MTH 311 Homework 8

Philip Warton

May 7, 2020

Proof. Let $f : X \to Y$. Wts that for any metric space the topological definition of continuity is equivalent to the ϵ-δ definition.
”$\Rightarrow$” Assume $f^{-1}(U)$ is open for every open set $U \subset Y$. Let $\epsilon > 0$, and let $x \in X$. Then let $y = f(x)$. It follows that $f^{-1}(V_\epsilon(f(x)))$ is open in X. Then since x must be in the set $f^{-1}(V_\epsilon(f(x)))$ there must exist some δ-neighborhood of x such that $V_\delta(x) \subset f^{-1}(V_\epsilon(f(x)))$. Hence $f(V_\delta(x) \subset V_\epsilon(f(x))$.
”$\Leftarrow$” Assume that for every ϵ-neighborhood of $f(x)$ there is some $V_\delta(x)$ whose image is contained in it. Let $O \subset Y$ be an open set in Y. Then for every point $y \in Y$ there is some neighborhood of y contained in O. It follows that $O = \bigcup_{y \in O} V_{\epsilon_y}(y)$. If $f^{-1}(y) = \emptyset$, we need not worry about this point. Otherwise there exists some $x \in X | f(x) = y$. Hence by assumption we can write

$$f^{-1}\left(\bigcup_{y \in O} V_{\epsilon_y}(y)\right) = \bigcup_{x \in f^{-1}(O)} V_{\delta_x}(x)$$

Since this is a union of open sets in X, we have $f^{-1}(O)$ is open and f is continuous. □

4.4.9

(a)

Show that a Lipschitz function is uniformly continuous.

Proof. Suppose $f : A \to \mathbb{R}$ is a Lipschitz function, i.e. there exists $M > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leqslant M$$

for all $x \neq y \in A$. We can multiply the above inequality and get $|f(x) - f(y)| \leqslant M|x - y|$. Let $\epsilon > 0$ be arbitrary, and let $\delta = \dfrac{\epsilon}{M}$. Then if we have $|x - y| < \dfrac{\epsilon}{M}$, we can mutliply by M and get

$$|f(x) - f(y)| \leqslant M|x - y| < M\delta = M\frac{\epsilon}{M} = \epsilon$$

□

(b)

Is the converse true? No.

Proof. Take the funciton $f(x) = \sqrt{x}$ on $[0, 1]$. We say that it is uniformly continuous because it is continuous on a compact set. Now, since the definition must apply for all $x, y \in A$ choose $x = \dfrac{1}{n} \in [0, 1]$ which works for any $n \in \mathbb{N}$ and $y = 0$. Then

$$\left|\frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}}\right| = \sqrt{n}$$

Since $\sqrt{n}$ is not bounded we say that the function is not Lipschitz. □

4.5.5 b

Proof. We continue this proof from where the book left off. We have an interval $I_1 = [a_1, b_1]$ such that $f(a_1) < 0$ and $f(b_1) \geqslant 0$. We generalize the process by taking $z_n = \frac{a_n + b_n}{2}$, and then if $f(z_n) > 0$ then $b_{n+1} = z_n$. If $f(z_n) < 0$ then $a_{n+1} = z_n$. Otherwise, $z_n \in [a, b]$ and $f(z_n) = 0$, and we are done. By the inductive step, we assume that we have $I_n = [a_n, b_n]$, where $f(a_n) < 0$ and $f(b_n) > 0$. Then, we can find a midpoint z_n and create the interval I_{n+1} such that $f(a_{n+1}) < 0$ and $f(b_{n+1}) > 0$.

We say that the length of this interval is equal to $\frac{a-b}{2^n}$, since each time n increases by one we cut the interval in half. Then we take the sequence of intervals I_n for all $n \in \mathbb{N}$, and it follows that the length of I_n approaches 0. Since the length of I_n approaches zero, and by the Nested Interval Property the infinite intersection of I_n is non-empty, we say that there is exactly one number c such that $\bigcap_{n \in \mathbb{N}} I_n = \{c\}$.

Suppose that $f(c) > 0$. Then there would be some $n \in \mathbb{N}$ such that $f(I_n) > 0$ i.e. $f(a_n) > 0$ and $f(b_n) > 0$. This is not possible by the process in which we chose a_n, therefore $f(c) \leqslant 0$. Now suppose $f(c) < 0$. Then, similarly there must be some interval where $f(b_n) < 0$, which again is not possible. Thus $f(c) \geqslant 0$. Since we have shown $f(c)$ not to be strictly positive or strictly negative, it follows that $f(c) = 0$. □

5.2.5

(a)

The function f is continuous at 0 for $a > 0$.

Proof. 0 is a limit point of $\mathbb{R}$. Thus, we can use the functional limit characterization of continuity and say that if $lim_{x \to 0} f(x) = f(0)$ then f is continuous at 0. Suppose $a < 0$. Then, this limit is undefined as it diverges. For $a = 0$, the limit is equal to $1 \neq 0$. Therefore a must be greater than 0. Assume that $a > 0$, then $lim_{x \to 0} x^a = 0 = f(0)$. Therefore the function is continuos at 0 assuming $a > 0$. □

(b)

The function f is differentiable at 0 for all $a > 1$.

Proof. We say that f is differentiable at 0 if $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists. This can be rewritten as

$$\lim_{x \to 0^-} \frac{0}{x} = \lim_{x \to 0^+} x^{a-1}$$

Suppose $a < 1$, then the right hand side of this limit diverges. Suppose $a = 1$, then the right hand side is zero, and the left hand side equals 1, therefore the limit does not exist, and f_1 is not differentiable at 0. Thus a must be strictly greater than 1. For any $a > 1$ it is clear that both sides will be equal to zero, and therefore the limit does exist for such values of a. □

(c)

The function f is twice differentiable at 0 if $a > 2$.

Proof. We say that f is twice differentiable at 0 if f' is differentaible at 0. We know from part b that $f'_a(0) = 0$ for all $a > 1$, so by the power rule, given that $a > 1$, we write

$$f'_a(x) = \begin{cases} ax^{a-1} & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

And by the left-right property of functional limits we say that $f'_a(x)$ is differentiable at 0 if

$$\lim_{x \to 0^-} \frac{0}{x} = \lim_{x \to 0^+} \frac{ax^{a-1}}{x} = \lim_{x \to 0^+} ax^{a-2} = 0 \Longrightarrow a > 2$$

$\square$

5.2.7

(a)

We choose $a = 3$. Since $a > 2$, we know from recitation that this means g_a will be differentiable at 0, and thus on all of $\mathbb{R}$. By the power rule and the chain rule, we say that

$$g_3'(x) = x\left(ax\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$$

We know that $\cos\left(\frac{1}{x}\right)$ is a function that is not differentiable at 0 as we saw in recitation, and by the Algebraic Differentiability Theorem one can show that the sum of a differentiable function and a non-differentiable function will be non-differentiable. Therefore we say that the function $ax\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ is non-differentiable, and thus $g_3'(x)$ has no derivative at 0. Since it is an infinite discontinuity we say the derivative is unbounded.

(b)

We use our proof from part $\boxed{\text{a}}$ as a sufficient answer for this part.

(c)

Choose $a = 4$. This guarentees that g' is continuous and differentiable, since we have

$$g_4'(x) = x^2\left(ax\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$$

Which we can show to be continuous by the squeeze theorem, and differentiable by the algebraic differentiability theorems. However, if we take the second derivative we get

$$lim_{x\to 0}x(ax\sin\frac{1}{x} - \cos\frac{1}{x})$$

We can show that this limit does not exist via methods from recitation.

5.3.1 a

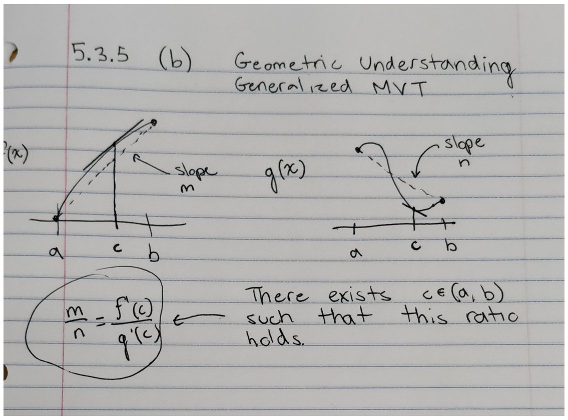
Proof. Suppose f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$. Without loss of generality, let $x < y \in [a, b]$. Then choose the interval $[x, y] \subset [a, b]$. Then by the mean value theorem there exists $c \in [x, y]$ such that $f'(c) = \frac{f(x) - f(y)}{x - y}$. Since f' is continuous on the compact set $[a, b]$ we say that there exists $M > 0$ such that $f'(x) \leqslant M$. Therefore we have

$$\frac{f(x) - f(y)}{x - y} = f'(c) \leqslant M$$

And we say that f is Lipschitz. $\square$

$$lim_{x\to 1}\frac{h(x^2) - h(1)}{x - 1} = lim_{x\to 1}\frac{h(x^2) - h(x)}{x - 1} + lim_{x\to 1}\frac{h(x) - h(1)}{x - 1} = lim_{x\to 1}\frac{h(x^2) - h(x)}{x - 1} + h'(1)$$

5.3.5 b


5.3.5 (b) Geometric Understanding
Generalized MVT
slope
m
g(x)
slope
n
a
c
b
a
c
b
m/n = f'(c)/g'(c)
There exists c ∈ (a, b)
such that this ratio
holds.