12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Algebra Test

1. [**3**] If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

2. [**3**] Let S be the sum of all the real coefficients of the expansion of $(1 + ix)^{2009}$. What is $\log_2(S)$?

3. [**4**] If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

4. [**4**] Suppose a, b and c are integers such that the greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $x + 1$ (in the set of polynomials in x with integer coefficients), and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $x^3 - 4x^2 + x + 6$. Find $a + b + c$.

5. [**4**] Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

6. [**5**] Let x and y be positive real numbers and θ an angle such that $\theta \neq \frac{\pi}{2}n$ for any integer n. Suppose
$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y}$$
and
$$\frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{97\sin 2\theta}{x^3 y + y^3 x}.$$
Compute $\frac{x}{y} + \frac{y}{x}$.

7. [**5**] Simplify the product
$$\prod_{m=1}^{100} \prod_{n=1}^{100} \frac{x^{n+m} + x^{n+m+2} + x^{2n+1} + x^{2m+1}}{x^{2n} + 2x^{n+m} + x^{2m}}.$$
Express your answer in terms of x.

8. [**7**] If a, b, x and y are real numbers such that $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, and $ax^4 + by^4 = 42$, find $ax^5 + by^5$.

9. [**7**] Let $f(x) = x^4 + 14x^3 + 52x^2 + 56x + 16$. Let z_1, z_2, z_3, z_4 be the four roots of f. Find the smallest possible value of $|z_a z_b + z_c z_d|$ where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

10. [**8**] Let $f(x) = 2x^3 - 2x$. For what positive values of a do there exist distinct b, c, d such that $(a, f(a))$, $(b, f(b))$, $(c, f(c))$, $(d, f(d))$ is a rectangle?