Gröbner Bases — Homework 2

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Problem 1

Consider $\mathbb{Q}[x, y, z, t]$, and let

$$\begin{aligned} f_1 &= x - 2y + z + t \\ f_2 &= x + y + 3z + t \\ f_3 &= 2x - y - z - t \\ f_4 &= 2x + 2y + z + t. \end{aligned}$$

Solve our four "important questions/goals" from the introduction for this set of polynomials.

Solution

1

Given some linear function $f \in \mathbb{Q}[x, y, z, t]$, is $f \in \langle f_1, f_2, f_3, f_4 \rangle$? We will start with row reduction,

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 2 & -1 & -1 & -1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 3 & -2 & -2 \\ 0 & 6 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to I$$

Since we have $\langle f_1, f_2, f_3, f_4 \rangle = \langle x, y, z, t \rangle$, it follows that our ideal I is equal to any linear combination of our four variables x, y, z, t. To check that some f is a member of $\langle x, y, z, t \rangle$, we wish to demonstrate that it is only a linear combination of these terms. That is to say, that it does not have a **constant term**. So we can simply check if $f(0, 0, 0, 0) = 0$ and we will then know if there is a constant term. So

$$f \in I \text{ if and only if } f(\mathbf{0}) = 0.$$

2

Let f be some function in our ideal $\langle x, y, z, t \rangle$. How do we write $f = a \cdot x + b \cdot y + c \cdot z + d \cdot t$. Then to compute the coefficients that generate f, we write

$$\begin{aligned} f &= a \cdot x + b \cdot y + c \cdot z + d \cdot t \\ a &= f(1, 0, 0, 0) \\ b &= f(0, 1, 0, 0) \\ c &= f(0, 0, 1, 0) \\ d &= f(0, 0, 0, 1). \end{aligned}$$

Which answers how we write f by the generators. Or we can just look at the coefficients :).

3

What are the coset representatives? If we take $\mathbb{Q}/\langle x, y, z, t \rangle$, then we are taking the quotient on all functions with no constant term. So then the cosets we are left with can be written as

$$c + \langle x, y, z, t \rangle \quad : \quad c \in \mathbb{Q}.$$

That is, they are distinguished only by the constant term. For any two $c \neq c'$ it follows that they represent two distinct cosets. If $c = c'$ then $c + \langle x, y, z, t \rangle = c' + \langle x, y, z, t \rangle$. So it follows that two functions are in the same coset if and only if the constant functions are the same.

4

What is a basis for our vector space $\mathbb{Q}[x, y, z, t] \;/\; \langle x, y, z, t\rangle$? We have a basis given by

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\} + \langle x, y, z, t\rangle.$$

Problem 2

Find a single generator for the ideal $I = \langle x^6 - 1, x^4 + 2x^3 + 2x^2 - 2x - 3\rangle$. Is $x^5 + x^3 + x^2 - 7$ in I? Is $x^4 + 2x^2 - 3$ in I?

Solution

We wish to find the greatest common denominator, so we use the "euclidean algorithm for polynomials" to do so. Let $f = x^6 - 1, g =$. Then in the first pass,

$$\begin{aligned} x^6 - 1 &\xrightarrow{x^4+2x^3+2x^2-2x-3} 2x^3 - 5x^2 - 2x + 5 \\ f &:= x^4 + 2x^3 + 2x^2 - 2x - 3 \\ g &:= 2x^3 - 5x^2 - 2x + 5. \end{aligned}$$

In the second pass,

$$\begin{aligned} x^4 + 2x^3 + 2x^2 - 3 &\xrightarrow{2x^3-5x^2-2x+5} \frac{57}{4}x^2 - \frac{57}{4} \\ f &:= 2x^3 - 5x^2 - 2x + 5 \\ g &:= \frac{57}{4}x^2 - \frac{57}{4}. \end{aligned}$$

In the third pass,

$$\begin{aligned} 2x^3 - 5x^2 - 2x + 5 &\xrightarrow{\frac{57}{4}x^2-\frac{57}{4}} 0 \\ f &:= \frac{57}{4}x^2 - \frac{57}{4} \\ g &:= 0. \end{aligned}$$

So finally let

$$f := \frac{1}{\frac{57}{4}}\left(\frac{57}{4}x^2 - \frac{57}{4}\right) = x^2 - 1.$$

And the algorithm is completed. So, we say that $I = \langle x^2 - 1\rangle$.

$\boxed{x^5 + x^3 + x^2 - 7}$ Recall that $f \in I = \langle g\rangle$ if and only if $f \xrightarrow{g}_+ 0$. So we will divide the one by the other, giving us

$$x^5 + x^3 + x^2 - 7 \xrightarrow{x^2-1} 2x - 6 \neq 0.$$

The function **does not** belong to the ideal.

$\boxed{x^4 + 2x^2 - 3}$ Again, we divide this polynomial by $x^2 - 1$ and see our remainder. That is,

$$x^4 + 2x^2 - 3 \xrightarrow{x^2-1} 0.$$

The function **does** belong to the ideal.