Applied Ordinary Differential Equations - Homework 1

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7.5.3

(b)

Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

$$\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \boldsymbol{x}$$

We begin by writing the characteristic polynomial,

$$(2-\lambda)(-2-\lambda) - (-1)(3) = \lambda^2$$

So we have two real and distinct eigenvalues $\lambda_1 = -1, \lambda_2 = 1$. We can write

$$[A - \lambda_1(I)]\boldsymbol{u} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 3u_1 - u_2 \\ 3u_1 - u_2 \end{pmatrix} = \boldsymbol{0}$$

This gives us an eigenvector of $\boldsymbol{u} = (1,\ 3)^T$. Then we write

$$[A - \lambda_2(I)]\boldsymbol{v} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ 3v_1 - 3v_2 \end{pmatrix} = \boldsymbol{0}$$

And then we have an eigenvector $\boldsymbol{v} = (1,\ \ -1)^T$. This leads to a general solution,

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

7.5.7

Find the general solution of the given system of equations

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \boldsymbol{x}$$

We begin by solving for eigenvalues,

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{pmatrix} &= (1-\lambda)[(2-\lambda)(1-\lambda) - 1] - (1)[(1)(1-\lambda) - (1)(2)] + (2)[(1)(1) - (2)(2-\lambda)] \\ &= -\lambda^3 + 4\lambda^2 + \lambda - 4 \end{aligned}$$

This gives us three eigenvalues $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 4$. After performing row reduction for the respective kernels of $A - \lambda I$. This yields the following eigenvectors,

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \ \boldsymbol{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \ \ \boldsymbol{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

7.5.20

Consider a 2×2 system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$. If we assume that $r_1 \neq r_2$, the general solution is $\boldsymbol{x} = c_1\xi^{(1)}e^{r_1 t} + c_2\xi^{(2)}e^{r_2 t}$, provided that $\xi^{(1)}$ and $\xi^{(2)}$ by assuming that they are linearly dependent and then showing that this leads to a contradiction.

(a)

Explain how we know that $\xi^{(1)}$ satisfies the matrix equation $(A - r_1 I)\xi^{(1)} = \mathbf{0}$. Similarly, explain why $(A - r_2 I)\xi^{(2)} = \mathbf{0}$.

We know that r_i is the eigenvalue corresponding to the eigenvector $\xi^{(i)}$. We also know that $A\xi^{(i)} = r_i\xi^{(i)}$. So then we expand our term to

$$(A - r_i I)\xi^{(i)} = A\xi^{(i)} - r_i I\xi^{(i)} = r_i\xi^{(i)} - r_i\xi^{(i)} = \mathbf{0}$$

(b)

Show that $(A - r_2 I)\xi^{(1)} = (r_1 - r_2)\xi^{(1)}$.

We start by expanding the LHS term to give us

$$\begin{aligned}(A - r_2 I)\xi^{(1)} &= A\xi^{(1)} - r_2 I\xi^{(1)} \\ &= r_1\xi^{(1)} - r_2\xi^{(1)} \\ &= (r_1 - r_2)\xi^{(1)}\end{aligned}$$

(c)

Suppose that $\xi^{(1)}$ and $\xi^{(2)}$ are linearly dependent. Then $c_1\xi^{(1)} + c_2\xi^{(2)} = \mathbf{0}$ and at least one of c_1 and c_2 (say, c_1) is not zero. Show that $(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)}) = \mathbf{0}$, and also show that $(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)}) = c_1(r_1 - r_2)\xi^{(1)}$. Hence $c_1 = 0$, which is a contradiction. Therefore, $\xi^{(1)}$ and $\xi^{(2)}$ are linearly independent.

We can expand a certain term as follows:

$$\begin{aligned}(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)}) &= Ac_1\xi^{(1)} + Ac_2\xi^{(2)} - r_2 I c_1\xi^{(1)} - r_2 I c_2\xi^{(2)} \\ &= c_1 A\xi^{(1)} + c_2 A\xi^{(2)} - r_2 c_1 I\xi^{(1)} - r_2 c_2 I\xi^{(2)} \\ &= c_1 r_1\xi^{(1)} + c_2 r_2\xi^{(2)} - c_1 r_2\xi^{(1)} - c_2 r_2\xi^{(2)} \\ &= c_1 r_1\xi^{(1)} - c_1 r_2\xi^{(1)} + c_2 r_2\xi^{(2)} - c_2 r_2\xi^{(2)} \\ &= c_1 r_1\xi^{(1)} - c_1 r_2\xi^{(1)} \\ &= c_1\xi^{(1)}(r_1 - r_2)\end{aligned}$$

However, by assumption, $c_1\xi^{(1)} + c_2\xi^{(2)} = \mathbf{0}$, so we know that

$$(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)}) = (A - r_2 I)(\mathbf{0}) = \mathbf{0}$$

Therefore we know that $\mathbf{0} = c_1\xi^{(1)}(r_1 - r_2)$, and we know that $r_1 - r_2$ is non-zero, so it follows that c_1 must 0. This is a contradiction to our assumption, so we conclude that $\xi^{(1)}$ and $\xi^{(2)}$ are linearly independent.

(d)

Modify the argument of part (c) if we assume that $c_2 \neq 0$.

The argument holds still.

(e)

Carry out a similar argument for the case where A is 3×3.

Proof. Assume that $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}$ are dependent. Then for any $i, j \in \{1, 2, 3\}$ we can write the following: Firstly, simlarly to part (a),

$$(A - r_i I)\xi^{(i)} = r_i\xi^{(i)} - r_i\xi^{(1)} = 0$$

Secondly, similarly to part (b),

$$(A - r_i I)\xi^{(j)} = r_j\xi^{(j)} - r_i\xi^{(j)} = (r_j - r_i)\xi^{(j)}$$

Finally, assume there exists some non-trivial solution to $c_1\xi^{(1)} + c_2\xi^{(2)} + c_3\xi^{(3)} = 0$. Without loss of generality, suppose that $c_1 \neq 0$. Now we write that

$$(A - r_3 I)(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)} + c_3\xi^{(3)}) = 0$$

And this is true trivially because $c_1\xi^{(1)} + c_2\xi^{(2)} + c_3\xi^{(3)} = 0$ by assumption. However, now we can also write

$$\begin{aligned}
&(A - r_3 I)(A - r_2 I)(c_1\xi^{(1)} + c_2\xi^{(2)} + c_3\xi^{(3)}) \\
\\
&= (A - r_3 I)(c_1 r_1\xi^{(1)} + c_2 r_2\xi^{(2)} + c_3 r_3\xi^{(3)} - c_1 r_2\xi^{(1)} - c_2 r_2\xi^{(2)} - c_3 r_2\xi^{(3)}) \\
\\
&= (A - r_3 I)(c_1(r_1 - r_2)\xi^{(1)} + c_3(r_3 - r_2)\xi^{(3)}) \\
\\
&= [c_1 r_1(r_1 - r_2)\xi^{(1)} + c_3 r_3(r_3 - r_2)\xi^{(3)}] \\
&\ - [c_1 r_3(r_1 - r_2)\xi^{(1)} + c_3 r_3(r_3 - r_2)\xi^{(3)}] \\
\\
&= c_1 r_1(r_1 - r_2)\xi^{(1)} - c_1 r_3(r_1 - r_2)\xi^{(1)} \\
\\
&= c_1(r_1 - r_3)(r_1 - r_2)\xi^{(1)}
\end{aligned}$$

So from this and from our previous assertion that the same term was equal to 0, we can write,

$$c_1(r_1 - r_3)(r_1 - r_2)\xi^{(1)} = 0 \implies c_1 = 0$$

This is true because we assume our eigenvalues to be distinct. However this clearly leads to the same contradiction as within our 2×2 case. So we conclude that our eigenvectors must be linearly independent. □

7.5.21

(a)

Take the ordinary differntial equation

$$ay'' + by' + cy = 0$$

Let $x_1 = y, x_2 = y'$. Then,

$$\begin{aligned}
ax_2' + bx_2 + cx_1 &= 0 \\
x_2' &= -(bx_2 + cx_1)/a \\
x_2' &= -\frac{b}{a}x_2 - \frac{c}{a}x_1
\end{aligned}$$

Also,

$$\begin{aligned}
ax_2' + bx_1' + cx_1 &= 0 \\
x_1' &= -(ax_2' + cx_1)/b \\
&= -(a(\frac{-bx_2}{a} - \frac{cx_1}{a}) + cx_1)/b \\
&= -(-bx_2 - cx_1 + cx_1)/b \\
&= bx_2/b \\
&= x_2
\end{aligned}$$

So we can now write this differential equation as a system of first order ODE's of the form $\boldsymbol{x}' = A\boldsymbol{x}$. We can write

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \boldsymbol{x}$$

(b)

To find the roots of the characteristic polynomial we want to compute $\det(A - \lambda I) = 0$. That is,

$$\begin{aligned}
\det \begin{pmatrix} 0-\lambda & 1 \\ -c/a & -b/a-\lambda \end{pmatrix} &= 0 \\
(-\lambda)(-b/a-\lambda) - (1)(-c/a) &= 0 \\
\lambda(b/a) + \lambda^2 + c/a &= 0 \\
\\
a\lambda^2 + b\lambda + c &= 0
\end{aligned}$$

This equation is the same as $ar^2 + br + c = 0$, the characteristic polynomial for our second order system of ordinary differential equations.