11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Team Round: B Division

Tropical Mathematics [95]

For real numbers x and y, let us consider the two operations $\oplus$ and $\odot$ defined by

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \odot y = x + y.$$

We also include ∞ in our set, and it satisfies $x \oplus \infty = x$ and $x \odot \infty = \infty$ for all x. When unspecified, $\odot$ precedes $\oplus$ in the order of operations.

1. [**10**] (Distributive law) Prove that $(x \oplus y) \odot z = x \odot z \oplus y \odot z$ for all $x, y, z \in \mathbb{R} \cup \{\infty\}$.

   Solution: This is equivalent to proving that

   $$\min(x, y) + z = \min(x + z, y + z).$$

   Consider two cases. If $x \leq y$, then $LHS = x+z$ and $RHS = x+z$. If $x > y$, then $LHS = y+z$ and $RHS = y + z$. It follows that $LHS = RHS$.

2. [**10**] (Freshman's Dream) Let z^n denote $z \odot z \odot z \odot \cdots \odot z$ with z appearing n times. Prove that $(x \oplus y)^n = x^n \oplus y^n$ for all $x, y \in \mathbb{R} \cup \{\infty\}$ and positive integer n.

   Solution: Without loss of generality, suppose that $x \leq y$, then $LHS = \min(x, y)^n = x^n = nx$, and $RHS = \min(x^n, y^n) = \min(nx, ny) = nx$.

3. [**35**] By a *tropical polynomial* we mean a function of the form

   $$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0,$$

   where exponentiation is as defined in the previous problem.

   Let p be a tropical polynomial. Prove that

   $$p\left(\frac{x+y}{2}\right) \geq \frac{p(x) + p(y)}{2}$$

   for all $x, y \in \mathbb{R} \cup \{\infty\}$. (This means that all tropical polynomials are concave.)

   Solution: First, note that for any $x_1, \ldots, x_n, y_1, \ldots, y_n$, we have

   $$\min\{x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\} \geq \min\{x_1, x_2, \ldots, x_n\} + \min\{y_1, y_2, \ldots, y_n\}.$$

   Indeed, suppose that $x_m + y_m = \min_i\{x_i + y_i\}$, then $x_m \geq \min_i x_i$ and $y_m \geq \min_i y_i$, and so $\min_i\{x_i + y_i\} = x_m + y_m \geq \min_i x_i + \min_i y_i$.

   Now, let us write a tropical polynomial in a more familiar notation. We have

   $$p(x) = \min_{0 \leq k \leq n}\{a_k + kx\}.$$

So

$$\begin{aligned}
p\left(\frac{x+y}{2}\right) &= \min_{0\le k\le n}\left\{a_k + k\left(\frac{x+y}{2}\right)\right\} \\
&= \frac{1}{2}\min_{0\le k\le n}\left\{(a_k+kx)+(a_k+ky)\right\} \\
&\ge \frac{1}{2}\left(\min_{0\le k\le n}\{a_k+kx\} + \min_{0\le k\le n}\{a_k+ky\}\right) \\
&= \frac{1}{2}\left(p(x)+p(y)\right).
\end{aligned}$$

4. [**40**] (Fundamental Theorem of Algebra) Let p be a tropical polynomial:

$$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0, \quad a_n \neq \infty$$

Prove that we can find $r_1, r_2, \ldots, r_n \in \mathbb{R} \cup \{\infty\}$ so that

$$p(x) = a_n \odot (x \oplus r_1) \odot (x \oplus r_2) \odot \cdots \odot (x \oplus r_n)$$

for all x.

Solution: Again, we have

$$p(x) = \min_{0\le k\le n}\{a_k + kx\}.$$

So the graph of $y = p(x)$ can be drawn as follows: first, draw all the lines $y = a_k + kx$, $k = 0, 1, \ldots, n$, then trace out the lowest broken line, which then is the graph of $y = p(x)$.

So $p(x)$ is piecewise linear and continuous, and has slopes from the set $\{0, 1, 2, \ldots, n\}$. We know from the previous problem that $p(x)$ is concave, and so its slope must be decreasing (this can also be observed simply from the drawing of the graph of $y = p(x)$). Then, let r_k denote the x-coordinate of the leftmost kink such that the slope of the graph is less than k to the right of this kink. Then, $r_n \le r_{n-1} \le \cdots \le r_1$, and for $r_{k-1} \le x \le r_k$, the graph of p is linear with slope k. Note that is if possible that $r_{k-1} = r_k$, if no segment of p has slope k. Also, since $a_n \neq \infty$, the leftmost piece of $p(x)$ must have slope n, and thus r_n exists, and thus all r_i exist.

Now, compare $p(x)$ with

$$\begin{aligned}
q(x) &= a_n \odot (x \oplus r_1) \odot (x \oplus r_2) \odot \cdots \odot (x \oplus r_n) \\
&= a_n + \min(x, r_1) + \min(x, r_2) + \cdots + \min(x, r_n).
\end{aligned}$$

For $r_{k-1} \le x \le r_k$, the slope of $q(x)$ is k, and for $x \le r_n$ the slope of q is n and for $x \ge r_1$ the slope of q is 0. So q is piecewise linear, and of course it is continuous. It follows that the graph of q coincides with that of p up to a translation. By taking any $x < r_n$, we see that $q(x) = a_n + nx = p(x)$, we see that the graphs of p and q coincide, and thus they must be the same function.

Juggling [125]

A *juggling sequence* of *length* n is a sequence $j(\cdot)$ of n nonnegative integers, usually written as a string

$$j(0)j(1)\ldots j(n-1)$$

such that the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(t) = t + j(\bar{t})$$

is a permutation of the integers. Here $\bar{t}$ denotes the remainder of t when divided by n. In this case, we say that f is the corresponding *juggling pattern*.

For a juggling pattern f (or its corresponding juggling sequence), we say that it has b *balls* if the permutation induces b infinite orbits on the set of integers. Equivalently, b is the maximum number such that we can find a set of b integers $\{t_1, t_2, \ldots, t_b\}$ so that the sets $\{t_i, f(t_i), f(f(t_i)), f(f(f(t_i))), \ldots\}$ are all infinite and mutually disjoint (i.e. non-overlapping) for $i = 1, 2, \ldots, b$. (This definition will become clear in a second.)

Now is probably a good time to pause and think about what all this has to do with juggling. Imagine that we are juggling a number of balls, and at time t, we toss a ball from our hand up to a height $j(\bar{t})$. This ball stays up in the air for $j(\bar{t})$ units of time, so that it comes back to our hand at time $f(t) = t + j(\bar{t})$. Then, the juggling pattern presents a simplified model of how balls are juggled (for instance, we ignore information such as which hand we use to toss the ball). A throw height of 0 (i.e., $j(\bar{t}) = 0$ and $f(t) = t$) represents that no thrown takes place at time t, which could correspond to an empty hand. Then, b is simply the minimum number of balls needed to carry out the juggling.

The following graphical representation may be helpful to you. On a horizontal line, an curve is drawn from t to $f(t)$. For instance, the following diagram depicts the juggling sequence 441 (or the juggling sequences 414 and 144). Then b is simply the number of contiguous "paths" drawn, which is 3 in this case.

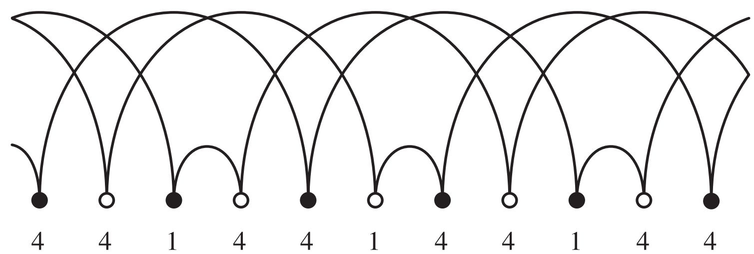


Figure 1: Juggling diagram of 441.

5. [**10**] Prove that 572 is not a juggling sequence.

   Solution: We are given $j(0) = 5$, $j(1) = 7$ and $j(2) = 2$. So $f(3) = 3 + j(0) = 8$ and $f(1) = 1 + j(1) = 8$. Thus $f(3) = f(1)$ and so f is not a permutation of $\mathbb{Z}$, and hence 572 is not a juggling pattern. (In other words, there is a "collision" at times $t \equiv 2 \pmod 3$.)

6. [**40**] Suppose that $j(0)j(1)\cdots j(n-1)$ is a valid juggling sequence. For $i = 0, 1, \dots, n-1$, Let a_i denote the remainder of $j(i) + i$ when divided by n. Prove that $(a_0, a_1, \dots, a_{n-1})$ is a permutation of $(0, 1, \dots, n-1)$.

   Solution: Suppose that $a_i = j(i) + i - b_i n$, where b_i is an integer. Note that $f(i - b_i n) = i - b_i n + j(i) = a_i$. Since $\{i - b_i n \mid i = 0, 1, \dots, n-1\}$ contains n distinct integers (as their residue mod n are all distinct), and f is a permutation, we see that after applying the map f, the resulting set $\{a_0, a_1, \dots, a_{n-1}\}$ is a set of n distinct integers. Since $0 \le a_i < n$ from definition, we see that $(a_0, a_1, \dots, a_{n-1})$ is a permutation of $(0, 1, \dots, n-1)$.

7. [**30**] Determine the number of juggling sequences of length n with exactly 1 ball.

   Answer: $\boxed{2^n - 1.}$ **Solution:** With 1 ball, we simply need to decide at times should the ball land in our hand. That is, we need to choose a non-empty subset of $\{0, 1, 2, \dots, n-1\}$ where the ball lands. It follows that the answer is $2^n - 1$.

8. [**40**] Prove that the number of balls b in a juggling sequence $j(0)j(1)\cdots j(n-1)$ is simply the average

$$b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n}.$$

   Solution: Consider the corresponding juggling diagram. Say the *length* of an curve from t to $f(t)$ is $f(t) - t$. Let us draw only the curves whose left endpoint lies inside $[0, Mn-1]$. For every single ball, the sum of the lengths of the arrows drawn corresponding to that ball is between $Mn - J$ and $Mn + J$, where $J = \max\{j(0), j(1), \dots, j(n-1)\}$. It follows that the sum of the lengths of the arrows drawn is between $b(Mn - J)$ and $b(Mn + J)$. Since the arrow drawn at t has length $j(\bar{t})$, the sum of the lengths of the arrows drawn is $M(j(0) + j(1) + \cdots + j(n-1))$. It follows that

$$b(Mn - J) \le M(j(0) + j(1) + \cdots + j(n-1)) \le b(Mn + J).$$

   Dividing by Mn, we get

$$b\left(1 - \frac{J}{nM}\right) \le \frac{j(0) + j(1) + \cdots + j(n-1)}{n} \le b\left(1 + \frac{J}{nM}\right).$$

   Since we can take M to be arbitrarily large, we must have

$$b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n},$$

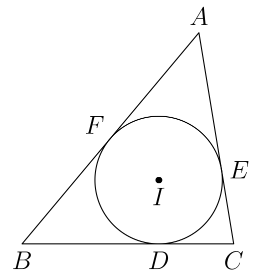
   as desired.

9. [**5**] Show that the converse of the previous statement is false by providing a non-juggling sequence $j(0)j(1)j(2)$ of length 3 where the average $\frac{1}{3}(j(0) + j(1) + j(2))$ is an integer. Show that your example works.

   Solution: One such example is 210. This is not a juggling sequence since $f(0) = f(1) = 2$.

Incircles [180]

In the following problems, ABC is a triangle with incenter I. Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB, respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. [**15**] Let a, b, c denote the side lengths of BC, CA, AB. Find the lengths of AE, BF, CD in terms of a, b, c.

    Solution: Let $x = AE = AF$, $y = BD = BF$, $z = CD = CE$. Since $BC = BD + CD$, we have $a = x + y$. Similarly with the other sides, we arrive at the following system of equations:

    $$a = y + z, \quad b = x + z, \quad c = x + y.$$

    Solving this system gives us

    $$\begin{aligned} AE = x &= \frac{b + c - a}{2}, \\ BF = y &= \frac{a + c - b}{2}, \\ CD = z &= \frac{a + b - c}{2}. \end{aligned}$$

11. [**15**] Show that lines AD, BE, CF pass through a common point.

    Solution: Using Ceva's theorem on triangle ABC, we see that it suffices to show that

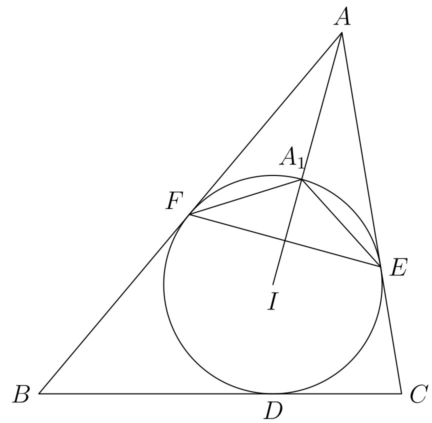
    $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

    Since $AF = AE$, $BD = BF$, and $CD = CE$ (due to equal tangents), we see that the LHS is indeed 1.

    Remark: The point of concurrency is known as the *Gergonne point*.

12. [**35**] Show that the incenter of triangle AEF lies on the incircle of ABC.

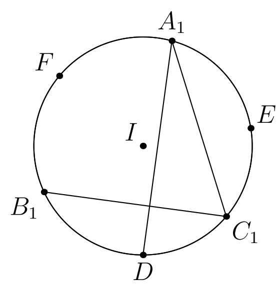
    Solution: Let segment AI meet the incircle at A_1. Let us show that A_1 is the incenter of AEF.



Since $AE = AF$ and AA' is the angle bisector of $\angle EAF$, we find that $A_1E = A_1F$. Using tangent-chord, we see that $\angle AFA_1 = \angle A_1EF = \angle A_1FE$. Therefore, A_1 lies on the angle bisector of $\angle AFE$. Since A_1 also lies on the angle bisector of $\angle EAF$, A_1 must be the incenter of AEF, as desired.

13. [**35**] Let A_1, B_1, C_1 be the incenters of triangle AEF, BDF, CDE, respectively. Show that A_1D, B_1E, C_1F all pass through the orthocenter of $A_1B_1C_1$.

Solution: Using the result from the previous problem, we see that A_1, B_1, C_1 are respectively the midpoints of the arc FE, FD, DF of the incircle. We have



$$\begin{aligned}
\angle DA_1C_1 + \angle B_1C_1A_1 &= \frac{1}{2}\angle DIC_1 + \frac{1}{2}\angle B_1IF + \frac{1}{2}\angle FIA_1 \\
&= \frac{1}{4}\left(\angle EID + \angle DIF + \angle FIE\right) \\
&= \frac{1}{4} \cdot 360^\circ \\
&= 90^\circ.
\end{aligned}$$

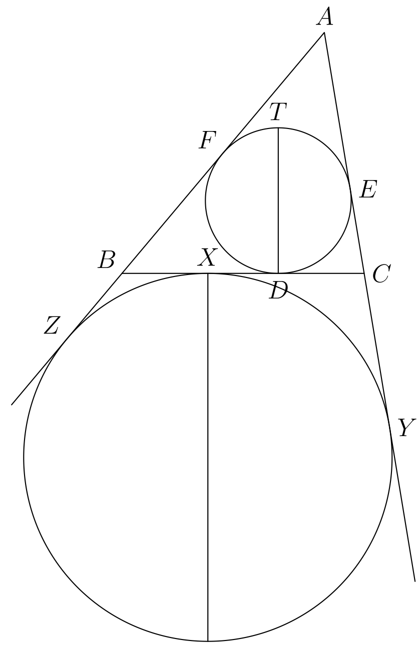
It follows that A_1D is perpendicular to B_1C_1, and thus A_1D passes through the orthocenter of $A_1B_1C_1$. Similarly, A_1D, B_1E, C_1F all pass through the orthocenter of $A_1B_1C_1$.

14. [**40**] Let X be the point on side BC such that $BX = CD$. Show that the excircle ABC opposite of vertex A touches segment BC at X.

Solution: Let the excircle touch lines BC, AC and AB at X', Y and Z, respectively. Using the equal tangent property repeatedly, we have

$$BX' - X'C = BZ - CY = (EY - CY) - (FZ - BZ) = CE - BF = CD - BD.$$

It follows that $BX' = CD$, and thus $X' = X$. So the excircle touches BC at X.

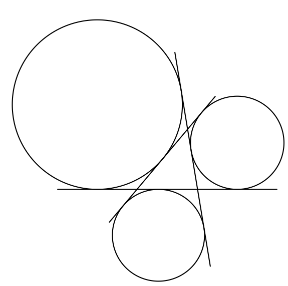


15. [**40**] Let X be as in the previous problem. Let T be the point diametrically opposite to D on on the incircle of ABC . Show that A, T, X are collinear.

    Solution: Consider a dilation centered at A that carries the incircle to the excircle. This dilation must send the diameter DT to some the diameter of excircle that is perpendicular to BC. The only such diameter is the one goes through X. It follows that T gets carried to X. Therefore, A, T, X are collinear.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC, tangent to all three sides.
- The *excircles* of a triangle ABC are the three circles on the exterior the triangle but tangent to all three lines AB, BC, CA.



- The *orthocenter* of a triangle is the point of concurrency of the three altitudes.

- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB, respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$