HMMT November 2012

Saturday 10 November 2012

Team Round

1. [**3**] Find the number of integers between 1 and 200 inclusive whose distinct prime divisors sum to 16. (For example, the sum of the distinct prime divisors of 12 is $2+3=5$.)

   Answer: $\boxed{6}$ The primes less than 16 are 2, 3, 5, 7, 11, and 13. We can write 16 as the sum of such primes in three different ways and find the integers less than 200 with those prime factors:

   - $13+3$: $3 \cdot 13 = 39$ and $3^2 \cdot 13 = 117$.
   - $11+5$: $5 \cdot 11 = 55$
   - $11+3+2$: $2 \cdot 3 \cdot 11 = 66$, $2^2 \cdot 3 \cdot 11 = 132$, and $2 \cdot 3^2 \cdot 11 = 198$.

   There are therefore 6 numbers less than 200 whose prime divisors sum to 16.

2. [**5**] Find the number of ordered triples of divisors (d_1, d_2, d_3) of 360 such that $d_1 d_2 d_3$ is also a divisor of 360.

   Answer: $\boxed{800}$ Since $360 = 2^3 \cdot 3^2 \cdot 5$, the only possible prime divisors of d_i are 2, 3, and 5, so we can write $d_i = 2^{a_i} \cdot 3^{b_i} \cdot 5^{c_i}$, for nonnegative integers a_i, b_i, and c_i. Then, $d_1 d_2 d_3 | 360$ if and only if the following three inequalities hold.

   $$\begin{aligned} a_1 + a_2 + a_3 &\leq 3 \\ b_1 + b_2 + b_3 &\leq 2 \\ c_1 + c_2 + c_3 &\leq 1 \end{aligned}$$

   Now, one can count that there are 20 assignments of a_i that satisfy the first inequality, 10 assignments of b_i that satisfy the second inequality, and 4 assignments of c_i that satisfy the third inequality, for a total of 800 ordered triples (d_1, d_2, d_3).

   (Alternatively, instead of counting, it is possible to show that the number of nonnegative-integer triples (a_1, a_2, a_3) satisfying $a_1 + a_2 + a_3 \leq n$ equals $\binom{n+3}{3}$, since this is equal to the number of nonnegative-integer quadruplets (a_1, a_2, a_3, a_4) satisfying $a_1 + a_2 + a_3 + a_4 = n$.)

3. [**6**] Find the largest integer less than 2012 all of whose divisors have at most two 1's in their binary representations.

   Answer: $\boxed{1536}$ Call a number *good* if all of its positive divisors have at most two 1's in their binary representations. Then, if p is an odd prime divisor of a good number, p must be of the form $2^k + 1$. The only such primes less than 2012 are 3, 5, 17, and 257, so the only possible prime divisors of n are 2, 3, 5, 17, and 257.

   Next, note that since $(2^i + 1)(2^j + 1) = 2^{i+j} + 2^i + 2^j + 1$, if either i or j is greater than 1, then there will be at least 3 1's in the binary representation of $(2^i + 1)(2^j + 1)$, so $(2^i + 1)(2^j + 1)$ cannot divide a good number. On the other hand, if $i = j = 1$, then $(2^1 + 1)(2^1 + 1) = 9 = 2^3 + 1$, so 9 is a good number and can divide a good number. Finally, note that since multiplication by 2 in binary just appends additional 0s, so if n is a good number, then $2n$ is also a good number.

   It therefore follows that any good number less than 2012 must be of the form $c \cdot 2^k$, where c belongs to $\{1, 3, 5, 9, 17, 257\}$ (and moreover, all such numbers are good). It is then straightforward to check that the largest such number is $1536 = 3 \cdot 2^9$.

4. [**3**] Let π be a permutation of the numbers from 2 through 2012. Find the largest possible value of $\log_2 \pi(2) \cdot \log_3 \pi(3) \cdots \log_{2012} \pi(2012)$.

Answer: $\boxed{1}$ Note that

$$\begin{aligned}\prod_{i=2}^{2012} \log_i \pi(i) &= \prod_{i=2}^{2012} \frac{\log \pi(i)}{\log i} \\ &= \frac{\prod_{i=2}^{2012} \log \pi(i)}{\prod_{i=2}^{2012} \log i} \\ &= 1,\end{aligned}$$

where the last equality holds since π is a permutation of the numbers 2 through 2012.

5. [**4**] Let π be a randomly chosen permutation of the numbers from 1 through 2012. Find the probability that $\pi(\pi(2012)) = 2012$.

Answer: $\boxed{\frac{1}{1006}}$ There are two possibilities: either $\pi(2012) = 2012$ or $\pi(2012) = i$ and $\pi(i) = 2012$ for $i \neq 2012$. The first case occurs with probability $2011!/2012! = 1/2012$, since any permutation on the remaining 2011 elements is possible. Similarly, for any fixed i, the second case occurs with probability $2010!/2012! = 1/(2011 \cdot 2012)$, since any permutation on the remaining 2010 elements is possible. Since there are 2011 possible values for i, and since our two possibilities are disjoint, the overall probability that $\pi(\pi(2012)) = 2012$ equals

$$\frac{1}{2012} + (2011)\frac{1}{2011 \cdot 2012} = \frac{1}{1006}.$$

6. [**6**] Let π be a permutation of the numbers from 1 through 2012. What is the maximum possible number of integers n with $1 \leq n \leq 2011$ such that $\pi(n)$ divides $\pi(n+1)$?

Answer: $\boxed{1006}$ Since any proper divisor of n must be less than or equal to $n/2$, none of the numbers greater than 1006 can divide any other number less than or equal to 2012. Since there are at most 1006 values of n for which $\pi(n) \leq 1006$, this means that there can be at most 1006 values of n for which $\pi(n)$ divides $\pi(n+1)$.

On the other hand, there exists a permutation for which $\pi(n)$ divides $\pi(n+1)$ for exactly 1006 values of n, namely the permutation:

$$(1, 2, 2^2, 2^3, \dots, 2^{10}, 3, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, \dots, 2^9 \cdot 3, 5, \dots)$$

Formally, for each odd number $\ell \leq 2012$, we construct the sequence $\ell, 2\ell, 4\ell, \dots, 2^k\ell$, where k is the largest integer such that $2^k\ell \leq 2012$. We then concatenate all of these sequences to form a permutation of the numbers 1 through ℓ (note that no number occurs in more than one sequence). It follows that if $\pi(n) \leq 1006$, then $\pi(n+1)$ will equal $2\pi(n)$, and therefore $\pi(n)$ will divide $\pi(n+1)$ for all 1006 values of n satisfying $1 \leq \pi(n) \leq 1006$.

7. [**8**] Let $A_1A_2 \dots A_{100}$ be the vertices of a regular 100-gon. Let π be a randomly chosen permutation of the numbers from 1 through 100. The segments $A_{\pi(1)}A_{\pi(2)}, A_{\pi(2)}A_{\pi(3)}, \dots, A_{\pi(99)}A_{\pi(100)}, A_{\pi(100)}A_{\pi(1)}$ are drawn. Find the expected number of pairs of line segments that intersect at a point in the interior of the 100-gon.

Answer: $\boxed{\frac{4850}{3}}$ By linearity of expectation, the expected number of total intersections is equal to the sum of the probabilities that any given intersection will occur.

Let us compute the probability $p_{i,j}$ that $A_{\pi(i)}A_{\pi(i+1)}$ intersects $A_{\pi(j)}A_{\pi(j+1)}$ (where $1 \leq i, j \leq 100$, $i \neq j$, and indices are taken modulo 100). Note first that if $j = i+1$, then these two segments share vertex $\pi(i+1)$ and therefore will not intersect in the interior of the 100-gon; similarly, if $i = j+1$, these two segments will also not intersect. On the other hand, if $\pi(i)$, $\pi(i+1)$, $\pi(j)$, and $\pi(j+1)$ are all distinct, then there is a 1/3 chance that $A_{\pi(i)}A_{\pi(i+1)}$ intersects $A_{\pi(j)}A_{\pi(j+1)}$; in any set of four points that form a convex quadrilateral, exactly one of the three ways of pairing the points into two pairs (two pairs of opposite sides and the two diagonals) forms two segments that intersect inside the quadrilateral (namely, the two diagonals).

Now, there are 100 ways to choose a value for i, and 97 ways to choose a value for j which is not i, $i+1$, or $i-1$, there are 9700 ordered pairs (i,j) where $p_{i,j} = 1/3$. Since each pair is counted twice (once as (i,j) and once as (j,i)), there are $9700/2 = 4850$ distinct possible intersections, each of which occurs with probability $1/3$, so the expected number of intersections is equal to $4850/3$.

8. [**4**] ABC is a triangle with $AB = 15$, $BC = 14$, and $CA = 13$. The altitude from A to BC is extended to meet the circumcircle of ABC at D. Find AD.

   Answer: $\boxed{\frac{63}{4}}$ Let the altitude from A to BC meet BC at E. The altitude AE has length 12; one way to see this is that it splits the triangle ABC into a $9-12-15$ right triangle and a $5-12-13$ right triangle; from this, we also know that $BE = 9$ and $CE = 5$.

   Now, by Power of a Point, $AE \cdot DE = BE \cdot CE$, so $DE = (BE \cdot CE)/AE = (9 \cdot 5)/(12) = 15/4$. It then follows that $AD = AE + DE = 63/4$.

9. [**5**] Triangle ABC satisfies $\angle B > \angle C$. Let M be the midpoint of BC, and let the perpendicular bisector of BC meet the circumcircle of $\triangle ABC$ at a point D such that points A, D, C, and B appear on the circle in that order. Given that $\angle ADM = 68°$ and $\angle DAC = 64°$, find $\angle B$.

   Answer: $\boxed{86°}$ Extend DM to hit the circumcircle at E. Then, note that since $ADEB$ is a cyclic quadrilateral, $\angle ABE = 180° - \angle ADE = 180° - \angle ADM = 180° - 68° = 112°$.

   We also have that $\angle MEC = \angle DEC = \angle DAC = 64°$. But now, since M is the midpoint of BC and since $EM \perp BC$, triangle BEC is isosceles. This implies that $\angle BEM = \angle MEC = 64°$, and $\angle MBE = 90° - \angle MEB = 26°$. It follows that $\angle B = \angle ABE - \angle MBE = 112° - 26° = 86°$.

10. [**6**] Triangle ABC has $AB = 4$, $BC = 5$, and $CA = 6$. Points A', B', C' are such that $B'C'$ is tangent to the circumcircle of $\triangle ABC$ at A, $C'A'$ is tangent to the circumcircle at B, and $A'B'$ is tangent to the circumcircle at C. Find the length $B'C'$.

    Answer: $\boxed{\frac{80}{3}}$ Note that by equal tangents, $B'A = B'C$, $C'A = C'B$, and $A'B = A'C$. Moreover, since the line segments $A'B'$, $B'C'$, and $C'A'$ are tangent to the circumcircle of ABC at C, A, and B respectively, we have that $\angle A'BC = \angle A'CB = \angle A$, $\angle B'AC = \angle B'CA = \angle B$, and $\angle C'BA = \angle C'AB = \angle C$. By drawing the altitudes of the isosceles triangles $BC'A$ and $AC'B$, we therefore have that $C'A = 2/\cos C$ and $B'A = 3/\cos B$.

    Now, by the Law of Cosines, we have that

$$\begin{aligned}
\cos B &= \frac{a^2+c^2-b^2}{2ac} = \frac{25+16-36}{2(5)(4)} = \frac{1}{8} \\
\cos C &= \frac{a^2+b^2-c^2}{2ab} = \frac{25+36-16}{2(5)(6)} = \frac{3}{4}.
\end{aligned}$$

    Therefore,

$$B'C' = C'A + B'A = 2\left(\frac{4}{3}\right) + 3(8) = \frac{80}{3}.$$