HMMT February 2018

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Algebra and Number Theory

1. For some real number c, the graphs of the equation $y = |x - 20| + |x + 18|$ and the line $y = x + c$ intersect at exactly one point. What is c?

   Proposed by: Henrik Boecken

   Answer: $\boxed{18}$

   We want to know the value of c so that the graph $|x-20|+|x+18|-x = c$ has one solution. The graph of the function $|x-20|+|x+18|-x$ consists of an infinite section of slope -3 for $x \in (-\infty, -18]$, then a finite section of slope -1 for $x \in [-18, 20]$, then an infinite section of slope 1 for $x \in [20, \infty)$. Notice that this graph is strictly decreasing on $(-\infty, 20]$ and strictly increasing on $[20, \infty)$. Therefore any horizontal line will intersect this graph 0 or 2 times, except the one that passes through the "vertex" $(20, |20 - 20| + |20 + 18| - 20) = (20, 18)$, giving a value of $c = 18$.

2. Compute the positive real number x satisfying

   $$x^{(2x^6)} = 3.$$

   Proposed by: Henrik Boecken

   Answer: $\boxed{\sqrt[6]{3}}$

   Let $t = x^6$, so $x^{2t} = 3$. Taking this to the third power gives $x^{6t} = 27$, or equivalently $t^t = 3^3$. We can see by inspection that $t = 3$, and this is the only solution as for $t > 1$, the function t^t is monotonically increasing, and if $0 < t < 1$, $t^t < 1$. Solving for x gives $x^6 = 3$, or $x = \sqrt[6]{3}$.

3. There are two prime numbers p so that $5p$ can be expressed in the form $\left\lfloor \frac{n^2}{5} \right\rfloor$ for some positive integer n. What is the sum of these two prime numbers?

   Proposed by: Kevin Sun

   Answer: $\boxed{52}$

   Note that the remainder when n^2 is divided by 5 must be 0, 1, or 4. Then we have that $25p = n^2$ or $25p = n^2 - 1$ or $25p = n^2 - 4$. In the first case there are no solutions. In the second case, if $25p = (n-1)(n+1)$, then we must have $n - 1 = 25$ or $n + 1 = 25$ as $n - 1$ and $n + 1$ cannot both be divisible by 5, and also cannot both have a factor besides 25. Similarly, in the third case, $25p = (n-2)(n+2)$, so we must have $n - 2 = 25$ or $n + 2 = 25$.

   Therefore the n we have to check are $23, 24, 26, 27$. These give values of $p = 21$, $p = 23$, $p = 27$, and $p = 29$, of which only 23 and 29 are prime, so the answer is $23 + 29 = 52$.

4. Distinct prime numbers p, q, r satisfy the equation

   $$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

   for some positive integer A. What is A?

   Proposed by: Kevin Sun

   Answer: $\boxed{1980}$

   Note that A is a multiple of p, q, and r, so $K = \frac{A}{pqr}$ is an integer. Dividing through, we have that

   $$K = 8 + \frac{12}{p} = 7 + \frac{55}{q} = 2 + \frac{50}{r}.$$

   Then $p \in \{2, 3\}$, $q \in \{5, 11\}$, and $r \in \{2, 5\}$. These values give $K \in \{14, 12\}$, $K \in \{18, 12\}$, and $K \in \{27, 12\}$, giving $K = 12$ and $(p, q, r) = (3, 11, 5)$. We can then compute $A = pqr \cdot K = 3 \cdot 11 \cdot 5 \cdot 12 = 1980$.

5. Let $\omega_1, \omega_2, \ldots, \omega_{100}$ be the roots of $\frac{x^{101}-1}{x-1}$ (in some order). Consider the set

$$S = \{\omega_1^1, \omega_2^2, \omega_3^3, \ldots, \omega_{100}^{100}\}.$$

Let M be the maximum possible number of unique values in S, and let N be the minimum possible number of unique values in S. Find $M - N$.

Proposed by: Henrik Boecken

Answer: $\boxed{98}$

Throughout this solution, assume we're working modulo 101.

First, $N = 1$. Let ω be a primitive 101st root of unity. We then let $\omega_n = \omega^{1/n}$, which we can do because 101 is prime, so $1/n$ exists for all nonzero n and $1/n = 1/m \implies m = n$. Thus the set contains only one distinct element, ω.

$M = 100$ is impossible. Fix ζ, a primitive 101st root of unity, and let $\omega_n = \zeta^{\pi(n)}$ for each n. Suppose that there are 100 distinct such $n\pi(n)$ exponents; then π permutes the set $\{1, 2, \cdots, 100\}$. Fix g, a primitive root of 101; write $n = g^{e_n}$ and $\pi(n) = g^{\tau(e_n)}$. Then $\{e_n\} = \{0, 1, 2, \ldots, 100\}$ and τ is a permutation of this set, as is $e_n + \tau(e_n)$. However, this is impossible: $\sum_{n=1}^{100} e_n + \tau(e_n) = 5050 + 5050 \equiv 5050 \pmod{100}$, which is a contradiction. Thus there cannot be 100 distinct exponents.

$M = 99$ is possible. Again, let ζ be a primitive root of unity and let $\omega_n = \zeta^{1/(n+1)}$, except when $n = 100$, in which case let ω_{100} be the last possible root. Notice that $\frac{n}{n+1} = \frac{m}{m+1}$ if and only if $n = m$, so this will produce 99 different elements in the set.

Thus $M - N = 99 - 1 = 98$.

6. Let α, β, and γ be three real numbers. Suppose that

$$\cos\alpha + \cos\beta + \cos\gamma = 1$$
$$\sin\alpha + \sin\beta + \sin\gamma = 1.$$

Find the smallest possible value of $\cos\alpha$.

Proposed by: Henrik Boecken

Answer: $\boxed{\frac{-1-\sqrt{7}}{4}}$

Let $a = \cos\alpha + i\sin\alpha$, $b = cos\beta + i\sin\beta$, and $c = \cos\gamma + i\sin\gamma$.

We then have

$$a + b + c = 1 + i$$

where a, b, c are complex numbers on the unit circle. Now, to minimize $\cos\alpha = \text{Re}[a]$, consider a triangle with vertices a, $1+i$, and the origin. We want a as far away from $1+i$ as possible while maintaining a nonnegative imaginary part. This is achieved when b and c have the same argument, so $|b + c| = |1 + i - a| = 2$. Now a, 0, and $1+i$ form a $1 - 2 - \sqrt{2}$ triangle. The value of $\cos\alpha$ is now the cosine of the angle between the 1 and $\sqrt{2}$ sides plus the $\frac{\pi}{4}$ angle from $1+i$. Call the first angle δ. Then

$$\cos\delta = \frac{1^2 + (\sqrt{2})^2 - 2^2}{2 \cdot 1 \cdot \sqrt{2}}$$
$$= \frac{-1}{2\sqrt{2}}$$

and

$$\begin{aligned}\cos\alpha &= \cos\left(\frac{\pi}{4}+\delta\right)\\ &= \cos\frac{\pi}{4}\cos\delta - \sin\frac{\pi}{4}\sin\delta\\ &= \frac{\sqrt{2}}{2}\cdot\frac{-1}{2\sqrt{2}} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{7}}{2\sqrt{2}}\\ &= \boxed{\frac{-1-\sqrt{7}}{4}}\end{aligned}$$

7. Rachel has the number 1000 in her hands. When she puts the number x in her left pocket, the number changes to $x+1$. When she puts the number x in her right pocket, the number changes to x^{-1}. Each minute, she flips a fair coin. If it lands heads, she puts the number into her left pocket, and if it lands tails, she puts it into her right pocket. She then takes the new number out of her pocket. If the expected value of the number in Rachel's hands after eight minutes is E, then compute $\lfloor \frac{E}{10} \rfloor$.

Proposed by: Kevin Sun

Answer: $\boxed{13}$

Call a real number very large if $x \in [1000, 1008]$, very small if $x \in [0, \frac{1}{1000}]$, and medium-sized if $x \in [\frac{1}{8}, 8]$. Every number Rachel is ever holding after at most 8 steps will fall under one of these categories. Therefore the main contribution to E will come from the probability that Rachel is holding a number at least 1000 at the end.

Note that if her number ever becomes medium-sized, it will never become very large or very small again. Therefore the only way her number ends up above 1000 is if the sequence of moves consists of $x \to x+1$ moves and consecutive pairs of $x \to x^{-1}$ moves. Out of the 256 possible move sequences, the number of ways for the number to stay above 1000 is the number of ways of partitioning 8 into an ordered sum of 1 and 2, or the ninth Fibonacci number $F_9 = 34$.

Therefore

$$\frac{34}{256}\cdot 1000 \le E \le \frac{34}{256}\cdot 1000 + 8,$$

where $\dfrac{34}{256}\cdot 1000 \approx 132.8$. Furthermore, the extra contribution will certainly not exceed 7, so we get that $\lfloor \frac{E}{10} \rfloor = 13$.

(The actual value of E is

$$\frac{1538545594943410132524842390483285519695831541468827074238984121209064525621}{11415831910281261197289931074429782903650103348754306523894286954489856000},$$

which is approximately equal to 134.77297. We can see that the extra contribution is about 2 and is very insignificant.)

8. For how many pairs of sequences of nonnegative integers $(b_1, b_2, \ldots, b_{2018})$ and $(c_1, c_2, \ldots, c_{2018})$ does there exist a sequence of nonnegative integers $(a_0, \ldots, a_{2018})$ with the following properties:

- For $0 \le i \le 2018$, $a_i < 2^{2018}$;
- For $1 \le i \le 2018$, $b_i = a_{i-1} + a_i$ and $c_i = a_{i-1} | a_i$;

where $|$ denotes the bitwise or operation?

(The *bitwise or* of two nonnegative integers $x = \cdots x_3x_2x_1x_0$ and $y = \cdots y_3y_2y_1y_0$ expressed in binary is defined as $x|y = \cdots z_3z_2z_1z_0$, where $z_i = 1$ if at least one of x_i and y_i is 1, and 0 otherwise.)

Proposed by: Kevin Sun

Answer: $\boxed{(2^{2019}-1)^{2018}}$

Define the *bitwise and* of two nonnegative integers $x = \cdots x_3x_2x_1x_0$ and $y = \cdots y_3y_2y_1y_0$ expressed in binary to be $x\&y = \cdots z_3z_2z_1z_0$, where $z_i = 1$ if both x_i and y_i are 1, and 0 otherwise.

Now, we can prove that from the definitions of $|$ and $\&$ that $x + y = (x|y) + (x\&y)$. Therefore it suffices to count pairs of sequences $(c_1, c_2, \ldots, c_{2018})$ and $(d_1, d_2, \ldots, d_{2018})$ such that $c_i = a_{i-1}|a_i$ and $d_i = a_{i-1}\&a_i$ for $0 \le a_i < 2^{2018}$.

Since both $|, \&$ are bitwise operations, it suffices to count the number of sequences $\{c_i\}$ and $\{d_i\}$ restricting each a_i to $\{0, 2^k\}$ for each $k \in [0, 2017]$ and multiply these counts together. Each sequence $(a_0, \ldots, a_{2018})$ leads to a unique $\{c_i\}$ and $\{d_i\}$ except for the sequences $(2^k, 0, 2^k, 0, \ldots, 2^k)$ and the sequences $(0, 2^k, 0, 2^k, \ldots, 0)$, which lead to the same $\{c_i\}$ and $\{d_i\}$.

Therefore for each k, there are $2^{2019} - 1$ ways to determine the k-th bits of each c_i and d_i. Multiplying this over all k gives a final count of $(2^{2019} - 1)^{2018}$.

9. Assume the quartic $x^4 - ax^3 + bx^2 - ax + d = 0$ has four real roots $\frac{1}{2} \le x_1, x_2, x_3, x_4 \le 2$. Find the maximum possible value of $\frac{(x_1+x_2)(x_1+x_3)x_4}{(x_4+x_2)(x_4+x_3)x_1}$ (over all valid choices of a, b, d).

   Proposed by: Allen Liu

   Answer: $\boxed{\frac{5}{4}}$

   We can rewrite the expression as

$$\frac{x_4^2}{x_1^2} \cdot \frac{(x_1+x_1)(x_1+x_2)(x_1+x_3)(x_1+x_4)}{(x_4+x_1)(x_4+x_2)(x_4+x_3)(x_4+x_4)}$$

$$\frac{x_4^2}{x_1^2} \cdot \frac{f(-x_1)}{f(-x_4)}$$

   where $f(x)$ is the quartic. We attempt to find a simple expression for $f(-x_1)$. We know that

$$f(-x_1) - f(x_1) = 2a \cdot x_1^3 + 2a \cdot x_1$$

   Since x_1 is a root, we have

$$f(-x_1) = 2a \cdot x_1^3 + 2a \cdot x_1$$

   Plugging this into our previous expression:

$$\frac{x_4^2}{x_1^2} \cdot \frac{x_1^3 + x_1}{x_4^3 + x_4}$$

$$\frac{x_1 + \frac{1}{x_1}}{x_4 + \frac{1}{x_4}}$$

   The expression $x + \frac{1}{x}$ is maximized at $x = 2, \frac{1}{2}$ and minimized at $x = 1$. We can therefore maximize the numerator with $x_1 = 2$ and minimize the denominator with $x_4 = 1$ to achieve the answer of $\frac{5}{4}$. It can be confirmed that such an answer can be achieved such as with $x_2 = x_3 = \frac{\sqrt{10}-1}{3}$.

10. Let S be a randomly chosen 6-element subset of the set $\{0, 1, 2, \ldots, n\}$. Consider the polynomial $P(x) = \sum_{i \in S} x^i$. Let X_n be the probability that $P(x)$ is divisible by some nonconstant polynomial $Q(x)$ of degree at most 3 with integer coefficients satisfying $Q(0) \ne 0$. Find the limit of X_n as n goes to infinity.

    Proposed by: Allen Liu

    Answer: $\boxed{\frac{10015}{20736}}$

    We begin with the following claims:

    Claim 1: There are finitely many $Q(x)$ that divide some $P(x)$ of the given form.

Proof: First of all the leading coefficient of Q must be 1, because if Q divides P then P/Q must have integer coefficients too. Note that if $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ with elements in increasing order, then

$$|P(x)| \geq |x^{s_6}| - |x^{s_5}| - |x^{s_4}| - \cdots - |x^{s_1}| = |x|^{s_6} - |x|^{s_5} - |x|^{s_4} - \cdots - |x|^{s_1},$$

so all the roots of P must have magnitude less than 2, and so do all the roots of Q. Therefore, all the symmetric expressions involving the roots of Q are also bounded, so by Vieta's Theorem all the coefficients of Q of a given degree are bounded, and the number of such Q is therefore finite.

Claim 2: If Q has a nonzero root that does not have magnitude 1, then the probability that it divides a randomly chosen P vanishes as n goes to infinity.

Proof: WLOG suppose that Q has a root r with $|r| > 1$ (similar argument will apply for $|r| < 1$). Then from the bound given in the proof of Claim 1, it is not difficult to see that $s_6 - s_5$ is bounded since

$$|P(r)| > |r|^{s_6} - 5|r|^{s_5} > |r|^{s_6 - s_5} - 5,$$

which approaches infinity as $s_6 - s_5$ goes to infinity. By similar argument we can show that $s_5 - s_4, s_4 - s_3, \ldots$ are all bounded. Therefore, the probability of choosing the correct coefficients is bounded above by the product of five fixed numbers divided by $n^5/5!$, which vanishes as n goes to infinity.

From the claims above, we see that we only need to consider polynomials with roots of magnitude 1, since the sum of all other possibilities vanishes as n goes to infinity. Moreover, this implies that we only need to consider roots of unity. Since Q has degree at most 3, the only possible roots are $-1, \pm i, \frac{-1\pm i\sqrt{3}}{2}, \frac{1\pm i\sqrt{3}}{2}$, corresponding to $x+1, x^2+1, x^2+x+1, x^2-x+1$ (note that eighth root of unity is impossible because x^4+1 cannot be factored in the rationals).

Now we compute the probability of $P(r) = 0$ for each possible root r. Since the value of x^s cycles with s, and we only care about $n \to \infty$, we may even assume that the exponents are chosen independently at random, with repetition allowed.

Case 1: When $r = -1$, the number of odd exponents need to be equal to the number of even exponents, which happens with probability $\frac{\binom{6}{3}}{2^6} = \frac{5}{16}$.

Case 2: When $r = \pm i$, the number of exponents that are 0 modulo 4 need to be equal to those that are 2 modulo 4, and same for 1 modulo 4 and 3 modulo 4, which happens with probability $\frac{\binom{6}{0}}{2^6} \cdot \frac{\binom{0}{0}\binom{6}{3}}{2^6} + \frac{\binom{6}{2}}{2^6} \cdot \frac{\binom{2}{1}\binom{4}{2}}{2^6} + \frac{\binom{6}{4}}{2^6} \cdot \frac{\binom{4}{2}\binom{2}{1}}{2^6} + \frac{\binom{6}{6}}{2^6} \cdot \frac{\binom{6}{3}\binom{0}{0}}{2^6} = \frac{25}{256}$. Note that Case 1 and Case 2 have no overlaps, since the former requires 3 even exponents, and the latter requires $0, 2, 4$, or 6 even exponents.

Case 3: When $r = \frac{-1\pm i\sqrt{3}}{2}$, the number of exponents that are $0, 1, 2$ modulo 3 need to be equal to each other, so the probability is $\frac{\binom{6}{2,2,2}}{3^6} = \frac{10}{81}$.

Case 4: When $r = \frac{1\pm i\sqrt{3}}{2}$, then if n_i is the number of exponents that are i modulo 6 ($i = 0, 1, 2, 3, 4, 5$), then $n_0 - n_3 = n_2 - n_5 = n_4 - n_1 = k$ for some k. Since $3k \equiv n_0 + n_1 + \cdots + n_5 = 6 \equiv 0 \pmod 2$, k must be one of $-2, 0, 2$. When $k = 0$, we have $n_0 + n_2 + n_4 = n_1 + n_3 + n_5$, which is the same as Case 1. When $k = 2$, we have $n_0 = n_2 = n_4 = 2$, which is covered in Case 3, and similar for $k = -2$. Therefore we do not need to consider this case.

Now we deal with over-counting. Since Case 1 and 2 deal with the exponents modulo 4 and Case 3 deal with exponents modulo 3, the probabilities are independent from each other. So by complementary counting, we compute the final probability as

$$1 - \left(1 - \frac{5}{16} - \frac{25}{256}\right)\left(1 - \frac{10}{81}\right) = 1 - \frac{151}{256} \cdot \frac{71}{81} = \frac{10015}{20736}.$$