Advanced Multivariable Calculus - Homework 5

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June 4, 2021

Problem 1

Let $f : I \to \mathbb{R}$ continuously where I is a product of real intervals. Then for any partition P on I, there exists some $J \in P$ where there exists $u, v \in J$ such that

$$0 \leqslant U(f,P) - L(f,P) \leqslant \text{Vol(I)}[f(u) - f(v)]$$

Proof. Since P is finite we can say let

$$\alpha = \max_{J \in P}\{M(f,J) - m(f,J)\}$$

Then it follows that

$$\sum_{J\in P} M(f,J)\text{Vol}(J) - \sum_{J\in P} m(f,J)\text{Vol}(J) = \sum_{J\in P} \text{Vol}(J)[M(f,J) - m(f,J)] \leqslant \sum_{J\in P} \text{Vol}(J)\alpha$$

Then since $\alpha \in \mathbb{R}$ is constant we can simply write

$$\sum_{J\in P} \text{Vol}(J)\alpha = \alpha \sum_{J\in P} \text{Vol}(J) = \alpha \text{Vol}(I)$$

So since each J is compact by the extreme value theorem we say there exists $u_J, v_J \in J$ for each J such that $f(u_J) = M(f,J), f(v_J) = m(f,J)$. So then finally we say that $\exists J \in P$ containing points u, v such that $f(u) - f(v) = \alpha$, and thus

$$0 \leqslant U(f,P) - L(f,P) \leqslant \text{Vol}(I)[f(u) - f(v)]$$

□

Problem 2

Let $f : \mathbb{R}^2 \to \mathbb{R}$ where f is defined by

$$f(x,y) = \begin{cases} 1 & x \in \mathbb{Q} \\ 2y & x \notin \mathbb{Q} \end{cases}$$

Then the following integral is true:

$$\int_0^1 \left(\int_0^1 f dy \right) dx = 1$$

However, f is not integrable.

Proof. Let us begin with the nested integrals. We say that $\int_0^1 f dy$ is equal to the following:

$$\begin{aligned} \int_{y=0}^{y=1} f dy &= \begin{cases} y & x \in \mathbb{Q} \\ y^2 & x \notin \mathbb{Q} \end{cases} \Bigg|_0^1 \\ &= 1 - 0 \quad \forall x \in [0,1] \\ &= 1 \end{aligned}$$

Then clearly $\int_0^1 (1) dx = 1$. So we say that

$$\int_0^1 \left(\int_0^1 f dy \right) dx = 1$$

Now we wish to show that f is not integrable. To do this we will show that $U(f,P) > L(f,P)$ thus they cannot be equal. We start with our upper sum. That is,

$$\begin{aligned} U(f,P) &= \sum_{J\in P} M(f,J)\text{Vol}(J) \\ &= \sum_{J\in P} \sup\{f(x,y) \mid (x,y)\in J\}\text{Vol}(J) \\ &= \sum_{J\in P} \max\{1, 2y \mid (x,y)\in J\}\text{Vol}(J) \end{aligned}$$

Then since we are partitioning the unit square, we know that there must exist some $J_1 \in P$ such that $(\frac{\pi}{4}, 1) \in J_1$. Therefore

$$\begin{aligned} U(f,P) &= \max\{1, 2y \mid (x,y)\in J_1\}\text{Vol}(J_1) + \sum_{J\in P\setminus\{J_1\}} \max\{1, 2y \mid (x,y)\in J\}\text{Vol}(J) \\ &= (2)\text{Vol}(J_1) + \sum_{J\in P\setminus\{J_1\}} \max\{1, 2y \mid (x,y)\in J\}\text{Vol}(J) \\ &> (1)\text{Vol}(J_1) + \sum_{J\in P\setminus\{J_1\}} (1)\text{Vol}(J) = \text{Vol}(I) = 1 \end{aligned}$$

Similarly $\exists J_0 \in P$ such that $(\frac{\pi}{4}, 0) \in J_0$. Then it follows that $\min\{1, 2y | (x,y) \in J_0\} = 0$. □

Problem 3

Let $f : [a,b]\times[a,b] \to \mathbb{R}$ continuously. Then

$$\int_a^b \left(\int_a^x f dy\right) dx = \int_a^b \left(\int_y^b f dx\right) dy$$

Proof. Define the set $D = \{(x,y)\in\mathbb{R}^2 | (x,y)\in[a,b]\times[a,b] \text{ and } x > y\}$. By Fubini's theorem, we can say that

$$\int_a^b \left(\int_a^x f dy\right) dx = \int_a^b \left(\int_{D(x)} f dy\right) dx = \int_D f dA = \int_a^b \left(\int_{D(y)} f dx\right) dy = \int_a^b \left(\int_y^b f dx\right) dy$$

□

Problem 4

Let $D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leqslant 1\}$. Then the integral of $f(x,y) = x^2y^2$ over D can be computed as

$$\int_D x^2y^2 dxdy = \frac{\pi}{24}$$

Proof. Firstly let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

Since Φ is continuously differentiable, it is a valid change of variables on $\mathbb{R}^2$. The boundary set of D can be shown to be equal to $\partial D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 = 1\}$. Then we define $C = \{(r,\theta)\in\mathbb{R}^2 \mid 0\leqslant r\leqslant 1 \text{ and } 0\leqslant\theta < 2\pi\}$. Similarly, in polar coordinates we have $\partial C = ([0,1]\times[0,2\pi]) \setminus \text{int}([0,1]\times[0,2\pi])$. Then, we compute that

$$\Phi(\partial C) = \{(\cos\theta, \sin\theta) \mid \theta\in[0,2\pi)\} = \partial D$$

So it follows that we have a valid change of variables moving from rectangular to polar coordinates, since Φ is both 1-1 and onto as well. Then we can write

$$\int_D x^2y^2 dxdy = \int_C r^4\cos^2\theta\sin^2\theta |J_\Phi| dr d\theta$$

Where $C = [0, 1] \times [0, 2\pi) \subset \mathbb{R}^2$. Then we compute $|J_\Phi(r, \theta)|$ by the following:

$$\begin{aligned}
|J_\Phi(r,\theta)| &= \left| \begin{pmatrix} \frac{\partial \Phi_1}{\partial r} & \frac{\partial \Phi_1}{\partial \theta} \\ \\ \frac{\partial \Phi_2}{\partial r} & \frac{\partial \Phi_2}{\partial \theta} \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| \\
&= (r\cos^2\theta) - (-r\sin^2\theta) \\
&= r(\cos^2\theta + \sin^2\theta) = r
\end{aligned}$$

Given that $|J_\Phi| = r$ we can now write

$$\int_D x^2 y^2 dx dy = \int_C r^4 \cos^2\theta \sin^2\theta r dr d\theta$$

Now all that is left is to compute the integral, which we will do now.

$$\begin{aligned}
\int_0^{2\pi}\int_0^1 r^5 \sin^2\theta\cos^2\theta dr d\theta &= \frac{1}{6}\int_0^{2\pi} \sin^2\theta\cos^2\theta d\theta \\
&= \frac{1}{6}\int_0^{2\pi} \sin^2\theta(1-\sin^2\theta) d\theta \\
&= \frac{1}{6}\int_0^{2\pi} \sin^2\theta - \sin^4\theta d\theta \\
&= \frac{1}{6}\left(\int_0^{2\pi} \sin^2\theta d\theta - \int_0^{2\pi} \sin^4\theta d\theta\right) \\
&= \frac{1}{6}\left(\int_0^{2\pi} \sin^2\theta d\theta - \left[\frac{-\cos\theta\sin^3\theta}{4}\bigg|_0^{2\pi} + \frac{3}{4}\int_0^{2\pi}\sin^2\theta d\theta\right]\right) \\
&= \frac{1}{24}\int_0^{2\pi} \sin^2\theta d\theta \\
&= \frac{1}{24}\int_0^{2\pi} \frac{1-\cos(2\theta)}{2} d\theta
\end{aligned}$$

Then we must perform a single variable u-substitution. Let $u = 2\theta, du = 2d\theta$. Then we can write

$$\begin{aligned}
&= \frac{1}{24}\int_0^{4\pi} \frac{1-\cos(u)}{4} du \\
&= \frac{1}{24}\left(\frac{4\pi}{4} - \frac{\sin u}{4}\bigg|_0^{4\pi}\right) \\
&= \frac{1}{24}(\pi - 0) = \frac{\pi}{24}
\end{aligned}$$

□

Problem 5

a)

Set $G = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$. Define $\Phi : G \to \mathbb{R}^2$ by $\Phi(x, y) = (x^2 - y^2, xy)$ for each $(x, y) \in G$. Then the function $\Phi : G \to \mathbb{R}^2$ is 1-1 on G, and that for each $(x, y) \in G$, the Jacobian matrix $J_\Phi(x, y)$ is invertible.

Proof. To show that Φ is 1-1, let $\Phi(x_1, y_1) = \Phi(x_2, y_2)$ be arbitrary. Then since both are points in $\mathbb{R}^2$ that are equal, their norms must be equal as well. That is,

$$\begin{aligned}
||\Phi(x_1,y_1)|| &= ||\Phi(x_2,y_2)|| \\
\sqrt{\Phi_1(x_1,y_1)^2+\Phi_2(x_1,y_1)^2} &= \sqrt{\Phi_1(x_2,y_2)^2+\Phi_2(x_2,y_2)^2} \\
\sqrt{(x_1^2-y_1^2)^2+(x_1y_1)^2} &= \sqrt{(x_2^2-y_2^2)^2+(x_2y_2)^2} \\
\sqrt{x_1^4-2x_1^2y_1^2+y_1^4+x_1^2y_1^2} &= \sqrt{x_2^4-2x_2^2y_2^2+y_2^4+x_2^2y_2^2} \\
\sqrt{x_1^4+2x_1^2y_1^2+y_1^4-3x_1^2y_1^2} &= \sqrt{x_2^4+2x_2^2y_2^2+y_2^4-3x_2^2y_2^2} \\
x_1^4+2x_1^2y_1^2+y_1^4-3x_1^2y_1^2 &= x_2^4+2x_2^2y_2^2+y_2^4-3x_2^2y_2^2
\end{aligned}$$

Since we know $x_1y_1 = x_2y_2$ it follows that $3x_1^2y_1^2 = 3x_2^2y_2^2$. So then we can write

$$\begin{aligned}
x_1^4+2x_1^2y_1^2+y_1^4 &= x_2^4+2x_2^2y_2^2+y_2^4 \\
(x_1^2+y_1^2)^2 &= (x_2^2+y_2^2)^2 \\
x_1^2+y_1^2 &= x_2^2+y_2^2
\end{aligned}$$

Given this fact, and that $x_1^2-y_1^2 = x_2^2-y_2^2$ it follows that $(x_1,y_1)=(x_2,y_2)$. So we say that Φ is 1-1.

To verify that J_Φ is invertible, we can simply check if the determinant is non-zero. So we write

$$\begin{aligned}
\det J_\Phi &= \det\begin{pmatrix} \frac{\partial\Phi_1}{\partial x} & \frac{\partial\Phi_1}{\partial y} \\ \\ \frac{\partial\Phi_2}{\partial x} & \frac{\partial\Phi_2}{\partial y}\end{pmatrix} \\
&= \det\begin{pmatrix} 2x & -2y \\ y & x\end{pmatrix} \\
&= 2x^2+2y^2
\end{aligned}$$

Then since we cannot have x or y equal to 0 it follows that $|J_\Phi| \neq 0$, and the Jacobian is thusforth invertible. □

b)

Using the change of variables given by Φ, and the given the set $D = \{(x,y)\in\mathbb{R}^2 \mid x>0, y>0, 1\leqslant x^2-y^2\leqslant 9, 2\leqslant xy\leqslant 4\}$. The following integral can be computed:

$$\int_D x^2+y^2dxdy = 8$$

Proof. First notice that $x^2+y^2 = \frac{|J_\Phi|}{2}$. Since Φ is a smooth transformation from D to C,

$$\int_C \frac{1}{2}dudv = \int_D \frac{|J_\Phi|}{2}dxdy$$

While the notation is a little bit backwards, the transformation is still as desired. So we want to write D in terms of $u = x^2-y^2$ and $v = xy$. This is simply $C = \{(u,v)\in\mathbb{R}^2 \mid 1\leqslant u\leqslant 9, 2\leqslant v\leqslant 4\}$. Then the integral is a simple compuation,

$$\begin{aligned}
\int_D \frac{|J_\Phi|}{2}dxdy &= \int_C \frac{1}{2}dudv \\
&= \int_2^4\int_1^9 \frac{1}{2}dudv \\
&= \frac{1}{2}(2)(8) = 8
\end{aligned}$$

□