2nd Annual Harvard-MIT November Tournament

Saturday 7 November 2009

Team Round

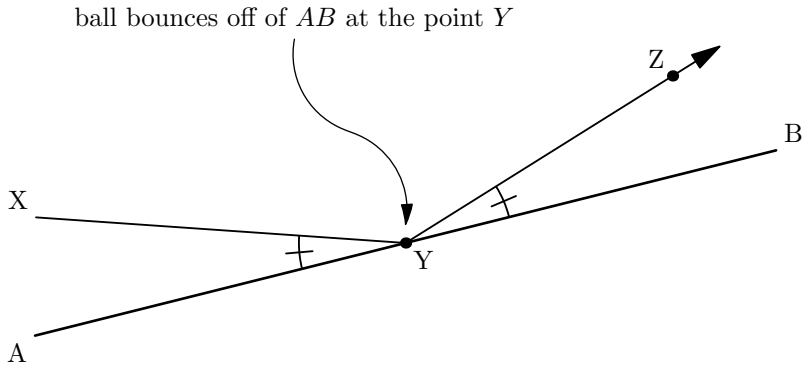
Down the Infinite Corridor

Consider an isosceles triangle T with base 10 and height 12. Define a sequence $\omega_1, \omega_2, \ldots$ of circles such that ω_1 is the incircle of T and ω_{i+1} is tangent to ω_i and both legs of the isosceles triangle for $i > 1$.

1. [**3**] Find the radius of ω_1.
2. [**3**] Find the ratio of the radius of ω_{i+1} to the radius of ω_i.
3. [**3**] Find the total area contained in all the circles.

Bouncy Balls

In the following problems, you will consider the trajectories of balls moving and bouncing off of the boundaries of various containers. The balls are small enough that you can treat them as points. Let us suppose that a ball starts at a point X, strikes a boundary (indicated by the line segment AB) at Y, and then continues, moving along the ray YZ. Balls always bounce in such a way that $\angle XYA = \angle BYZ$. This is indicated in the above diagram.



Balls bounce off of boundaries in the same way light reflects off of mirrors - if the ball hits the boundary at point P, the trajectory after P is the reflection of the trajectory before P through the perpendicular to the boundary at P.

A ball inside a rectangular container of width 7 and height 12 is launched from the lower-left vertex of the container. It first strikes the right side of the container after traveling a distance of $\sqrt{53}$ (and strikes no other sides between its launch and its impact with the right side).

4. [**2**] Find the height at which the ball first contacts the right side.
5. [**3**] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

Now a ball is launched from a vertex of an equilateral triangle with side length 5. It strikes the opposite side after traveling a distance of $\sqrt{19}$.

6. [**4**] Find the distance from the ball's point of first contact with a wall to the nearest vertex.
7. [**4**] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

In this final problem, a ball is again launched from the vertex of an equilateral triangle with side length 5.

8. [**6**] In how many ways can the ball be launched so that it will return again to a vertex for the first time after 2009 bounces?

Super Mario 64!

Mario is once again on a quest to save Princess Peach. Mario enters Peach's castle and finds himself in a room with 4 doors. This room is the first in a sequence of 2 indistinugishable rooms. In each room, 1 door leads to the next room in the sequence (or, for the second room, into Bowser's level), while the other 3 doors lead to the first room.

9. [**3**] Suppose that in every room, Mario randomly picks a door to walk through. What is the expected number of doors (not including Mario's initial entrance to the first room) through which Mario will pass before he reaches Bowser's level?

10. [**4**] Suppose that instead there are 6 rooms with 4 doors. In each room, 1 door leads to the next room in the sequence (or, for the last room, Bowser's level), while the other 3 doors lead to the first room. Now what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

11. [**5**] In general, if there are d doors in every room (but still only 1 correct door) and r rooms, the last of which leads into Bowser's level, what is the expected number of doors through which Mario will pass before he reaches Bowser's level?