11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Combinatorics Test

1. [**3**] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3 \times 3 \times 1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

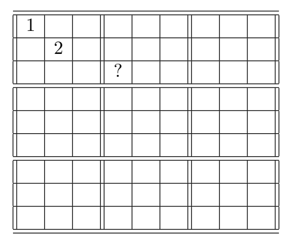
2. [**3**] Let $S = \{1, 2, \ldots, 2008\}$. For any nonempty subset $A \subset S$, define $m(A)$ to be the median of A (when A has an even number of elements, $m(A)$ is the average of the middle two elements). Determine the average of $m(A)$, when A is taken over all nonempty subsets of S.

3. [**4**] Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? (Assume that all animals are distinguishable from each other.)

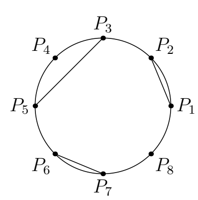
4. [**4**] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

5. [**5**] Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of non-negative integers in S less than 2008.

6. [**5**] A *Sudoku matrix* is defined as a 9×9 array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?



7. [**6**] Let $P_1, P_2, \ldots, P_8$ be 8 distinct points on a circle. Determine the number of possible configurations made by drawing a set of line segments connecting pairs of these 8 points, such that: (1) each P_i is the endpoint of at most one segment and (2) two no segments intersect. (The configuration with no edges drawn is allowed. An example of a valid configuration is shown below.)



8. [**6**] Determine the number of ways to select a sequence of 8 sets $A_1, A_2, \ldots, A_8$, such that each is a subset (possibly empty) of $\{1, 2\}$, and A_m contains A_n if m divides n.

9. [**7**] On an infinite chessboard (whose squares are labeled by (x, y), where x and y range over all integers), a king is placed at $(0, 0)$. On each turn, it has probability of 0.1 of moving to each of the four edge-neighboring squares, and a probability of 0.05 of moving to each of the four diagonally-neighboring squares, and a probability of 0.4 of not moving. After 2008 turns, determine the probability that the king is on a square with both coordinates even. An exact answer is required.

10. [**7**] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.