HMMT November 2021

November 13, 2021

General Round

1. A domino has a left end and a right end, each of a certain color. Alice has four dominos, colored red-red, red-blue, blue-red, and blue-blue. Find the number of ways to arrange the dominos in a row end-to-end such that adjacent ends have the same color. The dominos cannot be rotated.

   Proposed by: Sean Li

   Answer: $\boxed{4}$

   Solution: Without loss of generality assume that the the left end of the first domino is red. Then, we have two cases:

   If the first domino is red-red, this forces the second domino to be red-blue. The third domino cannot be blue-red, since the fourth domino would then be forced to be blue-blue, which is impossible. However, RR RB BB BR works.

   If the first domino is red-blue, then the second domino cannot be blue-red, since otherwise there is nowhere for the blue-blue domino to go. Therefore, the second domino is blue-blue, which forces the third to be blue-red, and forces the fourth to the red-red. This yields one possibility.

   Therefore, if the first color is red, there are 2 possibilities. We multiply by 2 to yield 4 total possibilities.

2. Suppose a and b are positive integers for which $8a^ab^b = 27a^bb^a$. Find $a^2 + b^2$.

   Proposed by: Sean Li

   Answer: $\boxed{117}$

   Solution: We have

$$8a^ab^b = 27a^bb^a \iff \frac{a^ab^b}{a^bb^a} = \frac{27}{8} \iff \frac{a^{a-b}}{b^{a-b}} = \frac{27}{8} \iff \left(\frac{a}{b}\right)^{a-b} = \frac{27}{8}.$$

   Since $27 = 3^3$ and $8 = 2^3$, there are only four possibilities:

   - $a/b = 3/2$ and $a - b = 3$, which yields $a = 9$ and $b = 6$;
   - $a/b = 27/8$ and $a - b = 1$, which yields no solutions;
   - $a/b = 2/3$ and $a - b = -3$, which yields $a = 6$ and $b = 9$;
   - $a/b = 8/27$ and $a - b = -1$, which yields no solutions.

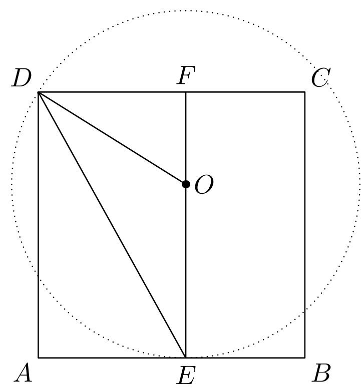
   Therefore $a^2 + b^2$ must equal $6^2 + 9^2 = 117$.

3. Let $ABCD$ be a unit square. A circle with radius $\frac{32}{49}$ passes through point D and is tangent to side AB at point E. Then $DE = \frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find $100m + n$.

   Proposed by: David Vulakh

   Answer: $\boxed{807}$

   Solution:



Let O be the center of the circle and let F be the intersection of lines OE and CD. Also let $r = 32/49$ and $x = DF$. Then we know

$$x^2 + (1-r)^2 = DF^2 + OF^2 = DO^2 = r^2,$$

which implies that $x^2 + 1 - 2r = 0$, or $1 + x^2 = 2r$. Now,

$$DE = \sqrt{DF^2 + EF^2} = \sqrt{1 + x^2} = \sqrt{2r} = \sqrt{64/49} = 8/7.$$

4. The sum of the digits of the time 19 minutes ago is two less than the sum of the digits of the time right now. Find the sum of the digits of the time in 19 minutes. (Here, we use a standard 12-hour clock of the form hh:mm.)

   Proposed by: Holden Mui

   Answer: $\boxed{11}$

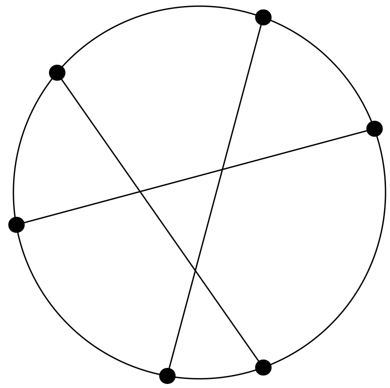
   Solution: Let's say the time 19 minutes ago is h hours and m minutes, so the sum of the digits is equivalent to $h + m$ mod 9. If $m \leq 40$, then the time right now is h hours and $m + 19$ minutes, so the sum of digits is equivalent mod 9 to $h + m + 19 \equiv h + m + 1 \pmod 9$, which is impossible. If $m > 40$ and $h < 12$, then the time right now is $h + 1$ hours and $m - 41$ minutes, so the sum of digits is equivalent to $h + m - 40 \equiv h + m + 5 \pmod 9$, which is again impossible. Therefore, we know that $h = 12$ and $m > 40$. Now, the sum of the digits 19 minutes ago is $3 + s(m)$, where $s(n)$ is the sum of the digits of n. On the other hand, the sum of the digits now is $1 + s(m - 41)$, meaning that $4 + s(m) = s(m - 41)$. The only m that satisfies this is $m = 50$, so the time right now is 1:09. In 19 minutes, the time will be 1:28, so the answer is 11.

5. A chord is drawn on a circle by choosing two points uniformly at random along its circumference. This is done two more times to obtain three total random chords. The circle is cut along these three lines, splitting it into pieces. The probability that one of the pieces is a triangle is $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find $100m + n$.

   Proposed by: Gabriel Wu

   Answer: $\boxed{115}$

   Solution: Instead of choosing three random chords, we instead first choose 6 random points on the circle and then choosing a random pairing of the points into 3 pairs with which to form chords. If the chords form a triangle, take a chord C. Any other chord C' must have its endpoints on different sides of C, since C and C' intersect. Therefore, the endpoints of C must be points that are opposite each other in the circle:



Conversely, if each point is connected to its opposite, the chords form a triangle unless these chords happen to be concurrent, which happens with probability 0. Therefore, out of the pairings, there is, almost always, exactly only one pairing that works. Since there are $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15$ ways to pair 6 points into three indistinguishable pairs, the probability is $1/15$.

6. Mario has a deck of seven pairs of matching number cards and two pairs of matching Jokers, for a total of 18 cards. He shuffles the deck, then draws the cards from the top one by one until he holds a pair of matching Jokers. The expected number of complete pairs that Mario holds at the end (including the Jokers) is $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find $100m + n$.

   Proposed by: Sean Li

   Answer: $\boxed{1003}$

   Solution: Considering ordering the nine pairs by the time they are first complete. Since the pairs are treated equally by the drawing process, this ordering is a uniform ordering. Therefore the problem becomes the following: consider ordering 7 N's and 2 J's randomly. What is the expected position of the first J?

   We may solve this by linearity of expectation. Every N has exactly a $1/3$ chance of being in front of the 2 J's, so the expected number of N's before the first J is $7/3$. Thus the expected position of the first J is $7/3 + 1 = 10/3$.

7. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{x^2+\sqrt{x^4+2x}} & \text{if } x \notin (-\sqrt[3]{2}, 0] \\ 0 & \text{otherwise} \end{cases}.$$

   The sum of all real numbers x for which $f^{10}(x) = 1$ can be written as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers, d is positive, c is square-free, and $\gcd(a, b, d) = 1$. Find $1000a + 100b + 10c + d$.

   (Here, $f^n(x)$ is the function $f(x)$ iterated n times. For example, $f^3(x) = f(f(f(x)))$.)

   Proposed by: Sean Li

   Answer: $\boxed{932}$

   Solution: If $x \in (-\sqrt[3]{2}, 0]$, it is evidently not a solution, so let us assume otherwise. Then, we find

$$f(x) = \frac{\sqrt{x^4 + 2x} - x^2}{2x},$$

   which implies that $xf(x)^2 + x^2 f(x) - 1/2 = 0$, by reverse engineering the quadratic formula. Therefore, if $x > 0$, $f(x)$ is the unique positive real t so that $xt^2 + x^2 t = 1/2$. However, then x is the unique positive real so that $xt^2 + x^2 t = 1/2$, so $f(t) = x$. This implies that if $x > 0$, then $f(f(x)) = x$.

Suppose that $f^{10}(x) = 1$. Then, since $f(x) > 0$, we find that $f(x) = f^{10}(f(x)) = f^{11}(x) = f(1)$. Conversely, if $f(x) = f(1)$, then $f^{10}(x) = f^9(f(x)) = f^9(f(1)) = 1$, so we only need to solve $f(x) = f(1)$.

This is equivalent to

$$x^2 + \sqrt{x^4 + 2x} = 1 + \sqrt{3} \iff \sqrt{x^4 + 2x} = 1 + \sqrt{3} - x^2 \implies x^4 + 2x = x^4 - 2(1+\sqrt{3})x^2 + (1+\sqrt{3})^2,$$

which is equivalent to

$$2(1+\sqrt{3})x^2 + 2x - (1+\sqrt{3})^2 = 0.$$

Obviously, if $x = 1$ then $f(x) = f(1)$, so we already know 1 is a root. This allows us to easily factor the quadratic and find that the other root is $-\frac{1+\sqrt{3}}{2}$. This ends up not being extraneous—perhaps the shortest way to see this is to observe that if $x = -\frac{1+\sqrt{3}}{2}$,

$$1 + \sqrt{3} - x^2 = (1+\sqrt{3})\left(1 - \frac{1+\sqrt{3}}{4}\right) > 0,$$

so since we already know

$$x^4 + 2x = (1 + \sqrt{3} - x^2)^2,$$

we have

$$\sqrt{x^4 + 2x} = 1 + \sqrt{3} - x^2.$$

Therefore, the sum of solutions is $\frac{1-\sqrt{3}}{2}$.

8. Eight points are chosen on the circumference of a circle, labelled $P_1, P_2, \ldots, P_8$ in clockwise order. A *route* is a sequence of at least two points $P_{a_1}, P_{a_2}, \ldots, P_{a_n}$ such that if an ant were to visit these points in their given order, starting at P_{a_1} and ending at P_{a_n}, by following $n-1$ straight line segments (each connecting each P_{a_i} and $P_{a_{i+1}}$), it would never visit a point twice or cross its own path. Find the number of routes.

   Proposed by: Gabriel Wu

   Answer: $\boxed{8744}$

   Solution 1: How many routes are there if we are restricted to n available points, and we must use all n of them? The answer is $n2^{n-2}$: first choose the starting point, then each move after that must visit one of the two neighbors of your expanding region of visited points (doing anything else would prevent you from visiting every point). Now simply sum over all possible sets of points that you end up visiting: $\binom{8}{8}(8 \cdot 2^6) + \binom{8}{7}(7 \cdot 2^5) + \cdots + \binom{8}{2}(2 \cdot 2^0) = 8744$.

   Solution 2: We use recursion. Let $f(n)$ be the answer for n points, with the condition that our path must start at P_n (so our final answer is $8f(8)$). Then $f(1) = 0$ and $f(2) = 1$.

   Now suppose $n \geq 3$ and suppose the second point we visit is P_i ($1 \leq i < n$). Then we can either stop the path there, yielding one possibility. Alternatively, we can continue the path. In this case, note that it may never again cross the chord P_iP_n. If the remainder of the path is among the points $P_1, \ldots, P_i$, there are $f(i)$ possible routes. Otherwise, there are $f(n-i)$ possible routes. As a result,

$$f(n) = \sum_{i=1}^{n-1} 1 + f(i) + f(n-i) = (n-1) + 2\sum_{i=1}^{n-1} f(i).$$

   From here we may compute:

n	1	2	3	4	5	6	7	8
$f(n)$	0	1	4	13	40	121	364	1093

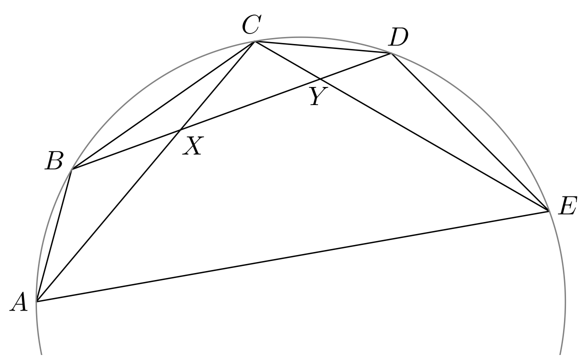
Therefore the answer is $8 \cdot 1093 = 8744$.

9. $ABCDE$ is a cyclic convex pentagon, and $AC = BD = CE$. AC and BD intersect at X, and BD and CE intersect at Y. If $AX = 6$, $XY = 4$, and $YE = 7$, then the area of pentagon $ABCDE$ can be written as $\frac{a\sqrt{b}}{c}$, where a, b, c are integers, c is positive, b is square-free, and $\gcd(a, c) = 1$. Find $100a + 10b + c$.

   Proposed by: Eric Shen

   Answer: $\boxed{2852}$

   Solution:



   Since $AC = BD$, $ABCD$ is an isosceles trapezoid. Similarly, $BCDE$ is also an isosceles trapezoid. Using this, we can now calculate that $CY = DY = DX - XY = AX - XY = 2$, and similarly $BX = CX = 3$. By applying Heron's formula we find that the area of triangle CXY is $\frac{3}{4}\sqrt{15}$.

   Now, note that
$$[ABC] = \frac{AC}{CX}[BXC] = 3[BXC] = 3\frac{XY}{BX}[CXY] = \frac{9}{4}[CXY].$$
   Similarly, $[CDE] = \frac{9}{4}[CXY]$. Also,
$$[ACE] = \frac{CA \cdot CE}{CX \cdot CY}[CXY] = \frac{81}{6}[CXY] = \frac{27}{2}[CXY].$$
   Thus, $[ABCDE] = (9/4 + 9/4 + 27/2)[CXY] = 18[CXY] = \frac{27}{2}\sqrt{15}$.

10. Real numbers x, y, z satisfy
$$x + xy + xyz = 1, \quad y + yz + xyz = 2, \quad z + xz + xyz = 4.$$
   The largest possible value of xyz is $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers, d is positive, c is square-free, and $\gcd(a, b, d) = 1$. Find $1000a + 100b + 10c + d$.

   Proposed by: Sean Li

   Answer: $\boxed{5272}$

   Solution 1: Let $p = xyz$ and $q = (x + 1)(y + 1)(z + 1)$. Then, we get
$$pq = [x(1 + y)] \cdot [y(1 + z)] \cdot [z(1 + x)] = (1 - p)(2 - p)(4 - p).$$
   Additionally, note that
$$q - p = xy + yz + zx + x + y + z + 1 = (x + xy) + (y + yz) + (z + xz) + 1 = 8 - 3p.$$

Therefore, we have $q = 8 - 2p$. Substituting this into our earlier equation gives us

$$p(8-2p) = (1-p)(2-p)(4-p).$$

We can rearrange this to get $(4-p)(2-5p+p^2) = 0$. Solving this gives us $p = 4, \frac{5\pm\sqrt{17}}{2}$. Thus, our maximum solution is $\frac{5+\sqrt{17}}{2}$, which yields an answer of 5272. To show that such a solution exists, see Solution 2.

Solution 2: Let $r = xyz - 1$. Observe that

$$rx = x(y + yz + xyz) - (x + xy + xyz) = 2x - 1 \iff x = \frac{1}{2-r}.$$

Similarly, $y = \frac{2}{4-r}$ and $z = \frac{4}{1-r}$. Therefore $8 = (1+r)(1-r)(2-r)(4-r)$. This factors as $r(r-3)(r^2-3r-2) = 0$, so the maximum possible value for r is $\frac{3+\sqrt{17}}{2}$.

Now let's check that this yields a valid solution for x, y, z. Let $r = \frac{3+\sqrt{17}}{2}$ and let $x = \frac{1}{2-r}, y = \frac{2}{4-r}, z = \frac{4}{1-r}$. Then $xyz - 1 = \frac{8}{(2-r)(4-r)(1-r)} - 1 = 1 + r - 1 = r$. Now, we may do our above computations in reverse to get

$$2x - 1 = 2x - (2-r)x = rx = x^2yz - x = x(y + yz + xyz) - (x - xy + xyz).$$

Repeating the same thing for y and z yields that

$$\begin{pmatrix} -1 & x & 0 \\ 0 & -1 & y \\ z & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 & x & 0 \\ 0 & -1 & y \\ z & 0 & -1 \end{pmatrix} \begin{pmatrix} x + xy + xyz \\ y + yz + xyz \\ z + xz + xyz \end{pmatrix}.$$

However, since $xyz - 1 \neq 0$, the determinant of the matrix is nonzero, so we may multiply by its inverse to find that

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} x + xy + xyz \\ y + yz + xyz \\ z + xz + xyz \end{pmatrix}.$$

Therefore this construction is valid.