

MTH 342 Homework 3

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1.

Consider a map $G : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ given by $G(u) = (x+1)u' - 2u$.

(a.)

Show that G is a linear map.

Proof. Let $u, v \in P_2(\mathbb{R})$. Let us check if $G(u+v) = G(u) + G(v)$. Taking the image of $u+v$ we get:

$$\begin{aligned} G(u+v) &= (x+1)(u+v)' - 2(u+v) \\ &= (x+1)(u' + v') - 2u - 2v \\ &= (x+1)u' + (x+1)v' - 2u - 2v \\ &= (x+1)u' - 2u + (x+1)v' - 2v \\ &= ((x+1)u' - 2u) + ((x+1)v' - 2v) \\ &= G(u) + G(v) \end{aligned}$$

Now we wish to verify that $G(cu) = cG(u)$ for all $u \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$. Let $u \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$. Now let us take $G(cu)$, giving us:

$$\begin{aligned} G(cu) &= (x+1)(cu)' - 2(cu) \\ &= c(x+1)u' - 2cu \\ &= c((x+1)u' - 2u) \\ &= cG(u) \end{aligned}$$

Since both of those equalities hold, we can say that G is a linear map. □

(b.)

Now we want to find the basis and rank of $\text{null}(G)$. Let the standard basis for $P_2(\mathbb{R})$ be $\{1, x, x^2\}$. Let us create the transformation matrix $[G]_{B_2 B_1}$. We can write $[G(1)_{B_2} \quad G(x)_{B_2} \quad G(x^2)_{B_2}] = [-2_{B_2} \quad (1-x)_{B_2} \quad (2x)_{B_2}] =$

$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Let us take the product of this matrix and the vector $(a + bx + cx^2)_{B_1}$ in the null space. We

get $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}$. This is equivalent to $\begin{bmatrix} -2a + b \\ -b + 2c \\ 0 \end{bmatrix}$. This implies that $a = c$, and $b = 2c$. From there we

have $\text{null}(G) = \left\{ c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$. Therefore we have $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ as the basis of $\text{null}(G)$ with nullity of G being 1.

(c.)

From the transformation matrix found in part 1(b.), we can find the basis of the range of G and $\text{rank}(G)$. We know that the range of G is equal to $\left\{ \begin{bmatrix} -2a+b \\ -b+2c \\ 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ as seen previously. This can be broken up into the set of all combinations of four vectors $G = \left\{ a \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$. Notice that the second vector can be written as a combination of the first and the third, and is therefore redundant. Putting all of our remaining vectors into a matrix as $\begin{bmatrix} -2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$, we get a matrix which is already in row-echelon form, and is linearly independent as all vectors are pivot columns. Therefore a valid basis for $\text{range}(G)$ would be $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. It follows that the rank of G is 2. This is confirmed by the rank nullity theorem which tells us $\dim(P_2(\mathbb{R})) = \text{rank}(G) + \text{nullity}(G)$, which in this case specifically is $3 = 2 + 1$.

(d.)

Since the dimension of $P_2(\mathbb{R})$ is 3 and the dimension of $\text{range}(G)$ is 2, we know that G is not monomorphic or isomorphic. However, with $\dim(\text{range}(G)) \neq \dim(P_2(\mathbb{R}))$ we know that the image of G cannot be equal to its co-domain, and the function is therefore not epimorphic.

2.

(a.)

Show that the set of matrices $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2 \times 2}(\mathbb{C}) : a + b + c + di = 0 \right\}$ is a subspace of $M_{2 \times 2}(\mathbb{C})$. //

Notice that $V \subset M_{2 \times 2}(\mathbb{C})$. Let $a, b, c, d = 0$ and our equation $a + b + c + di = 0$ still holds, so we do have the additive identity. Thus we must show only two things, closure under scaling, and closure under vector addition. //

Let $v, w \in V$. We want to show that $v + w \in V$. We can write $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}$. Since we know $v_1 + v_2 + v_3 + v_4i = 0$ and $w_1 + w_2 + w_3 + w_4i = 0$ we can say that

$$\begin{aligned} (v_1 + v_2 + v_3 + v_4i) + (w_1 + w_2 + w_3 + w_4i) &= 0 \\ (v_1 + w_1) + (v_2 + w_2) + (v_3 + w_3) + (v_4i + w_4i) &= \\ (v_1 + w_1) + (v_2 + w_2) + (v_3 + w_3) + (v_4 + w_4)i &= \end{aligned}$$

This satisfies the parameters of V so therefore $v + w = \begin{bmatrix} v_1 + w_1 & v_2 + w_2 \\ v_3 + w_3 & v_4 + w_4 \end{bmatrix} \in V$. // Let $v \in V$ and $z \in \mathbb{C}$. If

we take $zv = \begin{bmatrix} zv_1 & zv_2 \\ zv_3 & zv_4 \end{bmatrix}$, then we want to show that $zv_1 + zv_2 + zv_3 + zv_4i = 0$. By factoring out z we get $z(v_1 + v_2 + v_3 + v_4i) = 0$. Since $v \in V$ we know that $v_1 + v_2 + v_3 + v_4i = 0$, and therefore the equation $z(0) = 0$ holds, and $zv \in V$. We have shown that V is a subspace.

(b.)

Find a basis for V .

Take the equation $a + b + c + di = 0$ and rewrite it as $a = -b - c - di$. If we replace a with $-b - c - di$ we get $(-b - c - di) + b + c + di = 0$, which is true for all $a, b, c, d \in \mathbb{C}$. We can determine that $V =$

$\left\{ \begin{bmatrix} -b-c-di & b \\ c & d \end{bmatrix} : b, c, d \in \mathbb{C} \right\}$. This can be written as a combination of vectors in the $M_{2 \times 2}$ standard basis as

$$V = \left\{ b \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -i \\ 0 \\ 0 \\ 1 \end{bmatrix} : b, c, d \in \mathbb{C} \right\}.$$

It is simple to show that these vectors are independent by putting them into a matrix

$$\begin{bmatrix} -1 & -1 & -i \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and noting that it is already in row-echelon form with all vectors being pivot

columns. Therefore we can state the basis of V , $B_V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -i \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c.)

Find the matrix representation of linear map H where $H \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a+b)z^2 + (b+c)z + (c+d)$.

Let us take our basis vectors for V , and write their images in the basis of the co-domain. We have $H \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) = (0)z^2 + (1)z + (0)1$ which is equal to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in the basis $\{1, z, z^2\}$. Repeating this process we get $H \left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) =$

$(-1)z^2 + (1)z + (1)1$ which we can represent by $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Then we get $H \left(\begin{bmatrix} -i \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = (-i)z^2 + (0)z + (1)1$ which

we can represent by $\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}$. Thus we can write the transformation matrix $[H]_{B_2 B_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -i \end{bmatrix}$.

(d.)

If we take our transformation matrix for H and row reduce we get $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1-i \end{bmatrix}$. From there, it is clear that $\text{rank}(H) = 3$, and we know that the dimension of V is 3, so by the rank-nullity theorem it must be the case that the nullity is 0. This is a suitable answer for part (e.) as well.

3.

To get a basis for all 2×2 matrices where the sum of entries in each row is 0 we can write

$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = -b, c = -d, a, b, c, d \in \mathbb{R} \right\}$. This is equivalent to $V = \left\{ \begin{bmatrix} -b & b \\ -d & d \end{bmatrix} : a, b \in \mathbb{R} \right\} =$

$\left\{ b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} : b, d \in \mathbb{R} \right\}$, which has a basis of $B_V = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ when written as vectors in the

standard basis. For the sum of entries on columns to be zero, we simply take the transpose of the matrices represented

by vectors in our basis for V , which gives us the basis for W being $B_W = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ by switching the second and third rows. To get the basis for $V + W$, we must take the union of B_V and B_W and then take a linearly independent subset of that. We have $B_V \cup B_W = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$, which we can put in a matrix as $\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$. After row reduction this becomes $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since the last column is in the span of the first three, we can eliminate our fourth vector from $B_V \cup B_W$ to get a linearly independent subset that still spans $V + W$. This gives us $B_{V+W} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$.

4.

(a.)

Since the image of each basis vector is already provided See problem 4 on HW all that is left to do is write those as vectors in a matrix with respect to B_2 , ie $[f(v_1)_{B_2}, f(v_2)_{B_2}, \dots, f(v_7)_{B_2}]$. This can be done by observing the coefficients for each w_n vector and putting said coefficient in the n th row of the column for whichever vector $v \in B_1$

you are doing. This gives the following matrix:

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 & 1 & 2 \\ 1 & -1 & 2 & 0 & 1 & 1 & 0 \\ 0 & -1 & 5 & 1 & 0 & 2 & -6 \\ -1 & 0 & -4 & -1 & 4 & 3 & 2 \\ 0 & 1 & 7 & 0 & 5 & 5 & 1 \\ 2 & 4 & -1 & 1 & 3 & 0 & -1 \end{bmatrix} = [f]_{B_2 B_1}$$

(b.)

The following is my code to reduce the above matrix.

```
m = [1, 3, 0, 1, 0, 1, 2; 1, -1, 2, 0, 1, 1, 0; 0, -1, 5, 1, 0, 2, -6; -1, 0, -4, -1, 4, 3, 2; 0, 1, 7, 0, 5, 5, 1; 2, 4, 0, 1, 1, 3, 0, -1]
rref(m)
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From this the first 6 vectors are linearly independent but the 7th is not. Therefore the $\text{rank}(f) = 6$. From the rank nullity theorem we can say that since $\dim(V) = 7$ and $\text{rank}(f) = 6$ that $\text{nullity}(f) = 1$.