4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Theme Round

Fish

1. [**3**] Five of James' friends are sitting around a circular table to play a game of Fish. James chooses a place between two of his friends to pull up a chair and sit. Then, the six friends divide themselves into two disjoint teams, with each team consisting of three consecutive players at the table. If the order in which the three members of a team sit does not matter, how many possible (unordered) pairs of teams are possible?

2. [**3**] In a game of Fish, R2 and R3 are each holding a positive number of cards so that they are collectively holding a total of 24 cards. Each player gives an integer estimate for the number of cards he is holding, such that each estimate is an integer between 80% of his actual number of cards and 120% of his actual number of cards, inclusive. Find the smallest possible sum of the two estimates.

3. [**5**] In preparation for a game of Fish, Carl must deal 48 cards to 6 players. For each card that he deals, he runs through the entirety of the following process:

   1. He gives a card to a random player.
   2. A player Z is randomly chosen from the set of players who have at least as many cards as every other player (i.e. Z has the most cards or is tied for having the most cards).
   3. A player D is randomly chosen from the set of players other than Z who have at most as many cards as every other player (i.e. D has the fewest cards or is tied for having the fewest cards).
   4. Z gives one card to D.

   He repeats steps 1-4 for each card dealt, including the last card. After all the cards have been dealt, what is the probability that each player has exactly 8 cards?

4. [**6**] Toward the end of a game of Fish, the 2 through 7 of spades, inclusive, remain in the hands of three distinguishable players: DBR, RB, and DB, such that each player has at least one card. If it is known that DBR either has more than one card or has an even-numbered spade, or both, in how many ways can the players' hands be distributed?

5. [**8**] For any finite sequence of positive integers π, let $S(\pi)$ be the number of strictly increasing sub-sequences in π with length 2 or more. For example, in the sequence $\pi = \{3, 1, 2, 4\}$, there are five increasing sub-sequences: $\{3, 4\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 4\}$, and $\{1, 2, 4\}$, so $S(\pi) = 5$. In an eight-player game of Fish, Joy is dealt six cards of distinct values, which she puts in a random order π from left to right in her hand. Determine

$$\sum_{\pi} S(\pi),$$

   where the sum is taken over all possible orders π of the card values.

Circles and Tangents

6. [**3**] Let ABC be an equilateral triangle with $AB = 3$. Circle ω with diameter 1 is drawn inside the triangle such that it is tangent to sides AB and AC. Let P be a point on ω and Q be a point on segment BC. Find the minimum possible length of the segment PQ.

7. [**4**] Let XYZ be a triangle with $\angle XYZ = 40^\circ$ and $\angle YZX = 60^\circ$. A circle Γ, centered at the point I, lies inside triangle XYZ and is tangent to all three sides of the triangle. Let A be the point of tangency of Γ with YZ, and let ray $\overrightarrow{XI}$ intersect side YZ at B. Determine the measure of $\angle AIB$.

8. [**5**] Points D, E, F lie on circle O such that the line tangent to O at D intersects ray $\overrightarrow{EF}$ at P. Given that $PD = 4$, $PF = 2$, and $\angle FPD = 60^\circ$, determine the area of circle O.

9. [**6**] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC. The inscribed circles of triangles ABD and ACD are tangent to AD at P and Q, respectively, and are tangent to BC at X and Y, respectively. Let PX and QY meet at Z. Determine the area of triangle XYZ.

10. [**7**] Let Ω be a circle of radius 8 centered at point O, and let M be a point on Ω. Let S be the set of points P such that P is contained within Ω, or such that there exists some rectangle $ABCD$ containing P whose center is on Ω with $AB = 4$, $BC = 5$, and $BC \parallel OM$. Find the area of S.