# General Topology and Fundamental Groups - Homework 6

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## Problem 1

### a)

Let $\pi : X \to Y$ be continuous, and $\sigma : Y \to X$ such that $\pi \circ \sigma = Id_Y$. Show that $\pi$ is a quotient map.

*Proof.* This proof will consist of showing the following conditions: $\pi$ is surjective, $U \subset Y$ is open if and only if $\pi^{-1}(U) \subset X$ is open.

$\pi$ is a surjection.

Suppose by contradiction that $\pi$ is not a surjection. Then $\exists y \in Y$ such that $\pi^{-1}(y) = \emptyset$. However we know that

$$\pi \circ \sigma(y) = Id_Y(y) = y$$

However $\sigma(y)$ is an element of $X$ such that $\pi$ maps it to $y$ thus $\sigma(y) \in \pi^{-1}(y)$, and the set is not empty (contradiction). Thus $\pi$ is surjective.

$U \subset Y$ is open if and only if $\pi^{-1}(U) \subset X$ is open.

Since $\pi$ is assumed continuous, we have $U \subset Y$ open implies $\pi^{-1}(U) \subset X$ open already granted. Now assume that $U \subset Y$ is some set such that $\pi^{-1}(U)$ is open. Then since $\sigma$ is continuous it follows that $\sigma^{-1}(\pi^{-1}(U))$ is open in $Y$. Then we can write

$$\begin{aligned}
\sigma^{-1}(\pi^{-1}(U)) &= Id_Y(\sigma^{-1}(\pi^{-1}(U)) \\
&= \pi(\sigma(\sigma^{-1}(\pi^{-1}(U))) \\
&\subset \pi(\pi^{-1}(U)) = U && \text{(since } \pi \text{ is injective)}
\end{aligned}$$

Then by set theory we get the following result,

$$\begin{aligned}
U &= U \\
Id_Y(U) &= U \\
\pi(\sigma(U)) &= U \\
\pi^{-1}(\pi(\sigma(U))) &= \pi^{-1}(U) \\
\sigma^{-1}(\pi^{-1}(\pi(\sigma(U)))) &= \sigma^{-1}(\pi^{-1}(U)) \\
U \subset \sigma^{-1}(\sigma(U)) \subset \sigma^{-1}(\pi^{-1}(\pi(\sigma(U)))) &= \sigma^{-1}(\pi^{-1}(U))
\end{aligned}$$

Since the two are subsets of each other, we get

$$U = \sigma^{-1}(\pi^{-1}(U))$$

So it follows that, of course, $U$ is an open set in $Y$. Finally having shown both of these conditions, we say that $\pi$ is a quotient map. $\square$

### b)

Let $A \subset X$ be equipped with the subspace topology. A retraction of $X$ onto $A$ is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Show that any retraction $r$ is a quotient map.

*Proof.* We know that $a \in r^{-1}(a)$ for any $a \in A$, which gives us $r$ is an injection trivially. Then we wish to show that $r^{-1}(U)$ being open in $X$ implies $U$ is open in $A$. Note that since $r$ is surjective and for any subset $U \subset A, r(U) = U$, it must be the case that

$$\begin{aligned} U \cap A &= U \\ r(r^{-1}(U)) \cap r(A) &= r(U) \\ r^{-1}(U) \cap A &= U \end{aligned}$$

Then since $r^{-1}(U)$ is open by assumption, $U$ is open in the subspace topology of $A$ by definition. Thus it follows that $r$ is a quotient map by the critereon met for $\boxed{\text{1a}}$. $\square$

# Problem 2

> Let $\pi : X \to Y$ be a quotient map. Suppose that the saturation of any open set in $X$ is open. Show that $\pi$ is an open map. Does the analogous statement hold with closed sets and closed maps?

$\boxed{\text{Open}}$

*Proof.* Suppose by contradiction that $\pi$ is not an open map. Then $\exists U \subset X$ open such that $\pi(U)$ is not open. Denote $V = \pi(U)$. Then since $\pi$ is a quotient map we know that $V \subset Y$ is open if and only if $\pi^{-1}(V)$ is open. That is, if $V$ is not open then we have $\pi^{-1}(\pi(U))$ is not open in $X$ (contradiction). $\square$

Now for closed sets.

$\boxed{\text{Closed}}$

We repeat the same argument for $\pi$ being a closed maps, since the condition of quotient maps stating $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open yields a similar property of closed sets. Let us first prove this fact.

*Proof.* Let $q : X \to Y$ be a quotient map. We want to show that $C \subset Y$ is closed if and only if $q^{-1}(C) \subset X$ is closed.

$\boxed{\Rightarrow}$ Let $C \subset Y$ be an arbitrary closed set. We know that $q$ is continuous, thus $q^{-1}(C)$ is closed.

$\boxed{\Leftarrow}$ Let $q^{-1}(C)$ be a closed saturated set in $X$. Then we know

$$\begin{aligned} X \setminus q^{-1}(C) &= q^{-1}(Y) \setminus q^{-1}(C) \\ &= q^{-1}(Y \setminus C) \end{aligned}$$

Since $q^{-1}(Y \setminus C)$ is an open set in $X$ it must be the case that $Y \setminus C$ is open in $Y$, hence $C$ is closed. $\square$

Now the same argument for open sets should follow quite easily.

*Proof.* Suppose that $\pi$ is not a closed map. Then $\exists C \subset X$ that is closed such that $\pi(C)$ is not. Denote $D = \pi(C)$. Then we know that $D$ is closed if and only if $\pi^{-1}(D)$ is closed. Thus if $D$ is not closed $\pi^{-1}(D) = \pi^{-1}(\pi(C))$ is not closed. However, this contradicts the assumption that for every closed set $C$ its saturation $\pi^{-1}(\pi(C))$ is closed. $\square$

# Problem 3

## a)

> Let $\pi_i : X_i \to Y_i, i = 1, 2, \cdots, n$ be continuous, surjective, and open. Show that the map $\pi = \pi_1 \times \pi_2 \times \cdots \times \pi_n : X_1 \times X_2 \times \cdots \times X_n \to Y_1 \times Y_2 \times \cdots \times Y_n$ is a quotient map.

*Proof.* We will prove this for the case of a product of two spaces, and naturally since $\pi$ maps topological spaces, this will clearly extend to any finite product. We start by writing

$$\pi = \pi_1 \times \pi_2 : X_1 \times X_2 \to Y_1 \times Y_2$$

$\boxed{\text{Continuity}}$

Let $U \subset Y_1 \times Y_2$ be open, it can be written as $U = U_1 \times U_2$ where $U_1 \subset Y_1$ and $U_2 \subset Y_2$ are open. Then

$$\begin{aligned}\pi^{-1}(U) = \{(a,b) : \pi(a,b) \in U_1 \times U_2\} \\ &= \{(a,b) : a \in \pi_1^{-1}(U_1), b \in \pi_1^{-2}(U_2)\} \\ &= \pi_1^{-1}(U_1) \times \pi_2^{-1}(U_2)\end{aligned}$$

Since this is a product of open sets in $X_1, X_2$ respectively, it is an open set. Therefore $\pi$ is continuous.

$\boxed{\text{Surjectivity}}$

Let $(y_1, y_2) \in Y_1 \times Y_2$. Since $\pi_1, \pi_2$ are surejctive, $\exists x_1 \in X_1, x_2 \in X_2$ such that $\pi_1(x_1) = y_1, \pi_2(x_2) = y_2$ therefore $\pi(x_1, x_2) = (y_1, y_2)$. So we have shown that $\pi$ is surjective.

$\boxed{\text{Open Map Property}}$

Let $U \subset X_1, V \subset X_2$ be open and equivalently $U \times V \subset X_1 \times X_2$ be open. Then we say $\pi(U \times V) = \pi_1(U) \times \pi_2(V)$. Since $\pi_i$ are open maps, we have a product of open sets, thus an open set. Therefore $\pi$ is an open map. □

### b)

| Let $X$ be T2 and suppose that $K \subset X$ is compact. Show that $X/K$ is T2. |
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*Proof.* Let $x \neq y \in X/K$. If both are equal to $K$, then the points are not distinct. If neither are equal to $K$, then they must be singleton points in $X \setminus K$. Then since they are points in $X$ which is $T_2$, choose two neighborhoods $U, V$ of $x, y$ respectively such that they are disjoint. Then since $K$ is compact in a $T_2$ space, we say that $K$ must be closed. So we can take the following sets,

$$U \cap (X \setminus K), \quad V \cap (X \setminus K)$$

These will be open neighborhoods of $x$ and $y$ that are disjoint from each other, and also disjoint from $K$, so they will remain disjoint under this quotient topology. Thus we have two disjoint neighborhoods of $x$ and $y$ in $X/K$. Suppose that $x = K, y \neq K$. Then we must show that distinct neighborhoods around $K$, and some point $y \in X \setminus K$ both exist. Let $U(a)$ be a set around a point $a \in K$ such that it is disjoint from a set $V(y)$ which is a neighborhood of $Y$. Since $K$ is compact, we know

$$K = \bigcup_{a \in K} U(a) \cap K = \bigcup_{i=1}^{n} U(a_i) \cap K$$

Then for each set $U(a_i)$, we know that there is some $V_i(y)$ that is disjoint from it. So then it follows that the following sets will be the distinct open neighborhoods we require,

$$K \subset \bigcup_{i=1}^{n} U(a_i), \quad y \in \bigcap_{i=1}^{n} V_i(y)$$

Thus under the quotient map, these sets will remain disjoint because only one intersects $K$ and both are disjoint in $X$. Thus for any two distinct points in $X/K$ we have disjoint open neighborhoods of each, and we say that the space $X/K$ is itself $T_2$. □

## Problem 4

The $K$-topology $\mathbb{R}_K$ on the real axis is generated by the basis consisting of all open intervals $(a, b), a < b$, and the sets $(a, b) \setminus K$, where $K = \{1/n | n \in \mathbb{N}\}$. Let $\mathbb{R}_K/K$ be equipped with the quotient topology and let $\pi$ denote the quotient map.

### a)

| Show that $\mathbb{R}_K/K$ is T1 but not T2. |
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*Proof.* Let $x, y \in \mathbb{R}_K/K$. We want to show that there exists some neighborhood $U$ containing $x$ and not $y$. If $x, y \neq [K]$, then choose $U = \mathbb{R} \setminus \{y\}$. Since $[K] \in U$, we know that its pre-image $\pi^{-1}(U) = \mathbb{R} \setminus \{y\} = (-\infty, y) \cup (y, \infty)$ which is open in $\mathbb{R}_K$. Thus $U \subset \mathbb{R}_K/K$ is open in the quotient space and contains $x$ and not $y$. Now suppose that $x = [K]$. The same arugment still holds since $U = \mathbb{R} \setminus \{y\}$ still contains $K$. Suppose that $y = [K]$, then let $a < x < b$ such that $a, b \notin K$. Take the set $U = (a, b) \setminus K$. Clearly the set contains $x$ and not $y$. Since it is disjoint from $K$ its pre-image is equal to the set so $\pi^{-1}(U) = U = (a, b) \setminus K$. This set is clearly open in $\mathbb{R}_K$ thus $U \subset \mathbb{R}_K/K$ is open.

To show that it is not T2, take the points $0, K \in \mathbb{R}_K/K$. Since every $1/n$ is contained in $K$, for each of these we must have some interval of the form $(a, b) \ni 1/n$ that lies within any open neighborhood of $K$, since open intervals of the other form are disjoint from $K$. Then for any $\epsilon > 0$ there exists a point not of the form $1/n$ in the interval $(0, \epsilon)$. Thus it follows that any open neighborhood of $0$, even of the form $(a, b) \setminus K$ will still intersect any open neighborhood of $K$. Therefore these two points are not completely seperated, and $\mathbb{R}_K/K$ is not T2. $\square$

## b)

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| Show that $\pi \times \pi : \mathbb{R}_K \times \mathbb{R}_K \to \mathbb{R}_K/K \times \mathbb{R}_K/K$ is not a quotient map. |

*Proof.* We know (by a previous HW problem) that the diagonal, denoted by $\Delta$, of a product space is closed if and only if the original space is Hausdorff (that is, T2). So it follows that since $\mathbb{R}_K/K$ is not T2, $\Delta$ is not closed. Take $(\pi \times \pi)^{-1}(\Delta)$. This will be the set

$$\{(x, y) \in \mathbb{R}_K \times \mathbb{R}_K : \pi(x) = \pi(y)\}$$

This will consist of all pairs of identical points obviously, and also of all points in $K \times K$, since each of these are appended to each other. So we can write the set as $K \times K \cup \{(x, x) \in \mathbb{R}^2\}$. We argue that this set is equal to its closure in $\mathbb{R}^2_K$. Let $(x, y) \in \mathbb{R}^2_K \setminus (\pi \times \pi)^{-1}(\Delta)$. Then if neither $x$ or $y$ is in $K$, then pick two disjoint neighborhoods of $x$ and $y$ which are also both disjoint from $K$, that is pick $U \ni x, V \ni y$ such that $U \cap V = \emptyset, U \cap K = \emptyset, V \cap K = \emptyset$. This is guaranteed since $K$ is closed, so take two disjoint neighborhoods and intersect them with the complement of $K$. Then it follows that $(x, y) \in U \times V$ and that this is a neighborhood of that point disjoint from $(\pi \times \pi)^{-1}(\Delta)$. Suppose that $x$ or $y$ belongs to $K$. Then take two disjoint intervals $x \in (a, b), y \in (c, d)$ such that $y \notin (a, b), x \notin (c, d)$. Also we add that whichever interval contains the point not in $K$ be restricted to $K$'s complement. Then if $x$ belongs to $K$, we have the neighborhood

$$(a, b) \times ((c, d) \setminus K)$$

If $y \in K$ we take

$$((a, b) \setminus K) \times (c, d)$$

And it follows that these are disjoint from $(\pi \times \pi)^{-1}(\Delta)$ by construction. Therefore any point in the complement of the pre-image of the diagonal has a neighborhood that is contained within the complement of the pre-image of the diagonal. Therefore the pre-image of the diagonal is closed. So we say that $\Delta$ is not closed and $(\pi \times \pi)^{-1}(\Delta)$ is, therefore $\pi \times \pi$ is not a quotient map. $\square$