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Geometry

1. Let A, B, C, D be four points on a circle in that order. Also, $AB = 3$, $BC = 5$, $CD = 6$, and $DA = 4$. Let diagonals AC and BD intersect at P. Compute $\frac{AP}{CP}$.

   Proposed by: Sam Korsky

   Answer: $\boxed{\frac{2}{5}}$

   Note that $\triangle APB \sim \triangle DPC$ so $\frac{AP}{AB} = \frac{DP}{CD}$. Similarly, $\triangle BPC \sim \triangle APD$ so $\frac{CP}{BC} = \frac{DP}{DA}$. Dividing these two equations yields

   $$\frac{AP}{CP} = \frac{AB \cdot DA}{BC \cdot CD} = \boxed{\frac{2}{5}}$$

2. Let ABC be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Let ℓ be a line passing through two sides of triangle ABC. Line ℓ cuts triangle ABC into two figures, a triangle and a quadrilateral, that have equal perimeter. What is the maximum possible area of the triangle?

   Proposed by: Sam Korsky

   Answer: $\boxed{\frac{1323}{26}}$

   There are three cases: ℓ intersects AB, AC, ℓ intersects AB, BC, and ℓ intersects AC, BC. These cases are essentially identical, so let ℓ intersect segment AB at M and segment AC at N.

   Then the condition is equivalent to

   $$AM + MN + AN = MB + BC + CN + MN$$
   $$AM + AN = MB + CN + 15$$

   but $AN + CN = 14$ and $AM + BM = 13$, so that

   $$BM + CN = 27 - AM - AN = AM - AN - 15$$

   implying that $AM + AN = 21$.

   Now let $\angle BAC = \theta$ for convenience, so that

   $$[AMN] = \frac{1}{2} AM \cdot AN \cdot \sin\theta$$

   which is maximized when $AM = AN = \frac{21}{2}$. Further we can easily calculate $\sin\theta = \frac{12}{13}$ (e.g. by LOC); note that this is why the area is maximized in this case (we want to maximize $\sin\theta$, which is equivalent to maximizing θ, so θ should be opposite the largest side). Our answer is thus

   $$\frac{1}{2} \cdot \frac{21}{2} \cdot \frac{21}{2} \cdot \frac{12}{13} = \frac{1323}{26}$$

   Alternatively we could also calculate

   $$[AMN] = [ABC] \cdot \frac{AM}{AB} \cdot \frac{AN}{AC}$$
   $$= 84 \cdot \frac{\frac{21}{2}}{13} \cdot \frac{\frac{21}{2}}{14}$$

   which gives the same answer.

3. Let S be a set of 2017 distinct points in the plane. Let R be the radius of the smallest circle containing all points in S on either the interior or boundary. Also, let D be the longest distance between two of the points in S. Let a, b are real numbers such that $a \le \frac{D}{R} \le b$ for all possible sets S, where a is as large as possible and b is as small as possible. Find the pair (a, b).

   Proposed by: Yang Liu

   Answer: $\boxed{(\sqrt{3}, 2)}$

   It is easy to verify that the smallest circle enclosing all the points will either have some 2 points in S as its diameter, or will be the circumcircle of some 3 points in S who form an acute triangle.

   Now, clearly $\frac{D}{R} \le 2$. Indeed consider the two farthest pair of points S_1, S_2. Then $D = |S_1S_2| \le 2R$, as both points S_1, S_2 are inside a circle of radius R. We can achieve this upper bound by taking S to have essentially only 2 points, and the remaining 2015 points in S are at the same place as these 2 points.

   For the other direction, I claim $\frac{D}{R} \ge \sqrt{3}$. Recall that the smallest circle is either the circumcircle of 3 points, or has some 2 points as the diameter. In the latter case, say the diameter is S_1S_2. Then $D \ge |S_1S_2| = 2R$, so $\frac{D}{R} \ge 2$ in that case. Now say the points S_1, S_2, S_3 are the circumcircle. WLOG, say that S_1S_2 is the longest side of the triangle. As remarked above, we can assume this triangle is acute. Therefore, $\frac{\pi}{3} \le \angle S_1S_3S_2 \le \frac{\pi}{2}$. By the Law of Sines we have that

$$D \ge |S_1S_2| = 2R \sin \angle S_1S_3S_2 \ge 2R \sin \frac{\pi}{3} = R\sqrt{3}.$$

   This completes the proof. To achieve equality, we can take S to have 3 points in the shape of an equilateral triangle.

4. Let $ABCD$ be a convex quadrilateral with $AB = 5, BC = 6, CD = 7$, and $DA = 8$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 - PQ^2$.

   Proposed by: Sam Korsky

   Answer: $\boxed{13}$

   Draw in the diagonals of the quad and use the median formula three times to get MN^2 in terms of the diagonals. Do the same for PQ^2 and subtract, the diagonal length terms disappear and the answer is

$$\frac{BC^2 + DA^2 - AB^2 - CD^2}{2} = \boxed{13}$$

5. Let $ABCD$ be a quadrilateral with an inscribed circle ω and let P be the intersection of its diagonals AC and BD. Let R_1, R_2, R_3, R_4 be the circumradii of triangles APB, BPC, CPD, DPA respectively. If $R_1 = 31$ and $R_2 = 24$ and $R_3 = 12$, find R_4.

   Proposed by: Sam Korsky

   Answer: $\boxed{19}$

   Note that $\angle APB = 180^\circ - \angle BPC = \angle CPD = 180^\circ - \angle DPA$ so $\sin APB = \sin BPC = \sin CPD = \sin DPA$. Now let ω touch sides AB, BC, CD, DA at E, F, G, H respectively. Then $AB + CD = AE + BF + CG + DH = BC + DA$ so

$$\frac{AB}{\sin APB} + \frac{CD}{\sin CPD} = \frac{BC}{\sin BPC} + \frac{DA}{\sin DPA}$$

   and by the Extended Law of Sines this implies

$$2R_1 + 2R_3 = 2R_2 + 2R_4$$

   which immediately yields $R_4 = R_1 + R_3 - R_2 = \boxed{19}$.

6. In convex quadrilateral $ABCD$ we have $AB = 15$, $BC = 16$, $CD = 12$, $DA = 25$, and $BD = 20$. Let M and γ denote the circumcenter and circumcircle of $\triangle ABD$. Line CB meets γ again at F, line AF meets MC at G, and line GD meets γ again at E. Determine the area of pentagon $ABCDE$.

   Proposed by: Evan Chen

   Answer: $\boxed{396}$

   Note that $\angle ADB = \angle DCB = 90^\circ$ and $BC \parallel AD$. Now by Pascal theorem on $DDEBFA$ implies that B, M, E are collinear. So $[ADE] = [ABD] = 150$ and $[BCD] = 96$, so the total area is 396.

7. Let ω and Γ by circles such that ω is internally tangent to Γ at a point P. Let AB be a chord of Γ tangent to ω at a point Q. Let $R \neq P$ be the second intersection of line PQ with Γ. If the radius of Γ is 17, the radius of ω is 7, and $\frac{AQ}{BQ} = 3$, find the circumradius of triangle AQR.

   Proposed by: Sam Korsky

   Answer: $\boxed{\sqrt{170}}$

   Let r denote the circumradius of triangle AQR. By Archimedes Lemma, R is the midpoint of arc AB of Γ. Therefore $\angle RAQ = \angle RPB = \angle RPA$ so $\triangle RAQ \sim \triangle RPA$. By looking at the similarity ratio between the two triangles we have

$$\frac{r}{17} = \frac{AQ}{AP}$$

   Now, let AP intersect ω again at $X \neq P$. By homothety we have $XQ \parallel AR$ so

$$\frac{AX}{AP} = 1 - \frac{PQ}{PR} = 1 - \frac{7}{17} = \frac{10}{17}$$

   But we also know

$$AX \cdot AP = AQ^2$$

   so

$$\frac{10}{17} AP^2 = AQ^2$$

   Thus

$$\frac{r}{17} = \frac{AQ}{AP} = \sqrt{\frac{10}{17}}$$

   so we compute $r = \sqrt{170}$ as desired.

8. Let ABC be a triangle with circumradius $R = 17$ and inradius $r = 7$. Find the maximum possible value of $\sin \frac{A}{2}$.

   Proposed by: Sam Korsky

   Answer: $\boxed{\frac{17+\sqrt{51}}{34}}$

   Letting I and O denote the incenter and circumcenter of triangle ABC we have by the triangle inequality that

$$AO \leq AI + OI \implies R \leq \frac{r}{\sin \frac{A}{2}} + \sqrt{R(R-2r)}$$

   and by plugging in our values for r and R we get

$$\sin \frac{A}{2} \leq \frac{17 + \sqrt{51}}{34}$$

   as desired. Equality holds when ABC is isosceles and I lies between A and O.

9. Let ABC be a triangle, and let $BCDE, CAFG, ABHI$ be squares that do not overlap the triangle with centers X, Y, Z respectively. Given that $AX = 6$, $BY = 7$, and $CZ = 8$, find the area of triangle XYZ.

   Proposed by: Sam Korsky

   Answer: $\boxed{\frac{21\sqrt{15}}{4}}$

   By the degenerate case of Von Aubel's Theorem we have that $YZ = AX = 6$ and $ZX = BY = 7$ and $XY = CZ = 8$ so it suffices to find the area of a $6-7-8$ triangle which is given by $\boxed{\dfrac{21\sqrt{15}}{4}}$.

   To prove that $AX = YZ$, note that by LoC we get

$$YX^2 = \frac{b^2}{2} + \frac{c^2}{2} + bc\sin\angle A$$

   and

$$\begin{aligned} AX^2 &= b^2 + \frac{a^2}{2} - ab(\cos\angle C - \sin\angle C) \\ &= c^2 + \frac{a^2}{2} - ac(\cos\angle B - \sin\angle B) \\ &= \frac{b^2 + c^2 + a(b\sin\angle C + c\sin\angle B)}{2} \\ &= \frac{b^2}{2} + \frac{c^2}{2} + ah \end{aligned}$$

   where h is the length of the A-altitude of triangle ABC. In these calculations we used the well-known fact that $b\cos\angle C + c\cos\angle B = a$ which can be easily seen by drawing in the A-altitude. Then since $bc\sin\angle A$ and ah both equal twice the area of triangle ABC, we are done.

10. Let $ABCD$ be a quadrilateral with an inscribed circle ω. Let I be the center of ω let $IA = 12$, $IB = 16$, $IC = 14$, and $ID = 11$. Let M be the midpoint of segment AC. Compute $\frac{IM}{IN}$, where N is the midpoint of segment BD.

    Proposed by: Sam Korsky

    Answer: $\boxed{\frac{21}{22}}$

    Let points W, X, Y, Z be the tangency points between ω and lines AB, BC, CD, DA respectively. Now invert about ω. Then A', B', C', D' are the midpoints of segments ZW, WX, XY, YZ respectively. Thus by Varignon's Theorem $A'B'C'D'$ is a parallelogram. Then the midpoints of segments $A'C'$ and $B'D'$ coincide at a point P. Note that figure $IA'PC'$ is similar to figure $ICMA$ with similitude ratio $\frac{r^2}{IA\cdot IC}$ where r is the radius of ω. Similarly figure $IB'PD'$ is similar to figure $IDMB$ with similitude ratio $\frac{r^2}{IB\cdot ID}$. Therefore

$$IP = \frac{r^2}{IA\cdot IC}\cdot IM = \frac{r^2}{IB\cdot ID}\cdot IN$$

    which yields

$$\frac{IM}{IN} = \frac{IA\cdot IC}{IB\cdot ID} = \frac{12\cdot 14}{16\cdot 11} = \boxed{\frac{21}{22}}$$