The calculus of p-variation

Vygantas Butkus and Rimas Norvaiša
Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania

August 5, 2014

Abstract

The algorithm that calculates p-variation of the finite sample is presented in the article. This algorithm could be used to calculated the p-variation of any piecewise monotone functions. All the properties that were used in the article are formulated and proved.

1 Introduction

The p-variation is a generalization of the total variation of a function, namely, it is supremum of the sums of the p-th powers of absolute increments off over nonoverlapping intervals (the formal definition is given in Definition 2.2). The total variation is a special case of p-variation, then $p = 1$, i.e. the total variation is equal to 1-variation.

The p-variation was first defined by Wiener in 1924 [12]. In this work, the main focus was on 2-variation. Later on, Young extended the topic of p-variation in his work [4], there he analysed p-variations with $p \neq 2$. Lots and detailed information on the known properties of p-variation can be found in the books of Norvaiša and Dudley (see [2] and [3]). There is formulated and proved a lot of the properties of p-variation. In addition, the comprehensive bibliography of p-variation is include.

A regulated function f is called *rough* if its 1-variation value is infinite. The class of rough functions appears to be quite an important one. Probably the best know example of rough function is a trajectory of Wiener process. The Wiener is a well known process (see [5]) used in statistics, physics and economic ([1]). It is worth noting, that rough functions can not be investigated with regular tools of calculus as derivative or Lebesgue–Stieltjes integral. The p-variation plays an important role in the filed of rough function analysis. Firstly, the p-variation gives a convenient way to measure a degree of roughness. Secondly, it appears that p-variation is a very useful tool in calculus of rough functions. A good overview of the importance of p-variation in rough function analysis is given in [6].

The exact value of p-variation was not used to be a topic. Rather, the primary focus on p-variation properties is whether the p-variation bounded or unbounded. But the the work [7] presented by Norvaiša and Račkauskas greatly increased the motivation of obtaining the the exact value of p-variation. The work [7] presents necessary and sufficient conditions for the convergence in law of partial sum processes in p-variation norm. The result could be applied in the statistical data analysis, which uses p-variation as data statistics. For applying those theoretical result in data analysis it is necessary to have procedure that actually calculates the value of p-variation. Nonetheless, it seams that there is no proposed procedure of calculation of p-variation value.

The main goal of this article is to present the algorithm, which calculates p-variation of the sample (see Def. 2.16), and give all the mathematical proofs of the properties that were used in this algorithm. The algorithm is denoted as *pvar*. It is already realised in the R environment (see [11]) and it is publicly available in *pvar* package on CRAN[1]. It is worth noting, that this procedure could be used to calculate p-variation of any piecewise monotone function.

The mathematical properties that where used in the algorithm are presented and proved in the next section of this article. In addition, all known properties that are relevant to this work are listed with the reference to original works. Namely, in large extent we use the properties presented in J. Qian Ph.D. thesis [9][2]. And in the Section 3 we will present the *pvar* algorithm that calculates the p-variation of the sample.

2 Mathematical analysis

The main purpose of this sections is to ground all the properties of p-variations that were used in the calculation of p-variation. In this article, the Theorem 2.26 is considered to be the most important mathematical result, which is quite general and possible could be used in many applications. The other properties are specificity formulated to be used in algorithm, nevertheless, someone could find them to be useful as well.

Important notations

- $PP[a, b]$ – the set of all point partitions of $[a, b]$ (def. 2.1).

[1] http://cran.r-project.org/web/packages/pvar/index.html

[2] The main result of J. Qian Ph.D. thesis are presented in article [10].

- $s_p(f;\kappa)$ – p-variation sum (def. 2.2).
- $v_p\,(f)$ – p-variation of the function f (def. 2.2).
- $SP_p(f,[a,b])$ – the set of supreme partitions (def. 2.2).
- $\overline{SP}_p(f,[a,b])$ – a set of points that are in any supreme partition (def. 2.20).
- $PM[a,b]$ – a set of piecewise monotone functions (def. 2.6).
- $CPM[a,b]$ – a set of continuous piecewise monotone functions (def. 2.6).
- $K(f,[a,b])$ – minimal size of PM partitions (prop. 2.7).
- $X(f,[a,b])$ – the set of PM partitions with minimal size (def. 2.8).

2.1 General known properties

Definition 2.1 (Partition). Let $J=[a,b]$ be a closed interval of real numbers with $-\infty<a\leq b<+\infty$. If $a<b$, an ordered set $\kappa=\{x_i\}_{i=0}^n$ of points in $[a,b]$ such that a $a=x_0<x_1<x_2<...<x_n=b$ is called a *(point) partition*. The size of the partition is denoted $|\kappa|:=\#\kappa-1=n$. The set of all point partitions of $[a,b]$ is denoted by $PP[a,b]$.

Definition 2.2 (p-variation). Let $f:[a,b]\to\mathbb{R}$ be a real function from an interval $[a,b]$. If $a<b$, for $\kappa=\{x_i\}_{i=0}^n\in PP[a,b]$ the *p-variation sum* is

$$s_p(f,\kappa):=\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|^p, \tag{2.1}$$

where $0<p<\infty$. Thus, the *p-variation* of f over $[a,b]$ is 0 if $a=b$ and otherwise

$$v_p(f)=v_p(f,[a,b]):=\sup\left\{s_p(f,\kappa):\kappa\in PP[a,b]\right\}. \tag{2.2}$$

The partition κ is called *supreme partition* if it satisfies the property $v_p(f)=s_p(f,\kappa)$. The set of such partitions is denoted $SP_p(f,[a,b])$.

Lemma 2.3 (Elementary properties). Let $f:[a,b]\to\mathbb{R}$ and $0<p<\infty$. Then the following p-variation properties holds

a) $v_p(f,[a,b])\geq 0$,

b) $v_p(f,[a,b])=0\Leftrightarrow f\equiv Const.$,

c) $\forall C \in \mathbb{R} : v_p(f + C, [a, b]) = v_p(f, [a, b])$,

d) $\forall C \in \mathbb{R} : v_p(Cf, [a, b]) = C^p v_p(f, [a, b])$,

e) $\forall c \in [a, b] : v_p(f, [a, b]) \geq v_p(f, [a, c]) + v_p(f, [c, b])$,

f) $\forall [a', b'] \subset [a, b] : v_p(f, [a, b]) \geq v_p(f, [a', b'])$.

g) $\forall \kappa \in PP[a, b] : s_p(f; \kappa) \leq v_p(f, [a, b])$.

All listed properties are elementary derived directly form the p-variation definition.

Definition 2.4 (Regulated function). ([9], Def. 3.1) For any interval J, which may be open or closed at either end, real function f is called *regulated* on J if it has left and right limits $f(x-)$ and $f(x+)$ respectively at each point x in interior of J, a right limit at the left end point and a left limit at the right endpoint.

Proposition 2.5. ([9], Lemma 3.1) Let $1 \leq p < \infty$. If f is regulated then $v_p(f)$ remains the same if points $x+$, $x-$ are allowed as partition points x_i in the definition 2.2.

Definition 2.6 (Piecewise monotone functions). ([9], Def. 3.2) A regulated real-valued function f on closed interval $[a, b]$ will be called *piecewise monotone* (PM) if there are points $a = x_0 < \cdots < x_k = b$ for some finite k such that f is monotone on each interval $[x_{j-1}, x_j]$, $j = 1, \ldots, k$. Here for $j = 1, \ldots, k - 1$, x_j may be point $x-$ or $x+$. The set of all piecewise monotone functions is denoted $PM = PM[a, b]$.

In addition to PM, if f is continuous function we will call it continuous piecewise monotone(CPM). The set of such functions is denoted $CPM = CPM[a, b]$.

Proposition 2.7. ([9], Prop. 3.1) If f is PM, there is a minimal size of partition $|\kappa|$ for which the definition 2.6 holds. The minimal size of the the PM partition is denoted $K(f, [a, b]) = K(f)$, namely

$$K(f) := \min \left\{ n : \exists \{x_i\}_{i=0}^{n} \in PP[a, b] : f \text{ is monotonic in each } [x_{j-1}, x_j] \right\}. \tag{2.3}$$

Definition 2.8 (The set of PM partitions with minimal size). ([9], Def. 3.3) If f is PM, let $X(f) = X(f, [a, b])$ be the set of all $\{x_i\}_{i=0}^{K(f)}$ for which the definition of PM (def. 2.6) holds. $X(f)$ is called the *set of PM partitions with minimal size.*

Proposition 2.9. ([9], Prop. 3.3) Let f is PM then the numbers $\alpha_j(f) := f(x_j) - f(x_{j-1})$ for $\{x_j\}_{j=0}^{K(f)} \in X(f)$ and $j = 1, 2, \ldots K(f)$ are uniquely determined.

Proposition 2.10. ([9], Prop. 3.2) Let f is PM. For any partition $\{x_j\}_{j=0}^{K(f)} \in X(f)$ exactly one of the flowing stamens holds:

(a) $f(x_0) > f(x_1) < f(x_2) > \ldots$. Function f is not increasing in intervals $[x_{2j}, x_{2j+1}]$, then $2j+1 \leq K(f)$. Function f is not decreasing in intervals $[x_{2j-1}, x_{2j}]$, then $j \geq 1$ and $2j \leq K(f)$.

(b) (a) holds for a function $-f$; or

(c) $K(f, [a, b]) = 1$ and f is constant in interval $[a, b]$.

Definition 2.11 (The equality by PM). ([9], Def. 3.4) If f, g are two PM functions, possibly on different intervals, such that $K(f) = K(g)$ and $\alpha_j(f) = \alpha_j(g)$ for $j = 1, 2, \ldots, K(f)$, then we say that $f \overset{PM}{=} g$.

Proposition 2.12. ([9], Cor. 3.1) Let $p > 1$ and functions f and g are PM. If $f \overset{PM}{=} g$ or $f \overset{PM}{=} -g$, then $v_p(f) = v_p(g)$.

Proposition 2.13. ([9], Them. 3.1) Let f is PM, $\kappa \in X(f)$ and $1 \leq p < \infty$. Then the supremum of p-variation in Definition 2.2 is attained for some partition $r \subset \kappa$.

Corollary 2.14. The set $SP_p(f, [a, b])$ is not empty for all $f \in PM[a, b]$.

Definition 2.15 (Sample function). Suppose $X = \{X_i\}_{i=0}^n$ is any sequence real numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. Then the *sample function* $G_X : [0, n] \to \mathbb{R}$ is defined as

$$G_X(t) := X_{\lfloor t \rfloor},\ t \in [0, n], \tag{2.4}$$

where $\lfloor t \rfloor$ denotes floor function at point t.

Definition 2.16 (p-variation of the sequence). Let $X = \{X_i\}_{i=0}^n$. The p-variation of the sample X is defined as p-variation of the function $G_X(t)$, namely

$$v_p(X) := v_p(G_X(t), [0, n]). \tag{2.5}$$

2.2 General properties with proofs

Proposition 2.17. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{x_i\}_{i=0}^n \in PP[a, b]$ is any partition of interval $[a, b]$. Then the statement

$$\exists\kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa \tag{2.6}$$

is equivalent to

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{2.7}$$

Proof. Necessary. Let $f : [a, b] \to \mathbb{R}$ be PM, $\{x_i\}_{i=0}^n \in PP[a, b]$ and

$$\exists\kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa. \tag{2.8}$$

Points from the partition κ will be denoted t_i, i.e. $\kappa = \{t_i\}_{i=0}^m$. Then, according to definitions of SP_p and p-variation (def. 2.2) the following equation holds

$$v_p(f, [a, b]) = s_p(f; \kappa) = \sum_{j=1}^{m} |f(t_{j-1}) - f(t_j)|^p = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p, \tag{2.9}$$

where $h : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$ denotes a function from the set of index of x to the set of index of t, namely:

$$h(i) := (j_i : x_i = t_{j_i} = t_{h(i)}). \tag{2.10}$$

The equation (2.9) holds, because all the elements in the sum remains, we just grouped them.

Moreover, the inequality

$$\sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p \le v_p(f, [x_{i-1}, x_i]) \tag{2.11}$$

holds according to Lemma 2.3(g).

As a result of (2.9) and (2.11) we get

$$v_p(f, [a, b]) \le \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{2.12}$$

On the other hand, according to the same Lemma 2.3(e) the following inequality holds

$$v_p(f, [a, b]) \ge \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{2.13}$$

Finally, from the (2.12) and (2.13) follows

$$v_p(f,[a,b]) = \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{2.14}$$

Sufficiency. Suppose $f:[a,b]\to\mathbb{R}$ and

$$v_p(f,[a,b]) = \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{2.15}$$

According to Corollary 2.14, sets $SP_p(f,[x_{i-1},x_i]),\ i=1,\dots,n$ are not empty. Lets take any partition from each of the sets $SP_p(f,[x_{i-1},x_i]),\ i=1,\dots,n$ and denote it κ_i.

Then, let define a joint partition $\kappa := \cup_{i=1}^{n}\kappa_i$. Points from the partition κ will be denoted by t_i. In addition, we will use the function h, which is defined in (2.10). Then, continuing the equation (2.15) we get

$$v_p(f,[a,b]) = \sum_{i=1}^{n}\sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1})-f(t_j)|^p = \sum_{i=1}^{|\kappa|} |f(t_{i-1})-f(t_i)|^p. \tag{2.16}$$

This means that $\kappa\in MP_p(f,[a,b])$. Moreover, $\forall i: x_i\in\kappa$, because $\kappa = \cup_{i=1}^{n}\kappa_i$. ■

Lemma 2.18. Let $f:[a,b]\to\mathbb{R}$ be PM and $\{x_i\}_{i=0}^{n}\subset\kappa\in SP_p(f,[a,b])$. Then

$$\forall k,l : v_p(f,[x_k,x_l]) = \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]),\ 0\le k<l\le n. \tag{2.17}$$

Proof. Suppose $\{x_i\}_{i=0}^{n}\subset\kappa\in SP_p(f,[a,b])$. Let choose k and l such that $0\le k<l\le n$. Lets apply Proposition 2.17 for the partition $\{x_0,x_k,x_{k+1},\dots,x_{l-1},x_l,x_n\}$. Thus,

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]) + v_p(f,[x_l,x_n]). \tag{2.18}$$

In addition, we can apply the same proposition for the partition $\{x_0,x_k,x_l,x_n\}$, then

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + v_p(f,[x_k,x_l]) + v_p(f,[x_l,x_n]). \tag{2.19}$$

By subtracting one equation form the other we get the result that

$$v_p(f,[x_k,x_l]) = \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]). \tag{2.20}$$

■

Lemma 2.19. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f, [a, b])$. Then

$$\forall k, l : v_p(f, [t_k, t_l]) = s_p\left(f, \{t_i\}_{i=k}^l\right), \ 0 \leq k < l \leq n. \tag{2.21}$$

Proof. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f, [a, b])$. Let choose k and l such that $0 \leq k < l \leq n$. Then

$$\begin{aligned} v_p(f, [a, b]) &= s_p(f, \{t_i\}_{i=0}^n) & (2.22)\\ &= s_p(f, \{t_i\}_{i=0}^k) + s_p(f, \{t_i\}_{i=k}^l) + s_p(f, \{t_i\}_{i=l}^n) & (2.23)\\ &\leq v_p(f, [a, t_k]) + s_p(f, \{t_i\}_{i=k}^l) + v_p(f, [t_l, b]). & (2.24) \end{aligned}$$

The last inequality holds according to Lemma 2.3(g).

On the other hand, from Proposition 2.18 we get

$$v_p(f, [a, b]) = v_p(f, [a, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, b]). \tag{2.25}$$

Form (2.24) and (2.25) follows

$$v_p(f, [t_k, t_l]) \leq s_p(f, \{t_i\}_{i=k}^l). \tag{2.26}$$

Finally, from Lemma 2.3(g) we conclude that

$$v_p(f, [t_k, t_l]) = s_p(f, \{t_i\}_{i=k}^l). \tag{2.27}$$

■

Definition 2.20 (The point of supreme partition). Let $f : [a, b] \to \mathbb{R}$ be PM. The point x will be called the *point of supreme partition* if

$$\exists \kappa \in SP_p(f, [a, b]) : x \in \kappa. \tag{2.28}$$

The set of such points will be denoted by $\overline{SP}_p(f, [a, b])$.

Lemma 2.21. Suppose $f : [a, b] \to \mathbb{R}$ is PM, $x \in [a, b]$, $x \notin \overline{SP}_p(f, [a, b])$ and $\{t_i\}_{i=0}^n \in SP(f, [a, b])$ is any supreme partition. Then,

$$\exists j = 1, \ldots, n : x \in (t_{j-1}, t_j) \text{ and } x \notin \overline{SP}_p(f, [t_{j-1}, t_j]). \tag{2.29}$$

Proof. Suppose the assumptions of lemma is valid. Since $x \in [a, b]$ and $[a, b] = \cup_{i=1}^n [t_{i-1}, t_i]$, then

$$\exists j = 1, \ldots, n : x \in [t_{j-1}, t_j]. \tag{2.30}$$

Moreover, $x \notin \{t_i\}_{i=0}^n$, because $x \notin \overline{SP}_p(f, [a, b])$, thus, $x \neq t_{j-1}$ and $x \neq t_j$. In addition to (2.30) this means that $x \in (t_{j-1}, t_j)$.

Now, we will proof that $x \notin \overline{SP}_p(f, [t_{j-1}, t_j])$. Suppose to the contrary that $x \in \overline{SP}_p(f, [t_{j-1}, t_j])$. Then, according to definition of $\overline{SP}_p(f, [t_{j-1}, t_j])$,

$$\exists \kappa \in SP_p(f, [t_{j-1}, t_j]) : x \in \kappa. \tag{2.31}$$

Since, $\kappa \in SP_p(f, [t_{j-1}, t_j])$, then

$$v_p(f, [t_{j-1}, t_j]) = s_p(f, \kappa) \tag{2.32}$$

Applying Proposition 2.17 for partition $\{t_i\}_{i=0}^n$ we get

$$\begin{aligned} v_p(f, [a, b]) &= v_p(f, [t_0, t_{j-1}]) + v_p(f, [t_{j-1}, t_j]) + v_p(f, [t_j, t_n]) && (2.33)\\ &= s_p(f, \{t_i\}_{i=0}^{j-1}) + s_p(f, \kappa) + s_p(f, \{t_i\}_{i=j}^n) && (2.34) \end{aligned}$$

This means that the partition $r := \{t_i\}_{i=0}^{j-1} \cup \kappa \cup \{t_i\}_{i=j}^n$ is supreme partition, so $x \in r \in SP(f, [a, b])$, therefore, by definition $x \in \overline{SP}_p(f, [a, b])$. This contradict to initial assumption. ■

Definition 2.22 (f-join). Suppose $f : [a, b] \to \mathbb{R}$ is PM. We will say that points t_a and t_b ($t_a < t_b$) are *f-joined* in interval $[a, b]$ if

$$\exists \{x_j\}_{j=0}^n \in SP_p(f, [a, b]) : [t_a, t_b] = [x_{j-1}, x_j], \text{ with some } j. \tag{2.35}$$

Lemma 2.23. Suppose $f : [a, b] \to \mathbb{R}$ is PM and points t_a and t_b ($t_a < t_b$) are f-jointed in interval $[a, b]$. Then all following statements holds

a) $v_p(f, [t_a, t_b] = |f(t_a) - f(t_b)|^p$;

b) Let $x \in [t_a, t_b]$. If $f(t_a) \geq f(t_b)$, then $f(t_a) \geq f(x) \geq f(t_b)$. If $f(t_a) \leq f(t_b)$, then $f(t_a) \leq f(x) \leq f(t_b)$;

Proof.

a) Let $f : [a, b] \to \mathbb{R}$ be PM, $t_a < t_b$ and pair of points t_a, t_b are f-joined. Then exists $\{x_j\}_{j=0}^n$ and j from the Definition 2.22. Thus, according to Lemma 2.19

$$v_p(f, [t_{j-1}, t_j]) = s_p\left(f, \{t_{j-1}, t_j\}\right) = |f(t_a) - f(t_b)|^p. \tag{2.36}$$

b) Let $f : [a, b] \to \mathbb{R}$ be PM and points t_a and t_b are f-joined. Since $f \overset{PM}{=} -f$, with out loss of generality we can assume that $f(t_a) \leq f(t_b)$. Suppose to the contrary that $f(t_b)$ is not an extrema of the function in interval $[t_a, t_b]$. Hence, $\exists c \in [t_a, t_b] : f(c) > f(t_b)$. Therefore, $|f(c) - f(t_a)|^p > |f(t_b) - f(t_a)|^p$. According to (2.36), $v_p(f, [t_a, t_b]) = |f(t_b) - f(t_a)|^p$, thus, $|f(c) - f(t_a)|^p > v_p(f, [t_a, t_b])$, but this contradicts the definition of p-variation. So, point t_b must be an extrema in interval $[t_a, t_b]$. Symmetric arguments could be used for point t_a.

■

Lemma 2.24. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then function $f : [0, \infty) \to \mathbb{R}$ with the values

$$f(x) = (x + c_1)^p - x^p - C, \ x \in [0, \infty), \tag{2.37}$$

are non decreasing in interval $[0, \infty)$.

Proof. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then, for all $x \geq 0$, the derivative of the function f is

$$\begin{aligned} f'(x) &= p(x + c_1)^{p-1} - px^{p-1} \\ &\geq px^{p-1} - px^{p-1} = 0. \end{aligned}$$

The derivative of function f is non negative, thus the function f is non decreasing, if $x \geq 0$. ■

Corollary 2.25. Suppose $c_1 \geq 0$, $C \in \mathbb{R}$, $1 \leq p < \infty$ and $0 \leq x \leq y$. Then the following implication holds

$$|x + c_1|^p > x^p + C \Rightarrow |y + c_1|^p > y^p + C. \tag{2.38}$$

Proof. Suppose $0 \leq x \leq y$. Since f is non decreasing, $f(x) \leq f(y)$. Therefore, if $f(x) > 0$, then $f(y) > 0$. ■

Theorem 2.26. Suppose $f : [a, b] \to \mathbb{R}$ is PM and $x \in [a', b'] \subset [a, b]$. If $x \notin \overline{SP}_p(f, [a', b'])$, then $x \notin \overline{SP}_p(f, [a, b])$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be PM, $x \in [a', b'] \subset [a, b]$, and $x \notin \overline{SP}_p(f, [a', b'])$. Suppose to the contrary that $x \in \overline{SP}_p(f, [a, b])$.

Since $x \in \overline{SP}_p(f, [a, b])$, according to the Definition 2.20,

$$\exists \{t_i\}_{i=0}^n \in SP_p(f, [a, b]) : x \in \{t_i\}_{i=0}^n. \tag{2.39}$$

Let $\{y_i\}_{i=0}^n \in SP_p(f, [a', b'])$ be any supreme partition from the interval $[a', b']$. Then, according to Lemma 2.21,

$$\exists j = 1, \ldots, n : x \in (y_{j-1}, y_j) \text{ and } x \notin \overline{SP}_p(f, [y_{j-1}, y_j]). \tag{2.40}$$

Moreover,

$$x \notin \{a, y_{j-1}, y_j, b\}, \tag{2.41}$$

because $x \in (y_{j-1}, y_j) \subset [a, b]$. Thus, from (2.39) and (2.41) follows that

$$\exists r \in \{1, 2, \ldots, n-1\} : t_r = x. \tag{2.42}$$

Lets denote variables l and k by

$$\begin{aligned} l &:= \max\left\{i \in \{1, 2, \ldots, n-1\} : t_i \in (y_{j-1}, y_j)\right\}, \\ k &:= \min\left\{i \in \{1, 2, \ldots, n-1\} : t_i \in (y_{j-1}, y_j)\right\}. \end{aligned}$$

Since $x \in (y_{j-1}, y_j)$ and (2.42) holds, the values l and k always exists and $k \le r \le l$. According to l and k definitions the following inequality holds

$$t_{k-1} \le y_{j-1} < t_k \le x \le t_l < y_j \le t_{l+1}. \tag{2.43}$$

According to Lemma 2.3(e)

$$v_p(f, [y_{j-1}, y_j]) \ge v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]), \tag{2.44}$$

Firstly, let suppose

$$v_p(f, [y_{j-1}, y_j]) = v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]). \tag{2.45}$$

Lets take any supreme partitions from intervals $[y_{j-1}, t_k]$ and $[t_l, y_j]$, namely $\kappa_k \in SP_p(f, [y_{j-1}, t_k]$ and $\kappa_l \in SP_p(f, [t_l, y_j]$. In addition, since $\{t_i\}_{i=0}^n \in SP_p(f, [a, b])$, according to Lemma 2.19 $v_p(f, [t_k, t_l]) = s_p(f, \{t_i\}_{i=k}^l)$. Therefore,

$$v_p(f, [y_{j-1}, y_j]) = s_p(f, \kappa_k) + s_p(f, \{t_i\}_{i=k}^l) + v_p(f, \kappa_l).$$

This means, that

$$\kappa_k \cup \{t_i\}_{i=k}^l \cup \kappa_l \in SP_p(f, [y_{j-1}, y_j].$$

So, $x \in \overline{SP}_p(f, [y_{j-1}, y_j])$, because $x \in \{t_i\}_{i=k}^l$. This contradicts (2.40), thus, equality (2.45) is not valid. As a result, inequality (2.44) becomes

$$v_p(f, [y_{j-1}, y_j]) > v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]). \tag{2.46}$$

Since points y_{j-1} and y_j are f-joined, from the Lemma 2.23(a) we get

$$v_p(f,[y_{j-1},y_j]) = |f(y_{j-1}) - f(y_j)|^p,$$

therefore,

$$|f(y_{j-1}) - f(y_j)|^p > v_p(f,[y_{j-1},t_k]) + v_p(f,[t_k,t_l]) + v_p(f,[t_l,y_j]).$$

Moreover, according to Lemma 2.3(g), from the last statement follows

$$|f(y_{j-1})-f(y_j)|^p > |f(y_{j-1})-f(t_k)|^p+v_p(f,[t_k,t_l])+|f(t_l)-f(y_j)|^p. \tag{2.47}$$

Since $v_p(f) = v_p(-f)$ (see Proposition 2.12), with out loss of generality we can assume that $f(y_{j-1}) \geq f(y_j)$. Hence, from the Lemma 2.23(b) we get that

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j), \tag{2.48}$$
$$f(y_{j-1}) \geq f(t_k) \geq f(y_j). \tag{2.49}$$

In addition, the pairs of points $\{t_l, t_{l+1}\}$ and $\{t_{k-1}, t_k\}$ are also f-joined, therefore, by the Lemma 2.23(b) inequalities (2.48) and (2.49) could be extended as

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j) \geq f(t_{l+1}), \tag{2.50}$$
$$f(t_{k-1}) \geq f(y_{j-1}) \geq f(t_k) \geq f(y_j). \tag{2.51}$$

From the (2.50) the following inequalities holds

$$\begin{aligned} f(y_{j-1}) - f(t_l) &\geq 0, \\ f(t_l) - f(t_{l+1}) \geq f(t_l) - f(y_j) &\geq 0. \end{aligned}$$

Therefore, we can use Corollary 2.25. According to it, from the inequality (2.47) follows

$$\begin{aligned} |f(y_{j-1}) - f(t_l) + f(t_l) - f(y_j)|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f,[t_k,t_l]) + |f(t_l) - f(y_j)|^p, \\ |f(y_{j-1}) - f(t_l) + f(t_l) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f,[t_k,t_l]) + |f(t_l) - f(t_{l+1})|^p, \\ |f(y_{j-1}) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f,[t_k,t_l]) + v_p(f,[t_l,t_{l+1}]). \end{aligned}$$

The last inequality holds, because points t_l and t_{l+1} are f-joined, thus, according to Lemma 2.23(a),

$$v_p(f,[t_l,t_{l+1}]) = |f(t_l) - f(t_{l+1})|^p.$$

Symmetric argument could be used in other direction. Firstly lets modify last inequality

$$|f(y_{j-1})-f(t_k)+f(t_k)-f(t_{l+1})|^p > |f((y_{j-1})-f(t_k)|^p+v_p(f,[t_k,t_l])+v_p(f,[t_l,t_{l+1}]).$$

From the (2.51) we get that the following inequalities holds

$$\begin{aligned} f(t_k)-f(t_{l+1}) &\geq 0, \\ f(t_{k-1})-f(t_k) \geq f(y_{j-1})-f(t_k) &\geq 0. \end{aligned}$$

Thus, from Corollary 2.25 we get

$$\begin{aligned} |f(t_{k-1})-f(t_k)+f(t_k)-f(t_{l+1})|^p &> |f(t_{k-1})-f(t_k)|^p+v_p(f,[t_k,t_l])+v_p(f,[t_l,t_{l+1}]), \\ |f(t_{k-1})-f(t_{l+1})|^p &> v_p(f,[t_{k-1},t_k])+v_p(f,[t_k,t_l])+v_p(f,[t_l,t_{l+1}]). \end{aligned}$$

As previous, the last inequality holds, because t_{k-1} and t_k are f-joined.
Finally, using Lemma 2.18 we conclude that

$$|f(t_{k-1})-f(t_{l+1})|^p > v_p(f,[t_{k-1},t_{l+1}]).$$

This contradicts with the definition of p-variation.
■

2.3 Extra results

Definition 2.27 (Extremum). We will call the point $t \in [a,b]$ an *extrema* of the function f in interval $[a,b]$ if $f(t) = \sup\{f(z) : z \in [a,b]\}$ or $f(t) = \inf\{f(z) : z \in [a,b]\}$.

Proposition 2.28. Let $f : [a,b] \to \mathbb{R}$ is PM. If point $x \in [a,b]$ is extrema of the function f, then $x \in \overline{SP}_p(f,[a,b])$.

Proof. Let $f : [a,b] \to \mathbb{R}$ is PM and point $x \in [a,b]$ is an extrema of the function f. Let $\{t_i\}_{i=0}^n \in SP_p(f,[a,b]$ be any supreme partition. Then

$$\exists j \in 1,\dots,n : x \in [t_{j-1},t_j] \tag{2.52}$$

Since x is an extrema of function f in interval $[a,b] \supset [t_{j-1},t_j]$, the point x is an extrema in interval $[t_{j-1},t_j]$ as well. Therefore,

$$f(t_{j-1})-f(t_j)|^p \leq |f(t_{j-1})-f(x)|^p \tag{2.53}$$

or

$$|f(t_{j-1})-f(t_j)|^p \leq |f(x)-f(t_j)|^p. \tag{2.54}$$

As a result,

$$\begin{aligned}|f(t_{j-1}) - f(t_j)|^p &\leq |f(t_{j-1}) - f(x)|^p + |f(x) - f(t_j)|^p, \\ |f(t_{j-1}) - f(t_j)|^p &\leq v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]), \\ v_p(f, [t_{j-1}, t_j]) &\leq v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j]).\end{aligned}$$

The last inequality holds, because t_{j-1} and t_j are f-joined, so, $v_p(f, [t_{j-1}, t_j]) = |f(t_{j-1}) - f(t_j)|^p$. Since $v_p(f, [t_{j-1}, t_j]) < v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j])$ is not valid, the equation

$$v_p(f, [t_{j-1}, t_j]) = v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) \tag{2.55}$$

holds.

Moreover, applying Proposition 2.17 for the partition $\{a, t_{j-1}, t_j, b\}$ we have

$$\begin{aligned}v_p(f, [a, b]) &= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, t_j]) + v_p(f, [t_j, b]) \\ &= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) + v_p(f, [t_j, b])\end{aligned}$$

From the same proposition follows, that $\exists \kappa \in SP_p(f, [a, b]) : x \in \kappa$, so, by definition, $x \in \overline{SP}_p(f, [a, b])$. ■

Lemma 2.29. Let $f : [a, b] \to \mathbb{R}$ be PM. If $K(f)$ is even, then

$$\exists x \in (a, b) : x \in \overline{SP}_p(f, [a, b]). \tag{2.56}$$

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is PM. Lets take any $\{x_i\}_{i=0}^n \in X(f)$. If $n > 1$, then without loss of generality we can assume that Proposition 2.10(a) holds. So, $f(x_0) > f(x_1) < f(x_2) > \dots$, i.e. $f(x_{2i-1}) < (x_{2i})$, thus, if n is even, then $f(x_{n-1}) < (x_n)$, therefore, point x_n is not a global minimum. Since $f(x_0) > f(x_1)$, point x_0 is also not a global minimum. As a result, global minimum is in interval (a, b) and it is in $\overline{SP}_p(f, [a, b])$ according to Lemma 2.28. ■

Lemma 2.30. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in PP[a, b]$. Suppose

$$\{t_i\}_{i=0}^{n-1} \in SP_p(f, [a, t_{n-1}]) \tag{2.57}$$

and

$$\{t_i\}_{i=1}^{n} \in SP_p(f, [t_1, b]). \tag{2.58}$$

Then

$$\forall i \in \{1, \dots, n-1\} : t_i \in \overline{SP}_p(f, [a, b]) \tag{2.59}$$

or

$$\forall i \in \{1, \dots, n-1\} : t_i \notin \overline{SP}_p(f, [a, b]) \tag{2.60}$$

Proof. Let the assumptions of lemma be valid. Firstly, suppose

$$\exists j \in \{1, \dots, n-1\} : t_j \in \overline{SP}_p(f, [a,b]). \tag{2.61}$$

This means that $\exists \kappa \in SP_p(f, [a,b]) : t_j \in \kappa$. Then, by Proposition 2.17,

$$v_p(f, [a,b]) = v_p(f, [a, t_j]) + v_p(f, [t_j, b]) \tag{2.62}$$

Since (2.57) and (2.57) holds, by Lemma 2.19 we get

$$v_p(f, [a, t_j]) = s_p(f, \{t_i\}_{i=0}^{j}), \tag{2.63}$$

$$v_p(f, [t_j, b]) = s_p(f, \{t_i\}_{i=j}^{n}). \tag{2.64}$$

Therefore,

$$v_p(f, [a,b]) = s_p(f, \{t_i\}_{i=0}^{j}) + s_p(f, \{t_i\}_{i=j}^{n}) = s_p(f, \{t_i\}_{i=0}^{n}). \tag{2.65}$$

So, $\{t_i\}_{i=0}^{n} \in SP_p(f, [a,b])$, thus, statement (2.59) holds.
On the other hand, if (2.61) is not valid, then statement (2.60) holds.
■

3 p-variation calculus

In this chapter we will present the algorithm that calculates p-variation for the sample (see Def. 2.16). Nonetheless, this algorithm could be used to calculate the p-variation for arbitrary piecewise monotone function. This procedure will be called *pvar*. It is already realised in the R (see [11]) package *pvar* and is publicly available on CRAN[3].

Firstly, we will give some introductory notes about the procedure. Later on, the main schema of the algorithm will be presented in the Subsection 3.2. And finally, each step of the algorithm are disused in more details in Subsection 3.3.

3.1 Introductory notes

Suppose $X = \{X_i\}_{i=0}^{n}$ is any real-value sequence of numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size* and refer as $|X|$. The formal definition of p-variation of the sample is given in Definition 2.16, which states that

$$v_p(X) = v_p(G_X(t), [0, n]), \tag{3.1}$$

[3] http://cran.r-project.org/web/packages/pvar/index.html

where $G_X(t)$ is a sample function defined in Definition 2.15. On the other hand, p-variation of the sample could express as

$$v_p(X) = \max\left\{\sum_{i=1}^{k} |X_{j_i} - X_{j_{i-1}}|^p : 0 = j_0 < \cdots < j_k = n,\ k = 1, \ldots, n\right\}. \tag{3.2}$$

This expression could be verified from Proposition 2.13, which states, that p-variation is achieved in a subset of any partition $r \in X(f, [0, n])$. So, we can construct r from the subset $\{0, 1, \ldots, n\}$. If $j \in \{0, 1, \ldots, n\}$ then $G_X(j) = X_j$, thus, $G_X(j)$ could be replaced with X_j. As a result, we don't need to use $G_X(j)$ function, rather, the values of the sample X_j could be used directly. This is the way algorithm works – it actually operates the the values of the sample X, not involving $G_X(t)$ function. Nonetheless, then we will refer to properties of p-variation we actually have in mind function $G_X(t)$, since all the properties of p-variation are formulated for functions.

The members of the sample X that are in supreme partition are called *supreme points*. All the members that are not in supreme partition are refereed as *redundant* points – those points could be removed from original sample without any effect to the value of p-variation.

The main idea of the algorithm is to identify redundant points in sample X using known properties of p-variation. As soon as point was identified being redundant, it could be doped out from further consideration. The set of points under the consideration will be denoted by (sub) sample S. During the procedure this set is modified until it contains only the supreme points. The sample S might be refer as a partition of X, since it indicates a subset of X.

In the initial stage we consider any point being a supreme point, so, in the very begging $S = X$. In further stages some points might be doped from S. For this purpose, the most important property that grounds the algorithm is stated in the Theorem 2.26. It allows to consider points in S by analysing not the whole sample, but much smaller sub-samples. From the Theorem 2.26 follows, that if a point was identify as being redundant in any small interval, then the point is redundant in any larger interval as well. This property is used in several ways that will be revealed in further sections.

It is worth noting, that the procedure *pvar* could be used to calculate the p-variation of any piecewise monotone function. Let assume f is any piecewise monotone function. According to definition (see 2.6) there are points $a = x_1 < \cdots < x_n = b$ for some finite n such that f is monotone on each interval $[x_{j-1}, x_j]$, $j = 1, 2, \ldots, n$. We can calculate p-variation if we have the sample X, which contains the values of the function f at the points $\{x_i\}_{i=0}^n$, namely, $X = \{X_i\}_{i=0}^n := \{f(x_i)\}_{i=0}^n$. It is straight forward to see that $f \overset{PM}{=} G_X$, therefore, if $p \geq 1$, then $v_p(X) = v_p(G_X) = v_p(f)$.

3.2 Main function

The procedure *pvar* that calculates p-variation will be presented here. Firstly, we will introduce the main schema, further, each step will be discussed in more details. The main steps in *pvar* procedure goes as follows.

Procedure *pvar*. Input: sample X, scalar p.

1. *Initialisation and checking.* We consider any point being a supreme point, so, $S \leftarrow X$. In normal case we proceed to next step, but in special case there might be some exceptions. For example, if $|X| = 0$, then X has only 1 point. In this case p-variation is equal to 0 by Definition 2.2,so, further procedures are meaningless. Moreover, if $0 < p < 1$ then all the points must be included in supreme partitions. In further steps we will assume that $|X| > 0$ and $p > 1$.

2. *Removing monotonic points.* According to Proposition 2.13 all points that are not the end of monotonic interval could be excluded from further consideration.

3. *Checking all small sub-samples.* Every small sub-samples (with the predefined size) are checked if it is possible to identify redundant points. This operation is based on Theorem 2.26 which states that if $x \notin \overline{SP}_p(f, [a', b'])$, then $x \notin \overline{SP}_p(f, [a, b])$, where $[a', b'] \subset [a, b]$. So, if X_j appears to be redundant in any small sub-sample, then we do not need to consider possibility for X_j to be in supreme sample any more. More details are given in section 3.3.2.

4. *Merging small sub-samples.* Let $\kappa_{a,c} \in SP_p(G_X, [a, c])$ and $\kappa_{c,b} \in SP_p(G_X, [c, b])$. By using $\kappa_{a,c}$ ans $\kappa_{c,b}$, we can effectively find $\kappa_{a,b} \in SP_p(G_X, [a, b])$, since $\kappa_{a,b} \subset \kappa_{a,c} \cup \kappa_{c,b}$. Finding $\kappa_{a,b}$ from $\kappa_{a,c}$ and $\kappa_{c,b}$ is called *merging*. In this step we repeat merging of all small intervals until we get the finial supreme partition. The whole procedure is described in section 3.3.3.

5. *Calculating the value of p-variation.* If the supreme partition is obtain, then the value of p-variaion is caluculated using p-varion sum

$$v_p = \sum_{i=1}^{|S|} |S_{i-1} - S_i|^p$$

The corresponding pseudo-code of the main function goes as follows

Algorithm 1 The main function of *pvar* procedure that calculates p-variation of the sample.

Input: sample X, scalar p.
Require: $p > 0$.
Output: the value of p-variation of X

function PVAR(X, p)
  // 1. Initialization and checking
  $S \leftarrow X$;
  if $0 < p \leq 1$ **then**
    value $\leftarrow \sum |S_{i-1} - S_i|^p$;
    return value;
  end if
  if $|S| = 0$ **then**
    value $\leftarrow 0$;
    return value;
  end if
  // 2. Removing monotonic points
  $S \leftarrow FindCorners(S)$;
  // 3. Checking sub-samples. Check sub-intervals up to the size of M
  $M \leftarrow 3$;
  $S \leftarrow CheckSubSamples(S, p, M)$;
  // 4. Merging sub-samples starting form length $M + 1$
  $S \leftarrow MergeAll(S, p, M + 1)$;
  // 5. calculate the p-variation
  value $\leftarrow \sum |S_{i-1} - S_i|^p$;
  return value;
end function

Function *FindCorners* returns the subset of S that contains only the end of monotonic points (see Section 3.3.1). Function *CheckSubSamples* check all sequential sub-samples with size up to M and remove all the points that appeared to be redundant (see Section 3.3.2). Function *MergeAll* merges all small sub-samples until the final supreme partitions is obtain (see Section 3.3.3).

3.3 Detail explanation

3.3.1 Removing monotonic points

According to Proposition 2.13 the p-variation is achieved on the partition of monotonic intervals. Therefore the points that are not the end of monotonic interval are redundant and should be drop out. The points that are the end of monotonic intervals will be referred as *corners*. In a single loop we can find all the corners by checking if the sequence changed the direction form increasing to decreasing (or vice versa). In addition, points $X[0]$ and $X[n]$ always included in the array of corners by definition. The corresponding pseudo code looks as follows:

Algorithm 2 The function *FindCorners* that returns the subset of S, which contains only the corners of S.

Input: sample S
Output: sample $corners$ that is the subset of S, which contains only the corners

function FindCorners(S)
    $corners \leftarrow S_0$;
    $direction \leftarrow 0$;
    $n \leftarrow |S|$;
    for $i \leftarrow 1$ **to** $n-1$ **do**
        if $S_{i-1} < S_i$ **then**
            if $direction < 0$ **then**
                append $corners$ with element S_{i-1};
            end if
            $direction \leftarrow 1$;
        end if
        if $S_{i-1} > S_i$ **then**
            if $direction > 0$ **then**
                append $corners$ with element S_{i-1};
            end if
            $direction \leftarrow -1$;
        end if
    end for
    append $corners$ with element S_n;
    return corners;
end function

This procedure is quite simple and can by performed very quickly, since it does not include any cross checking and could be done in one loop. In *pvar* package all corners could be found with *ChangePoints* function.

3.3.2 Checking small sub-samples

Proposition 2.17 and Lemmas 2.30 and 2.29 gives an effective way to identify significant part of redundant points using quite simple operations. The pseudo code is given in the end the chapter and now we will reveal the main principles.

Suppose S is a sample without monotonic points. Let investigate a sub-sample of the size m ($m = 2, \dots, n-1$), namely, lets examine a sub-sample $\{S_i\}_{i=k}^{k+m}$ for any $k = 0, ..., n-m$. According to Proposition 2.17, If

$$\sum_{i=k+1}^{k+m} |S_i - S_{i-1}|^p < |S_j - S_{j+m}|^p, \tag{3.3}$$

then, partition $\{k, \dots, k+m\}$ can not be supreme partition in interval $[k, k+m]$. Therefore, some of the points $\{S_i\}_{i=k+1}^{k+m-1}$ must be redundant. Moreover, Lemmas 2.30 and 2.29 gives opportunity easily identify redundant points. To do so, we must systemically investigate small sub-samples starting form small m.

- Let $m = 2$. Then $|\{S_i\}_{i=k}^{k+2}| = 2$, thus according to Lemma 2.29, $\{S_i\}_{i=k}^{k+2}$ has at least one inner supreme point. Since it has only one inner point X_{k+1}, then this point must be supreme point. So, if $m = 2$, then (3.3) is always invalid, therefore, we don't need to do any checking.

- Let $m = 3$. Firstly, we need to check if (3.3) holds. If so, then at least one point must be redundant. In addition, Lemma 2.30 ensures that in this case both middle points S_{k+1}, S_{k+2} are redundant, therefore, they both should be removed.

  We do this checking for for all $k = 0, ..., n-2$. In this way we can identify quite an amount of redundant points. Moreover, after removing them from the sample the sample changes, so we can repeat the same procedure and find more redundant points. We do it until no new redundant points are identified.

- If $m = 4$, then $|\{S_i\}_{i=k}^{k+4}|$ is even, thus, according to Lemma 2.29, $\{S_i\}_{i=k}^{k+m}$ has at least one inner supreme point. So, all inner points are supreme points, based on Lemma 2.30, therefore, (3.3) do not hold. Lemma 2.29 ensure that this argument holds in all cases, then m is even. So, actually we need to investigate only the cases, then m is odd.

- Let $m = 5$ and all sub-samples $\{S_i\}_{i=k}^{k+3}$, $k = 0, ..., n-3$ are already checked. If (3.3) holds, then, based on Lemma 2.30, all middle points S_{k+1}, S_{k+2}, S_{k+3} and S_{k+4} are identified as redundant and should be removed. After removing points, the sample changes, thus, if we want to apply this checking again, then we should start checking from $m = 3$ again, because we must make sure that all sub-samples $\{X_i\}_{i=k}^{k+3}$, $j = 0, ..., n-3$ are already checked, because this is a requirement of the Lemma 2.30.

- If we want to go further and check sub-samples for $m = 7, 9, 11 \ldots$, every time we have to start from $m = 3$ and increase m by 2 only then (3.3) is not valid for all sub-samples.

We can continue this checking until m reaches an arbitrary threshold M. The pseudo-code of this procedure is given below

Algorithm 3 Procedure *CheckSubSamples*, which ensures that any sequential sub-sample of length M is a supreme partition in corresponding interval.

Input: sample S, scalar p, scalar M
Require: S must contain only the corners
Output: modified sample S such that any sequential sub-sample of length M is a supreme partition in corresponding interval

```
function CheckSubSamples(S, p, M)
    tempS ← NULL ;
    m ← 0 ;
    while tempS ≠ S and m < M do
        tempS ← S ;
        m ← 3 ;
        check all sub-samples of S with length m ;
        update S by dropping all redundant points ;
        while tempS ≠ S and m < M do
            m ← m + 2 ;
            check all sub-samples of S with length m ;
            update S by dropping all redundant points ;
        end while
    end while
    return S;
end function
```

With this procedure, we can actually find p-variation, setting M to sufficiently large value. But this way of finding p-variation is not effective, because in case of any modification of S we should go back and start checking from $m = 3$. So, it is reasonable to use this procedure only for small m and then go to next step of p-variation calculus. In *pvar* package all corners could be found with *CheckSmallIntervals* function.

3.3.3 Merging supreme sub-samples

Suppose it is already known that a sub-samples $\{S_i\}_{i=a}^{w}$ and $\{S_i\}_{i=w}^{b}$ are a supreme partitions in intervals $[S_a, S_w]$ and $[S_w, S_b]$ correspondingly. The *merge* operation of sub-samples $\{S_i\}_{i=a}^{w}$ and $\{S_i\}_{i=w}^{b}$ is an operation that finds supreme partition of the interval $[S_a, S_b]$ taking into account that $\{X_i\}_{i=a}^{w}$ and $\{X_i\}_{i=w}^{b}$ are a supreme partitions in corresponding intervals.

The final procedure in p-variation calculus will be presented in this subsection. It is based on merging operation. All the small sub-samples which where already checked in the previous procedure (see Section 3.3.2) are merged pair by pair until we end up having the supreme partition.

This procedure has two stages that will be discussed separately.

Merge two supreme samples This paragraph will present how two supreme sub-samples are merged . Suppose $\{S_i\}_{i=a}^{w}$ and $\{S_i\}_{i=w}^{b}$ are a supreme partitions in intervals $[S_a, S_w]$ and $[S_w, S_b]$ correspondingly. According to Theorem 2.26, the supreme partition in interval $[S_a, S_b]$ must be a subset of $\{S_i\}_{i=a}^{b} = \{S_i\}_{i=a}^{w} \cup \{S_i\}_{i=w}^{b}$.

In order to find supreme partition in interval $[a, b]$ we need to investigate all possible f-joints (Def. 2.22) between points $S_t, t \in \{a, \dots, w-1\}$ and $S_r, r \in \{w, \dots, b\}$ and choose pair of points (t, r) which maximise s_p function, namely

$$
\begin{aligned}
(t,r) &:= \operatorname*{arg\,max}_{(i,j)\in[a,w-1]\times[w,b]} \left\{ \sum_{k=a+1}^{i} |S_{k-1}+S_k|^p + |S_i - S_j|^p + \sum_{k=j+1}^{b} |S_{k-1}+S_k|^p \right\}, \\
&= \operatorname*{arg\,max}_{(i,j)\in[a,w-1]\times[w,b]} \left\{ |S_i - S_j|^p + \sum_{k=a+1}^{b} |S_{k-1}+S_k|^p - \sum_{l=i+1}^{j} |S_{l-1}+S_l|^p \right\}, \\
&= \operatorname*{arg\,max}_{(i,j)\in[a,w-1]\times[w,b]} \left\{ |S_i - S_j|^p - \sum_{l=i+1}^{j} |S_{l-1}+S_l|^p \right\}, \qquad (3.4)
\end{aligned}
$$

In the special case then $t = w - 1$ and $r = w$, the value of function in $\arg\max$ is zero and the supreme partition is $\{S_i\}_{i=a}^{b}$. So, in this case we do not need to do any modifications of the sample. This case is taken as baseline and we consider any modifications only then the value of function in $\arg\max$ is greater then zero.

In addition, the calculation of (t, r) in (3.4) could be optimised, since we actually do not need to check all $(i, j) \in [a, v-1] \times [v+1, b]$. Based on Lemma 2.23, points X_t and X_r could be f-joined only if

$$\forall i \in \{t, \ldots, r\} : S_t \geq S_i \geq S_r \text{ or } S_t \leq S_i \leq S_r. \tag{3.5}$$

And the last statement holds only if

$$S_t = \min(S_i : i \in \{t, \ldots, w\}) \text{ or } S_t = \max(S_i : i \in \{t, \ldots, w\}) \tag{3.6}$$

and

$$S_r = \min(S_i : i \in \{w, \ldots, r\}) \text{ or } S_r = \max(S_i : i \in \{w, \ldots, r\}). \tag{3.7}$$

Let T denotes a set of $i \in \{a, \ldots, w-1\}$ which satisfy (3.6) and R denotes a set of $j \in \{w+1, \ldots, b\}$ which satisfy (3.7). Then (3.4) could be expressed as

$$(t, r) := \operatorname*{arg\,max}_{(i,j) \in T \times R} \left\{ |S_i - S_j|^p - \sum_{l=i+1}^{j} |S_{l-1} + S_l|^p \right\}. \tag{3.8}$$

Since points S_t and S_r are f-joint, all the points in interval $t+1, \ldots, r-1$ are redundant and should be removed. And the remaining points are the supreme partition.

Algorithm 4 Procedure *MergeSupremeSamples*, which merges two supreme samples.

Input: sample A, sample B, scalar p
Require: samples A and B must be supreme partitions in corresponding intervals.
Output: sample S, which is the supreme partition of joined interval

```
function MergeSupremeSamples(A, B, p)
    S ← A ∪ B ;
    a ← 0 ;
    b ← |S| ;
    w ← |A| ;
    // Find the T list:
    initiate T as empty list ;
    cummax ← cummin ← S_w ;
    for i ← w − 1  to  a do
        if S_i > cummax then
            cummax ← S_i ;
            append T with element i ;
        end if
        if S_i < cummin then
            cummin ← S_i ;
            append T with element i ;
        end if
    end for
    // Find the R list:
    initiate R as empty list ;
    cummax ← cummin ← S_w ;
    for i ← w + 1  to  b do
        if S_i > cummax then
            cummax ← S_i ;
            append R with element i ;
        end if
        if X_i < cummin then
            cummin ← S_i ;
            append R with element i ;
        end if
    end for
```

```
    // Check all pairs in T x R:
    maxbalance ← 0 ;
    for all r ∈ R do
        for all t ∈ T do
            balance ← |S_t − S_r|^p − Σ_{i=t+1}^{r} |S_{i−1} − S_i|^p ;
            if balance > maxbalance then
                maxT ← t ;
                maxR ← r ;
                maxbalance ← balance ;
            end if
        end for
    end for
    if maxbalance > 0 then
        remove points maxT + 1, . . . , maxR − 1 from sample S ;
    end if
    return S;
end function
```

In *pvar* package the merge operation could be achieved with *AddPvar* function.

Merging all sub-samples. In the second stage of this procedure we get the final result by merging intervals pair by pair until the supreme partition of the whole sample is obtain. The pseudo-code is given below.

Algorithm 5 Procedure *MergeAll*, which merges all sub-samples of the size M into final supreme sample of X.

Input: sample S; scalar p; scalar M.
Require: any sub-sample of S of length M must be a supreme partition
Output: modified sample S that corresponds to supreme portion of the whole sample

function MergeAll(X, p, M)
  // construct the set $R_i, i = 1, \ldots,$ which contain supreme sub-samples
  for $i \leftarrow 1$ **to** $\left\lceil \frac{|S|}{M} \right\rceil$ **do**
    $R_i \leftarrow \{S_j\}_{j=(i-1)M}^{\min(iM,|S|))}$;
  end for

  // apply *MergeSupremeSamples* to pairs of R elements until all intervals are merged
  while $|R| > 1$ **do**
    $i \leftarrow 1$;
    while $i < |R|$ **do**
      $prt_i \leftarrow MergeSupremeSamples(prt_i, prt_{i+1}, p)$;
      $i \leftarrow i + 2$;
    end while
    delete all elements form R that have even index ;
  end while
  $S \leftarrow R_1$;
  return S;
end function

4 Conclusion

In this article we presented the algorithm *pvar* that calculates p-variations of the sample. This algorithm is based on mathematical properties that were formulated and proved in the article. The calculations of p-variation is already available in R package *pvar*. The core of functions are written in C++ and imported to R via *Rcpp* package, therefore, algorithm is quite fast can calculate p-variations for really large samples. It is worth noting that *pvar* procedure could be used to analyse any piecewise monotone function.

References

[1] J. Y. Campbell, A. W. Lo, A. Craig Mac Kinlay. The Econometrics of Financial Markets, Princeton University Press, ISBN 0-691-04301-9, 1997.

[2] R. M. Dudley, R. Norvaiša. An Introduction to p-variation and Young Integrals, Cambridge, Mass., 1998.

[3] R. M. Dudley, R. Norvaiša. Differentiability of Six Operators on Nonsmooth Functions and p-variation, Springer Berlin Heidelberg, Print ISBN 978-3-540-65975-4, Lecture Notes in Mathematics Vol. 1703, 1999.

[4] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration // Acta Mathematica, 67, 251-282, 1936.

[5] Karatsas, I. and Shreve, S. Brownian Motion and Stochastic Calculus, 2nd ed. New York: Springer-Verlag, 1997.

[6] R. Norvaiša. Rough functions: p-variation, calculus, and index estimation. Lithuanian Mathematical Journal, 46(1):102-128, 2005.

[7] R. Norvaiša, A. Račkauskas. Convergence in law of partial sum processes in p-variation norm // Lth. Math. J., Vol. 48, No. 2, 212–227, 2008.

[8] R. Norvaisa, D. M. Salopek. Estimating the p-variation index of a sample function: An application to financial data set // Methodology And Computing In Applied Probability Vol. 4, Issue 1 , 27-53, 2002.

[9] J. Qian. The p-variation of partial sum processes and the empirical process // Ph.D. thesis, Tufts University, 1997.

[10] J. Qian. The p-variation of partial sum processes and the empirical process // Ann. Probab. Volume 26, Number 3 (1998), 1370-1383.

[11] R Core Team (2014). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL http://www.R-project.org/.

[12] N. Wiener. The Quadratic Variation of a Function and Its Fourier Coefficients // Journ. Math. and Phys., MIT, 3, 73-94, 1924.