1^{st} Annual Harvard-MIT November Tournament

Saturday 8 November 2008

Team Round

Unit Fractions [100]

A *unit fraction* is a fraction of the form $\frac{1}{n}$, where n is a positive integer. In this problem, you will find out how rational numbers can be expressed as sums of these unit fractions. Even if you do not solve a problem, you may apply its result to later problems.

We say we *decompose* a rational number q into unit fractions if we write q as a sum of 2 or more **distinct** unit fractions. In particular, if we write q as a sum of k distinct unit fractions, we say we have decomposed q into k fractions. As an example, we can decompose $\frac{2}{3}$ into 3 fractions: $\frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$.

1. (a) Decompose 1 into unit fractions.
   (b) Decompose $\frac{1}{4}$ into unit fractions.
   (c) Decompose $\frac{2}{5}$ into unit fractions.
2. Explain how any unit fraction $\frac{1}{n}$ can be decomposed into other unit fractions.
3. (a) Write 1 as a sum of 4 distinct unit fractions.
   (b) Write 1 as a sum of 5 distinct unit fractions.
   (c) Show that, for any integer $k > 3$, 1 can be decomposed into k unit fractions.
4. Say that $\frac{a}{b}$ is a positive rational number in simplest form, with $a \neq 1$. Further, say that n is an integer such that:
$$\frac{1}{n} > \frac{a}{b} > \frac{1}{n+1}$$
   Show that when $\frac{a}{b} - \frac{1}{n+1}$ is written in simplest form, its numerator is smaller than a.
5. **An aside: the sum of all the unit fractions**
   It is possible to show that, given any real M, there exists a positive integer k large enough that:
$$\sum_{n=1}^{k} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \ldots > M$$
   Note that this statement means that the infinite harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, grows without bound, or diverges. For the specific example M = 5, find a value of k, *not necessarily the smallest*, such that the inequality holds. Justify your answer.
6. Now, using information from problems 4 and 5, prove that the following method to decompose any positive rational number will always terminate:

Step 1. Start with the fraction $\frac{a}{b}$. Let t_1 be the largest unit fraction $\frac{1}{n}$ which is less than or equal to $\frac{a}{b}$.

Step 2. If we have already chosen t_1 through t_k, and if $t_1 + t_2 + \ldots + t_k$ is still less than $\frac{a}{b}$, then let t_{k+1} be the largest unit fraction less than both t_k and $\frac{a}{b}$.

Step 3. If $t_1 + \ldots + t_{k+1}$ equals $\frac{a}{b}$, the decomposition is found. Otherwise, repeat step 2.

Why does this method never result in an infinite sequence of t_i?

Juicy Numbers [100]

A *juicy number* is an integer $j > 1$ for which there is a sequence $a_1 < a_2 < \ldots < a_k$ of positive integers such that $a_k = j$ and such that the sum of the reciprocals of all the a_i is 1. For example, 6 is a juicy number because $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, but 2 is not juicy.

In this part, you will investigate some of the properties of juicy numbers. Remember that if you do not solve a question, you can still use its result on later questions.

1. Explain why 4 is not a juicy number.
2. It turns out that 6 is the smallest juicy integer. Find the next two smallest juicy numbers, and show a decomposition of 1 into unit fractions for each of these numbers. You do not need to prove that no smaller numbers are juicy.
3. Let p be a prime. Given a sequence of positive integers b_1 through b_n, exactly one of which is divisible by p, show that when
$$\frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_n}$$
is written as a fraction in lowest terms, then its denominator is divisible by p. Use this fact to explain why no prime p is ever juicy.
4. Show that if j is a juicy integer, then $2j$ is juicy as well.
5. Prove that the product of two juicy numbers (not necessarily distinct) is always a juicy number. Hint: if j_1 and j_2 are the two numbers, how can you change the decompositions of 1 ending in $\frac{1}{j_1}$ or $\frac{1}{j_2}$ to make them end in $\frac{1}{j_1 j_2}$?