HMMT February 2020

February 15, 2020

Combinatorics

1. How many ways can the vertices of a cube be colored red or blue so that the color of each vertex is the color of the majority of the three vertices adjacent to it?

   Proposed by: Milan Haiman

   Answer: $\boxed{8}$

   Solution: If all vertices of the cube are of the same color, then there are 2 ways. Otherwise, look at a red vertex. Since it must have at least 2 red neighbors, there is a face of the cube containing 3 red vertices. The last vertex on this face must also be red. Similarly, all vertices on the opposite face must be blue. Thus, all vertices on one face of the cube are red while the others are blue. Since a cube has 6 faces, the answer is $2 + 6 = 8$.

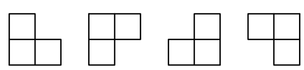
2. How many positive integers at most 420 leave different remainders when divided by each of 5, 6, and 7?

   Proposed by: Milan Haiman

   Answer: $\boxed{250}$

   Solution: Note that $210 = 5 \cdot 6 \cdot 7$ and 5, 6, 7 are pairwise relatively prime. So, by the Chinese Remainder Theorem, we can just consider the remainders n leaves when divided by each of 5, 6, 7. To construct an n that leaves distinct remainders, first choose its remainder modulo 5, then modulo 6, then modulo 7. We have $5 = 6 - 1 = 7 - 2$ choices for each remainder. Finally, we multiply by 2 because $420 = 2 \cdot 210$. The answer is $2 \cdot 5^3 = 250$.

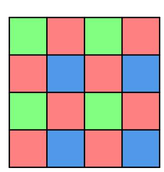
3. Each unit square of a 4×4 square grid is colored either red, green, or blue. Over all possible colorings of the grid, what is the maximum possible number of L-trominos that contain exactly one square of each color? (L-trominos are made up of three unit squares sharing a corner, as shown below.)



   Proposed by: Andrew Lin

   Answer: $\boxed{18}$

   Solution: Notice that in each 2×2 square contained in the grid, we can form 4 L-trominoes. By the pigeonhole principle, some color appears twice among the four squares, and there are two trominoes which contain both. Therefore each 2×2 square contains at most 2 L-trominoes with distinct colors. Equality is achieved by coloring a square (x, y) red if $x + y$ is even, green if x is odd and y is even, and blue if x is even and y is odd. Since there are nine 2×2 squares in our 4×4 grid, the answer is $9 \times 2 = 18$.



4. Given an 8×8 checkerboard with alternating white and black squares, how many ways are there to choose four black squares and four white squares so that no two of the eight chosen squares are in the same row or column?

   Proposed by: James Lin

   Answer: $\boxed{20736}$

   Solution: Number both the rows and the columns from 1 to 8, and say that black squares are the ones where the rows and columns have the same parity. We will use, e.g. "even rows" to refer to rows 2, 4, 6, 8. Choosing 8 squares all in different rows and columns is equivalent to matching rows to columns.

   For each of the 8 rows, we first decide whether they will be matched with a column of the same parity as itself (resulting in a black square) or with one of a different parity (resulting in a white square). Since we want to choose 4 squares of each color, the 4 rows matched to same-parity columns must contain 2 even rows and 2 odd rows. There are $\binom{4}{2}^2 = 6^2$ ways to choose 2 odd rows and 2 even rows to match with same-parity columns.

   After choosing the above, we have fixed which 4 rows should be matched with odd columns (while the other 4 should be matched with even columns). Then there are $(4!)^2 = 24^2$ ways to assign the columns to the rows, so the answer is $(6 \cdot 24)^2 = 144^2 = 20736$.

5. Let S be a set of intervals defined recursively as follows:

   - Initially, $[1, 1000]$ is the only interval in S.
   - If $l \neq r$ and $[l, r] \in S$, then both $\left[l, \left\lfloor \frac{l+r}{2} \right\rfloor\right], \left[\left\lfloor \frac{l+r}{2} \right\rfloor + 1, r\right] \in S$.

   (Note that S can contain intervals such as $[1, 1]$, which contain a single integer.) An integer i is chosen uniformly at random from the range $[1, 1000]$. What is the expected number of intervals in S which contain i?

   Proposed by: Benjamin Qi

   Answer: $\boxed{10.976}$

   Solution: The answer is given by computing the sum of the lengths of all intervals in S and dividing this value by 1000, where the length of an interval $[i, j]$ is given by $j - i + 1$. An interval may be categorized based on how many times $[1, 1000]$ must be split to attain it. An interval that is derived from splitting $[1, 1000]$ k times will be called a *k-split*.

   The only 0-split is $[1, 1000]$, with a total length of 1000. The 1-splits are $[1, 500]$ and $[501, 1000]$, with a total length of 1000. As long as none of the k-splits have length 1, the $(k+1)$-splits will have the same total length. Since the length of the intervals is reduced by half each time (rounded down), we find that the sum of the lengths of the k-splits is 1000 for $0 \leq k \leq 9$.

   Note that the 9-splits consist of $2^{10} - 1000$ intervals of length 1 and $1000 - 2^9$ intervals of length 2. Then the 10-splits consist of $2(1000 - 2^9)$ intervals of length 1, with total length $2(1000 - 2^9)$. The total interval length across all splits is equal to $12(1000) - 2^{10}$, so our answer is

$$12 - \frac{2^{10}}{1000} = 10.976.$$

6. Alice writes 1001 letters on a blackboard, each one chosen independently and uniformly at random from the set $S = \{a, b, c\}$. A move consists of erasing two distinct letters from the board and replacing them with the third letter in S. What is the probability that Alice can perform a sequence of moves which results in one letter remaining on the blackboard?

   Proposed by: Daniel Zhu

Answer: $\boxed{\frac{3-3^{-999}}{4}}$

Solution: Let n_a, n_b, and n_c be the number of a's, b's, and c's on the board, respectively The key observation is that each move always changes the parity of all three of n_a, n_b, and n_c. Since the final configuration must have n_a, n_b, and n_c equal to 1, 0, 0 in some order, Alice cannot leave one letter on the board if n_a, n_b, and n_c start with the same parity (because then they will always have the same parity). Alice also cannot leave one letter on the board if all the letters are initially the same (because she will have no moves to make).

We claim that in all other cases, Alice can make a sequence of moves leaving one letter on the board. The proof is inductive: the base cases $n_a + n_b + n_c \le 2$ are easy to verify, as the possible tuples are $(1, 0, 0)$, $(1, 1, 0)$, and permutations. If $n_a + n_b + n_c \ge 3$, assume without loss of generality that $n_a \ge n_b \ge n_c$. Then $n_b \ge 1$ (because otherwise all the letters are a) and $n_a \ge 2$ (because otherwise $(n_a, n_b, n_c) = (1, 1, 1)$, which all have the same parity). Then Alice will replace a and b by c, reducing to a smaller case.

We begin by computing the probability that n_a, n_b, and n_c start with the same parity. Suppose m letters are chosen at random in the same way (so that we are in the case $m = 1001$). Let x_m be the probability that n_a, n_b, and n_c all have the same parity. We have the recurrence $x_{m+1} = \frac{1}{3}(1 - x_m)$ because when when choosing the $(m + 1)$th letter, the n_i can only attain the same parity if they did not before, and the appropriate letter is drawn. Clearly $x_0 = 1$, which enables us to compute $x_m = \frac{1}{4}(1 + 3 \cdot (-3)^{-m})$. Then x_{1001} is the probability that n_a, n_b, and n_c have the same parity.

The probability that all the letters are initially the same is 3^{-1000}, as this occurs exactly when all the subsequent letters match the first. Thus our final answer is

$$1 - 3^{-1000} - \frac{1}{4}(1 + 3 \cdot (-3)^{-1001}) = \frac{3}{4} - \frac{1}{4 \cdot 3^{999}}.$$

7. Anne-Marie has a deck of 16 cards, each with a distinct positive factor of 2002 written on it. She shuffles the deck and begins to draw cards from the deck without replacement. She stops when there exists a nonempty subset of the cards in her hand whose numbers multiply to a perfect square. What is the expected number of cards in her hand when she stops?

Proposed by: Michael Ren

Answer: $\boxed{\frac{837}{208}}$

Solution: Note that $2002 = 2 \cdot 7 \cdot 11 \cdot 13$, so that each positive factor of 2002 is included on exactly one card. Each card can identified simply by whether or not it is divisible by each of the 4 primes, and we can uniquely achieve all of the 2^4 possibilities. Also, when considering the product of the values on many cards, we only care about the values of the exponents in the prime factorization modulo 2, as we have a perfect square exactly when each exponent is even.

Now suppose Anne-Marie has already drawn k cards. Then there are 2^k possible subsets of cards from those she has already drawn. Note that if any two of these subsets have products with the same four exponents modulo 2, then taking the symmetric difference yields a subset of cards in her hand where all four exponents are $0 \pmod 2$, which would cause her to stop. Now when she draws the $(k + 1)$th card, she achieves a perfect square subset exactly when the the exponents modulo 2 match those from a subset of the cards she already has. Thus if she has already drawn k cards, she will not stop if she draws one of $16 - 2^k$ cards that don't match a subset she already has.

Let p_k be the probability that Anne-Marie draws at least k cards. We have the recurrence

$$p_{k+2} = \frac{16 - 2^k}{16 - k} p_{k+1}$$

because in order to draw $k+2$ cards, the $(k+1)$th card, which is drawn from the remaining $16-k$ cards, must not be one of the $16-2^k$ cards that match a subset of Anne-Marie's first k cards. We now compute

$$\begin{aligned} p_1 &= 1, \\ p_2 &= \frac{15}{16}, \\ p_3 &= \frac{14}{15} p_2 = \frac{7}{8}, \\ p_4 &= \frac{12}{14} p_3 = \frac{3}{4}, \\ p_5 &= \frac{8}{13} p_4 = \frac{6}{13}, \\ p_6 &= 0. \end{aligned}$$

The expected number of cards that Anne-Marie draws is

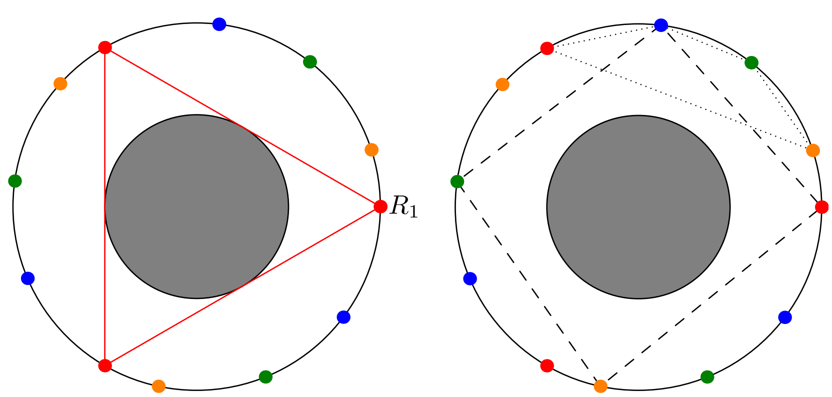
$$p_1 + p_2 + p_3 + p_4 + p_5 = 1 + \frac{15}{16} + \frac{7}{8} + \frac{3}{4} + \frac{6}{13} = \frac{837}{208}.$$

8. Let Γ_1 and Γ_2 be concentric circles with radii 1 and 2, respectively. Four points are chosen on the circumference of Γ_2 independently and uniformly at random, and are then connected to form a convex quadrilateral. What is the probability that the perimeter of this quadrilateral intersects Γ_1?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{22}{27}}$

Solution:



Define a *triplet* as three points on Γ_2 that form the vertices of an equilateral triangle. Note that due to the radii being 1 and 2, the sides of a triplet are all tangent to Γ_1. Rather than choosing four points on Γ_2 uniformly at random, we will choose four triplets of Γ_2 uniformly at random and then choose a random point from each triplet. (This results in the same distribution.) Assume without loss of generality that the first step creates 12 distinct points, as this occurs with probability 1.

In the set of twelve points, a segment between two of those points does not intersect Γ_1 if and only if they are at most three vertices apart. (In the diagram shown above, the segments connecting R_1 to the other red vertices are tangent to Γ_1, so the segments connecting R_1 to the six closer vertices do not intersect Γ_1.) There are two possibilities for the perimeter of the convex quadrilateral to not intersect Γ_1: either the convex quadrilateral contains Γ_1 or is disjoint from it.

Case 1: The quadrilateral contains Γ_1.

Each of the four segments of the quadrilateral passes at most three vertices, so the only possibility is that every third vertex is chosen. This is shown by the dashed quadrilateral in the diagram, and there are 3 such quadrilaterals.

Case 2: The quadrilateral does not contain Γ_1.

In this case, all of the chosen vertices are at most three apart. This is only possible if we choose four consecutive vertices, which is shown by the dotted quadrilateral in the diagram. There are 12 such quadrilaterals.

Regardless of how the triplets are chosen, there are 81 ways to pick four points and $12 + 3 = 15$ of these choices result in a quadrilateral whose perimeter does not intersect Γ_1. The desired probability is $1 - \frac{5}{27} = \frac{22}{27}$.

Remark. The problem can easily be generalized for a larger number of vertices, where Γ_1 and Γ_2 are the inscribed and circumscribed circles of a regular n-gon and $n + 1$ points are chosen uniformly at random on Γ_2. The probability that the perimeter of the convex $(n + 1)$-gon formed by those vertices intersects Γ_1 is $1 - \frac{n+2}{n^n}$.

9. Farmer James wishes to cover a circle with circumference 10π with six different types of colored arcs. Each type of arc has radius 5, has length either π or 2π, and is colored either red, green, or blue. He has an unlimited number of each of the six arc types. He wishes to completely cover his circle without overlap, subject to the following conditions:

   - Any two adjacent arcs are of different colors.
   - Any three adjacent arcs where the middle arc has length π are of three different colors.

   Find the number of distinct ways Farmer James can cover his circle. Here, two coverings are equivalent if and only if they are rotations of one another. In particular, two colorings are considered distinct if they are reflections of one another, but not rotations of one another.

   Proposed by: James Lin

   Answer: $\boxed{93}$

   Solution: Fix an orientation of the circle, and observe that the the problem is equivalent to finding the number of ways to color ten equal arcs of the circle such that each arc is one of three different colors, and any two arcs which are separated by exactly one arc are of different colors. We can consider every other arc, so we are trying to color just five arcs so that no two adjacent arcs are of the same color. This is independent from the coloring of the other five arcs.

   Let a_i be the number of ways to color i arcs in three colors so that no two adjacent arcs are the same color. Note that $a_1 = 3$ and $a_2 = 6$. We claim that $a_i + a_{i+1} = 3 \cdot 2^i$ for $i \geq 2$. To prove this, observe that a_i counts the number of ways to color $i + 1$ points in a line so that no two adjacent points are the same color, and the first and $(i + 1)$th points are the same color. Meanwhile, a_{i+1} counts the number of ways to color $i + 1$ points in a line so that no two adjacent points are the same color, and the first and $(i + 1)$th points are different colors. Then $a_i + a_{i+1}$ is the number of ways to color $i + 1$ points in a line so that no two adjacent points are the same color. There are clearly $3 \cdot 2^i$ ways to do this, as we pick the colors from left to right, with 3 choices for the first color and 2 for the rest. We then compute $a_3 = 6$, $a_4 = 18$, $a_5 = 30$. Then we can color the whole original circle by picking one of the 30 possible colorings for each of the two sets of 5 alternating arcs, for $30^2 = 900$ total.

   Now, we must consider the rotational symmetry. If a configuration has no rotational symmetry, then we have counted it 10 times. If a configuration has 180° rotational symmetry, then we have counted it 5 times. This occurs exactly when we have picked the same coloring from our 30 for both choices, and in exactly one particular orientation, so there are 30 such cases. Having 72° or 36° rotational symmetry is impossible, as arcs with exactly one arc between them must be different colors. Then after we correct for overcounting our answer is

$$\frac{900-30}{10}+\frac{30}{5}=93.$$

10. Max repeatedly throws a fair coin in a hurricane. For each throw, there is a 4% chance that the coin gets blown away. He records the number of heads H and the number of tails T before the coin is lost. (If the coin is blown away on a toss, no result is recorded for that toss.) What is the expected value of $|H-T|$?

    Proposed by: Krit Boonsiriseth

    Answer: $\boxed{\frac{24}{7}}$

    Solution 1: (Dilhan Salgado) In all solutions, $p=\frac{1}{25}$ will denote the probability that the coin is blown away. Let $D=|H-T|$. Note that if $D\neq 0$, the expected value of D is not changed by a coin flip, whereas if $D=0$, the expected value of D increases by 1. Therefore $\mathbf{E}(D)$ can be computed as the sum over all n of the probability that the nth coin flip occurs when $D=0$. This only occurs when $n=2k+1$ is odd, where the probability that the first n coin flips occur is $(1-p)^{2k+1}$ and the probability that $D=0$ after the first $n-1$ flips is $\frac{\binom{2k}{k}}{4^k}$. Therefore

$$\begin{aligned}\mathbf{E}(D)&=(1-p)\sum_{k=0}^{\infty}\left(\frac{1-p}{2}\right)^{2k}\binom{2k}{k}\\&=\frac{1-p}{\sqrt{1-(1-p)^2}}\end{aligned}$$

    using the generating function

$$\sum_{k=0}^{\infty}\binom{2k}{k}x^k=\frac{1}{\sqrt{1-4x}}.$$

    Plugging in $p=\frac{1}{25}$ yields $\mathbf{E}(D)=\frac{24}{7}$.

    Solution 2: For each $n\geqslant 0$, the probability that Max made n successful throws (not counting the last throw) is $p(1-p)^n$.

    Claim: Assuming Max made $n\geqslant 1$ throws, the expected value of $|H-T|$ is given by

$$\prod_{k=1}^{\lfloor (n-1)/2\rfloor}\frac{2k+1}{2k}.$$

    Proof. If n is odd then the expected value for $n+1$ will be equal to that for n; since $|H-T|$ will be nonzero, it will be equally likely to increase or decrease after the coin is flipped. Therefore, it suffices to compute the expected value for the n odd case. This is

$$\begin{aligned}\frac{\sum_{i=0}^{(n-1)/2}\binom{n}{i}\cdot(n-2i)}{2^{n-1}}&=n-\frac{\sum_{i=0}^{(n-1)/2}\binom{n}{i}\cdot 2i}{2^{n-1}}\\&=n\cdot\left(1-\frac{2\cdot\sum_{i=0}^{(n-3)/2}\binom{n-1}{i}}{2^{n-1}}\right)\\&=n\cdot\frac{\binom{n-1}{(n-1)/2}}{2^{n-1}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{n!}{(n-1)!!^2} \\
&= \frac{n!!}{(n-1)!!} \\
&= \prod_{k=1}^{(n-1)/2} \frac{2k+1}{2k},
\end{aligned}
$$

as desired. $\square$

Using the claim, we have

$$
\begin{aligned}
\mathbf{E}(|H-T|) &= p\left(\sum_{n=1}^{\infty}(1-p)^n \prod_{k=1}^{\lfloor (n-1)/2\rfloor} \frac{2k+1}{2k}\right) \\
&= p(1-p)(2-p)\sum_{m=0}^{\infty}\left((1-p)^{2m}\prod_{k=1}^{m}\frac{2k+1}{2k}\right) \\
&= p(1-p)(2-p)\big(1-(1-p)^2\big)^{-3/2} \\
&= \frac{1-p}{\sqrt{p(2-p)}}.
\end{aligned}
$$

Plugging in $p = \frac{1}{25}$ gives

$$
\mathbf{E}(|H-T|) = \frac{24}{25}\cdot 5 \cdot \frac{5}{7} = \frac{24}{7}.
$$

Solution 3: Let E_n be the expected value of $|H-T+n|$. By symmetry, $E_{-n} = E_n$ for all n. Considering what happens in the next throw gives

$$
2E_n = (1-p)E_{n-1} + (1-p)E_{n+1} + 2pn
$$

for all $n \geqslant 0$. Now let $\alpha = \frac{1-\sqrt{p(2-p)}}{1-p} < 1$ be the smaller root of $(1-p)x^2 - 2x + (1-p) = 0$. From

$$
\begin{aligned}
\sum_{n=1}^{\infty} 2\alpha^n E_n &= \sum_{n=1}^{\infty} \alpha^n\left((1-p)E_{n-1} + (1-p)E_{n+1} + 2pn\right) \\
&= \alpha(1-p)E_0 + \alpha^2(1-p)E_1 + \sum_{n=1}^{\infty} 2pn\alpha^n + \sum_{n=2}^{\infty}\big((1-p)\alpha^{n-1} + (1-p)\alpha^{n+1}\big)E_n \\
&= \alpha(1-p)E_0 + (2\alpha-(1-p))E_1 + \sum_{n=1}^{\infty} 2pn\alpha^n + \sum_{n=2}^{\infty} 2\alpha^n E_n,
\end{aligned}
$$

we have

$$
(1-p)E_1 - \alpha(1-p)E_0 = \sum_{n=1}^{\infty} 2pn\alpha^n = \frac{2p\alpha}{(1-\alpha)^2}.
$$

As $E_0 = (1-p)E_1$, this gives

$$
E_0\left(1-\alpha(1-p)\right) = \frac{2p\alpha}{(1-\alpha)^2}.
$$

Plugging in $p = \frac{1}{25}$ and $\alpha = \frac{3}{4}$ gives $E_0 = \frac{24}{7}$.