February 2017

February 18, 2017

Algebra and Number Theory

1. Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \le a_i < 3$ for all $0 \le i \le n$.

   Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

   Proposed by: Yang Liu

   Answer: $\boxed{86}$

   One can evaluate

   $$Q(\sqrt{3}) = (a_0 + 3a_2 + 3^2a_4 + \dots) + (a_1 + 3a_3 + 3^2a_5 + \dots)\sqrt{3}.$$

   Therefore, we have that

   $$(a_0 + 3a_2 + 3^2a_4 + \dots) = 20 \text{ and } (a_1 + 3a_3 + 3^2a_5 + \dots) = 17.$$

   This corresponds to the base-3 expansions of 20 and 17. This gives us that $Q(x) = 2+2x+2x^3+2x^4+x^5$, so $Q(2) = 86$.

2. Find the value of

   $$\sum_{1 \le a < b < c} \frac{1}{2^a 3^b 5^c}$$

   (i.e. the sum of $\frac{1}{2^a3^b5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$)

   Proposed by: Alexander Katz

   Answer: $\boxed{1/1624}$

   Let $x = b - a$ and $y = c - b$ so that $b = a + x$ and $c = a + x + y$. Then

   $$2^a3^b5^c = 2^a3^{a+x}5^{a+x+y} = 30^a15^x5^y$$

   and a, x, y are any positive integers. Thus

   $$\begin{aligned}
   \sum_{1 \le a \le b < c} \frac{1}{2^a3^b5^c} &= \sum_{1 \le a,x,y} \frac{1}{30^a15^x5^y} \\
   &= \sum_{1 \le a} \frac{1}{30^a} \sum_{1 \le x} \frac{1}{15^x} \sum_{1 \le y} \frac{1}{5^y} \\
   &= \frac{1}{29} \cdot \frac{1}{14} \cdot \frac{1}{4} \\
   &= \boxed{\frac{1}{1624}}
   \end{aligned}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x)f(y) = f(x - y)$. Find all possible values of $f(2017)$.

   Proposed by: Alexander Katz

   Let $P(x, y)$ be the given assertion. From $P(0, 0)$ we get $f(0)^2 = f(0) \implies f(0) = 0, 1$.

   From $P(x, x)$ we get $f(x)^2 = f(0)$. Thus, if $f(0) = 0$, we have $f(x) = 0$ for all x, which satisfies the given constraints. Thus $f(2017) = 0$ is one possibility.

Now suppose $f(0) = 1$. We then have $P(0, y) \implies f(-y) = f(y)$, so that $P(x, -y) \implies f(x)f(y) = f(x-y) = f(x)f(-y) = f(x+y)$. Thus $f(x-y) = f(x+y)$, and in particular $f(0) = f\left(\frac{x}{2} - \frac{x}{2}\right) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f(x)$. It follows that $f(x) = 1$ for all x, which also satisfies all given constraints.

Thus the two possibilities are $f(2017) = \boxed{0, 1}$.

4. Find all pairs (a, b) of positive integers such that $a^{2017} + b$ is a multiple of ab.

   Proposed by: Yang Liu

   Answer: $\boxed{(1,1) \text{ and } (2, 2^{2017}).}$

   We want $ab | a^{2017} + b$. This gives that $a|b$. Therefore, we can set $b = b_{2017}a$. Substituting this gives $b_{2017}a^2 | a^{2017} + b_{2017}a$, so $b_{2017}a | a^{2016} + b_{2017}$. Once again, we get $a | b_{2017}$, so we can set $b_{2017} = b_{2016}a$. Continuing this way, if we have $b_{i+1}a | a^i + b_{i+1}$, then $a | b_{i+1}$, so we can set $b_{i+1} = b_i a$ and derive $b_i a | a^{i-1} + b_i$. Continuing down to $i = 1$, we would have $b = b_1 a^{2017}$ so $ab_1 | 1 + b_1$. If $a \geq 3$, then $ab_1 > 1 + b_1$ for all $b_1 \geq 1$, so we need either $a = 1$ or $a = 2$. If $a = 1$, then $b | b+1$, so $b = 1$. This gives the pair $(1, 1)$. If $a = 2$, we need $2b | b + 2^{2017}$. Therefore, we get $b | 2^{2017}$, so we can write $b = 2^k$ for $0 \leq k \leq 2017$. Then we need $2^{k+1} | 2^k + 2^{2017}$. As $k \leq 2017$, we need $2 | 1 + 2^{2017-k}$. This can only happen is $k = 2017$. This gives the pair $(2, 2^{2017})$.

5. Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c \in \mathbb{N}}} (a + b + c),$$

   where $\mathbb{N}$ denotes the positive integers.

   Proposed by: Yang Liu

   Answer: $\boxed{49140}$

   Note that $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. The given sum clearly equals $3 \sum_{abc=2310} a$ by symmetry. The inner sum can be rewritten as

$$\sum_{a|2310} a \cdot \tau\left(\frac{2310}{a}\right),$$

   as for any fixed a, there are $\tau\left(\frac{2310}{a}\right)$ choices for the integers b, c.

   Now consider the function $f(n) = \sum_{a|n} a \cdot \tau\left(\frac{n}{a}\right)$. Therefore, $f = n * \tau$, where n denotes the function $g(n) = n$ and $*$ denotes Dirichlet convolution. As both n and τ are multiplicative, f is also multiplicative.

   It is easy to compute that $f(p) = p + 2$ for primes p. Therefore, our final answer is $3(2+2)(3+2)(5+2)(7+2)(11+2) = 49140$.

6. A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $\lfloor 2017P(2017) \rfloor$.

   Proposed by: Alexander Katz

   Answer: $\boxed{-9}$

   Let $Q(x) = x^2 P(x) - 1$. Then $Q(n) = n^2 P(n) - 1 = 0$ for $n = 1, 2, \ldots, 2016$, and Q has degree 2017. Thus we may write

$$Q(x) = x^2 P(x) - 1 = (x-1)(x-2)\ldots(x-2016)L(x)$$

   where $L(x)$ is some linear polynomial. Then $Q(0) = -1 = (-1)(-2)\ldots(-2016)L(0)$, so $L(0) = -\frac{1}{2016!}$.

Now note that

$$\begin{aligned} Q'(x) &= x^2P'(x) + 2xP(x) \\ &= \sum_{i=1}^{2016} (x-1)\dots(x-(i-1))(x-(i+1))\dots(x-2016)L(x) + (x-1)(x-2)\dots(x-2016)L'(x) \end{aligned}$$

Thus

$$Q'(0) = 0 = L(0)\left(\frac{2016!}{-1} + \frac{2016!}{-2} + \dots + \frac{2016!}{-2016}\right) + 2016!L'(0)$$

whence $L'(0) = L(0)\left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2016}\right) = -\frac{H_{2016}}{2016!}$, where H_n denotes the nth harmonic number. As a result, we have $L(x) = -\frac{H_{2016}x+1}{2016!}$. Then

$$Q(2017) = 2017^2P(2017) - 1 = 2016!\left(-\frac{2017H_{2016}+1}{2016!}\right)$$

which is $-2017H_{2016} - 1$. Thus

$$\boxed{P(2017) = \frac{-H_{2016}}{2017}}.$$

From which we get $2017P(2017) = -H_{2016}$. It remains to approximate H_{2016}. We alter the well known approximation

$$H_n \approx \int_1^n \frac{1}{x}dx = \log x$$

to

$$H_n \approx 1 + \frac{1}{2} + \int_3^n \frac{1}{x}dx = 1 + \frac{1}{2} + \log(2016) - \log(3) \approx \log(2016) + \frac{1}{2}$$

so that it suffices to lower bound $\log(2016)$. Note that $e^3 \approx 20$, which is close enough for our purposes. Then $e^6 \approx 400 \implies e^7 \approx 1080$, and $e^3 \approx 20 < 2^5 \implies e^{0.6} << 2 \implies e^{7.6} < 2016$, so that $\log(2016) > 7.6$. It follows that $H_{2016} \approx \log(2016) + 0.5 = 7.6 + 0.5 > 8$ (of course these are loose estimates, but more than good enough for our purposes). Thus $-9 < 2017P(2017) < -8$, making our answer $\boxed{-9}$.

Alternatively, a well-read contestant might know that $H_n \approx \log n + \gamma$, where $\gamma \approx .577$ is the Euler-Mascheroni constant. The above solution essentially approximates γ as 0.5 which is good enough for our purposes.

7. Determine the largest real number c such that for any 2017 real numbers $x_1, x_2, \dots, x_{2017}$, the inequality

$$\sum_{i=1}^{2016} x_i(x_i + x_{i+1}) \geq c \cdot x_{2017}^2$$

holds.

Proposed by: Pakawut Jiradilok

Answer: $\boxed{-\frac{1008}{2017}}$

Let $n = 2016$. Define a sequence of real numbers $\{p_k\}$ by $p_1 = 0$, and for all $k \geq 1$,

$$p_{k+1} = \frac{1}{4(1-p_k)}.$$

Note that, for every $i \geq 1$,

$$(1-p_i)\cdot x_i^2 + x_i x_{i+1} + p_{i+1}x_{i+1}^2 = \left(\frac{x_i}{2\sqrt{p_{i+1}}} + \sqrt{p_{i+1}}x_{i+1}\right)^2 \geq 0.$$

Summing from $i = 1$ to n gives

$$\sum_{i=1}^{n} x_i(x_i + x_{i+1}) \geq -p_{n+1}x_{n+1}^2.$$

One can show by induction that $p_k = \frac{k-1}{2k}$. Therefore, our answer is $-p_{2017} = -\frac{1008}{2017}$.

8. Consider all ordered pairs of integers (a, b) such that $1 \leq a \leq b \leq 100$ and

$$\frac{(a+b)(a+b+1)}{ab}$$

is an integer.

Among these pairs, find the one with largest value of b. If multiple pairs have this maximal value of b, choose the one with largest a. For example choose $(3, 85)$ over $(2, 85)$ over $(4, 84)$. Note that your answer should be an ordered pair.

Proposed by: Alexander Katz

Answer: $\boxed{(35,90)}$

Firstly note that $\frac{(a+b)(a+b+1)}{ab} = 2+\frac{a^2+b^2+a+b}{ab}$. Let c be this fraction so that $(a+b)(a+b+1) = ab(c+2)$ for some integers a, b, c. Suppose (a, b) with $a \geq b$ is a solution for some c. Consider the quadratic

$$x^2 - (bc-1)x + b^2 + b = 0$$

It has one root a, and the other root is therefore $bc - a - 1$. Furthermore the other root can also be expressed as $\frac{b^2+b}{a} \leq \frac{b^2+b}{b+1} = b$, so that $0 < bc - a - 1 \leq b$. In particular, $(b, bc - a - 1)$ is a solution as well.

Thus all solutions (a, b) reduce to a solution where $a = b$, at which point $c = 2 + \frac{2}{a}$. Since a, c are positive integers we thus have $a = 1, 2$, and so $c = \boxed{3, 4}$.

Through this jumping process, we iteratively find the solutions for $c = 3$:

$$(2,2) \to (2,3) \to (3,6) \to (6,14) \to (14,35) \to (35,90)$$

and $c = 4$:

$$(1,2) \to (2,6) \to (6,21) \to (21,77)$$

so that the desired pair is $\boxed{(35,90)}$.

9. The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Find the smallest positive integer m such that $F_m \equiv 0 \pmod{127}$ and $F_{m+1} \equiv 1 \pmod{127}$.

Proposed by: Sam Korsky

Answer: $\boxed{256}$

First, note that 5 is not a quadratic residue modulo 127. We are looking for the period of the Fibonacci numbers mod 127. Let $p = 127$. We work in $\mathbb{F}_{p^2}$ for the remainder of this proof. Let α and β be the roots of $x^2 - x - 1$. Then we know that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. Note that since $x \to x^p$ is an automorphism and since automorphisms cycle the roots of a polynomial we have that $\alpha^p = \beta$ and $\beta^p = \alpha$. Then $F_p = \frac{\alpha^p - \beta^p}{\alpha - \beta} = -1$ and $F_{p+1} = \frac{\alpha\beta - \beta\alpha}{\alpha - \beta} = 0$ and similarly we obtain $F_{2p+1} = 1$ and $F_{2p+2} = 0$. Thus since $2p + 2$ is a power of 2 and since the period does not divide $p + 1$, we must have the answer is $2p + 2 = \boxed{256}$.

10. Let $\mathbb{N}$ denote the natural numbers. Compute the number of functions $f : \mathbb{N} \to \{0, 1, \ldots, 16\}$ such that

$$f(x+17) = f(x) \quad \text{and} \quad f(x^2) \equiv f(x)^2 + 15 \pmod{17}$$

for all integers $x \geq 1$.

Proposed by: Yang Liu

Answer: $\boxed{12066}$

By plugging in $x = 0$, we get that $f(0)$ can be either $-1, 2$. As $f(0)$ is unrelated to all other values, we need to remember to multiply our answer by 2 at the end. Similarly, $f(1) = -1$ or 2.

Consider the graph $x \to x^2$. It is a binary tree rooted at -1, and there is an edge $-1 \to 1$, and a loop $1 \to 1$. Our first case is $f(1) = -1$. Note that if x, y satisfy $x^2 = y$, then $f(y) \neq 1$. Otherwise, we would have $f(x)^2 = 3 \pmod{17}$, a contradiction as 3 is a nonresidue. So only the 8 leaves can take the value 1. This contributes 2^8.

For $f(1) = 2$, we can once again propagate down the tree. While it looks like we have 2 choices at each node (for the square roots), this is wrong, as if $f(x) = -2$ and $y^2 = x$, then $f(y) = 0$ is forced.

Given this intuition, let a_n denote the answer for a binary tree of height n where the top is either -2 or 2. Therefore, $a_1 = 2, a_2 = 5$. You can show the recurrence $a_n = a_{n-1}^2 + 2^{2^n-4}$. This is because if the top is 2, then we get a contribution of a_{n-1}^2. If the top is -2, then both entries below it must be 0. After that, you can show that each of the remaining $2^n - 4$ vertices can be either of 2 possible square roots. Therefore, we get the recurrence as claimed. One can compute that $a_4 = 5777$, so we get the final answer $2(256 + 5777) = 12066$.