

# MTH 411 Post Midterm Notes

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## 1 Midterm Solutions and Review

### 1.1 Let $(M, d)$ be a metric space with the discrete metric. Show that any convergent sequence is eventually constant.

*Proof.* Let  $(x_n)$  be a convergent sequence in the space. Choose  $\epsilon = 1$ . Our sequence will eventually be in the epsilon ball of its limit, and therefore it will be eventually constant.  $\square$

### 1.2 The set $A = \{y \in M : d(x, y) \leq r\}$ is called the closed ball with radius $r$ about $x$ .

#### 1.2.1 Show that $A$ is closed.

*Proof.* Assume that  $(y_n)$  is a convergent sequence in  $A$ . We will show that its limit is in  $A$ . Let  $\epsilon > 0$  be arbitrary. Then,

$$d(x, y) \leq d(x, y_n) + d(y_n, y) \leq r + \epsilon$$

Since this is true for any  $\epsilon > 0$  we say that  $d(x, y) \leq r$ , and  $y \in A$ .  $\square$

#### 1.2.2 Give an example where $A$ is not the closure of the open ball.

Choose the space of integers, with an open ball radius 1 around 0. Then  $B_1(0) = \{0\}$  is already closed, and is a proper subset of  $A$ .

### 1.3 If $x_n \rightarrow x$ in a metric space, show that $d(x_n, y) \rightarrow d(x, y)$ .

*Proof.* By the reverse triangle inequality and the squeeze theorem, the result follows trivially.  $\square$

### 1.4 Show that the collection of polynomials with integer coefficients is countable.

*Proof.* Let  $\mathcal{P}$  be the set of all polynomials with integer coefficients,  $\mathcal{P}_n$  be the set of polynomials  $p(x) = \sum_{k=0}^n a_k x^k$  with integer coefficients and degree at most  $n$ . Then

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$$

To show that  $\mathcal{P}_n$  are countable, map  $\mathcal{P}_{n-1}$  onto  $\mathbb{Z}^n$  with the bijection:

$$f(z_1, z_2, \dots, z_n) = \sum_{k=1}^n z_k x^k$$

Then we assume that  $\mathbb{Q}^n$  is countable, and  $\mathbb{Z}^n \subset \mathbb{Q}^n$  and we say that  $\mathcal{P}$  must be countable.  $\square$

## 2 Continuity

## 3 Homeomorphisms

## 4 Connectedness

A space  $M$  is said to be disconnected if  $M = A \dot{\cup} B$ . That is to say, if  $M$  can be written as a disjoint union of open sets. Such a construction is called a disconnection of  $M$ , and  $M$  is connected if it yields no disconnection.

**Theorem 4.1.**  $M$  is connected if and only if  $M$  contains no nontrivial clopen sets.

## 5 Completeness

**Definition 5.1** (Totally Bounded). We define total boundedness to be the following: a set  $A$  in a metric space  $(M, d)$  is totally bounded  $\Leftrightarrow$

$$\forall \epsilon > 0, \exists n \in \mathbb{N}, x_1, \dots, x_n \in M : A \subset \bigcup_{j=1}^n B_\epsilon(x_j)$$

If we look at  $B_1(0) \in l_1$ , we find that although this set is bounded, it is not totally bounded.

**Theorem 5.1.** We can characterize total boundedness by:  $\forall \epsilon > 0 \exists n \in \mathbb{N}, A_1, \dots, A_n \subset A$  such that  $\text{diam}(A_j) < \epsilon, j = 1, \dots, n$  and  $A \subset \bigcup_{j=1}^n A_j$ .

The property of total boundedness can be considered as a generalization of compactness.

**Definition 5.2** (Bounded). We say that a set  $A \subset M$  is bounded if there exists some ball of finite radius such that  $A$  is contained in this ball.

**Lemma 5.1.** Let  $(x_n)$  be a sequence in  $(M, d)$  and  $A = \{x_n | n \in \mathbb{N}\}$  its range.

- (i) if  $(x_n)$  is Cauchy, then  $A$  is totally bounded
- (ii) if  $A$  is totally bounded, then  $x_n$  has a Cauchy subsequence

*Proof.* (i) Let  $\epsilon > 0$  be arbitrary. Since  $(x_n)$  is Cauchy, we say that for some  $N \in \mathbb{N}$ , for every  $m, n \geq N, d(x_m, x_n) < \epsilon$ . So we say that  $\bigcup_{n=1}^N B_\epsilon(x_n) \supset A$  and is a finite union of open balls, and is therefore open.

(ii) If  $A$  is finite, then every sequence  $(x_n) \in A$  has a constant subsequence. Otherwise,  $A$  will be infinite. □

**Definition 5.3.** A metric space  $(M, d)$  is complete if every Cauchy sequence in  $M$  converges to a point in  $M$ .

Of course the set of real numbers will be complete, however the set of rational numbers will not be complete. The Lebesgue space  $\ell_2$  is complete. To prove this is fairly difficult.

**Theorem 5.2.** For any metric space  $M$ , the following are equivalent

- (i)  $M$  is complete
- (ii) The Nested Set Property holds
- (iii) The Bolzano Weirstrass Property holds. That is, every totally bounded set has a limit point

This is another way to characterize completeness, this time for a normed vector space.

**Theorem 5.3.** A normed vector space  $V$  is complete if and only if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \Rightarrow \sum_{n=1}^{\infty} x_n \text{ converges in } V$$

Every absolutely summable series in  $V$  is summable.

*Proof.*  $\Rightarrow$  Assume  $V$  is complete, and let  $(x_n) \subset V$  be such that  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ . Let  $S_n$  be the sequence of partial sums. We wish to show that  $S_n$  is a Cauchy sequence.

$$\|S_n - S_m\| = \|\sum_{k=m+1}^n x_k\| \leq \sum_{k=m+1}^n \|x_k\| \rightarrow 0$$

Thus  $(S_n)$  is a Cauchy sequence in  $V$ . Since  $V$  is complete  $(S_n)$  converges to  $S = \sum_{k=1}^{\infty} x_k$ .

$\Leftarrow$  Now assume that  $\sum \|x_n\| < \infty \Rightarrow \sum x_n$  converges in  $v$  and let  $(x_n)$  be a Cauchy sequence in  $V$ . For  $k = 1, 2, \dots$  let  $N_k$

be such that  $\forall n > m \geq N_k : d(x_n, x_m) < 2^{-k}$ . Then let  $m = N_k \Rightarrow x_n \in B_{2^{-k}}(x_{N_k}) \forall n > N_k$ . Consider the subsequence  $y_k = x_{N_k}, k \in \mathbb{N}$ . Then  $y_{k+1} = x_{N_{k+1}} \in B_{2^{-k}}(x_{N_k}) = B_{2^{-k}}(y_k)$ . And  $\|y_{k+1} - y_k\| < 2^{-k}$ . Hence  $\sum_{k=1}^{\infty} \|y_{k+1} - y_k\|$  converges and therefore also  $\sum_{k=1}^{\infty} y_{k+1} - y_k$  converges. The partial sums for this series are  $S_n = \sum_{k=1}^n y_{k+1} - y_k = y_{n+1} - y_1$ . Therefore the sequence  $(y_k) = (x_{N_k})$  converges. Thus there exists some  $x \in M : x = \lim_{k \rightarrow \infty} x_{N_k}$  and  $(x_n)$  is Cauchy.  $\square$

Note: Banach Space is a complete normed vector space  $V$ .

**Definition 5.4.** A function  $f : (M, d) \rightarrow (N, s)$  is called Lipschitz if there is a constant  $k < \infty$  such that  $s(f(x), f(y)) \leq kd(x, y)$  for every  $x, y \in M$ .

Immediately it should be clear that a Lipschitz mapping will be continuous.

*Proof.* Let  $x_n \rightarrow x$  in  $M$ . Then  $d(x, x_n) \rightarrow 0$ . So  $s(f(x), f(x_n)) < kd(x, x_n) \rightarrow 0$ . Thus  $s(f(x), f(x_n)) \rightarrow 0$  and  $f$  is continuous.  $\square$

**Definition 5.5.** A map  $f : M \rightarrow M$  on a metric space  $(M, d)$  is called a contraction if there is  $0 \leq \alpha < 1$  such that  $d(f(x), f(y)) \leq \alpha d(x, y)$ .

Since a contraction is Lipschitz with  $k = \alpha$  it is continuous.

**Definition 5.6.** Let  $f : M \rightarrow M$ . Any  $x \in M$  such that  $f(x) = x$  is called a fixed point of  $f$ .

**Theorem 5.4.** (Contraction Mapping Theorem, Banach Fixed Point Theorem) Let  $(M, d)$  be a complete metric space and let  $f : M \rightarrow M$  be a contraction. Then,  $f$  has a unique fixed point. For any  $x_0 \in M$ , the iteration  $x_{n+1} = f(x_n)$  converges to  $x$ . One has  $d(x_n, x) \leq d(x_1, x_0) \frac{\alpha^n}{1-\alpha}$ .

**Definition 5.7.** Let  $f'(x) = f(x), f^{n+1}(x) = f(f^n(x))$ , i.e.  $f^n$  is the  $n$ -fold composition of  $f$  with itself.

*Proof.* The sequence  $x_n$  can be written as  $x_n = f^n(x_0)$ . Let  $x_0 \in M$  be arbitrary.

$$\begin{aligned}
 d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\
 &\leq \alpha d(x_n, x_{n-1}) = \alpha d(f(x_{n-1}), f(x_{n-2})) \\
 &\leq \alpha^2 d(x_{n-1}, x_{n-2}) \\
 &\vdots \\
 &\leq \alpha^n d(x_1, x_0) = c\alpha^n
 \end{aligned}
 \qquad c = d(x_1, x_0)$$

$\square$