Algebraic Topology — Homework 3

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Problem 1

Prove that if A is a compact subspace of a Hausdorff space X, then A is closed in X: $A \subset^{cl} X$.

Solution

Proof. Let $x \in X \setminus A$. Then for any point a in A, we have some neighborhood of a, U_a, that is disjoint from some neighborhood of x, V_a, since X is Hausdorff. Then it follows that

$$\bigcup_{a \in A} U_a$$

is an open cover cover of A. Since A is compact, it follows that there is some finite subcover of A. We will denote this by

$$\bigcup_{f \in F} U_f.$$

So then since F is finite, it follows that the intersection

$$\bigcap_{f \in F} V_f$$

is an open neighborhood of x that is disjoint from our finite cover of A, and is therefore disjoint from A. So if any arbitrary point $x \notin A$ has a neighborhood that is disjoint from A, then A is closed. $\square$

Problem 2

Prove that if $\phi : B^n X$ is a characteristic map for an n-cell c^n in a Hausdorff space X, then $\bar{c}^n = \phi(B^n)$ is the closure of the open cell $c^n = \phi(\text{int}(B^n))$ in X.

Solution

Proof. By $\boxed{\text{Problem 1}}$, we know that $\phi(B^n)$ is closed. This is because a continuous map carries compactness, and a compact subset of a Hausdorff space is closed. By basic properties of sets and maps, we know too that $\phi(int(B^n)) \subset \phi(B^n)$. Suppose that $\phi(B^n)$ is not the closure of $\phi(int(B^n))$, then it must be the case that $cl(\phi(int(B^n))) \subsetneq \phi(B^n)$. Then there must be some point $x \in \phi(B^n) \setminus cl(\phi(int(B^n)))$. Since x is not a limit point of $\phi(int(B^n))$, we know that $\phi^{-1}(x)$ is not a limit point of $int(B^n)$. This is a contradiction because any point in B^n is a limit point of $int(B^n)$. So we conclude that $\phi(B^n) = cl(c^n)$. □

What is left to show is that continuity preserves limit/accumulation points...

Let $f : X \to Y$ be continuous. Let $B \subset Y$ and let $y \in Y$ not be a limit point of B. Then $f^{-1}(y) = x$ is not a limit point of $f^{-1}(B)$.

Proof. Since y is not a limit point of B, there is some open neighborhood U of y such that U is disjoint from B. Therefore the pre-image $f^{-1}(U)$ is disjoint from the pre-image of B. So there is an open set $(f_{-1}(U))$ such that it is an open neighborhood of y and is disjoint from $f^{-1}(B)$. Therefore the pre-image of y is not a limit point of the pre-image of B. □

Problem 3

Let $\mathcal{F}$ be a family of closed subsets of a topological space X.

i. Prove that $\{A \subseteq X \ : \ A \cap F \subset^{cl} F \text{ for all } F \in \mathcal{F}\}$ is the set of closed sets for a topology on X. This is called the **weak topology** on X relative to $\mathcal{F}$. We write X_w to refer to the set X together with this weak topology.

ii. Prove that if $U \subseteq X$, then UX_w if and only if $U \cap F \subset^{op} F$ for all $F \in \mathcal{F}$.

iii. Prove that for each $F \in \mathcal{F}$, the subspace topology inherited from X_w is the same as the subspace topology inherited from X.

iv. Prove that if Y is any space, then a function $f : X_w \to Y$ is continuous if and only if $f|_F : F \to Y$ is continuous for all $F \in \mathcal{F}$.

v. Prove that the identity function $1_X : X_w \to X$ is continuous.

vi. Give an example to show that X_w need not be homeomorphic to X.

Solution

We denote $\{A \subset X \ : \ A \cap F \subset^{cl} F \text{for all} F \in \mathcal{F}\} = \mathcal{A}$.

i.

Proof. We wish to show that if we let $\mathcal{A}$ be our family of closed sets, that this will form a topology on X. To do this, one must show that the axioms of a topological space hold.

$\boxed{\emptyset, X \in \mathcal{A}}$ We begin with the empty set. For any $F \in \mathcal{F}$

$$\emptyset \cap F = \emptyset \subset^{cl} F.$$

For the universal set,

$$X \cap F = F \subset^{cl} F.$$

$\boxed{\text{Closure under arbitrary intersection}}$ Let

$$\bigcap_{i \in I} A_i$$

be some arbitrary intersection of sets in $\mathcal{A}$. Then let $F \in \mathcal{F}$ be arbitrary. It follows that

$$\left(\bigcap_{i \in I} A_i \right) \cap F = \bigcap_{i \in I} (A_i \cap F)$$

Since each $A_i \cap F$ is closed in the subspace of F, it follows that this intersection will also be closed.

$\boxed{\text{Closure under finite union}}$ Let

$$\bigcup_{j \in J} A_j$$

Be a finite union of sets in $\mathcal{A}$. Then let $F \in \mathcal{F}$ be arbitrary, and we write

$$\left(\bigcup_{j \in J} A_j \right) \cap F = \bigcup_{j \in J} (A_j \cap F)$$

Then each $A_j \cap F$ is closed in F so it follows that a finite union of sets closed in F is closed in F, and therefore this union belongs in $\mathcal{A}$. Having shown these three properties, this set of closed sets forms a topology on X. □

ii.

Proof. Suppose that $U \subset X$. Then we know that

$$\begin{aligned} U \subset^{op} X_w &\Leftrightarrow (X \setminus U) \subset^{cl} X_w \\ &\Leftrightarrow (X \setminus U) \in \mathcal{A} \\ &\Leftrightarrow (X \setminus U) \cap F \subset^{cl} F \quad \forall F \in \mathcal{F} \\ &\Leftrightarrow U \cap F \subset^{op} F \quad \forall F \in \mathcal{F} \end{aligned}$$

□

iii.

Proof. Let $F \in \mathcal{F}$ be arbitrary, without loss of generality. To show that the topologies are the same we will demonstrate that a set is closed in one topology if and only if it is closed in the other. Let $C \subset F$ be closed under the subspace topology induced by X_w. The set C is closed in F_w (slight abuse of notation, F_w is F equipped with the subspace topology induced by X_w) if and only if there exists some C' closed in X_w such that $C' \cap F = C$. Then

$$\begin{aligned} C' \subset^{cl} X_w &\Leftrightarrow C' \cap F \subset^{cl} F \quad \forall F \in \mathcal{F} \\ &\Leftrightarrow C' \cap F \subset^{cl} F \text{ for our fixed } F \\ &\Leftrightarrow C \subset^{cl} F \end{aligned}$$

So we conclude that $C \subset^{cl} F_w$ if and only if $C \subset^{cl} F$. □

iv.

Proof. Let Y be any space. Then a function $f : X_w \to Y$ is continuous if and only if closed sets in Y have closed pre-images. Closed sets $C \subset Y$ have closed pre-images in X_w if and only if $f^{-1}(C) \cap F \subset^{cl} F$ for every $F \in \mathcal{F}$. Then $f^{-1}(C) \cap F \subset^{cl} F$ for every $F \in \mathcal{F}$ if and only if $f^{-1}\big|_F(C) \subset^{cl} F$ for every $F \in \mathcal{F}$. And this is the case if and only if $f\big|_F$ is continuous for every $F \in \mathcal{F}$. □

v.

Proof. Let $C \subset^{cl} X$. Then we wish to show that its pre-image under the identity function is also closed. Let $F \in \mathcal{F}$ be arbitrary without loss of generality. Then by definition of subspace topology $C \cap F \subset^{cl} F$. Then it follows that if $C \cap F \subset^{cl} F$ for every F, then $C \subset^{cl} X_w$. □

vi.

Take $X = \mathbb{R}$. The let $\mathcal{F} = \{\{0\}\}$. Then what we are left with is the discrete, weakest, topology. That is, every set is both open and closed. If a set does not contain 0 , then its intersection with every $F \in \mathcal{F}$ is empty and is trivially closed. And if a set contains 0 then its intersection with every $F \in \mathcal{F}$ is $\{0\}$ and therefore closed in $\{0\}$. So $\mathbb{R}_w$ and $\mathbb{R}$ are not homeomorphic because $\mathbb{R}$ is connected and $\mathbb{R}_w$ is disconnected. A simple disconnection for $\mathbb{R}_w$ is $\mathbb{R}_w = \{0\} \cup (\mathbb{R} \setminus \{0\})$. Thus, we conclude that X and X_w need not be homeomorphic.

Problem 4

Prove that if $(X, \mathcal{C})$ is a CW complex, then the zero-skeleton X^0 is discrete.

Solution

Proof. Suppose that $(X, \mathcal{C})$ is a CW complex. The 0-skeleton X^0 is defined as

$$(X^0, \mathcal{C}^0) \text{ where } \mathcal{C}^0 = \{c \in \mathcal{C} \ : \ dim(c) \leq 0\}, X^0 = \bigcup_{dim(c) \leq 0} c$$

Since our dimension is 0, this means that all our open cells $c \in \mathcal{C}$ are of dimension 0 or lower. Since all cells of dimension $n < 0$ are non-existent (being images of the empty set). We conclude that all open cells $c \in \mathcal{C}$ are of dimension 0.

$$\forall c \in \mathcal{C}^0, dim(c) = 0$$

Let $c \in \mathcal{C}^0$ be arbitrary. Then by definition of a CW complex, $\{c\}$ is open. By definition of a 0-cell, c is simply a single point in X. Then, since X is Hausdorff, it follows $\{c\}$ is closed. Since the singleton set of any point in the 0-skeleton is both open and closed, it follows that X^0 is discrete. $\square$

Problem 5

Prove that $H_k(B^n, S^{n-1}) \cong \left\{ \begin{array}{cc} \mathbb{Z} & k=n \\ 0 & k \neq n \end{array} \right.$.

Solution

Proof. We know that for $n > 1$, for the homology of the sphere,

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

Let $n, k > 1$, then we have an exact sequence

$$0 = H_k(B^n) \to^f H_k(B^n, S^{n-1}) \to^g H_{k-1}(S^{n-1}) \to^h H_{k-1}(B^n) = 0$$

Then

$$H_{k-1}(S^{n-1}) = ker(h) = im(g) \Rightarrow g \text{ is surjective,}$$

and

$$ker(g) = im(f) = 0 \Rightarrow g \text{ is injective.}$$

So the middle mapping is a bijection therefore

$$H_k(B^n, S^{n-1}) \cong H_{k-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & k-1 = n-1 \\ 0 & k-1 \neq n-1 \end{cases}.$$

Critically $k - 1 = n - 1$ if and only if $k = n$. For $k = 1$,

$$0 = H_1(B^n) \to^f H_1(B^n, S^{n-1}) \to^g H_0(S^{n-1}) \to^h H_0(B^n) = \mathbb{Z}$$

If $n = 1$ then $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ so it follows that

$$\mathbb{Z} = ker(h) = im(g).$$

Then since

$$ker(g) = imk(g) = 0,$$

it follows that g is injective, and since it's image is $\mathbb{Z}$, $f^{-1}(\mathbb{Z}) = \mathbb{Z}$. Therefore,

$$H_1(B^1, S^0) = \mathbb{Z}.$$

If $n > 1$ then $H_0(S^n) = H_0(B^n) = \mathbb{Z}$ so h is an isomorphism. So then $0 = ker(h) = im(g)$, and therefore g is the 0 map. Since $im(g)$ is 0, it must be the case that $ker(f) = 0$, then $H_1(B^n, S^{n-1})$ must be 0 for $n > 1$. If $n = 0$, then

$$H_1(B^0, S^{-1}) = H_1(B^0, \emptyset) = H_1(B^0) = H_1(\{pt\}) = 0.$$

For $k = 0, n = 0$, we have

$$H_0(B^0, S^{-1}) = H_0(B^0, \emptyset) = H_0(B^0) = H_0(\{pt\}) = \mathbb{Z}.$$

□

This is incomplete. Edge cases are handled messily and incorrectly. Note: shouldn't $H_k(B^1, S^0)$ be isomorphic to $H_k(S^1)$, since we are quotienting the endpoints together in hand-wavey way?

Problem 6

Prove that if

$$C : C_m \overset{\partial_m}{\to} C_{m-1} \to \cdots \to C_1 \overset{\partial_1}{\to} C_0$$

is a chain complex consisting of finitely generated abelian groups $C_0, \ldots, C_m$, then

$$\sum_{i=0}^{m}(-1)^i \mathrm{rk}(C_i) = \sum_{i=0}^{m}(-1)^i \mathrm{rk}(H_i(C)).$$

Use this to prove that if $(X, \mathcal{C})$ is a finite CW complex and α_i is the number of i-cells in $\mathcal{C}$, then

$$\chi(X, \mathcal{C}) = \sum_{i=0}^{\dim(X)} (-1)^i \alpha_i$$

is an invariant of the homotopy type of the space X. In particular, this number is independent of the choice of (finite) cellular decomposition $\mathcal{C}$, so we denote this number by $\chi(X)$. It is the **Euler characteristic** of X.

Solution

Proof. First, notice that there are two important chain complexes

$$0 \to ker\partial_n \to C_n \to im\partial_n \to 0$$

and,

$$0 \to im\partial_{n+1} \to ker\partial_n \to H_n(C) \to 0.$$

By the rank nullity theorem

$$rk(C_n) = rk(ker\partial_n) + rk(im\partial_n)$$
$$rk(H_n(C)) = rk(ker\partial_n) - rk(im\partial_{n+1})$$

Then,

$$\begin{aligned}
\sum_{i=0}^{m}(-1)^i rk(C_n) &= \sum_{i=0}^{m}(-1)^i \left(rk(ker\partial_n) + rk(im\partial_n)\right) \\
&= \sum_{i=0}^{m}(-1)^i rk(ker\partial_n) + \sum_{i=0}^{m}(-1)^i rk(im\partial_n) \\
&= \sum_{i=0}^{m}(-1)^i rk(ker\partial_n) - \sum_{i=0}^{m}(-1)^{i-1} rk(im\partial_n) \\
&= \sum_{i=0}^{m}(-1)^i rk(ker\partial_n) - \sum_{i=-1}^{m}(-1)^i rk(im\partial_{n+1}) \\
&= \sum_{i=0}^{m}(-1)^i rk(ker\partial_n) - \sum_{i=0}^{m}(-1)^i rk(im\partial_{n+1}) \\
&= \sum_{i=0}^{m}(-1)^i rk(H_n(C))
\end{aligned}$$

□

Proof. Having proven the first property, proving that the Euler characteristic is invariant to the choice of finite cell decomposition $\mathcal{C}$ becomes relatively simple. First, notice that the cell decomposition $\mathcal{C}$ forms a chain complex,

$$0 \to C_d \to^{\partial} C_{d-1} \to^{\partial} C_{d-2} \to^{\partial} \cdots \to^{\partial} C_0 \to 0$$

Then, since we have a finite number of cells, it follows that these abelian groups are finitely generated, and that their rank is equal to the number of cells of that dimension. So we write,

$$\chi(X, \mathcal{C}) = \sum_{i=0}^{dim(X)} (-1)^i \alpha_i = \sum_{i=0}^{dim(X)} (-1)^i rk(C_i) = \sum_{i=0}^{dim(X)} (-1)^i rk(H_n(C)) = \chi(X)$$

□

Problem 7

Calculate $\chi(X)$ where

1. X is a finite contractible CW complex
2. $X = S^n$

Solution

1.

Suppose that X is a finite contractible CW complex. That X is homotopy equivalent to a singular point. So a cellular decomposition of a singular point is trivially,

$$\mathcal{C} = \mathcal{C}^0 = \{c\}$$

where c is a 0-cell. Then, we can simply compute,

$$\chi(\{pt\}, \mathcal{C}) = \sum_{i=0}^{0} (-1)^i \alpha_i = (-1)^0 \cdot 1 = 1$$

2.

For the n-sphere, it is slightly less trivial. We have a cellular decomposition given by one n-cell and one 0-cell, the point-n-cell pair that is quite useful. To compute α_n, it should be clear that

$$\alpha_n = \begin{cases} 1 & n \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}$$

Finally,

$$\begin{aligned} \chi(X, \mathcal{C}) &= \sum_{i=0}^{n} (-1)^n \alpha_i \\ &= (-1)^0 \alpha_0 + \cdots + (-1)^n \alpha_n \\ &= 1 + 0 + \cdots + 0 \pm 1 \\ &= \begin{cases} 2 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \end{aligned}$$