Advanced Multivariable Calculus - Homework 4

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Problem 1

Let $G \subset \mathbb{R}^n$ be open and let $f : G \to \mathbb{R}$ and suppose that ∂_{x_i} exists and is bounded for every $1 \leqslant i \leqslant n$. Show that f is continuous.

Proof. Let $\vec{x} \in G$ be a point arbitrarily. The since G is open we know that some ϵ-neighborhood of $\vec{x}$ is contained in G. Let $\epsilon > 0$ be arbitrary. To show that there exists some $\delta > 0$ such that $\vec{y} \in B_\delta(\vec{x}) \Rightarrow f(\vec{y}) \in B_\epsilon(f(\vec{x}))$ would suffice to show that f is continuous by the definition of continuity. Define

$$M = \max_{i \in \{1,2,\cdots,n\}} \{M_i\}$$

Where $M_i = \sup_{\vec{g} \in G}\{\partial_{x_i}\}$. Then define $0 < \delta < \frac{\epsilon}{nM}$ such that $B_\delta(\vec{x}) \subset G$. Write $\vec{x} = (x_1, x_2, \cdots, x_n)$ and let $\vec{y} \in B_\delta(\vec{x})$. Let $\vec{u}_k = (y_1, \cdots, y_k, x_{k+1}, \cdots, x_n)$ for $k = 1, 2, \cdots, n$. Then,

$$\begin{aligned}
|f(\vec{x}) - f(\vec{y})| &= |f(\vec{x}) - f(\vec{u}_1) + f(\vec{u}_1) - f(\vec{u}_2) + f(\vec{u}_2) - \cdots - f(\vec{u}_k) + f(\vec{u}_k) - f(\vec{y})| \\
&\leqslant |f(\vec{x}) - f(\vec{u}_1)| + |f(\vec{u}_1) - f(\vec{u}_2)| + \cdots + |f(\vec{u}_k) - f(\vec{y})| \\
&\leqslant M|x_1 - y_1| + M|x_2 - y_2| + \cdots + M|x_n - y_n| \\
&\leqslant nM\delta \\
&= nM\frac{\epsilon}{nM} = \epsilon
\end{aligned}$$

We know that $|f(\vec{u}_{i+1}) - f(\vec{u}_{i+2})| \leqslant M|x_1 - x_2|$ by the mean value theorem, applied to f in the i-th plane at our point $\vec{x}$. $\square$

Problem 2

Let

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq \mathbf{0} \\ 0 & (x,y) = \mathbf{0} \end{cases}$$

a)

Show that ∂_x, ∂_y exist and are bounded.

Suppose that $(x,y) \neq \mathbf{0}$. Then we write

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(3x^2) - (x^3)(2x)}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{x^4 + 2x^2y^2 + y^4}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - (x^3)(2y)}{(x^2+y^2)^2} = \frac{-x}{x^2+y^2} \cdot \frac{2xy}{x^2+y^2}$$

Then if $(x,y) = \mathbf{0}$ simply write $\partial_x(\mathbf{0}) = 1, \partial_y(\mathbf{0}) = 0$ by computing $\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$ and $\lim_{h \to 0} \frac{f(0,1)-f(0,0)}{h}$. To show that these are bounded we write

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{x^4}{x^4 + 2y^2x^2 + y^4} + \frac{3x^2y^2}{x^2 + 2x^2y^2 + y^4} \\
&\leqslant \frac{x^4}{x^4} + 3x^2y^2 2x^2y^2 \\
&\leqslant 1 + \frac{3}{2}
\end{aligned}$$

Because every term is squared and positive, it follows that $0 \leqslant \partial_x \leqslant \frac{5}{2}$ and is bounded. For our y partial, we write

$$0 \leqslant \frac{x^2}{x^2+y^2} \leqslant 1$$

Then we say that

$$\begin{aligned}
(|y|-|x|)^2 &\geqslant 0 \\
y^2 - 2|y||x| + x^2 &\geqslant 0 \\
y^2 + x^2 &\geqslant 2|x||y| \\
1 &\geqslant \frac{2|x||y|}{x^2+y^2} \\
1 &\geqslant \left|\frac{2xy}{x^2+y^2}\right|
\end{aligned}$$

So then we have $\partial_y = \frac{x^2}{x^2+y^2} \cdot \frac{2xy}{x^2+y^2}$ is a product of bounded functions, and is therefore bounded.

b)

Show that $\nabla_{\boldsymbol{u}}(f)$ exists at $\mathbf{0}$ for every unit vector $\boldsymbol{u}$.

Denote $\boldsymbol{u} = (u_1, u_2)$. We compute the directional derivative as

$$\begin{aligned}
\lim_{h\to 0} \frac{f(\mathbf{0}+h\boldsymbol{u}) - f(\mathbf{0})}{h} &= \lim_{h\to 0} \frac{\frac{hu_1^3}{u_1^2+u_2^2}}{h} \\
&= \lim_{h\to 0} \frac{u_1^3}{u_1^2+u_2^2}
\end{aligned}$$

We know that $u_1^2 + u_2^2 = 1$ since $\boldsymbol{u}$ is a unit vector, and we know that $|u_1| \leqslant 1$ for the same reason. So it follows that $|\nabla_{\boldsymbol{u}} f(\mathbf{0})| < 1$.

c)

Choose $\boldsymbol{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then we have

$$\nabla_{\boldsymbol{u}} f(\mathbf{0}) = \frac{\left(\frac{1}{\sqrt{2}}\right)^3}{\frac{1}{\sqrt{2}}^2 + \frac{1}{\sqrt{2}}^2} = \left(\frac{1}{\sqrt{2}}\right)^3 = 2^{-3/2}$$

However, note that

$$\nabla f(\mathbf{0}) \cdot \boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \neq \frac{1}{\sqrt{2}^3}$$

Problem 3

Let$G \subset \mathbb{R}^n$ be an open set, and assume$f : G \to \mathbb{R}$ is differentiable on G. Also assume f has a local maximum at $x \in G$. Prove that $\nabla f(x) = 0$.

Proof. Write $\boldsymbol{x} = (a_1, a_2, \cdots, a_n)$. Now write

$$f_i(t) = f(a_1, \cdots, a_{i-1}, t, a_{i+1}, \cdots, a_n)$$

For every $i = 1, 2, \cdots, n$. If $t \in (-\epsilon + a_i, a_i + \epsilon)$ then $(a_1, \cdots, a_{i-1}, t, a_{i+1}, \cdots, a_n) \in B_\epsilon(\boldsymbol{x})$. So it follows that for each such t in the ϵ-neighborhood of a_i, $f_i(t) \leqslant f(\boldsymbol{x}) = f_i(a_i)$. Thus we have that $t = x_i$ is a local maximum for f_i, and therefore

$$f_i'(a_i) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}) = \mathbf{0}$$

Since f is differentiable, we know that each partial derivative exists, and it must be the case that these partial derivatives for x_i are equal to the derivative of the restricted function f_i at the point $\boldsymbol{x}$ by definition of the partial derivative. □

Problem 4

Let $f(x, y) = (e^x \cos y, e^x \sin y)$.

a)

Compute the Jacobian J_f.

We are going to compute the appropriate derivatives as

$$\frac{\partial}{\partial x} e^x \cos y = e^x \cos y$$

$$\frac{\partial}{\partial y} e^x \cos y = -e^x \sin y$$

$$\frac{\partial}{\partial x} e^x \sin y = e^x \sin y$$

$$\frac{\partial}{\partial y} e^x \sin y = e^x \cos y$$

Then our matrix will clearly be

$$J_f = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

b)

Show that $J_f(x, y)^{-1}$ exists for every point in $\mathbb{R}^2$.

Proof. We know that $J_f(x, y)^{-1}$ exists if and only if $\det(J_f(x, y)) \neq 0$. So we compute

$$\begin{aligned} \det(J_f(x, y)) &= (e^x \cos y)(e^x \cos y) - ((-e^x) \sin y)(e^x \sin y) \\ &= (e^x)^2(\cos^2 y) + (e^x)^2(\sin^2 y) \\ &= (e^x)^2(\cos^2 y + \sin^2 y) \\ &= (e^x)^2 \end{aligned}$$

Since e^x is strictly positive, so is its square, so obviously $\det(J_f(x, y)) = (e^x)^2 > 0$ and we say that the matrix is invertible. $\square$

c)

Show that f is not invertible globally; that is, find different points of the domain that are mapped to the same value in the range.

Proof. Since e^x is a bijective map, we fix $x = 0$. Then we know that trigonometric functions are 2π-periodic, so pick the following two points $\boldsymbol{a} = (0, 0), \boldsymbol{b} = (0, 2\pi)$. Then we know that $f(\boldsymbol{a}) = (1, 0)$ and $f(\boldsymbol{b}) = (1, 0)$. So we know that the pre-image of $(1, 0)$ includes both $\boldsymbol{a}$ and $\boldsymbol{b}$ so then f^{-1} is not globally invertible. $\square$

Problem 5

Let $f_1(x, y_1, y_2) = 3y_1 + y_2^2 + 4x = 0, f_2(x, y_1, y_2) = 4y_1^3 + y_2 + x = 0$.

a)

Verify that $(x, y_1, y_2) = (-4, 0, 4)$ is a solution.

We have that

$$f_1(-4, 0, 4) = 3(0) + (4)^2 + 4(-4) = 16 - 16 = 0$$

And also that

$$f_2(-4,0,4) = 4(0)^3 + (4) + (-4) = 0$$

So then we know that $(-4,0,4)$ is a valid solution.

b)

Verify that $\exists g = (g_1, g_2)$ such that $g_1(x) = y_1, g_2(x) = y_2$ for all solutions near $(-4,0,4)$.

We write

$$\left(\frac{\partial f_i}{\partial y_j}\right) = \begin{pmatrix} 3 & 2y_2 \\ 12y_1^2 & 1 \end{pmatrix}$$

Then because the determinant is equal to $3 - (12y_1^2)(2y_2)$, if we plug in $\boldsymbol{s} = (-4,0,4)$ we will get $\det\left(\frac{\partial f_i}{\partial y_j}\right) = 3 \neq 0$. So by the implicit function theorem it follows that $\exists g : \mathbb{R} \to \mathbb{R}^2$ such that for every $(x, y_1, y_2) \in S \cap B_\epsilon(-4,0,4)$ where S is the set of solutions to our system of equations $(x, y_1, y_2) = (x, g(x)^t)$ therefore $g_1(x) = y_1$ and $g_2(x) = y_2$.

c)

So we use the final piece of the implicit function theorem for solutions around $(-4,0,4)$ since

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 & 8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_2}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So it follows that $\frac{\partial g_2}{\partial x} = -1$ and $\frac{\partial g_1}{\partial x} = \frac{4}{3}$.

Problem 6

Let $T = \{(x,y,u,v) \in \mathbb{R}^4 \mid x^2 + y^2 = 1,\ ux + vy = 0\}$. Then let

$$f(x_1, x_2, y_1, y_2) = (x_1^2 + y_1^2, x_2 x_1 + y_2 y_1)$$

Where $x_1 = x, y_1 = y, x_2 = u, y_2 = v$. Then we write $T = f^{-1}(1,0)$.

a)

Compute $\frac{\partial f_i}{\partial y_j}$. We simply take each of our partial derivatives,

$$\left(\frac{\partial f_i}{\partial y_j}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}$$

$$= \begin{pmatrix} 2y_1 & 0 \\ y_2 & y_1 \end{pmatrix}$$

b)

We know that $\det\left(\frac{\partial f_i}{\partial y_j}\right) = 2y_1^2$, so it follows that if $y_1 = y \neq 0$ then this determinant is non zero. So there exists by the implicit function theorem a continuously differentiable function $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that for all $(x_1, x_2, y_1, y_2) \in T$ we can write

$$(x_1, x_2, y_1, y_2) = (x_1, x_2, g(x_1, x_2))$$

which is entirely in terms of x and u since $x = x_1, u = x_2$.

c)

Find formulas for y and v in terms of x and u valid when $y \neq 0$.

If $y \neq 0$ then we can write $y = \pm\sqrt{1-x^2}$ and $v = \frac{ux}{y} = \frac{ux}{\pm\sqrt{1-x^2}}$. Then we know that our denominator is non-zero and we have a formula for y and v all in terms of u and x.

d)

If $x \neq 0$ then $x_1 \neq 0$ so we write $x = \pm\sqrt{1 - x^2}$ and $u = \frac{vy}{x} = \frac{vy}{\sqrt{1-x^2}}$. Since the denominator is guaranteed to be non-zero we have written x, u in terms of y, v.