MTH 312 Homework 2

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(1) 6.3.2

Let $h_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

(a)

Find the pointwise limit of h_n and prove that it converges uniformly.

$$\lim_{n\to\infty} h_n(x) = h(x) = \sqrt{x^2}$$

Before proving uniform convergence let us do some scratch work. Let a and b be real non-negative numbers. We know that $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$ by the triangle inequality. Since $\sqrt{a} \leqslant \sqrt{a+b}$, and of course $-\sqrt{b} \leqslant 0$, it follows that

$$\sqrt{a} - \sqrt{b} \leqslant \sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$$

Which is equivalent to the statement

$$\left|\sqrt{a+b} - \sqrt{a}\right| \leqslant \left|\sqrt{b}\right|$$

Proof. Let $\epsilon > 0$ be arbitrary. Then let $N > \frac{1}{\epsilon^2}$. Then for all $n \geqslant N$, $\sqrt{\frac{1}{n}} < \epsilon$.

$$|h_n(x) - h(x)| = \left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| \leqslant \sqrt{\frac{1}{n}} < \epsilon$$

□

(b)

Take the derivative of $h_n(x)$ and we get

$$h_n'(x) = \frac{x}{2\sqrt{x^2 + \frac{1}{n}}}$$

Note that $h_n'(0) = \frac{0}{2\sqrt{\frac{1}{n}}}$ for every natural number n. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we say that

$$\lim_{n\to\infty} h_n'(x) = g(x) = \begin{cases} \frac{x}{2\sqrt{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Since the pointwise limit $g(x)$ is not continuous at 0, it follows that $h_n'(x)$ cannot converge uniformly on any neighborhood of 0. This is the case because continuity is preserved under uniform convergence, yet each h_n' is continuous while g is not.

(2) 6.3.5

Let $g_n(x) = \dfrac{nx + x^2}{2n}$.

(a)

We can take the limit of $g_n(x)$ to be

$$\begin{aligned}
\lim_{n\in\mathbb{N}} g_n(x) &= \lim_{n\in\mathbb{N}} \frac{nx + x^2}{2n} \\
&= \lim_{n\in\mathbb{N}} \frac{nx}{2n} + \lim_{n\in\mathbb{N}} \frac{x^2}{2n} \\
&= \lim_{n\in\mathbb{N}} \frac{x}{2} + x^2 \lim_{n\in\mathbb{N}} \frac{1}{2n} \\
&= \frac{x}{2} + x^2(0) \\
&= \frac{1}{2}x
\end{aligned}$$

And thus we say,

$$\lim g_n(x) = g(x) = \frac{1}{2}x$$

Then we can use the power rule to compute the derivative

$$g'(x) = \frac{1}{2}$$

(b)

We want to show that $g_n'(x)$ converges uniformly on every interval $[-M, M]$.

Proof. Let us first compute what g_n' is. We know $g_n(x) = \dfrac{nx + x^2}{2n}$. Then since the function is continuous and differentiable on all of $\mathbb{R}$ we can use familiar rules such as the seperation and the power rules. We determine that $g_n'(x) = \dfrac{n + x}{2n}$. Now taking the pointwise limit we get $g(x) = \frac{1}{2}$. Let $M > 0, \epsilon > 0$ be arbitrary. Choose $N > \dfrac{M}{2\epsilon}$. Then for all $n \geqslant N$, $\quad \dfrac{M}{2n} < \epsilon \quad$ hence,

$$|g_n'(x) - g(x)| = \left|\frac{n + x}{2n} - \frac{1}{2}\right| = \left|\frac{1}{2} + \frac{x}{2n} - \frac{1}{2}\right| = \left|\frac{x}{2n}\right| \leqslant \left|\frac{M}{2n}\right| < \epsilon$$

This is true so long as $|x| \leqslant M$. Therefore on the interval $[-M, M], g_n'$ converges uniformly.

Choose the point $x = 0$, then $x \in [-M, M]$. Also for all $n \in \mathbb{N}$, $g_n(0) = \dfrac{n(0) + (0)^2}{2n} = 0$. This of course converges to $g(0) = 0$. By $\boxed{\text{Theorem 6.3.3}}$ (f_n) converges uniformly and the limit function $f = \lim f_n$ is differentiable. Also by this theorem we conclude that $g' = \lim g_n'$. $\square$