Stochastic Elements of Mathematical Biology - Homework 2

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Problem 1

For each time iteration, we have three possibilities

$$X_{k+1} = X_k - 1,$$
$$X_{k+1} = X_k,$$
$$X_{k+1} = X_k + 1$$

So for $X_{k+1} = X_k - 1$ then we must have the following conditions:

Select A individual to be replaced.
a) Replace with some a individual.
b) Replace with A and it mutates to a during cloning.

So we select A individual with probability of $\frac{X_t}{n}$ and then the probability of a) occuring given we have selected an A individual is $\frac{n-X_t}{n}$. Then the probability of b) occuring is $\frac{X_t}{n} \cdot \rho$. So we say that

$$p(X_{k+1} = X_k - 1) = \left(\frac{X_t}{n}\right) \cdot \left(\frac{n - X_t}{n} + \frac{X_k}{n} \cdot \rho\right)$$

And then for $X_{k+1} = X_k + 1$ we must have the following conditions:

Select a individual to be replaced.
Replace with A.
The individual A does not mutate to a during cloning.

To satisfy these conditions we pick an individual a with probability $\frac{n-X_t}{n}$ and the probability to choose some individual A is $\frac{X_k}{n}$, and the this individual mutates during cloning with a probability of $1 - \rho$. Then we say

$$p(X_{k+1} = X_k + 1) = \left(\frac{n - X_t}{n}\right) \cdot \left(\frac{X_k}{n}\right) \cdot (1 - \rho)$$

Then we are left with

$$p(X_{k+1} = X_k) = 1 - p(X_{k+1} = X_k - 1) - p(X_{k+1} = X_k + 1)$$

Our fixation time $T_f = \min\{t \in \{0, 1, 2, \cdots\} \, | X_t = 0$ or $X_t = n\}$.

Problem 2

To compute the fixation time we have to look at the following recurrence relation

$$\Delta\varphi(j) = \frac{q_j}{p_j}\Delta\varphi(j-1) - \frac{1}{p_j}$$

So if we extend this out to its final form we get

$$\Delta\varphi(j) = \prod_{k=0}^{j}\left(\frac{q_k}{p_k}\right)\Delta\varphi(0) - \sum_{k=0}^{j}\frac{\prod_{i=j-k+1}^{j} q_i}{\prod_{i=j-k}^{j} p_i}$$

Then we have clearly that $\Delta\varphi(0) = \varphi(1)$. Denote

$$g_p(i,j) = \prod_{k=i}^{j} p_k$$
$$g_q(i,j) = \prod_{k=i}^{j} q_k$$

Then we write

$$\Delta\varphi(j) = \varphi(1)\frac{g_q(0,j)}{g_p(0,j)} - \sum_{k=0}^{j} \frac{g_q(j-k+1,j)}{g_p(j-k,k)}$$

And it follows that we can write

$$\begin{aligned}\varphi(j) &= \sum_{m=0}^{j-1} \Delta\varphi(m) \\ &= \sum_{m=0}^{j-1} \left(\varphi(1)\frac{g_q(0,m)}{g_p(0,m)} - \sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right)\end{aligned}$$

If we choose $j = n$ then $\varphi(n) = 0$ so we write

$$\begin{aligned}0 &= \sum_{m=0}^{n-1} \left(\varphi(1)\frac{g_q(0,m)}{g_p(0,m)} - \sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right) \\ 0 &= \left(\sum_{m=0}^{n-1} \varphi(1)\frac{g_q(0,m)}{g_p(0,m)}\right) - \sum_{m=0}^{n-1} \left(\sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right) \\ \sum_{m=0}^{n-1} \left(\sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right) &= \left(\sum_{m=0}^{n-1} \varphi(1)\frac{g_q(0,m)}{g_p(0,m)}\right) \\ \sum_{m=0}^{n-1} \left(\sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right) &= (n-1)\varphi(1)\left(\sum_{m=0}^{n-1} \frac{g_q(0,m)}{g_p(0,m)}\right) \\ \frac{\sum_{m=0}^{n-1}\left(\sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right)}{(n-1)\left(\sum_{m=0}^{n-1} \frac{g_q(0,m)}{g_p(0,m)}\right)} &= \varphi(1)\end{aligned}$$

Now we can plug $\varphi(1)$ back into our $\Delta\varphi(j)$ and get

$$\Delta\varphi(j) = \frac{\sum_{m=0}^{n-1}\left(\sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right)}{(n-1)\left(\sum_{m=0}^{n-1} \frac{g_q(0,m)}{g_p(0,m)}\right)}\frac{g_q(0,j)}{g_p(0,j)} - \sum_{k=0}^{j} \frac{g_q(j-k+1,j)}{g_p(j-k,k)}$$

So for our final formula for $\varphi(j)$ we get

$$\begin{aligned}\varphi(j) &= \sum_{a=0}^{j-1} \Delta\varphi(a) \\ &= \sum_{a=0}^{j-1} \frac{\sum_{m=0}^{n-1}\left(\sum_{k=0}^{m} \frac{g_q(m-k+1,j)}{g_p(m-k,k)}\right)}{(n-1)\left(\sum_{m=0}^{n-1} \frac{g_q(0,m)}{g_p(0,m)}\right)}\frac{g_q(0,a)}{g_p(0,a)} - \sum_{k=0}^{a} \frac{g_q(a-k+1,j)}{g_p(a-k,k)}\end{aligned}$$

This is in fact quite messy, and most likely incorrect.