

# 14<sup>th</sup> Annual Harvard-MIT Mathematics Tournament

## Saturday 12 February 2011

1. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?
2. Let  $ABC$  be a triangle such that  $AB = 7$ , and let the angle bisector of  $\angle BAC$  intersect line  $BC$  at  $D$ . If there exist points  $E$  and  $F$  on sides  $AC$  and  $BC$ , respectively, such that lines  $AD$  and  $EF$  are parallel and divide triangle  $ABC$  into three parts of equal area, determine the number of possible integral values for  $BC$ .
3. Let  $a$ ,  $b$ , and  $c$  be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them:  $ax^2 + bx + c$ ,  $bx^2 + cx + a$ , and  $cx^2 + ax + b$ .
4. Josh takes a walk on a rectangular grid of  $n$  rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?
5. Let  $H$  be a regular hexagon of side length  $x$ . Call a hexagon in the same plane a “distortion” of  $H$  if and only if it can be obtained from  $H$  by translating each vertex of  $H$  by a distance strictly less than 1. Determine the smallest value of  $x$  for which every distortion of  $H$  is necessarily convex.
6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.
7. Let  $ABCDEF$  be a regular hexagon of area 1. Let  $M$  be the midpoint of  $DE$ . Let  $X$  be the intersection of  $AC$  and  $BM$ , let  $Y$  be the intersection of  $BF$  and  $AM$ , and let  $Z$  be the intersection of  $AC$  and  $BF$ . If  $[P]$  denotes the area of polygon  $P$  for any polygon  $P$  in the plane, evaluate  $[BXC] + [AYF] + [ABZ] - [MXZY]$ .
8. The integers from 1 to  $n$  are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.
9. Let  $ABCD$  be a square of side length 13. Let  $E$  and  $F$  be points on rays  $AB$  and  $AD$ , respectively, so that the area of square  $ABCD$  equals the area of triangle  $AEF$ . If  $EF$  intersects  $BC$  at  $X$  and  $BX = 6$ , determine  $DF$ .
10. Mike and Harry play a game on an  $8 \times 8$  board. For some positive integer  $k$ , Mike chooses  $k$  squares and writes an  $M$  in each of them. Harry then chooses  $k + 1$  squares and writes an  $H$  in each of them. After Harry is done, Mike wins if there is a sequence of letters forming “ $HMM$ ” or “ $MMH$ ,” when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of  $k$  for which Mike has a winning strategy.
11. Let  $ABCDEF$  be a convex equilateral hexagon such that lines  $BC$ ,  $AD$ , and  $EF$  are parallel. Let  $H$  be the orthocenter of triangle  $ABD$ . If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle  $HAD$  in degrees.
12. The ordered pairs  $(2011, 2), (2010, 3), (2009, 4), \dots, (1008, 1005), (1007, 1006)$  are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs  $(x_i, y_i)$  and  $(x_j, y_j)$ , with  $(x_i, y_i)$  left of  $(x_j, y_j)$ , erases them, and writes  $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j}\right)$  in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs  $(x, y)$  could appear on the blackboard after the process has come to a conclusion?

13. Let  $ABCD$  be a cyclic quadrilateral, and suppose that  $BC = CD = 2$ . Let  $I$  be the incenter of triangle  $ABD$ . If  $AI = 2$  as well, find the minimum value of the length of diagonal  $BD$ .
14. Let  $A = \{1, 2, \dots, 2011\}$ . Find the number of functions  $f$  from  $A$  to  $A$  that satisfy  $f(n) \leq n$  for all  $n$  in  $A$  and attain exactly 2010 distinct values.
15. Let  $f(x) = x^2 - r_2x + r_3$  for all real numbers  $x$ , where  $r_2$  and  $r_3$  are some real numbers. Define a sequence  $\{g_n\}$  for all nonnegative integers  $n$  by  $g_0 = 0$  and  $g_{n+1} = f(g_n)$ . Assume that  $\{g_n\}$  satisfies the following three conditions: (i)  $g_{2i} < g_{2i+1}$  and  $g_{2i+1} > g_{2i+2}$  for all  $0 \leq i \leq 2011$ ; (ii) there exists a positive integer  $j$  such that  $g_{i+1} > g_i$  for all  $i > j$ , and (iii)  $\{g_n\}$  is unbounded. If  $A$  is the greatest number such that  $A \leq |r_2|$  for any function  $f$  satisfying these properties, find  $A$ .
16. Let  $ABCD$  be a quadrilateral inscribed in the unit circle such that  $\angle BAD$  is 30 degrees. Let  $m$  denote the minimum value of  $CP + PQ + CQ$ , where  $P$  and  $Q$  may be any points lying along rays  $AB$  and  $AD$ , respectively. Determine the maximum value of  $m$ .
17. Let  $n$  be an odd positive integer, and suppose that  $n$  people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander's left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other  $n - 1$  members vote randomly.
18. Collinear points  $A$ ,  $B$ , and  $C$  are given in the Cartesian plane such that  $A = (a, 0)$  lies along the  $x$ -axis,  $B$  lies along the line  $y = x$ ,  $C$  lies along the line  $y = 2x$ , and  $AB/BC = 2$ . If  $D = (a, a)$ , the circumcircle of triangle  $ADC$  intersects  $y = x$  again at  $E$ , and ray  $AE$  intersects  $y = 2x$  at  $F$ , evaluate  $AE/EF$ .
19. Alice and Bob play a game in which two thousand and eleven  $2011 \times 2011$  grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers  $1, 2, \dots, 2011^2$  so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?
20. Let  $\omega_1$  and  $\omega_2$  be two circles that intersect at points  $A$  and  $B$ . Let line  $l$  be tangent to  $\omega_1$  at  $P$  and to  $\omega_2$  at  $Q$  so that  $A$  is closer to  $PQ$  than  $B$ . Let points  $R$  and  $S$  lie along rays  $PA$  and  $QA$ , respectively, so that  $PQ = AR = AS$  and  $R$  and  $S$  are on opposite sides of  $A$  as  $P$  and  $Q$ . Let  $O$  be the circumcenter of triangle  $ASR$ , and let  $C$  and  $D$  be the midpoints of major arcs  $AP$  and  $AQ$ , respectively. If  $\angle APQ$  is 45 degrees and  $\angle AQP$  is 30 degrees, determine  $\angle COD$  in degrees.