# $12^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

**Saturday 21 February 2009**

**Individual Round: Calculus Test**

1. [**3**] Let $f$ be a differentiable real-valued function defined on the positive real numbers. The tangent lines to the graph of $f$ always meet the $y$-axis 1 unit lower than where they meet the function. If $f(1) = 0$, what is $f(2)$?

2. [**3**] The differentiable function $F : \mathbb{R} \to \mathbb{R}$ satisfies $F(0) = -1$ and

$$\frac{d}{dx}F(x) = \sin(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x).$$

Find $F(x)$ as a function of $x$.

3. [**4**] Compute $e^A$ where $A$ is defined as

$$\int_{3/4}^{4/3} \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1} dx.$$

4. [**4**] Let $P$ be a fourth degree polynomial, with derivative $P'$, such that $P(1) = P(3) = P(5) = P'(7) = 0$. Find the real number $x \neq 1, 3, 5$ such that $P(x) = 0$.

5. [**4**] Compute

$$\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{3} + 4h\right) - 4\sin\left(\frac{\pi}{3} + 3h\right) + 6\sin\left(\frac{\pi}{3} + 2h\right) - 4\sin\left(\frac{\pi}{3} + h\right) + \sin\left(\frac{\pi}{3}\right)}{h^4}.$$

6. [**5**] Let $p_0(x), p_1(x), p_2(x), \ldots$ be polynomials such that $p_0(x) = x$ and for all positive integers $n$, $\frac{d}{dx}p_n(x) = p_{n-1}(x)$. Define the function $p(x) : [0, \infty) \to \mathbb{R}$ by $p(x) = p_n(x)$ for all $x \in [n, n+1)$. Given that $p(x)$ is continuous on $[0, \infty)$, compute

$$\sum_{n=0}^{\infty} p_n(2009).$$

7. [**5**] A line in the plane is called *strange* if it passes through $(a, 0)$ and $(0, 10 - a)$ for some $a$ in the interval $[0, 10]$. A point in the plane is called *charming* if it lies in the first quadrant and also lies below some strange line. What is the area of the set of all charming points?

8. [**7**] Compute

$$\int_1^{\sqrt{3}} x^{2x^2+1} + \ln\left(x^{2x^{2x^2+1}}\right) dx.$$

9. [**7**] Let $\mathcal{R}$ be the region in the plane bounded by the graphs of $y = x$ and $y = x^2$. Compute the volume of the region formed by revolving $\mathcal{R}$ around the line $y = x$.

10. [**8**] Let $a$ and $b$ be real numbers satisfying $a > b > 0$. Evaluate

$$\int_0^{2\pi} \frac{1}{a + b\cos(\theta)} d\theta.$$

Express your answer in terms of $a$ and $b$.