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1. **[4]** Submit an integer x as your answer to this problem. The number of points you receive will be $\max(0, 8 - |8x - 100|)$. (Non-integer answers will be given 0 points.)

2. **[4]** Let ABC be a triangle and ω be its circumcircle. The point M is the midpoint of arc BC not containing A on ω and D is chosen so that DM is tangent to ω and is on the same side of AM as C. It is given that $AM = AC$ and $\angle DMC = 38^\circ$. Find the measure of angle $\angle ACB$.

3. **[4]** Let ABC be a triangle and D, E, and F be the midpoints of sides BC, CA, and AB respectively. What is the maximum number of circles which pass through at least 3 of these 6 points?

4. **[4]** Compute the value of $\sqrt{105^3 - 104^3}$, given that it is a positive integer.

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5. **[5]** Alice, Bob, and Charlie roll a 4, 5, and 6-sided die, respectively. What is the probability that a number comes up exactly twice out of the three rolls?

6. **[5]** Two sides of a regular n-gon are extended to meet at a 28° angle. What is the smallest possible value for n?

7. **[5]** Ana and Banana are rolling a standard six-sided die. Ana rolls the die twice, obtaining a_1 and a_2, then Banana rolls the die twice, obtaining b_1 and b_2. After Ana's two rolls but before Banana's two rolls, they compute the probability p that $a_1b_1 + a_2b_2$ will be a multiple of 6. What is the probability that $p = \frac{1}{6}$?

8. **[5]** Tessa picks three real numbers x, y, z and computes the values of the eight expressions of the form $\pm x \pm y \pm z$. She notices that the eight values are all distinct, so she writes the expressions down in increasing order. For example, if $x = 2$, $y = 3$, $z = 4$, then the order she writes them down is

$$-x-y-z,\ +x-y-z,\ -x+y-z,\ -x-y+z,\ +x+y-z,\ +x-y+z,\ -x+y+z,\ +x+y+z.$$

How many possible orders are there?

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9. **[6]** Let $P(x)$ be the monic polynomial with rational coefficients of minimal degree such that $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \ldots, \frac{1}{\sqrt{1000}}$ are roots of P. What is the sum of the coefficients of P?

10. **[6]** Jarris is a weighted tetrahedral die with faces F_1, F_2, F_3, F_4. He tosses himself onto a table, so that the probability he lands on a given face is proportional to the area of that face (i.e. the probability he lands on face F_i is $\frac{[F_i]}{[F_1]+[F_2]+[F_3]+[F_4]}$ where $[K]$ is the area of K). Let k be the maximum distance any part of Jarris is from the table after he rolls himself. Given that Jarris has an inscribed sphere of radius 3 and circumscribed sphere of radius 10, find the minimum possible value of the expected value of k.

11. **[6]** Find the number of ordered pairs of positive integers (x, y) with $x, y \leq 2020$ such that $3x^2 + 10xy + 3y^2$ is the power of some prime.

12. **[6]** An 11×11 grid is labeled with consecutive rows $0, 1, 2, \ldots, 10$ and columns $0, 1, 2, \ldots, 10$ so that it is filled with integers from 1 to 2^{10}, inclusive, and the sum of all of the numbers in row n and in column n are both divisible by 2^n. Find the number of possible distinct grids.

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13. **[8]** Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 1$, and $CA = 4\sqrt{3}$. The angle trisectors of C intersect $\overline{AB}$ at D and E, and lines $\overline{AC}$ and $\overline{BC}$ intersect the circumcircle of $\triangle CDE$ again at X and Y, respectively. Find the length of XY.

14. **[8]** Let $\varphi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Let S be the set of positive integers n such that $\frac{2n}{\varphi(n)}$ is an integer. Compute the sum

$$\sum_{n \in S} \frac{1}{n}.$$

15. **[8]** You have six blocks in a row, labeled 1 through 6, each with weight 1. Call two blocks $x \leq y$ connected when, for all $x \leq z \leq y$, block z has not been removed. While there is still at least one block remaining, you choose a remaining block uniformly at random and remove it. The cost of this operation is the sum of the weights of the blocks that are connected to the block being removed, including itself. Compute the expected total cost of removing all the blocks.

16. **[8]** Determine all triplets of real numbers (x, y, z) satisfying the system of equations

$$\begin{aligned} x^2y + y^2z &= 1040 \\ x^2z + z^2y &= 260 \\ (x-y)(y-z)(z-x) &= -540. \end{aligned}$$

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17. **[10]** Let ABC be a triangle with incircle tangent to the perpendicular bisector of BC. If $BC = AE = 20$, where E is the point where the A-excircle touches BC, then compute the area of $\triangle ABC$.

18. **[10]** A *vertex-induced* subgraph is a subset of the vertices of a graph together with any edges whose endpoints are both in this subset.

    An undirected graph contains 10 nodes and m edges, with no loops or multiple edges. What is the minimum possible value of m such that this graph must contain a nonempty vertex-induced subgraph where all vertices have degree at least 5?

19. **[10]** The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exist unique positive integers $n_1, n_2, n_3, n_4, n_5, n_6$ such that

$$\sum_{i_1=0}^{100}\sum_{i_2=0}^{100}\sum_{i_3=0}^{100}\sum_{i_4=0}^{100}\sum_{i_5=0}^{100} F_{i_1+i_2+i_3+i_4+i_5} = F_{n_1} - 5F_{n_2} + 10F_{n_3} - 10F_{n_4} + 5F_{n_5} - F_{n_6}.$$

    Find $n_1 + n_2 + n_3 + n_4 + n_5 + n_6$.

20. **[10]** There exist several solutions to the equation

$$1 + \frac{\sin x}{\sin 4x} = \frac{\sin 3x}{\sin 2x},$$

    where x is expressed in degrees and $0^\circ < x < 180^\circ$. Find the sum of all such solutions.

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21. **[12]** We call a positive integer t *good* if there is a sequence $a_0, a_1, \ldots$ of positive integers satisfying $a_0 = 15, a_1 = t$, and

$$a_{n-1}a_{n+1} = (a_n - 1)(a_n + 1)$$

    for all positive integers n. Find the sum of all good numbers.

22. **[12]** Let A be a set of integers such that for each integer m, there exists an integer $a \in A$ and positive integer n such that $a^n \equiv m \pmod{100}$. What is the smallest possible value of $|A|$?

23. **[12]** A function $f\colon A \to A$ is called *idempotent* if $f(f(x)) = f(x)$ for all $x \in A$. Let I_n be the number of idempotent functions from $\{1, 2, \ldots, n\}$ to itself. Compute

$$\sum_{n=1}^{\infty} \frac{I_n}{n!}.$$

24. **[12]** In $\triangle ABC$, ω is the circumcircle, I is the incenter and I_A is the A-excenter. Let M be the midpoint of arc $\widehat{BAC}$ on ω, and suppose that X, Y are the projections of I onto MI_A and I_A onto MI, respectively. If $\triangle XYI_A$ is an equilateral triangle with side length 1, compute the area of $\triangle ABC$.

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25. **[15]** Let S be the set of 3^4 points in four-dimensional space where each coordinate is in $\{-1,0,1\}$. Let N be the number of sequences of points $P_1, P_2, \ldots, P_{2020}$ in S such that $P_iP_{i+1} = 2$ for all $1 \leq i \leq 2020$ and $P_1 = (0,0,0,0)$. (Here $P_{2021} = P_1$.) Find the largest integer n such that 2^n divides N.

26. **[15]** Let $ABCD$ be a cyclic quadrilateral, and let segments AC and BD intersect at E. Let W and Y be the feet of the altitudes from E to sides DA and BC, respectively, and let X and Z be the midpoints of sides AB and CD, respectively. Given that the area of AED is 9, the area of BEC is 25, and $\angle EBC - \angle ECB = 30°$, then compute the area of $WXYZ$.

27. **[15]** Let $\{a_i\}_{i\geq 0}$ be a sequence of real numbers defined by

$$a_{n+1} = a_n^2 - \frac{1}{2^{2020\cdot 2^n - 1}}$$

for $n \geq 0$. Determine the largest value for a_0 such that $\{a_i\}_{i\geq 0}$ is bounded.

28. **[15]** Let $\triangle ABC$ be a triangle inscribed in a unit circle with center O. Let I be the incenter of $\triangle ABC$, and let D be the intersection of BC and the angle bisector of $\angle BAC$. Suppose that the circumcircle of $\triangle ADO$ intersects BC again at a point E such that E lies on IO. If $\cos A = \frac{12}{13}$, find the area of $\triangle ABC$.

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29. **[18]** Let $ABCD$ be a tetrahedron such that its circumscribed sphere of radius R and its inscribed sphere of radius r are concentric. Given that $AB = AC = 1 \leq BC$ and $R = 4r$, find BC^2.

30. **[18]** Let $S = \{(x,y) \mid x > 0, y > 0, x+y < 200, \text{ and } x,y \in \mathbb{Z}\}$. Find the number of parabolas $\mathcal{P}$ with vertex V that satisfy the following conditions:

- $\mathcal{P}$ goes through both $(100,100)$ and at least one point in S,
- V has integer coordinates, and
- $\mathcal{P}$ is tangent to the line $x + y = 0$ at V.

31. **[18]** Anastasia is taking a walk in the plane, starting from $(1,0)$. Each second, if she is at (x,y), she moves to one of the points $(x-1,y)$, $(x+1,y)$, $(x,y-1)$, and $(x,y+1)$, each with $\frac{1}{4}$ probability. She stops as soon as she hits a point of the form (k,k). What is the probability that k is divisible by 3 when she stops?

32. **[18]** Find the smallest real constant α such that for all positive integers n and real numbers $0 = y_0 < y_1 < \cdots < y_n$, the following inequality holds:

$$\alpha \sum_{k=1}^{n} \frac{(k+1)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \sum_{k=1}^{n} \frac{k^2+3k+3}{y_k}.$$

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33. **[22]** Estimate

$$N = \prod_{n=1}^{\infty} n^{n^{-1.25}}.$$

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N) \rfloor$ points.

34. **[22]** For odd primes p, let $f(p)$ denote the smallest positive integer a for which there does not exist an integer n satisfying $p \mid n^2 - a$. Estimate N, the sum of $f(p)^2$ over the first 10^5 odd primes p.

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N)^3 \rfloor$ points.

35. **[22]** A collection $\mathcal{S}$ of 10000 points is formed by picking each point uniformly at random inside a circle of radius 1. Let N be the expected number of points of $\mathcal{S}$ which are vertices of the convex hull of the $\mathcal{S}$. (The convex hull is the smallest convex polygon containing every point of $\mathcal{S}$.) Estimate N.

An estimate of $E > 0$ will earn $\max(\lfloor 22 - |E - N| \rfloor, 0)$ points.

36. **[22]** A *snake of length k* is an animal which occupies an ordered k-tuple $(s_1, \dots, s_k)$ of cells in a $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k-1$. If the snake is currently occupying $(s_1, \dots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \dots, s_{k-1})$ instead.

Initially, a snake of length 4 is in the grid $\{1, 2, \dots, 30\}^2$ occupying the positions $(1,1), (1,2), (1,3), (1,4)$ with $(1,1)$ as its head. The snake repeatedly makes a move uniformly at random among moves it can legally make. Estimate N, the expected number of moves the snake makes before it has no legal moves remaining.

An estimate of $E > 0$ will earn $\lfloor 22 \min(N/E, E/N)^4 \rfloor$ points.