HMMT February 2023

February 18, 2023

Team Round

1. **[30]** For any positive integer a, let $\tau(a)$ be the number of positive divisors of a. Find, with proof, the largest possible value of $4\tau(n) - n$ over all positive integers n.

2. **[30]** Prove that there do not exist pairwise distinct complex numbers a, b, c, and d such that

$$a^3 - bcd = b^3 - cda = c^3 - dab = d^3 - abc.$$

3. **[35]** Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle BCD = \theta$ for some angle $\theta < 90^\circ$. Point X lies inside the quadrilateral such that $\angle XAD = \angle XDA = 90^\circ - \theta$. Prove that $BX = XC$.

4. **[35]** Philena and Nathan are playing a game. First, Nathan secretly chooses an ordered pair (x, y) of positive integers such that $x \leq 20$ and $y \leq 23$. (Philena knows that Nathan's pair must satisfy $x \leq 20$ and $y \leq 23$.) The game then proceeds in rounds; in every round, Philena chooses an ordered pair (a, b) of positive integers and tells it to Nathan; Nathan says YES if $x \leq a$ and $y \leq b$, and NO otherwise. Find, with proof, the smallest positive integer N for which Philena has a strategy that guarantees she can be certain of Nathan's pair after at most N rounds.

5. **[40]** Let S be the set of all points in the plane whose coordinates are positive integers less than or equal to 100 (so S has 100^2 elements), and let $\mathcal{L}$ be the set of all lines ℓ such that ℓ passes through at least two points in S. Find, with proof, the largest integer $N \geq 2$ for which it is possible to choose N distinct lines in $\mathcal{L}$ such that every two of the chosen lines are parallel.

6. **[50]** For any odd positive integer n, let $r(n)$ be the odd positive integer such that the binary representation of $r(n)$ is the binary representation of n written backwards. For example, $r(2023) = r(11111100111_2) = 11100111111_2 = 1855$. Determine, with proof, whether there exists a strictly increasing eight-term arithmetic progression $a_1, \ldots, a_8$ of odd positive integers such that $r(a_1), \ldots, r(a_8)$ is an arithmetic progression in that order.

7. **[55]** Let ABC be a triangle. Point D lies on segment BC such that $\angle BAD = \angle DAC$. Point X lies on the opposite side of line BC as A and satisfies $XB = XD$ and $\angle BXD = \angle ACB$. Analogously, point Y lies on the opposite side of line BC as A and satisfies $YC = YD$ and $\angle CYD = \angle ABC$. Prove that lines XY and AD are perpendicular.

8. **[60]** Find, with proof, all nonconstant polynomials $P(x)$ with real coefficients such that, for all nonzero real numbers z with $P(z) \neq 0$ and $P(\frac{1}{z}) \neq 0$, we have

$$\frac{1}{P(z)} + \frac{1}{P(\frac{1}{z})} = z + \frac{1}{z}.$$

9. **[75]** Let ABC be a triangle with $AB < AC$. The incircle of triangle ABC is tangent to side BC at D and intersects the perpendicular bisector of segment BC at distinct points X and Y. Lines AX and AY intersect line BC at P and Q, respectively. Prove that, if $DP \cdot DQ = (AC - AB)^2$, then $AB + AC = 3BC$.

10. **[90]** One thousand people are in a tennis tournament where each person plays against each other person exactly once, and there are no ties. Prove that it is possible to put all the competitors in a line so that each of the 998 people who are not at an end of the line either defeated both their neighbors or lost to both their neighbors.