THE HARVARD-MIT MATHEMATICS TOURNAMENT

TEAM

This test consists of ten problems to be solved by a team in one hour. The problems are unequally weighted with point values as shown in brackets. They are *not* necessarily in order of difficulty, though harder problems are generally worth more points. We do not expect most teams to get through all the problems.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted. If there is a chalkboard in the room, you may write on it *during* the round only, *not* before it starts.

Numerical answers should, where applicable, be simplified as much as reasonably possible and must be exact unless otherwise specified. Correct mathematical notation must be used. Your proctor *cannot* assist you in interpreting or solving problems but has our phone number and may help with any administrative difficulties.

All problems require full written proof/justification. Even if the answer is numerical, you must prove your result.

If you believe the test contains an error, please submit your protest in writing to the Science Center Lobby during lunchtime.

Enjoy!

HMMT 2014

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Team

1. [**10**] Let ω be a circle, and let A and B be two points in its interior. Prove that there exists a circle passing through A and B that is contained in the interior of ω.

2. [**15**] Let $a_1, a_2, \ldots$ be an infinite sequence of integers such that a_i divides a_{i+1} for all $i \geq 1$, and let b_i be the remainder when a_i is divided by 210. What is the maximal number of distinct terms in the sequence $b_1, b_2, \ldots$?

3. [**15**] There are n girls $G_1, \ldots, G_n$ and n boys $B_1, \ldots, B_n$. A pair (G_i, B_j) is called *suitable* if and only if girl G_i is willing to marry boy B_j. Given that there is exactly one way to pair each girl with a distinct boy that she is willing to marry, what is the maximal possible number of suitable pairs?

4. [**20**] Compute

$$\sum_{k=0}^{100} \left\lfloor \frac{2^{100}}{2^{50} + 2^k} \right\rfloor.$$

(Here, if x is a real number, then $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

5. [**25**] Prove that there exists a nonzero complex number c and a real number d such that

$$\left| \left| \frac{1}{1+z+z^2} \right| - \left| \frac{1}{1+z+z^2} - c \right| \right| = d$$

for all z with $|z| = 1$ and $1 + z + z^2 \neq 0$. (Here, $|z|$ denotes the absolute value of the complex number z, so that $|a + bi| = \sqrt{a^2 + b^2}$ for real numbers a, b.)

6. [**25**] Let n be a positive integer. A sequence $(a_0, \ldots, a_n)$ of integers is *acceptable* if it satisfies the following conditions:

   (a) $0 = |a_0| < |a_1| < \cdots < |a_{n-1}| < |a_n|$.

   (b) The sets $\{|a_1 - a_0|, |a_2 - a_1|, \ldots, |a_{n-1} - a_{n-2}|, |a_n - a_{n-1}|\}$ and $\{1, 3, 9, \ldots, 3^{n-1}\}$ are equal.

   Prove that the number of acceptable sequences of integers is $(n + 1)!$.

7. [**30**] Find the maximum possible number of diagonals of equal length in a convex hexagon.

8. [**35**] Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, and $BC = 6$. Let D be the foot of the altitude from A to BC, and E be the intersection of AO with BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Determine the length of BX.

9. [**35**] For integers $m, n \geq 1$, let $A(n, m)$ be the number of sequences $(a_1, \cdots, a_{nm})$ of integers satisfying the following two properties:

   (a) Each integer k with $1 \leq k \leq n$ occurs exactly m times in the sequence $(a_1, \cdots, a_{nm})$.

   (b) If i, j, and k are integers such that $1 \leq i \leq nm$ and $1 \leq j \leq k \leq n$, then j occurs in the sequence $(a_1, \cdots, a_i)$ at least as many times as k does.

   For example, if $n = 2$ and $m = 5$, a possible sequence is $(a_1, \cdots, a_{10}) = (1, 1, 2, 1, 2, 2, 1, 2, 1, 2)$. On the other hand, the sequence $(a_1, \cdots, a_{10}) = (1, 2, 1, 2, 2, 1, 1, 1, 2, 2)$ does not satisfy property (2) for $i = 5$, $j = 1$, and $k = 2$.

   Prove that $A(n, m) = A(m, n)$.

10. [**40**] Fix a positive real number $c > 1$ and positive integer n. Initially, a blackboard contains the numbers $1, c, \ldots, c^{n-1}$. Every minute, Bob chooses two numbers a, b on the board and replaces them with $ca + c^2 b$. Prove that after $n - 1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L} - 1}{c^{1/L} - 1}\right)^L,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $L = 1 + \log_\phi(c)$.

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2. [**15**]

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3. [**15**]

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4. [**20**]

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5. [**25**]

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6. [**25**]

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7. [**30**]

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8. [**35**]

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9. [**35**]

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10. [**40**]