a side length of 5 and ω is a circle with a radius of 2. The triangle and the circle have the same center. Let X be the area of the shaded region, and let Y be the area of the starred region. What is $X - Y$?



4. Let ABC be a triangle with $AB = 3$, $AC = 8$, $BC = 7$ and let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Point T is selected on side BC so that $AT = TC$. The circumcircles of triangles BAT, MAN intersect at D. Compute DC.

5. Nine pairwise noncongruent circles are drawn in the plane such that any two circles intersect twice. For each pair of circles, we draw the line through these two points, for a total of $\binom{9}{2} = 36$ lines. Assume that all 36 lines drawn are distinct. What is the maximum possible number of points which lie on at least two of the drawn lines?

6. Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc $\widehat{BAC}$ of Γ. Given that $AB = 5$, $AC = 8$, and $\angle A = 60^\circ$, compute the area of triangle XUV.

7. Let $S = \{(x, y) | x, y \in \mathbb{Z}, 0 \le x, y, \le 2016\}$. Given points $A = (x_1, y_1), B = (x_2, y_2)$ in S, define
$$d_{2017}(A, B) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \pmod{2017}.$$
The points $A = (5, 5), B = (2, 6), C = (7, 11)$ all lie in S. There is also a point $O \in S$ that satisfies
$$d_{2017}(O, A) = d_{2017}(O, B) = d_{2017}(O, C).$$
Find $d_{2017}(O, A)$.

8. For $i = 0, 1, \ldots, 5$ let l_i be the ray on the Cartesian plane starting at the origin, an angle $\theta = i\frac{\pi}{3}$ counterclockwise from the positive x-axis. For each i, point P_i is chosen uniformly at random from the intersection of l_i with the unit disk. Consider the convex hull of the points P_i, which will (with probability 1) be a convex polygon with n vertices for some n. What is the expected value of n?

9. In cyclic quadrilateral $ABCD$ with $AB = AD = 49$ and $AC = 73$, let I and J denote the incenters of triangles ABD and CBD. If diagonal $\overline{BD}$ bisects $\overline{IJ}$, find the length of IJ.

10. The incircle of a triangle ABC is tangent to BC at D. Let H and Γ denote the orthocenter and circumcircle of $\triangle ABC$. The *B-mixtilinear incircle*, centered at O_B, is tangent to lines BA and BC and internally tangent to Γ. The *C-mixtilinear incircle*, centered at O_C, is defined similarly. Suppose that $\overline{DH} \perp \overline{O_B O_C}$, $AB = \sqrt{3}$ and $AC = 2$. Find BC.