HMMT November 2012

Saturday 10 November 2012

Guts Round

1. [**5**]

   Answer: $\boxed{10}$ The prime numbers under 30 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. There are 10 in total.

2. [**5**]

   Answer: $\boxed{180}$ Albert's choice of burgers, sides, and drinks are independent events. Therefore, we can use the Multiplication Principle of Counting Independent Events to find the total number of different meals that Albert can get is:

$$5 \times 3 \times 12 = 180$$

3. [**5**]

   Answer: $\boxed{199\pi}$ The area of a circle of radius 100 is $100^2\pi$, and the area of a circle of radius 99 is $99^2\pi$. Therefore, the area of the region between them is $(100^2 - 99^2)\pi = (100+99)(100-99)\pi = 199\pi$.

4. [**6**]

   Answer: $\boxed{10}$ $\angle CBD = \angle CDB$ because $BC = CD$. Notice that $\angle BCD = 80 + 50 + 30 = 160$, so $\angle CBD = \angle CDB = 10$.

5. [**6**]

   Answer: $\boxed{\frac{20}{3}}$

$$\begin{aligned}
20 &= 4a^2 + 9b^2 \\
20 + 12ab &= 4a^2 + 12ab + 9b^2 \\
20 + 12ab &= (2a+3b)^2 \\
20 + 12ab &= 100 \\
12ab &= 80 \\
ab &= \frac{20}{3}
\end{aligned}$$

6. [**6**]

   Answer: $\boxed{25502400}$

$$\begin{aligned}
&(1^3 + 3\cdot 1^2 + 3\cdot 1) + (2^3 + 3\cdot 2^2 + 3\cdot 2) + \cdots + (99^3 + 3\cdot 99^2 + 3\cdot 99) \\
&= (1^3 + 3\cdot 1^2 + 3\cdot 1 + 1) + (2^3 + 3\cdot 2^2 + 3\cdot 2 + 1) + \cdots + (99^3 + 3\cdot 99^2 + 3\cdot 99 + 1) - 99 \\
&= (1+1)^3 + (2+1)^3 + ...(99+1)^3 - 99 \\
&= 1^3 + (1+1)^3 + (2+1)^3 + ...(99+1)^3 - 100 \\
&= \left(\frac{100(100+1)}{2}\right)^2 - 100 \\
&= 25502400
\end{aligned}$$

7. [**7**]

Answer: $\boxed{335}$

$$\begin{aligned} a_n &= 1 + 3 + 3^2 + ... + 3^n \\ 3a_n &= 3 + 3^2 + 3^3 + ... + 3^{n+1} \\ 3a_n &= 3^{n+1} - 1 + S_n \\ a_n &= \frac{3^{n+1} - 1}{2} \end{aligned}$$

Because $\gcd(2, 7) = 1$, to demonstrate that $7 | a_n = \frac{3^{n+1}-1}{2}$ is equivalent to showing that $3^{n+1} - 1 \equiv 0 \mod 7$

However $3^{n+1} - 1 = 0 \mod 7 \implies n + 1 \equiv 0 \mod 6$, hence the values of $n \leq 2012$ are 5, 11, 17, ..., 2009, which is in total

$$\frac{2009 + 1}{6} = 335$$

possible terms.

8. [**7**]

Answer: $\boxed{\frac{169}{425}}$ There are 13 cards of each suit. The probability that the second card has a different suit than the first is $\frac{3 \cdot 13}{52-1}$. The probability that the third card has a different suit than the first and second card is $\frac{2 \cdot 13}{52-2}$

Since the two events need to both occur, we use the multiplicative rule:

$$\frac{3 \cdot 13}{52 - 1} \cdot \frac{2 \cdot 13}{52 - 2}$$

$$= \frac{169}{425}$$

9. [**7**]

Answer: $\boxed{4}$ Because 23 is odd, we must have an odd number of odd numbers in our set. Since the smallest odd composite number is 9, we cannot have more than 2 odd numbers, as otherwise the sum would be at least 27. Therefore, the set has exactly one odd number.

The only odd composite numbers less than 23 are $9, 15$, and 21. If we include 21, then the rest of the set must include composite numbers that add up to 2, which is impossible. If we include 15, then the rest of the set must include distinct even composite numbers that add up to 8. The only possibility is the set $\{8\}$. If we include 9, the rest of the set must contain distinct even composite numbers that add to 14. The only possibilities are $\{14\}$, $\{4, 10\}$, and $\{6, 8\}$. We have exhausted all cases, so there are a total of 4 sets.

10. [**8**]

Answer: $\boxed{1006}$ Taking both sides modulo 2012, we see that $a_n \equiv a_{n-1} + n \pmod{2012}$. Therefore, $a_{2012} \equiv a_{2011} + 2012 \equiv a_{2010} + 2011 + 2012 \equiv ... \equiv 1 + 2 + ... + 2012 \equiv \frac{(2012)(2013)}{2} \equiv (1006)(2013) \equiv (1006)(1) \equiv 1006 \pmod{2012}$.

11. [**8**]

Answer: $\boxed{45}$ The number of zeroes that $n!$ ends with is the largest power of 10 dividing $n!$. The exponent of 5 dividing $n!$ exceeds the exponent of 2 dividing $n!$, so we simply seek the exponent of 5 dividing $n!$.

For a number less than 125, this exponent is just the number of multiples of 5, but not 25, less than n plus twice the number of multiples of 25 less than n.

Counting up, we see that 24! ends with 4 zeroes while 25! ends with 6 zeroes, so $n!$ cannot end with 5 zeroes. Continuing to count up, we see that the smallest n such that $n!$ ends with 10 zeroes is 45.

12. [**8**]

Answer: $\boxed{\frac{2\pi-\sqrt{3}}{2}}$ Let $A_\triangle$ be the equilateral triangle. Let A_1 be the area of the region outside of the equilateral triangle but inside the second and third circles. Define A_2, A_3 analogously. We have $A_1 = A_2 = A_3 = A_k =$

$$\left(\frac{1^2 \cdot \pi}{3} - \frac{1^2 \cdot \sin 120}{2}\right) = \frac{4\pi - 3\sqrt{3}}{12},$$

and

$$A_\triangle = \frac{1^2 \cdot \sin 60}{2} = \frac{\sqrt{3}}{4}.$$

Thus, the total area is

$$A_1 + A_2 + A_3 + A_\triangle = 3 \cdot \frac{4\pi - 3\sqrt{3}}{12} + \frac{\sqrt{3}}{4} = \frac{2\pi - \sqrt{3}}{2}.$$

13. [**9**]

Answer: $\boxed{6\sqrt{3}}$ Let AB be x. Then, $AC = 18 - 7 - x = 11 - x$. Using the law of cosines gives:

$$x^2 + (11-x)^2 - 2x(11-x)\cos 60^\circ = 7^2.$$
$$3x^2 - 33x + 72 = 0.$$
$$x = 3 \text{ or } 8.$$

Therefore, $AB = 8$ and $AC = 3$ or $AB = 3$ and $AC = 8$. In both cases, the area of the triangle is: $\frac{1}{2} \cdot 8 \cdot 3 \sin 60^\circ = \boxed{6\sqrt{3}}$.

14. [**9**]

Answer: $\boxed{32}$ First, we can fix the position of the 1. Then, by the condition that the numbers are increasing along each arc from 1, we know that the 2 must be adjacent to the 1; so we have two options for its placement. Similarly, we have two options for placing each of 3,4,5,6 in that order. Finally, the 7 must go in the remaining space, for a total of $2^5 = 32$ orderings.

15. [**9**]

Answer: $\boxed{49\pi}$ To find the region in question, we want to find (a, b) such that the discriminant of the quadratic is not positive. In other words, we want

$$4(a+b-7)^2 - 4(a)(2b) \leq 0 \Leftrightarrow a^2 + b^2 - 7a - 7b + 49 \leq 0 \Leftrightarrow (a-7)^2 + (b-7)^2 \leq 49,$$

which is a circle of radius 7 centered at $(7, 7)$ and hence has area 49π.

16. [**10**]

Answer: $\boxed{11}$ We divide this problem into cases based on the relative position of the two red beads:

- They are adjacent. Then, there are 4 possible placements of the green and blue beads: GGBB, GBBG, GBGB, BGGB.
- They are 1 bead apart. Then, there are two choices for the bead between then and 2 choices for the other bead of that color, for a total of 4.
- They are opposite. Then, there are three choices for the placement of the green beads.

This gives a total of 11 arrangements.

17. [**10**]

Answer: $\boxed{\left(\frac{5}{4}, \frac{1}{2}, \frac{1}{4}\right)}$ We can rewrite the equation in terms of $\ln 2, \ln 3, \ln 5$, to get

$$3\ln 2 + 3\ln 3 + 2\ln 5 = \ln 5400 = px + qy + rz = (p + 3q + r)\ln 2 + (p + 2q + 3r)\ln 3 + (p + q + r)\ln 5.$$

Consequently, since p, q, r are rational we want to solve the system of equations $p + 3q + r = 3, p + 2q + 3r = 3, p + q + r = 2$, which results in the ordered triple $\left(\frac{5}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

18. [**10**]

Answer: $\boxed{1 + \frac{\sqrt{3}}{3}}$ DBC is a right triangle with hypotenuse DC. Since $DE = EC, E$ is the midpoint of this right triangle's hypotenuse, and it follows that E is the circumcenter of the triangle. It follows that $BE = DE = CE$, as these are all radii of the same circle. A similar argument shows that $BD = DE = AD$.

Thus, $BD = DE = DE$, and triangle BDE is equilateral. So, $\angle DBE = \angle BED = \angle EDB = 60°$. We have $\angle BEC = 180° - \angle BED = 120°$. Because $BE = CE$, triangle BEC is isosceles and $\angle ECB = 30°$. Therefore, DBC is a right triangle with $\angle DBC = 90°$, $\angle BCD = 30°$, and $\angle CDB = 60°$. This means that $CD = \frac{2}{\sqrt{3}}BC$. Combined iwth $CD = \frac{2}{3}$, we have $BC = \frac{\sqrt{3}}{3}$. Similarly, $AB = \frac{\sqrt{3}}{3}$, so $AB + AC = 1 + \frac{\sqrt{3}}{3}$.

19. [**11**]

Answer: $\boxed{21}$ First, note that $525 = 3 \times 7 \times 5 \times 5$. Then, taking the equation modulo 7 gives that $7 \mid x$; let $x = 7x'$ for some nonnegative integer x'. Similarly, we can write $y = 5y'$ and $z = 3z'$ for some nonnegative integers y', z'. Then, after substitution and division of both sides by 105, the given equation is equivalent to $x' + y' + z' = 5$. This is the same as the problem of placing 2 dividers among 5 balls, so is $\binom{7}{2} = 21$.

20. [**11**]

Answer: $\boxed{\frac{81^{10}}{82^{10}}}$ The teacher will leave if the students from each pair are either both present or both not present; the probability that both are present is $\dfrac{81}{100}$ and the probability that neither are present is $\dfrac{1}{100}$. If the teacher leaves, then the probability that both students in any given pair did not get lost is $\dfrac{\frac{81}{100}}{\frac{81}{100} + \frac{1}{100}} = \dfrac{81}{82}$. Since there are ten pairs, the overall probability is $\left(\dfrac{81}{82}\right)^{10} = \dfrac{81^{10}}{82^{10}}$.

21. [**11**]

Answer: $\boxed{\frac{80\sqrt{3}}{9}}$ By shared bases, we know that

$$[EFC] = \left(\frac{5}{6}\right)[AEC] = \left(\frac{5}{6}\right)\left(\frac{4}{5}\right)[ADC] = \left(\frac{5}{6}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)[ABC].$$

By Heron's formula, we find that $[ABC] = \sqrt{(15)(8)(2)(5)} = 20\sqrt{3}$, so $[AEC] = \frac{80\sqrt{3}}{9}$

22. [**12**]

Answer: $\boxed{\frac{1}{2}}$ Let X_n be the nth number rolled. The number formed, $0.\overline{X_1X_2}\cdots$, is simply $\sum_{n=1}^{\infty}\frac{X_n}{10^n}$. By linearity of expectation, the expected value is $\sum_{n=1}^{\infty}\mathbb{E}\left(\frac{X_n}{10^n}\right)=\sum_{n=1}^{\infty}\frac{\mathbb{E}(X_n)}{10^n}$. However, the rolls are independent: for all n, $\mathbb{E}(X_n)=\frac{1}{6}(1+2+3+5+7+9)=\frac{9}{2}$. So, our answer is $\frac{9}{2}\cdot\sum_{n=1}^{\infty}\frac{1}{10^n}=\frac{9}{2}\cdot\frac{1}{9}=\frac{1}{2}$.

23. [**12**]

Answer: $\boxed{\frac{32}{11}}$ Since AP is a tangent to Ω, we know that $\angle PAB=\angle PCA$, so $\triangle PAB\sim\triangle PCA$, so we get that

$$\frac{PB}{PA}=\frac{4}{7}=\frac{PA}{PB+6}.$$

Solving, we get that $7PB=4PA$, so

$$4(PB+6)=7PA=\frac{49}{4}PB\Rightarrow\frac{33}{4}PB=24\Rightarrow PB=\frac{32}{11}.$$

24. [**12**]

Answer: $\boxed{105}$ Since 4 pieces of candy are distributed, there must be exactly 8 children who do not receive any candy; since no two consecutive children do receive candy, the 8 who do not must consist of 4 groups of consecutive children. We divide into cases based on the sizes of these groups:

- $\{5,1,1,1\}$: there are 12 places to begin the group of 5 children who do not receive any candy
- $\{4,2,1,1\}$: there are 12 places to begin the group of 4 children who do not receive candy and then 3 choices for the group of 2 children which does not receive candy, for a total of 36 choices
- $\{3,3,1,1\}$: these 8 children can either be bunched in the order 3,3,1,1, or in the order 3,1,3,1; the first has 12 positions in which to begin the first group of 3 non-candy receiving children and the second has 6 possibilities (due to symmetry), for a total of 18
- $\{3,2,2,1\}$: there are 12 places to begin the group of 3 children who do not receive candy and then 3 choices for the group of 1 child which does not receive candy, for a total of 36 choices
- $\{2,2,2,2\}$: there are $12/4=3$ ways in which this can occur

This gives a total of $12+36+18+36+3=105$

25. [**13**]

Answer: $\boxed{\frac{83}{80}}$ Using the Law of Sines, we have

$$\frac{\sin^2 B+\sin^2 C-\sin^2 A}{\sin B\sin C}=\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}-\frac{\sin A}{\sin B}\frac{\sin A}{\sin C}=\frac{AC}{AB}+\frac{AB}{AC}-\frac{BC}{AC}\frac{BC}{AB}=\frac{83}{80}.$$

26. [**13**]

Answer: $\boxed{2^{2013}-6036}$ Let $y_n=x_n-2^{n+1}$. Note that

$$x_{n+1}=2x_n-x_{n-1}+2^n\Leftrightarrow y_{n+1}=2y_n-y_{n-1}\Leftrightarrow y_{n+1}-y_n=y_n-y_{n-1}.$$

Using the values for x_1,x_2, we get that $y_1=-3$ and $y_2=-6$, so $y_{n+1}-y_n=(-6)-(-3)=-3$. By induction, $y_n=-3n$. Then, we get that $x_n=2^{n+1}-3n$, so $x_{2012}=2^{2013}-6036$.

27. [**13**]

Answer: $\boxed{1755}$ We write this as $(a-b)(a+b)+(a-b)(c)=(a-b)(a+b+c)=2012$. Since a,b,c are positive integers, $a-b<a+b+c$. So, we have three possibilities: $a-b=1$ and $a+b+c=2012$, $a-b=2$ and $a+b+c=1006$, and $a-b=4$ and $a+b+c=503$.

The first solution gives $a=b+1$ and $c=2011-2b$, so b can range from 1 through 1005, which determines a and c completely.

Similarly, the second solution gives $a=b+2$ and $c=1004-2b$, so b can range from 1 through 501.

Finally, the third solution gives $a=b+4$ and $c=499-2b$, so b can range from 1 through 249.

Hence, the total number of solutions is $1005+501+249=1755$.

28. [**15**]

Answer: $\boxed{(0,\frac{1+\sqrt{2}}{2}]}$ Suppose that $k=\frac{ab+b^2}{a^2+b^2}$ for some positive real a,b. We claim that k lies in $(0,\frac{1+\sqrt{2}}{2}]$. Let $x=\frac{a}{b}$. We have that $\frac{ab+b^2}{a^2+b^2}=\frac{\frac{a}{b}+1}{(\frac{a}{b})^2+1}=\frac{x+1}{x^2+1}$. Thus, $x+1=k(x^2+1)$, so the quadratic $kt^2-t+k-1=0$ has a positive real root. Thus, its discriminant must be nonnegative, so $1^2\geq 4(k-1)(k) \implies (2k-1)^2\leq 2$, which implies $\frac{1-\sqrt{2}}{2}\leq k\leq\frac{1+\sqrt{2}}{2}$. Since $x>0$, we also have $k>0$, so we know that k must lie in $(0,\frac{1+\sqrt{2}}{2}]$.

Now, take any k in the interval $(0,\frac{1+\sqrt{2}}{2}]$. We thus know that $1^2\geq 4k(k-1)$, so the quadratic $kt^2-t+k-1=0$ has a positive solution, $\frac{1+\sqrt{1-4k(k-1)}}{2k}$. Call this solution x. Then $k(x^2+1)=x+1$, so $\frac{x+1}{x^2+1}=k$. If we set $a=x$ and $b=1$, we get that $\frac{ab+b^2}{a^2+b^2}=k$. Thus, the set of all attainable values of $\frac{ab+b^2}{a^2+b^2}$ is the interval $(0,\frac{1+\sqrt{2}}{2}]$.

29. [**15**]

Answer: $\boxed{-\frac{25+5\sqrt{17}}{8}}$ First, note that $(x+1)(2x+1)(3x+1)(4x+1)=((x+1)(4x+1))((2x+1)(3x+1))=(4x^2+5x+1)(6x^2+5x+1)=(5x^2+5x+1-x^2)(5x^2+5x+1+x^2)=(5x^2+5x+1)^2-x^4$. Therefore, the equation is equivalent to $(5x^2+5x+1)^2=17x^4$, or $5x^2+5x+1=\pm\sqrt{17}x^2$.

If $5x^2+5x+1=\sqrt{17}x^2$, then $(5-\sqrt{17})x^2+5x+1=0$. The discriminant of this is $25-4(5-\sqrt{17})=5+4\sqrt{17}$, so in this case, there are two real roots and they sum to $-\frac{5}{5-\sqrt{17}}=-\frac{25+5\sqrt{17}}{8}$.

If $5x^2+5x+1=-\sqrt{17}x^2$, then $(5+\sqrt{17})x^2+5x+1=0$. The discriminant of this is $25-4(5+\sqrt{17})=5-4\sqrt{17}$. This is less than zero, so there are no real solutions in this case. Therefore, the sum of all real solutions to the equation is $-\frac{25+5\sqrt{17}}{8}$.

30. [**15**]

Answer: $\boxed{\frac{3}{7}}$ It suffices to assume that the monkey starts all over as soon as he has typed a string that ends in no prefix of either *abc* or *aaa*. For instance, if the monkey gets to *abb* we can throw these out because there's no way to finish one of those strings from this without starting all over.

Now, we draw the tree of all possible intermediate stages under this assumption; there are not many possibilities. The paths from the root "a" are:

a- aa- aaa

a- aa- aab- aabc

a- ab- abc

The first and last possibilities have probability 1/27 each and the middle one has probability 1/81, so in total the probability of getting the first before the second or the third is $\frac{1/27}{1/27+1/81+1/27}=\frac{3}{7}$.

31. [**17**]

Answer: $\boxed{\frac{3\sqrt{3}}{20}}$ We place the points in the coordinate plane. We let $A = (0, 0, \frac{\sqrt{6}}{3})$, $B = (0, \frac{\sqrt{3}}{3}, 0)$, $C = (-\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0)$, and $D = (\frac{1}{2}, \frac{\sqrt{3}}{6}, 0)$. The point P is the origin, while M is $(0, 0, \frac{\sqrt{6}}{6})$. The line through B and M is the line $x = 0$, $y = \frac{\sqrt{3}}{3} - z\sqrt{2}$. The plane through A, C, and D has equation $z = 2\sqrt{2}y + \sqrt{\frac{2}{3}}$. The coordinates of Q are the coordinates of the intersection of this line and this plane. Equating the equations and solving for y and z, we see that $y = -\frac{1}{5\sqrt{3}}$ and $z = \frac{\sqrt{6}}{5}$, so the coordinates of Q are $(0, -\frac{1}{5\sqrt{3}}, \frac{\sqrt{6}}{5})$.

Let N be the midpoint of CD, which has coordinates $(0, -\frac{\sqrt{3}}{6}, 0)$. By the distance formula, $QN = \frac{3\sqrt{3}}{10}$. Thus, the area of QCD is $\frac{QN \cdot CD}{2} = \frac{3\sqrt{3}}{20}$.

32. [**17**]

Answer: $\boxed{2530}$ Note that $\phi(23) = 22$ and $\phi(22) = 10$, so if $\operatorname{lcm}(23, 22, 10) = 2530|k$ then $f(n+k) \equiv f(n) \pmod{23}$ is always true. We show that this is necessary as well.

Choosing $n \equiv 0 \pmod{23}$, we see that $k \equiv 0 \pmod{23}$. Thus $n + k \equiv n \pmod{23}$ always, and we can move to the exponent by choosing n to be a generator modulo 23:

$(n + k)^{n+k} \equiv n^n \pmod{22}$

The choice of n here is independent of the choice $\pmod{23}$ since 22 and 23 are coprime. Thus we must have again that $22|k$, by choosing $n \equiv 0 \pmod{22}$. But then $n + k \equiv n \pmod{11}$ always, and we can go to the exponent modulo $\phi(11) = 10$ by choosing n a generator modulo 11:

$n + k \equiv n \pmod{10}$

From here it follows that $10|k$ as well. Thus $2530|k$ and 2530 is the minimum positive integer desired.

33. [**17**]

Answer: $\boxed{\tan^{-1}\left(\frac{1009}{1005}\right)}$ As per usual with reflection problems instead of bouncing off the sides of a 1×1 square we imagine the ball to travel in a straight line from origin in an infinite grid of 1×1 squares, "bouncing" every time it meets a line $x = m$ or $y = n$. Let the lattice point it first meets after leaving the origin be (a, b), so that $b > a$. Note that a and b are coprime, otherwise the ball will reach a vertex before the 2012th bounce. We wish to minimize the slope of the line to this point from origin, which is b/a.

Now, the number of bounces up to this point is $a - 1 + b - 1 = a + b - 2$, so the given statement is just $a + b = 2014$. To minimize b/a with a and b relatively prime, we must have $a = 1005$, $b = 1009$, so that the angle is $\tan^{-1}\left(\frac{1009}{1005}\right)$.

34. [**20**]

Answer: $\boxed{15612}$ Note that the number of integers between 1 and 2012 that have n as a divisor is $\lfloor \frac{2012}{n} \rfloor$. Therefore, if we sum over the possible divisors, we see that the sum is equivalent to $\sum_{d=1}^{2012} \lfloor \frac{2012}{d} \rfloor$. This can be approximated by $\sum_{d=1}^{2012} \frac{2012}{d} = 2012 \sum_{d=1}^{2012} \frac{1}{d} \approx 2012 \ln(2012)$. As it turns out, $2012 \ln(2012) \approx 15300$, which is worth 18 points. Using the very rough approximation $\ln(2012) \approx 7$ still gives 14 points.

35. [**20**]

Answer: $\boxed{3.1415875473}$ The answer is $1006 \sin \frac{\pi}{1006}$. Approximating directly by $\pi = 3.1415\ldots$ is worth only 3 points.

Using the third-degree Taylor polynomial for sin we can approximate $\sin x \approx x - \frac{x^3}{6}$. This gives an answer of 3.1415875473 worth full points. If during the calculation we use the approximation $\pi^3 \approx 30$, this gives an answer worth 9 points.

36. [**20**]

Answer: $\boxed{\text{x}}$ Submit solution