HMMT February 2023, February 18, 2023 — GUTS ROUND

Organization ______________________ Team ______________________ Team ID# ____________

1. **[10]** Suppose a and b are positive integers such that $a^b = 2^{2023}$. Compute the smallest possible value of b^a.

2. **[10]** Let n be a positive integer, and let s be the sum of the digits of the base-four representation of $2^n - 1$. If $s = 2023$ (in base ten), compute n (in base ten).

3. **[10]** Let $ABCD$ be a convex quadrilateral such that $\angle ABD = \angle BCD = 90^\circ$, and let M be the midpoint of segment BD. Suppose that $CM = 2$ and $AM = 3$. Compute AD.

4. **[10]** A *standard n-sided die* has n sides labeled 1 to n. Luis, Luke, and Sean play a game in which they roll a fair standard 4-sided die, a fair standard 6-sided die, and a fair standard 8-sided die, respectively. They lose the game if Luis's roll is less than Luke's roll, and Luke's roll is less than Sean's roll. Compute the probability that they lose the game.

HMMT February 2023, February 18, 2023 — GUTS ROUND

Organization ______________________ Team ______________________ Team ID# ____________

5. **[11]** If a and b are positive real numbers such that $a \cdot 2^b = 8$ and $a^b = 2$, compute $a^{\log_2 a} 2^{b^2}$.

6. **[11]** Let A, E, H, L, T, and V be chosen independently and at random from the set $\left\{0, \frac{1}{2}, 1\right\}$. Compute the probability that $\lfloor T \cdot H \cdot E \rfloor = L \cdot A \cdot V \cdot A$.

7. **[11]** Let Ω be a sphere of radius 4 and Γ be a sphere of radius 2. Suppose that the center of Γ lies on the surface of Ω. The intersection of the surfaces of Ω and Γ is a circle. Compute this circle's circumference.

8. **[11]** Suppose a, b, and c are distinct positive integers such that $\sqrt{a\sqrt{b\sqrt{c}}}$ is an integer. Compute the least possible value of $a + b + c$.

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Organization ____________________ Team ____________________ Team ID# ____________

9. **[13]** One hundred points labeled 1 to 100 are arranged in a 10×10 grid such that adjacent points are one unit apart. The labels are increasing left to right, top to bottom (so the first row has labels 1 to 10, the second row has labels 11 to 20, and so on).

   Convex polygon $\mathcal{P}$ has the property that every point with a label divisible by 7 is either on the boundary or in the interior of $\mathcal{P}$. Compute the smallest possible area of $\mathcal{P}$.

10. **[13]** The number

$$316990099009901 = \frac{32016000000000001}{101}$$

   is the product of two distinct prime numbers. Compute the smaller of these two primes.

11. **[13]** The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Given 15 wooden blocks of weights $F_2, F_3, \ldots, F_{16}$, compute the number of ways to paint each block either red or blue such that the total weight of the red blocks equals the total weight of the blue blocks.

12. **[13]** The number 770 is written on a blackboard. Melody repeatedly performs *moves*, where a move consists of subtracting either 40 or 41 from the number on the board. She performs moves until the number is not positive, and then she stops. Let N be the number of sequences of moves that Melody could perform. Suppose $N = a \cdot 2^b$ where a is an odd positive integer and b is a nonnegative integer. Compute $100a + b$.

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HMMT February 2023, February 18, 2023 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# ____________

13. **[14]** Suppose a, b, c, and d are pairwise distinct positive perfect squares such that $a^b = c^d$. Compute the smallest possible value of $a + b + c + d$.

14. **[14]** Acute triangle ABC has circumcenter O. The bisector of $\angle ABC$ and the altitude from C to side AB intersect at X. Suppose that there is a circle passing through B, O, X, and C. If $\angle BAC = n^\circ$, where n is a positive integer, compute the largest possible value of n.

15. **[14]** Let A and B be points in space for which $AB = 1$. Let $\mathcal{R}$ be the region of points P for which $AP \leq 1$ and $BP \leq 1$. Compute the largest possible side length of a cube contained within $\mathcal{R}$.

16. **[14]** The graph of the equation $x + y = \lfloor x^2 + y^2 \rfloor$ consists of several line segments. Compute the sum of their lengths.

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HMMT February 2023, February 18, 2023 — GUTS ROUND

Organization ______________ Team ______________ Team ID# ________

17. **[16]** An equilateral triangle lies in the Cartesian plane such that the x-coordinates of its vertices are pairwise distinct and all satisfy the equation $x^3 - 9x^2 + 10x + 5 = 0$. Compute the side length of the triangle.

18. **[16]** Elisenda has a piece of paper in the shape of a triangle with vertices A, B, and C such that $AB = 42$. She chooses a point D on segment AC, and she folds the paper along line BD so that A lands at a point E on segment BC. Then, she folds the paper along line DE. When she does this, B lands at the midpoint of segment DC. Compute the perimeter of the original unfolded triangle.

19. **[16]** Compute the number of ways to select 99 cells of a 19×19 square grid such that no two selected cells share an edge or vertex.

20. **[16]** Five people take a true-or-false test with five questions. Each person randomly guesses on every question. Given that, for each question, a majority of test-takers answered it correctly, let p be the probability that every person answers exactly three questions correctly. Suppose that $p = \frac{a}{2^b}$ where a is an odd positive integer and b is a nonnegative integer. Compute $100a + b$.

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HMMT February 2023, February 18, 2023 — GUTS ROUND

Organization ______________ Team ______________ Team ID# ________

21. **[18]** Let x, y, and N be real numbers, with y nonzero, such that the sets $\{(x+y)^2, (x-y)^2, xy, x/y\}$ and $\{4, 12.8, 28.8, N\}$ are equal. Compute the sum of the possible values of N.

22. **[18]** Let a_0, a_1, a_2, $\ldots$ be an infinite sequence where each term is independently and uniformly random in the set $\{1, 2, 3, 4\}$. Define an infinite sequence b_0, b_1, b_2, $\ldots$ recursively by $b_0 = 1$ and $b_{i+1} = a_i^{b_i}$. Compute the expected value of the smallest positive integer k such that $b_k \equiv 1 \pmod{5}$.

23. **[18]** A subset S of the set $\{1, 2, \ldots, 10\}$ is chosen randomly, with all possible subsets being equally likely. Compute the expected number of positive integers which divide the product of the elements of S. (By convention, the product of the elements of the empty set is 1.)

24. **[18]** Let $AXBY$ be a cyclic quadrilateral, and let line AB and line XY intersect at C. Suppose $AX \cdot AY = 6$, $BX \cdot BY = 5$, and $CX \cdot CY = 4$. Compute AB^2.

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Organization ____________________ Team ____________________ Team ID# ____________

25. **[20]** The *spikiness* of a sequence $a_1, a_2, \ldots, a_n$ of at least two real numbers is the sum $\sum_{i=1}^{n-1} |a_{i+1} - a_i|$. Suppose $x_1, x_2, \ldots, x_9$ are chosen uniformly and randomly from the interval $[0, 1]$. Let M be the largest possible value of the spikiness of a permutation of $x_1, x_2, \ldots, x_9$. Compute the expected value of M.

26. **[20]** Let $PABC$ be a tetrahedron such that $\angle APB = \angle APC = \angle BPC = 90^\circ$, $\angle ABC = 30^\circ$, and AP^2 equals the area of triangle ABC. Compute $\tan \angle ACB$.

27. **[20]** Suppose $m > n > 1$ are positive integers such that there exist n complex numbers $x_1, x_2, \ldots, x_n$ for which

    - $x_1^k + x_2^k + \cdots + x_n^k = 1$ for $k = 1, 2, \ldots, n-1$;
    - $x_1^n + x_2^n + \cdots + x_n^n = 2$; and
    - $x_1^m + x_2^m + \cdots + x_n^m = 4$.

    Compute the smallest possible value of $m + n$.

28. **[20]** Suppose $ABCD$ is a convex quadrilateral with $\angle ABD = 105^\circ$, $\angle ADB = 15^\circ$, $AC = 7$, and $BC = CD = 5$. Compute the sum of all possible values of BD.

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Organization ____________________ Team ____________________ Team ID# ____________

29. **[23]** Let $P_1(x)$, $P_2(x)$, $\ldots$, $P_k(x)$ be monic polynomials of degree 13 with integer coefficients. Suppose there are pairwise distinct positive integers n_1, n_2, $\ldots$, n_k for which, for all positive integers i and j less than or equal to k, the statement "n_i divides $P_j(m)$ for every integer m" holds if and only if $i = j$. Compute the largest possible value of k.

30. **[23]** Five pairs of twins are randomly arranged around a circle. Then they perform zero or more *swaps*, where each swap switches the positions of two adjacent people. They want to reach a state where no one is adjacent to their twin. Compute the expected value of the smallest number of swaps needed to reach such a state.

31. **[23]** Let
$$P = \prod_{i=0}^{2016} \left(i^3 - i - 1\right)^2.$$
The remainder when P is divided by the prime 2017 is not zero. Compute this remainder.

32. **[23]** Let ABC be a triangle with $\angle BAC > 90°$. Let D be the foot of the perpendicular from A to side BC. Let M and N be the midpoints of segments BC and BD, respectively. Suppose that $AC = 2$, $\angle BAN = \angle MAC$, and $AB \cdot BC = AM$. Compute the distance from B to line AM.

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Organization ______________________ Team ______________________ Team ID# ______________

33. **[25]** Given a function f, let $\pi(f) = f \circ f \circ f \circ f \circ f$. The attached sheet has the graphs of ten smooth functions from the interval $(0, 1)$ to itself. The left-hand side consists of five functions:

- $F_1(x) = 0.005 + \frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{8}\sin 8x + \frac{1}{16}\sin 16x + \frac{1}{32}\sin 32x$;
- $F_2(x) = F_1(F_1(x + 0.25))$;
- $F_3(x) = F_1((1 - x)F_1((1 - x)^2))$;
- $F_4(x) = F_1(x) + 0.05\sin(2\pi x)$;
- $F_5(x) = F_1(x + 1.45) + 0.65$.

The right-hand side consists of the five functions A, B, C, D, and E, which are $\pi(F_1), \ldots, \pi(F_5)$ in some order. Compute which of the functions $\{A, B, C, D, E\}$ correspond to $\pi(F_k)$ for $k = 1, 2, 3, 4, 5$.

Your answer should be a five-character string containing A, B, C, D, E, or X for blank. For instance, if you think $\pi(F_1) = A$ and $\pi(F_5) = E$, then you would answer $AXXXE$. If you attempt to identify n functions and get them **all** correct, then you will receive n^2 points. Otherwise, you will receive 0 points.

34. **[25]** The number 2027 is prime. For $i = 1, 2, \ldots, 2026$, let p_i be the smallest prime number such that $p_i \equiv i \pmod{2027}$. Estimate $\max(p_1, \ldots, p_{2026})$.

Submit a positive integer E. If the correct answer is A, you will receive $\lfloor 25\min((E/A)^8, (A/E)^8)\rfloor$ points. (If you do not submit a positive integer, you will receive zero points for this question.)

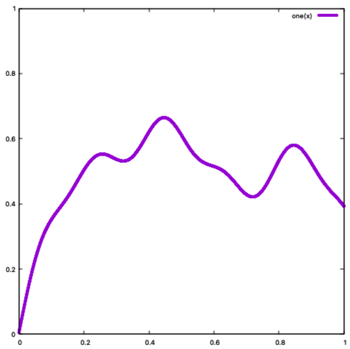
35. **[25]** The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Given 30 wooden blocks of weights $\sqrt[3]{F_2}$, $\sqrt[3]{F_3}$, $\ldots$, $\sqrt[3]{F_{31}}$, estimate the number of ways to paint each block either red or blue such that the total weight of the red blocks and the total weight of the blue blocks differ by at most 1.

Submit a positive integer E. If the correct answer is A, you will receive $\lfloor 25\min((E/A)^8, (A/E)^8)\rfloor$ points. (If you do not submit a positive integer, you will receive zero points for this question.)

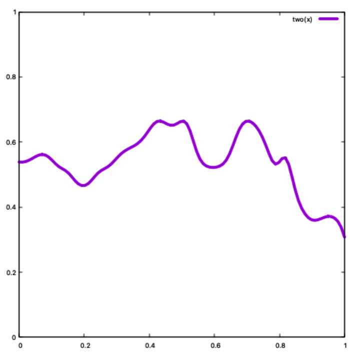
36. **[25]** After the Guts round ends, the HMMT organizers will calculate A, the total number of points earned over all participating teams on questions 33, 34, and 35 of this round (that is, the other estimation questions). Estimate A.

Submit a positive integer E. You will receive $\max(0, 25 - 3 \cdot |E - A|)$ points. (If you do not submit a positive integer, you will receive zero points for this question.)

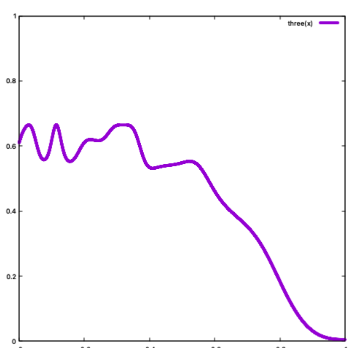
For your information, there are about 70 teams competing.



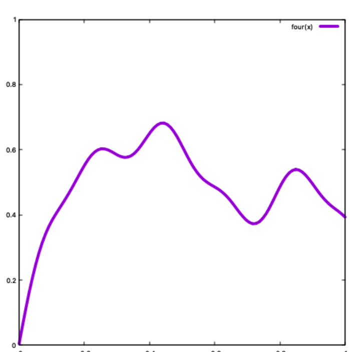
$F_1.$



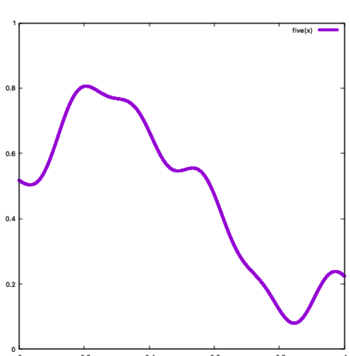
$F_2.$



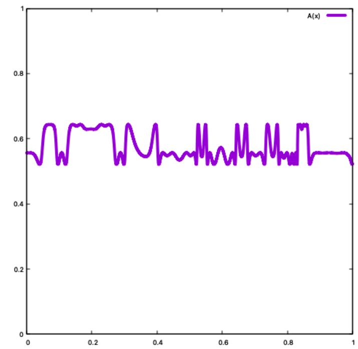
$F_3.$



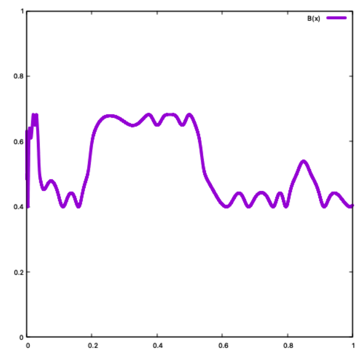
$F_4.$



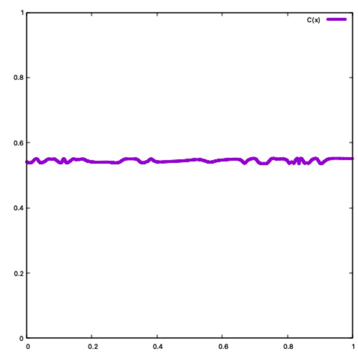
$F_5.$



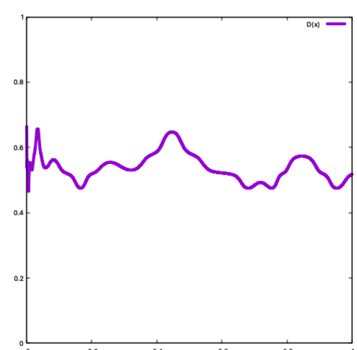
$A.$



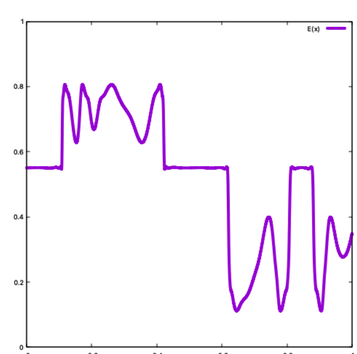
$B.$



$C.$



$D.$



$E.$