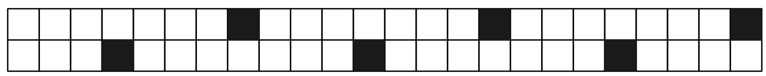
12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: General Test, Part 2

1. [**2**] How many ways can the integers from -7 to 7 be arranged in a sequence such that the absolute values of the numbers in the sequence are nonincreasing?

2. [**2**] How many ways can you tile the white squares of following 2×24 grid with dominoes? (A domino covers two adjacent squares, and a tiling is a non-overlapping arrangement of dominoes that covers every white square and does not intersect any black square.)



3. [**3**] Let S be the sum of all the real coefficients of the expansion of $(1+ix)^{2009}$. What is $\log_2(S)$?

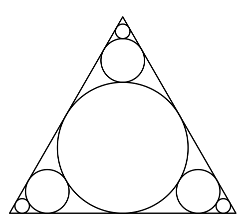
4. [**3**] A torus (donut) having inner radius 2 and outer radius 4 sits on a flat table. What is the radius of the largest spherical ball that can be placed on top of the center torus so that the ball still touches the horizontal plane? (If the $x-y$ plane is the table, the torus is formed by revolving the circle in the $x-z$ plane centered at $(3,0,1)$ with radius 1 about the z axis. The spherical ball has its center on the z-axis and rests on either the table or the donut.)

5. [**4**] Suppose a, b and c are integers such that the greatest common divisor of x^2+ax+b and x^2+bx+c is $x+1$ (in the set of polynomials in x with integer coefficients), and the least common multiple of x^2+ax+b and x^2+bx+c is x^3-4x^2+x+6. Find $a+b+c$.

6. [**4**] In how many ways can you rearrange the letters of "HMMTHMMT" such that the substring "HMMT" does not appear? (For instance, one such rearrangement is HMMHMTMT.)

7. [**5**] Let F_n be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Compute $\sum_{n=0}^{\infty} F_n/10^n$.

8. [**5**] The incircle ω of equilateral triangle ABC has radius 1. Three smaller circles are inscribed tangent to ω and the sides of ABC, as shown. Three smaller circles are then inscribed tangent to the previous circles and to each of two sides of ABC. This process is repeated an infinite number of times. What is the total length of the circumferences of all the circles?



9. [**6**] How many sequences of 5 positive integers (a,b,c,d,e) satisfy $abcde \leq a+b+c+d+e \leq 10$?

10. [**6**] Let T be a right triangle with sides having lengths 3, 4, and 5. A point P is called "awesome" if P is the center of a parallelogram whose vertices all lie on the boundary of T. What is the area of the set of awesome points?