HMMT February 2019

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Geometry

1. Let d be a real number such that every non-degenerate quadrilateral has at least two interior angles with measure less than d degrees. What is the minimum possible value for d?

   Proposed by: James Lin

   Answer: $\boxed{120}$

   The sum of the internal angles of a quadrilateral triangle is $360°$. To find the minimum d, we note the limiting case where three of the angles have measure d and the remaining angle has measure approaching zero. Hence, $d \geq 360°/3 = 120$. It is not difficult to see that for any $0 < \alpha < 120$, a quadrilateral of which three angles have measure α degrees and fourth angle has measure $(360 - 3\alpha)$ degrees can be constructed.

2. In rectangle $ABCD$, points E and F lie on sides AB and CD respectively such that both AF and CE are perpendicular to diagonal BD. Given that BF and DE separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the length of BD.

   Proposed by: Yuan Yao

   Answer: $\boxed{\sqrt{3}}$

   Observe that $AECF$ is a parallelogram. The equal area condition gives that $BE = DF = \frac{1}{3}AB$. Let $CE \cap BD = X$, then $\frac{EX}{CX} = \frac{BE}{CD} = \frac{1}{3}$, so that $BX^2 = EX \cdot CX = 3EX^2 \Rightarrow BX = \sqrt{3}EX \Rightarrow \angle EBX = 30°$. Now, $CE = 2BE = CF$, so CEF is an equilateral triangle and $CD = \frac{3}{2}CF = \frac{3}{2}$. Hence, $BD = \frac{2}{\sqrt{3}} \cdot \frac{3}{2} = \sqrt{3}$.

3. Let AB be a line segment with length 2, and S be the set of points P on the plane such that there exists point X on segment AB with $AX = 2PX$. Find the area of S.

   Proposed by: Yuan Yao

   Answer: $\boxed{\sqrt{3} + \frac{2\pi}{3}}$

   Observe that for any X on segment AB, the locus of all points P such that $AX = 2PX$ is a circle centered at X with radius $\frac{1}{2}AX$. Note that the point P on this circle where PA forms the largest angle with AB is where PA is tangent to the circle at P, such that $\angle PAB = \arcsin(1/2) = 30°$. Therefore, if we let Q and Q' be the tangent points of the tangents from A to the circle centered at B (call it ω) with radius $\frac{1}{2}AB$, we have that S comprises the two 30-60-90 triangles AQB and$AQ'B$, each with area $\frac{1}{2}\sqrt{3}$ and the $240°$ sector of ω bounded by BQ and BQ' with area $\frac{2}{3}\pi$. Therefore the total area is $\sqrt{3} + \frac{2\pi}{3}$.

4. Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.

   Proposed by: Andrew Lin

   Answer: $\boxed{196}$

   Since $ACDF$ and $ABDE$ have area 168, triangles ABD and ACD (which are each half a parallelogram) both have area 84. Thus, B and C are the same height away from AD, and since $ABCDEF$ is convex, B and C are on the same side of AD. Thus, BC is parallel to AD, and $ABCD$ is a trapezoid. In particular, we have that the area of ABG equals the area of CDG. Letting this quantity be x, we have that the area of BCG is $x - 10$, and the area of ADG is $84 - x$. Then notice that $\frac{[ABG]}{[CBG]} = \frac{AG}{GC} = \frac{[ADG]}{[CDG]}$. This means that $\frac{x}{x-10} = \frac{84-x}{x}$. Simplifying, we have $x^2 - 47x + 420 = 0$; this has solutions $x = 12$ and $x = 35$. The area of $ABCDEF$ is twice the area of trapezoid $ABCD$, or $2[x + (x - 10) + (84 - x) + x] = 4x + 148$; choosing $x = 12$, we get that the smallest possible area is $48 + 148 = 196$.

5. Isosceles triangle ABC with $AB = AC$ is inscribed in a unit circle Ω with center O. Point D is the reflection of C across AB. Given that $DO = \sqrt{3}$, find the area of triangle ABC.

   Proposed by: Lillian Zhang

   Answer: $\boxed{\frac{\sqrt{2}+1}{2} \text{ OR } \frac{\sqrt{2}-1}{2}}$

   Solution 1. Observe that

   $$\angle DBO = \angle DBA + \angle ABO = \angle CBA + \angle BAO = \frac{1}{2}(\angle CBA + \angle BCA) + \frac{1}{2}(\angle BAC) = \frac{1}{2}(180^\circ) = 90^\circ.$$

   Thus $BC = BD = \sqrt{2}$ by the Pythagorean Theorem on $\triangle DBO$. Then $\angle BOC = 90^\circ$, and the distance from O to BC is $\frac{\sqrt{2}}{2}$. Depending on whether A is on the same side of BC as O, the height from A to BC is either $1 + \frac{\sqrt{2}}{2}$ or $1 - \frac{\sqrt{2}}{2}$, so the area is $(\sqrt{2} \cdot (1 \pm \frac{\sqrt{2}}{2}))/2 = \frac{\sqrt{2}\pm 1}{2}$.

   Solution 2. One can observe that $\angle DBA = \angle CBA = \angle ACB$ by property of reflection and ABC being isosceles, hence DB is tangent to Ω and Power of a Point (and reflection property) gives $BC = BD = \sqrt{OD^2 - OB^2} = \sqrt{2}$. Proceed as in Solution 1.

   Note. It was intended, but not specified in the problem statement that triangle ABC is acute, so we accepted either of the two possible answers.

6. Six unit disks $C_1, C_2, C_3, C_4, C_5, C_6$ are in the plane such that they don't intersect each other and C_i is tangent to C_{i+1} for $1 \le i \le 6$ (where $C_7 = C_1$). Let C be the smallest circle that contains all six disks. Let r be the smallest possible radius of C, and R the largest possible radius. Find $R - r$.

   Proposed by: Daniel Liu

   Answer: $\boxed{\sqrt{3} - 1}$

   The minimal configuration occurs when the six circles are placed with their centers at the vertices of a regular hexagon of side length 2. This gives a radius of 3.

   The maximal configuration occurs when four of the circles are placed at the vertices of a square of side length 2. Letting these circles be C_1, C_3, C_4, C_6 in order, we place the last two so that C_2 is tangent to C_1 and C_3 and C_5 is tangent to C_4 and C_6. (Imagine pulling apart the last two circles on the plane; this is the configuration you end up with.) The resulting radius is $2 + \sqrt{3}$, so the answer is $\sqrt{3} - 1$.

   Now we present the proofs for these configurations being optimal. First, we rephrase the problem: given an equilateral hexagon of side length 2, let r be the minimum radius of a circle completely containing the vertices of the hexagon. Find the difference between the minimum and maximum values in r. (Technically this r is off by one from the actual problem, but since we want $R - r$ in the actual problem, this difference doesn't matter.)

   Proof of minimality. We claim the minimal configuration stated above cannot be covered by a circle with radius $r < 2$. If $r < 2$ and all six vertices $O_1, O_2, \dots, O_6$ are in the circle, then we have that $\angle O_1OO_2 > 60^\circ$ since O_1O_2 is the largest side of the triangle O_1OO_2, and similar for other angles $\angle O_2OO_3, \angle O_3OO_4, \dots$, but we cannot have six angles greater than 60° into 360°, contradiction. Therefore $r \ge 2$.

   Proof of maximality. Let $ABCDEF$ be the hexagon, and choose the covering circle to be centered at O, the midpoint of AD, and radius $\sqrt{3}+1$. We claim the other vertices are inside this covering circle. First, we will show the claim for B. Let M be the midpoint of AC. Since ABC is isosceles and $AM \ge 1$, we must have $BM \le \sqrt{4-1} = \sqrt{3}$. Furthermore, MO is a midline of ACD, so $MO = \frac{CD}{2} = 1$. Thus by the triangle inequality, $OB \le MB + OM = \sqrt{3} + 1$, proving the claim. A similar argument proves the claim for C, E, F. Finally, an analogous argument to above shows if we define P as the midpoint of BE, then $AP \le \sqrt{3} + 1$ and $DP \le \sqrt{3} + 1$, so by triangle inequality $AD \le 2(\sqrt{3} + 1)$. Hence $OA = OD \le \sqrt{3} + 1$, proving the claim for A and D. Thus the covering circle contains all six vertices of $ABCDEF$.

7. Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let H be the orthocenter of ABC. Find the radius of the circle with nonzero radius tangent to the circumcircles of AHB, BHC, CHA.

   Proposed by: Michael Ren

   Answer: $\boxed{\frac{65}{4}}$

   Solution 1. We claim that the circle in question is the circumcircle of the anticomplementary triangle of ABC, the triangle for which ABC is the medial triangle.

   Let $A'B'C'$ be the anticomplementary triangle of ABC, such that A is the midpoint of $B'C'$, B is the midpoint of $A'C'$, and C is the midpoint of $A'B'$. Denote by ω the circumcircle of $A'B'C'$. Denote by ω_A the circumcircle of BHC, and similarly define ω_B, ω_C.

   Since $\angle BA'C = \angle BAC = 180° - \angle BHC$, we have that ω_A passes through A'. Thus, ω_A can be redefined as the circumcircle of $A'BC$. Since triangle $A'B'C'$ is triangle $A'BC$ dilated by a factor of 2 from point A', ω is ω_A dilated by a factor of 2 from point A'. Thus, circles ω and ω_A are tangent at A'.

   By a similar logic, ω is also tangent to ω_B and ω_C. Therefore, the circumcircle of the anticomplementary triangle of ABC is indeed the circle that the question is asking for.

   Using the formula $R = \frac{abc}{4A}$, we can find that the circumradius of triangle ABC is $\frac{65}{8}$. The circumradius of the anticomplementary triangle is double of that, so the answer is $\frac{65}{4}$.

   Solution 2. It is well-known that the circumcircle of AHB is the reflection of the circumcircle of ABC over AB. In particular, the circumcircle of AHB has radius equal to the circumradius $R = \frac{65}{8}$. Similarly, the circumcircles of BHC and CHA have radii R. Since H lies on all three circles (in the question), the circle centered at H with radius $2R = \frac{65}{4}$ is tangent to each circle at the antipode of H in that circle.

8. In triangle ABC with $AB < AC$, let H be the orthocenter and O be the circumcenter. Given that the midpoint of OH lies on BC, $BC = 1$, and the perimeter of ABC is 6, find the area of ABC.

   Proposed by: Andrew Lin

   Answer: $\boxed{\frac{6}{7}}$

   Solution 1. Let $A'B'C'$ be the medial triangle of ABC, where A' is the midpoint of BC and so on. Notice that the midpoint of OH, which is the nine-point-center N of triangle ABC, is also the circumcircle of $A'B'C'$ (since the midpoints of the sides of ABC are on the nine-point circle). Thus, if N is on BC, then NA' is parallel to $B'C'$, so by similarity, we also know that OA is parallel to BC.

   Next, $AB < AC$, so B is on the minor arc AC. This means that $\angle OAC = \angle OCA = \angle C$, so $\angle AOC = 180 - 2\angle C$. This gives us the other two angles of the triangle in terms of angle C: $\angle B = 90 + \angle C$ and $\angle A = 90 - 2\angle C$. To find the area, we now need to find the height of the triangle from A to BC, and this is easiest by finding the circumradius R of the triangle.

   We do this by the Extended Law of Sines. Letting $AC = x$ and $AB = 5 - x$,

   $$\frac{1}{\sin(90 - 2C)} = \frac{x}{\sin(90 + C)} = \frac{5 - x}{\sin C} = 2R,$$

   which can be simplified to

   $$\frac{1}{\cos 2C} = \frac{x}{\cos C} = \frac{5 - x}{\sin C} = 2R.$$

   This means that

   $$\frac{1}{\cos 2C} = \frac{(x) + (5 - x)}{(\cos C) + (\sin C)} = \frac{5}{\cos C + \sin C}$$

   and the rest is an easy computation:

   $$\cos C + \sin C = 5\cos 2C = 5(\cos^2 C - \sin^2 C)$$

$$\frac{1}{5} = \cos C - \sin C$$

Squaring both sides,

$$\frac{1}{25} = \cos^2 C - 2\sin C \cos C + \sin^2 C = 1 - \sin 2C$$

so $\sin 2C = \frac{24}{25}$, implying that $\cos 2C = \frac{7}{25}$. Therefore, since $\frac{1}{\cos 2C} = 2R$ from above, $R = \frac{25}{14}$. Finally, viewing triangle ABC with $BC = 1$ as the base, the height is

$$\sqrt{R^2 - \left(\frac{BC}{2}\right)^2} = \frac{12}{7}$$

by the Pythagorean Theorem, yielding an area of $\frac{1}{2} \cdot 1 \cdot \frac{12}{7} = \frac{6}{7}$.

Solution 2. The midpoint of OH is the nine-point center N. We are given N lies on BC, and we also know N lies on the perpendicular bisector of EF, where E is the midpoint of AC and F is the midpoint of AB. The main observation is that N is equidistant from M and F, where M is the midpoint of BC.

Translating this into coordinates, we pick $B(-0.5, 0)$ and $C(0.5, 0)$, and arbitrarily set $A(a, b)$ where (without loss of generality) $b > 0$. We get $E(\frac{a+0.5}{2}, \frac{b}{2})$, $F(\frac{a-0.5}{2}, \frac{b}{2})$, $M(0, 0)$. Thus N must have x-coordinate equal to the average of those of E and F, or $\frac{a}{2}$. Since N lies on BC, we have $N(\frac{a}{2}, 0)$.

Since $MN = EN$, we have $\frac{a^2}{4} = \frac{1}{16} + \frac{b^2}{4}$. Thus $a^2 = b^2 + \frac{1}{4}$. The other equation is $AB + AC = 5$, which is just

$$\sqrt{(a+0.5)^2 + b^2} + \sqrt{(a-0.5)^2 + b^2} = 5.$$

This is equivalent to

$$\sqrt{2a^2 + a} + \sqrt{2a^2 - a} = 5$$
$$\sqrt{2a^2 + a} = 5 - \sqrt{2a^2 - a}$$
$$2a^2 + a = 25 - 10\sqrt{2a^2 - a} + 2a^2 - a$$
$$25 - 2a = 10\sqrt{2a^2 - a}$$
$$625 - 100a + 4a^2 = 200a^2 - 100a$$
$$196a^2 = 625$$

Thus $a^2 = \frac{625}{196}$, so $b^2 = \frac{576}{196}$. Thus $b = \frac{24}{14} = \frac{12}{7}$, so $[ABC] = \frac{b}{2} = \frac{6}{7}$.

9. In a rectangular box $ABCDEFGH$ with edge lengths $AB = AD = 6$ and $AE = 49$, a plane slices through point A and intersects edges BF, FG, GH, HD at points P, Q, R, S respectively. Given that $AP = AS$ and $PQ = QR = RS$, find the area of pentagon $APQRS$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{141\sqrt{11}}{2}}$

Let AD be the positive x-axis, AB be the positive y-axis, and AE be the positive z-axis, with A the origin. The plane, which passes through the origin, has equation $k_1 x + k_2 y = z$ for some undetermined parameters k_1, k_2. Because $AP = AS$ and $AB = AD$, we get $PB = SD$, so P and S have the same z-coordinate. But $P(0, 6, 6k_2)$ and $S(6, 0, 6k_1)$, so $k_1 = k_2 = k$ for some k. Then Q and R both have z-coordinate 49, so $Q(\frac{49}{k} - 6, 6, 49)$ and $R(6, \frac{49}{k} - 6, 49)$. The equation $QR^2 = RS^2$ then gives

$$\left(\frac{49}{k} - 6\right)^2 + (49 - 6k)^2 = 2\left(12 - \frac{49}{k} - 12\right)^2.$$

This is equivalent to

$$(49 - 6k)^2(k^2 + 1) = 2(49 - 12k)^2,$$

which factors as

$$(k - 7)(36k^3 - 336k^2 - 203k + 343) = 0.$$

This gives $k = 7$ as a root. Note that for Q and R to actually lie on FG and GH respectively, we must have $\frac{49}{6} \geq k \geq \frac{49}{12}$. Via some estimation, one can show that the cubic factor has no roots in this range (for example, it's easy to see that when $k = 1$ and $k = \frac{336}{36} = \frac{28}{3}$, the cubic is negative, and it also remains negative between the two values), so we must have $k = 7$.

Now consider projecting $APQRS$ onto plane $ABCD$. The projection is $ABCD$ save for a triangle $Q'CR'$ with side length $12 - \frac{49}{k} = 5$. Thus the projection has area $36 - \frac{25}{2} = \frac{47}{2}$. Since the area of the projection equals $[APQRS] \cdot \cos\theta$, where θ is the (smaller) angle between planes $APQRS$ and $ABCD$, and since the planes have normal vectors $(k, k, -1)$ and $(0, 0, 1)$ respectively, we get $\cos\theta = \frac{(k,k,-1)\cdot(0,0,1)}{\sqrt{k^2+k^2+1}} = \frac{1}{\sqrt{2k^2+1}} = \frac{1}{\sqrt{99}}$ and so

$$[APQRS] = \frac{47\sqrt{99}}{2} = \frac{141\sqrt{11}}{2}.$$

10. In triangle ABC, $AB = 13, BC = 14, CA = 15$. Squares ABB_1A_2, BCC_1B_2, CAA_1C_2 are constructed outside the triangle. Squares $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ are constructed outside the hexagon $A_1A_2B_1B_2C_1C_2$. Squares $A_3B_4B_5A_6$, $B_3C_4C_5B_6$, $C_3A_4A_5C_6$ are constructed outside the hexagon $A_4A_3B_4B_3C_4C_3$. Find the area of the hexagon $A_5A_6B_5B_6C_5C_6$.

Proposed by: Yuan Yao

Answer: $\boxed{19444}$

Solution 1.

We can use complex numbers to find synthetic observations. Let $A = a, B = b, C = c$. Notice that B_2 is a rotation by $-90°$ (counter-clockwise) of C about B, and similarly C_1 is a rotation by $90°$ of B about C. Since rotation by $90°$ corresponds to multiplication by i, we have $B_2 = (c - b)\cdot(-i) + b = b(1 + i) - ci$ and $C_1 = (b - c)\cdot i + c = bi + c(1 - i)$. Similarly, we get $C_2 = c(1 + i) - ai$, $A_1 = ci + a(1 - i)$, $A_2 = a(1 + i) - bi$, $B_1 = ai + b(1 - i)$. Repeating the same trick on $B_1B_2B_3B_4$ et. al, we get $C_4 = -a + b(-1 + i) + c(3 - i)$, $C_3 = a(-1 - i) - b + c(3 + i)$, $A_4 = -b + c(-1 + i) + a(3 - i)$, $A_3 = b(-1 - i) - c + a(3 + i)$, $B_4 = -c + a(-1 + i) + b(3 - i)$, $B_3 = c(-1 - i) - a + b(3 + i)$. Finally, repeating the same trick on the outermost squares, we get $B_6 = -a + b(3 + 5i) + c(-3 - 3i), C_5 = -a + b(-3 + 3i) + c(3 - 5i), C_6 = -b + c(3 + 5i) + a(-3 - 3i), A_5 = -b + c(-3 + 3i) + a(3 - 5i), A_6 = -c + a(3 + 5i) + b(-3 - 3i), B_5 = -c + a(-3 + 3i) + b(3 - 5i)$.

From here, we observe the following synthetic observations.

S1. $B_2C_1C_4B_3, C_2A_1A_4C_3, A_2B_1B_4A_3$ are trapezoids with bases of lengths $BC, 4BC$; $AC, 4AC$; $AB, 4AB$ and heights h_a, h_b, h_c respectively (where h_a is the length of the altitude from A to BC, and likewise for h_b, h_c)

S2. If we extend B_5B_4 and B_6B_3 to intersect at B_7, then $B_7B_4B_3 \cong BB_1B_2 \sim B_7B_5B_6$ with scale factor 1:5. Likewise when we replace all B's with A's or C's.

Proof of S1. Observe $C_1 - B_2 = c - b$ and $C_4 - B_3 = 4(c - b)$, hence $B_2C_1 \| B_3C_4$ and $B_3C_4 = 4B_2C_1$. Furthermore, since translation preserves properties of trapezoids, we can translate $B_2C_1C_4B_3$ such that B_2 coincides with A. Being a translation of $a - B_2$, we see that B_3 maps to $B_3' = 2b - c$ and C_4 maps to $C_4' = -2b + 3c$. Both $2b - c$ and $-2b + 3c$ lie on the line determined by b and c (since $-2 + 3 = 2 - 1 = 1$), so the altitude from A to BC is also the altitude from A to $B_3'C_4'$. Thus h_a equals the length of the altitude from B_2 to B_3C_4, which is the height of the trapezoid $B_2C_1C_4B_3$. This proves S1 for $B_2C_1C_4B_3$; the other trapezoids follow similarly.

Proof of S2. Notice a translation of $-a + 2b - c$ maps B_1 to B_4, B_2 to B_3, and B to a point $B_8 = -a + 3b - c$. This means $B_8B_3B_4 \cong BB_1B_2$. We can also verify that $4B_8 + B_6 = 5B_3$ and $4B_8 + B_5 = 5B_4$, showing that $B_8B_5B_6$ is a dilation of $B_8B_4B_3$ with scale factor 5. We also get B_8 lies on B_3B_6 and B_5B_4, so $B_8 = B_7$. This proves S2 for $B_3B_4B_5B_6$, and similar arguments prove the likewise part.

Now we are ready to attack the final computation. By S2, $[B_3B_4B_5B_6] + [BB_1B_2] = [B_7B_5B_6] = [BB_1B_2]$. But by the $\frac{1}{2}ac\sin B$ formula, $[BB_1B_2] = [ABC]$ (since $\angle B_1BB_2 = 180° - \angle ABC$). Hence,

$[B_3B_4B_5B_6] + [BB_1B_2] = 25[ABC]$. Similarly, $[C_3C_4C_5C_6] + [CC_1C_2] = 25[ABC]$ and $[A_3A_4A_5A_6] + [AA_1A_2] = 25[ABC]$. Finally, the formula for area of a trapezoid shows $[B_2C_1C_4B_3] = \frac{5BC}{2} \cdot h_a = 5[ABC]$, and similarly the other small trapezoids have area $5[ABC]$. The trapezoids thus contribute area $(75 + 3 \cdot 5) = 90[ABC]$. Finally, ABC contributes area $[ABC] = 84$.

By S1, the outside squares have side lengths $4BC, 4CA, 4AB$, so the sum of areas of the outside squares is $16(AB^2 + AC^2 + BC^2)$. Furthermore, a Law of Cosines computation shows $A_1A_2^2 = AB^2 + AC^2 + 2 \cdot AB \cdot AC \cdot \cos\angle BAC = 2AB^2 + 2AC^2 - BC^2$, and similarly $B_1B_2^2 = 2AB^2 + 2BC^2 - AC^2$ and $C_1C_2^2 = 2BC^2 + 2AC^2 - AB^2$. Thus the sum of the areas of $A_1A_2A_3A_4$ et. al is $3(AB^2 + AC^2 + BC^2)$. Finally, the small squares have area add up to $AB^2 + AC^2 + BC^2$. Aggregating all contributions from trapezoids, squares, and triangle, we get

$$[A_5A_6B_5B_6C_5C_6] = 91[ABC] + 20(AB^2 + AC^2 + BC^2) = 7644 + 11800 = 19444.$$

Solution 2. Let $a = BC, b = CA, c = AB$. We can prove S1 and S2 using some trigonometry instead.

Proof of S1. The altitude from B_3 to B_2C_1 has length $B_2B_3 \sin\angle BB_2B_1 = B_1B_2 \sin\angle BB_2B_1 = BB_1 \sin\angle B_1BB_2 = AB \sin\angle ABC = h_a$ using Law of Sines. Similarly, we find the altitude from C_4 to B_2C_1 equals h_a, thus proving $B_2C_1C_4B_3$ is a trapezoid. Using $B_1B_2 = \sqrt{2a^2 + 2c^2 - b^2}$ from end of Solution 1, we get the length of the projection of B_2B_3 onto B_3C_4 is $B_2B_3 \cos BB_2B_1 = \frac{(2a^2+2c^2-b^2)+a^2-c^2}{2a} = \frac{3a^2+c^2-b^2}{2a}$, and similarly the projection of C_1C_4 onto B_3C_4 has length $\frac{3a^2+b^2-c^2}{2a}$. It follows that $B_3C_4 = \frac{3a^2+c^2-b^2}{2a} + a + \frac{3a^2+b^2-c^2}{2a} = 4a$, proving S1 for $B_2C_1C_4B_3$; the other cases follow similarly.

Proof of S2. Define B_8 to be the image of B under the translation taking B_1B_2 to B_4B_3. We claim B_8 lies on B_3B_6. Indeed, $B_8B_4B_3 \cong BB_1B_2$, so $\angle B_8B_3B_4 = \angle BB_2B_1 = 180^\circ - \angle B_3B_2C_1 = \angle B_2B_3C_4$. Thus $\angle B_8B_3C_4 = \angle B_4B_3B_2 = 90^\circ$. But $\angle B_6B_3C_4 = 90^\circ$, hence B_8, B_3, B_6 are collinear. Similarly we can prove B_5B_4 passes through B_8, so $B_8 = B_7$. Finally, $\frac{B_7B_3}{B_7B_6} = \frac{B_7B_4}{B_7B_5} = \frac{1}{5}$ (using $B_3B_6 = 4a, B_4B_5 = 4c, B_7B_3 = a, B_7B_4 = c$) shows $B_7B_4B_3 \sim B_7B_5B_6$ with scale factor 1:5, as desired. The likewise part follows similarly.