15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Team B

1. [**10**] Triangle ABC has $AB = 5$, $BC = 3\sqrt{2}$, and $AC = 1$. If the altitude from B to AC and the angle bisector of angle A intersect at at D, what is BD?

   Answer: $\boxed{\frac{5}{3}}$ Let E be the foot of the perpendicular from B to line AC. By the Law of Cosines, $\cos\angle BAC = \frac{4}{5}$, and it follows that $BE = 3$ and $AE = 4$. Now, by the Angle Bisector Theorem, $\frac{BD}{BE} = \frac{AB}{AB+AE} = \frac{5}{9}$, so $BD = \frac{5}{3}$.

2. [**10**] You are given two line segments of length 2^n for each integer $0 \le n \le 10$. How many distinct nondegenerate triangles can you form with three of the segments? Two triangles are considered distinct if they are not congruent.

   Answer: $\boxed{55}$ First, observe that if we have three sticks of distinct lengths $2^a < 2^b < 2^c$, then $2^a + 2^b < 2^{b+1} \le 2^c$, so we cannot form a triangle. Thus, we must have (exactly) two of our sticks the same length, so that our triangle has side lengths $2^a, 2^a, 2^b$. This triangle is non-degenerate if and only if $2^{a+1} > 2^b$, and since $a \ne b$, this happens if and only if $a > b$. Clearly, there are $\binom{11}{2} = 55$ ways to choose such a, b.

3. [**10**] Mac is trying to fill 2012 barrels with apple cider. He starts with 0 energy. Every minute, he may rest, gaining 1 energy, or if he has n energy, he may expend k energy $(0 \le k \le n)$ to fill up to $n(k+1)$ barrels with cider. What is the minimal number of minutes he needs to fill all the barrels?

   Answer: $\boxed{46}$ First, suppose that Mac fills barrels during two consecutive minutes. Let his energy immediately before doing so be n, and the energy spent in the next two minutes be k_1, k_2, respectively. It is not difficult to check that he can fill at least as many barrels by spending $k_1 + k_2 + 1$ energy and resting for an additional minute before doing so, so that his starting energy is $n + 1$: this does not change the total amount of time. Furthermore, this does not affect the amount of energy Mac has remaining afterward. We may thus assume that Mac first rests for a (not necessarily fixed) period of time, then spends one minute filling barrels, and repeats this process until all of the barrels are filled.

   Next, we check that he only needs to fill barrels once. Suppose that Mac first rests for n_1 minutes, then spends k_1 energy, and next rests for n_2 minutes and spends k_2 energy. It is again not difficult to check that Mac can instead rest for $n_1 + n_2 + 1$ minutes and spend $k_1 + k_2 + 1$ energy, to increase the number of barrels filled, while not changing the amount of time nor the energy remaining. Iterating this operation, we can reduce our problem to the case in which Mac first rests for n minutes, then spends n energy filling all of the barrels.

   We need $n(n+1) \ge 2012$, so $n \ge 45$, and Mac needs a minimum of 46 minutes.

4. [**10**] A restaurant has some number of seats, arranged in a line. Its customers are in parties arranged in a queue. To seat its customers, the restaurant takes the next party in the queue and attempts to seat all of the party's member(s) in a contiguous block of unoccupied seats. If one or more such blocks exist, then the restaurant places the party in an arbitrarily selected block; otherwise, the party leaves.

   Suppose the queue has parties of sizes $6, 4, 2, 5, 3, 1$ from front to back, and all seats are initially empty. What is the minimal number of seats the restaurant needs to guarantee that it will seat all of these customers?

   Answer: $\boxed{29}$ First, note that if there are only 28 seats, it is possible for the seating not to be possible, in the following way. The party of six could be seated in such a way that the remaining contiguous regions have sizes 10 and 12. Then, the party of 4 is seated in the middle of the region of size 12, and the party of 2 is seated in the middle of the region of size 10, so that the remaining regions all of size 4. Now, it is impossible to seat the party of 5. It is clear that fewer than 28 seats also make it impossible to guarantee that everyone can be seated.

   Now, given 29 seats, clearly, the first party can be seated. Afterward, 23 seats remain in at most 2 contiguous regions, one of which has to have size at least 4. Next, 19 seats remain in at most 3 contiguous regions, one of which has size at least 2. 17 seats in at most 4 contiguous regions remain for the party of 5, and one region must have size at least 5. Finally, 12 seats in at most 5 contiguous regions are available for the party of 3, and the party of 1 can take any remaining seat. Our answer is therefore 29.

Remark: We can arrive at the answer of 29 by doing the above argument in reverse.

5. [**10**] Steph and Jeff each start with the number 4, and Travis is flipping a coin. Every time he flips a heads, Steph replaces her number x with $2x-1$, and Jeff replaces his number y with $y+8$. Every time he flips a tails, Steph replaces her number x with $\frac{x+1}{2}$, and Jeff replaces his number y with $y-3$. After some (positive) number of coin flips, Steph and Jeff miraculously end up with the same number below 2012. How many times was the coin flipped?

   Answer: $\boxed{137}$ Suppose that a heads and b tails are flipped. Jeff's number at the end is $4+8a-3b$. Note that the operations which Steph applies are inverses of each other, and as a result it is not difficult to check by induction that her final number is simply $1+3\cdot 2^{a-b}$.

   We now have $3+3(a-b)+5a = 3\cdot 2^{a-b}$. Letting $n = a-b$, we see that $2^n - n - 1$ must be divisible by 5, so that a is an integer. In particular, n is a positive integer. Furthermore, we have $1+3\cdot 2^n < 2012$, so that $n \leq 9$. We see that the only possibility is for $n = 7 = a-b$, and thus $4+8a-3b = 385$. Solving, we get $a = 72, b = 65$, so our answer is $72+65 = 137$.

6. [**20**] Let ABC be a triangle with $AB < AC$. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D. Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC, respectively. Prove that $BE = CF$.

   Answer: $\boxed{\text{see below}}$ Note that DE, DF are the distances from D to AB, AC, respectively, and because AD is the angle bisector of $\angle BAC$, we have $DE = DF$. Also, $DB = DC$ because D is on the perpendicular bisector of BC. Finally, $\angle DEB = \angle DFC = 90°$, so it follows that $DEB \cong DFC$, and $BE = CF$.

7. [**20**] For what positive integers n do there exist functions $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that for each $1 \leq i \leq n$, either $f(g(i)) = i$ or $g(f(i)) = i$, but not both?

   Answer: $\boxed{n \text{ even}}$ We claim that this is possible for all even n. First, a construction: set $f(2m-1) = f(2m) = 2m-1$ and $g(2m-1) = g(2m) = 2m$ for $m = 1, \ldots, \frac{n}{2}$. It is easy to verify that this solution works.

   Now, we show that this is impossible for odd n. Without loss of generality, suppose that $f(g(1)) = 1$ and that $g(1) = a \Rightarrow f(a) = 1$. Then, we have $g(f(a)) = g(a) = 1$. Consequently, $a \neq 1$. In this case, call 1 and a a *pair* (we likewise regard i and j as a pair when $g(f(i)) = i$ and $f(i) = j$). Now, to show that n is even it suffices to show that all pairs are disjoint. Suppose for the sake of contradiction that some integer $b \neq a$ is also in a pair with 1 (note that 1 is arbitrary). Then, we have $f(g(b)) = b, g(b) = 1$ or $g(f(b)) = b, f(b) = 1$. But we already know that $g(1) = a$, so we must have $f(g(b)) = b, g(b) = 1$. But that would mean that both $f(g(1)) = 1$ and $g(f(1)) = 1$, a contradiction.

8. [**20**] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

   (a) Suppose that a token can legally move to any horizontally or vertically adjacent square. Show that Bob has a winning strategy for this game.

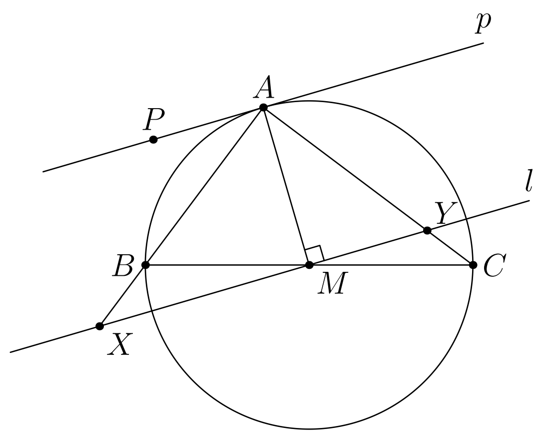
   (b) Suppose instead that a token can legally move to any horizontally, vertically, or diagonally adjacent square. Show that Alice has a winning strategy for this game.

   Answer: $\boxed{\text{see below}}$ For part (a), color the checkerboard in the standard way so that half of the squares are black and the other half are white. Bob's winning strategy is to place the two coins on the same color, so that Alice must always move her coin on to a square with the opposite color as the square containing Bob's coin.

For part (b), consider any starting configuration. By considering only the column that the tokens are in, it is easy to see that Alice can get to the same column as Bob (immediately after her move) in 7 rounds. (This is just a game on a 1×8 chessboard.) Following this, Alice can stay on the same column as Bob each turn, while getting to the same row as him. This too also takes at most 7 rounds. Thus, Alice can catch Bob in $14 < 2012$ rounds from any starting position.

9. [**20**] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC. Line l is drawn through M so that it is perpendicular to AM, and intersects line AB at point X and line AC at point Y. Prove that $\angle BAC = 90^\circ$ if and only if quadrilateral $XBYC$ is cyclic.

   Answer: $\boxed{\text{see below}}$



   First, note that $XBYC$ cyclic is equivalent to $\measuredangle BXM = \measuredangle ACB$. However, note that $\measuredangle BXM = 90^\circ - \measuredangle BAM$, so $XBYC$ cyclic is in turn equivalent to $\measuredangle BAM + \measuredangle ACB = 90^\circ$.

   Let the line tangent to the circumcircle of $\triangle ABC$ at A be p, and let P be an arbitrary point on p on the same side of AM as B. Note that $\measuredangle PAB = \measuredangle ACB$. If $\measuredangle ACB = 90^\circ - \measuredangle BAM$ we have $l \perp AM$ and thus the circumcenter O of $\triangle ABC$ lies on AM. Since $AB < AC$, we must have $O = M$, and $\measuredangle BAC = 90^\circ$. Conversely, if $\measuredangle BAC = 90^\circ$, $\measuredangle PAM = 90^\circ$, and it follows that $\measuredangle ACB = 90^\circ - \measuredangle BAM$.

10. [**20**] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes five pizzas, each with some subset of the five toppings. For each pizza, Arno states (with either a "yes" or a "no") if the pizza has any toppings that he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she has enough information to make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?

    Answer: $\boxed{\text{see below}}$ We claim the only way for Purineqa to deduce Arno's preferences is for each pizza to contain exactly one topping, with no topping be repeated. It is obvious that he can deduce the toppings in this case.

    We now claim that this is not possible with any other combination. Suppose that Arno tells Purineqa that he does not like any of the five pizzas. Then, Purineqa should be able to rule out at least one of the possibilities that Arno likes none of the toppings and that Arno likes exactly one of the toppings T. It is clear that this is possible if and only if there is a pizza with only T on it. This is true for all five toppings T, so we're done.