HMMT February 2016

February 20, 2016

Team

1. [**25**] Let a and b be integers (not necessarily positive). Prove that $a^3 + 5b^3 \neq 2016$.

   Proposed by: Evan Chen

   Since cubes are 0 or ± 1 modulo 9, by inspection we see that we must have $a^3 \equiv b^3 \equiv 0 \pmod 3$ for this to be possible. Thus a, b are divisible by 3. But then we get $3^3 \mid 2016$, which is a contradiction.

   One can also solve the problem in the same manner by taking modulo 7 exist, since all cubes are 0 or ± 1 modulo 7. The proof can be copied literally, noting that $7 \mid 2016$ but $7^3 \nmid 2016$.

2. [**25**] For positive integers n, let c_n be the smallest positive integer for which $n^{c_n} - 1$ is divisible by 210, if such a positive integer exists, and $c_n = 0$ otherwise. What is $c_1 + c_2 + \cdots + c_{210}$?

   Proposed by: Joy Zheng

   Answer: $\boxed{329}$

   In order for $c_n \neq 0$, we must have $\gcd(n, 210) = 1$, so we need only consider such n. The number $n^{c_n} - 1$ is divisible by 210 iff it is divisible by each of 2, 3, 5, and 7, and we can consider the order of n modulo each modulus separately; c_n will simply be the LCM of these orders. We can ignore the modulus 2 because order is always 1. For the other moduli, the sets of orders are

$$\begin{aligned} a &\in \{1,2\} \quad \text{mod } 3 \\ b &\in \{1,2,4,4\} \quad \text{mod } 5 \\ c &\in \{1,2,3,3,6,6\} \quad \text{mod } 7. \end{aligned}$$

   By the Chinese Remainder Theorem, each triplet of choices from these three multisets occurs for exactly one n in the range $\{1, 2, \ldots, 210\}$, so the answer we seek is the sum of $\operatorname{lcm}(a, b, c)$ over a, b, c in the Cartesian product of these multisets. For $a = 1$ this table of LCMs is as follows:

	1	2	3	3	6	6
1	1	2	3	3	6	6
2	2	2	6	6	6	6
4	4	4	12	12	12	12
4	4	4	12	12	12	12

   which has a sum of $21 + 56 + 28 + 56 = 161$. The table for $a = 2$ is identical except for the top row, where $1, 3, 3$ are replaced by $2, 6, 6$, and thus has a total sum of 7 more, or 168. So our answer is $161 + 168 = \boxed{\mathbf{329}}$.

   This can also be computed by counting how many times each LCM occurs:

   - 12 appears 16 times when $b = 4$ and $c \in \{3, 6\}$, for a contribution of $12 \times 16 = 192$;
   - 6 appears 14 times, 8 times when $c = 6$ and $b \leq 2$ and 6 times when $c = 3$ and $(a, b) \in \{(1, 2), (2, 1), (2, 2)\}$, for a contribution of $6 \times 14 = 84$;
   - 4 appears 8 times when $b = 4$ and $a, c \in \{1, 2\}$, for a contribution of $4 \times 8 = 32$;
   - 3 appears 2 times when $c = 3$ and $a = b = 1$, for a contribution of $3 \times 2 = 6$;
   - 2 appears 7 times when $a, b, c \in \{1, 2\}$ and $(a, b, c) \neq (1, 1, 1)$, for a contribution of $2 \times 7 = 14$;
   - 1 appears 1 time when $a = b = c = 1$, for a contribution of $1 \times 1 = 1$.

   The result is again $192 + 84 + 32 + 6 + 14 + 1 = 329$.

3. [**30**] Let ABC be an acute triangle with incenter I and circumcenter O. Assume that $\angle OIA = 90^\circ$. Given that $AI = 97$ and $BC = 144$, compute the area of $\triangle ABC$.

Proposed by: Evan Chen

Answer: $\boxed{14040}$

We present five different solutions and outline a sixth and seventh one. In what follows, let $a = BC$, $b = CA$, $c = AB$ as usual, and denote by r and R the inradius and circumradius. Let $s = \frac{1}{2}(a+b+c)$.

In the first five solutions we will only prove that

$$\angle AIO = 90^\circ \implies b + c = 2a.$$

Let us see how this solves the problem. This lemma implies that $s = 216$. If we let E be the foot of I on AB, then $AE = s - BC = 72$, consequently the inradius is $r = \sqrt{97^2 - 72^2} = 65$. Finally, the area is $sr = 216 \cdot 65 = \boxed{14040}$.

First Solution. Since $OI \perp DA$, $AI = DI$. Now, it is a well-known fact that $DI = DB = DC$ (this is occasionally called "Fact 5"). Then by Ptolemy's Theorem,

$$DB \cdot AC + DC \cdot AB = DA \cdot BC \implies AC + AB = 2BC. \qquad \square$$

Second Solution. As before note that I is the midpoint of AD. Let M and N be the midpoints of AB and AC, and let the reflection of M across BI be P; thus $BM = BP$. Also, $MI = PI$, but we know $MI = NI$ as I lies on the circumcircle of triangle AMN. Consequently, we get $PI = NI$; moreover by angle chasing we have

$$\angle INC = \angle AMI = 180^\circ - \angle BPI = \angle IPC.$$

Thus triangles INC and PIC are congruent (CI is a bisector) so we deduce $PC = NC$. Thus,

$$BC = BP + PC = BM + CN = \frac{1}{2}(AB + AC). \qquad \square$$

Third Solution. We appeal to Euler's Theorem, which states that $IO^2 = R(R - 2r)$.

Thus by the Pythagorean Theorem on $\triangle AIO$ (or by Power of a Point) we may write

$$(s-a)^2 + r^2 = AI^2 = R^2 - IO^2 = 2Rr = \frac{abc}{2s}$$

with the same notations as before. Thus, we derive that

$$\begin{aligned} abc &= 2s\left((s-a)^2 + r^2\right) \\ &= 2(s-a)\left(s(s-a) + (s-b)(s-c)\right) \\ &= \frac{1}{2}(s-a)\left((b+c)^2 - a^2 + a^2 - (b-c)^2\right) \\ &= 2bc(s-a). \end{aligned}$$

From this we deduce that $2a = b + c$, and we can proceed as in the previous solution. $\square$

Fourth Solution. From Fact 5 again ($DB = DI = DC$), drop perpendicular from I to AB at E; call M the midpoint of BC. Then, by AAS congruency on AIE and CDM, we immediately get that $CM = AE$. As $AE = \frac{1}{2}(AB + AC - BC)$, this gives the desired conclusion. $\square$

Fifth Solution. This solution avoids angle-chasing and using the fact that BI and CI are angle-bisectors. Recall the perpendicularity lemma, where

$$WX \perp YZ \iff WY^2 - WZ^2 = XY^2 - XZ^2.$$

Let B' be on the extension of ray CA such that $AB' = AB$. Of course, as in the proof of the angle bisector theorem, $BB' \parallel AI$, meaning that $BB' \perp IO$. Let I' be the reflection of I across A; of course, I' is then the incenter of triangle $AB'C'$. Now, we have $B'I^2 - BI^2 = B'O^2 - BO^2$ by the perpendicularity and by power of a point $B'O^2 - BO^2 = B'A \cdot B'C$. Moreover $BI^2 + B'I^2 = BI^2 + BI'^2 = 2BA^2 + 2AI^2$ by the median formula. Subtracting, we get $BI^2 = AI^2 + \frac{1}{2}(AB)(AB - AC)$. We have a similar expression for CI, and subtracting the two results in $BI^2 - CI^2 = \frac{1}{2}(AB^2 - AC^2)$. Finally,

$$BI^2 - CI^2 = \frac{1}{4}[(BC + AB - AC)^2 - (BC - AB + AC)^2]$$

from which again, the result $2BC = AB + AC$ follows. $\square$

Sixth Solution, outline. Use complex numbers, setting $I = ab+bc+ca$, $A = -a^2$, etc. on the unit circle (scale the picture to fit in a unit circle; we calculate scaling factor later). Set $a = 1$, and let $u = b + c$ and $v = bc$. Write every condition in terms of u and v, and the area in terms of u and v too. There should be two equations relating u and v: $2u + v + 1 = 0$ and $u^2 = \frac{130}{97}^2 v$ from the right angle and the 144 to 97 ratio, respectively. The square area can be computed in terms of u and v, because the area itself is antisymmetric so squaring it suffices. Use the first condition to homogenize (not coincidentally the factor $(1 - b^2)(1 - c^2) = (1 + bc)^2 - (b + c)^2 = (1 + v)^2 - u^2$ from the area homogenizes perfectly... because $AB \cdot AC = AI \cdot AI_A$, where I_A is the A-excenter, and of course the way the problem is set up $AI_A = 3AI$.), and then we find the area of the scaled down version. To find the scaling factor simply determine $|b - c|$ by squaring it, writing in terms again of u and v, and comparing this to the value of 144. $\square$

Seventh Solution, outline. Trigonometric solutions are also possible. One can you write everything in terms of the angles and solve the equations; for instance, the $\angle AIO = 90^\circ$ condition can be rewritten as $\frac{1}{2}\cos\frac{B-C}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2}$ and the 97 to 144 ratio condition can be rewritten as $\frac{2\sin\frac{B}{2}2\sin\frac{C}{2}}{\sin A} = \frac{97}{144}$. The first equation implies $\sin\frac{A}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2}$, which we can plug into the second equation to get $\cos\frac{A}{2}$. $\square$

4. [**30**] Let $n > 1$ be an odd integer. On an $n \times n$ chessboard the center square and four corners are deleted. We wish to group the remaining $n^2 - 5$ squares into $\frac{1}{2}(n^2 - 5)$ pairs, such that the two squares in each pair intersect at exactly one point (i.e. they are diagonally adjacent, sharing a single corner).

   For which odd integers $n > 1$ is this possible?

   Proposed by: Evan Chen

   Answer: $\boxed{3,5}$

   Constructions for $n = 3$ and $n = 5$ are easy. For $n > 5$, color the odd rows black and the even rows white. If the squares can be paired in the way desired, each pair we choose must have one black cell and one white cell, so the numbers of black cells and white cells are the same. The number of black cells is $\frac{n+1}{2}n - 4$ or $\frac{n+1}{2}n - 5$ depending on whether the removed center cell is in an odd row. The number of white cells is $\frac{n-1}{2}n$ or $\frac{n-1}{2}n - 1$. But

$$\left(\frac{n+1}{2}n - 5\right) - \frac{n-1}{2}n = n - 5$$

   so for $n > 5$ this pairing is impossible. Thus the answer is $n = 3$ and $n = 5$.

5. [**35**] Find all prime numbers p such that $y^2 = x^3 + 4x$ has exactly p solutions in integers modulo p.

In other words, determine all prime numbers p with the following property: there exist exactly p ordered pairs of integers (x, y) such that $x, y \in \{0, 1, \ldots, p-1\}$ and

$$p \text{ divides } y^2 - x^3 - 4x.$$

Proposed by: Evan Chen

Answer: $\boxed{p = 2 \text{ and } p \equiv 3 \pmod 4}$

Clearly $p = 2$ works with solutions $(0, 0)$ and $(1, 1)$ and not $(0, 1)$ or $(1, 0)$.

If $p \equiv 3 \pmod 4$ then -1 is not a quadratic residue, so for $x^3 + 4x \neq 0$, exactly one of $x^3 + 4x$ and $-x^3 - 4x$ is a square and gives two solutions (for positive and negative y), so there's exactly two solutions for each such pair $\{x, -x\}$. If x is such that $x^3 + 4x = 0$, there's exactly one solution.

If $p \equiv 1 \pmod 4$, let i be a square root of $-1 \pmod p$. The right hand side factors as $x(x+2i)(x-2i)$. For $x = 0, 2i, -2i$ this is zero, there is one choice of y, namely zero. Otherwise, the right hand side is nonzero. For any fixed x, there are either 0 or 2 choices for y. Replacing x by $-x$ negates the right hand side, again producing two choices for y since -1 is a quadratic residue. So the total number of solutions (x, y) is $3 \pmod 4$, and thus there cannot be exactly p solutions.

Remark: This is a conductor 36 elliptic curve with complex multiplication, and the exact formula for the number of solutions is given in `http://www.mathcs.emory.edu/~ono/publications-cv/pdfs/026.pdf`.

6. [**35**] A nonempty set S is called *well-filled* if for every $m \in S$, there are fewer than $\frac{1}{2}m$ elements of S which are less than m. Determine the number of well-filled subsets of $\{1, 2, \ldots, 42\}$.

Proposed by: Casey Fu

Answer: $\boxed{\binom{43}{21} - 1}$

Let a_n be the number of well-filled subsets whose maximum element is n (setting $a_0 = 1$). Then it's easy to see that

$$\begin{aligned} a_{2k+1} &= a_{2k} + a_{2k-1} + \cdots + a_0 \\ a_{2k+2} &= (a_{2k+1} - C_k) + a_{2k} + \cdots + a_0. \end{aligned}$$

where C_k is the number of well-filled subsets of size $k+1$ with maximal element $2k+1$.

We proceed to compute C_k. One can think of such a subset as a sequence of numbers $1 \leq s_1 < \cdots < s_{k+1} \leq 2k+1$ such that $s_i \geq 2i - 1$ for every $1 \leq i \leq k+1$. Equivalently, letting $s_i = i + 1 + t_i$ it's the number of sequences $0 \leq t_1 \leq \cdots \leq t_{k+1} \leq k+1$ such that $t_i \geq i$ for every i. This gives the list of x-coordinates of steps up in a Catalan path from $(0, 0)$ to $(k+1, k+1)$, so

$$C_k = \frac{1}{k+2}\binom{2(k+1)}{(k+1)}$$

is equal to the $(k+1)$th Catalan number.

From this we can solve the above recursion to derive that

$$a_n = \binom{n}{\lfloor (n-1)/2 \rfloor}.$$

Consequently, for even n,

$$a_0 + \cdots + a_n = a_{n+1} = \binom{n+1}{\lfloor n/2 \rfloor}.$$

Putting $n = 42$ gives the answer, after subtracting off the empty set (counted in a_0).

7. [**40**] Let $q(x) = q^1(x) = 2x^2 + 2x - 1$, and let $q^n(x) = q(q^{n-1}(x))$ for $n > 1$. How many negative real roots does $q^{2016}(x)$ have?

Proposed by: Ernest Chiu

Answer: $\boxed{\frac{2^{2017}+1}{3}}$

Define $g(x) = 2x^2 - 1$, so that $q(x) = -\frac{1}{2} + g\left(x + \frac{1}{2}\right)$. Thus

$$q^N(x) = 0 \iff \frac{1}{2} = g^N\left(x + \frac{1}{2}\right)$$

where $N = 2016$.

But, viewed as function $g : [-1, 1] \to [-1, 1]$ we have that $g(x) = \cos(2\arccos(x))$. Thus, the equation $q^N(x) = 0$ is equivalent to

$$\cos\left(2^{2016} \arccos\left(x + \frac{1}{2}\right)\right) = \frac{1}{2}.$$

Thus, the solutions for x are

$$x = -\frac{1}{2} + \cos\left(\frac{\pi/3 + 2\pi n}{2^{2016}}\right) \qquad n = 0, 1, \ldots, 2^{2016} - 1.$$

So, the roots are negative for the values of n such that

$$\frac{1}{3}\pi < \frac{\pi/3 + 2\pi n}{2^{2016}} < \frac{5}{3}\pi$$

which is to say

$$\frac{1}{6}(2^{2016} - 1) < n < \frac{1}{6}(5 \cdot 2^{2016} - 1).$$

The number of values of n that fall in this range is $\frac{1}{6}(5 \cdot 2^{2016} - 2) - \frac{1}{6}(2^{2016} + 2) + 1 = \frac{1}{6}(4 \cdot 2^{2016} + 2) = \frac{1}{3}(2^{2017} + 1)$.

8. [**40**] Compute

$$\int_0^\pi \frac{2\sin\theta + 3\cos\theta - 3}{13\cos\theta - 5} \mathrm{d}\theta.$$

Proposed by: Carl Lian

Answer: $\boxed{\frac{3\pi}{13} - \frac{4}{13}\log\frac{3}{2}}$

We have

$$\begin{aligned}
\int_0^\pi \frac{2\sin\theta + 3\cos\theta - 3}{13\cos\theta - 5} \mathrm{d}\theta &= 2\int_0^{\pi/2} \frac{2\sin 2x + 3\cos 2x - 3}{13\cos 2x - 5} \mathrm{d}x \\
&= 2\int_0^{\pi/2} \frac{4\sin x\cos x - 6\sin^2 x}{8\cos^2 x - 18\sin^2 x} \mathrm{d}x \\
&= 2\int_0^{\pi/2} \frac{\sin x(2\cos x - 3\sin x)}{(2\cos x + 3\sin x)(2\cos x - 3\sin x)} \mathrm{d}x \\
&= 2\int_0^{\pi/2} \frac{\sin x}{2\cos x + 3\sin x}.
\end{aligned}$$

To compute the above integral we want to write $\sin x$ as a linear combination of the denominator and its derivative:

$$
\begin{aligned}
2\int_0^{\pi/2} \frac{\sin x}{2\cos x + 3\sin x} &= 2\int_0^{\pi/2} \frac{-\frac{1}{13}[-3(2\cos x + 3\sin x) + 2(3\cos x - 2\sin x)]}{2\cos x + 3\sin x} \\
&= -\frac{2}{13}\left[\int_0^{\pi/2}(-3) + 2\int_0^{\pi} \frac{-2\sin x + 3\cos x}{2\cos x + 3\sin x}\right] \\
&= -\frac{2}{13}\left[-\frac{3\pi}{2} + 2\log(3\sin x + 2\cos x)\ |_0^{\pi/2}\right] \\
&= -\frac{2}{13}\left[-\frac{3\pi}{2} + 2\log\frac{3}{2}\right] \\
&= \frac{3\pi}{13} - \frac{4}{13}\log\frac{3}{2}.
\end{aligned}
$$

9. [**40**] Fix positive integers $r > s$, and let F be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r-1$ that shares at least s elements with each set in F.

Proposed by: Victor Wang

This is a generalization of 2002 ISL C5.

Solution 1. Say a set S *s-meets* F if it shares at least s elements with each set in F. Suppose no such set of size (at most) $r-1$ exists. (Each $S \in F$ s-meets F by the problem hypothesis.)

Let T be a maximal set such that $T \subseteq S$ for infinitely many $S \in F$, which form $F' \subseteq F$ (such T exists, since the empty set works). Clearly $|T| < r$, so by assumption, T does not s-meet F, and there exists $U \in F$ with $|U \cap T| \le s-1$. But U s-meets F', so by pigeonhole, there must exist $u \in U \setminus T$ belonging to infinitely many $S \in F'$, contradicting the maximality of T.

Comment. Let X be an infinite set, and $a_1, \ldots, a_{2r-2-s}$ elements not in X. Then $F = \{B \cup \{x\} : B \subseteq \{a_1, \ldots, a_{2r-2-s}\}, |B| = r-1, x \in X\}$ shows we cannot replace $r-1$ with any smaller number.

Solution 2. We can also use a more indirect approach (where the use of contradiction is actually essential).

Fix $S \in F$ and $a \in S$. By assumption, $S \setminus \{a\}$ does not s-meet F, so there exists $S' \in F$ such that S' contains at most $s-1$ elements of $S \setminus \{a\}$, whence $S \cap S'$ is an s-set containing a. We will derive a contradiction from the following lemma:

Lemma. Let F, G be families of r-sets such that any $f \in F$ and $g \in G$ share at least s elements. Then there exists a finite set H such that for any $f \in F$ and $g \in G$, $|f \cap g \cap H| \ge s$.

Proof. Suppose not, and take a counterexample with $r+s$ minimal; then F, G must be infinite and $r > s > 0$.

Take arbitrary $f_0 \in F$ and $g_0 \in G$; then the finite set $X = f_0 \cup g_0$ meets F, G. For every subset $Y \subseteq X$, let $F_Y = \{S \in F : S \cap X = Y\}$; analogously define G_Y. Then the F_Y, G_Y partition F, G, respectively. For any F_Y and $y \in Y$, define $F_Y(y) = \{S \setminus \{y\} : S \in F_Y\}$.

Now fix subsets $Y, Z \subseteq X$. If one of F_Y, G_Z is empty, define $H_{Y,Z} = \emptyset$.

Otherwise, if Y, Z are disjoint, take arbitrary $y \in Y$, $z \in Z$. By the minimality assumption, there exists finite $H_{Y,Z}$ such that for any $f \in F_Y(y)$ and $g \in G_Z(z)$, $|f \cap g \cap H_{Y,Z}| \ge s$.

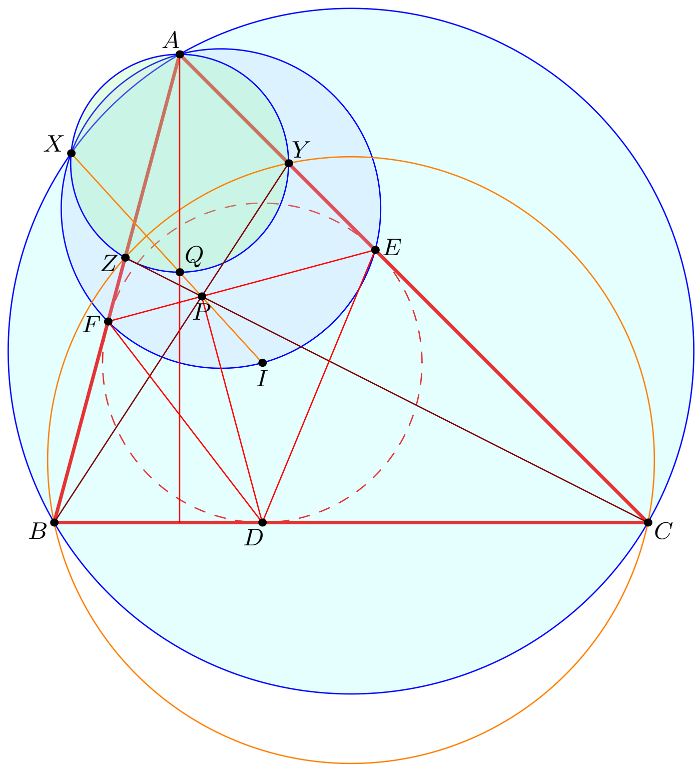
If Y, Z share an element a, and $s = 1$, take $H_{Y,Z} = \{a\}$. Otherwise, if $s \ge 2$, we find again by minimality a finite $H_{Y,Z}(a)$ such that for $f \in F_Y(a)$ and $g \in G_Z(a)$, $|f \cap g \cap H_{Y,Z}| \ge s-1$; then take $H_{Y,Z} = H_{Y,Z}(a) \cup \{a\}$.

Finally, we see that $H = \bigcup_{Y,Z \subseteq X} H_{Y,Z}$ shares at least s elements with each $f \cap g$ (by construction), contradicting our assumption.

10. [**50**] Let ABC be a triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F. Point P lies on $\overline{EF}$ such that $\overline{DP} \perp \overline{EF}$. Ray BP meets $\overline{AC}$ at Y and ray CP meets $\overline{AB}$ at Z. Point Q is selected on the circumcircle of $\triangle AYZ$ so that $\overline{AQ} \perp \overline{BC}$.

Prove that P, I, Q are collinear.

Proposed by: Evan Chen



The proof proceeds through a series of seven lemmas.

Lemma 1. Lines DP and EF are the internal and external angle bisectors of $\angle BPC$.

Proof. Since DEF the cevian triangle of ABC with respect to its Gregonne point, we have that

$$-1 = \left(\overline{EF} \cap \overline{BC}, D; B, C\right).$$

Then since $\angle DPF = 90^\circ$ we see P is on the Apollonian circle of BC through D. So the conclusion follows. $\square$

Lemma 2. Triangles BPF and CEP are similar.

Proof. Invoking the angle bisector theorem with the previous lemma gives

$$\frac{BP}{BF} = \frac{BP}{BD} = \frac{CP}{CD} = \frac{CP}{CE}.$$

But $\angle BFP = \angle CEP$, so $\triangle BFP \sim \triangle CEP$. $\square$

Lemma 3. Quadrilateral $BZYC$ is cyclic; in particular, line YZ is the antiparallel of line BC through $\angle BAC$.

Proof. Remark that $\angle YBZ = \angle PBF = \angle ECP = \angle YCZ$. $\square$

Lemma 4. The circumcircles of triangles AYZ, AEF, ABC are concurrent at a point X such that $\triangle XBF \sim \triangle XCE$.

Proof. Note that line EF is the angle bisector of $\angle BPZ = \angle CPY$. Thus

$$\frac{ZF}{FB} = \frac{ZP}{PB} = \frac{YP}{PC} = \frac{YE}{EC}.$$

Then, if we let X be the Miquel point of quadrilateral $ZYCB$, it follows that the spiral similarity mapping segment BZ to segment CY maps E to F; therefore the circumcircle of $\triangle AEF$ must pass through X too. $\square$

Lemma 5. Ray XP bisects $\angle FXE$.

Proof. The assertion amounts to

$$\frac{XF}{XE} = \frac{BF}{EC} = \frac{FP}{PE}.$$

The first equality follows from the spiral similarity $\triangle BFX \sim \triangle CEX$, while the second is from $\triangle BFP \sim \triangle CEP$. So the proof is complete by the converse of angle bisector theorem. $\square$

Lemma 6. Points X, P, I are collinear.

Proof. On one hand, $\angle FXI = \angle FAI = \frac{1}{2}\angle A$. On the other hand, $\angle FXP = \frac{1}{2}\angle FXE = \frac{1}{2}\angle A$. Hence, X, Y, I collinear. $\square$

Lemma 7. Points X, Q, I are collinear.

Proof. On one hand, $\angle AXQ = 90^\circ$, because we established earlier that line YZ was antiparallel to line BC through $\angle A$, hence $AQ \perp BC$ means exactly that $\angle AZQ = AYQ = 90^\circ$. On the other hand, $\angle AXI = 90^\circ$ according to the fact that X lies on the circle with diameter AI. This completes the proof of the lemma. $\square$

Finally, combining the final two lemmas solves the problem.