MTH 311 Homework 2

Philip Warton

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1.4.3

We want to show $\bigcap_{n\in N}(0, \frac{1}{n}) = \emptyset$.

Proof. By contradiction, suppose $x \in \mathbb{R}$ where $x \in \bigcap_{n\in N}(0, \frac{1}{n})$ arbitrarily. We know that $0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$. This is in contradiction to the archimedean property, which gives us $\exists n \in \mathbb{N} : \frac{1}{n} < x$. Therefore, there must not exist $x \in \mathbb{R} : x \in \bigcap_{n\in N}(0, \frac{1}{n})$ and thus $\bigcap_{n\in N}(0, \frac{1}{n}) = \emptyset$.

□

1.4.7

This is not a complete proof, see $\boxed{\text{Theorem 1.4.5}}$ and its proof in the text for the proof that we are completing.

Proof. By contradiction suppose $\alpha^2 > 2$. We know that the following is true:

$$\begin{aligned}
\alpha &> \alpha - \frac{1}{n} \\
\implies \alpha^2 &> (\alpha - \frac{1}{n})^2 \\
&= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\
&> \alpha^2 - \frac{2\alpha}{n}
\end{aligned}$$

Choose $n \in \mathbb{N} : n \geqslant \frac{2\alpha}{(\alpha^2 - 2)}$. Then we can write the inequality as follows:

$$\begin{aligned}
\alpha^2 - \frac{2\alpha}{n} &\geqslant \alpha^2 - \frac{2\alpha(\alpha^2 - 2)}{2\alpha} \\
&= \alpha^2 - (\alpha^2 - 2) \\
&= 2
\end{aligned}$$

Therefore we have $\alpha^2 > (\alpha - \frac{1}{n})^2 > \alpha^2 - \frac{2\alpha}{n} \geqslant 2$. Since $(\alpha - \frac{1}{n})^2 > 2$, we know that $\alpha - \frac{1}{n}$ is an upper bound for T that is less than α. Then by definition of least upper bound, α cannot be the least upper bound for T (contradiction). Since α^2 cannot be greater than 2, and cannot be less than 2, $\alpha^2 = 2$.

□

1.5.1

Let B be a countable infinite set. Let $A \subseteq B$ be an infite subset. A is countable.

Proof. Since B is countable, $\exists f : \mathbb{N} \to B$ where f is bijective. We want to inductively define a function $g : \mathbb{N} \to A$. Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$ and set $g(1) = f(n_1)$.

By induction, assume we have defined n_k for some $k \in \mathbb{N}$. Let $n_{k+1} = \min(\{n \in \mathbb{N} : f(n) \in A\} \setminus \{n_1, n_2, ..., n_k\})$. We have defined n_m $\forall m \in \mathbb{N}$. Define $g(m) = f(n_m)$. We must now check that $g(m)$ is bijective.

Let $k, l \in \mathbb{N}$ and suppose $g(k) = g(m)$. This can be rewritten by our definition of g as $f(n_k) = f(n_m)$. With f being injective, we have $n_k = n_m$. Since we defined n_p such that $n_p \notin \{n_1, n_2, ...n_{p-1}\}$ $\forall p \in \mathbb{N}$, we know $n_k = n_m \implies k = m$, and therefore g is injective.

Pick $a \in A$ arbitrarily. Since $A \subset B$, we have $a \in B$. This means that $\exists q \in \mathbb{N} : f(q) = a$. Let us find which n_s is associated with a. If $f^{-1}(a) \in \{n_1, n_2, ...n_{k-1}\}$, then choose the element n_s equal to $f^{-1}(a)$ where $s < k$. Otherwise we know that $f^{-1}(a) \in \{n \in \mathbb{N} : f(n) \in A\}$. If it is the minimum, then choose n_k. Otherwise choose some larger n_k, and eventually it will be $f^{-1}(a)$. Therefore $g(m)$ is surjective, and also bijective.

□

1.5.3

a.

Proof. We want to show that given sets A_1, A_2 where both are countable, their union is also countable. Let $B_2 = A_2\ A_2$. Note that $A_1 \cap B_2 = \emptyset$. Let us consider both cases where B_2 is finite and where B_2 is infinite.

$\boxed{B_2 \text{ is finite:}}$ Since A_1 is countable $\exists f : \mathbb{N} \to A_1$. Let $g : \mathbb{N} \to A_1 \cup B_2$. Since B_2 is finite let $B_2 = \{b_1, b_2, ..., b_i\}$ where $i = |B_2|$. Let us define our function

$$g(n) = \left\{ \begin{array}{ll} b_n & n \leqslant i \\ f(n-i) & n > i \end{array} \right\}$$

Now we must show $g(n)$ to be a bijection. To show that $g(n)$ is injective let $g(a) = g(b)$. Consider when $a, b > i$, we have $f(a - i) = f(b - i)$. Note this cannot equal the image of $f(0)$ because both $a, b > i$. Since f is bijective, $f(a - i) = f(b - i) \implies (a - i) = (b - i)$, and therefore $a = b$. If $a > i$ and $b \leqslant i$ then $f(a) \in A_1$ and $f(b) \in B_2$. These are disjoint sets and therefore $f(a) \neq f(b)$ (contradiction). Similarly the case where $a \leqslant i$ and $b > i$ is contradictory. Finally if $a, b \leqslant i$, then we have $b_a = b_b$. By defition of B_2 we have $a = b$. From this we konw $g(n)$ is a bijection, which is what was to be shown.

$\boxed{B_2 \text{ is infinite:}}$ We have a function $f : \mathbb{N} \to A_1$,, and a function $g : \mathbb{N} \to B_2$. Note that this is not g as defined in case 1, but a function granted by B_2 being infinite and countable. We want to show existence of a function $h : \mathbb{N} \to A_1 \cup B_2$. Let us define our function as

$$h(n) = \left\{ \begin{array}{ll} f(\frac{n}{2}) & \text{if } n \text{ is even} \\ g(\frac{n+1}{2}) & \text{if } n \text{ is odd} \end{array} \right\}$$

We wish to show $h(n)$ bijective. Let $h(a) = h(b)$. Since $A_1 \cap B_2 = \emptyset$, we know that a and b must have the same parity. Otherwise, $h(a) = h(b) \in A_1 \cap B_2$ (contradiction). Therefore either $f(\frac{a}{2}) = f(\frac{b}{2})$ or $g(\frac{a+1}{2}) = g(\frac{b+1}{2})$. With both f and g being bijective functions, in either case it must be true that $a = b$. To show that it is onto, choose an element d in $A_1 \cup B_2$ arbitrarily. If $d \in A_1$ then $\exists n \in \mathbb{N} : f(n) = d$, therefore $h(2n) = d$. If $d \in B_2$ then we know $\exists m \in \mathbb{N} : g(m) = d$. From there we know that $h(2m - 1) = d$. Therefore h is a bijection.

□

Now we have that given two coutable sets A and B, $A \cup B$ is countable. Therefore given any finite number of sets, it follows that their union is countable. Let $A = \{A_1, A_2, ..., A_n\}$ be a finite collection of coutable sets. We know that $A_1 \cup A_2$ is countable. We want to show that given any countable $\bigcup_{n=1}^{k} A_n$ we have $\bigcup_{n=1}^{k+1} A_n$ countable. By induction suppose we have $\bigcup_{n=1}^{k} A_n$ being countable. Since A_{k+1} is countable we can say that the union between these two countable sets is also countable. Therefore $\bigcup_{n=1}^{k} A_n$ holds for all $k \in \mathbb{N}$.