Real Analysis - Assignment 7

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Problem 8.17

> If M is compact, then M is seperable.

Proof. Since M is compact, it is totally bounded. For any $n \in \mathbb{N}$ there exists a finite collection of $(\frac{1}{n})$-balls such that

$$M = \bigcup_{y \in Y_n} B_{(\frac{1}{n})}(y)$$

Let $Y = \bigcup_{n \in \mathbb{N}} Y_n$, be the set of all points that are the center of our $(\frac{1}{n})$-balls for each n. The set Y is a countable union of finite sets, and must therefore be countable. This set is dense in M. To show this, let $x \in M$ be arbitrary. For every $n \in \mathbb{N}$, x must lie within at least one of our $\frac{1}{n}$-balls, therefore let

$$y_n \in \{y \in Y_n \mid x \in B_{(\frac{1}{n})}(y)\}$$

It follows that $y_n \in Y$ for every natural number n, and that $(y_n) \to x$ since $d(y_n, x) < \frac{1}{n}$ for every n. Thus Y is a countable dense set in M, and M is seprable. $\square$

Problem 8.23

> Let M be be a compact space. Let $f : M \to N$ be a continuous bijection. Then f is a homeomorphism.

Proof. To show that f is a homeomorphism we must show that f^{-1} is a continuous function. Since f is a bijection, f^{-1} is also a bijective function. Let $(y_n) \to y$ in the space N. Let (x_n) be a sequence in M that corresponds to (y_n), that is, $x_n = f^{-1}(y_n)$ for each n.

Suppose that x_n does not converge to $x = f^{-1}(y)$. Then $\exists \epsilon > 0$ such that for every $N \in \mathbb{N}$ there exists some $n \geqslant N$ where $x_n \notin B_\epsilon(x)$. Choose (x_{n_k}) to a be a subsequence of x_n such that no point x_{n_k} lies in $B_\epsilon(x)$. Since (x_{n_k}) is a sequence in a compact space M, it must have some subsequence $(x_{n_{k_m}})$ that converges to a point $x' \in M$. We know that $x' \neq x$ because each $x_{n_{k_m}} \notin B_\epsilon(x)$. Since f is continuous and bijective,

$$x_{n_{k_m}} \to x' \quad \implies \quad y_{n_{k_m}} \to f(x') \neq f(x) = y$$

However, this means that a subsequence of y_n converges to a point other than y, which contradicts our assumption. If follows that it must be the case that $x_n \to x$, or rather, $f^{-1}(y_n) \to f^{-1}(y)$. Hence f^{-1} is continuous. This makes f a continuous, bijective, open map, and therefore a homeomorphism. $\square$

Problem 8.48

First we prove the following:

> A sequence is Cauchy if and only if it is eventually in an arbitrary ϵ-neighborhood of some point in the sequence. Alternatively,
>
> $$(x_n) \text{ is Cauchy} \quad \iff \quad \forall \epsilon > 0,\ \exists N \in \mathbb{N} \mid \forall n \geqslant N, x_n \in B_\epsilon(x_N)$$

Proof. $\boxed{\Rightarrow}$ Let $(x_n) \subset M$ be a Cauchy sequence in some metric space M. Then let $\epsilon > 0$ be arbitrary, it follows that $\exists N \in \mathbb{N}$ such that $\forall m, n \geqslant N,\ d(x_m, x_n) < \epsilon$. Fix $m = N$, and then we have $d(x_N, x_n) < \epsilon\ \forall n \geqslant N$. Then for every $n \geqslant N$, clearly $x_n \in B_\epsilon(x_N)$.

$\boxed{\Leftarrow}$ Let (x_n) be a sequence such that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that for every $n \geqslant N, x_n \in B_\epsilon(x_N)$. Let $\epsilon > 0$ be arbitrary and choose $\delta = \frac{\epsilon}{2}$. Then there is some N such that every point at or beyond this index belongs to $B_{\frac{\epsilon}{2}}(x_N)$. Then it clearly follows that for any two such points at indices $m, n \geqslant N$, $d(x_m, x_n)$ must be less than ϵ. Thus the sequence is Cauchy. $\square$

Let $(M, d), (N, \rho)$ be metric spaces, and let $f : M \to N$ be uniformly continuous. Then the image of a Cauchy sequence $(x_n) \subset M$ is Cauchy in N.

Proof. Let $\epsilon > 0$ be arbitrary, since f is uniformly continuous there will exist some $\delta > 0$ such that $f(B^d_\delta(x)) \subset B^\rho_\epsilon(f(x))$. Now let $(x_n) \subset M$ be a Cauchy sequence. It follows that there exists some natural number N_δ such that $\forall m, n \geqslant N_\delta, d(x_m, x_n) < \delta$. Equivalently, we can say that $\forall n \geqslant N_\delta, x_n \in B^d_\delta(x_N)$. Since f is uniformly continuous it follows that $f(x_n) \in B^\rho_\epsilon(f(x_N))$ for every natural number $n \geqslant N$. This is equivalent to $(f(x_n))$ beign Cauchy. $\square$

Problem 8.54

For every bounded, non-compact subset $E \subset \mathbb{R}$, there exists some continuous function $f : E \to \mathbb{R}$ that is not uniformly continuous.

Proof. Since E is not compact, and it is bounded, it must not be closed (Heine-Borel Theorem). Therefore there exists some point $a \in \mathbb{R}$ such that a is a limit point of E but is not contained in E. Define the function

$$f(x) = \frac{1}{x - a}$$

Since f is clearly continuous on $\mathbb{R} \setminus \{a\}$, it is also continuous on E. This function is not, however, uniformly continuous. For every $\epsilon > 0$, there should exist some $\delta > 0$ such that

$$|x - y| < \delta \implies \left| \frac{1}{x - a} - \frac{1}{y - a} \right| < \epsilon$$

The right hand side of this implication can be written as

$$\left| \frac{(y - a) - (x - a)}{(x - a)(y - a)} \right| = \left| \frac{y - x}{(x - a)(y - a)} \right| < \epsilon$$

However, for any $\delta > 0$, we can simply fix some $y \in B_{\frac{\delta}{2}}(a) \cap E$ which must exist since a is a limit point of E. We can choose x arbitrarily close to a. Then since $a \notin E$ we know that $y - a$, will be fixed. Thus the numerator will approach this fixed value, while the the denominator will become arbitrarily small as x becomes arbitrarily close to a. Hence, the quantity is unbounded for every $\delta > 0$, and the implication can never hold; so f is not uniformly continuous. $\square$