February 2017

February 18, 2017

Team

1. [**15**] Let $P(x), Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers y, then $P(x) = Q(x)$ for all real numbers x.

Proposed by: Alexander Katz

Answer: $\boxed{}$

By the condition, we know that $|P(x) - Q(x)| \leq 1$ for all x. This can only hold if $P(x) - Q(x)$ is a constant polynomial. Now take a constant c such that $P(x) = Q(x) + c$. Without loss of generality, we can assume that $c \geq 0$. Assume that $c > 0$. By continuity, if $\deg P = \deg Q > 0$, we can select an integer r and a real number x_0 such that $Q(x_0) + c = r$. Then $\lfloor P(x_0) \rfloor = \lfloor Q(x_0) + c \rfloor = r$. On the other hand, $\lfloor Q(x_0) \rfloor = \lfloor r - c \rfloor < r$ as r was an integer. This is a contradiction. Therefore, $c = 0$ as desired.

2. [**25**] Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when x and y are both positive?

Proposed by: Alexander Katz

Answer: $\boxed{\text{No}}$

For any ϵ and positive x, $P(x, \epsilon) > 0$ and $P(x, -\epsilon) \leq 0$. Thus by continuity/IVT, $P(x, 0) = 0$ for all positive x. Similarly $P(0, y) = 0$ for all positive y. This implies $xy \mid P(x, y)$, and so we can write $P(x, y) = xyQ(x, y)$. But then this same logic holds for Q, and this cannot continue infinitely unless P is identically 0 – in which case the conditions do not hold. So no such polynomial exists.

3. [**30**] A polyhedron has $7n$ faces. Show that there exist $n+1$ of the polyhedron's faces that all have the same number of edges.

Proposed by: Alexander Katz

Let V, E, and F denote the number of vertices, edges, and faces respectively. Let a_k denote the number of faces with k sides, and let M be the maximum number of sides any face has.

Suppose that $a_k \leq n$ for all k and that $M > 8$. Note that each edge is part of exactly two faces, and each vertex is part of at least three faces. It follows that

$$\sum_{k=3}^{M} a_k = F$$

$$\sum_{k=3}^{M} \frac{k a_k}{2} = E$$

$$\sum_{k=3}^{M} \frac{k a_k}{3} \geq V$$

and in particular

$$\sum_{k=3}^{M} a_k \left(1 - \frac{k}{2} + \frac{k}{3}\right) \geq F - E + V = 2$$

by Euler's formula. But on the other hand, $a_k \leq n$ by assumption, so

$$\begin{aligned}
2 &\le \sum_{k=3}^{M} a_k \left(1 - \frac{k}{6}\right) \\
&\le \sum_{k=3}^{M} n \left(1 - \frac{k}{6}\right) \\
&= \sum_{k=3}^{8} n \left(1 - \frac{k}{6}\right) + \sum_{k=9}^{M} n \left(1 - \frac{k}{6}\right) \\
&\le \frac{1}{2}n - \frac{1}{2}n(M-8)
\end{aligned}$$

where the last step follows from the fact that $1 - \frac{k}{6} \le -\frac{1}{2}$ for $k \ge 9$. Thus

$$2 \le \frac{9}{2}n - \frac{1}{2}nM \implies M \le \frac{\frac{9}{2}n - 2}{\frac{1}{2}n} < 9$$

contradicting the fact that $M > 8$. It follows that $M \le 8$, and as each face has at least 3 edges, the result follows directly from Pigeonhole.

4. [**35**] Let $w = w_1 w_2 \ldots w_n$ be a word. Define a *substring* of w to be a word of the form $w_i w_{i+1} \ldots w_{j-1} w_j$, for some pair of positive integers $1 \le i \le j \le n$. Show that w has at most n distinct palindromic substrings.

   For example, $aaaaa$ has 5 distinct palindromic substrings, and $abcata$ has 5 (a, b, c, t, ata).

   Proposed by: Yang Liu

   For each palindrome substring appearing in w, consider only the leftmost position in which is appears. I claim that now, no two substrings share the same right endpoint. If some two do, then you can reflect the smaller one about the center of the larger one to move the smaller one left.

5. [**35**] Let ABC be an acute triangle. The altitudes BE and CF intersect at the orthocenter H, and point O denotes the circumcenter. Point P is chosen so that $\angle APH = \angle OPE = 90°$, and point Q is chosen so that $\angle AQH = \angle OQF = 90°$. Lines EP and FQ meet at point T. Prove that points A, T, O are collinear.

   Proposed by: Evan Chen

   Observe that T is the radical center of the circles with diameter OE, OF, AH. So T lies on the radical axis of (OE), (OF) which is the altitude from O to EF, hence passing through A.

   So ATO are collinear, done.

6. [**40**] Let r be a positive integer. Show that if a graph G has no cycles of length at most $2r$, then it has at most $|V|^{2016}$ cycles of length exactly $2016r$, where $|V|$ denotes the number of vertices in the graph G.

   Proposed by: Yang Liu

   The key idea is that there is at most 1 path of length r between any pair of vertices, or else you get a cycle of length $\le 2r$. Now, start at any vertex ($|V|$ choices) and walk 2015 times. There's at most $|V|^{2016}$ ways to do this by the previous argument. Now you have to go from the end to the start, and there's only one way to do this. So we're done.

7. [**45**] Let p be a prime. A *complete residue class modulo p* is a set containing at least one element equivalent to $k \pmod{p}$ for all k.

(a) (**20**) Show that there exists an n such that the nth row of Pascal's triangle forms a complete residue class modulo p.

(b) (**25**) Show that there exists an $n \le p^2$ such that the nth row of Pascal's triangle forms a complete residue class modulo p.

Proposed by: Alexander Katz

We use the following theorem of Lucas:

Theorem. Given a prime p and nonnegative integers a, b written in base p as $a = \overline{a_n a_{n-1} \dots a_0}_p$ and $b = \overline{b_n b_{n-1} \dots b_0}_p$ respectively, where $0 \le a_i, b_i \le p-1$ for $0 \le i \le n$, we have

$$\binom{a}{b} = \prod_{i=0}^{n} \binom{a_i}{b_i} \pmod{p}.$$

Now, let $n = (p-1) \times p + (p-2) = p^2 - 2$. For $k = pq + r$ with $0 \le q, r \le p-1$, applying Lucas's theorem gives

$$\binom{n}{k} \equiv \binom{p-1}{q}\binom{p-2}{r} \pmod{p}.$$

Note that

$$\binom{p-1}{q} = \prod_{i=1}^{q} \frac{p-i}{i} \equiv (-1)^q \pmod{p},$$

and

$$\binom{p-2}{r} = \prod_{i=1}^{r} \frac{p-1-i}{i} \equiv (-1)^r \frac{(r+1)!}{r!} = (-1)^r (r+1) \pmod{p}.$$

So for $2 \le i \le p$ we can take $k = (p+1)(i-1)$ and obtain $\binom{n}{k} \equiv i \pmod{p}$, while for $i = 1$ we can take $k = 0$. Thus this row satisfies the desired property.

8. [**45**] Does there exist an irrational number $\alpha > 1$ such that

$$\lfloor \alpha^n \rfloor \equiv 0 \pmod{2017}$$

for all integers $n \ge 1$?

Proposed by: Sam Korsky

Answer: $\boxed{\text{Yes}}$

Let $\alpha > 1$ and $0 < \beta < 1$ be the roots of $x^2 - 4035x + 2017$. Then note that $\lfloor \alpha^n \rfloor = \alpha^n + \beta^n - 1$. Let $x_n = \alpha^n + \beta^n$ for all nonnegative integers n. It's easy to verify that $x_n = 4035x_{n-1} - 2017x_{n-2} \equiv x_{n-1} \pmod{2017}$ so since $x_1 = 4035 \equiv 1 \pmod{2017}$ we have that $x_n \equiv 1 \pmod{2017}$ for all n. Thus α satisfies the problem.

9. [**65**] Let n be a positive odd integer greater than 2, and consider a regular n-gon $\mathcal{G}$ in the plane centered at the origin. Let a *subpolygon* $\mathcal{G}'$ be a polygon with at least 3 vertices whose vertex set is a subset of that of $\mathcal{G}$. Say $\mathcal{G}'$ is *well-centered* if its centroid is the origin. Also, say $\mathcal{G}'$ is *decomposable* if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.

Proposed by: Yang Liu

$\Rightarrow$, i.e. n has ≥ 3 prime divisors: Let $n = \prod p_i^{e_i}$. Note it suffices to only consider regular p_i-gons. Label the vertices of the n-gon $0, 1, \dots, n-1$. Let $S = \{\frac{xn}{p_1} : 0 \le x \le p_1 - 1\}$, and let $S_j = S + \frac{jn}{p_3}$ for $0 \le j \le p_3 - 2$. ($S + a = \{s + a : s \in S\}$.) Then let $S_{p_3-1} = \{\frac{xn}{p_2} : 0 \le x \le p_2 - 1\} + \frac{(p_3-1)n}{p_3}$. Finally, let $S' = \{\frac{xn}{p_3} : 0 \le x \le p_3 - 1\}$. Then I claim

$$\left(\bigsqcup_{i=0}^{p_3-1} S_i \right) \backslash S'$$

is well-centered but not decomposable. Well-centered follows from the construction: I only added and subtracted off regular polygons. To show that its decomposable, consider $\frac{n}{p_1}$. Clearly this is in the set, but isn't in S'. I claim that $\frac{n}{p_1}$ isn't in any more regular p_i-gons. For $i \geq 4$, this means that $\frac{n}{p_1} + \frac{n}{p_i}$ is in some set. But this is a contradiction, as we can easily check that all points we added in are multiples of $p_i^{e_i}$, while $\frac{n}{p_i}$ isn't.

For $i = 1$, note that 0 was removed by S'. For $i = 2$, note that the only multiples of $p_3^{e_3}$ that are in some S_j are $0, \frac{n}{p_1}, \ldots, \frac{(p_1-1)n}{p_1}$. In particular, $\frac{n}{p_1} + \frac{n}{p_2}$ isn't in any S_j. So it suffices to consider the case $i = 3$, but it is easy to show that $\frac{n}{p_1} + \frac{(p_3-1)n}{p_3}$ isn't in any S_i. So we're done.

$\Leftarrow$, i.e. n has ≤ 2 prime divisors: This part seems to require knowledge of cyclotomic polynomials. These will easily give a solution in the case $n = p^a$. Now, instead turn to the case $n = p^a q^b$. The next lemma is the key ingredient to the solution.

Lemma: Every well-centered subpolygon can be gotten by adding in and subtracting off regular polygons.

Note that this is weaker than the problem claim, as the problem claims that adding in polygons is enough.

Proof. It is easy to verify that $\phi_n(x) = \frac{(x^n-1)(x^{\frac{n}{pq}}-1)}{(x^{\frac{n}{p}}-1)(x^{\frac{n}{q}}-1)}$. Therefore, it suffices to check that there exist integer polynomials $c(x), d(x)$ such that

$$\frac{x^n - 1}{x^{\frac{n}{p}} - 1} \cdot c(x) + \frac{x^n - 1}{x^{\frac{n}{q}} - 1} \cdot d(x) = \frac{(x^n - 1)(x^{\frac{n}{pq}} - 1)}{(x^{\frac{n}{p}} - 1)(x^{\frac{n}{q}} - 1)}.$$

Rearranging means that we want

$$(x^{\frac{n}{q}} - 1) \cdot c(x) + (x^{\frac{n}{p}} - 1) \cdot d(x) = x^{\frac{n}{pq}} - 1.$$

But now, since $\gcd(n/p, n/q) = n/pq$, there exist positive integers s, t such that $\frac{sn}{q} - \frac{tn}{p} = \frac{n}{pq}$. Now choose $c(x) = \frac{x^{\frac{sn}{q}}-1}{x^{\frac{n}{q}}-1}, d(x) = \frac{x^{\frac{sn}{q}}-x^{\frac{n}{pq}}}{x^{\frac{n}{p}}-1}$ to finish. $\square$

Now we can finish combinatorially. Say we need subtraction, and at some point we subtract off a p-gon. All the points in the p-gon must have been added at some point. If any of them was added from a p-gon, we could just cancel both p-gons. If they all came from a q-gon, then the sum of those p q-gons would be a pq-gon, which could have been instead written as the sum of q p-gons. So we don't need subtraction either way. This completes the proof.

10. [**65**] Let LBC be a fixed triangle with $LB = LC$, and let A be a variable point on arc LB of its circumcircle. Let I be the incenter of $\triangle ABC$ and $\overline{AK}$ the altitude from A. The circumcircle of $\triangle IKL$ intersects lines KA and BC again at $U \neq K$ and $V \neq K$. Finally, let T be the projection of I onto line UV. Prove that the line through T and the midpoint of $\overline{IK}$ passes through a fixed point as A varies.

Proposed by: Sam Korsky

Answer: $\square$

Let M be the midpoint of arc BC not containing L and let D be the point where the incircle of triangle ABC touches BC. Also let N be the projection from I to AK. We claim that M is the desired fixed point.

By Simson's Theorem on triangle KUV and point I we have that points T, D, N are collinear and since quadrilateral $NKDI$ is a rectangle we have that line DN passes through the midpoint of IK. Thus it suffices to show that M lies on line DN.

Now, let I_a, I_b, I_c be the A, B, C-excenters of triangle ABC respectively. Then I is the orthocenter of triangle $I_aI_bI_c$ and ABC is the Cevian triangle of I with respect to triangle $I_aI_bI_c$. It's also well-known that M is the midpoint of II_a.

Let D' be the reflection of I over BC and let N' be the reflection of I over AK. Clearly K is the midpoint of $D'N'$. If we could prove that I_a, D', K, N' were collinear then by taking a homothety centered at I with ratio $\frac{1}{2}$ we would have that points M, D, N were collinear as desired. Thus it suffices to show that points I_a, D', K are collinear.

Let lines BC and I_bI_c intersect at R and let lines AI and BC intersect at S. Then it's well-known that $(I_b, I_c; A, R)$ is harmonic and projecting from C we have that $(I_a, I; S, A)$ is harmonic. But $KS \perp KA$ which means that KS bisects angle $\angle IKI_a$. But it's clear by the definition of D' that KS bisects angle $\angle IKD'$ which implies that points I_a, K, D' are collinear as desired. This completes the proof.