14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

Team Round B

Think Carefully! [55]

The problems in this section require only short answers.

1. [**55**] Tom, Dick, and Harry play a game in which they each pick an integer between 1 and 2011. Tom picks a number first and informs Dick and Harry of his choice. Then Dick picks a different number and informs Harry of his choice. Finally, Harry picks a number different from both Tom's and Dick's. After all the picks are complete, an integer is randomly selected between 1 and 2011. The player whose number is closest wins 2 dollars, unless there is a tie, in which case each of the tied players wins 1 dollar. If Tom knows that Dick and Harry will each play optimally and select randomly among equally optimal choices, there are two numbers Tom can pick to maximize his expected profit; what are they?

Answer: $\boxed{503, 1509}$ Let x denote the number Tom chooses. By the symmetry of the problem, picking x and picking $2012-x$ yield the same expected profit. If Tom picks 1006, Dick sees that if he picks 1007, Harry's best play is to pick 1005, and Dick will win with probability $\frac{1005}{2011}$, and clearly this is the best outcome he can achieve. So Dick will pick 1007 (or 1005) and Harry will pick 1005 (or 1007), and Tom will win with probability $\frac{1}{2011}$. Picking the number 2 will make the probability of winning at least $\frac{2}{2011}$ since Dick and Harry would both be foolish to pick 1, so picking 1006 is suboptimal. It is now clear that the answer to the problem is some pair $(x, 2012-x)$.

By the symmetry of the problem we can assume without loss of generality that $1 \le x \le 1005$. We will show that of these choices, $x = 503$ maximizes Tom's expected profit. The trick is to examine the relationship between Tom's choice and Dick's choice. We claim that (a) if $x > 503$, Dick's choice is $2013 - x$ and (b) if $x < 503$, Dick's choice is $1341 + \lfloor \frac{x+1}{3} \rfloor$. Let y denote the number Dick chooses.
Proof (a). Note first that if $x > 503$, $2013 - x$ is the smallest value of y for which Harry always chooses a number less than x. This is obvious, for if $y < 2011 - x$, a choice of $2012 - x$ will give Harry a greater expected profit than a choice of any number less than x, and if $y = 2012 - x$, Harry never chooses a number between x and y since $x - 1 > \frac{2012-x-x}{2} = 1006 - x$ holds for all *integers* x greater than 503, so, by symmetry, Harry chooses a number less than x exactly half the time. The desired result actually follows immediately because the fact that Harry always chooses a number less than x when $y = 2013 - x$ implies that there would be no point in Dick choosing a number greater than $2013 - x$, so it suffices to compare Dick's expected profit when $y = 2013 - x$ with that of all smaller values of y, which is trivial.
Proof (b). This case is somewhat more difficult. First, it should be obvious that if $x < 503$, Harry never chooses a number less than x. The proof is by contradiction. If $y \le 2011 - x$, Harry can obtain greater expected profit by choosing $2012 - x$ than by choosing any number less than x. If $y \ge 1002 + x$, Harry can obtain greater expected profit by choosing any number between x and y than by choosing a number less than x. Hence if Harry chooses a number less than x, x and y must satisfy $y > 2011 - x$ and $y < 1002 + x$, which implies $2x > 2009$, contradiction. We next claim that if $y \ge 1342 + \lfloor \frac{x+1}{3} \rfloor$, then Harry chooses a number between x and y. The proof follows from the inequality $\frac{1342+\lfloor \frac{x+1}{3} \rfloor - x}{2} > 2011 - (1342 + \lfloor \frac{x+1}{3} \rfloor)$, which is equivalent to $3\lfloor \frac{x+1}{3} \rfloor > x - 4$, which is obviously true. We also claim that if $y \le 1340 + \lfloor \frac{x+1}{3} \rfloor$ then Harry chooses a number greater than y. The proof follows from the inequality $\frac{1340+\lfloor \frac{x+1}{3} \rfloor - x}{2} < 2011 - (1340 + \lfloor \frac{x+1}{3} \rfloor)$, which is equivalent to $3\lfloor \frac{x+1}{3} \rfloor < x + 2$, which is also obviously true. Now if $x \not\equiv 1 \pmod 3$, then we have $3\lfloor \frac{x+1}{3} \rfloor > x - 1$, so, in fact, if $y = 1341 + \lfloor \frac{x+1}{3} \rfloor$, Harry still chooses a number between x and y. In this case it is clear that such a choice of y maximizes Dick's expected profit. It turns out that the same holds true even if $x \equiv 1 \pmod 3$; the computations are only slightly more involved. We omit them here because they bear tangential relation to the main proof.
We may conclude that if $x > 503$, the optimal choice for x is 504, and if $x < 503$, the optimal choice for x is 502. In the first case, $y = 1509$, and, in the second case, $y = 1508$. Since a choice of $x = 503$ and $y = 1509$ clearly outperforms both of these combinations when evaluated based on Tom's expected

profit, it suffices now to show that if Tom chooses 503, Dick chooses 1509. Since the computations are routine and almost identical to those shown above, the proof is left as an exercise to the reader.
Remark: For the more determined, intuition alone is sufficient to see the answer. Assuming Tom's number x is less than 1006, Dick will never pick a number y that makes Harry's best move $y+1$, so Harry's best move will either be $x-1$ or anything in between x and y. Clearly Tom does not want Harry to pick $x-1$, and the biggest number he can pick so that Harry will have to pick in between him and Dick is $\frac{1}{4}$th of the way along the set of numbers, which in this case is 503. In a rigorous solution, however, the calculations must be handled with grave care.

An interesting inquiry is, what is the optimal strategy with n players?

Complex Numbers [90]

The problems in this section require only short answers.

The *norm* of a complex number $z = a + bi$, denoted by $|z|$, is defined to be $\sqrt{a^2 + b^2}$. In the following problems, it may be helpful to note that the norm is multiplicative and that it obeys the triangle inequality. In other words, please observe that for all complex numbers x and y, $|xy| = |x||y|$ and $|x + y| \leq |x| + |y|$. (You may verify these facts for yourself if you like).

2. [**20**] Let a, b, and c be complex numbers such that $|a| = |b| = |c| = |a + b + c| = 1$. If $|a - b| = |a - c|$ and $b \neq c$, evaluate $|a + b||a + c|$.

Answer: $\boxed{2}$ *First solution.*

Since $|a| = 1$, a cannot be 0. Let $u = \frac{b}{a}$ and $v = \frac{c}{a}$. Dividing the given equations by $|a| = 1$ gives $|u| = |v| = |1 + u + v| = 1$ and $|1 - u| = |1 - v|$. The goal is to prove that $|1 + u||1 + v| = 2$.

By squaring $|1 - u| = |1 - v|$, we get $(1 - u)\overline{(1 - u)} = (1 - v)\overline{(1 - v)}$, and thus $1 - u - \bar{u} + |u|^2 = 1 - v - \bar{v} + |v|^2$, or $u + \bar{u} = v + \bar{v}$. This implies $\mathrm{Re}(u) = \mathrm{Re}(v)$. Since u and v are on the unit circle in the complex plane, u is equal to either v or $\bar{v}$. However, $b \neq c$ implies $u \neq v$, so $u = \bar{v}$.

Therefore, $1 = |1 + u + \bar{u}| = |1 + 2\,\mathrm{Re}(u)|$. Since $\mathrm{Re}(u)$ is real, we either have $\mathrm{Re}(u) = 0$ or $\mathrm{Re}(u) = -1$. The first case gives $u = \pm i$ and $|1 + u||1 + v| = |1 + i||1 - i| = 2$, as desired. It remains only to note that $\mathrm{Re}(u) = -1$ is in fact impossible because u is of norm 1 and $u = -1$ would imply $u = \bar{u} = v$.

Remark: by the rotational symmetry of the circle, it is acceptable to skip the first step of this solution and assume $a = 1$ without loss of generality.

Second solution.

Let a, b, and c, be the vertices of a triangle inscribed in the unit circle in the complex plane. Since the complex coordinate of the circumcenter is 0 and the complex coordinate of the centroid is $\frac{a+b+c}{3}$, it follows from well-known facts about the Euler line that the complex coordinate of the orthocenter is $a + b + c$. Hence the orthocenter lies on the unit circle as well. Is it not possible for the orthocenter not to be among the three vertices of the triangle, for, if it were, two opposite angles of the convex cyclic quadrilateral formed by the three vertices and the orthocenter would each measure greater than 90 degrees. It follows that the triangle is right. However, since $|a - b| = |a - c|$, the right angle cannot occur at b or c, so it must occur at a, and the desired conclusion follows immediately.

3. [**30**] Let x and y be complex numbers such that $|x| = |y| = 1$.

   (a) [**15**] Determine the maximum value of $|1 + x| + |1 + y| - |1 + xy|$.

   (b) [**15**] Determine the maximum value of $|1+x|+|1+xy|+|1+xy^2|+\ldots+|1+xy^{2011}|-1006|1+y|$.

Solution:

(a) **Answer:** $\boxed{2\sqrt{2}}$

(b) **Answer:** $\boxed{2012\sqrt{2}}$ We divide the terms into 1006 sums of the form

$$|1 + xy^{2k}| + |1 + xy^{2k+1}| - |1 + y|.$$

For each of these, we obtain, as in part a,

$$\begin{aligned}\left|1+xy^{2k}\right|+\left|1+xy^{2k+1}\right|-|1+y| &\le \left|1+xy^{2k}\right|+\left|xy^{2k+1}-y\right|\\ &=\left|1+xy^{2k}\right|+|y|\left|1-xy^{2k}\right|\\ &=\left|1+xy^{2k}\right|+\left|1-xy^{2k}\right|\\ &\le 2\sqrt{|1-x^2y^{4k}|}\end{aligned}$$

Again, this is maximized when $x^2y^4=-1$, which leaves a total sum of at most $2012\sqrt{2}$. If we let $x=i$ and $y=-1$, every sum of three terms will be $2\sqrt{2}$, for a total of $\boxed{2012\sqrt{2}}$, as desired.

4. [**40**] Let a, b, and c be complex numbers such that $|a|=|b|=|c|=1$. If

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=1$$

as well, determine the product of all possible values of $|a+b+c|$.

Answer: $\boxed{2}$ Let $s=a+b+c$. Then

$$\begin{aligned}s^3 &= a^3+b^3+c^3+3(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+6abc\\ &= abc\Big(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+3\big(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\big)+6\Big)\\ &= abc\Big(1+\big(3(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})-9\big)+6\Big)\\ &= abc\Big(3s\big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\big)-2\Big)\\ &= abc(3s\bar{s}-2)\quad (\text{because } \bar{s}=\bar{a}+\bar{b}+\bar{c}=1/a+1/b+1/c)\\ &= abc(3|s|^2-2)\end{aligned}$$

Taking absolute values, we find $|s|^3=|3|s|^2-2|$. It follows that $|s|$ must be a positive real root of $x^3-3x^2+2=0$ or $x^3+3x^2-2=0$. However, since the negative real roots of $x^3-3x^2+2=0$ are exactly the additive inverses of the positive real roots of $x^3-3x^2+2=0$, and all three roots of $x^3-3x^2+2=0$ are real ($x^3-3x^2+2=0$ may be factored as $(x-1)(x^2-2x-2)=0$, and the discriminant of x^2-2x-2 is positive), the product of all possible values of $|s|$ is $(-2)\cdot(-1)^n$, where n denotes the number of negative real roots of $x^3-3x^2+2=0$. By Descartes's Rule of Signs, we see that n is odd, so the answer is 2, as desired.

Warm Up Your Proof Skills! [40]

The problems in this section require complete proofs.

5. (a) [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $0<y-x<1$. The first player to write a number greater than or equal to 2010 wins. If Alice goes first, determine, with proof, who has the winning strategy.

   (b) [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $y-x\in(0,1)$. The first player to write a number greater than or equal to 2010 on her 2011th turn or later wins. If a player writes a number greater than or equal to 2010 on her 2010th turn or before, she loses immediately. If Alice goes first, determine, with proof, who has the winning strategy.

Solution: Each turn of the game is equivalent to adding a number between 0 and 1 to the number on the board.

(a) Barbara has the winning strategy: whenever Alice adds z to the number on the board, Barbara adds $1-z$. After Barbara's ith turn, the number on the board will be i. Therefore, after Barbara's 2009th turn, Alice will be forced to write a number between 2009 and 2010, after which Barbara can write 2010 and win the game.

(b) Alice has the winning strategy: she writes any number a on her first turn, and, after that, whenever Barbara adds z to the number on the board, Alice adds $1-z$. After Alice's ith turn, the number on the board will be $(i-1)+a$, so after Alice's 2010th turn, the number will be $2009+a$. Since Barbara cannot write a number greater than or equal to 2010 on her 2010th turn, she will be forced to write a number between $2009+a$ and 2010, after which Alice can write 2010 and win the game.

The Euler Totient Function [40]

The problems in this section require complete proofs.

Euler's *totient* function, denoted by φ, is a function whose domain is the set of positive integers. It is especially important in number theory, so it is often discussed on the radio or on national TV. (Just kidding). But what is it, exactly? For all positive integers k, $\varphi(k)$ is defined to be the number of positive integers less than or equal to k that are relatively prime to k. It turns out that φ is what you would call a *multiplicative function*, which means that if a and b are relatively prime positive integers, $\varphi(ab) = \varphi(a)\varphi(b)$. Unfortunately, the proof of this result is highly nontrivial. However, there is much more than that to φ, as you are about to discover!

6. [**10**] Let n be a positive integer such that $n > 2$. Prove that $\varphi(n)$ is even.

   Solution: Let A_n be the set of all positive integers $x \le n$ such that $\gcd(n, x) = 1$. Since $\gcd(n, x) = \gcd(n, n-x)$ for all x, if a is a positive integer in A_n, so is $n-a$. Moreover, if a is in A_n, a and $n-a$ are different since $\gcd(n, \frac{n}{2}) = \frac{n}{2} > 1$, meaning $\frac{n}{2}$ is not in A_n. Hence we may evenly pair up the elements of A_n that sum to n, so $\varphi(n)$, the number of elements of A_n, must be even, as desired.

7. [**10**] Let n be an even positive integer. Prove that $\varphi(n) \le \frac{n}{2}$.

   Solution: Again, let A_n be the set of all positive integers $x \le n$ such that $\gcd(n, x) = 1$. Since n is even, no element of A_n may be even, and, by definition, every element of A_n must be at most n. It follows that $\varphi(n)$, the number of elements of A_n, must be at most $\frac{n}{2}$, as desired.

8. [**20**] Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be a set of positive integers such that $a_1 = 2$ and $a_m = \varphi(a_{m+1})$ for all $1 \le m \le n-1$. Prove that $a_n \ge 2^{n-1}$.

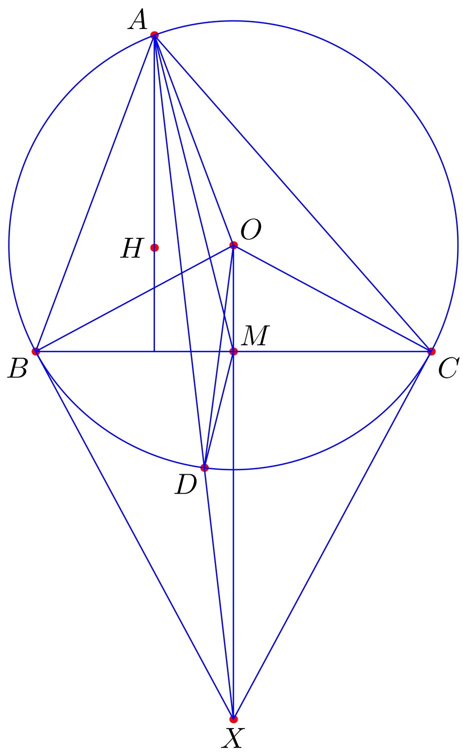
   Solution: From problem six, it follows that a_m is even for all $m \le n-1$. From problem seven, it follows that $a_m \ge 2a_{m-1}$ for all $m \le n-1$. We may conclude that $a_n \ge a_{n-1} \ge 2^{n-2}a_1 = 2^{n-1}$, as desired.

Introduction to the Symmedian [70]

The problems in this section require complete proofs.

If A, B, and C are three points in the plane that do not all lie on the same line, the symmedian from A in triangle ABC is defined to be the reflection of the median from A in triangle ABC about the bisector of angle A. Like the φ function, it turns out that the symmedian satisfies some interesting properties, too. For instance, just like how the medians from A, B, and C all intersect at the centroid of triangle ABC, the symmedians from A, B, and C all intersect at what is called (no surprises here) the symmedian point of triangle ABC. The proof of this fact is not easy, but it is unremarkable. In this section, you will investigate some surprising alternative constructions of the symmedian.

9. [**25**] Let ABC be a non-isosceles, non-right triangle, let ω be its circumcircle, and let O be its circumcenter. Let M be the midpoint of segment BC. Let the tangents to ω at B and C intersect at X. Prove that $\angle OAM = \angle OXA$. (Hint: use SAS similarity).



Solution:

Note that $\triangle OMC \sim \triangle OCX$ since $\angle OMC = \angle OCX = \frac{\pi}{2}$. Hence $\frac{OM}{OC} = \frac{OC}{OX}$, or, equivalently, $\frac{OM}{OA} = \frac{OA}{OX}$. By SAS similarity, it follows that $\triangle OAM \sim \triangle OXA$. Therefore, $\angle OAM = \angle OXA$.

10. [**15**] Let the circumcircle of triangle AOM intersect ω again at D. Prove that points A, D, and X are collinear.

    Solution: By the similarity $\triangle OAM \sim OXA$, we have that $\angle OAX = \angle OMA$. Since $AOMD$ is a cyclic quadrilateral, we have that $\angle OMA = \angle ODA$. Since $OA = OD$, we have that $\angle ODA = \angle OAD$. Combining these equations tells us that $\angle OAX = \angle OAD$, so A, D, and X are collinear, as desired.

11. [**10**] Let H be the intersection of the three altitudes of triangle ABC. (This point is usually called the orthocenter). Prove that $\angle DAH = \angle MAO$.

    Solution: Note that since both AH and OX are perpendicular to BC, it follows that $AH \parallel OX$, so $\angle DAH = \angle DXO = \angle AXO = \angle MAO$, as desired.

12. [**10**] Prove that line AD is the symmedian from A in triangle ABC by showing that $\angle DAB = \angle MAC$.

    Solution: First, we have that

    $$\angle OAC = \tfrac{1}{2}(\pi - \angle AOC) = \tfrac{\pi}{2} - \angle ABC = \angle HAB$$

    so, by the previous problem, we obtain

    $$\angle DAB = \angle DAH + \angle HAB = \angle MAO + \angle OAC = \angle MAC.$$

    It follows that the lines AD and AM are reflections of each other across the bisector of $\angle BAC$, so AD is the symmedian from A in triangle ABC, as desired.

13. [**10**] Prove that line AD is also the symmedian from D in triangle DBC.

    Solution: In the previous problems, we showed that the symmedian from A in triangle ABC is the line AX, where X is the intersection of the tangents to the circumcircle at B and C. Since the choice of triangle was arbitrary, we may conclude that DX is the symmedian from D in triangle DBC. Since A, D, and X, are collinear, this completes the proof.

Last Writes [65]

The problems in this section require complete proofs.

14. [**25**] Rachel and Brian are playing a game in a grid with 1 row of 2011 squares. Initially, there is one white checker in each of the first two squares from the left, and one black checker in the third square from the left. At each stage, Rachel can choose to either run or fight. If Rachel runs, she moves the black checker moves 1 unit to the right, and Brian moves each of the white checkers one unit to the right. If Rachel chooses to fight, she pushes the checker immediately to the left of the black checker 1 unit to the left; the black checker is moved 1 unit to the right, and Brian places a new white checker in the cell immediately to the left of the black one. The game ends when the black checker reaches the last cell. How many different final configurations are possible?

    Answer: $\boxed{2009}$ Both operations, run and fight, move the black checker exactly one square to the right, so the game will end after exactly 2008 moves regardless of Brian's choices. Furthermore, it is easy to see that the order of the operations does not matter, so two games with the same number of fights will end up in the same final configuration. Finally, note that each fight adds one white checker to the grid, so two games with different numbers of fights will end up in different final configurations. There are 2009 possible values for the number of fights, so there are 2009 possible final configurations.

15. [**40**] On Facebook, there is a group of people that satisfies the following two properties: (i) there exists a positive integers k such that any subset of $2k-1$ people in the group contains a subset of k people in the group who are all friends with each other, and (ii) every member of the group has 2011 friends or fewer.

    (a) [**15**] If $k = 2$, determine, with proof, the maximum number of people the group may contain.

    (b) [**25**] If $k = 776$, determine, with proof, the maximum number of people the group may contain.

    Solution:

    (a) **Answer:** $\boxed{4024}$ If $k = 2$, then among any three people at least two of them are friends. Clearly if we have 4024 people divided into two sets of 2012 such that everyone is friends with everyone in their set but no one in the other set, then any triple of three people will contain two people from the same set, who will automatically be friends. Therefore, the group may contain 4024 people.

    We now prove that 4024 is the maximum. Given any set with at least 4025 people, consider any two people who are not friends. Between them them have at most 4022 friends, so by the pigeonhole principle there exists someone who is not friends with either one. Therefore, there exists a triple of three people who are all not friends with each other, so the group may not contain 4025 people or more, so 4024 is the maximum, as we claimed.

    (b) **Answer:** $\boxed{4024}$ The answer remains the same. Considering the construction identical to that in the above solution, we see that the group may still contain 4024 people and satisfy the desired criterion.

    We now prove that 4024 is the maximum. Consider a group with at least 4025 people. From the previous part, we know there exist three people who are not friends. Pick such a threesome and call them X,Y, and Z. We now consider the rest of the people and successively pick pairs of persons A_i, B_i for $1 \le i \le 774$ as follows: Once we have picked A_i, B_i, of the remaining $4019 - 2i$ people, we find two who are not friends, which is always possible since $4019 - 2i > 2013$, and we name one of those people A_{i+1} and the other B_{i+1}. Once we have done this, consider the set containing X, Y, and Z as well as A_i and B_i for all $1 \le i \le 774$. This is a set of $2k - 1 = 1551$ people. Any subset where everyone is friends with each other can contain at most 1 of A_i, B_i for all i, and at most 1 of X, Y, and Z, meaning that any such subset may contain at most 775 people. Hence there exists a subset containing 1551 people that does not have 776 people who are all friends. Thus, the group may not contain 4025 people or more, so the answer is still 4024, as desired.