HMMT November 2023

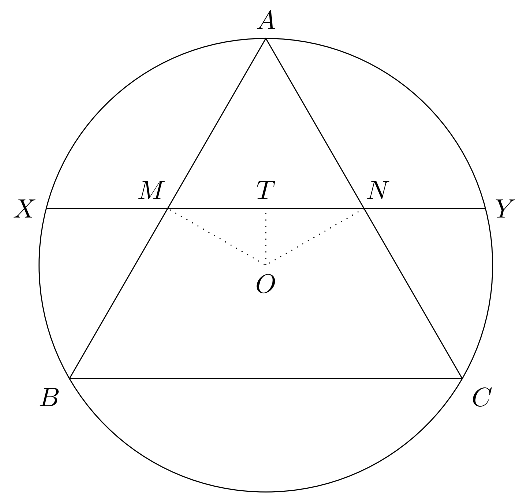
November 11, 2023

Team Round

1. **[20]** Let ABC be an equilateral triangle with side length 2 that is inscribed in a circle ω. A chord of ω passes through the midpoints of sides AB and AC. Compute the length of this chord.

   Proposed by: Rishabh Das

   Answer: $\boxed{\sqrt{5}}$



Solution 1: Let O and r be the center and the circumradius of $\triangle ABC$. Let T be the midpoint of the chord in question.

Note that $AO = \frac{AB}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Additionally, we have that AT is half the distance from A to BC, i.e. $AT = \frac{\sqrt{3}}{2}$. This means that $TO = AO - AT = \frac{\sqrt{3}}{6}$. By the Pythagorean Theorem, the length of the chord is equal to:

$$2\sqrt{r^2 - OT^2} = 2\sqrt{\frac{4}{3} - \frac{1}{12}} = 2\sqrt{\frac{5}{4}} = \boxed{\sqrt{5}}.$$

Solution 2: Let the chord be XY, and the midpoints of AB and AC be M and N, respectively, so that the chord has points X, M, N, Y in that order. Let $XM = NY = x$. Power of a point gives

$$1^2 = x(x+1) \implies x = \frac{-1 \pm \sqrt{5}}{2}.$$

Taking the positive solution, we have $XY = 2x + 1 = \boxed{\sqrt{5}}$.

2. **[20]** A real number x satisfies $9^x + 3^x = 6$. Compute the value of $16^{1/x} + 4^{1/x}$.

   Proposed by: Karthik Venkata Vedula

   Answer: $\boxed{90}$

   Solution: Setting $y = 3^x$ in the given equation yields

$$y^2 + y = 6 \implies y^2 + y - 6 = 0 \implies y = -3, 2.$$

Since $y > 0$ we must have

$$3^x = 2 \implies x = \log_3(2) \implies 1/x = \log_2(3).$$

This means that

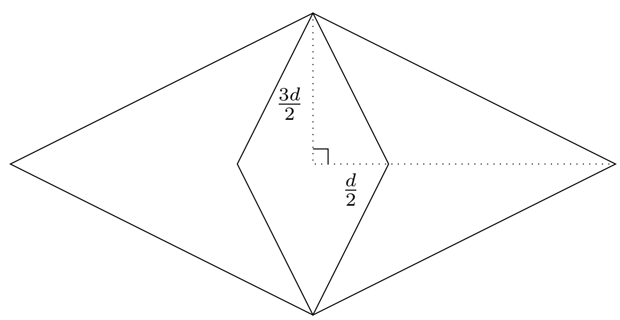
$$16^{1/x} + 4^{1/x} = (2^{1/x})^4 + (2^{1/x})^2 = 3^4 + 3^2 = \boxed{90}.$$

3. **[25]** Two distinct similar rhombi share a diagonal. The smaller rhombus has area 1, and the larger rhombus has area 9. Compute the side length of the larger rhombus.

Proposed by: Albert Wang

Answer: $\boxed{\sqrt{15}}$

Solution: Let d be the length of the smaller diagonal of the smaller rhombus. Since the ratio of the areas is $9 : 1$, the ratio of the lengths is $3 : 1$. This means that the smaller diagonal of the larger rhombus (which is also the longer diagonal of the smaller rhombus) has length $3d$.



(Diagram not to scale.)

Therefore, the smaller rhombus has diagonal lengths d and $3d$, so since it has area 1, we have

$$\frac{1}{2}d(3d) = 1 \implies d = \sqrt{\frac{2}{3}}.$$

By the Pythagorean Theorem, the side length of the smaller rhombus is

$$\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{3d}{2}\right)^2} = \sqrt{\frac{5d^2}{2}} = \sqrt{\frac{5}{3}}.$$

The side length of the larger rhombus is three times this, i.e. $\boxed{\sqrt{15}}$.

4. **[30]** There are six empty slots corresponding to the digits of a six-digit number. Claire and William take turns rolling a standard six-sided die, with Claire going first. They alternate with each roll until they have each rolled three times. After a player rolls, they place the number from their die roll into a remaining empty slot of their choice. Claire wins if the resulting six-digit number is divisible by 6, and William wins otherwise. If both players play optimally, compute the probability that Claire wins.

Proposed by: Serena An

Answer: $\boxed{\frac{43}{192}}$

Solution: A number being divisible by 6 is equivalent to the following two conditions:

- the sum of the digits is divisible by 3
- the last digit is even

Regardless of Claire and William's strategies, the first condition is satisfied with probability $\frac{1}{3}$. So Claire simply plays to maximize the chance of the last digit being even, while William plays to minimize this chance. In particular, clearly Claire's strategy is to place an even digit in the last position if she ever rolls one (as long as the last slot is still empty), and to try to place odd digits anywhere else. William's strategy is to place an odd digit in the last position if he ever rolls one (as long as the last slot is still empty), and to try to place even digits anywhere else.

To compute the probability that last digit ends up even, we split the game into the following three cases:

- If Claire rolls an even number before William rolls an odd number, then Claire immediately puts the even number in the last digit.
- If William rolls an odd number before Claire rolls an even number, then William immediately puts the odd number in the last digit.
- If William never rolls an odd number and Claire never rolls an even number, then since William goes last, he's forced to place his even number in the last slot.

The last digit ends up even in the first and third cases. The probability of the first case happening is $\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}$, depending on which turn Claire rolls her even number. The probability of the third case is $\frac{1}{2^6}$. So the probability the last digit is even is

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\frac{1}{2^6}=\frac{43}{64}.$$

Finally we multiply by the $\frac{1}{3}$ chance that the sum of all the digits is divisible by 3 (this is independent from the last-digit-even condition by e.g. Chinese Remainder Theorem), making our final answer

$$\frac{1}{3}\cdot\frac{43}{64}=\boxed{\frac{43}{192}}.$$

5. **[35]** A complex quartic polynomial Q is *quirky* if it has four distinct roots, one of which is the sum of the other three. There are four complex values of k for which the polynomial $Q(x)=x^4-kx^3-x^2-x-45$ is quirky. Compute the product of these four values of k.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{720}$

   Solution: Let the roots be a,b,c,d with $a+b+c=d$. Since $a+b+c=k-d$ by Vieta's formulas, we have $d=k/2$. Hence

$$0=P\left(\frac{k}{2}\right)=\left(\frac{k}{2}\right)^4-k\left(\frac{k}{2}\right)^3-\left(\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)-45=-\frac{k^4}{16}-\frac{k^2}{4}-\frac{k}{2}-45.$$

   We are told that there are four distinct possible values of k, which are exactly the four solutions to the above equation; by Vieta's formulas, their product $45\cdot 16=\boxed{720}$.

6. **[45]** The pairwise greatest common divisors of five positive integers are

$$2,3,4,5,6,7,8,p,q,r$$

in some order, for some positive integers p, q, r. Compute the minimum possible value of $p+q+r$.

Proposed by: Arul Kolla

Answer: $\boxed{9}$

Solution: To see that 9 can be achieved, take the set $\{6, 12, 40, 56, 105\}$, which gives

$$\{p, q, r\} = \{2, 3, 4\}.$$

Now we show it's impossible to get lower.

Notice that if m of the five numbers are even, then exactly $\binom{m}{2}$ of the gcd's will be even. Since we're shown four even gcd's and three odd gcd's, the only possibility is $m = 4$. Hence exactly two of p, q, r are even.

Similarly, if n of the five numbers are divisible by 3, then exactly $\binom{n}{2}$ of the gcd's will be divisible by 3. Since we're shown two gcd's that are multiples of 3 and five gcd's that aren't, the only possibility is $n = 3$. Hence exactly one of p, q, r is divisible by 3.

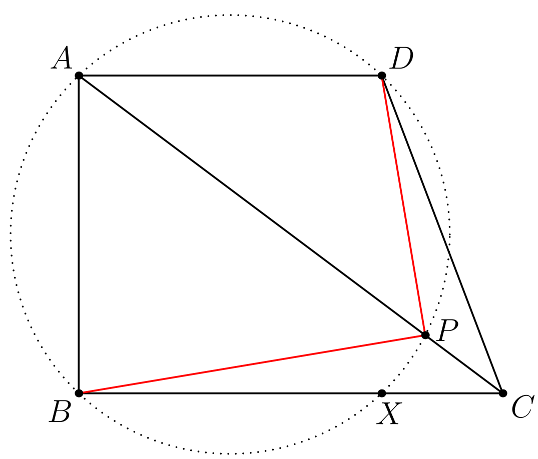
Similarly, if k of the five numbers are divisible by 4, then exactly $\binom{k}{2}$ of the gcd's will be divisible by 4. Since we're shown two gcd's that are multiples of 4 and five gcd's that aren't, the only possibility is $k = 3$. Hence exactly one of p, q, r is divisible by 4.

So two of p, q, r are even, one of them is divisible by 4, and one of them is divisible by 3. It's easy to see by inspection there are no possibilities where $p + q + r < 9$.

7. **[45]** Let $ABCD$ be a convex trapezoid such that $\angle BAD = \angle ADC = 90^\circ$, $AB = 20$, $AD = 21$, and $CD = 28$. Point $P \neq A$ is chosen on segment AC such that $\angle BPD = 90^\circ$. Compute AP.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{143}{5}}$



Solution 1: Construct the rectangle $ABXD$. Note that

$$\angle BAD = \angle BPD = \angle BXD = 90^\circ,$$

so $ABXPD$ is cyclic with diameter BD. By Power of a Point, we have $CX \cdot CD = CP \cdot CA$. Note that $CX = CD - XD = CD - AB = 8$ and $CA = \sqrt{AD^2 + DC^2} = 35$. Therefore,

$$CP = \frac{CX \cdot CD}{CA} = \frac{8 \cdot 28}{35} = \frac{32}{5},$$

and so

$$AP = AC - CP = 35 - \frac{32}{5} = \boxed{\frac{143}{5}}.$$

Solution 2: Since $ABPD$ is cyclic and $\angle BDP = \angle BAP = \angle ACD$, it follows that $\triangle BPD \sim \triangle ABC$. Thus, if $BD = 5x$, $BP = 4x$, and $DP = 3x$, then by Ptolemy's theorem, we have

$$AP \cdot (5x) = AD \cdot (3x) + AB \cdot (4x)$$
$$AP = \frac{21 \cdot 3 + 20 \cdot 4}{5} = \boxed{\frac{143}{5}}$$

8. **[55]** There are $n \geq 2$ coins, each with a different positive integer value. Call an integer m *sticky* if some subset of these n coins have total value m. We call the entire set of coins a *stick* if all the sticky numbers form a consecutive range of integers. Compute the minimum total value of a stick across all sticks containing a coin of value 100.

   Proposed by: Albert Wang

   Answer: $\boxed{199}$

   Solution: Sort a stick by increasing value. Note that all sticks must contain 1 by necessity, or the largest and second largest sticky values would not be consecutive. So, let's say a stick's highest coin value is a, and all the other terms have a value of S. If $a \geq S+2$, we cannot build $S+1$, but we can produce S and $S+2$, meaning that this cannot happen. So, $a \leq S+1$, and therefore $100 \leq S+1 \rightarrow S \geq 99$ giving a lower bound on the answer of $\boxed{199}$. This is easily achievable by picking any stick with $S = 99$. For instance, $\{1, 2, 3, 7, 12, 24, 50, 100\}$ is a construction.

9. **[60]** Let r_k denote the remainder when $\binom{127}{k}$ is divided by 8. Compute $r_1 + 2r_2 + 3r_3 + \cdots + 63r_{63}$.

   Proposed by: Rishabh Das

   Answer: $\boxed{8096}$

   Solution: Let $p_k = \frac{128-k}{k}$, so

$$\binom{127}{k} = p_1 p_2 \cdots p_k.$$

Now, for $k \leq 63$, unless $32 \mid \gcd(k, 128-k) = \gcd(k, 128)$, $p_k \equiv -1 \pmod 8$. We have $p_{32} = \frac{96}{32} = 3$. Thus, we have the following characterization:

$$r_k = \begin{cases} 1 & \text{if } k \text{ is even and } k \leq 31 \\ 7 & \text{if } k \text{ is odd and } k \leq 31 \\ 5 & \text{if } k \text{ is even and } k \geq 32 \\ 3 & \text{if } k \text{ is odd and } k \geq 32. \end{cases}$$

We can evaluate this sum as

$$\begin{aligned} &4 \cdot (0 + 1 + 2 + 3 + \cdots + 63) \\ &\quad + 3 \cdot (-0 + 1 - 2 + 3 - \cdots - 30 + 31) \\ &\quad + (32 - 33 + 34 - 35 + \cdots + 62 - 63) \\ &= 4 \cdot 2016 + 3 \cdot 16 + (-16) = 8064 + 32 = \boxed{8096}. \end{aligned}$$

10. **[65]** Compute the number of ways a non-self-intersecting concave quadrilateral can be drawn in the plane such that two of its vertices are $(0,0)$ and $(1,0)$, and the other two vertices are two distinct lattice points $(a,b),(c,d)$ with $0 \le a,c \le 59$ and $1 \le b,d \le 5$.

(A concave quadrilateral is a quadrilateral with an angle strictly larger than $180°$. A lattice point is a point with both coordinates integers.)

Proposed by: Julia Kozak

Answer: $\boxed{366}$

Solution: We instead choose points $(0,0),(1,0),(a,b),(c,d)$ with $0 \le a,c \le 59$ and $0 \le b,d \le 5$ with (c,d) in the interior of the triangle formed by the other three points. Any selection of these four points may be connected to form a concave quadrilateral in precisely three ways.

Apply Pick's theorem to this triangle. If I is the count of interior points, and B is the number of boundary lattice points, we have that the triangle's area is equal to

$$\frac{b}{2} = I + \frac{B}{2} - 1.$$

Let's first compute the number of boundary lattice points on the segment from $(0,0)$ to (a,b), not counting $(0,0)$. This is just $\gcd(a,b)$. Similarly, there are $\gcd(a-1,b)$ boundary lattice points from $(1,0)$ to (a,b). Adjusting for the overcounting at (a,b), we have

$$B = \gcd(a,b) + \gcd(a-1,b) - 1$$

and thus

$$I = \frac{b - \gcd(a,b) - \gcd(a-1,b) + 1}{2}$$

which we notice is periodic in a with period b. That is, the count of boundary points does not change between choices (a,b) and $(a+b,b)$.

We wanted to find the sum across all (a,b) of I, the number of interior points (c,d). Using casework on b, the periodicity allows us to just check I across points with $0 \le a < b$, and then multiply the count by $\frac{60}{b}$ to get the sum of I across the entire row of points.

For $b = 1, 2$, we always have $I = 0$.

For $b = 3$, we have $I = 0$ at $(0,3),(1,3)$ and $I = 1$ for $(2,3)$. Using periodicity, this y-coordinate has a total a total of

$$(0+0+1) \cdot \frac{60}{3} = 20.$$

For $b = 4$, we have $I = 0$ at $(0,4)$ and $(1,4)$, and $I = 1$ at both $(2,4)$ and $(3,4)$. Using periodicity, this y-coordinate has a total of

$$(0+0+1+1) \cdot \frac{60}{4} = 30.$$

For $b = 5$, we have $I = 0$ at $(0,5),(1,5)$ and $I = 2$ at $(2,5),(3,5),(4,5)$. Using periodicity, this y-coordinate has a total of

$$(0+0+2+2+2) \cdot \frac{60}{5} = 72.$$

Adding our cases, we have $20 + 30 + 72 = 122$ ways to choose the four points. Multiplying back by the number of ways to connect the quadrilateral gives an answer of $122 \cdot 3 = \boxed{366}$.