The calculus of p-variation

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Algorithm 1 Euclid's algorithm

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procedure Euclid(a, b)      // The g.c.d. of a and b
    r ← a mod b
    while r ≠ 0 do      // We have the answer if r is 0
        a ← b
        b ← r
        r ← a mod b
    end while
    return b      // The gcd is b
end procedure
```

Abstract

Your abstract goes here... ...

1 Introduction

The p-variation is a generalization of the total variation of a function, namely, it is supremum of the sums of the p-th powers of absolute increments off over nonoverlapping intervals (the formal definition is given in Definition 2.2). The total variation is a special case of p-variation, then $p = 1$, i.e. the total variation is equal to 1-variation.

The p-variation was first defined by Wiener in 1924 [12]. In this work, the main focus was on 2-variation. Later on, Young extended the topic of p-variation in his work [4], there he analysed p-variations with $p \neq 2$. Lots and detailed information on the known properties of p-variation can be found in the books of Norvaiša and Dudley (see [2] and [3]). There is formulated and proved a lot of the properties of p-variation. In addition, the comprehensive bibliography of p-variation is include.

A regulated function f is called *rough* if its 1-variation value is infinite. The class of rough functions appears to be quite an important one. Probably the best know example of rough function is a trajectory of Wiener process. The Wiener is a well known process (see [5]) are used in statistics, physics and economic ([1]). It is worth noting, that rough functions can not be investigated with regular tools of calculus as derivative or Lebesgue–Stieltjes integral. The p-variation plays an important role in the filed of rough function analysis. Firstly, the p-variation gives a convenient way to measure a degree of roughness. Secondly, it appears that p-variation is a very useful tool in calculus of rough functions. A good overview of the importance of p-variation in rough function analysis is given in [6].

The exact value of p-variation was not used to be a topic. Rather, the primary focus on p-variation properties is whether the p-variation bounded or unbounded. But the the work [7] presented by Norvaiša and Račkauskas greatly increased the motivation of obtaining the the exact value of p-variation. The work [7] presents necessary and sufficient conditions for the convergence in law of partial sum processes in p-variation norm. The result could be applied in the statistical data analysis, which uses p-variation as data statistics. For applying those theoretical result in data analysis it is necessary to have procedure that actually calculates the value of p-variation. Nonetheless, it seams that there is no proposed procedure of calculation of p-variation value.

The main goal of this article is to present the algorithm, which calculates p-variation of the sample (see Def. 2.16), and give all the mathematical proofs of the properties that were used in this algorithm. The algorithm is denoted as *pvar*. It is already realised in the R environment (see [11]) and it is publicly available in *pvar* package on CRAN[1]. It is worth noting, that this procedure could be used to calculate p-variation of any piecewise monotone function.

The mathematical properties that where used in the algorithm are presented and proved in the next section of this article. In addition, all known properties that are relevant to this work are listed with the reference to original works. Namely, in large extent we use the properties presented in J. Qian Ph.D. thesis [9][2]. And in the Section 3 we will present the *pvar* algorithm that calculates the p-variation of the sample.

[1] http://cran.r-project.org/web/packages/pvar/index.html

[2] The main result of J. Qian Ph.D. thesis are presented in article [10].

2 Mathematical analysis

- $PP[a,b]$ – the set of all point partitions of $[a,b]$ (def. 2.1).
- $s_p(f;\kappa)$ – p-variation sum (def. 2.2).
- $v_p(f)$ – p-variation of the function f (def. 2.2).
- $SP_p(f,[a,b])$ – the set of supreme partitions (def. 2.2).
- $\overline{SP}_p(f,[a,b])$ – a set of points that are in any supreme partition (def. 2.20).
- $PM[a,b]$ – a set of piecewise monotone functions (def. 2.6).
- $CPM[a,b]$ – a set of continuous piecewise monotone functions (def. 2.6).
- $K(f,[a,b])$ – minimal size of PM partitions (prop. 2.7).
- $X(f,[a,b])$ – the set of PM partitions with minimal size (def. 2.8).

2.1 General known properties

Definition 2.1 (Partition). Let $J=[a,b]$ be a closed interval of real numbers with $-\infty < a \le b < +\infty$. If $a<b$, an ordered set $\kappa = \{x_i\}_{i=0}^n$ of points in $[a,b]$ such that a $a = x_0 < x_1 < x_2 < ... < x_n = b$ is called a *(point) partition*. The size of the partition is denoted $|\kappa| := \#\kappa - 1 = n$. The set of all point partitions of $[a,b]$ is denoted by $PP[a,b]$.

Definition 2.2 (p-variation). Let $f:[a,b]\to\mathbb{R}$ be a real function from an interval $[a,b]$. If $a<b$, for $\kappa=\{x_i\}_{i=0}^n \in PP[a,b]$ the *p-variation sum* is

$$s_p(f,\kappa) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p, \tag{2.1}$$

where $0<p<\infty$. Thus, the *p-variation* of f over $[a,b]$ is 0 if $a=b$ and otherwise

$$v_p(f) = v_p(f,[a,b]) := \sup\left\{s_p(f,\kappa) : \kappa \in PP[a,b]\right\}. \tag{2.2}$$

The partition κ is called *supreme partition* if it satisfies the property $v_p(f) = s_p(f,\kappa)$. The set of such partitions is denoted $SP_p(f,[a,b])$.

Vidinis Komentaras (VK): Dar neaisku kaip pakrikstyti skaidini, kuris pasiekia supremuma.

Lemma 2.3 (Elementary properties). Let $f : [a, b] \to \mathbb{R}$ and $0 < p < \infty$. Then the following p-variation properties holds

a) $v_p(f, [a, b]) \geq 0$,

b) $v_p(f, [a, b]) = 0 \Leftrightarrow f \equiv Const.$,

c) $\forall C \in \mathbb{R} : v_p(f + C, [a, b]) = v_p(f, [a, b])$,

d) $\forall C \in \mathbb{R} : v_p(Cf, [a, b]) = C^p v_p(f, [a, b])$,

e) $\forall c \in [a, b] : v_p(f, [a, b]) \geq v_p(f, [a, c]) + v_p(f, [c, b])$,

f) $\forall [a', b'] \subset [a, b] : v_p(f, [a, b]) \geq v_p(f, [a', b'])$.

g) $\forall \kappa \in PP[a, b] : s_p(f; \kappa) \leq v_p(f, [a, b])$.

All listed properties are elementary derived directly form the p-variation definition.

Definition 2.4 (Regulated function). ([9], Def. 3.1) For any interval J, which may be open or closed at either end, real function f is called *regulated* on J if it has left and right limits $f(x-)$ and $f(x+)$ respectively at each point x in interior of J, a right limit at the left end point and a left limit at the right endpoint.

Proposition 2.5. ([9], Lemma 3.1) Let $1 \leq p < \infty$. If f is regulated then $v_p(f)$ remains the same if points $x+$, $x-$ are allowed as partition points x_i in the definition 2.2.

Definition 2.6 (Piecewise monotone functions). ([9], Def. 3.2) A regulated real-valued function f on closed interval $[a, b]$ will be called *piecewise monotone* (PM) if there are points $a = x_0 < \cdots < x_k = b$ for some finite k such that f is monotone on each interval $[x_{j-1}, x_j]$, $j = 1, \ldots, k$. Here for $j = 1, \ldots, k - 1$, x_j may be point $x-$ or $x+$. The set of all piecewise monotone functions is denoted $PM = PM[a, b]$.

In addition to PM, if f is continuous function we will call it continuous piecewise monotone(CPM). The set of such functions is denoted $CPM = CPM[a, b]$.

Proposition 2.7. ([9], Prop. 3.1) If f is PM, there is a minimal size of partition $|\kappa|$ for which the definition 2.6 holds. The minimal size of the the PM partition is denoted $K(f, [a, b]) = K(f)$, namely

$$K(f) := \min \left\{ n : \exists \{x_i\}_{i=0}^{n} \in PP[a, b] : f \text{ is monotonic in each } [x_{j-1}, x_j] \right\}. \tag{2.3}$$

Definition 2.8 (The set of PM partitions with minimal size). ([9], Def. 3.3) If f is PM, let $X(f) = X(f, [a, b])$ be the set of all $\{x_i\}_{i=0}^{K(f)}$ for which the definition of PM (def. 2.6) holds. $X(f)$ is called the *set of PM partitions with minimal size*.

Proposition 2.9. ([9], Prop. 3.3) Let f is PM then the numbers $\alpha_j(f) := f(x_j) - f(x_{j-1})$ for $\{x_j\}_{j=0}^{K(f)} \in X(f)$ and $j = 1, 2, \ldots K(f)$ are uniquely determined.

Proposition 2.10. ([9], Prop. 3.2) Let f is PM. For any partition $\{x_j\}_{j=0}^{K(f)} \in X(f)$ exactly one of the flowing stamens holds:

(a) $f(x_0) > f(x_1) < f(x_2) > \ldots$. Function f is not increasing in intervals $[x_{2j}, x_{2j+1}]$, then $2j+1 \leq K(f)$. Function f is not decreasing in intervals $[x_{2j-1}, x_{2j}]$, then $j \geq 1$ and $2j \leq K(f)$.

(b) (a) holds for a function $-f$; or

(c) $K(f, [a, b]) = 1$ and f is constant in interval $[a, b]$.

Definition 2.11 (The equality by PM). ([9], Def. 3.4) If f, g are two PM functions, possibly on different intervals, such that $K(f) = K(g)$ and $\alpha_j(f) = \alpha_j(g)$ for $j = 1, 2, \ldots, K(f)$, then we say that $f \overset{PM}{=} g$.

Proposition 2.12. ([9], Cor. 3.1) Let $p > 1$ and functions f and g are PM. If $f \overset{PM}{=} g$ or $f \overset{PM}{=} -g$, then $v_p(f) = v_p(g)$.

Proposition 2.13. ([9], Them. 3.1) Let f is PM, $\kappa \in X(f)$ and $1 \leq p < \infty$. Then the supremum of p-variation in Definition 2.2 is attained for some partition $r \subset \kappa$.

Corollary 2.14. The set $SP_p(f, [a, b])$ is not empty for all $f \in PM[a, b]$.

Definition 2.15 (Sample function). Suppose $X = \{X_i\}_{i=0}^{n}$ is any sequence real numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. Then the *sample function* $G_X : [0, n] \to \mathbb{R}$ is defined as

$$G_X(t) := X_{\lfloor t \rfloor},\ t \in [0, n], \tag{2.4}$$

where $\lfloor t \rfloor$ denotes floor function at point t.

Definition 2.16 (p-variation of the sequence). Let $X = \{X_i\}_{i=0}^{n}$. The p-variation of the sample X is defined as p-variation of the function $G_X(t)$, namely

$$v_p(X) := v_p(G_X(t), [0, n]). \tag{2.5}$$

2.2 General properties with proofs

Proposition 2.17. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{x_i\}_{i=0}^n \in PP[a, b]$ is any partition of interval $[a, b]$. Then the statement

$$\exists\kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa \tag{2.6}$$

is equivalent to

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{2.7}$$

Proof. Necessary. Let $f : [a, b] \to \mathbb{R}$ be PM, $\{x_i\}_{i=0}^n \in PP[a, b]$ and

$$\exists\kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa. \tag{2.8}$$

Points from the partition κ will be denoted t_i, i.e. $\kappa = \{t_i\}_{i=0}^m$. Then, according to definitions of SP_p and p-variation (def. 2.2) the following equation holds

$$v_p(f, [a, b]) = s_p(f; \kappa) = \sum_{j=1}^{m} |f(t_{j-1}) - f(t_j)|^p = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p, \tag{2.9}$$

where $h : \{0, 1, \dots, n\} \to \{0, 1, \dots, m\}$ denotes a function from the set of index of x to the set of index of t, namely:

$$h(i) := (j_i : x_i = t_{j_i} = t_{h(i)}). \tag{2.10}$$

The equation (2.9) holds, because all the elements in the sum remains, we just grouped them.

Moreover, the inequality

$$\sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p \leq v_p(f, [x_{i-1}, x_i]) \tag{2.11}$$

holds according to Lemma 2.3(g).

As a result of (2.9) and (2.11) we get

$$v_p(f, [a, b]) \leq \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{2.12}$$

On the other hand, according to the same Lemma 2.3(e) the following inequality holds

$$v_p(f, [a, b]) \geq \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{2.13}$$

Finally, from the (2.12) and (2.13) follows

$$v_p(f,[a,b]) = \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{2.14}$$

Sufficiency. Suppose $f : [a,b] \to \mathbb{R}$ and

$$v_p(f,[a,b]) = \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{2.15}$$

According to Corollary 2.14, sets $SP_p(f,[x_{i-1},x_i]),\ i = 1,\dots,n$ are not empty. Lets take any partition from each of the sets $SP_p(f,[x_{i-1},x_i]),\ i = 1,\dots,n$ and denote it κ_i.

Then, let define a joint partition $\kappa := \cup_{i=1}^{n}\kappa_i$. Points from the partition κ will be denoted by t_i. In addition, we will use the function h, which is defined in (2.10). Then, continuing the equation (2.15) we get

$$v_p(f,[a,b]) = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p = \sum_{i=1}^{|\kappa|} |f(t_{i-1}) - f(t_i)|^p. \tag{2.16}$$

This means that $\kappa \in MP_p(f,[a,b])$. Moreover, $\forall i : x_i \in \kappa$, because $\kappa = \cup_{i=1}^{n}\kappa_i$. ■

Lemma 2.18. Let $f : [a,b] \to \mathbb{R}$ be PM and $\{x_i\}_{i=0}^{n} \subset \kappa \in SP_p(f,[a,b])$. Then

$$\forall k,l : v_p(f,[x_k,x_l]) = \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]),\ 0 \le k < l \le n. \tag{2.17}$$

Proof. Suppose $\{x_i\}_{i=0}^{n} \subset \kappa \in SP_p(f,[a,b])$. Let choose k and l such that $0 \le k < l \le n$. Lets apply Proposition 2.17 for the partition $\{x_0, x_k, x_{k+1}, \dots, x_{l-1}, x_l, x_n\}$. Thus,

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]) + v_p(f,[x_l,x_n]). \tag{2.18}$$

In addition, we can apply the same proposition for the partition $\{x_0, x_k, x_l, x_n\}$, then

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + v_p(f,[x_k,x_l]) + v_p(f,[x_l,x_n]). \tag{2.19}$$

By subtracting one equation form the other we get the result that

$$v_p(f,[x_k,x_l]) = \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]). \tag{2.20}$$

■

Lemma 2.19. Let $f:[a,b]\to\mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f,[a,b])$. Then

$$\forall k,l : v_p(f,[t_k,t_l]) = s_p\left(f,\{t_i\}_{i=k}^l\right),\ 0\le k<l\le n. \tag{2.21}$$

Proof. Let $f:[a,b]\to\mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f,[a,b])$. Let choose k and l such that $0\le k<l\le n$. Then

$$\begin{aligned} v_p(f,[a,b]) &= s_p(f,\{t_i\}_{i=0}^n) && (2.22)\\ &= s_p(f,\{t_i\}_{i=0}^k)+s_p(f,\{t_i\}_{i=k}^l)+s_p(f,\{t_i\}_{i=l}^n) && (2.23)\\ &\le v_p(f,[a,t_k])+s_p(f,\{t_i\}_{i=k}^l)+v_p(f,[t_l,b]). && (2.24)\end{aligned}$$

The last inequality holds according to Lemma 2.3(g).

On the other hand, from Proposition 2.18 we get

$$v_p(f,[a,b]) = v_p(f,[a,t_k])+v_p(f,[t_k,t_l])+v_p(f,[t_l,b]). \tag{2.25}$$

Form (2.24) and (2.25) follows

$$v_p(f,[t_k,t_l]) \le s_p(f,\{t_i\}_{i=k}^l). \tag{2.26}$$

Finally, from Lemma 2.3(g) we conclude that

$$v_p(f,[t_k,t_l]) = s_p(f,\{t_i\}_{i=k}^l). \tag{2.27}$$

■

Definition 2.20 (The point of supreme partition). Let $f:[a,b]\to\mathbb{R}$ be PM. The point x will be called the *point of supreme partition* if

$$\exists\kappa\in SP_p(f,[a,b]) : x\in\kappa. \tag{2.28}$$

The set of such points will be denoted by $\overline{SP}_p(f,[a,b])$.

Lemma 2.21. Suppose $f:[a,b]\to\mathbb{R}$ is PM, $x\in[a,b]$, $x\notin\overline{SP}_p(f,[a,b])$ and $\{t_i\}_{i=0}^n\in SP(f,[a,b])$ is any supreme partition. Then,

$$\exists j=1,\dots,n : x\in(t_{j-1},t_j) \text{ and } x\notin\overline{SP}_p(f,[t_{j-1},t_j]). \tag{2.29}$$

Proof. Suppose the assumptions of lemma is valid. Since $x\in[a,b]$ and $[a,b]=\cup_{i=1}^n[t_{i-1},t_i]$, then

$$\exists j=1,\dots,n : x\in[t_{j-1},t_j]. \tag{2.30}$$

Moreover, $x\notin\{t_i\}_{i=0}^n$, because $x\notin\overline{SP}_p(f,[a,b])$, thus, $x\ne t_{j-1}$ and $x\ne t_j$. In addition to (2.30) this means that $x\in(t_{j-1},t_j)$.

Now, we will proof that $x \notin \overline{SP}_p(f, [t_{j-1}, t_j])$. Suppose to the contrary that $x \in \overline{SP}_p(f, [t_{j-1}, t_j])$. Then, according to definition of $\overline{SP}_p(f, [t_{j-1}, t_j])$,

$$\exists \kappa \in SP_p(f, [t_{j-1}, t_j]) : x \in \kappa. \tag{2.31}$$

Since, $\kappa \in SP_p(f, [t_{j-1}, t_j])$, then

$$v_p(f, [t_{j-1}, t_j]) = s_p(f, \kappa) \tag{2.32}$$

Applying Proposition 2.17 for partition $\{t_i\}_{i=0}^n$ we get

$$\begin{aligned} v_p(f, [a, b]) &= v_p(f, [t_0, t_{j-1}]) + v_p(f, [t_{j-1}, t_j]) + v_p(f, [t_j, t_n]) \quad (2.33) \\ &= s_p(f, \{t_i\}_{i=0}^{j-1}) + s_p(f, \kappa) + s_p(f, \{t_i\}_{i=j}^n) \quad (2.34) \end{aligned}$$

This means that the partition $r := \{t_i\}_{i=0}^{j-1} \cup \kappa \cup \{t_i\}_{i=j}^n$ is supreme partition, so $x \in r \in SP(f, [a, b])$, therefore, by definition $x \in \overline{SP}_p(f, [a, b])$. This contradict to initial assumption. ∎

Definition 2.22 (f-join). Suppose $f : [a, b] \to \mathbb{R}$ is PM. We will say that points t_a and t_b ($t_a < t_b$) are f*-joined* in interval $[a, b]$ if

$$\exists \{x_j\}_{j=0}^n \in SP_p(f, [a, b]) : [t_a, t_b] = [x_{j-1}, x_j], \text{ with some } j. \tag{2.35}$$

Lemma 2.23. Suppose $f : [a, b] \to \mathbb{R}$ is PM and points t_a and t_b ($t_a < t_b$) are f-jointed in interval $[a, b]$. Then all following statements holds

a) $v_p(f, [t_a, t_b] = |f(t_a) - f(t_b)|^p$;

b) Let $x \in [t_a, t_b]$. If $f(t_a) \geq f(t_b)$, then $f(t_a) \geq f(x) \geq f(t_b)$. If $f(t_a) \leq f(t_b)$, then $f(t_a) \leq f(x) \leq f(t_b)$;

Proof.

a) Let $f : [a, b] \to \mathbb{R}$ be PM, $t_a < t_b$ and pair of points t_a, t_b are f-joined. Then exists $\{x_j\}_{j=0}^n$ and j from the Definition 2.22. Thus, according to Lemma 2.19

$$v_p(f, [t_{j-1}, t_j]) = s_p\left(f, \{t_{j-1}, t_j\}\right) = |f(t_a) - f(t_b)|^p. \tag{2.36}$$

b) Let $f : [a, b] \to \mathbb{R}$ be PM and points t_a and t_b are f-joined. Since $f \overset{PM}{=} -f$, with out loss of generality we can assume that $f(t_a) \leq f(t_b)$. Suppose to the contrary that $f(t_b)$ is not an extrema of the function in interval $[t_a, t_b]$. Hence, $\exists c \in [t_a, t_b] : f(c) > f(t_b)$. Therefore, $|f(c) - f(t_a)|^p > |f(t_b) - f(t_a)|^p$. According to (2.36), $v_p(f, [t_a, t_b]) = |f(t_b) - f(t_a)|^p$, thus, $|f(c) - f(t_a)|^p > v_p(f, [t_a, t_b])$, but this contradicts the definition of p-variation. So, point t_b must be an extrema in interval $[t_a, t_b]$. Symmetric arguments could be used for point t_a.

■

Lemma 2.24. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then function $f : [0, \infty) \to \mathbb{R}$ with the values

$$f(x) = (x + c_1)^p - x^p - C, \; x \in [0, \infty), \tag{2.37}$$

are non decreasing in interval $[0, \infty)$.

Proof. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then, for all $x \geq 0$, the derivative of the function f is

$$\begin{aligned} f'(x) &= p(x + c_1)^{p-1} - px^{p-1} \\ &\geq px^{p-1} - px^{p-1} = 0. \end{aligned}$$

The derivative of function f is non negative, thus the function f is non decreasing, if $x \geq 0$. ■

Corollary 2.25. Suppose $c_1 \geq 0$, $C \in \mathbb{R}$, $1 \leq p < \infty$ and $0 \leq x \leq y$. Then the following implication holds

$$|x + c_1|^p > x^p + C \Rightarrow |y + c_1|^p > y^p + C. \tag{2.38}$$

Proof. Suppose $0 \leq x \leq y$. Since f is non decreasing, $f(x) \leq f(y)$. Therefore, if $f(x) > 0$, then $f(y) > 0$. ■

Proposition 2.26. Suppose $f : [a, b] \to \mathbb{R}$ is PM and $x \in [a', b'] \subset [a, b]$. If $x \notin \overline{SP}_p(f, [a', b'])$, then $x \notin \overline{SP}_p(f, [a, b])$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be PM, $x \in [a', b'] \subset [a, b]$, and $x \notin \overline{SP}_p(f, [a', b'])$. Suppose to the contrary that $x \in \overline{SP}_p(f, [a, b])$.

Since $x \in \overline{SP}_p(f, [a, b])$, according to the Definition 2.20,

$$\exists \{t_i\}_{i=0}^n \in SP_p(f, [a, b]) : x \in \{t_i\}_{i=0}^n. \tag{2.39}$$

Let $\{y_i\}_{i=0}^n \in SP_p(f,[a',b'])$ be any supreme partition from the interval $[a',b']$. Then, according to Lemma 2.21,

$$\exists j = 1,\dots,n : x \in (y_{j-1}, y_j) \text{ and } x \notin \overline{SP}_p(f,[y_{j-1},y_j]). \tag{2.40}$$

Moreover,

$$x \notin \{a, y_{j-1}, y_j, b\}, \tag{2.41}$$

because $x \in (y_{j-1}, y_j) \subset [a,b]$. Thus, from (2.39) and (2.41) follows that

$$\exists r \in \{1,2,\dots,n-1\} : t_r = x. \tag{2.42}$$

Lets denote variables l and k by

$$\begin{aligned} l &:= \max\{i \in \{1,2,\dots,n-1\} : t_i \in (y_{j-1},y_j)\}, \\ k &:= \min\{i \in \{1,2,\dots,n-1\} : t_i \in (y_{j-1},y_j)\}. \end{aligned}$$

Since $x \in (y_{j-1}, y_j)$ and (2.42) holds, the values l and k always exists and $k \le r \le l$. According to l and k definitions the following inequality holds

$$t_{k-1} \le y_{j-1} < t_k \le x \le t_l < y_j \le t_{l+1}. \tag{2.43}$$

According to Lemma 2.3(e)

$$v_p(f,[y_{j-1},y_j]) \ge v_p(f,[y_{j-1},t_k]) + v_p(f,[t_k,t_l]) + v_p(f,[t_l,y_j]), \tag{2.44}$$

Firstly, let suppose

$$v_p(f,[y_{j-1},y_j]) = v_p(f,[y_{j-1},t_k]) + v_p(f,[t_k,t_l]) + v_p(f,[t_l,y_j]). \tag{2.45}$$

Lets take any supreme partitions from intervals $[y_{j-1},t_k]$ and $[t_l,y_j]$, namely $\kappa_k \in SP_p(f,[y_{j-1},t_k]$ and $\kappa_l \in SP_p(f,[t_l,y_j]$. In addition, since $\{t_i\}_{i=0}^n \in SP_p(f,[a,b])$, according to Lemma 2.19 $v_p(f,[t_k,t_l]) = s_p(f,\{t_i\}_{i=k}^l)$. Therefore,

$$v_p(f,[y_{j-1},y_j]) = s_p(f,\kappa_k) + s_p(f,\{t_i\}_{i=k}^l) + v_p(f,\kappa_l).$$

This means, that

$$\kappa_k \cup \{t_i\}_{i=k}^l \cup \kappa_l \in SP_p(f,[y_{j-1},y_j].$$

So, $x \in \overline{SP}_p(f,[y_{j-1},y_j])$, because $x \in \{t_i\}_{i=k}^l$. This contradicts (2.40), thus, equality (2.45) is not valid. As a result, inequality (2.44) becomes

$$v_p(f,[y_{j-1},y_j]) > v_p(f,[y_{j-1},t_k]) + v_p(f,[t_k,t_l]) + v_p(f,[t_l,y_j]). \tag{2.46}$$

Since points y_{j-1} and y_j are f-joined, from the Lemma 2.23(a) we get

$$v_p(f, [y_{j-1}, y_j]) = |f(y_{j-1}) - f(y_j)|^p,$$

therefore,

$$|f(y_{j-1}) - f(y_j)|^p > v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]).$$

Moreover, according to Lemma 2.3(g), from the last statement follows

$$|f(y_{j-1}) - f(y_j)|^p > |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(y_j)|^p. \quad (2.47)$$

Since $v_p(f) = v_p(-f)$ (see Proposition 2.12), with out loss of generality we can assume that $f(y_{j-1}) \geq f(y_j)$. Hence, from the Lemma 2.23(b) we get that

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j), \quad (2.48)$$
$$f(y_{j-1}) \geq f(t_k) \geq f(y_j). \quad (2.49)$$

In addition, the pairs of points $\{t_l, t_{l+1}\}$ and $\{t_{k-1}, t_k\}$ are also f-joined, therefore, by the Lemma 2.23(b) inequalities (2.48) and (2.49) could be extended as

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j) \geq f(t_{l+1}), \quad (2.50)$$
$$f(t_{k-1}) \geq f(y_{j-1}) \geq f(t_k) \geq f(y_j). \quad (2.51)$$

From the (2.50) the following inequalities holds

$$\begin{aligned} f(y_{j-1}) - f(t_l) &\geq 0, \\ f(t_l) - f(t_{l+1}) \geq f(t_l) - f(y_j) &\geq 0. \end{aligned}$$

Therefore, we can use Corollary 2.25. According to it, from the inequality (2.47) follows

$$\begin{aligned} |f(y_{j-1}) - f(t_l) + f(t_l) - f(y_j)|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(y_j)|^p, \\ |f(y_{j-1}) - f(t_l) + f(t_l) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(t_{l+1})|^p, \\ |f(y_{j-1}) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]). \end{aligned}$$

The last inequality holds, because points t_l and t_{l+1} are f-joined, thus, according to Lemma 2.23(a),

$$v_p(f, [t_l, t_{l+1}]) = |f(t_l) - f(t_{l+1})|^p.$$

Symmetric argument could be used in other direction. Firstly lets modify last inequality

$$|f(y_{j-1})-f(t_k)+f(t_k)-f(t_{l+1})|^p > |f((y_{j-1})-f(t_k)|^p+v_p(f,[t_k,t_l])+v_p(f,[t_l,t_{l+1}]).$$

From the (2.51) we get that the following inequalities holds

$$\begin{aligned} f(t_k)-f(t_{l+1}) &\geq 0, \\ f(t_{k-1})-f(t_k) \geq f(y_{j-1})-f(t_k) &\geq 0. \end{aligned}$$

Thus, from Corollary 2.25 we get

$$\begin{aligned} |f(t_{k-1})-f(t_k)+f(t_k)-f(t_{l+1})|^p &> |f(t_{k-1})-f(t_k)|^p+v_p(f,[t_k,t_l])+v_p(f,[t_l,t_{l+1}]), \\ |f(t_{k-1})-f(t_{l+1})|^p &> v_p(f,[t_{k-1},t_k])+v_p(f,[t_k,t_l])+v_p(f,[t_l,t_{l+1}]). \end{aligned}$$

As previous, the last inequality holds, because t_{k-1} and t_k are f-joined.

Finally, using Lemma 2.18 we conclude that

$$|f(t_{k-1})-f(t_{l+1})|^p > v_p(f,[t_{k-1},t_{l+1}]).$$

This contradicts with the definition of p-variation.

■

2.3 Extra results

Definition 2.27 (Extremum). We will call the point $t \in [a,b]$ an *extrema* of the function f in interval $[a,b]$ if $f(t)=\sup\{f(z) : z \in [a,b]\}$ or $f(t)=\inf\{f(z) : z \in [a,b]\}$.

Proposition 2.28. Let $f:[a,b]\to\mathbb{R}$ is PM. If point $x\in[a,b]$ is extrema of the function f, then $x \in \overline{SP}_p(f,[a,b])$.

Proof. Let $f:[a,b]\to\mathbb{R}$ is PM and point $x\in[a,b]$ is an extrema of the function f. Let $\{t_i\}_{i=0}^n \in SP_p(f,[a,b]$ be any supreme partition. Then

$$\exists j \in 1,\dots,n : x \in [t_{j-1},t_j] \tag{2.52}$$

Since x is an extrema of function f in interval $[a,b]\supset[t_{j-1},t_j]$, the point x is an extrema in interval $[t_{j-1},t_j]$ as well. Therefore,

$$f(t_{j-1})-f(t_j)|^p \leq |f(t_{j-1})-f(x)|^p \tag{2.53}$$

or

$$|f(t_{j-1})-f(t_j)|^p \leq |f(x)-f(t_j)|^p. \tag{2.54}$$

As a result,

$$\begin{aligned}
|f(t_{j-1}) - f(t_j)|^p &\leq |f(t_{j-1}) - f(x)|^p + |f(x) - f(t_j)|^p, \\
|f(t_{j-1}) - f(t_j)|^p &\leq v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]), \\
v_p(f, [t_{j-1}, t_j]) &\leq v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j]).
\end{aligned}$$

The last inequality holds, because t_{j-1} and t_j are f-joined, so, $v_p(f, [t_{j-1}, t_j]) = |f(t_{j-1}) - f(t_j)|^p$. Since $v_p(f, [t_{j-1}, t_j]) < v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j])$ is not valid, the equation

$$v_p(f, [t_{j-1}, t_j]) = v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) \tag{2.55}$$

holds.

Moreover, applying Proposition 2.17 for the partition $\{a, t_{j-1}, t_j, b\}$ we have

$$\begin{aligned}
v_p(f, [a, b]) &= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, t_j]) + v_p(f, [t_j, b]) \\
&= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) + v_p(f, [t_j, b])
\end{aligned}$$

From the same proposition follows, that $\exists \kappa \in SP_p(f, [a, b]) : x \in \kappa$, so, by definition, $x \in \overline{SP}_p(f, [a, b])$. ■

Lemma 2.29. Let $f : [a, b] \to \mathbb{R}$ be PM. If $K(f)$ is even, then

$$\exists x \in (a, b) : x \in \overline{SP}_p(f, [a, b]). \tag{2.56}$$

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is PM. Lets take any $\{x_i\}_{i=0}^n \in X(f)$. If $n > 1$, then without loss of generality we can assume that Proposition 2.10(a) holds. So, $f(x_0) > f(x_1) < f(x_2) > \dots$, i.e. $f(x_{2i-1}) < (x_{2i})$, thus, if n is even, then $f(x_{n-1}) < (x_n)$, therefore, point x_n is not a global minimum. Since $f(x_0) > f(x_1)$, point x_0 is also not a global minimum. As a result, global minimum is in interval (a, b) and it is in $\overline{SP}_p(f, [a, b])$ according to Lemma 2.28. ■

Lemma 2.30. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in PP[a, b]$. Suppose

$$\{t_i\}_{i=0}^{n-1} \in SP_p(f, [a, t_{n-1}]) \tag{2.57}$$

and

$$\{t_i\}_{i=1}^{n} \in SP_p(f, [t_1, b]). \tag{2.58}$$

Then

$$\forall i \in \{1, \dots, n-1\} : t_i \in \overline{SP}_p(f, [a, b]) \tag{2.59}$$

or

$$\forall i \in \{1, \dots, n-1\} : t_i \notin \overline{SP}_p(f, [a, b]) \tag{2.60}$$

Proof. Let the assumptions of lemma be valid. Firstly, suppose

$$\exists j \in \{1, \dots, n-1\} : t_j \in \overline{SP}_p(f, [a,b]). \tag{2.61}$$

This means that $\exists \kappa \in SP_p(f, [a,b]) : t_j \in \kappa$. Then, by Proposition 2.17,

$$v_p(f, [a,b]) = v_p(f, [a, t_j]) + v_p(f, [t_j, b]) \tag{2.62}$$

Since (2.57) and (2.57) holds, by Lemma 2.19 we get

$$v_p(f, [a, t_j]) = s_p(f, \{t_i\}_{i=0}^{j}, \tag{2.63}$$
$$v_p(f, [t_j, b]) = s_p(f, \{t_i\}_{i=j}^{n}. \tag{2.64}$$

Therefore,

$$v_p(f, [a,b]) = s_p(f, \{t_i\}_{i=0}^{j}) + s_p(f, \{t_i\}_{i=j}^{n}) = s_p(f, \{t_i\}_{i=0}^{n}). \tag{2.65}$$

So, $\{t_i\}_{i=0}^{n} \in SP_p(f, [a,b])$, thus, statement (2.59) holds.
On the other hand, if (2.61) is not valid, then statement (2.60) holds.
■

3 p-variation calculus

In this chapter we will present the algorithm that calculates p-variation for the sample (see Def. 2.16). Nonetheless, this algorithm could be used to calculate the p-variation for arbitrary piecewise monotone function. This procedure will be called *pvar*. It is already realised in the R (see [11]) package *pvar* and is publicly available on CRAN[3].

Firstly, we will give some introductory notes about the procedure. Later on, the main schema of the algorithm will be presented in the Subsection 3.2. And finally, each step of the algorithm are disused in more details in Subsection 3.3.

3.1 Introductory notes

Suppose $X = \{X_i\}_{i=0}^{n}$ is any real-value sequence of numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. The formal definition of p-variation of the sample is given in Definition 2.16, which states that

$$v_p(X) = v_p(G_X(t), [0, n]), \tag{3.1}$$

[3] http://cran.r-project.org/web/packages/pvar/index.html

where $G_X(t)$ is a sample function defined in Definition 2.15. On the other hand, p-variation of the sample could express as

$$v_p(X) = \max\left\{\sum_{i=1}^{k} |X_{j_i} - X_{j_{i-1}}|^p : 0 = j_0 < \cdots < j_k = n,\ k = 1, \dots, n\right\}. \tag{3.2}$$

This expression could be verified from Proposition 2.13, which states, that p-variation is achieved in a subset of any partition $r \in X(f, [0, n])$. So, we can construct r from the subset $\{0, 1, \dots, n\}$. If $j \in \{0, 1, \dots, n\}$ then $G_X(j) = X_j$, thus, $G_X(j)$ could be replaced with X_j. As a result, we don't need to use $G_X(j)$ function, rather, the values of the sample X_j could be used directly. This is the way algorithm works – it actually operates the the values of the sample X, not involving $G_X(t)$ function. Nonetheless, then we will refer to properties of p-variation we actually have in mind function $G_X(t)$, since all the properties of p-variation are formulated for functions.

The members of the sample X that are in supreme partition are called *supreme points*. All the members that are not in supreme partition refereed as *redundant* points. Those points could be removed from original sample without any effect to the value of p-variation.

The main idea of the algorithm is to find the supreme partition, by identifying and dropping all the redundant points, i.e. the points that are not in supreme partition. For this purpose, the most important property is stated in the Proposition 2.26, which allows us to investigate possible partition by analysing not the whole sample, but smaller sub-samples of X. From the Proposition 2.26 follows, that if a point was identify as being redundant in any small interval, then the point is redundant in any larger interval.

Suppose $j \in \{0, 1, \dots, n\}$ and $j \notin SP_p(G_X, [0, n])$. This means that j is not in supreme partition. Therefore, sample member X_j is redundant and should be excluded from further consideration. Removing X_j means that we updating sample X with new sample X' which do not have X_j element, namely,

$$\{X'_i\}_{i=0}^{n-1} := \{X_0, \dots, X_{j-1}, X_{j+1}, \dots, X_n\}.$$

And in further analysis we investigate the sample X'.

It is worth noting, that the procedure *pvar* could be used to calculate the p-variation of any piecewise monotone function. Let assume f is any piecewise monotone function. According to definition (see 2.6) there are points $a = x_1 < \cdots < x_n = b$ for some finite n such that f is monotone on each interval $[x_{j-1}, x_j]$, $j = 1, 2, \dots, n$. We can calculate p-variation if we have the sample X, which contains the values of the function f at the points $\{x_i\}_{i=0}^n$, namely, $X = \{X_i\}_{i=0}^n := \{f(x_i)\}_{i=0}^n$. It is straight forward to see that $f \overset{PM}{=} G_X$, therefore, if $p \geq 1$, then $v_p(X) = v_p(G_X) = v_p(f)$.

3.2 Main function

The procedure *pvar* that calculates p-variation will be presented here. This procedure is already available in R package *pvar*. Firstly, we will introduce the main schema, further, each step will be discussed in more details. The main steps in *pvar* procedure goes as follows.

Procedure *pvar*. Input: sample X, scalar p.

1. *Removing monotonic points.* According to Proposition 2.13 all points that are not the end of monotonic interval could be excluded from further consideration.

2. *Checking all small intervals.* Every small intervals are checked if it is possible to identify insignificant points. This operation is based on Proposition 2.26 which states that If $x \notin \overline{SP}_p(f, [a', b'])$, then $x \notin \overline{SP}_p(f, [a, b])$. So, if X_j appears to be insignificant in any small sub-sample, then we do not need to consider possibility for X_j to be in supreme sample any more. More details are given in section 3.3.2.

3. *Merging small intervals.* Let $\kappa_{a,c} \in SP_p(G_X, [a, c])$ and $\kappa_{c,b} \in SP_p(G_X, [c, b])$. By using $\kappa_{a,c}$ ans $\kappa_{c,b}$, we can effectively find $\kappa_{a,b} \in SP_p(G_X, [a, b])$, since $\kappa_{a,b} \subset \kappa_{a,c} \cup \kappa_{c,b}$. Finding $\kappa_{a,b}$ from $\kappa_{a,c}$ and $\kappa_{c,b}$ is called *merging*. In this step we repeat merging of all small intervals until we get the finial supreme partition. The whole procedure is described in section 3.3.3.

The corresponding psiaudocode of the main function goes as follows

Algorithm 2 The main function of *pvar* procedure that calculates p-variation of the sample.

```
function PVAR(sample X, scalar p)
        // update X by removing monotonic points
    X ← sampleCorners(X);
        // update X by checking all sub-intervals and removing redundant points.    // check sub-intervals up to the length of 3
    LSI ← 3 ;
    X ← CheckSmallIntervals (X, p, LSI);
        // update X by merge all sub-intervals starting with the length of LSI + 1
    X ← merging (X, p, LSI + 1);
        // calculate the p-variation:
    value ← ∑ |X_{i−1} − X_i|^p ;
    return value;
end function
```

3.3 Detail explanation

3.3.1 Removing monotonic points

According to Proposition 2.13 the p-variation is achieved on the partition of monotonic intervals. Therefore the points that are not the end of monotonic interval are redundant and should be drop out. The points that are the end of monotonic intervals will be referred as *corners*. In a single loop we can find all the corners by checking if the sequence changed the direction form increasing to decreasing (or vice versa). In addition, points $X[0]$ and $X[n]$ always included in the array of corners by definition. The corresponding pseudo code looks as follows:

Algorithm 3 The function *SampleCorners* that returns the subset of X, which contains only the corners of X.

```
function SampleCorners(sample X)
Output: sample corners that is the subset of X, which contains only the
    corners
    corners ← X_0;
    direction ← 0;
    n ← counter(X);
    for i ← 1  to  n − 1 do
        if X_{i−1} < X_i then
            if direction < 0 then
                corners ← append(corners, X_{i−1});
            end if
            direction ← 1 ;
        end if
        if X_{i−1} > X_i then
            if direction > 0 then
                corners ← append(corners, X_{i−1});
            end if
            direction ← −1 ;
        end if
    end for
    corners ← append(corners, X_n);
    return corners;
end function
```

This procedure is quite simple and can by performed very quickly, since it does not include any cross checking and could be done in one loop. In *pvar* package all corners could be found with *ChangePoints* function.

3.3.2 Checking all small intervals

Proposition 2.17 and Lemmas 2.30 and 2.29 gives an effective way to identify significant part of redundant points using quite simple operations. The pseudo code is given in the end the chapter and now we will reviled it main principles.

Suppose X is a sample without monotonic points. Let investigate a sub-sample in any small interval of the length m ($m = 2, \dots, n-1$), namely, lets examine a sub-sample $\{X_i\}_{i=j}^{j+m}$ for any $j = 0, ..., n-m$. According to

Proposition 2.17, If

$$\sum_{i=j+1}^{j+m} |X_i - X_{i-1}|^p < |X_j - X_{j+m}|^p, \tag{3.3}$$

then, partition $\{j, \ldots, j+m\}$ can not be supreme partition in interval $[j, j+m]$. Therefore, some of the points $\{X_i\}_{i=j+1}^{j+m-1}$ must be redundant. Moreover, Lemmas 2.30 and 2.29 gives opportunity easily identify insignificant points. To do so, we must systemically investigate small intervals starting form small m.

- Let $m = 2$. Then $|\{X_i\}_{i=j}^{j+2}| = 2$, thus according to Lemma 2.29, $\{X_i\}_{i=j}^{j+2}$ has at least one inner significant point. Since it has only one inner point X_{j+1}, then this point must be significant. So, if $m = 2$, then (3.3) is always invalid, therefore, we don't need to do any checking.
- Let $m = 3$. Firstly, we need to check if (3.3) holds. If so, then at least one point must be redundant. In addition, Lemma 2.30 ensures that in this case both middle points X_{t+1}, X_{t+2} are redundant, therefore, they both should be removed.

  We do this checking for for all $j = 0, ..., n-2$. In this way we can identify quite an amount of redundant points. Moreover, after removing them from the sample the sample changes, so we can repeat the same procedure and find more redundant points. We do it until no new redundant points are identified.
- If $m = 4$, then $|\{X_i\}_{i=j}^{j+2}|$ is even, thus, according to Lemma 2.29, $\{X_i\}_{i=j}^{j+m}$ has at least one inner significant point. So, all inner points are significant, based on Lemma 2.30, therefore, (3.3) do not hold. Lemma 2.29 ensure that this argument holds in all cases, then m is even. So, actually we need to investigate only the cases, then m is odd.
- Let $m = 5$ and all sub-samples $\{X_i\}_{i=j}^{j+3}$, $j = 0, ..., n-3$ are already checked. If (3.3) holds, then, based on Lemma 2.30, all middle points X_{t+1}, X_{t+2}, X_{t+3} and X_{t+4} are identified as redundant and should be removed. After removing points, the sample changes, thus, if we want to apply this checking again, then we should start checking from $m = 3$ again, because we must make sure that all sub-samples $\{X_i\}_{i=j}^{j+3}$, $j = 0, ..., n-3$ are already checked, because this is a requirement of the Lemma 2.30.

- If we want to go further and check sub-samples for $m = 7, 9, 11\dots$, every time we have to start from $m = 3$ and increase m by 2 only then (3.3) is not valid for all sub-samples.

We can continue this checking until m reaches an arbitrary threshold M. The pseudo-code of this procedure is given below

Algorithm 4 Procedure *CheckSmallIntevals*, which ensures that any sequantal sub-sample of length M is a supreme partition in corresponding sub-interval.

```
function CheckSmallIntevals( sample X, constant p, constant M)
Require: sample X must contain only the corners (i.e. no monotonic points)
Output: modified sample X such that any sequential sub-sample of length
    M is a supreme partition in corresponding sub-interval

    tempX ← NULL ;
    m ← 0 ;
    while tempX ≠ X and  m < M do
        tempX ← X ;
        m ← 3 ;
        check all sub-samples of X with length m ;
        update X by dropping all redundant points ;
        while tempX ≠ X and  m < M do
            m ← m + 2 ;
            check all sub-samples of X with length m ;
            update X by dropping all redundant points ;
        end while
    end while
    return X;
end function
```

With this procedure, we can actually find p-variation, setting M to sufficiently large value. But this way of finding p-variation is not effective, because in case of any removing we should go back and start checking from $m = 3$. So, it is reasonable to use this procedure only for small m and then go to next step of p-variation calculus.

3.3.3 Merging small intervals

Suppose it is already known that a sub-samples $\{X_i\}_{i=a}^{w}$ and $\{X_i\}_{i=w}^{b}$ are a supreme partitions in intervals $[a, w]$ and $[w, b]$ correspondingly. The *merge* operation of intervals $[a, w]$ and $[w, b]$ is an operation that finds supreme partition of the interval $[a, b]$ taking into account that $\{X_i\}_{i=a}^{w}$ and $\{X_i\}_{i=w}^{b}$ are a supreme partitions.

The final procedure in p-variation calculus will be presented in this subsection. It is based on merging operation. All the small intervals which where already checked in the previous procedure are merged one by one until all the possible combinations are checked and we end up having the supreme partition and p-variation.

This procedure has two stages that will be discussed separately.

Merge two intervals This paragraph will present how two intervals are merged . Suppose $\{X_i\}_{i=a}^{w}$ and $\{X_i\}_{i=w}^{b}$ are a supreme partitions in intervals $[a, w]$ and $[w, b]$ correspondingly. According to Proposition 2.26, the supreme partition in interval $[a, b]$ must be a subset of $\{X_i\}_{i=a}^{b} = \{X_i\}_{i=a}^{w} \cup \{X_i\}_{i=w}^{b}$.

In order to find supreme partition in interval $[a, b]$ we need to investigate all possible f-joints (Def. 2.22) between points $X_t, t \in \{a, \dots, w-1\}$ and $X_r, r \in \{w, \dots, b\}$ and choose pair of points (t, r) which maximise s_p function, namely

$$\begin{aligned}
(t, r) \quad &:= \quad \underset{(i,j)\in[a,w-1]\times[w,b]}{\arg\max} \left\{ \sum_{k=a+1}^{i} |X_{k-1} + X_k|^p + |X_i - X_j|^p + \sum_{k=j+1}^{b} |X_{k-1} + X_k|^p \right\}, \\
&= \quad \underset{(i,j)\in[a,w-1]\times[w,b]}{\arg\max} \left\{ |X_i - X_j|^p + \sum_{k=a+1}^{b} |X_{k-1} + X_k|^p - \sum_{l=i+1}^{j} |X_{l-1} + X_l|^p \right\}, \\
&= \quad \underset{(i,j)\in[a,w-1]\times[w,b]}{\arg\max} \left\{ |X_i - X_j|^p - \sum_{l=i+1}^{j} |X_{l-1} + X_l|^p \right\}, \qquad (3.4)
\end{aligned}$$

In the special case then $t = w - 1$ and $r = w$, the value of function in $\arg\max$ is zero and the supreme partition is $\{X_i\}_{i=a}^{b}$. So, in this case we do not need to do any modifications of the sample. This case is taken as baseline and we consider any modifications only then the value of function in $\arg\max$ is greater then zero.

In addition, the calculation of (t, r) in (3.4) could be optimised, since we actually do not need to check all $(i, j) \in [a, v-1] \times [v+1, b]$. Based on Lemma 2.23, points X_t and X_r could be f-joined only if

$$\forall i \in \{t, \dots, r\} : X_t \geq X_i \geq X_r \text{ or } X_t \leq X_i \leq X_r. \qquad (3.5)$$

And the last statement holds only if

$$X_t = \min(X_i : i \in \{t, \dots, w\}) \text{ or } X_t = \max(X_i : i \in \{t, \dots, w\}) \qquad (3.6)$$

and

$$X_r = \min(X_i : i \in \{w, \ldots, r\}) \text{ or } X_r = \max(X_i : i \in \{w, \ldots, r\}). \tag{3.7}$$

Let T denotes a set of $i \in \{a, \ldots, w-1\}$ which satisfy (3.6) and R denotes a set of $j \in \{w+1, \ldots, b\}$ which satisfy (3.7). Then (3.4) could be expressed as

$$(t, r) := \underset{(i,j) \in T \times R}{\arg\max} \left\{ |X_i - X_j|^p - \sum_{l=i+1}^{j} |X_{l-1} + X_l|^p \right\}. \tag{3.8}$$

Since points X_t and X_r are f-joint, all the points in interval $t+1, \ldots, r-1$ are redundant and should be removed. And the remaining points are supreme partition.

Algorithm 5 Procedure *MergeTwoInt*, which merges two intervals represented by their supreme samples.

function MERGETWOINT(sample A, sample B, scalar p)

Require: samples A and B must be supreme partitions in intervals $[a, c]$ and $[c, b]$ correspondingly.

Output: sample X, which is the supreme partition of interval $[a, b]$

```
    X ← A ∪ B ;
    a ← 0 ;
    b ← counter(X) ;
    w ← counter(A) ;

    // Find the T list:
    initiate T as empty list ;
    cummax ← cummin ← X_w ;
    for i ← w − 1  to  a do
        if X_i > cummax then
            cummax ← X_i ;
            T ← append(T, i)
        end if
        if X_i < cummin then
            cummin ← X_i ;
            T ← append(T, i) ;
        end if
    end for

    // Find the R list:
    initiate R as empty list ;
    cummax ← cummin ← X_w ;
    for i ← w + 1  to  b do
        if X_i > cummax then
            cummax ← X_i ;
            R ← append(R, i)
        end if
        if X_i < cummin then
            cummin ← X_i ;
            R ← append(R, i) ;
        end if
    end for
```

```
    // Check all pairs in T x R:
    maxbalance ← 0 ;
    for r ∈ R do
        for t ∈ T do
            balance ← |x_t − X_r|^p − Σ_{i=t+1}^{r} |X_{i−1} − X_i|^p ;
            if balance > maxbalance then
                maxR ← r ;
                maxT ← t ;
                maxbalance ← balance ;
            end if
        end for
    end for
    if maxbalance > 0 then
        remove points t + 1, . . . , r − 1 from sample X ;
    end if
    return X;
end function
```

Merging all small intervals. In the second stage of this procedure we get the final result by merging intervals pair by pair until the supreme partition of the whole sample is obtain.

The pseudo code is given below and its illustration is given in Figure 1.

Algorithm 6 Procedure *MergeAll*, which merges all sub-intervals of the length M into final supreme sample

```
function MergeAll( sample X, scalar p, scalar M)
Require: sample X... of M ...
Output: sample S that corresponds to supreme portion.

    // Get array prt = {prt_i}_{i=1}^n of supreme sub-intervals
    for i ← 1 to ⌈counter(X)/M⌉ do
        prt_i ← {X_j}_{j=M(i-1)}^{min(Mi,counter(X))} ;
    end for

    // Apply pair merging to elements of prt until all intervals are merged
    while counter(prt) > 1 do
        i ← 1 ;
        while i < counter(prt) do
            prt_i ← mergeTwoInt(prt_i, prt_{i+1}) ;
            i ← i + 2 ;
        end whiledelete all elements form prt that have even index ;
    end while
    S ← prt_1 ;
    return S;
end function
```

4 Conclusion

In this article we presented the algorithm that calculates p-variations of the sample. This algorithm is based on mathematical properties that were formulated and proved in the article.

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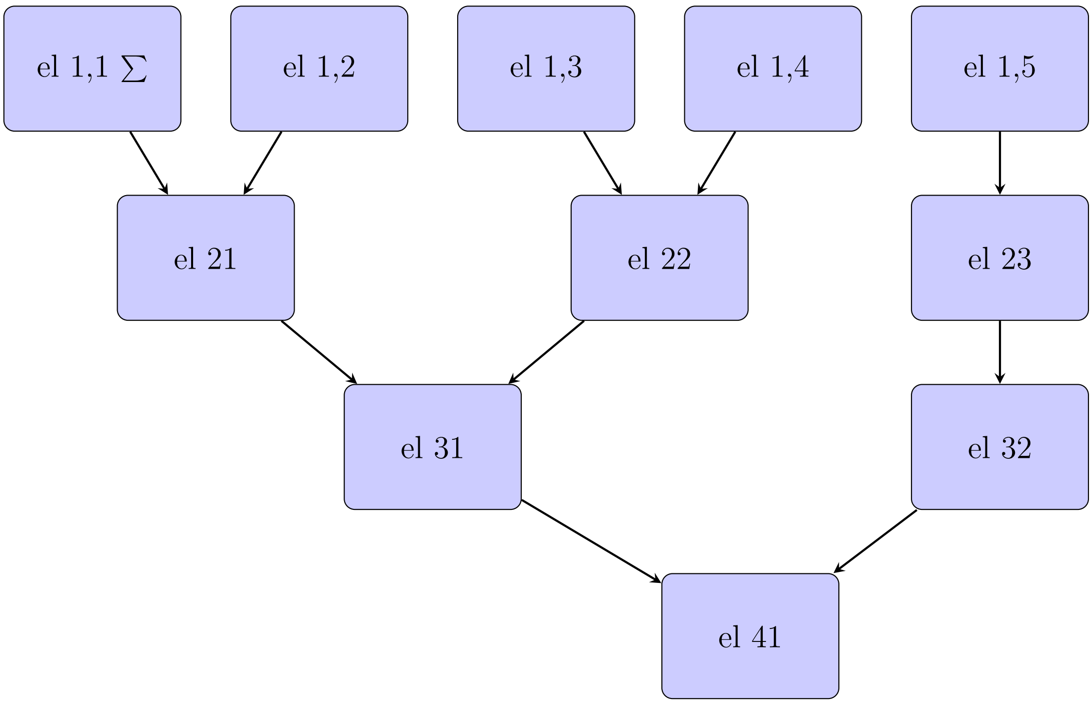

Figure 1: The illustration of the final stage of merging procedure.

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