15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Algebra Test

1. Let f be the function such that

$$f(x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

What is the total length of the graph of $\underbrace{f(f(\ldots f}_{2012 f's}(x)\ldots))$ from $x = 0$ to $x = 1$?

Answer: $\boxed{\sqrt{4^{2012}+1}}$ When there are n copies of f, the graph consists of 2^n segments, each of which goes $1/2^n$ units to the right, and alternately 1 unit up or down. So, the length is

$$2^n \sqrt{1 + \frac{1}{2^{2n}}} = \sqrt{4^n + 1}$$

Taking $n = 2012$, the answer is

$$\sqrt{4^{2012} + 1}$$

2. You are given an unlimited supply of red, blue, and yellow cards to form a hand. Each card has a point value and your score is the sum of the point values of those cards. The point values are as follows: the value of each red card is 1, the value of each blue card is equal to twice the number of red cards, and the value of each yellow card is equal to three times the number of blue cards. What is the maximum score you can get with fifteen cards?

Answer: $\boxed{168}$ If there are B blue cards, then each red card contributes $1 + 2B$ points (one for itself and two for each blue card) and each yellow card contributes $3B$ points. Thus, if $B > 1$, it is optimal to change all red cards to yellow cards. When $B = 0$, the maximum number of points is 15. When $B = 1$, the number of points is always 42. When $B > 1$, the number of points is $3BY$, where Y is the number of yellow cards. Since $B + Y = 15$, the desired maximum occurs when $B = 7$ and $Y = 8$, which gives 168 points.

3. Given points a and b in the plane, let $a \oplus b$ be the unique point c such that abc is an equilateral triangle with a, b, c in the clockwise orientation.

Solve $(x \oplus (0,0)) \oplus (1,1) = (1,-1)$ for x.

Answer: $\boxed{\left(\frac{1-\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right)}$ It is clear from the definition of $\oplus$ that $b \oplus (a \oplus b) = a$ and if $a \oplus b = c$ then $b \oplus c = a$ and $c \oplus a = b$. Therefore $x \oplus (0,0) = (1,1) \oplus (1,-1) = (1 - \sqrt{3}, 0)$. Now this means $x = (0,0) \oplus (1 - \sqrt{3}, 0) = (\frac{1-\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2})$.

4. During the weekends, Eli delivers milk in the complex plane. On Saturday, he begins at z and delivers milk to houses located at z^3, z^5, $z^7, \ldots,$ z^{2013}, in that order; on Sunday, he begins at 1 and delivers milk to houses located at z^2, z^4, $z^6, \ldots,$ z^{2012}, in that order. Eli always walks directly (in a straight line) between two houses. If the distance he must travel from his starting point to the last house is $\sqrt{2012}$ on both days, find the real part of z^2.

Answer: $\boxed{\frac{1005}{1006}}$ Note that the distance between two points in the complex plane, m and n, is $|m - n|$. We have that

$$\sum_{k=1}^{1006} \left| z^{2k+1} - z^{2k-1} \right| = \sum_{k=1}^{1006} \left| z^{2k} - z^{2k-2} \right| = \sqrt{2012}.$$

However, noting that

$$|z| \cdot \sum_{k=1}^{1006} \left|z^{2k} - z^{2k-2}\right| = \sum_{k=1}^{1006} \left|z^{2k+1} - z^{2k-1}\right|,$$

we must have $|z| = 1$. Then, since Eli travels a distance of $\sqrt{2012}$ on each day, we have

$$\sum_{k=1}^{1006} \left|z^{2k} - z^{2k-2}\right| = \left|z^2 - 1\right| \cdot \sum_{k=1}^{1006} \left|z^{2k-2}\right| = \left|z^2 - 1\right| \cdot \sum_{k=1}^{1006} |z|^{2k-2}$$

$$= 1006\left|z^2 - 1\right| = \sqrt{2012},$$

so $\left|z^2 - 1\right| = \frac{\sqrt{2012}}{1006}$. Since $|z| = 1$, we can write $z = \cos(\theta) + i\sin(\theta)$ and then $z^2 = \cos(2\theta) + i\sin(2\theta)$. Hence,

$$\left|z^2 - 1\right| = \sqrt{(\cos(2\theta) - 1)^2 + \sin^2(2\theta)} = \sqrt{2 - 2\cos(2\theta)} = \frac{\sqrt{2012}}{1006},$$

so $2 - 2\cos(2\theta) = \frac{2}{1006}$. The real part of z^2, $\cos(2\theta)$, is thus $\frac{1005}{1006}$.

5. Find all ordered triples (a, b, c) of positive reals that satisfy: $\lfloor a \rfloor bc = 3$, $a\lfloor b \rfloor c = 4$, and $ab\lfloor c \rfloor = 5$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Answer: $\boxed{\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{4}, \frac{2\sqrt{30}}{5}\right), \left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{2}, \frac{\sqrt{30}}{5}\right)}$ Write $p = abc$, $q = \lfloor a \rfloor \lfloor b \rfloor \lfloor c \rfloor$. Note that q is an integer. Multiplying the three equations gives:

$$p = \sqrt{\frac{60}{q}}$$

Substitution into the first equation,

$$p = 3\frac{a}{\lfloor a \rfloor} < 3\frac{\lfloor a \rfloor + 1}{\lfloor a \rfloor} \leq 6$$

Looking at the last equation:

$$p = 5\frac{c}{\lfloor c \rfloor} \geq 5\frac{\lfloor c \rfloor}{\lfloor c \rfloor} \geq 5$$

Here we've used $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, and also the apparent fact that $\lfloor a \rfloor \geq 1$. Now:

$$5 \leq \sqrt{\frac{60}{q}} \leq 6$$

$$\frac{12}{5} \geq q \geq \frac{5}{3}$$

Since q is an integer, we must have $q = 2$. Since q is a product of 3 positive integers, we must have those be 1,1, and 2 in some order, so there are three cases:

Case 1: $\lfloor a \rfloor = 2$. By the equations, we'd need $a = \frac{2}{3}\sqrt{30} = \sqrt{120/9} > 3$, a contradiction, so there are no solutions in this case.

Case 2: $\lfloor b \rfloor = 2$. We have the solution

$$\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{2}, \frac{\sqrt{30}}{5}\right)$$

Case 3: $\lfloor c \rfloor = 2$. We have the solution

$$\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{4}, \frac{2\sqrt{30}}{5}\right)$$

6. Let $a_0 = -2$, $b_0 = 1$, and for $n \geq 0$, let

$$a_{n+1} = a_n + b_n + \sqrt{a_n^2 + b_n^2},$$
$$b_{n+1} = a_n + b_n - \sqrt{a_n^2 + b_n^2}.$$

Find a_{2012}.

Answer: $\boxed{2^{1006}\sqrt{2^{2010}+2} - 2^{2011}}$ We have

$$a_{n+1} + b_{n+1} = 2(a_n + b_n)$$
$$a_{n+1}b_{n+1} = (a_n + b_n)^2 - a_n^2 - b_n^2 = 2a_nb_n$$

Thus,

$$a_n + b_n = -2^n$$
$$a_nb_n = -2^{n+1}$$

Using Viete's formula, a_{2012} and b_{2012} are the roots of the following quadratic, and, since the square root is positive, a_{2012} is the bigger root:

$$x^2 + 2^{2012}x - 2^{2013}$$

Thus,

$$a_{2012} = 2^{1006}\sqrt{2^{2010}+2} - 2^{2011}$$

7. Let $\otimes$ be a binary operation that takes two positive real numbers and returns a positive real number. Suppose further that $\otimes$ is continuous, commutative ($a \otimes b = b \otimes a$), distributive across multiplication ($a \otimes (bc) = (a \otimes b)(a \otimes c)$), and that $2 \otimes 2 = 4$. Solve the equation $x \otimes y = x$ for y in terms of x for $x > 1$.

Answer: $\boxed{y = \sqrt{2}}$ We note that $(a \otimes b^k) = (a \otimes b)^k$ for all positive integers k. Then for all rational numbers $\frac{p}{q}$ we have $a \otimes b^{\frac{p}{q}} = (a \otimes b^{\frac{1}{q}})^p = (a \otimes b)^{\frac{p}{q}}$. So by continuity, for all real numbers a, b, it follows that $2^a \otimes 2^b = (2 \otimes 2)^{ab} = 4^{ab}$. Therefore given positive reals x, y, we have $x \otimes y = 2^{\log_2(x)} \otimes 2^{\log_2(y)} = 4^{\log_2(x)\log_2(y)}$.

If $x = 4^{\log_2(x)\log_2(y)} = 2^{2\log_2(x)\log_2(y)}$ then $\log_2(x) = 2\log_2(x)\log_2(y)$ and $1 = 2\log_2(y) = \log_2(y^2)$. Thus $y = \sqrt{2}$ regardless of x.

8. Let $x_1 = y_1 = x_2 = y_2 = 1$, then for $n \geq 3$ let $x_n = x_{n-1}y_{n-2} + x_{n-2}y_{n-1}$ and $y_n = y_{n-1}y_{n-2} - x_{n-1}x_{n-2}$. What are the last two digits of $|x_{2012}|$?

Answer: $\boxed{84}$ Let $z_n = y_n + x_ni$. Then the recursion implies that:

$$z_1 = z_2 = 1 + i,$$
$$z_n = z_{n-1}z_{n-2}.$$

This implies that

$$z_n = (z_1)^{F_n},$$

where F_n is the n^{th} Fibonacci number ($F_1 = F_2 = 1$). So, $z_{2012} = (1+i)^{F_{2012}}$. Notice that

$$(1+i)^2 = 2i.$$

Also notice that every third Fibonnaci number is even, and the rest are odd. So:

$$z_{2012} = (2i)^{\frac{F_{2012}-1}{2}}(1+i)$$

Let $m = \frac{F_{2012}-1}{2}$. Since both real and imaginary parts of $1+i$ are 1, it follows that the last two digits of $|x_{2012}|$ are simply the last two digits of $2^m = 2^{\frac{F_{2012}-1}{2}}$.

By the Chinese Remainder Theorem, it suffices to evaluate 2^m modulo 4 and 25. Clearly, 2^m is divisible by 4. To evaluate it modulo 25, it suffices by Euler's Totient theorem to evaluate m modulo 20.

To determine $(F_{2012}-1)/2$ modulo 4 it suffices to determine F_{2012} modulo 8. The Fibonacci sequence has period 12 modulo 8, and we find

$$\begin{aligned} F_{2012} &\equiv 5 \pmod 8, \\ m &\equiv 2 \pmod 4. \end{aligned}$$

$2 * 3 \equiv 1 \pmod 5$, so

$$m \equiv 3F_{2012} - 3 \pmod 5.$$

The Fibonacci sequence has period 20 modulo 5, and we find

$$m \equiv 4 \pmod 5.$$

Combining,

$$\begin{aligned} m &\equiv 14 \pmod{20} \\ 2^m &\equiv 2^{14} = 4096 \equiv 21 \pmod{25} \\ |x_{2012}| &\equiv 4 \cdot 21 = 84 \pmod{100}. \end{aligned}$$

9. How many real triples (a,b,c) are there such that the polynomial $p(x) = x^4 + ax^3 + bx^2 + ax + c$ has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, and $\tan 3y$ for some real y?

Answer: $\boxed{18}$ Let p have roots r, r, s, t. Using Vieta's on the coefficient of the cubic and linear terms, we see that $2r + s + t = r^2 s + r^2 t + 2rst$. Rearranging gives $2r(1-st) = (r^2-1)(s+t)$.

If $r^2 - 1 = 0$, then since $r \neq 0$, we require that $1 - st = 0$ for the equation to hold. Conversely, if $1 - st = 0$, then since $st = 1, s+t = 0$ cannot hold for real s,t, we require that $r^2 - 1 = 0$ for the equation to hold. So one valid case is where both these values are zero, so $r^2 = st = 1$. If $r = \tan y$ (here we stipulate that $0 \le y < \pi$), then either $y = \frac{\pi}{4}$ or $y = \frac{3\pi}{4}$. In either case, the value of $\tan 2y$ is undefined. If $r = \tan 2y$, then we have the possible values $y = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$. In each of these cases, we must check if $\tan y \tan 3y = 1$. But this is true if $y + 3y = 4y$ is a odd integer multiple of $\frac{\pi}{2}$, which is the case for all such values. If $r = \tan 3y$, then we must have $\tan y \tan 2y = 1$, so that $3y$ is an odd integer multiple of $\frac{\pi}{2}$. But then $\tan 3y$ would be undefined, so none of these values can work.

Now, we may assume that $r^2 - 1$ and $1 - st$ are both nonzero. Dividing both sides by $(r^2-1)(1-st)$ and rearranging yields $0 = \frac{2r}{1-r^2} + \frac{s+t}{1-st}$, the tangent addition formula along with the tangent double angle formula. By setting r to be one of $\tan y$, $\tan 2y$, or $\tan 3y$, we have one of the following:

(a) $0 = \tan 2y + \tan 5y$

(b) $0 = \tan 4y + \tan 4y$

(c) $0 = \tan 6y + \tan 3y$.

We will find the number of solutions y in the interval $[0, \pi)$. Case 1 yields six multiples of $\frac{\pi}{7}$. Case 2 yields $\tan 4y = 0$, which we can readily check has no solutions. Case 3 yields eight multiples of $\frac{\pi}{9}$. In total, we have $4 + 6 + 8 = 18$ possible values of y.

10. Suppose that there are 16 variables $\{a_{i,j}\}_{0 \le i,j \le 3}$, each of which may be 0 or 1. For how many settings of the variables $a_{i,j}$ do there exist positive reals $c_{i,j}$ such that the polynomial

$$f(x,y) = \sum_{0 \le i,j \le 3} a_{i,j} c_{i,j} x^i y^j$$

$(x, y \in \mathbb{R})$ is bounded below?

Answer: $\boxed{126}$ For some choices of the $a_{i,j}$, let $S = \{(i,j) | a_{i,j} = 1\}$, and let $S' = S \cup \{(0,0)\}$. Let $C(S')$ denote the convex hull of S'. We claim that there exist the problem conditions are satisfied (there exist positive coefficients for the terms so that the polynomial is bounded below) if and only if the vertices of $C(S')$ all have both coordinates even.

For one direction, suppose that $C(S')$ has a vertex $v = (i', j')$ with at least one odd coordinate; WLOG, suppose it is i'. Since v is a vertex, it maximizes some objective function $ai + bj$ over $C(S')$ uniquely, and thus also over S'. Since $(0,0) \in S'$, we must have $ai' + bj' > 0$. Now consider plugging in $(x, y) = (-t^a, t^b)$ $(t > 0)$ into f. This gives the value

$$f(-t^a, t^b) = \sum_{(i,j) \in S} (-1)^i c_{i,j} t^{ai+bj}.$$

But no matter what positive $c_{i,j}$ we choose, this expression is not bounded below as t grows infinitely large, as there is a $-c_{i',j'} t^{ai'+bj'}$ term, with $ai' + bj' > 0$, and all other terms have smaller powers of t. So the polynomial cannot be bounded below.

For the other direction, suppose the vertices of $C(S')$ all have both coordinates even. If all points in S' are vertices of $C(S')$, then the polynomial is a sum of squares, so it is bounded below. Otherwise, we assume that some points in S' are not vertices of $C(S')$. It suffices to consider the case where there is exactly one such point. Call this point $w = (i', j')$. Let $V(S')$ denote the set of the vertices of $C(S')$, and let $n = |V(S')|$. Enumerate the points of $V(S')$ as $v_1, v_2, \ldots, v_n$. Let i_k, j_k denote the i and j coordinates of v_k, respectively.

Since $w \in C(S')$, there exist nonnegative constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k v_k = w$. (Here, we are treating the ordered pairs as vectors.) Then, by weighted AM-GM, we have

$$\sum_{k=1}^n \lambda_k |x|^{i_k} |y|^{j_k} \geq |x|^{i'} |y|^{j'}.$$

Let c be the λ-value associated with $(0,0)$. Then by picking $c_{i_k,j_k} = \lambda_k$ and $c_{i',j'} = 1$, we find that $p(x,y) \geq -c$ for all x, y, as desired.

We now find all possible convex hulls $C(S')$ (with vertices chosen from $(0,0), (0,2), (2,0)$, and $(2,2)$), and for each convex hull, determine how many possible settings of $a_{i,j}$ give that convex hull. There are 8 such possible convex hulls: the point $(0,0)$ only, 3 lines, 3 triangles, and the square. The point has 2 possible choices, each line has 4 possible choices, each triangle has 16 possible choices, and the square has 64 possible choices, giving $2 + 3 \cdot 4 + 3 \cdot 16 + 64 = 126$ total choices.