ST 421 Assignment 5

Philip Warton

November 2, 2020

Problem 3.148

We know that $E(Y) = \mu = \mu_1' = m^{(1)}(0)$. Then we know that $m(t) = \frac{pe^t}{1-qe^t}$. So we take the first derivative of $m(t)$ which gives us the following:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right) &= \frac{(1-qe^t)pe^t - (pe^t)(-qe^t)}{(1-qe^t)^2} \\
&= \frac{pe^t - pqe^te^t + qpe^te^t}{(1-qe^t)^2} \\
&= \frac{pe^t}{(1-qe^t)^2}
\end{aligned}$$

Now when $t = 0$ we know that $m'(t) = \frac{p}{(1-q)^2} = \frac{1}{p} = E(Y)$.

To find $E(Y^2)$ we take the second derivative of the moment generating function at 0. So we say:

$$\frac{d}{dt}\left(\frac{pe^t}{(1-qe^t)^2}\right) = \frac{(1-qe^t)^2pe^t - pe^t(2)(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

So we can look at $m''(0) = \frac{(1-q)^2p - p(2)(1-q)(-q)}{(q-q)^4} = \frac{p^3+2p^2q}{p^4} = \frac{2-p}{p^2} = \mu_2' = E(Y^2)$. Then simply take $V(Y) = E(Y^2) - E(Y)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$.

Problem 3.154

$m(t) = \left(\frac{e^t+2}{3}\right)^5$

We take the derivative of the moment generative funciton, giving us

$$m'(t) = 5\left(\frac{e^t+2}{3}\right)^4\left(\frac{e^t}{3}\right)$$

Then evaluated at 0 we have $m'(0) = 5(1)^4(\frac{1}{3}) = \frac{5}{3}$. Now we take the second derivative to be

$$m''(t) = 5\left[\left(\frac{e^t+2}{3}\right)^4\left(\frac{e^t}{3}\right) + 4\left(\frac{e^t+2}{3}\right)^3\left(\frac{e^t}{3}\right)\left(\frac{e^t}{3}\right)\right]$$

Then we compute $m''(0) = \frac{35}{9}$. From here we simple state that the mean is $\frac{5}{3}$, and that the variance is $\frac{35}{9} - \frac{25}{9} = \frac{10}{9}$.

$m(t) = \frac{e^t}{2-e^t}$

$$m'(t) = \frac{(2-e^t)(e^t) - e^t(-e^t)}{(2-e^t)^2} = \frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2} = \frac{2e^t}{(2-e^t)^2}$$

We evaluate $m'(0) = 2$.

$$m''(t) = \frac{2e^t(2-e^t)^2 - 2e^t(2)(2-e^t)(-e^t)}{(2-e^t)^4}$$

We write $m''(0) = 0 = 6$. Our mean will simply be $\mu = 2$ and our variance will be $\sigma^2 = 6 - 4 = 2$.

$m(t) = e^{2(e^t-1)}$

We do the same process once again, yielding the following:

$$m'(t) = e^{2(e^t-1)}(2e^t)$$

Which we evaluate at 0 and get $m'(0) = 2 = \mu$. Then we do the same for the second derivative.

$$m''(t) = e^{2(e^t-1)}(2e^t) + 2e^t(e^{2(e^t-1)}(2e^t))$$

This evaluated at 0 is $m''(0) = 2 + 4 = 6$. So we say that the mean is $\mu = 2$, and our variance is $\sigma^2 = 6 - 2^2 = 2$.

Problem 3.158

Proof. Let $m_Y(t) = E(e^{tY})$, $m_W(t) = E(e^{tW})$, and $W = aY + B$. We wish to show that $m_W(t) = e^{tb} m_Y(ta)$.

$$\begin{aligned}
m_W(t) &= E(e^{tW}) \\
&= E(e^{t(aY+b)}) \\
&= E(e^{taY+tb}) \\
&= E(e^{taY}e^{tb}) \\
&= e^{tb}E(e^{taY}) \\
&= e^{tb}m_Y(ta)
\end{aligned}$$

□

Problem 3.160

Suppose that Y is a binomial random variable based on n trials with success probability p and let $Y^* = n - Y$.

(a)

Show that $E(Y^*) = nq$ and $V(Y^*) = npq$ where $q = 1 - p$.

Proof. We begin by writing $E(Y^*) = E(n - Y) = E(n) - E(Y)$. Then using the result from $\boxed{\text{Problem 3.159}}$ that $E(n) - E(Y) = n - np = n(1 - p) = nq$. Then we say

$$\begin{aligned}
V(Y^*) &= V(n - Y) \\
&= V((-1)Y + n) \\
&= (-1)^2 V(Y) && \text{(Problem 3.159)} \\
&= npq
\end{aligned}$$

□

(b)

We want to show that $m^*(t) = (qe^t + p)^n$ is the moment generating function of Y^*.

Proof. We use the result from $\boxed{3.158}$ and state that $m^*(t) = e^{tn}m(-t) = e^{tn}(q + pe^{-t})^n$. Then we put rewrite as

$$\begin{aligned}
m^*(t) &= e^t(q + pe^{-t}) \\
&= (e^t(q + pe^{-t}))^n \\
&= (e^t q + p)^n
\end{aligned}$$

And we have the desired result. □

(c)

After computing the mean, variance, and moment generating function for Y^*, we see the similarities between Y and Y^*. The distribution of Y^* is clearly a binomial distribution with a probability of q for success of every trial.

(d)

We can interpret Y^* as the number of failed trials given a success probability p, given n trials.

(e)

These answers are now "obvious" because the probability of failure is simply $q = 1 - p$, so we can count the number of failures as the number of successes with the q probability rather than the p probability.