February 2017

February 18, 2017

Geometry

1. Let A, B, C, D be four points on a circle in that order. Also, $AB = 3$, $BC = 5$, $CD = 6$, and $DA = 4$. Let diagonals AC and BD intersect at P. Compute $\frac{AP}{CP}$.

2. Let ABC be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Let ℓ be a line passing through two sides of triangle ABC. Line ℓ cuts triangle ABC into two figures, a triangle and a quadrilateral, that have equal perimeter. What is the maximum possible area of the triangle?

3. Let S be a set of 2017 distinct points in the plane. Let R be the radius of the smallest circle containing all points in S on either the interior or boundary. Also, let D be the longest distance between two of the points in S. Let a, b are real numbers such that $a \leq \frac{D}{R} \leq b$ for all possible sets S, where a is as large as possible and b is as small as possible. Find the pair (a, b).

4. Let $ABCD$ be a convex quadrilateral with $AB = 5, BC = 6, CD = 7$, and $DA = 8$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 - PQ^2$.

5. Let $ABCD$ be a quadrilateral with an inscribed circle ω and let P be the intersection of its diagonals AC and BD. Let R_1, R_2, R_3, R_4 be the circumradii of triangles APB, BPC, CPD, DPA respectively. If $R_1 = 31$ and $R_2 = 24$ and $R_3 = 12$, find R_4.

6. In convex quadrilateral $ABCD$ we have $AB = 15$, $BC = 16$, $CD = 12$, $DA = 25$, and $BD = 20$. Let M and γ denote the circumcenter and circumcircle of $\triangle ABD$. Line CB meets γ again at F, line AF meets MC at G, and line GD meets γ again at E. Determine the area of pentagon $ABCDE$.

7. Let ω and Γ by circles such that ω is internally tangent to Γ at a point P. Let AB be a chord of Γ tangent to ω at a point Q. Let $R \neq P$ be the second intersection of line PQ with Γ. If the radius of Γ is 17, the radius of ω is 7, and $\frac{AQ}{BQ} = 3$, find the circumradius of triangle AQR.

8. Let ABC be a triangle with circumradius $R = 17$ and inradius $r = 7$. Find the maximum possible value of $\sin \frac{A}{2}$.

9. Let ABC be a triangle, and let $BCDE, CAFG, ABHI$ be squares that do not overlap the triangle with centers X, Y, Z respectively. Given that $AX = 6$, $BY = 7$, and $CZ = 8$, find the area of triangle XYZ.

10. Let $ABCD$ be a quadrilateral with an inscribed circle ω. Let I be the center of ω let $IA = 12$, $IB = 16$, $IC = 14$, and $ID = 11$. Let M be the midpoint of segment AC. Compute $\frac{IM}{IN}$, where N is the midpoint of segment BD.