12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Calculus Test Solutions

1. [**3**] Let f be a differentiable real-valued function defined on the positive real numbers. The tangent lines to the graph of f always meet the y-axis 1 unit lower than where they meet the function. If $f(1) = 0$, what is $f(2)$?

   Answer: $\boxed{\ln 2}$

   Solution: The tangent line to f at x meets the y-axis at $f(x) - 1$ for any x, so the slope of the tangent line is $f'(x) = \frac{1}{x}$, and so $f(x) = \ln(x) + C$ for some a. Since $f(1) = 0$, we have $C = 0$, and so $f(x) = \ln(x)$. Thus $f(2) = \ln(2)$.

2. [**3**] The differentiable function $F : \mathbb{R} \to \mathbb{R}$ satisfies $F(0) = -1$ and

$$\frac{d}{dx}F(x) = \sin(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x).$$

   Find $F(x)$ as a function of x.

   Answer: $\boxed{-\cos(\sin(\sin(\sin(x))))}$

   Solution: Substituting $u = \sin(\sin(\sin(x)))$, we find

$$F(x) = \int \sin(u)du = -\cos(u) + C.$$

   for some C. Since $F(0) = 1$ we find $C = 0$.

3. [**4**] Compute e^A where A is defined as

$$\int_{3/4}^{4/3} \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1} dx.$$

   Answer: $\boxed{\frac{16}{9}}$

   Solution: We can use partial fractions to decompose the integrand to $\frac{1}{x+1} + \frac{x}{x^2+1}$, and then integrate the addends separately by substituting $u = x + 1$ for the former and $u = x^2 + 1$ for latter, to obtain $\ln(x+1) + \frac{1}{2}\ln(x^2+1)\big|_{3/4}^{4/3} = \ln((x+1)\sqrt{x^2+1})\big|_{3/4}^{4/3} = \ln\frac{16}{9}$. Thus $e^A = 16/9$.

   Alternate solution: Substituting $u = 1/x$, we find

$$A = \int_{4/3}^{3/4} \frac{2u + u^2 + u^3}{1 + u + u^2 + u^3}(-\frac{1}{u^2})du = \int_{3/4}^{4/3} \frac{2/u + 1 + u}{1 + u + u^2 + u^3} du$$

   Adding this to the original integral, we find

$$2A = \int_{3/4}^{4/3} \frac{2/u + 2 + 2u + 2u^2}{1 + u + u^2 + u^3} du = \int_{3/4}^{4/3} \frac{2}{u} du$$

   Thus $A = \ln\frac{16}{9}$ and $e^A = \frac{16}{9}$.

4. [**4**] Let P be a fourth degree polynomial, with derivative P', such that $P(1) = P(3) = P(5) = P'(7) = 0$. Find the real number $x \neq 1, 3, 5$ such that $P(x) = 0$.

Answer: $\boxed{\frac{89}{11}}$

Solution: Observe that 7 is not a root of P. If r_1, r_2, r_3, r_4 are the roots of P, then $\frac{P'(7)}{P(7)} = \sum_i \frac{1}{7-r_i} = 0$. Thus $r_4 = 7 - \left(\sum_{i\neq 4} \frac{1}{7-r_i}\right)^{-1} = 7 + \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{2}\right)^{-1} = 7 + 12/11 = 89/11$.

5. [**4**] Compute

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{3} + 4h\right) - 4\sin\left(\frac{\pi}{3} + 3h\right) + 6\sin\left(\frac{\pi}{3} + 2h\right) - 4\sin\left(\frac{\pi}{3} + h\right) + \sin\left(\frac{\pi}{3}\right)}{h^4}$$

Answer: $\boxed{\frac{\sqrt{3}}{2}}$

Solution: The derivative of a function is defined as $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Iterating this formula four times yields

$$f^{(4)}(x) = \lim_{h\to 0} \frac{f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x)}{h^4}.$$

Substituting $f = \sin$ and $x = \pi/3$, the expression is equal to $\sin^{(4)}(\pi/3) = \sin(\pi/3) = \frac{\sqrt{3}}{2}$.

6. [**5**] Let $p_0(x), p_1(x), p_2(x), \ldots$ be polynomials such that $p_0(x) = x$ and for all positive integers n, $\frac{d}{dx}p_n(x) = p_{n-1}(x)$. Define the function $p(x) : [0, \infty) \to \mathbb{R}\ x$ by $p(x) = p_n(x)$ for all $x \in [n, n+1]$. Given that $p(x)$ is continuous on $[0, \infty)$, compute

$$\sum_{n=0}^{\infty} p_n(2009).$$

Answer: $\boxed{e^{2010} - e^{2009} - 1}$

Solution: By writing out the first few polynomials, one can guess and then show by induction that $p_n(x) = \frac{1}{(n+1)!}(x+1)^{n+1} - \frac{1}{n!}x^n$. Thus the sum evaluates to $e^{2010} - e^{2009} - 1$ by the series expansion of e^x.

7. [**5**] A line in the plane is called *strange* if it passes through $(a, 0)$ and $(0, 10 - a)$ for some a in the interval $[0, 10]$. A point in the plane is called *charming* if it lies in the first quadrant and also lies below some strange line. What is the area of the set of all charming points?

Answer: $\boxed{50/3}$

Solution: The strange lines form an envelope (set of tangent lines) of a curve $f(x)$, and we first find the equation for f on $[0, 10]$. Assuming the derivative f' is continuous, the point of tangency of the line ℓ through $(a, 0)$ and $(0, b)$ to f is the limit of the intersection points of this line with the lines ℓ_ϵ passing through $(a + \epsilon, 0)$ and $(0, b - \epsilon)$ as $\epsilon \to 0$. If these limits exist, then the derivative is indeed continuous and we can calculate the function from the points of tangency.

The intersection point of ℓ and ℓ_ϵ can be calculated to have x-coordinate $\frac{a(a-\epsilon)}{a+b}$, so the tangent point of ℓ has x-coordinate $\lim_{\epsilon\to 0} \frac{a(a-\epsilon)}{a+b} = \frac{a^2}{a+b} = \frac{a^2}{10}$. Similarly, the y-coordinate is $\frac{b^2}{10} = \frac{(10-a)^2}{10}$. Thus, solving for the y coordinate in terms of the x coordinate for $a \in [0, 10]$, we find $f(x) = 10 - 2\sqrt{10}\sqrt{x} + x$, and so the area of the set of charming points is

$$\int_0^{10} \left(10 - 2\sqrt{10}\sqrt{x} + x\right) dx = 50/3.$$

8. [**7**] Compute

$$\int_1^{\sqrt{3}} x^{2x^2+1} + \ln\left(x^{2x^{2x^2+1}}\right) dx.$$

Answer: $\boxed{13}$

Solution: Using the fact that $x = e^{\ln(x)}$, we evaluate the integral as follows:

$$\begin{aligned}\int x^{2x^2+1} + \ln\left(x^{2x^{2x^2+1}}\right) dx &= \int x^{2x^2+1} + x^{2x^2+1}\ln(x^2)dx \\ &= \int e^{\ln(x)(2x^2+1)}(1 + \ln(x^2))dx \\ &= \int xe^{x^2\ln(x^2)}(1 + \ln(x^2))dx\end{aligned}$$

Noticing that the derivative of $x^2 \ln(x^2)$ is $2x(1 + \ln(x^2))$, it follows that the integral evaluates to

$$\frac{1}{2}e^{x^2\ln(x^2)} = \frac{1}{2}x^{2x^2}.$$

Evaluating this from 1 to $\sqrt{3}$ we obtain the answer.

9. [**7**] let $\mathcal{R}$ be the region in the plane bounded by the graphs of $y = x$ and $y = x^2$. Compute the volume of the region formed by revolving $\mathcal{R}$ around the line $y = x$.

Answer: $\boxed{\frac{\sqrt{2}\pi}{60}}$

Solution: We integrate from 0 to 1 using the method of washers. Fix d between 0 and 1. Let the line $x = d$ intersect the graph of $y = x^2$ at Q, and let the line $x = d$ intersect the graph of $y = x$ at P. Then $P = (d, d)$, and $Q = (d, d^2)$. Now drop a perpendicular from Q to the line $y = x$, and let R be the foot of this perpendicular. Because PQR is a $45 - 45 - 90$ triangle, $QR = (d - d^2)/\sqrt{2}$. So the differential washer has a radius of $(d - d^2)/\sqrt{2}$ and a height of $\sqrt{2}dx$. So we integrate (from 0 to 1) the expression $[(x - x^2)/\sqrt{2}]^2\sqrt{2}dx$, and the answer follows.

10. [**8**] Let a and b be real numbers satisfying $a > b > 0$. Evaluate

$$\int_0^{2\pi} \frac{1}{a + b\cos(\theta)} d\theta.$$

Express your answer in terms of a and b.

Answer: $\boxed{\frac{2\pi}{\sqrt{a^2-b^2}}}$

Solution: Using the geometric series formula, we can expand the integral as follows:

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a+b\cos(\theta)} d\theta &= \frac{1}{a}\int_0^{2\pi} 1 + \frac{b}{a}\cos(\theta) + \left(\frac{b}{a}\right)^2 \cos^2(\theta) d\theta \\
&= \frac{1}{a}\sum_{n=0}^{\infty}\int_0^{2\pi} \left(\frac{b}{a}\right)^n \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^n d\theta \\
&= \frac{2\pi}{a}\sum_{n=0}^{\infty} \left(\frac{b^2}{a^2}\right)^n \frac{\binom{2n}{n}}{2^{2n}} d\theta
\end{aligned}$$

To evaluate this sum, recall that $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. The generating function for the Catalan numbers is

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x},$$

and taking the derivative of x times this generating function yields $\sum \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$. Thus the integral evaluates to $\frac{2\pi}{\sqrt{a^2-b^2}}$, as desired.