Real Analysis - Assignment 10

Philip Warton

December 6, 2020

A bounded subset of $\mathbb{R}$ is totally bounded.

Proof. Let $A \subset \mathbb{R}$ be bounded. That is, there exists some $M > 0 : |x| \leqslant M \forall x \in A$. Choose $\epsilon > 0$ randomly. Then $\exists k \in \mathbb{N} : k\epsilon > M$ (Archimedean Property). So take

$$S = \{-k\epsilon, -(k-\frac{1}{2})\epsilon, -(k-1)\epsilon, \cdots, -\epsilon, -\frac{1}{2}\epsilon, 0, \frac{1}{2}\epsilon, \epsilon, \cdots, (k-\frac{1}{2})\epsilon, k\epsilon\}$$

This is a finite set with a cardinality of $4k$ such that $A \subset \bigcup_{s \in S} B_\epsilon(s)$. Thus A is totally bounded. □

A bounded subset of $\mathbb{R}^n$ is totally bounded.

Proof. This will follow a similar formula. Let $A \subset \mathbb{R}^n : \exists M > 0$ where $A \subset B_M(0)$. Choose $\epsilon > 0$ arbitrarily, and $\exists k \in \mathbb{N} : k\epsilon > M$. Take S as defined in the previous proof, and then take $S \times S \times \cdots \times S$ so that you have an n-tuple of elements of S, and have each permutation. This set may be large, but it will be finite. Then $A \subset \bigcup_{s \in S^n} B_\epsilon(s)$. □

The closed unit ball in $\ell_p, 1 \leqslant p \leqslant \infty$ is not totally bounded.

Proof. Let $(x_n) \subset \ell_p : x_n = (0, \cdots, 0, 1, 0, \cdots)$ where there is a 1 at the n-th element and 0 elsewhere. For any two elements, for any $1 \leqslant p \leqslant \infty$ we know that $||x_m - x_n||_p = 1$. Therefore there is no Cauchy subsequence of this sequence, therefore this closed unit ball is not totally bounded. □