3^{rd} Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Team Round

Polyhedron Hopping

1. [**3**] Travis is hopping around on the vertices of a cube. Each minute he hops from the vertex he's currently on to the other vertex of an edge that he is next to. After four minutes, what is the probability that he is back where he started?

2. [**6**] In terms of k, for $k > 0$ how likely is he to be back where he started after $2k$ minutes?

3. [**3**] While Travis is having fun on cubes, Sherry is hopping in the same manner on an octahedron. An octahedron has six vertices and eight regular triangular faces. After five minutes, how likely is Sherry to be one edge away from where she started?

4. [**6**] In terms of k, for $k > 0$, how likely is it that after k minutes Sherry is at the vertex opposite the vertex where she started?

Circles in Circles

5. [**4**] Circle O has chord AB. A circle is tangent to O at T and tangent to AB at X such that $AX = 2XB$. What is $\frac{AT}{BT}$?

6. [**6**] AB is a diameter of circle O. X is a point on AB such that $AX = 3BX$. Distinct circles ω_1 and ω_2 are tangent to O at T_1 and T_2 and to AB at X. The lines T_1X and T_2X intersect O again at S_1 and S_2. What is the ratio $\frac{T_1T_2}{S_1S_2}$?

7. [**7**] ABC is a right triangle with $\angle A = 30^\circ$ and circumcircle O. Circles ω_1, ω_2, and ω_3 lie outside ABC and are tangent to O at T_1, T_2, and T_3 respectively and to AB, BC, and CA at S_1, S_2, and S_3, respectively. Lines T_1S_1, T_2S_2, and T_3S_3 intersect O again at A', B', and C', respectively. What is the ratio of the area of $A'B'C'$ to the area of ABC?

Linear? What's The Problem?

A function $f(x_1, x_2, \ldots, x_n)$ is said to be linear in each of its variables if it is a polynomial such that no variable appears with power higher than one in any term. For example, $1 + x + xy$ is linear in x and y, but $1 + x^2$ is not. Similarly, $2x + 3yz$ is linear in x, y, and z, but xyz^2 is not.

8. [**4**] A function $f(x, y)$ is linear in x and in y. $f(x, y) = \frac{1}{xy}$ for $x, y \in \{3, 4\}$. What is $f(5, 5)$?

9. [**5**] A function $f(x, y, z)$ is linear in x, y, and z such that $f(x, y, z) = \frac{1}{xyz}$ for $x, y, z \in \{3, 4\}$. What is $f(5, 5, 5)$?

10. [**6**] A function $f(x_1, x_2, \ldots, x_n)$ is linear in each of the x_i and $f(x_1, x_2, \ldots, x_n) = \frac{1}{x_1 x_2 \cdots x_n}$ when $x_i \in \{3, 4\}$ for all i. In terms of n, what is $f(5, 5, \ldots, 5)$?