HMMT November 2012

Saturday 10 November 2012

Team Round

Divisors

In this section, the word *divisor* is used to refer to a *positive* divisor of an integer; that is, a divisor of a positive integer n is a positive integer d such that $\frac{n}{d}$ is an integer.

1. [**3**] Find the number of integers between 1 and 200 inclusive whose distinct prime divisors sum to 16. (For example, the sum of the distinct prime divisors of 12 is $2 + 3 = 5$.)

2. [**5**] Find the number of ordered triples of divisors (d_1, d_2, d_3) of 360 such that $d_1 d_2 d_3$ is also a divisor of 360.

3. [**6**] Find the largest integer less than 2012 all of whose divisors have at most two 1's in their binary representations.

Permutations

A *permutation* π is defined as a function from a set of integers to itself that rearranges the elements of the set. For example, a possible permutation of the numbers from 1 through 4 is the function π given by $\pi(1) = 2$, $\pi(2) = 4$, $\pi(3) = 3$, $\pi(4) = 1$.

4. [**3**] Let π be a permutation of the numbers from 2 through 2012. Find the largest possible value of $\log_2 \pi(2) \cdot \log_3 \pi(3) \cdots \log_{2012} \pi(2012)$.

5. [**4**] Let π be a randomly chosen permutation of the numbers from 1 through 2012. Find the probability that $\pi(\pi(2012)) = 2012$.

6. [**6**] Let π be a permutation of the numbers from 1 through 2012. What is the maximum possible number of integers n with $1 \leq n \leq 2011$ such that $\pi(n)$ divides $\pi(n+1)$?

7. [**8**] Let $A_1 A_2 \ldots A_{100}$ be the vertices of a regular 100-gon. Let π be a randomly chosen permutation of the numbers from 1 through 100. The segments $A_{\pi(1)}A_{\pi(2)}, A_{\pi(2)}A_{\pi(3)}, \ldots, A_{\pi(99)}A_{\pi(100)}, A_{\pi(100)}A_{\pi(1)}$ are drawn. Find the expected number of pairs of line segments that intersect at a point in the interior of the 100-gon.

Circumcircles

The *circumcircle* of a triangle is the circle passing through all three vertices of the triangle.

8. [**4**] ABC is a triangle with $AB = 15$, $BC = 14$, and $CA = 13$. The altitude from A to BC is extended to meet the circumcircle of ABC at D. Find AD.

9. [**5**] Triangle ABC satisfies $\angle B > \angle C$. Let M be the midpoint of BC, and let the perpendicular bisector of BC meet the circumcircle of $\triangle ABC$ at a point D such that points A, D, C, and B appear on the circle in that order. Given that $\angle ADM = 68^\circ$ and $\angle DAC = 64^\circ$, find $\angle B$.

10. [**6**] Triangle ABC has $AB = 4$, $BC = 5$, and $CA = 6$. Points A', B', C' are such that $B'C'$ is tangent to the circumcircle of $\triangle ABC$ at A, $C'A'$ is tangent to the circumcircle at B, and $A'B'$ is tangent to the circumcircle at C. Find the length $B'C'$.