14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

Combinatorics & Geometry Individual Test

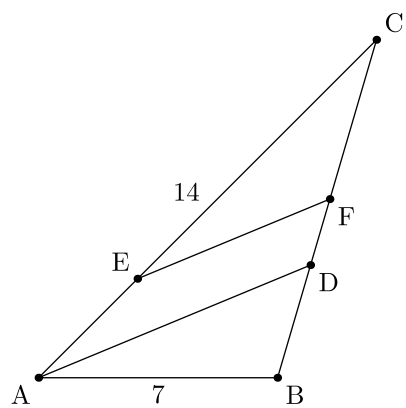
1. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

   Answer: $\boxed{2 \cdot 15!^2}$ If we color the desks of the class in a checkerboard pattern, we notice that all of one gender must go in the squares colored black, and the other gender must go in the squares colored white. There are 2 ways to pick which gender goes in which color, 15! ways to put the boys into desks and 15! ways to put the girls into desks. So the number of ways is $2 \cdot 15!^2$.

   (There is a little ambiguity in the problem statement as to whether the 15 boys and the 15 girls are distinguishable or not. If they are not distinguishable, the answer is clearly 2. Given the number of contestants who submitted the answer 2, the graders judged that there was enough ambiguity to justify accepting 2 as a correct answer. So both 2 and $2 \cdot 15!^2$ were accepted as correct answers.)

2. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

   Answer: $\boxed{13}$



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$

([] denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$d + 7 > 14$$
$$7 + 14 > d$$

Hence $7 < d < 21$ and there are 13 possible integral values for BC.

3. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: ax^2+bx+c, bx^2+cx+a, and cx^2+ax+b.

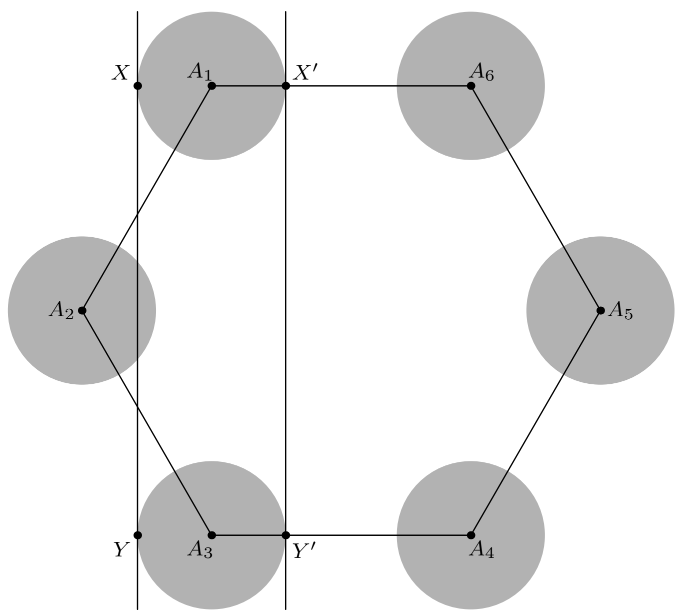
   Answer: $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc, b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a,b,c)=(1,5,6)$ give $-2,-3$ as roots to x^2+5x+6 and $-1,-\frac{1}{5}$ as roots to $5x^2+6x+1$.

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

   Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k+1$. So there are 2^{n-1} ways to get to the center square of row n.

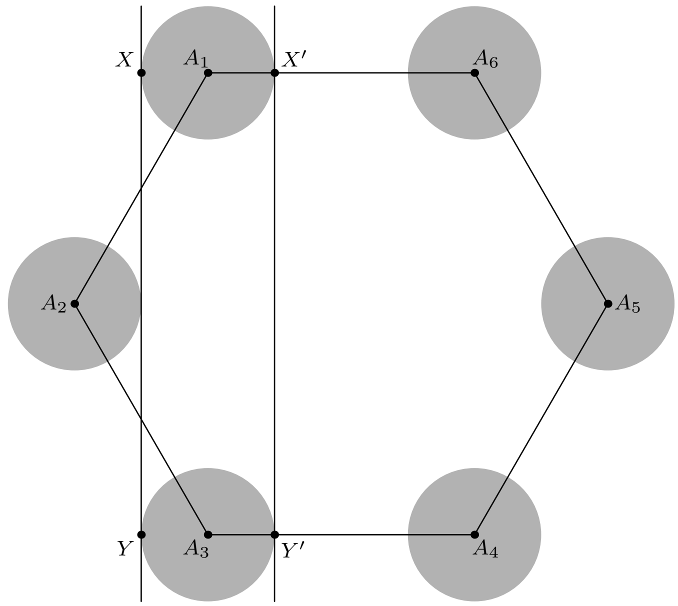
5. Let H be a regular hexagon of side length x. Call a hexagon in the same plane a "distortion" of H if and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

   Answer: $\boxed{4}$



Let $H=A_1A_2A_3A_4A_5A_6$ be the hexagon, and for all $1 \leq i \leq 6$, let points A_i' be considered such that $A_iA_i' < 1$. Let $H' = A_1'A_2'A_3'A_4'A_5'A_6'$, and consider all indices modulo 6. For any point P in the plane, let $D(P)$ denote the unit disk $\{Q|PQ<1\}$ centered at P; it follows that $A_i' \in D(A_i)$.

Let X and X' be points on line A_1A_6, and let Y and Y' be points on line A_3A_4 such that $A_1X = A_1X' = A_3Y = A_3Y' = 1$ and X and X' lie on opposite sides of A_1 and Y and Y' lie on opposite sides of A_3. If X' and Y' lie on segments A_1A_6 and A_3A_4, respectively, then segment $A_1'A_3'$ lies between the lines XY and $X'Y'$. Note that $\frac{x}{2}$ is the distance from A_2 to A_1A_3.



If $\frac{x}{2} \geq 2$, then $C(A_2)$ cannot intersect line XY, since the distance from XY to A_1A_3 is 1 and the distance from XY to A_2 is at least 1. Therefore, $A_1'A_3'$ separates A_2' from the other 3 vertices of the hexagon. By analogous reasoning applied to the other vertices, we may conclude that H' is convex.

If $\frac{x}{2} < 2$, then $C(A_2)$ intersects XY, so by choosing $A_1' = X$ and $A_3' = Y$, we see that we may choose A_2' on the opposite side of XY, in which case H' will be concave. Hence the answer is 4, as desired.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

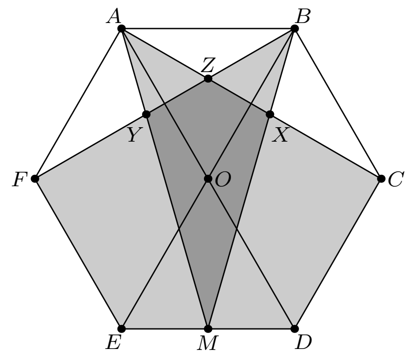
Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A, will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod 7$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

$$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6,\, l \neq k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^{6} x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \cdots = x_6$, and $6x_k = s$ for every k. Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.



Answer: $\boxed{0}$

Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

8. The integers from 1 to n are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.

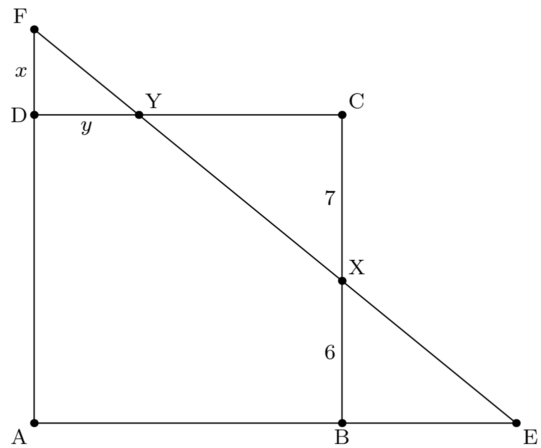
   Answer: $\boxed{4022}$ If n is odd and greater than 1, then Goliath makes the last move. No matter what two numbers are on the board, Goliath can combine them to make an even number. Hence Goliath has a winning strategy in this case.

   Now suppose n is even. We can replace all numbers on the board by their residues modulo 2. Initially the board reads $1, 0, 1, 0, \ldots, 1, 0$. David combines the rightmost 1 and 0 by addition to make 1, so now the board reads $1, 0, 1, 0, \ldots, 0, 1$. We call a board of this form a "good" board. When it is Goliath's turn, and there is a good board, no matter where he moves, David can make a move to restore a good board. Indeed, Goliath must combine a neighboring 0 and 1; David can then combine that number with a neighboring 1 to make 1 and create a good board with two fewer numbers.

   David can ensure a good board after his last turn. But a good board with one number is simply 1, so David wins. So David has a winning strategy if n is even. Therefore, the 2011th smallest positive integer greater than 1 for which David can guarantee victory is the 2011th even positive integer, which is 4022.

9. Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD, respectively, so that the area of square $ABCD$ equals the area of triangle AEF. If EF intersects BC at X and $BX = 6$, determine DF.

   Answer: $\boxed{\sqrt{13}}$



First Solution

Let Y be the point of intersection of lines EF and CD. Note that $[ABCD] = [AEF]$ implies that $[BEX] + [DYF] = [CYX]$. Since $\triangle BEX \sim \triangle CYX \sim \triangle DYF$, there exists some constant r such that $[BEX] = r \cdot BX^2$, $[YDF] = r \cdot CX^2$, and $[CYX] = r \cdot DF^2$. Hence $BX^2 + DF^2 = CX^2$, so $DF = \sqrt{CX^2 - BX^2} = \sqrt{49 - 36} = \sqrt{13}$.

Second Solution

Let $x = DF$ and $y = YD$. Since $\triangle BXE \sim \triangle CXY \sim \triangle DFY$, we have

$$\frac{BE}{BX} = \frac{CY}{CX} = \frac{DY}{DF} = \frac{y}{x}.$$

Using $BX = 6$, $XC = 7$ and $CY = 13 - y$ we get $BE = \frac{6y}{x}$ and $\frac{13-y}{7} = \frac{y}{x}$. Solving this last equation for y gives $y = \frac{13x}{x+7}$. Now $[ABCD] = [AEF]$ gives

$$\begin{aligned} 169 &= \frac{1}{2} AE \cdot AF = \frac{1}{2}\left(13 + \frac{6y}{x}\right)(13 + x)\,. \\ 169 &= 6y + 13x + \frac{78y}{x} \\ 13 &= \frac{6x}{x+7} + x + \frac{78}{x+7} \\ 0 &= x^2 - 13. \end{aligned}$$

Thus $x = \sqrt{13}$.

10. Mike and Harry play a game on an 8×8 board. For some positive integer k, Mike chooses k squares and writes an M in each of them. Harry then chooses $k+1$ squares and writes an H in each of them. After Harry is done, Mike wins if there is a sequence of letters forming "HMM" or "MMH," when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of k for which Mike has a winning strategy.

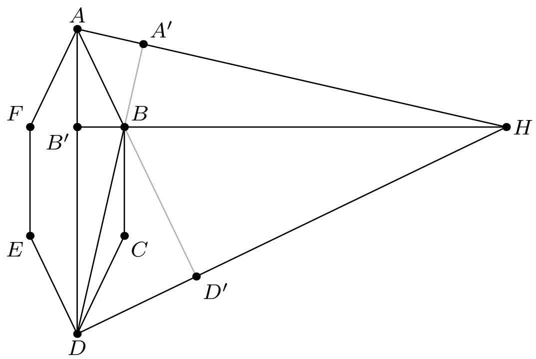
Answer: $\boxed{16}$ Suppose Mike writes k M's. Let a be the number of squares which, if Harry writes an H in, will yield either HMM or MMH horizontally, and let b be the number of squares which, if Harry writes an H in, will yield either HMM or MMH vertically. We will show that $a \leq k$ and $b \leq k$. Then, it will follow that there are at most $a + b \leq 2k$ squares which Harry cannot write an H in. There will be at least $64 - k - 2k = 64 - 3k$ squares which Harry can write in. If $64 - 3k \geq k + 1$, or $k \leq 15$, then Harry wins.

We will show that $a \leq k$ (that $b \leq k$ will follow by symmetry). Suppose there are a_i M's in row i. In each group of 2 or more consective M's, Harry cannot write H to the left or right of that group, giving at most 2 forbidden squares. Hence a_i is at most the number of M's in row i. Summing over the rows gives the desired result.

Mike can win by writing 16 M's according to the following diagram:

$$\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}.$$

11. Let $ABCDEF$ be a convex equilateral hexagon such that lines BC, AD, and EF are parallel. Let H be the orthocenter of triangle ABD. If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.



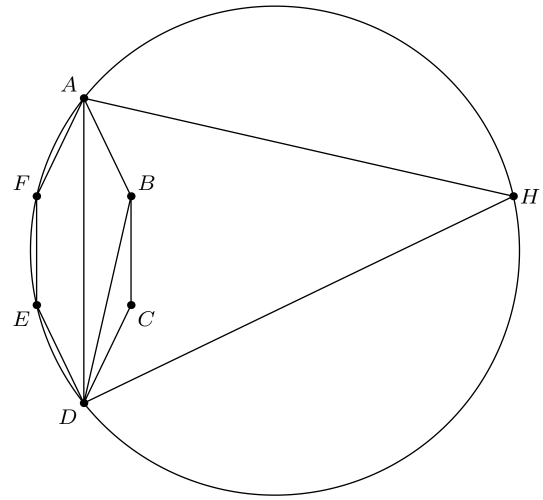
Answer: $\boxed{3}$

Note that $ABCD$ and $DEFA$ are isosceles trapezoids, so $\angle BAD = \angle CDA$ and $\angle FAD = \angle EDA$. In order for the hexagon to be convex, the angles at B, C, E, and F have to be obtuse, so $\angle A = \angle D = 4°$. Letting s be a side length of the hexagon, $AD = AB\cos\angle BAD + BC + CD\cos\angle CDA = s(1 + 2\cos\angle BAD)$, so $\angle BAD$ is uniquely determined by AD. Since the same equation holds for trapezoid $DEFA$, it follows that $\angle BAD = \angle FAD = \angle CDA = \angle EDA = 2°$. Then $\angle BCD = 180° - 2° = 178°$. Since $\triangle BCD$ is isosceles, $\angle CDB = 1°$ and $\angle BDA = 1°$. (One may also note that $\angle BDA = 1°$ by observing that equal lengths AB and BC must intercept equal arcs on the circumcircle of isosceles trapezoid $ABCD$).

Let A', B', and D' be the feet of the perpendiculars from A, B, and D to BD, DA, and AB, respectively. Angle chasing yields

$$\begin{aligned}
\angle AHD &= \angle AHB' + \angle DHB' = (90° - \angle A'AB') + (90° - \angle D'DB') \\
&= \angle BDA + \angle BAD = 1° + 2° = 3° \\
\angle HAD &= 90° - \angle AHB' = 89° \\
\angle HDA &= 90° - \angle DHB' = 88°
\end{aligned}$$

Hence the smallest angle in $\triangle HAD$ is $3°$.



It is faster, however, to draw the circumcircle of $DEFA$, and to note that since H is the orthocenter of triangle ABD, B is the orthocenter of triangle HAD. Then since F is the reflection of B across AD, quadrilateral $HAFD$ is cyclic, so $\angle AHD = \angle ADF + \angle DAF = 1^\circ + 2^\circ = 3^\circ$, as desired.

12. The ordered pairs $(2011, 2), (2010, 3), (2009, 4), \ldots,$ $(1008, 1005)$, $(1007, 1006)$ are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs (x_i, y_i) and (x_j, y_j), with (x_i, y_i) left of (x_j, y_j), erases them, and writes $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j}\right)$ in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs (x, y) could appear on the blackboard after the process has come to a conclusion?

Answer: $\boxed{504510}$ First, note that none of the numbers will ever be 0. Let $\star$ denote the replacement operation. For each pair on the board (x_i, y_i) define its *primary form* to be (x_i, y_i) and its *secondary form* to be $[x_i y_i, \frac{x_i}{y_i}]$. Note that the primary form determines the secondary form uniquely and vice versa. In secondary form,

$$[a_1, b_1] \star [a_2, b_2] = \left(\sqrt{a_1 b_1}, \sqrt{\frac{a_1}{b_1}}\right) \star \left(\sqrt{a_2 b_2}, \sqrt{\frac{a_2}{b_2}}\right) = \left(a_1 b_2, \frac{a_1}{b_2}\right) = [a_1^2, b_2^2].$$

Thus we may replace all pairs on the board by their secondary form and use the above rule for $\star$ instead. From the above rule, we see that if the leftmost number on the board is x, then after one minute it will be x or x^2 depending on whether it was erased in the intervening step, and similarly for the rightmost number. Let k be the number of times the leftmost pair is erased and n be the number of times the rightmost pair is erased. Then the final pair is

$$\left[4022^{2^k}, \left(\frac{1007}{1006}\right)^{2^n}\right]. \tag{2}$$

Any step except the last cannot involve both the leftmost and rightmost pair, so $k + n \le 1005$. Since every pair must be erased at least once, $k, n \ge 1$. Every pair of integers satisfying the above can occur, for example, by making $1005 - k - n$ moves involving only the pairs in the middle, then making $k - 1$ moves involving the leftmost pair, and finally n moves involving the rightmost pair.

In light of (2), the answer is the number of possible pairs (k, n), which is

$$\sum_{k=1}^{1004} \sum_{n=1}^{1005-k} 1 = \sum_{k=1}^{1004} 1005 - k = \sum_{k=1}^{1004} k = \frac{1004 \cdot 1005}{2} = 504510.$$

13. Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

Answer: $\boxed{2\sqrt{3}}$ Let T be the point where the incircle intersects AD, and let r be the inradius and R be the circumradius of $\triangle ABD$. Since $BC = CD = 2$, C is on the midpoint of arc BD on the opposite side of BD as A, and hence on the angle bisector of A. Thus A, I, and C are collinear. We have the following formulas:

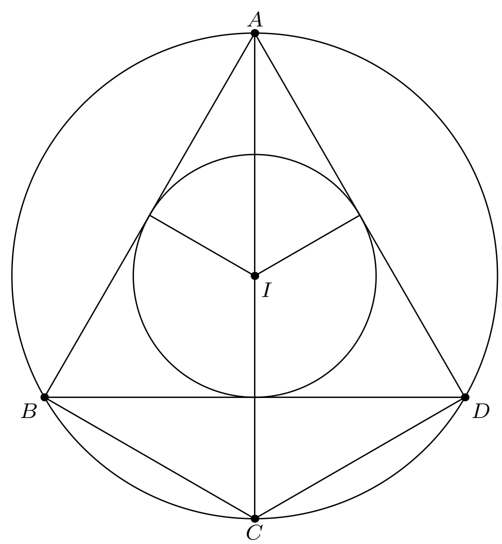
$$\begin{aligned} AI &= \frac{IM}{\sin \angle IAM} = \frac{r}{\sin \frac{A}{2}} \\ BC &= 2R \sin \frac{A}{2} \\ BD &= 2R \sin A \end{aligned}$$

The last two equations follow from the extended law of sines on $\triangle ABC$ and $\triangle ABD$, respectively.

Using $AI = 2 = BC$ gives $\sin^2 \frac{A}{2} = \frac{r}{2R}$. However, it is well-known that $R \geq 2r$ with equality for an equilateral triangle (one way to see this is the identity $1+\frac{r}{R} = \cos A + \cos B + \cos D$). Hence $\sin^2 \frac{A}{2} \leq \frac{1}{4}$ and $\frac{A}{2} \leq 30°$. Then

$$BD = 2R\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right) = BC \cdot 2\cos\frac{A}{2} \geq 2\left(2 \cdot \frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

with equality when $\triangle ABD$ is equilateral.



Remark: Similar but perhaps simpler computations can be made by noting that if AC intersects BD at X, then $AB/BX = AD/DX = 2$, which follows from the exterior angle bisector theorem; if I_A is the A-excenter of triangle ABC, then $AI_A/XI_A = 2$ since it is well-known that C is the circumcenter of cyclic quadrilateral $BIDI_A$.

14. Let $A = \{1, 2, \ldots, 2011\}$. Find the number of functions f from A to A that satisfy $f(n) \leq n$ for all n in A and attain exactly 2010 distinct values.

Answer: $\boxed{2^{2011} - 2012}$ Let n be the element of A not in the range of f. Let m be the element of A that is hit twice.

We now sum the total number of functions over n, m. Clearly $f(1) = 1$, and by induction, for $x \leq m, f(x) = x$. Also unless $n = 2011$, $f(2011) = 2011$ because f can take no other number to 2011. It follows from backwards induction that for $x > n$, $f(x) = x$. Therefore $n > m$, and there are only $n - m$ values of f that are not fixed.

Now $f(m+1) = m$ or $f(m+1) = m+1$. For $m < k < n$,given the selection of $f(1), f(2), \ldots, f(k-1)$, $k-1$ of the $k+1$ possible values of $f(k+1)$ $(1, 2, 3, \ldots, k$, and counting m twice) have been taken, so there are two distinct values that $f(k+1)$ can take (one of them is $k+1$, and the other is not, so they are distinct). For $f(n)$, when the other 2010 values of f have been assigned, there is only one missing, so $f(n)$ is determined.

For each integer in $[m, n)$, there are two possible values of f, so there are 2^{n-m-1} different functions f for a given m, n. So our answer is

$$\begin{aligned}
\sum_{m=1}^{2010} \sum_{n=m+1}^{2011} 2^{n-m-1} &= \sum_{m=1}^{2010} 2^{-m-1} \sum_{n=m+1}^{2011} 2^n \\
&= \sum_{m=1}^{2010} 2^{-m-1}(2^{2012} - 2^{m+1}) \\
&= \sum_{m=1}^{2010} 2^{2011-m} - 1 \\
&= \left(\sum_{m=1}^{2010} 2^m \right) - 2010 \\
&= 2^{2011} - 2012
\end{aligned}$$

15. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \le i \le 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \le |r_2|$ for any function f satisfying these properties, find A.

Answer: $\boxed{2}$ Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x, namely, for $x = g_{2i+1}, 0 \le i \le 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x-axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \ne b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \ge b$ then $f(g_i) \ge g_i \ge b$ so by induction, $g_{i+1} \ge g_i$ for all $i \ge j$. Hence $j > 1$; in fact $j \ge 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c. The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned}
f\left(\frac{b+a-1}{2}\right) &< c \\
\left(\frac{b+a-1}{2} - a\right)\left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a - 1 \\
\frac{1-(b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\
\frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\
4 &< (b-a-1)^2.
\end{aligned}$$

Then either $b - a - 1 < -2$ or $b - a - 1 > 2$, but $b > a$, so the latter must hold and $(b-a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b-a)^2$ (the square of the difference of the two roots) and $(r_2+1)^2 - 4r_3$ (from the coefficients), so $(r_2+1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \geq 0$ as follows. Let $N \geq 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x+2+\varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define G_i recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following. (i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \leq i \leq N-1$. First note $G_{2N} = -\sqrt{4+2\varepsilon} > -\sqrt{4+2\varepsilon+\varepsilon^2} = -2-\varepsilon$. Let l be the negative solution to $h(x) = x$. Note that $-2-\varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \geq -2-\varepsilon$, and it sends $(-2-\varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2-\varepsilon, l)$. It follows that the G_i, $0 \leq i \leq 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \leq i \leq N-1$ by backwards induction on i, so the desired inequalities follow. (ii) G_i is increasing for $i \geq 2N+1$. Indeed, if $x \geq 2+\varepsilon$, then $x^2 - x = x(x-1) > 2+\varepsilon$ so $h(x) > x$. Hence $2+\varepsilon = G_{2N+1} < G_{2N+2} < \cdots$. (iii) G_i is unbounded. This follows since $h(x) - x = x(x-2) - 2 - \varepsilon$ is increasing for $x > 2+\varepsilon$, so G_i increases faster and faster for $i \geq 2N+1$. Now define $f(x) = h(x+G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i.

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2+\varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε, it suffices to take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\cdots g}_{n}(2)\cdots) \to -1 \text{ as } n \to \infty.$$

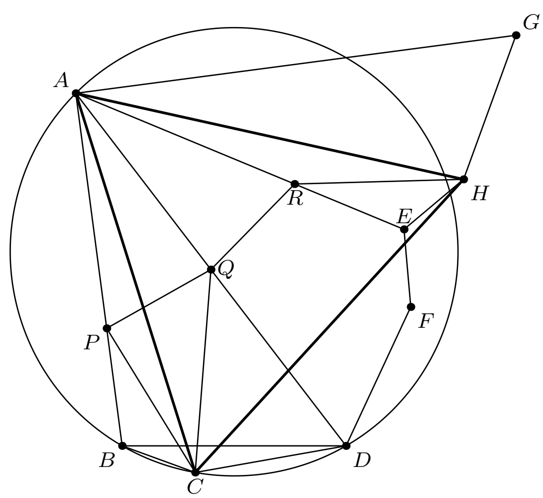
But note that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$g(-2\cos\theta) = -\sqrt{2-2\cos\theta} = -2\sin\left(\frac{\theta}{2}\right) = 2\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Hence by induction, $g^{(n)}(-2\cos\theta) = -2\cos\left(\frac{\pi}{2} - \frac{\pi}{4} + \cdots + (-1)^n\left(\theta - \frac{\pi}{2^n}\right)\right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2\cos 0)$ converges to $-2\cos(\frac{\pi}{2} - \frac{\pi}{4} + \cdots) = -2\cos(\frac{\pi}{3}) = -1$, as needed.

16. Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP + PQ + CQ$, where P and Q may be any points lying along rays AB and AD, respectively. Determine the maximum value of m.

Answer: $\boxed{2}$



For a fixed quadrilateral $ABCD$ as described, we first show that m, the minimum possible length of $CP + PQ + QC$, equals the length of AC. Reflect B, C, and P across line AD to points E, F, and R, respectively, and then reflect D and F across AE to points G and H, respectively. These two reflections combine to give a $60°$ rotation around A, so triangle ACH is equilateral. It also follows that RH is a $60°$ rotation of PC around A, so, in particular, these segments have the same length. Because $QR = QP$ by reflection,

$$CP + PQ + QC = CQ + QR + RH.$$

The latter is the length of a broken path $CQRH$ from C to H, and by the "shortest path is a straight line" principle, this total length is at least as long as $CH = CA$. (More directly, this follows from the triangle inequality: $(CQ+QR)+RH \geq CR+RH \geq CH$). Therefore, the lower bound $m \geq AC$ indeed holds. To see that this is actually an equality, note that choosing Q as the intersection of segment CH with ray AD, and choosing P so that its reflection R is the intersection of CH with ray AE, aligns path $CQRH$ with segment CH, thus obtaining the desired minimum $m = AC$.

We may conclude that the largest possible value of m is the largest possible length of AC, namely 2: the length of a diameter of the circle.

17. Let n be an odd positive integer, and suppose that n people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander's left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other $n-1$ members vote randomly.

Answer: $\boxed{\frac{2^n-1}{n2^{n-1}}}$ Let x be the probability Hermia is elected if Lysander votes for her, and let y be the probability that she wins if Lysander does not vote for her. We are trying to find x, and do so by first finding y. If Lysander votes for Hermia with probability $\frac{1}{2}$ then the probability that Hermia is elected chairman is $\frac{x}{2} + \frac{y}{2}$, but it is also $\frac{1}{n}$ by symmetry. If Lysander does not vote for Hermia, Hermia can get at most 1 vote, and then can only be elected if everyone gets one vote and she wins the tiebreaker. The probability she wins the tiebreaker is $\frac{1}{n}$, and chasing around the circle, the probability that every person gets 1 vote is $\frac{1}{2^{n-1}}$. (Everyone votes for the person to the left, or everyone votes for the person to the right.) Hence

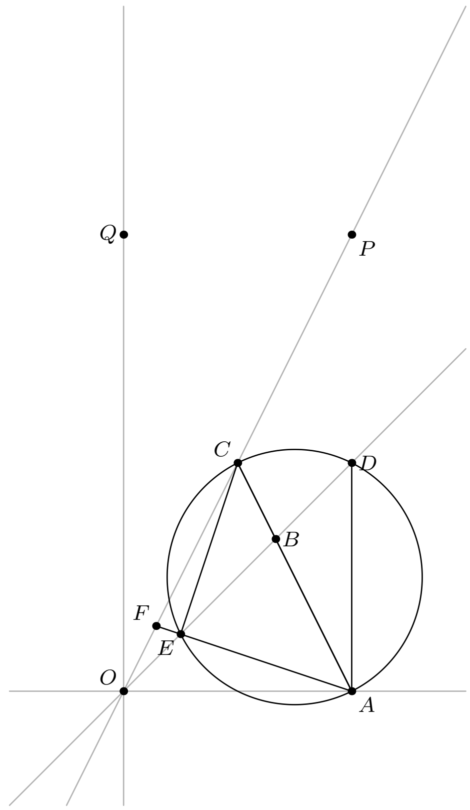
$$y = \frac{1}{n2^{n-1}}.$$

Then $\frac{x}{2} + \frac{1}{n2^n} = \frac{1}{n}$, so solving for x gives

$$x = \frac{2^n - 1}{n2^{n-1}}.$$

18. Collinear points A, B, and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x-axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $AB/BC = 2$. If $D = (a, a)$, the circumcircle of triangle ADC intersects $y = x$ again at E, and ray AE intersects $y = 2x$ at F, evaluate AE/EF.

Answer: $\boxed{7}$



Let points O, P, and Q be located at $(0,0)$, $(a,2a)$, and $(0,2a)$, respectively. Note that $BC/AB = 1/2$ implies $[OCD]/[OAD] = 1/2$, so since $[OPD] = [OAD]$, $[OCD]/[OPD] = 1/2$. It follows that $[OCD] = [OPD]$. Hence $OC = CP$. We may conclude that triangles OCQ and PCA are congruent, so $C = (a/2, a)$.
It follows that $\angle ADC$ is right, so the circumcircle of triangle ADC is the midpoint of AC, which is located at $(3a/4, a/2)$. Let $(3a/4, a/2) = H$, and let $E = (b, b)$. Then the power of the point O with respect to the circumcircle of ADC is $OD \cdot OE = 2ab$, but it may also be computed as $OH^2 - HA^2 = 13a/16 - 5a/16 = a/2$. It follows that $b = a/4$, so $E = (a/4, a/4)$.
We may conclude that line AE is $x + 3y = a$, which intersects $y = 2x$ at an x-coordinate of $a/7$. Therefore, $AE/EF = (a - a/4)/(a/4 - a/7) = (3a/4)/(3a/28) = 7$.

Remark: The problem may be solved more quickly if one notes from the beginning that lines OA, OD, OP, and OQ form a harmonic pencil because D is the midpoint of AP and lines OQ and AP are parallel.

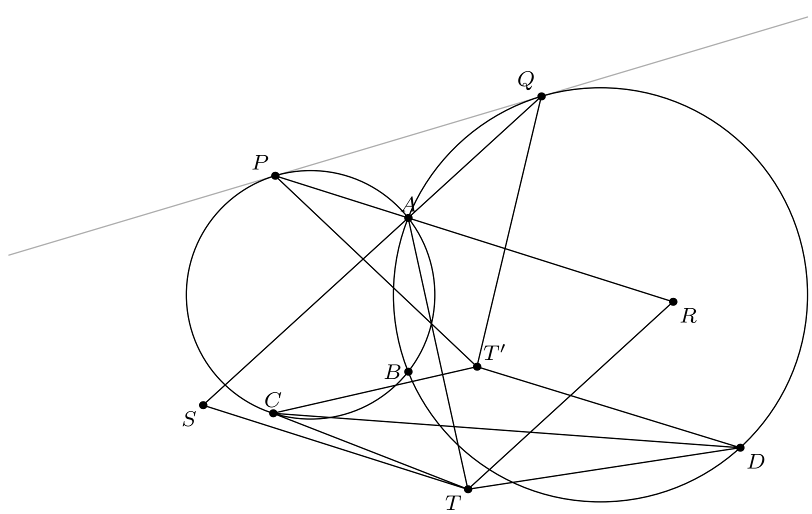
19. Alice and Bob play a game in which two thousand and eleven 2011×2011 grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers $1, 2, \ldots, 2011^2$ so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?

Answer: $\boxed{1}$ Consider the grid whose entries in the jth row are, in order, $2011j - 2010, 2011j - 2009, \ldots, 2011j$. Call this grid A_0. For $k = 1, 2 \ldots, 2010$, let grid A_k be the grid obtained from A_0 by swapping the rightmost entry of the kth row with the leftmost entry of the $k+1$st row. We claim

that if $A \in \{A_0, A_1, \ldots, A_{2010}\}$, then given any legally numbered grid B such that A and B differ in at least one entry, there exist two integers in the same column of B that occur in the same row of A. We first consider A_0. Assume for the sake of contradiction B is a legally numbered grid distinct from A_0, such that there do not exist two integers in the same column of B that occur in the same row of A_0. Since the numbers $1, 2, \ldots, 2011$ occur in the same row of A_0, they must all occur in different columns of B. Clearly 1 is the leftmost entry in B's first row. Let m be the smallest number that does not occur in the first row of B. Since each row is in order, m must be the first entry in its row. But then 1 and m are in the same column of B, a contradiction. It follows that the numbers $1, 2, \ldots, 2011$ all occur in the first row of B. Proceeding by induction, $2011j - 2010, 2011j - 2009, \ldots, 2011j$ must all occur in the jth row of B for all $1 \le j \le 2011$. Since A_0 is the only legally numbered grid satsifying this condition, we have reached the desired contradiction.
Now note that if $A \in \{A_1, \ldots, A_{2010}\}$, there exist two integers in the same column of A_0 that occur in the same row of A. In particular, if $A = A_k$ and $1 \le k \le 2010$, then the integers $2011k - 2010$ and $2011k + 1$ occur in the same column of A_0 and in the same row of A_k. Therefore, it suffices to show that for all $1 \le k \le 2010$, there is no legally numbered grid B distinct from A_k and A_0 such that there do not exist two integers in the same column of B that occur in the same row of A_0. Assume for the sake of contradiction that there does exist such a grid B. By the same logic as above, applied to the first $k-1$ rows and applied backwards to the last $2010-k-1$ rows, we see that B may only differ from A_k in the kth and $k+1$st rows. However, there are only two legally numbered grids that are identical to A_k outside of rows k and $k+1$, namely A_0 and A_k. This proves the claim.
It remains only to note that, by the pigeonhole principle, if one of Alice's grids is A_0, then there exists a positive integer k, $1 \le k \le 2010$, such that A_k is not one of the Alice's grids. Therefore, if Bob sets his initial grid to be A_0, he will require only one swap to switch his grid to A_k after examining Alice's grids. If A_0 is not among Alice's grids, then if Bob sets his initial grid to be A_0, he will not in fact require any swaps at all.

20. Let ω_1 and ω_2 be two circles that intersect at points A and B. Let line l be tangent to ω_1 at P and to ω_2 at Q so that A is closer to PQ than B. Let points R and S lie along rays PA and QA, respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and Q. Let O be the circumcenter of triangle ASR, and let C and D be the midpoints of major arcs AP and AQ, respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.

Answer: $\boxed{142.5}$



We use directed angles throughout the solution.

Let T denote the point such that $\angle TCD = 1/2\angle APQ$ and $\angle TDC = 1/2\angle AQP$. We claim that T is the circumcenter of triangle SAR.

Since $CP = CA$, $QP = RA$, and $\angle CPQ = \angle CPA + \angle APQ = \angle CPA + \angle ACP = \angle CAR$, we have $\triangle CPQ \cong \triangle CAR$. By spiral similarity, we have $\triangle CPA \sim \triangle CQR$.

Let T' denote the reflection of T across CD. Since $\angle TCT' = \angle APQ = \angle ACP$, we have $\triangle TCT' \sim \triangle ACP \sim \triangle RCQ$. Again, by spiral similarity centered at C, we have $\triangle CTR \sim \triangle CT'Q$. But $CT = CT'$, so $\triangle CTR \cong \triangle CT'Q$ and $TR = T'Q$. Similarly, $\triangle DTT' \sim \triangle DAQ$, and spiral similarity centered at D shows that $\triangle DTA \cong \triangle DT'Q$. Thus $TA = T'Q = TR$.

We similarly have $TA = T'P = TS$, so T is indeed the circumcenter. Therefore, we have $\angle COD = \angle CTD = 180^\circ - \frac{45^\circ}{2} - \frac{30^\circ}{2} = 142.5^\circ$.