Algebraic Topology - Homework 1

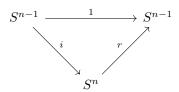
Philip Warton

October 5, 2021

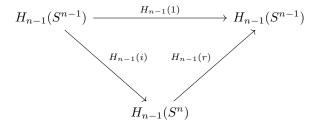
1 Problem 0.3

Assume, for $n \ge 1$, that $H_i(S^n) = \mathbb{Z}$ if i = 0, n, and that $H_i(S^n) = 0$ otherwise. Using the technique of the proof of Lemma 0.2, prove that the equator of the n-sphere is not a retract.

Proof. Assume that S^{n-1} is the equator of the *n*-sphere. Suppose by contradiction that S^{n-1} is a retract of S^n . Then it follows that there exists some retraction $r: S^n \to S^{n-1}$. Then, with $i: S^{n-1} \to S^n$ being the inclusion map, and with 1 being, of course, the identity map, it follows that we would have a commutative diagram:



To this diagram, we can apply our homology functor, giving us



We know by assumption that $H_{n-1}(S^n)=0$ since $n-1\neq n$ and that $H_n(S^{n-1})=\mathbb{Z}$ since n-1=n-1. This new diagram should continue to commute by the properties of our functor H_{n-1} . Since $H_{n-1}(S^n)=0$, it follows that its image under $H_{n-1}(r)$ must also be zero. That is, $H_{n-1}(S^{n-1})=0$. However, since this diagram commutes, it follows that the identity map takes a countable object to a trivial one (contradiction). Therefore, S^{n-1} is not a retract of S^n .

2 Problem 0.5

Let $f, g: I \to I \times I$ such that f(0) = (a, 0), f(1) = (b, 1), g(0) = (0, c), g(1) = (1, d). Show that there exists some point (s, t) such that f(s) = g(t).

Proof. Say that its the case that $f(s) \neq g(t)$ for all $(s,t) \in I^2$. Define

$$N(s,t) = \max\{|g_1(t) - f_1(s)|, |g_2(t) - f_2(s)|\}\$$

Then we define a function $F: I^2 \to I^2$ to be

$$F(s,t) = \left(\frac{g_1(t) - f_1(s)}{2N(s,t)} + 1, \frac{g_2(t) - f_2(s)}{2N(s,t)} + 1\right)$$

Then we have that $F(I^2) \subset \partial I^2$. Suppose we have a fixed point (x,y). Then we say that F(s,t) = (s,t) therefore $(s,t) \in \partial I^2$. This means that s = 0, s = 1, t = 0 or t = 1. Within any one of these cases, the fixed point does not hold so we no fixed point for F (justify). Then since I^2 is homeomorphic to D^2 , we say that a contradiction is reached by Brouwer fixed point.

3 Problem 0.7

Let $f \in \text{Hom}(A, B)$, and let $g, h \in \text{Hom}(B, A)$ such that $g \circ f = 1_A$ and that $f \circ h = 1_B$. Then g = h.

Proof. We write the following:

$$f = f$$

$$f \circ 1_A = 1_B \circ f \qquad \text{(identity in } hom(A, A) \text{ and } hom(B, B))$$

$$f \circ (g \circ f) = (f \circ h) \circ f \qquad \text{(by assumption)}$$

$$(f \circ g \circ f) = (f \circ h \circ f) \qquad \text{(associativity)}$$

$$h \circ (f \circ g \circ f) = h \circ (f \circ h \circ f) \qquad \text{(properties of equivalence relation)}$$

$$(h \circ f \circ g \circ f) \circ g = (h \circ f \circ h \circ f) \qquad \text{(properties of equivalence relation)}$$

$$(h \circ f) \circ g \circ (f \circ g) = (h \circ f) \circ h \circ (f \circ g) \qquad \text{(properties of equivalence relation)}$$

$$(h \circ f) \circ g \circ (f \circ g) = (h \circ f) \circ h \circ (f \circ g) \qquad \text{(associativity)}$$

$$1_B \circ g \circ 1_A = 1_B \circ h \circ 1_A \qquad \text{(by assumption)}$$

$$g = h \qquad \text{(identity in } hom(A, A) \text{ and } hom(B, B))$$

4 Problem 0.18

For an abelian group G, let

 $tG = \{x \in G : x \text{ has finite order}\}$

denote its torsion subgroup.

(ii)

Assume that t defines a functor and that $t(f) = f \Big|_{tG}$ for every homomorphism f. If f is injective, then t(f) is injective.

Proof. Let G, H be abelian groups, and let $f: G \to H$ be an injection. Then, for any $a, b \in G$ we know that $f(a) = f(b) \Longrightarrow a = b$. Choose some $x, y \in tG$ such that tf(x) = tf(y), or equivalently, $f\Big|_{tG}(x) = f\Big|_{tG}(y)$. Since both x and y belong to the torsion group, it follows that f(x) = f(y). Then, by the injectivity of f, we know that f(x) = f(y) and since both are in f(x) = f(y).

(iii)

Give an example of a surjective homomorphism f for which t(f) is not surjective.

Define a function $f: \mathbb{Z} \to \mathbb{Z}_2$ such that

$$f(x) = \begin{cases} [0] & x \text{ is even} \\ [1] & x \text{ is odd} \end{cases}$$

Since \mathbb{Z} contains both odd and even integers, f is clearly surjective. It can also be demonstrated that f is a group homomorphism. But, all non-zero integers have infinite order under addition, so we say that $t\mathbb{Z} = \{0\}$. Since \mathbb{Z}_2 is cyclic, we know that $t\mathbb{Z}_2 = \mathbb{Z}_2$. Take the element $[1] \in \mathbb{Z}_2$. Notice that it's pre-image under f is confined to only odd integers, so since 0 is not odd, it can't have a pre-image under t(f). Since there is an element of $t\mathbb{Z}_2$ with no pre-image under t(f), the function is not surjective.

5 Problem 0.19

Let p be a fixed prime in \mathbb{Z} . Define a functor $f: \mathbf{Ab} \to \mathbf{Ab}$ by F(G) = G/pG and $F(f): x + pG \mapsto f(x) + pH$ (where $f: G \to H$ is a homomorphism).

(i)

Show that if f is a surjection, then F(f) is a surjection.

Proof. Let $y + pH \in H/pH$. Then since $y + pH \in H/pH$ it follows that $y \in H$. Because we know that f is a surjection, it follows that if $y \in H$ there exists some $x \in G$ such that f(x) = y. So then take the point $x + pG \in G/pG$, and of course it must be the case that

$$F(f)(x + pG) = f(x) + pH = y + pH$$

And it has been proven that F(f) is surjective.

(ii)

Give an example of a group homomorphism f that is an injection such that F(f) is not an injection.

Let our prime number p be equal to 2. Then let $G = (\{0, 2\}, + \mod 4), H = \mathbb{Z}_4$. Finally, let f simply be the inclusion map. It follows that f is injective since it is the inclusion map. Then F(f)(4) = F(f)(2) = 2 + 2H, and we say that F(f) is not injective. This does not hold, try $p = 3, \{0, 2, 4, 6\}, \mathbb{Z}_8$ and the inclusion map.

6 Problem 0.20

(ii)

Show that there is a contravariant functor between Top and Ring, given by C(X).

(i) Let $X \in objTop$, then $C(X) \in objRing$ by Part (i)

(ii) Let $f: X \to Y$. Then let $u \in C(Y)$. We define $f^*: C(Y) \to C(X)$ by

$$f^*(u) = u \circ f$$

Then $f^*(u)$ is a continuous map from X to \mathbb{R} . That is, if $u \in C(Y)$ then $f^*(u) \in C(X)$, and f^* acts as a ring homomorphism. To demonstrate this, let $u, v \in C(Y)$ and $h: X \to Y$ where X, Y are topological spaces. Let $x \in X$ be arbitrary. Then we can write

$$h^*(f+g)(x) = ((f+g) \circ h)(x) \tag{1}$$

$$= (f+g)(h(x)) \tag{2}$$

$$= f(h(x)) + g(h(x)) \tag{3}$$

$$= (f \circ h)(x) + (g \circ h)(x) \tag{4}$$

$$= h^*(f(x)) + h^*(g(x)) \tag{5}$$

Finally we write

$$h^*(fg)(x) = (fg \circ h)(x) \tag{6}$$

$$= (fg)(h(x)) \tag{7}$$

$$= f(h(x))q(h(x)) \tag{8}$$

$$= (f \circ h)(x) \cdot (q \circ h)(x) \tag{9}$$

$$= h^*(g(x)) \cdot h^*(g(x)) \tag{10}$$

So the ring structure is preserved, adn the property is satisfied.

[iii) Let $f: X \to Y$ and $g: Y \to Z$ so that both are continuous and $g \circ f$ is defined and a morphism in Hom(X,Z). Let $u \in C(Z)$ be a continuous function on Z. Then we write

$$(g \circ f)^*(u) = u \circ (g \circ f)$$
 (definition of $(g \circ f)^*$) (11)

$$= (u \circ q) \circ f$$
 (associativity of functional composition) (12)

$$= g^*(u) \circ f \tag{definition of } g^*) \tag{13}$$

$$= f^*(g^*(u))$$
 (definition of f^*) (14)

$$= (f^* \circ g^*)(u)$$
 (definition of composition) (15)

(iv) Let $X \in objTop$. Then let $u \in C(X)$ be arbitrary. We then can write

(16)	(by definition of 1_X^*)	$(1_X)^*(u) = u \circ 1_X$
(17)	(identity in $Hom(X, X)$)	= u
(18)	(identity in $Hom(C(X), C(X))$	$=1_{C(X)}\circ u$
(19)	(definition of composition)	$=1_{C(X)}(u)$

Since all of these properties have been shown, we conclude that we have a contravariant functor from the class of topological spaces to the ring of continuous maps.