MTH 311 Homework 8

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4.4.9

(a)

Show that a Lipschitz function is uniformly continuous.

Proof. Suppose $f : A \to \mathbb{R}$ is a Lipschitz function, i.e. there exists $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leqslant M$$

for all $x \neq y \in A$. We can multiply the above inequality and get $|f(x) - f(y)| \leqslant M|x - y|$. Let $\epsilon > 0$ be arbitrary, and let $\delta = \dfrac{\epsilon}{M}$. Then if we have $|x - y| < \dfrac{\epsilon}{M}$, we can mutliply by M and get

$$|f(x) - f(y)| \leqslant M|x - y| < M\delta = M\frac{\epsilon}{M} = \epsilon$$

□

(b)

Is the converse true? No.

Proof. Take the funciton $f(x) = \sqrt{x}$ on $[0, 1]$. We say that it is uniformly continuous because it is continuous on a compact set. Now, since the definition must apply for all $x, y \in A$ choose $x = \dfrac{1}{n} \in [0, 1]$ which works for any $n \in \mathbb{N}$ and $y = 0$. Then

$$\left| \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}} \right| = \sqrt{n}$$

Since $\sqrt{n}$ is not bounded we say that the function is not Lipschitz. □

4.5.5 b

Proof. We continue this proof from where the book left off. We have an interval $I_1 = [a_1, b_1]$ such that $f(a_1) < 0$ and $f(b_1) \geqslant 0$. We generalize the process by taking $z_n = \frac{a_n + b_n}{2}$, and then if $f(z_n) > 0$ then $b_{n+1} = z_n$. If $f(z_n) < 0$ then $a_{n+1} = z_n$. Otherwise, $z_n \in [a, b]$ and $f(z_n) = 0$, and we are done. By the inductive step, we assume that we have $I_n = [a_n, b_n]$, where $f(a_n) < 0$ and $f(b_n) > 0$. Then, we can find a midpoint z_n and create the interval I_{n+1} such that $f(a_{n+1}) < 0$ and $f(b_{n+1}) > 0$.

We say that the length of this interval is equal to $\frac{a-b}{2^n}$, since each time n increases by one we cut the interval in half. Then we take the sequence of intervals I_n for all $n \in \mathbb{N}$, and it follows that the length of I_n approaches 0. Since the length of I_n approaches zero, and by the Nested Interval Property the infinite intersection of I_n is non-empty, we say that there is exactly one number c such that $\bigcap_{n \in \mathbb{N}} I_n = \{c\}$.

Suppose that $f(c) > 0$. Then there would be some $n \in \mathbb{N}$ such that $f(I_n) > 0$ i.e. $f(a_n) > 0$ and $f(b_n) > 0$. This is not possible by the process in which we chose $f(a_n)$, therefore $c \leqslant 0$. Now suppose $f(c) < 0$. Then, similarly there must be some interval where $f(b_n) < 0$, which again is not possible. Thus $f(c) \geqslant 0$. Since we have shown $f(c)$ not to be strictly positive or strictly negative, it follows that $f(c) = 0$. □

5.2.5

(a)

The function f is continuous at 0 for $a > 0$.

Proof. 0 is a limit point of $\mathbb{R}$. Thus, we can use the functional limit characterization of continuity and say that if $lim_{x\to 0} f(x) = f(0)$ then f is continuous at 0. Suppose $a < 0$. Then, this limit is undefined as it diverges. For $a = 0$, the limit is equal to $1 \neq 0$. Therefore a must be greater than 0. Assume that $a > 0$, then $lim_{x\to 0} x^a = 0 = f(0)$. Therefore the function is continuos at 0 assuming $a > 0$. □

(b)

$a > 1$.

5.2.7

Proof. □

5.3.1 a

Proof. Suppose f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$. Without loss of generality, let $x < y \in [a, b]$. Then choose the interval $[x, y] \subset [a, b]$. Then by the mean value theorem there exists $c \in [x, y]$ such that $f'(c) = \dfrac{f(x) - f(y)}{x - y}$. Since f' is continuous on the compact set $[a, b]$ we say that there exists $M > 0$ such that $f'(x) \leqslant M$. Therefore we have

$$\frac{f(x) - f(y)}{x - y} = f'(c) \leqslant M$$

And we say that f is Lipschitz. □

5.3.5 b