11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Algebra Test

1. [**3**] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

2. [**3**] Let $f(n)$ be the number of times you have to hit the $\sqrt{\ }$ key on a calculator to get a number less than 2 starting from n. For instance, $f(2) = 1, f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?

3. [**4**] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x.

4. [**4**] The function f satisfies
$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$
for all real numbers x, y. Determine the value of $f(10)$.

5. [**5**] Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$, and the roots of g are the squares of the roots of f. Find $g(9)$.

6. [**5**] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

7. [**5**] Compute $\displaystyle\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$.

8. [**6**] Compute $\arctan(\tan 65^\circ - 2 \tan 40^\circ)$. (Express your answer in degrees.)

9. [**7**] Let S be the set of points (a, b) with $0 \leq a, b \leq 1$ such that the equation
$$x^4 + ax^3 - bx^2 + ax + 1 = 0$$
has at least one real root. Determine the area of the graph of S.

10. [**8**] Evaluate the infinite sum
$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$