1^{st} Annual Harvard-MIT November Tournament

Saturday 8 November 2008

Team Round

Unit Fractions [100]

A *unit fraction* is a fraction of the form $\frac{1}{n}$, where n is a positive integer. In this problem, you will find out how rational numbers can be expressed as sums of these unit fractions. Even if you do not solve a problem, you may apply its result to later problems.

We say we *decompose* a rational number q into unit fractions if we write q as a sum of 2 or more **distinct** unit fractions. In particular, if we write q as a sum of k distinct unit fractions, we say we have decomposed q into k fractions. As an example, we can decompose $\frac{2}{3}$ into 3 fractions: $\frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$.

1. (a) Decompose 1 into unit fractions.

   Answer: $\boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{6}}$

   (b) Decompose $\frac{1}{4}$ into unit fractions.

   Answer: $\boxed{\frac{1}{8} + \frac{1}{12} + \frac{1}{24}}$

   (c) Decompose $\frac{2}{5}$ into unit fractions.

   Answer: $\boxed{\frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{30}}$

2. Explain how any unit fraction $\frac{1}{n}$ can be decomposed into other unit fractions.

   Answer: $\boxed{\frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}}$

3. (a) Write 1 as a sum of 4 distinct unit fractions.

   Answer: $\boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42}}$

   (b) Write 1 as a sum of 5 distinct unit fractions.

   Answer: $\boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{43\cdot 42}}$

   (c) Show that, for any integer $k > 3$, 1 can be decomposed into k unit fractions.

   Solution: If we can do it for k fractions, simply replace the last one (say $\frac{1}{n}$) with $\frac{1}{n+1} + \frac{1}{n(n+1)}$. Then we can do it for $k+1$ fractions. So, since we can do it for $k = 3$, we can do it for any $k > 3$.

4. Say that $\frac{a}{b}$ is a positive rational number in simplest form, with $a \neq 1$. Further, say that n is an integer such that:

$$\frac{1}{n} > \frac{a}{b} > \frac{1}{n+1}$$

   Show that when $\frac{a}{b} - \frac{1}{n+1}$ is written in simplest form, its numerator is smaller than a.

   Solution: $\frac{a}{b} - \frac{1}{n+1} = \frac{a(n+1)-b}{b(n+1)}$. Therefore, when we write it in simplest form, its numerator will be at most $a(n+1) - b$. We claim that $a(n+1) - b < a$. Indeed, this is the same as $an - b < 0 \iff an < b \iff \frac{b}{a} > n$, which is given.

5. **An aside: the sum of all the unit fractions**
   It is possible to show that, given any real M, there exists a positive integer k large enough that:
$$\sum_{n=1}^{k} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \ldots > M$$
   Note that this statement means that the infinite harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, grows without bound, or diverges. For the specific example M = 5, find a value of k, *not necessarily the smallest*, such that the inequality holds. Justify your answer.

   Solution: Note that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{1}{2n} + \ldots + \frac{1}{2n} = \frac{1}{2}$. Therefore, if we apply this to $n = 1, 2, 4, 8, 16, 32, 64, 128,$ we get
$$\left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{129} + \ldots + \frac{1}{256}\right) > \frac{1}{2} + \ldots + \frac{1}{2} = 4$$
   so, adding in $\frac{1}{1}$, we get
$$\sum_{n=1}^{256} \frac{1}{n} > 5$$
   so $k = 256$ will suffice.

6. Now, using information from problems 4 and 5, prove that the following method to decompose any positive rational number will always terminate:

Step 1. Start with the fraction $\frac{a}{b}$. Let t_1 be the largest unit fraction $\frac{1}{n}$ which is less than or equal to $\frac{a}{b}$.

Step 2. If we have already chosen t_1 through t_k, and if $t_1 + t_2 + \ldots + t_k$ is still less than $\frac{a}{b}$, then let t_{k+1} be the largest unit fraction less than both t_k and $\frac{a}{b}$.

Step 3. If $t_1 + \ldots + t_{k+1}$ equals $\frac{a}{b}$, the decomposition is found. Otherwise, repeat step 2.

   Why does this method never result in an infinite sequence of t_i?

   Solution: Let $\frac{a_k}{b_k} = \frac{a}{b} - t_1 - \ldots - t_k$, where $\frac{a_k}{b_k}$ is a fraction in simplest terms. Initially, this algorithm will have $t_1 = 1$, $t_2 = \frac{1}{2}$, $t_3 = \frac{1}{3}$, etc. until $\frac{a_k}{b_k} < \frac{1}{k+1}$. This will eventually happen by problem 5, since there exists a k such that $\frac{1}{1} + \ldots + \frac{1}{k+1} > \frac{a_k}{b_k}$. At that point, there is some n with $\frac{1}{n} < t_k$ such that $\frac{1}{n} > \frac{a_k}{b_k} > \frac{1}{n+1}$. In this case, $t_{k+1} = \frac{1}{n+1}$.

   Suppose that there exists n_k such that $\frac{1}{n_k} > \frac{a_k}{b_k} > \frac{1}{n_k+1}$ for some k. Then we have $t_{k+1} = \frac{1}{n_k+1}$ and $\frac{a_{k+1}}{b_{k+1}} < \frac{1}{n_k(n_k+1)}$. This shows that once we have found n_k such that $\frac{1}{n_k} > \frac{a_k}{b_k} > \frac{1}{n_k+1}$ and $\frac{1}{n_k} \leq t_k$, we no longer have to worry about t_{k+1} being less than t_k, since $t_{k+1} = \frac{1}{n_k+1} < \frac{1}{n_k} < t_k$, and also $n_{k+1} \geq n_k(n_k+1)$ while $\frac{1}{n_k(n_k+1)} \leq \frac{1}{n_k+1} = t_{k+1}$.

   On the other hand, once we have found such an n_k, the sequence $\{a_k\}$ must be decreasing by problem 4. Since the a_k are all integers, we eventually have to get to 0 (as there is no infinite decreasing sequence of positive integers). Therefore, after some finite number of steps the algorithm terminates with $a_{k+1} = 0$, so $0 = \frac{a_k}{b_k} = \frac{a}{b} - t_1 - \ldots - t_k$, so $\frac{a}{b} = t_1 + \ldots + t_k$, which is what we wanted.

Juicy Numbers [100]

A *juicy number* is an integer $j > 1$ for which there is a sequence $a_1 < a_2 < \ldots < a_k$ of positive integers such that $a_k = j$ and such that the sum of the reciprocals of all the a_i is 1. For example, 6 is a juicy number because $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, but 2 is not juicy.

In this part, you will investigate some of the properties of juicy numbers. Remember that if you do not solve a question, you can still use its result on later questions.

1. Explain why 4 is not a juicy number.

   Solution: If 4 were juicy, then we would have $1 = \ldots + \frac{1}{4}$. The $\ldots$ can only possible contain $\frac{1}{2}$ and $\frac{1}{3}$, but it is clear that $\frac{1}{2} + \frac{1}{4}$, $\frac{1}{3} + \frac{1}{4}$, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ are all not equal to 1.

2. It turns out that 6 is the smallest juicy integer. Find the next two smallest juicy numbers, and show a decomposition of 1 into unit fractions for each of these numbers. You do not need to prove that no smaller numbers are juicy.

   Answer: $\boxed{\text{12 and 15}}$ $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$, $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}$.

3. Let p be a prime. Given a sequence of positive integers b_1 through b_n, exactly one of which is divisible by p, show that when

   $$\frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_n}$$

   is written as a fraction in lowest terms, then its denominator is divisible by p. Use this fact to explain why no prime p is ever juicy.

   Solution: We can assume that b_n is the term divisible by p (i.e. $b_n = kp$) since the order of addition doesn't matter. We can then write

   $$\frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_{n-1}} = \frac{a}{b}$$

   where b is not divisible by p (since none of the b_i are). But then $\frac{a}{b} + \frac{1}{kp} = \frac{kpa+b}{kpb}$. Since b is not divisible by p, $kpa + b$ is not divisible by p, so we cannot remove the factor of p from the denominator. In particular, p cannot be juicy as 1 can be written as $\frac{1}{1}$, which has a denominator not divisible by p, whereas being juicy means we have a sum $\frac{1}{b_1} + \ldots + \frac{1}{b_n} = 1$, where $b_1 < b_2 < \ldots < b_n = p$, and so in particular none of the b_i with $i < n$ are divisible by p.

4. Show that if j is a juicy integer, then $2j$ is juicy as well.

   Solution: Just replace $\frac{1}{b_1} + \ldots + \frac{1}{b_n}$ with $\frac{1}{2} + \frac{1}{2b_1} + \frac{1}{2b_2} + \ldots + \frac{1}{2b_n}$. Since $n > 1$, $2b_1 > 2$.

5. Prove that the product of two juicy numbers (not necessarily distinct) is always a juicy number. Hint: if j_1 and j_2 are the two numbers, how can you change the decompositions of 1 ending in $\frac{1}{j_1}$ or $\frac{1}{j_2}$ to make them end in $\frac{1}{j_1 j_2}$?

   Solution: Let $1 = \frac{1}{b_1} + \ldots + \frac{1}{b_n} = \frac{1}{c_1} + \ldots + \frac{1}{c_m}$, where $b_n = j_1$ and $c_m = j_2$. Then

$$1 = \frac{1}{b_1} + \ldots + \frac{1}{b_{n-1}} + \left(\frac{1}{b_n c_1} + \frac{1}{b_n c_2} + \ldots + \frac{1}{b_n c_m} \right)$$

and so $j_1 j_2$ is juicy.