HMMT February 2020

February 15, 2020

Algebra and Number Theory

1. Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

2. Find the unique pair of positive integers (a, b) with $a < b$ for which

$$\frac{2020 - a}{a} \cdot \frac{2020 - b}{b} = 2.$$

3. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

4. For positive integers n and k, let $\mho(n, k)$ be the number of distinct prime divisors of n that are at least k. For example, $\mho(90, 3) = 2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mho(n, k)}{3^{n+k-7}}.$$

5. A positive integer N is *piquant* if there exists a positive integer m such that if n_i denotes the number of digits in m^i (in base 10), then $n_1 + n_2 + \cdots + n_{10} = N$. Let p_M denote the fraction of the first M positive integers that are piquant. Find $\lim_{M \to \infty} p_M$.

6. A polynomial $P(x)$ is a *base-n polynomial* if it is of the form $a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$, where each a_i is an integer between 0 and $n - 1$ inclusive and $a_d > 0$. Find the largest positive integer n such that for any real number c, there exists at most one base-n polynomial $P(x)$ for which $P(\sqrt{2} + \sqrt{3}) = c$.

7. Find the sum of all positive integers n for which

$$\frac{15 \cdot n!^2 + 1}{2n - 3}$$

is an integer.

8. Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

9. Let $P(x) = x^{2020} + x + 2$, which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

10. We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in $\mathbb{F}_{101}$ (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_{101}$ for each nonnegative integer k. For example, $(x+3)(100x+5) = 100x^2 + 2x + 15$ in $\mathbb{F}_{101}[x]$ because the corresponding coefficients are equal modulo 101.

    We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

    $$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x)$$

    in $\mathbb{F}_{101}[x]$. Find the number of lucky polynomials.