

HMMT February 2024

February 17, 2024

Combinatorics Round

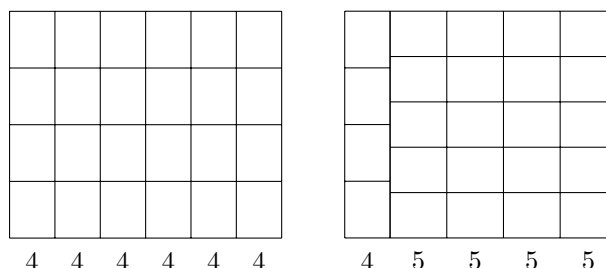
1. Compute the number of ways to divide a 20×24 rectangle into 4×5 rectangles. (Rotations and reflections are considered distinct.)

Proposed by: Derek Liu

Answer: 6

Solution: For convenience, say the edge of length 20 is vertical.

Consider some vertical line inside the rectangle. It must pass through rectangles of some heights adding to 20. In particular, these heights correspond to ways to add up to 20 with fours and fives, which is either $4 + 4 + 4 + 4 + 4$ or $5 + 5 + 5 + 5$. These options correspond to columns of rectangles with width 5 or 4, respectively. In particular, we need to span the width of the original 20×24 rectangle using these columns, meaning that we can just count the number of ways to add to 24 with fours and fives.



There are two ways to do this: either $4 + 4 + 4 + 4 + 4 + 4$ or $4 + 5 + 5 + 5 + 5$. Considering the orders in which the second sum can be written, we get an answer of $1 + 5 = \boxed{6}$.

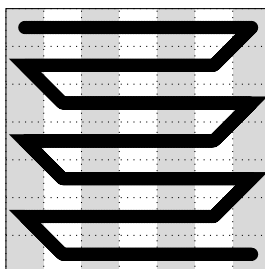
2. A *lame king* is a chess piece that can move from a cell to any cell that shares at least one vertex with it, except for the cells in the same column as the current cell.

A lame king is placed in the top-left cell of a 7×7 grid. Compute the maximum number of cells it can visit without visiting the same cell twice (including its starting cell).

Proposed by: Arul Kolla

Answer: 43

Solution: Color the columns all-black and all-white, alternating by column. Each move the lame king takes will switch the color it's on. Assuming the king starts on a black cell, there are 28 black and 21 white cells, so it can visit at most $22 + 21 = \boxed{43}$ cells in total, which is easily achievable:



3. Compute the number of ways there are to assemble 2 red unit cubes and 25 white unit cubes into a $3 \times 3 \times 3$ cube such that red is visible on exactly 4 faces of the larger cube. (Rotations and reflections are considered distinct.)

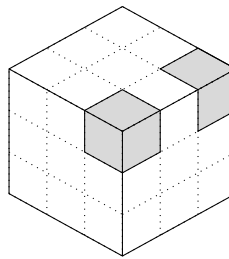
Proposed by: Albert Wang

Answer: 114

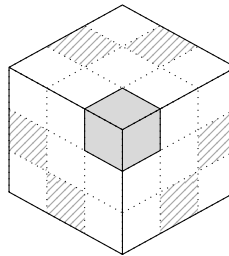
Solution:

We do casework on the two red unit cubes; they can either be in a corner, an edge, or the center of the face.

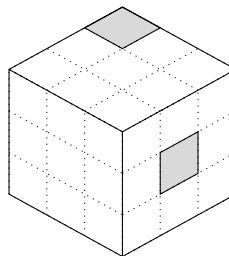
- If they are both in a corner, they must be adjacent – for each configuration, this corresponds to an edge, of which there are 12.



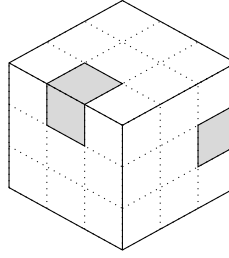
- If one is in the corner and the other is at an edge, we have 8 choices to place the corner. For the edge, the red edge square has to go on the boundary of the faces touching the red corner square, and there are six places here. Thus, we get $8 \cdot 6 = 48$ configurations.



- If one is a corner and the other is in the center of a face, we again have 8 choices for the corner and 3 choices for the center face (the faces not touching the red corner). This gives $8 \cdot 3 = 8 + 8 + 8 = 24$ options.



- We have now completed the cases with a red corner square! Now suppose we have two edges: If we chose in order, we have 12 choices for the first cube. For the second cube, we must place the edge so it covers two new faces, and thus we have five choices. Since we could have picked these edges in either order, we divide by two to avoid overcounting, and we have $12 \cdot 5/2 = 30$ in this case.

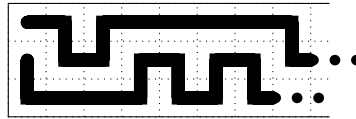


Now, since edges and faces only cover at most 2 and 1 face respectively, no other configuration works. Thus we have all the cases, and we add: $12 + 48 + 24 + 30 = \boxed{114}$.

4. Sally the snail sits on the 3×24 lattice of points (i, j) for all $1 \leq i \leq 3$ and $1 \leq j \leq 24$. She wants to visit every point in the lattice exactly once. In a move, Sally can move to a point in the lattice exactly one unit away. Given that Sally starts at $(2, 1)$, compute the number of possible paths Sally can take.

Proposed by: Isabella Zhu

Answer: $\boxed{4096}$



Solution 1:

On her first turn, Sally cannot continue moving down the middle row. She must turn either to the bottom row or the top row. WLOG, she turns to the top row, and enters the cell $(3, 1)$ and we will multiply by 2 later. Then, we can see that the path must finish in $(1, 1)$. So, we will follow these two branches of the path, one for the start and one for the end. These branches must both move one unit up, and then one of the paths must move into the center row. Both branches move up one unit, and then the path in the middle row must go back to fill the corner. After this, we have exactly the same scenario as before, albeit with two fewer rows. So, for each additional two rows, we have a factor of two and thus there are $2^{12} = \boxed{4096}$ paths.

Solution 2:

We solve this problem for a general 3 by $2n$ grid.

On her first turn, Sally cannot continue moving down the middle row. She must turn either to the topmost row or the bottommost row. WLOG, she turns to the top row.

Suppose Sally returns to the middle row k times. There are k "blocks". However, $2k$ of the squares are already occupied by Sally's row shifts. Thus, we are solving

$$2(x_1 + x_2 + \dots + x_k) = 2(n - k).$$

There are

$$\sum_{k=1}^n \binom{n-k+k-1}{k-1} = 2^{n-1}$$

solutions. We multiply by 2 to get 2^n .

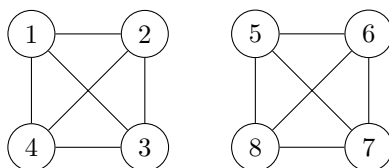
For $n = 12$, this evaluates to $2^{12} = \boxed{4096}$.

5. The country of HMMTland has 8 cities. Its government decides to construct several two-way roads between pairs of distinct cities. After they finish construction, it turns out that each city can reach exactly 3 other cities via a single road, and from any pair of distinct cities, either exactly 0 or 2 other cities can be reached from both cities by a single road. Compute the number of ways HMMTland could have constructed the roads.

Proposed by: Derek Liu

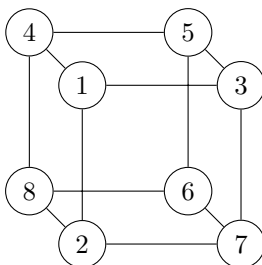
Answer: 875

Solution: Let the cities be numbered 1, 2, 3, 4, 5, 6, 7, 8. WLOG, 1 is connected to 2, 3, and 4. First suppose 2 and 3 are connected; then 3 and 1 share a second common neighbor, which must be 4 (as 1 is not connected to anything else). Likewise 2 and 4 are connected, and so 5, 6, 7, 8 are pairwise connected as well, so the graph consists of two disjoint copies of K_4 :



There are $\frac{1}{2} \binom{8}{4} = 35$ ways to partition the 8 vertices into two groups of 4, so there are 35 such graphs.

Otherwise, none of 2, 3, 4 are connected to each other. Then 2 and 3 must share a common neighbor, as must 3 and 4, and 2 and 4. If these are the same neighbor, this vertex would share all three neighbors with 1, so they must be pairwise distinct. The last vertex must then be connected to these three, creating a cube graph.



A cube has 48 symmetries, so the number of such graphs is $\frac{8!}{48} = 840$.

The total is $35 + 840 = \boxed{875}$.

6. In each cell of a 4×4 grid, one of the two diagonals is drawn uniformly at random. Compute the probability that the resulting 32 triangular regions can be colored red and blue so that any two regions sharing an edge have different colors.

Proposed by: Derek Liu

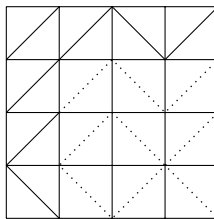
Answer: $\frac{1}{512}$

Solution: Give each cell coordinates from (1, 1) to (4, 4).

Claim. The grid has a desired coloring if and only if every vertex not on the boundary meets an even number of edges and diagonals.

Proof. If this were not the case, the odd number of regions around the vertex would have to alternate between the two colors, which is clearly impossible. In the event that every vertex has an even number of incident edges, it is not hard to show that the grid is always colorable. \square

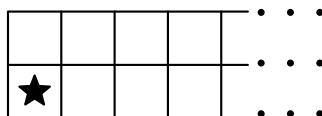
We claim the diagonals drawn in the cells of form $(1, a)$ and $(a, 1)$ for $1 \leq a \leq 4$ uniquely determine the rest (for a valid coloring to exist). Indeed, given the diagonals for any three cells around a vertex, we can uniquely determine the fourth one using the parity in the claim above. If $(1, 1)$, $(1, 2)$, $(2, 1)$ are fixed, so is $(2, 2)$; likewise so are $(2, 3)$ and $(2, 4)$, etc. until the whole grid is fixed.



The solid lines force the dotted lines as described above.

Thus, once the seven cells along the top row and leftmost column are determined, the remaining nine have a $\frac{1}{2^9} = \boxed{\frac{1}{512}}$ chance of being selected in a way that admits a coloring.

7. There is a grid of height 2 stretching infinitely in one direction. Between any two edge-adjacent cells of the grid, there is a door that is locked with probability $\frac{1}{2}$ independent of all other doors. Philip starts in a corner of the grid (in the starred cell). Compute the expected number of cells that Philip can reach, assuming he can only travel between cells if the door between them is unlocked.



Proposed by: Jacob Paltrowitz

Answer: $\boxed{\frac{32}{7}}$

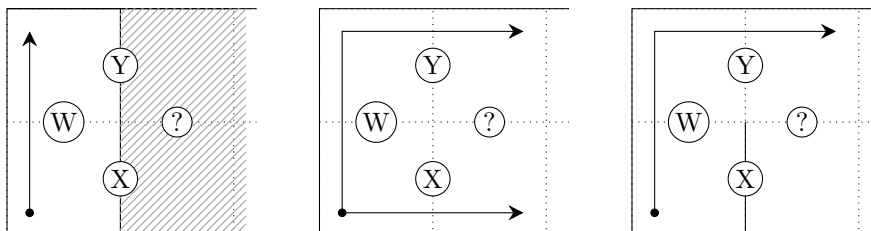
Solution: For clarity, we will number our grid, with $(0,0)$ being the corner that Philip starts in, and the grid stretching in the positive x direction, i.e. all elements of the grid are of the form (x, y) , with $y \in \{0, 1\}$ and $x \in \mathbb{N}$.

We will use recursion and casework. Let A be the expected number of reachable cells given that the door between $(0,0)$ and $(0,1)$ is unlocked, and B be the expected number of cells given that door is closed. Since that door is locked $\frac{1}{2}$ of the time, our answer is $\frac{A+B}{2}$.

We can write recurrence relations by considering the different configurations of the doors in the first 4 cells.

For the sake of writing, let W be the $(0,0) - (0,1)$ door, X be the $(0,0) - (1,0)$ door, Y be the $(0,1) - (1,1)$ door, and Z be the $(1,0) - (1,1)$ door.

Let's start with the case where W is unlocked and compute A :



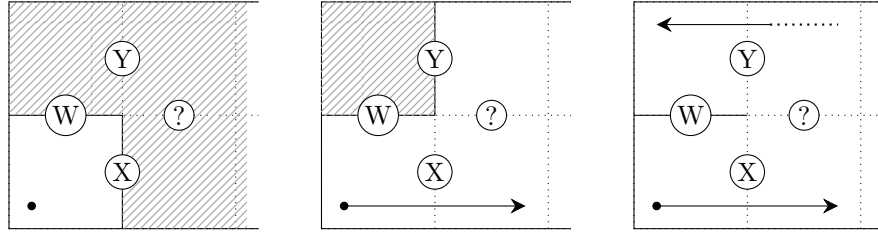
Case 1: W is unlocked. Shaded cells represent inaccessible cells, and the arrows show Philip's movements between cells.

- If X, Y are both locked, then Philip can reach exactly 2 rooms. This occurs with probability $\frac{1}{4}$.
- If both of X, Y are unlocked, then we have exactly the same case of A , except with the ability to reach two extra cells. This occurs with probability $\frac{1}{4}$.
- If exactly one of X, Y are unlocked, we have back to the original case, except with the ability to access two more cells, which occurs with probability $\frac{1}{2}$.

So, we get the equation:

$$A = \frac{1}{4}(2) + \frac{1}{4}(A + 2) + \frac{1}{2} \left(\frac{A + B}{2} + 2 \right)$$

Now, let's consider when W is locked.



Case 2: W is locked.

- If X is locked, Philip can only reach one cell. This occurs with probability $\frac{1}{2}$.
- If X is unlocked and Y is locked, we have exactly the original problem, except with the ability to reach one more cell. This occurs with probability $\frac{1}{4}$.
- In the case where X is unlocked and Y is unlocked, this is the same as the original configuration, except with the ability to reach one extra cell (the start) and possibly the cell at $(0, 1)$.

Now, let's compute the probability that Philip can reach $(0, 1)$ in this case. This is the probability that Philip can reach $(1, 1)$ since Y is unlocked. We can compute that the probability that Philip can reach $(1, 1)$ from $(1, 0)$ is equal to

$$\sum_{n=0}^{\infty} \frac{1}{2^{3n+1}}$$

by looking at the minimum distance Philip has to go to the right before getting back to $(1, 1)$. This is a geometric series with sum $\frac{4}{7}$. So, in this case, on average Philip can reach $1 + \frac{4}{7}$ more cells than the original case. This case occurs with probability $\frac{1}{4}$.

So, we can write the equation:

$$B = \frac{1}{2}(1) + \frac{1}{4} \left(\frac{A + B}{2} + 1 \right) + \frac{1}{4} \left(\frac{A + B}{2} + 1 + \frac{4}{7} \right)$$

Solving the system of these two linear equations, we get $A = \frac{40}{7}$, $B = \frac{24}{7}$ and $\frac{A+B}{2} = \boxed{\frac{32}{7}}$.

8. Rishabh has 2024 pairs of socks in a drawer. He draws socks from the drawer uniformly at random, without replacement, until he has drawn a pair of identical socks. Compute the expected number of unpaired socks he has drawn when he stops.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{4^{2024}}{\binom{4048}{2024}} - 2}$

Solution 1: We solve for the expected number of total socks drawn and subtract two at the end.

Let E_n be the expected number of socks drawn for n pairs of socks, so that $E_1 = 2$. Suppose there are n pairs of socks, Rishabh continued to draw socks until the drawer was empty, and without loss of generality let the last sock drawn be red. If we ignore the two red socks, the process is equivalent to drawing from a drawer with $n - 1$ pairs of socks. Let k be the number of socks drawn until a pair of identical socks is found, after ignoring the two red socks. Then the first red sock has probability $\frac{k}{2n-1}$ of being before this stopping point, so the expected value is $k + \frac{k}{2n-1} = k \cdot \frac{2n}{2n-1}$. Since the expected value of k is E_{n-1} , we have

$$E_n = \frac{2n}{2n-1} \cdot E_{n-1}.$$

Applying this recurrence, we get

$$E_n = \frac{(2n)!!}{(2n-1)!!} = \frac{2^n \cdot n!}{(2n-1)!!} = \frac{4^n \cdot (n!)^2}{(2n)!} = \frac{4^n}{\binom{2n}{n}}.$$

Subtracting two and plugging in $n = 2024$ gives a final answer of $\boxed{\frac{4^{2024}}{\binom{4048}{2024}} - 2}$.

Solution 2: Let $P(k)$ denote the probability that Rishabh draws more than k socks. We compute $P(k)$ for all $0 \leq k \leq 2024$ (and note $P(k) = 0$ for larger k).

The number of ways to draw k socks, none identical to each other, is

$$4048 \cdot 4046 \cdots (4050 - 2k) = 2^k \cdot \frac{2024!}{(2024 - k)!},$$

while the total number of ways to draw k socks is

$$4048 \cdot 4047 \cdots (4049 - k) = \frac{4048!}{(4048 - k)!}.$$

Thus,

$$P(k) = \frac{2^k \cdot \frac{2024!}{(2024-k)!}}{\frac{4048!}{(4048-k)!}} = \frac{2024!}{4048!} \cdot \frac{2^k (4048 - k)!}{(2024 - k)!} = \frac{1}{\binom{4048}{2024}} \cdot 2^k \binom{4048 - k}{2024}.$$

The expected number of socks drawn is

$$P(0) + P(1) + \cdots + P(2024) = \frac{1}{\binom{4048}{2024}} \sum_{k=0}^{2024} 2^k \binom{4048 - k}{2024}.$$

This sum is equivalent to **Putnam 2020 A2**. We claim that it is equal to 4^{2024} . We do this via a counting argument: we count how many ways there are to choose at least half of the elements from the set $\{1, 2, \dots, 4049\}$. On the one hand that is $\frac{2^{4049}}{2} = 4^{2024}$. On the other hand, letting $k + 1$ be the 2025th largest element chosen, there are $\binom{4048-k}{2024}$ ways to choose the elements larger than it, and 2^k ways to choose the elements smaller than it. Varying k , we get

$$\sum_{k=0}^{2024} 2^k \binom{4048 - k}{2024} = 4^{2024}.$$

This means the expected number of socks is

$$\frac{4^{2024}}{\binom{4048}{2024}},$$

and subtracting two for the matching pair gives $\boxed{\frac{4^{2024}}{\binom{4048}{2024}} - 2}$.

9. Compute the number of triples (f, g, h) of permutations on $\{1, 2, 3, 4, 5\}$ such that

$$\begin{aligned} f(g(h(x))) &= h(g(f(x))) = g(x), \\ g(h(f(x))) &= f(h(g(x))) = h(x), \text{ and} \\ h(f(g(x))) &= g(f(h(x))) = f(x) \end{aligned}$$

for all $x \in \{1, 2, 3, 4, 5\}$.

Proposed by: Rishabh Das

Answer: $\boxed{146}$

Solution: Let fg represent the composition of permutations f and g , where $(fg)(x) = f(g(x))$ for all $x \in \{1, 2, 3, 4, 5\}$.

Evaluating $fghfh$ in two ways, we get

$$f = gfh = (fgh)fh = fghfh = f(ghf)h = fhh,$$

so $hh = 1$. Similarly, we get f, g , and h are all involutions. Then

$$fgh = g \implies fg = gh,$$

so $fg = gh = hf$. Let $x := fg = gh = hf$. Then

$$x^3 = (fg)(gh)(hf) = 1.$$

We can also show that $fg = gh = hf$ along with f, g, h being involutions is enough to recover the initial conditions, so we focus on satisfying these new conditions.

If $x = 1$, then $f = g = h$ is an involution. There are $1 + \binom{5}{2} + \frac{1}{2} \binom{5}{2,2,1} = 26$ involutions, so this case gives 26 solutions.

Suppose $x \neq 1$. Then since $x^3 = 1$, x is composed of a 3-cycle and two fixed points, of which there are 20 choices. WLOG $x = (123)$. It can be checked that $\{1, 2, 3\}$ must map to itself for all of f, g, h and also $\{4, 5\}$. We can either have all of f, g, h map 4 and 5 to themselves or each other. Restricted to $\{1, 2, 3\}$, they are some rotation of $(12), (23), (13)$. Each of the 20 cases thus gives $2 \cdot 3 = 6$ triples, so overall we get $20 \cdot 6 = 120$.

The final answer is $26 + 120 = \boxed{146}$.

10. A *peacock* is a ten-digit positive integer that uses each digit exactly once. Compute the number of peacocks that are exactly twice another peacock.

Proposed by: Albert Wang

Answer: $\boxed{184320}$

Solution:

We begin with the following observation:

Claim 1. Let x be a peacock. Then, $2x$ is a peacock if and only if:

- the multiplication $x \cdot 2$ uses five carries,
- each of the pairs of digits $(0, 5), (1, 6), (2, 7), (3, 8), (4, 9)$ receives exactly one carry.
- The leading digit is not 5, 6, 7, 8, 9.

Proof. After the multiplication of $x \cdot 2$, we will have a ten digit number. Let's first consider the output without carrying. It consists of the digits 0, 2, 4, 6, 8 twice each, occupying positions where pairs of digits $(0, 5), (1, 6), (2, 7), (3, 8), (4, 9)$ were in x . However, we guaranteed that one digit from each pair received a carry, meaning all ten digits are present after adding in the carries. \square

We will now biject all peacocks to the following combination of objects:

- a queue of low digits 0, 1, 2, 3, 4, in any order with the constraint that 0 is not first,
- a queue of high digits 5, 6, 7, 8, 9, in any order, and
- of each of the pairs of digits $(0, 5), (1, 6), (2, 7), (3, 8), (4, 9)$ mark one of them to receive a carry, except we are not allowed to mark the final digit in the high queue.

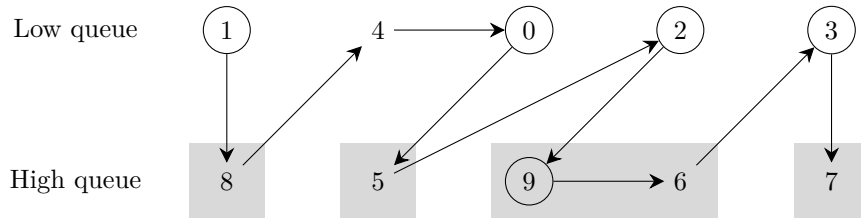
We construct a correspondence from these objects to peacocks by accumulating digits to an initially empty string. We'll say that we *poll* a queue by popping its front entry and appending it to the end of this string. First, poll the low queue. Then, if we have just polled a marked digit, poll the high queue; otherwise, poll the low queue. We repeat this until all queues are emptied.

As an example of this process, let our low queue be 1, 4, 0, 2, 3, our high queue be 8, 5, 9, 6, 7, and mark the digits 0, 1, 2, 3, 9 marked to receive a carry. Our steps are as follows:

- Poll the low queue, so that our string is now 1.
- Since 1 was marked to receive a carry, we poll the high queue, making our string 18.
- Since 8 was not marked, we poll the low queue to reach 184.
- Since 4 was not marked, we poll the low queue to reach 1840.
- Since 0 was marked, we poll the high queue to reach 18405.
- etc.

In the end, we will construct the peacock 1840529637, which is the one shown earlier to work.

Claim 2. Any string of digits x constructed through this process will be a peacock that satisfies the constraints outlined in **Claim 1**.



The order in which digits get polled to construct 1840529637; note the 4 connected components in the high queue. The circled digits are those that have been marked for carrying.

Proof. We first argue that all digits end up being polled. In particular, if a high digit is marked, let's connect it by an edge to the digit on its right (using the requirement that the last digit is not marked). If h of the high digits are marked, then we will have $5 - h$ connected components among these high digits. However, we then have $5 - h$ marked digits in the low queue, and every time we poll a marked low digit we will end up polling all digits from the next connected component in the high queue.

So, all digits end up being polled. Notice that our marked digits will always be followed immediately by a high digit, satisfying the first and second conditions of the claim. As we do not start with a high

digit, the third constraint is satisfied. Therefore any peacock x output by this process will also have $2x$ a peacock. \square

Since we always use all the digits, this process is evidently injective. To map from peacocks back to these sequences of digits, we can just let the queues be the order of appearances of the low and high digits in the peacock, and mark the carried digits accordingly. Indeed, we notice that this mapping is also injective.

Using this bijection, we just need to find the number of initial settings of the queues and marked digits. There are $4 \cdot 4!$ ways to order the low number queue. There are then $5!$ ways to order the high number queue. Finally, of each of the four pairs of digits not including the final high digit, there are 2^4 ways to mark them. This gives an answer of

$$4 \cdot 4! \cdot 5! \cdot 2^4 = \boxed{184320}.$$