12^{th} Annual Harvard-MIT Math Tournament

Saturday 21 February 2009

Solutions: Team Round - Division A

1. [**8**] Let $n \geq 3$ be a positive integer. A *triangulation* of a convex n-gon is a set of $n-3$ of its diagonals which do not intersect in the interior of the polygon. Along with the n sides, these diagonals separate the polygon into $n-2$ disjoint triangles. Any triangulation can be viewed as a graph: the vertices of the graph are the corners of the polygon, and the n sides and $n-3$ diagonals are the edges.

   For a fixed n-gon, different triangulations correspond to different graphs. Prove that all of these graphs have the same chromatic number.

   Solution: We will show that all triangulations have chromatic number 3, by induction on n. As a base case, if $n = 3$, a triangle has chromatic number 3. Now, given a triangulation of an n-gon for $n > 3$, every edge is either a side or a diagonal of the polygon. There are n sides and only $n-3$ diagonals in the edge-set, so the Pigeonhole Principle guarentees a triangle with two side edges. These two sides must be adjacent, so we can remove this triangle to leave a triangulation of an $(n-1)$-gon, which has chromatic number 3 by the inductive hypothesis. Adding the last triangle adds only one new vertex with two neighbors, so we can color this vertex with one of the three colors not used on its neighbors.

2. (a) [**6**] Let P be a graph with one vertex v_n for each positive integer n. If $a < b$, then an edge connects vertices v_a and v_b if and only if $\frac{b}{a}$ is a prime number. What is the chromatic number of P? Prove your answer.

   Answer: $\boxed{2}$

   Solution: At least two colors are needed in a good coloring of P. We show that two is sufficient. Write the positive integer n as $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, for distinct primes $p_1, p_2, \dots p_k$, and let $f(n) = e_1 + e_2 + \ldots + e_k$. Notice that if v_a and v_b are connected, then $f(a)$ and $f(b)$ have opposite parity. So, if we color v_n red if $f(n)$ is odd and blue otherwise, the two-coloring is good.

   (b) [**6**] Let T be a graph with one vertex v_n for every integer n. An edge connects v_a and v_b if $|a-b|$ is a power of two. What is the chromatic number of T? Prove your answer.

   Answer: $\boxed{3}$

   Solution: Since v_0, v_1, and v_2 are all connected to each other, three colors is necessary. Now, color v_n red if $n \equiv 0 \pmod 3$, blue if $n \equiv 1 \pmod 3$, and green otherwise. Since v_a and v_b are the same color only if $3|(a-b)$, no two connected vertices are the same color.

3. A graph is *finite* if it has a finite number of vertices.

   (a) [**6**] Let G be a finite graph in which every vertex has degree k. Prove that the chromatic number of G is at most $k+1$.

   Solution: We find a good coloring with $k+1$ colors. Order the vertices and color them one by one. Since each vertex has at most k neighbors, one of the $k+1$ colors has not been used on a neighbor, so there is always a good color for that vertex. In fact, we have shows that any graph in which every vertex has degree *at most* k can be colored with $k+1$ colors.

(b) [**10**] In terms of n, what is the minimum number of edges a finite graph with chromatic number n could have? Prove your answer.

Answer: $\boxed{\frac{n(n-1)}{2}}$

Solution: We prove this claim by induction - it holds for $n = 1$. Now assume the claim holds for n, and consider a graph of chromatic number $n + 1$. This graph must have at least one vertex of degree n, or else, by part a), it could be colored with only n colors.

Now, if we remove this vertex, the remaining graph must have chromatic number n or $n+1$ - if the chromatic number is $n-1$ or less, we can add the vertex back and give it a new color, creating a good coloring with only n colors. By the inductive hypothesis, the new graph has at least $\frac{n(n-1)}{2}$ edges, so the original graph had at least $\frac{n(n-1)}{2}+n = \frac{n(n+1)}{2}$ edges.

The complete graph on $n + 1$ vertices has exactly $\frac{n(n+1)}{2}$ edges, so the lower bound is tight and the inductive step is complete.

4. A *k-clique* of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent.

(a) [**4**] Find a graph with chromatic number 3 that does not contain any 3-cliques.

Solution: Consider a graph with 5 vertices arranged in a circle, with each vertex connected to its two neighbors. If only two colors are used, it is impossible to alternate colors to avoid using the same color on two adjacent vertices, so the chromatic number is 3.

(b) [**10**] Prove that, for all $n > 3$, there exists a graph with chromatic number n that does not contain any n-cliques.

Solution: We prove the claim by induction on n. The case $n = 3$ was adressed in (a). Let $n \geq 3$ and suppose G is a graph with chromatic number n containing no n-cliques. We produce a graph G' with chromatic number $n + 1$ containing no $(n + 1)$-cliques as follows. Add a vertex v to G, and add an edge from v to each vertex of G.

To see this graph has chromatic number $n + 1$, observe that any coloring of the vertices of G' restricts to a valid coloring of the vertices of G. So at least n distinct colors must be used among the vertices of G. In addition, another color must be used for v. By coloring v a new color, we have constructed a coloring of G' having $n + 1$ colors.

Lastly, any $(n+1)$-clique in G' must have at least n vertices in G which form an n-clique, which is impossible. Therefore, G' has no $(n + 1)$-cliques.

5. The *size* of a finite graph is the number of vertices in the graph.

(a) [**15**] Show that, for any $n > 2$, and any positive integer N, there are finite graphs with size at least N and with chromatic number n such that removing any vertex (and all its incident edges) from the graph decreases its chromatic number.

Solution: Let $k > 1$ be an odd number, and let G be a graph with k vertices arranged in a circle, with each vertex connected to its two neighbors. If $n = 3$, these graphs can be arbitrarily large, and are the graphs we need. If $n > 3$, let H be a complete graph on $n - 3$ vertices, and let J be the graph created by adding an edge from every vertex in G to every vertex in H. Then $n - 3$ colors are needed to color H and another 3 are needed to color G, so n colors is both necessary and sufficient for a good coloring of J. Now, say a vertex is removed from J. There are two cases:

If the vertex was removed from G, then the remaining vertices in G can be colored with 2 colors, because the cycle has been broken. A set of $n-3$ different colors can be used to color H, so only $n-1$ colors are needed to color the reduced graph. On the other hand, if the vertex was removed from H, then $n-4$ colors are used to color H and 3 used to color G. So removing any vertex decreases the chromatic number of J.

(b) [**15**] Show that, for any positive integers n and r, there exists a positive integer N such that for any finite graph having size at least N and chromatic number equal to n, it is possible to remove r vertices (and all their incident edges) in such a way that the remaining vertices form a graph with chromatic number at least $n-1$.

Solution: We claim that $N = nr$ is large enough. Take a graph with at least nr vertices and chromatic number n, and take a good n-coloring of the graph. Then by Pigeonhole, at least r of the vertices are the same color, which means that no pair of these r vertices is adjacent.

Remove this r vertices. If the resulting graph can be colored with only $n-2$ colors, then we can add the r vertices back in and color them with a new $(n-1)$st color, creating a good coloring of the graph with only $n-1$ colors. Since the original graph has chromatic number n, it must be impossible to color the smaller graph with $n-2$ colors, so we have removed r vertices without decreasing the chromatic number by 2 or more.

6. For any set of graphs $G_1, G_2, \ldots, G_n$ all having the same set of vertices V, define their *overlap*, denoted $G_1 \cup G_2 \cup \cdots \cup G_n$, to be the graph having vertex set V for which two vertices are adjacent in the overlap if and only if they are adjacent in at least one of the graphs G_i.

(a) [**10**] Let G and H be graphs having the same vertex set and let a be the chromatic number of G and b the chromatic number of H. Find, in terms of a and b, the largest possible chromatic number of $G \cup H$. Prove your answer.

Answer: $\boxed{ab}$

Solution: First, we show that we can always color $G \cup H$ using ab colors. Given a good coloring of G in a colors $c_1, \ldots, c_a$ and a good coloring of H using b colors $d_1, \ldots, d_b$, define ab new colors to be the ordered pairs (c_i, d_j). Label a vertex of $G \cup H$ with the color (c_i, d_j) if it is colored c_i in G and d_i in H. This gives a good coloring of $G \cup H$.

Now, it only remains to find graphs G and H such that $G \cup H$ has chromatic number ab. Consider the complete graph K_{ab} having ab vertices $v_1, \ldots, v_{ab}$ (every pair of vertices is adjacent). Let G be the graph with vertices $v_1, \ldots, v_{ab}$ such that v_i is connected to v_j if and only if $i-j$ is a multiple of a. Also, let H be the graph on $v_1, \ldots, v_{ab}$ such that two vertices are adjacent if and only if they are not adjacent in G. Then $G \cup H = K_{ab}$.

Note that G has chromatic number a since it is the disjoint union of b complete graphs on a vertices. Also, H has chromatic number b since we can color each set of vertices v_i with a color corresponding to i modulo b to obtain a good coloring. The chromatic number of $G \cup H$ is clearly ab, and so we have found such a pair of graphs.

(b) [**10**] Suppose G is a graph with chromatic number n. Suppose there exist k graphs $G_1, G_2, \ldots, G_k$ having the same vertex set as G such that $G_1 \cup G_2 \cup \cdots \cup G_k = G$ and each G_i has chromatic number at most 2. Show that $k \geq \lceil \log_2(n) \rceil$, and show that one can always find such a decomposition of G into $\lceil \log_2(n) \rceil$ graphs.

Solution: [NOTE: This problem differs from the problem statement in the test as administered at the 2009 HMMT. The reader is encouraged to try it before reading the solution.]

The bound on k follows from iterating part (a).

Let G be a graph with chromatic number n. Consider a coloring of G using n colors labeled $1, 2, \ldots, n$. For i from 1 to $\lceil \log_2(n) \rceil$, define G_i to be the graph on the vertices of G for which two vertices are connected by an edge if and only if the ith digit from the right in the binary expansions of their colors do not match. Clearly each of the graphs G_i have chromatic number at most 2, by coloring each node with the ith digit of the binary expansion of their color in G. Moreover, each edge occurs in some G_i, since if two vertices match in every digit they are not connected by an edge. Therefore $G_1 \cup G_2 \cup \cdots \cup G_{\lceil \log_2(n) \rceil} = G$, and so we have found such a decomposition of G.

7. [**20**] Let n be a positive integer. Let V_n be the set of all sequences of 0's and 1's of length n. Define G_n to be the graph having vertex set V_n, such that two sequences are adjacent in G_n if and only if they differ in either 1 or 2 places. For instance, if $n = 3$, the sequences $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ are mutually adjacent, but $(1, 0, 0)$ is not adjacent to $(0, 1, 1)$.

   Show that, if $n + 1$ is not a power of 2, then the chromatic number of G_n is at least $n + 2$.

   Solution: We will assume that there is a coloring with $n+1$ colors and derive a contradiction. For each string s, let T_s be the set consisting of all strings that differ from s in at most 1 place. Thus T_s has size $n + 1$ and all vertices in T_s are adjacent. In particular, if there is an $(n+1)$-coloring, then each color is used exactly once in T_s. Let c be one of the colors that we used. We will determine how many vertices are colored with c. We will do this by counting in two ways.

   Let k be the number of vertices colored with color c. Each such vertex is part of T_s for exactly $n + 1$ values of s. On the other hand, each T_s contains exactly one vertex with color c. It follows that $k(n + 1) = 2^n$. In particular, since k is an integer, $n + 1$ divides 2^n. This is a contradiction since $n + 1$ is now a power of 2 by assumption, so actually there can be no $n + 1$-coloring, as claimed.

8. [**30**] Two colorings are *distinct* if there is no way to relabel the colors to transform one into the other. Equivalently, they are distinct if and only if there is some pair of vertices which are the same color in one coloring but different colors in the other. For what pairs (n, k) of positive integers does there exist a finite graph with chromatic number n which has exactly k distinct good colorings using n colors?

   Answer: $\boxed{(1, 1),\ (2, 2^k) \text{ for integers } k \geq 0, \text{ and } (n, k) \text{ for } n > 2,\ k > 0}$

   Solution: If $n = 1$, there is only one coloring. If $n = 2$, then each connected component of the graph can be colored in two ways, because the color of any vertex in the graph determines the colors of all vertices connected to it. If the color scheme in one component is fixed, and there are k components, then there are 2^{k-1} ways to finish the coloring.

   Now say $n > 2$. We construct a graph with k different colorings. We begin with a complete graph G on n vertices, which can be colored in exactly one way. Let v_1, v_2, and v_3 be three vertices in the complete graph. If $k > 1$, add to the graph a row of vertices $w_1, w_2, \ldots w_{k-1}$, such that w_i is connected to w_{i+1} for $1 \leq k - 2$. Now, if $i \equiv 0$ (3), connect w_i to all the vertices in G except v_1 and v_2. If $i \equiv 1 \pmod 3$, connect w_i to all the vertices in G except v_2 and v_3, and if $i \equiv 2 \pmod 3$, connect w_i to all the vertices in G except v_1 and v_3.

We need to show that this graph can be colored with n colors in exactly k different ways. Say that v_1 is colored red, v_2 blue, and v_3 green. Then each of the w_i can be colored one of exactly two colors. Further, there is exactly one possible color that w_i and w_{i+1} could both be. Call the color w_i and w_{i+1} could both be w_i's *leading* color, and call w_i's other color its *lagging* color. Notice that w_i's lagging color is w_{i+1}'s leading color. So, if any w_i is colored with its lagging color, then all w_j with $j > i$ are also colored with their lagging colors.

So one possibility is that all the w_i are colored with their leading colors. Otherwise, some of them are colored with their lagging colors - these colorings are completely defined by which one of the $k - 1$ w_i is the first vertex colored with its lagging color. So there are $1 + k - 1$ or k colorings of this graph, as needed.