Applied Ordinary Differential Equations Notes

Philip Warton

September 22, 2021

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We begin with the following form of a second order autonomous ODE:

$$ay''(t) + by'(t) + cy(t) = f(t)$$

This equation can be transformed to a first order system of ODE's, if we define two equations to be

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{aligned}$$

Then the euqation can be rewritten in terms of x_1, x_2 as

$$ax_2'(t) + bx_2(t) + cx_1(t) = f(t)$$

Which immediately gives us this first order system of ODE's:

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{c}{a}x_1(t) - \frac{b}{a}x_2(t) + \frac{f(t)}{a} \end{aligned}$$

For notational purposes, write $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \vec{x'}(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$. Then we can write many first order systems of ODE's as

$$\boxed{\begin{aligned} A\vec{x}(t) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t) \end{pmatrix} \end{aligned}}$$

For our previous example we would have $A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$ ignoring the $f(t)$ term.

Now we study autonomous first order systems of ODE's. That is, $\vec{x'}(t) = A\vec{x}(t)$. These have solutions that can be represented as

$$\vec{x}(t) = e^{At}\vec{x_0}, \qquad e^{At} = \sum_{j=0}^{\infty} \frac{1}{j!} A^j t^j$$