General Topology and Fundamental Groups - Homework 1

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Problem 1

Let $f : X \to Y$ be a function. Let $A \subset X$ and $B \subset Y$. Let $U_\alpha \subset X, \alpha \in \mathcal{A}$ and $V_\beta \subset Y, \beta \in \mathcal{B}$.

(a)

$\boxed{f(\bigcup_{\alpha\in\mathcal{A}} U_\alpha) = \bigcup_{\alpha\in\mathcal{A}} f(U_\alpha)}$

Proof. $\boxed{\subset}$ Let $x \in f(\bigcup_{\alpha\in\mathcal{A}} U_\alpha)$. By definition of the image of a set, there exists $x' \in \bigcup_{\alpha\in\mathcal{A}} U_\alpha$ such that $f(x') = x$. Therefore $\exists\alpha_0 \in \mathcal{A}$ such that $x' \in U_{\alpha_0}$. It follows that $x \in f(U_{\alpha_0})$ and therefore $x \in \bigcup_{\alpha\in\mathcal{A}} f(U_\alpha)$.

$\boxed{\supset}$ Let $y \in \bigcup_{\alpha\in\mathcal{A}} f(U_\alpha)$. Then there exists some $\alpha_1 \in \mathcal{A}$ such that $y \in f(U_{\alpha_1})$. Thus $\exists y' \in U_{\alpha_1} : f(y') = y$. This element y' belongs to the union of all U_α such that $\alpha \in \mathcal{A}$ therefore y belongs to the image of this union. $\square$

(b)

$\boxed{f(\bigcap_{\alpha\in\mathcal{A}} U_\alpha) \subset \bigcap_{\alpha\in\mathcal{A}} f(U_\alpha)}$

Proof. Let $x \in f(\bigcap_{\alpha\in\mathcal{A}} U_\alpha) \implies \exists x' \in \bigcap_{\alpha\in\mathcal{A}} U_\alpha : f(x') = x \implies \forall\alpha \in \mathcal{A}, x' \in U_\alpha \implies \forall\alpha \in \mathcal{A}, x \in f(U_\alpha) \implies x \in \bigcap_{\alpha\in\mathcal{A}} f(U_\alpha)$ $\square$

(c)

$\boxed{f^{-1}(\bigcup_{\beta\in\mathcal{B}} V_\beta) = \bigcup_{\beta\in\mathcal{B}} f^{-1}(V_\beta)}$

Proof. $\boxed{\subset}$ Let $x \in f^{-1}(\bigcup_{\beta\in\mathcal{B}} V_\beta)$. Then $f(x) \in \bigcup_{\beta\in\mathcal{B}} V_\beta$. Thus $\exists\beta \in \mathcal{B} : f(x) \in V_\beta$. Therefore $x \in f^{-1}(V_\beta) \subset \bigcup_{\beta\in\mathcal{B}} f^{-1}(V_\beta)$.

$\boxed{\supset}$ Let $x \in \bigcup_{\beta\in\mathcal{B}} f^{-1}(V_\beta)$. Then $\exists\beta \in \mathcal{B} : x \in f^{-1}(V_\beta)$. So this means $f(x) \in V_\beta$ for some $\beta \in \mathcal{B}$. Then $f(x) \in \bigcup_{\beta\in\mathcal{B}} V_\beta$ therefore $x \in f^{-1}(\bigcup_{\beta\in\mathcal{B}} V_\beta)$. $\square$

(d)

$\boxed{f^{-1}(\bigcap_{\beta\in\mathcal{B}} V_\beta) = \bigcap_{\beta\in\mathcal{B}} f^{-1}(V_\beta)}$

Proof. $\boxed{\subset}$ Let $x \in f^{-1}(\bigcap_{\beta\in\mathcal{B}} V_\beta)$. Then $f(x) \in \bigcap_{\beta\in\mathcal{B}} V_\beta$, which means $f(x) \in V_\beta$ for every $\beta \in \mathcal{B}$. Thus $x \in f^{-1}(V_\beta)$ for every $\beta \in \mathcal{B}$, and the desired inclusion follows.

$\boxed{\supset}$ Let $x \in \bigcap_{\beta\in\mathcal{B}} f^{-1}(V_\beta)$. Then for every $\beta \in \mathcal{B}, x \in f^{-1}(V_\beta)$. So it must be the case that $f(x) \in V_\beta$ for every $\beta \in \mathcal{B}$. So it follows that $f(x) \in \bigcap_{\beta\in\mathcal{B}} V_\beta$ and thus we have $f^{-1}(\bigcap_{\beta\in\mathcal{B}} V_\beta) \supset \bigcap_{\beta\in\mathcal{B}} f^{-1}(V_\beta)$. $\square$

Problem 2

Let $f : X \to Y$ be a funciton. Prove that the following are equivalent.

$$\begin{gathered}(i)\ f \text{ is injective}\\ (ii)\ \forall A \subset X, f^{-1}(f(A)) = A\\ (iii)\ \forall A, B \subset X, f(A \cap B) = f(A) \cap f(B)\\ (iv)\ \forall A, B \subset X, A \cap B = \emptyset \Rightarrow f(A) \cap f(B) = \emptyset\\ (v)\ \forall A, B \subset X | B \subset A, \quad f(A \setminus B) = f(A) \setminus f(B)\end{gathered}$$

Proof. $\boxed{\text{(i)} \Rightarrow \text{(ii)}}$ Assume that f is injective. Let $A \subset X$ be arbitrary. We know that $f(x) = f(y) \implies x = y$. First let us show the claim that $x \in A \Leftrightarrow f(x) \in f(A)$. Clearly $x \in A$ implies $f(x) \in f(A)$. For the other direction, suppose that this was not the case. Then it could be the case that $f(x) \in f(A)$ but $x \notin A$. This would mean $\exists x_0 \in A$ such that $f(x) = f(x_0)$. However since f is injective $x = x_0$, so it is the case $x \in A$. Given this, we have $f^{-1}(f(A)) = \{x \in X | f(x) \in f(A)\} = \{x \in X | x \in A\} = A$.

$\boxed{\text{(ii)} \Rightarrow \text{(iii)}}$ Assume that for every subset of X, the pre-image of the image is equal to the set. Then let A and B be two subsets of X. We want to show $f(A \cap B) = f(A) \cap f(B)$. Let $x \in f(A \cap B)$. Then $f^{-1}(x) \in A \cap B$. Thus $f^{-1}(x) \in A$ and $f^{-1}(x) \in B$ therefore $x \in f(A)$ and $x \in f(B)$, so $x \in f(A) \cap f(B)$.

$\boxed{\text{(iii)} \Rightarrow \text{(iv)}}$ Assume (iii) to be true. Then let A and B be disjoint subsets of X. It follows that $f(A) \cap f(B) = f(A \cap B) = f(\emptyset) = \emptyset$.

$\boxed{\text{(iii)} \Rightarrow \text{(v)}}$ Assume (iii) to be true. Then Let $B \subset A \subset X$. First note that $f(A) \subset f(X)$, clearly. Then we can write

$$f(A \setminus B) = f(A \cap B^c) = f(A) \cap f(B^c) = f(A) \setminus f(B^c)^c = f(A) \setminus (f(B^c)^c \cap f(X)) = f(A) \setminus f(B)$$

$\boxed{\text{(v)} \Rightarrow \text{(i)}}$ Assume that for any two subsets of X, the image of their difference is the difference of their images. Then let $f(x) = f(y)$ for $x, y \in X$. Assuming that the function is well defined, we can say.

$$\begin{aligned} f(\{x\}) &= f(\{y\}) && (1)\\ \Rightarrow f(\{x\}) \setminus f(\{y\}) &= \emptyset && (2)\\ \Rightarrow f(\{x\} \setminus \{x\}) &= \emptyset && (3)\\ \Rightarrow \{x\} \setminus \{y\} &= \emptyset && (4)\\ \Rightarrow x &= y && (5)\end{aligned}$$

□

Problem 3

(a)

Let $\tau_\alpha : \alpha \in \mathcal{A}$ be a colleciton of topologies on X. Show that $\bigcap_{\alpha \in \mathcal{A}} \tau_\alpha = \{\mathcal{O} \subset X | \mathcal{O} \in \tau_\alpha \ \ \forall \alpha \in \mathcal{A}\}$ is a topology on X.

Proof. Since $\emptyset, X \in \tau_\alpha$ by the axioms of topological spaces for every $\alpha \in \mathcal{A}$ it follows that both belong also to their intersection. Let $\bigcup_{i \in I} \mathcal{O}_i$ be an arbitrary union of sets belonging to our intersect topology (which we will denote simply as τ). Then for every $i \in I$ and for every $\alpha \in \mathcal{A}$, we have $\mathcal{O}_i \in \tau_\alpha$. Therefore we must have $\bigcup_{i \in I} \mathcal{O}_i \in \tau_\alpha$ for every α, and finally it follows that this union must also belong to τ. Now let $\bigcap_{f \in F} \mathcal{O}_f$ be a finite intersection of open sets in τ. By the same logic as our arbitrary union, each open set belongs to each τ_α and it follows that since each τ_α is a proper topology, it will also contain $\bigcap_{f \in F} \mathcal{O}_f$. Since this is true for each $\alpha \in \mathcal{A}$, we have $\bigcap_{f \in F} \mathcal{O}_f \in \tau$. Thus the axioms of a topology are satisfied by our intersection of topologies. □

(b)

Let $\mathcal{F}$ be any family of subsets of a space X. Show that there is a smallest topology $\tau_{\mathcal{F}}$ on X containing $\mathcal{F}$.

Proof. Firstly note that there must be at least one topology containing $\tau_{\mathcal{F}}$, namely the discrete topology is guaranteed for any space. Thus it follows that the intersection of all topologies containing $\mathcal{F}$ will be non-empty. Suppose that there is some topology that is smaller than this intersection, call it τ_0. Then τ_0 should belong to the collection of all topologies containing $\mathcal{F}$ and it follows that τ_0 contains this intersection, and cannot be smaller than it. So a smallest topology $\tau_{\mathcal{F}}$ containing $\mathcal{F}$ exists and can be constructed by taking this intersection of all containing topologies. □

(c)

We can construct this topology by taking two collection

$$A = \{ \text{ collection of all arbitrary unions of sets in } \mathcal{F}\}, \quad B = \{ \text{ collection of all finite intersections of sets in } \mathcal{F}\}$$

Then simply take $\tau_{\mathcal{F}} = \mathcal{F} \cup A \cup B \cup X \cup \emptyset$.

Problem 4

(a)

Show by example that if A is a dense subset of a space X and $Y \subset X$, then $Y \cap A$ need not be dense in Y in the subspace topology.

Choose $X = \mathbb{R},\ \ Y = \mathbb{R} \setminus \mathbb{Q},\ \ A = \mathbb{Q}$. Then we have $A \cap Y = \emptyset$ which is not dense in Y.

(b)

Let A be dense in X and let $\mathcal{O} \subset X$ be open. Show that $\overline{A \cap \mathcal{O}} = \overline{\mathcal{O}}$.

Proof. $\boxed{\subset}$ Let $x \in \overline{A \cap \mathcal{O}}$. Then every neighborhood $U(x)$ intersects $A \cap \mathcal{O}$. This means that each neighborhood must intersect both A and $\mathcal{O}$. Then it follows that each neighborhood of x intersects $\mathcal{O}$. Thus $x \in \overline{O}$.

$\boxed{\supset}$ Let $x \in \overline{\mathcal{O}}$. Then every neighborhood $U(x)$ intersects $\mathcal{O}$. Let U be an arbitrary neighborhood of x. We know that it has a non-empty intersection with $\mathcal{O}$ (since $x \in \overline{\mathcal{O}}$) and with A (since A is dense in X). Then we know that $A \cap \mathcal{O}$ is non-empty since A is dense. So it follows that $U \cap A \cap \mathcal{O}$ is non-empty. Since this is true for any arbitrary neighborhood of x, we say that $x \in \overline{A \cap \mathcal{O}}$. □

(c)

Show that Ext$(A \cup B) =$ Ext $(A) \cap$ Ext (B)

Proof.

$$\begin{aligned} Ext(A \cup B) &= Int((A \cup B)^c) \\ &= Int(A^c \cap B^c) \\ &= Int(A^c) \cap Int(B^c) \\ &= Ext(A) \cap Ext(B) \end{aligned}$$

□

Show that $X \setminus \overline{A} =Ext(A)$.

Proof. Let $x \in (\overline{A})^c$. Then $x \notin \overline{A}$. This means that there exists some nieghborhood $U(x)$ such that U and A are disjoint. This means there is a neighborhood of x contained entirely in A^c, so we say that $x \in Int(A^c)$. Let $x \notin (\overline{A})^c$. Then it is the case that every neighborhood of x intersects A, and therefore $x \notin Int(A^c)$. Finally we say

$$(\overline{A})^c = Int(A^c) = Ext(A)$$

□

Problem 5

(a)

Show that $x \in \overline{S} \subset \mathbb{R}_\ell \iff \exists \{x_n\}_{n \in \mathbb{N}} \subset S : x_n \geqslant x$ and $\lim_{n \to \infty} x_n = x$ in $\mathbb{R}$.

Proof. $\boxed{\Rightarrow}$ Let $x \in \overline{S} \subset \mathbb{R}_\ell$. Then every neighborhood of x intersects S. By construction $[x, x+\frac{1}{n})$ is a neighborhood of x in $\mathbb{R}_\ell$ for any natural number n. Since each of these neighborhoods intersects S choose a sequence where x_n is some element of $[x, x+\frac{1}{n}) \cap S$ for each n (which we know such an element will always exist). Since x is a lower bound $[x, x+\frac{1}{n}) \cap S$, it follows that $x_n \geqslant x$ for every n. To show that the limit of the sequence converges to x, choose $\epsilon > 0$ arbitrarily. Then choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then for every $n \geqslant N$, clearly $x_n \in V_\epsilon(x)$ in $\mathbb{R}$.

$\boxed{\Leftarrow}$ Assume that there exists some sequence in S that is bounded below by x such that $(x_n) \to x$. Then for every $\epsilon > 0$ there exists a point in $[x, \epsilon) \cap S$. Then for every neighborhood of x, there is some ϵ such that $[x, x+\epsilon)$ is a subset of that neighborhood. Thus for every neighborhood of x, within it there lies some element of S, therefore $x \in \overline{A} \subset \mathbb{R}_\ell$. □

(b)

Show that $f : \mathbb{R}_\ell \to \mathbb{R}$ is continuous if and only if $\lim_{x \to a^+} f(x)$ exists for all $a \in \mathbb{R}$.

Proof. $\boxed{\Rightarrow}$ Let $x' \in \mathbb{R}$. Assume that f is continuous. Then every open set in $\mathbb{R}$ has an open pre-image in $\mathbb{R}_\ell$. Then there is some interval $[a, b) \subset f^{-1}(U)$. Then at least we know that there exists a sequence approaching x' from above that lies in $f^{-1}(U)$. If we take the positive limit of such a sequence, it follows that its image must approach $f(x')$.

$\boxed{\Leftarrow}$ Assume that $\lim_{x \to x'} f(x)$ exists and approaches $f(x')$. It follows that every neighborhood of x' in $\mathbb{R}_\ell$ will contain some element of the sequence approaching x' from above. Thus it must be the case that $f^{-1}(U)$ is open in $\mathbb{R}_\ell$. □

Problem 6

(a)

Show that all intervals on $\mathbb{R}$ in combination with neighborhoods of p form a basis for a topology on X.

Proof. If we take the intervals $(k, k+2)$ such that $k \in \mathbb{Z}$ and $(-1, 0) \cup \{p\} \cup (0, 1)$, together they form a cover on X. Let A, B be two sets from our collection. Let $x \in I = B_1 \cap B_2$. We want to show that $\exists C$ in our collection such that $x \in B_3 \subset I$. Write the endpoints for A and B as a_0, a_1, b_0, b_1. Choose some $x \in I$. If $x > 0$, choose $C = (c_0, c_1)$ such that $0 < c_0 < c_1 < \min\{a_1, b_1\}$. Then $C \subset I$. If $x < 0$ choose $C = (c_0, c_1)$ such that $\max\{a_0, b_0\} < c_0 < c_1 < 0$ If $x = 0$ we can choose (c_0, c_1) such that $\min\{a_0, b_0\} < c_0 < 0 < c_1 < \max\{a_1, b_1\}$. Finally if $x = p$, we can choose $(c_0, 0) \cup \{p\} \cup (0, c_1)$ such that $\min\{a_0, b_0\} < c_0 < 0 < c_1 < \max\{a_1, b_1\}$. □

(b)

Show that if U and V are any open sets containing 0 and p, then $U \cap V \neq \emptyset$.

Proof. There exists some $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset U$ and $(-\epsilon, 0) \cup \{p\} \cup (0, \epsilon) \subset V$. This is the case because there must be some basis set that lies within U and V respectively, of which we can find ϵ such that it is less than the absolute value of each end point of these two basis sets. Then it is guaranteed that $\frac{\epsilon}{2} \in U \cap V$. □

(c)

Show that $\mathbb{Q} \subset \mathbb{R}$ is dense in X.

Proof. We assume without proof that for any open set in $\mathbb{R}$ intersects $\mathbb{Q}$. Now choose some open set containing p. Then there is some open set $(-x, 0)$ that is contained within this neighborhood of p. Then we know that this open set in $\mathbb{R}$ must intersect $\mathbb{Q}$. Therefore any neighborhood of p must intersect $\mathbb{Q}$. Therefore $\mathbb{Q}$ is dense in X. □

(d)

Is the function $f : X \to \mathbb{R}$, $f(x) = x$ if $x \in \mathbb{R}$, $f(p) = 0$ continuous?

Choose $U \subset \mathbb{R}$. Then we claim $f^{-1}(U)$ is open in X. If $0 \notin U$ then clearly $f^{-1}(U) = U$ and is open in $\mathbb{R}$. If $0 \in U$ then we know $f^{-1}(U) = U \cup \{p\}$. Since $0 \in U$, there must also be some neighborhood of 0, which will also be in $f^{-1}(U)$. Then it follows that we have some neighborhood of $p \in U \cup \{p\}$. Simply modify the neighborhood of 0 to exclude 0 and include $\{p\}$. Thus for any point $x \in f^{-1}(U)$ we have a neighborhood around x that is contained in $f^{-1}(U)$. So the set is open, and f is continuous.

Problem 7

(a)

Proof. We wish to show $\{x \in X | f(x) < g(x) \text{ or } f(x) = g(x)\}$ is closed. Let us observe the that the complementary set is open,

$$\{x \in X | f(x) > g(x)\}$$

Choose some element x' in the set. Then we know that $f(x') > g(x')$. Choose some point $d \in Y$ such that $f(x') > d > g(x')$. Take the the intersection

$$A = f^{-1}\{y \in Y | y > d\} \cap g^{-1}\{y \in Y | y < d\}$$

We know that it must be non-empty since x' belongs to both. Then since they are both the pre-images of open sets under continuous funcitons, both are open, and so too is their intersection. Then $f(A) > d$ and $g(A) < d$, so the set must be contained in $\{x \in X | f(x) > g(x)\}$.

If such a point d does not exist, simply rewrite as $A = f^{-1}\{y \in Y | y > g(x')\} \cap g^{-1}\{y \in Y | y < f(x')\}$. Either $f(A) > g(A)$ or either f or g is not continuous. □

(b)

Show that $h : X \to Y, h(x) = \min\{f(x), g(x)\}$ is continuous.

Proof. Let $U \subset Y$ be an open set. Choose $x \in h^{-1}(U)$. Then either $h(x) = f(x)$ or $h(x) = g(x)$. There must exist some neighborhood $O(x)$ such that either $f(O(x)) < g(O(x))$ or $f(O(x)) > g(O(x))$. Then take the inverse under either f or g of the set $U \cap f(O(x))$ or $U \cap g(O(x))$. If some such neighborhood does not exist, then U is not open. □