100 Integrals

(Great for calc 1 and calc 2 students)

Video: https://youtu.be/dgm4-3-lv3s

March 1st, 2019

Video transcription by jackw11111

(Q1.) $\int \tan^5 x \sec^3 x \, dx$

$= \int \tan^4 x \sec^2 x \tan x \sec x \, dx$

$= \int (u^2 - 1)^2 u^2 \, du$

$= \int (u^6 - 2u^4 + u^2) \, du$

$= \boxed{\frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C}$

Aside

$\tan^4 x = (\tan^2 x)^2$

$= (\sec^2 x - 1)^2$

Let $u = \sec x$

$du = \sec x \tan x \, dx$

(Q2.) $\int \frac{\cos 2x}{\sin x + \cos x} \, dx$

$= \int \frac{(\cos x - \sin x)\cancel{(\cos x + \sin x)}}{\cancel{\sin x + \cos x}} \, dx$

$= \int (\cos x - \sin x) \, dx$

$= \boxed{\sin x + \cos x + C}$

Aside

$\cos(2x) = \cos^2 x - \sin^2 x$

$= (\cos x - \sin x)(\cos x + \sin x)$

(Q3.) $\int \frac{x^2 + 1}{x^4 - x^2 + 1} \, dx$

$= \int \frac{1 + \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}} \, dx$

$= \int \frac{du}{u^2 + 1}$

$= \boxed{\tan^{-1}\left(x - \frac{1}{x}\right) + C}$

Aside

$x^2 - 1 + \frac{1}{x^2} = x^2 - 2 + \frac{1}{x^2} + 1$

$= \left(x - \frac{1}{x}\right)^2 + 1$

Let $u = x - \frac{1}{x}$

$du = \left(1 + \frac{1}{x^2}\right) dx$

(Q4.) $\int (x + e^x)^2 \, dx$

$= \int (x^2 + 2xe^x + e^{2x}) \, dx$

$= \boxed{\frac{1}{3}x^3 + 2xe^x - 2e^x + \frac{1}{2}e^{2x} + C}$

Aside

	D	I
$+$	$2x$	e^x
$-$	2	e^x
$+$	0	e^x

(Q5.) $\int \csc^3 x \sec x \, dx$

$$= \int \frac{1}{\sin^3 x \cos x} \, dx$$

$$= \int \frac{\cancel{\sin^2 x}\ 1}{\sin^{\cancel{3}} x \cos x} + \frac{\cos^{\cancel{2}} x}{\sin^3 x \cancel{\cos x}} \, dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} + \frac{\cos x}{\sin^3 x} \, dx$$

$$= \int \frac{\sin^{\cancel{2}} x}{\cancel{\sin x} \cos x} + \frac{\cos^{\cancel{2}} x}{\sin x \cancel{\cos x}} + \frac{\cos x}{\sin^3 x} \, dx$$

$$= \int \frac{\sin x}{\cos x} \, dx + \int \frac{\cos x}{\sin x} \, dx + \int \frac{\cos x}{\sin^3 x} \, dx$$

$$= \ln|\sec x| + \ln|\sin x| - \frac{1}{2 \sin^2 x}$$

$$= \boxed{\ln|\tan x| - \frac{1}{2} \csc^2 x + C}$$

Aside

$$1 = \sin^2 x + \cos^2 x$$

$$\ln|\sec x| + \ln|\sin x| = \ln(|\sec x| \times |\sin x|)$$
$$= \ln|\tan x|$$

(Q6.) $\int \frac{\cos x}{\sin^2 x - 5 \sin x - 6} \, dx$

$$= \int \frac{1}{u^2 - 5u - 6} \, du$$

$$= \int \frac{\frac{1}{7}}{u-6} + \frac{-\frac{1}{7}}{u+1} \, du$$

$$= \frac{1}{7} \ln|u-6| - \frac{1}{7} \ln|u+1|$$

$$= \boxed{\frac{1}{7} \frac{\ln|\sin x - 6|}{\ln|\sin x + 1|} + C}$$

Aside

Let $u = \sin x$

$du = \cos x \, dx$

$$u^2 - 5u - 6 = (u-6)(u+1)$$

$u - 6 = 0$ when $u = 6$

sub $u = 6$ into $u + 1 = 7$

$u + 1 = 0$ when $u = -1$

sub $u = -1$ into $u - 6 = -7$

(Q7.) $\int \frac{1}{\sqrt{e^x}} \, dx$

$$= \int \frac{1}{e^{\frac{x}{2}}} \, dx$$

$$= \int e^{-\frac{x}{2}} \, dx$$

$$= -2e^{-\frac{x}{2}}$$

$$= \boxed{\frac{-2}{\sqrt{e^x}} + C}$$

Aside

$$\sqrt{n} = n^{\frac{1}{2}}$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$$

(Q8.) $\displaystyle\int \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx$

$= \displaystyle\int \frac{\cancel{e^x}\,u}{u^2+1+3}\,\frac{2\,u}{\cancel{e^x}}\,du$

$= 2\displaystyle\int \frac{u^2+4-4}{u^2+4}\,du$

$= 2\left(\displaystyle\int 1\,du - 4\int \frac{1}{u^2+2^2}\,du\right)$

$= 2\left(u - 4\,\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right)$

$= \boxed{2\sqrt{e^x-1}-4\tan^{-1}\left(\frac{\sqrt{e^x-1}}{2}\right)+C}$

Aside

Let $u = \sqrt{e^x-1}$

$du = \dfrac{e^x}{2\sqrt{e^x-1}}\,dx$

$dx = \dfrac{2\sqrt{e^x-1}}{e^x}\,du$

$u^2+1 = e^x$

(Q9.) $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$

$= \displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

$= \displaystyle\int \frac{1}{\cancel{\sqrt{x}}\times u}\,2\cancel{\sqrt{x}}\,du$

$= \boxed{2\ln(\sqrt{x}+1)+C}$

Aside

$\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx \neq \ln|x+\sqrt{x}|+C$

Let $u = \sqrt{x}+1$

$du = \dfrac{1}{2\sqrt{x}}\,dx$

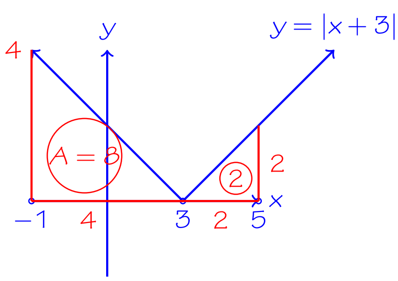
$dx = 2\sqrt{x}\,du$

(Q10.) $\displaystyle\int_{-1}^{5} |x-3|\,dx$

$= 8 + 2$

$= \boxed{10}$

Aside



(Q11.) $\displaystyle\int \frac{\sin x}{\sec^{2019} x}\,dx$

$$= \int \cos^{2019} x \sin x \, dx$$

$$= \int u^{2019}\cancel{\sin x}\, -\frac{du}{\cancel{\sin x}}$$

$$= \boxed{-\frac{1}{2020}\cos^{2020} x + C}$$

Aside

Let $u = \cos x$

$du = -\sin x \, dx$

$dx = -\dfrac{du}{\sin x}$

(Q12.) $\displaystyle\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx$

$$= -\sin^{-1} x\sqrt{1-x^2} + \int 1 \, dx$$

$$= \boxed{-\sin^{-1} x\sqrt{1-x^2} + x + C}$$

Aside

$$\begin{array}{ccc} & D & I \\ + & \sin^{-1} x & \frac{x}{\sqrt{1-x^2}} \\ - & \frac{1}{\sqrt{1-x^2}} & \to -\sqrt{1-x^2} \end{array}$$

$$\int u\,dv = \int -\frac{1}{\cancel{\sqrt{1-x^2}}} \times -\cancel{\sqrt{1-x^2}}\,dx$$

$$= \int 1 \, dx$$

(Q13.) $\displaystyle\int \frac{2\sin x}{\sin(2x)}\,dx$

$$= \int \frac{\cancel{2\sin x}}{\cancel{2\sin x}\cos x}\,dx$$

$$= \int \sec x \, dx$$

$$= \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$

Aside

$\sin 2x = 2\sin x \cos x$

Let $u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x \, dx$

$= \sec x(\tan x + \sec x)\,dx$

(Q14.) $\int \cos^2(2x)\,dx$

$$= \frac{1}{2}\int (1+\cos\ 4x)\,dx$$

$$= \frac{1}{2}(x+\frac{1}{4}\sin 4x)$$

$$= \boxed{\frac{1}{2}x+\frac{1}{8}\sin 4x + C}$$

Aside

$$\cos^2\ x = \frac{1}{2}(1+\cos(2\ x))$$

$$\cos^2 2x = \frac{1}{2}(1+\cos(4x))$$

Aside

$$(x^3+1) = (x+1)(x^2-x+1)$$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$x+1 = 0$ when $x = -1$

substituting $x = -1$ into $x^2 - x + 1 = 3$

$$A = \frac{1}{3}$$

used x = -1, now we sub x = 0

$$1 = \frac{\frac{1}{3}}{0+1} + \frac{B\times 0 + C}{0^2 - 0 + 1}$$

$$1 = \frac{1}{3} + C$$

$$C = \frac{2}{3}$$

use x = 1

$$\frac{1}{(1+1)(1^2-1+1)} = \frac{\frac{1}{3}}{1+1} + \frac{B\times 1+\frac{2}{3}}{1^2-1+1}$$

$$\frac{1}{2} = \frac{1}{6} + B + \frac{2}{3}$$

$$B = -\frac{1}{3}$$

$$\frac{d}{dx}(x^2-x+1) = 2x-1$$

$$x^2 - x + 1 = x^2 - x + \frac{1}{4} + \frac{3}{4}$$

(Q15.) $\int \frac{1}{x^3+1}\,dx$

$$= \frac{1}{3}\int \frac{1}{x+1}\,dx - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\int \frac{1}{x+1}\,dx - \frac{1}{3}\,\frac{1}{2}\int \frac{2(x-1-3)}{x^2-x+1}\,dx$$

$$= I_1 - \frac{1}{6}\int \left(\frac{2x-1}{x^2-x+1} - \frac{3}{(x^2+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right)dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}(\ln(x^2-x+1)) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \boxed{\frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) + C}$$

(Q16.) $\int x\sin^2 x\,dx$

$$= \frac{1}{2}\int x\,(1-\cos(2x))\,dx$$
$$= \frac{1}{2}(\int x\,dx + \int -x\cos(2x)\,dx)$$
$$= \frac{1}{2}(\frac{1}{2}x^2 - \frac{1}{2}x\sin(2x) - \frac{1}{4}\cos(2x))$$
$$= \boxed{\frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x) + C}$$

Aside

$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$

$$\begin{array}{ccc} & D & I \\ + & -x & \cos^{2x} \\ - & -1 & \frac{1}{2}\sin 2x \\ + & 0 & -\frac{1}{4}\cos 2x \end{array}$$

(Q17.) $\int \left(x+\frac{1}{x}\right)^2\,dx$

$$= \int x^2 + 2 + x^{-2}\,dx$$
$$= \boxed{\frac{1}{3}x^3 + 2x - \frac{1}{x} + C}$$

(Q18.) $\int \frac{3}{x^2+4x+29}\,dx$

$$=3\int \frac{1}{(x+2)^2+5^2}\,dx$$
$$= \boxed{\frac{3}{5}\tan^{-1}(\frac{x+2}{5}) + C}$$

Aside

$$x^2+4x+29 = x^2+4x+4+25$$
$$= (x+2)^2+5^2$$

(Q19.) $\int \cot^5 x\,dx$

$$= \int \frac{\cos^5 x}{\sin^5 x}\,dx$$
$$= \int \frac{(1-\sin^2 x)^2\cos x}{\sin^5 x}\,dx$$
$$= \int \frac{(1-u^2)^2}{u^5}\,du$$
$$= \int u^{-5} - 2u^{-3} + u^{-1}\,du$$
$$= \boxed{-\frac{1}{4}\csc^4 x + \csc^2 x + \ln|\sin x| + C}$$

Aside

$\cos^4 x = (1-\sin^2 x)^2$

Let $u = \sin x$

$du = \cos x$

$\frac{1}{\sin x} = \csc x$

(Q20.) $\displaystyle\int_{-1}^{1} \frac{\tan x}{x^4 - x^2 + 1}\, dx$

$= \boxed{0}$

Aside

$\dfrac{\text{odd } f(x)}{\text{even } f(x)} = \text{odd } f(x)$

$\displaystyle\int_{-a}^{a} \text{odd } f(x)\, dx = 0$

(Q21.) $\displaystyle\int \sin^3 x \cos^2 x\, dx$

$= \displaystyle\int (1 - \cos^2 x) \cos^2 x \sin x\, dx$

$= -\displaystyle\int (1 - u^2)\, u^2\, du$

$= -\displaystyle\int (u^2 - u^4)\, du$

$= \boxed{-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C}$

Aside

$\sin^3 x = \sin^2 x \times \sin x$

$= (1 - \cos^2 x) \times \sin x$

Let $u = \cos x$

$du = -\sin x\, dx$

(Q22.) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\, dx$

$= \displaystyle\int \frac{x^{-3}}{\sqrt{1+x^{-2}}}\, dx$

$= -\dfrac{1}{2}\displaystyle\int \frac{1}{\sqrt{u}}\, du$

$= -\dfrac{1}{2}\displaystyle\int u^{-\frac{1}{2}}\, du$

$= \boxed{-\sqrt{1 + \frac{1}{x^2}} + C}$

Aside

$x^2\sqrt{x^2+1} = x^2\sqrt{x^2\left(\frac{x^2}{x^2} + \frac{1}{x^2}\right)}$

$= x^2 x\sqrt{1 + \frac{1}{x^2}}$

$= \dfrac{x^{-3}}{\sqrt{1+x^{-2}}}$

Let $u = 1 + x^{-2}$

$du = 2x^{-3}dx$

(Q23.) $\displaystyle\int \sin x \sec x \tan x\, dx$

$= \displaystyle\int \tan^2 x\, dx$

$= \displaystyle\int (\sec^2 x - 1)\, dx$

$= \boxed{\tan x - x + C}$

Aside

$\sin x \sec x = \dfrac{\sin x}{\cos x}$

$= \tan x$

(Q24.) $\int \sec^3 x\,dx$

$= \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$

$= \boxed{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C}$

Aside

	D	I
$+$	$\sec x$	$\sec^2 x$
$-$	$\sec x \tan x$	$\tan x$
.		

$\tan^2 x = \sec^2 x - 1$

(Q25.) $\int \frac{1}{x\sqrt{9x^2-1}}\,dx$

$= \int \frac{1}{\frac{1}{3}\sec\theta\sqrt{\sec^2\theta - 1}}\,dx$

$= \int \frac{1}{\cancel{\frac{1}{3}\sec\theta\tan\theta}}\,\cancel{\frac{1}{3}\sec\theta\tan\theta}\,d\theta$

$= \int 1\,d\theta$

$= \theta$

$= \boxed{\sec^{-1}(3x) + C}$

Aside

$9x^2 = (3x)^2$

Let $3x = \sec\theta$

$x = \frac{1}{3}\sec\theta$

$dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$

$\sec^2\theta = \tan^2\theta + 1$

$\sec^2\theta - 1 = \tan^2\theta$

$\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta}$

$\sqrt{\sec^2\theta - 1} = \tan\theta$

$\sec\theta = 3x$

$\theta = \sec^{-1} 3x$

(Q26.) $\int \cos\sqrt{x}\,dx$

$= 2\int u\cos u\,du$

$= 2u\sin u + 2\cos u$

$= \boxed{2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}\,dx + C}$

Aside

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

	D	I
$+$	$2u$	$\cos u$
$-$	2	$\sin u$
$+$	0	$-\cos u$

(Q27.) $\int \csc x \, dx$

$$= \int \frac{\csc x \, (\csc x - \cot x)}{(\csc x - \cot x)} \, dx$$

$$= \int \frac{1}{u} \, du$$

$$= \boxed{\ln|\csc x - \cot x| + C}$$

Aside

Let $u = \csc x - \cot x$

$du = -\csc x \cot x + \cos^2 x \, dx$

(Q28.) $\int \sqrt{x^2 + 4x + 13} \, dx$

$$= \int \sqrt{(x+2)^2 + 3^2} \, dx$$

$$= \int \sqrt{(3\tan\theta)^2 + 3^2} \, (3\sec^2\theta) \, d\theta$$

$$= \int 3\sqrt{(\tan\theta)^2 + 1} \, (3\sec^2\theta) \, d\theta$$

$$= 9 \int \sec^3\theta \, d\theta$$

$$= 9\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right)$$

$$= \frac{\cancel{9}}{2} \frac{\sqrt{x^2+4x+13}}{\cancel{3}} \times \frac{(x+2)}{\cancel{3}} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+4x+13}}{3} + \frac{x+2}{3}\right| + C_1$$

$$= \boxed{\frac{1}{2}(x+2)\sqrt{x^2+4x+13} + \frac{9}{2}\ln\left(\sqrt{x^2+4x+13} + (x+2)\right) + C_2}$$

Aside

$\sqrt{x^2+4x+13} = \sqrt{x^2 + 4x + 4 + 9}$

$= \sqrt{(x+2)^2 + 3^2}$

Let $x + 2 = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

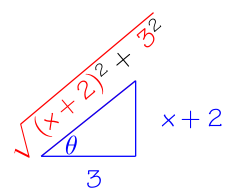
$\sqrt{\tan^2\theta + 1} = \sqrt{\sec^2\theta}$

$= \sec\theta$

recall $\int \sec^3 x \, dx$

$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$

$\tan\theta = \frac{x+2}{3}$



$C_2 = 3 + C_1$

(Q29.) $\int e^{2x}\cos x \, dx$

$$5\int e^{2x}\cos x = e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x \, dx + 4\int e^{2x}\cos x$$

$$\int e^{2x}\cos x \, dx = \boxed{\frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x + C}$$

Aside

	D	I
+	e^{2x}	$\cos x$
−	$2e^{2x}$	$\sin x$
+	$4e^{2x}$	$-\cos x$

(Q30.) $\int_3^5 (x-3)^9 \, dx$

$$= \int_{u=0}^{u=2} u^9 \, du$$

$$= \left[\frac{1}{10} u^{10} \right]_0^2$$

$$= \frac{1}{10} 2^{10} - \frac{1}{10} 0^{10}$$

$$= \frac{1}{10} 2^{10} - \frac{1}{10} 0^{10}$$

$$= \boxed{\frac{512}{5}}$$

Aside

Let $u = x - 3$

$du = dx$

(Q31.) $\int \frac{1}{\sqrt{x - x^{\frac{3}{2}}}} \, dx$

$$= \int \frac{1}{\sqrt{x}\sqrt{1-\sqrt{x}}} \, dx$$

$$= \int \frac{-2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\sqrt{u}} \, du$$

$$= -2 \int 2u^{-\frac{1}{2}} {}^{+1=\frac{1}{2}} \, du$$

$$= -4u^{\frac{1}{2}}$$

$$= \boxed{-4\sqrt{1-\sqrt{x}} + C}$$

Aside

$$\sqrt{x - x^{\frac{3}{2}}} = \sqrt{x(1 - x^{\frac{1}{2}})}$$

$$= \sqrt{x}\sqrt{1-\sqrt{x}}$$

Let $u = 1 - \sqrt{x}$

$du = \frac{-1}{2\sqrt{x}} \, dx$

$dx = -2\sqrt{x} \, du$

(Q32.) $\int \frac{1}{\sqrt{x - x^2}} \, dx$

$$= \int \frac{1}{\sqrt{x}\sqrt{1-x}} \, dx$$

$$= \int \frac{1}{\sqrt{x}\sqrt{1-(\sqrt{x})^2}} \, dx$$

$$= \int \frac{1}{\cancel{\sqrt{x}}\sqrt{1-u^2}} \, 2\cancel{\sqrt{x}} \, du$$

$$= 2 \sin^{-1}(u)$$

$$= \boxed{2 \sin^{-1}(\sqrt{x}) + C}$$

Aside

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} \, dx$

$dx = 2\sqrt{x} \, du$

(Q33.) $\int e^{2\ln x}\,dx$

$$= \int \cancel{e}^{\cancel{\ln}(x^2)}\,dx$$

$$= \int x^2\,dx$$

$$= \boxed{\frac{1}{3}x^3 + C}$$

<u>Aside</u>

$y\ln(x) = \ln(x^y)$

<u>Aside</u>

	D	I
$+$	$\ln x$	$x^{-\frac{1}{2}}$
$-$	$-\frac{1}{x}$	$2\sqrt{x}$

(Q34.) $\int \frac{\ln x}{\sqrt{x}}\,dx$

$$=2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx$$

$$= \boxed{2\sqrt{x}\ln(x) - 4\sqrt{x}\,dx + C}$$

$$-\frac{1}{x}\times 2\sqrt{x} = -2\frac{\sqrt{x}}{x}$$

$$= -2\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\sqrt{x}}$$

$$= -\frac{2}{\sqrt{x}}$$

(Q35.) $\int \frac{1}{e^x + e^{-x}}\,dx$

$$= \int \frac{1(e^x)}{e^x(e^x + e^{-x})}\,dx$$

$$= \int \frac{e^x}{(e^x)^2 + 1}\,dx$$

$$= \int \frac{u}{u^2+1}\,du$$

$$= \boxed{\tan^{-1}(e^x) + C}$$

<u>Aside</u>

Let $u = e^x$

$du = e^x\,dx$

(Q36.) $\displaystyle\int \log_2 x \, dx$

$$= \int \frac{\ln x}{\ln 2}\, dx$$

$$= \frac{1}{\ln 2}\int \ln x \, dx$$

$$= \frac{1}{\ln 2}(\, x\ln x - \, x)\; dx$$

$$= \frac{x\ln x}{\ln 2} - \frac{x}{\ln 2}\; dx$$

$$= \boxed{x\log_2(x) - \frac{x}{\ln 2} + C}$$

Aside

	D	I
$+$	$\ln x$	1
$-\int$	$\frac{1}{x}$	$x \; dx$

(Q37.) $\displaystyle\int x^3 \sin(2x)\, dx$

$$= \boxed{-\frac{1}{2}x^3\cos(2x) + \frac{3}{4}x^2\sin(2x) + \frac{3}{4}x\cos(2x) - \frac{3}{8}\sin(2x) + C}$$

Aside

	D	I
$+$	x^3	$\sin(2x)$
$-$	$3x^2$	$-\frac{1}{2}\cos 2x$
$+$	$6x$	$-\frac{1}{4}\sin 2x$
$-$	6	$\frac{1}{8}\cos 2x$
$+$	0	$\frac{1}{16}\sin 2x$

(Q38.) $\displaystyle\int x^2\sqrt[3]{1+x^3}\, dx$

$$= \frac{1}{3}\int u^{\frac{1}{3}}\, du$$

$$= \frac{\cancel{1}}{\cancel{3}}\,\frac{\cancel{3}}{4}u^{\frac{4}{3}}\, du$$

$$= \frac{1}{4}\sqrt[3]{(\,1+x^3)^4} + C$$

$$= \boxed{\frac{1}{4}(1+x^3)\sqrt[3]{1+x^3} + C}$$

Aside

Let $u = 1 + x^3$

$du = 3x^2\, dx$

$\frac{1}{3}du = x^2\, dx$

(Q39.) $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx$

$$= \int \frac{1}{(4\sec^2\theta)^2}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{(4\sec^2\theta)(\cancel{4\sec^2\theta})}\,\cancel{2\sec^2\theta}\,d\theta$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{8}\,\frac{1}{2}\int (1+\cos(2\theta))\,d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{1}{2}\sin(2\theta)\right)$$

$$= \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta$$

$$= \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{\cancel{16}\,8}\,\frac{x}{\sqrt{x^2+4}}\,\frac{\cancel{2}}{\sqrt{x^2+4}}$$

$$= \boxed{\frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\,\frac{x}{x^2+4} + C}$$

Aside

Let $x = 2\tan\theta$

$x^2+4 = 4\tan^2\theta + 4$

$x^2+4 = 4(\tan^2\theta + 1)$

$x^2+4 = 4\sec^2\theta$

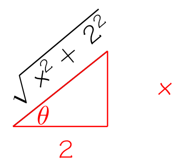
$dx = 2\sec^2\theta\,d\theta$

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

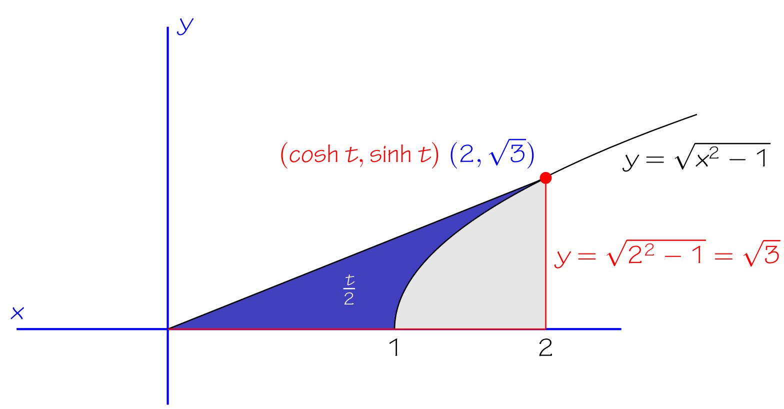
$\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

remember $\tan\theta = \frac{x}{2}$



Aside



(Q40.) $\displaystyle\int_1^2 \sqrt{x^2-1}\,dx$

$$= \boxed{\sqrt{3} - \frac{\cosh^{-1}(2)}{2}}$$

or

$$= \boxed{\sqrt{3} - \frac{\sinh^{-1}(\sqrt{3})}{2}}$$

When $x = 2, y = \sqrt{2^2-1} = \sqrt{3}$

Area of triangle $= \frac{1}{2}\times 2 \times \sqrt{3} - \frac{t}{2}$

$\cosh t = 2, \sinh t = \sqrt{3}$

$t = \cosh^{-1}(2), t = \sinh^{-1}(\sqrt{3})$

Area $= \sqrt{3} - \frac{\cosh^{-1}(2)}{2}$ or $\sqrt{3} - \frac{\sinh^{-1}(\sqrt{3})}{2}$

(Q41.) $\int \sinh x \, dx$

$= \int \frac{e^x - e^{-x}}{2} \, dx$

$= \frac{1}{2}(e^x + e^{-x})$

$= \boxed{\cosh x + C}$

(Q42.) $\int \sinh^2 x \, dx$

$= \frac{1}{2} \int -1 + \cosh(2x) \, dx$

$= \frac{1}{2}(-x + \frac{1}{2}\sinh(2x))$

$= \boxed{\frac{-1}{2}x + \frac{1}{4}\sinh(2x) + C}$

Aside

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\sinh^2 x = (\frac{e^x - e^{-x}}{2})^2$

$= \frac{1}{4}(e^x - e^{-x})^2$

$= \frac{1}{4}(e^{2x} - 2 + e^{-2x})$

$= \frac{1}{4} \times (-2) + \frac{1}{4}(e^{2x} + e^{-2x})$

$= -\frac{1}{2} + \frac{1}{2 \times 2}(e^{2x} + e^{-2x})$

$= -\frac{1}{2} + \frac{1}{2}(\frac{e^{2x} + e^{-2x}}{2})$

$= -\frac{1}{2} + \frac{1}{2}\cosh(2x)$

(Q43.) $\int \sinh^3 x \, dx$

$= \int \sinh^2 \sinh x \, dx$

$= \int (\cosh^2 x - 1) \sinh x \, dx$

$= \int (u^2 - 1) \, du$

$= \frac{1}{3}u^3 - u$

$= \boxed{\frac{1}{3}\cosh^3 x - \cosh x + C}$

Aside

$\cosh^2 x - \sinh^2 x = 1$

$\sinh^2 x = \cosh^2 x - 1$

Let $u = \cosh x$

$du = \sinh x \, dx$

(Q44.) $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

$$=\boxed{\sinh^{-1} x + C}$$

Aside

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

(Q45.) $\displaystyle\int \ln\left(x+\sqrt{1+x^2}\right)\,dx$

$$= x\sinh^{-1} x - \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$= \boxed{x\sinh^{-1} x - \sqrt{1+x^2} + C}$$

Aside

$$\begin{array}{ccc} & D & I \\ + & \sinh^{-1}(x) & 1 \\ - & \int \frac{1}{\sqrt{1+x^2}} & x \end{array}$$

(Q46.) $\displaystyle\int \tanh x\,dx$

$$= \int \frac{\sinh x}{\cosh x}\,dx$$

$$= \int \frac{du}{u}$$

$$= \boxed{\ln|\cosh x| + C}$$

Aside

Let $u = \cosh(x)$

$du = \sinh(x)\,dx$

(Q47.) $\displaystyle\int \operatorname{sech} x\,dx$

$$= \int \frac{1 \times \cosh x}{\cosh x \cosh x}\,dx$$

$$= \int \frac{\cosh x}{1+\sinh^2 x}\,dx$$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \boxed{\tan^{-1}(\sinh x) + C}$$

Aside

Let $u = \sinh x$

$du = \cosh x\,dx$

(Q48.) $\int \tanh^{-1} x \, dx$

$$= x\tanh^{-1} x - \int \frac{x}{1-x^2}\, dx$$

$$= \boxed{x\tanh^{-1} x + \frac{1}{2}\ln|1-x^2| + C}$$

Aside

	D	I
$+$	$\sinh^{-1}(x)$	1
$-$	$\int \frac{1}{1+x^2}$	x

(Q49.) $\int \sqrt{\tanh x}\, dx$

$$= \int u \times \frac{2u}{1-u^4}\, du$$

$$= \int \frac{u^2+1+u^2-1}{(1+u^2)(1-u^2)}\, du$$

$$= \int \frac{\cancel{u^2+1}}{\cancel{(1+u^2)}(1-u^2)} + \frac{-\cancel{(1-u^2)}}{(1+u^2)\cancel{(1-u^2)}}\, du$$

$$= \boxed{\tanh^{-1}(\sqrt{\tanh x}) - \tan^{-1}(\sqrt{\tanh x}) + C}$$

Aside

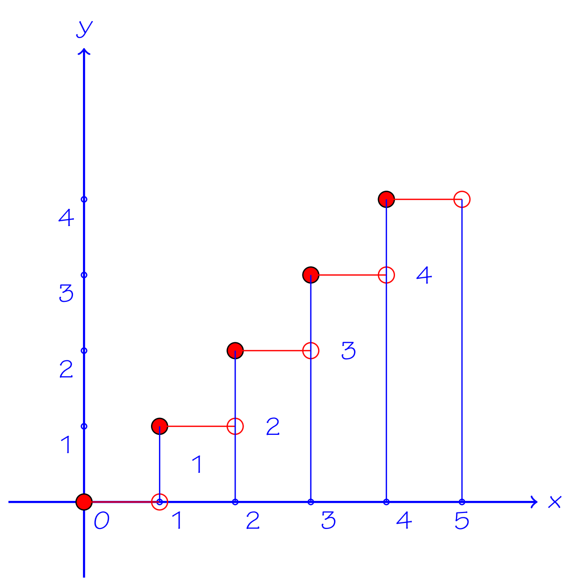
Let $u = \sqrt{\tanh x}$

$x = \tanh^{-1}(u^2)$

$dx = \frac{2u}{1-(u^2)^2}\, du$

$1 - u^4 = (1+u^2)(1-u^2)$

Aside



(Q50.) $\int_0^5 \lfloor x \rfloor \, dx$

$$= 1 + 2 + 3 + 4$$

$$= \boxed{10}$$

(Q51.) $\int \sec^6 x \, dx$

$$= \int (\tan^2 x + 1)^2 \sec^2 x \, dx$$

$$= \int (u^2 + 1)^2 \; du$$

$$= \int (u^4 + 2u^2 + 1) \, du$$

$$= \boxed{\frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C}$$

Aside

$u = \tan x$

$du = \sec^2 x \, dx$

(Q52.) $\int \frac{1}{(5x-2)^4} \, dx$

$$= \frac{1}{5}\int \frac{1}{u^4} \; du$$

$$= \frac{1}{5}\int -\frac{1}{3}u^{-4+1} \; du$$

$$=$$

$$= \boxed{-\frac{1}{15}\frac{1}{(5x+2)^3} + C}$$

Aside

Let $u = 5x - 2$

$du = 5 \, dx$

$\frac{1}{5} du = dx$

(Q53.) $\int \ln\left(1 + x^2\right) \, dx$

$$= x\ln(1+x^2) - 2\int \frac{1+x^2 - 1}{1+x^2} \, dx$$

$$= \boxed{x\ln(1+x^2) - 2x + 2\tan^{-1}(x) + C}$$

Aside

	D	I
+	$\ln(1+x^2)$	1
−	$\int \frac{-2x}{1+x^2}$	x

(Q54.) $\int \frac{1}{x^4 + x} \, dx$

$$= \int \frac{x^{-4}}{1 + x^{-3}} \, dx$$

$$= \int \frac{\cancel{x^{-4}}}{u} \; \frac{du}{-3\cancel{x^{-4}}}$$

$$= \boxed{-\frac{1}{3}\ln\left| 1 + x^{-3} \right| + C}$$

Aside

$x^4 + x = x^4 \left(1 + x^{-3}\right)$

Let $u = 1 + x^{-3}$

$du = -3x^{-4} \, dx$

$dx = \frac{du}{-3x^{-4}}$

(Q55.) $\displaystyle\int \frac{1-\tan x}{1+\tan x}\,dx$

$$= \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}\,dx$$

$$= \int \frac{\cos x - \sin x}{\cancel{\cos x}} \times \frac{\cancel{\cos x}}{\cos x + \sin x}\,dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \int \frac{du}{u}$$

$$= \boxed{\ln|\cos x + \sin x| + C}$$

Aside

Let $u = \cos x + \sin x$

$du = \cos x - \sin x\, dx$

(Q56.) $\displaystyle\int x \sec x \tan x\, dx$

$$= \boxed{x \sec x - \ln|\sec x + \tan x| + C}$$

Aside

	D	I
$+$	x	$\sec x \tan x$
$-$	1	$\sec x$
$+$	0	$\ln\lvert\sec x + \tan x\rvert$

(Q57.) $\displaystyle\int \sec^{-1} x\, dx$

$$= x\sec^{-1} x - \int \frac{\cancel{x}}{\cancel{x}\sqrt{x^2-1}}\,dx$$

$$= x\sec^{-1} x - \int \frac{1}{\sqrt{\sec^2\theta - 1}} \sec\theta \tan\theta\, d\theta$$

$$= x\sec^{-1} x - \int \frac{1}{\cancel{\tan\theta}} \sec\theta\cancel{\tan\theta}\, d\theta$$

$$= x\sec^{-1} x - \int \sec\theta\, d\theta$$

$$= x\sec^{-1} x - \ln|\sec\theta + \tan\theta|$$

$$= \boxed{x\sec^{-1} x - \ln\left|x + \sqrt{x^2-1}\right| + C}$$

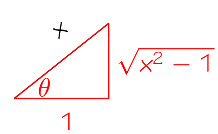
Aside

	D	I
$+$	$\sec^{-1} x$	1
$-$	$\frac{1}{x\sqrt{x^2-1}}$	x

Let $x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$\sec^2\theta - 1 = \tan\theta$



$\tan\theta = \sqrt{x^2-1}$

(Q58.) $\displaystyle\int \frac{1-\cos x}{1+\cos x}\,dx$

$\displaystyle = \int \frac{\cancel{2}\sin^2\left(\frac{x}{2}\right)}{\cancel{2}\cos^2\left(\frac{x}{2}\right)}\,dx$

$\displaystyle = \int \tan^2\left(\frac{x}{2}\right)\,dx$

$\displaystyle = \int \left(\sec^2\left(\frac{x}{2}\right) - 1\right)\,dx$

$\displaystyle = \boxed{2\tan\left(\frac{x}{2}\right) - x + C}$

Aside

$\displaystyle \cos^2\frac{x}{2} = \frac{1}{2}(1+\cos x)$

$\displaystyle \sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)$

(Q59.) $\displaystyle\int x^2\sqrt{x+4}\,dx$

$\displaystyle = \int (u-4)^2\sqrt{u}\;du$

$\displaystyle = \int (u^2 - 8u + 16)u^{\frac{1}{2}}\;du$

$\displaystyle = \int \left(\frac{2}{7}u^{\frac{5}{2}+1} - \frac{2}{5}8^{\frac{3}{2}+1} + 16\times\frac{2}{3}u^{\frac{1}{2}+1}\right)\,du$

$\displaystyle = \boxed{\frac{2}{7}(x+4)^{\frac{7}{2}} - \frac{16}{5}(x+4)^{\frac{5}{2}} + \frac{2}{3}(x+4)^{\frac{3}{2}} + C}$

Aside

Let $u = x+4$

$du = dx$

$x = u-4$

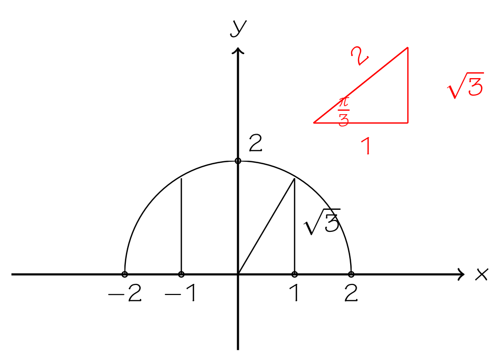
(Q60.) $\displaystyle\int_{-1}^{1}\sqrt{4-x^2}\,dx$

$\displaystyle =2\int_{0}^{1}\sqrt{4-x^2}\,dx$

$\displaystyle =2\left(\frac{1}{2}\times 1\times\sqrt{3} + \frac{1}{2}2^2\times\frac{\pi}{6}\right)$

$\displaystyle = \boxed{\sqrt{3} + \frac{2\pi}{3}}$

Aside



$\displaystyle \theta = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Area of Sector $\displaystyle = \frac{1}{2}r^2\theta$

(Q61.) $\int \sqrt{x^2+4x}\,dx$

$$= \int \sqrt{(x+2)^2 - 2^2}\,dx$$
$$= \int \sqrt{4\sec^2\theta - 4}\; 2\sec\theta\tan\theta\, d\theta$$
$$= 4\int \tan^2\theta \sec\theta\, d\theta$$
$$= 4\int (\sec^2\theta - 1)\sec\theta\, d\theta$$
$$= 4\int \sec^3\theta - \sec\theta\, d\theta$$
$$=4\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right)$$
$$=4\left(\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)$$
$$=\cancel{2}\,\frac{x+2}{2}\,\frac{\sqrt{x^2+4x}}{\cancel{2}} - 2\ln\left|\frac{x+2}{2}+\frac{\sqrt{x^2+4x}}{2}\right| + C_1$$
$$= \boxed{\frac{1}{2}(x+2)\sqrt{x^2+4x} - 2\ln|(x+2)+\sqrt{x^2+4x}| + C_2}$$

Aside

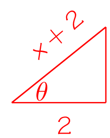
$x^2+4x+4-4 = (x+2)^2 - 2^2$

Let $x+2 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$4(\sec^2\theta - 1) = 4\tan^2\theta$

$\sec\theta = \dfrac{x+2}{2}$



$\sqrt{(x+2)^2 - 4} = \sqrt{x^2+4x}$

(Q62.) $\int x^2 e^{x^3}\,dx$

$$= \frac{1}{3}\int 3x^2 e^{x^3}\,dx$$
$$= \boxed{\frac{1}{3}e^{x^3} + C}$$

(Q63.) $\int x^3 e^{x^2}\,dx$

$$=\frac{1}{2}x^2e^{x^2} - \frac{1}{2}\int 2xe^{x^2}\,dx$$
$$= \boxed{\frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C}$$

Aside

	D	I
+	x^2	xe^{x^2}
−	$2x$	$\frac{1}{2}e^{x^2}$

(Q64.) $\int \tan x \ln(\cos x)\, dx$

$= \int \cancel{\tan x}\, u \frac{du}{\cancel{-\tan x}}$

$= -\int u\, du$

$= \boxed{-\frac{1}{2}(\ln(\cos x))^2 + C}$

Aside

Let $u = \ln(\cos x)$

$du = \frac{-\sin x}{\cos x} dx$

$du = \tan x\, dx$

$dx = \frac{du}{-\tan x}$

Aside

$x^3 - 4x^2 = x^2(x-4)$

$\frac{1}{x^3 - 4x^2} = \frac{Ax+B}{x^2} + \frac{C}{x-4}$

$\frac{1}{x^3 - 4x^2} = \frac{Ax}{x^2} + \frac{B}{x^2} + \frac{C}{x-4}$

$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$

When $x = 4$

$C = \frac{1}{16}$

When $x = 0$

$B = -\frac{1}{4}$

When $x = 1$

$-\frac{1}{3} = A - \frac{1}{4} - \frac{1}{48}$

$-\frac{16}{48} = A - \frac{12}{48} - \frac{1}{48}$

$A = -\frac{1}{16}$

(Q65.) $\int \frac{1}{x^3 - 4x^2}\, dx$

$= \int \frac{-\frac{1}{16}}{x} + \frac{-\frac{1}{4}}{x^2} + \frac{\frac{1}{16}}{x-4}\, dx$

$= \boxed{-\frac{1}{16}\ln|x| + \frac{1}{4x} + \frac{1}{16}\ln|x-4| + C}$

(Q66.) $\int \sin x \cos(2x)\, dx$

$= -\int (2u^2 - 1)\, du$

$= -(\frac{2u^3}{3} - u)$

$= \boxed{-\frac{2}{3}\cos^3 x + \cos x + C}$

Aside

$\cos(2x) = 2\cos^2 x - 1$

Let $u = \cos x$

$du = -\sin x\, dx$

(Q67.) $\int 2^{\ln x}\, dx$

$= \int (\, e^{\ln 2})^{\ln x}\, dx$

$= \int (\, e^{\ln x})^{\ln 2}\, dx$

$= \int (\, x)^{\ln 2}\, dx$

$= \int \frac{1}{(\,\ln 2 + 1)} x^{(\ln 2)\, +1}\, dx$

$= \frac{1}{1+\ln 2} x^{1+\ln x}$

$= \frac{1}{1+\ln 2}\, x \times\; x^{\ln x}$

$= \boxed{\frac{x\, 2^{\ln 2}}{1+\ln 2} + C}$

(Q68.) $\int \sqrt{1+\cos(2x)}\, dx$

$= \int \sqrt{2\cos^2 x}\, dx$

$= \sqrt{2} \int \cos x\, dx$

$= \boxed{\sqrt{2}\sin x + C}$

Aside

$2\cos(2x) = 1 + \cos^2 x$

(Q69.) $\int \frac{1}{1+\tan x}\, dx$

$= \frac{1}{2} \int \frac{1 - \tan x + 1 + \tan x}{1+\tan x}\, dx$

$= \frac{1}{2} \int \frac{1-\tan x}{1+\tan x} + \frac{1+\tan x}{1+\tan x}\, dx$

$=$

$= \boxed{\frac{1}{2}\ln|\cos x + \sin x| + \frac{1}{2}x + C}$

Aside

recall $\int \frac{1-\tan x}{1+\tan x} = \ln|\cos x + \sin x|$ from Q55

(Q70.) $\displaystyle\int_{\frac{1}{e}}^{e} \frac{\sqrt{1-(\ln x)^2}}{x}\,dx$

$\displaystyle = \int_{-1}^{1} \frac{\sqrt{1-u^2}}{\cancel{x}}\,\cancel{x}\,du$

$\displaystyle = \boxed{\frac{\pi}{2}}$

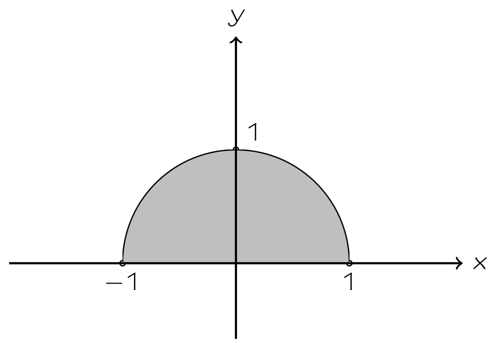
Aside

Let $u = \ln x$

$du = \dfrac{1}{x}\,dx$

When $x = \dfrac{1}{e}, u = -1$

When $x = e, u = 1$



$\text{Area} = \dfrac{\pi r^2}{2}$

$= \dfrac{\pi}{2}$

(Q71.) $\displaystyle\int \frac{1}{\sqrt[3]{x}+1}\,dx$

$\displaystyle = \int \frac{1}{u}\,3(u-1)^2\,du$

$\displaystyle = 3\int \frac{(u-1)^2}{u}\,du$

$\displaystyle = 3\int \frac{u^2-2u+1}{u}\,du$

$\displaystyle = 3\int (u-2+\frac{1}{u})\,du$

$\displaystyle = 3(\frac{1}{2}u^2 - 2u + \ln|u|)$

$\displaystyle = \boxed{\frac{3}{2}(\sqrt[3]{x}+1)^2 - 6(\sqrt[3]{x}+1) + \ln|\sqrt[3]{x}+1| + C}$

Aside

Let $u = \sqrt[3]{x}+1$

$x = (u-1)^3$

$dx = 3(u-1)^2$

(Q72.) $\int \frac{1}{\sqrt[3]{x+1}}\, dx$

$$= \frac{3}{2}\int u^{-\frac{1}{3}+1}\, du$$

$$= \frac{3}{2}(x+1)^{\frac{2}{3}}$$

$$= \boxed{\frac{3}{2}\sqrt[3]{(x+1)^2}+C}$$

Aside

Let $u = x+1$

$du = dx$

(Q73.) $\int (\sin x + \cos x)^2\, dx$

$$= \int \sin^2 x + 2\sin x \cos x + \cos^2 x\, dx$$

$$= \int (1+\sin(2x))\, dx$$

$$= \boxed{x - \frac{1}{2}\cos(2x) + C}$$

Aside

$\sin^2 x + \cos^2 x = 1$

$2\sin x \cos x = \sin 2x$

(Q74.) $\int 2x \ln(1+x)\, dx$

$$= x^2\ln(1+x) - \int \frac{x^2}{1+x^2}\, dx$$

$$= x^2\ln(1+x) - \int \left(x - 1 + \frac{1}{x+1}\right) dx$$

$$= \boxed{x^2\ln(1+x) - \frac{1}{2}x^2 + x - \ln(x+1) + C}$$

Aside

	D	I
$+$	$\ln(1+x)$	$2x$
$-$	$\frac{1}{1+x}$	x^2

$$\begin{array}{r} x-1 \\ = x+1 \overline{)\, x^2} \\ -(x^2+x) \\ \hline -x \\ -(-x-1) \\ \hline 1 \end{array}$$

$$(Q75.)\ \int \frac{1}{x\left(1+\sin^2(\ln x)\right)}\,dx$$

$$= \int \frac{\frac{1}{\cos^2 u}}{\frac{1}{\cos^2 u} + \frac{\sin^2(u)}{\cos^2 u}}\,du$$

$$= \int \frac{\sec^2 u}{\sec^2 u + \tan^2 u}\,du$$

$$= \int \frac{\sec^2 u}{2\tan^2 x + 1}\,du$$

$$= \frac{1}{2w^2+1}\,dw$$

$$= \frac{1}{(\sqrt{2}w)^2+1}\,dw$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}w)$$

$$= \boxed{\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan(\ln x)) + C}$$

Aside

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$\sec^2 x = \tan^2 x + 1$

Let $w = \tan u$

$dw = \sec^2 u\,du$

$$(Q76.)\ \int \sqrt{\frac{1-x}{1+x}}\,dx$$

$$= \int \sqrt{\frac{1-x\,(1-x)}{1+x\,(1-x)}}\,dx$$

$$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \boxed{\sin^{-1} x + \sqrt{1-x^2} + C}$$

$$(Q77.)\ \int x^{\frac{x}{\ln x}}\,dx$$

$$= \int (e^{\cancel{\ln x}})^{\frac{x}{\cancel{\ln x}}}\,dx$$

$$= \int e^x\,dx$$

$$= e^x$$

$$= \boxed{x^{\frac{x}{\ln x}} + C}$$

(Q78.) $\int \sin^{-1}(\sqrt{x})\,dx$

$$= \int 2u \sin^{-1} u \; du$$

$$= u^2 \sin^{-1} u - \int \frac{u^2}{\sqrt{1-u^2}}\,du$$

$$= \boxed{x \sin^{-1}\sqrt{x} - \frac{1}{2}\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x}\sqrt{1-x} + C}$$

<u>Aside</u>

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

$du = -\dfrac{1}{2\sqrt{x}}$

	D	I
$+$	$\sin^{-1} u$	$2u$
$-$	$\frac{1}{\sqrt{1-u^2}}$	u^2

$$-\int \frac{u^2}{\sqrt{1-u^2}}\,du$$

Let $u = \sin\theta$

$du = \cos\theta\, d\theta$

$$-\int \frac{\sin^2\theta}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta$$

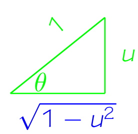
$$= -\frac{1}{2}\int 1 - \cos 2\theta \; d\theta$$

$$= -\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)$$

$$= -\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$\sin\theta = \dfrac{u}{1}$

$\theta = \sin^{-1} u$



(Q79.) $\int \tan^{-1} x\, dx$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)$$

$$= \boxed{x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C}$$

<u>Aside</u>

	D	I
$+$	$\tan^{-1} x$	1
$-$	$\frac{1}{1+x^2}$	x

(Q80.) $\displaystyle\int_0^5 f(x)dx$, where $f(x) = \begin{cases} 10 & \text{if } x \le 2 \\ 3x^2 - 2 & \text{if } x > 2 \end{cases} \quad dx$

$$= \int_0^2 10\, dx + \int_2^5 (3x^2 - 2)\, dx$$

$$= 20 + [x^3 - 2x]_2^5$$

$$= 20 + [5^3 - 10 - 2^3 + 4]$$

$$= \boxed{131}$$

(Q81.) $\displaystyle\int \frac{\sin\left(\frac{1}{x}\right)}{x^3}\, dx$

$$= \boxed{\frac{\cos\frac{1}{x}}{x} - \sin\left(\frac{1}{x}\right) + C}$$

Aside

	D	I
$+$	$\frac{1}{x}$	$\frac{\sin\frac{1}{x}}{x^2}$
$-$	$\frac{-1}{x^2}$	$\cos\frac{1}{x}$

(Q82.) $\displaystyle\int \frac{x-1}{x^4-1}\, dx$

$$= \int \frac{\frac{1}{2}}{x+1}\, dx - \frac{1}{2}\int \frac{x}{x^2+1}\, dx + \frac{1}{2}\int \frac{1}{x^2+1}\, dx$$

$$= \boxed{\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1} x + C}$$

Aside

$$x^4 - 1 = (x-1)(x+1)(x^2+1)$$

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$A = \frac{1}{2}$$

using $x = 0$, $C = \frac{1}{2}$

using $x = 1$, $B = -\frac{1}{2}$

(Q83.) $\displaystyle\int \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx$

$$= \int \sqrt{1+ x^2 - \frac{1}{2} + \frac{1}{16x^2}}\,dx$$

$$= \int \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\,dx$$

$$= \int \sqrt{(x+\frac{1}{4x})^2}\,dx$$

$$= \int x + \frac{1}{4x}\,dx$$

$$= \boxed{\frac{1}{2}x^2 + \frac{1}{4}\ln|x| + C}$$

(Q84.) $\displaystyle\int \frac{e^{\tan x}}{1-\sin^2 x}\,dx$

$$= \int sec^2\, x e^{\tan x}\,dx$$

$$= \boxed{e^{\tan x} + C}$$

Aside

$1-\sin^2 x = cos^2\, x$

(Q85.) $\displaystyle\int \frac{\tan^{-1} x}{x^2}\,dx$

$$= -\frac{\tan^{-1} x}{x} + \int \frac{1}{x(1+x^2)}\,dx$$

$$= -\frac{\tan^{-1} x}{x} + \int \frac{1}{x \times x^2(x^{-2}+1)}\,dx$$

$$= -\frac{\tan^{-1} x}{x} - \frac{1}{2}\int \frac{-2x^{-3}}{x^{-2}+1}\,dx$$

$$= \boxed{-\frac{\tan^{-1} x}{x} - \frac{1}{2}\ln(x^{-2}+1) + C}$$

Aside

	D	I
$+$	$\tan^{-1} x$	$\frac{1}{x^2}$
$-$	$\frac{1}{1+x^2}$	$-\frac{1}{x}$

(Q86.) $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx$

$$= \int u\,du$$

$$= \frac{1}{2}u^2$$

$$= \boxed{\frac{1}{2}(\tan^{-1} x)^2 + C}$$

Aside

Let $u = \tan^{-1} x$

$du = \dfrac{1}{1+x^2}\,dx$

$x = \tan u$

(Q87.) $\int (\ln x)^2\, dx$

$=x(\ln x)^2 - \int 2\ln x\, dx$

$=x(\ln x)^2 - (\, 2x\ln x - \int 2\, dx)$

$= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}$

Aside

	D	I
$+$	$(\ln x)^2$	1
$-$	$2\frac{\ln x}{x}$	x

	D	I
$+$	$2\ln x$	1
$-$	$\frac{2}{x}$	x

(Q88.) $\int \frac{\sqrt{x^2+4}}{x^2}\, dx$

$= \int \frac{\cancel{2}\sec\theta}{\cancel{4}\tan^2\theta}\, \cancel{2}\sec^2\theta\, d\theta$

$= \int \frac{1}{\cos^{\cancel{3}}\theta} \times \frac{\cancel{\cos^2\theta}}{\sin^2\theta} d\theta$

$= \int (\frac{\cancel{\sin^2\theta}}{\cos\theta\cancel{\sin^2\theta}} + \frac{\cos^{\cancel{2}}\theta}{\cancel{\cos\theta}\sin^2\theta})d\theta$

$= \ln|\sec\theta + \tan\theta| - \csc\theta$

$= \ln(\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}) - \frac{\sqrt{x^2+4}}{x} + C_1$

$= \boxed{\ln(\sqrt{x^2+4} + x) - \frac{\sqrt{x^2+4}}{x} + C_2}$

Aside

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

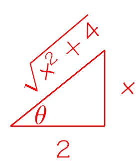
$\sqrt{(2\tan\theta)^2+4} = \sqrt{4(\tan^2\theta + 1)}$

$= \sqrt{4(\sec^2\theta)}$

$= 2\sec\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\tan\theta = \frac{x}{2}$



(Q89.) $\int \frac{\sqrt{x+4}}{x}\, dx$

$= \int \frac{u}{u^2-4} 2u\, du$

$= \int 2 + \frac{8}{u^2-4}\, du$

$= \int 2 + \frac{8}{(u-2)(u+2)}\, du$

$= \int 2 + \frac{2}{(u-2)} + \frac{-2}{(u+2)}\, du$

$=2u + 2\ln|u-2| - 2\ln|u+2|$

$= \boxed{2\sqrt{x-4} + 2\ln|\frac{\sqrt{x-4}-2}{\sqrt{x-4}+2}| + C}$

Aside

Let $u = \sqrt{x+4}$

$u^2 = x+4$

$u^2 - 4 = x$

$dx = 2u\, du$

$$\begin{array}{r} 2 \\ = u^2-4\,\overline{)\,2u^2} \\ -(\,2u^2-8) \\ \hline =8 \end{array}$$

(Q90.) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx$

$$= \boxed{\frac{\pi}{4}}$$

(Q91.) $\displaystyle\int \frac{x}{1+x^4}\, dx$

$$= \frac{1}{2}\int \frac{2x}{1+(x^2)^2}\, dx$$

$$= \boxed{\frac{1}{2}\tan^{-1}(x^2) + C}$$

(Q92.) $\displaystyle\int e^{\sqrt{x}}\, dx$

$$= \int 2ue^{u}\, du$$

$$= \boxed{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$$

Aside

Let $u = \sqrt{x}$

$u^2 = x$

$2u\, du = dx$

	D	I
+	$2u$	e^u
−	2	e^u
+	0	e^u

(Q93.) $\displaystyle\int \frac{1}{\csc^3 x}\, dx$

$$= \int \sin^3 x\, dx$$

$$= \int \sin^2 x \sin x\, dx$$

$$= \int (1-\cos^2 x)\sin x\, dx$$

$$= -\int (1-u^2)\, du$$

$$= -(u - \frac{1}{3}u^3)$$

$$= \boxed{\frac{1}{3}\cos^3 x - \cos x + C}$$

Aside

Let $u = \cos x$

$du = -\sin x\, dx$

(Q94.) $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

$\displaystyle = \int \frac{u\cancel{\cos u}}{\cancel{\cos u}}\, du$

$\displaystyle = \frac{1}{2}u^2$

$\displaystyle = \boxed{\frac{1}{2}(\sin^{-1} x)^2 + C}$

Aside

Let $u = \sin^{-1} x$

$x = \sin u$

$dx = \cos u \, du$

$\sqrt{1 - \sin^2 u} = \cos u$

(Q95.) $\displaystyle\int \sqrt{1 + \sin(2x)}\, dx$

$\displaystyle = \int \sqrt{\sin^2 x + 2\sin x \cos x + \cos^2 x}\, dx$

$\displaystyle = \int \sqrt{(\sin x + \cos x)^2}\, dx$

$\displaystyle = \int \sin x + \cos x \, dx$

$= \boxed{-\cos x + \sin x + C}$

(Q96.) $\displaystyle\int \sqrt[4]{x}\, dx$

$\displaystyle = \int \frac{4}{5} x^{\frac{1}{4}+1}\, dx$

$\displaystyle = \frac{4}{5} x^{\frac{5}{4}}$

$\displaystyle = \frac{4}{5}\sqrt[4]{x^5}$

$\displaystyle = \boxed{\frac{4}{5} x \sqrt[4]{x} + C}$

(Q97.) $\displaystyle\int \frac{1}{1+e^x}\, dx$

$\displaystyle = \int \frac{1 + e^x - e^x}{1+e^x}\, dx$

$\displaystyle = \int 1 dx - \int \frac{e^x}{1+e^x}\, dx$

$= \boxed{x - \ln(1+e^x) + C}$

(Q98.) $\int \sqrt{1+e^x}\,dx$

$$= \int u \frac{2u}{u^2-1}\,du$$

$$= \int 2 + \frac{2}{u^2-1}\,du$$

$$= \int 2 + \frac{2}{(u-1)(u+1)}\,du$$

$$= \int 2 + \frac{1}{(u-1)} - \frac{1}{(u+1)}\,du$$

$$= 2\,u + \ln\frac{|\,u-1|}{|\,u+1|}$$

$$= \boxed{2\,\sqrt{1+e^x} + \ln(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}) + C}$$

Aside

Let $u = \sqrt{1+e^x}$

$du = \dfrac{e^x}{2\sqrt{1+e^x}}\,dx$

$dx = \dfrac{2\sqrt{1+e^x}}{e^x}\,du$

$u^2 - 1 = e^x$

$$\begin{array}{r} 2 \\ u^2-1 \overline{)\, 2u^2} \\ \underline{-(\,2u^2-2)} \\ 2 \end{array}$$

(Q99.) $\int \dfrac{\sqrt{\tan x}}{\sin(2x)}\,dx$

$$= \int \frac{u}{2\sin x \cos x}\,dx$$

$$= \int \frac{u}{\cancel{2}\sin x \cos x} \times \frac{\cancel{2}u}{\sec^2 x}\,du$$

$$= \int \frac{u^2}{\sin x \cancel{\cos x}} \times \frac{\cos^{\cancel{2}} x}{1}\,du$$

$$= \int u^2 \cot x\,du$$

$$= \int u^2\,\frac{1}{u^2}\,du$$

$$= \int 1\,du$$

$$= u$$

$$= \boxed{\sqrt{\tan x} + C}$$

Aside

Let $u = \sqrt{\tan x}$

$du = \dfrac{\sec^2 x}{2\sqrt{\tan x}}\,dx$

$dx = \dfrac{2\sqrt{\tan x}}{\sec^2 x}\,du$

$\tan x = u^2$

$\dfrac{1}{\tan x} = \dfrac{1}{u^2}$

$$(Q100.)\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$$

$$=\int \frac{1(\ 1-\sin x)}{1+\sin x(\ 1-\sin x)}\,dx$$

$$=\int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$=\int (\sec^2 x-\sec x\tan x)\,dx$$

$$=[\tan x-\sec x]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{\sin x}{\cos x}-\frac{1}{\cos x}\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{\sin x-1(\ \sin x+1)}{\cos x(\ \sin x+1)}\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{-\cos^{\cancel{2}}\cancel{x}}{\cancel{\cos x}(\ 1+\sin x)}\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{-\cos x}{1+\sin x}\right]_0^{\frac{\pi}{2}}$$

$$=\boxed{1}$$

$$(Q101.)\int \left(\frac{\sin x}{x}+\ln x\cos x\right)\,dx$$

$$=\boxed{\sin x\ln x+C}$$