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1. [**4**] A random number generator will always output 7. Sam uses this random number generator once. What is the expected value of the output?

2. [**4**] Let A, B, C, D, E, F be 6 points on a circle in that order. Let X be the intersection of AD and BE, Y is the intersection of AD and CF, and Z is the intersection of CF and BE. X lies on segments BZ and AY and Y lies on segment CZ. Given that $AX = 3$, $BX = 2$, $CY = 4$, $DY = 10$, $EZ = 16$, and $FZ = 12$, find the perimeter of triangle XYZ.

3. [**4**] Find the number of pairs of integers (x, y) such that $x^2 + 2y^2 < 25$.

4. [**4**] Find the number of ordered triples of nonnegative integers (a, b, c) that satisfy

$$(ab + 1)(bc + 1)(ca + 1) = 84.$$

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5. [**6**] Find the number of ordered triples of positive integers (a, b, c) such that

$$6a + 10b + 15c = 3000.$$

6. [**6**] Let $ABCD$ be a convex quadrilateral with $AC = 7$ and $BD = 17$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 + PQ^2$

7. [**6**] An ordered pair of sets (A, B) is *good* if A is not a subset of B and B is not a subset of A. How many ordered pairs of subsets of $\{1, 2, \ldots, 2017\}$ are good?

8. [**6**] You have 128 teams in a single elimination tournament. The Engineers and the Crimson are two of these teams. Each of the 128 teams in the tournament is equally strong, so during each match, each team has an equal probability of winning.

   Now, the 128 teams are randomly put into the bracket.

   What is the probability that the Engineers play the Crimson sometime during the tournament?

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9. [**7**] Jeffrey writes the numbers 1 and $100000000 = 10^8$ on the blackboard. Every minute, if x, y are on the board, Jeffrey replaces them with

$$\frac{x+y}{2} \quad \text{and} \quad 2\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}.$$

After 2017 minutes the two numbers are a and b. Find $\min(a, b)$ to the nearest integer.

10. [**7**] Let ABC be a triangle in the plane with $AB = 13, BC = 14, AC = 15$. Let M_n denote the smallest possible value of $(AP^n + BP^n + CP^n)^{\frac{1}{n}}$ over all points P in the plane. Find $\lim_{n\to\infty} M_n$.

11. [**7**] Consider the graph in 3-space of

$$0 = xyz(x+y)(y+z)(z+x)(x-y)(y-z)(z-x).$$

This graph divides 3-space into N connected regions. What is N?

12. [**7**] In a certain college containing 1000 students, students may choose to major in exactly one of math, computer science, finance, or English. The *diversity ratio* $d(s)$ of a student s is the defined as number of students in a different major from s divided by the number of students in the same major as s (including s). The *diversity* D of the college is the sum of all the diversity ratios $d(s)$.

Determine all possible values of D.

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13. [**9**] The game of Penta is played with teams of five players each, and there are five roles the players can play. Each of the five players chooses two of five roles they wish to play. If each player chooses their roles randomly, what is the probability that each role will have exactly two players?

14. [**9**] Mrs. Toad has a class of 2017 students, with unhappiness levels $1, 2, \ldots, 2017$ respectively. Today in class, there is a group project and Mrs. Toad wants to split the class in exactly 15 groups. The unhappiness level of a group is the average unhappiness of its members, and the unhappiness of the class is the sum of the unhappiness of all 15 groups. What's the minimum unhappiness of the class Mrs. Toad can achieve by splitting the class into 15 groups?

15. [**9**] Start by writing the integers $1, 2, 4, 6$ on the blackboard. At each step, write the smallest positive integer n that satisfies both of the following properties on the board.

   - n is larger than any integer on the board currently.
   - n cannot be written as the sum of 2 distinct integers on the board.

   Find the 100-th integer that you write on the board. Recall that at the beginning, there are already 4 integers on the board.

16. [**9**] Let a and b be complex numbers satisfying the two equations

$$a^3 - 3ab^2 = 36$$
$$b^3 - 3ba^2 = 28i.$$

Let M be the maximum possible magnitude of a. Find all a such that $|a| = M$.

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17. [**10**] Sean is a biologist, and is looking at a string of length 66 composed of the letters A, T, C, G. A *substring* of a string is a contiguous sequence of letters in the string. For example, the string $AGTC$ has 10 substrings: $A, G, T, C, AG, GT, TC, AGT, GTC, AGTC$. What is the maximum number of distinct substrings of the string Sean is looking at?

18. [**10**] Let $ABCD$ be a quadrilateral with side lengths $AB = 2$, $BC = 3$, $CD = 5$, and $DA = 4$. What is the maximum possible radius of a circle inscribed in quadrilateral $ABCD$?

19. [**10**] Find (in terms of $n \geq 1$) the number of terms with odd coefficients after expanding the product:

$$\prod_{1 \leq i < j \leq n} (x_i + x_j)$$

e.g., for $n = 3$ the expanded product is given by $x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_2^2x_1 + x_3^2x_1 + x_3^2x_2 + 2x_1x_2x_3$ and so the answer would be 6.

20. [**10**] For positive integers a and N, let $r(a, N) \in \{0, 1, \ldots, N-1\}$ denote the remainder of a when divided by N. Determine the number of positive integers $n \leq 1000000$ for which

$$r(n, 1000) > r(n, 1001).$$

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21. [**12**] Let P and A denote the perimeter and area respectively of a right triangle with relatively prime integer side-lengths. Find the largest possible integral value of $\frac{P^2}{A}$

22. [**12**] Kelvin the Frog and 10 of his relatives are at a party. Every pair of frogs is either *friendly* or *unfriendly*. When 3 pairwise friendly frogs meet up, they will gossip about one another and end up in a *fight* (but stay *friendly* anyway). When 3 pairwise unfriendly frogs meet up, they will also end up in a *fight*. In all other cases, common ground is found and there is no fight. If all $\binom{11}{3}$ triples of frogs meet up exactly once, what is the minimum possible number of fights?

23. [**12**] Five points are chosen uniformly at random on a segment of length 1. What is the expected distance between the closest pair of points?

24. [**12**] At a recent math contest, Evan was asked to find $2^{2016} \pmod{p}$ for a given prime number p with $100 < p < 500$. Evan has forgotten what the prime p was, but still remembers how he solved it:

- Evan first tried taking 2016 modulo $p - 1$, but got a value e larger than 100.
- However, Evan noted that $e - \frac{1}{2}(p-1) = 21$, and then realized the answer was $-2^{21} \pmod{p}$.

What was the prime p?

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25. [**15**] Find all real numbers x satisfying the equation $x^3 - 8 = 16\sqrt[3]{x+1}$.

26. [**15**] Kelvin the Frog is hopping on a number line (extending to infinity in both directions). Kelvin starts at 0. Every minute, he has a $\frac{1}{3}$ chance of moving 1 unit left, a $\frac{1}{3}$ chance of moving 1 unit right and $\frac{1}{3}$ chance of getting eaten. Find the expected number of times Kelvin returns to 0 (not including the start) before getting eaten.

27. [**15**] Find the smallest possible value of $x + y$ where $x, y \geq 1$ and x and y are integers that satisfy $x^2 - 29y^2 = 1$

28. [**15**] Let $\ldots, a_{-1}, a_0, a_1, a_2, \ldots$ be a sequence of positive integers satisfying the following relations: $a_n = 0$ for $n < 0$, $a_0 = 1$, and for $n \geq 1$,

$$a_n = a_{n-1} + 2(n-1)a_{n-2} + 9(n-1)(n-2)a_{n-3} + 8(n-1)(n-2)(n-3)a_{n-4}.$$

Compute

$$\sum_{n\geq 0} \frac{10^n a_n}{n!}.$$

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29. [**17**] Yang has the sequence of integers $1, 2, \ldots, 2017$. He makes 2016 *swaps* in order, where a swap changes the positions of two integers in the sequence. His goal is to end with $2, 3, \ldots, 2017, 1$. How many different sequences of swaps can Yang do to achieve his goal?

30. [**17**] Consider an equilateral triangular grid G with 20 points on a side, where each row consists of points spaced 1 unit apart. More specifically, there is a single point in the first row, two points in the second row, ..., and 20 points in the last row, for a total of 210 points. Let S be a closed non-self-intersecting polygon which has 210 vertices, using each point in G exactly once. Find the sum of all possible values of the area of S.

31. [**17**] A baseball league has 6 teams. To decide the schedule for the league, for each pair of teams, a coin is flipped. If it lands head, they will play a game this season, in which one team wins and one team loses. If it lands tails, they don't play a game this season. Define the *imbalance* of this schedule to be the minimum number of teams that will end up undefeated, i.e. lose 0 games. Find the expected value of the imbalance in this league.

32. [**17**] Let a, b, c be non-negative real numbers such that $ab + bc + ca = 3$. Suppose that

$$a^3b + b^3c + c^3a + 2abc(a + b + c) = \frac{9}{2}.$$

What is the minimum possible value of $ab^3 + bc^3 + ca^3$?

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33. [**20**] Welcome to the **USAYNO**, where each question has a yes/no answer. Choose any subset of the following six problems to answer. If you answer n problems and get them **all** correct, you will receive $\max(0,(n-1)(n-2))$ points. If any of them are wrong, you will receive 0 points.

    Your answer should be a six-character string containing 'Y' (for yes), 'N' (for no), or 'B' (for blank). For instance if you think 1, 2, and 6 are 'yes' and 3 and 4 are 'no', you would answer YYNNBY (and receive 12 points if all five answers are correct, 0 points if any are wrong).

    (a) a,b,c,d,A,B,C, and D are positive real numbers such that $\frac{a}{b} > \frac{A}{B}$ and $\frac{c}{d} > \frac{C}{D}$. Is it necessarily true that $\frac{a+c}{b+d} > \frac{A+C}{B+D}$?

    (b) Do there exist irrational numbers α and β such that the sequence $\lfloor \alpha \rfloor + \lfloor \beta \rfloor, \lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor, \lfloor 3\alpha \rfloor + \lfloor 3\beta \rfloor, \ldots$ is arithmetic?

    (c) For any set of primes $\mathbb{P}$, let $S_{\mathbb{P}}$ denote the set of integers whose prime divisors all lie in $\mathbb{P}$. For instance $S_{\{2,3\}} = \{2^a 3^b \mid a,b \geq 0\} = \{1,2,3,4,6,8,9,12,\ldots\}$. Does there exist a finite set of primes $\mathbb{P}$ and integer polynomials P and Q such that $\gcd(P(x),Q(y)) \in S_{\mathbb{P}}$ for all x,y?

    (d) A function f is called **P-recursive** if there exists a positive integer m and real polynomials $p_0(n), p_1(n), \ldots, p_m(n)$ satisfying

    $$p_m(n)f(n+m) = p_{m-1}(n)f(n+m-1) + \ldots + p_0(n)f(n)$$

    for all n. Does there exist a P-recursive function f satisfying $\lim_{n\to\infty} \frac{f(n)}{n^{\sqrt{2}}} = 1$?

    (e) Does there exist a **nonpolynomial** function $f : \mathbb{Z} \to \mathbb{Z}$ such that $a-b$ divides $f(a)-f(b)$ for all integers $a \neq b$?

    (f) Do there exist periodic functions $f,g : \mathbb{R} \to \mathbb{R}$ such that $f(x)+g(x) = x$ for all x?

34. [**20**]

    (a) Can 1000 queens be placed on a 2017×2017 chessboard such that every square is attacked by some queen? A square is attacked by a queen if it lies in the same row, column, or diagonal as the queen.

    (b) A 2017×2017 grid of squares originally contains a 0 in each square. At any step, Kelvin the Frog chooses two adjacent squares (two squares are adjacent if they share a side) and increments the numbers in both of them by 1. Can Kelvin make every square contain a different power of 2?

    (c) A *tournament* consists of single games between every pair of players, where each game has a winner and loser with no ties. A set of people is *dominated* if there exists a player who beats all of them. Does there exist a tournament in which every set of 2017 people is dominated?

    (d) Every cell of a 19×19 grid is colored either red, yellow, green, or blue. Does there necessarily exist a rectangle whose sides are parallel to the grid, all of whose vertices are the same color?

    (e) Does there exist a $c \in \mathbb{R}^+$ such that $\max(|A \cdot A|, |A+A|) \geq c|A|\log^2|A|$ for all finite sets $A \subset \mathbb{Z}$?

    (f) Can the set $\{1,2,\ldots,1093\}$ be partitioned into 7 subsets such that each subset is sum-free (i.e. no subset contains a,b,c with $a+b=c$)?

35. [**20**]

    (a) Does there exist a finite set of points, not all collinear, such that a line between any two points in the set passes through a third point in the set?

(b) Let ABC be a triangle and P be a point. The *isogonal conjugate* of P is the intersection of the reflection of line AP over the A-angle bisector, the reflection of line BP over the B-angle bisector, and the reflection of line CP over the C-angle bisector. Clearly the incenter is its own isogonal conjugate. Does there exist another point that is its own isogonal conjugate?

(c) Let F be a convex figure in a plane, and let P be the largest pentagon that can be inscribed in F. Is it necessarily true that the area of P is at least $\frac{3}{4}$ the area of F?

(d) Is it possible to cut an equilateral triangle into 2017 pieces, and rearrange the pieces into a square?

(e) Let ABC be an acute triangle and P be a point in its interior. Let D, E, F lie on BC, CA, AB respectively so that PD bisects $\angle BPC$, PE bisects $\angle CPA$, and PF bisects $\angle APB$. Is it necessarily true that $AP + BP + CP \geq 2(PD + PE + PF)$?

(f) Let P_{2018} be the surface area of the 2018-dimensional unit sphere, and let P_{2017} be the surface area of the 2017-dimensional unit sphere. Is $P_{2018} > P_{2017}$?

36. [**20**]

(a) Does $\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \pmod{p^2}$ for all odd prime numbers p? (Note that $\frac{1}{i}$ denotes the number such that $i \cdot \frac{1}{i} \equiv 1 \pmod{p^2}$)

(b) Do there exist 2017 positive perfect cubes that sum to a perfect cube?

(c) Does there exist a right triangle with rational side lengths and area 5?

(d) A *magic square* is a 3×3 grid of numbers, all of whose rows, columns, and major diagonals sum to the same value. Does there exist a magic square whose entries are all prime numbers?

(e) Is $\prod_p \frac{p^2+1}{p^2-1} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \ldots$ a rational number?

(f) Do there exist an infinite number of pairs of *distinct* integers (a, b) such that a and b have the same set of prime divisors, and $a + 1$ and $b + 1$ also have the same set of prime divisors?