HMMT 2014

Saturday 22 February 2014

Team

1. [**10**] Let ω be a circle, and let A and B be two points in its interior. Prove that there exists a circle passing through A and B that is contained in the interior of ω.

   Answer: $\boxed{\text{N/A}}$ WLOG, suppose $OA \geq OB$. Let ω' be the circle of radius OA centered at O. We have that B lies inside ω'. Thus, it is possible to scale ω' down about the point A to get a circle ω'' passing through both A and B. Since ω'' lies inside ω' and ω' lies inside ω, ω'' lies inside ω.

   Alternative solution 1: WLOG, suppose $OA \geq OB$. Since $OA \geq OB$, the perpendicular bisector of AB intersects segment OA at some point C. We claim that the circle ω' passing through A and B and centered at C lies entirely in ω. Let $x = OA$ and $y = AC = BC$. Note that y is the length of the radius of ω'. By definition, any point P contained in ω' is of distance at most y from C. Applying the triangle inequality to OCP, we see that $OP \leq OC + CP \leq (x - y) + y = x$, so P lies in ω. Since P was arbitrary, it follows that ω' lies entirely in ω.

   Alternative solution 2: Draw line AB, and let it intersect ω at A' and B', where A and A' are on the same side of B. Choose X inside the segment AB so that $A'X/AX = B'X/BX$; such a point exists by the intermediate value theorem. Notice that X is the center of a dilation taking $A'B'$ to AB - the same dilation carries ω to ω' which goes through A and B. Since ω' is ω dilated with respect to a point in its interior, it's clear that ω' must be contained entirely within ω, and so we are done.

2. [**15**] Let $a_1, a_2, \ldots$ be an infinite sequence of integers such that a_i divides a_{i+1} for all $i \geq 1$, and let b_i be the remainder when a_i is divided by 210. What is the maximal number of distinct terms in the sequence $b_1, b_2, \ldots$?

   Answer: $\boxed{127}$ It is clear that the sequence $\{a_i\}$ will be a concatenation of sequences of the form $\{v_i\}_{i=1}^{N_0}, \{w_i \cdot p_1\}_{i=1}^{N_1}, \{x_i \cdot p_1p_2\}_{i=1}^{N_2}, \{y_i \cdot p_1p_2p_3\}_{i=1}^{N_3}$, and $\{z_i \cdot p_1p_2p_3p_4\}_{i=1}^{N_4}$, for some permutation (p_1, p_2, p_3, p_4) of $(2, 3, 5, 7)$ and some sequences of integers $\{v_i\}.\{w_i\}.\{x_i\}.\{y_i\}.\{z_i\}$, each coprime with 210.

   In $\{v_i\}_{i=1}^{N_0}$, there are a maximum of $\phi(210)$ distinct terms mod 210. In $\{w_i \cdot p_1\}_{i=1}^{N_1}$, there are a maximum of $\phi(\frac{210}{p_1})$ distinct terms mod 210. In $\{x_i \cdot p_1p_2\}_{i=1}^{N_2}$, there are a maximum of $\phi(\frac{210}{p_1p_2})$ distinct terms mod 210. In $\{y_i \cdot p_1p_2p_3\}_{i=1}^{N_3}$, there are a maximum of $\phi(\frac{210}{p_1p_2p_3})$ distinct terms mod 210. In $\{z_i \cdot p_1p_2p_3p_4\}_{i=1}^{N_4}$, there can only be one distinct term mod 210.

   Therefore we wish to maximize $\phi(210) + \phi(\frac{210}{p_1}) + \phi(\frac{210}{p_1p_2}) + \phi(\frac{210}{p_1p_2p_3}) + 1$ over all permutations (p_1, p_2, p_3, p_4) of $(2, 3, 5, 7)$. It's easy to see that the maximum occurs when we take $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$ for an answer of $\phi(210) + \phi(105) + \phi(35) + \phi(7) + 1 = 127$. This upper bound is clearly attainable by having the v_i's cycle through the $\phi(210)$ integers less than 210 coprime to 210, the w_i's cycle through the $\phi(\frac{210}{p_1})$ integers less than $\frac{210}{p_1}$ coprime to $\frac{210}{p_1}$, etc.

3. [**15**] There are n girls $G_1, \ldots, G_n$ and n boys $B_1, \ldots, B_n$. A pair (G_i, B_j) is called *suitable* if and only if girl G_i is willing to marry boy B_j. Given that there is exactly one way to pair each girl with a distinct boy that she is willing to marry, what is the maximal possible number of suitable pairs?

   Answer: $\boxed{\frac{n(n+1)}{2}}$ We represent the problem as a graph with vertices $G_1, \ldots, G_n, B_1, \ldots, B_n$ such that there is an edge between vertices G_i and B_j if and only if (G_i, B_j) is suitable, so we want to maximize the number of edges while having a unique matching.

   We claim the answer is $\frac{n(n+1)}{2}$. First, note that this can be achieved by having an edge between G_i and B_j for all pairs $j \leq i$, because the only possible matching in this case is pairing G_i with B_i for all i. To prove that this is maximal, we first assume without loss of generality that our unique matching consists of pairing G_i with B_i for all i, which takes n edges. Now, note that for any i, j, at most one of the two edges G_iB_j and G_jB_i can be added, because if both were added, we could pair G_i with B_j

and G_j with B_i instead to get another valid matching. Therefore, we may add at most $\binom{n}{2} \cdot 1 = \frac{n(n-1)}{2}$ edges, so the maximal number of edges is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ as desired.

4. **[20]** Compute

$$\sum_{k=0}^{100} \left\lfloor \frac{2^{100}}{2^{50} + 2^k} \right\rfloor .$$

(Here, if x is a real number, then $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

Answer: $\boxed{101 \cdot 2^{49} - 50}$ Let $a_k = \frac{2^{100}}{2^{50}+2^k}$. Notice that, for $k = 0, 1, \ldots, 49$,

$$a_k + a_{100-k} = \frac{2^{100}}{2^{50} + 2^k} + \frac{2^{100}}{2^{50} + 2^{100-k}} = \frac{2^{100}}{2^{50} + 2^k} + \frac{2^{50+k}}{2^k + 2^{50}} = 2^{50}$$

It is clear that for $k = 0, 1, \ldots, 49$, $a_k, a_{100-k} \notin \mathbb{Z}$, so $\lfloor a_k \rfloor + \lfloor a_{100-k} \rfloor = 2^{50} - 1$ (since the sum of floors is an integer less than $a_k + a_{100-k}$ but greater than $a_k - 1 + a_{100-k} - 1$). Thus,

$$\sum_{k=0}^{100} \lfloor a_k \rfloor = 50 \cdot (2^{50} - 1) + 2^{49} = 101 \cdot 2^{49} - 50$$

5. **[25]** Prove that there exists a nonzero complex number c and a real number d such that

$$\left| \left| \frac{1}{1+z+z^2} \right| - \left| \frac{1}{1+z+z^2} - c \right| \right| = d$$

for all z with $|z| = 1$ and $1 + z + z^2 \neq 0$. (Here, $|z|$ denotes the absolute value of the complex number z, so that $|a + bi| = \sqrt{a^2 + b^2}$ for real numbers a, b.)

Answer: $\boxed{\frac{4}{3}}$ Let $f(z) = \left| \frac{1}{1+z+z^2} \right|$. Parametrize $z = e^{it} = \cos t + i \sin t$ and let $g(t) = f(e^{it})$, $0 \leq t < 2\pi$. Writing out $\frac{1}{1+z+z^2}$ in terms of t and simplifying, we find that $g(t) = \frac{\cos t - i \sin t}{1 + 2\cos t}$. Letting $x(t) = \Re(g(t))$ and $y(t) = \Im(g(t))$ (the real and imaginary parts of $g(t)$, respectively), what we wish to prove is equivalent to showing that $\{(x(t), y(t)) \mid 0 \leq t < 2\pi\}$ is a hyperbola with one focus at $(0, 0)$. However

$$9\left(x(t) - \frac{2}{3}\right)^2 - 3y(t)^2 = 1$$

holds for all t, so from this equation we find that the locus of points $(x(t), y(t))$ is a hyperbola, with center $(\frac{2}{3}, 0)$ and focal length $\frac{2}{3}$, so the foci are at $(0, 0)$ and $(\frac{4}{3}, 0)$. Hence $c = \frac{4}{3}$.

6. **[25]** Let n be a positive integer. A sequence $(a_0, \ldots, a_n)$ of integers is *acceptable* if it satisfies the following conditions:

(a) $0 = |a_0| < |a_1| < \cdots < |a_{n-1}| < |a_n|$.

(b) The sets $\{|a_1 - a_0|, |a_2 - a_1|, \ldots, |a_{n-1} - a_{n-2}|, |a_n - a_{n-1}|\}$ and $\{1, 3, 9, \ldots, 3^{n-1}\}$ are equal.

Prove that the number of acceptable sequences of integers is $(n + 1)!$.

Answer: $\boxed{\text{N/A}}$ We actually prove a more general result via strong induction on n.

First, we state the more general result we wish to prove.

For $n > 0$, define a *great sequence* to be a sequence of integers $(a_0, \ldots, a_n)$ such that
1. $0 = |a_0| < |a_1| < \cdots < |a_{n-1}| < |a_n|$
2. Let $(b_1, b_2, \ldots, b_n)$ be a set of positive integers such that $3b_i \leq b_{i+1}$ for all i. The sets $\{|a_1 - a_0|, |a_2 - a_1|, \ldots, |a_{n-1} - a_{n-2}|, |a_n - a_{n-1}|\}$ and $\{b_1, b_2, \ldots, b_n\}$ are equal.

Then, the number of great sequences is $(n + 1)!$.

If we prove this statement, then we can just consider the specific case of $b_1 = 1$, $3b_i = b_{i+1}$ to solve our problem.

Before we proceed, we will prove a lemma.

Lemma: Let $(b_1, b_2, \ldots, b_n)$ be a set of positive integers such that $3b_i \leq b_{i+1}$ for all i. Then $b_i > 2\sum_{k=1}^{i-1} b_k$.

Proof: $\frac{b_i}{2} = \frac{b_i}{3} + \frac{b_i}{9} + \cdots \geq \sum_{k=1}^{i-1} b_k$. Equality only occurs when the sequence b_i is infinite, which is not the case, so the inequality holds.

We can now proceed with the induction. The base case is obvious. Assume it is true up to $n = j-1$. Next, consider some permutation of the set $B = \{b_1, b_2, b_3, \cdots, b_j\}$. Denote it as $C = \{c_1, c_2, \ldots, c_j\}$. Find the element $c_k = b_j$. For the first $k-1$ elements of C, we can put them in order and apply the inductive hypothesis. The number of great sequences such that the set of differences $\{|a_1 - a_0|, \ldots, |a_{k-1} - a_{k-2}|\}$ is equal to $\{c_1, c_2, \ldots, c_{k-1}\}$ is $k!$.

Then, a_k can be either $a_{k-1} + c_k$ or $a_{k-1} - c_k$. This is because, by the lemma, $c_k > 2\sum_{m=1}^{k-1} c_m = 2\sum_{m=1}^{k-1} |a_m - a_{m-1}| \geq 2|a_{k-1} - a_{k-2} + a_{k-2} - a_{k-3} + \cdots + a_1 - a_0| = 2|a_{k-1}|$. It is easy to check that for either possible value of a_k, $|a_k| > |a_{k-1}|$. After that, there is only one possible value for $a_{k+1}, \ldots, a_j$ because only one of $a_i \pm c_{i+1}$ will satisfy $|a_i| > |a_{i-1}|$.

There are $\binom{j-1}{k-1}$ possible ways to choose $c_1, c_2, \ldots, c_{k-1}$ from B. Given those elements of C, there are $k!$ ways to make a great sequence $(a_0, a_1, \ldots, a_{k-1})$. Then, there are 2 possible values for a_k. After that, there are $(j-k)!$ ways to order the remaining elements of C, and for each such ordering, there is exactly 1 possible great sequence $(a_0, a_1, \ldots, a_j)$.

Now, counting up all the possible ways to do this over all values of k, we get that the number of great sequences is equal to $\sum_{k=1}^{j} \binom{j-1}{k-1} k! 2(j-k)! = \sum_{k=1}^{j} 2(j-1)!(k) = 2(j-1)!\frac{j(j+1)}{2} = (j+1)!$. The induction is complete, and this finishes the proof.

Alternate solution: Another method of performing the induction is noting that any acceptable sequence $(a_0, \ldots, a_n)$ can be matched with $n+2$ acceptable sequences of length $n+2$ because we can take $(3a_0, \ldots, 3a_n)$ and add an element with a difference of 1 in any of $n+2$ positions.

7. [**30**] Find the maximum possible number of diagonals of equal length in a convex hexagon.

Answer: $\boxed{7}$ First, we will prove that 7 is possible. Consider the following hexagon $ABCDEF$ whose vertices are located at $A(0,0)$, $B(\frac{1}{2}, 1 - \frac{\sqrt{3}}{2})$, $C(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $D(0,1)$, $E(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $F(-\frac{1}{2}, 1 - \frac{\sqrt{3}}{2})$. One can easily verify that all diagonals but BE and CF have length 1.

Now suppose that there are at least 8 diagonals in a certain convex hexagon $ABCDEF$ whose lengths are equal. There must be a diagonal such that, with this diagonal taken out, the other 8 have equal length. There are two cases.

Case I. The diagonal is one of AC, BD, CE, DF, EA, FB. WLOG, assume it is AC. We have $EC = EB = FB = FC$. Thus, B and C are both on the perpendicular bisector of EF. Since $ABCDEF$ is convex, both B and C must be on the same side of line EF, but this is impossible as one of B or C, must be contained in triangle CEF. Contradiction.

Case II: The diagonal is one of AD, BE, CF. WLOG, assume it is AD. Again, we have $EC = EB = FB = FC$. By the above reasoning, this is a contradiction.

Thus, 7 is the maximum number of possible diagonals.

8. [**35**] Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, and $BC = 6$. Let D be the foot of the altitude from A to BC, and E be the intersection of AO with BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Determine the length of BX.

Answer: $\boxed{96/41}$ Let AX intersect the circumcircle of $\triangle ABC$ again at K. Let OY intersect AK and BC at T and L, respectively. We have $\angle LOA = \angle OYX = \angle TDX = \angle LAK$, so AL is tangent to the circumcircle. Furthermore, $OL \perp AK$, so $\triangle ALK$ is isosceles with $AL = AK$, so AK is also tangent to the circumcircle. Since BC and the tangents to the circumcircle at A and K all intersect at the same point L, CL is a symmedian of $\triangle ACK$. Then AK is a symmedian of $\triangle ABC$. Then we can use $\frac{BX}{XC} = \frac{(AB)^2}{(AC)^2}$ to compute $BX = \frac{96}{41}$.

9. [**35**] For integers $m, n \geq 1$, let $A(n,m)$ be the number of sequences $(a_1, \cdots, a_{nm})$ of integers satisfying the following two properties:

(a) Each integer k with $1 \leq k \leq n$ occurs exactly m times in the sequence $(a_1, \cdots, a_{nm})$.

(b) If i, j, and k are integers such that $1 \leq i \leq nm$ and $1 \leq j \leq k \leq n$, then j occurs in the sequence $(a_1, \cdots, a_i)$ at least as many times as k does.

For example, if $n = 2$ and $m = 5$, a possible sequence is $(a_1, \cdots, a_{10}) = (1,1,2,1,2,2,1,2,1,2)$. On the other hand, the sequence $(a_1, \cdots, a_{10}) = (1,2,1,2,2,1,1,1,2,2)$ does not satisfy property (2) for $i = 5$, $j = 1$, and $k = 2$.

Prove that $A(n,m) = A(m,n)$.

Answer: $\boxed{\text{N/A}}$ Solution 1: We show that $A(n,m)$ is equal to the the number of standard Young tableaux with n rows and m columns (i.e. fillings of an $n \times m$ matrix with the numbers $1, 2, ..., nm$ so that numbers are increasing in each row and column). Consider the procedure where every time a k appears in the sequences, you add a number to the leftmost empty spot of the k-th row. Doing this procedure will result in a valid standard Young tableau. The entries are increasing along every row because new elements are added from left to right. The elements are also increasing along every column. This is because the condition about the sequences implies that there will always be at least as many elements in row i as there are in row j for $i < j$. At the end of this procedure, the Young Tableaux has been filled because each of the n numbers have been added m times.

Now, consider a $n \times m$ standard Young tableau. If the number p is in row k, you add a k to the sequence. This will produce a valid sequence. To see this, suppose that p appears in the entry (x, y). Then all of the entries (q, y) where $q < x$ have already been added to the sequence because they must contain entries less than p. Thus, the numbers 1 through $x - 1$ have all already been added at least y times Then, when we process p, we are adding the y-th x, which is valid. At the end of this procedure, n numbers have been added m times to the sequence.

The two procedures given above are inverses of each other. Thus, $A(n,m)$ is equal to the number of $n \times m$ standard Young tableaux. For every $n \times m$ Tableaux, we can transpose it to form an $m \times n$ tableau. The number of such tableaux is $A(m,n)$. Thus, $A(n,m) = A(m,n)$.

Solution 2: We can also form a direct bijection to show $A(n,m) = A(m,n)$, as follows. Suppose that $a = (a_1, \ldots, a_{mn})$ is a sequence satisfying properties 1 and 2. We will define a sequence $f(a) = (b_1, \ldots, b_{nm})$ satisfying the same properties 1 and 2, but with m and n switched.

The bijection f is simple: just define b_i, for $1 \leq i \leq nm$, to be equal to the number of j with $1 \leq j \leq i$ such that $a_j = a_i$. In other words, to obtain $f(a)$ from a, replace the kth occurrence of each number with the number k. For example, if $n = 2$ and $m = 5$, then $f(1,1,2,1,2,2,1,2,1,2) = (1,2,1,3,2,3,4,4,5,5)$.

First, it is clear that $f(a)$ satisfies property 1 with m and n switched. Indeed, it follows directly from the definition of f that for each pair (k_1, k_2) with $1 \leq k_1 \leq n$ and $1 \leq k_2 \leq m$, there is exactly one index i for which $(a_i, b_i) = (k_1, k_2)$. This implies the desired result.

Second, we show that $f(a)$ satisfies property 2 with m and n switched. For this, suppose that i, j, k are integers with $1 \leq i \leq nm$ and $1 \leq j \leq k \leq m$. Then, the number of times that k appears in the sequence $(b_1, \ldots, b_i)$ is exactly equal to the number of integers ℓ with $1 \leq \ell \leq n$ such that ℓ appears at least k times in the sequence $(a_1, \ldots, a_i)$. Similarly, the number of times that j appears in the sequence $(b_1, \ldots, b_i)$ is exactly equal to the number of integers ℓ with $1 \leq j \leq n$ such that ℓ appears at least j

times in the sequence $(a_1, \ldots, a_i)$. In particular, the number j appears in the sequence $(b_1, \ldots, b_i)$ at least as many times as k does.

Now, we show that f is a bijection. We claim that f is actually an involution; that is, $f(f(a)) = a$. (Note in particular that this implies that f is a bijection, and its inverse is itself.)

Fix an index i with $1 \le i \le mn$. Let $\ell = a_i$ and $k = b_i$; then, k is the number of times that the term ℓ appears in the sequence $(a_1, \ldots, a_i)$. It is enough to show that ℓ is the number of times that the term k appears in the sequence $(b_1, \ldots, b_i)$. First, note that by definition of f, the number of times k appears in the sequence $(b_1, \ldots, b_i)$ is equal to the number of integers j such that the sequence $(a_1, \ldots, a_i)$ contains the number j at least k times.

Now, ℓ occurs exactly k times in the sequence $(a_1, \ldots, a_i)$. So by property 2, each j with $j \le \ell$ appears at least k times in the sequence $(a_1, \ldots, a_i)$. Furthermore, each j with $j > \ell$ appears at most $k-1$ times in the sequence $(a_1, \ldots, a_{i-1})$ and thus appears at most $k-1$ times in the sequence $(a_1, \ldots, a_i)$ as well (because $a_i = \ell$). Therefore, the number of integers j such that the sequence $(a_1, \ldots, a_i)$ contains the number j at least k times is exactly equal to ℓ. This shows the desired result.

Note: The bijections given in solutions 1 and 2 can actually be shown to be the same.

10. [**40**] Fix a positive real number $c > 1$ and positive integer n. Initially, a blackboard contains the numbers $1, c, \ldots, c^{n-1}$. Every minute, Bob chooses two numbers a, b on the board and replaces them with $ca + c^2 b$. Prove that after $n-1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L}-1}{c^{1/L}-1}\right)^L,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $L = 1 + \log_\phi(c)$.

Answer: $\boxed{\text{N/A}}$ By a simple reverse induction, we can show that at any instant, any number on the board takes the form $\sum_{\alpha \in A} c^{r_\alpha} c^\alpha$ for a certain $A \subseteq \{0, \ldots, n-1\}$ for non-negative integer weights r_α satisfying $\sum_{\alpha \in A} \phi^{-r_\alpha} = 1$, *where these subsets A partition $\{0, \ldots, n-1\}$, and after each minute the number of parts decreases by* 1. (Note that $1 = \phi^{-1} + \phi^{-2}$, and when we "merge" two of these sets A, B corresponding to $a, b \to ca + c^2 b$, we add 1 to all the weights of A, and 2 to all the weights of B). Therefore, the final number takes the form $\sum_{i=0}^{n-1} c^{t_i} c^i$ for positive integer weights $t_0, \ldots, t_{n-1}$ satisfying $\sum_{i=0}^{n-1} \phi^{-t_i} = 1$.

Let $M = \log_\phi(c) > 0$ (so $L = 1 + M$). Then $c = \phi^M$, so by Holder's inequality,

$$\left(\sum_{i=0}^{n-1} c^{t_i} c^i\right)^1 \left(\sum_{i=0}^{n-1} \phi^{-t_i}\right)^M \ge \left(\sum_{i=0}^{n-1} [c^i (c\phi^{-M})^{t_i}]^{1/L}\right)^L = \left(\frac{c^{n/L}-1}{c^{1/L}-1}\right)^L,$$

as desired.

Comment: Note that the same method applies for any choice of initial numbers $x_1, x_2, \ldots, x_n$ (the choice $c^0, c^1, \ldots, c^{n-1}$ simply gives a relatively clean closed form). The invariant $\sum \phi^{-t_i} = 1$ was inspired by the corresponding invariant $\sum 2^{-t_i} = 1$ for ISL 2007 A5, where the relevant operation is $\{a, b\} \mapsto \{c^1 a + c^1 b\}$ (specifically, for $c = 2$, but this is not important) rather than $\{a, b\} \mapsto \{c^1 a + c^2 b\}$.

Alternate solution: We have $cx^L + c^2 y^L \ge (x+y)^L$ for any nonnegative reals x, y, since $(cx^L + c^2 y^L)(c^{-1/(L-1)} + c^{-2/(L-1)})^{L-1} \ge (x+y)^L$ by Holder's inequality, $c^{1/(L-1)} = c^{1/\log_\phi(c)} = \phi$, and of course $\phi^{-1} + \phi^{-2} = 1$.

In particular, $(ca + c^2 b)^{1/L} \ge a^{1/L} + b^{1/L}$. Thus $\sum_{i=0}^{n-1} (c^i)^{1/L} \le N^{1/L}$, where N is the final (nonnegative) number on the board. Hence $N^{1/L} \ge \frac{c^{n/L}-1}{c^{1/L}-1}$.