3^{rd} Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Guts Round

1. [**5**] David, Delong, and Justin each showed up to a problem writing session at a random time during the session. If David arrived before Delong, what is the probability that he also arrived before Justin?

2. [**5**] A circle of radius 6 is drawn centered at the origin. How many squares of side length 1 and integer coordinate vertices intersect the interior of this circle?

3. [**5**] Jacob flipped a fair coin five times. In the first three flips, the coin came up heads exactly twice. In the last three flips, the coin also came up heads exactly twice. What is the probability that the third flip was heads?

4. [**6**] Let x be a real number. Find the maximum value of $2^{x(1-x)}$.

5. [**6**] An icosahedron is a regular polyhedron with twenty faces, all of which are equilateral triangles. If an icosahedron is rotated by θ degrees around an axis that passes through two opposite vertices so that it occupies exactly the same region of space as before, what is the smallest possible positive value of θ?

6. [**6**] How many ordered pairs (S, T) of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are there whose union contains exactly three elements?

7. [**7**] Let $f(x, y) = x^2 + 2x + y^2 + 4y$. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) be the vertices of a square with side length one and sides parallel to the coordinate axes. What is the minimum value of $f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) + f(x_4, y_4)$?

8. [**7**] What is the sum of all four-digit numbers that are equal to the cube of the sum of their digits (leading zeros are not allowed)?

9. [**7**] How many functions $f : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 10\}$ satisfy the property that $f(i) + f(j) = 11$ for all values of i and j such that $i + j = 11$.

10. [**8**] What is the smallest integer greater than 10 such that the sum of the digits in its base 17 representation is equal to the sum of the digits in its base 10 representation?

11. [**8**] How many nondecreasing sequences $a_1, a_2, \ldots, a_{10}$ are composed entirely of at most three distinct numbers from the set $\{1, 2, \ldots, 9\}$ (so $1, 1, 1, 2, 2, 2, 3, 3, 3, 3$ and $2, 2, 2, 2, 5, 5, 5, 5, 5, 5$ are both allowed)?

12. [**8**] An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point $(5, 5)$?

13. [**8**] How many sequences of ten binary digits are there in which neither two zeroes nor three ones ever appear in a row?

14. [**8**] The positive integer i is chosen at random such that the probability of a positive integer k being chosen is $\frac{3}{2}$ times the probability of $k+1$ being chosen. What is the probability that the i^{th} digit after the decimal point of the decimal expansion of $\frac{1}{7}$ is a 2?

15. [**8**] Distinct points A, B, C, D are given such that triangles ABC and ABD are equilateral and both are of side length 10. Point E lies inside triangle ABC such that $EA = 8$ and $EB = 3$, and point F lies inside triangle ABD such that $FD = 8$ and $FB = 3$. What is the area of quadrilateral $AEFD$?

16. [**9**] Triangle ABC is given in the plane. Let AD be the angle bisector of $\angle BAC$; let BE be the altitude from B to AD, and let F be the midpoint of AB. Given that $AB = 28, BC = 33, CA = 37$, what is the length of EF?

17. [**9**] A triangle with side lengths $5, 7, 8$ is inscribed in a circle C. The diameters of C parallel to the sides of lengths 5 and 8 divide C into four sectors. What is the area of either of the two smaller ones?

18. [**9**] Jeff has a 50 point quiz at 11 am. He wakes up at a random time between 10 am and noon, then arrives at class 15 minutes later. If he arrives on time, he will get a perfect score, but if he arrives more than 30 minutes after the quiz starts, he will get a 0, but otherwise, he loses a point for each minute he's late (he can lose parts of one point if he arrives a nonintegral number of minutes late). What is Jeff's expected score on the quiz?

19. [**11**] How many 8-digit numbers begin with 1, end with 3, and have the property that each successive digit is either one more or two more than the previous digit, considering 0 to be one more than 9?

20. [**11**] Given a permutation π of the set $\{1, 2, \ldots, 10\}$, define a rotated cycle as a set of three integers i, j, k such that $i < j < k$ and $\pi(j) < \pi(k) < \pi(i)$. What is the total number of rotated cycles over all permutations π of the set $\{1, 2, \ldots, 10\}$?

21. [**11**] George, Jeff, Brian, and Travis decide to play a game of hot potato. They begin by arranging themselves clockwise in a circle in that order. George and Jeff both start with a hot potato. On his turn, a player gives a hot potato (if he has one) to a randomly chosen player among the other three (if a player has two hot potatoes on his turn, he only passes one). If George goes first, and play proceedes clockwise, what is the probability that Travis has a hot potato after each player takes one turn?

22. [**12**] Let $g_1(x) = \frac{1}{3}(1 + x + x^2 + \cdots)$ for all values of x for which the right hand side converges. Let $g_n(x) = g_1(g_{n-1}(x))$ for all integers $n \geq 2$. What is the largest integer r such that $g_r(x)$ is defined for some real number x?

23. [**12**] Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers such that for integers $n > 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. How many such sequences $\{a_n\}$ are there such that $a_{2010} \leq 2^{2012}$?

24. [**12**] Let $P(x)$ be a polynomial of degree at most 3 such that $P(x) = \frac{1}{1+x+x^2}$ for $x = 1, 2, 3, 4$. What is $P(5)$?

25. [**14**] Triangle ABC is given with $AB = 13$, $BC = 14$, $CA = 15$. Let E and F be the feet of the altitudes from B and C, respectively. Let G be the foot of the altitude from A in triangle AFE. Find AG.

26. [**14**] w, x, y, z are real numbers such that

$$\begin{aligned} w + x + y + z &= 5 \\ 2w + 4x + 8y + 16z &= 7 \\ 3w + 9x + 27y + 81z &= 11 \\ 4w + 16x + 64y + 256z &= 1 \end{aligned}$$

What is the value of $5w + 25x + 125y + 625z$?

27. [**14**] Let $f(x) = -x^2 + 10x - 20$. Find the sum of all 2^{2010} solutions to $\underbrace{f(f(\ldots(x)\ldots))}_{2010 fs} = 2$.

28. [**17**] In the game of set, each card has four attributes, each of which takes on one of three values. A set deck consists of one card for each of the 81 possible four-tuples of attributes. Given a collection of 3 cards, call an attribute *good* for that collection if the three cards either all take on the same value of that attribute or take on all three different values of that attribute. Call a collection of 3 cards *two-good* if exactly two attributes are good for that collection. How many two-good collections of 3 cards are there? The order in which the cards appear does not matter.

29. [**17**] In the game of Galactic Dominion, players compete to amass cards, each of which is worth a certain number of points. Say you are playing a version of this game with only two kinds of cards, planet cards and hegemon cards. Each planet card is worth 2010 points, and each hegemon card is

worth four points per planet card held. You start with no planet cards and no hegemon cards, and, on each turn, starting at turn one, you take either a planet card or a hegemon card, whichever is worth more points given the hand you currently hold. Define a sequence $\{a_n\}$ for all positive integers n by setting a_n to be 0 if on turn n you take a planet card and 1 if you take a hegemon card. What is the smallest value of N such that the sequence $a_N, a_{N+1}, \ldots$ is necessarily periodic (meaning that there is a positive integer k such that $a_{n+k} = a_n$ for all $n \geq N$)?

30. [**17**] In the game of projective set, each card contains some nonempty subset of six distinguishable dots. A projective set deck consists of one card for each of the 63 possible nonempty subsets of dots. How many collections of five cards have an even number of each dot? The order in which the cards appear does not matter.

31. [**20**] What is the perimeter of the triangle formed by the points of tangency of the incircle of a 5-7-8 triangle with its sides?

32. [**20**] Let T be the set of numbers of the form $2^a 3^b$ where a and b are integers satisfying $0 \leq a, b \leq 5$. How many subsets S of T have the property that if n is in S then all positive integer divisors of n are in S?

33. [**20**] Convex quadrilateral $BCDE$ lies in the plane. Lines EB and DC intersect at A, with $AB = 2$, $AC = 5$, $AD = 200$, $AE = 500$, and $\cos\angle BAC = \frac{7}{9}$. What is the largest number of nonoverlapping circles that can lie in quadrilateral $BCDE$ such that all of them are tangent to both lines BE and CD?

34. [**25**] Estimate the sum of all the prime numbers less than $1,000,000$. If the correct answer is X and you write down A, your team will receive $\min\left(\lfloor\frac{25X}{A}\rfloor, \lfloor\frac{25A}{X}\rfloor\right)$ points, where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

35. [**25**] A mathematician M' is called a descendent of mathematician M if there is a sequence of mathematicians $M = M_1, M_2, \ldots, M_k = M'$ such that M_i was M_{i+1}'s doctoral advisor for all i. Estimate the number of descendents that the mathematician who has had the largest number of descendents has had, according to the Mathematical Genealogy Project. Note that the Mathematical Genealogy Project has records dating back to the 1300s. If the correct answer is X and you write down A, your team will receive $\max\left(25 - \lfloor\frac{|X-A|}{100}\rfloor, 0\right)$ points, where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

36. [**25**] Paul Erdős was one of the most prolific mathematicians of all time and was renowned for his many collaborations. The Erdős number of a mathematician is defined as follows. Erdős has an Erdős number of 0, a mathematician who has coauthored a paper with Erdős has an Erdős number of 1, a mathematician who has not coauthored a paper with Erdős, but has coauthored a paper with a mathematician with Erdős number 1 has an Erdős number of 2, etc. If no such chain exists between Erdős and another mathematician, that mathematician has an Erdős number of infinity. Of the mathematicians with a finite Erdős number (including those who are no longer alive), what is their average Erdős number according to the Erdős Number Project? If the correct answer is X and you write down A, your team will receive $\max(25 - \lfloor 100|X - A|\rfloor, 0)$ points where $\lfloor x\rfloor$ is the largest integer less than or equal to x.