11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

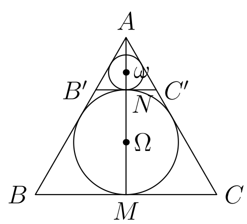
Individual Round: Geometry Test

1. [**3**] How many different values can $\angle ABC$ take, where A,B,C are distinct vertices of a cube?

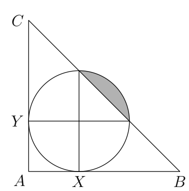
   Answer: $\boxed{5}$. In a unit cube, there are 3 types of triangles, with side lengths $(1, 1, \sqrt{2})$, $(1, \sqrt{2}, \sqrt{3})$ and $(\sqrt{2}, \sqrt{2}, \sqrt{2})$. Together they generate 5 different angle values.

2. [**3**] Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

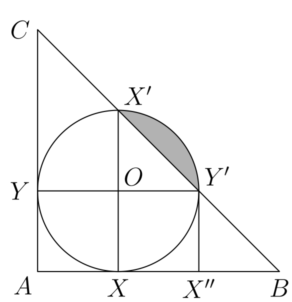
   Answer: $\boxed{3}$ Label the diagram as shown below, where Ω and ω also denote the center of the corresponding circles. Note that AM is a median and Ω is the centroid of the equilateral triangle. So $AM = 3M\Omega$. Since $M\Omega = N\Omega$, it follows that $AM/AN = 3$, and triangle ABC is the image of triangle $AB'C'$ after a scaling by a factor of 3, and so the two incircles must also be related by a scale factor of 3.



3. [**4**] Let ABC be a triangle with $\angle BAC = 90°$. A circle is tangent to the sides AB and AC at X and Y respectively, such that the points on the circle diametrically opposite X and Y both lie on the side BC. Given that $AB = 6$, find the area of the portion of the circle that lies outside the triangle.



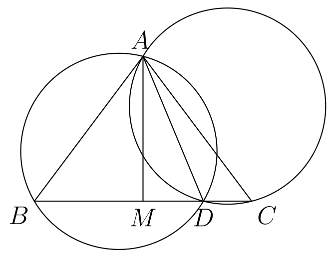
   Answer: $\boxed{\pi - 2}$ Let O be the center of the circle, and r its radius, and let X' and Y' be the points diametrically opposite X and Y, respectively. We have $OX' = OY' = r$, and $\angle X'OY' = 90°$. Since triangles $X'OY'$ and BAC are similar, we see that $AB = AC$. Let X'' be the projection of Y' onto AB. Since $X''BY'$ is similar to ABC, and $X''Y' = r$, we have $X''B = r$. It follows that $AB = 3r$, so $r = 2$.



Then, the desired area is the area of the quarter circle minus that of the triangle $X'OY'$. And the answer is $\frac{1}{4}\pi r^2 - \frac{1}{2}r^2 = \pi - 2$.

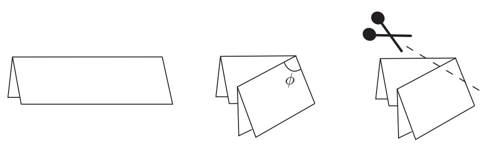
4. [**4**] In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

   Answer: $\boxed{108}$



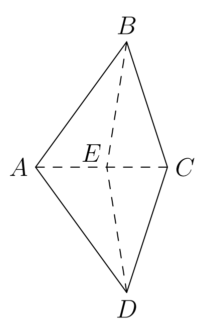
The fact that the two circumcircles are congruent means that the chord AD must subtend the same angle in both circles. That is, $\angle ABC = \angle ACB$, so ABC is isosceles. Drop the perpendicular M from A to BC; we know $MC = 9$ and so $MD = 5$ and by Pythagoras on AMD, $AM = 12$. Therefore, the area of ABC is $\frac{1}{2}(AM)(BC) = \frac{1}{2}(12)(18) = 108$.

5. [**5**] A piece of paper is folded in half. A second fold is made such that the angle marked below has measure ϕ ($0° < \phi < 90°$), and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O, and also that $\angle XOY = 144°$, where X and Y are the the vertices of the hole adjacent to O. Find the value(s) of ϕ (in degrees).

Answer: $\boxed{81°}$ Try actually folding a piece of paper. We see that the cut out area is a kite, as shown below. The fold was made on AC, and then BE and DE. Since DC was folded onto DA, we have $\angle ADE = \angle CDE$.



Either A or C is the center of the circle. If it's A, then $\angle BAD = 144°$, so $\angle CAD = 72°$. Using $CA = DA$, we see that $\angle ACD = \angle ADC = 54°$. So $\angle EDA = 27°$, and thus $\phi = 72° + 27° = 99°$, which is inadmissible, as $\phi < 90°$.

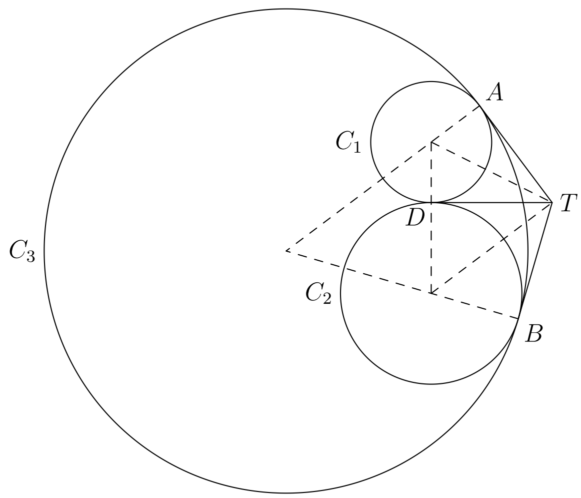
So C is the center of the circle. Then, $\angle CAD = \angle CDA = 54°$, $\angle ADE = 27°$, and $\phi = 54° + 27° = 81°$.

6. [**5**] Let ABC be a triangle with $\angle A = 45°$. Let P be a point on side BC with $PB = 3$ and $PC = 5$. Let O be the circumcenter of ABC. Determine the length OP.

Answer: $\boxed{\sqrt{17}}$ Using extended Sine law, we find the circumradius of ABC to be $R = \frac{BC}{2\sin A} = 4\sqrt{2}$. By considering the power of point P, we find that $R^2 - OP^2 = PB \cdot PC = 15$. So $OP = \sqrt{R^2 - 15} = \sqrt{16 \cdot 2 - 15} = \sqrt{17}$.

7. [**6**] Let C_1 and C_2 be externally tangent circles with radius 2 and 3, respectively. Let C_3 be a circle internally tangent to both C_1 and C_2 at points A and B, respectively. The tangents to C_3 at A and B meet at T, and $TA = 4$. Determine the radius of C_3.

   Answer: $\boxed{8}$ Let D be the point of tangency between C_1 and C_2. We see that T is the radical center of the three circles, and so it must lie on the radical axis of C_1 and C_2, which happens to be their common tangent TD. So $TD = 4$.



   We have

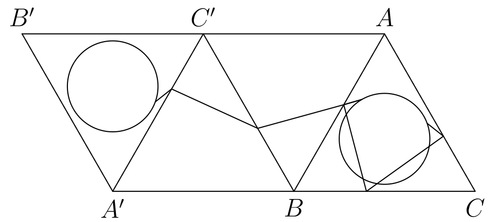
$$\tan \frac{\angle ATD}{2} = \frac{2}{TD} = \frac{1}{2}, \qquad \text{and} \qquad \tan \frac{\angle BTD}{2} = \frac{3}{TD} = \frac{3}{4}.$$

   Thus, the radius of C_3 equals to

$$\begin{aligned} TA \tan \frac{\angle ATB}{2} &= 4 \tan \left(\frac{\angle ATD + \angle BTD}{2} \right) \\ &= 4 \cdot \frac{\tan \frac{\angle ATD}{2} + \tan \frac{\angle BTD}{2}}{1 - \tan \frac{\angle ATD}{2} \tan \frac{\angle BTD}{2}} \\ &= 4 \cdot \frac{\frac{1}{2} + \frac{3}{4}}{1 - \frac{1}{2} \cdot \frac{3}{4}} \\ &= 8. \end{aligned}$$

8. [**6**] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ, visits all three sides of ABC, and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

   Answer: $\boxed{\sqrt{\frac{28}{3}} - 1}$ Suppose that the path visits sides AB, BC, CA in this order. Construct points A', B', C' so that C' is the reflection of C across AB, A' is the reflection of A across BC', and B' is the reflection of B across $A'C'$. Finally, let Γ' be the circle with radius $\frac{1}{2}$ centered at the center of $A'B'C'$. Note that Γ' is the image of Γ after the three reflections: $AB, BC', C'A'$.

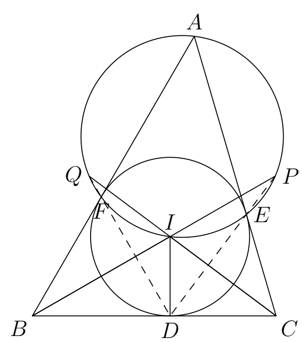


When the path hits AB, let us reflect the rest of the path across AB and follow this reflected path. When we hit BC', let us reflect the rest of the path across BC', and follow the new path. And when we hit $A'C'$, reflect the rest of the path across $A'C'$ and follow the new path. We must eventually end up at Γ'.

It is easy to see that the shortest path connecting some point on Γ to some point on Γ' lies on the line connecting the centers of the two circles. We can easily find the distance between the two centers to be $\sqrt{3^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{28}{3}}$. Therefore, the length of the shortest path connecting Γ to Γ' has length $\sqrt{\frac{28}{3}} - 1$. By reflecting this path three times back into ABC, we get a path that satisfies our conditions.

9. [**7**] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

Answer: $\boxed{\frac{75}{64}}$

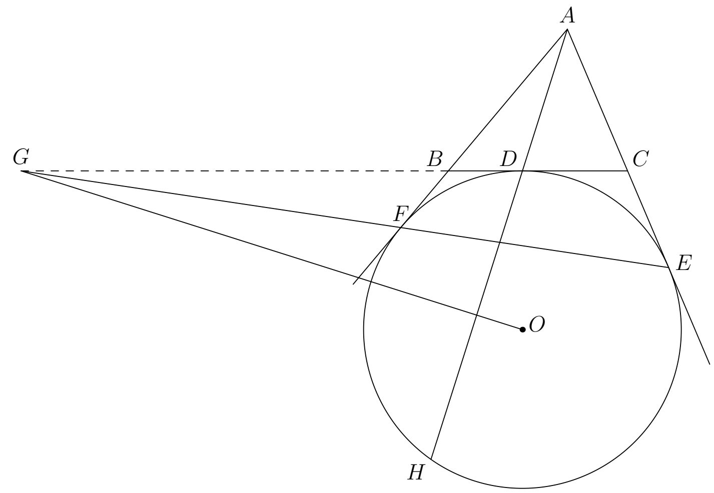


Let the incircle touch sides AC and AB at E and F respectively. Note that E and F both lie on the circle with diameter AI since $\angle AEI = \angle AFI = 90°$. The key observation is that D, E, P are collinear. To prove this, suppose that P lies outside the triangle (the other case is analogous), then $\angle PEA = \angle PIA = \angle IBA + \angle IAB = \frac{1}{2}(\angle B + \angle A) = 90° - \frac{1}{2}\angle C = \angle DEC$, which implies that D, E, P are collinear. Similarly D, F, Q are collinear. Then, by Power of a Point, $DE \cdot DP = DF \cdot DQ$. So $DP/DQ = DF/DE$.

Now we compute DF/DE. Note that $DF = 2DB \sin \angle DBI = 2\sqrt{6^2 - 3^2}\left(\frac{3}{6}\right) = 3\sqrt{3}$, and $DE = 2DC \sin \angle DCI = 2\sqrt{5^2 - 3^2}\left(\frac{3}{5}\right) = \frac{24}{5}$. Therefore, $DF/DE = \frac{5\sqrt{3}}{8}$.

10. [**7**] Let ABC be a triangle with $BC = 2007$, $CA = 2008$, $AB = 2009$. Let ω be an excircle of ABC that touches the line segment BC at D, and touches extensions of lines AC and AB at E and F, respectively (so that C lies on segment AE and B lies on segment AF). Let O be the center of ω. Let ℓ be the line through O perpendicular to AD. Let ℓ meet line EF at G. Compute the length DG.

Answer: $\boxed{2014024}$ Let line AD meet ω again at H. Since AF and AE are tangents to ω and ADH is a secant, we see that $DEHF$ is a harmonic quadrilateral. This implies that the pole of AD with respect to ω lies on EF. Since $\ell \perp AD$, the pole of AD lies on ℓ. It follows that the pole of AD is G.



Thus, G must lie on the tangent to ω at D, so C, D, B, G are collinear. Furthermore, since the pencil of lines $(AE, AF; AD, AG)$ is harmonic, by intersecting it with the line BC, we see that $(C, B; D, G)$ is harmonic as well. This means that

$$\frac{BD}{DC} \cdot \frac{CG}{GB} = -1.$$

(where the lengths are directed.) The semiperimeter of ABC is $s = \frac{1}{2}(2007 + 2008 + 2009) = 3012$. So $BD = s - 2009 = 1003$ and $CD = s - 2008 = 1004$. Let $x = DG$, then the above equations gives

$$\frac{1003}{1004} \cdot \frac{x + 1004}{x - 1003} = 1.$$

Solving gives $x = 2014024$.

Remark: If you are interested to learn about projective geometry, check out the last chapter of *Geometry Revisited* by Coxeter and Greitzer or *Geometric Transformations* III by Yaglom.