HMMT February 2019

February 16, 2019

Guts Round

1. [**3**] Find the sum of all real solutions to $x^2 + \cos x = 2019$.

   Proposed by: Evan Chen

   Answer: $\boxed{0}$

   The left-hand side is an even function, hence for each x that solves the equation, $-x$ will also be a solution. Pairing the solutions up in this way, we get that the sum must be 0.

2. [**3**] There are 100 people in a room with ages $1, 2, \ldots, 100$. A pair of people is called *cute* if each of them is at least seven years older than half the age of the other person in the pair. At most how many pairwise disjoint cute pairs can be formed in this room?

   Proposed by: Yuan Yao

   Answer: $\boxed{43}$

   For a *cute* pair (a, b) we would have

$$a \geq \frac{b}{2} + 7,\ b \geq \frac{a}{2} + 7.$$

   Solving the system, we get that a and b must both be at least 14. However 14 could only be paired with itself or a smaller number; therefore, only people with age 15 or above can be paired with someone of different age. Pairing consecutive numbers $(15, 16), (17, 18), \ldots, (99, 100)$ works, giving $\frac{100-14}{2} = 43$ pairs.

3. [**3**] Let $S(x)$ denote the sum of the digits of a positive integer x. Find the maximum possible value of $S(x + 2019) - S(x)$.

   Proposed by: Alec Sun

   Answer: $\boxed{12}$

   We note that $S(a + b) \leq S(a) + S(b)$ for all positive a and b, since carrying over will only decrease the sum of digits. (A bit more rigorously, one can show that $S(x + a \cdot 10^b) - S(x) \leq a$ for $0 \leq a \leq 9$.) Hence we have $S(x + 2019) - S(x) \leq S(2019) = 12$, and equality can be achieved with $x = 100000$ for example.

4. [**3**] Tessa has a figure created by adding a semicircle of radius 1 on each side of an equilateral triangle with side length 2, with semicircles oriented outwards. She then marks two points on the boundary of the figure. What is the greatest possible distance between the two points?

   Proposed by: Yuan Yao

   Answer: $\boxed{3}$

   Note that both points must be in different semicircles to reach the maximum distance. Let these points be M and N, and O_1 and O_2 be the centers of the two semicircles where they lie respectively. Then

$$MN \leq MO_1 + O_1O_2 + O_2N.$$

   Note that the the right side will always be equal to 3 ($MO_1 = O_2N = 1$ from the radius condition, and $O_1O_2 = 1$ from being a midline of the equliateral triangle), hence MN can be at most 3. Finally, if the four points are collinear (when M and N are defined as the intersection of line O_1O_2 with the two semicircles), then equality will hold. Therefore, the greatest possible distance between M and N is 3.

5. [**4**] Call a positive integer n *weird* if n does not divide $(n-2)!$. Determine the number of weird numbers between 2 and 100 inclusive.

   Proposed by: Yuan Yao

   Answer: $\boxed{26}$

   We claim that all the weird numbers are all the prime numbers and 4. Since no numbers between 1 and $p-2$ divide prime p, $(p-2)!$ will not be divisible by p. We also have $2! = 2$ not being a multiple of 4.

   Now we show that all other numbers are not weird. If $n = pq$ where $p \neq q$ and $p, q \geq 2$, then since p and q both appear in $1, 2, \ldots, n-2$ and are distinct, we have $pq \mid (n-2)!$. This leaves the only case of $n = p^2$ for prime $p \geq 3$. In this case, we can note that p and $2p$ are both less than $p^2 - 2$, so $2p^2 \mid (n-2)!$ and we are similarly done.

   Since there are 25 prime numbers not exceeding 100, there are $25 + 1 = 26$ weird numbers.

6. [**4**] The pairwise products ab, bc, cd, and da of positive integers a, b, c, and d are $64, 88, 120$, and 165 in some order. Find $a + b + c + d$.

   Proposed by: Anders Olsen

   Answer: $\boxed{42}$

   The sum $ab + bc + cd + da = (a+c)(b+d) = 437 = 19 \cdot 23$, so $\{a+c, b+d\} = \{19, 23\}$ as having either pair sum to 1 is impossible. Then the sum of all 4 is $19 + 23 = 42$. (In fact, it is not difficult to see that the only possible solutions are $(a, b, c, d) = (8, 8, 11, 15)$ or its cyclic permutations and reflections.)

7. [**4**] For any real number α, define

$$\operatorname{sign}(\alpha) = \begin{cases} +1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases}$$

   How many triples $(x, y, z) \in \mathbb{R}^3$ satisfy the following system of equations

$$\begin{aligned} x &= 2018 - 2019 \cdot \operatorname{sign}(y+z), \\ y &= 2018 - 2019 \cdot \operatorname{sign}(z+x), \\ z &= 2018 - 2019 \cdot \operatorname{sign}(x+y)? \end{aligned}$$

   Proposed by: Pakawut Jiradilok

   Answer: $\boxed{3}$

   Since $\operatorname{sign}(x+y)$ can take one of 3 values, z can be one of 3 values: $4037, 2018$, or -1. The same is true of x and y. However, this shows that $x + y$ cannot be 0, so z can only be 4037 or -1. The same is true of x and y. Now note that, if any two of x, y, z are -1, then the third one must be 4037. Furthermore, if any one of x, y, z is 4037, then the other two must be -1. Thus, the only possibility is to have exactly two of x, y, z be -1 and the third one be 4037. This means that the only remaining triples are $(-1, -1, 4037)$ and its permutations. These all work, so there are exactly 3 ordered triples.

8. [**4**] A regular hexagon $PROFIT$ has area 1. Every minute, greedy George places the largest possible equilateral triangle that does not overlap with other already-placed triangles in the hexagon, with ties broken arbitrarily. How many triangles would George need to cover at least 90% of the hexagon's area?

   Proposed by: Yuan Yao

   Answer: $\boxed{46}$

   It's not difficult to see that the first triangle must connect three non-adjacent vertices (e.g. POI), which covers area $\frac{1}{2}$, and leaves three 30-30-120 triangles of area $\frac{1}{6}$ each. Then, the next three triangles cover $\frac{1}{3}$ of the respective small triangle they are in, and leave six 30-30-120 triangles of area $\frac{1}{18}$ each.

This process continues, doubling the number of 30-30-120 triangles each round and the area of each triangle is divided by 3 each round. After $1+3+6+12+24=46$ triangles, the remaining area is $\frac{3\cdot 2^4}{6\cdot 3^4}=\frac{48}{486}=\frac{8}{81}<0.1$, and the last triangle removed triangle has area $\frac{1}{486}$, so this is the minimum number necessary.

9. [**5**] Define $P=\{\text{S},\text{T}\}$ and let $\mathcal{P}$ be the set of all proper subsets of P. (A *proper subset* is a subset that is not the set itself.) How many ordered pairs $(\mathcal{S},\mathcal{T})$ of proper subsets of $\mathcal{P}$ are there such that

   (a) $\mathcal{S}$ is not a proper subset of $\mathcal{T}$ and $\mathcal{T}$ is not a proper subset of $\mathcal{S}$; and

   (b) for any sets $S\in\mathcal{S}$ and $T\in\mathcal{T}$, S is not a proper subset of T and T is not a proper subset of S?

   Proposed by: Yuan Yao

   Answer: $\boxed{7}$

   For ease of notation, we let $0=\varnothing, 1=\{\text{S}\}, 2=\{\text{T}\}$. Then both $\mathcal{S}$ and $\mathcal{T}$ are proper subsets of $\{0,1,2\}$. We consider the following cases:

   Case 1. If $\mathcal{S}=\varnothing$, then $\mathcal{S}$ is a proper subset of anyset except the empty set, so we must have $\mathcal{T}=\varnothing$.

   Case 2. If $\mathcal{S}=\{0\}$, then $\mathcal{T}$ cannot be empty, nor can it contain either 1 or 2, so we must have $\mathcal{T}=\{0\}$. This also implies that if $\mathcal{S}$ contains another element, then there would be no choice of $\mathcal{T}$ because $\{0\}$ would be a proper subset.

   Case 3. If $\mathcal{S}=\{1\}$, then $\mathcal{T}$ cannot contain 0, and cannot contain both 1 and 2 (or it becomes a proper superset of $\mathcal{S}$), so it can only be $\{1\}$ or $\{2\}$, and both work. The similar apply when $\mathcal{S}=\{2\}$.

   Case 4. If $\mathcal{S}=\{1,2\}$, then since $\mathcal{T}$ cannot contain 0, it must contain both 1 and 2 (or it becomes a proper subset of $\mathcal{S}$), so $\mathcal{T}=\{1,2\}$.

   Hence, all the possibilities are

   $$(\mathcal{S},\mathcal{T})=(\varnothing,\varnothing),(\{0\},\{0\}),(\{1\},\{1\}),(\{1\},\{2\}),(\{2\},\{1\}),(\{2\},\{2\}),(\{1,2\},\{1,2\}),$$

   for 7 possible pairs in total.

10. [**5**] Let

   $$A=(1+2\sqrt{2}+3\sqrt{3}+6\sqrt{6})(2+6\sqrt{2}+\sqrt{3}+3\sqrt{6})(3+\sqrt{2}+6\sqrt{3}+2\sqrt{6})(6+3\sqrt{2}+2\sqrt{3}+\sqrt{6}),$$

   $$B=(1+3\sqrt{2}+2\sqrt{3}+6\sqrt{6})(2+\sqrt{2}+6\sqrt{3}+3\sqrt{6})(3+6\sqrt{2}+\sqrt{3}+2\sqrt{6})(6+2\sqrt{2}+3\sqrt{3}+\sqrt{6}).$$

   Compute the value of A/B.

   Proposed by: Yuan Yao

   Answer: $\boxed{1}$

   Note that

   $$A=((1+2\sqrt{2})(1+3\sqrt{3}))((2+\sqrt{3})(1+3\sqrt{2}))((3+\sqrt{2})(1+2\sqrt{3}))((3+\sqrt{3})(2+\sqrt{2})),$$

   $$B=((1+3\sqrt{2})(1+2\sqrt{3}))((2+\sqrt{2})(1+3\sqrt{3}))((3+\sqrt{3})(1+2\sqrt{2}))((2+\sqrt{3})(3+\sqrt{2})).$$

   It is not difficult to check that they have the exact same set of factors, so $A=B$ and thus the ratio is 1.

11. [**5**] In the Year 0 of Cambridge there is one squirrel and one rabbit. Both animals multiply in numbers quickly. In particular, if there are m squirrels and n rabbits in Year k, then there will be $2m+2019$ squirrels and $4n-2$ rabbits in Year $k+1$. What is the first year in which there will be strictly more rabbits than squirrels?

   Proposed by: Yuan Yao

   Answer: $\boxed{13}$

In year k, the number of squirrels is

$$2(2(\cdots(2\cdot 1+2019)+2019)+\cdots)+2019 = 2^k + 2019\cdot(2^{k-1}+2^{k-2}+\cdots+1) = 2020\cdot 2^k - 2019$$

and the number of rabbits is

$$4(4(\cdots(4\cdot 1-2)-2)-\cdots)-2 = 4^k - 2\cdot(4^{k-1}+4^{k-2}+\cdots+1) = \frac{4^k+2}{3}.$$

For the number of rabbits to exceed that of squirrels, we need

$$4^k + 2 > 6060\cdot 2^k - 6057 \Leftrightarrow 2^k > 6059.$$

Since $2^{13} > 6059 > 2^{12}$, $k = 13$ is the first year for which there are more rabbits than squirrels.

12. [**5**] Bob is coloring lattice points in the coordinate plane. Find the number of ways Bob can color five points in $\{(x,y) \mid 1 \le x, y \le 5\}$ blue such that the distance between any two blue points is *not* an integer.

Proposed by: Michael Ren

Answer: $\boxed{80}$

We can see that no two blue points can have the same x or y coordinate. The blue points then must make a permutation of $1,2,3,4,5$ that avoid the pattern of 3-4-5 triangles. It is not hard to use complementary counting to get the answer from here.

There are 8 possible pairs of points that are a distance of 5 apart while not being in the same row or column (i.e. a pair that is in the 3-4-5 position). If such a pair of points is included in the choice of five points, then there are $3! = 6$ possibilities for the remaining three points, yielding $8\times 6 = 48$ configurations that have violations. However, we now need to consider overcounting.

The only way to have more than one violation in one configuration is to have a corner point and then two points adjacent to the opposite corner, e.g. $(1,1),(4,5),(5,4)$. In each such case, there are exactly $2! = 2$ possibilities for the other two points, and there are exactly two violations so there are a total of $2\times 4 = 8$ configurations that are double-counted.

Therefore, there are $48-8 = 40$ permutations that violate the no-integer-condition, leaving $120-40 = 80$ good configurations.

13. [**7**] Reimu has 2019 coins $C_0, C_1, \ldots, C_{2018}$, one of which is fake, though they look identical to each other (so each of them is equally likely to be fake). She has a machine that takes any two coins and picks one that is not fake. If both coins are not fake, the machine picks one uniformly at random. For each $i = 1, 2, \ldots, 1009$, she puts C_0 and C_i into the machine once, and machine picks C_i. What is the probability that C_0 is fake?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{2^{1009}}{2^{1009}+1009}}$

Let E denote the event that C_0 is fake, and let F denote the event that the machine picks C_i over C_0 for all $i = 1, 2, \ldots 1009$. By the definition of conditional probability, $P(E|F) = \frac{P(E\cap F)}{P(F)}$. Since E implies F, $P(E\cap F) = P(E) = \frac{1}{2019}$. Now we want to compute $P(F)$. If C_0 is fake, F is guaranteed to happen. If C_i is fake for some $1 \le i \le 1009$, then F is impossible. Finally, if C_i is fake for some $1010 \le i \le 2018$, then F occurs with probability 2^{-1009}, since there is a $\frac{1}{2}$ probability for each machine decision. Therefore, $P(F) = \frac{1}{2019}\cdot 1 + \frac{1009}{2019}\cdot 0 + \frac{1009}{2019}\cdot 2^{-1009} = \frac{2^{1009}+1009}{2019\cdot 2^{1009}}$. Therefore, $P(E|F) = \frac{2^{1009}}{2^{1009}+1009}$.

14. [**7**] Let ABC be a triangle where $AB = 9, BC = 10, CA = 17$. Let Ω be its circumcircle, and let A_1, B_1, C_1 be the diametrically opposite points from A, B, C, respectively, on Ω. Find the area of the convex hexagon with the vertices A, B, C, A_1, B_1, C_1.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1155}{4}}$

We first compute the circumradius of ABC: Since $\cos A = \frac{9^2-17^2-10^2}{2\cdot 9\cdot 17} = -\frac{15}{17}$, we have $\sin A = \frac{8}{17}$ and $R = \frac{a}{2\sin A} = \frac{170}{16}$. Moreover, we get that the area of triangle ABC is $\frac{1}{2}bc\sin A = 36$.

Note that triangle ABC is obtuse, The area of the hexagon is equal to twice the area of triangle ABC (which is really $[ABC] + [A_1B_1C_1]$) plus the area of rectangle ACA_1C_1. The dimensions of ACA_1C_1 are $AC = 17$ and $A_1C = \sqrt{(2R)^2 - AC^2} = \frac{51}{4}$, so the area of the hexagon is $36 \cdot 2 + 17 \cdot \frac{51}{4} = \frac{1155}{4}$.

15. [**7**] Five people are at a party. Each pair of them are *friends*, *enemies*, or *frenemies* (which is equivalent to being both *friends* and *enemies*). It is known that given any three people A, B, C:

- If A and B are friends and B and C are friends, then A and C are friends;
- If A and B are enemies and B and C are enemies, then A and C are friends;
- If A and B are friends and B and C are enemies, then A and C are enemies.

How many possible relationship configurations are there among the five people?

Proposed by: Yuan Yao

Answer: $\boxed{17}$

If A and B are frenemies, then regardless of whether another person C is friends or enemies with A, C will have to be frenemies with B and vice versa. Therefore, if there is one pair of frenemies then all of them are frenemies with each other, and there is only one possibility.

If there are no frenemies, then one can always separate the five people into two possibly "factions" (one of which may be empty) such that two people are friends if and only if they belong to the same faction. Since the factions are unordered, there are $2^5/2 = 16$ ways to assign the "alignments" that each gives a unique configuration of relations. So in total there are $16 + 1 = 17$ possibilities.

16. [**7**] Let $\mathbb{R}$ be the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers x and y, we have

$$f(x^2) + f(y^2) = f(x+y)^2 - 2xy.$$

Let $S = \sum_{n=-2019}^{2019} f(n)$. Determine the number of possible values of S.

Proposed by: Yuan Yao

Answer: $\boxed{2039191 \text{ OR } \binom{2020}{2} + 1}$

Letting $y = -x$ gives

$$f(x^2) + f(x^2) = f(0)^2 + 2x^2$$

for all x. When $x = 0$ the equation above gives $f(0) = 0$ or $f(0) = 2$.

If $f(0) = 2$, then $f(x) = x + 2$ for all nonnegative x, so the LHS becomes $x^2 + y^2 + 4$, and RHS becomes $x^2 + y^2 + 4x + 4y + 4$ for all $x + y \geq 0$, which cannot be equal to LHS if $x + y > 0$.

If $f(0) = 0$ then $f(x) = x$ for all nonnegative x. Moreover, letting $y = 0$ gives

$$f(x^2) = f(x)^2 \Rightarrow f(x) = \pm x$$

for all x. Since negative values are never used as inputs on the LHS and the output on the RHS is always squared, we may conclude that for all negative x, $f(x) = x$ and $f(x) = -x$ are both possible (and the values are independent). Therefore, the value of S can be written as

$$S = f(0) + (f(1) + f(-1)) + (f(2) + f(-2)) + \cdots + (f(2019) + f(-2019)) = 2\sum_{i=1}^{2019} i\delta_i$$

for $\delta_1, \delta_2, \ldots, \delta_{2019} \in \{0, 1\}$. It is not difficult to see that $\frac{S}{2}$ can take any integer value between 0 and $\frac{2020\cdot 2019}{2} = 2039190$ inclusive, so there are 2039191 possible values of S.

17. [**9**] Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Let A_1, A_2 be points on side BC, B_1, B_2 be points on side CA, and C_1, C_2 be points on side AB. Suppose that there exists a point P such that PA_1A_2, PB_1B_2, and PC_1C_2 are congruent equilateral triangles. Find the area of convex hexagon $A_1A_2B_1B_2C_1C_2$.

Proposed by: Michael Ren

Answer: $\boxed{\frac{12+22\sqrt{3}}{15}}$

Since P is the shared vertex between the three equilateral triangles, we note that P is the incenter of ABC since it is equidistant to all three sides. Since the area is 6 and the semiperimeter is also 6, we can calculate the inradius, i.e. the altitude, as 1, which in turn implies that the side length of the equilateral triangle is $\frac{2}{\sqrt{3}}$. Furthermore, since the incenter is the intersection of angle bisectors, it is easy to see that $AB_2 = AC_1, BC_2 = BA_1$, and $CA_2 = CB_1$. Using the fact that the altitudes from P to AB and CB form a square with the sides, we use the side lengths of the equilateral triangle to compute that $AB_2 = AC_1 = 2 - \frac{1}{\sqrt{3}}$, $BA_1 = BC_2 = 1 - \frac{1}{\sqrt{3}}$, and $CB_1 = CA_2 = 3 - \frac{1}{\sqrt{3}}$. We have that the area of the hexagon is therefore

$$6 - \left(\frac{1}{2}\left(2 - \frac{1}{\sqrt{3}}\right)^2 \cdot \frac{4}{5} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)^2 + \frac{1}{2}\left(3 - \frac{1}{\sqrt{3}}\right)^2 \cdot \frac{3}{5}\right) = \frac{12 + 22\sqrt{3}}{15}.$$

18. [**9**] 2019 points are chosen independently and uniformly at random on the interval $[0, 1]$. Tairitsu picks 1000 of them randomly and colors them black, leaving the remaining ones white. Hikari then computes the sum of the positions of the leftmost white point and the rightmost black point. What is the probability that this sum is at most 1?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1019}{2019}}$

Note that each point is chosen uniformly and independently from 0 to 1, so we can apply symmetry. Given any coloring, suppose that we flip all the positions of the black points: then the problem becomes computing the probability that the leftmost white point is to the left of the leftmost black point, which is a necessary and sufficient condition for the sum of the original leftmost white point and the original rightmost black point being at most 1. This condition, however, is equivalent to the leftmost point of all 2019 points being white. Since there are 1019 white points and 1000 black points and each point is equally likely to be the leftmost, this happens with probability $\frac{1019}{2019}$.

19. [**9**] Complex numbers a, b, c form an equilateral triangle with side length 18 in the complex plane. If $|a + b + c| = 36$, find $|bc + ca + ab|$.

Proposed by: Henrik Boecken

Answer: $\boxed{432}$

Using basic properties of vectors, we see that the complex number $d = \frac{a+b+c}{3}$ is the center of the triangle. From the given, $|a+b+c| = 36 \implies |d| = 12$. Then, let $a' = a - d, b' = b - d$, and $c' = c - d$. Due to symmetry, $|a' + b' + c'| = 0$ and $|b'c' + c'a' + a'b'| = 0$.

Finally, we compute

$$\begin{aligned} |bc + ca + ab| &= |(b' + d)(c' + d) + (c' + d)(a' + d) + (a' + d)(b' + d)| \\ &= |b'c' + c'a' + a'b' + 2d(a' + b' + c') + 3d^2| \\ &= |3d^2| = 3 \cdot 12^2 = 432. \end{aligned}$$

20. [**9**] On floor 0 of a weird-looking building, you enter an elevator that only has one button. You press the button twice and end up on floor 1. Thereafter, every time you press the button, you go up by one floor with probability $\frac{X}{Y}$, where X is your current floor, and Y is the total number of times you have pressed the button thus far (not including the current one); otherwise, the elevator does nothing.

Between the third and the 100^{th} press inclusive, what is the expected number of pairs of consecutive presses that both take you up a floor?

Proposed by: Kevin Yang

Answer: $\boxed{\frac{97}{3}}$

By induction, we can determine that after n total button presses, your current floor is uniformly distributed from 1 to $n-1$: the base case $n=2$ is trivial to check, and for the $n+1$th press, the probability that you are now on floor i is $\frac{1}{n-1}(1-\frac{i}{n})+\frac{1}{n-1}(\frac{i-1}{n})=\frac{1}{n}$ for $i=1,2,\ldots,n$, finishing the inductive step.

Hence, the probability that the $(n+1)$-th and $(n+2)$-th press both take you up a floor is

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{i}{n}\cdot\frac{i+1}{n+1}=\frac{\sum_{i=1}^{n-1}i^2+i}{(n-1)n(n+1)}=\frac{\frac{(n-1)n(2n-1)}{6}+\frac{n(n-1)}{2}}{(n-1)n(n+1)}=\frac{1}{3}.$$

Since there are $100-3=97$ possible pairs of consecutive presses, the expected value is $\frac{97}{3}$.

21. [**12**] A regular hexagon $ABCDEF$ has side length 1 and center O. Parabolas $P_1, P_2, \ldots, P_6$ are constructed with common focus O and directrices AB, BC, CD, DE, EF, FA respectively. Let χ be the set of all distinct points on the plane that lie on at least two of the six parabolas. Compute

$$\sum_{X\in\chi}|OX|.$$

(Recall that the focus is the point and the directrix is the line such that the parabola is the locus of points that are equidistant from the focus and the directrix.)

Proposed by: Yuan Yao

Answer: $\boxed{35\sqrt{3}}$

Recall the focus and the directrix are such that the parabola is the locus of points equidistant from the focus and the directrix. We will consider pairs of parabolas and find their points of intersections (we label counterclockwise):

(1): $P_1\cap P_2$, two parabolas with directrices adjacent edges on the hexagon (sharing vertex A). The intersection inside the hexagon can be found by using similar triangles: by symmetry this X must lie on OA and must have that its distance from AB and FA are equal to $|OX|=x$, which is to say

$$\sin 60^\circ=\frac{\sqrt{3}}{2}=\frac{x}{|OA|-x}=\frac{x}{1-x}\implies x=2\sqrt{3}-3.$$

By symmetry also, the second intersection point, outside the hexagon, must lie on OD. Furthermore, X must have that its distance AB and FA are equal to $|OX|$. Then again by similar triangles

$$\sin 60^\circ=\frac{\sqrt{3}}{2}=\frac{x}{|OA|+x}=\frac{x}{1+x}\implies x=2\sqrt{3}+3.$$

(2): $P_1\cap P_3$, two parabolas with directrices edges one apart on the hexagon, say AB and CD. The intersection inside the hexagon is clearly immediately the circumcenter of triangle BOC (equidistance condition), which gives

$$x=\frac{\sqrt{3}}{3}.$$

Again by symmetry the X outside the hexagon must lie on the lie through O and the midpoint of EF; then one can either observe immediately that $x=\sqrt{3}$ or set up

$$\sin 30^\circ=\frac{1}{2}=\frac{x}{x+\sqrt{3}}\implies x=\sqrt{3}$$

where we notice $\sqrt{3}$ is the distance from O to the intersection of AB with the line through O and the midpoint of BC.

(3): $P_1 \cap P_4$, two parabolas with directrices edges opposite on the hexagon, say AB and DE. Clearly the two intersection points are both inside the hexagon and must lie on CF, which gives

$$x = \frac{\sqrt{3}}{2}.$$

These together give that the sum desired is

$$6(2\sqrt{3}-3) + 6(2\sqrt{3}+3) + 6\left(\frac{\sqrt{3}}{3}\right) + 6(\sqrt{3}) + 6\left(\frac{\sqrt{3}}{2}\right) = 35\sqrt{3}.$$

22. [**12**] Determine the number of subsets S of $\{1, 2, \ldots, 1000\}$ that satisfy the following conditions:

- S has 19 elements, and
- the sum of the elements in any non-empty subset of S is not divisible by 20.

Proposed by: Alec Sun

Answer: $\boxed{8 \cdot \binom{50}{19}}$

First we prove that each subset must consist of elements that have the same residue mod 20. Let a subset consist of elements $a_1, \ldots, a_{19}$, and consider two lists of partial sums

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_{19}$$

$$a_2, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_{19}.$$

The residues mod 20 of the partial sums in each list must be pairwise distinct, otherwise subtracting the sum with less terms from the sum with more terms yields a subset whose sum of elements is 0 (mod 20). Since the residues must also be nonzero, each list forms a complete nonzero residue class mod 20. Since the latter 18 sums in the two lists are identical, $a_1 \equiv a_2 \pmod{20}$. By symmetric arguments, $a_i \equiv a_j \pmod{20}$ for any i, j.

Furthermore this residue $1 \le r \le 20$ must be relatively prime to 20, because if $d = \gcd(r, 20) > 1$ then any $20/d$ elements of the subset will sum to a multiple of 20. Hence there are $\varphi(20) = 8$ possible residues. Since there are 50 elements in each residue class, the answer is $\binom{50}{19}$. We can see that any such subset whose elements are a relatively prime residue $r \pmod{20}$ works because the sum of any $1 \le k \le 19$ elements will be $kr \not\equiv 0 \pmod{20}$.

23. [**12**] Find the smallest positive integer n such that

$$\underbrace{2^{2^{2^{\cdots^{2}}}}}_{n\text{ 2's}} > \underbrace{((\cdots((100!)!)!\cdots)!)!}_{100\text{ factorials}}.$$

Proposed by: Zack Chroman

Answer: $\boxed{104}$

Note that $2^{2^{2^{2}}} > 100^2$. We claim that $a > b^2 \implies 2^a > (b!)^2$, for $b > 2$. This is because

$$2^a > b^{2b} \iff a > 2b\log_2(b),$$

and $\log_2(b) < b^2/2$ for $b > 2$. Then since $b^b > b!$ this bound works. Then

$$\underbrace{(2^{2^{2^{\cdots^{2}}}})}_{m\text{ 2's}} > \underbrace{((((100!)!)!)!\ldots)^2}_{m-4\text{ factorials}}$$

for all $m \geq 4$ by induction. So $n = 104$ works. The lower bound follows from the fact that $n! > 2^n$ for $n > 3$, and since $100 > 2^{2^2}$, we have

$$\underbrace{(((100!)!)!)!\dots)}_{100 \text{ factorials}} > \underbrace{2^{2^{\cdots^{2^{100}}}}}_{100 \text{ 2's}} > \underbrace{2^{2^{\cdots^{2}}}}_{103 \text{ 2's}}.$$

24. [**12**] Let S be the set of all positive factors of 6000. What is the probability of a random quadruple $(a, b, c, d) \in S^4$ satisfies

$$\operatorname{lcm}(\gcd(a, b), \gcd(c, d)) = \gcd(\operatorname{lcm}(a, b), \operatorname{lcm}(c, d))?$$

Proposed by: Yuan Yao

Answer: $\boxed{\frac{41}{512}}$

For each prime factor, let the greatest power that divides a, b, c, d be p, q, r, s. WLOG assume that $p \leq q$ and $r \leq s$, and further WLOG assume that $p \leq r$. Then we need $r = \min(q, s)$. If $q = r$ then we have $p \leq q = r \leq s$, and if $r = s$ then we have $p \leq r = s \leq q$, and in either case the condition reduces to the two "medians" among p, q, r, s are equal. (It is not difficult to see that this condition is also sufficient.)

Now we compute the number of quadruples (p, q, r, s) of integers between 0 and n inclusive that satisfy the above condition. If there are three distinct numbers then there are $\binom{n+1}{3}$ ways to choose the three numbers and $4!/2 = 12$ ways to assign them (it must be a 1-2-1 split). If there are two distinct numbers then there are $\binom{n+1}{2}$ ways to choose the numbers and $4 + 4 = 8$ ways to assign them (it must be a 3-1 or a 1-3 split). If there is one distinct number then there are $n + 1$ ways to assign. Together we have

$$12\binom{n+1}{3} + 8\binom{n+1}{2} + (n+1) = 2(n+1)n(n-1) + 4(n+1)n + (n+1) = (n+1)(2n(n+1)+1)$$

possible quadruples. So if we choose a random quadruple then the probability that it satisfies the condition is $\frac{(n+1)(2n(n+1)+1)}{(n+1)^4} = \frac{2n(n+1)+1}{(n+1)^3}$.

Since $6000 = 2^4 \cdot 5^3 \cdot 3^1$ and the power of different primes are independent, we plug in $n = 4, 3, 1$ to get the overall probability to be

$$\frac{41}{125} \cdot \frac{25}{64} \cdot \frac{5}{8} = \frac{41}{512}.$$

25. [**15**] A 5 by 5 grid of unit squares is partitioned into 5 pairwise incongruent rectangles with sides lying on the gridlines. Find the maximum possible value of the product of their areas.

Proposed by: Yuan Yao

Answer: $\boxed{2304}$

The greatest possible value for the product is $3 \cdot 4 \cdot 4 \cdot 6 \cdot 8 = 2304$, achieved when the rectangles are $3 \times 1, 1 \times 4, 2 \times 2, 2 \times 3, 4 \times 2$. To see that this is possible, orient these rectangles so that the first number is the horizontal dimension and the second number is the vertical dimension. Then, place the bottom-left corners of these rectangles at $(2, 4), (4, 0), (2, 2), (0, 2), (0, 0)$ respectively on the grid.

We will now prove that no larger product can be achieved. Suppose that there is at least one rectangle of area at most 2. Then the product is at most $2 \cdot 5.75^4 = 2 \cdot 33.0625^2 < 2 \cdot 1100 = 2200$ by AM-GM. Now suppose that there is at least one rectangle of area at least 9. Then the product is at most $9 \cdot 4^4 = 2304$ by AM-GM. (Neither of these is tight, since you cannot have non-integer areas, nor can you have four rectangles all of area 4.)

Now consider the last possibility that is not covered by any of the above: that there are no rectangles of size at most 2 and no rectangles of area at least 9. There can be at most one rectangle of area $3, 5, 6, 8$ each, at most two rectangles of area 4, and no rectangles of area 7. The only way to achieve a sum of 25 with these constraints is $3, 4, 4, 6, 8$, which produces a product of 2304. We have shown through the earlier cases that a larger product cannot be achieved, so this is indeed the maximum.

26. [**15**] Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let I_A, I_B, I_C be the A, B, C excenters of this triangle, and let O be the circumcenter of the triangle. Let $\gamma_A, \gamma_B, \gamma_C$ be the corresponding excircles and ω be the circumcircle. X is one of the intersections between γ_A and ω. Likewise, Y is an intersection of γ_B and ω, and Z is an intersection of γ_C and ω. Compute

$$\cos \angle OXI_A + \cos \angle OYI_B + \cos \angle OZI_C.$$

Proposed by: Andrew Gu

Answer: $\boxed{-\frac{49}{65}}$

Let r_A, r_B, r_C be the exradii. Using $OX = R, XI_A = r_A, OI_A = \sqrt{R(R+2r_A)}$ (Euler's theorem for excircles), and the Law of Cosines, we obtain

$$\cos \angle OXI_A = \frac{R^2 + r_A^2 - R(R+2r_A)}{2Rr_A} = \frac{r_A}{2R} - 1.$$

Therefore it suffices to compute $\frac{r_A+r_B+r_C}{2R} - 3$. Since

$$r_A + r_B + r_C - r = 2K\left(\frac{1}{-a+b+c} + \frac{1}{a-b+c} + \frac{1}{a+b-c} - \frac{1}{a+b+c}\right) = 2K\frac{8abc}{(4K)^2} = \frac{abc}{K} = 4R$$

where $K = [ABC]$, this desired quantity the same as $\frac{r}{2R} - 1$. For this triangle, $r = 4$ and $R = \frac{65}{8}$, so the answer is $\frac{4}{65/4} - 1 = -\frac{49}{65}$.

27. [**15**] Consider the eighth-sphere $\{(x, y, z) \mid x, y, z \geq 0,\ x^2 + y^2 + z^2 = 1\}$. What is the area of its projection onto the plane $x + y + z = 1$?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{\pi\sqrt{3}}{4}}$

Consider the three flat faces of the eighth-*ball*. Each of these is a quarter-circle of radius 1, so each has area $\frac{\pi}{4}$. Furthermore, the projections of these faces cover the desired area without overlap. To find the projection factor one can find the cosine of the angle θ between the planes, which is the same as the angle between their normal vectors. Using the dot product formula for the cosine of the angle between two vectors, $\cos\theta = \frac{(1,0,0)\cdot(1,1,1)}{|(1,0,0)||(1,1,1)|} = \frac{1}{\sqrt{3}}$. Therefore, each area is multiplied by $\frac{1}{\sqrt{3}}$ by the projection, so the area of the projection is $3 \cdot \frac{\pi}{4} \cdot \frac{1}{\sqrt{3}} = \frac{\pi\sqrt{3}}{4}$.

28. [**15**] How many positive integers $2 \leq a \leq 101$ have the property that there exists a positive integer N for which the last two digits in the decimal representation of a^{2^n} is the same for all $n \geq N$?

Proposed by: Pakawut Jiradilok

Answer: $\boxed{36}$

Solution 1. It suffices to consider the remainder mod 100. We start with the four numbers that have the same last two digits when squared: 0, 1, 25, 76.

We can now go backwards, repeatedly solving equations of the form $x^2 \equiv n \pmod{100}$ where n is a number that already satisfies the condition.

0 and 25 together gives all multiples of 5, for 20 numbers in total.

1 gives $1, 49, 51, 99$, and 49 then gives $7, 43, 57, 93$. Similarly 76 gives $24, 26, 74, 76$, and 24 then gives $18, 32, 68, 82$, for 16 numbers in total.

Hence there are $20 + 16 = 36$ such numbers in total.

Solution 2. An equivalent formulation of the problem is to ask for how many elements of $\mathbb{Z}_{100}$ the map $x \mapsto x^2$ reaches a fixed point. We may separately solve this modulo 4 and modulo 25.

Modulo 4, it is easy to see that all four elements work.

Modulo 25, all multiples of 5 will work, of which there are 5. For the remaining 25 elements that are coprime to 5, we may use the existence of a primitive root to equivalently ask for how many elements of $\mathbb{Z}_{20}$ the map $y \mapsto 2y$ reaches a fixed point. The only fixed point is 0, so the only valid choices are the multiples of 5 again. There are $5 + 4 = 9$ solutions here.

Finally, the number of solutions modulo 100 is $4 \times 9 = 36$.

29. [**20**] Yannick picks a number N randomly from the set of positive integers such that the probability that n is selected is 2^{-n} for each positive integer n. He then puts N identical slips of paper numbered 1 through N into a hat and gives the hat to Annie. Annie does not know the value of N, but she draws one of the slips uniformly at random and discovers that it is the number 2. What is the expected value of N given Annie's information?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1}{2\ln 2-1}}$

Let S denote the value drawn from the hat. The probability that 2 is picked is $\frac{1}{n}$ if $n \geq 2$ and 0 if $n = 1$. Thus, the total probability X that 2 is picked is

$$P(S = 2) = \sum_{k=2}^{\infty} \frac{2^{-k}}{k}.$$

By the definition of conditional probability, $P(N = n|S = 2) = \frac{P(N=n,S=2)}{P(S=2)} = \frac{2^{-n}/n}{X}$ if $n \geq 2$ and 0 if $n = 1$. Thus the conditional expectation of N is

$$\mathbb{E}[N|S = 2] = \sum_{n=1}^{\infty} n \cdot P(N = n|S = 2) = \sum_{n=2}^{\infty} n \cdot \frac{2^{-n}/n}{X} = \frac{1}{X}\sum_{n=2}^{\infty} 2^{-n} = \frac{1}{2X}.$$

It remains to compute X. Note that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$. Integrating both sides with respect to x yields

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1 - x) + C$$

for some constant C, and plugging in $x = 0$ shows that $C = 0$. Plugging in $x = \frac{1}{2}$ shows that $\sum_{k=1}^{\infty} \frac{2^{-k}}{k} = \ln 2$. Note that X is exactly this summation but without the first term. Thus, $X = \ln 2 - \frac{1}{2}$, so $\frac{1}{2X} = \frac{1}{2\ln 2-1}$.

30. [**20**] Three points are chosen inside a unit cube uniformly and independently at random. What is the probability that there exists a cube with side length $\frac{1}{2}$ and edges parallel to those of the unit cube that contains all three points?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1}{8}}$

Let the unit cube be placed on a xyz-coordinate system, with edges parallel to the x, y, z axes. Suppose the three points are labeled A, B, C. If there exists a cube with side length $\frac{1}{2}$ and edges parallel to the edges of the unit cube that contain all three points, then there must exist a segment of length $\frac{1}{2}$ that contains all three projections of A, B, C onto the x-axis. The same is true for the y- and z- axes. Likewise, if there exists segments of length $\frac{1}{2}$ that contains each of the projections of A, B, C onto the x,y, and z axes, then there must exist a unit cube of side length $\frac{1}{2}$ that contains A, B, C. It is easy to see that the projection of a point onto the x-axis is uniform across a segment of length 1, and that each of the dimensions are independent. The problem is therefore equivalent to finding the cube of the probability that a segment of length $\frac{1}{2}$ can cover three points chosen randomly on a segment of length 1.

Note that selecting three numbers $p < q < r$ uniformly and independently at random from 0 to 1 splits the number line into four intervals. That is, we can equivalently sample four positive numbers a, b, c, d

uniformly satisfying $a+b+c+d=1$ (here, we set $a=p, b=q-p, c=r-q, d=1-r$). The probability that the points p, q, r all lie on a segment of length $\frac{1}{2}$ is the probability that $r-q \leq \frac{1}{2}$, or $b+c \leq \frac{1}{2}$. Since $a+d$ and $b+c$ are symmetric, we have that this probability is $\frac{1}{2}$ and our final answer is $(\frac{1}{2})^3 = \frac{1}{8}$.

31. [**20**] Let ABC be a triangle with $AB=6$, $AC=7$, $BC=8$. Let I be the incenter of ABC. Points Z and Y lie on the interior of segments AB and AC respectively such that YZ is tangent to the incircle. Given point P such that

$$\angle ZPC = \angle YPB = 90^\circ,$$

find the length of IP.

Proposed by: Zack Chroman

Answer: $\boxed{\frac{\sqrt{30}}{2}}$

Solution 1. Let PU, PV tangent from P to the incircle. We will invoke the *dual of the Desargues Involution Theorem*, which states the following:

Given a point P in the plane and four lines $\ell_1, \ell_2, \ell_3, \ell_4$, consider the set of conics tangent to all four lines. Then we define a function on the pencil of lines through P by mapping one tangent from P to each conic to the other. This map is well defined and is a projective involution, and in particular maps $PA \to PD, PB \to PE, PC \to PF$, where $ABCDEF$ is the complete quadrilateral given by the pairwise intersections of $\ell_1, \ell_2, \ell_3, \ell_4$. ■

An overview of the projective background behind the (Dual) Desargues Involution Theorem can be found here: `https://www.scribd.com/document/384321704/Desargues-Involution-Theorem`, and a proof can be found at `https://www2.washjeff.edu/users/mwoltermann/Dorrie/63.pdf`.

Now, we apply this to the point P and the lines AB, AC, BC, YZ, to get that the pairs

$$(PU, PV), (PY, PB), (PZ, PC)$$

are swapped by some involution. But we know that the involution on lines through P which rotates by 90° swaps the latter two pairs, thus it must also swap the first one and $\angle UPV = 90$. It follows by equal tangents that $IUPV$ is a square, thus $IP = r\sqrt{2}$ where r is the inradius of ABC. Since $r = \frac{2K}{a+b+c} = \frac{21\sqrt{15}/2}{21} = \frac{\sqrt{15}}{2}$, we have $IP = \frac{\sqrt{30}}{2}$.

Solution 2. Let H be the orthocenter of ABC.

Lemma. $HI^2 = 2r^2 - 4R^2\cos(A)\cos(B)\cos(C)$, where r is the inradius and R is the circumradius.

Proof. This follows from barycentric coordinates or the general result that for a point X in the plane,

$$aXA^2 + bXB^2 + cXC^2 = (a+b+c)XI^2 + aAI^2 + bBI^2 + cCI^2,$$

which itself is a fact about vectors that follows from barycentric coordinates. This can also be computed directly using trigonometry.

Let $E = BH \cap AC, F = CH \cap AB$, then note that B, P, E, Y are concyclic on the circle of diameter BY, and C, P, F, Z are concyclic on the circle of diameter CZ. Let Q be the second intersection of these circles. Since $BCYZ$ is a tangential quadrilateral, the midpoints of BY and CZ are collinear with I (this is known as *Newton's theorem*), which implies that $IP = IQ$ by symmetry. Note that as $BH \cdot HE = CH \cdot HF$, H lies on the radical axis of the two circles, which is PQ. Thus, if $IP = IQ = x$, $BH \cdot HE$ is the power of H with respect to the circle centered at I with radius x, which implies $BH \cdot HE = x^2 - HI^2$.

As with the first solution, we claim that $x = r\sqrt{2}$, which by the lemma is equivalent to $BH \cdot HE = 4R^2\cos(A)\cos(B)\cos(C)$. Then note that

$$BH \cdot HE = BH \cdot CH\cos(A) = (2R\cos(B))(2R\cos(C))\cos(A),$$

so our claim holds and we finish as with the first solution.

Note. Under the assumption that the problem is well-posed (the answer does not depend on the choice of Y, Z, or P), then here is an alternative method to obtain $IP = r\sqrt{2}$ by making convenient choices. Let U be the point where YZ is tangent to the incircle, and choose U so that $IU \parallel BC$ (and therefore $YZ \perp BC$). Note that $YZ \cap BC$ is a valid choice for P, so assume that P is the foot from U to BC. If D is the point where BC is tangent to the incircle, then $IUPD$ is a square so $IP = r\sqrt{2}$. (This disregards the condition that Y and Z are in the interior of segments AC and AB, but there is no reason to expect that this condition is important.)

32. [**20**] For positive integers a and b such that a is coprime to b, define $\text{ord}_b(a)$ as the least positive integer k such that $b \mid a^k - 1$, and define $\varphi(a)$ to be the number of positive integers less than or equal to a which are coprime to a. Find the least positive integer n such that

$$\text{ord}_n(m) < \frac{\varphi(n)}{10}$$

for all positive integers m coprime to n.

Proposed by: Andrew Gu

Answer: $\boxed{240}$

The maximum order of an element modulo n is the Carmichael function, denoted $\lambda(n)$. The following properties of the Carmichael function are established:

- For primes $p > 2$ and positive integers k, $\lambda(p^k) = (p-1)p^{k-1}$.
- For a positive integer k,

$$\lambda(2^k) = \begin{cases} 2^{k-2} & \text{if } k \geq 3 \\ 2^{k-1} & \text{if } k \leq 2 \end{cases}.$$

- For a positive integer n with prime factorization $n = \prod p_i^{k_i}$,

$$\lambda(n) = \text{lcm}\left(\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots\right)$$

Meanwhile, for $n = \prod p_i^{k_i}$, we have $\varphi(n) = \prod (p_i - 1)p_i^{k_i - 1}$. Hence the intuition is roughly that the $(p_i - 1)p_i^{k_i-1}$ terms must share divisors in order to reach a high value of $\frac{\varphi(n)}{\lambda(n)}$.

We will now show that $n \geq 240$ by doing casework on the prime divisors of $z = \frac{\varphi(n)}{\lambda(n)}$. Suppose $p \mid z$ and $p > 2$. This requires two terms among $\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots$ to be multiples of p because $\lambda(n)$ is the lcm of the terms whereas the product of these numbers has the same number of factors of p as $\varphi(n)$ (note that this does not hold for $p = 2$ because $\lambda(2^k) \neq 2^{k-1}$ in general). These correspond to either $p^2 \mid n$ or $q \mid n$ with $q \equiv 1 \pmod{p}$. Therefore

$$n \geq \max(p^2(2p+1), (2p+1)(4p+1))$$

because the smallest primes congruent to $1 \pmod{p}$ are at least $2p+1$ and $4p+1$. For $p \geq 5$ this gives $n > 240$, so we may assume $p \leq 3$.

First we address the case $p = 3$. This means that two numbers among $9, 7, 13, 19, 31, 37, \dots$ divide n. As $7 \times 37 > 240$, we discard primes greater than 31. Of the remaining numbers, we have

$$\lambda(9) = 6, \lambda(7) = 6, \lambda(13) = 12, \lambda(19) = 18, \lambda(31) = 30.$$

No candidate value of n is the product of just two of these numbers as the gcd of any two of the associated λ values is at most 6. Furthermore, multiplying by just 2 will not affect $\varphi(n)$ or $\lambda(n)$, so we must multiply at least two of these numbers by a number greater than 2. Throwing out numbers greater than 240, this leaves only $3 \times 9 \times 7$, which does not work. (A close candidate is $3 \times 7 \times 13 = 273$, for which $\varphi(n) = 144, \lambda(n) = 12$.)

The remaining case is when the only prime divisors of $\frac{\varphi(n)}{\lambda(n)}$ are 2. It is not hard to see that $\lambda(n) \geq 4$ when $n \nmid 24$ (and when $n \mid 24$ it's clear that $\phi(n) \leq 8$, so we do not need to consider them). When $\lambda(n) = 4$, we need $\varphi(n) \geq 4 \cdot 2^4 = 64$ and $v_2(n) \leq 4$, so the smallest such integer is $n = 2^4 \cdot 3 \cdot 5 = 240$, which we can check does indeed satisfy $\frac{\varphi(n)}{\lambda(n)} > 10$. It is not difficult to check that higher values of $\lambda(n)$ will not yield any n below 240, so 240 is indeed the smallest possible n.

Note: The sequence $\frac{\varphi(n)}{\lambda(n)}$ is given by A034380 in the OEIS.

H_1. Let $r = H_1$ be the answer to this problem. Given that r is a nonzero real number, what is the value of $r^4 + 4r^3 + 6r^2 + 4r$?

Answer: $\boxed{-1}$

Since H_1 is the answer, we know $r^4 + 4r^3 + 6r^2 + 4r = r \Rightarrow (r+1)^4 = r + 1$. Either $r + 1 = 0$, or $(r+1)^3 = 1 \Rightarrow r = 0$. Since r is nonzero, $r = -1$.

H_2. Given two distinct points A, B and line ℓ that is not perpendicular to AB, what is the maximum possible number of points P on ℓ such that ABP is an isosceles triangle?

Answer: $\boxed{5}$

In an isosceles triangle, one vertex lies on the perpendicular bisector of the opposite side. Thus, either P is the intersection of AB and ℓ, or P lies on the circle centered at A with radius AB, or P lies on the circle centered at B with radius AB. Each circle-line intersection has at most two solutions, and the line-line intersection has at most one, giving 5. This can be easily constructed by taking any $\overline{AB}$, and taking ℓ that isn't a diameter but intersects both relevant circles twice.

H_3. Let $A = H_1, B = H_6 + 1$. A real number x is chosen randomly and uniformly in the interval $[A, B]$. Find the probability that $x^2 > x^3 > x$.

Answer: $\boxed{\frac{1}{4}}$

$A = -1, B = 3$. For $x^3 > x$, either $x > 1$ or $-1 < x < 0$. However, for $x > 1$, $x^2 < x^3$, so there are no solutions. $-1 < x < 0$ also satisfies $x^2 > x^3$, so our answer is $1/4$.

H_4. Let $A = \lceil 1/H_3 \rceil, B = \lceil H_5/2 \rceil$. How many ways are there to partition the set $\{1, 2, \ldots, A + B\}$ into two sets U and V with size A and B respectively such that the probability that a number chosen from U uniformly at random is greater than a number chosen from V uniformly at random is exactly $\frac{1}{2}$?

Answer: $\boxed{24}$

$A = 4, B = 7$. There are 28 total ways of choosing an element from U and V, so there must be 14 ways where U's is larger. If we relabel the elements to be $0, 1, \cdots, 10$, then element i is greater than exactly i elements in the set. However, we overcount other elements in U, so the four elements in $U = \{a, b, c, d\}$ must satisfy

$$(a - 0) + (b - 1) + (c - 2) + (d - 3) = 14 \Rightarrow a + b + c + d = 20.$$

To remove the uniqueness condition, we subtract 1 from b, 2 from c, and 3 from d, so we wish to find solutions $a \leq b \leq c \leq d \leq 7$ to $a + b + c + d = 14$. From here, we do casework. If $a = 0$, $b = 0, 1, 2, 3, 4$ give $1, 1, 2, 2, 3$ solutions, respectively. If $a = 1$, $b = 1, 2, 3, 4$ give $2, 2, 3, 1$ solutions, respectively. If $a = 2$, $b = 2, 3, 4$ give $3, 2, 1$ solutions, respectively. If $a = 3$, the only solution is $3, 3, 4, 4$. Thus, the answer is $(1 + 1 + 2 + 2 + 3) + (2 + 2 + 3 + 1) + (3 + 2 + 1) + 1 = 24$.

H_5. Let $A = H_2, B = H_7$. Two circles with radii A and B respectively are given in the plane. If the length of their common external tangent is twice the length of their common internal tangent (where both tangents are considered as segments with endpoints being the points of tangency), find the distance between the two centers.

Answer: $\boxed{\frac{2\sqrt{429}}{3}}$

Let the distance between the centers be d. The length of the common external tangent is $E = \sqrt{d^2 - (7-5)^2} = \sqrt{d^2 - 4}$, and the length of the internal tangent is $I = \frac{12}{5}\sqrt{(\frac{5}{12}d)^2 - 5^2}$. Solving the equation $E = 2I$ gives $d = \frac{2\sqrt{429}}{3} (\approx 13.8)$.

H_6. How many ways are there to arrange the numbers $21, 22, 33, 35$ in a row such that any two adjacent numbers are relatively prime?

Answer: $\boxed{2}$

21 cannot be adjacent to 33 or 35, so it must be on one end bordering 22. 33 cannot be adjacent to 21 or 22, so it must be on the other end bording 35. Thus, there are only 2 orderings: 21, 22, 35, 33, and 33, 35, 22, 21.

H_7. How many pairs of integers (x, y) are there such that $|x^2 - 2y^2| \leq 1$ and $|3x - 4y| \leq 1$?

Answer: $\boxed{7}$

Note that if (x, y) is a solution, so is $(-x, -y)$. Thus, we consider $x \geq 0$.

When $x \equiv 0 \pmod 4$, $y = 3x/4$ by inequality 2. Inequality 1 gives $|x^2/9| \leq 1$, so $x \leq 3$, so $x = 0$.

When $x \equiv 1 \pmod 4$, $y = (3x+1)/4$ by inequality 2. Beyond $x = 1$, $2y^2 - x^2 > 1$, so there are no more solutions.

When $x \equiv 2 \pmod 4$, there are no solutions for y.

When $x \equiv 3 \pmod 4$, $y = (3x-1)/4$ by inequality 2. Beyond $x = 7$, $2y^2 - x^2 > 1$, so there are no more solutions.

Thus, the solutions are $(0, 0), (1, 1), (3, 2), (7, 5)$, and the negations of the latter three, giving 7 solutions.

M_1. Let $S = M_{10}$. Determine the number of ordered triples (a, b, c) of nonnegative integers such that $a + 2b + 4c = S$.

Answer: $\boxed{196}$

$S = 53$. Firstly, the number of solutions is the same as the number of solutions to $a + 2b + 4c = 52$, since $2b, 4c$ are both even. Then, $a + 2b = 2x$ has $x + 1$ solutions in nonnegative integers, so we wish to find $27 + 25 + \cdots + 1$. This is the sum of the first 14 odd numbers, which is $14^2 = 196$.

M_2. Let $S = \lfloor M_5 \rfloor$. Two integers m and n are chosen between 1 and S inclusive uniformly and independently at random. What is the probability that $m^n = n^m$?

Answer: $\boxed{\frac{7}{72}}$

$S = 12$. The solutions are (x, x) for all x, and $(2, 4), (4, 2)$. Thus, there are $S + 2 = 14$ solutions out of $S^2 = 196$ possibilities, so the answer is $14/144 = 7/72$.

M_3. Let $S = \lceil M_7 \rceil$. In right triangle ABC, $\angle C = 90°$, $AC = 27, BC = 36$. A circle with radius S is tangent to both AC and BC and intersects AB at X and Y. Find the length of XY.

Answer: $\boxed{16\sqrt{3}}$

$S = 14$. We first note that the distance from the center of the circle to AB (which has length 45 by Pythagorean theorem) is $\frac{27 \cdot 36 - 27 \cdot 14 - 36 \cdot 14}{45} = 2$, so the length of the chord XY is equal to $2\sqrt{14^2 - 2^2} = 16\sqrt{3}$.

M_4. Let $S = M_{13} + 5$. Compute the product of all positive divisors of S.

Answer: $\boxed{810000}$

$S = 30 = 2 \cdot 3 \cdot 5$. The divisors of S are $1, 2, 3, 5, 6, 10, 15, 30$. Each prime factor appears 4 times, so the product is $2^4 3^4 5^4 = 30^4 = 810000$.

M_5. Let $A = \sqrt{M_1}, B = \lceil M_{11} \rceil$. Given complex numbers x and y such that $x + \frac{1}{y} = A$, $\frac{1}{x} + y = B$, compute the value of $xy + \frac{1}{xy}$.

Answer: $\boxed{12}$

$A = 14, B = 1$. Multiplying the two given equations gives $xy + 1/(xy) + 2 = 14$, so the answer is $14 - 2 = 12$.

M_6. Let $A = \lfloor 1/M_2 \rfloor, B = \lfloor M_3^2/100 \rfloor$. Let P and Q both be quadratic polynomials. Given that the real roots of $P(Q(x)) = 0$ are $0, A, B, C$ in some order, find the sum of all possible values of C.

Answer: $\boxed{17}$

$A = 10, B = 7$. Let the roots of $P(x) = 0$ be p, q. Then, the roots of $P(Q(x)) = 0$ are when $Q(x) = p$ or $Q(x) = q$. If these are r, s, t, u in order, then note that $(r+s)/2 = (t+u)/2$ must both be the center of the parabola Q. Thus, $0, 10, 7, C$ must divide into two pairs with equal sum. When we consider the three possible groupings, we get that the possible values are $10+7-0 = 17, 10+0-7 = 3, 7+0-10 = -3$. Therefore the sum is $17 + (-3) + 3 = 17$.

M_7. Let $A = \lceil \log_2 M_4 \rceil, B = M_{12} + 1$. A 5-term sequence of positive reals satisfy that the first three terms and the last three terms both form an arithmetic sequence and the middle three terms form a geometric sequence. If the first term is A and the fifth term is B, determine the third term of the sequence.

Answer: $\boxed{\frac{40}{3}}$

$A = 20, B = 8$. If the middle term is x, then $(x + 20)/2, x, (x + 8)/2$ forms a geometric series.This means that $(x+20)/2 \cdot (x+8)/2 = x^2$, which upon solving gives $x = 40/3$ or $x = -4$ (which we discard because $x > 0$).

M_8. Let $A = \lfloor M_5^2 \rfloor, B = \lfloor M_6^2 \rfloor$. A regular A-gon, a regular B-gon, and a circle are given in the plane. What is the greatest possible number of regions that these shapes divide the plane into?

Answer: $\boxed{1156}$

$A = 144, B = 289$. First, note that with only the circle, there are 2 regions. If the three shapes never coincide at a point, then each intersection adds precisely one region. Optimistically, we wish to have the maximal number of intersections where all intersections have both shapes. The maximum number of intersections between the 289-gon and the circle is 578, since each side can only intersect the circle twice. Similarly, the 144-gon and the circle add at most 288. Finally, each side of the 144-gon can only intersect the 289-gon twice, so this adds another 288. This maximum can be achieved when all three shapes have the same circumcenter and circumradius, and are rotated slightly. The answer is $2 + 598 + 288 + 288 = 1156$.

M_9. Let A and B be the unit digits of $\lceil 7M_6 \rceil$ and $\lfloor 6M_7 \rfloor$ respectively. When all the positive integers not containing digit A or B are written in increasing order, what is the 2019^{th} number in the list?

Answer: $\boxed{3743}$

$A = 9, B = 0$. First, there are 8 numbers with 1 digit, 64 with two digits, and 512 with three digits. This leaves $2019 - 512 - 64 - 8 = 1435$ of four-digit numbers we have to go through, starting with 1111. Since we don't have two digits 0 and 9, we are basically counting in base 8, where the digits 01234567 in base 8 are actually 12345678, and $1111_{10} = 0000_8$. Converting 1435 to base 8, we get 2633_8, and mapping this back, we get 3744. However, we must remember that 1111 is 0_8 under our mapping, so the 1435^{th} four-digit number is actually 3743.

M_{10}. What is the smallest positive integer with remainder $2, 3, 4$ when divided by $3, 5, 7$ respectively?

Answer: $\boxed{53}$

We note that if we double the number then it leaves a remainder of 1 when divided by all of 3, 5, and 7. The smallest even number satisfying this is $3 \cdot 5 \cdot 7 + 1 = 106$, so the smallest possible number is $106/2 = 53$.

M_{11}. An equiangular hexagon has side lengths $1, 2, 3, 4, 5, 6$ in some order. Find the nonnegative difference between the largest and the smallest possible area of this hexagon.

Answer: $\boxed{\frac{\sqrt{3}}{2}}$

Extending three sides of the equiangular hexagon gives an equilateral triangle. Thus, if the sides are a, b, c, d, e, f, in order, then $a + b + c = a + f + e \Rightarrow b + c = e + f \Rightarrow f - c = b - e$. By a symmetric argument, we see that $d - a = f - c = b - e$ holds, which means that they must be separated into three groups of two with equal differences. If the grouping is $(1, 2), (3, 4), (5, 6)$, then we have $1, 4, 5, 2, 3, 6$ around the hexagon. If the grouping is $(1, 4), (2, 5), (3, 6)$, then we get $1, 5, 3, 4, 2, 6$ as the other possibility. Finally, we can use our equilateral triangle trick to find the areas. For the first, we get a big triangle of side $1 + 4 + 5 = 10$, and must subtract smaller triangles of sides $1, 5, 3$. This gives $(100 - 1 - 25 - 9)\sqrt{3}/4 = 65\sqrt{3}/4$. For the other, we get $(81 - 1 - 9 - 4)\sqrt{3}/4 = 67\sqrt{3}/4$. The positive difference between these is $\sqrt{3}/2$.

M_{12}. Determine the second smallest positive integer n such that $n^3 + n^2 + n + 1$ is a perfect square.

Answer: $\boxed{7}$

$n^3 + n^2 + n + 1 = (n + 1)(n^2 + 1)$. Note that $\gcd(n^2 + 1, n + 1) = \gcd(2, n + 1) = 1$ or 2, and since $n^2 + 1$ is not a perfect square for $n \geq 1$, we must have $n^2 + 1 = 2p^2$ and $n + 1 = 2q^2$ for some integers p and q. The first equation is a variant of Pell's equation, which (either by brute-forcing small cases or using the known recurrence) gives solutions $(n, p) = (1, 1), (7, 5), \ldots$. Incidentally, both smallest solutions $n = 1$ and $n = 7$ allows an integer solution to the second equation, so $n = 7$ is the second smallest integer that satisfy the condition.

M_{13}. Given that A, B are nonzero base-10 digits such that $A \cdot \overline{AB} + B = \overline{BB}$, find $\overline{AB}$.

Answer: $\boxed{25}$

We know $A \cdot \overline{AB}$ ends in 0. Since neither is 0, they must be 2, 5 in some order. We easily find that $A = 2$, $B = 5$ works while the opposite doesn't, so $\overline{AB} = 25$.

T_1. Let S, P, A, C, E be (not necessarily distinct) decimal digits where $E \neq 0$. Given that $N = \sqrt{\overline{ESCAPE}}$ is a positive integer, find the minimum possible value of N.

Answer: $\boxed{319}$

Since $E \neq 0$, the 6-digit number $\overline{ESCAPE}$ is at least 10^5, so $N \geq 317$. If N were 317 or 318, the last digit of N^2 would not match the first digit of N^2, which contradicts the condition. However, $N = 319$ will work, since the first and last digit of N^2 are both 1.

T_2. Let $X = \lfloor T_1/8 \rfloor, Y = T_3 - 1, Z = T_4 - 2$. A point P lies inside the triangle ABC such that $PA = X, PB = Y, PC = Z$. Find the largest possible area of the triangle.

Answer: $\boxed{1344}$

$X = 39, Y = 33, Z = 25$. Fix some position for P, A, and B, and we shall find the optimal position for C. Letting $\overline{AB}$ be the base of the triangle, we wish to maximize the height. The legal positions for C are a subset of the circle with center P and radius PC, so the height is maximized when $\overline{PC}$ is orthogonal to $\overline{AB}$. Symmetrically, we deduce that P is the orthocenter of ABC when the area is maximized (moreover, P must be inside the triangle). If ray BP intersects AC at E, then since AEB is similar to PEC, we have

$$\frac{\sqrt{39^2 - x^2}}{33 + x} = \frac{\sqrt{25^2 - x^2}}{x} \Rightarrow x^2(x + 33)^2 = (39^2 - x^2)(25^2 - x^2) \Rightarrow 66x^3 + 3235x^2 - 950625 = 0.$$

The LHS factors to $(x - 15)(66x^2 + 4225x + 63375)$, meaning that $x = 15$ is the only positive solution, giving $AE = 36, BE = 20$, and therefore the maximum area of triangle ABC is $(33 + 15)(36 + 20)/2 = 1344$.

T_3. How many ways can one tile a 2×8 board with 1×1 and 2×2 tiles? Rotations and reflections of the same configuration are considered distinct.

Answer: $\boxed{34}$

Let $f(n)$ denote the number of ways to fill a $2 \times n$ board. One can fill the leftmost column with two 1×1 tiles, leaving $f(n-1)$ ways, or one can fill the leftmost two columns with one 2×2 tile, leaving $f(n-2)$ ways. Therefore, $f(n) = f(n-1) + f(n-2)$. One can also directly verify that $f(0) = f(1) = 1$. Therefore, $f(n) = F_{n+1}$, where F_n is the n^{th} Fibonacci number. Easy calculation shows that the desired quantity is $f(8) = F_9 = 34$.

T_4. Let $S = T_5$. Given real numbers a, b, c such that $a^2 + b^2 + c^2 + (a+b+c)^2 = S$, find the maximum possible value of $(a+b)(b+c)(c+a)$.

Answer: $\boxed{27}$

Notice that $S = 27 = a^2 + b^2 + c^2 + (a+b+c)^2 = (a+b)^2 + (b+c)^2 + (c+a)^2$. By AM-GM, $\frac{S}{3} \geq ((a+b)(b+c)(c+a))^{2/3}$ with equality if and only if $a+b = b+c = c+a$, i.e. $a = b = c$. Thus, the maximum possible value is $\left(\frac{S}{3}\right)^{3/2} = 27$, achieved at $a = b = c = \frac{3}{2}$.

T_5. A regular tetrahedron has volume 8. What is the volume of the set of all the points in the space (*not necessarily inside the tetrahedron*) that are closer to the center of the tetrahedron than any of the four vertices?

Answer: $\boxed{27}$

Let h denote the height of the tetrahedron. The center of the tetrahedron is a distance $\frac{h}{4}$ from each face. Therefore, the perpendicular bisector plane of the segment connecting the center to a vertex lies a distance $\frac{3}{8}h$ away from both the vertex and the center. Symmetrical considerations with the other three vertices will thus show that the desired region is also a regular tetrahedron, with the center of the original tetrahedron a distance $\frac{3}{8}h$ away from each face. Based on the distance from the center to a face, one can see that the scale factor of this tetrahedron is $\frac{3h}{8} : \frac{h}{4} = 3 : 2$ relative to the original tetrahedron, so its volume is $8 \cdot \left(\frac{3}{2}\right)^3 = 27$.

33. [**25**] Determine the value of H_4.

Proposed by: Yuan Yao

Answer: $\boxed{24}$

34. [**25**] Determine the value of M_8.

Proposed by: Yuan Yao

Answer: $\boxed{1156}$

35. [**25**] Determine the value of M_9.

Proposed by: Yuan Yao

Answer: $\boxed{3743}$

36. [**25**] Determine the value of T_2.

Proposed by: Yuan Yao

Answer: $\boxed{1344}$