# $4^{\text{th}}$ Annual Harvard-MIT November Tournament

**Saturday 12 November 2011**

## Guts Round

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### Round 1

1. [**5**] Determine the remainder when $1 + 2 + \cdots + 2014$ is divided by 2012.

2. [**5**] Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 4$. Let $E$ be the point on $BC$ with $BE = 3$, and let $F$ be the point on segment $AE$ such that $F$ lies halfway between the segments $AB$ and $CD$. If $G$ is the point of intersection of $DF$ and $BC$, find $BG$.

3. [**5**] Let $x$ be a real number such that $2^x = 3$. Determine the value of $4^{3x+2}$.

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### Round 2

4. [**6**] Determine which of the following numbers is smallest in value: $54\sqrt{3}, 144, 108\sqrt{6} - 108\sqrt{2}$.

5. [**6**] Charlie folds an $\frac{17}{2}$-inch by 11-inch piece of paper in half twice, each time along a straight line parallel to one of the paper's edges. What is the smallest possible perimeter of the piece after two such folds?

6. [**6**] To survive the coming Cambridge winter, Chim Tu doesn't wear one T-shirt, but instead wears up to FOUR T-shirts, all in different colors. An *outfit* consists of three or more T-shirts, put on one on top of the other in some order, such that two outfits are distinct if the sets of T-shirts used are different or the sets of T-shirts used are the same but the order in which they are worn is different. Given that Chim Tu changes his outfit every three days, and otherwise never wears the same outfit twice, how many days of winter can Chim Tu survive? (Needless to say, he only has four t-shirts.)

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### Round 3

7. [**7**] How many ordered triples of positive integers $(a, b, c)$ are there for which $a^4b^2c = 54000$?

8. [**7**] Let $a, b, c$ be not necessarily distinct integers between 1 and 2011, inclusive. Find the smallest possible value of $\dfrac{ab + c}{a + b + c}$.

9. [**7**] Unit circle $\Omega$ has points $X, Y, Z$ on its circumference so that $XYZ$ is an equilateral triangle. Let $W$ be a point other than $X$ in the plane such that triangle $WYZ$ is also equilateral. Determine the area of the region inside triangle $WYZ$ that lies outside circle $\Omega$.

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## Round 4

10. [**8**] Determine the number of integers $D$ such that whenever $a$ and $b$ are both real numbers with $-1/4 < a, b < 1/4$, then $|a^2 - Db^2| < 1$.

11. [**8**] For positive integers $m, n$, let $\gcd(m, n)$ denote the largest positive integer that is a factor of both $m$ and $n$. Compute
$$\sum_{n=1}^{91} \gcd(n, 91).$$

12. [**8**] Joe has written 5 questions of different difficulties for a test with problems numbered 1 though 5. He wants to make sure that problem $i$ is harder than problem $j$ whenever $i - j \geq 3$. In how many ways can he order the problems for his test?

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## Round 5

13. [**8**] Tac is dressing his cat to go outside. He has four indistinguishable socks, four indistinguishable shoes, and 4 indistinguishable show-shoes. In a hurry, Tac randomly pulls pieces of clothing out of a door and tries to put them on a random one of his cat's legs; however, Tac never tries to put more than one of each type of clothing on each leg of his cat. What is the probability that, after Tac is done, the snow-shoe on each of his cat's legs is on top of the shoe, which is on top of the sock?

14. [**8**] Let $AMOL$ be a quadrilateral with $AM = 10$, $MO = 11$, and $OL = 12$. Given that the perpendicular bisectors of sides $AM$ and $OL$ intersect at the midpoint of segment $AO$, find the length of side $LA$.

15. [**8**] For positive integers $n$, let $L(n)$ be the largest factor of $n$ other than $n$ itself. Determine the number of ordered pairs of composite positive integers $(m, n)$ for which $L(m)L(n) = 80$.

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## Round 6

16. [**10**] A small fish is holding 17 cards, labeled 1 through 17, which he shuffles into a random order. Then, he notices that although the cards are not currently sorted in ascending order, he can sort them into ascending order by removing one card and putting it back in a *different* position (at the beginning, between some two cards, or at the end). In how many possible orders could his cards currently be?

17. [**10**] For a positive integer $n$, let $p(n)$ denote the product of the positive integer factors of $n$. Determine the number of factors $n$ of 2310 for which $p(n)$ is a perfect square.

18. [**10**] Consider a cube $ABCDEFGH$, where $ABCD$ and $EFGH$ are faces, and segments $AE, BF, CG, DH$ are edges of the cube. Let $P$ be the center of face $EFGH$, and let $O$ be the center of the cube. Given that $AG = 1$, determine the area of triangle $AOP$.

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## Round 7

19. [**10**] Let $ABCD$ be a rectangle with $AB = 3$ and $BC = 7$. Let $W$ be a point on segment $AB$ such that $AW = 1$. Let $X, Y, Z$ be points on segments $BC, CD, DA$, respectively, so that quadrilateral $WXYZ$ is a rectangle, and $BX < XC$. Determine the length of segment $BX$.

20. [**10**] The UEFA Champions League playoffs is a 16-team soccer tournament in which Spanish teams always win against non-Spanish teams. In each of 4 rounds, each remaining team is *randomly* paired against one other team; the winner advances to the next round, and the loser is permanently knocked out of the tournament. If 3 of the 16 teams are Spanish, what is the probability that there are 2 Spanish teams in the final round?

21. [**10**] Let $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$ be a polynomial with roots $r_1, r_2, r_3, r_4$. Let $Q$ be the quartic polynomial with roots $r_1^2, r_2^2, r_3^2, r_4^2$, such that the coefficient of the $x^4$ term of $Q$ is 1. Simplify the quotient $Q(x^2)/P(x)$, leaving your answer in terms of $x$. (You may assume that $x$ is not equal to any of $r_1, r_2, r_3, r_4$).

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## Round 8

22. [**12**] Let $ABC$ be a triangle with $AB = 23$, $BC = 24$, and $CA = 27$. Let $D$ be the point on segment $AC$ such that the incircles of triangles $BAD$ and $BCD$ are tangent. Determine the ratio $CD/DA$.

23. [**12**] Let $N = \overline{5AB37C2}$, where $A, B, C$ are digits between 0 and 9, inclusive, and $N$ is a 7-digit positive integer. If $N$ is divisible by 792, determine all possible ordered triples $(A, B, C)$.

24. [**12**] Three not necessarily distinct positive integers between 1 and 99, inclusive, are written in a row on a blackboard. Then, the numbers, without including any leading zeros, are concatenated to form a new integer $N$. For example, if the integers written, in order, are 25, 6, and 12, then $N = 25612$ (and not $N = 250612$). Determine the number of possible values of $N$.

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## Round 9

*The answers to the following three problems are mutually dependent, although your answer to each will be graded independently. Let $A$ be the answer to problem 25, $B$ the answer to problem 26, and $C$ be the answer to problem 27.*

25. [**12**] Let $XYZ$ be an equilateral triangle, and let $K, L, M$ be points on sides $XY, YZ, ZX$, respectively, such that $XK/KY = B$, $YL/LZ = 1/C$, and $ZM/MX = 1$. Determine the ratio of the area of triangle $KLM$ to the area of triangle $XYZ$.

26. [**12**] Determine the positive real value of $x$ for which

$$\sqrt{2 + AC + 2Cx} + \sqrt{AC - 2 + 2Ax} = \sqrt{2(A + C)x + 2AC}.$$

27. [**12**] In-Young generates a string of $B$ zeroes and ones using the following method:

- First, she flips a fair coin. If it lands heads, her first digit will be a 0, and if it lands tails, her first digit will be a 1.
- For each subsequent bit, she flips an unfair coin, which lands heads with probability $A$. If the coin lands heads, she writes down the number (zero or one) different from previous digit, while if the coin lands tails, she writes down the previous digit again.

What is the expected value of the number of zeroes in her string?

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## Round 10

28. [**14**] Determine the value of

$$\sum_{k=1}^{2011} \frac{k-1}{k!(2011-k)!}.$$

29. [**14**] Let $ABC$ be a triangle with $AB = 4$, $BC = 8$, and $CA = 5$. Let $M$ be the midpoint of $BC$, and let $D$ be the point on the circumcircle of $ABC$ so that segment $AD$ intersects the interior of $ABC$, and $\angle BAD = \angle CAM$. Let $AD$ intersect side $BC$ at $X$. Compute the ratio $AX/AD$.

30. [**14**] Let $S$ be a set of consecutive positive integers such that for any integer $n$ in $S$, the sum of the digits of $n$ is not a multiple of 11. Determine the largest possible number of elements of $S$.

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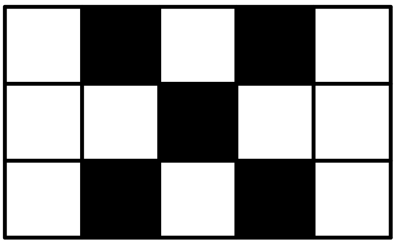
## Round 11

31. [**17**] Each square in a $3 \times 10$ grid is colored black or white. Let $N$ be the number of ways this can be done in such a way that no five squares in an 'X' configuration (as shown by the black squares below) are all white or all black. Determine $\sqrt{N}$.



32. [**17**] Find all real numbers $x$ satisfying

$$x^9 + \frac{9}{8}x^6 + \frac{27}{64}x^3 - x + \frac{219}{512} = 0.$$

33. [**17**] Let $ABC$ be a triangle with $AB = 5$, $BC = 8$, and $CA = 7$. Let $\Gamma$ be a circle internally tangent to the circumcircle of $ABC$ at $A$ which is also tangent to segment $BC$. $\Gamma$ intersects $AB$ and $AC$ at points $D$ and $E$, respectively. Determine the length of segment $DE$.

## Round 12

34. [**20**] The integer 843301 is prime. The *primorial* of a prime number $p$, denoted $p\#$, is defined to be the product of all prime numbers less than or equal to $p$. Determine the number of digits in $843301\#$. Your score will be

$$\max\left\{\left\lfloor 60\left(\frac{1}{3}-\left|\ln\left(\frac{A}{d}\right)\right|\right)\right\rfloor, 0\right\},$$

where $A$ is your answer and $d$ is the actual answer.

35. [**20**] Let $G$ be the number of Google hits of "guts round" at 10:31PM on October 31, 2011. Let $B$ be the number of Bing hits of "guts round" at the same time. Determine $B/G$. Your score will be

$$\max\left(0, \left\lfloor 20\left(1-\frac{20|a-k|}{k}\right)\right\rfloor\right),$$

where $k$ is the actual answer and $a$ is your answer.

36. [**20**] Order any subset of the following twentieth century mathematical achievements chronologically, from earliest to most recent. If you correctly place at least six of the events in order, your score will be $2(n-5)$, where $n$ is the number of events in your sequence; otherwise, your score will be zero. Note: if you order any number of events with one error, your score will be zero.

A). Axioms for Set Theory published by Zermelo

B). Category Theory introduced by Mac Lane and Eilenberg

C). Collatz Conjecture proposed

D). Erdos number defined by Goffman

E). First United States delegation sent to International Mathematical Olympiad

F). Four Color Theorem proven with computer assistance by Appel and Haken

G). Harvard-MIT Math Tournament founded

H). Hierarchy of grammars described by Chomsky

I). Hilbert Problems stated

J). Incompleteness Theorems published by Godel

K). Million dollar prize for Millennium Problems offered by Clay Mathematics Institute

L). Minimum number of shuffles needed to randomize a deck of cards established by Diaconis

M). Nash Equilibrium introduced in doctoral dissertation

N). Proof of Fermat's Last Theorem completed by Wiles

O). Quicksort algorithm invented by Hoare

Write your answer as a list of letters, without any commas or parentheses.

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**Saturday 12 November 2011**

## Guts Round

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School ____________________ Team ____________________ Team ID# __________

1. **[5]** ____________________
2. **[5]** ____________________
3. **[5]** ____________________

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School ____________________ Team ____________________ Team ID# __________

4. **[6]** ____________________
5. **[6]** ____________________
6. **[6]** ____________________

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School ____________________ Team ____________________ Team ID# __________

7. **[7]** ____________________
8. **[7]** ____________________
9. **[7]** ____________________

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School ____________________ Team ____________________ Team ID# __________

10. **[8]** ____________________
11. **[8]** ____________________
12. **[8]** ____________________

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School ____________________ Team ____________________ Team ID# __________

13. **[8]** ____________________
14. **[8]** ____________________
15. **[8]** ____________________

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School ____________________________ Team ____________________________ Team ID# __________

16. [**10**] ____________________

17. [**10**] ____________________

18. [**10**] ____________________

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School ____________________________ Team ____________________________ Team ID# __________

19. [**10**] ____________________

20. [**10**] ____________________

21. [**10**] ____________________

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School ____________________________ Team ____________________________ Team ID# __________

22. [**12**] ____________________

23. [**12**] ____________________

24. [**12**] ____________________

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School ____________________________ Team ____________________________ Team ID# __________

25. [**12**] ____________________

26. [**12**] ____________________

27. [**12**] ____________________

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School ____________________________ Team ____________________________ Team ID# __________

28. [**14**] ____________________

29. [**14**] ____________________

30. [**14**] ____________________

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School ______________________________ Team ______________________________ Team ID# ___________

31. [**17**] ______________________

32. [**17**] ______________________

33. [**17**] ______________________

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School ______________________________ Team ______________________________ Team ID# ___________

34. [**20**] ______________________

35. [**20**] ______________________

36. [**20**] ______________________