2$^{\text{nd}}$ Annual Harvard-MIT November Tournament
Saturday 7 November 2009
Team Round

Down the Infinite Corridor

Consider an isosceles triangle T with base 10 and height 12. Define a sequence $\omega_1, \omega_2, \ldots$ of circles such that ω_1 is the incircle of T and ω_{i+1} is tangent to ω_i and both legs of the isosceles triangle for $i > 1$.

1. [**3**] Find the radius of ω_1.

   Answer: $\boxed{\frac{10}{3}}$ Using the Pythagorean theorem, we see that the legs of T each have length 13. Let r be the radius of ω_1. We can divide T into three triangles, each with two vertices at vertices of T and one vertex at the center of ω_1. These triangles all have height r and have bases 13, 13, and 10. Thus their total area is $\frac{13r}{2} + \frac{13r}{2} + \frac{10r}{2} = 18r$. However, T has height 12 and base 10, so its area is 60. Thus $18r = 60$, so $r = \frac{10}{3}$.

2. [**3**] Find the ratio of the radius of ω_{i+1} to the radius of ω_i.

3. [**3**] Find the total area contained in all the circles.

   Answer: $\boxed{\frac{180\pi}{13}}$ Using the notation from the previous solution, the area contained in the ith circle is equal to πr_i^2. Since the radii form a geometric sequence, the areas do as well. Specifically, the areas form a sequence with initial term $\pi \cdot \frac{100}{9}$ and common ratio $\frac{16}{81}$, so their sum is then $\pi \cdot \frac{\frac{100}{9}}{\frac{65}{81}} = \frac{180\pi}{13}$.

Bouncy Balls

In the following problems, you will consider the trajectories of balls moving and bouncing off of the boundaries of various containers. The balls are small enough that you can treat them as points. Let us suppose that a ball starts at a point X, strikes a boundary (indicated by the line segment AB) at Y, and then continues, moving along the ray YZ. Balls always bounce in such a way that $\angle XYA = \angle BYZ$. This is indicated in the above diagram.

ball bounces off of AB at the point Y
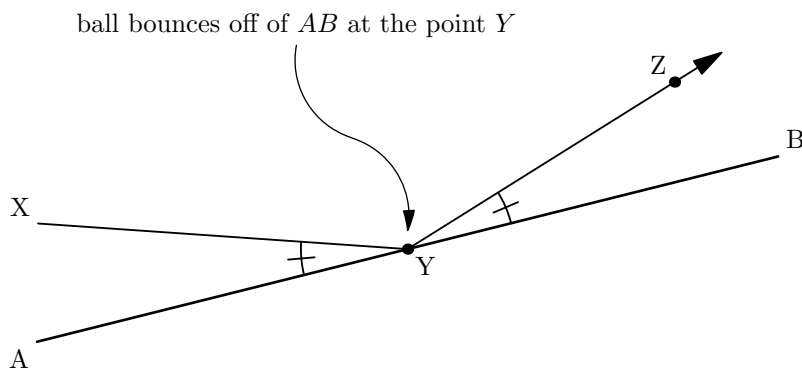
Balls bounce off of boundaries in the same way light reflects off of mirrors - if the ball hits the boundary at point P, the trajectory after P is the reflection of the trajectory before P through the perpendicular to the boundary at P.

A ball inside a rectangular container of width 7 and height 12 is launched from the lower-left vertex of the container. It first strikes the right side of the container after traveling a distance of $\sqrt{53}$ (and strikes no other sides between its launch and its impact with the right side).

4. [**2**] Find the height at which the ball first contacts the right side.

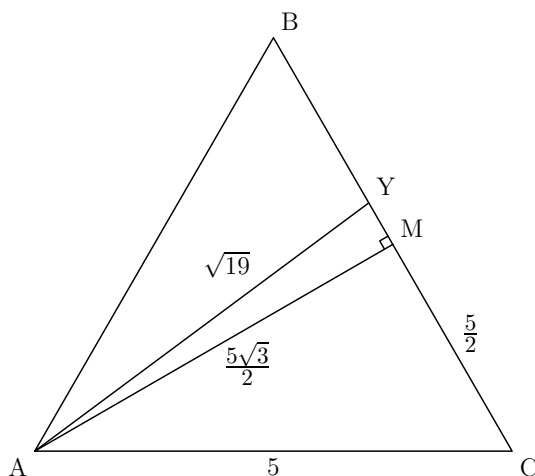**Answer:** ☐2 Let h be this height. Then, using the Pythagorean theorem, we see that $h^2 + 7^2 = 53$, so $h = 2$.

5. [**3**] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

Answer: ☐5 Every segment the ball traverses between bounces takes it 7 units horizontally and 2 units up. Thus, after 5 bounces it has traveled up 10 units, and the final segment traversed takes it directly to the upper right vertex of the rectangle.

Now a ball is launched from a vertex of an equilateral triangle with side length 5. It strikes the opposite side after traveling a distance of $\sqrt{19}$.

6. [**4**] Find the distance from the ball's point of first contact with a wall to the nearest vertex.

Answer: ☐2

Consider the diagram above, where M is the midpoint of BC. Then AM is perpendicular to BC since ABC is equilateral, so by the Pythagorean theorem $AM = \frac{5\sqrt{3}}{2}$. Then, using the Pythagorean theorem again, we see that $MY = \frac{1}{2}$, so that $BY = 2$.

7. [**4**] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

Answer: ☐7 The key idea is that, instead of reflecting the line AY off of BC, we will reflect ABC about BC and extend AY beyond $\triangle ABC$. We keep doing this until the extension of AY hits a vertex of one of our reflected triangles. This is illustrated in the diagram below:
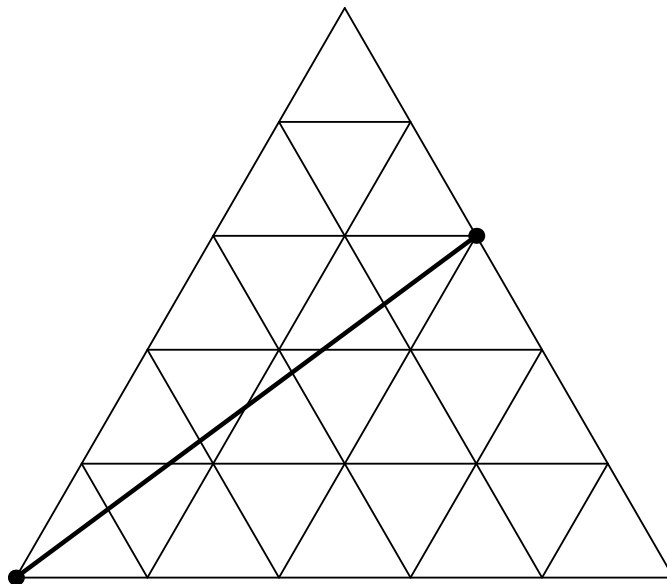
We can calculate that the line AY has slope $\frac{\frac{3\sqrt{3}}{2}}{\frac{7}{2}} = \frac{3\sqrt{3}}{7}$, so that (as indicated in the diagram), AY first intersects a vertex at the point $\left(\frac{35}{2}, \frac{15\sqrt{3}}{2}\right)$. To get there, it has to travel through 2 horizontal lines, 1 upward sloping line, and 4 downward sloping lines, so it bounces $2 + 1 + 4 = 7$ times total.

In this final problem, a ball is again launched from the vertex of an equilateral triangle with side length 5.

8. [**6**] In how many ways can the ball be launched so that it will return again to a vertex for the first time after 2009 bounces?

Answer: $\boxed{502}$ We will use the same idea as in the previous problem. We first note that every vertex of a triangle can be written uniquely in the form $a(5,0) + b\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$, where a and b are non-negative integers. Furthermore, if a ball ends at $a(5,0) + b\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$, then it bounches off of a wall $2(a+b) - 3$ times. Therefore, the possible directions that you can launch the ball in correspond to solutions to $2(a+b) - 3 = 2009$, or $a+b = 1006$. However, if a and b have a common factor, say, k, then the ball will pass through the vertex corresponding to $\frac{a}{k}$ and $\frac{b}{k}$ before it passes through the vertex corresponding to a and b. Therefore, we must discount all such pairs a, b. This corresponds to when a is even or $a = 503$, so after removing these we are left with 502 remaining possible values of a, hence 502 possible directions in which to launch the ball.

Super Mario 64!

Mario is once again on a quest to save Princess Peach. Mario enters Peach's castle and finds himself in a room with 4 doors. This room is the first in a sequence of 2 indistinugishable rooms. In each room, 1 door leads to the next room in the sequence (or, for the second room, into Bowser's level), while the other 3 doors lead to the first room.

9. [**3**] Suppose that in every room, Mario randomly picks a door to walk through. What is the expected number of doors (not including Mario's initial entrance to the first room) through which Mario will pass before he reaches Bowser's level?

Answer: $\boxed{20}$ Let E_i be the expected number of doors through which Mario will pass in the future if he is currently in room i for $i = 1, 2, 3$ (we will set $E_3 = 0$). We claim that $E_i = 1 + \frac{3}{4}E_1 + \frac{1}{4}E_{i+1}$. Indeed, the 1 at the beginning comes from the fact that we need to pass through a door to leave the room, the $\frac{3}{4}E_1$ comes from the fact that there is a $\frac{3}{4}$ chance of ending up in room 1, and the $\frac{1}{4}E_{i+1}$ corresponds to the fact that there is a $\frac{1}{4}$ chance of ending up in E_{i+1}. Using this, we get $E_1 = 1 + \frac{3}{4}E_1 + \frac{1}{4}E_2$, or $E_1 = 4 + E_2$. We also get $E_2 = 1 + \frac{3}{4}E_1$. Solving this system of equations yields $E_1 = 20$.

10. [**4**] Suppose that instead there are 6 rooms with 4 doors. In each room, 1 door leads to the next room in the sequence (or, for the last room, Bowser's level), while the other 3 doors lead to the first room. Now what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

Answer: $\boxed{5460}$ This problem works in the same general way as the last problem, but it can be more succintly solved using the general formula, which is provided below in the solution to the next problem.

11. [**5**] In general, if there are d doors in every room (but still only 1 correct door) and r rooms, the last of which leads into Bowser's level, what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

Answer: $\boxed{\dfrac{d(d^r - 1)}{d - 1}}$ Let E_i be the expected number of doors through which Mario will pass in the future if he is currently in room i for $i = 1, 2, \ldots, r + 1$ (we will set $E_{r+1} = 0$). We claim that $E_i = 1 + \frac{d-1}{d}E_1 + \frac{1}{d}E_{i+1}$. This is because, as before, there is a $\frac{d-1}{d}$ chance of ending up in room 1, and a $\frac{1}{d}$ chance of ending up in room $i + 1$. Note that we can re-write this equation as $dE_i = d + (d - 1)E_1 + E_{i+1}$.

We can solve this system of equation as follows: let $E_1 = E_i + c_i$. Then we can re-write each equation as $dE_1 - d \cdot c_i = d + (d - 1)E_1 + E_1 - c_{i+1}$, so $c_{i+1} = d \cdot c_i + d$, and $c_1 = 0$. We can then see that $c_i = d\frac{d^{i-1}-1}{d-1}$ (either by finding the pattern or by using more advanced techniques, such as those given at http://en.wikipedia.org/wiki/Recurrence_relation). Solving for E_1 using $E_1 = E_r + c_r$ and $E_r = 1 + \frac{d-1}{d}E_1$, we get $E_1 = \frac{d(d^r - 1)}{d-1}$.