HMMT February 2016

February 20, 2016

Geometry

1. Dodecagon $QWARTZSPHINX$ has all side lengths equal to 2, is not self-intersecting (in particular, the twelve vertices are all distinct), and moreover each interior angle is either $90°$ or $270°$. What are all possible values of the area of $\triangle SIX$?

   Proposed by: Evan Chen

   Answer: $\boxed{2, 6}$

   The dodecagon has to be a "plus shape" of area 20, then just try the three non-congruent possibilities.

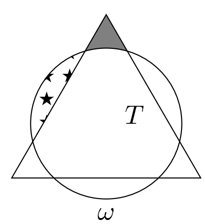
2. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let H be the orthocenter of ABC. Find the distance between the circumcenters of triangles AHB and AHC.

   Proposed by: Evan Chen

   Answer: $\boxed{14}$

   Let H_B be the reflection of H over AC and let H_C be the reflection of H over AB. The reflections of H over AB, AC lie on the circumcircle of triangle ABC. Since the circumcenters of triangles AH_CB, AH_BC are both O, the circumcenters of AHB, AHC are reflections of O over AB, AC respectively. Moreover, the lines from O to the circumcenters in question are the perpendicular bisectors of AB and AC. Now we see that the distance between the two circumcenters is simply twice the length of the midline of triangle ABC that is parallel to BC, meaning the distance is $2(\frac{1}{2}BC) = 14$.

3. In the below picture, T is an equilateral triangle with a side length of 5 and ω is a circle with a radius of 2. The triangle and the circle have the same center. Let X be the area of the shaded region, and let Y be the area of the starred region. What is $X - Y$?



   Proposed by:

   Answer: $\boxed{\frac{25\sqrt{3}}{12} - \frac{4\pi}{3}}$

   Let $K(P)$ denote the area of P. Note that $K(T) - K(\omega) = 3(X - Y)$, which gives our answer.

4. Let ABC be a triangle with $AB = 3$, $AC = 8$, $BC = 7$ and let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Point T is selected on side BC so that $AT = TC$. The circumcircles of triangles BAT, MAN intersect at D. Compute DC.

   Proposed by: Evan Chen

   Answer: $\boxed{\frac{7\sqrt{3}}{3}}$

   We note that D is the circumcenter O of ABC, since $2\angle C = \angle ATB = \angle AOB$. So we are merely looking for the circumradius of triangle ABC. By Heron's Formula, the area of the triangle is $\sqrt{9 \cdot 6 \cdot 1 \cdot 2} = 6\sqrt{3}$, so using the formula $\frac{abc}{4R} = K$, we get an answer of $\frac{3 \cdot 8 \cdot 7}{4 \cdot 6\sqrt{3}} = \frac{7\sqrt{3}}{3}$. Alternatively, one can compute the circumradius using trigonometric methods or the fact that $\angle A = 60°$.

5. Nine pairwise noncongruent circles are drawn in the plane such that any two circles intersect twice. For each pair of circles, we draw the line through these two points, for a total of $\binom{9}{2} = 36$ lines. Assume that all 36 lines drawn are distinct. What is the maximum possible number of points which lie on at least two of the drawn lines?

   Proposed by: Evan Chen

   Answer: $\boxed{462}$

   The lines in question are the radical axes of the 9 circles. Three circles with noncollinear centers have a radical center where their three pairwise radical axes concur, but all other intersections between two of the $\binom{9}{2}$ lines can be made to be distinct. So the answer is

$$\binom{\binom{9}{2}}{2} - 2\binom{9}{3} = 462$$

   by just counting pairs of lines, and then subtracting off double counts due to radical centers (each counted three times).

6. Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc $\widehat{BAC}$ of Γ. Given that $AB = 5$, $AC = 8$, and $\angle A = 60°$, compute the area of triangle XUV.

   Proposed by: Evan Chen

   Answer: $\boxed{\frac{21\sqrt{3}}{8}}$

   Let segments AI and EF meet at K. Extending AK to meet the circumcircle again at Y, we see that X and Y are diametrically opposite, and it follows that AX and EF are parallel. Therefore the height from X to $\overline{UV}$ is merely AK. Observe that $AE = AF$, so $\triangle AEF$ is equilateral; since MN, MP are parallel to AF, AE respectively, it follows that $\triangle MVU, \triangle UEN, \triangle FPV$ are equilateral as well. Then $MV = MP - PV = \frac{1}{2}AC - FP = \frac{1}{2}AC - AF + AP = \frac{1}{2}AC - AF + \frac{1}{2}AB = \frac{1}{2}BC$, since E, F are the tangency points of the incircle. Since $\triangle MVU$ is equilateral, we have $UV = MU = MV = \frac{1}{2}BC$.

   Now we can compute $BC = 7$, whence $UV = \frac{7}{2}$ and

$$AK = \frac{AB + AC - BC}{2} \cdot \cos 30° = \frac{3\sqrt{3}}{2}.$$

   Hence, the answer is $\frac{21\sqrt{3}}{8}$.

7. Let $S = \{(x, y) | x, y \in \mathbb{Z}, 0 \le x, y, \le 2016\}$. Given points $A = (x_1, y_1), B = (x_2, y_2)$ in S, define

$$d_{2017}(A, B) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \pmod{2017}.$$

   The points $A = (5, 5), B = (2, 6), C = (7, 11)$ all lie in S. There is also a point $O \in S$ that satisfies

$$d_{2017}(O, A) = d_{2017}(O, B) = d_{2017}(O, C).$$

   Find $d_{2017}(O, A)$.

   Proposed by: Yang Liu

   Answer: $\boxed{1021}$

   Note that the triangle is a right triangle with right angle at A. Therefore, $R^2 = \frac{(7-2)^2+(11-6)^2}{4} = \frac{25}{2} = (25)(2^{-1}) \equiv 1021 \pmod{2017}$. (An equivalent approach works for general triangles; the fact that the triangle is right simply makes the circumradius slightly easier to compute.)

8. For $i = 0, 1, \ldots, 5$ let l_i be the ray on the Cartesian plane starting at the origin, an angle $\theta = i\frac{\pi}{3}$ counterclockwise from the positive x-axis. For each i, point P_i is chosen uniformly at random from the intersection of l_i with the unit disk. Consider the convex hull of the points P_i, which will (with probability 1) be a convex polygon with n vertices for some n. What is the expected value of n?

   Proposed by:

   Answer: $\boxed{2 + 4\ln(2)}$

   A vertex P_i is part of the convex hull if and only if it is not contained in the triangle formed by the origin and the two adjacent vertices. Let the probability that a given vertex is contained in the aforementioned triangle be p. By linearity of expectation, our answer is simply $6(1-p)$. Say $|P_0| = a, |P_2| = b$. Stewart's Theorem and the Law of Cosines give that p is equal to the probability that $|P_1| < \sqrt{ab - ab\frac{a^2+b^2+ab}{(a+b)^2}} = \frac{ab}{a+b}$; alternatively this is easy to derive using coordinate methods. The corresponding double integral evaluates to $p = \frac{2}{3}(1 - \ln(2))$, thus telling us our answer.

9. In cyclic quadrilateral $ABCD$ with $AB = AD = 49$ and $AC = 73$, let I and J denote the incenters of triangles ABD and CBD. If diagonal $\overline{BD}$ bisects $\overline{IJ}$, find the length of IJ.

   Proposed by: Evan Chen

   Answer: $\boxed{\frac{28}{5}\sqrt{69}}$

   Let O be circumcenter, R the circumradius and r the common inradius. We have $IO^2 = JO^2 = R(R-2r)$ by a result of Euler; denote x for the common value of IO and JO. Additionally, we know $AJ = AB = AD = 49$ (angle chase to find that $\angle BJA = \angle JBA$). Since A is the midpoint of the arc $\widehat{BD}$ not containing C, both J and A lie on the angle bisector of angle $\angle BCD$, so C, J, A are collinear. So by Power of a Point we have

$$R^2 - x^2 = 2Rr = AJ \cdot JC = 49 \cdot 24.$$

   Next, observe that the angle bisector of angle BAD contains both I and O, so A, I, O are collinear. Let M be the midpoint of IJ, lying on $\overline{BD}$. Let K be the intersection of IO and BD. Observing that the right triangles $\triangle IMO$ and $\triangle IKM$ are similar, we find $IM^2 = IK \cdot IO = rx$, so $IJ^2 = 4rx$. Now apply Stewart's Theorem to $\triangle AOJ$ to derive

$$R\left(x(R-x) + 4rx\right) = 49^2 x + x^2(R-x).$$

   Eliminating the common factor of x and rearranging gives

$$49^2 - (R-x)^2 = 4Rr = 48 \cdot 49$$

   so $R - x = 7$. Hence $R + x = \frac{49 \cdot 24}{7} = 168$, and thus $2R = 175$, $2x = 161$. Thus $r = \frac{49 \cdot 24}{175} = \frac{168}{25}$.

   Finally, $IJ = 2\sqrt{rx} = 2\sqrt{\frac{84 \cdot 161}{25}} = \frac{28\sqrt{69}}{5}$.

10. The incircle of a triangle ABC is tangent to BC at D. Let H and Γ denote the orthocenter and circumcircle of $\triangle ABC$. The B-*mixtilinear incircle*, centered at O_B, is tangent to lines BA and BC and internally tangent to Γ. The C-*mixtilinear incircle*, centered at O_C, is defined similarly. Suppose that $\overline{DH} \perp \overline{O_B O_C}$, $AB = \sqrt{3}$ and $AC = 2$. Find BC.

   Proposed by: Evan Chen

   Answer: $\boxed{\sqrt{\frac{1}{3}(7 + 2\sqrt{13})}}$

   Let the B-mixtilinear incircle ω_B touch Γ at T_B, BA at B_1 and BC at B_2. Define $T_C \in \Gamma$, $C_1 \in CB$, $C_2 \in CA$, and ω_C similarly. Call I the incenter of triangle ABC, and γ the incircle.

   We first identify two points on the radical axis of the B and C mixtilinear incircles:

- The midpoint M of arc BC of the circumcircle of ABC. This follows from the fact that M, B_1, T_B are collinear with
$$MB^2 = MC^2 = MB_1 \cdot MT_B$$
and similarly for C.
- The midpoint N of ID. To see this, first recall that I is the midpoint of segments B_1B_2 and C_1C_2. From this, we can see that the radical axis of ω_B and γ contains N (since it is the line through the midpoints of the common external tangents of ω_B, γ). A similar argument for C shows that the midpoint of ID is actually the radical center of the ω_B, ω_C, γ.

Now consider a homothety with ratio 2 at I. It sends line MN to the line through D and the A-excenter I_A (since M is the midpoint of II_A, by "Fact 5"). Since DH was supposed to be parallel to line MN, it follows that line DH passes through I_A; however a homothety at D implies that this occurs only if H is the midpoint of the A-altitude.

Let $a = BC$, $b = CA = 2$ and $c = AB = \sqrt{3}$. So, we have to just find the value of a such that the orthocenter of ABC lies on the midpoint of the A-altitude. This is a direct computation with the Law of Cosines, but a more elegant solution is possible using the fact that H has barycentric coordinates $(S_BS_C : S_CS_A : S_AS_B)$, where $S_A = \frac{1}{2}(b^2 + c^2 - a^2)$ and so on. Indeed, as H is on the A-midline we deduce directly that

$$S_BS_C = S_A(S_B + S_C) = a^2 S_A \implies \frac{1}{4}(a^2 - 1)(a^2 + 1) = \frac{1}{2}a^2(7 - a^2).$$

Solving as a quadratic in a^2 and taking the square roots gives

$$3a^4 - 14a^2 - 1 = 0 \implies a = \sqrt{\frac{1}{3}(7 + 2\sqrt{13})}$$

as desired. 1