Gröbner Bases — Homework 6

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Problem 1

In $\mathbb{Q}[x, y, x]$ using lex with $x > y > z$, let $I = \langle x^2+z, xy+y^2+z, xz-y^3-2yz, y^4+3y^2z+z^2\rangle$ and $J = \langle x^2+z, xy+y^2+z, x^3-yz\rangle$.

1. Compute the reduced Gröbner bases for I and J and use them to determine whether $I = J$.
2. Show that $y^4 + 3y^2z + z^2 \in J$ and compute the $u_1, u_2, u_3 \in \mathbb{Q}[x, y, x]$ such that $y^4 + 3y^2z + z^2 = u_1(x^2 + z) + u_2(xy + y^2 + z) + u_3(x^3 - yz)$.

Solution

Using computer algebra from sage math, we find a Gröbner basis for I given by

$$\{x^2 + z, xy + y^2 + z, xz - y^3 - 2yz, y^4 + 3y^2z + z^2\}$$

and a Gröbner basis for J given by

$$\{x^2 + z, xy + y^2 + z, xz + yz, y^3 + 3yz, y^2z, z^2\}.$$

We can see that this differ greatly even up to cardinality, so the two are not equal. Using computer algebra once again we can see that $y^4 + 3y^2z + z^2 \in J$, and for u_i we get

$$\begin{aligned} u_1 &= y^2 \\ u_2 &= -xy + y^2 + z \\ u_3 &= 0 \end{aligned}$$

Problem 2

In $\mathbb{Q}[x, y, z]$, let $A = \langle x^4y^2 + z^2 - 4xy^3z - 2y^5z, x^2 + 2xy^2 + y^4\rangle$. Let $f = yz - x^3$.

1. Show that $f \in \sqrt{A}$ using our theorem from class (instead of the definition).
2. Find the least power of f that lies in A (i.e., show $f \in \sqrt{A}$ using the definition).

Solution

We take the ideal generated by $\langle A, 1 - wf\rangle$ and verify that it has a reduced Gröbner basis of exactly $\{1\}$. Using sage, we do just that, and verify that this is true. We check that both f and f^2 are not in A. However, f^3 is, so the least power of f such that $f^n \in A$ is $n = 3$.