HMMT February 2016

February 20, 2016

Guts

1. [**5**] Let x and y be complex numbers such that $x+y=\sqrt{20}$ and $x^2+y^2=15$. Compute $|x-y|$.

   Proposed by: Evan Chen

   Answer: $\boxed{\sqrt{10}}$

   We have $(x-y)^2+(x+y)^2=2(x^2+y^2)$, so $(x-y)^2=10$, hence $|x-y|=\sqrt{10}$.

2. [**5**] Sherry is waiting for a train. Every minute, there is a 75% chance that a train will arrive. However, she is engrossed in her game of sudoku, so even if a train arrives she has a 75% chance of not noticing it (and hence missing the train). What is the probability that Sherry catches the train in the next five minutes?

   Proposed by:

   Answer: $\boxed{1-\left(\frac{13}{16}\right)^5}$

   During any given minute, the probability that Sherry doesn't catch the train is $\frac{1}{4}+\left(\frac{3}{4}\right)^2=\frac{13}{16}$. The desired probability is thus one minus the probability that she doesn't catch the train for the next five minutes: $1-\left(\frac{13}{16}\right)^5$.

3. [**5**] Let $PROBLEMZ$ be a regular octagon inscribed in a circle of unit radius. Diagonals MR, OZ meet at I. Compute LI.

   Proposed by: Evan Chen

   Answer: $\boxed{\sqrt{2}}$

   If W is the center of the circle then I is the incenter of $\triangle RWZ$. Moreover, $PRIZ$ is a rhombus. It follows that PI is twice the inradius of a 1-1-$\sqrt{2}$ triangle, hence the answer of $2-\sqrt{2}$. So $LI=\sqrt{2}$.

   Alternatively, one can show (note, really) that the triangle OIL is isosceles.

4. [**5**] Consider a three-person game involving the following three types of fair six-sided dice.

   - Dice of type A have faces labelled 2, 2, 4, 4, 9, 9.
   - Dice of type B have faces labelled 1, 1, 6, 6, 8, 8.
   - Dice of type C have faces labelled 3, 3, 5, 5, 7, 7.

   All three players simultaneously choose a die (more than one person can choose the same type of die, and the players don't know one another's choices) and roll it. Then the score of a player P is the number of players whose roll is less than P's roll (and hence is either 0, 1, or 2). Assuming all three players play optimally, what is the expected score of a particular player?

   Proposed by:

   Answer: $\boxed{\frac{8}{9}}$

   Short version: third player doesn't matter; against 1 opponent, by symmetry, you'd both play the same strategy. Type A beats B, B beats C, and C beats A all with probability 5/9. It can be determined that choosing each die with probability 1/3 is the best strategy. Then, whatever you pick, there is a 1/3 of dominating, a 1/3 chance of getting dominated, and a 1/3 chance of picking the same die (which gives a $1/3\cdot 2/3+1/3\cdot 1/3=1/3$ chance of rolling a higher number). Fix your selection; then the expected payout is then $1/3\cdot 5/9+1/3\cdot 4/9+1/3\cdot 1/3=1/3+1/9=4/9$. Against 2 players, your EV is just $E(p1)+E(p2)=2E(p1)=8/9$

5. [**6**] Patrick and Anderson are having a snowball fight. Patrick throws a snowball at Anderson which is shaped like a sphere with a radius of 10 centimeters. Anderson catches the snowball and uses the snow from the snowball to construct snowballs with radii of 4 centimeters. Given that the total volume of the snowballs that Anderson constructs cannot exceed the volume of the snowball that Patrick threw, how many snowballs can Anderson construct?

   Proposed by:

   Answer: $\boxed{15}$

$$\left\lfloor \left(\frac{10}{4}\right)^3 \right\rfloor = \left\lfloor \frac{125}{8} \right\rfloor = 15.$$

6. [**6**] Consider a $2 \times n$ grid of points and a path consisting of $2n - 1$ straight line segments connecting all these $2n$ points, starting from the bottom left corner and ending at the upper right corner. Such a path is called *efficient* if each point is only passed through once and no two line segments intersect. How many efficient paths are there when $n = 2016$?

   Proposed by: Casey Fu

   Answer: $\boxed{\binom{4030}{2015}}$

   The general answer is $\binom{2(n-1)}{n-1}$: Simply note that the points in each column must be taken in order, and anything satisfying this avoids intersections, so just choose the steps during which to be in the first column.

7. [**6**] A contest has six problems worth seven points each. On any given problem, a contestant can score either 0, 1, or 7 points. How many possible total scores can a contestant achieve over all six problems?

   Proposed by: Evan Chen

   Answer: $\boxed{28}$

   For $0 \le k \le 6$, to obtain a score that is $k \pmod 6$ exactly k problems must get a score of 1. The remaining $6 - k$ problems can generate any multiple of 7 from 0 to $7(6 - k)$, of which there are $7 - k$. So the total number of possible scores is $\sum_{k=0}^{6}(7-k) = 28$.

8. [**6**] For each positive integer n and non-negative integer k, define $W(n, k)$ recursively by

$$W(n,k) = \begin{cases} n^n & k = 0 \\ W(W(n,k-1),k-1) & k > 0. \end{cases}$$

   Find the last three digits in the decimal representation of $W(555, 2)$.

   Proposed by:

   Answer: $\boxed{875}$

   For any n, we have

$$W(n,1) = W(W(n,0),0) = (n^n)^{n^n} = n^{n^{n+1}}.$$

   Thus,

$$W(555,1) = 555^{555^{556}}.$$

   Let $N = W(555, 1)$ for brevity, and note that $N \equiv 0 \pmod{125}$, and $N \equiv 3 \pmod 8$. Then,

$$W(555,2) = W(N,1) = N^{N^{N+1}}$$

   is $0 \pmod{125}$ and $3 \pmod 8$.

   From this we can conclude (by the Chinese Remainder Theorem) that the answer is 875.

9. [**7**] Victor has a drawer with two red socks, two green socks, two blue socks, two magenta socks, two lavender socks, two neon socks, two mauve socks, two wisteria socks, and 2000 copper socks, for a total of 2016 socks. He repeatedly draws two socks at a time from the drawer at random, and stops if the socks are of the same color. However, Victor is red-green colorblind, so he also stops if he sees a red and green sock.

   What is the probability that Victor stops with two socks of the same color? Assume Victor returns both socks to the drawer at each step.

   Proposed by: Evan Chen

   Answer: $\boxed{\frac{1999008}{1999012}}$

   There are $\binom{2000}{2} + 8\binom{2}{2} = 1999008$ ways to get socks which are matching colors, and four extra ways to get a red-green pair, hence the answer.

10. [**7**] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let O be the circumcenter of ABC. Find the distance between the circumcenters of triangles AOB and AOC.

    Proposed by: Evan Chen

    Answer: $\boxed{\frac{91}{6}}$

    Let S, T be the intersections of the tangents to the circumcircle of ABC at A, C and at A, B respectively. Note that $ASCO$ is cyclic with diameter SO, so the circumcenter of AOC is the midpoint of OS, and similarly for the other side. So the length we want is $\frac{1}{2}ST$. The circumradius R of ABC can be computed by Heron's formula and $K = \frac{abc}{4R}$, giving $R = \frac{65}{8}$. A few applications of the Pythagorean theorem and similar triangles gives $AT = \frac{65}{6}, AS = \frac{39}{2}$, so the answer is $\boxed{\dfrac{91}{6}}$.

11. [**7**] Define $\phi^!(n)$ as the product of all positive integers less than or equal to n and relatively prime to n. Compute the remainder when

    $$\sum_{\substack{2 \le n \le 50 \\ \gcd(n,50)=1}} \phi^!(n)$$

    is divided by 50.

    Proposed by: Evan Chen

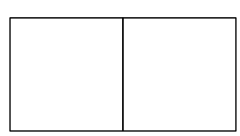
    Answer: $\boxed{12}$

    First, $\phi^!(n)$ is even for all odd n, so it vanishes modulo 2.

    To compute the remainder modulo 25, we first evaluate $\phi^!(3) + \phi^!(7) + \phi^!(9) \equiv 2 + 5 \cdot 4 + 5 \cdot 3 \equiv 12 \pmod{25}$. Now, for $n \ge 11$ the contribution modulo 25 vanishes as long as $5 \nmid n$.

    We conclude the answer is 12.

12. [**7**] Let R be the rectangle in the Cartesian plane with vertices at $(0,0), (2,0), (2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.

    Compute the number of ways to choose one or more of the seven edges such that the resulting figure is traceable without lifting a pencil. (Rotations and reflections are considered distinct.)

    Proposed by: Joy Zheng

    Answer: $\boxed{61}$

We have two cases, depending on whether we choose the middle edge. If so, then either all the remaining edges are either to the left of or to the right of this edge, or there are edges on both sides, or neither; in the first two cases there are 6 ways each, in the third there are $16+1=17$ ways, and in the last there is 1 way. Meanwhile, if we do not choose the middle edge, then we have to choose a beginning and endpoint, plus the case where we have a loop, for a total of $6 \cdot 5 + 1 = 31$ cases. This gives a total of $6+6+17+1+31 = 61$ possible cases.

13. [**9**] A right triangle has side lengths a, b, and $\sqrt{2016}$ in some order, where a and b are positive integers. Determine the smallest possible perimeter of the triangle.

    Proposed by: Evan Chen

    Answer: $\boxed{48+\sqrt{2016}}$

    There are no integer solutions to $a^2+b^2=2016$ due to the presence of the prime 7 on the right-hand side (by Fermat' s Christmas Theorem). Assuming $a<b$, the minimal solution $(a,b)=(3,45)$ which gives the answer above.

14. [**9**] Let ABC be a triangle such that $AB=13$, $BC=14$, $CA=15$ and let E, F be the feet of the altitudes from B and C, respectively. Let the circumcircle of triangle AEF be ω. We draw three lines, tangent to the circumcircle of triangle AEF at A, E, and F. Compute the area of the triangle these three lines determine.

    Proposed by: Christopher Shao

    Answer: $\boxed{\frac{462}{5}}$

    Note that $AEF \sim ABC$. Let the vertices of the triangle whose area we wish to compute be P,Q,R, opposite A,E,F respectively. Since H,O are isogonal conjugates, line AH passes through the circumcenter of AEF, so $QR \parallel BC$.

    Let M be the midpoint of BC. We claim that $M=P$. This can be seen by angle chasing at E,F to find that $\angle PFB = \angle ABC$, $\angle PEC = \angle ACB$, and noting that M is the circumcenter of $BFEC$. So, the height from P to QR is the height from A to BC, and thus if K is the area of ABC, the area we want is $\frac{QR}{BC}K$.

    Heron's formula gives $K=84$, and similar triangles QAF, MBF and RAE, MCE give $QA = \frac{BC}{2}\frac{\tan B}{\tan A}$, $RA = \frac{BC}{2}\frac{\tan C}{\tan A}$, so that $\frac{QR}{BC} = \frac{\tan B + \tan C}{2\tan A} = \frac{\tan B \tan C - 1}{2} = \frac{11}{10}$,

    since the height from A to BC is 12. So our answer is $\boxed{\dfrac{462}{5}}$.

15. [**9**] Compute $\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right)$.

    Proposed by: Alexander Katz

    Answer: $\boxed{\sqrt{7}}$

    Consider the polynomial $P(z) = z^7 - 1$. Let $z = e^{ix} = \cos x + i \sin x$. Then

$$\begin{aligned} z^7 - 1 = &\left(\cos^7 x - \binom{7}{2}\cos^5 x \sin^2 x + \binom{7}{4}\cos^3 x \sin^4 x - \binom{7}{6}\cos x \sin^6 x - 1\right) \\ &+ i\left(-\sin^7 x + \binom{7}{2}\sin^5 x \cos^2 x - \binom{7}{4}\sin^3 x \cos^4 x + \binom{7}{6}\sin x \cos 6x\right) \end{aligned}$$

Consider the real part of this equation. We may simplify it to $64\cos^7 x - \ldots - 1$, where the middle terms are irrelevant. The roots of P are $x = \frac{2\pi}{7}, \frac{4\pi}{7}, \ldots$, so $\prod_{k=1}^{7} \cos\left(\frac{2\pi k}{7}\right) = \frac{1}{64}$. But

$$\prod_{k=1}^{7} \cos\left(\frac{2\pi k}{7}\right) = \left(\prod_{k=1}^{3} \cos\left(\frac{k\pi}{7}\right)\right)^2$$

so $\prod_{k=1}^{3} \cos\left(\frac{k\pi}{7}\right) = \frac{1}{8}$.

Now consider the imaginary part of this equation. We may simplify it to $-64\sin^{11} x + \ldots + 7\sin x$, where again the middle terms are irrelevant. We can factor out $\sin x$ to get $-64\sin^{10} x + \ldots + 7$, and this polynomial has roots $x = \frac{2\pi}{7}, \ldots, \frac{12\pi}{7}$ (but not 0). Hence $\prod_{k=1}^{6} \sin\left(\frac{2\pi k}{7}\right) = -\frac{7}{64}$. But, like before, we have

$$\prod_{k=1}^{6} \sin\left(\frac{2\pi k}{7}\right) = -\left(\prod_{k=1}^{3} \sin\left(\frac{2\pi k}{7}\right)\right)^2$$

hence $\prod_{k=1}^{3} \sin\left(\frac{k\pi}{7}\right) = \frac{\sqrt{7}}{8}$. As a result, our final answer is $\frac{\frac{\sqrt{7}}{8}}{\frac{1}{8}} = \boxed{\sqrt{7}}$.

16. [**9**] Determine the number of integers $2 \leq n \leq 2016$ such that $n^n - 1$ is divisible by 2, 3, 5, 7.

Proposed by: Evan Chen

Answer: $\boxed{9}$

Only $n \equiv 1 \pmod{210}$ work. Proof: we require $\gcd(n, 210) = 1$. Note that $\forall p \leq 7$ the order of $n \pmod{p}$ divides $p - 1$, hence is relatively prime to any $p \leq 7$. So $n^n \equiv 1 \pmod{p} \iff n \equiv 1 \pmod{p}$ for each of these p.

17. [**11**] Compute the sum of all integers $1 \leq a \leq 10$ with the following property: there exist integers p and q such that p, q, $p^2 + a$ and $q^2 + a$ are all distinct prime numbers.

Proposed by: Evan Chen

Answer: $\boxed{20}$

Odd a fail for parity reasons and $a \equiv 2 \pmod{3}$ fail for mod 3 reasons. This leaves $a \in \{4, 6, 10\}$. It is easy to construct p and q for each of these, take $(p, q) = (3, 5)$, $(5, 11)$, $(3, 7)$, respectively.

18. [**11**] Alice and Bob play a game on a circle with 8 marked points. Alice places an apple beneath one of the points, then picks five of the other seven points and reveals that none of them are hiding the apple. Bob then drops a bomb on any of the points, and destroys the apple if he drops the bomb either on the point containing the apple or on an adjacent point. Bob wins if he destroys the apple, and Alice wins if he fails. If both players play optimally, what is the probability that Bob destroys the apple?

Proposed by:

Answer: $\boxed{\frac{1}{2}}$

Let the points be $0, \ldots, 7 \pmod{8}$, and view Alice's reveal as revealing the three possible locations of the apple. If Alice always picks $0, 2, 4$ and puts the apple randomly at 0 or 4, by symmetry Bob cannot achieve more than $\frac{1}{2}$. Here's a proof that $\frac{1}{2}$ is always possible.

Among the three revealed indices a, b, c, positioned on a circle, two must (in the direction in which they're adjacent) have distance at least 3, so without loss of generality the three are $0, b, c$ where $1 \leq b < c \leq 5$. Modulo reflection and rotation, the cases are: $(0, 1, 2)$: Bob places at 1 and wins. $(0, 1, 3)$: Bob places at 1 half the time and 3 half the time, so wherever the apple is Bob wins with probability $\frac{1}{2}$. $(0, 1, 4)$: Bob places at 1 or 4, same as above. $(0, 2, 4)$: Bob places at 1 or 3, same as above. $(0, 2, 5)$: Bob places at 1 or 5, same as above.

These cover all cases, so we're done.

19. [**11**] Let

$$A = \lim_{n\to\infty} \sum_{i=0}^{2016} (-1)^i \cdot \frac{\binom{n}{i}\binom{n}{i+2}}{\binom{n}{i+1}^2}$$

Find the largest integer less than or equal to $\frac{1}{A}$.

The following decimal approximation might be useful: $0.6931 < \ln(2) < 0.6932$, where ln denotes the natural logarithm function.

Proposed by: Pakawut Jiradilok

Answer: $\boxed{1}$

Note

$$\sum_{i=0}^{2016}(-1)^i\cdot\frac{\binom{n}{i}\binom{n}{i+2}}{\binom{n}{i+1}^2}=\sum_{i=0}^{2016}(-1)^i\cdot\frac{(i+1)(n-i-1)}{(i+2)(n-i)},$$

So

$$\lim_{n\to\infty}\sum_{i=0}^{2016}(-1)^i\cdot\frac{\binom{n}{i}\binom{n}{i+2}}{\binom{n}{i+1}^2}=\sum_{i=0}^{2016}(-1)^i\cdot\frac{(i+1)}{(i+2)}=1-\sum_{i=2}^{2016}\frac{(-1)^i}{i}\approx\ln(2).$$

Then $\frac{1}{A}\approx\frac{1}{\ln(2)}\approx 1.44$, so the answer is 1.

20. [**11**] Let ABC be a triangle with $AB=13$, $AC=14$, and $BC=15$. Let G be the point on AC such that the reflection of BG over the angle bisector of $\angle B$ passes through the midpoint of AC. Let Y be the midpoint of GC and X be a point on segment AG such that $\frac{AX}{XG}=3$. Construct F and H on AB and BC, respectively, such that $FX\parallel BG\parallel HY$. If AH and CF concur at Z and W is on AC such that $WZ\parallel BG$, find WZ.

Proposed by: Ritesh Ragavender

Answer: $\boxed{\frac{1170\sqrt{37}}{1379}}$

Observe that BG is the B-symmedian, and thus $\frac{AG}{GC}=\frac{c^2}{a^2}$. Stewart's theorem gives us

$$BG=\sqrt{\frac{2a^2c^2b}{b(a^2+c^2)}-\frac{a^2b^2c^2}{a^2+c^2}}=\frac{ac}{a^2+c^2}\sqrt{2(a^2+c^2)-b^2}=\frac{390\sqrt{37}}{197}.$$

Then by similar triangles,

$$ZW=HY\frac{ZA}{HA}=BG\frac{YC}{GC}\frac{ZA}{HA}=BG\frac{1}{2}\frac{6}{7}=\boxed{\frac{1170\sqrt{37}}{1379}}$$

where $\frac{ZA}{HA}$ is found with mass points or Ceva.

21. [**12**] Tim starts with a number n, then repeatedly flips a fair coin. If it lands heads he subtracts 1 from his number and if it lands tails he subtracts 2. Let E_n be the expected number of flips Tim does before his number is zero or negative. Find the pair (a,b) such that

$$\lim_{n\to\infty}(E_n-an-b)=0.$$

Proposed by:

Answer: $\boxed{\left(\frac{2}{3},\frac{2}{9}\right)}$

We have the recurrence $E_n=\frac{1}{2}(E_{n-1}+1)+\frac{1}{2}(E_{n-2}+1)$, or $E_n=1+\frac{1}{2}(E_{n-1}+E_{n-2})$, for $n\geq 2$.

Let $F_n=E_n-\frac{2}{3}n$. By directly plugging this into the recurrence for E_n, we get the recurrence $F_n=\frac{1}{2}(F_{n-1}+F_{n-1})$. The roots of the characteristic polynomial of this recurrence are 1 and $-\frac{1}{2}$, so $F_n=A+B(-\frac{1}{2})^n$ for some A and B depending on the initial conditions. But clearly we have $E_0=0$ and $E_1=1$ so $F_0=0$ and $F_1=\frac{1}{3}$ so $A=\frac{2}{9}$ and $B=-\frac{2}{9}$.

Hence, $E_n=\frac{2}{3}n+\frac{2}{9}-\frac{2}{9}(-\frac{1}{2})^n$, so $\lim_{n\to\infty}(E_n-\frac{2}{3}n-\frac{2}{9})=0$. Hence $(\frac{2}{3},\frac{2}{9})$ is the desired pair.

22. [**12**] On the Cartesian plane $\mathbb{R}^2$, a circle is said to be *nice* if its center is at the origin $(0,0)$ and it passes through at least one lattice point (i.e. a point with integer coordinates). Define the points $A=(20,15)$ and $B=(20,16)$. How many nice circles intersect the open segment AB?

For reference, the numbers 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691 are the only prime numbers between 600 and 700.

Proposed by:

Answer: $\boxed{10}$

The square of the radius of a nice circle is the sum of the square of two integers.

The nice circle of radius r intersects (the open segment) $\overline{AB}$ if and only if a point on $\overline{AB}$ is a distance r from the origin. $\overline{AB}$ consists of the points $(20, t)$ where t ranges over $(15, 16)$. The distance from the origin is $\sqrt{20^2 + t^2} = \sqrt{400 + t^2}$. As t ranges over $(15, 16)$, $\sqrt{400 + t^2}$ ranges over $(\sqrt{625}, \sqrt{656})$, so the nice circle of radius r intersects $\overline{AB}$ if and only if $625 < r^2 < 656$.

The possible values of r^2 are those in this range that are the sum of two perfect squares, and each such value corresponds to a unique nice circle. By Fermat's Christmas theorem, an integer is the sum of two squares if an only if in its prime factorization, each prime that is 3 mod 4 appears with an even exponent (possibly 0.) In addition, since squares are 0, 1, or 4 mod 8, we can quickly eliminate integers that are 3, 6, or 7 mod 8.

Now I will list all the integers that aren't 3, 6, or 7 mod 8 in the range and either supply the bad prime factor or write "nice" with the prime factorization.

626: nice $(2 \cdot 313)$

628: nice $(2^2 \cdot 157)$

629: nice $(17 \cdot 37)$

632: 79

633: 3

634: nice $(2 \cdot 317)$

636: 3

637: nice $(7^2 \cdot 13)$

640: nice $(2^7 \cdot 5)$

641: nice (641)

642: 3

644: 7

645: 3

648: nice $(2^3 \cdot 3^4)$

649: 11

650: nice $(2 \cdot 5^2 \cdot 13)$

652: 163

653: nice (653). There are 10 nice circles that intersect $\overline{AB}$.

23. [**12**] Let $t = 2016$ and $p = \ln 2$. Evaluate in closed form the sum

$$\sum_{k=1}^{\infty} \left(1 - \sum_{n=0}^{k-1} \frac{e^{-t} t^n}{n!}\right) (1-p)^{k-1} p.$$

Proposed by: Aaron Landesman

Answer: $\boxed{1 - \left(\frac{1}{2}\right)^{2016}}$

Let $q = 1 - p$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty}\left(1-\sum_{n=0}^{k-1}\frac{e^{-t}t^n}{n!}\right)q^{k-1}p &= \sum_{k=1}^{\infty}q^{k-1}p - \sum_{k=1}^{\infty}\sum_{n=0}^{k-1}\frac{e^{-t}t^n}{n!}q^{k-1}p \\
&= 1 - \sum_{k=1}^{\infty}\sum_{n=0}^{k-1}\frac{e^{-t}t^n}{n!}q^{k-1}p \\
&= 1 - \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\frac{e^{-t}t^n}{n!}q^{k-1}p \\
&= 1 - \sum_{n=0}^{\infty}\frac{e^{-t}t^n}{n!}q^n \\
&= 1 - \sum_{n=0}^{\infty}\frac{e^{-t}(qt)^n}{n!} = 1 - e^{-t}e^{qt} = 1 - e^{-pt}.
\end{aligned}$$

Thus the answer is $1 - \left(\frac{1}{2}\right)^{2016}$.

24. [**12**] Let ΔA_1B_1C be a triangle with $\angle A_1B_1C = 90^\circ$ and $\frac{CA_1}{CB_1} = \sqrt{5}+2$. For any $i \geq 2$, define A_i to be the point on the line A_1C such that $A_iB_{i-1} \perp A_1C$ and define B_i to be the point on the line B_1C such that $A_iB_i \perp B_1C$. Let Γ_1 be the incircle of ΔA_1B_1C and for $i \geq 2$, Γ_i be the circle tangent to Γ_{i-1}, A_1C, B_1C which is smaller than Γ_{i-1}.

How many integers k are there such that the line A_1B_{2016} intersects Γ_k?

Proposed by:

Answer: $\boxed{4030}$

We claim that Γ_2 is the incircle of $\triangle B_1A_2C$. This is because $\triangle B_1A_2C$ is similar to A_1B_1C with dilation factor $\sqrt{5}-2$, and by simple trigonometry, one can prove that Γ_2 is similar to Γ_1 with the same dilation factor. By similarities, we can see that for every k, the incircle of $\triangle A_kB_kC$ is Γ_{2k-1}, and the incircle of $\triangle B_kA_{k+1}C$ is Γ_{2k}. Therefore, A_1B_{2016} intersects all $\Gamma_1, \ldots, \Gamma_{4030}$ but not Γ_k for any $k \geq 4031$.

25. [**14**] A particular coin can land on heads (H), on tails (T), or in the middle (M), each with probability $\frac{1}{3}$. Find the expected number of flips necessary to observe the contiguous sequence HMMTHMMT...HMMT, where the sequence HMMT is repeated 2016 times.

Proposed by: Ritesh Ragavender

Answer: $\boxed{\frac{3^{8068}-81}{80}}$

Let E_0 be the expected number of flips needed. Let E_1 be the expected number more of flips needed if the first flip landed on H. Let E_2 be the expected number more if the first two landed on HM. In general, let E_k be the expected number more of flips needed if the first k flips landed on the first k values of the sequence HMMTHMMT...HMMT.

We have

$$E_i = \begin{cases} 1 + \frac{1}{3}E_{i+1} + \frac{1}{3}E_1 + \frac{1}{3}E_0 & i \not\equiv 0 \pmod 4 \\ 1 + \frac{1}{3}E_{i+1} + \frac{2}{3}E_0 & i \equiv 0 \pmod 4 \end{cases}$$

Using this relation for $i = 0$ gives us $E_1 = E_0 - 3$. Let $F_i = \frac{1}{3^i}E_i$. By simple algebraic manipulations we have

$$F_{i+1} - F_i = \begin{cases} -\frac{2}{3^{i+1}} \cdot E_0 & i \not\equiv 0 \pmod 4 \\ -\frac{1}{3^i} - \frac{2}{3^{i+1}} \cdot E_0 & i \equiv 0 \pmod 4 \end{cases}$$

We clearly have $F_{2016\cdot 4} = 0$ and $F_0 = E_0$. So adding up the above relations for $i = 0$ to $i = 2016 \cdot 4 - 1$ gives

$$
\begin{aligned}
-E_0 &= -2E_0 \sum_{i=1}^{2016\cdot 4} \frac{1}{3^i} - \sum_{k=0}^{2015} \frac{1}{3^{4k}} \\
&= E_0 \left(\frac{1}{3^{2016\cdot 4}} - 1 \right) - \frac{1 - \frac{1}{3^{2016\cdot 4}}}{\frac{80}{81}}
\end{aligned}
$$

so $E_0 = \frac{3^{8068}-81}{80}$.

26. [**14**] For positive integers a, b, $a \uparrow\uparrow b$ is defined as follows: $a \uparrow\uparrow 1 = a$, and $a \uparrow\uparrow b = a^{a \uparrow\uparrow (b-1)}$ if $b > 1$.

   Find the smallest positive integer n for which there exists a positive integer a such that $a \uparrow\uparrow 6 \not\equiv a \uparrow\uparrow 7 \mod n$.

   Proposed by: Sammy Luo

   Answer: $\boxed{283}$

   We see that the smallest such n must be a prime power, because if two numbers are distinct mod n, they must be distinct mod at least one of the prime powers that divide n. For $k \geq 2$, if $a \uparrow\uparrow k$ and $a \uparrow\uparrow (k+1)$ are distinct mod p^r, then $a \uparrow\uparrow (k-1)$ and $a \uparrow\uparrow k$ must be distinct mod $\phi(p^r)$. In fact they need to be distinct mod $\frac{\phi(p^r)}{2}$ if $p = 2$ and $r \geq 3$ because then there are no primitive roots mod p^r.

   Using this, for $1 \leq k \leq 5$ we find the smallest prime p such that there exists a such that $a \uparrow\uparrow k$ and $a \uparrow\uparrow (k+1)$ are distinct mod p. The list is: $3, 5, 11, 23, 47$. We can easily check that the next largest prime for $k = 5$ is 139, and also any prime power other than 121 for which $a \uparrow\uparrow 5$ and $a \uparrow\uparrow 6$ are distinct is also larger than 139.

   Now if $a \uparrow\uparrow 6$ and $a \uparrow\uparrow 7$ are distinct mod p, then $p - 1$ must be a multiple of 47 or something that is either 121or at least 139. It is easy to see that 283 is the smallest prime that satisfies this.

   If n is a prime power less than 283 such that $a \uparrow\uparrow 6$ and $a \uparrow\uparrow 7$ are distinct mod n, then the prime can be at most 13 and clearly this doesn't work because $\phi(p^r) = p^{r-1}(p-1)$.

   To show that 283 works, choose a so that a is a primitive root mod $283, 47, 23, 11, 5$ and 3. This is possible by the Chinese Remainder theorem, and it is easy to see that this a works by induction.

27. [**14**] Find the smallest possible area of an ellipse passing through $(2,0)$, $(0,3)$, $(0,7)$, and $(6,0)$.

   Proposed by: Calvin Deng

   Answer: $\boxed{\frac{56\pi\sqrt{3}}{9}}$

   Let Γ be an ellipse passing through $A = (2,0), B = (0,3), C = (0,7), D = (6,0)$, and let $P = (0,0)$ be the intersection of AD and BC. $\frac{\text{Area of }\Gamma}{\text{Area of }ABCD}$ is unchanged under an affine transformation, so we just have to minimize this quantity over situations where Γ is a circle and $\frac{PA}{PD} = \frac{1}{3}$ and $\frac{PB}{BC} = \frac{3}{7}$. In fact, we may assume that $PA = \sqrt{7}, PB = 3, PC = 7, PD = 3\sqrt{7}$. If $\angle P = \theta$, then we can compute lengths to get

$$
r = \frac{\text{Area of }\Gamma}{\text{Area of }ABCD} = \pi \frac{32 - 20\sqrt{7}\cos\theta + 21\cos^2\theta}{9\sqrt{7}\cdot\sin^3\theta}
$$

   Let $x = \cos\theta$. Then if we treat r as a function of x,

$$
0 = \frac{r'}{r} = \frac{3x}{1-x^2} + \frac{42x - 20\sqrt{7}}{32 - 20x\sqrt{7} + 21x^2}
$$

   which means that $21x^3 - 40x\sqrt{7} + 138x - 20\sqrt{7} = 0$. Letting $y = x\sqrt{7}$ gives

$$
0 = 3y^3 - 40y^2 + 138y - 140 = (y-2)(3y^2 - 34y + 70)
$$

   The other quadratic has roots that are greater than $\sqrt{7}$, which means that the minimum ratio is attained when $\cos\theta = x = \frac{y}{\sqrt{7}} = \frac{2}{\sqrt{7}}$. Plugging that back in gives that the optimum $\frac{\text{Area of }\Gamma}{\text{Area of }ABCD}$ is

$\frac{28\pi\sqrt{3}}{81}$, so putting this back into the original configuration gives Area of $\Gamma \geq \frac{56\pi\sqrt{3}}{9}$. If you want to check on Geogebra, this minimum occurs when the center of Γ is $(\frac{8}{3}, \frac{7}{3})$.

28. [**14**] Among citizens of Cambridge there exist 8 different types of blood antigens. In a crowded lecture hall are 256 students, each of whom has a blood type corresponding to a distinct subset of the antigens; the remaining of the antigens are foreign to them.

Quito the Mosquito flies around the lecture hall, picks a subset of the students uniformly at random, and bites the chosen students in a random order. After biting a student, Quito stores a bit of any antigens that student had. A student bitten while Quito had k blood antigen foreign to him/her will suffer for k hours. What is the expected total suffering of all 256 students, in hours?

Proposed by: Sammy Luo

Answer: $\boxed{\frac{2^{135}-2^{128}+1}{2^{119}\cdot 129}}$

Let $n = 8$.

First, consider any given student S and an antigen a foreign to him/her. Assuming S has been bitten, we claim the probability S will suffer due to a is

$$1 - \frac{2^{2^{n-1}+1} - 1}{2^{2^{n-1}}(2^{n-1}+1)}.$$

Indeed, let $N = 2^{n-1}$ denote the number of students with a. So considering just these students and summing over the number bitten, we obtain a probability

$$\frac{1}{2^N} \sum_{t=0}^{N} \binom{N}{t}\binom{N}{t} \frac{t}{t+1} = \frac{1}{2^N} \frac{2^N N - 2^N + 1}{N+1}.$$

We now use linearity over all pairs (S, a) of students S and antigens a foreign to them. Noting that each student is bitten with probability $\frac{1}{2}$, and retaining the notation $N = 2^{n-1}$, we get

$$\frac{1}{2} \sum_{k=0}^{n} \left[\binom{n}{k} \cdot k \left(\frac{2^N N - 2^N + 1}{2^N(N+1)}\right)\right] = \frac{nN(2^N N - 2^N + 1)}{2^{N+1}(N+1)}.$$

Finally, setting $n = 8 = 2^3$ and $N = 2^{n-1} = 2^7 = 128$, we get the claimed answer.

29. [**16**] Katherine has a piece of string that is 2016 millimeters long. She cuts the string at a location chosen uniformly at random, and takes the left half. She continues this process until the remaining string is less than one millimeter long. What is the expected number of cuts that she makes?

Proposed by:

Answer: $\boxed{1 + \log(2016)}$

Letting $f(x)$ be the expected number of cuts if the initial length of the string is x, we get the integral equation $f(x) = 1 + \frac{1}{x}\int_1^x f(y)dy$. Letting $g(x) = \int_1^x f(y)dy$, we get $dg/dx = 1 + \frac{1}{x}g(x)$. Using integrating factors, we see that this has as its solution $g(x) = x\log(x)$, and thus $f(x) = 1 + \log(x)$.

30. [**16**] Determine the number of triples $0 \leq k, m, n \leq 100$ of integers such that

$$2^m n - 2^n m = 2^k.$$

Proposed by: Casey Fu

Answer: $\boxed{22}$

First consider when $n \geq m$, so let $n = m + d$ where $d \geq 0$. Then we have $2^m(m + d - 2^d m) = 2^m(m(1 - 2^d) + d)$, which is non-positive unless $m = 0$. So our first set of solutions is $m = 0, n = 2^j$.

Now, we can assume that $m > n$, so let $m = n + d$ where $d > 0$. Rewrite $2^m n - 2^n m = 2^{n+d}n - 2^n(n+d) = 2^n((2^d - 1)n - d)$. In order for this to be a power of 2, $(2^d - 1)n - d$ must be a power of 2. This implies that for some j, $2^j \equiv -d \pmod{2^d - 1}$. But notice that the powers of 2 $\pmod{2^d - 1}$ are $1, 2, 4, \ldots, 2^{d-1}$ ($2^d \equiv 1$ so the cycle repeats).

In order for the residues to match, we need $2^j + d = c(2^d - 1)$, where $0 \le j \le d-1$ and $c \ge 1$. In order for this to be true, we must have $2^{d-1} + d \ge 2^d - 1 \iff d + 1 \ge 2^{d-1}$. This inequality is only true for $d = 1, 2, 3$. We plug each of these into the original expression $(2^d - 1)n - d$.

For $d = 1$: $n - 1$ is a power of 2. This yields the set of solutions $(2^j + 2, 2^j + 1)$ for $j \ge 0$.

For $d = 2$: $3n - 2$ is a power of 2. Note that powers of 2 are $-2 \pmod 3$ if and only if it is an even power, so $n = \frac{2^{2j}+2}{3}$. This yields the solution set $(\frac{2^{2j}+8}{3}, \frac{2^{2j}+2}{3}), j \ge 0$.

For $d = 3$: $7n - 3$ is a power of 2. Powers of 2 have a period of 3 when taken $\pmod 7$, so inspection tells us $7n - 3 = 2^{3j+2}$, yielding the solution set $(\frac{2^{3j+2}+24}{7}, \frac{2^{3j+2}+3}{7}), j \ge 0$.

Therefore, all the solutions are of the form

$$(m, n) = (0, 2^j), (2^j + 2, 2^j + 1)$$
$$(\frac{2^{2j}+8}{3}, \frac{2^{2j}+2}{3}), (\frac{2^{3j+2}+24}{7}, \frac{2^{3j+2}+3}{7})$$

for $j \ge 0$.

Restricting this family to $m, n \le 100$ gives $7 + 7 + 5 + 3 = 22$.

31. [**16**] For a positive integer n, denote by $\tau(n)$ the number of positive integer divisors of n, and denote by $\phi(n)$ the number of positive integers that are less than or equal to n and relatively prime to n. Call a positive integer n *good* if $\varphi(n) + 4\tau(n) = n$. For example, the number 44 is good because $\varphi(44) + 4\tau(44) = 44$.

Find the sum of all good positive integers n.

Proposed by: Lawrence Sun

Answer: $\boxed{172}$

We claim that $44, 56, 72$ are the only good numbers. It is easy to check that these numbers work.

Now we prove none others work. First, remark that as $n = 1, 2$ fail so we have $\varphi(n)$ is even, thus n is even. This gives us $\varphi(n) \le n/2$. Now remark that $\tau(n) < 2\sqrt{n}$, so it follows we need $n/2 + 8\sqrt{n} > n \implies n \le 256$. This gives us a preliminary bound. Note that in addition we have $8\tau(n) > n$.

Now, it is easy to see that powers of 2 fail. Thus let $n = 2^a p_1^b$ where p_1 is an odd prime. From $8\tau(n) > n$ we get $8(a+1)(b+1) > 2^a p_1^b \ge 2^a 3^b$ from which we get that (a, b) is one of

$$(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1).$$

Remark that $p_1 \le \sqrt[b]{\frac{8(a+1)(b+1)}{2^a}}$. From this we can perform some casework:

- If $a = 1, b = 1$ then $p_1 - 1 + 16 = 2p_1$ but then $p = 15$, absurd.
- If $a = 1, b = 2$ then we have $p_1 \le 5$ which is obviously impossible.
- If $a = 1, b = 3$ then $p_1 \le 4$ which is impossible.
- If $a = 2, b = 1$ then $p_1 \le 12$ and it is easy to check that $p_1 = 11$ and thus $n = 44$ is the only solution.
- If $a = 2, b = 2$ then $p_1 \le 4$ which is impossible.
- If $a = 3, b = 1$ then $p_1 \le 8$ and only $p_1 = 7$ or $n = 56$ works.
- If $a = 3, b = 2$ then $p_1 \le 3$ and $p_1 = 3, n = 72$ works.
- If $a = 4, b = 1$ then $p_1 \le 1$ which is absurd.

Now suppose n is the product of 3 distinct primes, so $n = 2^a p_1^b p_2^c$ so we have $8(a+1)(b+1)(c+1) > 2^a 3^b 5^c$ then we must have (a, b, c) equal to one of

$$(1, 1, 1), (1, 2, 1), (2, 1, 1), (3, 1, 1).$$

Again, we can do some casework:

- If $a = b = c = 1$ then $8\tau(n) = 64 > 2p_1p_2$ but then $p_1 = 3, p_2 = 5$ or $p_1 = 3, p_2 = 7$ is forced neither of which work.
- If $a = 1, b = 2, c = 1$ then $8\tau(n) = 96 > 2p_1^2p_2$ but then $p_1 = 3, p_2 = 5$ is forced which does not work.
- If $a = 2, b = 1, c = 1$ then $8\tau(n) = 96 > 4p_1p_2$ forces $p_1 = 3, p_2 = 5$ or $p_1 = 3, p_2 = 7$ neither of which work.
- If $a = 3, b = 1, c = 1$ then $8\tau(n) = 108 > 8p_1p_2$ which has no solutions for p_1, p_2.

Finally, take the case where n is the product of at least 4 distinct primes. But then $n \geq 2 \cdot 3 \cdot 5 \cdot 7 = 210$ and as $2 \cdot 3 \cdot 5 \cdot 11 > 256$, it suffices to check only the case of 210. But 210 clearly fails, so it follows that $44, 56, 72$ are the only good numbers so we are done.

32. [**16**] How many equilateral hexagons of side length $\sqrt{13}$ have one vertex at $(0, 0)$ and the other five vertices at lattice points?

(A lattice point is a point whose Cartesian coordinates are both integers. A hexagon may be concave but not self-intersecting.)

Proposed by: Casey Fu

Answer: $\boxed{216}$

We perform casework on the point three vertices away from $(0, 0)$. By inspection, that point can be $(\pm 8, \pm 3)$, $(\pm 7, \pm 2)$, $(\pm 4, \pm 3)$, $(\pm 3, \pm 2)$, $(\pm 2, \pm 1)$ or their reflections across the line $y = x$. The cases are as follows:

If the third vertex is at any of $(\pm 8, \pm 3)$ or $(\pm 3, \pm 8)$, then there are 7 possible hexagons. There are 8 points of this form, contributing 56 hexagons.

If the third vertex is at any of $(\pm 7, \pm 2)$ or $(\pm 2, \pm 7)$, there are 6 possible hexagons, contributing 48 hexagons.

If the third vertex is at any of $(\pm 4, \pm 3)$ or $(\pm 3, \pm 4)$, there are again 6 possible hexagons, contributing 48 more hexagons.

If the third vertex is at any of $(\pm 3, \pm 2)$ or $(\pm 2, \pm 3)$, then there are again 6 possible hexagons, contributing 48 more hexagons.

Finally, if the third vertex is at any of $(\pm 2, \pm 1)$, then there are 2 possible hexagons only, contributing 16 hexagons.

Adding up, we get our answer of $\boxed{216}$.

33. [**20**] **(Lucas Numbers)** The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for every $n \geq 0$. There are N integers $1 \leq n \leq 2016$ such that L_n contains the digit 1. Estimate N.

An estimate of E earns $\lfloor 20 - 2|N - E| \rfloor$ or 0 points, whichever is greater.

Proposed by: Evan Chen

Answer: $\boxed{1984}$

```
lucas_ones n = length . filter (elem '1') $ take (n + 1) lucas_strs
  where
    lucas = 2 : 1 : zipWith (+) lucas (tail lucas)
    lucas_strs = map show lucas

main = putStrLn . show $ lucas_ones 2016
```

34. [**20**] **(Caos)** A cao [sic] has 6 legs, 3 on each side. A walking pattern for the cao is defined as an ordered sequence of raising and lowering each of the legs exactly once (altogether 12 actions), starting and ending with all legs on the ground. The pattern is safe if at any point, he has at least 3 legs on the ground and not all three legs are on the same side. Estimate N, the number of safe patterns.

An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N)^4 \rfloor$ points.

Proposed by:

Answer: $\boxed{1416528}$

```
# 1 = on ground, 0 = raised, 2 = back on ground
cache = {}

def pangzi(legs):
  if legs == (2,2,2,2,2,2): return 1
  elif legs.count(0) > 3: return 0
  elif legs[0] + legs[1] + legs[2] == 0: return 0
  elif legs[3] + legs[4] + legs[5] == 0: return 0
  elif cache.has_key(legs): return cache[legs]

  cache[legs] = 0
  for i in xrange(6): # raise a leg
    if legs[i] == 1:
      new = list(legs)
      new[i] = 0
      cache[legs] += pangzi(tuple(new))
    elif legs[i] == 0: # lower a leg
      new = list(legs)
      new[i] = 2
      cache[legs] += pangzi(tuple(new))
  return cache[legs]

print pangzi((1,1,1,1,1,1))
```

35. [**20**] **(Maximal Determinant)** In a 17×17 matrix M, all entries are ± 1. The maximum possible value of $|\det M|$ is N. Estimate N.

An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N)^2 \rfloor$ points.

Proposed by: Evan Chen

Answer: $\boxed{327680 \cdot 2^{16}}$

This is Hadamard's maximal determinant problem. There's an upper bound of $n^{\frac{1}{2}n}$ which empirically seems to give reasonably good estimates, but in fact this is open for general n.

36. [**20**] **(Self-Isogonal Cubics)** Let ABC be a triangle with $AB = 2$, $AC = 3$, $BC = 4$. The *isogonal conjugate* of a point P, denoted P^*, is the point obtained by intersecting the reflection of lines PA, PB, PC across the angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively.

Given a point Q, let $\mathfrak{K}(Q)$ denote the unique cubic plane curve which passes through all points P such that line PP^* contains Q. Consider:

(a) the M'Cay cubic $\mathfrak{K}(O)$, where O is the circumcenter of $\triangle ABC$,

(b) the Thomson cubic $\mathfrak{K}(G)$, where G is the centroid of $\triangle ABC$,

(c) the Napoleon-Feurerbach cubic $\mathfrak{K}(N)$, where N is the nine-point center of $\triangle ABC$,

(d) the Darboux cubic $\mathfrak{K}(L)$, where L is the de Longchamps point (the reflection of the orthocenter across point O),

(e) the Neuberg cubic $\mathfrak{K}(X_{30})$, where X_{30} is the point at infinity along line OG,

(f) the nine-point circle of $\triangle ABC$,

(g) the incircle of $\triangle ABC$, and

(h) the circumcircle of $\triangle ABC$.

Estimate N, the number of points lying on at least two of these eight curves. An estimate of E earns $\lfloor 20 \cdot 2^{-|N-E|/6} \rfloor$ points.

Proposed by: Evan Chen

Answer: $\boxed{49}$

The first main insight is that all the cubics pass through the points A, B, C, H (orthocenter), O, and the incenter and three excenters. Since two cubics intersect in at most nine points, this is all the intersections of a cubic with a cubic.

On the other hand, it is easy to see that among intersections of circles with circles, there are exactly 3 points; the incircle is tangent to the nine-point circle at the Feurerbach point while being contained completely in the circumcircle; on the other hand for this obtuse triangle the nine-point circle and the circumcircle intersect exactly twice.

All computations up until now are exact, so it remains to estimate:

- Intersection of the circumcircle with cubics. Each cubic intersects the circumcircle at an even number of points, and moreover we already know that A, B, C are among these, so the number of additional intersections contributed is either 1 or 3; it is the former only for the Neuberg cubic which has a "loop". Hence the actual answer in this case is $1+3+3+3+3 = 13$ (but an estimate of $3 \cdot 5 = 15$ is very reasonable).
- Intersection of the incircle with cubics. Since $\angle A$ is large the incircle is small, but on the other hand we know I lies on each cubic. Hence it's very likely that each cubic intersects the incircle twice (once "coming in" and once "coming out"). This is the case, giving $2 \cdot 5 = 10$ new points.
- Intersection of the nine-point with cubics. We guess this is close to the 10 points of the incircle, as we know the nine-point circle and the incircle are tangent to each other. In fact, the exact count is 14 points; just two additional branches appear.

In total, $N = 9 + 3 + 13 + 10 + 14 = 49$.