The calculus of p-variation

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- $PP[a,b]$ – the set of all point partitions of $[a,b]$ (def. 1.1).
- $s_p(f;\kappa)$ – p-variation sum (def. 1.2).
- $v_p(f)$ – p-variation of the function f (def. 1.2).
- $SP_p(f,[a,b])$ – the set of supreme partitions (def. 1.2).
- $\overline{SP}_p(f,[a,b])$ – a set of points that are in any supreme partition (def. 1.24).
- $PM[a,b]$ – a set of piecewise monotone functions (def. 1.6).
- $CPM[a,b]$ – a set of continuous piecewise monotone functions (def. 1.6).
- $K(f,[a,b])$ – minimal size of PM partitions (prop. 1.7).
- $X(f,[a,b])$ – the set of PM partitions with minimal size (def. 1.8).

1 Mathematical analysis

1.1 General known properties

Definition 1.1 (Partition). Let $J=[a,b]$ be a closed interval of real numbers with $-\infty<a\leq b<+\infty$. If $a<b$, an ordered set $\kappa=\{x_i\}_{i=0}^n$ of points in $[a,b]$ such that a $a=x_0<x_1<x_2<...<x_n=b$ is called a *(point) partition*. The size of the partition is denoted $|\kappa|:=\#\kappa-1=n$. The set of all point partitions of $[a,b]$ is denoted by $PP[a,b]$.

Definition 1.2 (p-variation). Let $f:[a,b]\to\mathbb{R}$ be a real function from an interval $[a,b]$. If $a<b$, for $\kappa=\{x_i\}_{i=0}^n\in PP[a,b]$ the *p-variation sum* is

$$s_p(f,\kappa):=\sum_{i=1}^n|f(x_i)-f(x_{i-1})|^p, \tag{1.1}$$

where $0 < p < \infty$. Thus, the *p-variation* of f over $[a, b]$ is 0 if $a = b$ and otherwise

$$v_p(f) = v_p(f, [a, b]) := \sup \{s_p(f, \kappa) : \kappa \in PP[a, b]\}. \tag{1.2}$$

The partition κ is called *supreme partition* if it satisfies the property $v_p(f) = s_p(f, \kappa)$. The set of such partitions is denoted $SP_p(f, [a, b])$.

Vidinis Komentaras (VK): Dar neaisku kaip pakrikstyti skaidini, kuris pasiekia supremuma.

Lemma 1.3 (Elementary properties). Let $f : [a, b] \to \mathbb{R}$ and $0 < p < \infty$. Then the following p-variation properties holds

a) $v_p(f, [a, b]) \geq 0$,

b) $v_p(f, [a, b]) = 0 \Leftrightarrow f \equiv Const.$,

c) $\forall C \in \mathbb{R} : v_p(f + C, [a, b]) = v_p(f, [a, b])$,

d) $\forall C \in \mathbb{R} : v_p(Cf, [a, b]) = C^p v_p(f, [a, b])$,

e) $\forall c \in [a, b] : v_p(f, [a, b]) \geq v_p(f, [a, c]) + v_p(f, [c, b])$,

f) $\forall [a', b'] \subset [a, b] : v_p(f, [a, b]) \geq v_p(f, [a', b'])$.

g) $\forall \kappa \in PP[a, b] : s_p(f; \kappa) \leq v_p(f, [a, b])$.

All listed properties are elementary derived directly form the p-variation definition.

Definition 1.4 (Regulated function). ([1], Def. 3.1) For any interval J, which may be open or closed at either end, real function f is called *regulated* on J if it has left and right limits $f(x-)$ and $f(x+)$ respectively at each point x in interior of J, a right limit at the left end point and a left limit at the right endpoint.

Proposition 1.5. ([1], Lemma 3.1) Let $1 \leq p < \infty$. If f is regulated then $v_p(f)$ remains the same if points $x+$, $x-$ are allowed as partition points x_i in the definition 1.2.

Definition 1.6 (Piecewise monotone functions). ([1], Def. 3.2) A regulated real-valued function f on closed interval $[a, b]$ will be called *piecewise monotone* (PM) if there are points $a = x_0 < \cdots < x_k = b$ for some finite k such that f is monotone on each interval $[x_{j-1}, x_j]$, $j = 1, \ldots, k$. Here for $j = 1, \ldots, k-1$, x_j may be point $x-$ or $x+$. The set of all piecewise monotone functions is denoted $PM = PM[a, b]$.

In addition to PM, if f is continuous function we will call it continuous piecewise monotone(CPM). The set of such functions is denoted $CPM = CPM[a, b]$.

Proposition 1.7. ([1], Prop. 3.1) If f is PM, there is a minimal size of partition $|\kappa|$ for which the definition 1.6 holds. The minimal size of the the PM partition is denoted $K(f, [a, b]) = K(f)$, namely

$$K(f) := \min\left\{n : \exists\{x_i\}_{i=0}^{n} \in PP[a, b] : f \text{ is monotonic in each } [x_{j-1}, x_j]\right\}. \tag{1.3}$$

Definition 1.8 (The set of PM partitions with minimal size). ([1], Def. 3.3) If f is PM, let $X(f) = X(f, [a, b])$ be the set of all $\{x_i\}_{i=0}^{K(f)}$ for which the definition of PM (def. 1.6) holds. $X(f)$ is called the *set of PM partitions with minimal size.*

Proposition 1.9. ([1], Prop. 3.3) Let f is PM then the numbers $\alpha_j(f) := f(x_j) - f(x_{j-1})$ for $\{x_j\}_{j=0}^{K(f)} \in X(f)$ and $j = 1, 2, \ldots K(f)$ are uniquely determined.

Proposition 1.10. ([1], Prop. 3.2) Let f is PM. For any partition $\{x_j\}_{j=0}^{K(f)} \in X(f)$ exactly one of the flowing stamens holds:

(a) $f(x_0) > f(x_1) < f(x_2) > \ldots$. Function f is not increasing in intervals $[x_{2j}, x_{2j+1}]$, then $2j+1 \leq K(f)$. Function f is not decreasing in intervals $[x_{2j-1}, x_{2j}]$, then $j \geq 1$ and $2j \leq K(f)$.

(b) (a) holds for a function $-f$; or

(c) $K(f, [a, b]) = 1$ and f is constant in interval $[a, b]$.

Definition 1.11 (The equality by PM). ([1], Def. 3.4) If f, g are two PM functions, possibly on different intervals, such that $K(f) = K(g)$ and $\alpha_j(f) = \alpha_j(g)$ for $j = 1, 2, \ldots, K(f)$, then we say that $f \overset{PM}{=} g$.

Proposition 1.12. ([1], Cor. 3.1) Let $p > 1$ and functions f and g are PM. If $f \overset{PM}{=} g$ or $f \overset{PM}{=} -g$, then $v_p(f) = v_p(g)$.

Proposition 1.13. ([1], Them. 3.1) Let f is PM, $\kappa \in X(f)$ and $1 \leq p < \infty$. Then the supremum of p-variation in Definition 1.2 is attained for some partition $r \subset \kappa$.

Corollary 1.14. The set $SP_p(f, [a, b])$ is not empty for all $f \in PM[a, b]$.

Definition 1.15 (Sample function). Suppose $X = \{X_i\}_{i=0}^n$ is any sequence real numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. Then the *sample function* $G_X : [0, n] \to \mathbb{R}$ is defined as

$$G_X(t) := X_{\lfloor t \rfloor},\ t \in [0, n], \tag{1.4}$$

where $\lfloor t \rfloor$ denotes floor function at point t.

Definition 1.16 (p-variation of the sequence). Let $X = \{X_i\}_{i=0}^n$. The p-variation of the sample X is defined as p-variation of the function $G_X(t)$, namely

$$v_p(X) := v_p(G_X(t), [0, n]). \tag{1.5}$$

Definition 1.17 (Piecewise linear function). Let $X = \{X_i\}_{i=0}^n$ be any real-value sequence. The function $L_X : [0, n] \to \mathbb{R}$ is defined as

$$L_X(t) := (1 + \lfloor t \rfloor - t)X_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)X_{\lfloor t \rfloor + 1},\ t \in [0, n] \tag{1.6}$$

is called *piecewise linear function*.

Definition 1.18 (Partial sum). Let $X_1, X_2, \ldots, X_n$ be any sequence of real numbers. The *partial sum* of the first j terms is defined by

$$S_j := \sum_{i=1}^{j} X_i,\ j = 1, 2, \ldots, n. \tag{1.7}$$

In addition, lets denote $S_0 = 0$.

1.2 General properties with proofs

Proposition 1.19. For all $f \in PM$ exists $g \in CPM$ such that $f \overset{PM}{=} g$.

Proof. Let f be PM. According to Proposition 1.9 the numbers $\alpha_j(f) := f(x_j) - f(x_{j-1})$, $j = 1, 2, \ldots K(f)$ are uniquely determined. Then, the partial sums of the sequence $\alpha_j(f)$ are

$$S_j := \sum_{i=1}^{j} \alpha_i(f). \tag{1.8}$$

Lets connect points S_j by piecewise linear function (def. 1.17), namely

$$L_S(t) := (1 + \lfloor t \rfloor - t)S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)S_{\lfloor t \rfloor + 1},\ t \in [0, K(f)] \tag{1.9}$$

Function L_S is CPM. In addition, it is straight forward to see that

$$\alpha_j(L_S) = S_j - S_{j-1} = \alpha_j(f), \tag{1.10}$$

hence, by Definition 1.11 $f \overset{PM}{=} L_S$. ■

Corollary 1.20. By applying last proof to the function $G_X(t)$ (from Definition 1.15), we get that

$$G_X(t) \overset{PM}{=} L_X(t). \tag{1.11}$$

Therefore, according to Proposition 1.12, if $p > 1$ then

$$v_p(X) := v_p(G_X(t), [0, n]) = v_p(L_X(t), [0, n]) \tag{1.12}$$

Proposition 1.21. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{x_i\}_{i=0}^n \in PP[a, b]$ is any partition of interval $[a, b]$. Then the statement

$$\exists \kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa \tag{1.13}$$

is equivalent to

$$v_p(f, [a, b]) = \sum_{i=1}^{n} v_p(f, [x_{i-1}, x_i]). \tag{1.14}$$

Proof. Necessary. Let $f : [a, b] \to \mathbb{R}$ be PM, $\{x_i\}_{i=0}^n \in PP[a, b]$ and

$$\exists \kappa \in SP_p(f, [a, b]) : \forall i, x_i \in \kappa. \tag{1.15}$$

Points from the partition κ will be denoted t_i, i.e. $\kappa = \{t_i\}_{i=0}^m$. Then, according to definitions of SP_p and p-variation (def. 1.2) the following equation holds

$$v_p(f, [a, b]) = s_p(f; \kappa) = \sum_{j=1}^{m} |f(t_{j-1}) - f(t_j)|^p = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p, \tag{1.16}$$

where $h : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$ denotes a function from the set of index of x to the set of index of t, namely:

$$h(i) := (j_i : x_i = t_{j_i} = t_{h(i)}). \tag{1.17}$$

The equation (1.16) holds, because all the elements in the sum remains, we just grouped them.

Moreover, the inequality

$$\sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1}) - f(t_j)|^p \leq v_p(f, [x_{i-1}, x_i]) \tag{1.18}$$

holds according to Lemma 1.3(g).

As a result of (1.16) and (1.18) we get

$$v_p(f,[a,b]) \le \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{1.19}$$

On the other hand, according to the same Lemma 1.3(e) the following inequality holds

$$v_p(f,[a,b]) \ge \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{1.20}$$

Finally, from the (1.19) and (1.20) follows

$$v_p(f,[a,b]) = \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{1.21}$$

Sufficiency. Suppose $f:[a,b]\to\mathbb{R}$ and

$$v_p(f,[a,b]) = \sum_{i=1}^{n} v_p(f,[x_{i-1},x_i]). \tag{1.22}$$

According to Corollary 1.14, sets $SP_p(f,[x_{i-1},x_i]),\ i=1,\dots,n$ are not empty. Lets take any partition from each of the sets $SP_p(f,[x_{i-1},x_i]),\ i=1,\dots,n$ and denote it κ_i.

Then, let define a joint partition $\kappa := \cup_{i=1}^{n}\kappa_i$. Points from the partition κ will be denoted by t_i. In addition, we will use the function h, which is defined in (1.17). Then, continuing the equation (1.22) we get

$$v_p(f,[a,b]) = \sum_{i=1}^{n} \sum_{j=h(i-1)+1}^{h(i)} |f(t_{j-1})-f(t_j)|^p = \sum_{i=1}^{|\kappa|} |f(t_{i-1})-f(t_i)|^p. \tag{1.23}$$

This means that $\kappa \in MP_p(f,[a,b])$. Moreover, $\forall i: x_i \in \kappa$, because $\kappa = \cup_{i=1}^{n}\kappa_i$. ■

Lemma 1.22. Let $f:[a,b]\to\mathbb{R}$ be PM and $\{x_i\}_{i=0}^{n} \subset \kappa \in SP_p(f,[a,b])$. Then

$$\forall k,l: v_p(f,[x_k,x_l]) = \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]),\ 0\le k<l\le n. \tag{1.24}$$

Proof. Suppose $\{x_i\}_{i=0}^{n} \subset \kappa \in SP_p(f,[a,b])$. Let choose k and l such that $0\le k<l\le n$. Lets apply Proposition 1.21 for the partition $\{x_0,x_k,x_{k+1},\dots,x_{l-1},x_l,x_n\}$. Thus,

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]) + v_p(f,[x_l,x_n]). \tag{1.25}$$

In addition, we can apply the same proposition for the partition $\{x_0, x_k, x_l, x_n\}$, then

$$v_p(f,[a,b]) = v_p(f,[x_0,x_k]) + v_p(f,[x_k,x_l]) + v_p(f,[x_l,x_n]). \tag{1.26}$$

By subtracting one equation form the other we get the result that

$$v_p(f,[x_k,x_l]) = \sum_{i=k+1}^{l} v_p(f,[x_{i-1},x_i]). \tag{1.27}$$

■

Lemma 1.23. Let $f:[a,b]\to\mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f,[a,b])$. Then

$$\forall k,l : v_p(f,[t_k,t_l]) = s_p\left(f,\{t_i\}_{i=k}^l\right),\ 0\le k<l\le n. \tag{1.28}$$

Proof. Let $f:[a,b]\to\mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in SP_p(f,[a,b])$. Let choose k and l such that $0\le k<l\le n$. Then

$$\begin{aligned} v_p(f,[a,b]) &= s_p(f,\{t_i\}_{i=0}^n) && (1.29)\\ &= s_p(f,\{t_i\}_{i=0}^k) + s_p(f,\{t_i\}_{i=k}^l) + s_p(f,\{t_i\}_{i=l}^n) && (1.30)\\ &\le v_p(f,[a,t_k]) + s_p(f,\{t_i\}_{i=k}^l) + v_p(f,[t_l,b]). && (1.31)\end{aligned}$$

The last inequality holds according to Lemma 1.3(g).

On the other hand, from Proposition 1.22 we get

$$v_p(f,[a,b]) = v_p(f,[a,t_k]) + v_p(f,[t_k,t_l]) + v_p(f,[t_l,b]). \tag{1.32}$$

Form (1.31) and (1.32) follows

$$v_p(f,[t_k,t_l]) \le s_p(f,\{t_i\}_{i=k}^l). \tag{1.33}$$

Finally, from Lemma 1.3(g) we conclude that

$$v_p(f,[t_k,t_l]) = s_p(f,\{t_i\}_{i=k}^l). \tag{1.34}$$

■

Definition 1.24 (The point of supreme partition). Let $f:[a,b]\to\mathbb{R}$ be PM. The point x will be called the *point of supreme partition* if

$$\exists \kappa \in SP_p(f,[a,b]) : x\in\kappa. \tag{1.35}$$

The set of such points will be denoted by $\overline{SP}_p(f,[a,b])$.

Lemma 1.25. Suppose $f : [a,b] \to \mathbb{R}$ is PM, $x \in [a,b]$, $x \notin \overline{SP}_p(f,[a,b])$ and $\{t_i\}_{i=0}^n \in SP(f,[a,b])$ is any supreme partition. Then,

$$\exists j = 1,\dots,n : x \in (t_{j-1},t_j) \text{ and } x \notin \overline{SP}_p(f,[t_{j-1},t_j]). \tag{1.36}$$

Proof. Suppose the assumptions of lemma is valid. Since $x \in [a,b]$ and $[a,b] = \cup_{i=1}^n [t_{i-1},t_i]$, then

$$\exists j = 1,\dots,n : x \in [t_{j-1},t_j]. \tag{1.37}$$

Moreover, $x \notin \{t_i\}_{i=0}^n$, because $x \notin \overline{SP}_p(f,[a,b])$, thus, $x \neq t_{j-1}$ and $x \neq t_j$. In addition to (1.37) this means that $x \in (t_{j-1},t_j)$.

Now, we will proof that $x \notin \overline{SP}_p(f,[t_{j-1},t_j])$. Suppose to the contrary that $x \in \overline{SP}_p(f,[t_{j-1},t_j])$. Then, according to definition of $\overline{SP}_p(f,[t_{j-1},t_j])$,

$$\exists \kappa \in SP_p(f,[t_{j-1},t_j]) : x \in \kappa. \tag{1.38}$$

Since, $\kappa \in SP_p(f,[t_{j-1},t_j])$, then

$$v_p(f,[t_{j-1},t_j]) = s_p(f,\kappa) \tag{1.39}$$

Applying Proposition 1.21 for partition $\{t_i\}_{i=0}^n$ we get

$$\begin{aligned} v_p(f,[a,b]) &= v_p(f,[t_0,t_{j-1}]) + v_p(f,[t_{j-1},t_j]) + v_p(f,[t_j,t_n]) && (1.40)\\ &= s_p(f,\{t_i\}_{i=0}^{j-1}) + s_p(f,\kappa) + s_p(f,\{t_i\}_{i=j}^n) && (1.41) \end{aligned}$$

This means that the partition $r := \{t_i\}_{i=0}^{j-1} \cup \kappa \cup \{t_i\}_{i=j}^n$ is supreme partition, so $x \in r \in SP(f,[a,b])$, therefore, by definition $x \in \overline{SP}_p(f,[a,b])$. This contradict to initial assumption. ■

Definition 1.26 (f-join). Suppose $f : [a,b] \to \mathbb{R}$ is PM. We will say that points t_a and t_b ($t_a < t_b$) are *f-joined* in interval $[a,b]$ if

$$\exists \{x_j\}_{j=0}^n \in SP_p(f,[a,b]) : [t_a,t_b] = [x_{j-1},x_j], \text{ with some } j. \tag{1.42}$$

Lemma 1.27. Suppose $f : [a,b] \to \mathbb{R}$ is PM and points t_a and t_b ($t_a < t_b$) are f-jointed in interval $[a,b]$. Then all following statements holds

a) $v_p(f,[t_a,t_b] = |f(t_a) - f(t_b)|^p$;

b) Let $x \in [t_a,t_b]$. If $f(t_a) \geq f(t_b)$, then $f(t_a) \geq f(x) \geq f(t_b)$;

Proof.

a) Let $f : [a,b] \to \mathbb{R}$ be PM, $t_a < t_b$ and pair of points t_a, t_b are f-joined. Then exists $\{x_j\}_{j=0}^n$ and j from the Definition 1.26. Thus, according to Lemma 1.23

$$v_p(f, [t_{j-1}, t_j]) = s_p\left(f, \{t_{j-1}, t_j\}\right) = |f(t_a) - f(t_b)|^p. \tag{1.43}$$

b) Let $f : [a,b] \to \mathbb{R}$ be PM and points t_a and t_b are f-joined. Since $f \overset{PM}{=} -f$, with out loss of generality we can assume that $f(t_a) \leq f(t_b)$. Suppose to the contrary that $f(t_b)$ is not an extrema of the function in interval $[t_a, t_b]$. Hence, $\exists c \in [t_a, t_b] : f(c) > f(t_b)$. Therefore, $|f(c) - f(t_a)|^p > |f(t_b) - f(t_a)|^p$. According to (1.43), $v_p(f, [t_a, t_b]) = |f(t_b) - f(t_a)|^p$, thus, $|f(c) - f(t_a)|^p > v_p(f, [t_a, t_b])$, but this contradicts the definition of p-variation. So, point t_b must be an extrema in interval $[t_a, t_b]$. Symmetric arguments could be used for point t_a.

■

Lemma 1.28. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then function $f : [0, \infty) \to \mathbb{R}$ with the values

$$f(x) = (x + c_1)^p - x^p - C, \ x \in [0, \infty), \tag{1.44}$$

are non decreasing in interval $[0, \infty)$.

Proof. Suppose $C \in \mathbb{R}$, $c_1 \geq 0$ and $1 \leq p < \infty$. Then, for all $x \geq 0$, the derivative of the function f is

$$\begin{aligned} f'(x) &= p(x + c_1)^{p-1} - px^{p-1} \\ &\geq px^{p-1} - px^{p-1} = 0. \end{aligned}$$

The derivative of function f is non negative, thus the function f is non decreasing, if $x \geq 0$. ■

Corollary 1.29. Suppose $c_1 \geq 0$, $C \in \mathbb{R}$, $1 \leq p < \infty$ and $0 \leq x \leq y$. Then the following implication holds

$$|x + c_1|^p > x^p + C \Rightarrow |y + c_1|^p > y^p + C. \tag{1.45}$$

Proof. Suppose $0 \leq x \leq y$. Since f is non decreasing, $f(x) \leq f(y)$. Therefore, if $f(x) > 0$, then $f(y) > 0$. ■

Proposition 1.30. Suppose $f : [a,b] \to \mathbb{R}$ is PM and $x \in [a', b'] \subset [a,b]$. If $x \notin \overline{SP}_p(f, [a', b'])$, then $x \notin \overline{SP}_p(f, [a,b])$.

Proof. Let $f:[a,b]\to\mathbb{R}$ be PM, $x\in[a',b']\subset[a,b]$, and $x\notin\overline{SP}_p(f,[a',b'])$. Suppose to the contrary that $x\in\overline{SP}_p(f,[a,b])$.

Since $x\in\overline{SP}_p(f,[a,b])$, according to the Definition 1.24,

$$\exists\{t_i\}_{i=0}^n\in SP_p(f,[a,b]):x\in\{t_i\}_{i=0}^n. \tag{1.46}$$

Let $\{y_i\}_{i=0}^n\in SP_p(f,[a',b'])$ be any supreme partition from the interval $[a',b']$. Then, according to Lemma 1.25,

$$\exists j=1,\dots,n:x\in(y_{j-1},y_j)\text{ and }x\notin\overline{SP}_p(f,[y_{j-1},y_j]). \tag{1.47}$$

Moreover,

$$x\notin\{a,y_{j-1},y_j,b\}, \tag{1.48}$$

because $x\in(y_{j-1},y_j)\subset[a,b]$. Thus, from (1.46) and (1.48) follows that

$$\exists r\in\{1,2,\dots,n-1\}:t_r=x. \tag{1.49}$$

Lets denote variables l and k by

$$\begin{aligned} l &:= \max\{i\in\{1,2,\dots,n-1\}:t_i\in(y_{j-1},y_j)\},\\ k &:= \min\{i\in\{1,2,\dots,n-1\}:t_i\in(y_{j-1},y_j)\}. \end{aligned}$$

Since $x\in(y_{j-1},y_j)$ and (1.49) holds, the values l and k always exists and $k\le r\le l$. According to l and k definitions the following inequality holds

$$t_{k-1}\le y_{j-1}<t_k\le x\le t_l<y_j\le t_{l+1}. \tag{1.50}$$

According to Lemma 1.3(e)

$$v_p(f,[y_{j-1},y_j])\ge v_p(f,[y_{j-1},t_k])+v_p(f,[t_k,t_l])+v_p(f,[t_l,y_j]), \tag{1.51}$$

Firstly, let suppose

$$v_p(f,[y_{j-1},y_j])=v_p(f,[y_{j-1},t_k])+v_p(f,[t_k,t_l])+v_p(f,[t_l,y_j]). \tag{1.52}$$

Lets take any supreme partitions from intervals $[y_{j-1},t_k]$ and $[t_l,y_j]$, namely $\kappa_k\in SP_p(f,[y_{j-1},t_k]$ and $\kappa_l\in SP_p(f,[t_l,y_j]$. In addition, since $\{t_i\}_{i=0}^n\in SP_p(f,[a,b])$, according to Lemma 1.23 $v_p(f,[t_k,t_l])=s_p(f,\{t_i\}_{i=k}^l)$. Therefore,

$$v_p(f,[y_{j-1},y_j])=s_p(f,\kappa_k)+s_p(f,\{t_i\}_{i=k}^l)+v_p(f,\kappa_l).$$

This means, that

$$\kappa_k\cup\{t_i\}_{i=k}^l\cup\kappa_l\in SP_p(f,[y_{j-1},y_j].$$

So, $x \in \overline{SP}_p(f, [y_{j-1}, y_j])$, because $x \in \{t_i\}_{i=k}^{l}$. This contradicts (1.47), thus, equality (1.52) is not valid. As a result, inequality (1.51) becomes

$$v_p(f, [y_{j-1}, y_j]) > v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]). \qquad (1.53)$$

Since points y_{j-1} and y_j are f-joined, from the Lemma 1.27(a) we get

$$v_p(f, [y_{j-1}, y_j]) = |f(y_{j-1}) - f(y_j)|^p,$$

therefore,

$$|f(y_{j-1}) - f(y_j)|^p > v_p(f, [y_{j-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, y_j]).$$

Moreover, according to Lemma 1.3(g), from the last statement follows

$$|f(y_{j-1}) - f(y_j)|^p > |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(y_j)|^p. \qquad (1.54)$$

Since $v_p(f) = v_p(-f)$ (see Proposition 1.12), with out loss of generality we can assume that $f(y_{j-1}) \geq f(y_j)$. Hence, from the Lemma 1.27(b) we get that

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j), \qquad (1.55)$$
$$f(y_{j-1}) \geq f(t_k) \geq f(y_j). \qquad (1.56)$$

In addition, the pairs of points $\{t_l, t_{l+1}\}$ and $\{t_{k-1}, t_k\}$ are also f-joined, therefore, by the Lemma 1.27(b) inequalities (1.55) and (1.56) could be extended as

$$f(y_{j-1}) \geq f(t_l) \geq f(y_j) \geq f(t_{l+1}), \qquad (1.57)$$
$$f(t_{k-1}) \geq f(y_{j-1}) \geq f(t_k) \geq f(y_j). \qquad (1.58)$$

From the (1.57) the following inequalities holds

$$\begin{aligned} f(y_{j-1}) - f(t_l) &\geq 0, \\ f(t_l) - f(t_{l+1}) \geq f(t_l) - f(y_j) &\geq 0. \end{aligned}$$

Therefore, we can use Corollary 1.29. According to it, from the inequality (1.54) follows

$$\begin{aligned} |f(y_{j-1}) - f(t_l) + f(t_l) - f(y_j)|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(y_j)|^p, \\ |f(y_{j-1}) - f(t_l) + f(t_l) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + |f(t_l) - f(t_{l+1})|^p, \\ |f(y_{j-1}) - f(t_{l+1})|^p &> |f(y_{j-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]). \end{aligned}$$

The last inequality holds, because points t_l and t_{l+1} are f-joined, thus, according to Lemma 1.27(a),

$$v_p(f, [t_l, t_{l+1}]) = |f(t_l) - f(t_{l+1})|^p.$$

Symmetric argument could be used in other direction. Firstly lets modify last inequality

$$|f(y_{j-1})-f(t_k)+f(t_k)-f(t_{l+1})|^p > |f((y_{j-1})-f(t_k)|^p+v_p(f, [t_k, t_l])+v_p(f, [t_l, t_{l+1}]).$$

From the (1.58) we get that the following inequalities holds

$$\begin{aligned} f(t_k) - f(t_{l+1}) &\geq 0, \\ f(t_{k-1}) - f(t_k) \geq f(y_{j-1}) - f(t_k) &\geq 0. \end{aligned}$$

Thus, from Corollary 1.29 we get

$$\begin{aligned} |f(t_{k-1}) - f(t_k) + f(t_k) - f(t_{l+1})|^p &> |f(t_{k-1}) - f(t_k)|^p + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]), \\ |f(t_{k-1}) - f(t_{l+1})|^p &> v_p(f, [t_{k-1}, t_k]) + v_p(f, [t_k, t_l]) + v_p(f, [t_l, t_{l+1}]). \end{aligned}$$

As previous, the last inequality holds, because t_{k-1} and t_k are f-joined.

Finally, using Lemma 1.22 we conclude that

$$|f(t_{k-1}) - f(t_{l+1})|^p > v_p(f, [t_{k-1}, t_{l+1}]).$$

This contradicts with the definition of p-variation.

■

1.3 Extra results

Definition 1.31 (Extremum). We will call the point $t \in [a, b]$ an *extrema* of the function f in interval $[a, b]$ if $f(t) = \sup \{f(z) : z \in [a, b]\}$ or $f(t) = \inf \{f(z) : z \in [a, b]\}$.

Proposition 1.32. Let $f : [a, b] \to \mathbb{R}$ is PM. If point $x \in [a, b]$ is extrema of the function f, then $x \in \overline{SP}_p(f, [a, b])$.

Proof. Let $f : [a, b] \to \mathbb{R}$ is PM and point $x \in [a, b]$ is an extrema of the function f. Let $\{t_i\}_{i=0}^n \in SP_p(f, [a, b]$ be any supreme partition. Then

$$\exists j \in 1, \ldots, n : x \in [t_{j-1}, t_j] \tag{1.59}$$

Since x is an extrema of function f in interval $[a, b] \supset [t_{j-1}, t_j]$, the point x is an extrema in interval $[t_{j-1}, t_j]$ as well. Therefore,

$$f(t_{j-1}) - f(t_j)|^p \leq |f(t_{j-1}) - f(x)|^p \tag{1.60}$$

or

$$|f(t_{j-1}) - f(t_j)|^p \le |f(x) - f(t_j)|^p. \tag{1.61}$$

As a result,

$$\begin{aligned} |f(t_{j-1}) - f(t_j)|^p &\le |f(t_{j-1}) - f(x)|^p + |f(x) - f(t_j)|^p, \\ |f(t_{j-1}) - f(t_j)|^p &\le v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]), \\ v_p(f, [t_{j-1}, t_j]) &\le v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j]). \end{aligned}$$

The last inequality holds, because t_{j-1} and t_j are f-joined, so, $v_p(f, [t_{j-1}, t_j]) = |f(t_{j-1}) - f(t_j)|^p$. Since $v_p(f, [t_{j-1}, t_j]) < v_p(f, [t_{j-1}), x]) + v_p(f, [x, t_j])$ is not valid, the equation

$$v_p(f, [t_{j-1}, t_j]) = v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) \tag{1.62}$$

holds.

Moreover, applying Proposition 1.21 for the partition $\{a, t_{j-1}, t_j, b\}$ we have

$$\begin{aligned} v_p(f, [a, b]) &= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, t_j]) + v_p(f, [t_j, b]) \\ &= v_p(f, [a, t_j]) + v_p(f, [t_{j-1}, x]) + v_p(f, [x, t_j]) + v_p(f, [t_j, b]) \end{aligned}$$

From the same proposition follows, that $\exists \kappa \in SP_p(f, [a, b]) : x \in \kappa$, so, by definition, $x \in \overline{SP}_p(f, [a, b])$. ■

Lemma 1.33. Let $f : [a, b] \to \mathbb{R}$ be PM. If $K(f)$ is even, then

$$\exists x \in (a, b) : x \in \overline{SP}_p(f, [a, b]). \tag{1.63}$$

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is PM. Lets take any $\{x_i\}_{i=0}^n \in X(f)$. If $n > 1$, then without loss of generality we can assume that Proposition 1.10(a) holds. So, $f(x_0) > f(x_1) < f(x_2) > \dots$, i.e. $f(x_{2i-1}) < (x_{2i})$, thus, if n is even, then $f(x_{n-1}) < (x_n)$, therefore, point x_n is not a global minimum. Since $f(x_0) > f(x_1)$, point x_0 is also not a global minimum. As a result, global minimum is in interval (a, b) and it is in $\overline{SP}_p(f, [a, b])$ according to Lemma 1.32. ■

Lemma 1.34. Let $f : [a, b] \to \mathbb{R}$ be PM and $\{t_i\}_{i=0}^n \in PP[a, b]$. Suppose

$$\{t_i\}_{i=0}^{n-1} \in SP_p(f, [a, t_{n-1}]) \tag{1.64}$$

and

$$\{t_i\}_{i=1}^n \in SP_p(f, [t_1, b]). \tag{1.65}$$

Then

$$\forall i \in \{1, \dots, n-1\} : t_i \in \overline{SP}_p(f, [a, b]) \tag{1.66}$$

or

$$\forall i \in \{1, \dots, n-1\} : t_i \notin \overline{SP}_p(f, [a, b]) \tag{1.67}$$

Proof. Let the assumptions of lemma be valid. Firstly, suppose

$$\exists j \in \{1, \dots, n-1\} : t_j \in \overline{SP}_p(f, [a, b]). \tag{1.68}$$

This means that $\exists \kappa \in SP_p(f, [a, b]) : t_j \in \kappa$. Then, by Proposition 1.21,

$$v_p(f, [a, b]) = v_p(f, [a, t_j]) + v_p(f, [t_j, b]) \tag{1.69}$$

Since (1.64) and (1.64) holds, by Lemma 1.23 we get

$$v_p(f, [a, t_j]) = s_p(f, \{t_i\}_{i=0}^{j}, \tag{1.70}$$
$$v_p(f, [t_j, b]) = s_p(f, \{t_i\}_{i=j}^{n}. \tag{1.71}$$

Therefore,

$$v_p(f, [a, b]) = s_p(f, \{t_i\}_{i=0}^{j}) + s_p(f, \{t_i\}_{i=j}^{n}) = s_p(f, \{t_i\}_{i=0}^{n}). \tag{1.72}$$

So, $\{t_i\}_{i=0}^{n} \in SP_p(f, [a, b])$, thus, statement (1.66) holds.
On the other hand, if (1.68) is not valid, then statement (1.67) holds.
■

2 p-variation calculus

This chapter will present the algorithm that calculates p-variation for the sample (see Def. 1.16). Nonetheless, this algorithm could be used to calculate the p-variation for arbitrary piecewise monotone function. This algorithm will be called *pvar*. It is already realised in the R (see [2]) package *pvar* and is publicly available on CRAN[1].

Suppose $X = \{X_i\}_{i=0}^{n}$ is any real-value sequence of numbers. We will call such sequence a *sample*, whereas n will be referred to as a *sample size*. The formal definition of p-variation of the sample is given in Definition 1.16, but it is more intuitive to consider a sample as continuous piecewise monotone (CPM) function, namely, Proposition 1.20 states that if $p \geq 1$ then

$$v_p(X) = v_p(L_X(t), [0, n]), \tag{2.1}$$

where $L_X(t)$ is piecewise linear function connecting sample points (Def. 1.17). Since $L_X(t)$ is CPM, all the properties proved in this chapter are applicable to it.

On the other hand, the algorithm does not use $L_X(t)$ function directly, but it actually operates the sample X. Therefore, in the context of algorithm

[1] http://cran.r-project.org/web/packages/pvar/index.html

it is more convenient to use the sample X rather then the function $L_X(t)$. Thus in this part we will be using X, but please don't be mislead then we will refer to properties of p-variation that were proved to functions. In that case we actually have in mind function $L_X(t)$.

The $L_X(t)$ is CPM and all its break points could be only at points $t = 0, 1, \ldots, n$, i.e. at points where $L_X(t) = X_0, X_1, \ldots, X_n$. Therefore, according to Prop. **??**, the p-variation of the sample could be expressed as

$$v_p(X) = \max\left\{\sum_{i=1}^{k} |X_{j_i} - X_{j_{i-1}}|^p : 0 = j_0 < \cdots < j_k = n,\ k = 1, \ldots, n\right\}. \tag{2.2}$$

It is worth noting, that the procedure *pvar* could be used to calculate the p-variation of any piecewise monotone function. Let assume f is any piecewise monotone function. According to definition (see 1.6) there are points $a = x_1 < \cdots < x_n = b$ for some finite n such that f is monotone on each interval $[x_{j-1}, x_j],\ j = 1, 2, \ldots, n$. Let the sample X contain the values of the function f at the points $\{x_i\}_{i=0}^n$, namely, $X = \{X_i\}_{i=0}^n := \{f(x_i)\}_{i=0}^n$. It is straight forward to see that $f \overset{PM}{=} L_X$, therefore, if $p \geq 1$ then $v_p(X) = v_p(L_X) = v_p(f)$.

2.1 Main function

The procedure *pvar* that calculates p-variation will be presented here. Firstly, we will introduce the main schema, further, each step will be discussed in more details.

The main principal of the whole procedure is to identify meaningless points using known properties of p-variation. If a single point X_i is identified as meaningless, then we could exude it from further consideration. As long as X_i is meaningless, the actual value of X_i has no effect on p-variation since X_i do not participate in the sum of p-variation. Actually, X_i could be ignored, but it is more convenient to drop it form the sample, because it is hard to apply properties then not all points are under consideration. The *drop* operation means that we updating sample X with new sample X' which do not have X_i element, namely,

$$\{X'_i\}_{i=0}^{n-1} := \{X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\}.$$

The most important properties which were used in algorithm are ... ??? Later on, we will discus in great detail how we actually using them. The main steps in *pvar* procedure goes as follows.

Procedure *pvar*. Input: sample X, scalar p.

1. *Removing monotonic points.* According to proposition ??? all points that are not the end of monotonic interval could be excluded from further consideration.

2. *Checking all small intervals.* Every small intervals are checked if it is possible to identify meaningless points. All points that are found to be meaningless are excluded. This operation is based on proposition ??? which states that if point is not meaningful in small interval, then it will not be meaningful in big interval as well.

3. *Merging small intervals.*

The corresponding psiaudocode of the main functions goes as follows

```
pvar <- function(x, p){
  partition <- Remove_Monotonic_Points(x)
  partition <- Test_Points_In_Small_Intervals(partition, p, LSI
   = 3)
  partition <- Merge_Small_Intervals(partition, p, LSI=3+1)
  pvar <- sum(abs(diff(partition))^p)
  return(pvar)
}
```

2.2 Detail explanation

2.2.1 Removing monotonic points

According to Proposition 1.13 the p-variation is achieved on the partition of monotonic intervals. Therefore the points that are not the end of monotonic interval could be drop out.

There are multiple options to perform this checking. The method that was actuality applied involves two steps:

1. *Removing constant intervals.* If $X_t = X_{t+1},\ t = 1, \ldots, n-1$ then X_t is removed since both points X_t and X_{t+1} cannot be in break points partition. Let Label the new sample as $\{Y_i\}_{i=0}^{m}$.

2. *Finding break point.* If ΔY_t and ΔY_{t+1} $(t = 1, \ldots, m-1)$ has alternating sign, then point Y_t is considered to be break point. In addition points Y_0 and Y_m is also the break points since end points of the interval is always included in partition (see Definition 1.1). All other point are identified as meaningless and removed from sample.

This procedure can by performed very quickly, since it does not include any interdependent checking and could be done with vectorised functions. In *pvar* package all break points could be found with *ChangePointsId* function.

2.2.2 Test points in small intervals

Proposition **??** is very important for speeding up the calculus of p-variation. It states that if the point was not significant in any small interval, then it will not be significant in the full sample as well. Moreover, Proposition 1.21, Lemmas ?? and ?? gives an effective way to identify some meaningless points. Those property allows to drop significant amount of points using quite simple operations.

Suppose X is a sample without monotonic points. Let consider any small interval with $m+1$ points ($m = 2, \dots, n-1$), namely, lets examine a sub-sample $\{X_i\}_{i=j}^{j+m}$ for any $j = 0, ..., n-m$. Based on Proposition 1.21, if

$$\sum_{i=j+1}^{j+m} |X_i - X_{i-1}|^p < |X_j - X_{j+m}|^p, \tag{2.3}$$

then partition $\{j, \dots, j+m\}$ could not be meaningful in interval $[j, j+m]$, therefore, some of the points $\{X_i\}_{i=j+1}^{j+m-1}$ must be meaningless. Those meaningless points can be easily identify if we systemically check all intervals starting from small m.

If $m = 2$ then $|\{X_i\}_{i=j}^{j+2}| = 2$, thus according to Lemma **??**, $\{X_i\}_{i=j}^{j+m}$ has at least one inner significant point. Since it only has one inner point, then this point must be significant. So, we don't need to do any checkings here.

If $m = 3$ and (2.3) holds, then at least one point must be insignificant. Moreover, according to Lemma **??**, in this case both middle points X_{t+1} and X_{t+2} must be meaningless.

So, some insignificant points could be identified by applying this kind of checking for all sub-samples $\{X_i\}_{i=j}^{j+3}$, $j = 0, ..., n-3$. Then all identified meaningless points could be droped out. As a result, the whole sample changes, hence, we can apply the same procedure again and try to find even more meaningless points. We should repeat the iterations until no new meaningless points are identified.

If $m = 4$ then $|\{X_i\}_{i=j}^{j+2}|$ is even, thus, according to Lemma **??**, $\{X_i\}_{i=j}^{j+m}$ has at least one inner significant point. So, all inner points are significant, based on Lemma **??**. We can conclude, that this argument holds in all cases, then m is even, so we don't need to check it any more.

So, the can go further and apply same checking with $m = 5$. Suppose all sub-samples $\{X_i\}_{i=j}^{j+3}$, $j = 0, ..., n-3$ are already checked. If (2.3) holds, then based on the same Lemma, all middle points X_{t+1}, X_{t+2}, X_{t+3} and X_{t+4} are identified as meaningless and could be dropped out. If we want to apply this checking in new itaration, then we should start checking from $m = 3$ again, because we must make sure that all sub-samples $\{X_i\}_{i=j}^{j+3}$, $j = 0, ..., n-3$ are already checked if we want to apply Lemma ??

We can continue checking by this principle, and actually find p-variation. But it is not effective way, because in any case of drop out we should go back and start checking from $m = 3$, so, it is actually effective for small m. From Monte-Carlo experiment we made a recommendation to check up till $m = 7$.

The final psiaudocode of this procedure is given below

```

SET dum_X to NULL
WHILE dum_X is not equal to X
  SET dum_X = X
  SET d=3
  CHECK all sub-samples with m=d
  UPDATE X by dropping all meaningless points
  WHILE dum_X is equal to X
    SET d = d + 2
    CHECK all sub-samples with m=d points
    UPDATE x by dropping all meaningless points
  ENDWHILE
ENDWHILE
```

2.2.3 Merging small intervals

Adding an extra point to interval Let suppose ... Then x_0 could make a f-join with any other partition point.

Test reasinable posibilities. If find - delete all midpoints

Merge two intervals

```

1. Find potential points from one side an the other

2. Merging
for i in
  AddPoint(i)
EndFor
```

Merging all small intervals

```
while the length of in > 1

  for
    int[i] = merge(int[i], int[i+1])
    i+2;
  end for
  delete all even int



end
```

3 Conclusion

References

[1] J. Qian. The p-variation of Partial Sum Processes and the Empirical Process // Ph.D. thesis, Tufts University, 1997.

[2] R