# $14^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

**Saturday 12 February 2011**

1. [**4**] Let $ABC$ be a triangle with area 1. Let points $D$ and $E$ lie on $AB$ and $AC$, respectively, such that $DE$ is parallel to $BC$ and $DE/BC = 1/3$. If $F$ is the reflection of $A$ across $DE$, find the area of triangle $FBC$.

2. [**4**] Let $a \star b = \sin a \cos b$ for all real numbers $a$ and $b$. If $x$ and $y$ are real numbers such that $x \star y - y \star x = 1$, what is the maximum value of $x \star y + y \star x$?

3. [**4**] Evaluate $2011 \times 20122012 \times 201320132013 - 2013 \times 20112011 \times 201220122012$.

4. [**4**] Let $p$ be the answer to this question. If a point is chosen uniformly at random from the square bounded by $x = 0$, $x = 1$, $y = 0$, and $y = 1$, what is the probability that at least one of its coordinates is greater than $p$?

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5. [**5**] Rachelle picks a positive integer $a$ and writes it next to itself to obtain a new positive integer $b$. For instance, if $a = 17$, then $b = 1717$. To her surprise, she finds that $b$ is a multiple of $a^2$. Find the product of all the possible values of $\frac{b}{a^2}$.

6. [**5**] Square $ABCD$ is inscribed in circle $\omega$ with radius 10. Four additional squares are drawn inside $\omega$ but outside $ABCD$ such that the lengths of their diagonals are as large as possible. A sixth square is drawn by connecting the centers of the four aforementioned small squares. Find the area of the sixth square.

7. [**6**] For any positive real numbers $a$ and $b$, define $a \circ b = a + b + 2\sqrt{ab}$. Find all positive real numbers $x$ such that $x^2 \circ 9x = 121$.

8. [**6**] Find the smallest $k$ such that for any arrangement of 3000 checkers in a $2011 \times 2011$ checkerboard, with at most one checker in each square, there exist $k$ rows and $k$ columns for which every checker is contained in at least one of these rows or columns.

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9. [**6**] Segments $AA'$, $BB'$, and $CC'$, each of length 2, all intersect at a point $O$. If $\angle AOC' = \angle BOA' = \angle COB' = 60^\circ$, find the maximum possible value of the sum of the areas of triangles $AOC'$, $BOA'$, and $COB'$.

10. [**6**] In how many ways can one fill a $4 \times 4$ grid with a 0 or 1 in each square such that the sum of the entries in each row, column, and long diagonal is even?

11. [**8**] Rosencrantz and Guildenstern play a game in which they repeatedly flip a fair coin. Let $a_1 = 4$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. On the $n$th flip, if the coin is heads, Rosencrantz pays Guildenstern $a_n$ dollars, and, if the coin is tails, Guildenstern pays Rosencrantz $a_n$ dollars. If play continues for 2010 turns, what is the probability that Rosencrantz ends up with more money than he started with?

12. [**8**] A sequence of integers $\{a_i\}$ is defined as follows: $a_i = i$ for all $1 \leq i \leq 5$, and $a_i = a_1 a_2 \cdots a_{i-1} - 1$ for all $i > 5$. Evaluate $a_1 a_2 \cdots a_{2011} - \sum_{i=1}^{2011} a_i^2$.

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13. [**8**] Let $a$,$b$, and $c$ be the side lengths of a triangle, and assume that $a \le b$ and $a \le c$. Let $x = \frac{b+c-a}{2}$. If $r$ and $R$ denote the inradius and circumradius, respectively, find the minimum value of $\frac{ax}{rR}$.

14. [**8**] Danny has a set of 15 pool balls, numbered $1, 2, \ldots, 15$. In how many ways can he put the balls in 8 indistinguishable bins such that the sum of the numbers of the balls in each bin is 14, 15, or 16?

15. [**10**] Find all irrational numbers $x$ such that $x^3 - 17x$ and $x^2 + 4x$ are both rational numbers.

16. [**10**] Let $R$ be a semicircle with diameter $XY$. A trapezoid $ABCD$ in which $AB$ is parallel to $CD$ is circumscribed about $R$ such that $AB$ contains $XY$. If $AD = 4$, $CD = 5$, and $BC = 6$, determine $AB$.

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17. [**10**] Given positive real numbers $x$, $y$, and $z$ that satisfy the following system of equations:

$$\begin{aligned} x^2 + y^2 + xy &= 1, \\ y^2 + z^2 + yz &= 4, \\ z^2 + x^2 + zx &= 5, \end{aligned}$$

find $x + y + z$.

18. [**10**] In how many ways can each square of a $4 \times 2011$ grid be colored red, blue, or yellow such that no two squares that are diagonally adjacent are the same color?

19. [**12**] Find the least positive integer $N$ with the following property: If all lattice points in $[1, 3] \times [1, 7] \times [1, N]$ are colored either black or white, then there exists a rectangular prism, whose faces are parallel to the $xy$, $xz$, and $yz$ planes, and whose eight vertices are all colored in the same color.

20. [**12**] Let $ABCD$ be a quadrilateral circumscribed about a circle with center $O$. Let $O_1$, $O_2$, $O_3$, and $O_4$ denote the circumcenters of $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle DOA$. If $\angle A = 120°, \angle B = 80°$, and $\angle C = 45°$, what is the acute angle formed by the two lines passing through $O_1O_3$ and $O_2O_4$?

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21. [**12**] Let $ABCD$ be a quadrilateral inscribed in a circle with center $O$. Let $P$ denote the intersection of $AC$ and $BD$. Let $M$ and $N$ denote the midpoints of $AD$ and $BC$. If $AP = 1$, $BP = 3$, $DP = \sqrt{3}$, and $AC$ is perpendicular to $BD$, find the area of triangle $MON$.

22. [**12**] Find the number of ordered triples $(a, b, c)$ of pairwise distinct integers such that $-31 \le a, b, c \le 31$ and $a + b + c > 0$.

23. [**14**] Let $S$ be the set of points $(x, y, z)$ in $\mathbb{R}^3$ such that $x$, $y$, and $z$ are positive integers less than or equal to 100. Let $f$ be a bijective map between $S$ and the $\{1, 2, \ldots, 1000000\}$ that satisfies the following property: if $x_1 \le x_2$, $y_1 \le y_2$, and $z_1 \le z_2$, then $f(x_1, y_1, z_1) \le f(x_2, y_2, z_2)$. Define

$$A_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(i, j, k),$$

$$B_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, i, k),$$

$$\text{and } C_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, k, i).$$

Determine the minimum value of $A_{i+1} - A_i + B_{j+1} - B_j + C_{k+1} - C_k$.

24. [**14**] In how many ways may thirteen beads be placed on a circular necklace if each bead is either blue or yellow and no two yellow beads may be placed in adjacent positions? (Beads of the same color are considered to be identical, and two arrangements are considered to be the same if and only if each can be obtained from the other by rotation).

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25. [**14**] Let $n$ be an integer greater than 3. Let $R$ be the set of lattice points $(x, y)$ such that $0 \le x, y \le n$ and $|x - y| \le 3$. Let $A_n$ be the number of paths from $(0, 0)$ to $(n, n)$ that consist only of steps of the form $(x, y) \to (x, y + 1)$ and $(x, y) \to (x + 1, y)$ and are contained entirely within $R$. Find the smallest positive real number that is greater than $\frac{A_{n+1}}{A_n}$ for all $n$.

26. [**14**] In how many ways can 13 bishops be placed on an $8 \times 8$ chessboard such that (i) a bishop is placed on the second square in the second row, (ii) at most one bishop is placed on each square, (iii) no bishop is placed on the same diagonal as another bishop, and (iv) every diagonal contains a bishop? (For the purposes of this problem, consider all diagonals of the chessboard to be diagonals, not just the main diagonals).

27. [**16**] Find the number of polynomials $p(x)$ with integer coefficients satisfying $p(x) \ge \min\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ and $p(x) \le \max\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ for all $x \in \mathbb{R}$.

28. [**16**] Let $ABC$ be a triangle, and let points $P$ and $Q$ lie on $BC$ such that $P$ is closer to $B$ than $Q$ is. Suppose that the radii of the incircles of triangles $ABP$, $APQ$, and $AQC$ are all equal to 1, and that the radii of the corresponding excircles opposite $A$ are 3, 6, and 5, respectively. If the radius of the incircle of triangle $ABC$ is $\frac{3}{2}$, find the radius of the excircle of triangle $ABC$ opposite $A$.

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29. [**16**] Let $ABC$ be a triangle such that $AB = AC = 182$ and $BC = 140$. Let $X_1$ lie on $AC$ such that $CX_1 = 130$. Let the line through $X_1$ perpendicular to $BX_1$ at $X_1$ meet $AB$ at $X_2$. Define $X_2, X_3, \ldots$, as follows: for $n$ odd and $n \ge 1$, let $X_{n+1}$ be the intersection of $AB$ with the perpendicular to $X_{n-1}X_n$ through $X_n$; for $n$ even and $n \ge 2$, let $X_{n+1}$ be the intersection of $AC$ with the perpendicular to $X_{n-1}X_n$ through $X_n$. Find $BX_1 + X_1X_2 + X_2X_3 + \ldots$.

30. [**16**] How many ways are there to color the vertices of a $2n$-gon with three colors such that no vertex has the same color as its either of its two neighbors or the vertex directly across from it?

31. [**18**] Let $A = \{1, 2, 3, \ldots, 9\}$. Find the number of bijective functions $f : A \to A$ for which there exists at least one $i \in A$ such that
$$|f(i) - f^{-1}(i)| > 1.$$

32. [**18**] Let $p$ be a prime positive integer. Define a mod-$p$ recurrence of degree $n$ to be a sequence $\{a_k\}_{k\ge0}$ of numbers modulo $p$ satisfying a relation of the form $a_{i+n} = c_{n-1}a_{i+n-1} + ... + c_1a_{i+1} + c_0a_i$ for all $i \ge 0$, where $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \not\equiv 0 \pmod{p}$. Compute the number of distinct linear recurrences of degree at most $n$ in terms of $p$ and $n$.

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33. [**25**] Find the number of sequences consisting of 100 $R$'s and 2011 $S$'s that satisfy the property that among the first $k$ letters, the number of $S$'s is strictly more than 20 times the number of $R$'s for all $1 \le k \le 2111$.

34. [**25**] Let $w = w_1, w_2, \ldots, w_6$ be a permutation of the integers $\{1, 2, \ldots, 6\}$. If there do not exist indices $i < j < k$ such that $w_i < w_j < w_k$ or indices $i < j < k < l$ such that $w_i > w_j > w_k > w_l$, then $w$ is said to be *exquisite*. Find the number of exquisite permutations.

35. [**25**] An *independent set* of a graph $G$ is a set of vertices of $G$ such that no two vertices among these are connected by an edge. If $G$ has 2000 vertices, and each vertex has degree 10, find the maximum possible number of independent sets that $G$ can have.

36. [**25**] An *ordering* of a set of $n$ elements is a bijective map between the set and $\{1, 2, \ldots, n\}$. Call an ordering $\rho$ of the 10 unordered pairs of distinct integers from the set $\{1, 2, 3, 4, 5\}$ admissible if, for any $1 \leq a < b < c \leq 5$, either $p(\{a, b\}) < p(\{a, c\}) < p(\{b, c\})$ or $p(\{b, c\}) < p(\{a, c\}) < p(\{a, b\})$. Find the total number of admissible orderings.

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