15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Team A

1. [**20**] Let ABC be a triangle with $AB < AC$. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D. Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC, respectively. Prove that $BE = CF$.

   Answer: $\boxed{\text{see below}}$ Note that DE, DF are the distances from D to AB, AC, respectively, and because AD is the angle bisector of $\angle BAC$, we have $DE = DF$. Also, $DB = DC$ because D is on the perpendicular bisector of BC. Finally, $\angle DEB = \angle DFC = 90°$, so it follows that $DEB \cong DFC$, and $BE = CF$.

2. [**20**] For what positive integers n do there exist functions $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that for each $1 \le i \le n$, either $f(g(i)) = i$ or $g(f(i)) = i$, but not both?

   Answer: $\boxed{n \text{ even}}$ We claim that this is possible for all even n. First, a construction: set $f(2m-1) = f(2m) = 2m-1$ and $g(2m-1) = g(2m) = 2m$ for $m = 1, \ldots, \frac{n}{2}$. It is easy to verify that this solution works.

   Now, we show that this is impossible for odd n. Without loss of generality, suppose that $f(g(1)) = 1$ and that $g(1) = a \Rightarrow f(a) = 1$. Then, we have $g(f(a)) = g(a) = 1$. Consequently, $a \neq 1$. In this case, call 1 and a a *pair* (we likewise regard i and j as a pair when $g(f(i)) = i$ and $f(i) = j$). Now, to show that n is even it suffices to show that all pairs are disjoint. Suppose for the sake of contradiction that some integer $b \neq a$ is also in a pair with 1 (note that 1 is arbitrary). Then, we have $f(g(b)) = b, g(b) = 1$ or $g(f(b)) = b, f(b) = 1$. But we already know that $g(1) = a$, so we must have $f(g(b)) = b, g(b) = 1$. But that would mean that both $f(g(1)) = 1$ and $g(f(1)) = 1$, a contradiction.

3. [**20**] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

   (a) Suppose that a token can legally move to any horizontally or vertically adjacent square. Show that Bob has a winning strategy for this game.

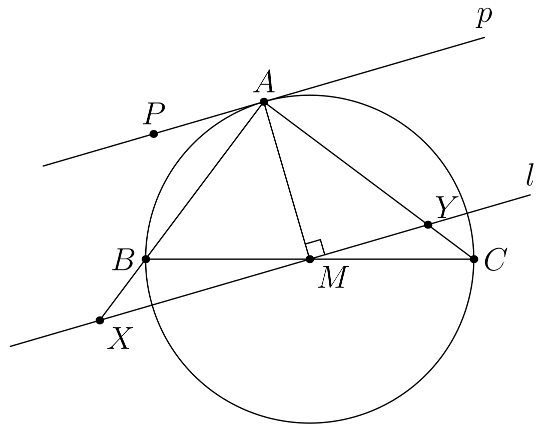
   (b) Suppose instead that a token can legally move to any horizontally, vertically, or diagonally adjacent square. Show that Alice has a winning strategy for this game.

   Answer: $\boxed{\text{see below}}$ For part (a), color the checkerboard in the standard way so that half of the squares are black and the other half are white. Bob's winning strategy is to place the two coins on the same color, so that Alice must always move her coin on to a square with the opposite color as the square containing Bob's coin.

   For part (b), consider any starting configuration. By considering only the column that the tokens are in, it is easy to see that Alice can get to the same column as Bob (immediately after her move) in 7 rounds. (This is just a game on a 1×8 chessboard.) Following this, Alice can stay on the same column as Bob each turn, while getting to the same row as him. This too also takes at most 7 rounds. Thus, Alice can catch Bob in $14 < 2012$ rounds from any starting position.

4. [**20**] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC. Line l is drawn through M so that it is perpendicular to AM, and intersects line AB at point X and line AC at point Y. Prove that $\angle BAC = 90°$ if and only if quadrilateral $XBYC$ is cyclic.

   Answer: $\boxed{\text{see below}}$



First, note that $XBYC$ cyclic is equivalent to $\measuredangle BXM = \measuredangle ACB$. However, note that $\measuredangle BXM = 90^\circ - \measuredangle BAM$, so $XBYC$ cyclic is in turn equivalent to $\measuredangle BAM + \measuredangle ACB = 90^\circ$.

Let the line tangent to the circumcircle of $\triangle ABC$ at A be p, and let P be an arbitrary point on p on the same side of AM as B. Note that $\measuredangle PAB = \measuredangle ACB$. If $\measuredangle ACB = 90^\circ - \measuredangle BAM$ we have $l \perp AM$ and thus the circumcenter O of $\triangle ABC$ lies on AM. Since $AB < AC$, we must have $O = M$, and $\measuredangle BAC = 90^\circ$. Conversely, if $\measuredangle BAC = 90^\circ$, $\measuredangle PAM = 90^\circ$, and it follows that $\measuredangle ACB = 90^\circ - \measuredangle BAM$.

5. [**20**] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes five pizzas, each with some subset of the five toppings. For each pizza, Arno states (with either a "yes" or a "no") if the pizza has any toppings that he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she has enough information to make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?

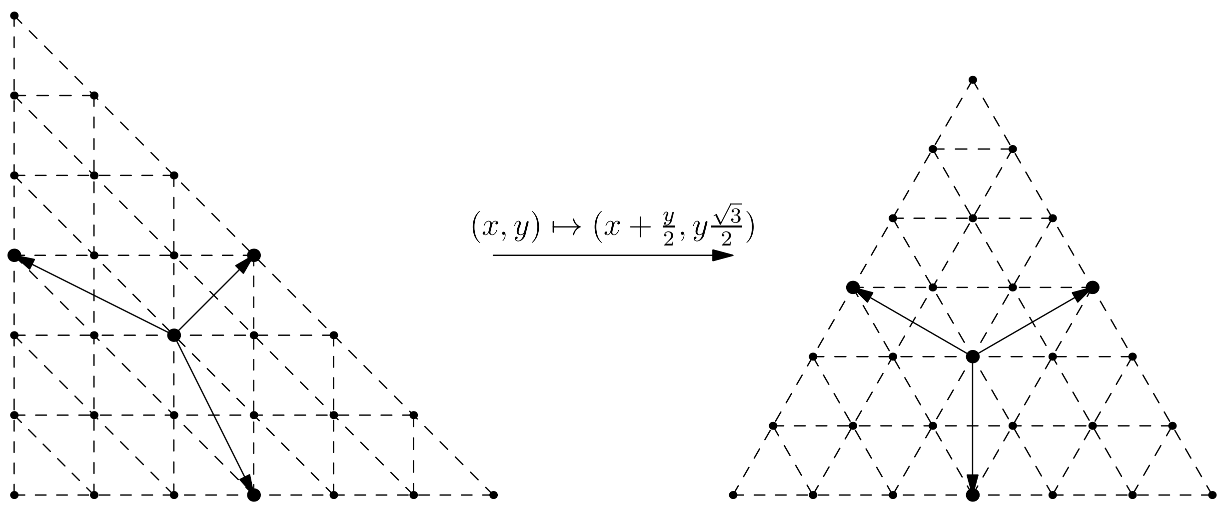
   Answer: $\boxed{\text{see below}}$ We claim the only way for Purineqa to deduce Arno's preferences is for each pizza to contain exactly one topping, with no topping be repeated. It is obvious that he can deduce the toppings in this case.

   We now claim that this is not possible with any other combination. Suppose that Arno tells Purineqa that he does not like any of the five pizzas. Then, Purineqa should be able to rule out at least one of the possibilities that Arno likes none of the toppings and that Arno likes exactly one of the toppings T. It is clear that this is possible if and only if there is a pizza with only T on it. This is true for all five toppings T, so we're done.

6. [**30**] It has recently been discovered that the right triangle with vertices (0,0), (0,2012), and (2012,0) is a giant pond that is home to many frogs. Frogs have the special ability that, if they are at a lattice point (x, y), they can hop to any of the three lattice points $(x+1, y+1)$, $(x-2, y+1)$, and $(x+1, y-2)$, assuming the given lattice point lies in or on the boundary of the triangle.

   Frog Jeff starts at the corner $(0, 0)$, while Frog Kenny starts at the corner $(0, 2012)$. Show that the set of points that Jeff can reach is equal in size to the set of points that Kenny can reach.

   Answer: $\boxed{\text{see below}}$



We transform the triangle as follows: map each lattice point (x, y) to the point

$$x(1, 0) + y(1/2, \sqrt{3}/2) = (x + y/2, y\sqrt{3}/2).$$

This transforms the right triangle into an equilateral triangle as shown above.

Now, the three allowed movements

$$\begin{aligned}(x, y) &\mapsto (x + 1, y + 1),\\ (x, y) &\mapsto (x - 2, y + 1),\\ (x, y) &\mapsto (x + 1, y - 2)\end{aligned}$$

become the movements

$$\begin{aligned}(x, y) &\mapsto (x + 3/2, y + \sqrt{3}/2),\\ (x, y) &\mapsto (x - 3/2, y + \sqrt{3}/2),\\ (x, y) &\mapsto (x, y - \sqrt{3})\end{aligned}$$

That is, each step is a movement of $\sqrt{3}$ in any of these three directions, which are separated by $120°$ angles. The pond is now completely symmetrical with respect to $120°$ rotations, so it does not matter which vertex you start at. The lower left vertex corresponds to the original point $(0, 0)$, and the top vertex corresponds to the original point $(0, 2012)$.

7. [**30**] Five points are chosen on a sphere of radius 1. What is the maximum possible volume of their convex hull?

   Answer: $\boxed{\frac{\sqrt{3}}{2}}$ Let the points be A, B, C, X, Y so that X and Y are on opposite sides of the plane defined by triangle ABC. The volume is 1/3 the product of the area of ABC and sum of the distances from X and Y to the plane defined by ABC. The area of ABC is maximized when the plane containing it intersects the sphere in the largest possible cross section, and ABC is equilateral: this gives an area of $3\sqrt{3}/4$. Then, the sum of the distances from X and Y to the plane of ABC is at most 2. This is clearly obtainable when A, B, and C form an equilateral triangle circumscribed by the equator of the sphere, and X and Y are at the poles, and we get a volume of $\sqrt{3}/2$.

8. [**30**] For integer $n, m \geq 1$, let $A(n, m)$ denote the number of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that $f(j) - f(i) \leq j - i$ for all $1 \leq i < j \leq n$, and let $B(n, m)$ denote the number of functions $g : \{0, 1, \ldots, 2n + m\} \to \{0, 1, \ldots, m\}$ such that $g(0) = 0$, $g(2n + m) = m$, and $|g(i) - g(i - 1)| = 1$ for all $1 \leq i \leq 2n + m$. Prove that $A(n, m) = B(n, m)$.

Answer: $\boxed{\text{see below}}$ We first note that the condition for f is equivalent to $i - f(i) \le j - f(j)$ for all $1 \le i < j \le n$. Letting $f'(x) = x - f(x)$, we see this is equivalent to saying that f' is decreasing. Thus, we only need that $f'(x) \le f'(x+1)$; in other words, we only require the statement to be true for $j = i + 1$.

Fix m, n. For any function g satisfying the conditions for B, we construct a function f satisfying the conditions for A as follows. For a given function g and $1 \le i \le 2n + m$, say that g has a *up step* at i if $g(i) - g(i-1) = 1$, and say it has a *down step* at i otherwise. We see that g must be composed of $m+n$ up steps and n down steps. Let $i_1, i_2, \ldots, i_n$ be the indices for which down steps occur, in ascending order. Let f be the function such that $f(j) = (m+1) - g(i_j)$ for $1 \le j \le n$. By our argument in the first paragraph, it suffices to show that $f(k) - f(k-1) \le 1$ for $1 < k \le n$, or that $g(i_{k-1}) - 1 \le g(i_k)$. If this were not the case for some k, then there would be at least 1 down step in between i_{k-1} and i_k, a contradiction, so the condition indeed holds.

We now claim that this construction is a bijection. For injectivity, note that for any two distinct g, g', there exists a k for which the values of $g(i_k), g'(i'_k)$ are distinct, in which case the functions f, f' must be distinct. For surjectivity, consider any suitable f. Let f' be the function such that $f'(k) = (m+1) - f(k)$ for all $1 \le k \le n$. (The range of this function is still $\{1, 2, \ldots, m\}$.) Then, we can find a g as follows: for each $1 \le k \le n$ in sequence, have g make up steps until it reaches the value $f'(k)$, then take one down step. This is always possible, as $f'(k+1) - f'(k) \le 1$. Thus, our claim is true, and our proof is complete.

9. [**40**] For any positive integer n, let $N = \varphi(1) + \varphi(2) + \ldots + \varphi(n)$. Show that there exists a sequence

$$a_1, a_2, \ldots, a_N$$

containing exactly $\varphi(k)$ instances of k for all positive integers $k \le n$ such that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_N a_1} = 1.$$

Answer: $\boxed{\text{see below}}$ We write all fractions of the form b/a, where a and b are relatively prime, and $0 \le b \le a \le n$, in ascending order. For instance, for $n = 5$, this is the sequence

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

This sequence is known as the *Farey sequence*.

Now, if we look at the the sequence of the denominators of the fractions, we see that k appears $\varphi(k)$ times when $k > 1$, although 1 appears twice. Thus, there are $N + 1$ elements in the Farey sequence. Let the Farey sequence be

$$\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_{N+1}}{a_{N+1}}$$

Now, $a_{N+1} = 1$, so the sequence $a_1, a_2, \ldots, a_N$ contains $\varphi(k)$ instances of k for every $1 \le k \le n$. We claim that this sequence also satisfies

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_N a_1} = 1.$$

Since $a_1 = a_{N+1} = 1$, we have

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_N a_1} = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_N a_{N+1}}.$$

Now, it will suffice to show that $\frac{1}{a_i a_{i+1}} = \frac{b_{i+1}}{a_{i+1}} - \frac{b_i}{a_i}$. Once we have shown this, the above sum will telescope to $\frac{b_{N+1}}{a_{N+1}} - \frac{b_1}{a_1} = 1 - 0 = 1$.

To see why $\frac{1}{a_i a_{i+1}} = \frac{b_{i+1}}{a_{i+1}} - \frac{b_i}{a_i}$ holds, we note that this is equivalent to $1 = b_{i+1} a_i - b_i a_{i+1}$. We can prove this fact geometrically: consider the triangle in the plane with vertices $(0, 0)$, (a_i, b_i), and

(a_{i+1}, b_{i+1}). This triangle contains these three boundary points, but it contains no other boundary or interior points since a_i and a_{i+1} are relatively prime to b_i and b_{i+1}, respectively, and since no other fraction with denominator at most n lies between $\frac{b_i}{a_i}$ and $\frac{b_{i+1}}{a_{i+1}}$. Thus, by Pick's theorem, this triangle has area $1/2$. But the area of the triangle can also be computed as the cross product $\frac{1}{2}(b_{i+1}a_i - b_i a_{i+1})$; hence $b_{i+1}a_i - b_i a_{i+1} = 1$ and we are done.

10. [**40**] For positive odd integer n, let $f(n)$ denote the number of matrices A satisfying the following conditions:

- A is $n \times n$.
- Each row and column contains each of $1, 2, \ldots, n$ exactly once in some order.
- $A^T = A$. (That is, the element in row i and column j is equal to the one in row j and column i, for all $1 \leq i, j \leq n$.)

Prove that $f(n) \geq \frac{n!(n-1)!}{\varphi(n)}$.

Answer: $\boxed{\text{see below}}$ We first note that main diagonal (the squares with row number equal to column number) is a permutation of $1, 2, \ldots, n$. This is because each number i ($1 \leq i \leq n$) appears an even number of times off the main diagonal, so must appear an odd number of times on the main diagonal. Thus, we may assume that the main diagonal's values are $1, 2, \ldots, n$ in that order. Call any matrix satifsying this condition and the problem conditions *good*. Let $g(n)$ denote the number of good matrices. It now remains to show that $g(n) \geq \frac{(n-1)!}{\varphi(n)}$.

Now, consider a round-robin tournament with n teams, labeled from 1 through n, with the matches spread over n days such that on day i, all teams except team i play exactly one match (so there are $\frac{n-1}{2}$ pairings), and at the end of n days, each pair of teams has played exactly once. We consider two such tournaments distinct if there is some pairing of teams i, j which occurs on different days in the tournaments. We claim that the tournaments are in bijection with the good matrices.
Proof of Claim: Given any good matrix A, we construct a tournament by making day k have matches between team i and j for each i, j such that $A_{i,j} = k$, besides $(i, j) = (k, k)$. Every pair will play some day, and since each column and row contains exactly one value of each number, no team will play more than once a day. Furthermore, given two distinct good matrices, there exists a value (off the main diagonal) on which they differ; this value corresponds to the same pair playing on different dates, so the corresponding tournaments must be distinct. For the other direction, take any tournament. Make a matrix A with the main diagonal as $1, 2, \ldots, n$, and for each k, set $A_{i,j} = k$ for each i, j such that teams i, j play each other on day k. This gives a good matrix. Similarly, given any two distinct tournaments, there exists a team pair i, j which play each other on different days; this corresponds to a differing value on the corresponding good matrices.

It now suffices to exhibit $\frac{(n-1)!}{\varphi(n)}$ distinct tournaments. (It may be helpful here to think of the days in the tournament as an unordered collection of sets of pairings, with the order implictly imposed by the team not present in the set of pairings.) For our construction, consider a regular n-gon with center O. Label the points as $A_1, A_2, \ldots, A_n$ as an arbitrary permutation (so there are $n!$ possible labelings). The team k will be represented by A_k. For each k, consider the line A_kO. The remaining $n-1$ vertices can be paired into $\frac{n-1}{2}$ groups which are perpendicular to this line; use these pairings for day k. Of course, this doesn't generate $n!$ distinct tournaments– but how many does it make?

Consider any permutation of labels. Starting from an arbitrary point, let the points of the polygon be $A_{\pi(1)}, A_{\pi(2)}, \ldots, A_{\pi(n)}$ in clockwise order. Letting $\pi(0) = \pi(n)$ and $\pi(n+1) = \pi(1)$, we note that $\pi(i-1)$ and $\pi(i+1)$ play each other on day $\pi(i)$. We then see that any other permutation of labels representing the same tournament must have $A_{\pi(i-1)}A_{\pi(i)} = A_{\pi(i)}A_{\pi(i+1)}$ for all i. Thus, if $A_{\pi(1)}$ is k vertices clockwise of $A_{\pi(0)}$, then $A_{\pi(2)}$ is k vertices clockwise of $A_{\pi(1)}$, and so on all the way up to $A_{\pi(n-1)}$ being k vertices clockwise of $A_{\pi(n)}$. This is only possible if k is relatively prime to n,so there are $\varphi(n)$ choices of k. There are n choices of the place to put $A_{\pi(1)}$, giving $n\varphi(n)$ choices of permutations meeting this condition. It is clear that each permutation meeting this condition provides the same tournament, so the $n!$ permutations can be partitioned into equivalence classes of size $n\varphi(n)$ each. Thus, there are $\frac{n!}{n\varphi(n)}$ distinct equivalence classes, and we are done.