Advanced Multivariable Calculus - Notes

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1 Introduction to $\mathbb{R}^n$

Definition 1.1 (n-dimensional Vector)**.** An n-dimensional vector is an ordered tuple

$$\vec{x} = (x_1, x_2, \cdots, x_n)$$

With $x_i \in \mathbb{R}$ for every $i \in \{1, 2, \cdots, n\}$.

Then it is simple to say that $\mathbb{R}^n$ is the set of all n-dimensional vectors. Given some scalar constant $c \in \mathbb{R}$ and a vector $\vec{x} \in \mathbb{R}^n$ we write

$$c\vec{x} = (cx_1, cx_2, \cdots, cx_n)$$

We also define a norm on $\mathbb{R}^n$ which is given by $||\vec{x}|| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.

Definition 1.2 (Dot Product)**.** The dot product maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. Given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ we define their dot product to be $\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i$.

Then we have $||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}$. The angle between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{||\vec{x}||\,||\vec{y}||}\right)$$

Theorem 1.1 (Cauchy Shwarz Inequality)**.** *Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then we say*

$$|\vec{x} \cdot \vec{y}| \leqslant ||\vec{x}||\,||\vec{y}||$$

2 Functions

Definition 2.1 (Function)**.** For $m, n \in \mathbb{N}, D \subset \mathbb{R}^n$, a function $F : D \to \mathbb{R}^m$ assigns to each $\vec{x} \in D$ a unique point $\vec{y} \in \mathbb{R}^m$. We write $F(\vec{x}) = \vec{y}$. For each $\vec{x} \in D$, we can write

$$\vec{y} = F(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \cdots, f_n(\vec{x}))$$

Where $f_j : D \to \mathbb{R} \quad \forall j, 1 \leqslant j \leqslant m$.

We can call f_j the j-th component function of F. Of course D is the domain of F and $F(D)$ is the image of F. Now there are lots of great examples of functions that map $\mathbb{R}^m \to \mathbb{R}^n$. But we care mostly about the following property:

Definition 2.2 (Continuity)**.** Let $D \subset \mathbb{R}^n$. Then $f : D \to \mathbb{R}^m$ is continuous at $x \in D$ if given any $\epsilon > 0$ there exists some $\delta > 0$ such that $||\vec{x} - \vec{y}|| < \delta$ implies $||f(\vec{x}) - f(\vec{y})|| < \epsilon$.