4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Team Round

p-Polynomials

1. [**2**] Find the number of positive integers x less than 100 for which

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

2. [**4**] Determine the set of all real numbers p for which the polynomial $Q(x) = x^3 + px^2 - px - 1$ has three distinct real roots.

3. [**6**] Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, and $P(17) = 23$.

4. [**7**] Determine the number of quadratic polynomials $P(x) = p_1x^2 + p_2x - p_3$, where p_1, p_2, p_3 are not necessarily distinct (positive) prime numbers less than 50, whose roots are distinct rational numbers.

C Coloring

5. [**3**] Sixteen wooden Cs are placed in a 4-by-4 grid, all with the same orientation, and each is to be colored either red or blue. A *quadrant operation* on the grid consists of choosing one of the four two-by-two subgrids of Cs found at the corners of the grid and moving each C in the subgrid to the adjacent square in the subgrid that is 90 degrees away in the clockwise direction, without changing the orientation of the C. Given that two colorings are the considered same if and only if one can be obtained from the other by a series of quadrant operations, determine the number of distinct colorings of the Cs.

C	C	C	C
C	C	C	C
C	C	C	C
C	C	C	C

6. [**5**] Ten Cs are written in a row. Some Cs are upper-case and some are lower-case, and each is written in one of two colors, green and yellow. It is given that there is at least one lower-case C, at least one green C, and at least one C that is both upper-case and yellow. Furthermore, no lower-case C can be followed by an upper-case C, and no yellow C can be followed by a green C. In how many ways can the Cs be written?

7. [**7**] Julia is learning how to write the letter C. She has 6 differently-colored crayons, and wants to write Cc Cc Cc Cc Cc. In how many ways can she write the ten Cs, in such a way that each upper case C is a different color, each lower case C is a different color, and in each pair the upper case C and lower case C are different colors?

[GG]eometry

8. [**4**] Let $G, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, B_5$ be ten points on a circle such that $GA_1A_2A_3A_4$ is a regular pentagon and $GB_1B_2B_3B_4B_5$ is a regular hexagon, and B_1 lies on minor arc GA_1. Let B_5B_3 intersect B_1A_2 at G_1, and let B_5A_3 intersect GB_3 at G_2. Determine the degree measure of $\angle GG_2G_1$.

9. [**4**] Let ABC be a triangle with $AB = 9$, $BC = 10$, and $CA = 17$. Let B' be the reflection of the point B over the line CA. Let G be the centroid of triangle ABC, and let G' be the centroid of triangle $AB'C$. Determine the length of segment GG'.

10. [**8**] Let $G_1G_2G_3$ be a triangle with $G_1G_2 = 7$, $G_2G_3 = 13$, and $G_3G_1 = 15$. Let G_4 be a point outside triangle $G_1G_2G_3$ so that ray $\overrightarrow{G_1G_4}$ cuts through the interior of the triangle, $G_3G_4 = G_4G_2$, and $\angle G_3G_1G_4 = 30^\circ$. Let G_3G_4 and G_1G_2 meet at G_5. Determine the length of segment G_2G_5.