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1. [**5**] Find the sum of all solutions for x:

$$\begin{aligned} xy &= 1 \\ x + y &= 3 \end{aligned}$$

2. [**5**] Evaluate the sum
$$1 - 2 + 3 - 4 + \cdots + 2007 - 2008.$$

3. [**5**] What is the largest x such that x^2 divides $24 \cdot 35 \cdot 46 \cdot 57$?

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4. [**6**] What is the smallest prime divisor of $5^{7^{10^{7^{10}}}} + 1$?

5. [**6**] What is the sum of all integers x such that $|x + 2| \leq 10$?

6. [**6**] Sarah is deciding whether to visit Russia or Washington, DC for the holidays. She makes her decision by rolling a regular 6-sided die. If she gets a 1 or 2, she goes to DC. If she rolls a $3, 4$, or 5, she goes to Russia. If she rolls a 6, she rolls again. What is the probability that she goes to DC?

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7. [**7**] Compute $2009^2 - 2008^2$.

8. [**7**] Alice rolls two octahedral dice with the numbers $2, 3, 4, 5, 6, 7, 8, 9$. What's the probability the two dice sum to 11?

9. [**7**] Let $a_0 = \frac{6}{7}$, and
$$a_{n+1} = \begin{cases} 2a_n & \text{if } a_n < \frac{1}{2} \\ 2a_n - 1 & \text{if } a_n \geq \frac{1}{2}. \end{cases}$$
Find a_{2008}.

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10. [**8**] Find the sum of all positive integers n such that n divides $n^2 + n + 2$.

11. [**8**] Al has a rectangle of integer side lengths a and b, and area 1000. What is the smallest perimeter it could have?

12. [**8**] Solve the following system of equations for w.

$$\begin{aligned} 2w + x + y + z &= 1 \\ w + 2x + y + z &= 2 \\ w + x + 2y + z &= 2 \\ w + x + y + 2z &= 1. \end{aligned}$$

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13. [**9**] Find the number of distinct primes dividing $1 \cdot 2 \cdot 3 \cdots 9 \cdot 10$.

14. [**9**] You have a 2×3 grid filled with integers between 1 and 9. The numbers in each row and column are distinct, the first row sums to 23, and the columns sum to $14, 16$, and 17 respectively.

	14	16	17
23	a	b	c
	x	y	z

What is $x + 2y + 3z$?

15. [**9**] A cat is going up a stairwell with ten stairs. However, instead of walking up the stairs one at a time, the cat jumps, going either two or three stairs up at each step (though if necessary, it will just walk the last step). How many different ways can the cat go from the bottom to the top?

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16. [**10**] If p and q are positive integers and $\frac{2008}{2009} < \frac{p}{q} < \frac{2009}{2010}$, what is the minimum value of p?

17. [**10**] Determine the last two digits of 17^{17}, written in base 10.

18. [**10**] Find the coefficient of x^6 in the expansion of

$$(x+1)^6 \cdot \sum_{i=0}^{6} x^i$$

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19. [**11**] Let P be a polynomial with $P(1) = P(2) = \cdots = P(2007) = 0$ and $P(0) = 2009!$. $P(x)$ has leading coefficient 1 and degree 2008. Find the largest root of $P(x)$.

20. [**11**] You have a die with faces labelled 1 through 6. On each face, you draw an arrow to an adjacent face, such that if you start on a face and follow the arrows, after 6 steps you will have passed through every face once and will be back on your starting face. How many ways are there to draw the arrows so that this is true?

21. [**11**] Call a number *overweight* if it has at least three positive integer divisors (including 1 and the number), and call a number *obese* if it has at least four positive integer divisors (including 1 and the number). How many positive integers between 1 and 200 are overweight, but not obese?

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22. [**12**] Sandra the Maverick has 5 pairs of shoes in a drawer, each pair a different color. Every day for 5 days, Sandra takes two shoes out and throws them out the window. If they are the same color, she treats herself to a practice problem from a past HMMT. What is the expected value (average number) of practice problems she gets to do?

23. [**12**] If x and y are real numbers such that $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$, find the largest possible value of $\frac{x^2}{4} + \frac{y^2}{9}$.

24. [**12**] Let $f(x) = \frac{1}{1-x}$. Let $f^{k+1}(x) = f(f^k(x))$, with $f^1(x) = f(x)$. What is $f^{2008}(2008)$?

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25. [**13**] Evaluate the sum
$$\cos\left(\frac{2\pi}{18}\right) + \cos\left(\frac{4\pi}{18}\right) + \cdots + \cos\left(\frac{34\pi}{18}\right).$$

26. [**13**] John M. is sitting at $(0,0)$, looking across the aisle at his friends sitting at (i,j) for each $1 \leq i \leq 10$ and $0 \leq j \leq 5$. Unfortunately, John can only see a friend if the line connecting them doesn't pass through any other friend. How many friends can John see?

27. [**13**] $ABCDE$ is a regular pentagon inscribed in a circle of radius 1. What is the area of the set of points inside the circle that are farther from A than they are from any other vertex?

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28. [**14**] Johnny the grad student is typing all the integers from 1 to ∞, in order. The 2 on his computer is broken however, so he just skips any number with a 2. What's the 2008th number he types?

29. [**14**] Let $p(x)$ be the polynomial of degree 4 with roots $1,2,3,4$ and leading coefficient 1. Let $q(x)$ be the polynomial of degree 4 with roots $1, \frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$ and leading coefficient 1. Find $\lim_{x\to 1} \frac{p(x)}{q(x)}$.

30. [**14**] Alice has an equilateral triangle ABC of area 1. Put D on BC, E on CA, and F on AB, with $BD = DC$, $CE = 2EA$, and $2AF = FB$. Note that AD, BE, and CF pass through a single point M. What is the area of triangle EMC?

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31. [**15**] Find the sum of all primes p for which there exists a prime q such that $p^2 + pq + q^2$ is a square.

32. [**15**] Pirate ships Somy and Lia are having a tough time. At the end of the year, they are both one pillage short of the minimum required for maintaining membership in the Pirate Guild, so they decide to pillage each other to bring their counts up. Somy by tradition only pillages $28 \cdot 3^k$ coins for integers k, and Lia by tradition only pillages $82 \cdot 3^j$ coins for integers j. Note that each pillage can have a different k or j. Soma and Lia work out a system where Somy pillages Lia n times, Lia pillages Somy n times, and after both sets of pillages Somy and Lia are financially even.

    What is the smallest n can be?

33. [**15**] The polynomial $ax^2 - bx + c$ has two distinct roots p and q, with a, b, and c positive integers and with $0 < p, q < 1$. Find the minimum possible value of a.

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34. [**20**] How many hits did *math tournament* get on Google the morning of November 8, 2008? If you submit integer N, and the correct answer is A, you will receive $\lfloor 20 \cdot \min\{\frac{N}{A}, \frac{A}{N}\} \rfloor$ points for this problem.

35. [**25**] Find $\max\{\text{Perimeter}(T)\}$ for T a triangle contained in a regular heptagon (7-sided figure) of unit edge length. Write your answer N to 2 places after the decimal. If the correct answer rounded to 2 decimal places is A, you will receive 0 points if $N > A$ and $\lfloor \max\{0, 25 - 50 \cdot (A - N)\} \rfloor$ points otherwise.

36. [**25**] How many numbers less than 1,000,000 are the product of exactly 2 distinct primes? You will receive $\lfloor 25 - 50 \cdot |\frac{N}{A} - 1| \rfloor$ points, if you submit N and the correct answer is A.

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