HMMT November 2023
November 11, 2023

Guts Round

1. **[5]** The formula to convert Celsius to Fahrenheit is

$$F^\circ = 1.8 \cdot C^\circ + 32.$$

In Celsius, it is 10° warmer in New York right now than in Boston. In Fahrenheit, how much warmer is it in New York than in Boston?

Proposed by: Rishabh Das

Answer: $\boxed{18^\circ}$

Solution: Let x and y be the temperatures in New York and Boston, respectively, in Fahrenheit. Then $x - y = 10$, so we compute

$$(1.8 \cdot x + 32) - (1.8 \cdot y + 32) = 1.8 \cdot (x - y) = 18.$$

2. **[5]** Compute the number of dates in the year 2023 such that when put in MM/DD/YY form, the three numbers are in strictly increasing order.

For example, 06/18/23 is such a date since $6 < 18 < 23$, while today, 11/11/23, is not.

Proposed by: William Hu

Answer: $\boxed{186}$

Solution: January contains 21 such dates, February contains 20, and so on, until December contains 10. The answer is

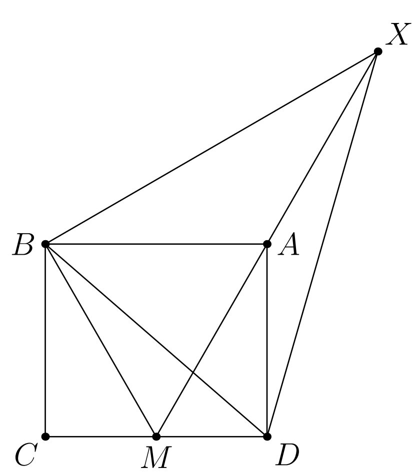
$$21 + 20 + \cdots + 10 = 186.$$

3. **[5]** Let $ABCD$ be a rectangle with $AB = 20$ and $AD = 23$. Let M be the midpoint of CD, and let X be the reflection of M across point A. Compute the area of triangle XBD.

Proposed by: Daniel Xianzhe Hong

Answer: $\boxed{575}$

Solution:



Observe that $[XBD] = [BAD] + [BAX] + [DAX]$. We will find the area of each of these triangles individually.

- We have $[ABD] = \frac{1}{2}[ABCD]$.
- Because $AM = AX$, $[BAX] = [BAM]$ as the triangles have the same base and height. Thus, as $[BAM]$ have the same base and height as $ABCD$, $[BAX] = [BAM] = \frac{1}{2}[ABCD]$.
- From similar reasoning, we know that $[DAX] = [DAM]$. We have that DAM has the same base and half the height of the rectangle. Thus, $[DAX] = [DAM] = \frac{1}{4}[ABCD]$.

Hence, we have

$$\begin{aligned}[XBD] &= [BAD] + [BAX] + [DAX] \\ &= \frac{1}{2}[ABCD] + \frac{1}{2}[ABCD] + \frac{1}{4}[ABCD] \\ &= \frac{5}{4}[ABCD]\end{aligned}$$

Thus, our answer is $\frac{5}{4}[ABCD] = \frac{5}{4}(20 \cdot 23) = 575$.

4. **[6]** The number 5.6 may be expressed uniquely (ignoring order) as a product $\underline{a}.\underline{b} \times \underline{c}.\underline{d}$ for digits a, b, c, d all nonzero. Compute $\underline{a}.\underline{b} + \underline{c}.\underline{d}$.

   Proposed by: Albert Wang, Karthik Venkata Vedula

   Answer: $\boxed{5.1}$

   Solution: We want $\overline{ab} \times \overline{cd} = 560 = 2^4 \times 5 \times 7$. To avoid a zero digit, we need to group the 5 with the 7 to get 3.5 and 1.6, and our answer is $3.5 + 1.6 = 5.1$.

5. **[6]** Let $ABCDE$ be a convex pentagon such that

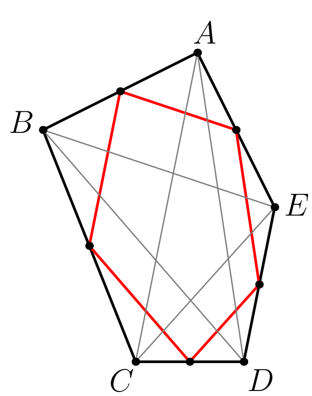
$$\begin{aligned}AB + BC + CD + DE + EA &= 64 \text{ and} \\ AC + CE + EB + BD + DA &= 72.\end{aligned}$$

   Compute the perimeter of the convex pentagon whose vertices are the midpoints of the sides of $ABCDE$.

   Proposed by: Arul Kolla

   Answer: $\boxed{36}$

   Solution:



By the midsegment theorem on triangles ABC, BCD, $\ldots$, DEA, the side lengths of the said pentagons are $AC/2$, $BD/2$, $CE/2$, $DA/2$, and $EB/2$. Thus, the answer is

$$\frac{AC+BD+CE+DA+EB}{2} = \frac{72}{2} = 36.$$

6. **[6]** There are five people in a room. They each simultaneously pick two of the other people in the room independently and uniformly at random and point at them. Compute the probability that there exists a group of three people such that each of them is pointing at the other two in the group.

   Proposed by: Neil Shah

   Answer: $\boxed{\frac{5}{108}}$

   Solution: The desired probability is the number of ways to pick the two isolated people times the probability that the remaining three point at each other. So,

$$P = \binom{5}{2} \cdot \left(\frac{\binom{2}{2}}{\binom{4}{2}}\right)^3 = 10 \cdot \left(\frac{1}{6}\right)^3 = \boxed{\frac{5}{108}}$$

   is the desired probability.

7. **[7]** Suppose a and b be positive integers not exceeding 100 such that

$$ab = \left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)^2.$$

   Compute the largest possible value of $a+b$.

   Proposed by: Rishabh Das

   Answer: $\boxed{78}$

   Solution: For any prime p and a positive integer n, let $\nu_p(n)$ be the largest nonnegative integer k for which p^k divides n. Taking ν_p on both sides of the given equation, we get

$$\nu_p(a) + \nu_p(b) = 2 \cdot |\nu_p(a) - \nu_p(b)|,$$

   which means $\frac{\nu_p(a)}{\nu_p(b)} \in \left\{3, \frac{1}{3}\right\}$ for all primes p. Using this with $a, b \le 100$, we get that

   - We must have $(\nu_2(a), \nu_2(b)) \in \{(0,0), (1,3), (3,1), (2,6), (6,2)\}$ because a and b cannot be divisible by 2^7.
   - We must have $(\nu_3(a), \nu_3(b)) \in \{(0,0), (1,3), (3,1)\}$ because a and b cannot be divisible by $3^6 > 100$.
   - a and b cannot be divisible by any prime $p \ge 5$, because if not, then one of a and b must be divisible by $p^3 \ge 5^3 > 100$.

   If $(\nu_2(a), \nu_2(b)) = (2,6)$ (and similarly with $(6,2)$), then we must have $(a,b) = (4,64)$, so the sum is 68.

   If $(\nu_3(a), \nu_3(b)) = (1,3)$ (and similarly with $(3,1)$), then we must have $\nu_2(b) \le 1$ (otherwise, $b \ge 2^2 \cdot 3^3 > 100$). Hence, the optimal pair is $(a,b) = (2^3 \cdot 3^1, 2^1 \cdot 3^3) = (24, 54)$, so the sum is $24 + 54 = 78$.

   If neither of the above happens, then $a + b \le 2^1 + 2^3 \le 10$, which is clearly not optimal.

   Hence, the optimal pair is $(24, 54)$, and the answer is 78.

8. **[7]** Six standard fair six-sided dice are rolled and arranged in a row at random. Compute the expected number of dice showing the same number as the sixth die in the row.

Proposed by: Holden Mui

Answer: $\boxed{11/6}$

Solution: For each $i = 1, 2, \ldots, 6$, let X_i denote the indicator variable of whether the i-th die shows the same number as the sixth die. Clearly, $X_6 = 1$ always. For all other i, X_i is 1 with probability $\frac{1}{6}$ and 0 otherwise, so $\mathbb{E}[X_i] = \frac{1}{6}$. By linearity of expectation, the answer is

$$\mathbb{E}[X_1 + \cdots + X_6] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_6] = 5 \cdot \frac{1}{6} + 1 = \frac{11}{6}.$$

9. **[7]** The largest prime factor of 101 101 101 101 is a four-digit number N. Compute N.

Proposed by: Arul Kolla

Answer: $\boxed{9901}$

Solution: Note that

$$\begin{aligned} 101\,101\,101\,101 &= 101 \cdot 1\,001\,001\,001 \\ &= 101 \cdot 1001 \cdot 1\,000\,001 \\ &= 101 \cdot 1001 \cdot (100^3 + 1) \\ &= 101 \cdot 1001 \cdot (100 + 1)(100^2 - 100 + 1) \\ &= 101 \cdot 1001 \cdot 101 \cdot 9901 \\ &= 101^2 \cdot 1001 \cdot 9901 \\ &= (7 \cdot 11 \cdot 13) \cdot 101^2 \cdot 9901, \end{aligned}$$

and since we are given that the largest prime factor must be four-digit, it must be 9901. One can also check manually that it is prime.

10. **[8]** A real number x is chosen uniformly at random from the interval $(0, 10)$. Compute the probability that $\sqrt{x}$, $\sqrt{x+7}$, and $\sqrt{10-x}$ are the side lengths of a non-degenerate triangle.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{22}{25}}$

Solution 1: For any positive reals a, b, c, numbers a, b, c is a side length of a triangle if and only if

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) > 0 \iff \sum_{\text{cyc}} (2a^2b^2 - a^4) > 0,$$

(to see why, just note that if $a \geq b + c$, then only the factor $-a + b + c$ is negative). Therefore, x works if and only if

$$\begin{aligned} 2(x+7)(10-x) + 2x(x+7) + 2x(10-x) &> x^2 + (x+7)^2 + (10-x)^2 \\ -5x^2 + 46x - 9 &> 0 \\ x &\in \left(\frac{1}{5}, 9\right), \end{aligned}$$

giving the answer $\frac{22}{25}$.

Solution 2: Note that $\sqrt{x} < \sqrt{x+7}$, so $\sqrt{x}$ cannot be the maximum. Thus, x works if and only if the following equivalent inequalities hold.

$$\begin{aligned}
\sqrt{x} &> \left|\sqrt{x+7} - \sqrt{10-x}\right| \\
x &> (x+7) + (10-x) - 2\sqrt{(x+7)(10-x)} \\
\sqrt{4(x+7)(10-x)} &> 17 - x \\
4(x+7)(10-x) &> x^2 - 34x + 289 \\
4(70 + 3x - x^2) &> x^2 - 34x + 289 \\
0 &> 5x^2 - 46x + 9 \\
0 &> (5x-1)(x-9),
\end{aligned}$$

so the range is $x \in \left(\frac{1}{5}, 9\right)$, and the answer is

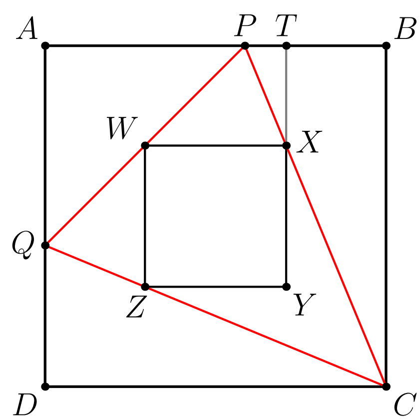
$$\frac{9 - \frac{1}{5}}{10} = \frac{22}{25}.$$

11. **[8]** Let $ABCD$ and $WXYZ$ be two squares that share the same center such that $WX \parallel AB$ and $WX < AB$. Lines CX and AB intersect at P, and lines CZ and AD intersect at Q. If points P, W, and Q are collinear, compute the ratio AB/WX.

Proposed by: Edward Yu

Answer: $\boxed{\sqrt{2}+1}$

Solution:



Without loss of generality, let $AB = 1$. Let $x = WX$. Then, since $BPWX$ is a parallelogram, we have $BP = x$. Moreover, if $T = XY \cap AB$, then we have $BT = \frac{1-x}{2}$, so $PT = x - \frac{1-x}{2} = \frac{3x-1}{2}$. Then, from $\triangle PXT \sim \triangle PBC$, we have

$$\begin{aligned}
\frac{PT}{XT} = \frac{PB}{BC} &\implies \frac{\frac{3x-1}{2}}{\frac{1-x}{2}} = \frac{x}{1} \\
&\implies 3x - 1 = x(1-x) \\
&\implies x = \pm\sqrt{2} - 1.
\end{aligned}$$

Selecting only positive solution gives $x = \sqrt{2} - 1$. Thus, the answer is $\frac{1}{\sqrt{2}-1} = \sqrt{2} + 1$.

12. **[8]** A jar contains 97 marbles that are either red, green, or blue. Neil draws two marbles from the jar without replacement and notes that the probability that they would be the same color is $\frac{5}{12}$. After Neil puts his marbles back, Jerry draws two marbles from the jar with replacement. Compute the probability that the marbles that Jerry draws are the same color.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{41}{97}}$

Solution 1: Note that $\frac{5}{12} = \frac{40 \cdot 97}{97 \cdot 96}$. Of all of the original ways we could've drawn marbles, we are adding 97 ways, namely drawing the same marble twice, all of which work. Thus, the answer is

$$\frac{40 \cdot 97 + 97}{97 \cdot 96 + 97} = \frac{41}{97}.$$

Solution 2: Let there be a, b, and c marbles of each type. We know $a + b + c = 97$ and

$$\frac{a^2 - a + b^2 - b + c^2 - c}{97 \cdot 96} = \frac{a^2 + b^2 + c^2 - 97}{97 \cdot 96} = \frac{5}{12} = \frac{40}{96}.$$

This means $a^2 + b^2 + c^2 = 41 \cdot 97$. Then the probability Bob's marbles are the same color is

$$\frac{a^2 + b^2 + c^2}{97^2} = \frac{41}{97}.$$

Remark. A triple (a, b, c) that works is $(12, 32, 53)$.

13. **[9]** Suppose x, y, and z are real numbers greater than 1 such that

$$\begin{aligned} x^{\log_y z} &= 2, \\ y^{\log_z x} &= 4, \text{ and} \\ z^{\log_x y} &= 8. \end{aligned}$$

Compute $\log_x y$.

Proposed by: Rishabh Das

Answer: $\boxed{\sqrt{3}}$

Solution: Taking $\log_2$ both sides of the first equation gives

$$\begin{aligned} \log_2 x \log_y z &= 1 \\ \frac{\log_2 x \log_2 z}{\log_2 y} &= 1. \end{aligned}$$

Performing similar manipulations on other two equations, we get

$$\begin{aligned} \frac{\log_2 x \log_2 z}{\log_2 y} &= 1 \\ \frac{\log_2 y \log_2 x}{\log_2 z} &= 2 \\ \frac{\log_2 z \log_2 y}{\log_2 x} &= 3. \end{aligned}$$

Multiplying the first and second equation gives $(\log_2 x)^2 = 2$ or $\log_2 x = \pm\sqrt{2}$. Multiplying the second and third equation gives $(\log_2 y)^2 = 6$ or $\log_2 y = \pm\sqrt{6}$. Thus, we have

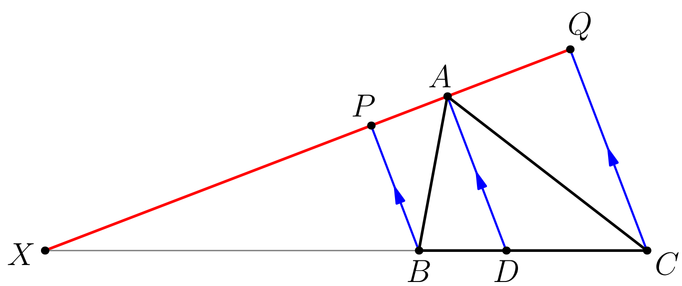
$$\log_x y = \frac{\log_2 y}{\log_2 x} = \pm\frac{\sqrt{6}}{\sqrt{2}} = \pm\sqrt{3}.$$

14. **[9]** Suppose that point D lies on side BC of triangle ABC such that AD bisects $\angle BAC$, and let ℓ denote the line through A perpendicular to AD. If the distances from B and C to ℓ are 5 and 6, respectively, compute AD.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{60}{11}}$

Solution:



Let ℓ, the external angle bisector, intersect BC at X. By the external angle bisector theorem, $AB : AC = XB : XC = 5 : 6$, so $BD : DC = 5 : 6$ by the angle bisector theorem. Then AD is a weighted average of the distances from B and C to ℓ, namely

$$\frac{6}{11} \cdot 5 + \frac{5}{11} \cdot 6 = \frac{60}{11}.$$

15. **[9]** Lucas writes two distinct positive integers on a whiteboard. He decreases the smaller number by 20 and increases the larger number by 23, only to discover the product of the two original numbers is equal to the product of the two altered numbers. Compute the minimum possible sum of the original two numbers on the board.

Proposed by: Andrew Wen

Answer: $\boxed{321}$

Solution: Let the original numbers be $m < n$. We know

$$mn = (m-20)(n+23) = mn - 20n + 23m - 460 \implies 23m - 20n = 460.$$

Furthermore, $23m < 23n$ hence $460 < 3n \implies n \geq 154$. Furthermore, we must have $23 \mid n$ hence the least possible value of n is 161 which corresponds $m = 160$.

This yields a minimum sum of $161 + 160 = 321$.

16. **[10]** Compute the number of tuples $(a_0, a_1, a_2, a_3, a_4, a_5)$ of (not necessarily positive) integers such that $a_i \leq i$ for all $0 \leq i \leq 5$ and

$$a_0 + a_1 + \cdots + a_5 = 6.$$

Proposed by: Rishabh Das

Answer: $\boxed{2002}$

Solution: Let $b_i = i - a_i$, so $b_i \geq 0$. Then

$$15 - (b_0 + b_1 + \cdots + b_5) = 6 \implies b_0 + b_1 + \cdots + b_5 = 9.$$

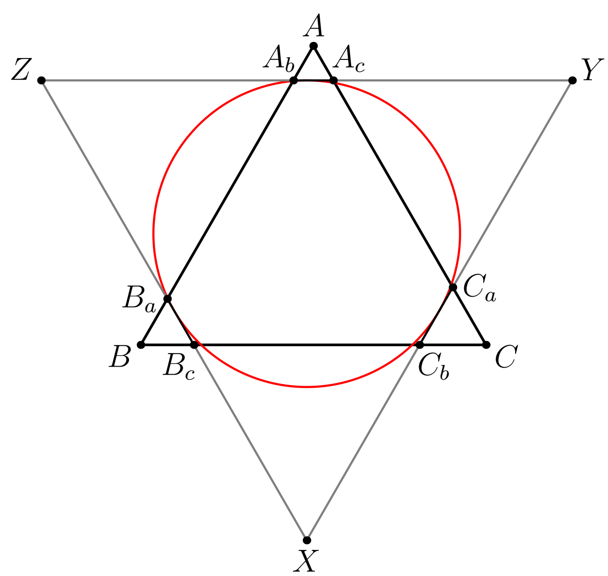
By stars and bars, the answer is $\binom{14}{5} = 2002$.

17. **[10]** Let ABC be an equilateral triangle of side length 15. Let A_b and B_a be points on side AB, A_c and C_a be points on side AC, and B_c and C_b be points on side BC such that $\triangle AA_bA_c$, $\triangle BB_cB_a$, and $\triangle CC_aC_b$ are equilateral triangles with side lengths 3, 4, and 5, respectively. Compute the radius of the circle tangent to segments $\overline{A_bA_c}$, $\overline{B_aB_c}$, and $\overline{C_aC_b}$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{3\sqrt{3}.}$

Solution:



Let $\triangle XYZ$ be the triangle formed by lines A_bA_c, B_aB_c, and C_aC_b. Then, the desired circle is the incircle of $\triangle XYZ$, which is equilateral. We have

$$\begin{aligned} YZ &= YA_c + A_cA_b + A_bZ \\ &= A_cC_a + A_cA_b + A_bB_a \\ &= (15 - 3 - 5) + 3 + (15 - 3 - 4) \\ &= 18, \end{aligned}$$

and so the inradius is $\frac{1}{2\sqrt{3}} \cdot 18 = 3\sqrt{3}$.

18. **[10]** Over all real numbers x and y such that

$$x^3 = 3x + y \qquad \text{and} \qquad y^3 = 3y + x,$$

compute the sum of all possible values of $x^2 + y^2$.

Proposed by: Rishabh Das

Answer: $\boxed{15}$

Solution 1: First, we eliminate easy cases.

- if $x = -y$, then $x^3 = 3x - x = 2x$, so $x \in \{0, \sqrt{2}, -\sqrt{2}\}$. Therefore, we get $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$, and $(0, 0)$.
- if $x = y \neq 0$, then $x^3 = 3x + x = 4x$, so $x \in \{2, -2\}$. Therefore, we get $(2, 2)$ and $(-2, -2)$.

Otherwise, adding two equations gives

$$\begin{aligned} x^3 + y^3 &= 4x + 4y \\ (x+y)(x^2 - xy + y^2) &= 4(x+y) \\ x^2 - xy + y^2 &= 4, \end{aligned}$$

and subtracting the two equations gives

$$\begin{aligned} x^3 - y^3 &= 2x - 2y \\ (x-y)(x^2 + xy + y^2) &= 2(x-y) \\ x^2 + xy + y^2 &= 2. \end{aligned}$$

We have $x^2 - xy + y^2 = 4$ and $x^2 + xy + y^2 = 2$, so adding these gives $x^2 + y^2 = 3$. One can also see that $xy = -1$, so the solution obtained will be real.

The final answer is $4 + 8 + 0 + 3 = 15$.

Solution 2: Let $x = a + \frac{1}{a}$ and $y = b + \frac{1}{b}$ for nonzero a and b. Then $x^3 - 3x = a^3 + \frac{1}{a^3}$ and $y^3 - 3y = b^3 + \frac{1}{b^3}$, so

$$a^3 + \frac{1}{a^3} = b + \frac{1}{b} \qquad \text{and} \qquad b^3 + \frac{1}{b^3} = a + \frac{1}{a}.$$

These imply $b \in \{a^3, 1/a^3\}$ and $a \in \{b^3, 1/b^3\}$. These mean $a^9 = a$ or $a^9 = \frac{1}{a}$, so $a^8 = 1$ or $a^{10} = 1$. We can WLOG $b = a^3$ for simplicity.

First suppose $a^8 = 1$. Now $a = 1$ gives $(2, 2)$, $a = -1$ gives $(-2, -2)$, $a = \operatorname{cis}(\pm\pi/4)$ gives $(\sqrt{2}, -\sqrt{2})$, and $a = \operatorname{cis}(\pm 3\pi/4)$ gives $(-\sqrt{2}, \sqrt{2})$. Finally $a = \pm i$ gives $(0, 0)$. This gives $8 + 4 + 0 = 12$ to our sum.

Now suppose $a^{10} = 1$; we can assume $a \neq 1, -1$ since these were covered by the previous case. It can be seen that for all other values of a, the pairs (a, b) that work can be get by either swapping a and b or negating one of the variables, all of which give the same value of $x^2 + y^2$, so we only need to work on one of them. Suppose $a = \operatorname{cis}(\pi/5)$ and $b = \operatorname{cis}(3\pi/5)$, so $x = 2\cos(\pi/5) = \frac{1+\sqrt{5}}{2}$ and $y = 2\cos(3\pi/5) = \frac{1-\sqrt{5}}{2}$. Then $x^2 + y^2 = 3$.

The final answer is $12 + 3 = 15$.

19. **[11]** Suppose a, b, and c are real numbers such that

$$\begin{aligned} a^2 - bc &= 14, \\ b^2 - ca &= 14, \text{ and} \\ c^2 - ab &= -3. \end{aligned}$$

Compute $|a+b+c|$.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{17}{5}}$

Solution: Subtracting the first two equations gives $(a-b)(a+b+c)=0$, so either $a=b$ or $a+b+c=0$. However, subtracting first and last equations gives $(a-c)(a+b+c)=17$, so $a+b+c\neq 0$. This means $a=b$.

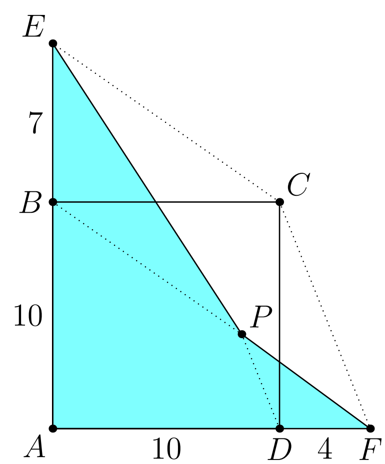
Now adding all three equations gives $(a-c)^2=25$, so $a-c=\pm 5$. Then $a+b+c=\pm\frac{17}{5}$.

20. **[11]** Let $ABCD$ be a square of side length 10. Point E is on ray $\overrightarrow{AB}$ such that $AE=17$, and point F is on ray $\overrightarrow{AD}$ such that $AF=14$. The line through B parallel to CE and the line through D parallel to CF meet at P. Compute the area of quadrilateral $AEPF$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{100}$

Solution:



From $BP \parallel CE$, we get that $[BPE]=[BPC]$. From $DP \parallel CF$, we get that $[DPF]=[DPC]$. Thus,

$$\begin{aligned}[AEPF] &= [BACP]+[BPE]+[DPF] \\ &= [BACP]+[BPC]+[DPC] \\ &= [ABCD] \\ &= 10^2 = \boxed{100}.\end{aligned}$$

21. **[11]** An integer n is chosen uniformly at random from the set $\{1,2,3,\ldots,2023!\}$. Compute the probability that

$$\gcd(n^n+50, n+1)=1.$$

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{265}{357}.}$

Solution: If n is even, we need $\gcd(n+1, 51) = 1$. If n is odd, we need $\gcd(n+1, 49) = 1$. Thus, the answer is

$$\frac{1}{2}\left(\frac{\varphi(49)}{49} + \frac{\varphi(51)}{51}\right) = \frac{265}{357}.$$

22. **[12]** There is a 6×6 grid of lights. There is a switch at the top of each column and on the left of each row. A light will only turn on if the switches corresponding to both its column and its row are in the "on" position. Compute the number of different configurations of lights.

Proposed by: Jacob Paltrowitz

Answer: $\boxed{3970}$

Solution: Take any configuration of switches such that there exists at least one row and one column which are switched on. There are $(2^6-1)^2 = 3969$ such configurations.

We prove that any two such configurations A and B lead to a different set of lights. Without loss of generality assume A has row r switched on and B doesn't have row r switched on. Thus, configuration A will contain at least one light turned on in row r (since there exists at least one column switch which is turned on), while configuration B contains zero such lights turned on. Thus configuration A and B lead to different sets of lights.

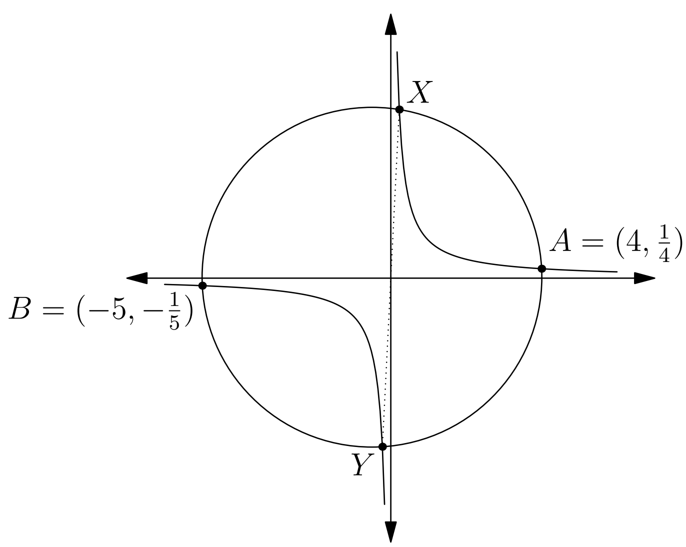
All configurations where all columns or all rows are turned off lead to all lights being turned off. We add 1 extra option to account for this case, getting $3969 + 1 = 3970$ total possibilities.

23. **[12]** The points $A = (4, \frac{1}{4})$ and $B = (-5, -\frac{1}{5})$ lie on the hyperbola $xy = 1$. The circle with diameter AB intersects this hyperbola again at points X and Y. Compute XY.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\sqrt{\frac{401}{5}}}$

Solution:



Let $A = (a, 1/a)$, $B = (b, 1/b)$, and $X = (x, 1/x)$. Since X lies on the circle with diameter $\overline{AB}$, we have $\angle AXB = 90°$. Thus, $\overline{AX}$ and $\overline{BX}$ are perpendicular, and so the product of their slopes must be -1. We deduce:

$$\frac{a-x}{\frac{1}{a}-\frac{1}{x}}\frac{b-x}{\frac{1}{b}-\frac{1}{x}} = -1 \implies (ax)(bx) = -1,$$

so $x = \pm\sqrt{-ab}$. Plugging in $a = 4$ and $b = -5$ gives $X = (\sqrt{20}, 1/\sqrt{20})$ and $Y = (-\sqrt{20}, -1/\sqrt{20})$, giving the answer.

24. **[12]** Compute the smallest positive integer k such that 49 divides $\binom{2k}{k}$.

Proposed by: Edward Yu

Answer: $\boxed{25}$

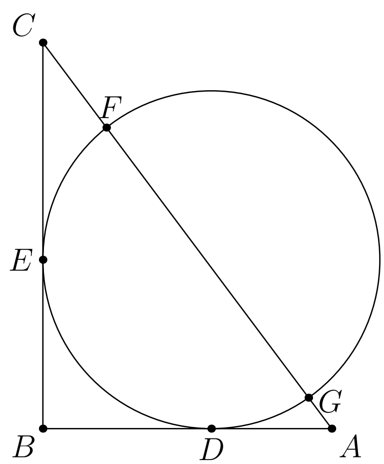
Solution: The largest a such that $7^a | \binom{2k}{k}$ is equal to the number of carries when you add $k + k$ in base 7, by Kummer's Theorem. Thus, we need two carries, so $2k$ must have at least 3 digits in base 7. Hence, $2k \geq 49$, so $k \geq 25$. We know $k = 25$ works because $25+25 = 34_7+34_7 = 101_7$ has two carries.

25. **[13]** A right triangle and a circle are drawn such that the circle is tangent to the legs of the right triangle. The circle cuts the hypotenuse into three segments of lengths 1, 24, and 3, and the segment of length 24 is a chord of the circle. Compute the area of the triangle.

Proposed by: Karthik Venkata Vedula

Answer: $\boxed{192}$

Solution 1:



Let the triangle be $\triangle ABC$, with AC as the hypotenuse, and let D, E, F, G be on sides AB, BC, AC, AC, respectively, such that they all lie on the circle. We have $AG = 1$, $GF = 24$, and $FC = 3$.

By power of a point, we have

$$AD = \sqrt{AG \cdot AF} = \sqrt{1(1+24)} = 5$$
$$CE = \sqrt{CF \cdot CG} = \sqrt{3(3+24)} = 9.$$

Now, let $BD = BE = x$. By the Pythagorean Theorem, we get that

$$\begin{aligned}(x+5)^2 + (x+9)^2 &= 28^2 \\ (x+5)^2 + (x+9)^2 - \big((x+9)-(x+5)\big)^2 &= 28^2 - 4^2 \\ 2(x+5)(x+9) &= 768 \\ (x+5)(x+9) &= 384.\end{aligned}$$

The area of $\triangle ABC$ is $\frac{1}{2}(x+5)(x+9) = \frac{1}{2} \cdot 384 = 192$.

26. **[13]** Compute the smallest multiple of 63 with an odd number of ones in its base two representation.

Proposed by: Holden Mui

Answer: $\boxed{4221}$

Solution: Notice that $63 = 2^6 - 1$, so for any a we know

$$63a = 64a - a = 2^6(a-1) + (64-a).$$

As long as $a \leq 64$, we know $a-1$ and $64-a$ are both integers between 0 and 63, so the binary representation of $63a$ is just $a-1$ followed by $64-a$ in binary (where we append leading 0s to make the latter 6 digits).

Furthermore, $a-1$ and $64-a$ sum to $63 = 111111_2$, so $a-1$ has 1s in binary where $64-a$ has 0s, and vice versa. Thus, together, they have six 1s, so $63a$ will always have six 1s in binary when $a \leq 64$.

We can also check $63 \cdot 65 = 2^{12} - 1$ has twelve 1s, while $63 \cdot 66 = 2(63 \cdot 33)$ has the same binary representation with an extra 0 at the end, so it also has six 1s. Finally,

$$63 \cdot 67 = 2^{12} + 125 = 1000001111101_2$$

has seven 1s, so the answer is $63 \cdot 67 = 4221$.

27. **[13]** Compute the number of ways to color the vertices of a regular heptagon red, green, or blue (with rotations and reflections distinct) such that no isosceles triangle whose vertices are vertices of the heptagon has all three vertices the same color.

Proposed by: Ethan Liu

Answer: $\boxed{294}$

Solution: Number the vertices 1 through 7 in order. Then, the only way to have three vertices of a regular heptagon that do not form an isosceles triangle is if they are vertices $1, 2, 4$, rotated or reflected. Thus, it is impossible for have four vertices in the heptagon of one color because it is impossible for all subsets of three vertices to form a valid scalene triangle. We then split into two cases:

Case 1: Two colors with three vertices each, one color with one vertex. There is only one way to do this up to permutations of color and rotations and reflections; if vertices $1, 2, 4$ are the same color, of the remaining 4 vertices, only $3, 5, 6$ form a scalene triangle. Thus, we have 7 possible locations for the vertex with unique color, 3 ways to pick a color for that vertex, and 2 ways to assign the remaining two colors to the two triangles, for a total of 42 ways.

Case 2: Two colors with two vertices each, one color with three vertices. There are 3 choices of color for the set of three vertices, 14 possible orientations of the set of three vertices, and $\binom{4}{2}$ choices of which pair of the remaining four vertices is of a particular remaining color; as there are only two of each color, any such assignment is valid. This is a total of total of $3 \cdot 14 \cdot 6 = 252$ ways.

Thus, the final total is $42 + 252 = 294$.

28. **[15]** There is a unique quadruple of positive integers (a, b, c, k) such that c is not a perfect square and $a + \sqrt{b + \sqrt{c}}$ is a root of the polynomial $x^4 - 20x^3 + 108x^2 - kx + 9$. Compute c.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{7}$

Solution: There are many ways to do this, including bashing it out directly.

The four roots are $a \pm \sqrt{b \pm \sqrt{c}}$, so the sum of roots is 20, so $a = 5$. Next, we compute the sum of squares of roots:

$$\left(a + \sqrt{b \pm \sqrt{c}}\right)^2 + \left(a - \sqrt{b \pm \sqrt{c}}\right)^2 = 2a^2 + 2b \pm 2\sqrt{c},$$

so the sum of squares of roots is $4a^2 + 4b$. However, from Vieta, it is $20^2 - 2 \cdot 108 = 184$, so $100 + 4b = 184 \implies b = 21$. Finally, the product of roots is

$$(a^2 - (b + \sqrt{c}))(a^2 - (b - \sqrt{c})) = (a^2 - b)^2 - c = 16 - c,$$

so we have $c = \boxed{7}$.

Remark. Here we provide a justification that there is a unique quadruple. Let $P(x) = x^4 - 20x^3 + 108x^2 - kx + 9$ and $r = a + \sqrt{b + \sqrt{c}}$. Then, note that

- r cannot be an integer because if not, then $b + \sqrt{c}$ must be an integer, so c must be a perfect square.
- r cannot be a root of an irreducible cubic. One can check this directly because the remaining factor must be linear, so it must be $x \pm 1$, $x \pm 3$, or $x \pm 9$. We can also argue using the theory of field extensions: $[\mathbb{Q}(r) : \mathbb{Q}] = 3$. However, we have the tower,

  $$\mathbb{Q}(r) \supset \mathbb{Q}(b + \sqrt{c}) \overset{\deg 2}{\supset} \mathbb{Q},$$

  and by multiplicativity of degree of field extensions, this forces $[\mathbb{Q}(r) : \mathbb{Q}(b + \sqrt{c})]$ to not be an integer.
- r cannot be a root of quadratic. If not, then P factors into two quadratic polynomials $(x^2 - mx + p)(x^2 - nx + q)$. We then have $m + n = 20$, $pq = 9$, and $mn + p + q = 108$. By AM-GM, we have $mn \leq 100$, so $p + q \geq 8$, which forces $(p, q) = (1, 9)$, but this makes $mn = 98$, which is impossible.

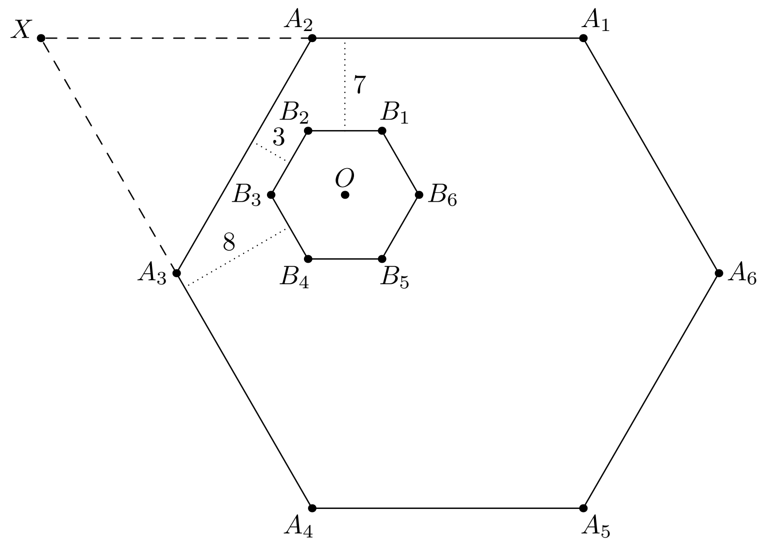
Thus, the minimal polynomial of r must be the quartic $P(x)$. This means that all roots of P are $a \pm \sqrt{b \pm c}$, and we can proceed as in the solution.

29. **[15]** Let $A_1A_2 \dots A_6$ be a regular hexagon with side length $11\sqrt{3}$, and let $B_1B_2 \dots B_6$ be another regular hexagon completely inside $A_1A_2 \dots A_6$ such that for all $i \in \{1, 2, \dots, 5\}$, A_iA_{i+1} is parallel to B_iB_{i+1}. Suppose that the distance between lines A_1A_2 and B_1B_2 is 7, the distance between lines A_2A_3 and B_2B_3 is 3, and the distance between lines A_3A_4 and B_3B_4 is 8. Compute the side length of $B_1B_2 \dots B_6$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{3\sqrt{3}}$

Solution:



Let $X = A_1A_2 \cap A_3A_4$, and let O be the center of $B_1B_2 \dots B_6$. Let p be the apothem of hexagon B. Since OA_2XA_3 is a convex quadrilateral, we have

$$\begin{aligned}[A_2A_3X] &= [A_2XO] + [A_3XO] - [A_2A_3O] \\ &= \frac{11\sqrt{3}(7+p)}{2} + \frac{11\sqrt{3}(8+p)}{2} - \frac{11\sqrt{3}(3+p)}{2} \\ &= \frac{11\sqrt{3}(12+p)}{2}.\end{aligned}$$

Since $[A_2A_3X] = (11\sqrt{3})^2\frac{\sqrt{3}}{4}$, we get that

$$\frac{12+p}{2} = (11\sqrt{3})\frac{\sqrt{3}}{4} = \frac{33}{4} \implies p = \frac{9}{2}.$$

Thus, the side length of hexagon B is $p \cdot \frac{2}{\sqrt{3}} = 3\sqrt{3}$.

30. **[15]** An HMMT party has m MIT students and h Harvard students for some positive integers m and h, For every pair of people at the party, they are either friends or enemies. If every MIT student has 16 MIT friends and 8 Harvard friends, and every Harvard student has 7 MIT enemies and 10 Harvard enemies, compute how many pairs of friends there are at the party.

Proposed by: Reagan Choi

Answer: $\boxed{342}$

Solution: We count the number of MIT-Harvard friendships. Each of the m MIT students has 8 Harvard friends, for a total of $8m$ friendships. Each of the h Harvard students has $m-7$ MIT friends, for a total of $h(m-7)$ friendships. So, $8m = h(m-7) \implies mh - 8m - 7h = 0 \implies (m-7)(h-8) = 56$.

Each MIT student has 16 MIT friends, so $m \geq 17$. Each Harvard student has 10 Harvard enemies, so $h \geq 11$. This means $m - 7 \geq 10$ and $h - 8 \geq 3$. The only such pair $(m-7, h-8)$ that multiplies to 56 is $(14, 4)$, so there are 21 MIT students and 12 Harvard students.

We can calculate the number of friendships as $\frac{16m}{2} + 8m + \frac{(h-1-10)h}{2} = 168 + 168 + 6 = \boxed{342}$.

31. **[17]** Let $s(n)$ denote the sum of the digits (in base ten) of a positive integer n. Compute the number of positive integers n at most 10^4 that satisfy

$$s(11n) = 2s(n).$$

Proposed by: Rishabh Das

Answer: $\boxed{2530}$

Solution 1: Note $2s(n) = s(10n) + s(n) = s(11n)$, so there cannot be any carries when adding n and $10n$. This is equivalent to saying no two consecutive digits of n sum to greater than 9.

We change the problem to nonnegative integers less than 10^4 (as both 0 and 10^4 satisfy the condition) so that we simply consider 4-digit numbers, possibly with leading 0s. Letting our number be $abcd$, we need $a+b \le 9$, $b+c \le 9$, and $c+d \le 9$. Letting $b' = 9-b$ and $d' = 9-d$, this means $a \le b' \ge c \le d'$. Summing over all possible values of b' and d', we want

$$\sum_{x,y=1}^{10} x \cdot \min(x,y).$$

The sum over pairs (x,y) with $x > y$ is

$$\frac{(1+2+\cdots+10)^2 - (1^2+2^2+\cdots+10^2)}{2} = \frac{55^2 - 55\cdot 7}{2} = 55 \cdot 24.$$

The sum over pairs $x \le y$ is

$$\sum_{k=1}^{10} k^2(11-k) = 11\cdot 55\cdot 7 - 55^2 = 55\cdot 22.$$

The final answer is $55 \cdot (24+22) = 55 \cdot 46 = 2530$.

Solution 2: Here is another way to calculate the sum. By doing casework on (b,c), the sum is

$$\sum_{b+c\le 9} (10-b)(10-c)$$

This is the coefficient of x^9 of

$$\begin{aligned}\left(\sum_{n\ge 0}(10-n)x^n\right)^2\left(\sum_{n\ge 0} x^n\right) &= \frac{(10-11x)^2}{(1-x)^5} \\ &= (10-11x)^2 \sum_{n\ge 0}\binom{n+4}{4} x^n\end{aligned}$$

Thus, the answer is

$$100\binom{13}{4} - 220\binom{12}{4} + 121\binom{11}{4} = \boxed{2530}$$

32. **[17]** Compute

$$\sum_{\substack{a+b+c=12\\ a\ge 6,\ b,c\ge 0}} \frac{a!}{b!c!(a-b-c)!},$$

where the sum runs over all triples of nonnegative integers (a,b,c) such that $a+b+c=12$ and $a \ge 6$.

Proposed by: Rishabh Das

Answer: $\boxed{2731}$

Solution: We tile a 1×12 board with red 1×1 pieces, blue 1×2 pieces, and green 1×2 pieces. Suppose we use a total pieces, b blue pieces, and c green pieces. Then we must have $a + b + c = 12$, and the number of ways to order the pieces is

$$\binom{a}{b, c, a-b-c}.$$

Thus, the desired sum is the number of ways to do this.

Let a_n be the number of ways to do this on a $1 \times n$ board. Then we have the recursion $a_n = a_{n-1} + 2a_{n-2}$ by casework on the first piece: if it is 1×1, we are left with a $1 \times n - 1$ board, and otherwise we are left with a $1 \times n - 2$ board. We also know $a_1 = 1$ and $a_2 = 3$, so the characteristic polynomial for this recursion is $t^2 - t - 2 = 0$, which has roots 2 and -1. Thus,

$$a_n = A \cdot (-1)^n + B \cdot 2^n.$$

Then plugging in $n = 1$ and $n = 2$ gives $A = -\frac{1}{3}$ and $B = \frac{2}{3}$, so

$$a_n = \frac{2^{n+1} + (-1)^n}{3}.$$

With $n = 12$, this evaluates to our answer of 2731.

33. **[17]** Let ω_1 and ω_2 be two non-intersecting circles. Suppose the following three conditions hold:

- The length of a common internal tangent of ω_1 and ω_2 is equal to 19.
- The length of a common external tangent of ω_1 and ω_2 is equal to 37.
- If two points X and Y are selected on ω_1 and ω_2, respectively, uniformly at random, then the expected value of XY^2 is 2023.

Compute the distance between the centers of ω_1 and ω_2.

Proposed by: Nilay Mishra

Answer: $\boxed{38}$

Solution 1: The key claim is that $\mathbb{E}[XY^2] = d^2 + r_1^2 + r_2^2$.

To prove this claim, choose an arbitrary point B on ω_2. Let r_1, r_2 be the radii of ω_1, ω_2 respectively, and O_1, O_2 be the centers of ω_1, ω_2 respectively. Thus, by the law of cosines, $\overline{O_1B} = \sqrt{d^2 + r_2^2 - 2r_2 d \cos(\theta)}$, where $\theta = \angle O_1O_2B$. Since the average value of $\cos(\theta)$ is 0, the average value of $\overline{O_1B}^2$ is $d^2 + r_2^2$.

Now suppose A is an arbitrary point on ω_1. By the law of cosines, $\overline{AB}^2 = \overline{O_1B}^2 + r_1^2 - 2r_1 d \cos(\theta)$, where $\theta = \angle AO_1B$. Thus, the expected value of $\overline{AB}^2$ is the expected value of $\overline{O_1B}^2 + r_1^2$ which becomes $d^2 + r_1^2 + r_2^2$. This proves the key claim.

Thus, we have $d^2 + r_1^2 + r_2^2 = 2023$. The lengths of the internal and the external tangents give us $d^2 - (r_1 + r_2)^2 = 361$, and $d^2 - (r_1 - r_2)^2 = 1369$. Thus,

$$d^2 - (r_1^2 + r_2^2) = \frac{(d^2 - (r_1 + r_2)^2) + (d^2 - (r_1 - r_2)^2)}{2} = \frac{361 + 1369}{2} = 865.$$

Thus, $d^2 = \frac{865+2023}{2} = 1444 \implies d = 38$.

Solution 2: We present another way of showing that $\mathbb{E}[XY^2] = d^2 + r_1^2 + r_2^2$ using complex numbers. The finish is the same as Solution 1.

Let the center of ω_1 and ω_2 be 0 and k, respectively. Select Z_1 and Z_2 uniformly random on unit circle. Then,

$$\begin{aligned}
\mathbb{E}[XY^2] &= \mathbb{E}|k + r_1 Z_1 + r_2 Z_2|^2 \\
&= \mathbb{E}(k + r_1 Z_1 + r_2 Z_2)(\overline{k} + r_1 \overline{Z_1} + r_2 \overline{Z_2})
\end{aligned}$$

Then, observe that

$$\mathbb{E}[Z_1] = \mathbb{E}[Z_2] = \mathbb{E}[Z_1\overline{Z_2}] = \mathbb{E}[Z_2\overline{Z_1}] = 0,$$

so when expanding, six terms vanish, leaving only

$$\mathbb{E}[XY^2] = \mathbb{E}[k\overline{k} + r_1^2 Z_1\overline{Z_1} + r_2^2 Z_2\overline{Z_2}] = d^2 + r_1^2 + r_2^2$$

34. **[20]** Compute the smallest positive integer that does not appear in any problem statement on any round at HMMT November 2023.

Submit a positive integer A. If the correct answer is C, you will receive $\max(0, 20 - 5|A - C|)$ points.

Proposed by: Arul Kolla

Answer: $\boxed{22}$

Solution: The number 22 does not appear on any round. On the other hand, the numbers 1 through 21 appear as follows.

Number	Round	Problem
1	Guts	21
2	Guts	13
3	Guts	17
4	Guts	13
5	Guts	14
6	Guts	2
7	Guts	10
8	Guts	13
9	Guts	28
10	Guts	10
11	General	3
12	Guts	32
13	Theme	8
14	Guts	19
15	Guts	17
16	Guts	30
17	Guts	20
18	Guts	2
19	Guts	33
20	Guts	3
21	Team	7

35. **[20]** Dorothea has a 3×4 grid of dots. She colors each dot red, blue, or dark gray. Compute the number of ways Dorothea can color the grid such that there is no rectangle whose sides are parallel to the grid lines and whose vertices all have the same color.

Submit a positive integer A. If the correct answer is C and your answer is A, you will receive $\left\lfloor 20\left(\min\left(\frac{A}{C}, \frac{C}{A}\right)\right)^2 \right\rfloor$ points.

Proposed by: Amy Feng, Isabella Quan, Pitchayut Saengrungkongka, Rishabh Das, Vidur Jasuja

Answer: $\boxed{284688}$

Solution: To find an appropriate estimate, we will lower bound the number of rectangles. Let $P(R)$ be the probability a random 3 by 4 grid will have a rectangle with all the same color in the grid. Let $P(r)$ be the probability that a specific rectangle in the grid will have the same color. Note $P(r) = \frac{3}{3^4} = \frac{1}{27}$. Observe that there are $\binom{4}{2}\binom{3}{2} = 18$ rectangles in the grid. Hence, we know that $P(R) \leq 18 \cdot P(r) = \frac{18}{27} = \frac{2}{3}$. Thus, $1 - P(R)$, the probability no such rectangle is in the grid, is at most $\frac{1}{3}$. This implies that our answer should be at least $\frac{3^{12}}{3} = 3^{11}$, which is enough for around half points. Closer estimations can be obtained by using more values of Inclusion-Exclusion.

```
n = 4
cnt = 0
for i in range(3**(3*n)):
    mask = i
    a = [[], [], []]
    for x in range(3):
        for y in range(n):
            a[x].append(mask % 3)
            mask //= 3
    pairs = [set() for i in range(3)]
    works = True
    for i in range(n):
        for j,k in [(0,1), (0,2), (1,2)]:
            if a[j][i] == a[k][i]:
                if (j,k) in pairs[a[j][i]]:
                    works = False
                else:
                    pairs[a[j][i]].add((j,k))
    if works:
        cnt += 1
print(cnt)
```

36. **[20]** Isabella writes the expression $\sqrt{d}$ for each positive integer d not exceeding $8!$ on the board. Seeing that these expressions might not be worth points on HMMT, Vidur simplifies each expression to the form $a\sqrt{b}$, where a and b are integers such that b is not divisible by the square of a prime number. (For example, $\sqrt{20}$, $\sqrt{16}$, and $\sqrt{6}$ simplify to $2\sqrt{5}$, $4\sqrt{1}$, and $1\sqrt{6}$, respectively.) Compute the sum of $a + b$ across all expressions that Vidur writes.

Submit a positive real number A. If the correct answer is C and your answer is A, you get $\max\left(0, \left\lceil 20\left(1 - |\log(A/C)|^{1/5}\right)\right\rceil\right)$ points.

Proposed by: Isabella Quan, Pitchayut Saengrungkongka, Rishabh Das, Vidur Jasuja

Answer: $\boxed{534810086}$

Solution: Let $\sqrt{n}$ simplifies to $a_n\sqrt{b_n}$, and replace $8!$ by x. First, notice that $\sum_{n\leq x} a_n$ is small ($O(x^{3/2})$ in particular) because each term cannot exceed $\sqrt{x}$. On the other hand, $\sum_{n\leq x} b_n$ will be large; we have $b_n = n$ when n is squarefree, and squarefree numbers occurs $\frac{6}{\pi^2}$ over the time. Thus, it suffices to consider $\sum_{n\leq x} b_n$.

We first explain how to derive the formula heuristically. Then, we will provide a rigorous proof that

$$B(x) := \sum_{n\leq x} b_n = \frac{\pi^2}{30}x^2 + O(x^{3/2}).$$

For heuristic explanation, we first rewrite the sum as

$$B(x) = \sum_{\substack{a^2 b \le x \\ b \text{ squarefree}}} b = \sum_{a \le x} \sum_{\substack{b \le x/a^2 \\ b \text{ squarefree}}} b.$$

We estimate the inner sum as follows: first, recall that the density of squarefree numbers is $\frac{6}{\pi^2}$. The sum of first k positive integers is approximately $k^2/2$, so the sum of squarefree numbers from $1, 2, \ldots, k$ should roughly be about $\frac{6}{\pi^2} \cdot \frac{k^2}{2} = \frac{3}{\pi^2} k^2$. Knowing this, we estimate

$$\begin{aligned} B(x) &\approx \sum_{a \le x} \frac{3}{\pi^2} \left(\frac{x}{a^2} \right)^2 \\ &= x^2 \sum_{a \le x} \frac{3}{\pi^2} \frac{1}{a^4} \\ &\approx \frac{3}{\pi^2} x^2 \sum_{a=1}^{\infty} \frac{1}{a^4} \\ &= \frac{3}{\pi^2} x^2 \cdot \frac{\pi^4}{90} = \frac{\pi^2}{30} x^2. \end{aligned}$$

The estimate $\frac{\pi^2}{30} \cdot (8!)^2 = 534\,834\,652$ is good enough for 18 points.

We now give a rigorous proof, which is essentially the above proof, but the errors are properly treated. To do that, we need several lemmas and some standard techniques in analytic number theory.

Lemma 1. The number of squarefree integers not exceeding x is $\frac{6}{\pi^2} x + O(\sqrt{x})$.

Proof. This is a standard result in analytic number theory, but we give the full proof for completeness.

$$\mu(n) = \begin{cases} (-1)^r & n \text{ is the product of } r \ge 0 \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Inclusion-Exclusion, we have

$$\begin{aligned} \#\{\text{squarefree } \le x\} &= \sum_{k \le \sqrt{x}} \mu(k) \left\lfloor \frac{x}{k^2} \right\rfloor \\ &= \sum_{k \le \sqrt{x}} \mu(k) \frac{x}{k^2} + O(\sqrt{x}) \\ &= x \left(\sum_{k \le \sqrt{x}} \frac{\mu(k)}{k^2} \right) + O(\sqrt{x}). \end{aligned}$$

The inner summation is

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} + O\left(\sum_{k \ge \sqrt{x}} \frac{1}{k^2} \right) &= \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2} \right) + O\left(\frac{1}{\sqrt{x}} \right) \\ &= \frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots} + O\left(\frac{1}{\sqrt{x}} \right) \\ &= \frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}} \right), \end{aligned}$$

so putting it together, we get that

$$\#\{\text{squarefree } \le x\} = x \left(\frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}} \right) \right) + O(\sqrt{x}) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

□

Lemma 2. We have

$$\sum_{\substack{n \text{ squarefree} \\ n \leq x}} n = \frac{3}{\pi^2}x^2 + O(x^{3/2}).$$

Proof. We apply Abel's summation formula on the sequence $a_n = \mathbf{1}_{\text{squarefree}}(n)$ and weight $\phi(n) = n$. Define the partial summation $A(x) = \sum_{n \leq x} a_n$. Applying Abel's summation, we get that

$$\begin{aligned}
\sum_{\substack{n \text{ squarefree} \\ n \leq x}} n &= \sum_{n \leq x} a_n \phi(n) \\
&= A(x)\phi(x) - \int_1^x A(t)\phi'(t)\,dt \\
&= \left(\frac{6}{\pi^2}x + O(\sqrt{x})\right)x - \int_1^x \left(\frac{6}{\pi^2}t + O(\sqrt{t})\right)\,dt \\
&= \left(\frac{6}{\pi^2 x} + O(x^{3/2})\right) - \left(\frac{6}{\pi^2} \cdot \frac{x^2}{2} + O(x^{3/2})\right) \\
&= \frac{3}{\pi^2}x^2 + O(x^{3/2}).
\end{aligned}$$

□

Main Proof. Once we have Lemma 2., it is easy to get the desired estimate. We have

$$\begin{aligned}
B(x) &= \sum_{\substack{a^2 b \leq x \\ b \text{ squarefree}}} b \\
&= \sum_{a \leq x} \sum_{\substack{b \leq x/a^2 \\ b \text{ squarefree}}} b \\
&= \sum_{a \leq x} \frac{3}{\pi^2}\frac{x^2}{a^4} + O\left(\frac{x^{3/2}}{a^3}\right) \\
&= \frac{3}{\pi^2}x^2\left(\sum_{a \leq x}\frac{1}{a^4}\right) + O\left(x^{3/2}\sum_{a \leq x}\frac{1}{a^3}\right).
\end{aligned}$$

Since $\sum_{a=1}^{\infty} \frac{1}{a^3}$ converges, we get that the big-O term is indeed $O(x^{3/2})$. Now, we only need to deal with the main term. Note the estimate

$$\sum_{a \leq x} \frac{1}{a^4} = \sum_{a=1}^{\infty} \frac{1}{a^4} - \sum_{a \geq x} \frac{1}{a^4} = \frac{\pi^4}{90} + O\left(\frac{1}{x^3}\right).$$

Hence, we have

$$B(x) = \frac{3}{\pi^2}x^2 \cdot \frac{\pi^4}{90} + O(x^{3/2}) = \frac{\pi^2}{30}x^2 + O(x^{3/2}),$$

as desired.

Here is the code that gives the exact answer:

```
import math
n = 40320
a = [1 for i in range(n+1)]
for d in range(1, math.ceil(math.sqrt(n))):
    for x in range(d**2, n+1, d**2):
        a[x] = d
b = [i//a[i]**2 for i in range(1, n+1)]
print(sum(a)+sum(b))
```