MTH 351 Homework 1

Philip Warton

January 16, 2020

1

a.

We first must take the nth derivative of $f(x)$.

$$
\begin{aligned}
f(t) &= \frac{1}{1-t} \\
f(t) &= (1-t)^{-1} \\
f'(t) &= (-1)(1-t)^{-2}(-1) \\
f''(t) &= (-1)(-2)(1-t)^{-3}(-1)(-1) \\
f'''(t) &= (-1)(-2)(-3)(1-t)^{-4}(-1)(-1) \\
f^{n}(t) &= (n!)(1-t)^{-(n+1)}
\end{aligned}
$$

We can then determine our Taylor polynomial of degree n as follows $p_n(t) = 1 + t + t^2 + t^3 ... + t^n$

b.

Now, we use Lagrange's theorem to find n sufficiently large such that our error $R_n(t) < \epsilon = 10^{-4}$. We have so far that $f^{(n+1)}(c) = (n+1)!(1-c)^{-(n+2)}$. Then we can say that

$$
\begin{aligned}
R_k(t) &= (1-c)^{-(k+2)} t^{(k+1)} \\
R_k(t) &\leqslant (1-\frac{1}{3})^{-(k+2)} (\frac{1}{3})^{k+1} \\
&= (\frac{2}{3})^{-k-2} (\frac{1}{3})^{k+1}
\end{aligned}
$$

From here we can use a calculator in order to find $k : (\frac{2}{3})^{-k-2}(\frac{1}{3})^{k+1} < \epsilon$. This gives us $k \geqslant 13$.

2

a.

To get the Taylor polynomial of $g(x) = \frac{1}{2+3x}$ we can start by taking a few derivatives.

$$
\begin{aligned}
g(x) &= (2+3x)^{-1} \\
g'(x) &= (-1)(2_3x)^{-2}(3) \\
g''(x) &= (-1)(-2)(2+3x)^{-3}(3)(3) \\
g'''(x) &= (-1)(-2)(-3)(2+3x)^{-4}(3)(3)(3) \\
&... \\
g^n(x) &= (-1)(-2)...(-n) * (2+3x)^{-(n+1)}(3)^n
\end{aligned}
$$

From this we can write out the polynomial as $q_n(x) = \frac{1}{2} - \frac{3}{2^2}x + \frac{3^2}{2^3}x^2 + ... + \frac{(-3x)^n}{2^{n+1}}$.

b.

We can then construct a Langrangian error term in the form $R_n(x) = \frac{g^{n+1}(c)}{(n+1)!}(x - x_0)^{n+1}$. This gives us

$$
\begin{aligned}
R_n(x) &= \frac{(-1)^{n+1}(n+1)!(2+3x)^{-(n+2)}3^{n+1}}{(n+1)!}(x-x_0)^{n+1} \\
\leqslant |R_n(x)| &= \frac{(n+1)!(2+3x)^{-(n+2)}3^{n+1}}{(n+1)!}|(x-x_0)|^{n+1} \\
&= \frac{3^{n+1}}{(2+3x)^{n+2}} \\
&= (\frac{3x}{2+3c})^{n+1}(\frac{1}{2+3c})
\end{aligned}
$$

Since we know $0 < x < \frac{1}{5}$, and we have positive x in the numerator, we can choose $x = \frac{1}{5}$ to bound our error. With c lying between $x_0 = 0$ and $x = \frac{1}{5}$, we can choose $c = 0$ to bound our error when it is largest. From this, we get

$$
\begin{aligned}
R_n(x) &\leqslant (\frac{3(1/5)}{2+3(0)})^{n+1}(\frac{1}{2+3(0)}) \\
&= (\frac{3}{10})^{n+1}(\frac{1}{2})
\end{aligned}
$$

Let this term be less than epsilon, and n must be at least 8.

3

a

For this problem we will use substitution to get our Taylor polynomial and error term. Let $\frac{1}{1-t} = \frac{1}{1+x^2}$, by solving for t we get $t = -x^2$. By plugging this into our polynomial from $\boxed{\text{1a.)}}$ we get $p_n(-x^2) = \sum_{k=1}^{n}(-x^2)^k$. Written out, this looks like $h(x) = 1 - x^2 + x^{2^2} - x^{2^3} + ... + (-x^2)^n$.

b

Using this same method in order to find our Langrange error term we take our error term from $\boxed{\text{1b.}}$, $R_n(t) = (1-c)^{-(n+2)}t^{(n+1)}$ and let $t = -x^2$. This gives us $R_n(-x^2) = (1-c)^{-(n+2)} - x^{2^{(n+1)}}$. Rewriting this we get

$$\begin{aligned} \frac{(-x^2)^{n+1}}{(1-c)^n+2} &\leqslant \frac{|(-x^2)^{n+1}|}{(1-c)^n+2} \\ &= \frac{(x^2)^{n+1}}{(1-c)^n+2} \end{aligned}$$

To maximize our bound, let $x = 0.5$ and $c = 0$, so that it lies between x and x_0 (given $x_0 = 0$). Now we have $\frac{((\frac{1}{2})^2)^{n+1}}{(1-0)^{n+2}} = (\frac{1}{4})^{n+1}$. We must choose a sufficiently large n so that $(\frac{1}{4})^{n+1} < \epsilon$ where $\epsilon = 10^-5$. Using a calculator, it can be shown that $n = 8$ is large enough to make our error term smaller than ϵ.

4

See matlab file attached on canvas.