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1. **[5]** Two hexagons are attached to form a new polygon P. Compute the minimum number of sides that P can have.

2. **[5]** Let a be a positive integer such that $2a$ has units digit 4. What is the sum of the possible units digits of $3a$?

3. **[5]** How many six-digit multiples of 27 have only 3, 6, or 9 as their digits?

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4. **[6]** Ainsley and Buddy play a game where they repeatedly roll a standard fair six-sided die. Ainsley wins if two multiples of 3 in a row are rolled before a non-multiple of 3 followed by a multiple of 3, and Buddy wins otherwise. If the probability that Ainsley wins is $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

5. **[6]** The points $(0,0), (1,2), (2,1), (2,2)$ in the plane are colored red while the points $(1,0), (2,0), (0,1), (0,2)$ are colored blue. Four segments are drawn such that each one connects a red point to a blue point and each colored point is the endpoint of some segment. The smallest possible sum of the lengths of the segments can be expressed as $a + \sqrt{b}$, where a, b are positive integers. Compute $100a + b$.

6. **[6]** If x, y, z are real numbers such that $xy = 6$, $x - z = 2$, and $x + y + z = 9$, compute $\frac{x}{y} - \frac{z}{x} - \frac{z^2}{xy}$.

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7. **[7]** Compute the maximum number of sides of a polygon that is the cross-section of a regular hexagonal prism.

8. **[7]** A small village has n people. During their yearly elections, groups of three people come up to a stage and vote for someone in the village to be the new leader. After every possible group of three people has voted for someone, the person with the most votes wins.

   This year, it turned out that everyone in the village had the exact same number of votes! If $10 \le n \le 100$, what is the number of possible values of n?

9. **[7]** A fair coin is flipped eight times in a row. Let p be the probability that there is exactly one pair of consecutive flips that are both heads and exactly one pair of consecutive flips that are both tails. If $p = \frac{a}{b}$, where a, b are relatively prime positive integers, compute $100a + b$.

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10. **[8]** The number 3003 is the only number known to appear eight times in Pascal's triangle, at positions

$$\binom{3003}{1}, \binom{3003}{3002}, \binom{a}{2}, \binom{a}{a-2}, \binom{15}{b}, \binom{15}{15-b}, \binom{14}{6}, \binom{14}{8}.$$

   Compute $a + b(15 - b)$.

11. **[8]** Two diameters and one radius are drawn in a circle of radius 1, dividing the circle into 5 sectors. The largest possible area of the smallest sector can be expressed as $\frac{a}{b}\pi$, where a, b are relatively prime positive integers. Compute $100a + b$.

12. **[8]** In a single-elimination tournament consisting of $2^9 = 512$ teams, there is a strict ordering on the skill levels of the teams, but Joy does not know that ordering. The teams are randomly put into a bracket and they play out the tournament, with the better team always beating the worse team. Joy is then given the results of all 511 matches and must create a list of teams such that she can guarantee that the third-best team is on the list. What is the minimum possible length of Joy's list?

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13. **[9]** Wendy is playing darts with a circular dartboard of radius 20. Whenever she throws a dart, it lands uniformly at random on the dartboard. At the start of her game, there are 2020 darts placed randomly on the board. Every turn, she takes the dart farthest from the center, and throws it at the board again. What is the expected number of darts she has to throw before all the darts are within 10 units of the center?

14. **[9]** A point (x, y) is selected uniformly at random from the unit square $S = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. If the probability that $(3x+2y, x+4y)$ is in S is $\frac{a}{b}$, where a, b are relatively prime positive integers, compute $100a + b$.

15. **[9]** For a real number r, the quadratics $x^2 + (r - 1)x + 6$ and $x^2 + (2r + 1)x + 22$ have a common real root. The sum of the possible values of r can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

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16. **[10]** Three players play tic-tac-toe together. In other words, the three players take turns placing an "A", "B", and "C", respectively, in one of the free spots of a 3×3 grid, and the first player to have three of their label in a row, column, or diagonal wins. How many possible final boards are there where the player who goes third wins the game? (Rotations and reflections are considered different boards, but the order of placement does not matter.)

17. **[10]** Let $\mathbb{N}_{>1}$ denote the set of positive integers greater than 1. Let $f : \mathbb{N}_{>1} \to \mathbb{N}_{>1}$ be a function such that $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}_{>1}$. If $f(101!) = 101!$, compute the number of possible values of $f(2020 \cdot 2021)$.

18. **[10]** Suppose Harvard Yard is a 17×17 square. There are 14 dorms located on the perimeter of the Yard. If s is the minimum distance between two dorms, the maximum possible value of s can be expressed as $a - \sqrt{b}$ where a, b are positive integers. Compute $100a + b$.

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19. **[11]** Three distinct vertices of a regular 2020-gon are chosen uniformly at random. The probability that the triangle they form is isosceles can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

20. **[11]** Let ω_1 be a circle of radius 5, and let ω_2 be a circle of radius 2 whose center lies on ω_1. Let the two circles intersect at A and B, and let the tangents to ω_2 at A and B intersect at P. If the area of $\triangle ABP$ can be expressed as $\frac{a\sqrt{b}}{c}$, where b is square-free and a, c are relatively prime positive integers, compute $100a + 10b + c$.

21. **[11]** Let $f(n)$ be the number of distinct prime divisors of n less than 6. Compute

$$\sum_{n=1}^{2020} f(n)^2.$$

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22. **[12]** In triangle ABC, $AB = 32$, $AC = 35$, and $BC = x$. What is the smallest positive integer x such that $1 + \cos^2 A$, $\cos^2 B$, and $\cos^2 C$ form the sides of a non-degenerate triangle?

23. **[12]** Two points are chosen inside the square $\{(x, y) \mid 0 \leq x, y \leq 1\}$ uniformly at random, and a unit square is drawn centered at each point with edges parallel to the coordinate axes. The expected area of the union of the two squares can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

24. **[12]** Compute the number of positive integers less than 10! which can be expressed as the sum of at most 4 (not necessarily distinct) factorials.

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25. **[13]** Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers where $a_1 = \sum_{i=0}^{100} i!$ and $a_i + a_{i+1}$ is an odd perfect square for all $i \geq 1$. Compute the smallest possible value of a_{1000}.

26. **[13]** Two players play a game where they are each given 10 indistinguishable units that must be distributed across three locations. (Units cannot be split.) At each location, a player wins at that location if the number of units they placed there is at least 2 more than the units of the other player. If both players distribute their units randomly (i.e. there is an equal probability of them distributing their units for any attainable distribution across the 3 locations), the probability that at least one location is won by one of the players can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

27. **[13]** In $\triangle ABC$, D and E are the midpoints of BC and CA, respectively. AD and BE intersect at G. Given that $GECD$ is cyclic, $AB = 41$, and $AC = 31$, compute BC.

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28. **[15]** Bernie has 2020 marbles and 2020 bags labeled $B_1, \ldots, B_{2020}$ in which he randomly distributes the marbles (each marble is placed in a random bag independently). If E the expected number of integers $1 \leq i \leq 2020$ such that B_i has at least i marbles, compute the closest integer to $1000E$.

29. **[15]** In acute triangle ABC, let H be the orthocenter and D the foot of the altitude from A. The circumcircle of triangle BHC intersects AC at $E \neq C$, and AB at $F \neq B$. If $BD = 3$, $CD = 7$, and $\frac{AH}{HD} = \frac{5}{7}$, the area of triangle AEF can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

30. **[15]** Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers that satisfies

$$\sum_{n=k}^{\infty} \binom{n}{k} a_n = \frac{1}{5^k},$$

for all positive integers k. The value of $a_1 - a_2 + a_3 - a_4 + \cdots$ can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$.

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31. **[17]** For some positive real α, the set S of positive real numbers x with $\{x\} > \alpha x$ consists of the union of several intervals, with total length 20.2. The value of α can be expressed as $\frac{a}{b}$, where a, b are relatively prime positive integers. Compute $100a + b$. (Here, $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.)

32. **[17]** The numbers $1, 2, \ldots, 10$ are written in a circle. There are four people, and each person randomly selects five consecutive integers (e.g. $1, 2, 3, 4, 5$, or $8, 9, 10, 1, 2$). If the probability that there exists some number that was not selected by any of the four people is p, compute $10000p$.

33. **[17]** In quadrilateral $ABCD$, there exists a point E on segment AD such that $\frac{AE}{ED} = \frac{1}{9}$ and $\angle BEC$ is a right angle. Additionally, the area of triangle CED is 27 times more than the area of triangle AEB. If $\angle EBC = \angle EAB$, $\angle ECB = \angle EDC$, and $BC = 6$, compute the value of AD^2.

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34. **[20]** Let a be the proportion of teams that correctly answered problem 1 on the Guts round. Estimate $A = \lfloor 10000a \rfloor$. An estimate of E earns $\max(0, \lfloor 20 - |A - E|/20 \rfloor)$ points. If you have forgotten, question 1 was the following:

    Two hexagons are attached to form a new polygon P. What is the minimum number of sides that P can have?

35. **[20]** Estimate A, the number of times an 8-digit number appears in Pascal's triangle. An estimate of E earns $\max(0, \lfloor 20 - |A - E|/200 \rfloor)$ points.

36. **[20]** Let p_i be the ith prime. Let

$$f(x) = \sum_{i=1}^{50} p_i x^{i-1} = 2 + 3x + \cdots + 229x^{49}.$$

If a is the unique positive real number with $f(a) = 100$, estimate $A = \lfloor 100000a \rfloor$. An estimate of E will earn $\max(0, \lfloor 20 - |A - E|/250 \rfloor)$ points.