Systems of ODE's - Notes

Philip Warton

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1 Solutions to systems of ODE's

Suppose we have an autonomous system

$$x'(t) = F(x(t))$$

The solution $x = x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is a function

$$X : I \to \mathbb{R}^n$$

Definition 1.1. An equilibrium solution of $X' = F(x)$ is a constant solution, i.e., it takes the form

$$X(t) = X^* \in \mathbb{R}^n \quad \forall t \in I$$

where $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T \in \mathbb{R}^n$ is a constant vector.

The implication of this is that $X'(t) = 0 \Rightarrow F(x^*) = 0$. Sometimes these are called transient solutions.

Exponential growth/decay ODE:

$$x' := \frac{dx}{dt} = rx, r \in \mathbb{R}$$

This is solved using the seperation of variables technique, that is

$$\frac{dx}{dt} = rx \tag{1}$$
$$dx = rxdt \tag{2}$$
$$x^{-1}dx = rdt \tag{3}$$
$$\int x^{-1}dx = \int rdt \tag{4}$$
$$\ln(x) + c_0 = rt + c_1 \tag{5}$$
$$\ln(x) = rt + c \tag{6}$$
$$x = Ke^{rt} \tag{7}$$

This is a one parameter (K) family of solutions ($\forall K \in \mathbb{R}$ or $\mathbb{R}^+ \cup \{0\}$). If we are given some initial value such as $x_0 = x(0)$ then we have one unique solution based on this value. Specifically we have the solution $x(t) = x_0 e^{rt}$. If we ahve $x_0 = 0$ then $x(t) = 0$ for all $r \in \mathbb{R}$. This constant solution is called an equilibrium or steady state solution. Equilibrium solutions or equilibria are constant solutions to the ODE.

Logistic Initial Value Problem

$$x' = rx(1 - \frac{x}{K}), r > 0, K > 0$$

Where $x(0) = x_0, x_0 \in \mathbb{R}^+ \cup \{0\}$. The solution to this can be derived using seperation of variables, giving us $x(t) = \frac{Kx_0}{x_0 + (k - x_0)e^{-rt}}$. If $x_0 = 0$ then we have an equilibria of $x(t) = 0$. If we have $x_0 = K$ we also have a constant solution $x(t) = K$.

2 Systems

Let $n \in \mathbb{N}$. What are the equilibrium solutions of the linear system? That is,

$$X' = AX$$

By definition and its consequence equilibrium solutions are solutions to $AX = 0$. If $\det(A) \neq 0$, then $X(t) = 0$ is the only equilibrium solution. We look at planar systems of linear ODE's. Then we break it down into several cases, based on our different eigenvalues for a $A \in M_{2\times 2}(\mathbb{R})$ matrix. Our main cases are

$$\text{Real and distinct eigenvalues} \tag{8}$$
$$\text{Real repeated eigenvalues} \tag{9}$$
$$\text{Complex eigenvalues} \tag{10}$$

We often get a solution of the form $X(t) = \begin{bmatrix} \alpha e^{\lambda_1 t} \\ \beta e^{\lambda_2 t} \end{bmatrix}$. Understanding the phase plane and portrait.