THE HARVARD-MIT MATHEMATICS TOURNAMENT

GEOMETRY TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted other than the official translation sheets. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Our goal is that a closed form answer equivalent to the correct answer will be accepted. However, we do not always have the resourses to determine whether a complicated or strange answer is equivalent to ours. To assist us in awarding you all the points that you deserve, you answers should be simplified as much as possible. Answers must be exact unless otherwise specified.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to Science Center 109 during lunchtime.

Enjoy!

15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Geometry Test

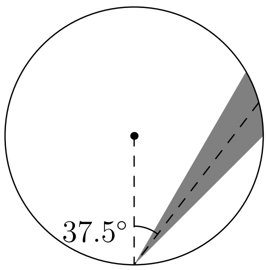
1. ABC is an isosceles triangle with $AB = 2$ and $\measuredangle ABC = 90°$. D is the midpoint of BC and E is on AC such that the area of $AEDB$ is twice the area of ECD. Find the length of DE.

2. Let ABC be a triangle with $\angle A = 90°$, $AB = 1$, and $AC = 2$. Let ℓ be a line through A perpendicular to BC, and let the perpendicular bisectors of AB and AC meet ℓ at E and F, respectively. Find the length of segment EF.

3. Let ABC be a triangle with incenter I. Let the circle centered at B and passing through I intersect side AB at D and let the circle centered at C passing through I intersect side AC at E. Suppose DE is the perpendicular bisector of AI. What are all possible measures of angle BAC in degrees?

4. There are circles ω_1 and ω_2. They intersect in two points, one of which is the point A. B lies on ω_1 such that AB is tangent to ω_2. The tangent to ω_1 at B intersects ω_2 at C and D, where D is the closer to B. AD intersects ω_1 again at E. If $BD = 3$ and $CD = 13$, find EB/ED.

5. A mouse lives in a circular cage with completely reflective walls. At the edge of this cage, a small flashlight with vertex on the circle whose beam forms an angle of $15°$ is centered at an angle of $37.5°$ away from the center. The mouse will die in the dark. What fraction of the total area of the cage can keep the mouse alive?



6. Triangle ABC is an equilateral triangle with side length 1. Let $X_0, X_1, \ldots$ be an infinite sequence of points such that the following conditions hold:

   - X_0 is the center of ABC
   - For all $i \geq 0$, X_{2i+1} lies on segment AB and X_{2i+2} lies on segment AC.
   - For all $i \geq 0$, $\measuredangle X_i X_{i+1} X_{i+2} = 90°$.
   - For all $i \geq 1$, X_{i+2} lies in triangle $AX_i X_{i+1}$.

   Find the maximum possible value of $\sum_{i=0}^{\infty} |X_i X_{i+1}|$, where $|PQ|$ is the length of line segment PQ.

7. Let S be the set of the points $(x_1, x_2, \ldots, x_{2012})$ in 2012-dimensional space such that $|x_1| + |x_2| + \cdots + |x_{2012}| \leq 1$. Let T be the set of points in 2012-dimensional space such that $\max_{i=1}^{2012} |x_i| = 2$. Let p be a randomly chosen point on T. What is the probability that the closest point in S to p is a vertex of S?

8. Hexagon $ABCDEF$ has a circumscribed circle and an inscribed circle. If $AB = 9$, $BC = 6$, $CD = 2$, and $EF = 4$. Find $\{DE, FA\}$.

9. Let O, O_1, O_2, O_3, O_4 be points such that O_1, O, O_3 and O_2, O, O_4 are collinear in that order, $OO_1 = 1, OO_2 = 2, OO_3 = \sqrt{2}, OO_4 = 2$, and $\measuredangle O_1 O O_2 = 45°$. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the circles with respective centers O_1, O_2, O_3, O_4 that go through O. Let A be the intersection of ω_1 and ω_2, B be the intersection of ω_2 and ω_3, C be the intersection of ω_3 and ω_4, and D be the intersection of ω_4 and ω_1, with A, B, C, D all distinct from O. What is the largest possible area of a convex quadrilateral $P_1P_2P_3P_4$ such that P_i lies on O_i and that A, B, C, D all lie on its perimeter?

10. Let C denote the set of points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \leq 1$. A sequence $A_i = (x_i, y_i) | i \geq 0$ of points in $\mathbb{R}^2$ is 'centric' if it satisfies the following properties:

- $A_0 = (x_0, y_0) = (0, 0)$, $A_1 = (x_1, y_1) = (1, 0)$.
- For all $n \geq 0$, the circumcenter of triangle $A_n A_{n+1} A_{n+2}$ lies in C.

Let K be the maximum value of $x_{2012}^2 + y_{2012}^2$ over all centric sequences. Find all points (x, y) such that $x^2 + y^2 = K$ and there exists a centric sequence such that $A_{2012} = (x, y)$.

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Geometry Test

Name ______________________________ Team ID# ________

School ____________________ Team ____________________

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Score: __________