

# 11<sup>th</sup> Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

## Individual Round: General Test, Part 1

- [2] Let  $ABCD$  be a unit square (that is, the labels  $A, B, C, D$  appear in that order around the square). Let  $X$  be a point outside of the square such that the distance from  $X$  to  $AC$  is equal to the distance from  $X$  to  $BD$ , and also that  $AX = \frac{\sqrt{2}}{2}$ . Determine the value of  $CX^2$ .
- [3] Find the smallest positive integer  $n$  such that  $107n$  has the same last two digits as  $n$ .
- [3] There are 5 dogs, 4 cats, and 7 bowls of milk at an animal gathering. Dogs and cats are distinguishable, but all bowls of milk are the same. In how many ways can every dog and cat be paired with either a member of the other species or a bowl of milk such that all the bowls of milk are taken?
- [3] Positive real numbers  $x, y$  satisfy the equations  $x^2 + y^2 = 1$  and  $x^4 + y^4 = \frac{17}{18}$ . Find  $xy$ .
- [4] The function  $f$  satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

for all real numbers  $x, y$ . Determine the value of  $f(10)$ .

- [4] In a triangle  $ABC$ , take point  $D$  on  $BC$  such that  $DB = 14$ ,  $DA = 13$ ,  $DC = 4$ , and the circumcircle of  $ADB$  is congruent to the circumcircle of  $ADC$ . What is the area of triangle  $ABC$ ?
- [5] The equation  $x^3 - 9x^2 + 8x + 2 = 0$  has three real roots  $p, q, r$ . Find  $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$ .
- [5] Let  $S$  be the smallest subset of the integers with the property that  $0 \in S$  and for any  $x \in S$ , we have  $3x \in S$  and  $3x + 1 \in S$ . Determine the number of positive integers in  $S$  less than 2008.
- [5] A *Sudoku matrix* is defined as a  $9 \times 9$  array with entries from  $\{1, 2, \dots, 9\}$  and with the constraint that each row, each column, and each of the nine  $3 \times 3$  boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?

1								
	2							
			?					

- [6] Let  $ABC$  be an equilateral triangle with side length 2, and let  $\Gamma$  be a circle with radius  $\frac{1}{2}$  centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on  $\Gamma$ , visits all three sides of  $ABC$ , and ends somewhere on  $\Gamma$  (not necessarily at the starting point). Express your answer in the form of  $\sqrt{p} - q$ , where  $p$  and  $q$  are rational numbers written as reduced fractions.