Systems of ODE's - Homework 3

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Problem 3.2

(iv)

We have a system of ODE's given by $X' = AX$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. We begin by computing the eigenvalues using the characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - (1)(-1) = (\lambda - 2)^2 \implies \lambda_1, \lambda_2 = 2$$

So then we can compute the eigenvectors as follows

$$\begin{aligned} (A - \lambda I)V &= \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} -x + y \\ -x + y \end{bmatrix} \end{aligned}$$

For $(A - \lambda I)V = 0$ it follows that $x = y$, so we say that we have only one linearly independent eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Choose $W = \langle 1, 0 \rangle$ to be another vector that is independent from V. We then wish to solve for μ in the following equation

$$\begin{aligned} AW &= \mu V + vW \\ \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \mu V + vW \\ \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \implies \mu &= -1 \end{aligned}$$

Now we write $U = \frac{1}{\mu} W = \langle -1, 0 \rangle$, giving us the transformation matrix $T = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. We know that $T^{-1}AT = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. Seeing this matrix in canonical form, we can solve this system easily, giving us

$$Y(t) = \alpha e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta e^{2t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Then we can solve for $X(T)$ using the fact that $X(t) = TY(t)$:

$$TY(t) = X(t) = \alpha e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{2t} \begin{bmatrix} t - 1 \\ t \end{bmatrix}$$

All phase portraits can be found on the attached scan.

(vi)

Finding eigenvalues using the characteristic polynomial we get

$$\det(A - \lambda I) = (\lambda - \sqrt{2})(\lambda + \sqrt{2})$$

The eigenvalues will be $\lambda_1 = \sqrt{2}, \lambda_2 = -\sqrt{2}$. Solving for eigenvectors using the kernel of $A - \lambda I$ we get eigenvectors $V_1 = \langle 1 + \sqrt{2}, 1 \rangle, V_2 = \langle 1 - \sqrt{2}, 1 \rangle$ corresponding to λ_1, λ_2 respectively. So we use the transfomation matrix $T = \begin{bmatrix} 1 + \sqrt{2} & 1 - \sqrt{2} \\ 1 & 1 \end{bmatrix}$ such that $T^{-1}AT = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$. This gives us the solution

$$Y(t) = \alpha e^{\sqrt{2}t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta e^{-\sqrt{2}t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then we can compute the proper solution $X(t)$ by applying our transformation

$$TY(t) = X(t) = \alpha e^{\sqrt{2}t} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} + \beta e^{-\sqrt{2}t} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$

See attached scan for phase portraits.

Problem 3.5

We start by computing the eigenvalues of the system, so that we can see which category our solutions will fall into. We take the characteristic polynomial to be

$$(a-\lambda)(2-\lambda) - 2a = \lambda(\lambda - (2+a))$$

This gives us two eigenvalues, $\lambda_1 = 1, \lambda_2 = 2+a$. In the case where $a = -2$ we have repeated eigenvalues with an eigenspace equivalent to the null space of A (which by rank-nullity will have dimension 1). Otherwise we have real and distinct eigenvalues, with either a positive eigenvalue with 0 or a negative eigenvalue with 0.

Problem 3.6

Let us first observe what our eigenvalues will be for this system. We have $A = \begin{bmatrix} 2a & b \\ b & 0 \end{bmatrix}$.Then we take the characteristic polynomial to be

$$\lambda^2 - 2a\lambda - b^2$$

So it follows that we have eigenvalues of

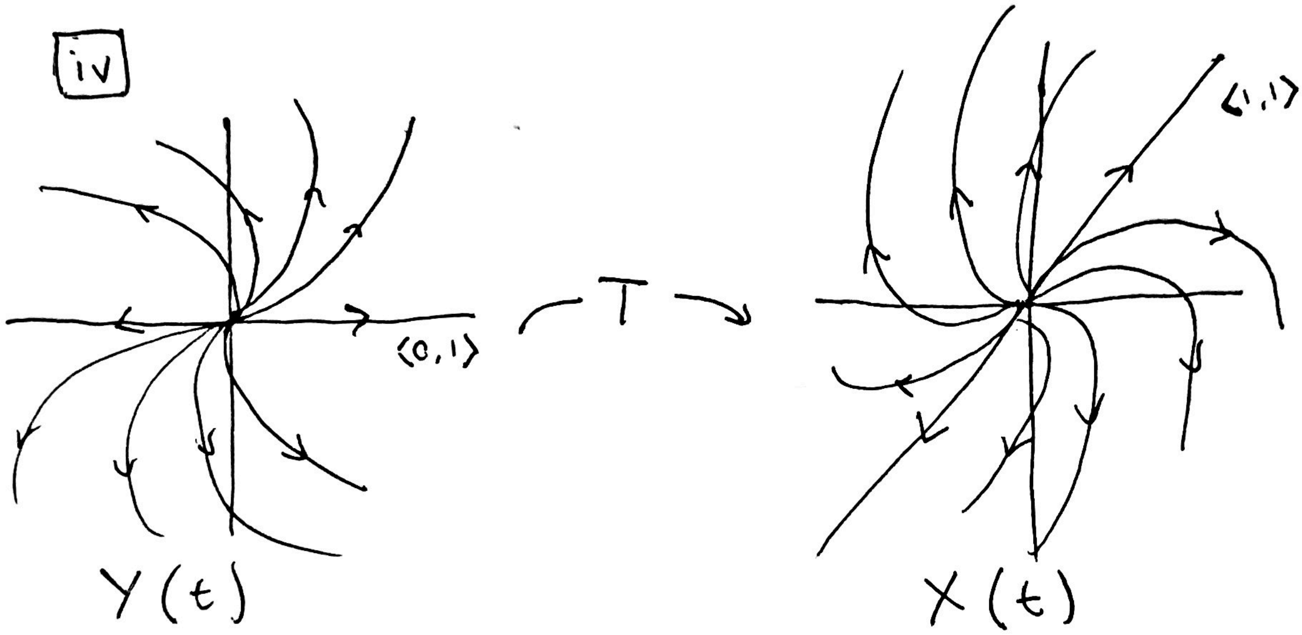
$$\begin{aligned} \lambda &= \frac{2a \pm \sqrt{4a^2 - 4(1)(-b^2)}}{2} \\ &= a \pm \sqrt{a^2+b^2} \end{aligned}$$

It should be clear that this will always provide two real and distinct eigenvalues for any $a, b \in \mathbb{R}$. What may be of notice is that will always have one positive and one negative eigenvalue since $\sqrt{a^2+b^2} \geqslant a$ for any $b \in \mathbb{R}$.

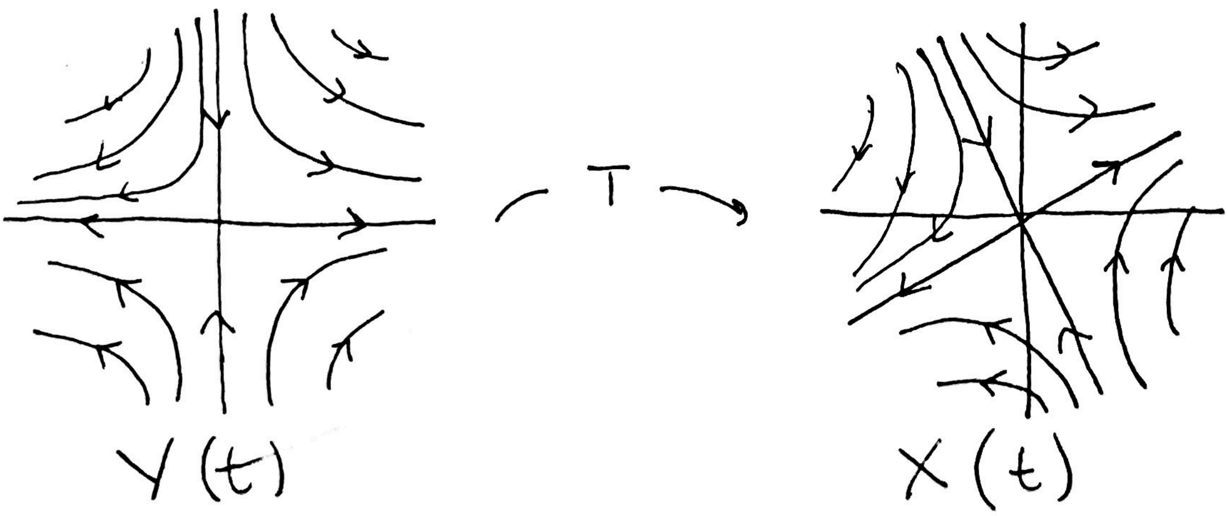
Phase Portraits

Problem 3.2

$\boxed{iv}$

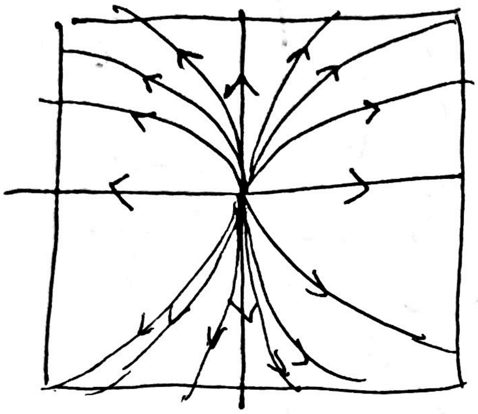


$\boxed{vi}$



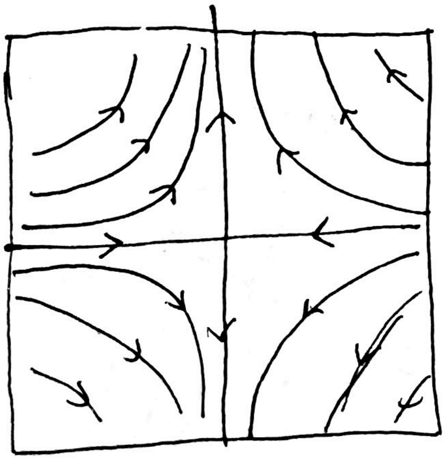
Problem 3.5

$\boxed{a > -2}$



Under some linear transformation.

$\boxed{a < -2}$



Under some linear transformation. (It could even look as though x and y axis are flipped).

Problem 3.6