HMMT February 2016

February 20, 2016

Combinatorics

1. For positive integers n, let S_n be the set of integers x such that n distinct lines, no three concurrent, can divide a plane into x regions (for example, $S_2 = \{3, 4\}$, because the plane is divided into 3 regions if the two lines are parallel, and 4 regions otherwise). What is the minimum i such that S_i contains at least 4 elements?

   Proposed by:

   Answer: $\boxed{4}$

   For S_3, either all three lines are parallel (4 regions), exactly two are parallel (6 regions), or none are parallel (6 or seven regions, depending on whether they all meet at one point), so $|S_3| = 3$. Then, for S_4, either all lines are parallel (5 regions), exactly three are parallel (8 regions), there are two sets of parallel pairs (9 regions), exactly two are parallel (9 or 10 regions), or none are parallel (8, 9, 10, or 11 regions), so $|S_4| = 4$.

2. Starting with an empty string, we create a string by repeatedly appending one of the letters H, M, T with probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, respectively, until the letter M appears twice consecutively. What is the expected value of the length of the resulting string?

   Proposed by:

   Answer: $\boxed{6}$

   Let E be the expected value of the resulting string. Starting from the empty string,

   - We have a $\frac{1}{2}$ chance of not selecting the letter M; from here the length of the resulting string is $1 + E$.
   - We have a $\frac{1}{4}$ chance of selecting the letter M followed by a letter other than M, which gives a string of length $2 + E$.
   - We have a $\frac{1}{4}$ chance of selecting M twice, for a string of length 2.

   Thus, $E = \frac{1}{2}(1 + E) + \frac{1}{4}(2 + E) + \frac{1}{4}(2)$. Solving gives $E = 6$.

3. Find the number of ordered pairs of integers (a, b) such that a, b are divisors of 720 but ab is not.

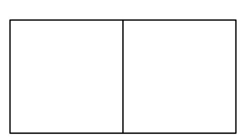
   Proposed by: Casey Fu

   Answer: $\boxed{2520}$

   First consider the case $a, b > 0$. We have $720 = 2^4 \cdot 3^2 \cdot 5$, so the number of divisors of 720 is $5 * 3 * 2 = 30$. We consider the number of ways to select an ordered pair (a, b) such that a, b, ab all divide 720. Using the balls and urns method on each of the prime factors, we find the number of ways to distribute the factors of 2 across a and b is $\binom{6}{2}$, the factors of 3 is $\binom{4}{2}$, the factors of 5 is $\binom{3}{2}$. So the total number of ways to select (a, b) with a, b, ab all dividing 720 is $15 * 6 * 3 = 270$. The number of ways to select any (a, b) with a and b dividing 720 is $30 * 30 = 900$, so there are $900 - 270 = 630$ ways to select a and b such that a, b divide 720 but ab doesn't.

   Now, each $a, b > 0$ corresponds to four solutions $(\pm a, \pm b)$ giving the final answer of 2520. (Note that $ab \neq 0$.)

4. Let R be the rectangle in the Cartesian plane with vertices at $(0, 0), (2, 0), (2, 1)$, and $(0, 1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.



How many subsets of these seven edges form a connected figure?

Proposed by: Joy Zheng

Answer: $\boxed{81}$

We break this into cases. First, if the middle edge is not included, then there are $6 * 5 = 30$ ways to choose two distinct points for the figure to begin and end at. We could also allow the figure to include all or none of the six remaining edges, for a total of 32 connected figures not including the middle edge. Now let's assume we are including the middle edge. Of the three edges to the left of the middle edge, there are 7 possible subsets we can include (8 total subsets, but we subtract off the subset consisting of only the edge parallel to the middle edge since it's not connected). Similarly, of the three edges to the right of the middle edge, there are 7 possible subsets we can include. In total, there are 49 possible connected figures that include the middle edge. Therefore, there are $32 + 49 = 81$ possible connected figures.

5. Let a, b, c, d, e, f be integers selected from the set $\{1, 2, \ldots, 100\}$, uniformly and at random with replacement. Set
$$M = a + 2b + 4c + 8d + 16e + 32f.$$
What is the expected value of the remainder when M is divided by 64?

Proposed by: Evan Chen

Answer: $\boxed{\frac{63}{2}}$

Consider M in binary. Assume we start with $M = 0$, then add a to M, then add $2b$ to M, then add $4c$ to M, and so on. After the first addition, the first bit (defined as the rightmost bit) of M is toggled with probability $\frac{1}{2}$. After the second addition, the second bit of M is toggled with probability $\frac{1}{2}$. After the third addition, the third bit is toggled with probability $\frac{1}{2}$, and so on for the remaining three additions. As such, the six bits of M are each toggled with probability $\frac{1}{2}$ - specifically, the k^{th} bit is toggled with probability $\frac{1}{2}$ at the k^{th} addition, and is never toggled afterwards. Therefore, each residue from 0 to 63 has probability $\frac{1}{64}$ of occurring, so they are all equally likely. The expected value is then just $\frac{63}{2}$.

6. Define the sequence $a_1, a_2 \ldots$ as follows: $a_1 = 1$ and for every $n \geq 2$,
$$a_n = \begin{cases} n - 2 & \text{if } a_{n-1} = 0 \\ a_{n-1} - 1 & \text{if } a_{n-1} \neq 0 \end{cases}$$
A non-negative integer d is said to be *jet-lagged* if there are non-negative integers r, s and a positive integer n such that $d = r + s$ and that $a_{n+r} = a_n + s$. How many integers in $\{1, 2, \ldots, 2016\}$ are jet-lagged?

Proposed by: Pakawut Jiradilok

Let $N = n+r$, and $M = n$. Then $r = N - M$, and $s = a_N - a_M$, and $d = r+s = (a_N + N) - (a_M + M)$. So we are trying to find the number of possible values of $(a_N + N) - (a_M + M)$, subject to $N \geq M$ and $a_N \geq a_M$.

Divide the a_i into the following "blocks":

- $a_1 = 1$, $a_2 = 0$,
- $a_3 = 1$, $a_4 = 0$,
- $a_5 = 3$, $a_6 = 2$, $a_7 = 1$, $a_8 = 0$,
- $a_9 = 7$, $a_{10} = 6$, $\ldots$, $a_{16} = 0$,

and so on. The k^{th} block contains a_i for $2^{k-1} < i \leq 2^k$. It's easy to see by induction that $a_{2^k} = 0$ and thus $a_{2^k+1} = 2^k - 1$ for all $k \geq 1$. Within each block, the value $a_n + n$ is constant, and for the kth block ($k \geq 1$) it equals 2^k. Therefore, $d = (a_N + N) - (a_M + M)$ is the difference of two powers of 2, say $2^n - 2^m$. For any $n \geq 1$, it is clear there exists an N such that $a_N + N = 2^n$ (consider the n^{th}

block). We can guarantee $a_N \geq a_M$ by setting $M = 2^m$. Therefore, we are searching for the number of integers between 1 and 2016 that can be written as $2^n - 2^m$ with $n \geq m \geq 1$. The pairs (n, m) with $n > m \geq 1$ and $n \leq 10$ all satisfy $1 \leq 2^n - 2^m \leq 2016$ (45 possibilities). In the case that $n = 11$, we have that $2^n - 2^m \leq 2016$ so $2^m \geq 32$, so $m \geq 5$ (6 possibilities). There are therefore $45 + 6 = 51$ jetlagged numbers between 1 and 2016.

7. Kelvin the Frog has a pair of standard fair 8-sided dice (each labelled from 1 to 8). Alex the sketchy Kat also has a pair of fair 8-sided dice, but whose faces are labelled differently (the integers on each Alex's dice need not be distinct). To Alex's dismay, when both Kelvin and Alex roll their dice, the probability that they get any given sum is equal!

   Suppose that Alex's two dice have a and b total dots on them, respectively. Assuming that $a \neq b$, find all possible values of $\min\{a, b\}$.

   Proposed by: Alexander Katz

   Answer: $\boxed{24, 28, 32}$

   Ed. note: I'm probably horribly abusing notation

   Define the *generating function* of an event A as the polynomial

$$g(A, x) = \sum p_i x^i$$

   where p_i denotes the probability that i occurs during event A. We note that the generating is multiplicative; i.e.

$$g(A \text{ AND } B, x) = g(A)g(B) = \sum p_i q_j x^{i+j}$$

   where q_j denotes the probability that j occurs during event B.

   In our case, events A and B are the rolling of the first and second dice, respectively, so the generating functions are the same:

$$g(\text{die}, x) = \frac{1}{8}x^1 + \frac{1}{8}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 + \frac{1}{8}x^5 + \frac{1}{8}x^6 + \frac{1}{8}x^7 + \frac{1}{8}x^8$$

   and so

$$g(\text{both dice rolled}, x) = g(\text{die}, x)^2 = \frac{1}{64}(x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^2$$

   where the coefficient of x^i denotes the probability of rolling a sum of i.

   We wish to find two alternate dice, C and D, satisfying the following conditions:

   - C and D are both 8-sided dice; i.e. the sum of the coefficients of $g(C, x)$ and $g(D, x)$ are both 8 (or $g(C, 1) = g(D, 1) = 8$).
   - The faces of C and D are all labeled with a positive integer; i.e. the powers of each term of $g(C, x)$ and $g(D, x)$ are positive integer (or $g(C, 0) = g(D, 0) = 0$).
   - The probability of rolling any given sum upon rolling C and D is equal to the probability of rolling any given sum upon rolling A and B; i.e. $g(C, x)g(D, x) = g(A, x)g(B, x)$.

   Because the dice are "fair" – i.e. the probability of rolling any face is $\frac{1}{8}$ – we can multiply $g(A, x), g(B, x), g(C, x)$ and $g(D, x)$ by 8 to get integer polynomials; as this does not affect any of the conditions, we can assume $g(C, x)$ and $g(D, x)$ are integer polynomials multiplying to $(x^1 + x^2 + \ldots + x^8)^2$ (and subject to the other two conditions as well). Since $\mathbb{Z}$ is a UFD (i.e. integer polynomials can be expressed as the product of integer polynomials in exactly one way, up to order and scaling by a constant), all factors of $g(C, x)$ and $g(D, x)$ must also be factors of $x^1 + x^2 + \ldots + x^8$. Hence it is useful to factor $x^1 + x^2 + \ldots + x^8 = x(x + 1)(x^2 + 1)(x^4 + 1)$.

We thus have $g(C,x)g(D,x) = x^2(x+1)^2(x^2+1)^2(x^4+1)^2$. We know that $g(C,0) = g(D,0) = 0$, so $x \mid g(C,x), g(D,x)$. It remains to distribute the remaining term $(x+1)^2(x^2+1)^2(x^4+1)^2$; we can view each of these 6 factors as being "assigned" to either C or D. Note that since $g(C,1) = g(D,1) = 8$, and each of the factors $x+1, x^2+1, x^4+1$ evaluates to 2 when $x = 1$, exactly three factors must be assigned to C and exactly three to D. Finally, assigning $x+1$, x^2+1, and x^4+1 to C results in the standard die, with $a = b = 28$.. This gives us the three cases (and their permutations):

- $g(C,x) = x(x+1)^2(x^2+1)$, $g(D,x) = x(x^2+1)(x^4+1)^2$. In this case we get $g(C,x) = x^5 + 2x^4 + 2x^3 + 2x^2 + x$ and $g(D,x) = x^{11} + x^9 + 2x^7 + 2x^5 + x^3 + x$, so the "smaller" die has faces $5, 4, 4, 3, 3, 2, 2$, and 1 which sum to 24.
- $g(C,x) = x(x+1)(x^2+1)^2$, $g(D,x) = x(x+1)(x^4+1)^2$. In this case we have $g(C,x) = x^6 + x^5 + 2x^4 + 2x^3 + x^2 + x$ and $g(D,x) = x^{10} + x^9 + 2x^6 + 2x^5 + x^2 + x$, so the "smaller" die has faces $6, 5, 4, 4, 3, 3, 2$ and 1 which sum to 28.
- $g(C,x) = x(x^2+1)^2(x^4+1), g(D,x) = x(x+1)^2(x^4+1)$. In this case we have $g(C,x) = x^9 + 2x^7 + 2x^5 + 2x^3 + x$ and $g(D,x) = x^7 + 2x^6 + x^5 + x^3 + 2x^2 + x$, so the "smaller die" has faces $7, 6, 6, 5, 3, 2, 2, 1$ which sum to 32.

Therefore, $\min\{a,b\}$ is equal to $\boxed{24, 28, \text{ or } 32}$.

8. Let X be the collection of all functions $f : \{0, 1, \ldots, 2016\} \to \{0, 1, \ldots, 2016\}$. Compute the number of functions $f \in X$ such that

$$\max_{g \in X} \left(\min_{0 \le i \le 2016} \big(\max(f(i), g(i)) \big) - \max_{0 \le i \le 2016} \big(\min(f(i), g(i)) \big) \right) = 2015.$$

Proposed by:

Answer: $\boxed{2 \cdot (3^{2017} - 2^{2017})}$

For each $f, g \in X$, we define

$$d(f,g) := \min_{0 \le i \le 2016} \big(\max(f(i), g(i)) \big) - \max_{0 \le i \le 2016} \big(\min(f(i), g(i)) \big)$$

Thus we desire $\max_{g \in X} d(f,g) = 2015$.

First, we count the number of functions $f \in X$ such that

$$\exists g : \min_i \max\{f(i), g(i)\} \ge 2015 \text{ and } \exists g : \min_i \max\{f(i), g(i)\} = 0.$$

That means for every value of i, either $f(i) = 0$ (then we pick $g(i) = 2015$) or $f(i) \ge 2015$ (then we pick $g(i) = 0$). So there are $A = 3^{2017}$ functions in this case.

Similarly, the number of functions such that

$$\exists g : \min_i \max\{f(i), g(i)\} = 2016 \text{ and } \exists g : \min_i \max\{f(i), g(i)\} \le 1$$

is also $B = 3^{2017}$.

Finally, the number of functions such that

$$\exists g : \min_i \max\{f(i), g(i)\} = 2016 \text{ and } \exists g : \min_i \max\{f(i), g(i)\} = 0$$

is $C = 2^{2017}$.

Now $A + B - C$ counts the number of functions with $\max_{g \in X} d(f,g) \ge 2015$ and C counts the number of functions with $\max_{g \in X} d(f,g) \ge 2016$, so the answer is $A + B - 2C = 2 \cdot (3^{2017} - 2^{2017})$.

9. Let $V = \{1, \ldots, 8\}$. How many permutations $\sigma : V \to V$ are automorphisms of some tree?

   (A *graph* consists of a some set of vertices and some edges between pairs of distinct vertices. It is *connected* if every two vertices in it are connected by some path of one or more edges. A *tree* G on V is a connected graph with vertex set V and exactly $|V| - 1$ edges, and an *automorphism* of G is a permutation $\sigma : V \to V$ such that vertices $i, j \in V$ are connected by an edge if and only if $\sigma(i)$ and $\sigma(j)$ are.)

   Proposed by: Mitchell Lee

   Answer: $\boxed{30212}$

   We decompose into cycle types of σ. Note that within each cycle, all vertices have the same degree; also note that the tree has total degree 14 across its vertices (by all its seven edges).

   For any permutation that has a 1 in its cycle type (i.e it has a fixed point), let $1 \leq a \leq 8$ be a fixed point. Consider the tree that consists of the seven edges from a to the seven other vertices - this permutation (with a as a fixed point) is an automorphism of this tree.

   For any permutation that has cycle type $2 + 6$, let a and b be the two elements in the 2-cycle. If the 6-cycle consists of c, d, e, f, g, h in that order, consider the tree with edges between a and b, c, e, g and between b and d, f, h. It's easy to see σ is an automorphism of this tree.

   For any permutation that has cycle type $2+2+4$, let a and b be the two elements of the first two-cycle. Let the other two cycle consist of c and d, and the four cycle be e, f, g, h in that order. Then consider the tree with edges between a and b, a and c, b and d, a and e, b and f, a and g, b and h. It's easy to see σ is an automorphism of this tree.

   For any permutation that has cycle type $2 + 3 + 3$, let a and b be the vertices in the 2-cycle. One of a and b must be connected to a vertex distinct from a, b (follows from connectedness), so there must be an edge between a vertex in the 2-cycle and a vertex in a 3-cycle. Repeatedly applying σ to this edge leads to a cycle of length 4 in the tree, which is impossible (a tree has no cycles). Therefore, these permutations cannot be automorphisms of any tree.

   For any permutation that has cycle type $3 + 5$, similarly, there must be an edge between a vertex in the 3-cycle and a vertex in the 5-cycle. Repeatedly applying σ to this edge once again leads to a cycle in the tree, which is not possible. So these permutations cannot be automorphisms of any tree.

   The only remaining possible cycle types of σ are $4 + 4$ and 8. In the first case, if we let x and y be the degrees of the vertices in each of the cycles, then $4x + 4y = 14$, which is impossible for integer x, y. In the second case, if we let x be the degree of the vertices in the 8-cycle, then $8x = 14$, which is not possible either.

   So we are looking for the number of permutations whose cycle type is not $2+2+3$, 8, $4+4$, $3+5$. The number of permutations with cycle type $2+2+3$ is $\binom{8}{2}\frac{1}{2}\binom{6}{3}(2!)^2 = 1120$, with cycle type 8 is $7! = 5040$, with cycle type $4 + 4$ is $\frac{1}{2}\binom{8}{4}(3!)^2 = 1260$, with cycle type $3 + 5$ is $\binom{8}{3}(2!)(4!) = 2688$. Therefore, by complementary counting, the number of permutations that ARE automorphisms of some tree is $8! - 1120 - 1260 - 2688 - 5040 = 30212$.

10. Kristoff is planning to transport a number of indivisible ice blocks with positive integer weights from the north mountain to Arendelle. He knows that when he reaches Arendelle, Princess Anna and Queen Elsa will name an ordered pair (p, q) of nonnegative integers satisfying $p+q \leq 2016$. Kristoff must then give Princess Anna *exactly* p kilograms of ice. Afterward, he must give Queen Elsa *exactly* q kilograms of ice.

    What is the minimum number of blocks of ice Kristoff must carry to guarantee that he can always meet Anna and Elsa's demands, regardless of which p and q are chosen?

    Proposed by: Pakawut Jiradilok

    Answer: $\boxed{18}$

    The answer is 18.

First, we will show that Kristoff must carry at least 18 ice blocks. Let

$$0 < x_1 \le x_2 \le \cdots \le x_n$$

be the weights of ice blocks he carries which satisfy the condition that for any $p, q \in \mathbb{Z}_{\ge 0}$ such that $p + q \le 2016$, there are disjoint subsets I, J of $\{1, \dots, n\}$ such that $\sum_{\alpha \in I} x_\alpha = p$ and $\sum_{\alpha \in J} x_\alpha = q$.

Claim: For any i, if $x_1 + \cdots + x_i \le 2014$, then

$$x_{i+1} \le \left\lfloor \frac{x_1 + \cdots + x_i}{2} \right\rfloor + 1.$$

Proof. Suppose to the contrary that $x_{i+1} \ge \lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 2$. Consider when Anna and Elsa both demand $\lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 1$ kilograms of ice (which is possible as $2 \times \left(\lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 1\right) \le x_1 + \cdots + x_i + 2 \le 2016$). Kristoff cannot give any ice x_j with $j \ge i + 1$ (which is too heavy), so he has to use from $x_1, \dots, x_i$. Since he is always able to satisfy Anna's and Elsa's demands, $x_1 + \cdots + x_i \ge 2 \times \left(\lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 1\right) \ge x_1 + \cdots + x_i + 1$. A contradiction. $\square$

It is easy to see $x_1 = 1$, so by hand we compute obtain the inequalities $x_2 \le 1$, $x_3 \le 2$, $x_4 \le 3$, $x_5 \le 4$, $x_6 \le 6$, $x_7 \le 9$, $x_8 \le 14$, $x_9 \le 21$, $x_{10} \le 31$, $x_{11} \le 47$, $x_{12} \le 70$, $x_{13} \le 105$, $x_{14} \le 158$, $x_{15} \le 237$, $x_{16} \le 355$, $x_{17} \le 533$, $x_{18} \le 799$. And we know $n \ge 18$; otherwise the sum $x_1 + \cdots + x_n$ would not reach 2016.

Now we will prove that $n = 18$ works. Consider the 18 numbers named above, say $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $\dots$, $a_{18} = 799$. We claim that with $a_1, \dots, a_k$, for any $p, q \in \mathbb{Z}_{\ge 0}$ such that $p + q \le a_1 + \cdots + a_k$, there are two disjoint subsets I, J of $\{1, \dots, k\}$ such that $\sum_{\alpha \in I} x_\alpha = p$ and $\sum_{\alpha \in J} x_\alpha = q$. We prove this by induction on k. It is clear for small $k = 1, 2, 3$. Now suppose this is true for a certain k, and we add in a_{k+1}. When Kristoff meets Anna first and she demands p kilograms of ice, there are two cases.

Case I: if $p \ge a_{k+1}$, then Kristoff gives the a_{k+1} block to Anna first, then he consider $p' = p - a_{k+1}$ and the same unknown q. Now $p' + q \le a_1 + \cdots + a_k$ and he has $a_1, \dots, a_k$, so by induction he can successfully complete his task.

Case II: if $p < a_{k+1}$, regardless of the value of q, he uses the same strategy as if $p + q \le a_1 + \cdots + a_k$ and he uses ice from $a_1, \dots, a_k$ without touching a_{k+1}. Then, when he meets Elsa, if $q \le a_1 + \cdots + a_k - p$, he is safe. If $q \ge a_1 + \cdots + a_k - p + 1$, we know $q - a_{k+1} \ge a_1 + \cdots + a_k - p + 1 - \left(\lfloor \frac{a_1 + \cdots + a_k}{2} \rfloor + 1\right) \ge 0$. So he can give the a_{k+1} to Elsa first then do as if $q' = q - a_{k+1}$ is the new demand by Elsa. He can now supply the ice to Elsa because $p + q' \le a_1 + \cdots + a_k$. Thus, we finish our induction.

Therefore, Kristoff can carry those 18 blocks of ice and be certain that for any $p + q \le a_1 + \cdots + a_{18} = 2396$, there are two disjoint subsets $I, J \subseteq \{1, \dots, 18\}$ such that $\sum_{\alpha \in I} a_\alpha = p$ and $\sum_{\alpha \in J} a_\alpha = q$. In other words, he can deliver the amount of ice both Anna and Elsa demand.