14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

Algebra & Combinatorics Individual Test

1. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

   Answer: $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc, b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

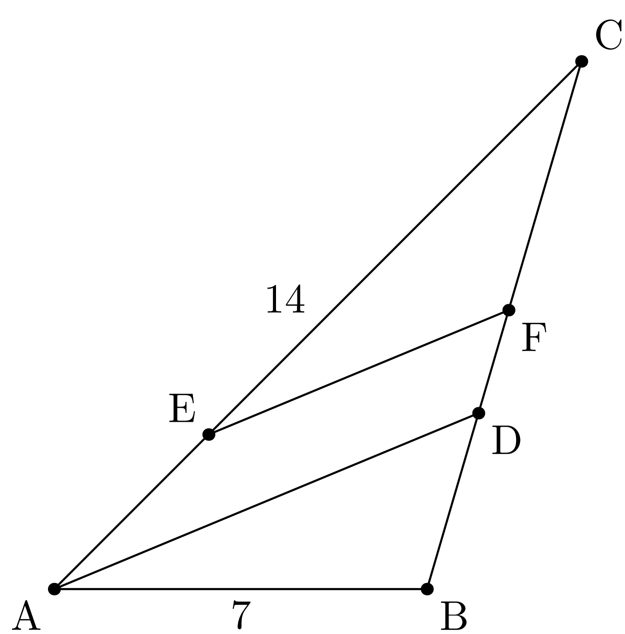
2. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

   Answer: $\boxed{2 \cdot 15!^2}$ If we color the desks of the class in a checkerboard pattern, we notice that all of one gender must go in the squares colored black, and the other gender must go in the squares colored white. There are 2 ways to pick which gender goes in which color, 15! ways to put the boys into desks and 15! ways to put the girls into desks. So the number of ways is $2 \cdot 15!^2$.

   (There is a little ambiguity in the problem statement as to whether the 15 boys and the 15 girls are distinguishable or not. If they are not distinguishable, the answer is clearly 2. Given the number of contestants who submitted the answer 2, the graders judged that there was enough ambiguity to justify accepting 2 as a correct answer. So both 2 and $2 \cdot 15!^2$ were accepted as correct answers.)

3. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

   Answer: $\boxed{13}$



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$

([] denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$\begin{aligned} d + 7 &> 14 \\ 7 + 14 &> d \end{aligned}$$

Hence $7 < d < 21$ and there are 13 possible integral values for BC.

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

   Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k+1$. So there are 2^{n-1} ways to get to the center square of row n.

5. Let $a \star b = ab + a + b$ for all integers a and b. Evaluate $1 \star (2 \star (3 \star (4 \star \ldots (99 \star 100) \ldots)))$.

   Answer: $\boxed{101! - 1}$ We will first show that $\star$ is both commutative and associative.

   - Commutativity: $a \star b = ab + a + b = b \star a$
   - Associativity: $a \star (b \star c) = a(bc + b + c) + a + bc + b + c = abc + ab + ac + bc + a + b + c$ and $(a \star b) \star c = (ab + a + b)c + ab + a + b + c = abc + ab + ac + bc + a + b + c$. So $a \star (b \star c) = (a \star b) \star c$.

   So we need only calculate $((\ldots(1 \star 2) \star 3) \star 4) \ldots \star 100)$. We will prove by induction that

   $$((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n+1)! - 1.$$

   - Base case ($n = 2$): $(1 \star 2) = 2 + 1 + 2 = 5 = 3! - 1$
   - Inductive step:
     Suppose that
     $$(((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n+1)! - 1.$$
     Then,
     $$\begin{aligned} ((((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) \star (n+1)) &= ((n+1)! - 1) \star (n+1) \\ &= (n+1)!(n+1) - (n+1) + (n+1)! - 1 + (n+1) \\ &= (n+2)! - 1 \end{aligned}$$

   Hence, $((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n+1)! - 1$ for all n. For $n = 100$, this results to $101! - 1$.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

   Answer: $\boxed{\frac{5}{11}}$ For $1 \le k \le 6$, let x_k be the probability that the current player, say A, will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod 7$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \le k \le 6$,

$$x_k = \frac{1}{6} \sum_{1 \le l \le 6,\, l \ne k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^{6} x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \cdots = x_6$, and $6x_k = s$ for every k. Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

   Answer: $\boxed{-3, 2, 5}$ Let $f(x) = 2x^2 + x - 6 = (2x - 3)(x + 2)$. Suppose a positive integer a divides both $2x - 3$ and $x + 2$. Then a must also divide $2(x + 2) - (2x - 3) = 7$. Hence, a can either be 1 or 7. As a result, $2x - 3 = 7^n$ or -7^n for some positive integer n, or either $x + 2$ or $2x - 3$ is ± 1. We consider the following cases:

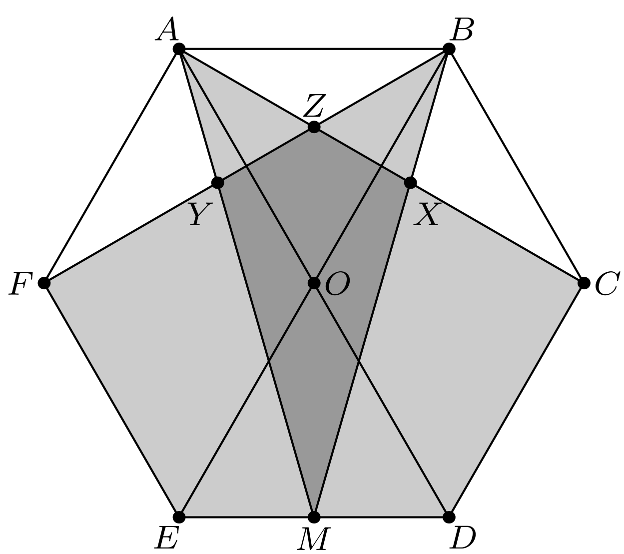
   - $(2x - 3) = 1$. Then $x = 2$, which yields $f(x) = 4$, a prime power.
   - $(2x - 3) = -1$. Then $x = 1$, which yields $f(x) = -3$, not a prime power.
   - $(x + 2) = 1)$. Then $x = -1$, which yields $f(x) = -5$ not a prime power.
   - $(x + 2) = -1$. Then $x = -3$, which yields $f(x) = 9$, a prime power.
   - $(2x - 3) = 7$. Then $x = 5$, which yields $f(x) = 49$, a prime power.
   - $(2x - 3) = -7$. Then $x = -2$, which yields $f(x) = 0$, not a prime power.
   - $(2x - 3) = \pm 7^n$, for $n \ge 2$. Then, since $x + 2 = \dfrac{(2x - 3) + 7}{2}$, we have that $x + 2$ is divisible by 7 but not by 49. Hence $x + 2 = \pm 7$, yielding $x = 5, -9$. The former has already been considered, while the latter yields $f(x) = 147$.

   So x can be either -3, 2 or 5.

   (Note: In the official solutions packet we did not list the answer -3. This oversight was quickly noticed on the day of the test, and only the answer $-3, 2, 5$ was marked as correct.

8. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

   Answer: $\boxed{0}$



Let O be the center of the hexagon. The desired area is $[ABCDEF]-[ACDM]-[BFEM]$. Note that $[ADM]=[ADE]/2=[ODE]=[ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM]=[ACD]+[ADM]=[ACD]+[ABC]=[ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

9. For all real numbers x, let
$$f(x)=\frac{1}{\sqrt[2011]{1-x^{2011}}}.$$
Evaluate $(f(f(\ldots(f(2011))\ldots)))^{2011}$, where f is applied 2010 times.

Answer: $\boxed{2011^{2011}}$ Direct calculation shows that $f(f(x))=\dfrac{\sqrt[2011]{1-x^{2011}}}{-x}$ and $f(f(f(x)))=x$. Hence $(f(f(\ldots(f(x))\ldots)))=x$, where f is applied 2010 times. So $(f(f(\ldots(f(2011))\ldots)))^{2011}=2011^{2011}$.

10. The integers from 1 to n are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.

Answer: $\boxed{4022}$ If n is odd and greater than 1, then Goliath makes the last move. No matter what two numbers are on the board, Goliath can combine them to make an even number. Hence Goliath has a winning strategy in this case.

Now suppose n is even. We can replace all numbers on the board by their residues modulo 2. Initially the board reads $1,0,1,0,\ldots,1,0$. David combines the rightmost 1 and 0 by addition to make 1, so now the board reads $1,0,1,0,\ldots,0,1$. We call a board of this form a "good" board. When it is Goliath's turn, and there is a good board, no matter where he moves, David can make a move to restore a good board. Indeed, Goliath must combine a neighboring 0 and 1; David can then combine that number with a neighboring 1 to make 1 and create a good board with two fewer numbers.

David can ensure a good board after his last turn. But a good board with one number is simply 1, so David wins. So David has a winning strategy if n is even. Therefore, the 2011th smallest positive integer greater than 1 for which David can guarantee victory is the 2011th even positive integer, which is 4022.

11. Let $f(x)=x^2+6x+c$ for all real numbers x, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

Answer: $\boxed{\frac{11-\sqrt{13}}{2}}$ Suppose f has only one distinct root r_1. Then, if x_1 is a root of $f(f(x))$, it must be the case that $f(x_1)=r_1$. As a result, $f(f(x))$ would have at most two roots, thus not satisfying the problem condition. Hence f has two distinct roots. Let them be $r_1 \neq r_2$.

Since $f(f(x))$ has just three distinct roots, either $f(x)=r_1$ or $f(x)=r_2$ has one distinct root. Assume without loss of generality that r_1 has one distinct root. Then $f(x)=x^2+6x+c=r_1$ has one root, so that $x^2+6x+c-r_1$ is a square polynomial. Therefore, $c-r_1=9$, so that $r_1=c-9$. So $c-9$ is a root of f. So $(c-9)^2+6(c-9)+c=0$, yielding $c^2-11c+27=0$, or $(c-\frac{11}{2})^2=\frac{13}{2}$. This results to $c=\frac{11\pm\sqrt{13}}{2}$.

If $c=\frac{11-\sqrt{13}}{2}$, $f(x)=x^2+6x+\frac{11-\sqrt{13}}{2}=(x+\frac{7+\sqrt{13}}{2})(x+\frac{5-\sqrt{13}}{2})$. We know $f(x)=\frac{-7-\sqrt{13}}{2}$ has a double root, -3. Now $\frac{-5+\sqrt{13}}{2}>\frac{-7-\sqrt{13}}{2}$ so the second root is above the vertex of the parabola, and is hit twice.

If $c=\frac{11+\sqrt{13}}{2}$, $f(x)=x^2+6x+\frac{11+\sqrt{13}}{2}=(x+\frac{7-\sqrt{13}}{2})(x+\frac{5+\sqrt{13}}{2})$. We know $f(x)=\frac{-7+\sqrt{13}}{2}$ has a double root, -3, and this is the value of f at the vertex of the parabola, so it is its minimum value. Since $\frac{-5-\sqrt{13}}{2}<\frac{-7+\sqrt{13}}{2}$, $f(x)=\frac{-5-\sqrt{13}}{2}$ has no solutions. So in this case, f has only one real root.

So the answer is $c = \frac{11-\sqrt{13}}{2}$.

Note: In the solutions packet we had both roots listed as the correct answer. We noticed this oversight on the day of the test and awarded points only for the correct answer.

12. Mike and Harry play a game on an 8×8 board. For some positive integer k, Mike chooses k squares and writes an M in each of them. Harry then chooses $k+1$ squares and writes an H in each of them. After Harry is done, Mike wins if there is a sequence of letters forming "HMM" or "MMH," when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of k for which Mike has a winning strategy.

Answer: $\boxed{16}$ Suppose Mike writes k M's. Let a be the number of squares which, if Harry writes an H in, will yield either HMM or MMH horizontally, and let b be the number of squares which, if Harry writes an H in, will yield either HMM or MMH vertically. We will show that $a \le k$ and $b \le k$. Then, it will follow that there are at most $a + b \le 2k$ squares which Harry cannot write an H in. There will be at least $64 - k - 2k = 64 - 3k$ squares which Harry can write in. If $64 - 3k \ge k + 1$, or $k \le 15$, then Harry wins.

We will show that $a \le k$ (that $b \le k$ will follow by symmetry). Suppose there are a_i M's in row i. In each group of 2 or more consective M's, Harry cannot write H to the left or right of that group, giving at most 2 forbidden squares. Hence a_i is at most the number of M's in row i. Summing over the rows gives the desired result.

Mike can win by writing 16 M's according to the following diagram:

$$\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & M & M & \cdot & \cdot & M & M & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

13. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \le P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

Answer: $\boxed{86400000}$ For each nonnegative integer i, let $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$. (Define $x^{\underline{0}} = 1$.)

Lemma: Each polynomial with integer coefficients f can be uniquely written in the form

$$f(x) = a_n x^{\underline{n}} + \ldots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}, a_n \neq 0.$$

Proof: Induct on the degree. The base case (degree 0) is clear. If f has degree m with leading coefficient c, then by matching leading coefficients we must have $m = n$ and $a_n = c$. By the induction hypothesis, $f(x) - cx^{\underline{n}}$ can be uniquely written as $a_{n-1}x^{\underline{n-1}}(x) + \ldots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}$.

There are 120 possible choices for a_0, namely any integer in $[0, 120)$. Once $a_0, \ldots, a_{i-1}$ have been chosen so $0 \le P(0), \ldots, P(i-1) < 120$, for some $0 \le i \le 5$, then we have

$$P(i) = a_i i! + a_{i-1} i^{\underline{i-1}} + \cdots + a_0$$

so by choosing a_i we can make $P(i)$ any number congruent to $a_{i-1} i^{\underline{i-1}} + \cdots + a_0$ modulo $i!$. Thus there are $\frac{120}{i!}$ choices for a_i. Note the choice of a_i does not affect the value of $P(0), \ldots, P(i-1)$. Thus all polynomials we obtain in this way are valid. The answer is

$$\prod_{i=0}^{5} \frac{120}{i!} = 86400000.$$

Note: Their is also a solution involving finite differences that is basically equivalent to this solution. One proves that for $i = 0, 1, 2, 3, 4, 5$ there are $\frac{5!}{i!}$ ways to pick the ith finite difference at the point 0.

14. The ordered pairs $(2011, 2), (2010, 3), (2009, 4), \ldots, (1008, 1005), (1007, 1006)$ are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs (x_i, y_i) and (x_j, y_j), with (x_i, y_i) left of (x_j, y_j), erases them, and writes $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j}\right)$ in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs (x, y) could appear on the blackboard after the process has come to a conclusion?

Answer: $\boxed{504510}$ First, note that none of the numbers will ever be 0. Let $\star$ denote the replacement operation. For each pair on the board (x_i, y_i) define its *primary form* to be (x_i, y_i) and its *secondary form* to be $[x_i y_i, \frac{x_i}{y_i}]$. Note that the primary form determines the secondary form uniquely and vice versa. In secondary form,

$$[a_1, b_1] \star [a_2, b_2] = \left(\sqrt{a_1 b_1}, \sqrt{\frac{a_1}{b_1}}\right) \star \left(\sqrt{a_2 b_2}, \sqrt{\frac{a_2}{b_2}}\right) = \left(a_1 b_2, \frac{a_1}{b_2}\right) = [a_1^2, b_2^2].$$

Thus we may replace all pairs on the board by their secondary form and use the above rule for $\star$ instead. From the above rule, we see that if the leftmost number on the board is x, then after one minute it will be x or x^2 depending on whether it was erased in the intervening step, and similarly for the rightmost number. Let k be the number of times the leftmost pair is erased and n be the number of times the rightmost pair is erased. Then the final pair is

$$\left[4022^{2^k}, \left(\frac{1007}{1006}\right)^{2^n}\right]. \tag{2}$$

Any step except the last cannot involve both the leftmost and rightmost pair, so $k + n \leq 1005$. Since every pair must be erased at least once, $k, n \geq 1$. Every pair of integers satisfying the above can occur, for example, by making $1005 - k - n$ moves involving only the pairs in the middle, then making $k - 1$ moves involving the leftmost pair, and finally n moves involving the rightmost pair.

In light of (2), the answer is the number of possible pairs (k, n), which is $\sum_{k=1}^{1004} \sum_{n=1}^{1005-k} 1 = \sum_{k=1}^{1004} 1005 - k = \sum_{k=1}^{1004} k = \frac{1004 \cdot 1005}{2} = 504510$.

15. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A.

Answer: $\boxed{2}$ Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x, namely, for $x = g_{2i+1}, 0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x-axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c.

The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned} f\left(\frac{b+a-1}{2}\right) &< c \\ \left(\frac{b+a-1}{2}-a\right)\left(\frac{b+a-1}{2}-b\right)+\frac{b+a-1}{2} &< a-1 \\ \frac{1-(b-a)^2}{4}+\frac{b-a+1}{2} &< 0 \\ \frac{3}{4} &< \frac{(b-a)^2}{4}-\frac{b-a}{2} \\ 4 &< (b-a-1)^2. \end{aligned}$$

Then either $b-a-1<-2$ or $b-a-1>2$, but $b>a$, so the latter must hold and $(b-a)^2>9$. Now, the discriminant of $f(x)-x$ equals $(b-a)^2$ (the square of the difference of the two roots) and $(r_2+1)^2-4r_3$ (from the coefficients), so $(r_2+1)^2>9+4r_3$. But $r_3=g_1>g_0=0$ so $|r_2|>2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i\geq 0$ as follows. Let $N\geq 2012$ be an integer. For $\varepsilon>0$ let $h(x)=x^2-2-\varepsilon, g_\varepsilon(x)=-\sqrt{x+2+\varepsilon}$ and $G_{2N+1}=2+\varepsilon$, and define G_i recursively by $G_i=g_\varepsilon(G_{i+1}), G_{i+1}=h(G_i)$. (These two equations are consistent.) Note the following.

(i) $G_{2i}<G_{2i+1}$ and $G_{2i+1}>G_{2i+2}$ for $0\leq i\leq N-1$. First note $G_{2N}=-\sqrt{4+2\varepsilon}>-\sqrt{4+2\varepsilon+\varepsilon^2}=-2-\varepsilon$. Let l be the negative solution to $h(x)=x$. Note that $-2-\varepsilon<G_{2N}<l<0$ since $h(G_{2N})>0>G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x\geq -2-\varepsilon$, and it sends $(-2-\varepsilon,l)$ into $(l,0)$ and $(l,0)$ into $(-2-\varepsilon,l)$. It follows that the G_i, $0\leq i\leq 2N$ are well-defined; moreover, $G_{2i}<l$ and $G_{2i+1}>l$ for $0\leq i\leq N-1$ by backwards induction on i, so the desired inequalities follow.

(ii) G_i is increasing for $i\geq 2N+1$. Indeed, if $x\geq 2+\varepsilon$, then $x^2-x=x(x-1)>2+\varepsilon$ so $h(x)>x$. Hence $2+\varepsilon=G_{2N+1}<G_{2N+2}<\cdots$.

(iii) G_i is unbounded. This follows since $h(x)-x=x(x-2)-2-\varepsilon$ is increasing for $x>2+\varepsilon$, so G_i increases faster and faster for $i\geq 2N+1$.

Now define $f(x)=h(x+G_0)-G_0=x^2+2G_0x+G_0^2-G_0-2-\varepsilon$. Note $G_{i+1}=h(G_i)$ while $g_{i+1}=f(g_i)=h(g_i+G_0)-G_0$, so by induction $g_i=G_i-G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i.

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2=-2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2+\varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε, it suffices to take $\varepsilon=0$ and show that (letting $g=g_0$)

$$g^{(n)}(2)=\underbrace{g(\cdots g}_{n}(2)\cdots)\to -1 \text{ as } n\to\infty.$$

But note that for $0\leq\theta\leq\frac{\pi}{2}$,

$$g(-2\cos\theta)=-\sqrt{2-2\cos\theta}=-2\sin\left(\frac{\theta}{2}\right)=2\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right).$$

Hence by induction, $g^{(n)}(-2\cos\theta)=-2\cos\left(\frac{\pi}{2}-\frac{\pi}{4}+\cdots+(-1)^n\left(\theta-\frac{\pi}{2^n}\right)\right)$. Hence $g^{(n)}(2)=g^{(n-1)}(-2\cos 0)$ converges to $-2\cos(\frac{\pi}{2}-\frac{\pi}{4}+\cdots)=-2\cos(\frac{\pi}{3})=-1$, as needed.

16. Let $A=\{1,2,\ldots,2011\}$. Find the number of functions f from A to A that satisfy $f(n)\leq n$ for all n in A and attain exactly 2010 distinct values.

Answer: $\boxed{2^{2011}-2012}$ Let n be the element of A not in the range of f. Let m be the element of A that is hit twice.

We now sum the total number of functions over n, m. Clearly $f(1) = 1$, and by induction, for $x \leq m, f(x) = x$. Also unless $n = 2011$, $f(2011) = 2011$ because f can take no other number to 2011. It follows from backwards induction that for $x > n$, $f(x) = x$. Therefore $n > m$, and there are only $n - m$ values of f that are not fixed.

Now $f(m+1) = m$ or $f(m+1) = m+1$. For $m < k < n$,given the selection of $f(1), f(2), \ldots, f(k-1)$, $k-1$ of the $k+1$ possible values of $f(k+1)$ $(1, 2, 3, \ldots, k$, and counting m twice) have been taken, so there are two distinct values that $f(k+1)$ can take (one of them is $k+1$, and the other is not, so they are distinct). For $f(n)$, when the other 2010 values of f have been assigned, there is only one missing, so $f(n)$ is determined.

For each integer in $[m, n)$, there are two possible values of f, so there are 2^{n-m-1} different functions f for a given m, n. So our answer is

$$\begin{aligned}
\sum_{m=1}^{2010} \sum_{n=m+1}^{2011} 2^{n-m-1} &= \sum_{m=1}^{2010} 2^{-m-1} \sum_{n=m+1}^{2011} 2^n \\
&= \sum_{m=1}^{2010} 2^{-m-1}(2^{2012} - 2^{m+1}) \\
&= \sum_{m=1}^{2010} 2^{2011-m} - 1 \\
&= \left(\sum_{m=1}^{2010} 2^m \right) - 2010 \\
&= 2^{2011} - 2012
\end{aligned}$$

17. Let $z = \cos \frac{2\pi}{2011} + i \sin \frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \ldots \frac{2008 \cdot 2009}{2} x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3) \ldots P(z^{2010})$.

Answer: $\boxed{2011^{2009} \cdot (1005^{2011} - 1004^{2011})}$ Multiply $P(x)$ by $x - 1$ to get

$$P(x)(x-1) = x^{2009} + 2x^{2008} + \ldots + 2009x - \frac{2009 \cdot 2010}{2},$$

or,

$$P(x)(x-1) + 2010 \cdot 1005 = x^{2009} + 2x^{2008} + \ldots + 2009x + 2010.$$

Multiplying by $x - 1$ once again:

$$\begin{aligned}
(x-1)(P(x)(x-1) + \frac{2010 \cdot 2011}{2}) &= x^{2010} + x^{2009} + \ldots + x - 2010, \\
&= (x^{2010} + x^{2009} + \ldots + x + 1) - 2011.
\end{aligned}$$

Hence,

$$P(x) = \frac{\frac{(x^{2010} + x^{2009} + \ldots + x + 1) - 2011}{x-1} - 2011 \cdot 1005}{x-1}$$

Note that $x^{2010}+x^{2009}+\ldots+x+1$ has $z, z^2, \ldots, z^{2010}$ as roots, so they vanish at those points. Plugging those 2010 powers of z into the last equation, and multiplying them together, we obtain

$$\prod_{i=1}^{2010} P(z^i) = \frac{(-2011)\cdot 1005 \cdot (x - \frac{1004}{1005})}{(x-1)^2}.$$

Note that $(x-z)(x-z^2)\ldots(x-z^{2010}) = x^{2010} + x^{2009} + \ldots + 1$. Using this, the product turns out to be $2011^{2009}\cdot(1005^{2011} - 1004^{2011})$.

18. Let n be an odd positive integer, and suppose that n people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander's left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other $n-1$ members vote randomly.

    Answer: $\boxed{\frac{2^n-1}{n2^{n-1}}}$ Let x be the probability Hermia is elected if Lysander votes for her, and let y be the probability that she wins if Lysander does not vote for her. We are trying to find x, and do so by first finding y. If Lysander votes for Hermia with probability $\frac{1}{2}$ then the probability that Hermia is elected chairman is $\frac{x}{2}+\frac{y}{2}$, but it is also $\frac{1}{n}$ by symmetry. If Lysander does not vote for Hermia, Hermia can get at most 1 vote, and then can only be elected if everyone gets one vote and she wins the tiebreaker. The probability she wins the tiebreaker is $\frac{1}{n}$, and chasing around the circle, the probability that every person gets 1 vote is $\frac{1}{2^{n-1}}$. (Everyone votes for the person to the left, or everyone votes for the person to the right.) Hence

$$y = \frac{1}{n2^{n-1}}.$$

    Then $\frac{x}{2}+\frac{1}{n2^n} = \frac{1}{n}$, so solving for x gives

$$x = \frac{2^n-1}{n2^{n-1}}.$$

19. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2$; $b_0 = 2$, and $a_{n+1} = a_n\sqrt{1+a_n^2+b_n^2} - b_n$; $b_{n+1} = b_n\sqrt{1+a_n^2+b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4.

    Answer: $\boxed{1000001100111222 \text{ and } 2211100110000012}$
    Note first that $\sqrt{1+a_n^2+b_n^2} = 3^{2^n}$. The proof is by induction; the base case follows trivially from what is given. For the inductive step, note that $1+a_{n+1}^2+b_{n+1}^2 = 1+a_n^2(1+a_n^2+b_n^2)+b_n^2-2a_nb_n\sqrt{1+a_n^2+b_n^2}+b_n^2(1+a_n^2+b_n^2)+a_n^2+2a_nb_n\sqrt{1+a_n^2+b_n^2} = 1+(a_n^2+b_n^2)(1+a_n^2+b_n^2)+a_n^2+b_n^2 = (1+a_n^2+b_n^2)^2$. Invoking the inductive hypothesis, we see that $\sqrt{1+a_{n+1}^2+b_{n+1}^2} = (3^{2^n})^2 = 3^{2^{n+1}}$, as desired.
    The quickest way to finish from here is to consider a sequence of complex numbers $\{z_n\}$ defined by $z_n = a_n + b_n i$ for all nonnegative integers n. It should be clear that $z_0 = 2+2i$ and $z_{n+1} = z_n(3^{2^n}+i)$. Therefore, $z_4 = (2+2i)(3^{2^0}+i)(3^{2^1}+i)(3^{2^2}+i)(3^{2^3}+i)$. This product is difficult to evaluate in the decimal number system, but in ternary the calculation is a cinch! To speed things up, we will use *balanced ternary*[1], in which the three digits allowed are $-1, 0$, and 1 rather than $0, 1$, and 2. Let $x+yi = (3^{2^0}+i)(3^{2^1}+i)(3^{2^2}+i)(3^{2^3}+i)$, and consider the balanced ternary representation of x and y. For all $0 \le j \le 15$, let x_j denote the digit in the 3^j place of x, let y_j denote the digit in the 3^j place of y, and let $b(j)$ denote the number of ones in the binary representation of j. It should be clear that $x_j = -1$ if $b(j) \equiv 2 \pmod 4$, $x_j = 0$ if $b(j) \equiv 1 \pmod 2$, and $x_j = 1$ if $b(j) \equiv 0 \pmod 4$. Similarly, $y_j = -1$ if $b(j) \equiv 1 \pmod 4$, $y_j = 0$ if $b(j) \equiv 0 \pmod 2$, and $y_j = 1$ if $b(j) \equiv 3 \pmod 4$. Converting to ordinary ternary representation, we see that $x = 221211221122001_3$ and $y = 110022202212120_3$. It

[1] http://en.wikipedia.org/wiki/Balanced_ternary

remains to note that $a_4 = 2x - 2y$ and $b_4 = 2x + 2y$ and perform the requisite arithmetic to arrive at the answer above.

20. Alice and Bob play a game in which two thousand and eleven 2011×2011 grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers $1, 2, \ldots, 2011^2$ so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?

    Answer: $\boxed{1}$
    Consider the grid whose entries in the jth row are, in order, $2011j - 2010, 2011j - 2009, \ldots, 2011j$. Call this grid A_0. For $k = 1, 2 \ldots, 2010$, let grid A_k be the grid obtained from A_0 by swapping the rightmost entry of the kth row with the leftmost entry of the $k{+}1$st row. We claim that if $A \in \{A_0, A_1, \ldots, A_{2010}\}$, then given any legally numbered grid B such that A and B differ in at least one entry, there exist two integers in the same column of B that occur in the same row of A.
    We first consider A_0. Assume for the sake of contradiction B is a legally numbered grid distinct from A_0, such that there do not exist two integers in the same column of B that occur in the same row of A_0. Since the numbers $1, 2, \ldots, 2011$ occur in the same row of A_0, they must all occur in different columns of B. Clearly 1 is the leftmost entry in B's first row. Let m be the smallest number that does not occur in the first row of B. Since each row is in order, m must be the first entry in its row. But then 1 and m are in the same column of B, a contradiction. It follows that the numbers $1, 2, \ldots, 2011$ all occur in the first row of B. Proceeding by induction, $2011j - 2010, 2011j - 2009, \ldots, 2011j$ must all occur in the jth row of B for all $1 \leq j \leq 2011$. Since A_0 is the only legally numbered grid satsifying this condition, we have reached the desired contradiction.
    Now note that if $A \in \{A_1, \ldots, A_{2010}\}$, there exist two integers in the same column of A_0 that occur in the same row of A. In particular, if $A = A_k$ and $1 \leq k \leq 2010$, then the integers $2011k - 2010$ and $2011k + 1$ occur in the same column of A_0 and in the same row of A_k. Therefore, it suffices to show that for all $1 \leq k \leq 2010$, there is no legally numbered grid B distinct from A_k and A_0 such that there do not exist two integers in the same column of B that occur in the same row of A_0. Assume for the sake of contradiction that there does exist such a grid B. By the same logic as above, applied to the first $k - 1$ rows and applied backwards to the last $2010 - k - 1$ rows, we see that B may only differ from A_k in the kth and $k + 1$st rows. However, there are only two legally numbered grids that are identical to A_k outside of rows k and $k + 1$, namely A_0 and A_k. This proves the claim.
    It remains only to note that, by the pigeonhole principle, if one of Alice's grids is A_0, then there exists a positive integer k, $1 \leq k \leq 2010$, such that A_k is not one of the Alice's grids. Therefore, if Bob sets his initial grid to be A_0, he will require only one swap to switch his grid to A_k after examining Alice's grids. If A_0 is not among Alice's grids, then if Bob sets his initial grid to be A_0, he will not in fact require any swaps at all.