HMMT November 2014

Saturday 15 November 2014

General Test

1. Two circles ω and γ have radii 3 and 4 respectively, and their centers are 10 units apart. Let x be the shortest possible distance between a point on ω and a point on γ, and let y be the longest possible distance between a point on ω and a point on γ. Find the product xy.

   Answer: $\boxed{51}$ Let ℓ be the line connecting the centers of ω and γ. Let A and B be the intersections of ℓ with ω, and let C and D be the intersections of ℓ with γ, so that A, B, C, and D are collinear, in that order. The shortest distance between a point on ω and a point on γ is $BC = 3$. The longest distance is $AD = 3 + 10 + 4 = 17$. The product is 51.

2. Let ABC be a triangle with $\angle B = 90°$. Given that there exists a point D on AC such that $AD = DC$ and $BD = BC$, compute the value of the ratio $\frac{AB}{BC}$.

   Answer: $\boxed{\sqrt{3}}$ D is the circumcenter of ABC because it is the midpoint of the hypotenuse. Therefore, $DB = DA = DC$ because they are all radii of the circumcircle, so DBC is an equilateral triangle, and $\angle C = 60°$. This means that ABC is a $30° - 60° - 90°$ triangle, with $\dfrac{AB}{BC} = \boxed{\sqrt{3}}$.

3. Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$.

   Answer: $\boxed{15}$ Let $d = \gcd(a, b)$ for some $a, b \in \mathbb{Z}^+$.
   Then, we can write $d = ax - by$, where $x, y \in \mathbb{Z}^+$, and

   $$2^a - 1 \mid 2^{ax} - 1 \tag{1}$$
   $$2^b - 1 \mid 2^{by} - 1 \tag{2}$$

   Multiplying the right-hand side of (2) by 2^d, we get,

   $$2^b - 1 \mid 2^{ax} - 2^d$$

   Thus, $\gcd(2^a - 1, 2^b - 1) = 2^d - 1 = 2^{\gcd(a,b)} - 1$.

   Using $a = 16$ and $b = 36$, we get

   $$\gcd(2^{16} - 1, 2^{36} - 1) = 2^{\gcd(16,36)} - 1 = 2^4 - 1 = 15$$

4. In rectangle $ABCD$ with area 1, point M is selected on $\overline{AB}$ and points X, Y are selected on $\overline{CD}$ such that $AX < AY$. Suppose that $AM = BM$. Given that the area of triangle MXY is $\frac{1}{2014}$, compute the area of trapezoid $AXYB$.

   Answer: $\boxed{\frac{1}{2} + \frac{1}{2014} \text{ OR } \frac{504}{1007}}$ Notice that $[AMX] + [BYM] = \frac{1}{2}[ABCD] = \frac{1}{2}$. Thus,

   $$[AXYB] = [AMX] + [BYM] + [MXY] = \frac{1}{2} + \frac{1}{2014} = \frac{504}{1007}$$

5. Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2^{100} wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move?

   (By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

   Answer: $\boxed{33}$ We will work in the binary system in this solution.

Let multiplying the stored value by 2 and adding 1 be $Move\ A$ and multiplying the stored value by 4 and adding 3 be $Move\ B$. Let the stored value be S. Then, $Move\ A$ affixes one 1 to S, while $Move\ B$ affixes two 1s. The goal is to have greater than or equal to 101 1s. If any player makes the number of 1s in S congruent to $2 \mod 3$, then no matter what the other player does, he will lose, since the number of 1s in S reaches 101 or 102 only from $99 \equiv 0 \pmod 3$ or $100 \equiv 1 \pmod 3$.

Mark's winning strategy: Do $Move\ A$. In the succeeding moves, if William does $Move\ B$, then Mark does $Move\ A$, and vice versa, which in total, affixes three 1s to S. This ensures that William always takes his turn while the number of 1s in S is congruent to $2 \mod 3$. Note that Mark has to follow this strategy because once he does not, then William can follow the same strategy and make Mark lose, a contradiction to the required optimal play. Since S starts out with one 1, this process gives William a maximum of 33 moves.

6. Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The angle bisector of C intersects side AB at X. Points M and N are drawn on sides BC and AC, respectively, such that $\overline{XM} \parallel \overline{AC}$ and $\overline{XN} \parallel \overline{BC}$. Compute the length MN.

   Answer: $\boxed{\frac{3\sqrt{14}}{5}}$ By Stewart's Theorem on the angle bisector,

$$CX^2 = AC \cdot BC \left(1 - \frac{AB}{AC + BC}^2\right)$$

Thus,

$$CX^2 = 4 \cdot 6 \left(1 - \frac{5}{10}^2\right) = 18$$

Since $\overline{XM} \parallel \overline{AC}$ and $\overline{XN} \parallel \overline{BC}$, we produce equal angles. So, by similar triangles, $XM = XN = \frac{4 \cdot 6}{10} = \frac{12}{5}$.
Moreover, triangles MCX and NCX are congruent isosceles triangles with vertices M and N, respectively. Since CX is an angle bisector, then CX and MN are perpendicular bisectors of each other. Therefore,

$$MN^2 = 4(XN^2 - (CX/2)^2) = 4 \cdot \left(\frac{12}{5}\right)^2 - 18 = \frac{126}{25}$$

and

$$MN = \frac{3\sqrt{14}}{5}$$

7. Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is *perfect* if for any distinct indices i, j, k, the three numbers a_i, a_j, a_k do *not* form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

   Answer: $\boxed{780}$ There are two situations.

   1. The multiset is $aabbc$; the only condition here is $c \neq \frac{1}{2}(a+b)$, for $(\binom{5}{3} - |S|) \cdot \binom{3}{1} = 18$ such triples, where S is the set of unordered triples (a, b, c) which do not satisfy the condition, and $S = \{(1, 2, 3),\ (2, 3, 4),\ (3, 4, 5),\ (1, 3, 5)\}$. Each one gives $\frac{5!}{2!2!} = 30$ orderings, so $18 \cdot 30 = 540$ in this case.

   2. There are four distinct elements in the tuple. Then, the elements must be $\{1, 2, 4, 5\}$. All of them work, for an additional $4 \cdot 60 = 240$.

   Therefore, there are $540 + 240 = 780$ such tuples.

8. Let a, b, c, x be reals with $(a+b)(b+c)(c+a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \quad \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \quad \text{and} \quad \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

   Compute x.

Answer: $\boxed{-34}$ Note that

$$\begin{aligned}\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}-\frac{a^2}{c+a}-\frac{b^2}{a+b}-\frac{c^2}{b+c} &= \frac{a^2-b^2}{a+b}+\frac{b^2-c^2}{b+c}+\frac{c^2-a^2}{c+a} \\ &= (a-b)+(b-c)+(c-a) \\ &= 0.\end{aligned}$$

Thus, when we sum up all the given equations, we get that
$20+14+x=0$. Therefore, $x=-34$.

9. For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 10100_2 \oplus 1110_2 = 11010_2 = 26$. (The operation $\oplus$ is called the *exclusive or*.) Compute the sum

$$\sum_{k=0}^{2^{2014}-1}\left(k \oplus \left\lfloor \frac{k}{2} \right\rfloor\right).$$

Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

Answer: $\boxed{2^{2013}(2^{2014}-1) \text{ OR } 2^{4027}-2^{2013}}$ Let $k = a_{2013}a_{2012}...a_0$ in base 2. Then $\left\lfloor \frac{k}{2} \right\rfloor = \overline{0a_{2013}...a_1}$ in base 2. So the leftmost digit of $k \oplus \left\lfloor \frac{k}{2} \right\rfloor$ is 1 if and only if $a_{2013}=1$, and the nth digit from the right is 1 if and only if $a_n \neq a_{n-1}$ $(1 \le n \le 2013)$.

In either case, the probability of each digit being 1 is $\frac{1}{2}$. Therefore, the sum of all such numbers is

$$\frac{1}{2}\cdot 2^{2014}\cdot \underbrace{11\ldots11_2}_{2014\text{ digits}} = 2^{2013}(2^{2014}-1).$$

10. Suppose that m and n are integers with $1 \le m \le 49$ and $n \ge 0$ such that m divides $n^{n+1}+1$. What is the number of possible values of m?

Answer: $\boxed{29}$ If n is even, $n+1 \mid n^{n+1}+1$, so we can cover all odd m.

If m is even and $m \mid n^{n+1}+1$, then n must be odd, so $n+1$ is even, and m cannot be divisible by 4 or any prime congruent to 3 (mod 4). Conversely, if $m/2$ has all factors 1 (mod 4), then by CRT there exists $N \equiv 1 \pmod 4$ such that $m \mid N^2+1 \mid N^{N+1}+1$ (note $(N+1)/2$ is odd).

So the only bad numbers take the form $2k$, where $1 \le k \le 24$ is divisible by at least one of $2,3,7,11,19,23,31,\ldots$. We count $k=2,4,\ldots,24$ (there are 12 numbers here), $k=3,9,15,21$ (another four), $k=7,11,19,23$ (another four), giving a final answer of $49-12-4-4=29$.