HMMT 2014

Saturday 22 February 2014

Guts

1. [**4**] Compute the prime factorization of 159999.

   Answer: $\boxed{3 \cdot 7 \cdot 19 \cdot 401}$ We have $159999 = 160000 - 1 = 20^4 - 1 = (20-1)(20+1)(20^2+1) = 3 \cdot 7 \cdot 19 \cdot 401$.

2. [**4**] Let $x_1, \ldots, x_{100}$ be defined so that for each i, x_i is a (uniformly) random integer between 1 and 6 inclusive. Find the expected number of integers in the set $\{x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_{100}\}$ that are multiples of 6.

   Answer: $\boxed{\frac{50}{3}}$ Note that for any i, the probability that $x_1 + x_2 + \ldots + x_i$ is a multiple of 6 is $\frac{1}{6}$ because exactly 1 value out of 6 possible values of x_i works. Because these 100 events are independent, the expected value is $100 \cdot \frac{1}{6} = \frac{50}{3}$.

3. [**4**] Let $ABCDEF$ be a regular hexagon. Let P be the circle inscribed in $\triangle BDF$. Find the ratio of the area of circle P to the area of rectangle $ABDE$.

   Answer: $\boxed{\frac{\pi\sqrt{3}}{12}}$ Let the side length of the hexagon be s. The length of BD is $s\sqrt{3}$, so the area of rectangle $ABDE$ is $s^2\sqrt{3}$. Equilateral triangle BDF has side length $s\sqrt{3}$. The inradius of an equilateral triangle is $\sqrt{3}/6$ times the length of its side, and so has length $\frac{s}{2}$. Thus, the area of circle P is $\frac{\pi s^2}{4}$, so the ratio is $\frac{\pi s^2/4}{s^2\sqrt{3}} = \frac{\pi\sqrt{3}}{12}$.

4. [**4**] Let D be the set of divisors of 100. Let Z be the set of integers between 1 and 100, inclusive. Mark chooses an element d of D and an element z of Z uniformly at random. What is the probability that d divides z?

   Answer: $\boxed{\frac{217}{900}}$ As $100 = 2^2 \cdot 5^2$, there are $3 \cdot 3 = 9$ divisors of 100, so there are 900 possible pairs of d and z that can be chosen.

   If d is chosen, then there are $\frac{100}{d}$ possible values of z such that d divides z, so the total number of valid pairs of d and z is $\sum_{d|100} \frac{100}{d} = \sum_{d|100} d = (1+2+2^2)(1+5+5^2) = 7 \cdot 31 = 217$. The answer is therefore $\frac{217}{900}$.

5. [**5**] If four fair six-sided dice are rolled, what is the probability that the lowest number appearing on any die is exactly 3?

   Answer: $\boxed{175/1296}$ The probability that all the die rolls are at least 3 is $\frac{4}{6}^4$. The probability they are all at least 4 is $\frac{3}{6}^4$. The probability of being in the former category but not the latter is thus $\frac{4}{6}^4 - \frac{3}{6}^4 = \frac{256-81}{1296} = \frac{175}{1296}$.

6. [**5**] Find all integers n for which $\dfrac{n^3+8}{n^2-4}$ is an integer.

   Answer: $\boxed{0,\ 1,\ 3,\ 4,\ 6}$ We have $\frac{n^3+8}{n^2-4} = \frac{(n+2)(n^2-2n+4)}{(n+2)(n-2)} = \frac{n^2-2n+4}{n-2}$ for all $n \neq -2$. Then $\frac{n^2-2n+4}{n-2} = n + \frac{4}{n-2}$, which is an integer if and only if $\frac{4}{n-2}$ is an integer. This happens when $n - 2 = -4, -2, -1, 1, 2, 4$, corresponding to $n = -2, 0, 1, 3, 4, 6$, but we have $n \neq -2$ so the answers are $0, 1, 3, 4, 6$.

7. [**5**] The Evil League of Evil is plotting to poison the city's water supply. They plan to set out from their headquarters at $(5, 1)$ and put poison in two pipes, one along the line $y = x$ and one along the line $x = 7$. However, they need to get the job done quickly before Captain Hammer catches them. What's the shortest distance they can travel to visit both pipes and then return to their headquarters?

   Answer: $\boxed{4\sqrt{5}}$ After they go to $y = x$, we reflect the remainder of their path in $y = x$, along with the second pipe and their headquarters. Now, they must go from $(5, 1)$ to $y = 7$ crossing $y = x$, and then go to $(1,5)$. When they reach $y=7$, we reflect the remainder of their path again, so now their reflected headquarters is at $(1,9)$. Thus, they just go from $(5,1)$ to $(1,9)$ in some path that inevitably crosses $y=x$ and $y=7$. The shortest path they can take is a straight line with length $\sqrt{4^2+8^2}=4\sqrt{5}$.

Comment. These ideas can be used to prove that the orthic triangle of an acute triangle has the smallest possible perimeter of all inscribed triangles.

Also, see if you can find an alternative solution using Minkowski's inequality!

8. [**5**] The numbers $2^0, 2^1, \cdots, 2^{15}, 2^{16} = 65536$ are written on a blackboard. You repeatedly take two numbers on the blackboard, subtract one from the other, erase them both, and write the result of the subtraction on the blackboard. What is the largest possible number that can remain on the blackboard when there is only one number left?

   Answer: $\boxed{131069}$ If we reverse the order of the numbers in the final subtraction we perform, then the final number will be negated. Thus, the possible final numbers come in pairs with opposite signs. Therefore, the largest possible number is the negative of the smallest possible number. To get the smallest possible number, clearly we can take the smallest number originally on the board and subtract all of the other numbers from it (you can make this rigorous pretty easily if needed), so the smallest possible number is $1-\sum_{k=1}^{16} 2^k = 1-131070 = -131069$, and thus the largest possible number is 131069.

9. [**6**] Compute the side length of the largest cube contained in the region

$$\{(x,y,z) : x^2+y^2+z^2 \le 25 \text{ and } x \ge 0\}$$

   of three-dimensional space.

   Answer: $\boxed{\frac{5\sqrt{6}}{3}}$ The given region is a hemisphere, so the largest cube that can fit inside it has one face centered at the origin and the four vertices of the opposite face on the spherical surface. Let the side length of this cube be s. Then, the radius of the circle is the hypotenuse of a triangle with side lengths s and $\frac{\sqrt{2}}{2}s$. So, by the Pythagorean Theorem, the radius equals $\frac{\sqrt{6}}{2}s$. Since the radius of the hemisphere is 5, the side length of the cube is $\frac{5\sqrt{6}}{3}$.

10. [**6**] Find the number of nonempty sets $\mathcal{F}$ of subsets of the set $\{1,\ldots,2014\}$ such that:

    (a) For any subsets $S_1, S_2 \in \mathcal{F}$, $S_1 \cap S_2 \in \mathcal{F}$.

    (b) If $S \in \mathcal{F}$, $T \subseteq \{1,\ldots,2014\}$, and $S \subseteq T$, then $T \in \mathcal{F}$.

    Answer: $\boxed{2^{2014}}$ For a subset S of $\{1,\ldots,2014\}$, let $\mathcal{F}_S$ be the set of all sets T such that $S \subseteq T \subseteq \{1,\ldots,2014\}$. It can be checked that the sets $\mathcal{F}_S$ satisfy the conditions 1 and 2. We claim that the $\mathcal{F}_S$ are the only sets of subsets of $\{1,\ldots,2014\}$ satisfying the conditions 1 and 2. (Thus, the answer is the number of subsets S of $\{1,\ldots,2014\}$, which is 2^{2014}.)

    Suppose that $\mathcal{F}$ satisfies the conditions 1 and 2, and let S be the intersection of all the sets of $\mathcal{F}$. We claim that $\mathcal{F} = \mathcal{F}_S$. First, by definition of S, all elements $T \in \mathcal{F}$ are supersets of S, so $\mathcal{F} \subseteq \mathcal{F}_S$. On the other hand, by iterating condition 1, it follows that S is an element of $\mathcal{F}$, so by condition 2 any set T with $S \subseteq T \subseteq \{1,\ldots,2014\}$ is an element of $\mathcal{F}$. So $\mathcal{F} \supseteq \mathcal{F}_S$. Thus $\mathcal{F} = \mathcal{F}_S$.

11. [**6**] Two fair octahedral dice, each with the numbers 1 through 8 on their faces, are rolled. Let N be the remainder when the product of the numbers showing on the two dice is divided by 8. Find the expected value of N.

    Answer: $\boxed{\frac{11}{4}}$ If the first die is odd, which has $\frac{1}{2}$ probability, then N can be any of 0, 1, 2, 3, 4, 5, 6, 7 with equal probability, because multiplying each element of $\{0,\ldots,7\}$ with an odd number and taking modulo 8 results in the same numbers, as all odd numbers are relatively prime to 8. The expected value in this case is 3.5.

If the first die is even but not a multiple of 4, which has $\frac{1}{4}$ probability, then using similar reasoning, N can be any of 0, 2, 4, 6 with equal probability, so the expected value is 3.

If the first die is 4, which has $\frac{1}{8}$ probability, then N can by any of 0, 4 with equal probability, so the expected value is 2.

Finally, if the first die is 8, which has $\frac{1}{8}$ probability, then $N = 0$. The total expected value is $\frac{1}{2}(3.5) + \frac{1}{4}(3) + \frac{1}{8}(2) + \frac{1}{8}(0) = \frac{11}{4}$.

12. [**6**] Find a nonzero monic polynomial $P(x)$ with integer coefficients and minimal degree such that $P(1 - \sqrt[3]{2} + \sqrt[3]{4}) = 0$. (A polynomial is called *monic* if its leading coefficient is 1.)

Answer: $\boxed{x^3 - 3x^2 + 9x - 9}$ Note that $(1 - \sqrt[3]{2} + \sqrt[3]{4})(1 + \sqrt[3]{2}) = 3$, so $1 - \sqrt[3]{2} + \sqrt[3]{4} = \frac{3}{1+\sqrt[3]{2}}$.

Now, if $f(x) = x^3 - 2$, we have $f(\sqrt[3]{2}) = 0$, so if we let $g(x) = f(x-1) = (x-1)^3 - 2 = x^3 - 3x^2 + 3x - 3$, then $g(1+\sqrt[3]{2}) = f(\sqrt[3]{2}) = 0$. Finally, we let $h(x) = g(\frac{3}{x}) = \frac{27}{x^3} - \frac{27}{x^2} + \frac{9}{x} - 3$ so $h(\frac{3}{1+\sqrt[3]{2}}) = g(1+\sqrt[3]{2}) = 0$. To make this a monic polynomial, we multiply $h(x)$ by $-\frac{x^3}{3}$ to get $x^3 - 3x^2 + 9x - 9$.

13. [**8**] An auditorium has two rows of seats, with 50 seats in each row. 100 indistinguishable people sit in the seats one at a time, subject to the condition that each person, except for the first person to sit in each row, must sit to the left or right of an occupied seat, and no two people can sit in the same seat. In how many ways can this process occur?

Answer: $\boxed{\binom{100}{50}2^{98}}$ First, note that there are 2^{49} ways a single row can be filled, because each of the 49 people after the first in a row must sit to the left or to the right of the current group of people in the row, so there are 2 possibilities for each of these 49 people.

Now, there are $\binom{100}{50}$ ways to choose the order in which people are added to the rows, and 2^{49} ways to fill up each row separately, for a total of $\binom{100}{50}2^{98}$ ways to fill up the auditorium.

14. [**8**] Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $\angle D = 90°$. Suppose that there is a point E on CD such that $AE = BE$ and that triangles AED and CEB are similar, but not congruent. Given that $\frac{CD}{AB} = 2014$, find $\frac{BC}{AD}$.

Answer: $\boxed{\sqrt{4027}}$ Let M be the midpoint of AB. Let $AM = MB = ED = a$, $ME = AD = b$, and $AE = BE = c$. Since $\triangle BEC \sim \triangle DEA$, but $\triangle BEC$ is not congruent to $\triangle DAE$, we must have $\triangle BEC \sim \triangle DEA$. Thus, $BC/BE = AD/DE = b/a$, so $BC = bc/a$, and $CE/EB = AE/ED = c/a$, so $EC = c^2/a$. We are given that $CD/AB = \frac{\frac{c^2}{a}+a}{2a} = \frac{c^2}{2a^2} + \frac{1}{2} = 2014 \Rightarrow \frac{c^2}{a^2} = 4027$. Thus, $BC/AD = \frac{bc/a}{b} = c/a = \sqrt{4027}$.

15. [**8**] Given a regular pentagon of area 1, a *pivot line* is a line not passing through any of the pentagon's vertices such that there are 3 vertices of the pentagon on one side of the line and 2 on the other. A *pivot point* is a point inside the pentagon with only finitely many non-pivot lines passing through it. Find the area of the region of pivot points.

Answer: $\boxed{\frac{1}{2}(7 - 3\sqrt{5})}$ Let the pentagon be labeled $ABCDE$. First, no pivot point can be on the same side of AC as vertex B. Any such point P has the infinite set of non-pivot lines within the hourglass shape formed by the acute angles between lines PA and PC. Similar logic can be applied to points on the same side of BD as C, and so on. The set of pivot points is thus a small pentagon with sides on AC, BD, CE, DA, EB. The side ratio of this small pentagon to the large pentagon is

$$(2\cos(72°))^2 = \frac{3 - \sqrt{5}}{2},$$

so the area of the small pentagon is

$$\left(\frac{3 - \sqrt{5}}{2}\right)^2 = \frac{1}{2}(7 - 3\sqrt{5}).$$

16. [**8**] Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

Answer: $\boxed{\frac{72+32\sqrt{2}}{7}}$ Let $u = x^2 + 2y^2$. By AM-GM, $u \geq \sqrt{8}xy$, so $xy \leq \dfrac{u}{\sqrt{8}}$. If we let $xy = ku$ where $k \leq \dfrac{1}{\sqrt{8}}$, then we have

$$u(1-k) = 8$$
$$u(1+k) = x^2 + xy + 2y^2$$

that is, $u(1+k) = 8 \cdot \dfrac{1+k}{1-k}$. It is not hard to see that the maximum value of this expression occurs at $k = \dfrac{1}{\sqrt{8}}$, so the maximum value is $8 \cdot \dfrac{1+\frac{1}{\sqrt{8}}}{1-\frac{1}{\sqrt{8}}} = \dfrac{72+32\sqrt{2}}{7}$.

17. [**11**] Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following conditions:

(a) $f(1) = 1$.

(b) $f(a) \leq f(b)$ whenever a and b are positive integers with $a \leq b$.

(c) $f(2a) = f(a) + 1$ for all positive integers a.

How many possible values can the 2014-tuple $(f(1), f(2), ..., f(2014))$ take?

Answer: $\boxed{1007}$ Note that $f(2014) = f(1007)+1$, so there must be exactly one index $1008 \leq i \leq 2014$ such that $f(i) = f(i-1)+1$, and for all $1008 \leq j \leq 2014, j \neq i$ we must have $f(j) = f(j-1)$. We first claim that each value of i corresponds to exactly one 2014-tuple $(f(1), \ldots, f(2014))$. To prove this, note that $f(1024) = 11$, so each i uniquely determines the values of $f(1007), \ldots, f(2014)$. Then all of $f(1), \ldots, f(1006)$ can be uniquely determined from these values because for any $1 \leq k \leq 1006$, there exists a unique n such that $1007 \leq k \cdot 2^n \leq 2014$. It's also clear that these values satisfy the condition that f is nondecreasing, so we have a correspondence from each $1008 \leq i \leq 2014$ to a unique 2014-tuple.

Also, given any valid 2014-tuple $(f(1), \ldots, f(2014))$, we know that $f(1), \ldots, f(1006)$ can be uniquely determined by $f(1007), \ldots, f(2014)$, which yields some $1008 \leq i \leq 2014$ where $f(i) = f(i-1)+1$, so we actually have a bijection between possible values of i and 2014-tuples. Therefore, the total number of possible 2014-tuples is 1007.

18. [**11**] Find the number of ordered quadruples of positive integers (a, b, c, d) such that $a, b, c,$ and d are all (not necessarily distinct) factors of 30 and $abcd > 900$.

Answer: $\boxed{1940}$ Since $abcd > 900 \iff \frac{30}{a}\frac{30}{b}\frac{30}{c}\frac{30}{d} < 900$, and there are $\binom{4}{2}^3$ solutions to $abcd = 2^2 3^2 5^2$, the answer is $\frac{1}{2}(8^4 - \binom{4}{2}^3) = 1940$ by symmetry.

19. [**11**] Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at E, the bisectors of $\angle ABC$ and $\angle BCD$ meet at F, the bisectors of $\angle BCD$ and $\angle CDA$ meet at G, and the bisectors of $\angle DAB$ and $\angle ABC$ meet at H. Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle ABH has area 25. Find the area of triangle CDG.

Answer: $\boxed{\frac{256}{7}}$ Let M, N be the midpoints of AD, BC respectively. Since AE and DE are bisectors of supplementary angles, the triangle AED is right with right angle E. Then EM is the median of a right triangle from the right angle, so triangles EMA and EMD are isosceles with vertex M. But then $\angle MEA = \angle EAM = \angle EAB$, so $EM \parallel AB$. Similarly, $FN \parallel BA$. Thus, both E and F are on the midline of this trapezoid. Let the length of EF be x. Triangle EFH has area 1 and is similar to triangle ABH, which has area 25, so $AB = 5x$. Then, letting the heights of trapezoids $EABF$ and $EDCF$ be h (they are equal since EF is on the midline), the area of trapezoid $EABF$ is $\frac{6xh}{2} = 24$. So the area of trapezoid $EDCF$ is $36 = \frac{9xh}{2}$. Thus $DC = 8x$. Then, triangle GEF is similar to and has $\frac{1}{64}$ times the area of triangle CDG. So the area of triangle CDG is $\frac{64}{63}$ times the area of quadrilateral $EDCF$, or $\frac{256}{7}$.

20. [**11**] A deck of 8056 cards has 2014 ranks numbered 1–2014. Each rank has four suits—hearts, diamonds, clubs, and spades. Each card has a rank and a suit, and no two cards have the same rank and the same suit. How many subsets of the set of cards in this deck have cards from an odd number of distinct ranks?

Answer: $\boxed{\frac{1}{2}(16^{2014} - 14^{2014})}$ There are $\binom{2014}{k}$ ways to pick k ranks, and 15 ways to pick the suits in each rank (because there are 16 subsets of suits, and we must exclude the empty one). We therefore want to evaluate the sum $\binom{2014}{1}15^1 + \binom{2014}{3}15^3 + \cdots + \binom{2014}{2013}15^{2013}$.

Note that $(1+15)^{2014} = 1 + \binom{2014}{1}15^1 + \binom{2014}{2}15^2 + \ldots + \binom{2014}{2013}15^{2013} + 15^{2014}$ and $(1-15)^{2014} = 1 - \binom{2014}{1}15^1 + \binom{2014}{2}15^2 - \ldots - \binom{2014}{2013}15^{2013} + 15^{2014}$, so our sum is simply $\frac{(1+15)^{2014} - (1-15)^{2014}}{2} = \frac{1}{2}(16^{2014} - 14^{2014})$.

21. [**14**] Compute the number of ordered quintuples of nonnegative integers $(a_1, a_2, a_3, a_4, a_5)$ such that $0 \le a_1, a_2, a_3, a_4, a_5 \le 7$ and 5 divides $2^{a_1} + 2^{a_2} + 2^{a_3} + 2^{a_4} + 2^{a_5}$.

Answer: $\boxed{6528}$ Let $f(n)$ denote the number of n-tuples $(a_1, \ldots, a_n)$ such that $0 \le a_1, \ldots, a_n \le 7$ and $5 \mid 2^{a_1} + \ldots + 2^{a_n}$. To compute $f(n+1)$ from $f(n)$, we note that given any n-tuple $(a_1, \ldots, a_n)$ such that $0 \le a_1, \ldots, a_n \le 7$ and $5 \nmid 2^{a_1} + \ldots + 2^{a_n}$, there are exactly two possible values for a_{n+1} such that $0 \le a_{n+1} \le 7$ and $5 \mid 2^{a_1} + \ldots + 2^{a_{n+1}}$, because $2^n \equiv 1, 2, 4, 3, 1, 2, 4, 3 \pmod 5$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$ respectively.

Also, given any valid $(n+1)$-tuple $(a_1, \ldots, a_{n+1})$, we can remove a_{n+1} to get an n-tuple $(a_1, \ldots, a_n)$ such that $0 \le a_1, \ldots, a_n \le 7$ and $5 \nmid 2^{a_1} + \ldots + 2^{a_n}$, so these are in bijection. There are a total of 8^n n-tuples, $f(n)$ of which satisfy $5 \mid 2^{a_1} + \ldots + 2^{a_n}$, so there are $8^n - f(n)$ for which $5 \nmid 2^{a_1} + \ldots + 2^{a_n}$. Therefore, $f(n+1) = 2(8^n - f(n))$.

We now have $f(1) = 0$, $f(2) = 2(8-0) = 16$, $f(3) = 2(64-16) = 96$, $f(4) = 2(512-96) = 832$, $f(5) = 2(4096 - 832) = 6528$.

22. [**14**] Let ω be a circle, and let $ABCD$ be a quadrilateral inscribed in ω. Suppose that BD and AC intersect at a point E. The tangent to ω at B meets line AC at a point F, so that C lies between E and F. Given that $AE = 6, EC = 4, BE = 2$, and $BF = 12$, find DA.

Answer: $\boxed{2\sqrt{42}}$ By power of a point, we have $ED \cdot EB = EA \cdot EC$, whence $ED = 12$. Additionally, by power of a point, we have $144 = FB^2 = FC \cdot FA = FC(FC+10)$, so $FC = 8$. Note that $\angle FBC = \angle FAB$ and $\angle CFB = \angle AFB$, so $\triangle FBC \sim \triangle FAB$. Thus, $AB/BC = FA/FB = 18/12 = 3/2$, so $AB = 3k$ and $BC = 2k$ for some k. Since $\triangle BEC \sim \triangle AED$, we have $AD/BC = AE/BE = 3$, so $AD = 3BC = 6k$. By Stewart's theorem on $\triangle EBF$, we have

$$(4)(8)(12) + (2k)^2(12) = (2)^2(8) + (12)^2(4) \implies 8 + k^2 = 8/12 + 12,$$

whence $k^2 = 14/3$. Thus,

$$DA = 6k = 6\sqrt{14/3} = 6\frac{\sqrt{42}}{3} = 2\sqrt{42}.$$

23. [**14**] Let $S = \{-100, -99, -98, \ldots, 99, 100\}$. Choose a 50-element subset T of S at random. Find the expected number of elements of the set $\{|x| : x \in T\}$.

Answer: $\boxed{\frac{8825}{201}}$ Let us solve a more generalized version of the problem: Let S be a set with $2n+1$ elements, and partition S into sets $A_0, A_1, \ldots, A_n$ such that $|A_0| = 1$ and $|A_1| = |A_2| = \cdots = |A_n| = 2$. (In this problem, we have $A_0 = \{0\}$ and $A_k = \{k, -k\}$ for $k = 1, 2, \ldots, 100$.) Let T be a randomly chosen m-element subset of S. What is the expected number of A_k's that have a representative in T?

For $k = 0, 1, \ldots, n$, let $w_k = 1$ if $T \cap A_k \ne \emptyset$ and 0 otherwise, so that the number of A_k's that have a representative in T is equal to $\sum_{k=0}^n w_k$. It follows that the expected number of A_k's that have a representative in T is equal to

$$\mathrm{E}[w_0 + w_1 + \cdots + w_n] = \mathrm{E}[w_0] + \mathrm{E}[w_1] + \cdots + \mathrm{E}[w_n] = \mathrm{E}[w_0] + n\,\mathrm{E}[w_1],$$

since $\mathrm{E}[w_1] = \mathrm{E}[w_2] = \cdots = \mathrm{E}[w_n]$ by symmetry.

Now $\mathrm{E}[w_0]$ is equal to the probability that $T \cap A_0 \neq \emptyset$, that is, the probability that the single element of A_0 is in T, which is $T|/|S| = m/(2n+1)$. Similarly, $\mathrm{E}[w_1]$ is the probability that $T \cap A_1 \neq \emptyset$, that is, the probability that at least one of the two elements of A_1 is in T. Since there are $\binom{2n-1}{m}$ m-element subsets of S that exclude both elements of A_1, and there are $\binom{2n+1}{m}$ m-element subsets of S in total, we have that

$$\mathrm{E}[w_1] = 1 - \frac{\binom{2n-1}{m}}{\binom{2n+1}{m}} = 1 - \frac{(2n-m)(2n-m+1)}{2n(2n+1)}.$$

Putting this together, we find that the expected number of A_k's that have a representative in T is

$$\frac{m}{2n+1} + n - \frac{(2n-m+1)(2n-m)}{2(2n+1)}.$$

In this particular problem, we have $n = 100$ and $m = 50$, so substituting these values gives our answer of $\boxed{\frac{8825}{201}}$.

24. [**14**] Let $A = \{a_1, a_2, \ldots, a_7\}$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of A is an integer. Find the smallest possible value of the sum of the elements in A.

Answer: $\boxed{1267}$ For $2 \leq i \leq 6$, we claim that $a_1 \equiv \ldots \equiv a_7 \pmod{i}$. This is because if we consider any $i-1$ of the 7 numbers, the other $7-(i-1) = 8-i$ of them must all be equal modulo i, because we want the sum of all subsets of size i to be a multiple of i. However, $8-i \geq 2$, and this argument applies to any $8-i$ of the 7 integers, so in fact all of them must be equal modulo i.

We now have that all of the integers are equivalent modulo all of $2, \ldots, 6$, so they are equivalent modulo 60, their least common multiple. Therefore, if the smallest integer is k, then the other 6 integers must be at least $k+60, k+60\cdot 2, \ldots, k+60\cdot 6$. This means the sum is $7k + 60\cdot 21 \geq 7 + 60\cdot 21 = 1267$. 1267 is achievable with $\{1, 1+60, \ldots, 1+60\cdot 6\}$, so it is the answer.

25. [**17**] Let ABC be an equilateral triangle of side length 6 inscribed in a circle ω. Let A_1, A_2 be the points (distinct from A) where the lines through A passing through the two trisection points of BC meet ω. Define B_1, B_2, C_1, C_2 similarly. Given that $A_1, A_2, B_1, B_2, C_1, C_2$ appear on ω in that order, find the area of hexagon $A_1A_2B_1B_2C_1C_2$.

Answer: $\boxed{\frac{846\sqrt{3}}{49}}$ Let A' be the point on BC such that $2BA' = A'C$. By law of cosines on triangle $AA'B$, we find that $AA' = 2\sqrt{7}$. By power of a point, $A'A_1 = \frac{2*4}{2\sqrt{7}} = \frac{4}{\sqrt{7}}$. Using side length ratios, $A_1A_2 = 2\frac{AA_1}{AA'} = 2\frac{2\sqrt{7}+\frac{4}{\sqrt{7}}}{2\sqrt{7}} = \frac{18}{7}$.

Now our hexagon can be broken down into equilateral triangle $A_1B_1C_1$ and three copies of triangle $A_1C_1C_2$. Since our hexagon has rotational symmetry, $\angle C_2 = 120$, and using law of cosines on this triangle with side lengths $\frac{18}{7}$ and 6, a little algebra yields $A_1C_2 = \frac{30}{7}$ (this is a 3-5-7 triangle with an angle 120).

The area of the hexagon is therefore $\frac{6^2\sqrt{3}}{4} + 3 * \frac{1}{2}\frac{18}{7}\frac{30}{7}\frac{\sqrt{3}}{2} = \frac{846\sqrt{3}}{49}$

26. [**17**] For $1 \leq j \leq 2014$, define

$$b_j = j^{2014} \prod_{i=1, i\neq j}^{2014} (i^{2014} - j^{2014})$$

where the product is over all $i \in \{1, \ldots, 2014\}$ except $i = j$. Evaluate

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{2014}}.$$

Answer: $\boxed{\frac{1}{2014!^{2014}}}$ We perform Lagrange interpolation on the polynomial $P(x) = 1$ through the points $1^{2014}, 2^{2014}, \ldots, 2014^{2014}$. We have

$$1 = P(x) = \sum_{j=1}^{2014} \frac{\prod_{i=1,i\neq j}^{2014}(x - i^{2014})}{\prod_{i=1,i\neq j}^{2014}(j^{2014} - i^{2014})}.$$

Thus,

$$1 = P(0) = \sum_{j=1}^{2014} \frac{((-1)^{2013})\frac{2014!^{2014}}{j^{2014}}}{(-1)^{2013}\prod_{i=1,i\neq j}^{2014}(i^{2014} - j^{2014})},$$

which equals

$$2014!^{2014} \sum_{j=1}^{2014} \frac{1}{j^{2014}\prod_{i=1,i\neq j}^{2014}(i^{2014} - j^{2014})} = 2014!^{2014}\left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{2014}}\right),$$

so the desired sum is $\frac{1}{2014!^{2014}}$.

27. [**17**] Suppose that $(a_1, \ldots, a_{20})$ and $(b_1, \ldots, b_{20})$ are two sequences of integers such that the sequence $(a_1, \ldots, a_{20}, b_1, \ldots, b_{20})$ contains each of the numbers $1, \ldots, 40$ exactly once. What is the maximum possible value of the sum

$$\sum_{i=1}^{20}\sum_{j=1}^{20} \min(a_i, b_j)?$$

Answer: $\boxed{5530}$ Let x_k, for $1 \le k \le 40$, be the number of integers i with $1 \le i \le 20$ such that $a_i \ge k$. Let y_k, for $1 \le k \le 40$, be the number of integers j with $1 \le j \le 20$ such that $b_j \ge k$. It follows from the problem statement that $x_k + y_k$ is the number of elements of the set $\{1, \ldots, 40\}$ which are greater than or equal to 40, which is just $41 - k$.

Note that if $1 \le i, j \le 20$, and $1 \le k \le 40$, then $\min(a_i, b_j) \ge k$ if and only if $a_i \ge k$ and $b_j \ge k$. So for a fixed k with $1 \le k \le 40$, the number of pairs (i, j) with $1 \le i, j \le 20$ such that $\min(a_i, b_j) \ge k$ is equal to $x_k y_k$. So we can rewrite

$$\sum_{i=1}^{20}\sum_{j=1}^{20} \min(a_i, b_j) = \sum_{k=1}^{40} x_k y_k.$$

Since $x_k + y_k = 41 - k$ for $1 \le k \le 40$, we have

$$x_k y_k \le \left\lfloor \frac{41-k}{2} \right\rfloor \left\lceil \frac{41-k}{2} \right\rceil$$

by a convexity argument. So

$$\sum_{i=1}^{20}\sum_{j=1}^{20} \min(a_i, b_j) \le \sum_{k=1}^{40} \left\lfloor \frac{41-k}{2} \right\rfloor \left\lceil \frac{41-k}{2} \right\rceil = 5530.$$

Equality holds when $(a_1, \ldots, a_{20}) = (2, 4, \ldots, 38, 40)$ and $(b_1, \ldots, b_{20}) = (1, 3, \ldots, 37, 39)$.

28. [**17**] Let $f(n)$ and $g(n)$ be polynomials of degree 2014 such that $f(n) + (-1)^n g(n) = 2^n$ for $n = 1, 2, \ldots, 4030$. Find the coefficient of x^{2014} in $g(x)$.

Answer: $\boxed{\frac{3^{2014}}{2^{2014} \cdot 2014!}}$ Define the polynomial functions h_1 and h_2 by $h_1(x) = f(2x) + g(2x)$ and $h_2(x) = f(2x-1) - g(2x-1)$. Then, the problem conditions tell us that $h_1(x) = 2^{2x}$ and $h_2(x) = 2^{2x-1}$ for $x = 1, 2, \ldots, 2015$.

By the Lagrange interpolation formula, the polynomial h_1 is given by

$$h_1(x) = \sum_{i=1}^{2015} 2^{2i} \prod_{\substack{j=1 \\ i \neq j}}^{2015} \frac{x-j}{i-j}.$$

So the coefficient of x^{2014} in $h_1(x)$ is

$$\sum_{i=1}^{2015} 2^{2i} \prod_{\substack{j=1 \\ i \neq j}}^{2015} \frac{1}{i-j} = \frac{1}{2014!} \sum_{i=1}^{2015} 2^{2i}(-1)^{2015-i} \binom{2014}{i-1} = \frac{4 \cdot 3^{2014}}{2014!}$$

where the last equality follows from the binomial theorem. By a similar argument, the coefficient of x^{2014} in $h_2(x)$ is $\frac{2 \cdot 3^{2014}}{2014!}$.

We can write $g(x) = \frac{1}{2}(h_1(x/2) - h_2((x+1)/2))$. So, the coefficient of x^{2014} in $g(x)$ is

$$\frac{1}{2}\left(\frac{4 \cdot 3^{2014}}{2^{2014} \cdot 2014!} - \frac{2 \cdot 3^{2014}}{2^{2014} \cdot 2014!}\right) = \frac{3^{2014}}{2^{2014} \cdot 2014!}.$$

29. **[20]** Natalie has a copy of the unit interval $[0,1]$ that is colored white. She also has a black marker, and she colors the interval in the following manner: at each step, she selects a value $x \in [0,1]$ uniformly at random, and

(a) If $x \leq \frac{1}{2}$ she colors the interval $[x, x+\frac{1}{2}]$ with her marker.

(b) If $x > \frac{1}{2}$ she colors the intervals $[x, 1]$ and $[0, x-\frac{1}{2}]$ with her marker.

What is the expected value of the number of steps Natalie will need to color the entire interval black?

Answer: $\boxed{5}$ The first choice always wipes out half the interval. So we calculate the expected value of the amount of time needed to wipe out the other half.

Solution 1 (non-calculus):

We assume the interval has $2n$ points and we start with the last n colored black. We let $f(k)$ be the expected value of the number of turns we need if there are k white points left. So we must calculate $f(n)$.

We observe that

$$f(k) = 1 + \frac{(n-k+1) \cdot 0 + (n-k+1) \cdot f(k) + 2\sum_{i=1}^{k-1} f(i)}{2n}$$

$$f(k)\frac{n+k-1}{2n} = 1 + \frac{\sum_{i=1}^{k-1} f(i)}{n}$$

$$f(k+1)\frac{n+k}{2n} = 1 + \frac{\sum_{i=1}^{k} f(i)}{n}$$

$$f(k+1) = f(k)\frac{n+k+1}{n+k}$$

$$f(k) = f(1)\frac{n+k}{n+1}$$

And note that $f(1) = 2$ so $f(n) = \frac{4n}{n+1}$ and $\lim_{n\to\infty} f(n) = 4$.

Therefore adding the first turn, the expected value is 5.

Solution 2 (calculus):

We let $f(x)$ be the expected value with length x uncolored. Like above, $\lim_{x\to 0} f(x) = 2$.

Similarly we have the recursion

$$f(x) = 1 + (\frac{1}{2} - x)f(x) + 2\int_0^x f(y)dy$$

$$f'(x) = 0 + \frac{1}{2}f'(x) - f(x) - xf'(x) + 2f(x)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x + \frac{1}{2}}$$

And solving yields $f(x) = c(\frac{1}{2} + x)$ and since $\lim_{x\to 0} f(x) = 2$, $c = 4$. So $f(x) = 2 + 4x$ and $f(\frac{1}{2}) = 4$. Therefore adding the first turn, our expected value is 5.

30. [**20**] Let ABC be a triangle with circumcenter O, incenter I, $\angle B = 45°$, and $OI \parallel BC$. Find $\cos\angle C$.

Answer: $\boxed{1 - \frac{\sqrt{2}}{2}}$ Let M be the midpoint of BC, and D the foot of the perpendicular of I with BC. Because $OI||BC$, we have $OM = ID$. Since $\angle BOC = 2\angle A$, the length of OM is $OA\cos\angle BOM = OA\cos A = R\cos A$, and the length of ID is r, where R and r are the circumradius and inradius of $\triangle ABC$, respectively.

Thus, $r = R\cos A$, so $1 + \cos A = (R + r)/R$. By Carnot's theorem, $(R + r)/R = \cos A + \cos B + \cos C$, so we have $\cos B + \cos C = 1$. Since $\cos B = \frac{\sqrt{2}}{2}$, we have $\cos C = 1 - \frac{\sqrt{2}}{2}$.

31. [**20**] Compute

$$\sum_{k=1}^{1007} \left(\cos\left(\frac{\pi k}{1007}\right)\right)^{2014}.$$

Answer: $\boxed{\frac{2014(1+\binom{2013}{1007})}{2^{2014}}}$ Let $\omega = e^{\frac{2\pi i}{2014}}$. We have $\omega^{2014} = 1$. Note that $\cos(\frac{\pi k}{1007}) = \frac{1}{2}(\omega^k + \omega^{-k})$. Our desired expression is

$$\frac{1}{2^{2014}} \sum_{k=1}^{1007} (\omega^k + \omega^{-k})^{2014}$$

Using binomial expansion and switching the order of the resulting summation, this is equal to

$$\frac{1}{2^{2014}} \sum_{j=0}^{2014} \binom{2014}{j} \sum_{k=1}^{1007} (\omega^{2014-2j})^k$$

.

Note that unless $\omega^{2014-2j} = 1$, the summand

$$\sum_{k=1}^{1007} (\omega^{2014-2j})^k$$

is the sum of roots of unity spaced evenly around the unit circle in the complex plane (in particular the 1007th, 19th, and 53rd roots of unity), so it is zero. Thus, we must only sum over those j for which $\omega^{2014-2j} = 1$, which holds for $j = 0, 1007, 2014$. This yields the answer

$$\frac{1}{2^{2014}}\left(1007 + 1007\binom{2014}{1007} + 1007\right) = \frac{2014\left(1 + \binom{2013}{1007}\right)}{2^{2014}}.$$

32. [**20**] Find all ordered pairs (a, b) of complex numbers with $a^2 + b^2 \neq 0$, $a + \frac{10b}{a^2+b^2} = 5$, and $b + \frac{10a}{a^2+b^2} = 4$.

Answer: $\boxed{(1, 2), (4, 2), (\frac{5}{2}, 2 \pm \frac{3}{2}i)}$ **Solution 1.** First, it is easy to see that $ab \neq 0$. Thus, we can write

$$\frac{5 - a}{b} = \frac{4 - b}{a} = \frac{10}{a^2 + b^2}.$$

Then, we have

$$\frac{10}{a^2+b^2} = \frac{4a-ab}{a^2} = \frac{5b-ab}{b^2} = \frac{4a+5b-2ab}{a^2+b^2}.$$

Therefore, $4a+5b-2ab = 10$, so $(2a-5)(b-2) = 0$. Now we just plug back in and get the four solutions: $(1,2), (4,2), (\frac{5}{2}, 2 \pm \frac{3}{2}i)$. It's not hard to check that they all work.

Solution 2. The first equation plus i times the second yields $5+4i = a+bi+\frac{10(b+ai)}{a^2+b^2} = a+bi-\frac{10i}{a+bi}$, which is equivalent to $a+bi = \frac{(5\pm 3)+4i}{2}$ by the quadratic formula.

Similarly, the second equation plus i times the first yields $4+5i = b+ai-\frac{10i}{b+ai}$, which is equivalent to $b+ai = \frac{4+(5\pm 3)i}{2}$.

Letting $\epsilon_1, \epsilon_2 \in \{-1,1\}$ be the signs in $a+bi$ and $b+ai$, we get $(a,b) = \frac{1}{2}(a+bi, b+ai) - \frac{1}{2}i(b+ai, a+bi) = (\frac{10+(\epsilon_1+\epsilon_2)3}{4}, \frac{8+(\epsilon_2-\epsilon_1)3i}{4})$.

Comment. Many alternative approaches are possible. For instance, $\frac{5-a}{b} = \frac{4-b}{a} \implies b-2 = \epsilon\sqrt{(a-1)(a-4)}$ for some $\epsilon \in \{-1,1\}$, and substituting in and expanding gives $0 = (-2a^2+5a)\epsilon\sqrt{(a-1)(a-4)}$.

More symmetrically, we may write $a = \lambda(4-b)$, $b = \lambda(5-a)$ to get $(a,b) = \frac{\lambda}{1-\lambda^2}(4-5\lambda, 5-4\lambda)$, and then plug into $a^2+b^2 = 10\lambda$ to get $0 = 10(\lambda^4+1) - 41(\lambda^3+\lambda) + 60\lambda^2 = (\lambda-2)(2\lambda-1)(5\lambda^2-8\lambda+5)$.

33. **[25]** An *up-right path* from $(a,b) \in \mathbb{R}^2$ to $(c,d) \in \mathbb{R}^2$ is a finite sequence $(x_1,y_1), \ldots, (x_k,y_k)$ of points in $\mathbb{R}^2$ such that $(a,b) = (x_1,y_1)$, $(c,d) = (x_k,y_k)$, and for each $1 \le i < k$ we have that either $(x_{i+1}, y_{i+1}) = (x_i+1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i+1)$.

Let S be the set of all up-right paths from $(-400,-400)$ to $(400,400)$. What fraction of the paths in S do not contain any point (x,y) such that $|x|, |y| \le 10$? Express your answer as a decimal number between 0 and 1.

If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1-10|C-A|), 0\} \rceil$.

Answer: $\boxed{0.2937156494680644\ldots}$ Note that any up-right path must pass through exactly one point of the form $(n,-n)$ (i.e. a point on the upper-left to lower-right diagonal), and the number of such paths is $\binom{800}{400-n}^2$ because there are $\binom{800}{400-n}$ up-right paths from $(-400,-400)$ to $(n,-n)$ and another $\binom{800}{400-n}$ from $(n,-n)$ to $(400,400)$. An up-right path contains a point (x,y) with $|x|,|y| \le 10$ if and only if $-10 \le n \le 10$, so the probability that this happens is

$$\frac{\sum_{n=-10}^{10} \binom{800}{400-n}^2}{\sum_{n=-400}^{400} \binom{800}{400-n}^2} = \frac{\sum_{n=-10}^{10} \binom{800}{400-n}^2}{\binom{1600}{800}}$$

To estimate this, recall that if we normalize $\binom{800}{n}$ to be a probability density function, then it will be approximately normal with mean 400 and variance $800 \cdot \frac{1}{4} = 200$. If this is squared, then it is proportional to a normal distribution with half the variance and the same mean, because the probability density function of a normal distribution is proportional to $e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ is the mean and σ^2 is the variance.

Therefore, the $\binom{800}{n}^2$ probability density function is roughly proportional to a normal distribution with mean 400 and variance 100, or standard deviation 10. So $\sum_{n=-10}^{10} \binom{800}{400-n}^2$ represents roughly one standard deviation. Recall that approximately 68 percent of a normal distribution lies within one standard deviation of the mean (look up the 68-95-99.7 rule to read more), so a good guess would be around .32. This guess can be improved by noting that we're actually summing 21 values instead of 20, so you'd have approximately $.68 \cdot \frac{21}{20} \approx .71$ of the normal distribution, giving an answer of .29.

34. **[25]** Consider a number line, with a lily pad placed at each integer point. A frog is standing at the lily pad at the point 0 on the number line, and wants to reach the lily pad at the point 2014 on the number line. If the frog stands at the point n on the number line, it can jump directly to either point

$n+2$ or point $n+3$ on the number line. Each of the lily pads at the points $1, \cdots, 2013$ on the number line has, independently and with probability $1/2$, a snake. Let p be the probability that the frog can make some sequence of jumps to reach the lily pad at the point 2014 on the number line, without ever landing on a lily pad containing a snake. What is $p^{1/2014}$? Express your answer as a decimal number.

If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1-20|C-A|), 0\} \rceil$.

Answer: $\boxed{0.9102805441016536}$

First, we establish a rough upper bound for the probability p. Let q be the probability that the frog can reach the lily pad at the point 2014 on the number line if it is allowed to jump from a point n on the number line to the point $n+1$, in addition to the points $n+2$ and $n+3$. Clearly, $p \leq q$. Furthermore, p is approximated by q; it should be easy to convince one's self that jumps from a point n to the point $n+1$ are only useful for reaching the lily pad at point 2014 in very few situations.

Now we compute q. We note that, if the frog can jump from points n to points $n+1$, $n+2$, and $n+3$, then it can reach the lily pad at the point 2014 on the number line if and only if each snake-free lily pad is at most 3 units away from the closest snake-free lily pad on the left.

Define the sequence $\{a_m\}_{m=1}^{\infty}$ by $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, and $a_{m+3} = a_{m+2} + a_{m+1} + a_m$ for $m \geq 0$. Then, it can be shown by induction that a_m is the number of possible arrangements of snakes on lily pads at points $1, \cdots, m-1$ so that the frog can make some sequence of jumps (of size 1, 2, or 3) from the lily pad at point 0 to the lily pad at point m without landing on a lily pad containing a snake. It follows that $q = a_{2014}/2^{2013}$. So

$$p^{1/2014} \approx q^{1/2014} = (a_{2014})^{1/2014}/2^{2013/2014} \approx (a_{2014})^{1/2014}/2.$$

Analyzing the recurrence relation $a_{m+3} = a_{m+2} + a_{m+1} + a_m$ yields that $(a_{2014})^{1/2014}$ is approximately equal to the largest real root r of the characteristic polynomial equation $r^3 - r^2 - r - 1 = 0$. So to roughly approximate p, it suffices to find the largest real root of this equation.

For this, we apply Newton's method, or one of many other methods for computing the roots of a polynomial. With an initial guess of 2, one iteration of Newton's method yields $r \approx 13/7$, so $p \approx r/2 \approx 13/14 \approx 0.928571$. A second iteration yields $r \approx 1777/966$, so $p \approx r/2 \approx 1777/1932 \approx 0.919772$. (It turns out that the value of r is $1.839286\ldots$, yielding $p \approx r/2 = 0.919643\ldots$.)

Using tools from probability theory, we can get an even better estimate for p. We model the problem using a discrete-time Markov chain. The state of the Markov chain at time n, for $n = 0, 1, \ldots, 2013$, indicates which of the lily pads at positions $n-2, n-1, n$ are reachable by the frog. It is clear that the state of the Markov chain at time n only depends (randomly) on its state at time $n-1$. There are $2^3 = 8$ possible states for this Markov chain, because each of the lily pads at positions $n-2, n-1, n$ can be either reachable or unreachable by the frog. Number each state using the number $1 + d_2 + 2d_1 + 4d_0$, where d_i is 1 if the lily pad at point $n-i$ is reachable, and 0 otherwise. So, for example, at time $n = 0$, the lily pad at point n is reachable ($d_0 = 1$) whereas the lily pads at points $n-1$ and $n-2$ are unreachable ($d_1 = d_2 = 0$), so the Markov chain is in state number $1 + d_2 + 2d_1 + 4d_0 = 5$.

The transition matrix M for the Markov chain can now be computed directly from the conditions of the problem. It is equal to

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

(The verification of this transition matrix is left as an exercise for the reader.) So the state vector v for the Markov chain at time 2013 is

$$v := M^{2014}[0, 1, 0, 0, 0, 0, 0, 0]^t.$$

Now, the lily pad at point 2014 is reachable by the frog if and only if the Markov chain is in state $3, 4, 5, 6, 7$, or 8 at time 2013. This happens with probability

$$p = [0, 0, 1, 1, 1, 1, 1, 1]\, v.$$

By expanding $[0, 1, 0, 0, 0, 0, 0, 0]^t$ in an eigenbasis for M, we find that $p^{1/2014}$ is approximately equal to the second-largest real eigenvalue of the matrix M. The characteristic polynomial of M is

$$\det(\lambda I - M) = -\frac{\lambda^3}{8} + \frac{3\lambda^4}{8} + \frac{\lambda^6}{4} - \frac{3\lambda^7}{2} + \lambda^8,$$

so its eigenvalues are the roots of this polynomial. The largest real root of this characteristic polynomial is $\lambda = 1$, and the second-largest real root is $0.9105247383471604\ldots$ (which can be found, again, using Newton's method, after factoring out $(\lambda - 1)\lambda^3$ from the polynomial), which is a good approximation for p.

35. [**25**] How many times does the letter "e" occur in all problem statements in this year's HMMT February competition?

If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1 - |\log_2(C/A)|), 0\} \rceil$.

Answer: $\boxed{1661}$ It is possible to arrive at a good estimate using Fermi estimation. See `http://en.wikipedia.org/wiki/Fermi_problem` for more details.

For example, there are 76 problems on the HMMT this year. You might guess that the average number of words in a problem is approximately 40, and the average number of of letters in a word is about 5. The frequency of the letter "e" in the English language is about 10%, resulting in an estimate of

$$76 \cdot 40 \cdot 5 \cdot 0.1 = 1520.$$

This is remarkably close to the actual answer.

36. [**25**] We have two concentric circles C_1 and C_2 with radii 1 and 2, respectively. A random chord of C_2 is chosen. What is the probability that it intersects C_1?

Your answer to this problem must be expressed in the form $\frac{m}{n}$, where m and n are positive integers. If your answer is in this form, your score for this problem will be $\lfloor \frac{25 \cdot X}{Y} \rfloor$, where X is the total number of teams who submit the answer $\frac{m}{n}$ (including your own team), and Y is the total number of teams who submit a valid answer. Otherwise, your score is 0. (Your answer is *not* graded based on correctness, whether your fraction is in lowest terms, whether it is at most 1, etc.)

Answer: $\boxed{\text{N/A}}$ The question given at the beginning of the problem statement is a famous problem in probability theory widely known as Bertrand's paradox. Depending on the interpretation of the phrase "random chord," there are at least three different possible answers to this question:

- If the random chord is chosen by choosing two (uniform) random endpoints on circle C_2 and taking the chord joining them, the answer to the question is $1/3$.
- If the random chord is chosen by choosing a (uniformly) random point P the interior of C_2 (other than the center) and taking the chord with midpoint P, the answer to the question becomes $1/4$.
- If the random chord is chosen by choosing a (uniformly) random diameter d of C, choosing a point P on d, and taking the chord passing through P and perpendicular to d, the answer to the question becomes $1/2$. (This is also the answer resulting from taking a uniformly random *horizontal* chord of C_2.)

You can read more about Bertrand's paradox online at `http://en.wikipedia.org/wiki/Bertrand_paradox_(probability)`. We expect that many of the valid submissions to this problem will be equal to $1/2, 1/3$, or $1/4$.

However, your score on this problem is not based on correctness, but is rather proportional to the number of teams who wrote the same answer as you! Thus, this becomes a problem of finding what is known in game theory as the "focal point," or "Schelling point." You can read more about focal points at `http://en.wikipedia.org/wiki/Focal_point_(game_theory)` or in economist Thomas Schelling's book *The Strategy Of Conflict*.