MTH 463 Assignment 4

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November 16, 2020

Problem 1

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{8}{x^3}, & x \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

First we check that $f(x)$ is non-negative. We have $x > 0 \Rightarrow x^3 > 0 \Rightarrow \frac{8}{x^3} > 0$ so our probability is strictly positive for $x \geq 2$, and 0 otherwise. Next we check that $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_2^{\infty} f(x)dx \\
&= \int_2^{\infty} \frac{8}{x^3}dx \\
&= \frac{-4}{x^2}\bigg|_2^{\infty} \\
&= \lim_{n\to\infty} \frac{-4}{x^2}\bigg|_2^{n} \\
&= \lim_{n\to\infty} \frac{-4}{n^2} - \frac{-4}{4} \\
&= 1 - \lim_{n\to\infty} \frac{4}{n^2} \\
&= 1
\end{aligned}$$

The integral is equal to 1 so we have a probability density function.

$$P(X > 5) = 1 - P(X < 5) = 1 - \int_2^5 f(x)dx = 1 - \left[\frac{-4}{25} - \frac{-4}{4}\right] = \frac{4}{25} = .16$$

$$\begin{aligned}
E[X] &= \int_2^{\infty} \frac{8}{x^2}dx \\
&= \frac{-8}{x}\bigg|_2^{\infty} \\
&= \lim_{n\to\infty} \left[\frac{-8}{n} - \frac{-8}{2}\right] \\
&= 4 - \lim_{n\to\infty} \frac{8}{n} \\
&= 4
\end{aligned}$$

Problem 2

Find c and $E[X]$. To find c we make sure the function integrates to 1 over $\mathbb{R}$.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{1}^{2} c(x-1)^4 dx \\
&= \int_{1}^{2} c(x^4 - 4x^3 + 6x^2 - 4x + 1)dx \\
&= c(\frac{x^5}{4} - x^4 + 2x^3 - 2x^2 + x)_1^2 \\
&= c[(\frac{2^5}{4} - 2^4 + 2 \cdot 2^3 - 2 \cdot 2^2 + 2) - (\frac{1}{4} - 1 + 2 \cdot 1 - 2 \cdot 1 + 1)] \\
&= c\frac{7}{4}
\end{aligned}$$

To make this equal 1, let $c = \frac{4}{7}$. To find $E[X]$ we do the following:

$$\begin{aligned}
E[X] &= \int_{1}^{2} \frac{4}{7}x(x^4 - 4x^3 + 6x^2 - 4x + 1)dx \\
&= \frac{4}{7}\int_{1}^{2}(x^5 - 4x^4 + 6x^3 - 4x^2 + x)dx \\
&= \frac{4}{7}\left[\frac{x^6}{6} - \frac{4x^5}{5} + \frac{3x^4}{2} - \frac{4x^3}{3} + \frac{x^2}{2}\right]_1^2 \\
&= \frac{4}{7}\left[\left(\frac{2^6}{6} - \frac{4 \cdot 2^5}{5} + \frac{3 \cdot 2^4}{2} - \frac{4 \cdot 2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{1}{6} - \frac{4}{5} + \frac{3}{2} - \frac{4}{3} + \frac{1}{2}\right)\right] \\
&= \frac{22}{105} = .2095
\end{aligned}$$

Problem 3

We have a system of equations and we want to solve for a and b. First we have $\int_{-\infty}^{\infty} f(x)dx = 1$, second we have $\int_{-\infty}^{\infty} xf(x)dx = .75$.

$$\begin{aligned}
1 &= \int_0^1 ax^2 + bxdx \\
1 &= \frac{ax^3}{3} + \frac{bx^2}{2}\Big|_0^1 \\
1 &= \frac{a}{3} + \frac{b}{2}
\end{aligned}$$

Then we have

$$\begin{aligned}
.75 &= \int_0^1 ax^3 + bx^2dx \\
.75 &= \frac{ax^4}{4} + \frac{bx^3}{3}\Big|_0^1 \\
.75 &= \frac{a}{4} + \frac{b}{3}
\end{aligned}$$

With these two equations we can simply solve for a and b.

$$1 = \frac{a}{3} + \frac{b}{2}$$
$$\implies b = 2 - \frac{2a}{3}$$

$$.75 = \frac{a}{4} + \frac{b}{3}$$
$$.75 = \frac{a}{4} + \frac{2 - \frac{2a}{3}}{3}$$
$$9 = 3a + 4(2 - \frac{2a}{3})$$
$$9 = 3a + 8 - \frac{8a}{3}$$
$$1 = a(3 - \frac{8}{3})$$
$$\frac{1}{3 - \frac{8}{3}} = a = 3 \Rightarrow b = 0$$

Then to compute $E[X^2]$ and $Var(x)$ we must integrate for the expecation of X^2.

$$E[X^2] = \int_0^1 x^2 3x^2 dx$$
$$= \int_0^1 3x^4 dx$$
$$= \frac{3x^5}{5}\Big|_0^1$$
$$= \frac{3}{5} = .6$$

Then we say that $Var(X) = E[X^2] - E[X]^2 = .6 - (.75^2) = .0375$.

Problem 4

Find $P(1 < X < 4)$ and $E[X]$.

$$P(1 < X < 4) = F(4) - F(1) = 1 - (4+1)^{-2} - [1 - (1+1)^{-2}] = .21$$

To find the expectation, we will take the derivative of our CDF to get our PDF, then compute expectation from there.

$$F'(x) = f(x) = \begin{cases} 2(x+1)^{-3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Then we can compute the expectation

$$E[X] = \int_0^\infty 2x(x+1)^{-3} dx$$
$$= 2\int_0^\infty x(x+1)^{-3} dx$$
$$= 2(\frac{1}{2})$$
$$= 1$$

Problem 5

$$x = \frac{-4Y \pm \sqrt{4^2Y^2 - 4(4)(6-Y)}}{8}$$

This will be real valued if the inside of the square root is non-negative, which is the case when

$$\begin{aligned}
0 &\leq 4^2Y^2 - 4(4)(6-Y) \\
0 &\leq Y^2 - 6 + Y \\
0 &\leq Y^2 + Y - 6 \\
\implies Y &\geq 2
\end{aligned}$$

So to compute the probability of this being the case we want $P(Y > 2)$, for an exponential with $\lambda = 3$. This will be equal to the following:

$$\begin{aligned}
P(Y > 2) &= 1 - \int_0^2 3e^{-3x}dx \\
&= 1 - \left[-3\frac{1}{3}e^{-3x}\right]_0^2 \\
&= 1 - \left[-e^{-6} + e^0\right] \\
&= e^{-6} = .00248
\end{aligned}$$

The result is the probability that the polynomial $4x^2 + 4xY - Y + 6 = 0$ has real solutions.

Problem 6

Show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

Proof. We write

$$\begin{aligned}
\alpha\Gamma(\alpha) &= \alpha\left(\int_0^\infty e^{-y}y^{\alpha-1}\right) \\
&= \alpha\left(\frac{y^\alpha}{e^y\alpha}\bigg|_0^\infty - \int_0^\infty \frac{y^\alpha}{\alpha}(-e^{-y})dy\right) \\
&= \frac{y^\alpha}{e^y}\bigg|_0^\infty + \int_0^\infty e^{-y}y^\alpha dy \\
&= \frac{y^\alpha}{e^y}\bigg|_0^\infty + \Gamma(\alpha+1) \\
&= \lim_{n\to\infty}\frac{n^\alpha}{e^n} + \Gamma(\alpha+1) \\
&= 0 + \Gamma(\alpha+1) = \Gamma(\alpha+1)
\end{aligned}$$

□

Compute $\Gamma(1)$.

$$\Gamma(1) = \int_0^\infty e^{-y}y^{\alpha-1}dy = \int_0^\infty e^{-y}dy = 1$$

We know that the integral at the end of that chain must compute to 1 because it is the same as the total probability of an exponential density function with $\lambda = 1$.

Show that $\Gamma(k) = (k-1)!$ for all $k \in \mathbb{N}$.

Proof. We do proof by induction, for the base case $k = 1$ we know $\Gamma(1) = 1 = 0! = (1-1)!$. By induction assume that $\Gamma(k) = (k-1)!$, then it follows that

$$\Gamma(k+1) = k\Gamma(k) = k(k-1)! = k!$$

□

Problem 7

Show that if X is an exponential random variable with $\lambda > 0$,

$$E[X^k] = \frac{k!}{\lambda^k}$$

For all positive integer $k = 1, 2, \cdots$.

Proof. We begin by writing

$$\begin{aligned}
E[X^k] &= \int_0^\infty x^k f(x) dx \\
&= \int_0^\infty x^k \lambda e^{-\lambda x} dx \\
&= \int_0^\infty x^k \frac{\lambda^k}{\lambda^{k-1}} e^{-\lambda x} dx \\
&= \frac{1}{\lambda^{k-1}} \int_0^\infty (x\lambda)^k e^{-\lambda x} dx \\
&= \frac{1}{\lambda^{k-1}} \Gamma(k+1) \frac{1}{\lambda} \\
&= \frac{k!}{\lambda^k}
\end{aligned}$$

□

Problem 8

Let X be a gamma distributed random variable with $\alpha > 0, \lambda > 0$. Compute $E[e^{-X}]$.

$$\begin{aligned}
E[e^{-X}] &= \int_0^\infty e^{-x} \frac{1}{\Gamma(\alpha)} \lambda e^{-\lambda x} (\lambda x)^{\alpha-1} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda e^{-x-\lambda x} (\lambda x)^{\alpha-1} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-x(1+\lambda)} x^{\alpha-1} dx
\end{aligned}$$

Now let $y = x(\lambda+1), dy = dx(\lambda+1)$, and we make the following substitution

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-x(1+\lambda)} x^{\alpha-1} dx &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-y} (\frac{y}{\lambda+1})^{\alpha-1} \frac{1}{\lambda+1} dy \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda+1)^\alpha} \int_0^\infty e^{-y} y^{\alpha-1} dy \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda+1)^\alpha} \cdot \Gamma(\alpha) \\
&= \frac{\lambda^\alpha}{(\lambda+1)^\alpha} \\
&= \left(\frac{\lambda}{\lambda+1}\right)^\alpha
\end{aligned}$$

Problem 9

Let X be an exponential random variable, show that its hazard funciton $h(t)$ will be constant.

Proof. We wish to show that

$$h(t) = \frac{f(t)}{1-F(t)} = x, \quad \forall t$$

Let us first write out $f(t)$ and $F(t)$ explicitly,

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & 0 \le t < \infty \\ 0, & \text{otherwise} \end{cases}$$

Then if $y \leq 0, F(y) = 0$, and otherwise

$$\begin{aligned}
F(t) &= \int_{-\infty}^{t} f(y)dy \\
&= \int_{0}^{t} \lambda e^{-\lambda y} dy \\
&= \lambda \int_{0}^{t} e^{-\lambda y} dy \\
&= \lambda \int_{0}^{-\lambda t} -\frac{1}{\lambda} e^{u} du \\
&= -\int_{0}^{-\lambda t} e^{u} du \\
&= -[e^{-\lambda t} - e^{0}] \\
&= -[e^{-\lambda t} - 1] \\
&= 1 - e^{-\lambda t}
\end{aligned}$$

So we say

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\lambda t} & t \geq 0 \end{cases}$$

For any non-negative value of t, we have

$$\begin{aligned}
h(t) &= \frac{\lambda e^{-\lambda t}}{1 - [1 - e^{-\lambda t}]} \\
&= \lambda \frac{e^{-\lambda t}}{e^{-\lambda t}} \\
&= \lambda e^{-\lambda t + \lambda t} \\
&= \lambda
\end{aligned}$$

And we say that $h(t)$ is constant. $\square$