HMMT February 2018

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Combinatorics

1. Consider a 2×3 grid where each entry is one of 0, 1, and 2. For how many such grids is the sum of the numbers in every row and in every column a multiple of 3? One valid grid is shown below.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Proposed by: Henrik Boecken

Answer: $\boxed{9}$

Any two elements in the same row fix the rest of the grid, so $3^2 = 9$.

2. Let a and b be five-digit palindromes (without leading zeroes) such that $a < b$ and there are no other five-digit palindromes strictly between a and b. What are all possible values of $b - a$? (A number is a palindrome if it reads the same forwards and backwards in base 10.)

Proposed by: Kevin Sun

Answer: $\boxed{100, 110, 11}$

Let $\overline{xyzyx}$ be the digits of the palindrome a. There are three cases. If $z < 9$, then the next palindrome greater than $\overline{xyzyx}$ is $\overline{xy(z+1)yx}$, which differs by 100. If $z = 9$ but $y < 9$, then the next palindrome up is $\overline{x(y+1)0(y+1)x}$, which differs from $\overline{xy9yx}$ by 110. Finally, if $y = z = 9$, then the next palindrome after $\overline{x999x}$ is $\overline{(x+1)000(x+1)}$, which gives a difference of 11. Thus, the possible differences are $11, 100, 110$.

3. A 4×4 window is made out of 16 square windowpanes. How many ways are there to stain each of the windowpanes, red, pink, or magenta, such that each windowpane is the same color as exactly two of its neighbors? Two different windowpanes are neighbors if they share a side.

Proposed by: Kevin Sun

Answer: $\boxed{24}$

For the purpose of explaining this solution, let's label the squares as

11 12 13 14
21 22 23 24
31 32 33 34
41 42 43 44

Note that since the corner squares $11, 14, 41, 44$ each only have two neighbors, each corner square is the same color as both of its neighbors (for example, $11, 12$, and 21 are the same color, $31, 41$, and 42 are the same color, etc.). This corner square constraint heavily limits the possible colorings. We will now use casework.

Case 1: Suppose two corner squares on the same side (without loss of generality, let them be 11 and 14) have the same color (without loss of generality, red). Then $21, 11, 12, 13, 14, 24$ are all red, and 12 has two red neighbors (11 and 13) so its third neighbor (22) is a color different from red (without loss of generality, magenta). But 22 has two red neighbors (12 and 21), so its other two neighbors (23 and 32)must be magenta. Applying the same logic symmetrically, we find that all four interior squares $(22, 23, 32, 33)$ have the same color. Furthermore, 21 has one magenta neighbor 22, so 31 must be red. Symmetrically, 34 is red, and by the corner square constraint we have that all the exterior squares are the same color. Thus in general, this case is equivalent to a window taking the following form (with distinct colors A and B)

$$\begin{matrix} A & A & A & A \\ A & B & B & A \\ A & B & B & A \\ A & A & A & A \end{matrix}$$

The number of choices of A and B is $3 \cdot 2 = 6$.

Case 2: No two corner squares on the same side have the same color.

Then from the corner square constraint 12 has neighbor 11 of the same color and neighbor 13 of a different color, so its neighbor 22 must be the same color as 12. Therefore, this case is equivalent to coloring each quadrant entirely in one color such two quadrants sharing a side have different colors. (A quadrant refers to the four squares on one vertical half and one horizontal half, e.g. $13, 14, 23, 24$).

If only two colors are used, the window will take the form (with distinct colors A and B):

$$\begin{matrix} A & A & B & B \\ A & A & B & B \\ B & B & A & A \\ B & B & A & A \end{matrix}$$

Again there are $3 \cdot 2 = 6$ ways to chose A and B.

If all three colors are used, the window will take the form (with distinct colors A, B and C)

$$\begin{matrix} A & A & B & B \\ A & A & B & B \\ C & C & A & A \\ C & C & A & A \end{matrix}$$

or

$$\begin{matrix} A & A & B & B \\ A & A & B & B \\ B & B & C & C \\ B & B & C & C \end{matrix}$$

There are $3 \cdot 2 \cdot 1 = 6$ ways to select colors for each of these forms.

Therefore, there are 6 colorings in Case 1 and $6 + 6 + 6$ in Case 2, for a total of 24 colorings.

4. How many ways are there for Nick to travel from $(0, 0)$ to $(16, 16)$ in the coordinate plane by moving one unit in the positive x or y direction at a time, such that Nick changes direction an odd number of times?

   Proposed by: Huaiyu Wu

   Answer: $\boxed{2 \cdot \binom{30}{15} = 310235040}$

   This condition is equivalent to the first and last step being in different directions, as if you switch directions an odd number of times, you must end in a different direction than you started. If the first step is in the x direction and the last step is in the y direction, it suffices to count the number of paths from $(1, 0)$ to $(16, 15)$, of which there are $\binom{30}{15}$. Similarly, in the other case, it suffices to count the number of paths from $(0,1)$ to $(15,16)$, of which there are also $\binom{30}{15}$. Therefore the total number of paths is $2 \cdot \binom{30}{15}$.

5. A bag contains nine blue marbles, ten ugly marbles, and one special marble. Ryan picks marbles randomly from this bag with replacement until he draws the special marble. He notices that none of the marbles he drew were ugly. Given this information, what is the expected value of the number of total marbles he drew?

   Proposed by: Kevin Sun

   Answer: $\boxed{\frac{20}{11}}$

   The probability of drawing k marbles is the probability of drawing $k-1$ blue marbles and then the special marble, which is $p_k = \left(\frac{9}{20}\right)^{k-1} \times \frac{1}{20}$. The probability of drawing no ugly marbles is therefore $\sum_{k=1}^{\infty} p_k = \frac{1}{11}$.

   Then given that no ugly marbles were drawn, the probability that k marbles were drawn is $11p_k$. The expected number of marbles Ryan drew is

   $$\sum_{k=1}^{\infty} k(11p_k) = \frac{11}{20}\sum_{k=1}^{\infty} k\left(\frac{9}{20}\right)^{k-1} = \frac{11}{20} \times \frac{400}{121} = \frac{20}{11}.$$

   (To compute the sum in the last step, let $S = \sum_{k=1}^{\infty} k\left(\frac{9}{20}\right)^{k-1}$ and note that $\frac{9}{20}S = S - \sum_{k=1}^{\infty}\left(\frac{9}{20}\right)^{k-1} = S - \frac{20}{11}$).

6. Sarah stands at $(0,0)$ and Rachel stands at $(6,8)$ in the Euclidean plane. Sarah can only move 1 unit in the positive x or y direction, and Rachel can only move 1 unit in the negative x or y direction. Each second, Sarah and Rachel see each other, independently pick a direction to move at the same time, and move to their new position. Sarah catches Rachel if Sarah and Rachel are ever at the same point. Rachel wins if she is able to get to $(0,0)$ without being caught; otherwise, Sarah wins. Given that both of them play optimally to maximize their probability of winning, what is the probability that Rachel wins?

   Proposed by: Rachel Zhang

   Answer: $\boxed{\frac{63}{64}}$

   We make the following claim: In a game with $n \times m$ grid where $n \leq m$ and $n \equiv m \pmod 2$, the probability that Sarah wins is $\frac{1}{2^n}$ under optimal play.

   Proof: We induct on n. First consider the base case $n = 0$. In this case Rachel is confined on a line, so Sarah is guaranteed to win.

   We then consider the case where $n = m$ (a square grid). If Rachel and Sarah move in parallel directions at first, then Rachel can win if she keep moving in this direction, since Sarah will not be able to catch Rachel no matter what. Otherwise, the problem is reduced to a $(n-1) \times (n-1)$ grid. Therefore, the optimal strategy for both players is to choose a direction completely randomly, since any bias can be abused by the other player. So the reduction happens with probability $\frac{1}{2}$, and by induction hypothesis Sarah will with probability $\frac{1}{2^{n-1}}$, so on a $n \times n$ grid Sarah wins with probability $\frac{1}{2^n}$.

   Now we use induction to show that when $n < m$, both player will move in the longer (m) direction until they are at corners of a square grid (in which case Sarah wins with probability $\frac{1}{2^n}$. If Sarah moves in the n direction and Rachel moves in the m (or n) direction, then Rachel can just move in the n direction until she reaches the other side of the grid and Sarah will not be able to catch her. If Rachel moves in the n direction and Sarah moves in the m direction, then the problem is reduced to a $(n-1) \times (m-1)$ grid, which means that Sarah's winning probability is now doubled to $\frac{1}{2^{n-1}}$ by induction hypothesis. Therefore it is suboptimal for either player to move in the shorter (n) direction. This shows that the game will be reduced to $n \times n$ with optimal play, and thus the claim is proved.

   From the claim, we can conclude that the probability that Rachel wins is $1 - \frac{1}{2^6} = \frac{63}{64}$.

7. A tourist is learning an incorrect way to sort a permutation $(p_1, \ldots, p_n)$ of the integers $(1, \ldots, n)$. We define a *fix* on two adjacent elements p_i and p_{i+1}, to be an operation which swaps the two elements if $p_i > p_{i+1}$, and does nothing otherwise. The tourist performs $n-1$ rounds of fixes, numbered $a = 1, 2, \ldots, n-1$. In round a of fixes, the tourist fixes p_a and p_{a+1}, then p_{a+1} and p_{a+2}, and so on, up to p_{n-1} and p_n. In this process, there are $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$ total fixes performed.

   How many permutations of $(1, \ldots, 2018)$ can the tourist start with to obtain $(1, \ldots, 2018)$ after performing these steps?

   Proposed by: Kevin Sun

   Answer: $\boxed{1009! \cdot 1010!}$

   Note that the given algorithm is very similar to the well-known Bubble Sort algorithm for sorting an array. The exception is that in the i-th round through the array, the first $i-1$ pairs are not checked.

   We claim a necessary and sufficient condition for the array to be sorted after the tourist's process is: for all i, after i rounds, the numbers $1, \cdots, i$ are in the correct position. Firstly, this is necessary because these indices of the array are not touched in future rounds - so if a number was incorrect, then it would stay incorrect. On the other hand, suppose this condition holds. Then, we can "add" the additional fixes during each round (of the first $i-1$ pairs during the i-th round) to make the process identical to bubble sort. The tourist's final result won't change because by our assumption these swaps won't do anything. However, this process is now identical to bubble sort, so the resulting array will be sorted. Thus, our condition is sufficient.

   Now, there are two positions the 1 can be in (p_1, p_2). There are three positions the 2 can be in $(p_1, \cdots, p_4$ except for the position of 1). Similarly, for $1 \le i \le 1009$ there are $2i - (i-1) = i+1$ positions i can be in, and after that the remaining 1009 numbers can be arranged arbitrarily. Thus, the answer is $1010! \cdot 1009!$.

8. A permutation of $\{1, 2, \ldots, 7\}$ is chosen uniformly at random. A partition of the permutation into contiguous blocks is correct if, when each block is sorted independently, the entire permutation becomes sorted. For example, the permutation $(3, 4, 2, 1, 6, 5, 7)$ can be partitioned correctly into the blocks $[3, 4, 2, 1]$ and $[6, 5, 7]$, since when these blocks are sorted, the permutation becomes $(1, 2, 3, 4, 5, 6, 7)$.

   Find the expected value of the maximum number of blocks into which the permutation can be partitioned correctly.

   Proposed by: Mehtaab Sawhney

   Answer: $\boxed{\frac{151}{105}}$

   Let σ be a permutation on $\{1, \ldots, n\}$. Call $m \in \{1, \ldots, n\}$ a breakpoint of σ if $\{\sigma(1), \ldots, \sigma(m)\} = \{1, \ldots, m\}$. Notice that the maximum partition is into k blocks, where k is the number of breakpoints: if our breakpoints are $m_1, \ldots, m_k$, then we take $\{1, \ldots, m_1\}, \{m_1+1, \ldots, m_2\}, \ldots, \{m_{k-1}+1, \ldots, m_k\}$ as our contiguous blocks.

   Now we just want to find
$$\mathbb{E}[k] = \mathbb{E}[X_1 + \cdots + X_n],$$
   where $X_i = 1$ if i is a breakpoint, and $X_i = 0$ otherwise. We use linearity of expectation and notice that
$$\mathbb{E}[X_i] = \frac{i!(n-i)!}{n!},$$
   since this is the probability that the first i numbers are just $1, \ldots, i$ in some order. Thus,
$$\mathbb{E}[k] = \sum_{i=1}^{n} \frac{i!(n-i)!}{n!} = \sum_{i=1}^{n} \binom{n}{i}^{-1}.$$
   We can compute for $n = 7$ that the answer is $\boxed{\frac{151}{105}}$.

9. How many ordered sequences of 36 digits have the property that summing the digits to get a number and taking the last digit of the sum results in a digit which is not in our original sequence? (Digits range from 0 to 9.)

Proposed by: Kevin Sun

Answer: $\boxed{9^{36}+4}$

We will solve this problem for 36 replaced by n. We use $[n]$ to denote $\{1,2,\ldots,n\}$ and σ_s to denote the last digit of the sum of the digits of s.

Let D be the set of all sequences of n digits and let S_i be the set of digit sequences s such that $s_i=\sigma_s$, the i^{th} digit of s. The quantity we are asked to compute is equal to $\left|D\setminus\bigcup_{i=1}^{n}S_i\right|$. We use the principle of inclusion-exclusion to compute this:

$$\left|D\setminus\bigcup_{i=1}^{n}S_i\right|=\sum_{J\subseteq[n]}(-1)^{|J|}\left|\bigcap_{j\in J}S_j\right|$$

Note that a digit sequence is in S_i if and only if the $n-1$ digits which are not i sum to a multiple of 10. This gives that $|S_i|=10\cdot 10^{n-2}=10^{n-1}$ as there are 10 ways to pick the i^{th} digit and 10^{n-2} ways to pick the other digits.

Similarly, given a subset $J\subseteq[n]$, we can perform a similar analysis. If a string s is in $\bigcap_{j\in J}S_j$, we must have that $s_j=\sigma_s$ for all $j\in J$. There are 10 ways to pick σ_s, which determines s_j for all $j\in J$. From there, there are $10^{(n-|J|)-1}$ ways to pick the remaining digits as if we fix all but one, the last digit is uniquely determined. This gives $10^{n-|J|}$ choices.

However, this breaks down when $|J|=n$, as not all choices of σ_s lead to any valid solutions. When $|J|=n$, $J=[n]$ and we require that the last digit of $n\sigma_s$ is σ_s, which happens for $\gcd(n-1,10)$ values of σ_s.

We now compare our expression from the principle of inclusion-exclusion to the binomial expansion of $(10-1)^n$. By the binomial theorem,

$$9^n=(10-1)^n=\sum_{J\subseteq[n]}(-1)^{|J|}10^{n-|J|}.$$

These agree on every term except for the term where $J=[n]$. In this case, we need to add an extra $(-1)^n\gcd(n-1,10)$ and subtract $(-1)^n$.

Thus our final value for $\left|D\setminus\bigcup_{i=1}^{n}S_i\right|$ is $9^n+(-1)^n(\gcd(n-1,10)-1)$, which is $9^{36}+4$ for $n=36$.

10. Lily has a 300×300 grid of squares. She now removes 100×100 squares from each of the four corners and colors each of the remaining 50000 squares black and white. Given that no 2×2 square is colored in a checkerboard pattern, find the maximum possible number of (unordered) pairs of squares such that one is black, one is white and the squares share an edge.

Proposed by: Allen Liu

Answer: $\boxed{49998}$

First we show an upper bound. Define a grid point as a vertex of one of the squares in the figure. Construct a graph as follows. Place a vertex at each grid point and draw an edge between two adjacent points if that edge forms a black-white boundary. The condition of there being no 2×2 checkerboard is equivalent to no vertex having degree more than 2. There are $101^2+4\cdot99^2=49405$ vertices that are allowed to have degree 2 and $12\cdot99=1188$ vertices (on the boundary) that can have degree 1.

This gives us an upper bound of 49999 edges. We will show that exactly this many edges is impossible. Assume for the sake of contradiction that we have a configuration achieving exactly this many edges.

Consider pairing up the degree 1 vertices so that those on a horizontal edge pair with the other vertex in the same column and those on a vertical edge pair with the other vertex in the same row. If we combine the pairs into one vertex, the resulting graph must have all vertices with degree exactly 2. This means the graph must be a union of disjoint cycles. However all cycles must have even length and there are an odd number of total vertices so this is impossible. Thus we have an upper bound of 49998.

We now describe the construction. The top row alternates black and white. The next 99 rows alternate between all black and all white. Lets say the second row from the top is all white. The 101^{st} row alternates black and white for the first 100 squares, is all black for the next 100 and alternates between white and black for the last 100 squares. The next 98 rows alternate between all black and all white (the 102^{nd} row is all white). Finally, the bottom 101 rows are a mirror of the top 101 rows with the colors reversed. We easily verify that this achieves the desired. We illustrate the construction for 300 replaced by 12.