HMMT February 2019

February 16, 2019

Team Round

1. [**20**] Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and AC intersects XY at point Z. Prove that
$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{AC}{AZ}.$$

2. [**20**] Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all positive integers, and let f be a bijection from $\mathbb{N}$ to $\mathbb{N}$. Must there exist some positive integer n such that $(f(1), f(2), \ldots, f(n))$ is a permutation of $(1, 2, \ldots, n)$?

3. [**25**] For any angle $0 < \theta < \pi/2$, show that
$$0 < \sin\theta + \cos\theta + \tan\theta + \cot\theta - \sec\theta - \csc\theta < 1.$$

4. [**35**] Find all positive integers n for which there do not exist n consecutive composite positive integers less than $n!$.

5. [**40**] Find all positive integers n such that the unit segments of an $n \times n$ grid of unit squares can be partitioned into groups of three such that the segments of each group share a common vertex.

6. [**45**] Scalene triangle ABC satisfies $\angle A = 60°$. Let the circumcenter of ABC be O, the orthocenter be H, and the incenter be I. Let D, T be the points where line BC intersects the internal and external angle bisectors of $\angle A$, respectively. Choose point X on the circumcircle of $\triangle IHO$ such that $HX \parallel AI$. Prove that $OD \perp TX$.

7. [**50**] A convex polygon on the plane is called *wide* if the projection of the polygon onto any line in the same plane is a segment with length at least 1. Prove that a circle of radius $\frac{1}{3}$ can be placed completely inside any wide polygon.

8. [**50**] Can the set of lattice points $\{(x, y) | x, y \in \mathbb{Z}, 1 \le x, y \le 252, x \ne y\}$ be colored using 10 distinct colors such that for all $a \ne b, b \ne c$, the colors of (a, b) and (b, c) are distinct?

9. [**55**] Let $p > 2$ be a prime number. $\mathbb{F}_p[x]$ is defined as the set of all polynomials in x with coefficients in $\mathbb{F}_p$ (the integers modulo p with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_p$ for each nonnegative integer k. For example, $(x + 2)(2x + 3) = 2x^2 + 2x + 1$ in $\mathbb{F}_5[x]$ because the corresponding coefficients are equal modulo 5.

   Let $f, g \in \mathbb{F}_p[x]$. The pair (f, g) is called *compositional* if
$$f(g(x)) \equiv x^{p^2} - x$$
   in $\mathbb{F}_p[x]$. Find, with proof, the number of compositional pairs (in terms of p).

10. [**60**] Prove that for all positive integers n, all complex roots r of the polynomial
$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$
    lie on the unit circle (i.e. $|r| = 1$).