MTH 311 Homework 7

Philip Warton

March 5, 2020

4.2.5

(a)

Show that $\lim_{x\to 2}(3x+4) = 10$.

Proof. Let $\epsilon > 0$ be arbitrary. Let $\delta = \dfrac{\epsilon}{3}$. Then, if $0 < |x-2| < \delta$, we have

$$0 < |x-2| < \frac{\epsilon}{3}$$
$$|3x-6| < \epsilon$$
$$|(3x+4)-10| < \epsilon$$

Therefore $\lim_{x\to 2}(3x+4) = 10$. ☐

(b)

Show that $\lim_{x\to 0} x^3 = 0$.

Proof. Let $\epsilon > 0$ be arbitrary. Let $\delta = \sqrt[3]{\epsilon}$. Then, if $0 < |x-0| < \delta$, it follows that $|x^3| < \epsilon$. Therefore, $\lim_{x\to 0} x^3 = 0$. ☐

4.2.7

Let $g : A \to \mathbb{R}$. Let f be a function such that $\exists M > 0 : |f(x)| \leqslant M \ \forall x \in A$. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x)$ is also 0.

Proof. Assume that $\lim_{x\to c} g(x) = 0$. Let $\frac{\epsilon}{M}$ be arbitrary, where M is a bound for f on A. Then there exists δ such that if $0 < |x-c| < \delta$, then $|g(x)-0| < \frac{\epsilon}{M}$. It Follows that $|Mg(x)| < \epsilon$, and since $f(x) \leqslant M$ for all $x \in A$, we write

$$|f(x)g(x)| \leqslant |Mg(x)| < \epsilon$$
$$\Rightarrow \quad |(f(x)g(x)) - 0| < \epsilon$$

Therefore $\lim_{x\to c} g(x)f(x) = 0$. ☐

4.3.3

(a)

Prove theorem 4.3.9 using epsilon delta continuity.

The theorem we wish to prove: Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ assume that the range $f(A) = \{f(x) : x \in A\}$ si contained in the domain B so that $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f(c)$ is continuous at c.

Proof. Let $\epsilon_g > 0$ be arbitrary. Since g is continuous at $f(c)$, there exists $\delta_g > 0$ such that for all x where $|x - f(c)| < \delta_g$, we know $|g(x) - g(f(c))| < \epsilon_g$. Choose $\epsilon_f = \delta_g$, then, since f is continuous at c, there exists $\delta_f > 0$ where if $|x - c| < \delta_f$ then $|f(x) - f(c)| < \epsilon_f = \delta_g$. Then, since $|f(x) - f(c)| < \delta_g$, it follows that $|g(f(x)) - g(f(c))| < \epsilon_g$. Therefore for any arbitrary $\epsilon_g > 0$ there exists some δ_f, where $|x - c| < \delta_f \Rightarrow |g(f(x)) - g(f(c))| < \epsilon_g$. □

(b)

We must now proof this same theorem using the sequential characterization of continuity.

Proof. Since f is continuous at c, for all $(x_n) \to c$ with $x_n \in A$, $f(x_n) \to f(c)$. Then, since g is continuous at $f(c)$, and $f(x_n) \to f(c)$, it follows that $g(f(x_n)) \to g(f(c))$. □

4.3.5

Show using the epsilon delta definition of continuity that if c is an isolated point of $A \subset \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Proof. Let $\epsilon > 0$. Since c is an isolated point of A, there exists some $V_\delta(c) = \{c\}$. Suppose $x \in A$, and $|x - c| < \delta$, then x must be equal to c. Then, for all $x \in A$ where $|x - c| < \delta$, we know that $|f(x) - f(c)|$ is equivalent to $|f(c) - f(c)| = 0 < \epsilon$. Thus, f is continuous at c. □

4.4.3

Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the interval $[1, \infty)$ but not on the set $(0, 1]$. Let us begin with the first interval, the closed set $[1, \infty)$.

Proof. First, notice that for all x, y in the interval, $x \leqslant 1$ and $y \leqslant 1$. Also note that $\dfrac{1}{x^2y^2} \leqslant 1$. Then,

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2}y^2\right| = \left|\frac{(x+y)(x-y)}{x^2y^2}\right| = \left|\left(\frac{x}{x^2y^2} + \frac{y}{x^2y^2}\right)(x-y)\right| \leqslant \left(\frac{1}{x^2y^2} + \frac{1}{x^2y^2}\right)|x-y| \leqslant 2|x-y|$$

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \dfrac{\epsilon}{2}$. Then, for all $x, y \in (0, 1]$, if $|x - y| < \delta$, then

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leqslant 2|x-y| < 2\delta = 2\left(\frac{\epsilon}{2}\right) = \epsilon$$

Thus we have shown uniform continuity on the interval $(0, 1]$. □

Now we want to show that we do not have uniform continuity on $(0, 1]$.

Proof. Assume by contradiction that f is uniformly continuous. Choose $\epsilon = 1$, and there should exist some $\delta > 0$ such that for all $x, y \in (0, 1]$, if $|x - y| < \delta$ then $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| < 1$. Let $x < \delta$, and let $y = \dfrac{x}{\sqrt{2}}$. Then,

$$|x - y| = \left|x - \frac{x}{\sqrt{2}}\right| = \left|\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)x\right| < |x| < \delta$$

And since $|x - y| < \delta$, it should follow that $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| < 1$. We write

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{1}{x^2} - \frac{1}{(\frac{x}{\sqrt{2}})^2}\right| \left|\frac{1}{x^2} - \frac{2}{x^2}\right| = \left|\frac{-1}{x^2}\right| = \frac{1}{x^2} > 1$$

Since $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| < 1$, and $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| > 1$ we have a contradiction, and our assumption that f is uniformly continuous must be false. Therefore f is not uniformly continuous on $(0, 1]$. □