12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Algebra Test

1. [**3**] If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

   Answer: $\boxed{47}$

   Solution: We can factor the equation as $(a - b^2)(a + b^2) = 41 \cdot 49$, from which it is evident that $a = 45$ and $b = 2$ is a possible solution. By examining the factors of 2009, one can see that there are no other solutions.

2. [**3**] Let S be the sum of all the real coefficients of the expansion of $(1 + ix)^{2009}$. What is $\log_2(S)$?

   Answer: $\boxed{1004}$

   Solution: The sum of all the coefficients is $(1 + i)^{2009}$, and the sum of the real coefficients is the real part of this, which is $\frac{1}{2}\left((1 + i)^{2009} + (1 - i)^{2009}\right) = 2^{1004}$. Thus $\log_2(S) = 1004$.

3. [**4**] If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

   Answer: $\boxed{20}$

   Solution: We have $\cot x + \cot y = \frac{\tan x + \tan y}{\tan x \tan y}$, so $\tan x \tan y = \frac{4}{5}$. Thus, by the tan sum formula, $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = 20$.

4. [**4**] Suppose a, b and c are integers such that the greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $x + 1$ (in the ring of polynomials in x with integer coefficients), and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $x^3 - 4x^2 + x + 6$. Find $a + b + c$.

   Answer: $\boxed{-6}$

   Solution: Since $x+1$ divides x^2+ax+b and the constant term is b, we have $x^2+ax+b = (x+1)(x+b)$, and similarly $x^2+bx+c = (x+1)(x+c)$. Therefore, $a = b+1 = c+2$. Furthermore, the least common multiple of the two polynomials is $(x+1)(x+b)(x+b-1) = x^3-4x^2+x+6$, so $b = -2$. Thus $a = -1$ and $c = -3$, and $a + b + c = -6$.

5. [**4**] Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

   Answer: $\boxed{-2}$

   Solution: We can substitute $x = y - 1$ to obtain a polynomial having roots $a+1$, $b+1$, $c+1$, namely, $(y - 1)^3 - (y - 1) + 1 = y^3 - 3y^2 + 2y + 1$. The sum of the reciprocals of the roots of this polynomial is, by Viete's formulas, $\frac{2}{-1} = -2$.

6. [**5**] Let x and y be positive real numbers and θ an angle such that $\theta \neq \frac{\pi}{2}n$ for any integer n. Suppose

$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y}$$

and

$$\frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{97\sin 2\theta}{x^3 y + y^3 x}.$$

Compute $\frac{x}{y}+\frac{y}{x}$.

Answer: $\boxed{4}$

Solution: From the first relation, there exists a real number k such that $x = k\sin\theta$ and $y = k\cos\theta$. Then we have

$$\frac{\cos^4\theta}{\sin^4\theta}+\frac{\sin^4\theta}{\cos^4\theta}=\frac{194\sin\theta\cos\theta}{\sin\theta\cos\theta(\cos^2\theta+\sin^2\theta)}=194.$$

Notice that if $t=\frac{x}{y}+\frac{y}{x}$ then $(t^2-2)^2-2=\frac{\cos^4\theta}{\sin^4\theta}+\frac{\sin^4\theta}{\cos^4\theta}=194$ and so $t=4$.

7. [**5**] Simplify the product

$$\prod_{m=1}^{100}\prod_{n=1}^{100}\frac{x^{n+m}+x^{n+m+2}+x^{2n+1}+x^{2m+1}}{x^{2n}+2x^{n+m}+x^{2m}}.$$

Express your answer in terms of x.

Answer: $\boxed{x^{9900}\left(\frac{1+x^{100}}{2}\right)^2 \text{ (OR } \frac{1}{4}x^{9900}+\frac{1}{2}x^{10000}+\frac{1}{4}x^{10100})}$

Solution: We notice that the numerator and denominator of each term factors, so the product is equal to

$$\prod_{m=1}^{100}\prod_{n=1}^{100}\frac{(x^m+x^{n+1})(x^{m+1}+x^n)}{(x^m+x^n)^2}.$$

Each term of the numerator cancels with a term of the denominator except for those of the form (x^m+x^{101}) and $(x^{101}+x^n)$ for $m,n=1,\ldots,100$, and the terms in the denominator which remain are of the form (x^1+x^n) and (x^1+x^m) for $m,n=1,\ldots,100$. Thus the product simplifies to

$$\left(\prod_{m=1}^{100}\frac{x^m+x^{101}}{x^1+x^m}\right)^2$$

Reversing the order of the factors of the numerator, we find this is equal to

$$\begin{aligned}\left(\prod_{m=1}^{100}\frac{x^{101-m}+x^{101}}{x^1+x^m}\right)^2 &= \left(\prod_{m=1}^{100}x^{100-m}\frac{x^1+x^{m+1}}{x^1+x^m}\right)^2\\ &= \left(\frac{x^1+x^101}{x^1+x^1}\prod_{m=1}^{100}x^{100-m}\right)^2\\ &= (x^{\frac{99\cdot 100}{2}})^2\left(\frac{1+x^{100}}{2}\right)^2\end{aligned}$$

as desired.

8. [**7**] If a, b, x, and y are real numbers such that $ax+by=3$, $ax^2+by^2=7$, $ax^3+by^3=16$, and $ax^4+by^4=42$, find ax^5+by^5.

Answer: $\boxed{20.}$

Solution: We have $ax^3+by^3=16$, so $(ax^3+by^3)(x+y)=16(x+y)$ and thus

$$ax^4+by^4+xy(ax^2+by^2)=16(x+y)$$

It follows that

$$42 + 7xy = 16(x + y) \tag{1}$$

From $ax^2 + by^2 = 7$, we have $(ax^2 + by^2)(x + y) = 7(x + y)$ so $ax^3 + by^3 + xy(ax^2 + by^2) = 7(x + y)$. This simplifies to

$$16 + 3xy = 7(x + y) \tag{2}$$

We can now solve for $x + y$ and xy from (1) and (2) to find $x + y = -14$ and $xy = -38$. Thus we have $(ax^4 + by^4)(x + y) = 42(x + y)$, and so $ax^5 + by^5 + xy(ax^3 + by^3) = 42(x + y)$. Finally, it follows that $ax^5 + by^5 = 42(x + y) - 16xy = 20$ as desired.

9. [**7**] Let $f(x) = x^4 + 14x^3 + 52x^2 + 56x + 16$. Let z_1, z_2, z_3, z_4 be the four roots of f. Find the smallest possible value of $|z_a z_b + z_c z_d|$ where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

Answer: $\boxed{8}$

Solution: Note that $\frac{1}{16} f(2x) = x^4 + 7x^3 + 13x^2 + 7x + 1$. Because the coefficients of this polynomial are symmetric, if r is a root of $f(x)$ then $\frac{4}{r}$ is as well. Further, $f(-1) = -1$ and $f(-2) = 16$ so $f(x)$ has two distinct roots on $(-2, 0)$ and two more roots on $(-\infty, -2)$. Now, if σ is a permutation of $\{1, 2, 3, 4\}$:

$|z_{\sigma(1)} z_{\sigma(2)} + z_{\sigma(3)} z_{\sigma(4)}| \leq \frac{1}{2}(z_{\sigma(1)} z_{\sigma(2)} + z_{\sigma(3)} z_{\sigma(4)} + z_{\sigma(4)} z_{\sigma(3)} + z_{\sigma(2)} z_{\sigma(1)})$

Let the roots be ordered $z_1 \leq z_2 \leq z_3 \leq z_4$, then by rearrangement the last expression is at least:

$\frac{1}{2}(z_1 z_4 + z_2 z_3 + z_3 z_2 + z_4 z_1)$

Since the roots come in pairs $z_1 z_4 = z_2 z_3 = 4$, our expression is minimized when $\sigma(1) = 1, \sigma(2) = 4, \sigma(3) = 3, \sigma(4) = 2$ and its minimum value is 8.

10. [**8**] Let $f(x) = 2x^3 - 2x$. For what positive values of a do there exist distinct b, c, d such that $(a, f(a))$, $(b, f(b))$, $(c, f(c))$, $(d, f(d))$ is a rectangle?

Answer: $\boxed{[\frac{\sqrt{3}}{3}, 1]}$

Solution: Say we have four points $(a, f(a))$, $(b, f(b))$, $(c, f(c))$, $(d, f(d))$ on the curve which form a rectangle. If we interpolate a cubic through these points, that cubic will be symmetric around the center of the rectangle. But the unique cubic through the four points is $f(x)$, and $f(x)$ has only one point of symmetry, the point $(0, 0)$.

So every rectangle with all four points on $f(x)$ is of the form $(a, f(a))$, $(b, f(b))$, $(-a, f(-a))$, $(-b, f(-b))$, and without loss of generality we let $a, b > 0$. Then for any choice of a and b these points form a parallelogram, which is a rectangle if and only if the distance from $(a, f(a))$ to $(0, 0)$ is equal to the distance from $(b, f(b))$ to $(0, 0)$. Let $g(x) = x^2 + (f(x))^2 = 4x^6 - 8x^4 + 5x^2$, and consider $g(x)$ restricted to $x \geq 0$. We are looking for all the values of a such that $g(x) = g(a)$ has solutions other than a.

Note that $g(x) = h(x^2)$ where $h(x) = 4x^3 - 8x^2 + 5x$. This polynomial $h(x)$ has a relative maximum of 1 at $x = \frac{1}{2}$ and a relative minimum of 25/27 at $x = \frac{5}{6}$. Thus the polynomial $h(x) - h(1/2)$ has the double root 1/2 and factors as $(4x^2 - 4x + 1)(x - 1)$, the largest possible value of a^2 for which $h(x^2) = h(a^2)$ is $a^2 = 1$, or $a = 1$. The smallest such value is that which evaluates to 25/27 other than 5/6, which is similarly found to be $a^2 = 1/3$, or $a = \frac{\sqrt{3}}{3}$. Thus, for a in the range $\frac{\sqrt{3}}{3} \leq a \leq 1$ the equation $g(x) = g(a)$ has nontrivial solutions and hence an inscribed rectangle exists.