HMMT February 2023

February 18, 2023

Algebra and Number Theory Round

1. Suppose $P(x)$ is a cubic polynomial with integer coefficients such that $P(\sqrt{5}) = 5$ and $P(\sqrt[3]{5}) = 5\sqrt[3]{5}$. Compute $P(5)$.

   Proposed by: Sean Li

   Answer: $\boxed{-95}$

   Solution: Write $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are integers. Then we have that

$$P(\sqrt{5}) - 5 = (5a + c)\sqrt{5} + (5b + d - 5) = 0,$$
$$P(\sqrt[3]{5}) - 5\sqrt[3]{5} = (5a + d) + (c - 5)\sqrt[3]{5} + b\sqrt[3]{25} = 0.$$

   Recall that $\sqrt{5}$ is irrational. In particular, since $(5a+c)\sqrt{5} + (5b+d-5) = 0$, we must have $5a + c = 0$ and $5b + d - 5 = 0$. Similarly, from the condition on $\sqrt[3]{5}$, we must have $5a + d = c - 5 = b = 0$.

   This is enough to imply $(a, b, c, d) = (-1, 0, 5, 5)$, so $P(x) = -x^3 + 5x + 5$. Hence, our final answer is $P(5) = -125 + 25 + 5 = -95$.

2. Compute the number of positive integers $n \leq 1000$ such that $\operatorname{lcm}(n, 9)$ is a perfect square. (Recall that lcm denotes the least common multiple.)

   Proposed by: Luke Robitaille

   Answer: $\boxed{43}$

   Solution: Suppose $n = 3^a m$, where $3 \nmid m$. Then

$$\operatorname{lcm}(n, 9) = 3^{\max(a,2)} m.$$

   In order for this to be a square, we require m to be a square, and a to either be even or 1. This means n is either a square (if a is even) or of the form $3k^2$ where $3 \nmid k$ (if $a = 1$).

   There are 31 numbers of the first type, namely

$$1^2, 2^2, 3^2, 4^2, \ldots, 30^2, 31^2.$$

   There are 12 numbers of the second type, namely

$$3 \cdot 1^2, 3 \cdot 2^2, 3 \cdot 4^2, 3 \cdot 5^2, \ldots, 3 \cdot 16^2, 3 \cdot 17^2.$$

   Overall, there are $31 + 12 = 43$ such n.

3. Suppose x is a real number such that $\sin(1 + \cos^2 x + \sin^4 x) = \frac{13}{14}$. Compute $\cos(1 + \sin^2 x + \cos^4 x)$.

   Proposed by: Ankit Bisain, Luke Robitaille, Maxim Li, Milan Haiman, Sean Li

   Answer: $\boxed{-\frac{3\sqrt{3}}{14}}$

   Solution: We first claim that $\alpha := 1 + \cos^2 x + \sin^4 x = 1 + \sin^2 x + \cos^4 x$. Indeed, note that

$$\sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = \sin^2 x - \cos^2 x,$$

   which is the desired after adding $1 + \cos^2 x + \cos^4 x$ to both sides.

Hence, since $\sin\alpha = \frac{13}{14}$, we have $\cos\alpha = \pm\frac{3\sqrt{3}}{14}$. It remains to determine the sign. Note that $\alpha = t^2 - t + 2$ where $t = \sin^2 x$. We have that t is between 0 and 1. In this interval, the quantity $t^2 - t + 2$ is maximized at $t \in \{0, 1\}$ and minimized at $t = 1/2$, so α is between $7/4$ and 2. In particular, $\alpha \in (\pi/2, 3\pi/2)$, so $\cos\alpha$ is negative. It follows that our final answer is $-\frac{3\sqrt{3}}{14}$.

Remark. During the official contest, 258 contestants put the (incorrect) positive version of the answer and 105 contestants answered correctly. This makes $\frac{3\sqrt{3}}{14}$ the second most submitted answer to an Algebra/Number Theory problem, beat only by the correct answer to question 1.

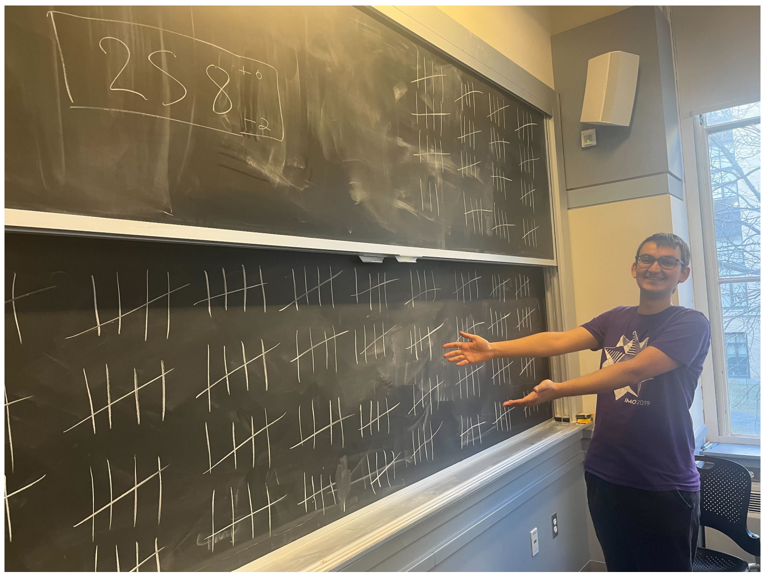

Figure: Milan.

Here is how the problem was written:

- Luke wanted a precalculus/trigonometry problem on the test and enlisted the help of the other problem authors, much to Sean's dismay.
- Maxim proposed "given $\sin x + \cos^2 x$, find $\sin^2 x + \cos^4 x$."
- Ankit observed that $\sin^2 x + \cos^4 x = \cos^2 x + \sin^4 x$ and proposed "given $\cos^2 x + \sin^4 x$, find $\sin^2 x + \cos^4 x$."
- Luke suggested wrapping both expressions with an additional trig function and proposed "given $\sin(\cos^2 x + \sin^4 x) = \frac{\sqrt{3}}{2}$, find $\cos(\sin^2 x + \cos^4 x)$."
- Milan suggested using the fact that fixing $\sin\alpha$ only determines $\cos\alpha$ up to sign and proposed "given $\sin(1 + \cos^2 x + \sin^4 x) = \frac{\sqrt{3}}{2}$, find $\cos(1 + \sin^2 x + \cos^4 x)$."
- Sean noted that the function $\sin(1+\cos^2 x+\sin^4 x)$ only achieves values in the range $[\sin 2, \sin 7/4]$ and suggested to make the answer contain radicals, and proposed "given $\sin(1+\cos^2 x+\sin^4 x) = \frac{13}{14}$, find $\cos(1 + \sin^2 x + \cos^4 x)$."

4. Suppose $P(x)$ is a polynomial with real coefficients such that $P(t) = P(1)t^2 + P(P(1))t + P(P(P(1)))$ for all real numbers t. Compute the largest possible value of $P(P(P(P(1))))$.

   Proposed by: Raymond Feng

   Answer: $\boxed{1/9}$

Solution: Let $(a,b,c) := (P(1), P(P(1)), P(P(P(1))))$, so $P(t) = at^2+bt+c$ and we wish to maximize $P(c)$. Then we have that

$$\begin{aligned} a &= P(1) = a+b+c, \\ b &= P(a) = a^3+ab+c, \\ c &= P(b) = ab^2+b^2+c. \end{aligned}$$

The first equation implies $c=-b$. The third equation implies $b^2(a+1)=0$, so $a=-1$ or $b=0$. If $b=0$, then $(a,b,c)=(0,0,0)$. If $a=-1$, then $b=(-1)^3+(-1)b+(-b)$ or $b=-\frac{1}{3}$, so $c=\frac{1}{3}$ and $(a,b,c)=(-1,-\frac{1}{3},\frac{1}{3})$.

The first tuple gives $P(c)=0$, while the second tuple gives $P(c)=-\frac{1}{3^2}-\frac{1}{3^2}+\frac{1}{3}=\frac{1}{9}$, which is the answer.

5. Suppose E, I, L, V are (not necessarily distinct) nonzero digits in base ten for which

- the four-digit number $\underline{E}\ \underline{V}\ \underline{I}\ \underline{L}$ is divisible by 73, and
- the four-digit number $\underline{V}\ \underline{I}\ \underline{L}\ \underline{E}$ is divisible by 74.

Compute the four-digit number $\underline{L}\ \underline{I}\ \underline{V}\ \underline{E}$.

Proposed by: Sean Li

Answer: $\boxed{9954}$

Solution: Let $\underline{E}=2k$ and $\underline{V}\ \underline{I}\ \underline{L}=n$. Then $n \equiv -2000k \pmod{73}$ and $n \equiv -k/5 \pmod{37}$, so $n \equiv 1650k \pmod{2701}$. We can now exhaustively list the possible cases for k:

- if $k=1$, then $n \equiv 1650$ which is not possible;
- if $k=2$, then $n \equiv 2\cdot 1650 \equiv 599$, which gives $E=4$ and $n=599$;
- if $k=3$, then $n \equiv 599+1650 \equiv 2249$ which is not possible;
- if $k=4$, then $n \equiv 2249+1650 \equiv 1198$ which is not possible.

Hence, we must have $(E,V,I,L)=(4,5,9,9)$, so $\underline{L}\ \underline{I}\ \underline{V}\ \underline{E}=9954$.

6. Suppose $a_1, a_2, \ldots, a_{100}$ are positive real numbers such that

$$a_k = \frac{ka_{k-1}}{a_{k-1}-(k-1)}$$

for $k=2, 3, \ldots, 100$. Given that $a_{20}=a_{23}$, compute a_{100}.

Proposed by: Sean Li, Vidur Jasuja

Answer: $\boxed{215}$

Solution: If we cross multiply, we obtain $a_na_{n-1}=na_{n-1}+(n-1)a_n$, which we can rearrange and factor as $(a_n-n)(a_{n-1}-(n-1))=n(n-1)$.

Let $b_n=a_n-n$. Then, $b_nb_{n-1}=n(n-1)$. If we let $b_1=t$, then we have by induction that $b_n=nt$ if n is odd and $b_n=n/t$ if n is even. So we have

$$a_n = \begin{cases} nt+n & \text{if } n \text{ odd} \\ n/t+n & \text{if } n \text{ even} \end{cases}$$

for some real number t. We have $20/t+20=23t+23$, so $t\in\{-1, 20/23\}$. But if $t=-1$, then $a_1=0$ which is not positive, so $t=20/23$ and $a_{100}=100/t+100=215$.

7. If a, b, c, and d are pairwise distinct positive integers that satisfy $\operatorname{lcm}(a,b,c,d) < 1000$ and $a+b = c+d$, compute the largest possible value of $a+b$.

Proposed by: Ankit Bisain

Answer: $\boxed{581}$

Solution: Let $a' = \frac{\operatorname{lcm}(a,b,c,d)}{a}$. Define b', c', and d' similarly. We have that a', b', c', and d' are pairwise distinct positive integers that satisfy

$$\frac{1}{a'} + \frac{1}{b'} = \frac{1}{c'} + \frac{1}{d'}.$$

Let T be the above quantity. We have

$$a + b = T \operatorname{lcm}(a,b,c,d),$$

so we try to maximize T. Note that since $\frac{1}{2} + \frac{1}{3} < \frac{1}{1}$, we cannot have any of a', b', c', and d' be 1. At most one of them can be 2, so at least one side of the equation must have both denominators at least 3. Hence, the largest possible value of T is

$$T = \frac{1}{3} + \frac{1}{4} = \frac{1}{2} + \frac{1}{12} = \frac{7}{12},$$

and the second largest possible value of T is

$$T = \frac{1}{3} + \frac{1}{5} = \frac{1}{2} + \frac{1}{30} = \frac{8}{15}.$$

Taking $T = \frac{7}{12}$ and $\operatorname{lcm}(a,b,c,d) = 996 = 12 \cdot 83$, we get $a + b = 581$. Since the next best value of T gives $8/15 \cdot 1000 < 534 < 581$, this is optimal.

8. Let S be the set of ordered pairs (a,b) of positive integers such that $\gcd(a,b) = 1$. Compute

$$\sum_{(a,b)\in S} \left\lfloor \frac{300}{2a+3b} \right\rfloor.$$

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{7400}$

Solution: The key claim is the following.

Claim: The sum in the problem is equal to the number of solutions of $2x + 3y \le 300$ where x, y are positive integers.

Proof. The sum in the problem is the same as counting the number of triples (a,b,d) of positive integers such that $\gcd(a,b) = 1$ and $d(2a+3b) \le 300$. Now, given such (a,b,d), we biject it to the pair (x,y) described in the claim by $x = da$ and $x = db$. This transformation can be reversed by $d = \gcd(x,y)$, $a = x/d$, and $b = y/d$, implying that it is indeed a bijection, so the sum is indeed equal to the number of such (x,y). $\square$

Hence, we wish to count the number of positive integer solutions to $2x+3y \le 300$. One way to do this is via casework on y, which we know to be an integer less than 100:

- If y is even, then $y = 2k$ for $1 \le k \le 49$. Fixing k, there are exactly $\frac{300-6k}{2} = 150 - 3k$ values of x which satisfy the inequality, hence the number of solutions in this case is

$$\sum_{k=1}^{49}(150 - 3k) = \frac{150 \cdot 49}{2} = 3675.$$

- If y is odd, then $y = 2k - 1$ for $1 \le k \le 50$. Fixing y, there are exactly $\frac{302-6k}{2} = 151 - 3k$ values of x which satisfy the inequality, hence the number of solutions in this case is

$$\sum_{k=1}^{50}(151 - 3k) = \frac{149 \cdot 50}{2} = 3725.$$

The final answer is $3675 + 3725 = 7400$.

9. For any positive integers a and b with $b > 1$, let $s_b(a)$ be the sum of the digits of a when it is written in base b. Suppose n is a positive integer such that

$$\sum_{i=1}^{\lfloor \log_{23} n \rfloor} s_{20}\left(\left\lfloor \frac{n}{23^i} \right\rfloor\right) = 103 \quad \text{and} \quad \sum_{i=1}^{\lfloor \log_{20} n \rfloor} s_{23}\left(\left\lfloor \frac{n}{20^i} \right\rfloor\right) = 115.$$

Compute $s_{20}(n) - s_{23}(n)$.

Proposed by: Luke Robitaille, Raymond Feng

Answer: $\boxed{81}$

Solution: First we will prove that

$$s_a(n) = n - (a-1)\left(\sum_{i=1}^{\infty} \left\lfloor \frac{n}{a^i} \right\rfloor\right).$$

If $n = (n_k n_{k-1} \cdots n_1 n_0)_a$, then the digit n_i contributes n_i to the left side of the sum, while it contributes

$$n_i(a^i - (a-1)(a^{i-1} + a^{i-2} + \cdots + a^1 + a^0)) = n_i$$

to the right side, so the two are equal as claimed.

Now we have

$$\begin{aligned} 103 &= \sum_{i=1}^{\infty} s_{20}\left(\left\lfloor \frac{n}{23^i} \right\rfloor\right) \\ &= \sum_{i=1}^{\infty}\left(\left\lfloor \frac{n}{23^i} \right\rfloor - 19\left(\sum_{j=1}^{\infty}\left\lfloor \frac{\lfloor n/23^i \rfloor}{20^j} \right\rfloor\right)\right) \\ &= \sum_{i=1}^{\infty}\left\lfloor \frac{n}{23^i} \right\rfloor - 19\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\lfloor \frac{n}{20^j \cdot 23^i} \right\rfloor, \end{aligned}$$

where we have used the fact that $\left\lfloor \frac{\lfloor n/p \rfloor}{q} \right\rfloor = \left\lfloor \frac{n}{pq} \right\rfloor$ for positive integers n, p, q. Similarly,

$$115 = \sum_{j=1}^{\infty}\left\lfloor \frac{n}{20^j} \right\rfloor - 22\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\lfloor \frac{n}{20^j \cdot 23^i} \right\rfloor.$$

Let

$$A = \sum_{j=1}^{\infty}\left\lfloor \frac{n}{20^j} \right\rfloor, \qquad B = \sum_{i=1}^{\infty}\left\lfloor \frac{n}{23^i} \right\rfloor, \qquad \text{and} \qquad X = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\lfloor \frac{n}{20^j \cdot 23^i} \right\rfloor.$$

Then we have $103 = B - 19X$ and $115 = A - 22X$.

Thus, we have

$$\begin{aligned} s_{20}(n) - s_{23}(n) &= \left(n - 19 \sum_{j=1}^{\infty} \left\lfloor \frac{n}{20^j} \right\rfloor \right) - \left(n - 22 \sum_{i=1}^{\infty} \left\lfloor \frac{n}{23^i} \right\rfloor \right) \\ &= 22B - 19A \\ &= 22(103 + 19X) - 19(115 + 22X) \\ &= 22 \cdot 103 - 19 \cdot 115 = 81. \end{aligned}$$

Remark. The value $n = 22399976$ satisfies both equations, so a valid solution to the system exists. It seems infeasible to compute this solution by hand.

10. Let $\zeta = e^{2\pi i/99}$ and $\omega = e^{2\pi i/101}$. The polynomial

$$x^{9999} + a_{9998}x^{9998} + \cdots + a_1 x + a_0$$

has roots $\zeta^m + \omega^n$ for all pairs of integers (m, n) with $0 \le m < 99$ and $0 \le n < 101$. Compute $a_{9799} + a_{9800} + \cdots + a_{9998}$.

Proposed by: Sean Li

Answer: $\boxed{14849 - \frac{9999}{200}\binom{200}{99}}$

Solution: Let $b_k := a_{9999-k}$ for sake of brevity, so we wish to compute $b_1 + b_2 + \cdots + b_{200}$. Let p_k be the sum of the k-th powers of $\zeta^m + \omega^n$ over all ordered pairs (m, n) with $0 \le m < 99$ and $0 \le n < 101$. Recall that *Newton's sums* tells us that

$$\begin{aligned} p_1 + b_1 &= 0 \\ p_2 + p_1 b_1 + 2b_2 &= 0 \\ p_3 + p_2 b_1 + p_1 b_2 + 3b_3 &= 0 \\ &\vdots \end{aligned}$$

and in general $kb_k + \sum_{j=1}^{k} p_j b_{k-j} = 0$. The key idea is that p_k is much simpler to compute than b_k, and we can relate the two with Newton's sums.

The roots of unity filter identity tells us that if $P(s, t)$ is a two-variable polynomial, then $z(P) := \frac{1}{9999} \sum P(\zeta^m, \omega^n)$ over all $0 \le m < 99$ and $0 \le n < 101$ is exactly the sum of the coefficients of the terms $s^{99a}t^{101b}$. Suppose that $P_k(s, t) = (s + t)^k$. Then $z(P_k)$ is precisely $p_k/9999$. So one can check that

- if $k \ne 99, 101, 198, 200$, then $(s + t)^{98}$ has no terms of the form $s^{99a}t^{101b}$ and so $p_k = 0$.
- if $k = 99, 101, 198$, then $z(P_k) = 1$ and $p_k = 9999$.
- if $k = 200$, then $z(P_k) = \binom{200}{99}$ and $p_k = 9999\binom{200}{99}$.

We can now compute b_k using Newton's sums identities:

- we have $b_k = 0$ for $k \ne 99, 101, 198, 200$.
- since $p_{99} + 99b_{99} = 0$, we have $b_{99} = -101$;
- since $p_{101} + 101b_{101} = 0$, we have $b_{101} = -99$;
- since $p_{198} + p_{99}b_{99} + 198b_{198} = 0$, we have

$$b_{198} = \frac{1}{198}(-p_{198} - p_{99}b_{99}) = \frac{1}{198}(-9999 + 9999 \cdot 101) = 5050.\cdot$$

- since $p_{200} + p_{101}b_{99} + p_{99}b_{101} + 200b_{200} = 0$, we have

$$\begin{aligned} b_{200} &= \frac{1}{200}(-p_{200} - p_{101}b_{99} - p_{99}b_{101}) \\ &= \frac{1}{200}\left(-9999\binom{200}{99} + 9999 \cdot 101 + 9999 \cdot 99\right) \\ &= 9999 - \frac{9999}{200}\binom{200}{99}. \end{aligned}$$

Hence, we have

$$\begin{aligned} b_1 + b_2 + \cdots + b_{200} &= -101 - 99 + 5050 + 9999 - \frac{9999}{200}\binom{200}{99} \\ &= 14849 - \frac{9999}{200}\binom{200}{99}. \end{aligned}$$