THEME ROUND

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems are unequally weighted with point values as shown in brackets on the answer form. The maximum possible score is 50 points. There is no point penalty for guessing, though in the case of a tie it is slightly more advantageous not to answer than to answer incorrectly.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Answers should be simplified as much as is reasonably possible and must be exact unless otherwise specified. Rational numbers should be written in lowest terms, although denominators of irrationals need not be rationalized. An nth root should be simplified so that the radicand is not divisible by the nth power of any prime.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to the Science Center Lobby during lunchtime.

Enjoy!

HMMT November 2012
Saturday 10 November 2012
## Theme Round

Power Towers

In a tower of exponents, the exponents are computed from the top down. For example, $2^{2^3} = 2^8 = 256$.

1. **[3]** If $4^{4^4} = \sqrt[128]{2^{2^{2^n}}}$, find n.

2. **[4]** If $x^x = 2012^{2012^{2013}}$, find x.

3. **[4]** Find the smallest positive integer n such that $\underbrace{2^{2^{\cdot^{\cdot^2}}}}_{n} > 3^{3^{3^3}}$. (The notation $\underbrace{2^{2^{\cdot^{\cdot^2}}}}_{n}$ is used to denote a power tower with n 2's. For example, $\underbrace{2^{2^{\cdot^{\cdot^2}}}}_{n}$ with $n = 4$ would equal $2^{2^{2^2}}$.)

4. **[7]** Find the sum of all real solutions for x to the equation $(x^2 + 2x + 3)^{(x^2 + 2x + 3)^{(x^2 + 2x + 3)}} = 2012$.

5. **[7]** Given any positive integer, we can write the integer in base 12 and add together the digits of its base 12 representation. We perform this operation on the number $7^{6^{5^{4^{3^{2^1}}}}}$ repeatedly until a single base 12 digit remains. Find this digit.

Rock-paper-scissors

In the game of rock-paper-scissors, two players each choose one of rock, paper, or scissors to play. Rock beats scissors, scissors beats paper, and paper beats rock. If the players play the same thing, the match is considered a draw.

6. **[3]** A rectangular piece of paper with vertices $ABCD$ is being cut by a pair of scissors. The pair of scissors starts at vertex A, and then cuts along the angle bisector of DAB until it reaches another edge of the paper. One of the two resulting pieces of paper has 4 times the area of the other piece. What is the ratio of the longer side of the original paper to the shorter side?

7. **[4]** The game of rock-scissors is played just like rock-paper-scissors, except that neither player is allowed to play paper. You play against a poorly-designed computer program that plays rock with 50% probability and scissors with 50% probability. If you play optimally against the computer, find the probability that after 8 games you have won at least 4.

8. **[4]** In the game of rock-paper-scissors-lizard-Spock, rock defeats scissors and lizard, paper defeats rock and Spock, scissors defeats paper and lizard, lizard defeats paper and Spock, and Spock defeats rock and scissors, as shown in the below diagram. As before, if two players choose the same move, then there is a draw. If three people each play a game of rock-paper-scissors-lizard-Spock at the same time by choosing one of the five moves at random, what is the probability that one player beats the other two?
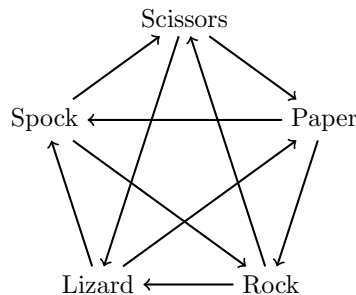
9. [**6**]   64 people are in a single elimination rock-paper-scissors tournament, which consists of a 6-round knockout bracket. Each person has a different rock-paper-scissors skill level, and in any game, the person with the higher skill level will always win. For how many players P is it possible that P wins the first four rounds that he plays?

(A *6-round knockout bracket* is a tournament which works as follows:

   (a) In the first round, all 64 competitors are paired into 32 groups, and the two people in each group play each other. The winners advance to the second round, and the losers are eliminated.

   (b) In the second round, the remaining 32 players are paired into 16 groups. Again, the winner of each group proceeds to the next round, while the loser is eliminated.

   (c) Each round proceeds in a similar way, eliminating half of the remaining players. After the sixth round, only one player will not have been eliminated. That player is declared the champion.)

10. [**8**]   In a game of rock-paper-scissors with n people, the following rules are used to determine a champion:

   (a) In a round, each person who has not been eliminated randomly chooses one of rock, paper, or scissors to play.

   (b) If at least one person plays rock, at least one person plays paper, and at least one person plays scissors, then the round is declared a tie and no one is eliminated. If everyone makes the same move, then the round is also declared a tie.

   (c) If exactly two moves are represented, then everyone who made the losing move is eliminated from playing in all further rounds (for example, in a game with 8 people, if 5 people play rock and 3 people play scissors, then the 3 who played scissors are eliminated).

   (d) The rounds continue until only one person has not been eliminated. That person is declared the champion and the game ends.

If a game begins with 4 people, what is the expected value of the number of rounds required for a champion to be determined?

Name _____ Team ID# _____

School _____ Team _____

1. [**3**] _____

2. [**4**] _____

3. [**4**] _____

4. [**7**] _____

5. [**7**] _____

6. [**3**] _____

7. [**4**] _____

8. [**4**] _____

9. [**6**] _____

10. [**8**] _____

Score: _____