The calculus of p-variation

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- $PP[a,b]$ – the set of all point partitions of $[a,b]$ (def. 1.1).
- $s_p(f;\kappa)$ – p-variation sum (def. 1.2).
- $v_p(f)$ – p-variation of the function f (def. 1.2).
- $MP_p(f,[a,b])$ – the set of meaningful partitions (def. 1.2).
- $PM[a,b]$ – a set of piecewise monotone functions (def. 1.6).
- $CPM[a,b]$ – a set of continuous piecewise monotone functions (def. 1.6).
- $K(f,[a,b])$ – minimal size of PM partitions (prop. 1.7).
- $X(f,[a,b])$ – the set of PM partitions with minimal size (def. 1.8).
- $\overline{BP}(f,[a,b])$ – the set of break points (definition 1.25).
- $\overline{BP}(f,(a,b))$ – the set of inner break points (def. 1.25).
- $BP(f,[a,b])$ – the set of partitions that contains only break points (def. 1.25).
- $BP(f,(a,b))$ – the set of partitions that contains only break points, but excludes the partition $\{a,b\}$. (def. 1.25).
- $\overline{MB}_p(f,[a,b])$ – the set of meaningfully break points (def. 1.27).
- $MB_p(f,[a,b])$ – the set of partitions that contains only meaningfully break points (def. 1.27).

1 Mathematical analysis

1.1 General known properties

Definition 1.1 (Partition). Let $J = [a, b]$ be a closed interval of real numbers with $-\infty < a \leq b < +\infty$. If $a < b$, an ordered set $\kappa = \{x_i\}_{i=0}^n$ of points in $[a, b]$ such that a $a = x_0 < x_1 < x_2 < ... < x_n = b$ is called a *(point) partition.* The size of the partition is denoted $|\kappa| := \#\kappa - 1 = n$. The set of all point partitions of $[a, b]$ is denoted by $PP[a, b]$.

Definition 1.2 (p-variation). Let $f : [a, b] \to \mathbb{R}$ be a real function from an interval $[a, b]$. If $a < b$, for $\kappa = \{x_i\}_{i=0}^n \in PP[a, b]$ the *p-variation sum* is

$$s_p(f, \kappa) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p, \tag{1.1}$$

where $0 < p < \infty$. Thus, the *p-variation* of f over $[a, b]$ is 0 if $a = b$ and otherwise

$$v_p(f) = v_p(f, [a, b]) := \sup \{s_p(f, \kappa) : \kappa \in PP[a, b]\}. \tag{1.2}$$

The partition κ is called *meaningful* (in the content of p-variation) if it satisfies the property $v_p(f) = s_p(f, \kappa)$. The set of such partitions is denoted $MP_p(f, [a, b])$.

Lemma 1.3 (Elementary properties). Let $f : [a, b] \to \mathbb{R}$ and $0 < p < \infty$. Then the following p-variation properties holds

a) $v_p(f, [a, b]) \geq 0$,

b) $v_p(f, [a, b]) = 0 \Leftrightarrow f \equiv Const.$,

c) $\forall C \in \mathbb{R} : v_p(f + C, [a, b]) = v_p(f, [a, b])$,

d) $\forall C \in \mathbb{R} : v_p(Cf, [a, b]) = C^p v_p(f, [a, b])$,

e) $\forall c \in [a, b] : v_p(f, [a, b]) \geq v_p(f, [a, c]) + v_p(f, [c, b])$,

f) $\forall [a', b'] \subset [a, b] : v_p(f, [a, b]) \geq v_p(f, [a', b'])$.

g) $\forall \kappa \in PP[a, b] : s_p(f; \kappa) \leq v_p(f, [a, b])$.

All listed properties are elementary derived directly form the p-variation definition.

Definition 1.4 (Regulated function). ([1], Def. 3.1) For any interval J, which may be open or closed at either end, bound or unbound, real function f is called *regulated* on J if it has left and right limits $f(x-)$ and $f(x+)$ respectively at each point x in interior of J, a right limit at the left end point and a left limit at the right endpoint.

Proposition 1.5. ([1], Lemma 3.1) Let $1 \leq p < \infty$. If f is regulated then $v_p(f)$ remains the same if points $x+$, $x-$ are allowed as partition points x_i in the definition 1.2.

Definition 1.6 (Piecewise monotone functions). ([1], Def. 3.2) A regulated real-valued function f on closed interval $[a, b]$ will be called *piecewise monotone* (PM) if there are points $a = x_0 < \cdots < x_k = b$ for some finite k such that f is monotone on each interval $[x_{j-1}, x_j]$, $j = 1, \ldots, k$. Here for $j = 1, \ldots, k-1$, x_j may be point $x-$ or $x+$. The set of all piecewise monotone functions is denoted $PM = PM[a, b]$.

In addition to PM, if f is continuous function we will call it continuous piecewise monotone(CPM). The set of such functions is denoted $CPM = CPM[a, b]$.

Proposition 1.7. ([1], Prop. 3.1) If f is PM, there is a minimal size of partition $|\kappa|$ for which the definition 1.6 holds. The minimal size of the the PM partition is denoted $K(f, [a, b]) = K(f)$, namely

$$K(f) := \min \left\{ n : \exists \{x_i\}_{i=0}^{n} \in PP[a, b] : f \text{ is monotonic in each } [x_{j-1}, x_j] \right\}. \tag{1.3}$$

Definition 1.8 (The set of PM partitions with minimal size). ([1], Def. 3.3) If f is PM, let $X(f) = X(f, [a, b])$ be the set of all $\{x_i\}_{i=0}^{K(f)}$ for which the definition of PM (def. 1.6) holds. $X(f)$ is called the *set of PM partitions with minimal size.*

Proposition 1.9. ([1], Prop. 3.3) Let f is PM then the numbers $\alpha_j(f) := f(x_j) - f(x_{j-1})$ for $\{x_j\}_{j=0}^{K(f)} \in X(f)$ and $j = 1, 2, \ldots K(f)$ are uniquely determined.

Definition 1.10 (The equality by PM). ([1], Def. 3.4) If f, g are two PM functions, possibly on different intervals, such that $K(f) = K(g)$ and $\alpha_j(f) = \alpha_j(g)$ for $j = 1, 2, \ldots, K(f)$, then we say that $f \overset{PM}{=} g$.

Proposition 1.11. ([1], Cor. 3.1) Let $p > 1$ and functions f and g are PM. If $f \overset{PM}{=} g$ or $f \overset{PM}{=} -g$, then $v_p(f) = v_p(g)$.

Proposition 1.12. ([1], Them. 3.1) Let f is PM, $\kappa \in X(f)$ and $1 \leq p < \infty$. Then the supremum of p-variation in Definition 1.2 is attained for some partition $r \subset \kappa$.

Corollary 1.13. The set $MP_p(f, [a, b])$ is not empty for all $f \in PM[a, b]$.

Definition 1.14 (Partial sum). Let $X_1, X_2, \ldots, X_n$ be any real-value sequence. The *partial sum* of the first j terms is defined by

$$S_j := \sum_{i=1}^{j} X_i,\ j = 1, 2, \ldots, n. \tag{1.4}$$

In addition, lets denote $S_0 = 0$.

Definition 1.15 (Sample function). Suppose $X = \{X_i\}_{i=0}^n$ is any real-value sequence. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. Then the *sample function* $G_X : [0, n] \to \mathbb{R}$ is defined as

$$G_X(t) := X_{\lfloor t \rfloor},\ t \in [0, n]. \tag{1.5}$$

Definition 1.16 (p-variation of the sequence). Let $X = \{X_i\}_{i=0}^n$. The p-variation of the sample X is defined as p-variation of the function $G_X(t)$, namely

$$v_p(X) := v_p(G_X(t), [0, n]). \tag{1.6}$$

Definition 1.17 (Piecewise linear function). Let $X = \{X_i\}_{i=0}^n$ be any real-value sequence. The function $L_X : [0, n] \to \mathbb{R}$ is defined as

$$L_X(t) := (1 + \lfloor t \rfloor - t)X_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)X_{\lfloor t \rfloor + 1},\ t \in [0, n] \tag{1.7}$$

is called *piecewise linear function*.

1.2 General properties with proofs

Proposition 1.18. For all $f \in PM$ exists $g \in CPM$ such that $f \overset{PM}{=} g$.

Proof. Let f be PM. According to Proposition 1.9 the numbers $\alpha_j(f) := f(x_j) - f(x_{j-1})$, $j = 1, 2, \ldots K(f)$ are uniquely determined. Then, the partial sums of the sequence $\alpha_j(f)$ are

$$S_j := \sum_{i=1}^{j} \alpha_i(f). \tag{1.8}$$

Lets connect points S_j by piecewise linear function (def. 1.17), namely

$$L_S(t) := (1 + \lfloor t \rfloor - t)S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)S_{\lfloor t \rfloor + 1},\ t \in [0, K(f)] \tag{1.9}$$

Function L_S is CPM. In addition, It is straight forward to see that

$$\alpha_j(L_S) = S_j - S_{j-1} = \alpha_j(f), \tag{1.10}$$

hence, by Definition 1.10 $f \overset{PM}{=} L_S$. ■

Corollary 1.19. Similar argument could be used to show that

$$G_X(t) \overset{PM}{=} L_X(t). \tag{1.11}$$

Therefore, according to Proposition 1.11, if $p > 1$ then

$$v_p(X) := v_p(G_X(t), [0, n]) = v_p(L_X(t), [0, n]) \tag{1.12}$$

Proposition 1.20. Suppose $f : [a, b] \to \mathbb{R}$ is CPM. Let $\{x_i\}_{i=0}^n \in PP[a, b]$ be any partition of interval $[a, b]$. Then the proposition

$$\exists \kappa \in MP_p(f, [a, b]) : \forall i, x_i \in \kappa \tag{1.13}$$

is equivalent to

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.14}$$

Proof. Necessary. Let $f : [a, b] \to \mathbb{R}$ be CPM, $\{x_i\}_{i=0}^n \in PP[a, b]$ and

$$\exists \kappa \in MP_p(f, [a, b]) : \forall i, x_i \in \kappa. \tag{1.15}$$

Points from the partition κ will be denoted t_i, i.e. $\kappa = \{t_i\}_{i=0}^m$. Then, according to definitions of MP_p and p-variation (def. 1.2) the following equation holds

$$v_p(f, [a, b]) = s_p(f; \kappa) = \sum_{i=1}^{|\kappa|} |f(t_{i-1}) - f(t_i)|^p = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p, \tag{1.16}$$

where $h : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$ denotes a function from the set of index of x to the set of index of t, namely:

$$h(i) := (j_i : x_i = t_{j_i} = t_{h(i)}). \tag{1.17}$$

The equation (1.16) holds, because it's just a rearranged sum.

Moreover, the inequality

$$\sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p \le v_p(f, [x_{i-1}, x_i]) \tag{1.18}$$

holds according to Lemma 1.3(g).

As a result of (1.16) and (1.18) we get

$$v_p(f, [a, b]) \le \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.19}$$

On the other hand, according to the same Lemma 1.3(g) the following inequality holds

$$v_p(f, [a, b]) \ge \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.20}$$

Finally, from the (1.19) and (1.20) follows

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.21}$$

Sufficiency. Suppose $f : [a, b] \to \mathbb{R}$ is CPM and

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.22}$$

Lets take any partition from each of the sets $MP_p(f, [x_{i-1}, x_i])$, $i = 1, \ldots, n$ and denote it κ_i. Then, let define a joint partition $\kappa := \cup_{i=1}^{n} \kappa_i$. Points from the partition κ will be denoted by t_i. In addition, we will use the function h, which is defined in (1.17). Then, continuing the equation (1.22) we get

$$v_p(f, [a, b]) = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p = \sum_{i=1}^{|\kappa|} |f(t_{i-1}) - f(t_i)|^p. \tag{1.23}$$

This means that $\kappa \in MP_p(f, [a, b])$. Moreover, $\forall i : x_i \in \kappa$, because $\kappa = \cup_{i=1}^{n} \kappa_i$. ■

1.3 Meaningful break points

Definition 1.21 (Lexicographical order). Suppose f is CPM. Let κ and r denotes two partitions with minimal size, namely $\kappa = \{x_j\}_{j=0}^{K(f)} \in X(f)$ and $r = \{y_j\}_{j=0}^{K(f)} \in X(f)$. Then partitions κ and r could be compare using *lexicographical order*, namely we will say that $\kappa \succ r$, if

- $\exists j : x_j > y_j$; and
- $\forall i < j : x_i = y_i$.

We will say $\kappa = r$ if $\forall j : x_j = y_j$.
We will say $\kappa \succeq r$ if $\kappa \succ r$ or $\kappa = r$.

Proposition 1.22. Binary relation $\succeq$ is a total order, i.e. the relationship is transitive, antisymmetric and total.

Proof. Need reference ... ■

Proposition 1.23. Let $f : [a, b] \to \mathbb{R}$ is CPM. Then there is a partition κ_f such that

$$\forall r \in X(f) : \kappa_f \succeq r. \tag{1.24}$$

Proof. Firstly, we will construct κ_f, latter, we will proof that the property (1.24) holds.

Suppose, $f : [a, b] \to \mathbb{R}$ is CPM. Let denote

$$x_1 := \sup \left\{ x \in [a, b] : |f(y_1) - f(a)| \le |f(y_2) - f(a)|, \forall a \le y_1 < y_2 \le x \right\}.$$

Other $x_i,\ i = 2, \dots, K(f)$ will be defined by induction. Suppose x_{i-1} is defined and $x_{i-1} < b$, then

$$x_i := \sup \left\{ x \in [x_{i-1}, b] : |f(y_1) - f(x_{i-1})| \le |f(y_2) - f(x_{i-1})|, \forall x_{i-1} \le y_1 < y_2 \le x \right\}.$$

According condition used in the definition of x_i, all $(x_{i-1}, x_i),\ i = 1, \dots, K(f)$ are monotonic intervals.

In addition, points x_i have the greatest values, that satisfies the condition. Therefore, if the values x_i would increased, the condition would no longer be valid, i.e.

$$\forall i, \exists \delta : |f(x_i) - f(x_{i-1})| > |f(x_i + \varepsilon) - f(x_{i-1})|, \text{ if } \varepsilon \in (0, \delta). \tag{1.25}$$

We will show that $\kappa_f := \{x_i\}_{i=1}^n$ satisfies the condition (1.24). Suppose to the contrary that

$$\exists r = \{t_i\}_{i=1}^n \in X(f) : r \succ \kappa_s. \tag{1.26}$$

According to Definition of $\succ$ we get

$$\exists j : t_j > x_j, \text{ and } t_i = x_i, \text{ if } i < j. \tag{1.27}$$

Since $r \in X(f)$, the intervals $[t_{j-1}, t_j]$ are monotonic intervals, i.e.

$$|f(y_1) - f(t_{j-1})| \le |f(y_2) - f(t_{j-1})|, \tag{1.28}$$

if $t_{j-1} \le y_1 < y_2 \le t_j$. But (1.28) contradicts (1.25), then $y_1 = x_j$ and $y_2 = x_j + \varepsilon$. ■

Lemma 1.24. Suppose $f : [a,b] \to \mathbb{R}$ is CPM and $\kappa_f = \{x_i\}_{i=1}^{K(f)}$ is the partition from (1.24). Then, for all $x_i, i = 1, \dots, K(f) - 1$, the following proposition holds

$$\forall \varepsilon > 0 : f \text{ is not a constant in } [x_i, x_i + \varepsilon]. \tag{1.29}$$

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is CPM and $\kappa_f = \{x_i\}_{i=1}^{K(f)}$ is the partition from (1.24). Suppose to the contrary, that exists x_j and $\varepsilon > 0$ such that f is not a constant in $[x_j, x_j + \varepsilon]$. Since f is constant in $[x_j, x_j + \varepsilon]$, the point $x_j + \varepsilon$ is also the end of monotone interval (def. 1.6).

Lets define the partition $\{y_i\}_{i=1}^{K(f)} \in X(f)$, which is

$$y_i := \begin{cases} x_i & , \text{if } j \neq i \\ x_i + \varepsilon & , \text{if } j = i \end{cases}, \tag{1.30}$$

where $\varepsilon \in (0, x_{j+1} - x_j)$.

It is straight forward to see that $\{y_i\}_{i=1}^{K(f)} \succ \kappa_f$, since $y_j > x_j$ and $y_i = x_i, i < j$. This contradict the definition of κ_f. ■

Definition 1.25 (Break points). Let κ_f be a partition form Proposition 1.23. The points $x_j \in \kappa_f$ will be called *break points* and the set of such points are denoted $\overline{BP}(f, [a,b])$, which is non empty and finite. The set of partitions that contains only break points are denoted $BP(f, [a,b])$, i.e.

$$BP(f) = BP(f, [a,b]) := \left\{ \kappa \in PP[a,b] : \kappa \subset \overline{BP}(f, [a,b]) \right\}. \tag{1.31}$$

$BP(f, [a,b])$ is also non empty and finite.

For the convenience let define the subsets of the sets $\overline{BP}$ and BP that excludes the ends of the interval, namely

$$\overline{BP}(f, (a,b)) \ := \ \overline{BP}(f, [a,b]) \setminus \{a, b\}, \tag{1.32}$$
$$BP(f, (a,b)) \ := \ BP(f, [a,b]) \setminus \{\{a, b\}\}. \tag{1.33}$$

Proposition 1.26. Let f is CPM function defined in $[a,b]$ and $1 \le p < \infty$. Then the p-variations could be expressed as

$$v_p(f) = v_p(f, [a,b]) = \max \{ s_p(f; \kappa) : \kappa \in BP(f, [a,b]) \}. \tag{1.34}$$

Proof. This result follows directly from the Proposition 1.12. ■

Definition 1.27 (Meaningfull break points). Suppose f is CPM and $1 \leq p < \infty$. Let denote the set of *partitions of meaningful break* (points)

$$MB_p(f) = MB_p(f, [a, b]) := MP_p(f, [a, b]) \cap BP(f, [a, b]). \tag{1.35}$$

The set $MB_p(f, [a, b])$ is not empty according to Proposition 1.26.

The point x will be called *meaningful break* if $\exists \kappa \in MB_p(f) : x \in \kappa$. The set of such points are denoted $\overline{MB}_p(f) = \overline{MB}_p(f, [a, b])$. The point a is called *meaningless* if $a \notin \overline{MB}_p(f, [a, b])$.

Lemma 1.28. Let $\{x_i\}_{i=0}^n \subset \kappa \in MB_p(f, [a, b])$, then

$$\forall i, j : v_p(f, [x_i, x_j]) = \sum_{k=i+1}^{j} v_p(f, [x_{k-1}, x_k]),\ 0 \leq i < j \leq n. \tag{1.36}$$

Proof. Suppose $\{x_i\}_{i=0}^n \subset \kappa \in MB_p(f, [a, b])$. Let choose i and j such that $0 \leq i < j \leq n$. Because $MB_p \subset MP_p$, we can apply Proposition 1.20 for the partition $\{x_0, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_n\}$. Thus,

$$v_p(f, [a, b]) = v_p(f, [x_0, x_i]) + \sum_{k=i+1}^{j} v_p(f, [x_{k-1}, x_k]) + v_p(f, [x_j, x_n]). \tag{1.37}$$

In addition, we can apply the same proposition for the partition $\{x_0, x_i, x_j, x_n\}$, then

$$v_p(f, [a, b]) = v_p(f, [x_0, x_i]) + v_p(f, [x_i, x_j]) + v_p(f, [x_j, x_n]). \tag{1.38}$$

By subtracting one equation form the other we get the result that

$$v_p(f, [x_i, x_j]) = \sum_{k=i+1}^{j} v_p(f, [x_{k-1}, x_k]). \tag{1.39}$$

■

Proposition 1.29. Suppose $f : [a, b] \to \mathbb{R}$ is CPM. Then

$$\exists \kappa_{ab} \in MB_p(f, [a, b]) : \forall \kappa \in MB_p(f, [a, b])[|\kappa_{ab}| \geq |\kappa|]. \tag{1.40}$$

Proof. Suppose f is CPM, then $\forall \kappa \in BP(f) : |\kappa| \leq K(f)$. Since $MB_p(f) \subset BP(f)$, then $\forall \kappa \in MP_p(f) : |\kappa| \leq K(f)$. Therefore, the size of partitions from $MP_p(f)$ are bounded. As a result the size of the partition $|\kappa|$ is bounded natural number, whereas such set has the larges element. ■

Definition 1.30. The set of partitions κ_{ab} which satisfies 1.40 will be denoted $MB_p^m(f,[a,b]) = MB_p^m(f)$. This set is non empty according to Proposition 1.29.

In addition it could be shown (see Proposition 1.45) that the set $MB_p^m(f)$ has exactly one element and

$$\forall \kappa \in MB_p(f) : \kappa \subset \kappa_{ab} \in MB_p^m(f) \tag{1.41}$$

Lemma 1.31. Let f is CMD. Then $\{a,b\} \in MB_p^m(f,[a,b])$ if and only if

$$\forall \kappa \in BP(f,(a,b)) : v_p(f,[a,b]) > s_p(f,\kappa).$$

Here the $BP(f,(a,b))$ is the set of partitions of break points excluding $\{a,b\}$ (see def. 1.33).

Proof. Necessary. Let assume on the contrary that $\{a,b\} \in MP_p^m(f,[a,b])$, but

$$\exists \kappa \in BP(f,(a,b)) : v_p(f,[a,b]) = s_p(f,\kappa).$$

The partition $\kappa \in BP(f,(a,b))$ must have more points then the end of the intervals a and b, because if $\kappa = \{a,b\}$ then it contradicts to the definition of $BP(f,(a,b))$. Therefore, $|\kappa| > |\{a,b\}|$, but this contradicts to assumptions that $\{a,b\} \in MP_p^m(f,[a,b])$.

Sufficiency. Suppose

$$\forall \kappa \in BP(f,(a,b)) : v_p(f,[a,b]) > s_p(f,\kappa). \tag{1.42}$$

According to Proposition 1.26, the p-variation is achieved in the set $BP(f,[a,b])$, therefore, from the (1.42) follows that $\{a,b\}$ is the partition of meaningful break points, i.e. $\{a,b\} \in MP_p(f,[a,b])$. In addition, this partition is the only one, therefore it is the biggest partition, thus $\{a,b\} \in MP_p^m(f,[a,b])$. ■

Lemma 1.32. Let $f : [a,b] \to \mathbb{R}$ be CPM. Suppose $\{x_i\}_{i=0}^n \in MB_p^m(f,[a,b])$. Then

$$\forall i \in \{1,2,\ldots,n\} : \{x_{i-1},x_i\} \in MB_p^m(f,[x_{i-1},x_i]). \tag{1.43}$$

Proof. Suppose to the contrary that

$$\exists j \in \{1,2,\ldots,n\} : \{x_{j-1},x_j\} \notin MB_p^m(f,[x_{j-1},x_j]). \tag{1.44}$$

Then, according to Lemma 1.31

$$\exists \kappa \in BP(f,(x_{j-1},x_j)) : v_p(f,[x_{j-1},x_j]) = s_p(f,\kappa). \tag{1.45}$$

On the other hand, from the Proposition 1.20 we get

$$\begin{aligned} v_p(f,[x_0,x_n]) &= v_p(f,[x_0,x_{j-1}]) + v_p(f,[x_{j-1},x_j]) + v_p(f,[x_j,x_n]) \\ &= s_p(f,\{x_i\}_{i=0}^{j-1}) + s_p(f,\kappa) + s_p(f,\{x_i\}_{i=j}^{n}). \end{aligned}$$

Thus, $\{x_i\}_{i=0}^{j-1} \cup \kappa \cup \{x_i\}_{i=j}^{n} \in MB_p(f,[a,b])$. Therefore, $\{x_i\}_{i=0}^{n} \notin MB_p^m(f,[a,b])$, because

$$|\{x_i\}_{i=0}^{j-1} \cup \kappa \cup \{x_i\}_{i=j}^{n}| > |\{x_i\}_{i=0}^{n}|, \tag{1.46}$$

This contradicts the initial assumption. ■

1.4 f-join

Definition 1.33 (f-join). Suppose $f:[a,b]\to\mathbb{R}$ is CPM. We will say that points t_a and t_b ($t_a<t_b$) are *f-joined* in interval $[a,b]$ if

$$\exists \{x_j\}_{j=0}^{n} \in MB_p(f,[a,b]) : [t_a,t_b] = [x_{j-1},x_j], \text{ with some } j. \tag{1.47}$$

Lemma 1.34. Suppose $f:[a,b]\to\mathbb{R}$ is CPM and $a\le t_a<t_b\le b$. If points t_a and t_b are f-joined, then

$$v_p(f,[t_a,t_b]) = |f(t_b)-f(t_a)|^p. \tag{1.48}$$

Proof. Let $f:[a,b]\to\mathbb{R}$ is CMD, $t_a<t_b$ and points t_a and t_b are f-joined. Then exists $\{x_j\}_{j=0}^{n}$ and j from the Definition 1.33. According to definition of the p-variation and properties of s_p function we get

$$v_p(f,[a,b]) = s_p(f,\{x_i\}_{i=0}^{K(f)}) = s_p(f,\{x_i\}_{i=0}^{j-1}) + s_p(f,\{x_{j-1},x_j\}) + s_p(f,\{x_i\}_{i=j}^{K(f)}). \tag{1.49}$$

Suppose to the contrary that $\exists r \in PP_p(f,[x_{j-1},x_j]) : s_p(f,r) > s_p(f,\{x_{j-1},x_j\})$, thus

$$v_p(f,[a,b]) < s_p(f,\{x_i\}_{i=0}^{j-1}) + s_p(f,r) + s_p(f,\{x_i\}_{i=j}^{K(f)}) = s_p(f,\kappa), \tag{1.50}$$

where $\kappa := \{x_i\}_{i=0}^{j-1} \cup r \cup \{x_i\}_{i=j}^{K(f)}$. The last inequality contradicts the definition of p-variation, therefore $\{x_{j-1},x_j\}$ is the partition of meaningful break points in interval $[x_{j-1},x_j]$, thus

$$v_p(f,[x_{j-1},x_j]) = s_p(f,\{x_{j-1},x_j\}) = |f(x_j)-f(x_{j-1})|^p. \tag{1.51}$$

■

Definition 1.35 (Extremum). We will call the point $t\in[a,b]$ an *extremum* of the function f in interval $[a,b]$ if $f(t)=\sup\{f(z):z\in[a,b]\}$ or $f(t)=\inf\{f(z):z\in[a,b]\}$.

Definition 1.36 (Pseudo-monotonic function). A function f is called *pseudo-monotonic* in interval $[a, b]$ if

$$\forall x \in [a, b] : f(x) \in [\min(f(a), f(b)), \max(f(a), f(b))]. \tag{1.52}$$

Proposition 1.37. Let $f : [a, b] \to \mathbb{R}$ be CPM and points t_a and t_b are f-joined. Then points t_a are t_b extremums of the function in the interval $[t_a, t_b]$. In addition, there is no other break point $d \in BP(f, (t_a, t_b))$ that is an extremum of the function in the interval $[t_a, t_b]$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be CPM and points t_a and t_b are f-joined. Since $f \stackrel{PM}{=} -g$, with out loss of generality we can assume that $f(t_a) \leq f(t_b)$. Suppose to the contrary that $f(t_b)$ is not an extremum of the function in interval $[t_a, t_b]$. Hence, $\exists c \in [t_a, t_b] : f(c) > f(t_b)$. Therefore, $|f(c) - f(t_a)|^p > |f(t_b) - f(t_a)|^p$. According to Proposition 1.34, $v_p(f, [t_a, t_b]) = |f(t_b) - f(t_a)|^p$, thus, $|f(c) - f(t_a)|^p > v_p(f, [t_a, t_b])$, but this contradicts the definition of p-variation. So, point t_b must be an extremum in interval $[t_a, t_b]$. Completely symmetric arguments could be used for point t_a.

In addition, it could be shown that if $d \in BP(f, (t_a, t_b))$, then $f(d) < f(t_b)$. Suppose to the contrary that $\exists d \in BP(f, (t_a, t_b)) : f(d) = f(t_b)$. Then, $|f(d) - f(t_a)|^p = |f(t_b) - f(t_a)|^p$, therefore, $v_p(f, [t_a, d]) = v_p(f, [t_a, t_b])$. Since, $v_p(f, [t_a, t_b]) \geq v_p(f, [t_a, d]) + v_p(f, [d, t_b])$, we get that $v_p(f, [d, t_b]) \leq 0$. So, $v_p(f, [d, t_b]) = 0$, because $v_p(f, [d, t_b]) < 0$ is not valid. According to Lemma 1.3(b), $v_p(f, [d, t_b]) = 0$ iff function is constant in $[d, t_b]$. But this contradicts to the fact that d is a break point (see Lemma 1.24). ■

Corollary 1.38. Let $f : [a, b] \to \mathbb{R}$ be CPM and points t_a and t_b are f-joined. Then function is pseudo-monotonic in interval $[t_a, t_b]$. So, if $f(t_a) \leq f(t_b)$, then

$$\forall x \in (t_a, t_b) : f(t_a) \leq f(x) \leq f(t_b). \tag{1.53}$$

In addition,

$$\forall d \in BP(f, (a, b)) : f(a) < f(d) < f(b). \tag{1.54}$$

Proof. The result follows directly from the Proposition 1.37. ■

Proposition 1.39. Let $f : [a, b] \to \mathbb{R}$ is CPM. If point $d \in [a, b]$ is a break point, which is an extremums of the function f, then d is a meaningful break point.

Proof. Let $f : [a, b] \to \mathbb{R}$ is CPM. Suppose to the contrary that exists $d \in \overline{BP}(f, [a, b])$, which is an extremum of the function, but $d \notin \overline{MB}_p(f, [a, b])$.

Let $\{x_j\}_{j=0}^n \in MB(f,[a,b]$ be any partition from $MB(f)$. Since $d \in [a,b]$ and $d \notin \overline{MB}_p(f,[a,b])$,

$$\exists j \in 1,\dots,n : d \in (x_{j-1},x_j). \tag{1.55}$$

The points x_{j-1} and x_j are f-joined and d is break point, hence, according Corollary 1.38

$$f(x_{j-1}) < f(d) < f(x_j). \tag{1.56}$$

But this contradicts the assumption that d is the extremum. ■

1.5 Main result

Lemma 1.40. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then function $f : [0,\infty) \to \mathbb{R}$ with the values

$$f(x) = (x + c_1)^p - x^p - C,\ x \in [0,\infty), \tag{1.57}$$

are non decreasing in interval $[0,\infty)$.

Proof. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. The the derivative of the function f is

$$\begin{aligned} f'(x) &= p(x+c_1)^{p-1} - px^{p-1} \\ &\geq p(x)^{p-1} - px^{p-1} = 0. \end{aligned}$$

The derivative of function f is non negative, thus the function f is non decreasing. ■

Corollary 1.41. Suppose $c_1 \geq 0$, $1 \leq p < \infty$ and $0 \leq x \leq y$. Then the following implication holds

$$|x + c_1|^p > x^p + C \Rightarrow |y + c_1|^p > y^p + C. \tag{1.58}$$

Proof. Suppose $0 \leq x \leq y$. Since f is non decreasing, $f(x) \leq f(y)$. Therefore, if $f(x) > 0$, then $f(y) > 0$. ■

Proposition 1.42. Let $f : [a,b] \to \mathbb{R}$ is CPM, $1 \leq p < \infty$, $[a',b'] \subset [a,b]$. If

$$\{a',b'\} \in MB_p^m(f,[a',b']), \tag{1.59}$$

then

$$\forall x \in (a',b') : x \notin \overline{MB}_p(f,[a,b]) \tag{1.60}$$

Proof. Suppose to the contrary that the assumptions of the proposition is valid and

$$\exists x \in (a', b') : x \in \overline{MB}_p(f, [a, b]). \tag{1.61}$$

According to the definition of $\overline{MB}_p(f, [a, b])$, the (1.61) means that

$$\exists \{t_i\}_{i=0}^n \in MB_p(f, [a, b]) : x \in \{t_i\}_{i=0}^n. \tag{1.62}$$

Moreover,

$$x \notin \{a, a', b', b\}, \tag{1.63}$$

because $x \in (a', b')$.

Thus, from (1.63) and (1.62) it follows that

$$\exists j \in \{1, 2, \dots, n-1\} : x = t_j. \tag{1.64}$$

Lets denote variables l and k by

$$l := \max \{i \in \{1, 2, \dots, n-1\} : t_i \in (a', b')\}, \tag{1.65}$$

$$k := \min \{i \in \{1, 2, \dots, n-1\} : t_i \in (a', b')\}. \tag{1.66}$$

Since (1.64) holds, the values l and k always exists. According to l and k definitions the following inequality holds

$$t_{k-1} \leq a' < t_k \leq t_l < b' \leq t_{l+1}. \tag{1.67}$$

Points a' and b' are f-joined according to (1.59), therefore, function f is pseudo-monotonic in interval $[a', b']$. Since $v_p(f) = v_p(-f)$ (see Proposition 1.11), with out loss of generality we can assume that interval $[a', b']$ is decreasing, i.e.

$$f(a') > f(b').$$

Moreover, points t_l and t_k are breakpoints, because $\{t_i\}_{i=0}^n \in MB_p(f, [a, b]) \subset BP(f, [a, b])$. Therefore, according to Corollary 1.38 the values $f(t_l)$ and $f(t_k)$ can be only between $f(a')$ and $f(b')$. So,

$$f(a') > f(t_l) > f(b'), \tag{1.68}$$

$$f(a') > f(t_k) > f(b'). \tag{1.69}$$

In addition, since the pairs of points $\{t_l, t_{l+1}\}$ and $\{t_k, t_{k+1}\}$ are also f-joined, the inequalities (1.68) and (1.69) could be extended as follows

$$f(a') > f(t_l) > f(b') \geq f(t_{l+1}), \tag{1.70}$$

$$f(t_{k-1}) \geq f(a') > f(t_k) > f(b'). \tag{1.71}$$

Since $\{a', b'\} \in MB_p^m(f, [a', b'])$, by the Lemma 1.31, any other partition in interval $[a', b']$ is not a meaningful partition, thus

$$|f(a') - f(b')|^p = v_p(f, [a', b']) > v_p(f, [a', t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, b']). \tag{1.72}$$

According to Lemma 1.3(g), from the following proposition follows

$$\begin{aligned} |f(a') - f(b')|^p &> |f(a') - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(b')|^p \\ |f(a') - f(t_l) + f(t_l) - f(b')|^p &> |f(a') - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(b')|^p. \end{aligned}$$

From the (1.70) the following inequalities holds

$$\begin{aligned} f(a') - f(t_l) &> 0, \\ f(t_l) - f(t_{l+1}) \geq f(t_l) - f(b') &> 0. \end{aligned}$$

Therefore, we can use Corollary 1.41. According to it, from the inequality (1.73) follows

$$\begin{aligned} |f(a') - f(t_l) + f(t_l) - f(t_{l+1})|^p &> |f(a') - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(t_{l+1})|^p \\ |f(a') - f(t_{l+1})|^p &> |f(a') - f(t_k)|^p + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]). \end{aligned}$$

Absolutely symmetric argument could be used in other direction. Firstly lets modify last inequality

$$|f(a') - f(t_k) + f(t_k) - f(t_{l+1})|^p > |f(a') - f(t_k)|^p + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]),$$

then form the (1.71) holds the following inequalities

$$\begin{aligned} f(t_k) - f(t_{l+1}) &> 0, \\ f(t_{k-1}) - f(t_k) \geq f(a') - f(t_k) &> 0. \end{aligned}$$

Thus, from Corollary 1.41 we get

$$\begin{aligned} |f(t_{k-1}) - f(t_k) + f(t_k) - f(t_{l+1})|^p &> |f(t_{k-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]) \\ |f(t_{k-1}) - f(t_{l+1})|^p &> v_p(f, [t_{k-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]) \end{aligned}$$

Finlay, using Lemma 1.28 we conclude that

$$|f(t_{k-1}) - f(t_{l+1})|^p > v_p(f, [t_{k-1}, t_{l+1}]).$$

This contradicts with the definition of p-variation. ■

Proposition 1.43. Suppose $f : [a, b] \to \mathbb{R}$ is CPM, $1 \le p < \infty$ and $x \in [a, b]$. Then the proposition

$$T := \exists (a', b') \in \Big\{(u, v) \in \mathbb{R}^2 : x \in [u, v] \subset [a, b]\Big\} : x \notin \overline{MB}_p(f, [a', b']) \tag{1.73}$$

is equivalent to

$$S := x \notin \overline{MB}_p(f, [a, b]). \tag{1.74}$$

Proof. Sufficiency. Suppose $x \notin \overline{MB}_p(f, [a, b])$. Then the proposition (1.73) is obviously true then $[a', b'] = [a, b]$.

Necessity. Suppose $f : [a, b] \to \mathbb{R}$ is CPM, $1 \le p < \infty$, $x \in [a, b]$ and (1.73) holds.

Let $\{t_i\}_{i=0}^{n} \in MB_p^m(f, [a', b'])$. Since $x \in [a', b']$, then $\exists j : x \in (t_{j-1}, t_j)$. In addition, the Lemma 1.32 ensures that $\{t_{j-1}, t_j\} \in MB_p^m(f, [t_{j-1}, t_j])$. Thus, we can apply Proposition 1.42 for interval $[a, b]$ and point x. According to it $x \notin \overline{MB}_p(f, [a, b])$. ■

Corollary 1.44. Since $(T \Leftrightarrow S) \Leftrightarrow (\neg T \Leftrightarrow \neg S)$, we get

$$x \in \overline{MB}_p(f, [a, b]) \Leftrightarrow \forall [a', b'] \subset [a, b], x \in [a', b'] : x \in \overline{MB}_p(f, [a', b']).$$

1.6 Extra conclusions

Proposition 1.45. Suppose $f : [a, b] \to \mathbb{R}$ is CPM, $1 \le p < \infty$. Let $\kappa_{ab} \in MP_p^m(f, [a, b])$, then

$$x \in \overline{MB}_p(f, [a, b]) \Leftrightarrow x \in \kappa_{ab}. \tag{1.75}$$

Proof. Sufficiency. If $x \in \kappa_{ab} \in MP_p^m(f, [a, b])$, then $x \in \overline{MB}_p(f, [a, b])$ according to the definition of $\overline{MB}_p(f, [a, b])$ (see Definition 1.27).

Necessary. Suppose to the contrary that $x \in \overline{MB}(f, [a, b])$, but $x \notin \kappa_{ab} \in MB_p^m(f, [a, b])$. The points of the partition κ_{ab} will be denoted t_i, i.e. $\kappa_{ab} = \{t_i\}_{i=0}^{n}$. Then

$$\exists j \in \{1, 2, \ldots, n\} : x \in (t_{j-1}, t_j).$$

Since $\{(t_{j-1}, t_j\} \in MB_p^m(f, [t_{j-1}, t_j])$, $x \notin \overline{MB}_p(f, [t_{j-1}, t_j])$. Therefore, we can apply Proposition 1.43. From it follows that

$$x \notin \overline{MB}_p(f, [a, b]).$$

This is the contradiction that proves the proposition. ■

Corollary 1.46. According to the definition of MB (see Definition 1.27), from the last proposition directly follows that

$$\forall \kappa \in MB_p(f,[a,b]) : \kappa \subset \kappa_{ab},$$

this means

$$\exists! \kappa_{ab} \in MB_p^m(f,[a,b]).$$

Lemma 1.47. Let $C \geq 0$ and $c_i \geq 0,\ i = 1,\dots,n.$ Suppose $1 \leq p < q < \infty$. Then the implication holds

$$C^p > \sum_{i=1}^{n} c_i^p \Rightarrow C^q > \sum_{i=1}^{n} c_i^q.$$

Proof. Let denote the function $f(p)$ as

$$f(p) = C^p - \left[\sum_{i=1}^{n} c_i^p\right].$$

The derivative of function f is

$$\begin{aligned} f'(p) &= C^p \log p - \left[\sum_{i=1}^{n} c_i^p \log p\right] \\ &= \left(C^p - \left[\sum_{i=1}^{n} c_i^p\right]\right) \log p. \end{aligned}$$

Thus, if $1 \leq p < \infty$ and $C^p > \sum_{i=1}^{n} c_i^p$, then f derivative is non negative. This means that function f is non decreasing, whereas this proof the Lemma 1.47
■

Proposition 1.48. Suppose $f : [a,b] \to \mathbb{R}$ is CPM and $1 \leq p < q < \infty$. Then

$$\overline{MB}_q(f,[a,b]) \subset \overline{MB}_p(f,[a,b]).$$

Proof. Suppose to the contrary that

$$\exists x \in [a,b] : x \in \overline{MB}_q(f,[a,b]),\ x \notin \overline{MB}_p(f,[a,b]).$$

Let $\{t_i\}_{i=0}^{n} \in MB_p^m(f,[a,b])$. Because $x \in [a,b]$ and $x \notin \overline{MB}_p(f,[a,b])$, then

$$\exists j \in \{1,2,\dots,n\} : x \in (t_{j-1},t_j). \tag{1.76}$$

Let arbitrary choose $\kappa \in BP(f,(a,b))$. Then according to Lemma 1.31, the following inequality holds

$$|f(t_{j-1}) - f(t_j)|^p > s_p(f,\kappa).$$

Whereas, according to Lemma 1.47,

$$|f(t_{j-1}) - f(t_j)|^q > s_q(f, \kappa).$$

This means that

$$\{t_{j-1}, t_j\} \in MB_q^m(f, [t_{j-1}, t_j]).$$

Finlay, from the Proposition 1.42 follows that $x \notin \overline{MB}_q(f, [a, b])$, but this contradicts the assumptions. ■

2 p-variation calculus

This chapter will present the algorithm that calculates the p-variation for arbitrary piecewise monotone function. This algorithm will be called *pvar*. It is already realised in the R (see ???) package *pvar* and is publicly available on CRAN.

Let assume f is any piecewise monotone function. According to definition (see 1.6) there are points $a = x_1 < \cdots < x_n = b$ for some finite n such that f is monotone on each interval $[x_{j-1}, x_j]$, $j = 1, 2, \ldots, n$. The sequence of values of the function f at the points of partition $\{x_i\}_{i=0}^n$ will be referred as *sample* and denoted $X = \{X_i\}_{i=0}^n := \{f(x_i)\}_{i=0}^n$. The value of n will be called *sample size*.

The algorithm *pvar* do not use f directly, rather it uses the sample X as an input and operates it to find the p-variation.

2.1 Algorithm

The procedure *pvar* that calculates p-variation will be presented here. Firstly, we will introduce the main schema, further, each step will be discussed in more details.

Procedure *pvar*. Input: sample X, scalar p.

1. *Removing monotonic points.* According to proposition ??? all points that are not the end of monotonic interval could be excluded from further consideration.

2. *Checking all small intervals.* Every sm

3. *Partition spliting.*

4. *Indipenden intervals assesments.*

5. *Joining results.*

Pseudo-code

```
pvar <- function(x, p){
  xNew = Remove_Monotonic_Points(x)
  xNew = Test_Points_In_Small_Intervals(xNew, p, sizeN = 7)
  IntervalList = Split_By_Extremums_All(xNew)
  for(Interval in IntervalList){
    Partition = Join(Partition, pvarMon(Interval, p, sizeN))
  }
  pvar = Pvar_Sum(Partition, p)
  return(pvar)
}
```

3 p-variation for pseudo-monotonic sample

Let suppose that the sample is

- Pseudo monotonic.
- There are no monotonic points.
- All possible small intervals are checked.

1. Check if the given interval is short.

Pseudo kodas:

```
pvarMon <- function(Interval, p, sizeN){
  # Jei tai trumpas intervalas, tai jis jau buvo patikrintas
  if(length(Interval)<=sizeN){
    return(Interval) # tiesiog grazinamas intervalas
  }

  # Kitu atveju skaiciuojam giliau
  NewIntervalList = Split_By_CumExtremums(x[a:b])
  for(Int in NewIntervalList){
    (int1, int2) = Split_MinMax(Int)
    Partition1 = pvarMon(int1)
    Partition2 = pvarMon(int2)
    PassiblePartition = Join(PassiblePartition, Partition1,
   Partition2)
  }

```

```
  # Patikriname pontecialius skaidinio tasku,
  # is eiles tikrindami galimas jungtis
  PointList = EndPoints(NewIntervalList)
  for(Point in PointList){
    SubInt = Subset(PassiblePartition, from=1, to=Point)
    if(Pvar_Sum(SubInt, p)<Pvar_Sum({1,Point}, p)){
      DropPoints = SetDiff(PassiblePartition, {1,Point})
      PassiblePartition = SetDiff(PassiblePartition, DropPoints)
    }
  }
  return(PassiblePartition)
}
```

References

[1] J. Qian. The p-variation of Partial Sum Processes and the Empirical Process // Ph.D. thesis, Tufts University, 1997.