General Topology and Fundamental Group - Notes

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1 Introduction to Algebraic Topology

Let $Top = \{\text{all topological spaces}\}$ and let $Alg = \{\text{all groups/rings/} \cdots\}$. Then we have a functor $F : Top \to Alg$. Top is equipped with continuous functions. If $f : X \to Y$ is continuous then there exists $f_* : F(X) \to F(Y)$. The function f_* must satisfy:

$$f = id \implies f_* = id \tag{1}$$
$$f : X \to Y, g : Y \to Z \implies (g \circ f)_* = g_* \circ f_* \tag{2}$$

If f is homeomorphic then f_* is isomorphic.

Definition 1.1 (Path)**.** Let X be a topological space with $x_0, x_1 \in X$. A path α is a function $\alpha : I \to X$ such that $\alpha(0) = x_0, \alpha(1) = x_1$ is a path in X.

The constant path $\alpha(t) = x \forall t$ can be written simply as $\alpha = x$.

Definition 1.2 (Path Homotopy)**.** Two paths from x_0 to x_1 are said to be homotopic if $\exists F : I \times I \to X$ such that

$$F(t, 0) = \alpha(t) \tag{3}$$
$$F(t, 1) = \beta(t) \tag{4}$$
$$F(0, u) = x_0 \tag{5}$$
$$F(1, u) = x_1 \tag{6}$$

and F is continuous.

For path homotopy we write $\alpha \simeq_p^F \beta$. Now this is an equivalence relation, that is,

$$\alpha \simeq_p \alpha \tag{7}$$
$$\alpha \simeq_p \beta \implies \beta \simeq_p \alpha \tag{8}$$
$$\alpha \simeq_p \beta, \beta \simeq_p \gamma \implies \alpha \simeq_p \beta \qquad \text{by glueing lemma} \tag{9}$$

We show this now.

Proof. Assume that $\alpha \simeq_p \beta$. □

We can define a concatenation of two paths α, β such that $\alpha(1) = \beta(0)$ as

$$\alpha * \beta : I \to X, \alpha * \beta(t) = \begin{cases} x_0, & t \in [0, 1/2] \\ \alpha(2t - 1), & t \in (1/2, 1] \end{cases}$$

One can show that $x_0 * \alpha \simeq_p \alpha$.