Real Analysis Assignment 5

Philip Warton

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Preamble

Characterization of the Closure

Let A be a set, and $\mathcal{L}$ be the set of limit points of A, that is,

$$\mathcal{L} = \{l \in M : \forall \epsilon > 0, \quad B_\epsilon(l) \setminus \{l\} \cap A \neq \emptyset\}$$

We claim that $A \cup \mathcal{L} = \overline{A}$.

Proof. In order to show this we wish to show that both $A \cup \mathcal{L} \subset \overline{A}$ and $A \cup \mathcal{L} \supset \overline{A}$.

$\boxed{\subset}$ Let $x \in A \cup \mathcal{L}$ be arbitrary. If $x \in A$, then $x \in \overline{A}$ since we know that $A \subset \overline{A}$ by definition. Otherwise we know that $x \in \mathcal{L} \setminus A$. Suppose by contradiction that $x \in \left(\overline{A}\right)^c$. Since $\overline{A}$ is closed, its complement is open. Then if x is in the open set $\left(\overline{A}\right)^c$, it follows that $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset \left(\overline{A}\right)^c$. However since x is a limit point of A, we know that every ϵ-ball of x intersects A at some point other than x and we have a contradiction. Therefore $x \notin \left(\overline{A}\right)^c$, and $x \in \overline{A} \implies A \cup \mathcal{L} \subset \overline{A}$.

$\boxed{\supset}$ Let $x \in \overline{A}$ be arbitrary. Then we know that every closed set containing A contains x. If $x \in A$, then trivially $x \in A \cup \mathcal{L}$. Suppose that $x \notin A$, then by contradiction suppose $x \notin \mathcal{L}$. Then $\exists \epsilon > 0$ such that $B_\epsilon(x) \cap A = \emptyset$. We can exclude the possibility of having an isolated point since $x \notin A$ by assumption. Then we know that $B_\epsilon(x)^c$ is going to be a closed set containing A. Since there exists a closed set containing A such that x does not belong to it, we say $x \notin \overline{A}$ (contradiction).

Having shown both inclusions, it follows that $A \cup \mathcal{L} = \overline{A}$. □

Problem 4.46 (Characterizations of Dense Sets)

A set A is dense in M if and only if the following conditions hold:

(a) Every point in M is the limit of a sequence in A.
(b) For every point in M, every ϵ-neighborhood intersects A.
(c) Every non-empty open set intersects A.
(d) The interior of the complement of A is empty.

Proof. $\boxed{\text{Dense} \Rightarrow (a)}$ To show this, assume that A is dense in M, that is, $\overline{A} = M$. Let $x \in M$ be arbitrary. Then $x \in \overline{A}$. Then we know that either $x \in A$ or $x \in \mathcal{L}$ where $\mathcal{L}$ is the set of limit points of A. If x is a limit point of A, simply construct a sequence (x_n) where for every x_n choose some element $y \in B_{\frac{1}{n}}(x) \setminus \{x\} \cap A$ which we know to exist by the definition of limit point. Then $\forall n \in \mathbb{N}, x_n \in A$ and $(x_n) \to A$. So if x is not a limit point of A, then $x \in A$. Simply choose the constant sequence (x_n) such that $x_n = x \forall n \in \mathbb{N}$ and trivially $(x_n) \to x$ and the sequence is contained in A.

$\boxed{(a) \Rightarrow (b)}$ Assume that every point in M is the limit of a sequence contained in A. Let $x \in M$ be arbitrary, and let $\epsilon > 0$ be arbitrary. Since there exists a sequence in A that converges to x, call this sequence x_n, we know that this sequence is eventually in $B_\epsilon(x)$ and therefore $B_\epsilon(x) \cap A$ is not empty.

$\boxed{(b) \Rightarrow (c)}$ Assume that $\forall x \in M, \forall \epsilon > 0, \quad B_\epsilon(x) \cap A \neq \emptyset$. Let $U \subset M$ be an open set that is non-empty. Then we know that there exists some ϵ-neighborhood of some point $x \in M$ such that $B_\epsilon(x) \subset U$. By assumption this ball intersects A, therefore U also intersects A.

$\boxed{(c) \Rightarrow (d)}$ Assume that every non-empty open set in M intersects A. Suppose by contradiciton there exists some x that belongs to the interior of A^c. Since the interior must be open, there exists some ϵ-neighborhood of x that is contained within the interior of A^c. However since this is a non-empty open set in M, it follows that $B_\epsilon(x)$ intersects A, and therefore cannot be contained in $int(A^c) \subset A^c$ (contradiciton). Since supposing that $\exists x \in int(A^c)$ leads to a contradiction, $int(A^c)$ must be empty.

$\boxed{(d) \Rightarrow \text{Dense}}$ Assume that $int(A^c)$ is empty. We wish to show that $\overline{A} = M$. Since M is our unviersal set, we say that $\overline{A} \subset M$ by definition. To show that $\overline{A} \supset M$, let $x \in M$ be arbitrary. Suppose by contradiction that $x \notin \overline{A}$. Then since we know the closure of A is closed, its complement must be open. This means that $\exists B_\epsilon(x) \subset \left(\overline{A}\right)^c \subset A^c$. However this would imply that $int(A^c)$ is non-empty (contradiction). Thus $\overline{A} = M$.

Having shown each implication, we can now characterize a set as being totally dense by any one of these conditions. □

Problem 5.7

Question (a)

If $f : M \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$, show that the sets $A = \{x : f(x) > a\}$ and $B = \{x : f(x) < a\}$ are open subsets of M.

Proof. Since f is a continuous function, we say that for any open set in $\mathbb{R}$, its pre-image must be open in M. We can write

$$A = \{x \in M : f(x) \in (a, \inf)\} = f^{-1}(a, \inf) \qquad B = \{x \in M : f(x) \in (-\inf, a)\} = f^{-1}(-\inf, a)$$

Since both intervals are open sets in $\mathbb{R}$, A and B are open in M. □

Question (b)

Now we must show the converse. That is, show that if $f^{-1}(-\inf, a)$ and $f^{-1}(a, \inf)$ are open in M for every $a \in \mathbb{R}$, then f is continuous.

Proof. Let $B(a)$ denote $f^{-1}(-\inf, a)$ and $A(a)$ denote $f^{-1}(a, \inf)$. Let $U \in \mathbb{R}$ be an open set, we wish to show that $f^{-1}(U)$ is open in M. Notice that we can write any interval in $\mathbb{R}$ as

$$(x, y) = (-\inf, y) \cap (x, \inf) = B(y) \cap A(x)$$

This means that $f^{-1}(x, y) = f^{-1}(B(y)) \cap f^{-1}(A(x))$. This means that the pre-image of the interval (x, y) is a finite intersection of open sets, and therefore is open. Since any open set in $\mathbb{R}$ can be written as a union of open intervals (we have shown that the set of such intervals with rational endpoints is a valid basis for the standard topology on $\mathbb{R}$ in a previous homework, so it would follow that the set of all open intervals would be as well), we say

$$U = \bigcup_{x \in X, y \in Y} (x, y) \qquad \text{and} \qquad f^{-1}(U) = f^{-1}\left(\bigcup_{x \in X, y \in Y} (x, y)\right) = \bigcup_{x \in X, y \in Y} (f^{-1}(x, y))$$

Then since every set $f^{-1}(x, y)$ is open in M, a union of such sets must also be open. Thus $f^{-1}(U)$ is open in M for an arbitrary open set $U \subset \mathbb{R}$, and we say that f is continuous. □

Question (c)

Show that f is continuous even if we take the sets $A(a)$ and $B(a)$ upon rational numbers.

Proof. The proof is the same, except we say that any rational interval $(p, a) = A(p) \cap B(q)$. Then it follows that any interval of this form will be an open set in M since $f^{-1}(p, a) = f^{-1}A(p) \cap f^{-1}B(q)$. Then since we know that the set of all rational intervals form a basis for the standard topology on $\mathbb{R}$. We write for any $U \in \mathbb{R}$,

$$U = \bigcup_{p \in P, q \in Q} (p, q)$$

And the pre-image of U is equal to the union of the all the pre-images of intervals of the form (p, q), thus $f^{-1}(U)$ is a union of open sets in M and is therefore open. □

Problem 5.17

Let $f, g : (M, d) \to (N, \rho)$ be continuous, and let D be a dense subset of M. If $f(x) = g(x)$ for every $x \in D$, show that $f(x) = g(x)$ for all $x \in M$. If f is onto, show that $f(D)$ is dense in N.

Proof. Assume that $f(x) = g(x)$ for every $x \in D$. We wish to show that for every $x \notin D$, we still have $f(x) = g(x)$. Let $x \notin D$ be arbitrary. Then by our characterization of dense sets, we know that there exists some sequence $(x_n) \in D$ such that $(x_n) \to x$. Since f and g are both continuous we have both

$$f(x_n) \to f(x), \qquad g(x_n) \to g(x)$$

Then we know that for every element of D, its image under f and g is the same, so we say that for every natural number n we have $f(x_n) = g(x_n)$. We have two convergent sequences that are equal to each other, so it must be the case that their limits are equal, thus $f(x) = g(x)$. □

Now that we have proved that $f(x) = g(x)$ for every point $x \in M$, we must prove that if f is onto, then $f(D)$ is dense in N.

Proof. Assume that f is onto. Let $x \notin f(D)$, we wish to show that any neighborhood of x intersects $f(D)$. Let $B_\epsilon(x)$ be arbitrary. Since f is continuous, we know that $f^{-1}(B_\epsilon(x))$ is an open set in M. Since f is onto we know that this set is non-empty. Then since we have a non-empty open set in M we say $f^{-1}(B_\epsilon(x)) \cap D \neq \emptyset$. Let y be an element of this intersection. Then $f(y) \in f(D) \cap B_\epsilon(x)$, and we say that $f(D)$ is dense in N. □

Problem 5.56

Let $f : (M, d) \to (N, \rho)$.

(i)

Provide examples that show that continuity does not imply an open map, and the converse. Let $f : (\mathbb{Q}, d) \to (\mathbb{R}, d)$ such that $f(x) = x$ (identity map). Then we say that f is continuous but f is not an open map. Since both metric spaces share the same metric, and $f(x) = x$, it follows that if $d(x_n, x) \to 0$ in $(\mathbb{Q}, d)$ then it would do the same in $(\mathbb{R}, d)$. Thus f is continuous. Let U be a non-empty open set in $(\mathbb{Q}, d)$. The image of this set $f(U)$ will be a set of rational points in $\mathbb{R}$. Since $\mathbb{Q}^c$ is dense in $\mathbb{R}$, it follows that the interior of $\mathbb{Q}$ is empty, and therefore no subset of $\mathbb{Q}$ can be open in $\mathbb{R}$.

To show that the converse does not hold let $g : (\mathbb{R}, d) \to (\mathbb{Z}, d')$ where d is the standard metric on the real numbers, and d' is the discrete metric. We give a formula for g as

$$g(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then since every set in $(\mathbb{Z}, d')$ is both open and closed, we say g must be an open map. However, we say that g can not be continuous because there exists an open set in $(\mathbb{Z}, d')$ such that its pre-image is closed in $(\mathbb{R}, d)$. The open ball $B_{\frac{1}{2}}(1)$ is open in $(\mathbb{Z}, d')$, but its pre-image $\{1\}$ is closed in $(\mathbb{R}, d)$. Thus f is not

(ii)

Qualifying Exam Problem 2

(a)

Given $a \in \mathbb{R}$ denote by $\{a\}$ the fractional part of a; that is,

$$\{a\} = \min\{a - n : n \in \mathbb{Z}, n \leq a\}$$

Suppose that α is a real irrational number. Prove that the set

$$A_\alpha = \{\{n\alpha\} : n \in \mathbb{Z}\}$$

is dense in $[0, 1]$.

Proof. To show that A_α is dense in the closed unit interval, we can use our characterizations of dense sets. Let $x \in A_\alpha^c$, let $\epsilon > 0$, we wish to show that $B_\epsilon(x)$ itersects the set A_α. We can write the ϵ-ball about x as $(x - \epsilon, x + \epsilon)$. Then we claim that for some y in this interval, $y = n\alpha$ for some integer number n. □