14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

1. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

2. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f(1) = 1$, and $|f'(x)| \le 2$ for all real numbers x. If a and b are real numbers such that the set of possible values of $\displaystyle\int_0^1 f(x)\,dx$ is the open interval (a, b), determine $b - a$.

4. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

5. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

6. Let $a \star b = ab + a + b$ for all integers a and b. Evaluate $1 \star (2 \star (3 \star (4 \star \ldots (99 \star 100) \ldots)))$.

7. Let $f : [0, 1) \to \mathbb{R}$ be a function that satisfies the following condition: if
$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} = .a_1a_2a_3\ldots$$
is the decimal expansion of x and there does not exist a positive integer k such that $a_n = 9$ for all $n \ge k$, then
$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^{2n}}.$$
Determine $f'\left(\frac{1}{3}\right)$.

8. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

9. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

10. For all real numbers x, let
$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}.$$
Evaluate $(f(f(\ldots(f(2011))\ldots)))^{2011}$, where f is applied 2010 times.

11. Evaluate $\displaystyle\int_1^{\infty} \left(\frac{\ln x}{x}\right)^{2011} dx$.

12. Let $f(x) = x^2 + 6x + c$ for all real numbers x, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

13. Sarah and Hagar play a game of darts. Let O_0 be a circle of radius 1. On the nth turn, the player whose turn it is throws a dart and hits a point p_n randomly selected from the points of O_{n-1}. The player then draws the largest circle that is centered at p_n and contained in O_{n-1}, and calls this circle O_n. The player then colors every point that is inside O_{n-1} but not inside O_n her color. Sarah goes first, and the two players alternate turns. Play continues indefinitely. If Sarah's color is red, and Hagar's color is blue, what is the expected value of the area of the set of points colored red?

14. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \leq P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

15. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(f(x)) = 1$ for all $x \in [0, 1]$. Determine the set of possible values of $\displaystyle\int_0^1 f(x)\,dx$.

16. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A.

17. Let $f : (0, 1) \to (0, 1)$ be a differentiable function with a continuous derivative such that for every positive integer n and odd positive integer $a < 2^n$, there exists an odd positive integer $b < 2^n$ such that $f\left(\frac{a}{2^n}\right) = \frac{b}{2^n}$. Determine the set of possible values of $f'\left(\frac{1}{2}\right)$.

18. Let $z = \cos\dfrac{2\pi}{2011} + i \sin\dfrac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \ldots \frac{2008 \cdot 2009}{2} x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3)\ldots P(z^{2010})$.

19. Let

$$F(x) = \frac{1}{(2 - x - x^5)^{2011}},$$

and note that F may be expanded as a power series so that $F(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Find an ordered pair of positive real numbers (c, d) such that $\displaystyle\lim_{n\to\infty} \frac{a_n}{n^d} = c$.

20. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2$; $b_0 = 2$, and $a_{n+1} = a_n \sqrt{1 + a_n^2 + b_n^2} - b_n$; $b_{n+1} = b_n \sqrt{1 + a_n^2 + b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4.