MTH 463 Assignment 3

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Problem 1

Expectation and variance of a discrete probability mass function. Let $p_X(0) = \frac{1}{3}, p_X(2) = \frac{1}{2}, p_X(3) = \frac{1}{6}$. Compute the following:

$E(X)$

To compute the expectation, we say $E(X) = \sum x p_X(x)$.

$$(0)\frac{1}{3} + (2)\frac{1}{2} + (3)\frac{1}{6} = \frac{3}{2} = 1.5$$

$Var(X)$

We know that to compute the variance we simply need to compute $E(X^2) - E(X)^2$.

$$(0^2)\frac{1}{3} + (2^2)\frac{1}{2} + (3^2)\frac{1}{6} - \frac{3}{2}^2 = \frac{7}{2} - \frac{9}{4} = \frac{5}{4} = 1.25$$

$E(|X - E(X)|)$

First note that for the expectation of a constant value c, $E(c) = c$. Since expectation is distributive over addition, we have two cases.
$\boxed{\text{Case 1: } X - E(X) > 0}$ In this case we say

$$E(|X - E(X)|) = E(X - E(X)) = E(X) - E(E(X)) = E(X) - E(X) = 0$$

$\boxed{\text{Case 2: } X - E(X) \leqslant 0}$ In this case the subtraction is reversed when we remove the absolute value, and the difference is still 0.

$E(2^X)$

By the Law of the Unconscious Statistician, we say $E(f(X)) = \sum p_X(x) f(x)$. This gives us the following result:

$$E(2^X) = \sum 2^x p_X(x) = (2^0)\frac{1}{3} + (2^2)\frac{1}{2} + (2^3)\frac{1}{6} = \frac{1}{3} + 2 + \frac{4}{3} = \frac{11}{3} = 3.67$$

Problem 2

Compute $E(\frac{1}{X+1})$ for a Poisson distribution with $\lambda > 0$.

$$\begin{aligned}
E\left(\frac{1}{X+1}\right) &= \sum_{i=0}^{\infty} \frac{1}{i+1} e^{-\lambda} \frac{\lambda^i}{i!} \\
&= e^{-\lambda} \left(\sum_{i=0}^{\infty} \frac{1}{i+1} \frac{\lambda^i}{i!} \right) \\
&= e^{-\lambda} \left(\sum_{i=0}^{\infty} \frac{\lambda^i}{(i+1)!} \right) \\
&= e^{-\lambda} \left(\frac{e^{\lambda} - 1}{\lambda} \right) && \text{(by Taylor expansion of exponential)} \\
&= e^{-\lambda} \frac{e^{\lambda}}{\lambda} - e^{-\lambda} \frac{1}{\lambda} \\
&= \frac{1}{\lambda}(1 - \frac{1}{e^{\lambda}})
\end{aligned}$$

Problem 3

Proof. Let $q = (1-p)$ for readability.

$$\begin{aligned}
E(\frac{1}{X+1}) &= \sum i = 0^n \frac{1}{i+1}\binom{n}{i} p^i q^{n-1} \\
&= \sum i = 0^n \binom{n+1}{i+1}\frac{1}{n+1} p^i q^{n-i} \\
&= \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i+1} p^i q^{n-i} \\
&= \frac{1}{p(n+1)}\sum_{i=0}^{n}\binom{n+1}{i+1} p^{i+1} q^{n-i} \\
&= \frac{1}{p(n+1)}\sum_{i=1}^{n+1}\binom{n+1}{i} p^{i} q^{n-i+1} \\
&= \frac{1}{p(n+1)}\left[\left(\sum_{i=0}^{n+1}\binom{n+1}{i} p^{i} q^{n-i+1}\right) - \binom{n+1}{0} p^0 q^{n-0+1}\right] \\
&= \frac{1}{p(n+1)}\left[\left((p+q)^{n+1}\right) - q^{n+1}\right] \\
&= \frac{1}{p(n+1)}\left[1 - (1-p)^{n+1}\right]
\end{aligned}$$

□

Problem 4

Proof. We write the following

$$\begin{aligned}
\sum_{j=1}^{\infty} P(X \geqslant j) &= \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} P(X = k) \\
&= P(X=1) + P(X=2) + P(X=3) + \cdots \\
&\qquad\qquad + P(X=2) + P(X=3) + \cdots \\
&\qquad\qquad\qquad\qquad + P(X=3) + \cdots \\
&= P(X=1) + P(X=2) + P(X=2) + P(X=3) + P(X=3) + P(X=3) + \cdots \\
&= (1)P(X=1) + (2)P(X=2) + (3)P(X=3) + \cdots \\
&= \sum_{j=1}^{\infty} jP(X=j) \\
&= E(X)
\end{aligned}$$

□

Problem 5

Proof. Let us compute $P(W > j)$ first. We write

$$\begin{aligned}
P(W > j) &= P(W = j+1) + P(W = j+2) + \cdots \\
&= \sum_{n=j+1}^{\infty} P(W = n) \\
&= \sum_{n=j+1}^{\infty} p(1-p)^{n-1} \\
&= p \sum_{n=j+1}^{\infty} (1-p)^{n-1} \\
&= p \sum_{n=j}^{\infty} (1-p)^{n} \\
&= p \left[\frac{1}{1-(1-p)} - \sum_{n=0}^{j-1} (1-p)^n \right] \\
&= p \left[\frac{1}{p} - \frac{1-(1-p)^j}{1-(1-p)} \right] \\
&= 1 - (1 - (1-p)^j) \\
&= (1-p)^j
\end{aligned}$$

Now we use our result from $\boxed{\text{Problem 4}}$ to say that

$$E(W) = \sum_{j=0}^{\infty} (1-p)^j = \frac{1}{1-(1-p)} = \frac{1}{p}$$

□

Problem 6

Let X be a random variable such that $P(X = 1) = p$ and $P(X = -1) = 1 - p$. Find $a \neq 1$ such that $E[a^x] = 1$. We can rewrite $E[a^x]$ as $E[a^x] = p(a^1) + (1-p)(a^{-1})$, since those are the only two events in this sample space. We want to solve for values of a such that this equation is equal to 1, so we set the equation equal to 1 and solve for a.

$$\begin{aligned}
(p)a^1 + (1-p)a^{-1} &= 1 \\
(p)a^2 + (1-p) &= a \\
(p)a^2 - a + (1-p) &= 0
\end{aligned}$$

From here we can say that by the quadratic formula, $a = \frac{1+-\sqrt{1-4(p)(1-p)}}{2p} = \frac{1+-\sqrt{1-4p+4p^2}}{2p}$.

Problem 7

We wish to show that $E(X^2) \geq E(X)^2$.

Proof. We have $E(X^2) - E(X)^2 = E(X^2) - 2E(X)^2 + E(X)^2$. Then we can replace $E(X)$ with μ for some of the terms giving us $E[X^2 - 2E(X)^2 + E(X)^2] = E(X^2) - 2E(X)\mu + (\mu)^2$. By the distributive property of expectation, this is equal to $E[X^2 - 2X\mu + \mu^2] = E[(x-\mu)^2]$. Then we write

$$E[(x-\mu)^2] = \sum_{1 \leq j \leq \infty} p(x_j)(x_j - \mu)^2$$

It is clear that since our probability is non-negative and that $(a-b)^2$ is non-negative that $E[(x-\mu)^2] \geqslant 0$. So we have $E(X^2) - E(X)^2 = E[(X-\mu)^2] \geqslant 0 \Longrightarrow E(X^2) \geqslant E(X)^2$. □

Problem 8

Let X be a random variable, and $Y = \frac{X-\mu}{\sigma}$. We write

$$\begin{aligned}
E[Y] &= E[\frac{X-\mu}{\sigma}] \\
&= E[\frac{X}{\sigma}] - E[\frac{\mu}{\sigma}] \\
&= \frac{1}{\sigma}E[X] - \frac{1}{\sigma}E[\mu] \\
&= \frac{1}{\sigma}\mu - \frac{1}{\sigma}E[\mu] \\
&= 0
\end{aligned}$$

And thus our answer is 0.

Problem 9

We wish to find what values of p would cause a 3-engine rocket to be more reliable than a 5-engine rocket, where each engine fails with a probability of p. The success of each rocket will be a simple binomial distribution. For the three engine rocket we have the probability

$$S_3 = p^3\binom{3}{0} + p^2(1-p)\binom{3}{1}$$

We omit the terms where we have fewer than 2 rockets that work. For the 5-engine rocket we have the probability

$$S_5 = p^5\binom{5}{0} + p^4(1-p)\binom{5}{1} + p^3(1-p)^2\binom{5}{2}$$

From here, we wish to find p such that the value of the first sum is greater than that of the second. We want to find $p \in [0,1] : S_3 > S_5$.

$$\begin{gathered}
p^3\binom{3}{0} + p^2(1-p)\binom{3}{1} > p^5\binom{5}{0} + p^4(1-p)\binom{5}{1} + p^3(1-p)^2\binom{5}{2} \\
p^3 + 3p^2(1-p) > p^5 + 5p^4(1-p) + 10p^3(1-p)^2
\end{gathered}$$

This is true for $p \in (0, \frac{1}{2})$.