MTH 483 Homework 3

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Problem 2.21

Assume f and $\overline{f}$ are holomorphic on a region $G \subset \mathbb{C}$. Show that f is constant.

Proof. Let f be a complex function such that f and $\overline{f}$ are holomorphic on the region $G \subset \mathbb{C}$. Denote $f(x, y) = u(x, y) + v(x, y)i$. Since f is holomorphic on G, by the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Then since $\overline{f(x, y)} = u(x, y) - v(x, y)i$ is holomorphic we similarly have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

It follows that $\frac{\partial u}{\partial x} = 0$ and that $\frac{\partial v}{\partial x} = 0$. By the Cauchy-Riemann equations $f'(x, y) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 + 0i$. Hence f is constant. □

Problem 3.33

Describe the set $f(A)$ where $f(z) = \exp(z)$ and A is a subset of $\mathbb{C}$.

(a)

Let $A = \{iy \ : \ 0 \leqslant y \leqslant 2\pi\}$. Then $f(A) = \{\exp(iy) \ : \ 0 \leqslant y \leqslant 2\pi\} = S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Interpreting this set in polar form, it becomes clear that this set is the unit circle.

(b)

Let $A = \{1 + iy \in \mathbb{C} : 0 \leqslant y \leqslant 2\pi\}$. By properties of the exponential it follows that $\exp(1 + iy) = \exp(1) \cdot \exp(iy) = e \cdot \exp(iy)$. This is a circle around 0 of radius e.

(c)

Let $A = \{x + yi \in \mathbb{C} : x \in [0, 1], y \in [0, 2\pi]\}$. By properties of the exponential we write $\exp(x + yi) = \exp(x) \cdot \exp(yi)$. We know that $\exp(x)$ is real valued and that $\exp(iy)$ is a point on the unit circle. This means our set $\exp(A)$ will be a ring consisting of all circles with $1 \leqslant r \leqslant e$ centered at 0.

Problem 3.40

Find the principle values of the following.

(a)

$\text{Log}(2i) = \ln|2i| + i\text{Arg}(2i) = \ln 2 + i\frac{\pi}{2}$.

(b)

$(-1)^i = \exp(iLog(-1)) = \exp(i[ln| - 1| + i\text{Arg}(-1)]) = \exp(i(0) + (i^2)\pi) = \exp(-\pi)$.

(c)

$\text{Log}(-1+i) = \ln|-1+i| + i\text{Arg}(-1+i) = \ln\sqrt{2}^{-1} + i\frac{3\pi}{4}$.

Problem 3.41

Convert the following complex numbers to the form $x + yi$.

(c)

i^i. For this we can use the principle value definition of a^b which gives us

$$i^i = \exp(i\text{Log}(i)) = \exp(i[\ln|i| + i\text{Arg}(i)]) = \exp(i[0 + \frac{\pi}{2}i]) = \exp(-\frac{\pi}{2}) = e^{-\frac{\pi}{2}} + 0i$$

(e)

$\exp(\text{Log}(3+4i))$. By the properties of the complex logarithm we have $\exp(\text{Log}(3+4i)) = 3+4i$

(f)

$(1+i)^{\frac{1}{2}}$. We will use our principle value definition again here

$$\begin{aligned}(1+i)^{\frac{1}{2}} &= \exp\left(\frac{1}{2}\left(\text{Log}(1+i)\right)\right)\\ &= \exp\left(\frac{1}{2}\left[\ln\left(\sqrt{2}\right) + i\left(\frac{\pi}{4}\right)\right]\right)\\ &= \exp\left(\frac{\ln(\sqrt{2})}{2} + i\left(\frac{\pi}{8}\right)\right)\\ &= e^{\frac{\ln\sqrt{2}}{2}} \cdot e^{i\frac{\pi}{8}}\end{aligned}$$

Then we can use the properties of the expontential to write $e^{\frac{\ln\sqrt{2}}{2}} = \sqrt[4]{2}$. Which in combination with a conversion from polar to rectangular gives use the following

$$\begin{aligned}(1+i)^{\frac{1}{2}} &= \left[e^{\frac{\ln\sqrt{2}}{2}}\cos\left(\frac{\pi}{8}\right)\right] + \left[e^{\frac{\ln\sqrt{2}}{2}}\sin\left(\frac{\pi}{8}\right)\right]i\\ &= \left[\sqrt[4]{2}\cos\left(\frac{\pi}{8}\right)\right] + \left[\sqrt[4]{2}\sin\left(\frac{\pi}{8}\right)\right]i\end{aligned}$$

Problem 3.45

Find all solutions to the following.

(b)

$\text{Log}(z) = \frac{3\pi i}{2}$. This is equivalent to $\ln|z| + i\text{Arg}(z) = \frac{3\pi i}{2}$. Since the right hand side has no real part $\ln|z| = 0 \longrightarrow |z| = 1$. From there it follows that $\text{Arg}(z) = \frac{3\pi i}{2}$. Solutions to the equation are $z = e^{i(\frac{3\pi}{2}+2\pi k)}$ where $k \in \mathbb{Z}$.

(c)

$\exp(z) = \pi i$. This is equivalent to writing $e^x e^{iy} = \pi i$. To have the modulus be equal we must have $|e^x e^{iy}| = |\pi i| \Longrightarrow e^x = \pi$ and it follows that $x = \ln\pi$. Then we must have $e^{iy} = i$ which means that $y = \frac{\pi}{2} + 2\pi k$ where $k \in \mathbb{Z}$. So $z = \ln\pi + (\frac{\pi}{2} + 2\pi k)i$ where $k \in \mathbb{Z}$.

(e)

$\cos(z) = 0$. For this we write $\frac{1}{2}(e^{iz} + e^{-iz}) = 0$. Then it must be the case that $e^{iz} = -e^{-iz}$. Denote $z = x + yi$ and we have

$$e^{i(x+yi)} = -e^{-i(x+yi)} \iff e^{-y}e^{ix} = -e^{y}e^{-ix}$$

From what we know about polar form it follows that to have the modulus of the left and right hand side be equal we must have $e^{-y} = e^{y} \Rightarrow y = 0$. It follows that $z = x \in \mathbb{R}$. Since $z \in \mathbb{R}$ we can use our real-valued trig function to assert that $z = \frac{\pi}{2} + \pi k$ where $k \in \mathbb{Z}$.

Problem 4.2

Compute the lengths of the following paths:

(a)

The circle C[1+i,1]. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ where $\gamma(t) = e^{it} + 1 + i$. We have length$(\gamma) = \int_0^{2\pi} |ie^{it}|dt = \int_0^{2\pi} 1dt = 2\pi$.

(b)

The line segment from $-1-i$ to $2i$. This can be parameterized by $\gamma : [0, 1] \to \mathbb{C}$ where $\gamma(t) = (t-1)+(3t-1)i$. Then $\gamma'(t) = 1+3i$. So we have length$(\gamma) = \int_0^1 |1 + 3i|dt = \int_0^1 \sqrt{10}dt = \sqrt{10}$.

(c)

The top half of the circle $C[0, 34]$. Let $\gamma : [0, \pi] \to \mathbb{C}$ where $\gamma(t) = 34e^{it}$. Then $\gamma'(t) = i34e^{it}$. It follows that length$(\gamma) = \int_0^{\pi} |i34e^{it}|dt = \int_0^{\pi} 34dt = 34\pi$.

Problem 4.4

Compute $\int_\gamma \frac{dz}{z}$ where γ is the counter clock-wise unit circle.

We can paramaterize this circle with $\gamma : [0, 2\pi] \to \mathbb{C}$ where $\gamma(t) = e^{it}$. Using this parameterization we compute the integral to be

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it}dt = \int_0^{2\pi} idt = 2\pi i$$

Now show that for any circle $C[w, r]$ we have the result $\int_{C[w,r]} \frac{dz}{z-w} = 2\pi i$.

For the circle generally we paramaterize it by the function $\gamma : [0, 2\pi] \to \mathbb{C}$ where $\gamma(t) = re^{it} + w$. Now we compute the integral and get the desired result.

$$\int_\gamma \frac{dz}{z-w} = \int_0^{2\pi} \frac{1}{[re^{it} + w] - w} \cdot ire^{it}dt = \int_0^{2\pi} \frac{1}{re^{it}} \cdot ire^{it}dt = \int_0^{2\pi} idt = 2\pi i$$