The calculus of p-variation

June 24, 2014

- $PP[a, b]$ – the set of all point partitions of $[a, b]$ (def. 1.1).
- $s_p(f; \kappa)$ – p-variation sum (def. 1.2).
- $v_p(f)$ – p-variation of the function f (def. 1.2).
- $SP_p(f, [a, b])$ – the set of supreme partitions (def. 1.2).
- $\overline{SP}_p(f, [a, b])$ – a set of points that are in any supreme partition (def. 1.20).
- $PM[a, b]$ – a set of piecewise monotone functions (def. 1.6).
- $CPM[a, b]$ – a set of continuous piecewise monotone functions (def. 1.6).
- $K(f, [a, b])$ – minimal size of PM partitions (prop. 1.7).
- $X(f, [a, b])$ – the set of PM partitions with minimal size (def. 1.8).

1 Mathematical analysis

1.1 General known properties

Definition 1.1 (Partition). Let $J = [a, b]$ be a closed interval of real numbers with $-\infty < a \leq b < +\infty$. If $a < b$, an ordered set $\kappa = \{x_i\}_{i=0}^{n}$ of points in $[a, b]$ such that a $a = x_0 < x_1 < x_2 < ... < x_n = b$ is called a *(point) partition*. The size of the partition is denoted $|\kappa| := \#\kappa - 1 = n$. The set of all point partitions of $[a, b]$ is denoted by $PP[a, b]$.

Definition 1.2 (p-variation). Let $f : [a, b] \to \mathbb{R}$ be a real function from an interval $[a, b]$. If $a < b$, for $\kappa = \{x_i\}_{i=0}^{n} \in PP[a, b]$ the *p-variation sum* is

$$s_p(f, \kappa) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p, \tag{1.1}$$

where $0 < p < \infty$. Thus, the *p-variation* of f over $[a,b]$ is 0 if $a = b$ and otherwise

$$v_p(f) = v_p(f,[a,b]) := \sup\left\{s_p(f,\kappa) : \kappa \in PP[a,b]\right\}. \tag{1.2}$$

The partition κ is called *supreme partition* if it satisfies the property $v_p(f) = s_p(f,\kappa)$. The set of such partitions is denoted $SP_p(f,[a,b])$.

Vidinis Komentaras (VK): Dar neaisku kaip pakrikstyti skaidini, kuris pasiekia supremuma.

Lemma 1.3 (Elementary properties). Let $f : [a,b] \to \mathbb{R}$ and $0 < p < \infty$. Then the following p-variation properties holds

a) $v_p(f,[a,b]) \geq 0$,

b) $v_p(f,[a,b]) = 0 \Leftrightarrow f \equiv Const.$,

c) $\forall C \in \mathbb{R} : v_p(f + C,[a,b]) = v_p(f,[a,b])$,

d) $\forall C \in \mathbb{R} : v_p(Cf,[a,b]) = C^p v_p(f,[a,b])$,

e) $\forall c \in [a,b] : v_p(f,[a,b]) \geq v_p(f,[a,c]) + v_p(f,[c,b])$,

f) $\forall [a',b'] \subset [a,b] : v_p(f,[a,b]) \geq v_p(f,[a',b'])$.

g) $\forall \kappa \in PP[a,b] : s_p(f;\kappa) \leq v_p(f,[a,b])$.

All listed properties are elementary derived directly form the p-variation definition.

Definition 1.4 (Regulated function). ([1], Def. 3.1) For any interval J, which may be open or closed at either end, real function f is called *regulated* on J if it has left and right limits $f(x-)$ and $f(x+)$ respectively at each point x in interior of J, a right limit at the left end point and a left limit at the right endpoint.

Proposition 1.5. ([1], Lemma 3.1) Let $1 \leq p < \infty$. If f is regulated then $v_p(f)$ remains the same if points $x+$, $x-$ are allowed as partition points x_i in the definition 1.2.

Definition 1.6 (Piecewise monotone functions). ([1], Def. 3.2) A regulated real-valued function f on closed interval $[a,b]$ will be called *piecewise monotone* (PM) if there are points $a = x_0 < \cdots < x_k = b$ for some finite k such that f is monotone on each interval $[x_{j-1},x_j]$, $j = 1,\ldots,k$. Here for $j = 1, \ldots, k-1$, x_j may be point $x-$ or $x+$. The set of all piecewise monotone functions is denoted $PM = PM[a,b]$.

In addition to PM, if f is continuous function we will call it continuous piecewise monotone(CPM). The set of such functions is denoted $CPM = CPM[a,b]$.

Proposition 1.7. ([1], Prop. 3.1) If f is PM, there is a minimal size of partition $|\kappa|$ for which the definition 1.6 holds. The minimal size of the the PM partition is denoted $K(f,[a,b]) = K(f)$, namely

$$K(f) := \min\left\{n : \exists\{x_i\}_{i=0}^{n} \in PP[a,b] : f \text{ is monotonic in each } [x_{j-1}, x_j]\right\}. \tag{1.3}$$

Definition 1.8 (The set of PM partitions with minimal size). ([1], Def. 3.3) If f is PM, let $X(f) = X(f,[a,b])$ be the set of all $\{x_i\}_{i=0}^{K(f)}$ for which the definition of PM (def. 1.6) holds. $X(f)$ is called the *set of PM partitions with minimal size.*

Proposition 1.9. ([1], Prop. 3.3) Let f is PM then the numbers $\alpha_j(f) := f(x_j) - f(x_{j-1})$ for $\{x_j\}_{j=0}^{K(f)} \in X(f)$ and $j = 1, 2, \ldots K(f)$ are uniquely determined.

Proposition 1.10. ([1], Prop. 3.2) Let f is PM. For any partition $\{x_j\}_{j=0}^{K(f)} \in X(f)$ exactly one of the flowing stamens holds:

(a) $f(x_0) > f(x_1) < f(x_2) > \ldots$. Function f is not increasing in intervals $[x_{2j}, x_{2j+1}]$, then $2j+1 \leq K(f)$. Function f is not decreasing in intervals $[x_{2j-1}, x_{2j}]$, then $j \geq 1$ and $2j \leq K(f)$.

(b) (a) holds for a function $-f$; or

(c) $K(f,[a,b]) = 1$ and f is constant in interval $[a,b]$.

Definition 1.11 (The equality by PM). ([1], Def. 3.4) If f, g are two PM functions, possibly on different intervals, such that $K(f) = K(g)$ and $\alpha_j(f) = \alpha_j(g)$ for $j = 1, 2, \ldots, K(f)$, then we say that $f \overset{PM}{=} g$.

Proposition 1.12. ([1], Cor. 3.1) Let $p > 1$ and functions f and g are PM. If $f \overset{PM}{=} g$ or $f \overset{PM}{=} -g$, then $v_p(f) = v_p(g)$.

Proposition 1.13. ([1], Them. 3.1) Let f is PM, $\kappa \in X(f)$ and $1 \leq p < \infty$. Then the supremum of p-variation in Definition 1.2 is attained for some partition $r \subset \kappa$.

Corollary 1.14. The set $SP_p(f, [a, b])$ is not empty for all $f \in PM[a, b]$.

Definition 1.15 (Sample function). Suppose $X = \{X_i\}_{i=0}^n$ is any sequence real numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. Then the *sample function* $G_X : [0, n] \to \mathbb{R}$ is defined as

$$G_X(t) := X_{\lfloor t \rfloor},\ t \in [0, n], \tag{1.4}$$

where $\lfloor t \rfloor$ denotes floor function at point t.

Definition 1.16 (p-variation of the sequence). Let $X = \{X_i\}_{i=0}^n$. The p-variation of the sample X is defined as p-variation of the function $G_X(t)$, namely

$$v_p(X) := v_p(G_X(t), [0, n]). \tag{1.5}$$

1.2 General properties with proofs

Proposition 1.17. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{x_i\}_{i=0}^n \in PP[a, b]$ is any partition of interval $[a, b]$. Then the statement

$$\exists \kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa \tag{1.6}$$

is equivalent to

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.7}$$

Proof. Necessary. Let $f : [a, b] \to \mathbb{R}$ be PM, $\{x_i\}_{i=0}^n \in PP[a, b]$ and

$$\exists \kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa. \tag{1.8}$$

Points from the partition κ will be denoted t_i, i.e. $\kappa = \{t_i\}_{i=0}^m$. Then, according to definitions of SP_p and p-variation (def. 1.2) the following equation holds

$$v_p(f, [a, b]) = s_p(f; \kappa) = \sum_{j=1}^{m} |f(t_{j-1}) - f(t_j)|^p = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p, \tag{1.9}$$

where $h : \{0, 1, \dots, n\} \to \{0, 1, \dots, m\}$ denotes a function from the set of index of x to the set of index of t, namely:

$$h(i) := (j_i : x_i = t_{j_i} = t_{h(i)}). \tag{1.10}$$

The equation (1.9) holds, because all the elements in the sum remains, we just grouped them.

Moreover, the inequality

$$\sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p \leq v_p(f, [x_{i-1}, x_i]) \tag{1.11}$$

holds according to Lemma 1.3(g).

As a result of (1.9) and (1.11) we get

$$v_p(f, [a, b]) \leq \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.12}$$

On the other hand, according to the same Lemma 1.3(e) the following inequality holds

$$v_p(f, [a, b]) \geq \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.13}$$

Finally, from the (1.12) and (1.13) follows

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.14}$$

Sufficiency. Suppose $f : [a, b] \to \mathbb{R}$ and

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.15}$$

According to Corollary 1.14, sets $SP_p(f, [x_{i-1}, x_i]),\ i = 1, \ldots, n$ are not empty. Lets take any partition from each of the sets $SP_p(f, [x_{i-1}, x_i]),\ i = 1, \ldots, n$ and denote it κ_i.

Then, let define a joint partition $\kappa := \cup_{i=1}^{n} \kappa_i$. Points from the partition κ will be denoted by t_i. In addition, we will use the function h, which is defined in (1.10). Then, continuing the equation (1.15) we get

$$v_p(f, [a, b]) = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p = \sum_{i=1}^{|\kappa|} |f(t_{i-1}) - f(t_i)|^p. \tag{1.16}$$

This means that $\kappa \in MP_p(f, [a, b])$. Moreover, $\forall i : x_i \in \kappa$, because $\kappa = \cup_{i=1}^{n} \kappa_i$. ■

Lemma 1.18. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{x_i\}_{i=0}^{n} \subset \kappa \in SP_p(f, [a, b])$. Then

$$\forall k, l : v_p(f, [x_k, x_l]) = \sum_{i=k+1}^{l} v_p(f, [x_{i-1}, x_i]),\ 0 \leq k < l \leq n. \tag{1.17}$$

Proof. Suppose $\{x_i\}_{i=0}^n \subset \kappa \in SP_p(f,[a,b])$. Let choose k and l such that $0 \leq k < l \leq n$. Lets apply Proposition 1.17 for the partition $\{x_0, x_k, x_{k+1}, \ldots, x_{l-1}, x_l, x_n\}$. Thus,

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]) + v_p(f,[x_l,x_n]). \tag{1.18}$$

In addition, we can apply the same proposition for the partition $\{x_0, x_k, x_l, x_n\}$, then

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + v_p(f,[x_k,x_l]) + v_p(f,[x_l,x_n]). \tag{1.19}$$

By subtracting one equation form the other we get the result that

$$v_p(f,[x_k,x_l]) = \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]). \tag{1.20}$$

■

Lemma 1.19. Let $f : [a,b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f,[a,b])$. Then

$$\forall k,l : v_p(f,[t_k,t_l]) = s_p\left(f, \{t_i\}_{i=k}^l\right),\ 0 \leq k < l \leq n. \tag{1.21}$$

Proof. Let $f : [a,b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f,[a,b])$. Let choose k and l such that $0 \leq k < l \leq n$. Then

$$\begin{aligned} v_p(f,[a,b]) &= s_p(f,\{t_i\}_{i=0}^n) && (1.22)\\ &= s_p(f,\{t_i\}_{i=0}^k) + s_p(f,\{t_i\}_{i=k}^l) + s_p(f,\{t_i\}_{i=l}^n) && (1.23)\\ &\leq v_p(f,[a,t_k]) + s_p(f,\{t_i\}_{i=k}^l) + v_p(f,[t_l,b]). && (1.24) \end{aligned}$$

The last inequality holds according to Lemma 1.3(g).

On the other hand, from Proposition 1.18 we get

$$v_p(f,[a,b]) = v_p(f,[a,t_k]) + v_p(f,[t_k,t_l]) + v_p(f,[t_l,b]). \tag{1.25}$$

Form (1.24) and (1.25) follows

$$v_p(f,[t_k,t_l]) \leq s_p(f,\{t_i\}_{i=k}^l). \tag{1.26}$$

Finally, from Lemma 1.3(g) we conclude that

$$v_p(f,[t_k,t_l]) = s_p(f,\{t_i\}_{i=k}^l). \tag{1.27}$$

■

Definition 1.20 (The point of supreme partition). Let $f : [a,b] \to \mathbb{R}$ be PM. The point x will be called the *point of supreme partition* if

$$\exists \kappa \in SP_p(f,[a,b]) : x \in \kappa. \tag{1.28}$$

The set of such points will be denoted by $\overline{SP}_p(f,[a,b])$.

Lemma 1.21. Suppose $f : [a,b] \to \mathbb{R}$ is PM, $x \in [a,b]$, $x \notin \overline{SP}_p(f,[a,b])$ and $\{t_i\}_{i=0}^n \in SP(f,[a,b])$ is any supreme partition. Then,

$$\exists j = 1,\dots,n : x \in (t_{j-1},t_j) \text{ and } x \notin \overline{SP}_p(f,[t_{j-1},t_j]). \tag{1.29}$$

Proof. Suppose the assumptions of lemma is valid. Since $x \in [a,b]$ and $[a,b] = \cup_{i=1}^n [t_{i-1},t_i]$, then

$$\exists j = 1,\dots,n : x \in [t_{j-1},t_j]. \tag{1.30}$$

Moreover, $x \notin \{t_i\}_{i=0}^n$, because $x \notin \overline{SP}_p(f,[a,b])$, thus, $x \neq t_{j-1}$ and $x \neq t_j$. In addition to (1.30) this means that $x \in (t_{j-1},t_j)$.

Now, we will proof that $x \notin \overline{SP}_p(f,[t_{j-1},t_j])$. Suppose to the contrary that $x \in \overline{SP}_p(f,[t_{j-1},t_j])$. Then, according to definition of $\overline{SP}_p(f,[t_{j-1},t_j])$,

$$\exists \kappa \in SP_p(f,[t_{j-1},t_j]) : x \in \kappa. \tag{1.31}$$

Since, $\kappa \in SP_p(f,[t_{j-1},t_j])$, then

$$v_p(f,[t_{j-1},t_j]) = s_p(f,\kappa) \tag{1.32}$$

Applying Proposition 1.17 for partition $\{t_i\}_{i=0}^n$ we get

$$\begin{aligned} v_p(f,[a,b]) &= v_p(f,[t_0,t_{j-1}]) + v_p(f,[t_{j-1},t_j]) + v_p(f,[t_j,t_n]) &(1.33)\\ &= s_p(f,\{t_i\}_{i=0}^{j-1}) + s_p(f,\kappa) + s_p(f,\{t_i\}_{i=j}^n) &(1.34)\end{aligned}$$

This means that the partition $r := \{t_i\}_{i=0}^{j-1} \cup \kappa \cup \{t_i\}_{i=j}^n$ is supreme partition, so $x \in r \in SP(f,[a,b])$, therefore, by definition $x \in \overline{SP}_p(f,[a,b])$. This contradict to initial assumption. ■

Definition 1.22 (*f*-join). Suppose $f : [a,b] \to \mathbb{R}$ is PM. We will say that points t_a and t_b $(t_a < t_b)$ are *f-joined* in interval $[a,b]$ if

$$\exists \{x_j\}_{j=0}^n \in SP_p(f,[a,b]) : [t_a,t_b] = [x_{j-1},x_j], \text{ with some } j. \tag{1.35}$$

Lemma 1.23. Suppose $f : [a,b] \to \mathbb{R}$ is PM and points t_a and t_b $(t_a < t_b)$ are f-jointed in interval $[a,b]$. Then all following statements holds

a) $v_p(f,[t_a,t_b] = |f(t_a) - f(t_b)|^p$;

b) Let $x \in [t_a,t_b]$. If $f(t_a) \geq f(t_b)$, then $f(t_a) \geq f(x) \geq f(t_b)$. If $f(t_a) \leq f(t_b)$, then $f(t_a) \leq f(x) \leq f(t_b)$;

Proof.

a) Let $f : [a, b] \to \mathbb{R}$ be PM, $t_a < t_b$ and pair of points t_a, t_b are f-joined. Then exists $\{x_j\}_{j=0}^{n}$ and j from the Definition 1.22. Thus, according to Lemma 1.19

$$v_p(f, [t_{j-1}, t_j]) = s_p(f, \{t_{j-1}, t_j\}) = |f(t_a) - f(t_b)|^p . \tag{1.36}$$

b) Let $f : [a, b] \to \mathbb{R}$ be PM and points t_a and t_b are f-joined. Since $f \overset{PM}{=} -f$, with out loss of generality we can assume that $f(t_a) \leq f(t_b)$. Suppose to the contrary that $f(t_b)$ is not an extrema of the function in interval $[t_a, t_b]$. Hence, $\exists c \in [t_a, t_b] : f(c) > f(t_b)$. Therefore, $|f(c) - f(t_a)|^p > |f(t_b) - f(t_a)|^p$. According to (1.36), $v_p(f, [t_a, t_b]) = |f(t_b) - f(t_a)|^p$, thus, $|f(c) - f(t_a)|^p > v_p(f, [t_a, t_b])$, but this contradicts the definition of p-variation. So, point t_b must be an extrema in interval $[t_a, t_b]$. Symmetric arguments could be used for point t_a.

■

Lemma 1.24. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then function $f : [0, \infty) \to \mathbb{R}$ with the values

$$f(x) = (x + c_1)^p - x^p - C, \ x \in [0, \infty), \tag{1.37}$$

are non decreasing in interval $[0, \infty)$.

Proof. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then, for all $x \geq 0$, the derivative of the function f is

$$\begin{aligned} f'(x) &= p(x + c_1)^{p-1} - px^{p-1} \\ &\geq px^{p-1} - px^{p-1} = 0. \end{aligned}$$

The derivative of function f is non negative, thus the function f is non decreasing, if $x \geq 0$. ■

Corollary 1.25. Suppose $c_1 \geq 0$, $C \in \mathbb{R}$, $1 \leq p < \infty$ and $0 \leq x \leq y$. Then the following implication holds

$$|x + c_1|^p > x^p + C \Rightarrow |y + c_1|^p > y^p + C. \tag{1.38}$$

Proof. Suppose $0 \leq x \leq y$. Since f is non decreasing, $f(x) \leq f(y)$. Therefore, if $f(x) > 0$, then $f(y) > 0$. ■

Proposition 1.26. Suppose $f : [a, b] \to \mathbb{R}$ is PM and $x \in [a', b'] \subset [a, b]$. If $x \notin \overline{SP}_p(f, [a', b'])$, then $x \notin \overline{SP}_p(f, [a, b])$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be PM, $x \in [a', b'] \subset [a, b]$, and $x \notin \overline{SP}_p(f, [a', b'])$. Suppose to the contrary that $x \in \overline{SP}_p(f, [a, b])$.

Since $x \in \overline{SP}_p(f, [a, b])$, according to the Definition 1.20,

$$\exists \{t_i\}_{i=0}^{n} \in SP_p(f, [a, b]) : x \in \{t_i\}_{i=0}^{n}. \tag{1.39}$$

Let $\{y_i\}_{i=0}^{n} \in SP_p(f, [a', b'])$ be any supreme partition from the interval $[a', b']$. Then, according to Lemma 1.21,

$$\exists j = 1, \ldots, n : x \in (y_{j-1}, y_j) \text{ and } x \notin \overline{SP}_p(f, [y_{j-1}, y_j]). \tag{1.40}$$

Moreover,

$$x \notin \{a, y_{j-1}, y_j, b\}, \tag{1.41}$$

because $x \in (y_{j-1}, y_j) \subset [a, b]$. Thus, from (1.39) and (1.41) follows that

$$\exists r \in \{1, 2, \ldots, n-1\} : t_r = x. \tag{1.42}$$

Lets denote variables l and k by

$$\begin{aligned} l &:= \max \{i \in \{1, 2, \ldots, n-1\} : t_i \in (y_{j-1}, y_j)\}, \\ k &:= \min \{i \in \{1, 2, \ldots, n-1\} : t_i \in (y_{j-1}, y_j)\}. \end{aligned}$$

Since $x \in (y_{j-1}, y_j)$ and (1.42) holds, the values l and k always exists and $k \le r \le l$. According to l and k definitions the following inequality holds

$$t_{k-1} \le y_{j-1} < t_k \le x \le t_l < y_j \le t_{l+1}. \tag{1.43}$$

According to Lemma 1.3(e)

$$v_p(f, [y_{j-1}, y_j]) \ge v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]), \tag{1.44}$$

Firstly, let suppose

$$v_p(f, [y_{j-1}, y_j]) = v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]). \tag{1.45}$$

Lets take any supreme partitions from intervals $[y_{j-1}, t_k]$ and $[t_l, y_j]$, namely $\kappa_k \in SP_p(f, [y_{j-1}, t_k]$ and $\kappa_l \in SP_p(f, [t_l, y_j]$. In addition, since $\{t_i\}_{i=0}^{n} \in SP_p(f, [a, b])$, according to Lemma 1.19 $v_p(f, [t_k, t_l]) = s_p(f, \{t_i\}_{i=k}^{l})$. Therefore,

$$v_p(f, [y_{j-1}, y_j]) = s_p(f, \kappa_k) + s_p(f, \{t_i\}_{i=k}^{l}) + v_p(f, \kappa_l).$$

This means, that

$$\kappa_k \cup \{t_i\}_{i=k}^{l} \cup \kappa_l \in SP_p(f, [y_{j-1}, y_j].$$

So, $x \in \overline{SP}_p(f, [y_{j-1}, y_j])$, because $x \in \{t_i\}_{i=k}^{l}$. This contradicts (1.40), thus, equality (1.45) is not valid. As a result, inequality (1.44) becomes

$$v_p(f, [y_{j-1}, y_j]) > v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]). \qquad (1.46)$$

Since points y_{j-1} and y_j are f-joined, from the Lemma 1.23(a) we get

$$v_p(f, [y_{j-1}, y_j]) = |f(y_{j-1}) - f(y_j)|^p,$$

therefore,

$$|f(y_{j-1}) - f(y_j)|^p > v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]).$$

Moreover, according to Lemma 1.3(g), from the last statement follows

$$|f(y_{j-1}) - f(y_j)|^p > |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(y_j)|^p. \qquad (1.47)$$

Since $v_p(f) = v_p(-f)$ (see Proposition 1.12), with out loss of generality we can assume that $f(y_{j-1}) \geq f(y_j)$. Hence, from the Lemma 1.23(b) we get that

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j), \qquad (1.48)$$
$$f(y_{j-1}) \geq f(t_k) \geq f(y_j). \qquad (1.49)$$

In addition, the pairs of points $\{t_l, t_{l+1}\}$ and $\{t_{k-1}, t_k\}$ are also f-joined, therefore, by the Lemma 1.23(b) inequalities (1.48) and (1.49) could be extended as

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j) \geq f(t_{l+1}), \qquad (1.50)$$
$$f(t_{k-1}) \geq f(y_{j-1}) \geq f(t_k) \geq f(y_j). \qquad (1.51)$$

From the (1.50) the following inequalities holds

$$\begin{aligned} f(y_{j-1}) - f(t_l) &\geq 0, \\ f(t_l) - f(t_{l+1}) \geq f(t_l) - f(y_j) &\geq 0. \end{aligned}$$

Therefore, we can use Corollary 1.25. According to it, from the inequality (1.47) follows

$$\begin{aligned} |f(y_{j-1}) - f(t_l) + f(t_l) - f(y_j)|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(y_j)|^p, \\ |f(y_{j-1}) - f(t_l) + f(t_l) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(t_{l+1})|^p, \\ |f(y_{j-1}) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]). \end{aligned}$$

The last inequality holds, because points t_l and t_{l+1} are f-joined, thus, according to Lemma 1.23(a),

$$v_p(f,[t_l,t_{l+1}]) = |f(t_l) - f(t_{l+1})|^p.$$

Symmetric argument could be used in other direction. Firstly lets modify last inequality

$$|f(y_{j-1})-f(t_k)+f(t_k)-f(t_{l+1})|^p > |f((y_{j-1})-f(t_k)|^p+v_p(f,[t_k,t_l])+v_p(f,[t_l,t_{l+1}]).$$

From the (1.51) we get that the following inequalities holds

$$\begin{aligned} f(t_k) - f(t_{l+1}) &\geq 0, \\ f(t_{k-1}) - f(t_k) \geq f(y_{j-1}) - f(t_k) &\geq 0. \end{aligned}$$

Thus, from Corollary 1.25 we get

$$\begin{aligned} |f(t_{k-1}) - f(t_k) + f(t_k) - f(t_{l+1})|^p &> |f(t_{k-1}) - f(t_k)|^p + v_p(f,[t_k,t_l]) + v_p(f,[t_l,t_{l+1}]), \\ |f(t_{k-1}) - f(t_{l+1})|^p &> v_p(f,[t_{k-1},t_k]) + v_p(f,[t_k,t_l]) + v_p(f,[t_l,t_{l+1}]). \end{aligned}$$

As previous, the last inequality holds, because t_{k-1} and t_k are f-joined.

Finally, using Lemma 1.18 we conclude that

$$|f(t_{k-1}) - f(t_{l+1})|^p > v_p(f,[t_{k-1},t_{l+1}]).$$

This contradicts with the definition of p-variation.

■

1.3 Extra results

Definition 1.27 (Extremum). We will call the point $t \in [a,b]$ an *extrema* of the function f in interval $[a,b]$ if $f(t) = \sup\{f(z) : z \in [a,b]\}$ or $f(t) = \inf\{f(z) : z \in [a,b]\}$.

Proposition 1.28. Let $f : [a,b] \to \mathbb{R}$ is PM. If point $x \in [a,b]$ is extrema of the function f, then $x \in \overline{SP}_p(f,[a,b])$.

Proof. Let $f : [a,b] \to \mathbb{R}$ is PM and point $x \in [a,b]$ is an extrema of the function f. Let $\{t_i\}_{i=0}^n \in SP_p(f,[a,b]$ be any supreme partition. Then

$$\exists j \in 1,\dots,n : x \in [t_{j-1},t_j] \tag{1.52}$$

Since x is an extrema of function f in interval $[a,b] \supset [t_{j-1},t_j]$, the point x is an extrema in interval $[t_{j-1},t_j]$ as well. Therefore,

$$f(t_{j-1}) - f(t_j)|^p \leq |f(t_{j-1}) - f(x)|^p \tag{1.53}$$

or

$$|f(t_{j-1}) - f(t_j)|^p \leq |f(x) - f(t_j)|^p. \tag{1.54}$$

As a result,

$$\begin{aligned} |f(t_{j-1}) - f(t_j)|^p &\leq |f(t_{j-1}) - f(x)|^p + |f(x) - f(t_j)|^p, \\ |f(t_{j-1}) - f(t_j)|^p &\leq v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]), \\ v_p(f, [t_{j-1}, t_j]) &\leq v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j]). \end{aligned}$$

The last inequality holds, because t_{j-1} and t_j are f-joined, so, $v_p(f, [t_{j-1}, t_j]) = |f(t_{j-1}) - f(t_j)|^p$. Since $v_p(f, [t_{j-1}, t_j]) < v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j])$ is not valid, the equation

$$v_p(f, [t_{j-1}, t_j]) = v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) \tag{1.55}$$

holds.

Moreover, applying Proposition 1.17 for the partition $\{a, t_{j-1}, t_j, b\}$ we have

$$\begin{aligned} v_p(f, [a, b]) &= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, t_j]) + v_p(f, [t_j, b]) \\ &= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) + v_p(f, [t_j, b]) \end{aligned}$$

From the same proposition follows, that $\exists \kappa \in SP_p(f, [a, b]) : x \in \kappa$, so, by definition, $x \in \overline{SP}_p(f, [a, b])$. ■

Lemma 1.29. Let $f : [a, b] \to \mathbb{R}$ be PM. If $K(f)$ is even, then

$$\exists x \in (a, b) : x \in \overline{SP}_p(f, [a, b]). \tag{1.56}$$

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is PM. Lets take any $\{x_i\}_{i=0}^n \in X(f)$. If $n > 1$, then without loss of generality we can assume that Proposition 1.10(a) holds. So, $f(x_0) > f(x_1) < f(x_2) > \dots$, i.e. $f(x_{2i-1}) < (x_{2i})$, thus, if n is even, then $f(x_{n-1}) < (x_n)$, therefore, point x_n is not a global minimum. Since $f(x_0) > f(x_1)$, point x_0 is also not a global minimum. As a result, global minimum is in interval (a, b) and it is in $\overline{SP}_p(f, [a, b])$ according to Lemma 1.28. ■

Lemma 1.30. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in PP[a, b]$. Suppose

$$\{t_i\}_{i=0}^{n-1} \in SP_p(f, [a, t_{n-1}]) \tag{1.57}$$

and

$$\{t_i\}_{i=1}^{n} \in SP_p(f, [t_1, b]). \tag{1.58}$$

Then

$$\forall i \in \{1, \dots, n-1\} : t_i \in \overline{SP}_p(f, [a, b]) \tag{1.59}$$

or

$$\forall i \in \{1, \dots, n-1\} : t_i \notin \overline{SP}_p(f, [a, b]) \tag{1.60}$$

Proof. Let the assumptions of lemma be valid. Firstly, suppose

$$\exists j \in \{1, \ldots, n-1\} : t_j \in \overline{SP}_p(f, [a, b]). \tag{1.61}$$

This means that $\exists \kappa \in SP_p(f, [a, b]) : t_j \in \kappa$. Then, by Proposition 1.17,

$$v_p(f, [a, b]) = v_p(f, [a, t_j]) + v_p(f, [t_j, b]) \tag{1.62}$$

Since (1.57) and (1.57) holds, by Lemma 1.19 we get

$$v_p(f, [a, t_j]) = s_p(f, \{t_i\}_{i=0}^{j}, \tag{1.63}$$

$$v_p(f, [t_j, b]) = s_p(f, \{t_i\}_{i=j}^{n}. \tag{1.64}$$

Therefore,

$$v_p(f, [a, b]) = s_p(f, \{t_i\}_{i=0}^{j}) + s_p(f, \{t_i\}_{i=j}^{n}) = s_p(f, \{t_i\}_{i=0}^{n}). \tag{1.65}$$

So, $\{t_i\}_{i=0}^{n} \in SP_p(f, [a, b])$, thus, statement (1.59) holds.
On the other hand, if (1.61) is not valid, then statement (1.60) holds.
■

2 p-variation calculus

This chapter will present the algorithm that calculates p-variation for the sample (see Def. 1.16). Nonetheless, this algorithm could be used to calculate the p-variation for arbitrary piecewise monotone function. This algorithm will be called *pvar*. It is already realised in the R (see [2]) package *pvar* and is publicly available on CRAN[1].

Suppose $X = \{X_i\}_{i=0}^{n}$ is any real-value sequence of numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. The formal definition of p-variation of the sample is given in Definition 1.16, which states that

$$v_p(X) = v_p(G_X(t), [0, n]), \tag{2.1}$$

where $G_X(t)$ is sample function defined in Definition 1.15.

On the other hand, the algorithm does not use $G_X(t)$ function directly, but it actually operates the sample X. Therefore, in the context of algorithm, it is more convenient to use the sample X rather then the function $G_X(t)$. Thus, please don't be mislead then in the context of sample we will refer to properties of p-variation that were proved to functions. In that case we actually have in mind function $G_X(t)$.

[1] http://cran.r-project.org/web/packages/pvar/index.html

It is worth noting, that the procedure *pvar* could be used to calculate the p-variation of any piecewise monotone function. Let assume f is any piecewise monotone function. According to definition (see 1.6) there are points $a = x_1 < \cdots < x_n = b$ for some finite n such that f is monotone on each interval $[x_{j-1}, x_j],\ j = 1, 2, \ldots, n$. Lets construct the the sample X, which contains the values of the function f at the points $\{x_i\}_{i=0}^n$, namely, $X = \{X_i\}_{i=0}^n := \{f(x_i)\}_{i=0}^n$. It is straight forward to see that $f \overset{PM}{=} G_X$, therefore, if $p \geq 1$ then $v_p(X) = v_p(G_X) = v_p(f)$.

2.1 Main function

The procedure *pvar* that calculates p-variation will be presented here. Firstly, we will introduce the main schema, further, each step will be discussed in more details.

The main goal of the procedure is to find any supreme partition $\kappa \in SP_p(G_X)$ and calculate p-variation by $v_p(X) = s_p(G_X, \kappa)$. The main principal of the procedure is to identify points that are not in supreme partition and drop them out from further consideration.

For the continence, it is worth pointing out, that p-variation of the sample could express as

$$v_p(X) = \max\left\{\sum_{i=1}^{k} |X_{j_i} - X_{j_{i-1}}|^p : 0 = j_0 < \cdots < j_k = n,\ k = 1, \ldots, n\right\}. \tag{2.2}$$

This expression could be verified from Proposition 1.13, which states, that p-variation is achieved in a subset of any partition $r \in X(f, [0, n])$. So, we choose partition r to be a subset of $\{0, 1, \ldots, n\}$. This partition exists, since all the values of function G_X are generate from points on $\{0, 1, \ldots, n\}$, thus, all the remaining points are redundant for p-variation calculus. If $j \in \{0, 1, \ldots, n\}$ then $G_X(j) = X_j$. Therefore, instead of using $G_X(j)$ we can directly use the sample member X_j.

The sample members X_0, X_n are in supreme partition according to definition. All the other members $X_1, \ldots, X_{n-1}$ may or may not be in supreme partition. The number of all possible combinations are 2^{n-1}, but we will not investigate all of them. Rather we will use the properties of p-variation to identify the points that are not in supreme partition and drop them out from further consideration.

Suppose $j \in \{0, 1, \ldots, n\}$ and $j \notin SP_p(G_X, [0, n])$. This means that j can not be in supreme partition. Therefore, sample member X_j should be excluded from further consideration. Removing X_j means that we updating sample X with new sample X' which do not have X_j element, namely,

$$\{X'_i\}_{i=0}^{n-1} := \{X_0, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}.$$

The members of the sample X that are in supreme partition are called *significant*. All the members that are not in supreme partition are called *insignificant* and could be removed from original sample without any effect to the value of p-variation.

The most important property which were used in algorithm is stated in Proposition 1.26. This property allows to investigate small intervals and find insignificant points. Later on we will discus in great detail how we actually using is. The main steps in *pvar* procedure goes as follows.

Procedure *pvar*. Input: sample X, scalar p.

1. *Removing monotonic points.* According to Proposition 1.13 all points that are not the end of monotonic interval could be excluded from further consideration.

2. *Checking all small intervals.* Every small intervals are checked if it is possible to identify insignificant points. This operation is based on Proposition 1.26 which states that If $x \notin \overline{SP}_p(f, [a', b'])$, then $x \notin \overline{SP}_p(f, [a, b])$. So, if X_j appears to be insignificant in any small sub-sample, then we do not need to consider possibility for X_j to be in supreme sample any more. More details are given in section 2.2.2.

3. *Merging small intervals.* Let $\kappa_{a,c} \in SP_p(G_X, [a, c])$ and $\kappa_{c,b} \in SP_p(G_X, [c, b])$. By using $\kappa_{a,c}$ ans $\kappa_{c,b}$, we can effectively find $\kappa_{a,b} \in SP_p(G_X, [a, b])$, since $\kappa_{a,b} \subset \kappa_{a,c} \cup \kappa_{c,b}$. Finding $\kappa_{a,b}$ from $\kappa_{a,c}$ and $\kappa_{c,b}$ is called *merging*. In this step we repeat merging of all small intervals until we get the finial supreme partition. The whole procedure is described in section 2.2.3.

The corresponding psiaudocode of the main function goes as follows

```
pvar <- function(x, p){
  partition <- Remove_Monotonic_Points(x)
  partition <- Test_Points_In_Small_Intervals(partition, p, LSI
    = 3)
  partition <- Merge_Small_Intervals(partition, p, LSI=3+1)
  pvar <- sum(abs(diff(partition))^p)
  return(pvar)
}
```

2.2 Detail explanation

2.2.1 Removing monotonic points

According to Proposition 1.13 the p-variation is achieved on the partition of monotonic intervals. Therefore the points that are not the end of monotonic interval are insignificant and could be drop out. The points that are the end of monotonic intervals will be referred as *corners*. In one loop we can find all the corners by checking if the sequence changed the direction form increasing to decreasing (or vice versa). The corresponding pseudo code looks as follows:

```
INCLUDE x[0] to set of CORNERS
SET direction to 0
for(i = 1; i < n; ++i) {
  if(x[i-1]<x[i]){
    if(direction<0){
      INCLUDE x[i-1] to set of CORNERS
    }
    SET direction to 1
  }
  if(x[i-1]>x[i]){
    if(direction>0){
      INCLUDE x[i-1] to set of CORNERS
    }
    SET direction to -1
  }
}
INCLUDE x[n] to set of CORNERS
```

This procedure is quite simple and can by performed very quickly, since it does not include any cross checking and could be done in one loop. In *pvar* package all corners could be found with *ChangePoints* function.

2.2.2 Checking all small intervals

Proposition 1.17 and Lemmas 1.30 and 1.29 gives an effective way to identify part of insignificant points using quite simple operations. The pseudo code is given in the end the chapter and now we will reviled it main principles.

Suppose X is a sample without monotonic points. Let investigate a sub-sample in any small interval of the length m ($m = 2, \dots, n-1$), namely, lets examine a sub-sample $\{X_i\}_{i=j}^{j+m}$ for any $j = 0, ..., n-m$. According to Proposition 1.17, If

$$\sum_{i=j+1}^{j+m} |X_i - X_{i-1}|^p < |X_j - X_{j+m}|^p, \tag{2.3}$$

then,, partition $\{j, \ldots, j+m\}$ can not be supreme partition in interval $[j, j+m]$. Therefore, some of the points $\{X_i\}_{i=j+1}^{j+m-1}$ must be insignificant. Moreover, Lemmas 1.30 and 1.29 gives opportunity easily identify insignificant points. To do so, we must systemically investigate investigate small intervals starting form small m.

- Let $m = 2$. Then $|\{X_i\}_{i=j}^{j+2}| = 2$, thus according to Lemma 1.29, $\{X_i\}_{i=j}^{j+2}$ has at least one inner significant point. Since it has only one inner point X_{j+1}, then this point must be significant. So, if $m = 2$, then (2.3) is always invalid, therefore, we don't need to do any checking.

- Let $m = 3$. Firstly, we need to check if (2.3) holds. If so, then at least one point must be insignificant. In addition, Lemma 1.30 ensures that in this case both middle points X_{t+1}, X_{t+2} are insignificant, therefore, they should be removed.

  We do this checking for for all $j = 0, ..., n-2$. In this way we can identify quite an amount of insignificant points. Moreover, after removing them from the sample the sample changes, so we can repeat the same procedure and find more insignificant points. We do it until no new insignificant points are identified.

- If $m = 4$, then $|\{X_i\}_{i=j}^{j+2}|$ is even, thus, according to Lemma 1.29, $\{X_i\}_{i=j}^{j+m}$ has at least one inner significant point. So, all inner points are significant, based on Lemma 1.30, therefore, (2.3) do not hold. Lemma 1.29 ensure that this argument holds in all cases, then m is even. So, actually we need to investigate only the cases, then m is odd.

- Let $m = 5$ and all sub-samples $\{X_i\}_{i=j}^{j+3}$, $j = 0, ..., n-3$ are already checked. If (2.3) holds, then, based on Lemma 1.30, all middle points X_{t+1}, X_{t+2}, X_{t+3} and X_{t+4} are identified as insignificant and should be removed. After removing points, the sample changes, thus, if we want to apply this checking in again, then we should start checking from $m = 3$ again, because we must make sure that all sub-samples $\{X_i\}_{i=j}^{j+3}$, $j = 0, ..., n-3$ are already checked, because this is a requirement of the Lemma 1.30.

- If we want to go further and check sub-samples for $m = 7, 9, 11 \ldots$, every time we have to start from $m = 3$ and increase m by 2 only then (2.3) is not valid for all sub-samples.

We can continue this checking until m reaches an arbitrary threshold M. The pseudo-code of this procedure is given below

```
SET dum_X to NULL
WHILE dum_X is not equal to X AND m<M
  SET dum_X to X
  SET m = 3
  CHECK all sub−samples with m=d
  UPDATE X by dropping all insignificant points
  WHILE dum_X is equal to X AND m<M
    SET m = m + 2
    CHECK all sub−samples with m=d points
    UPDATE X by dropping all insignificant points
  ENDWHILE
ENDWHILE
```

With this procedure, we can actually find p-variation, setting M to sufficiently large value. But this way of finding p-variation is not effective, because in case of any removing we should go back and start checking from $m = 3$. So, it is reasonable to use this procedure only for small m and then go to final step of p-variation calculus.

2.2.3 Merging small intervals

Suppose it is already known that a sub-samples $\{X_i\}_{i=a}^{w}$ and $\{X_i\}_{i=w}^{b}$ are a supreme partitions in intervals $[a, w]$ and $[w, b]$ correspondingly. The *merge* operation of intervals $[a, w]$ and $[w, b]$ is an operation that finds supreme partition of the interval $[a, b]$ taking into account that $\{X_i\}_{i=a}^{w}$ and $\{X_i\}_{i=w}^{b}$ are a supreme partitions.

The final procedure in p-variation calculus will be presented in this subsection. It is based on merging operation. All the small intervals which where already checked in the previous procedure are merged one by one until all the possible combinations are checked and we end up having the supreme partition and p-variation.

This procedure have two levels that will be discussed separately.

Merge two intervals This paragraph will present how two intervals are merged . Suppose $\{X_i\}_{i=a}^{w}$ and $\{X_i\}_{i=w}^{b}$ are a supreme partitions in intervals $[a, w]$ and $[w, b]$ correspondingly. In order to find supreme partition in interval $[a, b]$ we need to investigate all possible f-joints between points $X_t, t \in \{a, \dots, v-1\}$ and $X_r, r \in \{v, \dots, b\}$ and choose pair of points (t, r)

which maximise s_p function, namely

$$\begin{aligned}
(t,r) &:= \operatorname*{arg\,max}_{(i,j)\in[a,v-1]\times[v,b]} \left\{ \sum_{k=1}^{i} |X_{k-1}+X_k|^p + |X_i - X_j|^p + \sum_{k=j+1}^{b} |X_{k-1}+X_k|^p \right\}, \\
&= \operatorname*{arg\,max}_{(i,j)\in[a,v-1]\times[v,b]} \left\{ |X_i - X_j|^p + \sum_{k=a+1}^{b} |X_{k-1}+X_k|^p - \sum_{l=i+1}^{j} |X_{l-1}+X_l|^p \right\}, \\
&= \operatorname*{arg\,max}_{(i,j)\in[a,v-1]\times[v,b]} \left\{ |X_i - X_j|^p - \sum_{l=i+1}^{j} |X_{l-1}+X_l|^p \right\}, \qquad (2.4)
\end{aligned}$$

In addition, the calculation of (t,r) in (2.4) could be optimised, since we actually do not need to check all $(i,j) \in [a, v-1] \times [v,b]$. Based on Lemma 1.23, points X_t and X_r could be f-joined only if

$$\forall i \in \{t,\dots,r\} : X_t \geq X_i \geq X_r \text{ or } X_t \leq X_i \leq X_r. \qquad (2.5)$$

And the last statement holds only if

$$X_t = \{\min(X_i : i \in \{t,\dots,w-1\}), \max(X_i : i \in \{t,\dots,w-1\})\} \qquad (2.6)$$

and

$$X_r = \{\min(X_i : i \in \{w,\dots,r\}), \max(X_i : i \in \{w,\dots,r\})\}. \qquad (2.7)$$

Let T denotes a set of $i \in \{a,\dots,w-1\}$ which satisfy (2.6) and R denotes a set of $j \in \{w,\dots,b\}$ which satisfy (2.7). Then (2.4) could be expressed as

$$(t,r) := \operatorname*{arg\,max}_{(i,j)\in T\times R} \left\{ |X_i - X_j|^p - \sum_{l=i+1}^{j} |X_{l-1}+X_l|^p \right\}. \qquad (2.8)$$

Since points X_t and X_r are f-joint, all the pints in interval $t+1,\dots,r-1$ are insignificant and should be removed. And the remaining points are supreme partition.

```

while the length of in > 1

  for
    int[i] = merge(int[i], int[i+1])
    i+2;
  end for
  delete all even int



end
```

Merging all small intervals dfd

```

while the length of in > 1

  for
    int[i] = merge(int[i], int[i+1])
    i+2;
  end for
  delete all even int



end
```

3 Conclusion

References

[1] J. Qian. The p-variation of Partial Sum Processes and the Empirical Process // Ph.D. thesis, Tufts University, 1997.

[2] R