HMMT February 2023

February 18, 2023

Geometry Round

1. Let $ABCDEF$ be a regular hexagon, and let P be a point inside quadrilateral $ABCD$. If the area of triangle PBC is 20, and the area of triangle PAD is 23, compute the area of hexagon $ABCDEF$.

   Proposed by: Ankit Bisain

   Answer: $\boxed{189}$

   Solution: If s is the side length of the hexagon, h_1 is the length of the height from P to BC, and h_2 is the length of the height from P to AD, we have $[PBC] = \frac{1}{2}s \cdot h_1$ and $[PAD] = \frac{1}{2}(2s) \cdot h_2$. We also have $h_1 + h_2 = \frac{\sqrt{3}}{2}s$. Therefore,

$$2[PBC] + [PAD] = s(h_1 + h_2) = \frac{\sqrt{3}}{2}s^2.$$

   The area of a hexagon with side length s is $\frac{3\sqrt{3}}{2}s^2$, giving a final answer of

$$6[PBC] + 3[PAD] = 6 \cdot 20 + 3 \cdot 23 = \boxed{189}.$$

2. Points X, Y, and Z lie on a circle with center O such that $XY = 12$. Points A and B lie on segment XY such that $OA = AZ = ZB = BO = 5$. Compute AB.

   Proposed by: Rishabh Das

   Answer: $\boxed{2\sqrt{13}}$

   Solution: Let the midpoint of XY be M. Because $OAZB$ is a rhombus, $OZ \perp AB$, so M is the midpoint of AB as well. Since $OM = \frac{1}{2}OX$, $\triangle OMX$ is a $30-60-90$ triangle, and since $XM = 6$, $OM = 2\sqrt{3}$. Since $OA = 5$, the Pythagorean theorem gives $AM = \sqrt{13}$, so $AB = 2\sqrt{13}$.

3. Suppose $ABCD$ is a rectangle whose diagonals meet at E. The perimeter of triangle ABE is 10π and the perimeter of triangle ADE is n. Compute the number of possible integer values of n.

   Proposed by: Luke Robitaille

   Answer: $\boxed{47}$

   Solution: For each triangle $\mathcal{T}$, we let $p(\mathcal{T})$ to denote the perimeter of $\mathcal{T}$.

   First, we claim that $\frac{1}{2}p(\triangle ABE) < p(\triangle ADE) < 2p(\triangle ABE)$. To see why, observe that

$$p(\triangle ADE) = EA + ED + AD < 2(EA + ED) = 2(EA + EB) < 2p(\triangle ABE),$$

   Similarly, one can show that $p(\triangle ABE) < 2p(\triangle ADE)$, proving the desired inequality.

   This inequality limits the possibility of n to only those in $(5\pi, 20\pi) \subset (15.7, 62.9)$, so n could only range from $16, 17, 18, \ldots, 62$, giving 47 values. These values are all achievable because

   - when AD approaches zero, we have $p(\triangle ADE) \to 2EA$ and $p(\triangle ABE) \to 4EA$, implying that $p(\triangle ADE) \to \frac{1}{2}p(\triangle ABE) = 5\pi$;
   - similarly, when AB approaches zero, we have $p(\triangle ADE) \to 2p(\triangle ABE) = 20\pi$; and
   - by continuously rotating segments AC and BD about E, we have that $p(\triangle ADE)$ can reach any value between $(5\pi, 20\pi)$.

Hence, the answer is 47.

4. Let $ABCD$ be a square, and let M be the midpoint of side BC. Points P and Q lie on segment AM such that $\angle BPD = \angle BQD = 135^\circ$. Given that $AP < AQ$, compute $\frac{AQ}{AP}$.

   Proposed by: Ankit Bisain, Luke Robitaille

   Answer: $\boxed{\sqrt{5}}$

   Solution: Notice that $\angle BPD = 135^\circ = 180^\circ - \frac{\angle BAD}{2}$ and P lying on the opposite side of BD as C means that P lies on the circle with center C through B and D. Similarly, Q lies on the circle with center A through B and D.

   Let the side length of the square be 1. We have $AB = AQ = AD$, so $AQ = 1$. To compute AP, let E be the reflection of D across C. We have that E lies both on AM and the circle centered at C through B and D. Since AB is tangent to this circle,

$$AB^2 = AP \cdot AE$$

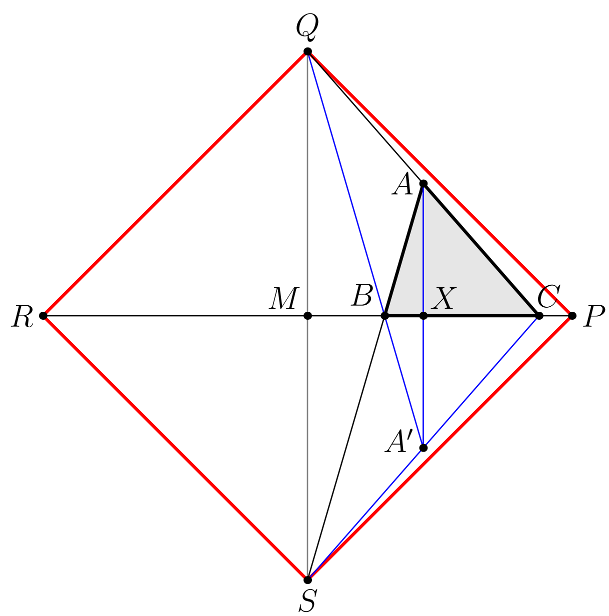
   by power of a point. Thus, $1^2 = AP \cdot \sqrt{5} \implies AP = \frac{1}{\sqrt{5}}$. Hence, the answer is $\sqrt{5}$.

5. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Suppose $PQRS$ is a square such that P and R lie on line BC, Q lies on line CA, and S lies on line AB. Compute the side length of this square.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{42\sqrt{2}}$

   Solution:



   Let A' be the reflection of A across BC. Since Q and S are symmetric across BC, we get that $Q \in BA'$, $S \in CA'$. Now, let X and M be the midpoints of AA' and PR. Standard altitude computation gives $BX = 5$, $CX = 9$, $AX = 12$. Moreover, from similar triangles, $CX : CY = AA' : PR = BX : BM$, so $BM : CM = 5 : 9$, so we easily get that $BM = 35/2$. Now, $PM = \frac{12}{9} \cdot BY = 42$, so the side length is $42\sqrt{2}$.

6. Convex quadrilateral $ABCD$ satisfies $\angle CAB = \angle ADB = 30^\circ$, $\angle ABD = 77^\circ$, $BC = CD$, and $\angle BCD = n^\circ$ for some positive integer n. Compute n.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{68}$

   Solution: Let O be the circumcenter of $\triangle ABD$. From $\angle ADB = 30^\circ$, we get that $\triangle AOB$ is equilateral. Moreover, since $\angle BAC = 30^\circ$, we have that AC bisects $\angle BAO$, and thus must be the perpendicular bisector of BO. Therefore, we have $CB = CD = CO$, so C is actually the circumcenter of $\triangle BDO$. Hence,

$$\begin{aligned}\angle BCD &= 2(180^\circ - \angle BOD)\\ &= 2(180^\circ - 2\angle BAD)\\ &= 2(180^\circ - 146^\circ) = 68^\circ\end{aligned}$$

7. Quadrilateral $ABCD$ is inscribed in circle Γ. Segments AC and BD intersect at E. Circle γ passes through E and is tangent to Γ at A. Suppose that the circumcircle of triangle BCE is tangent to γ at E and is tangent to line CD at C. Suppose that Γ has radius 3 and γ has radius 2. Compute BD.

   Proposed by: Eric Shen, Luke Robitaille

   Answer: $\boxed{\frac{9\sqrt{21}}{7}}$

   Solution: The key observation is that $\triangle ACD$ is equilateral. This is proven in two steps.

   - From tangency at C, we have
  $$\angle DCA = \angle DCE = \angle EBC = \angle DBC = \angle DAC,$$
  implying that $CA = CD$.
   - Consider the common tangent of γ and Γ at A. By homothety at E, this line is parallel to the tangent of $\odot(EBC)$ at C, which is line CD. This implies that $AC = AD$.

   Once we have this, compute

$$\begin{aligned}AC &= 2R_\Gamma \cdot \sin 60^\circ = 3\sqrt{3}\\ AE &= 2R_\gamma \cdot \sin 60^\circ = 2\sqrt{3}\end{aligned}$$

   There are now many ways to finish. One way is to use Stewart's theorem on $\triangle ADC$ to get $ED = \sqrt{21}$, then use Power of Point to get $EB = \frac{AE\cdot EC}{ED} = \frac{2\sqrt{21}}{7}$. The final answer is $BD = BE + ED = \frac{9\sqrt{21}}{7}$.

8. Triangle ABC with $\angle BAC > 90^\circ$ has $AB = 5$ and $AC = 7$. Points D and E lie on segment BC such that $BD = DE = EC$. If $\angle BAC + \angle DAE = 180^\circ$, compute BC.

   Proposed by: Maxim Li

   Answer: $\boxed{\sqrt{111}}$

Solution: Let M be the midpoint of BC, and consider dilating about M with ratio $-\frac{1}{3}$. This takes B to E, C to D, and A to some point A' on AM with $AM = 3A'M$. Then the angle condition implies $\angle DAE + \angle EA'D = 180^\circ$, so $ADA'E$ is cyclic. Then by power of a point, we get

$$\frac{AM^2}{3} = AM \cdot A'M = DM \cdot EM = \frac{BC^2}{36}.$$

But we also know $AM^2 = \frac{2AB^2+2AC^2-BC^2}{4}$, so we have $\frac{2AB^2+2AC^2-BC^2}{12} = \frac{BC^2}{36}$, which rearranges to $BC^2 = \frac{3}{2}(AB^2 + AC^2)$. Plugging in the values for AB and AC gives $BC = \sqrt{111}$.

9. Point Y lies on line segment XZ such that $XY = 5$ and $YZ = 3$. Point G lies on line XZ such that there exists a triangle ABC with centroid G such that X lies on line BC, Y lies on line AC, and Z lies on line AB. Compute the largest possible value of XG.

   Proposed by: Luke Robitaille

   Answer: $\boxed{\frac{20}{3}}$

   Solution: The key claim is that we must have $\frac{1}{XG} + \frac{1}{YG} + \frac{1}{ZG} = 0$ (in directed lengths).

   We present three proofs of this fact.

   Proof 1: By a suitable affine transformation, we can assume without loss of generality that ABC is equilateral. Now perform an inversion about G with radius $GA = GB = GC$. Then the images of X, Y, Z (call them X', Y', Z') lie on $(GBC), (GAC), (GAB)$, so they are the feet of the perpendiculars from A_1, B_1, C_1 to line XYZ, where A_1, B_1, C_1 are the respective antipodes of G on $(GBC), (GAC), (GAB)$. But now $A_1B_1C_1$ is an equilateral triangle with medial triangle ABC, so its centroid is G. Now the centroid of (degenerate) triangle $X'Y'Z'$ is the foot of the perpendicular of the centroid of $A_1B_1C_1$ onto the line, so it is G. Thus $X'G + Y'G + Z'G = 0$, which yields the desired claim. ■

   Proof 2: Let W be the point on line XYZ such that $WG = 2GX$ (in directed lengths). Now note that $(Y, Z; G, W)$ is a harmonic bundle, since projecting it through A onto BC gives $(B, C; M_{BC}, \infty_{BC})$. By harmonic bundle properties, this yields that $\frac{1}{YG} + \frac{1}{ZG} = \frac{2}{WG}$ (in directed lengths), which gives the desired. ■

   Proof 3: Let $P \neq G$ be an arbitrary point on the line XYZ. Now, in directed lengths and signed areas, $\frac{GP}{GX} = \frac{[GBP]}{[GBX]} = \frac{[GCP]}{[GCX]}$, so $\frac{GP}{GX} = \frac{[GBP]-[GCP]}{[GBX]-[GCX]} = \frac{[GBP]-[GCP]}{[GBC]} = \frac{3([GBP]-[GCP])}{[ABC]}$. Writing analogous equations for $\frac{GP}{GY}$ and $\frac{GP}{GZ}$ and summing yields $\frac{GP}{GX} + \frac{GP}{GY} + \frac{GP}{GZ} = 0$, giving the desired. ■

   With this lemma, we may now set $XG = g$ and know that

   $$\frac{1}{g} + \frac{1}{g-5} + \frac{1}{g-8} = 0$$

   Solving the quadratic gives the solutions $g = 2$ and $g = \frac{20}{3}$; the latter hence gives the maximum (it is not difficult to construct an example for which XG is indeed $20/3$).

10. Triangle ABC has incenter I. Let D be the foot of the perpendicular from A to side BC. Let X be a point such that segment AX is a diameter of the circumcircle of triangle ABC. Given that $ID = 2$, $IA = 3$, and $IX = 4$, compute the inradius of triangle ABC.

    Proposed by: Maxim Li

    Answer: $\boxed{\frac{11}{12}}$

Solution: Let R and r be the circumradius and inradius of ABC, let AI meet the circumcircle of ABC again at M, and let J be the A-excenter. We can show that $\triangle AID \sim \triangle AXJ$ (e.g. by $\sqrt{bc}$ inversion), and since M is the midpoint of IJ and $\angle AMX = 90°$, $IX = XJ$. Thus, we have $\frac{2R}{IX} = \frac{XA}{XJ} = \frac{IA}{ID}$, so $R = \frac{IX \cdot IA}{2ID} = 3$. But we also know $R^2 - 2Rr = IO^2 = \frac{2XI^2+2AI^2-AX^2}{4}$. Thus, we have

$$r = \frac{1}{2R}\left(R^2 - \frac{2IX^2 + 2IA^2 - 4R^2}{4}\right) = \frac{11}{12}.$$