15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Geometry Test

1. ABC is an isosceles triangle with $AB = 2$ and $\measuredangle ABC = 90^\circ$. D is the midpoint of BC and E is on AC such that the area of $AEDB$ is twice the area of ECD. Find the length of DE.

   Answer: $\boxed{\frac{\sqrt{17}}{3}}$ Let F be the foot of the perpendicular from E to BC. We have $[AEDB]+[EDC] = [ABC] = 2 \Rightarrow [EDC] = \frac{2}{3}$. Since we also have $[EDC] = \frac{1}{2}(EF)(DC)$, we get $EF = FC = \frac{4}{3}$. So $FD = \frac{1}{3}$, and $ED = \frac{\sqrt{17}}{3}$ by the Pythagorean Theorem.

2. Let ABC be a triangle with $\angle A = 90^\circ$, $AB = 1$, and $AC = 2$. Let ℓ be a line through A perpendicular to BC, and let the perpendicular bisectors of AB and AC meet ℓ at E and F, respectively. Find the length of segment EF.

   Answer: $\boxed{\frac{3\sqrt{5}}{4}}$ Let M, N be the midpoints of AB and AC, respectively. Then we have $\angle EAB = \angle ACB$ and $\angle EAC = \angle ABC$, so $AEM \sim CBA \Rightarrow AE = \frac{\sqrt{5}}{4}$ and $FAN \sim CBA \Rightarrow AF = \sqrt{5}$. Consequently, $EF = AF - AE = \frac{3\sqrt{5}}{4}$.

3. Let ABC be a triangle with incenter I. Let the circle centered at B and passing through I intersect side AB at D and let the circle centered at C passing through I intersect side AC at E. Suppose DE is the perpendicular bisector of AI. What are all possible measures of angle BAC in degrees?

   Answer: $\boxed{\frac{540}{7}}$ Let $\alpha = \measuredangle BAC$. DE is the perpendicular bisector of AI, so $DA = DI$, and $\angle DIA = \angle DAI = \alpha/2$. Thus, $\angle IDB = \angle DIB = \alpha$, since $BD = BI$. This gives $\angle DBI = 180^\circ - 2\alpha$, so that $\angle ABC = 360^\circ - 4\alpha$. Similarly, $\angle ACB = 360^\circ - 4\alpha$. Now, summing the angles in ABC, we find $720^\circ - 7\alpha = 180^\circ$, so that $\alpha = \frac{540}{7}^\circ$.

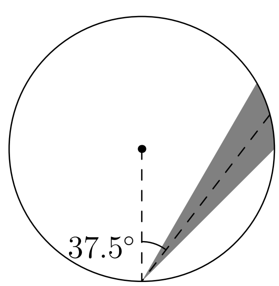
4. There are circles ω_1 and ω_2. They intersect in two points, one of which is the point A. B lies on ω_1 such that AB is tangent to ω_2. The tangent to ω_1 at B intersects ω_2 at C and D, where D is the closer to B. AD intersects ω_1 again at E. If $BD = 3$ and $CD = 13$, find EB/ED.

   Answer: $\boxed{4\sqrt{3}/3}$

   [diagram]

   By power of a point, $BA = \sqrt{BD \cdot BC} = 4\sqrt{3}$. Also, $DEB \sim DBA$, so $EB/ED = BA/BD = 4\sqrt{3}/3$.

5. A mouse lives in a circular cage with completely reflective walls. At the edge of this cage, a small flashlight with vertex on the circle whose beam forms an angle of 15° is centered at an angle of 37.5° away from the center. The mouse will die in the dark. What fraction of the total area of the cage can keep the mouse alive?



Answer: $\boxed{\frac{3}{4}}$ We claim that the lit region is the entire cage except for a circle of half the radius of the cage in the center, along with some isolated points on the boundary of the circle and possibly minus a set of area 0. Note that the region is the same except for a set of area 0 if we disallow the light paths at the very edge of the beam. In that case, we can see that the lit region is an open subset of the disk, as clearly the region after k bounces is open for each k and the union of open sets is again open. We will then show that a dense subset of the claimed region of the cage is lit.

First, let us show that no part of the inner circle is lit. For any given light path, each chord of the circle is the same length, and in particular the minimum distance from the center of the circle is the same on each chord of the path. Since none of the initial chords can come closer than half the cage's radius to the center, the circle with half the cage's radius is indeed dark.

Now we need to show that for each open subset of the outer region, there is a light path passing through it, which will imply that the unlit region outside the small circle contains no open set, and thus has area 0. To do this, simply consider a light path whose angle away from the center is irrational such that the distance d from the center of the cage to the midpoint of the first chord is sufficiently close to the distance from the center of the cage to a point in the open set we're considering. Each successive chord of the light path can be seen as a rotation of the original one, and since at each step it is translated by an irrational angle, we obtain a dense subset of all the possible chords. This means that we obtain a dense subset of the circumference of the circle of radius d centered at the center of the cage, and in particular a point inside the open set under consideration, as desired.

Therefore, the lit region of the cage is the area outside the concentric circle of half the radius plus or minus some regions of area 0, which tells us that $\frac{3}{4}$ of the cage is lit.

6. Triangle ABC is an equilateral triangle with side length 1. Let $X_0, X_1, \ldots$ be an infinite sequence of points such that the following conditions hold:

   - X_0 is the center of ABC
   - For all $i \geq 0$, X_{2i+1} lies on segment AB and X_{2i+2} lies on segment AC.
   - For all $i \geq 0$, $\measuredangle X_i X_{i+1} X_{i+2} = 90^\circ$.
   - For all $i \geq 1$, X_{i+2} lies in triangle $AX_i X_{i+1}$.

   Find the maximum possible value of $\sum_{i=0}^{\infty} |X_i X_{i+1}|$, where $|PQ|$ is the length of line segment PQ.

   Answer: $\boxed{\frac{\sqrt{6}}{3}}$ Let Y be the foot of the perpendicular from A to X_0X_1: note that the sum we wish to minimize is simply $X_0Y + YA$. However, it is not difficult to check (for example, by AM-GM) that $AY + YX_0 \geq \sqrt{2}AX_0 = \sqrt{6}/3$. This may be achieved by making $\angle YX_0A = 45^\circ$, so that $\angle AX_1X_0 = 105^\circ$.

7. Let S be the set of the points $(x_1, x_2, \ldots, x_{2012})$ in 2012-dimensional space such that $|x_1| + |x_2| + \cdots + |x_{2012}| \leq 1$. Let T be the set of points in 2012-dimensional space such that $\max_{i=1}^{2012} |x_i| = 2$. Let p be a randomly chosen point on T. What is the probability that the closest point in S to p is a vertex of S?

   Answer: $\boxed{\frac{1}{2^{2011}}}$ Note that T is a hypercube in 2012-dimensional space, containing the rotated hyperoctahedron S. Let v be a particular vertex of S, and we will consider the set of points x on T such that v is the closest point to x in S. Let w be another point of S and let ℓ be the line between v and w. Then in order for v to be the closest point to x in S, it must also be so on the region of ℓ contained in S. This condition is then equivalent to the projection of x lying past v on the line ℓ, or alternatively that v lies in the opposite halfspace of w defined by the hyperplane perpendicular to ℓ and passing through v. This can be written algebraically as $(x - v) \cdot (w - v) \leq 0$. Therefore, v is the closest point to x if and only if $(x - v) \cdot (w - v) \leq 0$ for all w in S.

   Note that these conditions do not depend on where w is on the line, so for each line intersecting S nontrivially, let us choose w such that w lies on the hyperplane H containing all the vertices of S except for v and $-v$. We can further see that the conditions are linear in w, so them holding for all w

in H is equivalent to them holding on the vertices of the region $S \cap H$, which are simply the vertices of S except for v and $-v$. Let us now compute what these conditions look like.

Without loss of generality, let $v = (1, 0, \ldots, 0)$ and $w = (0, 1, 0, \ldots, 0)$. Then the equation is of the form $(x - v) \cdot (-1, 1, 0, \ldots, 0) \leq 0$, which we can rewrite as $(x_1 - 1, x_2, x_3, \ldots, x_2 012) \cdot (-1, 1, 0, \ldots, 0) = 1 - x_1 + x_2 \leq 0$. For the other choices of w, we get the similar conditions that $1 - x_1 + x_i \leq 0$ and also $1 - x_1 - x_i \leq 0$ for each $i \in \{2, \ldots, 2012\}$. Note that if any $x_i \in \{2, -2\}$ for $i \neq 1$, then one of these conditions trivially fails, as it would require $3 - x_1 \leq 0$. Therefore, the only face of T where x can lie is the face defined by $x_1 = 2$, which gives us the conditions that $-1 + x_i \leq 0$ and $-1 - x_i \leq 0$, so $x_i \in [-1, 1]$ for all $i \in \{2, \ldots, 2012\}$. This defines a 2011-dimensional hypercube of side length 2 on the face of T defined by $x_1 = 2$, and we obtain similar regions on each of the other faces corresponding to the other vertices of S. Therefore, the volume of the set of x for which x is closest to a vertex is $2 \cdot 2012 \cdot 2^{2011}$ and the volume of all the choices of x is $2 \cdot 2012 \cdot 4^{2011}$, so the desired probability is $\frac{1}{2^{2011}}$.

8. Hexagon $ABCDEF$ has a circumscribed circle and an inscribed circle. If $AB = 9$, $BC = 6$, $CD = 2$, and $EF = 4$. Find $\{DE, FA\}$.

Answer: $\boxed{\{\frac{9+\sqrt{33}}{2}, \frac{9-\sqrt{33}}{2}\}}$ By Brianchon's Theorem, AD, BE, CF concur at some point P. Also, it follows from the fact that tangents from a point to a circle have equal lengths that $AB + CD + EF = BC + DE + FA$. Let $DE = x$, so that $FA = 9 - x$.

Note that $APB \sim EPD$, $BPC \sim FPE$, and $CPD \sim APF$. The second similarty gives $BP/FP = 3/2$, so that $BP = 3y, FP = 2y$ for some y. From here, the first similarity gives $DP = xy/3$. Now, the third similarity gives $4y = \frac{(9-x)xy}{3}$, so that $x^2 - 9x + 12 = 0$. It follows that $x = \dfrac{9 \pm \sqrt{33}}{2}$, giving our answer.

9. Let O, O_1, O_2, O_3, O_4 be points such that O_1, O, O_3 and O_2, O, O_4 are collinear in that order, $OO_1 = 1, OO_2 = 2, OO_3 = \sqrt{2}, OO_4 = 2$, and $\measuredangle O_1OO_2 = 45°$. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the circles with respective centers O_1, O_2, O_3, O_4 that go through O. Let A be the intersection of ω_1 and ω_2, B be the intersection of ω_2 and ω_3, C be the intersection of ω_3 and ω_4, and D be the intersection of ω_4 and ω_1, with A, B, C, D all distinct from O. What is the largest possible area of a convex quadrilateral $P_1P_2P_3P_4$ such that P_i lies on O_i and that A, B, C, D all lie on its perimeter?

Answer: $\boxed{8 + 4\sqrt{2}}$ We first maximize the area of triangle P_1OP_2, noting that the sum of the area of P_1OP_2 and the three other analogous triangles is the area of $P_1P_2P_3P_4$. Note that if $A \neq P_1, P_2$, without loss of generality say $\angle OAP_1 < 90°$. Then, $\angle OO_1P_1 = 2\angle OAP_1$, and since $\angle OAP_2 = 180° - \angle OAP_1 > 90°$, we see that $\angle OO_2P_2 = 2\angle OAP_1$ as well, and it follows that $OO_1P_1 \sim OO_2P_2$. This is a spiral similarity, so $OO_1O_2 \sim OP_1P_2$, and in particular $\angle P_1OP_2 = \angle O_1OO_2$, which is fixed. By the sine area formula, to maximize $OP_1 \cdot OP_2$, which is bounded above by the diameters $2(OO_1), 2(OO_2)$. In a similar way, we want P_3, P_4 to be diametrically opposite O_3, O_4 in their respective circles.

When we take these P_i, we indeed have $A \in P_1P_2$ and similarly for B, C, D, since $\angle OAP_1 = \angle OAP_2 = 90°$. To finish, the area of the quadrilateral is the sum of the areas of the four triangles, which is

$$\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot 2^2 \cdot (1 \cdot 2 + 2 \cdot \sqrt{2} + \sqrt{2} \cdot 2 + 2 \cdot 1) = 8 + 4\sqrt{2}.$$

10. Let C denote the set of points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \leq 1$. A sequence $A_i = (x_i, y_i) | i \geq 0$ of points in $\mathbb{R}^2$ is 'centric' if it satisfies the following properties:

- $A_0 = (x_0, y_0) = (0, 0)$, $A_1 = (x_1, y_1) = (1, 0)$.
- For all $n \geq 0$, the circumcenter of triangle $A_nA_{n+1}A_{n+2}$ lies in C.

Let K be the maximum value of $x_{2012}^2 + y_{2012}^2$ over all centric sequences. Find all points (x, y) such that $x^2 + y^2 = K$ and there exists a centric sequence such that $A_{2012} = (x, y)$.

Answer: $\boxed{(-1006, 1006\sqrt{3}), (-1006, -1006\sqrt{3})}$ Consider any triple of points $\triangle A_n A_{n+1} A_{n+2}$ with circumcenter P_n. By the Triangle Inequality we have $A_n P_n \leq A_n A_0 + A_0 P_n \leq A_n A_0 + 1$. Since P_n is the circumcenter, we have $P_n A_n = P_n A_{n+2}$. Finally we have $A_{n+2} a_0 \leq P_n A_{n+2} + 1 = A_n P_n + 1 \leq A_n A_0 + 2$. Therefore $\sqrt{x_{n+2}^2 + y_{n+2}^2} \leq \sqrt{x_n^2 + y_n^2} + 2$.

It is also clear that equality occurs if and only if A_n, A_0, P_n, A_{n+2} are all collinear and P_n lies on the unit circle.

From the above inequality it is clear that $\sqrt{x_{2012}^2 + y_{2012}^2} \leq 2012$. So the maximum value of K is 2012^2.

Now we must find all points A_2 that conforms to the conditions of the equality case. P_0 must lie on the unit circle, so it lies on the intersection of the unit circle with the perpendicular bisector of $A_0 A_1$, which is the line $x = \frac{1}{2}$. Thus P_0 must be one of $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$. From now on we assume that we take the positive root, as the negative root just reflects all successive points about the x-axis.

If $P_0 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ then A_0, P_0, A_2 must be colinear, so $A_2 = (1, \sqrt{3})$.

Then since we must have $A_0, P_{2n}, A_{2n}, A_{2n+2}$ colinear and P_{2n} on the unit circle, it follows that $P_{2n} = (-1)^n \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Then by induction we have $A_{2n} = (-1)^{n+1}(n, n\sqrt{3})$. To fill out the rest of the sequence, we may take $A_{2n+1} = (-1)^n(n+1, -n\sqrt{3})$ and $P_{2n+1} = (-1)^{n+1}\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ so that we get the needed properties.

Therefore the answer is

$$A_{2012} \in \{(-1006, 1006\sqrt{3}), (-1006, -1006\sqrt{3})\}$$

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