MTH 430 Homework 1

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Problem 1

Let $f : X \to Y$ be a function.

(a)

Show that for all $A_1, A_2 \subset X, \quad f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Proof. Let $A_1, A_2 \subset X$.

$\boxed{\subset}$ Let $y \in Y$ such that $y \in f(A_1 \cup A_2)$. Then $\exists x \in A_1 \cup A_2$ such that $f(x) = y$. If $x \in A_1$, then $f(x) = y \in f(A_1) \subset f(A_1) \cup f(A_2)$. If $x \notin A_1$ then $x \in A_2$, and similarly it follows that $y \in f(A_1) \cup f(A_2)$.

$\boxed{\supset}$ Now, let $y \in Y$ such that $y \in f(A_1) \cup f(A_2)$. Then either $y \in f(A_1)$ or $y \in f(A_2)$. If $y \in f(A_1)$ then $\exists a_1 \in A_1$ such that $f(a_1) = y$. Thus, $a_1 \in A_1 \cup A_2$ and $f(a_1) = y \in f(A_1 \cup A_2)$ Otherwise, $y \in f(A_2)$, and then $\exists a_2 \in A_2 : f(a_2) = y$, and thus $f(a_2) = y \in f(A_1 \cup A_2)$. □

(b)

Show that for all $A_1, A_2 \subset X, \quad f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

Proof. Let $A_1, A_2 \subset X$ be arbitrary. Let $y \in Y$ such that $y \in f(A_1 \cap A_2)$. Then, there exists $a \in A_1 \cap A_2$ such that $f(a) = y$. Since $a \in A_1$, $f(a) \in f(A_1)$, and similarly $f(a) \in f(A_2)$. Thus $f(a) = y \in f(A_1) \cap f(A_2)$. □

Problem 2

(a)

Show that for all $B_1, B_2 \in Y, \quad f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

Proof. Let $B_1, B_2 \subset Y$.

$\boxed{\subset}$ Let $x \in X$ such that $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$. If $f(x) \in B_1$ then $x \in f^{-1}(B_1) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$. Otherwise, $f(x) \in B_2$ thus $x \in f^{-1}(B_2) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$.

$\boxed{\supset}$ Let $x \in X$ such that $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. If $x \in f^{-1}(B_1)$, then $f(x) \in B_1 \subset B_1 \cup B_2$, thus $x \in f^{-1}(B_1 \cup B_2)$. Otherwise, $x \in f^{-1}(B_2)$, and it follows that $f(x) \in B_2 \subset B_1 \cup B_2$ so $x \in f^{-1}(B_1 \cup B_2)$. □

(b)

Show that for all $B_1, B_2 \in Y, \quad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

Proof. Let $B_1, B_2 \subset Y$.

$\boxed{\subset}$ Let $x \in X$ such that $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$, thus $f(x) \in B_1$ and $f(x) \in B_2$. Since $f(x) \in B_1, x \in f^{-1}(B_1)$, and similarly $x \in f^{-1}(B_2)$. Therefore $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.

$\boxed{\supset}$ Let $x \in X$ such that $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Since $x \in f^{-1}(B_1), f(x) \in B_1$, and since $x \in f^{-1}(B_2), f(x) \in B_2$. Since $f(x) \in B_1$ and $f(x) \in B_2$ and thus $f(x) \in B_1 \cap B_2$, it follows that $x \in f^{-1}(B_1 \cap B_2)$. $\square$

Problem 3

(b)

We wish to show that (a) and (b) are equivalent.

Proof. Let $A \subset X$ be arbitrary.

"$\Rightarrow$" Assume f is injective. Let $a \in A$, then $f(a) \in f(A)$. Since $f(a) \in f(A)$, by definition $a \in f^{-1}(f(A))$. Thus $\forall a \in A, a \in f^{-1}(f(A))$ and we say $A \subset f^{-1}(f(A))$. Now let $a \in f^{-1}(f(A))$ be arbitrary, then $f(a) \in f(A)$. Since $f(a) \in f(A)$, then $\exists a_0 \in A$ such that $f(a_0) = f(a)$. We know that f is injective therefore $a = a_0 \in A$. Thus $f^{-1}(f(A)) \subset A$, and $f^{-1}(f(A)) \supset A$, so $f^{-1}(f(A)) = A$.

"$\Leftarrow$" Assume that $f^{-1}(f(A)) = A \quad \forall A \subset X$. Let $a, b \in X$ such that $f(a) = f(b)$ and let $A = \{a\}$. Then $f(A) = \{f(a)\}$ and since $f(a) = f(b)$ it follows that $f(b) \in f^{-1}(f(A))$. Therefore by our assumption that $f^{-1}(f(A)) = A$, we have $b \in A$, and thus $b = a$. $\square$

(c)

We wish to show that (a) and (c) are equivalent.

Proof. A function f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$.

"$\Rightarrow$" Assume that f is injective. We wish to show that $f(A \cap B) = f(A) \cap f(B)$. Let $y \in f(A \cap B)$, then $\exists x \in A \cap B$ such that $f(x) = y$. Since $x \in A, f(x) = y \in f(A)$. Similarly $y \in f(B)$, thus $y \in f(A) \cap f(B)$, and we say $f(A \cap B) \subset f(A) \cap f(B)$.

Now let $y \in f(A) \cap f(B)$, then $\exists x_1 \in A : f(x_1) = y$. Similarly $\exists x_2 \in B : f(x_2) = y$. Since f is an injection we can say $x_1 = x_2 = x$. Thus $x \in A$ and $x \in B$ so $x \in A \cap B$ and it follows that $y = f(x) \in f(A \cap B)$.

"$\Leftarrow$" Assume that $f(A \cap B) = f(A) \cap f(B) \quad \forall A, B \subset X$. Let $a, b \in X$ such that $f(a) = y = f(b)$. Let $A = \{a\}$ and $B = \{b\}$, then $f(A) = \{y\} = f(B) = f(A) \cap f(B) = f(A \cap B)$. Since $y \in f(A \cap B)$, then there must exist some $x \in A \cap B$ such that $f(x) = y$. Therefore $x \in \{a\} \cap \{b\}$ and $a = x = b$. $\square$

(d)

We wish to show that (c) and (d) are equivalent.

Proof. We want to show $f(A \cap B) = f(A) \cap f(B)$ if and only if $A \cap B = \emptyset \Rightarrow f(A) \cap f(B) = \emptyset$.

"$\Rightarrow$" Assume that for all $A, B \subset X$ that $f(A \cap B) = f(A) \cap f(B)$. Let $A, B \subset X$ such that $A \cap B = \emptyset$. Then $f(A \cap B) = \emptyset = f(A) \cap f(B)$, and the desired implication holds.

"$\Leftarrow$" Assume that for all $A, B \subset X$ that $A \cap B = \emptyset \Rightarrow f(A) \cap f(B) = \emptyset$. Let $a \in \{a\} = A \subset X$ and $b \in \{b\} = B \subset X$, and suppose $a \neq b$. Then $A \cap B = \emptyset = f(A) \cap f(B)$, which means that $f(a) \neq f(b)$. Since $a \neq b \Rightarrow f(a) \neq f(b)$, it follows that f is injective, which is equivalent to $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subset X$. $\square$

(e)

We wish to show that f is injective if and only if $\forall B \subset A \subset X, f(A \setminus B) = f(A) \setminus f(B)$.

Proof. "$\Rightarrow$" Assume that f is injective. Let $B \subset A \subset X$ be arbitrary. We want to show that $f(A \setminus B) = f(A) \setminus f(B)$.

$\boxed{\subset}$ Let $y \in Y : y \in f(A \setminus B)$. Since f is injective, there exists a unique $x \in A \setminus B$ such that $f(x) = y$. Since $x \in A, f(x) = y \in f(A)$. We know that for all $b \in B, f(b) \neq y$, because our unique solution $x \notin B$. Since $\nexists b \in B : f(b) = y, y \notin f(B)$. Then with $y \in f(A)$ and $y \notin f(B)$, $y \in f(A) \setminus f(B)$.

$\boxed{\supset}$ Let $y \in Y$ such that $y \in f(A) \setminus f(B)$. Then $\nexists b \in B : f(b) = y$, and $\exists x \in A : f(x) = y$. It follows that if $x \in A$ and $f(x) = y$ that $x \in A \setminus B$. Therefore $f(x) = y \in f(A \setminus B)$.

"$\Leftarrow$" Assume that for all $B \subset A \subset X \quad f(A \setminus B) = f(A) \setminus f(B)$. Let $a, b \in X$ such that $f(a) = y = f(b)$. We want to show that $a = b$. Suppose by contradiction that $a \neq b$. Let $B = \{b\} \subset A = \{a, b\} \subset X$. Then $A \setminus B = \{a\}$, and then $f(A \setminus B) = \{f(a)\} = \{y\}$. However, we also know that $f(A) = \{y\}$ and $f(B) = \{y\}$ so then $f(A) \setminus f(B) = \emptyset$. By assumption $f(A \setminus B) = f(A) \setminus f(B)$, therefore $\{y\} = \emptyset$ (contradiction). It must then be the case that $a = b$. $\square$