Taxicab Geometry Chapter 2 Exercise Responses

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2.2

Let the taxicab distance be written as $d = \Sigma d_x + \Sigma d_y$ where Σd_s denotes total distance in the s direction. Given two points $B = (3, 3)$ and $A = (-3, -1)$ we can calculate the distance by taking the distance in the x direction and summing it with the distance in the y direction. In this case, we have $d_x(A, B) = 3 - (-3) = 6$ and $d_y(A, B) = 3 - (-1) = 4$, therefore $d(A, B) = 6 + 4 = 10$. To find a point C to satisfy the requirement that distance $d(A, C) + d(C, B)$ is as small as possible, there are exactly two properties which this point must have:
1.) C_x lies between A_x and B_x
2.) C_y lies betweenA_y and B_y

2.3

Midpoints

Now, we must find all the points which minimize combined distance as in $\boxed{2.2}$ and also make it so that $d(A, C) \leqslant d(C, B)$. Let us find all the points where these distances are equal (find the midpoints), then determine what will cause the less or equal statement to hold. To find a midpoint C we can satisfy the following three requirements:
1.) C_x lies between A_x and B_x
2.) C_y lies betweenA_y and B_y
3.) $d(A, C) = \frac{d(A,B)}{2}$ or $d(B, C) = \frac{d(A,B)}{2} \iff d(A, C) = d(B, C)$

Restated, this means $C_x \in [A_x, B_x]$, $C_y \in [A_y, B_y]$, and $C = (C_x, C_y) : (B_x - C_x) + (B_y - C_y) = \frac{d(A,B)}{2}$. In the case of our specific A and B this means that we have:

$$\begin{aligned}(3 - C_x) + (3 - C_y) &= 5\\ 6 - C_x - C_y &= 5\\ 6 - (C_x + C_y) &= 5\\ 6 - 5 &= (C_x + C_y)\\ 1 &= C_x + C_y\end{aligned}$$

The points that satisfy these requirements are $(-2, 3), (2, -1)$ and everything on the Euclidean line between them.

Favoring Point A

Now we need to find all points D such that the distance $d(A, D) + d(D, B)$ is minimized and where D is equidistant or closer to A than it is to B. By replacing our third requirement we get:
1.) D_x lies between A_x and B_x
2.) D_y lies betweenA_y and B_y
3.) $d(A, D) \leqslant d(B, D)$

The results for point D will be anywhere on or above the line of midpoints that we found while still meeting conditions 1 and 2, where above means a ray going straight in the negative y direction will eventually reach the line. This is the case because requirements 1 and 2 already minimize the sum of distances $d(A, D) + d(D, B)$, and so long as we are on the line of midpoints or closer to A (above the line), the inequality will hold.

2.5

Let us change our desires once again. We must find a set of points $\mathbf{F} = \{$all points that are equidistant from A and $B\}$. The sum of distances $d(A, F) + d(F, B)$ can now be greater than 10. Our line of midpoints is a subset of our new set of points, but our set of points does not simply extend this line. Let us consider a one unit change in each direction from a midpoint on the edge of our line segment, $(-2, 3)$. Move one unit in the positive x direction, and we will be closer to B than to A. Move one unit in the negative x direction, and it will be the other way around. However, if we move one unit in the poistive y direction to $(-2, 4)$, our distances each increase by exactly one. This can be done starting from our new point $(-2, 4)$, and the same result can be found. The reason for this is that our last midpoint was on the top edge of the magic rectangle of shortest paths, and its x coordinate was still between A_x and B_x. As we move out of our rectangle, we must go upward, because the y coordinate will be greater or equal to both A_y and B_y, meaning that distance will be added to both $d(F, A)$ and $d(F, B)$, whereas moving in the x direction would offset this balance. Similarly, from our other edge case midpoint we go downward, as our y coordinate is lesser than both A_y and B_y, and we are on the bottom edge of our magic rectangle.