12^{th} Annual Harvard-MIT Math Tournament

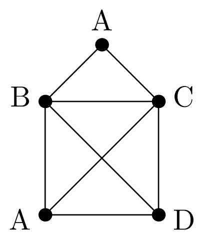
Saturday 21 February 2009

Team Round - Division B

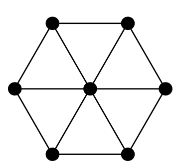
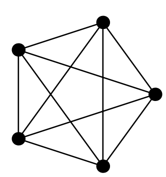
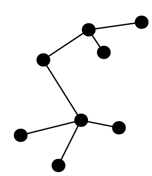
A *graph* consists of a set of vertices and a set of edges connecting pairs of vertices. If vertices u and v are connected by an edge, we say they are *adjacent*. The *degree* of a vertex u is the number of vertices v which are adjacent to u. A *finite* graph has a finite number of vertices.

A *coloring* of a graph is an assignment of a color to each vertex of the graph. A coloring is *good* if no two adjacent vertices are the same color. The *chromatic number* of a graph is the minimum number of colors needed in any good coloring of that graph.

For example, the chromatic number of the following graph is 4. The good coloring shown below uses four colors, and there is no good coloring using only three colors because four of the vertices are mutually adjacent.



1. [**6**] What are the chromatic numbers of each of the three graphs shown below? Draw a coloring having the minimum number of colors for each. Label the vertices with symbols to indicate the color of each vertex. (For example, you may mark a vertex "R" if you wish to indicate that it is colored red.)

2. [**6**] In a *connected* graph, it is possible to reach any vertex from any other vertex by following the edges. A *tree* is a connected graph with n vertices and $n-1$ edges for some positive integer n. Suppose $n \geq 2$. What is the chromatic number of a tree having n vertices? Prove your answer.

3. [**8**] Let $n \geq 3$ be a positive integer. A *triangulation* of a convex n-gon is a set of $n-3$ of its diagonals which do not intersect in the interior of the polygon. Along with the n sides, these diagonals separate the polygon into $n-2$ disjoint triangles. Any triangulation can be viewed as a graph: the vertices of the graph are the corners of the polygon, and the n sides and $n-3$ diagonals are the edges.

   For a fixed n-gon, different triangulations correspond to different graphs. Prove that all of these graphs have the same chromatic number.

4. [**10**] Let G be a finite graph in which every vertex has degree less than or equal to k. Prove that the chromatic number of G is less than or equal to $k+1$.

5. [**10**] A *k-clique* of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent. The *clique number* of a graph is the size of the largest clique in the graph. Does there exist a graph which has a clique number smaller than its chromatic number?

6. (a) [**5**] If a single vertex is removed from a finite graph, show that the graph's chromatic number cannot decrease by more than 1.

   (b) [**15**] Show that, for any $n > 2$, there are infinitely many graphs with chromatic number n such that removing any vertex from the graph decreases its chromatic number.