MTH 483 Homework 5

Philip Warton

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Problem 5.3

d

Let $f(z) = e^z, w = 0$.

$$\begin{aligned}
\int_{C[0,3]} \frac{e^z}{z^3} dz &= \int_{C[0,3]} \frac{e^z}{(z-0)^3} dz \\
&= \pi i \cdot \frac{1}{\pi i} \int_{C[0,3]} \frac{e^z}{(z-0)^3} dz \\
&= \pi i \cdot f''(w) \\
&= \pi i \cdot e^0 \\
&= \pi i
\end{aligned}$$

f

Let $f(z) = zi^{z-3}, w = 0$.

$$\begin{aligned}
\int_{C[0,3]} i^{z-3} dz &= \int_{C[0,3]} \frac{zi^{z-3}}{z} dz \\
&= 2\pi i \cdot \frac{1}{2\pi i} \int_{C[0,3]} \frac{zi^{z-3}}{z} dz \\
&= 2\pi i \cdot f(w) \\
&= 2\pi i \cdot 0 i^{0-3} \\
&= 0
\end{aligned}$$

g

Let γ_1 and γ_2 'cut the circle in half' by both being simply closed paths around a semicircle of $C[0,3]$ and along the real axis. Let $f(z) = \frac{\sin(z)}{z - \frac{i}{\sqrt{2}}}$ and $g(z) = \frac{\sin(z)}{z + \frac{i}{\sqrt{2}}}$. We will need to take their derivatives as well, those being

$$f'(z) = \frac{(z - \frac{i}{\sqrt{2}})\cos(z) - \sin(z)}{(z - \frac{i}{\sqrt{2}})^2}, \quad g'(z) = \frac{(z + \frac{i}{\sqrt{2}})\cos(z) - \sin(z)}{(z + \frac{i}{\sqrt{2}})^2}$$

Then finally let $w_f = -\frac{i}{\sqrt{2}}, w_g = \frac{i}{\sqrt{2}}$. Now we can compute the integral using some manipulation and Cauchy's integral formula.

$$
\begin{aligned}
\int_{C[0,3]} \frac{\sin z}{(z^2+\frac{1}{2})^2} dz &= \int_{\gamma_1} \frac{\sin z}{(z^2+\frac{1}{2})^2} dz + \int_{\gamma_2} \frac{\sin z}{(z^2+\frac{1}{2})^2} dz \\
&= \int_{\gamma_1} \frac{\frac{\sin z}{z-\frac{1}{\sqrt{2}}i}}{(z+\frac{1}{\sqrt{2}}i)^2} dz + \int_{\gamma_2} \frac{\frac{\sin z}{z+\frac{1}{\sqrt{2}}i}}{(z-\frac{1}{\sqrt{2}}i)^2} dz \\
&= \int_{\gamma_1} \frac{f(z)}{(z-w_f)^2} dz + \int_{\gamma_2} \frac{g(z)}{(z-w_g)^2} dz \\
&= 2\pi f'(w_f) + 2\pi g'(w_g) \\
&= 2\pi (f'(w_f) + g'(w_g)) \\
&= 2\pi \left(\frac{(\frac{-2i}{\sqrt{2}})\cos(\frac{-i}{\sqrt{2}}) - \sin(\frac{-i}{\sqrt{2}})}{(\frac{-2i}{\sqrt{2}})^2} + \frac{(\frac{2i}{\sqrt{2}})\cos(\frac{i}{\sqrt{2}}) - \sin(\frac{i}{\sqrt{2}})}{(\frac{2i}{\sqrt{2}})^2} \right) \\
&= 2\pi \left(2\left((\frac{2i}{\sqrt{2}})\cos(\frac{-i}{\sqrt{2}}) - \sin(\frac{-i}{\sqrt{2}}) \right) - 2\left((\frac{2i}{\sqrt{2}})\cos(\frac{i}{\sqrt{2}}) - \sin(\frac{i}{\sqrt{2}}) \right) \right) \\
&= 2\pi(0) \\
&= 0
\end{aligned}
$$

So we say that the integral evaluates to 0.

Problem 5.18

Let σ_R be the union of γ_R, the semicircle Re^{it} where $t \in [0, \pi]$, and the line segment on the real axis $[-R, R]$. Since these paths are connected we can write.

$$\int_{\sigma_R} \frac{dz}{z^4+1} = \int_{[-R,R]} \frac{dz}{z^4+1} + \int_{\gamma_R} \frac{dz}{z^4+1}$$

Now we can compute two of the integrals in this equation, granting us the third. For the integral along the semicircle γ_R, we know

$$\begin{aligned} \left| \int_{\gamma_R} \frac{dz}{z^4+1} \right| &\leqslant \max_{z \in \gamma_R} \left| \frac{1}{z^4+1} \right| \cdot \text{length}(\gamma_R) \\ &= \max_{z \in \gamma_R} \left| \frac{1}{z^4+1} \right| \pi R \\ &= \frac{\pi R}{R^4 - 1} \end{aligned}$$

So then

$$\lim_{R \to \infty} \left| \int_{\gamma_R} \frac{dz}{z^4+1} \right| \leqslant \lim_{R \to \infty} \frac{\pi R}{R^4-1}$$

Since the right limit evaluates to 0 the integral must be equal to 0. So the $\int_{\sigma_R} f dz = \int_{[-R,R]} f dz$. So now we wish to evaluate $\int_{\sigma_R}$. First let us note that

$$z^4 + 1 = (z^2+i)(z^2-i) = (z+\sqrt{i})(z-\sqrt{i})(z+i\sqrt{i})(z-i\sqrt{i})$$

Using this factorization we can rewrite or integral

$$\int_{\sigma_R} \frac{dz}{z^4+1} = \int_{\sigma_R} \frac{\frac{1}{(z-\sqrt{i})(z^2+i)}}{z+\sqrt{i}} + \int_{\sigma_R} \frac{\frac{1}{(z+\sqrt{i})(z^2+i)}}{z-\sqrt{i}} + \int_{\sigma_R} \frac{\frac{1}{(z^2-i)(z+i\sqrt{i})}}{z-i\sqrt{i}} + \int_{\sigma_R} \frac{\frac{1}{(z^2-i)(z-i\sqrt{i})}}{z+i\sqrt{i}}$$

Now we can rewrite this as

$$2\pi i \left(f_1(-\sqrt{i}) + f_2(\sqrt{i}) + f_3(i\sqrt{i}) + f_4(-i\sqrt{i}) \right) = 2\pi i \left(\frac{1}{2\sqrt{2}i} \right) = \frac{\pi}{\sqrt{2}}$$

Problem 2.26

Is $u(x, y) = \frac{x}{x^2+y^2}$ harmonic on $\mathbb{C}$?

To check if the function is harmonic we check if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Let us compute each partial derivative and then check if this is true.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} \\
&= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \\
&= \frac{y^2-x^2}{(x^2+y^2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)\left(2(x^2+y^2)(2x)\right)}{(x^2+y^2)^4} \\
&= \frac{(x^4+2x^2y^2+y^4)(-2x) - (y^4-x^4)(4x)}{(x^2+y^2)^4} \\
&= \frac{(2x)\left[(x^4+2x^2y^2+y^4)(-1) - 2y^4 + 2x^4\right]}{(x^2+y^2)^4} \\
&= \frac{(2x)(x^4-2x^2y^2-3y^4)}{(x^2+y^2)^4}
\end{aligned}$$

We now compute the second partial derivative with respect to y.

$$\begin{aligned}
\frac{\partial u}{\partial y} &= x\frac{\partial}{\partial y}\frac{1}{x^2+y^2} \\
&= x(-1)(x^2+y^2)^{-2}(2y) \\
&= \frac{-2xy}{(x^2+y^2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= -2x\frac{\partial}{\partial y}\frac{y}{(x^2+y^2)^2} \\
&= -2x\frac{(x^2+y^2)^2(1) - (y)(2)(x^2+y^2)(2y)}{(x^2+y^2)^4} \\
&= \frac{(-2x)\left[(x^4+2x^2y^2+y^4) - 4y^2(x^2+y^2)\right]}{(x^2+y^2)^4} \\
&= \frac{(-1)(2x)(x^4-2x^2y^2-3y^4)}{(x^2+y^2)^4}
\end{aligned}$$

Then it becomes clear that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for every $(x, y) \in \mathbb{C}$, and therefore u is harmonic on $\mathbb{C}$.

Is $u(x,y) = \frac{x^2}{x^2+y^2}$ harmonic on $\mathbb{C}$?

Once again, we compute the two second partial derivatives and check that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{(x^2+y^2)(2x) - (x^2)(2x)}{(x^2+y^2)^2} \\
&= \frac{2x^3 + 2xy^2 - 2x^3}{(x^2+y^2)^2} \\
&= \frac{2xy^2}{(x^2+y^2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{(x^2+y^2)^2(2y^2) - (2xy^2)(2(x^2+y^2)(2x))}{(x^2+y^2)^4} \\
&= \frac{(x^4+2x^2y^2+y^4)(2y^2) - 8x^2y^2(x^2+y^2)}{(x^2+y^2)^4} \\
&= \frac{(2y^2)\left[(x^4+2x^2y^2+y^4) - 4x^2(x^2+y^2)\right]}{(x^2+y^2)^4} \\
&= \frac{(2y^2)\left[-3x^4 - 2x^2y^2 + y^4\right]}{(x^2+y^2)^4}
\end{aligned}$$

And now we compute the partial derivatives with respect to y.

$$\begin{aligned}
\frac{\partial u}{\partial y} &= x^2 \frac{\partial}{\partial y} \frac{1}{x^2+y^2} \\
&= x^2(-1)(x^2+y^2)^{-2}(2y) \\
&= \frac{2x^y}{(x^2+y^2)^2} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{(x^2+y^2)^2(2x^2) - (2x^2y)(2(x^2+y^2)(2y))}{(x^2+y^2)^4} \\
&\vdots \\
&= \frac{(2x^2)(-3x^4 - 2x^2y^2 + y^4)}{(x^2+y^2)^4}
\end{aligned}$$

Since there exists $(x,y) \in \mathbb{C}$ such that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0$, u is not harmonic on all of $\mathbb{C}$.

$\boxed{\text{Alternative evidence:}}$ In both definitions, $u'(0)$ is undefined therefore the function cannot be harmonic on $\mathbb{C}$.

Problem 7.25

Find a power series and its region of convergence for the following functions.

a

We want to find a power series to represent the function

$$\frac{1}{1+4z}$$

Let us start with the familiar geometric series and replace w with $-4z$.

$$\begin{aligned}\sum_{k\geqslant 0} w^k &= \frac{1}{1-w}\\ \sum_{k\geqslant 0}(-4z)^k &= \frac{1}{1-(-4z)}\\ &= \frac{1}{1+4z}\end{aligned}$$

This series will converge when $|w| = |-4z| < 1$, which is when $|z| < \frac{1}{4}$. The region of convergence will be $D[0, \frac{1}{4}]$.

b

Now we must find a series to represent

$$\frac{1}{3-(\frac{z}{2})}$$

Once again we will manipulate the geometric series, this time replacing w with $z = -2 + \frac{z}{2}$.

$$\begin{aligned}\sum_{k\geqslant 0} w^k &= \frac{1}{1-z}\\ \sum_{k\geqslant 0}\left(-2+\frac{z}{2}\right)^k &= \frac{1}{1-(-2+\frac{z}{2})}\\ \sum_{k\geqslant 0}\left(-2+\frac{z}{2}\right)^k &= \frac{1}{3-\frac{z}{2}}\end{aligned}$$

Since this is the geometric series it will converge when $|w| = |-2+\frac{z}{2}| < 1$. This is equivalent to the statement $|-4+z| < 2$, which is a disk centered at -4 with radius 2, which is the region of convergence for this series.

Problem 7.33

Find the radius of convergence for the following series.

b

We want to find the radius of convergence for the series $\sum_{k\geqslant 0} k^n z^k$ where $n \in \mathbb{Z}$. We can use the root test in order to compute the radius of convergence.

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{|c_k|} &= \lim_{k\to\infty} \sqrt[k]{|k^n|} \\
&= \lim_{k\to\infty} \sqrt[k]{k^n} \\
&= \lim_{k\to\infty} e^{\ln(k)\frac{n}{k}} \\
&= \lim_{k\to\infty} e^{\frac{n}{k}\ln(k)} \\
&= e^{\lim_{k\to\infty}\frac{n}{k}\ln(k)} \\
&= e^0 \\
&= 1
\end{aligned}$$

Since the limit exists and is non-zero, we take the inverse of it to be the radius of convergence, so $R = 1$.

g

We want to calculate the radius of convergence for the following series:

$$\sum_{k\geqslant 0} 4^k (z-2)^k$$

We will use the root test once again to compute the radius of convergence.

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{|c_k|} &= \lim_{k\to\infty} \sqrt[k]{|4^k|} \\
&= \lim_{k\to\infty} 4 \\
&= 4
\end{aligned}$$

So the radius of convergence is $R = \frac{1}{4}$.

Problem 7.34

Find a function for the power series.

a

Find a function for $\sum_{k\geqslant 0}\frac{z^{2k}}{k!}$ By properties of the exponent we know this is equal to $\sum_{k\geqslant 0}\frac{(z^2)^k}{k!}$. Let $w=z^2$, and we have

$$\sum_{k\geqslant 0}\frac{w^k}{k!}=e^w=e^{(z^2)}$$

b

Now we wish to find a function for $\sum_{k\geqslant 1}k(z-1)^{k-1}$. Now we can take the integral of the series to determine the function that it is equal to.

$$\begin{aligned}
\sum_{k\geqslant 1}k(z-1)^{k-1}&=f(z)\\
\int\sum_{k\geqslant 1}k(z-1)^{k-1}dz&=\int f(z)dz\\
\sum_{k\geqslant 1}\left(\int k(z-1)^{k-1}dz\right)&=\int f(z)dz\\
\sum_{k\geqslant 1}(z-1)^k&=\int f(z)dz\\
\frac{1}{1-(z-1)}&=\int f(z)dz\\
\frac{1}{2-z}&=\int f(z)dz\\
\frac{d}{dz}\frac{1}{2-z}&=\frac{d}{dz}\int f(z)dz\\
\frac{1}{(2-z)^2}&=f(z)
\end{aligned}$$

c

We want to find a function for the series $\sum_{k\geqslant 2}k(k-1)z^k$. Lets start with the geometric series.

$$\begin{aligned}
\sum_{k\geqslant 0}z^k&=\frac{1}{1-z}\\
\Rightarrow\frac{d}{dz}\sum_{k\geqslant 0}z^k&=\frac{d}{dz}\frac{1}{1-z}\\
\sum_{k\geqslant 1}kz^{k-1}&=\frac{-1}{(1-z)^2}\\
\Rightarrow\frac{d}{dz}\sum_{k\geqslant 1}kz^{k-1}&=\frac{d}{dz}\frac{-1}{(1-z)^2}\\
\sum_{k\geqslant 2}k(k-1)z^{k-2}&=\frac{2}{(1-z)^3}\\
\sum_{k\geqslant 2}k(k-1)z^{k}&=\frac{2z^2}{(1-z)^3}
\end{aligned}$$

And now we have a function for our power series.

Problem 8.3

Find the power series for $f(z) = \sin(z)$ centered at π. We know that $\sin(z) = \sum_{k \geqslant 0} c_k (z-\pi)^k$ where $c_k = \frac{f^k(\pi)}{k!}$. We know that $f^k(\pi) = 0$ when k is even, and is either -1 or 1 if k is odd, alternating between the two. This gives us the series

$$\begin{aligned}\sin(z) &= 0 + \frac{-1}{1!}(z-\pi)^1 + 0 + \frac{1}{3!}(z-\pi)^3 + \cdots \\ &= \sum_{k \geqslant 0} (-1)^{k+1} \frac{(z-\pi)^{2k+1}}{(2k+1)!}\end{aligned}$$

Problem 8.5

Compute the power series up to order 3.

a

$f(z) = \frac{1}{1+z^2}, z_0 = 1$. Let us compute the first three derivatives and the image of z_0 under them.

$$\begin{aligned}
f(z) &= \frac{1}{1+z^2} & f(z_0) &= \frac{1}{2} \\
f'(z) &= (-1)\frac{2z}{(1+z^2)^2} & f'(z_0) &= -\frac{1}{2} \\
f''(z) &= (-1)(-2)\frac{(2z)^2}{(1+z^2)^3} & f''(z_0) &= 1 \\
f'''(z) &= (-1)(-2)(-3)\frac{(2z)^3}{(1+z^2)^4} & f'''(z_0) &= -3
\end{aligned}$$

So the first three terms of the power series are

$$\frac{1}{2} - \frac{z-1}{2} + \frac{(z-1)^2}{2} - \frac{(z-1)^3}{2}$$

b

$f(z) = \frac{1}{e^z+1}, z_0 = 0$. We will compute the first three derivatives again.

$$\begin{aligned}
f(z) &= \frac{1}{e^z+1} & f(0) &= \frac{1}{2} \\
f'(z) &= (-1)(e^z+1)^{-2}(e^z) & f'(0) &= -\frac{1}{4} \\
f''(z) &= (-1)(-2)(e^z+1)^{-3}(e^z) + (-1)(e^z+1)^{-2}(e^z) & f''(0) &= \frac{1}{4} - \frac{1}{4} = 0 \\
f'''(z) &= (-1)(-2)(-3)(e^z+1)^{-4}(e^z)^2 + (-1)(-2)(e^z+1)^{-3}(e^z) + & & \\
&\quad (-1)(-2)(e^z+1)^{-3}(e^z)^2 + (-1)(e^z+1)^{-2}(e^z) & f'''(0) &= \frac{-6}{16} + \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = \frac{5}{8}
\end{aligned}$$

So for the terms of the power series up to order 3 we get

$$\frac{1}{2} - \frac{z}{4} + 0 + \frac{5z^3}{48}$$

c

$f(z) = (1+z)^{\frac{1}{2}}, z_0 = 0$. Let's compute the first three derivatives.

$$\begin{aligned}
f(z) &= (1+z)^{\frac{1}{2}} & f(0) &= 1 \\
f'(z) &= \frac{1}{2}(1+z)^{-\frac{1}{2}} & f'(0) &= \frac{1}{2} \\
f''(z) &= \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)(1+z)^{-\frac{1}{2}} & f''(0) &= -\frac{1}{4} \\
f'''(z) &= \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)(1+z)^{-\frac{5}{2}} & f'''(0) &= \frac{3}{8}
\end{aligned}$$

This gives us the solution

$$1 + \frac{z}{2} - \frac{z^2}{8} + \frac{3z^3}{48}$$

d

$f(z) = e^{z^2}$, $z_0 = i$. We compute the derivatives and then use it to compute the first three terms of the power series.

$$\begin{aligned} f(z) &= e^{z^2} & f(i) &= e^{-1} \\ f'(z) &= e^{z^2}(2z) & f'(i) &= 2ie^{-1} \\ f''(z) &= e^{z^2}(2z)^2 + e^{z^2}(2) & f''(i) &= -2e^{-1} \\ f'''(z) &= e^{z^2}8z^3 + e^{z^2}8z + e^{z^2}4z & f'''(i) &= 4ie^{-1} \end{aligned}$$

This makes our solution

$$\frac{1}{e} + \frac{2i(z-i)}{e} - \frac{(z-i)^2}{e} + \frac{2i(z-i)^3}{3e}$$