14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

Guts Round

1. [**4**] Let ABC be a triangle with area 1. Let points D and E lie on AB and AC, respectively, such that DE is parallel to BC and $DE/BC = 1/3$. If F is the reflection of A across DE, find the area of triangle FBC.

   Answer: $\boxed{\frac{1}{3}}$

   Let AF intersect BC at H. Since $DE/BC = 1/3$ and F and A are equidistant from DE, we have $AF = \frac{2}{3}AH$ and $FH = AH - AF = \frac{1}{3}AH$. Furthermore, since AF is perpendicular to DE, we have AH and FH are the altitudes of triangles ABC and FBC respectively. Therefore the area of triangle FBC is $\frac{1}{2} \cdot FH \cdot BC = \frac{1}{2} \cdot \frac{1}{3} \cdot AH \cdot BC = \frac{1}{3}$.

2. [**4**] Let $a \star b = \sin a \cos b$ for all real numbers a and b. If x and y are real numbers such that $x \star y - y \star x = 1$, what is the maximum value of $x \star y + y \star x$?

   Answer: $\boxed{1}$

   We have $x \star y + y \star x = \sin x \cos y + \cos x \sin y = \sin(x + y) \le 1$. Equality is achieved when $x = \frac{\pi}{2}$ and $y = 0$. Indeed, for these values of x and y, we have $x \star y - y \star x = \sin x \cos y - \cos x \sin y = \sin(x - y) = \sin \frac{\pi}{2} = 1$.

3. [**4**] Evaluate $2011 \times 20122012 \times 201320132013 - 2013 \times 20112011 \times 201220122012$.

   Answer: $\boxed{0}$

   Both terms are equal to $2011 \times 2012 \times 2013 \times 1 \times 10001 \times 100010001$.

4. [**4**] Let p be the answer to this question. If a point is chosen uniformly at random from the square bounded by $x = 0$, $x = 1$, $y = 0$, and $y = 1$, what is the probability that at least one of its coordinates is greater than p?

   Answer: $\boxed{\frac{\sqrt{5}-1}{2}}$

   The probability that a randomly chosen point has both coordinates less than p is p^2, so the probability that at least one of its coordinates is greater than p is $1 - p^2$. Since p is the answer to this question, we have $1 - p^2 = p$, and the only solution of p in the interval $[0, 1]$ is $\frac{\sqrt{5}-1}{2}$.

5. [**5**] Rachelle picks a positive integer a and writes it next to itself to obtain a new positive integer b. For instance, if $a = 17$, then $b = 1717$. To her surprise, she finds that b is a multiple of a^2. Find the product of all the possible values of $\frac{b}{a^2}$.

   Answer: $\boxed{77}$

   Suppose a has k digits. Then $b = a(10^k + 1)$. Thus a divides $10^k + 1$. Since $a \ge 10^{k-1}$, we have $\frac{10^k+1}{a} \le 11$. But since none of 2, 3, or 5 divide $10^k + 1$, the only possibilities are 7 and 11. These values are obtained when $a = 143$ and $a = 1$, respectively.

6. [**5**] Square $ABCD$ is inscribed in circle ω with radius 10. Four additional squares are drawn inside ω but outside $ABCD$ such that the lengths of their diagonals are as large as possible. A sixth square is drawn by connecting the centers of the four aforementioned small squares. Find the area of the sixth square.

   Answer: $\boxed{144}$

   Let $DEGF$ denote the small square that shares a side with AB, where D and E lie on AB. Let O denote the center of ω, K denote the midpoint of FG, and H denote the center of $DEGF$. The area of the sixth square is $2 \cdot OH^2$.

   Let $KF = x$. Since $KF^2 + OK^2 = OF^2$, we have $x^2 + (2x + 5\sqrt{2})^2 = 10^2$. Solving for x, we get $x = \sqrt{2}$. Thus, we have $OH = 6\sqrt{2}$ and $2 \cdot OH^2 = 144$.

7. [**6**] For any positive real numbers a and b, define $a \circ b = a + b + 2\sqrt{ab}$. Find all positive real numbers x such that $x^2 \circ 9x = 121$.

Answer: $\boxed{\frac{31-3\sqrt{53}}{2}}$

Since $a \circ b = (\sqrt{a} + \sqrt{b})^2$, we have $x^2 \circ 9x = (x + 3\sqrt{x})^2$. Moreover, since x is positive, we have $x + 3\sqrt{x} = 11$, and the only possible solution is that $\sqrt{x} = \frac{-3+\sqrt{53}}{2}$, so $x = \frac{31-3\sqrt{53}}{2}$.

8. [**6**] Find the smallest k such that for any arrangement of 3000 checkers in a 2011×2011 checkerboard, with at most one checker in each square, there exist k rows and k columns for which every checker is contained in at least one of these rows or columns.

Answer: $\boxed{1006}$

If there is a chip in every square along a main diagonal, then we need at least 1006 rows and columns to contain all these chips. We are left to show that 1006 is sufficient.

Take the 1006 rows with greatest number of chips. Assume without loss of generality they are the first 1006 rows. If the remaining 1005 rows contain at most 1005 chips, then we can certainly choose 1006 columns that contain these chips. Otherwise, there exists a row that contains at least 2 chips, so every row in the first 1006 rows must contain at least 2 chips. But this means that there are at least $2 \times 1006 + 1006 = 3018$ chips in total. Contradiction.

9. [**6**] Segments AA', BB', and CC', each of length 2, all intersect at a point O. If $\angle AOC' = \angle BOA' = \angle COB' = 60^\circ$, find the maximum possible value of the sum of the areas of triangles AOC', BOA', and COB'.

Answer: $\boxed{\sqrt{3}}$

Extend OA to D and OC' to E such that $AD = OA'$ and $C'E = OC$. Since $OD = OE = 2$ and $\angle DOE = 60^\circ$, we have ODE is an equilateral triangle. Let F be the point on DE such that $DF = OB$ and $EF = OB'$. Clearly we have $\triangle DFA \cong \triangle OBA'$ and $\triangle EFC' \cong OB'C$. Thus the sum of the areas of triangles AOC', BOA', and COB' is the same as the sum of the areas of triangle DFA, FEC', and OAC', which is at most the area of triangle ODE. Since ODE is an equilateral triangle with side length 2, its area is $\sqrt{3}$. Equality is achieved when $OC = OA' = 0$.

10. [**6**] In how many ways can one fill a 4×4 grid with a 0 or 1 in each square such that the sum of the entries in each row, column, and long diagonal is even?

Answer: $\boxed{256}$

First we name the elements of the square as follows:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}$$

We claim that for any given values of $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{32}$, and a_{33} (the $+$ signs in the diagram below), there is a unique way to assign values to the rest of the entries such that all necessary sums are even.

$$\begin{array}{cccc} + & + & + & a_{14} \\ + & + & + & a_{24} \\ a_{31} & + & + & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}$$

Taking additions mod 2, we have

$$\begin{aligned} a_{14} &= a_{11} + a_{12} + a_{13} \\ a_{24} &= a_{21} + a_{22} + a_{23} \\ a_{44} &= a_{11} + a_{22} + a_{33} \\ a_{42} &= a_{12} + a_{22} + a_{32} \\ a_{43} &= a_{13} + a_{23} + a_{33} \end{aligned}$$

Since the 4th column, the 4th row, and the 1st column must have entries that sum to 0, we have

$$\begin{aligned} a_{34} &= a_{14} + a_{24} + a_{44} = a_{12} + a_{13} + a_{21} + a_{23} + a_{33} \\ a_{41} &= a_{42} + a_{43} + a_{44} = a_{11} + a_{12} + a_{13} + a_{23} + a_{32} \\ a_{31} &= a_{11} + a_{21} + a_{41} = a_{12} + a_{13} + a_{21} + a_{23} + a_{32} \end{aligned}$$

It is easy to check that the sum of entries in every row, column, and the main diagonal is even. Since there are $2^8 = 256$ ways to assign the values to the initial 8 entries, there are exactly 256 ways to fill the board.

11. [**8**] Rosencrantz and Guildenstern play a game in which they repeatedly flip a fair coin. Let $a_1 = 4$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. On the nth flip, if the coin is heads, Rosencrantz pays Guildenstern a_n dollars, and, if the coin is tails, Guildenstern pays Rosencrantz a_n dollars. If play continues for 2010 turns, what is the probability that Rosencrantz ends up with more money than he started with?

Answer: $\boxed{\frac{1}{2} - \frac{1}{2^{1341}}}$

Since Rosencrantz and Guildenstern have an equal chance of winning each toss, both have the same probability of ending up with a positive amount of money. Let x denote the probability that they both end up with zero dollars. We wish to find $\frac{1-x}{2}$.

We have x is equal to the probability that

$$s_{2010} := i_1 a_1 + i_2 a_2 + \cdots i_{2010} a_{2010} = 0,$$

where i_n has an equal probability of being either 1 or -1.

We claim that $s_{2010} = 0$ if and only if $i_{3n} = -i_{3n-1} = -i_{3n-2}$ for all $n \leq 670$. We start with the following lemma.

Lemma. We have $a_n > \sum_{k=1}^{n-3} a_k$ for all $n \geq 4$.

Proof: For the case $n = 4$, $a_4 = a_3 + a_2 = 2a_2 + a_1 > a_1$. In case $n > 4$, we have

$$a_n \; = \; a_{n-2} + a_{n-1} \; > \; a_{n-2} + \sum_{k=1}^{n-4} a_k \; = \; a_{n-4} + \sum_{k=1}^{n-3} a_k \; > \; \sum_{k=1}^{n-3} a_k.$$

It suffices to show that $s_{3n} = 0$ only if $i_{3k} = -i_{3k-1} = -i_{3k-2}$ for all $k \leq n$. The triangle inequality implies the following:

$$\begin{aligned} 0 \; &\leq \; \Big| \big| i_{3n-2} a_{3n-2} + s_{3n-3} \big| - \big| i_{3n-1} a_{3n-1} + i_{3n} a_{3n} \big| \Big| \; \leq \; |s_{3n}| \; = \; 0 \\ 0 \; &\leq \; \Big| \big| i_{3n-1} a_{3n-1} + s_{3n-3} \big| - \big| i_{3n-2} a_{3n-2} + i_{3n} a_{3n} \big| \Big| \; \leq \; |s_{3n}| \; = \; 0 \end{aligned}$$

By the lemma, we have

$$\begin{array}{rcl} a_{3n-2} + \sum_{k=1}^{3n-3} a_k < a_{3n-2} + a_{3n} & < & a_{3n-1} + a_{3n} \\ a_{3n-1} + \sum_{k=1}^{3n-3} a_k < a_{3n-1} + a_{3n-3} + a_{3n-4} + a_{3n-2} & = & a_{3n-2} + a_{3n} \end{array}$$

$i_{3n} = i_{3n-1}$ implies $\left|a_{3n-2} + \sum_{k=1}^{3n-3} a_k\right| < |i_{3n-1}a_{3n-1} + i_{3n}a_{3n}|$ and $i_{3n} = i_{3n-2}$ implies $\left|a_{3n-1} + \sum_{k=1}^{3n-3} a_k\right| < |i_{3n-2}a_{3n-2} + i_{3n}a_{3n}|$, which are both contradictions; therefore, we must have $i_{3n} = -i_{3n-1}$ and $i_{3n} = -i_{3n-2}$.

The probability that $i_{3n} = -i_{3n-1} = -i_{3n-2}$ is $\frac{1}{4}$, so $x = \left(\frac{1}{4}\right)^{670} = \frac{1}{2^{1340}}$, and $\frac{1-x}{2} = \frac{1}{2} - \frac{1}{2^{1341}}$.

12. [**8**] A sequence of integers $\{a_i\}$ is defined as follows: $a_i = i$ for all $1 \le i \le 5$, and $a_i = a_1 a_2 \cdots a_{i-1} - 1$ for all $i > 5$. Evaluate $a_1 a_2 \cdots a_{2011} - \sum_{i=1}^{2011} a_i^2$.

Answer: $\boxed{-1941}$

For all $i \ge 6$, we have $a_i = a_1 a_2 \cdots a_{i-1} - 1$. So

$$\begin{aligned} a_{i+1} &= a_1 a_2 \cdots a_i - 1 \\ &= (a_1 a_2 \cdots a_{i-1}) a_i - 1 \\ &= (a_i + 1) a_i - 1 \\ &= a_i^2 + a_i - 1. \end{aligned}$$

Therefore, for all $i \ge 6$, we have $a_i^2 = a_{i+1} - a_i + 1$, and we obtain that

$$\begin{aligned} & a_1 a_2 \cdots a_{2011} - \sum_{i=1}^{2011} a_i^2 \\ = \;& a_{2012} + 1 - \sum_{i=1}^{5} a_i^2 - \sum_{i=6}^{2011} a_i^2 \\ = \;& a_{2012} + 1 - \sum_{i=1}^{5} i^2 - \sum_{i=6}^{2011} (a_{i+1} - a_i + 1) \\ = \;& a_{2012} + 1 - 55 - (a_{2012} - a_6 + 2006) \\ = \;& a_6 - 2060 \\ = \;& -1941 \end{aligned}$$

13. [**8**] Let a,b, and c be the side lengths of a triangle, and assume that $a \le b$ and $a \le c$. Let $x = \frac{b+c-a}{2}$. If r and R denote the inradius and circumradius, respectively, find the minimum value of $\frac{ax}{rR}$.

Answer: $\boxed{3}$

It is well-known that both $\frac{abc}{4R}$ and $\frac{r(a+b+c)}{2}$ are equal to the area of triangle ABC. Thus $\frac{abc}{4R} = \frac{r(a+b+c)}{2}$, and

$$Rr = \frac{abc}{2(a+b+c)}.$$

Since $a \le b$ and $a \le c$, we have $\frac{a^2}{bc} \le 1$. We thus obtain that

$$\begin{aligned}
\frac{ax}{rR} &= \frac{a(b+c-a)/2}{\frac{abc}{2(a+b+c)}} \\
&= \frac{(a+b+c)(b+c-a)}{bc} \\
&= \frac{(b+c)^2-a^2}{bc} \\
&= \frac{(b+c)^2}{bc} - \frac{a^2}{bc} \\
&= \frac{b}{c} + \frac{c}{b} + 2 - \frac{a^2}{bc} \\
&\ge \frac{b}{c} + \frac{c}{b} + 2 - 1 \\
&\ge 2+2-1 \\
&= 3
\end{aligned}$$

Equality is achieved when $a = b = c$.

14. [**8**] Danny has a set of 15 pool balls, numbered $1, 2, \ldots, 15$. In how many ways can he put the balls in 8 indistinguishable bins such that the sum of the numbers of the balls in each bin is 14, 15, or 16?

Answer: $\boxed{122}$

Clearly, the balls numbered $15, 14, \ldots, 9, 8$ must be placed in separate bins, so we number the bins $15, 14, \ldots, 9, 8$. Note that bins 15 and 14 may contain only one ball while all other bins must contain at least two balls. We have two cases to examine.

Case 1: Only one bin contains exactly one ball. Let a_i denote the number of ways to place the balls numbered $1, 2, \ldots, i-1$ into the bins numbered $15, 14, \ldots, 15-i+1$. We can place either $i-1$ or $i-2$ into the bin numbered $15-i+1$. If we place $i-1$ in there, then there are a_{i-1} ways to finish packing the rest. If we place $i-2$ in this bin, then $i-1$ must be placed in the bin numbered $15-i+2$, so there are a_{i-2} ways to place the rest of the balls. Therefore $a_i = a_{i-1} + a_{i-2}$. Since $a_1 = 2$ and $a_2 = 3$, the sequence $\{a_i\}$ is the Fibonacci sequence, and $a_7 = 34$.

Case 2: Both bins 14 and 15 contain only one ball. A pair of balls from 1-7 must be put together to one of the bins numbered 8 through 13. This pair has sum at most 8, so we can count for all the cases.

Balls	Number of packings
1, 2	16
1, 3	10
1, 4	12
1, 5	12
1, 6	10
1, 7	8
2, 3	6
2, 4	4
2, 5	4
2, 6	3
3, 4	2
3, 5	1

Therefore, there are 88 possibilities in this case, and the total number of possibilities is 122.

15. [**10**] Find all irrational numbers x such that $x^3 - 17x$ and $x^2 + 4x$ are both rational numbers.

Answer: $\boxed{-2 \pm \sqrt{5}}$

From $x^2+4x \in \mathbb{Q}$, we deduce that $(x+2)^2 = x^2+4x+4$ is also rational, and hence $x = -2 \pm \sqrt{y}$, where y is rational. Then $x^3 - 17x = (26-6y) \pm (y-5)\sqrt{y}$, which forces y to be 5. Hence $x = -2 \pm \sqrt{5}$. It is easy to check that both values satisfy the problem conditions.

16. [**10**] Let R be a semicircle with diameter XY. A trapezoid $ABCD$ in which AB is parallel to CD is circumscribed about R such that AB contains XY. If $AD = 4$, $CD = 5$, and $BC = 6$, determine AB.

    Answer: $\boxed{10}$

    We claim that $AB = AD + BC$. Let O denote the center of R. Since DA and DC are both tangent to R, we have $\angle ADO = \angle ODC$. Since CD is parallel to AB, we also have $\angle ODC = \angle DOA$. Thus $\angle ADO = \angle DOA$, and it follows that $AD = AO$. Similarly, we have $BC = BO$. Therefore, we

    $$AB = AO + BO = AD + BC = 10.$$

17. [**10**] Given positive real numbers x, y, and z that satisfy the following system of equations:

    $$\begin{aligned} x^2 + y^2 + xy &= 1, \\ y^2 + z^2 + yz &= 4, \\ z^2 + x^2 + zx &= 5, \end{aligned}$$

    find $x + y + z$.

    Answer: $\boxed{\sqrt{5+2\sqrt{3}}}$

    Let O denote the origin. Construct vectors OA, OB, and OC as follows: The lengths of OA, OB, and OC are x, y, and z, respectively, and the angle between any two vectors is $120°$. By the Law of Cosines, we have $AB = 1$, $BC = 2$, and $AC = \sqrt{5}$. Thus ABC is a right triangle, and O is its Fermat point. The area of triangle ABC is equal to $\frac{1}{2} \cdot 1 \cdot 2 = 1$. But the area is also equal to the sum of the areas of triangles AOB, BOC, and COA, which is equal to $\frac{1}{2} \cdot xy \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot yz \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot zx \cdot \frac{\sqrt{3}}{2}$. We thus obtain $xy + yz + zx = \frac{4}{\sqrt{3}}$. Adding the three equations given in the problem and subtracting both sides by $xy+yz+zx$, we obtain $x^2+y^2+z^2 = 5 - \frac{2}{\sqrt{3}}$. Therefore $(x+y+z)^2 = (x^2+y^2+z^2) + 2(xy+yz+zx) = 5 + 2\sqrt{3}$.

18. [**10**] In how many ways can each square of a 4×2011 grid be colored red, blue, or yellow such that no two squares that are diagonally adjacent are the same color?

    Answer: $\boxed{64 \cdot 3^{4020}}$

    If we first color the board in a checkerboard pattern, it is clear that the white squares are independent of the black squares in diagonal coloring, so we calculate the number of ways to color the white squares of a $4 \times n$ board and then square it.

    Let a_n be the number of ways to color the white squares of a $4 \times n$ board in this manner such that the two squares in the last column are the same color, and b_n the number of ways to color it such that they are different. We want to find their sum x_n. We have $a_1 = 3$, $b_1 = 6$. Given any filled $4 \times n - 1$ grid with the two white squares in the last column different, there is only 1 choice for the middle square in the nth row, and two choices for the outside square, 1 choice makes them the same color, 1 makes them different. If the two white squares are the same, there are 2 choices for the middle square and the outer square, so 4 choices. Of these, in 2 choices, the two new squares are the same color, and in the other 2, the two squares are different. It follows that $a_n = 2a_{n-1} + b_{n-1}$ and $b_n = 2a_{n-1} + b_{n-1}$, so $a_n = b_n$ for $n \geq 2$. We have $x_n = 8 \cdot 3^{n-1}$ and $x_{2011} = 8 \cdot 3^{2010}$. So the answer is $64 \cdot 3^{4020}$.

19. [**12**] Find the least positive integer N with the following property: If all lattice points in $[1,3] \times [1,7] \times [1,N]$ are colored either black or white, then there exists a rectangular prism, whose faces are parallel to the xy, xz, and yz planes, and whose eight vertices are all colored in the same color.

    Answer: $\boxed{127}$

First we claim that if the lattice points in $[1,3] \times [1,7]$ are colored either black or white, then there exists a rectangle whose faces are parallel to the x and y axes, whose vertices are all the same color (a.k.a. monochromatic). Indeed, in every row $y = i$, $1 \le i \le 7$, there are two lattice points with the same color. Note there are 3 combinations of 2 columns to choose from (for the two similarly-colored lattice points to be in), and 2 colors to choose from. By the Pigeonhole Principle, in the $2 \cdot 3 + 1 = 7$ rows two rows must have a pair of similarly-colored lattice points in the same columns, i.e. there is a monochromatic rectangle.

This shows that in each cross section $z = i$, $1 \le i \le N$ there is a monochromatic rectangle. Next, note there are $\binom{3}{2}\binom{7}{2}$ possibilities for this rectangle ($\binom{3}{2}$ ways to choose the 2 x-coordinates and $\binom{7}{2}$ ways to choose the 2 y-coordinates), and 2 possible colors. Thus if $N = 2\binom{3}{2}\binom{7}{2} + 1 = 127$, then by the Pigeonhole Principle there are two values of i such that the same-colored rectangle has the same x and y coordinates in the plane $z = i$, i.e. there is a monochromatic rectangular prism.

For $N = 126$ the assertion is not true. In each cross section $z = i$, we can color so that there is exactly 1 monochromatic rectangle, and in the 126 cross sections, have all 126 possible monochromatic rectangles represented. To do this, in each cross section we color so that each row has exactly 2 lattice points of the same color, and such that 6 of the rows give all possible combinations of 2 points having the same color. This way, there will be exactly 1 monochromatic rectangle in each cross section; we can obviously vary it for the different cross sections.

20. [**12**] Let $ABCD$ be a quadrilateral circumscribed about a circle with center O. Let O_1, O_2, O_3, and O_4 denote the circumcenters of $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle DOA$. If $\angle A = 120°, \angle B = 80°$, and $\angle C = 45°$, what is the acute angle formed by the two lines passing through O_1O_3 and O_2O_4?

Answer: $\boxed{82.5}$

Lemma: Given a triangle $\triangle ABC$, let I be the incenter, I_A be the excenter opposite A, and $\check{S}$ be the second intersection of AI with the circumcircle. Then $\check{S}$ is the center of the circle through B, I, C, and I_A.

Proof. First, note

$$\angle IBI_A = \angle IBC + \angle CBI_A = \frac{\angle ABC}{2} + \frac{180° - \angle ABC}{2} = 90°.$$

Similarly $\angle ICI_A = 90°$. Therefore $BICI_A$ is cyclic. Now note that A, I, $\check{S}$, and I_A are collinear because they are all on the angle bisector of $\angle BAC$. Hence

$$\angle CI\check{S} = 180° - \angle CIA = \angle CAI + \angle ACI = \angle BC\check{S} + \angle ICB = \angle IC\check{S}.$$

(Note $\angle CAI = \angle BA\check{S} = \angle BC\check{S}$ since A, B, $\check{S}$, and C are concyclic.) Hence $\check{S}C = \check{S}I$. Similarly $\check{S}B = \check{S}I$. Thus $\check{S}$ is the center of the circle passing through B, I, and C, and therefore I_A as well. □

Let BA and CD intersect at E and DA and CB intersect at F. We first show that F, O_1, O, and O_3 are collinear.

Let O_1' and O_3' denote the intersections of FO with the circumcircles of triangles FAB and FDC. Since O is the excenter of triangle FAB, by the lemma O_1' is the circumcenter of $\triangle ABO$; since O is incenter of triangle FDC, by the lemma O_3' is the circumcenter of $\triangle DOC$. Hence $O_1' = O_1$ and $O_3' = O_3$. Thus, points F, O_1, O, and O_3 are collinear, and similarly, we have E, O_2, O, and O_3 are collinear. Now $\angle BEC = 55°$ and $\angle DFC = 20°$ so considering quadrilateral $EOFC$, the angle between O_1O_3 and O_2O_4 is

$$\begin{aligned} \angle EOF &= \angle OEC + \angle OFC + \angle FCE \\ &= \frac{\angle BEC}{2} + \frac{\angle DFC}{2} + \angle FCE \\ &= 27.5° + 10° + 45° = 82.5°. \end{aligned}$$

21. [**12**] Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Let P denote the intersection of AC and BD. Let M and N denote the midpoints of AD and BC. If $AP = 1$, $BP = 3$, $DP = \sqrt{3}$, and AC is perpendicular to BD, find the area of triangle MON.

Answer: $\boxed{\frac{3}{4}}$

We first prove that $ONPM$ is a parallelogram. Note that APD and BPC are both $30° - 60° - 90°$ triangles. Let M' denote the intersection of MP and BC. Since $\angle BPM' = \angle MPD = 30°$, we have $MP \perp BC$. Since ON is the perpendicular bisector of BC, we have $MP//NO$. Similarly, we have $MO//NP$. Thus $ONPM$ is a parallelogram. It follows that the area of triangle MON is equal to the area of triangle MPN, which is equal to $\frac{1}{2} \cdot 1 \cdot 3 \cdot \sin \angle MPN = \frac{1}{2} \cdot 1 \cdot 3 \cdot \sin 150° = \frac{3}{4}$.

22. [**12**] Find the number of ordered triples (a, b, c) of pairwise distinct integers such that $-31 \le a, b, c \le 31$ and $a + b + c > 0$.

Answer: $\boxed{117690}$

We will find the number of such triples with $a < b < c$. The answer to the original problem will then be six times what we will get. By symmetry, the number of triples (a, b, c) with $a + b + c > 0$ is equal to the number of those with $a + b + c < 0$. Our main step is thus to find the number of triples with sum 0.

If $b = 0$, then $a = -c$, and there are 31 such triples. We will count the number of such triples with $b > 0$ since the number of those with $b < 0$ will be equal by symmetry.

For all positive n such that $1 \le n \le 15$, if $a = -2n$, there are $n - 1$ pairs (b, c) such that $a + b + c = 0$ and $b > 0$, and for all positive n such that $1 \le n \le 16$, if $a = -2n + 1$, there are also $n - 1$ such pairs (b, c). In total, we have $1 + 1 + 2 + 2 + 3 + 3 + ... + 14 + 14 + 15 = 225$ triples in the case $b > 0$ (and hence likewise for $b < 0$.)

In total, there are $31 + 225 + 225 = 481$ triples such that $a < b < c$ and $a + b + c = 0$. Since there are $\binom{63}{3} = 39711$ triples (a, b, c) such that $-31 \le a < b < c \le 31$, the number of triples with the additional restriction that $a + b + c > 0$ is $\frac{39711-481}{2} = 19615$. So the answer to the original problem is $19615 \times 6 = 117690$.

23. [**14**] Let S be the set of points (x, y, z) in $\mathbb{R}^3$ such that x, y, and z are positive integers less than or equal to 100. Let f be a bijective map between S and the $\{1, 2, \ldots, 1000000\}$ that satisfies the following property: if $x_1 \le x_2$, $y_1 \le y_2$, and $z_1 \le z_2$, then $f(x_1, y_1, z_1) \le f(x_2, y_2, z_2)$. Define

$$A_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(i, j, k),$$

$$B_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, i, k),$$

$$\text{and } C_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, k, i).$$

Determine the minimum value of $A_{i+1} - A_i + B_{j+1} - B_j + C_{k+1} - C_k$.

Answer: $\boxed{30604}$

We examine the 6 planes, their intersections and the lines between 2 points in one of the three pairs of parallel planes. The expression is equivalent to summing differences in values along all these lines. We examine the planes intersections. There is one cube, $3 \cdot 98$ squares and $3 \cdot 98 \cdot 98$ lines. The minimum value of the difference along a line is 1. For a square, to minimize the differences we take four consecutive numbers, and the minimum value is 6. To find the minimum value along a cube, we take 8 consecutive numbers. Since we are taking differences, we can add or subtract any constant to the numbers, so we assume the numbers are 1-8. Examining the cube, we see there's 1 spot where the number is multiplied by -3, 3 spots where the number is multiplied by -1, 1 spot where the number is multiplied by 3, and 3 spots where the number is multiplied by 1. 1 and 8 must go in the corners, and 2,3,4,5 must go in spots multiplied by -1, -1, 1,1, respectively. To minimize the differences we put 5 in the final spot multiplied by -1, and 4 in the spot multiplied by 1 opposite 5. Then the sum of all the differences is 28, so the minimum for a cube is 28. So the answer is $28+18(100-2)+3(100-2)^2 = 30000+600+12-36+28 = 30604$. It is clear that this value can be obtained.

24. [**14**] In how many ways may thirteen beads be placed on a circular necklace if each bead is either blue or yellow and no two yellow beads may be placed in adjacent positions? (Beads of the same color are considered to be identical, and two arrangements are considered to be the same if and only if each can be obtained from the other by rotation).

Answer: $\boxed{41}$

Let t_n be the number of arrangements of n beads in a row such that bead i and $i+1$ are not both yellow for $1 \le i < n$. Let a_n and b_n be the number of arrangements satisfying the additional condition that beads n and 1 are not both yellow, and that beads n and 1 are both yellow, respectively. Clearly $t_n = a_n + b_n$.

First consider t_n. If bead n is blue, there are t_{n-1} ways to choose the remaining colors. If bead n is yellow, then bead $n-1$ must be blue, and there are t_{n-2} ways to choose the remaining colors. Hence

$$t_n = t_{n-1} + t_{n-2}.$$

Next consider b_n. In this case beads 1 and n are both yellow, so beads 2 and $n-1$ must be blue; the remaining colors can be chosen in t_{n-4} ways. Hence

$$b_n = t_{n-4}.$$

From $t_0 = 1$ and $t_1 = 2$ we see that $t_n = F_{n+2}$ and $b_n = F_{n-2}$ and so $a_n = F_{n+2} - F_{n-2}$, where F_n denotes the nth Fibonacci number.

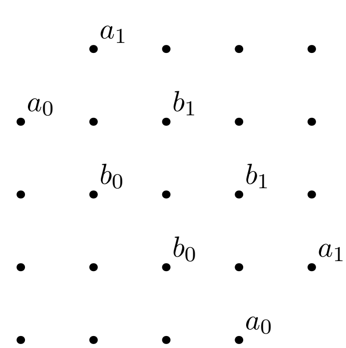
Next, since 13 is prime, a circular arrangement corresponds to exactly 13 straight-line arrangements, except for when all beads are the same color. The all-blue's arrangement satisfies our condition, while the all-yellow does not. Hence our answer is

$$\frac{a_{13}-1}{13} + 1 = \frac{F_{15} - F_{11} + 12}{13} = \frac{610 - 89 + 12}{13} = 41.$$

25. [**14**] Let n be an integer greater than 3. Let R be the set of lattice points (x, y) such that $0 \le x, y \le n$ and $|x - y| \le 3$. Let A_n be the number of paths from $(0,0)$ to (n,n) that consist only of steps of the form $(x,y) \to (x, y+1)$ and $(x,y) \to (x+1, y)$ and are contained entirely within R. Find the smallest positive real number that is greater than $\frac{A_{n+1}}{A_n}$ for all n.

Answer: $\boxed{2+\sqrt{2}}$

We first find A_n in terms of n. Let a_n be the number of ways to get to the point $(n, n+3)$, and let b_n be the number of ways to get to the point $(n+1, n+2)$. By symmetry, a_n is also the number of ways to get to $(n+3, n)$ and b_n is also the number of ways to get to the point $(n+2, n+1)$.



We can easily see that $a_0 = 1$ and $b_0 = 3$. This also means that $A_n = a_{n-3} + 3b_{n-3} + 3b_{n-3} + a_{n-3} = 2a_{n-3} + 6b_{n-3}$.

We also get the recurrence:

$$\begin{aligned} a_{i+1} &= a_i + b_i \\ b_{i+1} &= a_i + 3b_i \end{aligned}$$

We have both $3a_{i+1} = 3a_i + 3b_i$ and $a_{i+2} = a_{i+1} + b_{i+1}$. Subtracting these gives

$$\begin{aligned} a_{i+2} - 3a_{i+1} &= a_{i+1} - 3a_i + b_{i+1} - 3b_i \\ a_{i+2} - 3a_{i+1} &= a_{i+1} - 3a_i + a_i \\ a_{i+2} &= 4a_{i+1} - 2a_i \end{aligned}$$

Now we can solve this recurrence using its characteristic polynomial $x^2 - 4x + 2$, which has roots of $2 + \sqrt{2}$ and $2 - \sqrt{2}$. We can then write $a_i = A(2+\sqrt{2})^i + B(2-\sqrt{2})^i$ for some constants A and B. Now, $a_0 = a$ and $a_1 = a_0 + b_0 = 4$. Using this, we solve for A and B to get

$$a_i = \left(\frac{1+\sqrt{2}}{2}\right)(2+\sqrt{2})^i + \left(\frac{1-\sqrt{2}}{2}\right)(2-\sqrt{2})^i$$

Then,

$$\begin{aligned} b_i &= a_{i+1} - a_i \\ &= \left(\frac{1+\sqrt{2}}{2}\right)((2+\sqrt{2})^{i+1} - (2+\sqrt{2})^i) + \left(\frac{1-\sqrt{2}}{2}\right)((2-\sqrt{2})^{i+1} - (2-\sqrt{2})^i) \\ &= \left(\frac{1+\sqrt{2}}{2}\right)(1+\sqrt{2})(2+\sqrt{2})^i + \left(\frac{1-\sqrt{2}}{2}\right)(1-\sqrt{2})(2-\sqrt{2})^i \\ &= \left(\frac{3+2\sqrt{2}}{2}\right)(2+\sqrt{2})^i + \left(\frac{3-2\sqrt{2}}{2}\right)(2-\sqrt{2})^i \end{aligned}$$

Therefore,

$$A_n = 2a_{n-3} + 6b_{n-3} = (10 + 7\sqrt{2})(2+\sqrt{2})^{n-3} + (10 - 7\sqrt{2})(2-\sqrt{2})^{n-3}.$$

We can then easily see that $A_n < (2+\sqrt{2})A_{n-1}$. Also, since $2 - \sqrt{2} < 1$, as n approaches infinity, the ratio $\frac{A_n}{A_{n-1}}$ approaches $2 + \sqrt{2}$. Hence the least upper bound of $\frac{A_n}{A_{n-1}}$ is $2 + \sqrt{2}$.

26. [**14**] In how many ways can 13 bishops be placed on an 8×8 chessboard such that (i) a bishop is placed on the second square in the second row, (ii) at most one bishop is placed on each square, (iii) no bishop is placed on the same diagonal as another bishop, and (iv) every diagonal contains a bishop? (For the purposes of this problem, consider all diagonals of the chessboard to be diagonals, not just the main diagonals).

Answer: $\boxed{1152}$

We color the squares of the chessboard white and black such that B2 (the second square in the second row) is black. Note that at most 7 bishops can go on the white squares, and if there is a bishop on b2, at most 5 more can be on the white squares. So of the other 12 bishops, 7 go on white squares and 5 go on black squares.

Consider the long diagonal on the white squares, and the 6 white diagonals parallel to it. Of the 7 bishops placed on the white squares, exactly one must go on each of these diagonals (this also proves that at most 7 can go on the white squares). Of these diagonals there is 1 of length 8, and 2 of length 2,4, and 6. There are 2 ways to place 2 bishops on the diagonals of length 2, then 2 ways to place 2 bishops on the diagonals of length 4, then 2 ways to place 2 bishops on the diagonals of length 2, then the long diagonal bishop can go on either corner. So there are 16 ways to place 7 bishops on the white squares.

Now we can divide the black squares of the board into the 6 diagonals parallel to the long white diagonal, and the long black diagonal. The bishop on b2 accounts for two of these diagonals. We are left with a diagonal of length 3, and two diagonals of length 5,7. There are 3 ways to pick the bishop on the diagonal of length 3, 6 ways to pick two bishop for the diagonals of length 5, and 6 ways to pick the bishop on the diagonals of length 7. So there are 72 ways to pick 5 other bishops for the black squares. So the answer is $72 \cdot 16 = 1152$.

27. [**16**] Find the number of polynomials $p(x)$ with integer coefficients satisfying $p(x) \geq \min\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ and $p(x) \leq \max\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ for all $x \in \mathbb{R}$.

Answer: $\boxed{4}$

We first find the intersection points of $f(x) = 2x^4 - 6x^2 + 1$ and $g(x) = 4 - 5x^2$. If $2x^4 - 6x^2 + 1 = 4 - 5x^2$, then $2x^4 - x^2 - 3 = 0$, so $(2x^2 - 3)(x^2 + 1) = 0$, and $x = \pm\sqrt{\frac{3}{2}}$. Note that this also demonstrates that $g(x) \geq f(x)$ if and only if $|x| \leq \sqrt{\frac{3}{2}}$ and that $p(x)$ must satisfy $p(x) \leq g(x)$ iff $|x| \leq \sqrt{\frac{3}{2}}$. We must have $p(\sqrt{\frac{3}{2}}) = -\frac{7}{2}$, so $p(x) = -\frac{7}{2} + (2x^2 - 3)q(x)$ for a polnomial q of degree $0, 1$ or 2. We now examine cases.

Case 1: q is constant. We have $q = \frac{n}{2}$ for an integer n since $1 \leq p(0) \leq 4$, $n = -\frac{3}{2}$ or $-\frac{5}{2}$. Clearly $q = -\frac{5}{2}$ is an appropriate choice because then $p = g$. Let $p_1(x) = -\frac{7}{2} + -\frac{3}{2}(2x^2 - 3) = 1 - 3x^2$. We have $p_1(x)$ is $\leq g(x)$ and $\leq f(x)$ in the right places, so this function works. Thus, there are 2 solutions.

Case 2: q is linear. We have $p(x)$ is a cubic, so $p(x) - g(x)$ is also a cubic, which means it can't be positive for both arbitrarily large positive values of x and arbitrarily large negative values of x. Thus, there are no solutions.

Case 3: q is quadratic. We have $q(x) = ax^2 + bx + c$. Apply the same argument for the case when q is a constant, we have $c = -\frac{3}{2}$ or $-\frac{5}{2}$. Since p must have integer coefficients, we have b must be an integer. Since $f(x) \geq p(x) \geq g(x)$ for large x, the leading coefficient of f must be greater than or equal to the leading coefficient of p, which must be greater than 0. Thus $a = 1$ or $a = \frac{1}{2}$. However, if $a = \frac{1}{2}$, then the quadratic term of p is not an integer, so $a = 1$.

Now if $c = -\frac{5}{2}$, then $p(0) = 4 = g(0)$. But this is the maximum value of $g(x)$, so it must be a local maximum of p. Thus p must not have a linear term odd (otherwise the function behaves like $-3bx + 4$ around $x = 0$). So $p(x)$ must be $2x^4 - 8x^2 + 4$. This is indeed bounded between $f(x)$ and $g(x)$ at all points.

Now suppose $c = -\frac{3}{2}$. We have a must equal to 1. If $b \neq 0$, then p will have a cubic term, which means $f(x) - p(x)$ can't be positive for both arbitrarily large positive x and arbitrarily large negative x, so $b = 0$. Therefore $p(x) = -\frac{7}{2} + (2x^2 - 3)(x^2 - \frac{3}{2}) = f(x)$. It is easy to check that this choice of p is indeed bounded between $f(x)$ and $g(x)$.

Therefore, there are 4 solutions in total.

28. [**16**] Let ABC be a triangle, and let points P and Q lie on BC such that P is closer to B than Q is. Suppose that the radii of the incircles of triangles ABP, APQ, and AQC are all equal to 1, and that the radii of the corresponding excircles opposite A are 3, 6, and 5, respectively. If the radius of the incircle of triangle ABC is $\frac{3}{2}$, find the radius of the excircle of triangle ABC opposite A.

Answer: $\boxed{135}$

Let t denote the radius of the excircle of triangle $\triangle ABC$.

Lemma: Let $\triangle ABC$ be a triangle, and let r and r_A be the inradius and exradius opposite A. Then

$$\frac{r}{r_A} = \tan\frac{B}{2}\tan\frac{C}{2}.$$

Proof. Let I and J denote the incenter and the excenter with respect to A. Let D and E be the foot of the perpendicular from I and J to BC, respectively. Then

$$\begin{aligned} r = ID &= BI \sin \frac{B}{2} \\ r_A = JE &= BJ \sin \frac{180^\circ - B}{2} = BJ \cos \frac{B}{2} \\ BI &= BJ \tan \angle AJB = BY \tan \frac{C}{2}. \end{aligned}$$

The last equation followed from

$$\angle AJB = 180^\circ - \angle ABJ - \angle JAB = \frac{180^\circ - B}{2} - \frac{A}{2} = \frac{C}{2}.$$

Hence

$$\frac{r}{r_A} = \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} \cdot \frac{BI}{BJ} = \tan \frac{B}{2} \cdot \tan \frac{C}{2}.$$

□

Noting $\tan \frac{\angle APB}{2} \tan \frac{\angle APQ}{2} = \tan \frac{\angle AQP}{2} \tan \frac{\angle AQC}{2} = 1$ and applying the lemma to $\triangle ABC$, $\triangle ABP$, $\triangle APQ$, and $\triangle AQC$ give

$$\begin{aligned} \frac{3/2}{t} &= \tan \frac{\angle ABC}{2} \cdot \tan \frac{\angle ACB}{2} \\ &= \left(\tan \frac{\angle ABC}{2} \cdot \tan \frac{\angle APB}{2} \right) \cdot \left(\tan \frac{\angle APQ}{2} \cdot \tan \frac{\angle AQP}{2} \right) \cdot \left(\tan \frac{\angle AQC}{2} \cdot \tan \frac{\angle ACB}{2} \right) \\ &= \frac{1}{3} \cdot \frac{1}{6} \cdot \frac{1}{5} \end{aligned}$$

Therefore, $t = 135$.

29. [**16**] Let ABC be a triangle such that $AB = AC = 182$ and $BC = 140$. Let X_1 lie on AC such that $CX_1 = 130$. Let the line through X_1 perpendicular to BX_1 at X_1 meet AB at X_2. Define $X_2, X_3, \ldots$, as follows: for n odd and $n \geq 1$, let X_{n+1} be the intersection of AB with the perpendicular to $X_{n-1}X_n$ through X_n; for n even and $n \geq 2$, let X_{n+1} be the intersection of AC with the perpendicular to $X_{n-1}X_n$ through X_n. Find $BX_1 + X_1X_2 + X_2X_3 + \ldots$.

Answer: $\boxed{\frac{1106}{5}}$

Let M and N denote the perpendiculars from X_1 and A to BC, respectively. Since triangle ABC is isosceles, we have M is the midpoint of BC. Moreover, since AM is parallel to X_1N, we have $\frac{NC}{X_1C} = \frac{MC}{AC} \Leftrightarrow \frac{X_1N}{130} = \frac{70}{182} = \frac{5}{13}$, so $NC = 50$. Moreover, since $X_1N \perp BC$, we find $X_1C = 120$ by the Pythagorean Theorem. Also, $BN = BC - NC = 140 - 50 = 90$, so by the Pythagorean Theorem, $X_1B = 150$.

We want to compute $X_2X_1 = X_1B \tan(\angle ABX_1)$. We have

$$\tan(\angle ABX_1) = \tan(\angle ABC - \angle X_1BC) = \frac{1 + \tan(\angle ABC)\tan(\angle X_1BC)}{\tan(\angle ABC) - \tan(\angle X_1BC)} = \frac{\left(\frac{12}{5}\right) - \left(\frac{4}{3}\right)}{1 + \left(\frac{12}{5}\right)\left(\frac{4}{3}\right)}$$

$$= \frac{\frac{16}{15}}{\frac{63}{15}} = \frac{16}{63}.$$

Hence $X_2X_1 = 150 \cdot \frac{16}{63}$, and by the Pythagorean Theorem again, $X_2B = 150 \cdot \frac{65}{63}$.

Next, notice that $\frac{AX_n}{AX_{n+2}}$ is constant for every nonnegative integer n (where we let $B = X_0$). Indeed, since X_nX_{n+1} is parallel to $X_{n+2}X_{n+3}$ for each n, the dilation taking X_n to X_{n+2} for some n also takes X_k to X_{k+2} for all k.

Since $\triangle AX_{n+2}X_{n+3} \sim \triangle AX_nX_{n+1}$ with ratio $\frac{AX_n}{AX_{n+2}}$ for each even n, we can compute that $\frac{X_{n+2}X_{n+3}}{X_nX_{n+1}} = \frac{AX_{n+2}}{AX_n} = 1 - \frac{150 \cdot \frac{65}{63}}{182}$ for every nonnegative integer n. Notice we use all three sides of the above similar triangles.

We now split our desired sum into two geometric series, one with the even terms and one with the odd terms, to obtain

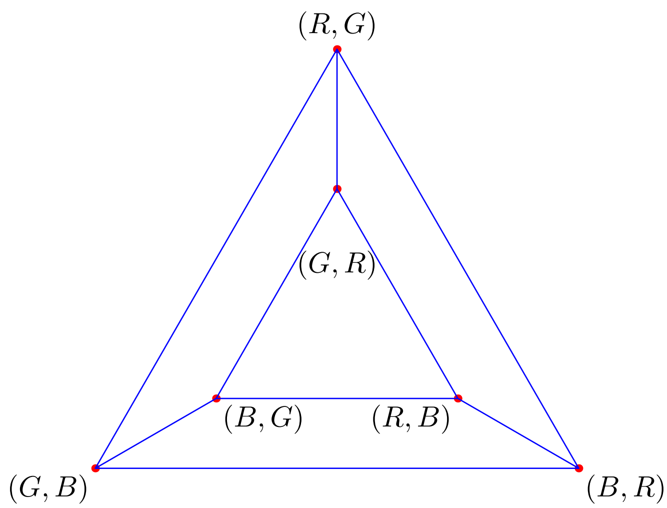
$$BX_1 + X_1X_2 + \ldots = (BX_1 + X_2X_3 + \ldots) + (X_1X_2 + X_3X_4 + \ldots) = \frac{150}{\frac{150 \cdot \frac{65}{63}}{182}} + \frac{150 \cdot \frac{16}{63}}{\frac{150 \cdot \frac{65}{63}}{182}}$$

$$= \frac{\frac{79}{63} \cdot 150}{\frac{150 \cdot \frac{65}{63}}{182}} = \frac{1106}{5}.$$

30. [**16**] How many ways are there to color the vertices of a $2n$-gon with three colors such that no vertex has the same color as its either of its two neighbors or the vertex directly across from it?

Answer: $\boxed{3^n + (-2)^{n+1} - 1}$

Let the $2n$-gon have vertices $A_1, A_2, ..., A_{2n}$, in that order. Consider the diagonals $d_1 = (A_1, A_{n+1})$, $d_2 = (A_2, A_{n+2})$, $\cdots$, $d_n = (A_n, A_{2n})$. Suppose the three colors are red (R), green (G), and blue (B). Each diagonal can either be colored (R, G), (G, R), (G, B), (B, G), (B, R), or (R, B). We first choose one of the six colorings for d_1, which then constrains the possible colorings for d_2, which constrains the possible colorings for d_3, and so on. This graph shows the possible configurations; two pairs of colors are connected by an edge if they can be the colors for d_i and d_{i+1} for any $1 \le i \le n-1$.

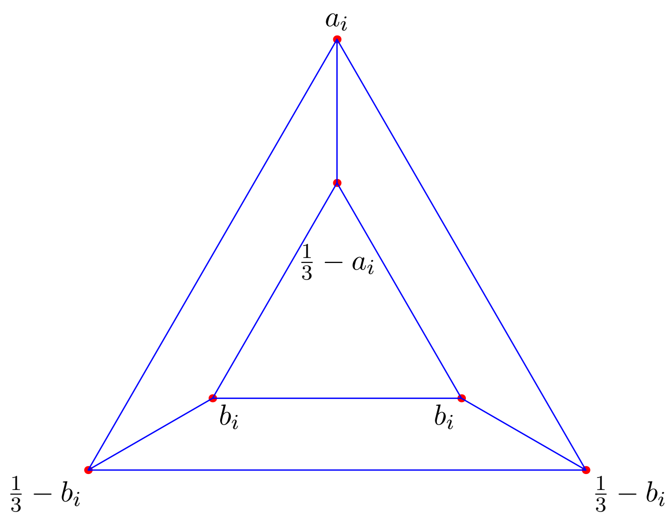


Suppose without loss of generality that d_1 is colored (R, G). (At the end, we multiply our answer by 6.) Then d_n must be either (R, G), (B, G), or (R, B). Now, we simply need to count the number of paths of length $n - 1$ within this graph from (R, G) to one of these three points.

Suppose we are making a random walk of $n - 1$ steps, where at each move we pick one of the three possible edges with probability $\frac{1}{3}$. We will calculate the probability that the walk ends at one of (R, G), (B, G), or (R, B).

Let a_i and b_i be the probability that, after i steps, we are at (R, G) and (B, G), respectively. By symmetry, b_i is also the probability that we are at (R, B) after i steps.

Observe that after each move, the probability of arriving at either (R, G) or (G, R) will always be $\frac{1}{3}$. Therefore, the probability of being at (G, R) after i steps is $\frac{1}{3} - a_i$. Similarly, the probability of being at (G, B) is $\frac{1}{3} - b_i$ and the probability of being at (B, R) is $\frac{1}{3} - b_i$.



Now, for $i \geq 1$ we have the recurrences

$$\begin{aligned} a_{i+1} &= \frac{1}{3}\left(\left(\frac{1}{3} - a_i\right) + \left(\frac{1}{3} - b_i\right) + \left(\frac{1}{3} - b_i\right)\right) \\ &= \frac{1}{3} - \frac{1}{3}a_i - \frac{2}{3}b_i \end{aligned}$$

$$\begin{aligned} b_{i+1} &= \frac{1}{3}\left(\left(\frac{1}{3} - a_i\right) + \left(\frac{1}{3} - b_i\right) + b_i\right) \\ &= \frac{2}{9} - \frac{1}{3}a_i \end{aligned}$$

So then

$$\begin{aligned} a_{i+2} &= \frac{1}{3} - \frac{1}{3}a_{i+1} - \frac{2}{3}b_{i+1} \\ a_{i+2} &= \frac{1}{3} - \frac{1}{3}a_{i+1} - \frac{2}{3}\left(\frac{2}{9} - \frac{1}{3}a_i\right) \\ a_{i+2} &= \frac{5}{27} - \frac{1}{3}a_{i+1} + \frac{2}{9}a_i \\ \left(a_{i+2} - \frac{1}{6}\right) &= -\frac{1}{3}\left(a_{i+1} - \frac{1}{6}\right) + \frac{2}{9}\left(a_i - \frac{1}{6}\right) \end{aligned}$$

This recurrence has a characteristic polynomial $x^2 + \frac{1}{3}x - \frac{2}{9}$, which has roots $\frac{1}{3}$ and $-\frac{2}{3}$. We can write $a_i = \frac{1}{6} + A(\frac{1}{3})^i + B(-\frac{2}{3})^i$ for some constants A and B for $i \geq 1$. Since $a_1 = 0$ and $a_2 = \frac{1}{3}$, we can solve for A and B and get

$$a_i = \frac{1}{6} + \frac{1}{6}\left(\frac{1}{3}\right)^i + \frac{1}{3}\left(-\frac{2}{3}\right)^i$$

The answer to the problem is then

$$\begin{aligned}
6\cdot 3^{n-1}(a_{n-1}+2b_{n-1}) &= 6\cdot 3^{n-1}\left(a_{n-1}+1-a_{n-1}-3a_n\right)\\
&= 6\cdot 3^{n-1}(1-3a_n)\\
&= 6\cdot 3^{n-1}\left(1-3\left(\frac{1}{6}+\frac{1}{6}\left(\frac{1}{3}\right)^n+\frac{1}{3}\left(-\frac{2}{3}\right)^n\right)\right)\\
&= 6\cdot 3^{n-1}\left(\frac{1}{2}-\frac{1}{2}\left(\frac{1}{3}\right)^n-\left(-\frac{2}{3}\right)^n\right)\\
&= 3^n+(-2)^{n+1}-1.
\end{aligned}$$

31. [**18**] Let $A=\{1,2,3,\dots,9\}$. Find the number of bijective functions $f:A\to A$ for which there exists at least one $i\in A$ such that

$$|f(i)-f^{-1}(i)|>1.$$

Answer: $\boxed{359108}$

We count the complement — the number of functions f such that for all $i\in A$, $|f(i)-f^{-1}(i)|\le 1$.

The condition is equivalent to $|f(f(i))-i|\le 1$ for all $i\in A$. If $f(j)=j$, the inequality is automatically satisfied for $i=j$. Otherwise, if $f(f(j))=j$ but $f(j)=k\neq j$, then we will have $f(f(k))=k$, allowing the inequality to be satisfied for $i=j,k$. Else, if $f(f(i))\neq i$, say $f(f(i))=i+1$ and $f(i)=k$, then $f(f(k))=f(i+1)=k+1$ or $k-1$. Thus the function f allows us to partition the elements of A into three groups:

(a) those such that $f(i)=i$,

(b) those that form pairs $\{i,j\}$ such that $f(i)=j$ and $f(j)=i$, and

(c) those that form quartets $\{i,i+1,j,j+1\}$ such that f permutes them as $(i\quad j\quad i+1\quad j+1)$ or $(i\quad j+1\quad i+1\quad j)$, in cycle notation.

Let a be the number of elements of the second type. Note that a is even.

Case 1: There are no elements of the third type. If $a=8$, there are $9\cdot 7\cdot 5\cdot 3=945$ possibilities. If $a=6$, there are $\binom{9}{3}\cdot 5\cdot 3=1260$ possibilities. If $a=4$, there are $\binom{9}{5}\cdot 3=378$ possibilities. If $a=2$, there are $\binom{9}{7}=36$ possibilities. If $a=0$, there is 1 possibility. In total, case 1 offers $945+1260+378+36+1=2620$ possibilities.

Case 2: There are 4 elements of the third type. There are 21 ways to choose the quartet $\{i,i+1,j,j+1\}$. For each way, there are two ways to assign the values of the function to each element (as described above). For the remaining 5 elements, we divide into cases according to the value of a. If $a=4$, there are $5\times 3=15$ possibilities. If $a=2$, there are $\binom{5}{3}=10$ possibilities. If $a=0$, there is one possibility. In total, case 2 offers $21\times 2\times(15+10+1)=1092$ possibilities.

Case 3: There are 8 elements of the third type. There are 5 ways to choose the unique element not of the third type. Of the remaining eight, there are 3 ways to divide them into two quartets, and for each quartet, there are 2 ways to assign values of f. In total, case 3 offers $5\times 3\times 2^2=60$ possibilities.

Therefore, the number of functions $f:A\to A$ such that for at least one $i\in A$, $|f(i)-f^{-1}(i)|>1$ is $9!-2620-1092-60=359108$.

32. [**18**] Let p be a prime positive integer. Define a mod-p recurrence of degree n to be a sequence $\{a_k\}_{k\ge 0}$ of numbers modulo p satisfying a relation of the form $a_{i+n}=c_{n-1}a_{i+n-1}+\dots+c_1a_{i+1}+c_0a_i$ for all $i\ge 0$, where $c_0,c_1,\dots,c_{n-1}$ are integers and $c_0\not\equiv 0 \pmod{p}$. Compute the number of distinct linear recurrences of degree at most n in terms of p and n.

Answer: $\boxed{1-n\frac{p-1}{p+1}+\frac{p^2(p^{2n}-1)}{(p+1)^2}}$

In the solution all polynomials are taken modulo p. Call a polynomial *nice* if it is monic with nonzero constant coefficient. We can associate each recurrence relation with a polynomial: associate

$$c_n a_{i+n} + c_{n-1} a_{i+n-1} + \dots + c_1 a_{i+1} + c_0 a_i = 0$$

with

$$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0.$$

Let D_i be the set of mod-p recurrences $\{a_k\}_{k\geq 0}$ where i is the least integer so that $\{a_k\}_{k\geq 0}$ has degree i, and let $d_i = |D_i|$.

Let S_n be the set of pairs $(\{a_k\}_{k\geq 0}, P)$ where $\{a_k\}_{k\geq 0}$ is a mod-p recurrence, and P is a nice polynomial associated to a recurrence relation of degree at most n satisfied by $\{a_k\}_{k\geq 0}$. To find d_n generally, we count the number of elements in S_n in two ways.

One the one hand, for each sequence $\{a_k\}_{k\geq 0}$ in D_i, there exist p^{n-i} polynomials P such that $(\{a_k\}_{k\geq 0}, P) \in S$. Indeed, $\{a_k\}_{k\geq 0}$ satisfies any recurrence relation associated with a polynomial multiple of P. When $j = i$ there is just one nice degree j polynomial that is a multiple of P, P itself. For $j > i$, there are $(p-1)p^{j-i-1}$ nice polynomials of degree j that are multiples of P, namely QP where Q is a nice polynomial of degree $j-i$. (There are p choices for the coefficients of $x, \dots, x^{j-i-1}$ and $p-1$ choices for the constant term.) So the number of nice polynomial multiples of degree at most n is

$$1 + \sum_{j=i+1}^{n} (p-1)p^{j-i-1} = 1 + (p-1)\left(\frac{p^{n-i}-1}{p-1}\right) = p^{n-i}.$$

Hence

$$|S_n| = \sum_{i=0}^{n} d_i p^{n-i}. \tag{1}$$

On the other hand, given a monic polynomial P of degree i, there are p^i recurrences $\{a_k\}_{k\geq 0}$ such that $(\{a_k\}_{k\geq 0}, P) \in S$, since $a_0, \dots, a_{i-1}$ can be chosen arbitrarily and the rest of the terms are determined. Since there are $(p-1)p^{i-1}$ nice polynomials of degree $i \neq 0$ (and 1 nice polynomial for $i = 0$), summing over i gives

$$|S_n| = 1 + \sum_{i=1}^{n} (p-1)p^{2i-1} \tag{2}$$

Now clearly $d_0 = 1$. Setting (1) and (2) equal for n and $n+1$ give

$$\sum_{i=0}^{n+1} d_i p^{n+1-i} = 1 + (p-1)\sum_{i=1}^{n+1} p^{2i-1} \tag{3}$$

$$\sum_{i=0}^{n} d_i p^{n-i} = 1 + (p-1)\sum_{i=1}^{n} p^{2i-1}$$

$$\implies \sum_{i=0}^{n} d_i p^{n+1-i} = p + (p-1)\sum_{i=1}^{n} p^{2i}. \tag{4}$$

Subtracting (4) from (3) yields:

$$\begin{aligned}
d_{n+1} &= 1 - p + (p-1)\sum_{i=1}^{2n+1}(-1)^{i+1}p^i \\
&= (p-1)\sum_{i=0}^{2n+1}(-1)^{i+1}p^i \\
&= (p-1)^2\sum_{i=0}^{n}p^{2m} \\
&= (p-1)^2\left(\frac{p^{2n+2}-1}{p^2-1}\right) \\
&= \frac{(p-1)(p^{2n+2}-1)}{p+1}
\end{aligned}$$

Thus the answer is

$$\begin{aligned}
\sum_{i=0}^{n} d_i &= 1 + \frac{p-1}{p+1}\sum_{i=1}^{n}(p^{2i}-1) \\
&= 1 + \frac{p-1}{p+1}\left(-n + p^2\cdot\frac{p^{2n}-1}{p^2-1}\right) \\
&= 1 - n\frac{p-1}{p+1} + \frac{p^2(p^{2n}-1)}{(p+1)^2}
\end{aligned}$$

33. [**25**] Find the number of sequences consisting of 100 R's and 2011 S's that satisfy the property that among the first k letters, the number of S's is strictly more than 20 times the number of R's for all $1 \le k \le 2111$.

Answer: $\boxed{\frac{11}{2111}\binom{2111}{100}}$

Given positive integers r and s such that $s \ge 20r$, let $N(s,r)$ denote the number of sequences of s copies of S and r copies of R such that for all $1 \le k \le r+s-1$, among the first k letters, the number of S's is strictly more than 20 times the number of R's. We claim that

$$N(s,r) = \frac{s-20r}{s+r}\binom{s+r}{s}.$$

We induct on $s+r$. For the base case, note that $N(20r,r) = 0$ for all r and $N(s,0) = 1$ for all s.

Assume the formula holds up to $r+s-1$. We now construct a path in the Cartesian plane as follows: Let R represent moving one unit to the left and S represent moving one unit up. This induces a bijection between sequences of S's and Rs and lattice paths from the origin to the point (r,s) that are above the line $y = 20x$. In order to reach the point (r,s), we must go through either $(r-1,s)$ or $(r,s-1)$. Therefore, we have $N(r,s) = N(s,r-1) + N(s-1,r) = \frac{s-20(r-1)}{s+r-1}\binom{s+r-1}{s} + \frac{s-1-20r}{s+r-1}\binom{s+r-1}{s} = \frac{s-20r}{s+r}\binom{s+r}{s}$.

Remark This is a special case of the ballot problem, first studied in 1887 by Joseph Bertrand, generalizing the Catalan numbers. A good expository article on this problem is "Four Proofs of the Ballot Problem" by Marc Renault available on his website at
`http://webspace.ship.edu/msrenault/ballotproblem/`.

In 2009, Yufei Zhao studied a variant problem called bidirectional ballot sequences, which he used to construct More-Sums-Then-Differences sets in additive combinatorics. His paper is available on his website at
`http://web.mit.edu/yufeiz/www/mstd_construction.pdf`.

34. [**25**] Let $w = w_1, w_2, \dots, w_6$ be a permutation of the integers $\{1, 2, \dots, 6\}$. If there do not exist indices $i < j < k$ such that $w_i < w_j < w_k$ or indices $i < j < k < l$ such that $w_i > w_j > w_k > w_l$, then w is said to be *exquisite*. Find the number of exquisite permutations.

    Answer: $\boxed{25}$

    Given a permutation $w = w_1, \dots, w_n$ for some n, call a sequence $w_{i_1}, w_{i_2}, \dots, w_{i_m}$ an increasing subsequence if $i_1 < \cdots < i_m$ and $w_{i_1} < \cdots < w_{i_m}$. Define decreasing subsequences similarly. Let $is(w)$ denote the length of the longest increasing sequence and $ds(w)$ denote the length of the longest decreasing sequence. We wish to find the number of permutations for $n = 6$ such that $is(w) \leq 2$ and $ds(w) \leq 3$. We note here that $6 = 2 \times 3$ is not a coincidence.

    Erdos and Szekeres first studied problems on the longest increasing and decreasing subsequences. In 1935, they showed that for any permutation w of $\{1, 2, \dots, pq + 1\}$, either $is(w) > p$ or $ds(w) > q$, which later appeared on the Russian Math Olympiad.

    In 1961, Schensted proved that the bound $pq + 1$ is sharp, and he enumerated the number of permutations for $n = pq$ such that $is(w) \leq p$ and $ds(w) \leq q$ (exquisite permutations for simplicity), with an elegant combinatorial proof based on the RSK-algorithm relating Young Tableux and permutations.

    The main idea of his proof is as follows. Consider a $p \times q$ rectangle. A Young Tableau is an assignment of $1, 2, \dots, pq$, one to each unit square of the rectangle, such that every row and column is in increasing order. There is a bijection between set of exquisite permutations and pairs of Young Tableaux. Since the number of ways to write $1, 2, \dots, 6$ on a 2×3 rectangle with every row and column in increasing order is 5, there are exactly 25 exquisite permutations.

    For a thorough exposition of increasing and decreasing subsequences and a collection of interesting open questions, see Richard Stanley's note for his undergraduate research students at MIT, "Increasing and decreasing subsequences and their variants" available at
    `http://www.math.mit.edu/~rstan/papers/ids.pdf`.

35. [**25**] An *independent set* of a graph G is a set of vertices of G such that no two vertices among these are connected by an edge. If G has 2000 vertices, and each vertex has degree 10, find the maximum possible number of independent sets that G can have.

    Answer: $\boxed{2047^{100}}$

    The upper bound is obtained when G is a disjoint union of bipartite graphs, each of which has 20 vertices with 10 in each group such that every pair of vertices not in the same group are connected.

    In 1991, during his study of the Cameron—Erdos conjecture on the number of sum-free sets, Noga Alon came across this problem on independent sets and conjectured that our construction gives the best bound.

    This problem received considerable attention due to its application to combinatorial group theory and statistical mechanics, but no solution was found until 2009, when Yufei Zhao resolved Alon's conjecture with a beautiful and elementary approach.

    For the reader to enjoy the full insight of Yufei's argument, we omit the proof here and refer to his paper "The Number of Independent Sets in a Regular Graph" available on his website at
    `http://web.mit.edu/yufeiz/www/indep_reg.pdf`.

36. [**25**] An *ordering* of a set of n elements is a bijective map between the set and $\{1, 2, \dots, n\}$. Call an ordering ρ of the 10 unordered pairs of distinct integers from the set $\{1, 2, 3, 4, 5\}$ admissible if, for any $1 \leq a < b < c \leq 5$, either $p(\{a, b\}) < p(\{a, c\}) < p(\{b, c\})$ or $p(\{b, c\}) < p(\{a, c\}) < p(\{a, b\})$. Find the total number of admissible orderings.

    Answer: $\boxed{768}$

    This problem is a special case of the *higher Bruhat order*, a class of combinatorial object widely studied for its connection to an assortment of mathematical areas such as algebraic geometry, algebraic combinatorics, and computational geometry.

An admissble order in our problem—the higher Bruhat order $B(5, 2)$—are best viewed as the set of *reduced decompositions* of the permutation 4321. Loosely speaking, a reduced decomposition is a sequence of adjacent transpositions that changes the permutation $n, n-1, \ldots, 1$ to $1, 2, \ldots, n$. For example, for $n = 4$, the sequence (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) induces the following reduced decomposition:

$$43\mathbf{21} \to 4\mathbf{31}2 \to \mathbf{41}32 \to 14\mathbf{32} \to 1\mathbf{42}3 \to 12\mathbf{43} \to 1234.$$

For each permutation above, switching the two bolded numbers yields the next permutation in the chain. For example, switching 3 and 1 in 4**31**2 yields 4132.

Readers interested in the connection between the higher Bruhat order and reduced decompositions are referred to Delong Meng's paper "Reduced decompositions and permutation patterns generalized to the higher Bruhat order" for background as well as recent development of this subject. The paper is available at
`http://web.mit.edu/delong13/www/papers.html`.

The number of reduced decompositions of $n, n-1, \ldots, 1$ is equal to the number of $(n-1)$st *standard Young Tableaux* of staircase shape, given by the formula

$$\frac{\binom{n}{2}!}{1^{n-1} \cdot 3^{n-2} \cdots (2n-3)}.$$

When $n = 5$, the formula gives 768.

Young Tableaux are one of the most important tools in algebraic combinatorics, especially for problems involving permutations, integer partitions, and posets. For a concise and accessible introduction to the Young Tableaux and reduced decompositions (and much more cool combinatorial stuff!), see Section 7 of Richard Stanley's note "A Combinatorial Miscellany" at
`http://math.mit.edu/~rstan/papers/comb.pdf`.

. . . .