HMMT November 2024

November 09, 2024

Team Round

1. **[20]** The integers from 1 to 9 are arranged in a 3×3 grid. The rows and columns of the grid correspond to 6 three-digit numbers, reading rows from left to right, and columns from top to bottom. Compute the least possible value of the largest of the 6 numbers.

   Proposed by: Srinivas Arun

   Answer: $\boxed{523}$

   Solution: The 5 cells that make up the top row and left column are all leading digits of the three-digit numbers. Therefore, the largest number has leading digit at least 5, achievable only if 6, 7, 8, and 9 are placed in the bottom right 2×2 square. Then, the only three-digit numbers with tens digit less than 6 are the top row and the left column, so unless 5 is in the top left corner, the three-digit number starting with 5 will be at least 560.

   Now observe 5 is next to two other digits; if they are not 1 or 2 in some order, then either the top row or left column will read at least 530. Thus we can assume 5 is next to 1 or 2. The next-smallest remaining digit is 3, so the three-digit number starting with 52 must be at least $\boxed{523}$. This is achievable as shown below.

5	2	3
1	6	7
4	8	9

2. **[20]** Compute the sum of all positive integers x such that $(x-17)\sqrt{x-1}+(x-1)\sqrt{x+15}$ is an integer.

   Proposed by: Edward Yu

   Answer: $\boxed{11}$

   Solution: First, we prove the following claim.

   Claim 1. If integers a, b, c, d, n satisfy a and c are nonzero, b and d are nonnegative, and $a\sqrt{b}+c\sqrt{d} = n$, then either $n = 0$ or both b and d are perfect squares.

   Proof. We know $a\sqrt{b} = n - c\sqrt{d}$. Squaring both sides, we get

$$a^2 b = (n - c\sqrt{d})^2 = n^2 + c^2 d - 2nc\sqrt{d},$$

   so $2nc\sqrt{d}$ is an integer. If $\sqrt{d}$ is not an integer, then it is not rational, so $2nc\sqrt{d} = 0$. Since c and $\sqrt{d}$ are nonzero, we must have $n = 0$.

   Similarly, if $\sqrt{b}$ is not an integer, $n = 0$. Thus either $n = 0$ or both b and d are perfect squares. $\square$

   Applying our claim to the given expression, we get three cases.

   - **Case 1:** Either $x - 17$ or $x - 1$ is zero. It is easy to verify $x = 1$ is a solution, while $x = 17$ is not.
   - **Case 2:** $(x-17)\sqrt{x-1} + (x-1)\sqrt{x+15} = 0$. Then

$$(x-17)^2(x-1) = (x-1)^2(x+15) \implies (x-1)(-48x+304) = 0.$$

     Thus $x = 1$ or $x = \frac{304}{48}$, and the latter isn't an integer.

- **Case 3:** $x-1$ and $x+15$ are both perfect squares. Let $x+15=y^2$ and $x-1=z^2$ for nonnegative integers y and z. Then,

  $$(y+z)(y-z)=y^2-z^2=(x+15)-(x-1)=16.$$

  Observe that $y+z$ is nonnegative and $y+z \geq y-z$, so we have the following cases for $y+z$ and $y-z$:

  - $y+z=16$, $y-z=1 \implies y=\frac{17}{2}$, $z=\frac{15}{2}$
  - $y+z=8$, $y-z=2 \implies y=5$, $z=3$
  - $y+z=4$, $y-z=4 \implies y=4$, $z=0$.

  The only nonnegative integer solutions are $(y,z)=(5,3)$ or $(y,z)=(4,0)$, which correspond to $x=10$ and $x=1$. Both of these are indeed solutions.

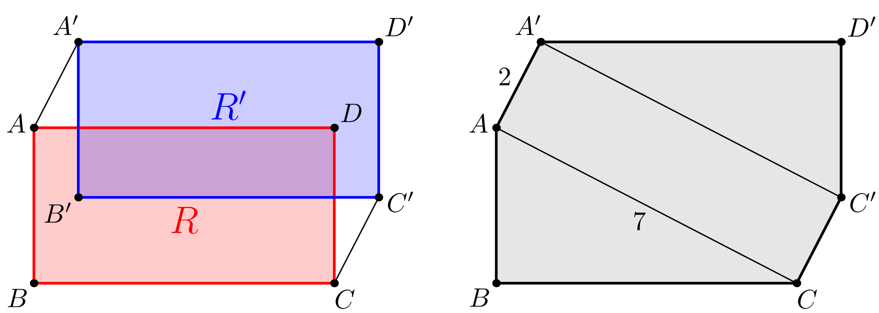
Hence, the only x that work are $x=1$ and $x=10$, for a total of $\boxed{11}$.

3. **[30]** Rectangle R with area 20 and diagonal of length 7 is translated 2 units in some direction to form a new rectangle R'. The vertices of R and R' that are not contained in the other rectangle form a convex hexagon. Compute the maximum possible area of this hexagon.

   Proposed by: Ethan Liu

   Answer: $\boxed{34}$

   Solution:



Dissect the hexagon as shown above, so that it consists of a parallelogram and two triangles which are each half the original rectangle. The parallelogram has side lengths 7 and 2, so its maximum possible area is 14. As the two triangles combined always have the same area as the original rectangle, 20, the answer is $14+20=\boxed{34}$.

4. **[35]** Albert writes down all of the multiples of 9 between 9 and 999, inclusive. Compute the sum of the digits he wrote.

   Proposed by: Jackson Dryg

   Answer: $\boxed{1512}$

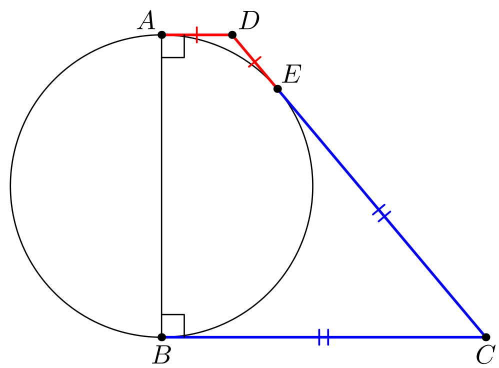
   Solution: If x is a multiple of 9, so is $999-x$, and their digits always sum to 27. The 112 multiples from 0 to 999 form 56 such pairs, so the sum of their digits is $56 \cdot 27 = \boxed{1512}$.

5. **[40]** Let $ABCD$ be a convex quadrilateral with area 202, $AB=4$, and $\angle A = \angle B = 90^\circ$ such that there is exactly one point E on line CD satisfying $\angle AEB = 90^\circ$. Compute the perimeter of $ABCD$.

   Proposed by: Benjamin Shimabukuro

Answer: $\boxed{206}$

Solution:



The locus of point E such that $\angle AEB = 90^\circ$ is the circle ω with diameter AB. Thus, if there exists unique point E, the circle ω must intersect line CD at exactly one point and hence line CD must be tangent to ω.

Now, since $\angle DAB = 90^\circ$, we get that AD is tangent to ω, so $DA = DE$ by equal tangents property. Similarly, $CB = CE$. Thus,

$$CD = CE + DE = AD + BC.$$

However, equating the given area of the quadrilateral gives

$$\tfrac{1}{2}(AD + BC) \cdot AB = 202 \implies AD + BC = 101.$$

Hence, the final answer is

$$CD + (AD + BC) + AB = 101 + 101 + 4 = \boxed{206}.$$

6. **[45]** There are 5 people who start with 1, 2, 3, 4, and 5 cookies, respectively. Every minute, two different people are chosen uniformly at random. If they have a and b cookies and $a \neq b$, the person with more cookies eats $|a - b|$ of their own cookies. If $a = b$, the minute still passes with nothing happening.

   Compute the expected number of minutes until all 5 people have an equal number of cookies.

   Proposed by: Jacob Paltrowitz

   Answer: $\boxed{25/3}$

   Solution: Each person's cookie count can never increase or go below 1, so in order for everyone to have the same number of cookies, everyone must have exactly 1 cookie. The only time the number of people with exactly 1 cookie increases is when one person with exactly 1 cookie and one person with more than 1 cookie are selected.

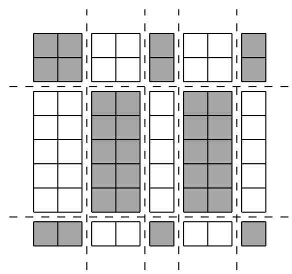
   Suppose there are currently k people with exactly 1 cookie. Then there are $k(5-k)$ ways for the above to happen. There are 10 possible pairs of people that could be selected, so the expected number of minutes before the number of people with exactly 1 cookie increases to $k+1$ is $\frac{10}{k(5-k)}$. Initially, there is one person with exactly one cookie, and the process ends when all five people do; by linearity of expectation, the total expected time is the sum of the expected times to get from k to $k+1$ for $k = 1$, 2, 3, and 4:

$$\frac{10}{4} + \frac{10}{6} + \frac{10}{6} + \frac{10}{4} = \boxed{\frac{25}{3}}.$$

7. **[50]** A *weird checkerboard* is a coloring of an 8×8 grid constructed by making some (possibly none or all) of the following 14 cuts:

- the 7 vertical cuts along a gridline through the entire height of the board,
- and the 7 horizontal cuts along a gridline through the entire width of the board.

The divided rectangles are then colored black and white such that the bottom left corner of the grid is black, and no two rectangles adjacent by an edge share a color. Compute the number of weird checkerboards that have an equal amount of area colored black and white.



Proposed by: Sebastian Attlan

Answer: $\boxed{7735}$

Solution: We can focus on only the black cells of the grid, which we need 32 of. Moreover, the number of black squares in the bottom row and leftmost column uniquely determine the total number of black squares. Suppose that there are x black cells in the bottom row and y black cells in the leftmost column. Then, each of the x rows with black leftmost cell is identical to the bottom row and has y black cells, while the remaining $8-x$ rows are inverted and have $8-y$ black cells, so the total number of black cells is

$$(8-x)(8-y) + xy = 32.$$

This rearranges as

$$2(x-4)(y-4) = 0,$$

which tells us we have 32 black cells exactly when either the bottom row or leftmost column (or both) contains 4 black cells.

The bottom-left corner is already black. There are $\binom{7}{3}$ ways to choose three more cells in the bottom row or leftmost column to be black, and 2^7 ways to color the remaining cells in the bottom row or leftmost column with no restrictions. Hence, there are $2^7\binom{7}{3}$ ways for the bottom row to have 4 black cells, $2^7\binom{7}{3}$ ways for the leftmost column to have 4 black cells, and $\binom{7}{3}^2$ ways for both to occur. The answer is

$$2(2^7)\binom{7}{3} - \binom{7}{3}^2 = \boxed{7735}.$$

8. **[50]** Compute the unique real number $x < 3$ such that

$$\sqrt{(3-x)(4-x)} + \sqrt{(4-x)(6-x)} + \sqrt{(6-x)(3-x)} = x.$$

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{23/8 = 2.875}$

Solution 1: Let $a = \sqrt{3-x}$, $b = \sqrt{4-x}$, $c = \sqrt{6-x}$, so $x = ab + bc + ca$. Then

$$(a+b)(a+c) = a^2 + ab + bc + ca = (3-x) + x = 3.$$

Likewise, we get that

$$\begin{aligned}(a+b)(a+c) &= 3\\ (b+a)(b+c) &= 4\\ (c+a)(c+b) &= 6.\end{aligned}$$

By multiplying these equations and taking the square root, we get that

$$(a+b)(b+c)(c+a) = \sqrt{72} = 6\sqrt{2},$$

so

$$b+c = 2\sqrt{2}, \quad c+a = \tfrac{3\sqrt{2}}{2}, \quad a+b = \sqrt{2},$$

and hence

$$2a = \tfrac{3\sqrt{2}}{2} + \sqrt{2} - 2\sqrt{2} \implies a = \tfrac{\sqrt{2}}{4}.$$

Since $a = \sqrt{3-x}$, it follows that $x = \boxed{\frac{23}{8}}$.

Solution 2: The given equation implies

$$\begin{aligned}&\left(\sqrt{3-x} + \sqrt{4-x} + \sqrt{6-x}\right)^2\\ &= (3-x) + (4-x) + (6-x)\\ &\qquad + 2\left(\sqrt{(3-x)(4-x)} + \sqrt{(4-x)(6-x)} + \sqrt{(6-x)(3-x)}\right)\\ &= 13 - 3x + 2x = 13 - x.\end{aligned}$$

As $\sqrt{3-x} + \sqrt{4-x} + \sqrt{6-x}$ is nonnegative,

$$\sqrt{3-x} + \sqrt{4-x} + \sqrt{6-x} = \sqrt{13-x}.$$

We repeatedly rearrange, square both sides, and simplify:

$$\begin{aligned}\sqrt{3-x} + \sqrt{4-x} &= \sqrt{13-x} - \sqrt{6-x}\\ 7 - 2x + 2\sqrt{(3-x)(4-x)} &= 19 - 2x - 2\sqrt{(13-x)(6-x)}\\ \sqrt{(3-x)(4-x)} &= 6 - \sqrt{(13-x)(6-x)}\\ (3-x)(4-x) &= 36 - 12\sqrt{(13-x)(6-x)} + (13-x)(6-x)\\ 2x - 17 &= 2\sqrt{(13-x)(6-x)}\\ 4x^2 - 68x + 289 &= 4(13-x)(6-x)\\ 8x = 23 &\implies x = \boxed{\tfrac{23}{8}}.\end{aligned}$$

9. **[55]** Let P be a point inside isosceles trapezoid $ABCD$ with $AB \parallel CD$ such that

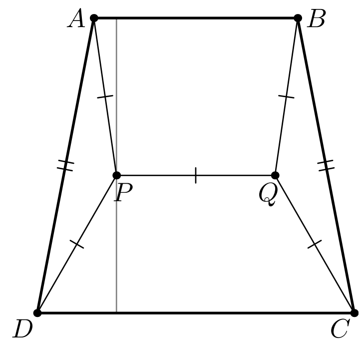
$$\angle PAD = \angle PDA = 90^\circ - \angle BPC.$$

If $PA = 14$, $AB = 18$, and $CD = 28$, compute the area of $ABCD$.

Proposed by: Karthik Venkata Vedula, Pitchayut Saengrungkongka

Answer: $\boxed{345\sqrt{3}}$

Solution:



Let Q be the circumcenter of $\triangle BPC$. Thus, $\angle QBC = \angle QCB = 90^\circ - \angle BPC$, and so $\triangle PAD$ and $\triangle QBC$ are congruent. This means that PQ, AB, and CD share the common perpendicular bisector.

We now find the area by determining the altitude. Note that we have all four side lengths of isosceles trapezoids $PQAB$ and $PQCD$. Thus, one can compute their altitudes via Pythagorean theorem:

$$\text{distance}(P, AB) = \sqrt{14^2 - \left(\tfrac{18-14}{2}\right)^2} = 8\sqrt{3}$$
$$\text{distance}(P, CD) = \sqrt{14^2 - \left(\tfrac{28-14}{2}\right)^2} = 7\sqrt{3},$$

so the altitude of trapezoid $ABCD$ is $15\sqrt{3}$, so the final answer is $\frac{1}{2} \cdot (14 + 18) \cdot 15\sqrt{3} = \boxed{345\sqrt{3}}$.

10. **[55]** For each positive integer n, let $f(n)$ be either the unique integer $r \in \{0, 1, \ldots, n-1\}$ such that n divides $15r - 1$, or 0 if such r does not exist. Compute

$$f(16) + f(17) + f(18) + \cdots + f(300).$$

Proposed by: Jordan Lefkowitz, Pitchayut Saengrungkongka

Answer: $\boxed{11856}$

Solution: Note we only need to sum $f(n)$ for n relatively prime to 15. For any such $n > 1$, there exists a positive integer b such that $15f(n) - 1 = bn$. Since $bn \le 15(n-1) - 1 < 15n$ it follows that $b \in \{1, \ldots, 14\}$. Moreover, we have $bn \equiv 1 \pmod{15}$. These two conditions uniquely determine b.

Now, we are in business to compute the sum. Let S be the set of positive integers $a \in \{0, 1, 2, \ldots, 14\}$ such that $\gcd(a, 15) = 1$. For each $a \in S$, we first sum $f(n)$ over $n = 15k + a$ for $k \in \{1, 2, \ldots, 19\}$. Given a, we can uniquely determine b from $b \equiv -1/a \pmod{15}$. Thus, the sum of $f(n)$ across all numbers of the form $n = 15k + a$ is

$$\begin{aligned}\sum_{k=1}^{19} f(15k + a) &= \sum_{k=1}^{19} \frac{b(15k + a) + 1}{15} \\ &= 19 \cdot \frac{ab + 1}{15} + b\sum_{k=0}^{19} k \\ &= 19 \cdot \frac{ab + 1}{15} + 190b.\end{aligned}$$

We now sum the above expression across all $a \in S$.

- The sum of the first term $19 \cdot \frac{ab+1}{15}$ needs to be evaluated manually. The pairs (a, b) that works are $(1, 14)$, $(2, 7)$, $(4, 11)$, $(8, 13)$, and four other pairs obtained by swapping a and b. Thus, the sum of the first term over all $a \in S$ is

$$19 \cdot 2 \cdot \left(\tfrac{1 \cdot 14 + 1}{15} + \tfrac{2 \cdot 7 + 1}{15} + \tfrac{4 \cdot 11 + 1}{15} + \tfrac{8 \cdot 13 + 1}{15}\right) = 19 \cdot 2 \cdot (1 + 1 + 3 + 7) = 456.$$

- To evaluate the sum of the second term $190b$, we note that as a varies through S, b attains every value in S exactly once. We have $|S| = \varphi(15) = 8$ and $b \in S \iff 15 - b \in S$, so the sum of elements of S is $\frac{15 \cdot 8}{2} = 60$. Hence, the sum of the second term over all $a \in S$ is $190 \cdot 60 = 11400$.

Hence, the answer is $11400 + 456 = \boxed{11856}$.

Remark. Two versions of this problem were submitted independently (!) by Jordan Lefkowitz and Pitchayut Saengrungkongka within a few hours. The version that was selected for the exam is Pitchayut's. The following is Jordan's problem:

For integers $2 \le n \le 100$, let $f(n)$ denote the unique integer $1 \le a < n$ such that $101a \equiv 1 \pmod{n}$. Estimate

$$\sum_{n=2}^{100} \frac{f(n)}{n}.$$