# HMMT February 2022

**February 19, 2022**

## Combinatorics Round

1. Sets $A$, $B$, and $C$ satisfy $|A| = 92, |B| = 35, |C| = 63, |A \cap B| = 16, |A \cap C| = 51, |B \cap C| = 19$. Compute the number of possible values of $|A \cap B \cap C|$.

2. Compute the number of ways to color 3 cells in a $3 \times 3$ grid so that no two colored cells share an edge.

3. Michel starts with the string $HMMT$. An operation consists of either replacing an occurrence of $H$ with $HM$, replacing an occurrence of $MM$ with $MOM$, or replacing an occurrence of $T$ with $MT$. For example, the two strings that can be reached after one operation are $HMMMT$ and $HMOMT$. Compute the number of distinct strings Michel can obtain after exactly 10 operations.

4. Compute the number of nonempty subsets $S \subseteq \{-10, -9, -8, \ldots, 8, 9, 10\}$ that satisfy $|S| + \min(S) \cdot \max(S) = 0$.

5. Five cards labeled $1, 3, 5, 7, 9$ are laid in a row in that order, forming the five-digit number 13579 when read from left to right. A swap consists of picking two distinct cards, and then swapping them. After three swaps, the cards form a new five-digit number $n$ when read from left to right. Compute the expected value of $n$.

6. The numbers $1, 2, \ldots, 10$ are randomly arranged in a circle. Let $p$ be the probability that for every positive integer $k < 10$, there exists an integer $k' > k$ such that there is at most one number between $k$ and $k'$ in the circle. If $p$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers $a$ and $b$, compute $100a + b$.

7. Let $S = \{(x, y) \in \mathbb{Z}^2 \mid 0 \le x \le 11, 0 \le y \le 9\}$. Compute the number of sequences $(s_0, s_1, \ldots, s_n)$ of elements in $S$ (for any positive integer $n \ge 2$) that satisfy the following conditions:

   - $s_0 = (0, 0)$ and $s_1 = (1, 0)$,
   - $s_0, s_1, \ldots, s_n$ are distinct,
   - for all integers $2 \le i \le n$, $s_i$ is obtained by rotating $s_{i-2}$ about $s_{i-1}$ by either $90°$ or $180°$ in the clockwise direction.

8. Random sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are chosen so that every element in each sequence is chosen independently and uniformly from the set $\{0, 1, 2, 3, \ldots, 100\}$. Compute the expected value of the smallest nonnegative integer $s$ such that there exist positive integers $m$ and $n$ with

$$s = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

9. Consider permutations $(a_0, a_1, \ldots, a_{2022})$ of $(0, 1, \ldots, 2022)$ such that

   - $a_{2022} = 625$,
   - for each $0 \le i \le 2022$, $a_i \ge \frac{625i}{2022}$,
   - for each $0 \le i \le 2022$, $\{a_i, \ldots, a_{2022}\}$ is a set of consecutive integers (in some order).

   The number of such permutations can be written as $\frac{a!}{b!c!}$ for positive integers $a, b, c$, where $b > c$ and $a$ is minimal. Compute $100a + 10b + c$.

10. Let $S$ be a set of size 11. A random 12-tuple $(s_1, s_2, \ldots, s_{12})$ of elements of $S$ is chosen uniformly at random. Moreover, let $\pi \colon S \to S$ be a permutation of $S$ chosen uniformly at random. The probability that $s_{i+1} \ne \pi(s_i)$ for all $1 \le i \le 12$ (where $s_{13} = s_1$) can be written as $\frac{a}{b}$ where $a$ and $b$ are relatively prime positive integers. Compute $a$.