HMMT February 2015

Saturday 21 February 2015

Team

1. [**5**] The complex numbers x, y, z satisfy

$$\begin{aligned} xyz &= -4 \\ (x+1)(y+1)(z+1) &= 7 \\ (x+2)(y+2)(z+2) &= -3. \end{aligned}$$

Find, with proof, the value of $(x+3)(y+3)(z+3)$.

Answer: $\boxed{-28}$ **Solution 1.** Consider the cubic polynomial $f(t) = (x+t)(y+t)(z+t)$. By the theory of finite differences, $f(3) - 3f(2) + 3f(1) - f(0) = 3! = 6$, since f is monic. Thus $f(3) = 6 + 3f(2) - 3f(1) + f(0) = 6 + 3(-3) - 3(7) + (-4) = -28$.

Solution 2. Alternatively, note that the system of equations is a (triangular) linear system in $w := xyz$, $v := xy + yz + zx$, and $u := x + y + z$. The unique solution (u, v, w) to this system is $\left(-\frac{27}{2}, \frac{47}{2}, -4\right)$. Plugging in yields

$$\begin{aligned} (x+3)(y+3)(z+3) &= w + 3v + 9u + 27 \\ &= -4 + 3 \cdot \frac{47}{2} + 9 \cdot \left(-\frac{27}{2}\right) + 27 \\ &= -28. \end{aligned}$$

Remark. Since $f(0) < 0$, $f(1) > 0$, $f(2) < 0$, $f(+\infty) > 0$, the intermediate value theorem tells us the roots $-x, -y, -z$ of f are real numbers in $(0,1)$, $(1,2)$, and $(2,+\infty)$, in some order. With a little more calculation, one finds that x, y, z are the three distinct zeroes of the polynomial $X^3 + \frac{27}{2}X^2 + \frac{47}{2}X + 4$. The three zeroes are approximately $-11.484, -1.825$, and -0.191.

2. [**10**] Let P be a (non-self-intersecting) polygon in the plane. Let $C_1, \ldots, C_n$ be circles in the plane whose interiors cover the interior of P. For $1 \le i \le n$, let r_i be the radius of C_i. Prove that there is a single circle of radius $r_1 + \cdots + r_n$ whose interior covers the interior of P.

Answer: $\boxed{\text{N/A}}$ If $n = 1$, we are done. Suppose $n > 1$. Since P is connected, there must be a point x on the plane which lies in the interiors of two circles, say C_i, C_j. Let O_i, O_j, respectively, be the centers of C_i, C_j. Since $O_iO_j < r_i + r_j$, we can choose O to be a point on segment O_iO_j such that $O_iO \le r_j$ and $O_jO \le r_i$. Replace the two circles C_i and C_j with the circle C centered at O of radius $r_i + r_j$. Note that C covers both C_i and C_j. Induct to finish.

3. [**15**] Let $z = a + bi$ be a complex number with integer real and imaginary parts $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$ (i.e. z is a Gaussian integer). If p is an odd prime number, show that the real part of $z^p - z$ is an integer divisible by p.

Answer: $\boxed{\text{N/A}}$ **Solution 1.** We directly compute/expand

$$\begin{aligned} \operatorname{Re}(z^p - z) &= \operatorname{Re}\left((a+bi)^p - (a+bi)\right) \\ &= \left[a^p - \binom{p}{2}a^{p-2}b^2 + \binom{p}{4}a^{p-4}b^4 - \cdots\right] - a. \end{aligned}$$

Since $\binom{p}{i}$ is divisible by p for all $i = 2, 4, 6, \ldots$ (since $1 \le i \le p-1$), we have

$$[a^p - \binom{p}{2}a^{p-2}b^2 + \binom{p}{4}a^{p-4}b^4 - \cdots] - a \equiv a^p - a \equiv 0 \pmod{p}$$

by Fermat's little theorem. Thus p divides the real part of $z^p - z$.

Remark. The fact that $\binom{p}{i}$ is divisible by p (for $1 \le i \le p-1$) is essentially equivalent to the polynomial congruence $(rX+sY)^p \equiv rX^p + sY^p \pmod p$ (here the coefficients are taken modulo p), a fundamental result often called that "Frobenius endomorphism".

Solution 2. From the Frobenius endomorphism,

$$z^p = (a+bi)^p \equiv a^p + (bi)^p = a^p \pm b^p i \equiv a \pm bi \pmod{p \cdot \mathbb{Z}[i]},$$

where we're using congruence of Gaussian integers (so that $u \equiv v \pmod p$ if and only if $\frac{u-v}{p}$ is a Gaussian integer). This is equivalent to the simultaneous congruence of the real and imaginary parts modulo p, so the real part of z^p is congruent to a, the real part of z. So indeed p divides the real part of $z^p - z$.

4. [**15**] (Convex) quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle Ω; the diagonals of $ABCD$ meet at X. Suppose $AD < AB$, the circumcircle of triangle BCX intersects segment AB at a point $Y \ne B$, and **ray** $\overrightarrow{CY}$ meets Ω again at a point $Z \ne C$. Prove that ray $\overrightarrow{DY}$ bisects angle ZDB.

   (We have only included the conditions $AD < AB$ and that Z lies on **ray** $\overrightarrow{CY}$ for everyone's convenience. With slight modifications, the problem holds in general. But we will only grade your team's solution in this special case.)

   Answer: $\boxed{\text{N/A}}$ This is mostly just angle chasing. The conditions $AD < AB$ (or $\angle ABD < \angle ADB$) and the assumption $Z \in \overrightarrow{CY}$ are not crucial, as long as we're careful with configurations (for example, DY may only be an external angle bisector of $\angle ZDB$ in some cases), [1] but it's the easiest to visualize.

   In this case Y and Z lie between A and B, on the respective segment/arc. We'll prove Y is the incenter of $\triangle ZDB$; it will follow that ray $\overrightarrow{DY}$ indeed (internally) bisects $\angle ZDB$. It suffices to prove the following two facts:

   - BY is the internal angle bisector of $\angle DBZ$.[2]
   - ZY is the internal angle bisector of $\angle BZD$, since $CB = CD$. Indeed (more explicitly), arcs BC and CD are equal, so $\angle BZC = \angle CZD$, i.e. YZ bisects $\angle BZD$.

5. [**20**] For a convex quadrilateral P, let D denote the sum of the lengths of its diagonals and let S denote its perimeter. Determine, with proof, all possible values of $\frac{S}{D}$.

   Answer: $\boxed{1 < \frac{S}{D} < 2}$ Suppose we have a convex quadrilateral $ABCD$ with diagonals AC and BD intersecting at E (convexity is equivalent to having E on the interiors of segments AC and BD).

   To prove the lower bound, note that by the **triangle inequality**, $AB+BC > AC$ and $AD+DC > AC$, so $S = AB+BC+AD+DC > 2AC$. Similarly, $S > 2BD$, so $2S > 2AC+2BD = 2D$ gies $S > D$.

   To prove the upper bound, note that again by the **triangle inequality**, $AE+EB > AB$, $CE+BE > BC$, $AE+ED > AD$, $CE+ED > CD$. Adding these yields

   $$2(AE+EC+BE+ED) > AB+BC+AD+CD = S.$$

   Now since $ABCD$ is convex, E is inside the quadrilateral, so $AE+EC = AC$ and $BE+ED = BD$. Thus $2(AC+BD) = D > S$.

   To achieve every real value in this range, first consider a square $ABCD$. This has $\frac{S}{D} = \sqrt{2}$. Suppose now that we have a rectangle with $AB = CD = 1$ and $BC = AD = x$, where $0 < x \le 1$. As x

[1] The cleanest way to see this is via directed angles, or more rigorously by checking that the problem is equivalent to a four-variable polynomial identity in complex numbers.

[2] This is true in general; it doesn't require $CB = CD$. It's part of the spiral similarity configuration centered at $B : YX \to ZA$ and $B : ZY \to AX$, due to $YZ \cap AX = C$ and $B = (CYX) \cap (CZA)$.

More explicitly, this follows from the angle chase

$$\angle DBA = \angle XBY = \angle XCY = \angle ACZ = \angle ABZ.$$

approaches 0 (i.e. **our rectangle gets thinner**), $\frac{S}{D}$ gets arbitrarily close to 1, so by intermediate value theorem, we hit every value $\frac{S}{D} \in (1, \sqrt{2}]$.

To achieve the other values, we let $AB = BC = CD = DA = 1$ and let $\theta = m\angle ABE$ vary from 45° down to 0° (i.e. **a rhombus that gets thinner**). This means $AC = 2\sin\theta$ and $BD = 2\cos\theta$. We have $S = 4$ and $D = 2(\sin\theta + \cos\theta)$. When $\theta = 45^\circ$, $\frac{S}{D} = \sqrt{2}$, and when $\theta = 0^\circ$, $\frac{S}{D} = 2$. Thus by the intermediate value theorem, we are able to choose θ to obtain any value in the range $[\sqrt{2}, 2)$.

Putting this construction together with the strict upper and lower bounds, we find that all possible values of $\frac{S}{D}$ are all real values in the open interval $(1, 2)$.

6. [**30**] \$indy has \$100 in pennies (worth \$0.01 each), nickels (worth \$0.05 each), dimes (worth \$0.10 each), and quarters (worth \$0.25 each). Prove that she can split her coins into two piles, each with total value exactly \$50.

Answer: $\boxed{\text{N/A}}$ **Solution 1.** First, observe that if there are pennies in the mix, there must be a multiple of 5 pennies (since $5, 10, 25 \equiv 0 \pmod 5$), so we can treat each group of 5 pennies as nickels and reduce the problem to the case of no pennies, treated below.

Indeed, suppose there are no pennies and that we are solving the problem with only quarters, dimes, and nickels. So we have $25q + 10d + 5n = 10000$ (or equivalently, $5q + 2d + n = 2000$), where q is the number of quarters, d is the number of dimes, and n is the number of nickels. Notice that if $q \geq 200$, $d \geq 500$, or $n \geq 1000$, then we can just take a subset of 200 quarters, 500 dimes, or 1000 nickels, respectively, and make that our first pile and the remaining coins our second pile.

Thus, we can assume that $q \leq 199$, $d \leq 499$, $n \leq 999$. Now if there are only quarters and dimes, we have $5q + 2d < 1000 + 1000 = 2000$, so there must be nickels in the mix. By similar reasoning, there must be dimes and quarters in the mix.

So to make our first pile, we **throw all the quarters in** that pile. The total money in the pile is at most \$49.75, so now,

- **if possible, we keep adding in dimes until we get an amount equal to or greater than** \$50. If it is \$50, then we are done. Otherwise, it must be \$50.05 since the total money value before the last dime is either \$49.90 or \$49.95. In that case, we take out the last dime and replace it with a nickel, giving us \$50.
- **if not possible (i.e. we run out of dimes before reaching** \$50**)**, then all the quarters and dimes have been used up, so we only have nickels left. Of course, our money value now has a hundredth's place of 5 or 0 and is less than \$50, so we simply keep adding nickels until we are done.

Solution 2. First, observe that if there are pennies in the mix, there must be a multiple of 5 pennies (since $5, 10, 25 \equiv 0 \pmod 5$), so we can treat each group of 5 pennies as nickels and reduce the problem to the case of no pennies, treated below. Similarly, we can treat each group of 2 nickels as dimes, and reduce the problem to the case of at most one nickel. We have two cases:

Case 1. There are no nickels. Then, either the dimes alone total to at least \$50, in which case \$indy can form a pile with \$50 worth of dimes and a pile with the rest of the money, or the quarters alone total to at least \$50, in which case \$indy can form a pile with \$50 worth of quarters and a pile with the rest of the money.

Case 2. There is exactly one nickel. By examining the value of the money modulo \$0.25, we find that there must be at least two dimes. Then, we treat the nickel and two dimes as a quarter, and reduce the problem to the previously solved case.

Remark. There are probably many other solutions/algorithms for this specific problem. For example, many teams did casework based on the parities of q, d, n, after "getting rid of pennies".

Remark. It would certainly be interesting to investigate the natural generalizations of this problem (see e.g. the knapsack problem and partition problem). For instance, it's true that given a set of positive integers ranging from 1 to n with sum no less than $2n!$, there exists a subset of them summing up to exactly $n!$. (See here and here for discussion on two AoPS blogs.)

Remark. The problem author (Carl Lian) says the problem comes from algebraic geometry!

7. [**35**] Let $f : [0,1] \to \mathbb{C}$ be a *nonconstant* complex-valued function on the real interval $[0,1]$. Prove that there exists $\epsilon > 0$ (possibly depending on f) such that for any polynomial P with complex coefficients, there exists a complex number z with $|z| \le 1$ such that $|f(|z|) - P(z)| \ge \epsilon$.

Answer: $\boxed{\text{N/A}}$ Here's a brief summary of how we'll proceed: Use a suitably-centered high-degree roots of unity filter, or alternatively integrate over suitably-centered circles ("continuous roots of unity filter").

We claim we can choose $\epsilon(f) = \max(|f(x_1) - f(x_2)|)/2$. Fix P and suppose for the sake of contradiction that for all z with $|z| \le 1$ it is the case that $|f(z) - P(z)| < \epsilon$. We can write $z = re^{i\theta}$ so that

$$|f(r) - P(re^{i\theta})| < \epsilon.$$

Let p be a prime larger than $\deg(P)$ and set $\theta = 0, 2\pi/p, 4\pi/p, \ldots$ in the above. Averaging, and using triangle inequality,

$$|f(r) - \frac{1}{p}\sum_{k=0}^{p-1} P(re^{2\pi ik/p})| < \epsilon.$$

The sum inside the absolute value is a roots of unity filter; since $p > \deg(P)$ it is simply the constant term of P - denote this value by p_0. Thus for all r,

$$|f(r) - p_0| < \epsilon.$$

Setting $r = x_1, x_2$ and applying triangle inequality gives the desired contradiction.

Remark. This is more or less a concrete instance of Runge's approximation theorem from complex analysis. However, do not be intimidated by the fancy names here: in fact, the key ingredient behind this more general theorem (namely, or similar[3]) uses the same core idea of the "continuous roots of unity filter".

8. [**40**] Let π be a permutation of $\{1, 2, \ldots, 2015\}$. With proof, determine the maximum possible number of ordered pairs $(i,j) \in \{1, 2, \ldots, 2015\}^2$ with $i < j$ such that $\pi(i) \cdot \pi(j) > i \cdot j$.

Answer: $\boxed{\binom{2014}{2}}$ Let $n = 2015$. The only information we will need about n is that $n > 5$.[4] For the construction, take π to be the n-cycle defined by

$$\pi(k) = \begin{cases} k+1 & \text{if } 1 \le k \le n-1 \\ 1 & \text{if } k = n. \end{cases}$$

Then $\pi(i) > i$ for $1 \le i \le n-1$. So $\pi(i)\pi(j) > ij$ for at least $\binom{n-1}{2}$ pairs $i < j$.

For convenience let $z_i = \frac{\pi(i)}{i}$, so that we are trying to maximize the number of pairs $(i,j), i < j$ with $z_i z_j > 1$. Notice that over any cycle $c = (i_1\ i_2\ \cdots\ i_k)$ in the cycle decomposition of π we have $\prod_{i \in c} z_i = 1$. In particular, multiplying over all such cycles gives $\prod_{i=1}^n z_i = 1$.

Construct a graph G on vertex set $V = [n]$ such that there is an edge between i and j whenever $\pi(i)\pi(j) > ij$. For any cycle $C = (v_1, v_2, \ldots, v_k)$ in this graph we get $1 < \prod_{i=1}^k z_{v_i} z_{v_{i+1}} = \prod_{v \in C} z_v^2$. So, we get in particular that G is non-Hamiltonian.

By the contrapositive of Ore's theorem,[5] there are two distinct indices $u, v \in [n]$ such that $d(u) + d(v) \le n - 1$. The number of edges is then at most $\binom{n-2}{2} + (n-1) = \binom{n-1}{2} + 1$.

[3] which generally underlies a great deal of complex analysis

[4] If you want, you can try running a computer check for $n \le 5$. One of the problem czars believes the answer is still $\binom{4}{2} = 6$ (compare with the Remark at the end for a looser general problem) based on computer program, but didn't check carefully.

[5] see also this StackExchange thread

If equality is to hold, we need $G \setminus \{u, v\}$ to be a complete graph. We also need $d(u) + d(v) = n - 1$, with uv not an edge. This implies $d(u), d(v) \geq 1$. Since $n > 5$, the pigeonhole principle gives that at least one of u, v has degree at least 3. WLOG $d(u) \geq 3$.

Let w be a neighbor of v and let a, b, c be neighbors of u; WLOG $w \neq a, b$. Since $G \setminus \{u, v, w\}$ is a complete graph, we can pick a Hamiltonian path in $G \setminus \{u, v, w\}$ with endpoints a, b. Connecting u to the ends of this path forms an $(n-2)$-cycle C.

This gives us $\prod_{x \in C} z_x^2 > 1$. But we also have $z_v z_w > 1$, so $1 = \prod_{i=i}^n z_i > 1$, contradiction.

So, $\binom{n-1}{2} + 1$ cannot be attained, and $\binom{n-1}{2}$ is indeed the maximum number of pairs possible.

Remark. Let's think about the more general problem where we forget about the permutation structure (and only remember $\prod z_i = 1$ and $z_i > 0$). If $n = 4$ it's easy to show by hand that $\binom{n-1}{2}$ is still the best possible. If $n = 5$, then we can get $1 + \binom{n-1}{2} = 7$: consider exponentiating the zero-sum set $\{t, t, t^2 - t, t^2 - t, -2t^2\}$ for some sufficiently small t. A computer check would easily determine whether a construction exists in the original permutation setting. (If the following code is correct, the answer is 'no', i.e. 6 is the best: see `http://codepad.org/Efsv8Suy`.)

9. [**40**] Let $z = e^{\frac{2\pi i}{101}}$ and let $\omega = e^{\frac{2\pi i}{10}}$. Prove that

$$\prod_{a=0}^{9} \prod_{b=0}^{100} \prod_{c=0}^{100} (\omega^a + z^b + z^c)$$

is an integer and find (with proof) its remainder upon division by 101.

Answer: $\boxed{13}$ **Solution 1.** Let $p = 101$ and $r = 10$. Note that $p \nmid r$.

In the sequel, we will repeatedly use the polynomial identities $\prod_{k \pmod p} (x - z^k) = x^p - 1$, and $\prod_{j \pmod r} (x - \omega^j) = x^r - 1$.

The product is an integer by standard symmetric sum theory (concrete precursor to Galois theory, though one doesn't need the full language). More precisely, it expands into an integer-coefficient polynomial symmetric in the ω^j (for the r residues $j \pmod r$), and also symmetric in the z^k (for the p residues $k \pmod p$). So by the fundamental theorem of symmetric sums, it can be written as an integer-coefficient polynomial in the symmetric sums of the ω^j together with the symmetric sums of the z^k. But these symmetric sums are integers themselves, so the original expression is indeed an integer.

To actually compute the remainder modulo p, note[6] by the Frobenius endomorphism $(u+v)^p \equiv u^p + v^p \pmod p$ that

$$\begin{aligned}\prod_{a,b,c} (\omega^a + z^b + z^c) &= \prod_{a,b} (-1)^p ((-\omega^a - z^b)^p - 1) \\ &\equiv \prod_{a,b} (\omega^{pa} + z^{pb} - (-1)^p) = \prod_{a,b} (\omega^{pa} + 2) \pmod p,\end{aligned}$$

where we use $p \equiv 1 \pmod 2$ in the last step. This simplifies further to

$$\begin{aligned}\prod_a (-1)^p (-2 - \omega^{pa})^p &= \prod_a (-1)^p (-2 - \omega^a)^p \\ &= (-1)^{pr} [(-2)^r - 1]^p \equiv (-1)^{pr} [(-2)^r - 1] \pmod p,\end{aligned}$$

where we've used the fact that $\{pa \pmod r\} = \{a \pmod r\}$ (since p is invertible mod r), as well as Fermat's little theorem on $(-2)^r - 1$.

Finally, we plug in the specific numbers:

$$(-1)^{101 \cdot 10} [(-2)^{10} - 1] = 1023 \equiv 13 \pmod{101}.$$

[6]Here we are using congruence of *algebraic integers* modulo p, so that $\alpha \equiv \beta \pmod p$ if and only if $\frac{\alpha - \beta}{p}$ is an algebraic integer. In particular, for integers u, v, $\frac{u-v}{p}$ is an algebraic integer if and only if $\frac{u-v}{p}$ is an integer.

Remark. By repeatedly getting rid of "z" terms and applying the Frobenius map, one can show that $\prod(\omega^a + z^{b_1} + \cdots + z^{b_m})$ is an integer congruent to something like $m^r + 1$ modulo p (as long as p is odd, etc.).

Solution 2 (sketch). Here we sketch a better solution found by several teams, but missed by the author (primarily due to blindness from having first found the first solution). The proof of the first part (that the product is an integer) is the same, so we only sketch different proofs of the second part (computation of the remainder mod p).

For example, we can use the algebraic integer formulation from the previous solution. Indeed, note that $D := \prod_{a,b,c}(\omega^a + z^b + z^c) - \prod_{a,b,c}(\omega^a + 1^b + 1^c)$ is an algebraic integer divisible by $z-1$. But D is also a difference of integers, hence an integer itself. The only way $\frac{D}{z-1}$ can be an algebraic integer is if $p \mid D$ (Why?), so it simply remains to compute the remainder when the integer $\prod_{a,b,c}(\omega^a + 1^b + 1^c)$ is divided by p, which is quite easy.

Alternatively (as some contestants/teams found), we can get rid of ω first (rather than z as in the previous solution), so (up to sign) we want to evaluate $\prod_{b,c}(1 + (z^b + z^c)^r) \pmod p$. But we have a polynomial identity $\prod_{b,c}(1 + (T^b + T^c)^r) \in \Phi_p(T) \cdot \mathbb{Z}[x] + M$ for some constant integer M, where $\Phi_p(T) = T^{p-1} + \cdots + T + 1$ denotes the pth cyclotomic polynomial. But note that $\Phi_p(z) = 0$ is congruent to $\Phi_p(1) = p$ modulo p, so

$$\prod_{b,c}(1 + (z^b + z^c)^r) - \prod_{b,c}(1 + (1^b + 1^c)^r)$$

is an integer divisible by p. The rest is easy.

Remark. Without using the Frobenius map, this second solution gives another proof that $\prod(\omega^a + z^{b_1} + \cdots + z^{b_m})$ is an integer congruent to something like $m^r + 1$ modulo p (as long as p is odd, etc.). Can you reconcile this difference (e.g. is the use of Frobenius in the first solution equivalent to a step of the second solution)?

Remark. The central idea behind the second solution is that if $f(z)$ is an integer for some polynomial $f \in \mathbb{Z}[x]$, then it's congruent to (the integer) $f(1)$ modulo p. Can you generalize this, say when z is not necessarily a root of unity?

10. [**40**] The sequences of real numbers $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ satisfy $a_{n+1} = (a_{n-1} - 1)(b_n + 1)$ and $b_{n+1} = a_n b_{n-1} - 1$ for $n \geq 2$, with $a_1 = a_2 = 2015$ and $b_1 = b_2 = 2013$. Evaluate, with proof, the infinite sum

$$\sum_{n=1}^{\infty} b_n \left(\frac{1}{a_{n+1}} - \frac{1}{a_{n+3}} \right).$$

Answer: $\boxed{1 + \frac{1}{2014 \cdot 2015} \text{ OR } \frac{4058211}{4058210}}$ First note that a_n and b_n are weakly increasing and tend to infinity. In particular, $a_n, b_n \notin \{0, -1, 1\}$ for all n.

For $n \geq 1$, we have $a_{n+3} = (a_{n+1} - 1)(b_{n+2} + 1) = (a_{n+1} - 1)(a_{n+1} b_n)$, so

$$\frac{b_n}{a_{n+3}} = \frac{1}{a_{n+1}(a_{n+1} - 1)} = \frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+1}}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n}{a_{n+1}} - \frac{b_n}{a_{n+3}} &= \sum_{n=1}^{\infty} \frac{b_n}{a_{n+1}} - \left(\frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+1}} \right) \\ &= \sum_{n=1}^{\infty} \frac{b_n + 1}{a_{n+1}} - \frac{1}{a_{n+1} - 1}. \end{aligned}$$

Furthermore, $b_n + 1 = \frac{a_{n+1}}{a_{n-1}-1}$ for $n \geq 2$. So the sum over $n \geq 2$ is

$$\begin{aligned}
\sum_{n=2}^{\infty} \left(\frac{1}{a_{n-1}-1} - \frac{1}{a_{n+1}-1} \right) &= \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{a_{n-1}-1} - \frac{1}{a_{n+1}-1} \right) \\
&= \frac{1}{a_1 - 1} + \frac{1}{a_2 - 1} - \lim_{N\to\infty} \left(\frac{1}{a_N - 1} + \frac{1}{a_{N+1} - 1} \right) \\
&= \frac{1}{a_1 - 1} + \frac{1}{a_2 - 1}.
\end{aligned}$$

Hence the final answer is

$$\left(\frac{b_1 + 1}{a_2} - \frac{1}{a_2 - 1} \right) + \left(\frac{1}{a_1 - 1} + \frac{1}{a_2 - 1} \right).$$

Cancelling the common terms and putting in our starting values, this equals

$$\frac{2014}{2015} + \frac{1}{2014} = 1 - \frac{1}{2015} + \frac{1}{2014} = 1 + \frac{1}{2014 \cdot 2015}.$$