Applied Ordinary Differential Equations — Homework 6

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Section 5.4 – Problem 25

Find all values of β for which all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.

Solution

We begin by letting $y = x^r$, making the assumption that solutions will take this form. Then the derivatives of y can be computed.

$$y = x^r \quad \longrightarrow \quad y' = rx^{r-1} \quad \longrightarrow \quad y' = r(r-1)x^{r-2}$$

These can be substituted into the differential eqution, and then this can be simplified as follows.

$$\begin{aligned} x^2(r(r-1)x^{r-2}) + \beta(x^r) &= 0 \\ r(r-1)x^r + \beta x^r &= 0 \\ x^r(r(r-1) + \beta) &= 0 \\ x^r(r^2 - r + \beta) &= 0 \end{aligned}$$

The roots of the inner polynomial can now be computed by the quadratic formula. This yields

$$r = \frac{1 \pm \sqrt{1-4\beta}}{2}$$

Since we say that $y = x^r$, it follows that if $x \to 0$ then $x^r \to 0$ if and only if r has a positive real part. In order to have complex eigenvalues, it must be the case that $1 - 4\beta < 0$. Equivalently,

$$\begin{aligned} 1 - 4\beta &< 0 \\ 1 &< 4\beta \\ \frac{1}{4} &< \beta \end{aligned}$$

And in this scenario the real part will be guaranteed to be positive. In the case that we have real eigenvalues, for r_1, r_2 to be positive, it must be the case that $1 - \sqrt{1-4\beta} > 0$. Equivalently,

$$\begin{aligned} 1 - \sqrt{1-4\beta} &> 0 \\ 1 &> \sqrt{1-4\beta} \\ 1 &> 1 - 4\beta \\ 0 &> -4\beta \\ 0 &< \beta \end{aligned}$$

In combination, we can guarantee a postive real part for both eigenvalues simply by restricting β to be strictly positive. That is, if $\beta > 0$ then all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.

Section 5.4 – Problem 28

Using the method of reduction of order, show that if r_1 is a repeated root of $r(r-1)+\alpha r+\beta=0$, then x^{r_1} and $x^{r_1}\ln x$ are solutions of $x^2y''+\alpha xy'+\beta y=0$ for $x>0$.

Solution

It can be reasonably assumed that, to find a solution, we can substitute $y=x^r$ once again. These can all be differentiated, giving the following.

$$y=x^r \quad\longrightarrow\quad y'=rx^{r-1} \quad\longrightarrow\quad y'=r(r-1)x^{r-2}$$

This can be substituted into our original equation

$$\begin{aligned} x^2r(r-1)x^{r-2}+\alpha xrx^{r-1}+\beta x^r &= 0\\ r(r-1)x^r+\alpha rx^r+\beta x^r &= 0\\ x^r(r(r-1)+\alpha r+\beta) &= 0 \end{aligned}$$

If r_1 is a root for $r(r-1)+\alpha r+\beta$, then we clearly have a solution for the differential equation given by $y=x^{r_1}$, since having the term in parenthesis go to 0 causes the entire left-hand side to be 0 and thus the equation is satisfied. If r_1 is a repeated root, then we can determine an equality involving α and β.

$$\begin{aligned} r(r-1)+\alpha r+\beta &= 0\\ r^2-r+1+\alpha r+\beta &= 0\\ r^2+(\alpha-1)r+(\beta+1) &= 0 \end{aligned}$$

The quadratic equation can be used to solve for the roots.

$$r=\frac{(1-\alpha)\pm\sqrt{(\alpha-1)^2-4(\beta+1)}}{2}$$

Assuming that we have a repeated root r_1 we know that $(\alpha-1)^2-4(\beta+1)=0$. Then we know that r_1 is of the following form.

$$r_1=\frac{1-\alpha}{2}$$

Set $y=v(x)x^r$, by the method of reduction of order, then the first and second derivative can be taken.

$$y=vx^r \quad\longrightarrow\quad y'=v'x^r+vrx^{r-1} \quad\longrightarrow\quad y''=v''x^r+v'(2rx^{r-1})+v(r(r-1)x^{r-2})$$

Then we will substitute in these for y, y', and y''.

$$x^2\left[v''x^r+v'(2rx^{r-1})+v(r(r-1)x^{r-2}\right]+\alpha x\left[v'x^r+vrx^{r-1}\right]+\beta\left[vx^r\right]$$

This can be simplified with some algebra.

$$\begin{aligned} 0 &= x^2\left[v''x^r+v'(2rx^{r-1})+v(r(r-1)x^{r-2}\right]+\alpha x\left[v'x^r+vrx^{r-1}\right]+\beta\left[x^r\right]\\ 0 &= \left[v''x^{r+2}+v'2rx^{r+1}+v(r(r-1)x^r)\right]+\left[v'\alpha x^{r+1}+v\alpha rx^r\right]+v[\beta x^r]\\ 0 &= v''(x^{r+2})+v'(x^{r+1}(2r+\alpha))+v(x^r(r(r-1)+\alpha r+\beta)) \end{aligned}$$

Since r_1 is a root of $r(r-1)+\alpha r+\beta$, the coefficient of v is equal to 0. Since $r_1=\dfrac{1-\alpha}{2}$, we can simplify the coefficient on the term with v'.

$$\begin{aligned} 2r_1+\alpha &= 2\frac{1-\alpha}{2}+\alpha\\ &= 1-\alpha+\alpha\\ &= 1 \end{aligned}$$

So then our equation simplfies further.

$$\begin{aligned} 0 &= v''(x^{r+2})+v'(x^{r+1}(2r+\alpha))+v(x^r(r(r-1)+\alpha r+\beta))\\ 0 &= v''(x^{r+2})+v'(x^{r+1}) \end{aligned}$$

Let $w = v'$, and we can reduce the order of our differential equation. Substituting in our new variable w, and $w' = v''$, we can rewrite the equation.

$$0 = w'x^{r+2} + wx^{r+1}$$

This can be solved using the seperation of variables method.

$$\begin{aligned}
0 &= w'x^{r+2} + wx^{r+1} \\
-wx^{r+1} &= w'x^{r+2} \\
\frac{-w}{x} &= w' \\
\frac{-w}{x} &= \frac{dw}{dx} \\
\frac{-1}{x} &= \frac{1}{w} \cdot \frac{dw}{dx} \\
\frac{-1}{x}dx &= \frac{1}{w}dw \\
-\ln|x| + c &= \ln|w| \\
e^{-\ln|x|+c} &= e^{\ln|w|} \\
\frac{e^c}{e^{\ln|x|}} &= w \\
\frac{k}{x} &= w
\end{aligned}$$

Since $w = v'$, we can integrate w to solve for v.

$$\begin{aligned}
\frac{k}{x} &= w \\
\frac{k}{x} &= v' \\
\int \frac{k}{x}dx &= \int v'dx \\
k\ln|x| + c &= v
\end{aligned}$$

If we assume that our constants of integration were both zero, then $k = e^{c_1} = e^0 = 1$ and $c_2 = 1$. Then v can be substituted into our original solution $y = vx^r$, giving us

$$y = x^r \ln x$$

Section 5.5 – Problem 10

The Bessel equation of order 0 is

$$x^2y'' + xy' + x^2y = 0$$

a. Show that $x = 0$ is a regular singular point.

b. Show that the roots of the indicial equation are $r_1 = r_2 = 0$.

c. Show that one solution for $x > 0$ is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

The function J_0 is known as the Bessel function of the first kind of order zero.

d. Show that the series for $J_0(x)$ converges for all x.

Solution

a.

Proof. Firstly, it must be identified that $P(x) = x^2, Q(x) = x, R(x) = x^2$. The point $x_0 = 0$ must be a singular point since

$$P(x_0) = P(0) = 0^2 = 0$$

To check that it is regular, it must be the case that the following limits are both finite.

1. $\lim_{x\to 0} x\frac{Q(x)}{P(x)}$

2. $\lim_{x\to 0} x^2\frac{R(x)}{P(x)}$

The first can be evaulated simply.

$$\begin{aligned}\lim_{x\to 0} x\frac{Q(x)}{P(x)} &= \lim_{x\to 0} x\frac{x}{x^2} \\ &= \lim_{x\to 0} 1 \\ &= 1\end{aligned}$$

Similarly we can also evaluate the second limit somewhat trivially.

$$\begin{aligned}\lim_{x\to 0} x^2\frac{R(x)}{P(x)} &= \lim_{x\to 0} x^2\frac{x^2}{x^2} \\ &= \lim_{x\to 0} x^2 \\ &= 0\end{aligned}$$

Since both limits are finite, we conclude the point $x = 0$ is a regular singular point. $\square$

b.

Proof. We assume that we have some solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}$$

This can be differentiated twice to find y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^{r+n} \quad\longrightarrow\quad y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1} \quad\longrightarrow\quad y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}$$

These power series representations can be substituted in to our original order 0 Bessel equation.

$$x^2\left[\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}\right] + x\left[\sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1}\right] + x^2\left[\sum_{n=0}^{\infty} a_n x^{r+n}\right] = 0$$
$$\left[\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n}\right] + \left[\sum_{n=0}^{\infty} a_n(r+n)x^{r+n}\right] + \left[\sum_{n=0}^{\infty} a_n x^{r+n+2}\right] = 0$$
$$\left[\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n}\right] + \left[\sum_{n=0}^{\infty} a_n(r+n)x^{r+n}\right] + \left[\sum_{n=2}^{\infty} a_{n-2} x^{r+n}\right] = 0$$
$$\left[\sum_{n=0}^{2} a_n(r+n)^2x^{r+n}\right] + \left[\sum_{n=2}^{\infty} x^{r+n}(a_n(r+n)(r+n-1) + a_n(r+n) + a_{n-2})\right] = 0$$
$$a_0r^2x^r + a_1(r+1)^2x^{r+1} + \left[\sum_{n=2}^{\infty} x^{r+n}(a_n(r+n)^2 + a_{n-2})\right] = 0$$

Since we know that our term x^r cannot be zero, and it must be the case that $a_0 \neq 0$, then our indical equation is

$$r^2 = 0.$$

Clearly, the roots for this equation must both be 0. That is, $r_1 = r_2 = 0$. □

c.

Proof. For the terms that remain within the sum, we know that they must also be equal to 0 for this sum to be equal to 0. This gives us an equation that can be turned into an equivalence relation.

$$\begin{aligned} a_n(r+n)^2 + a_{n-2} &= 0 \\ a_n(r+n)^2 &= -a_{n-2} \\ a_n &= \frac{-a_{n-2}}{(r+n)^2} \end{aligned}$$

So we can compute some of our power series coefficients in terms of a_1 and a_2.

$$\begin{aligned} a_2 &= \frac{(-1)}{(r+2)^2} \cdot a_0 \\ a_4 &= \frac{(-1)^2}{(r+4)^2(r+2)^2} \cdot a_0 \\ a_6 &= \frac{(-1)^3}{(r+6)^2(r+4)^2(r+2)^2} \cdot a_0 \\ &\vdots \\ a_3 &= \frac{(-1)}{(r+3)^2} \cdot a_1 \\ a_5 &= \frac{(-1)^2}{(r+5)^2(r+3)^2} \cdot a_1 \\ a_7 &= \frac{(-1)^3}{(r+7)^2(r+5)^2(r+3)^2} \cdot a_1 \end{aligned}$$

Let $r = 0$, and let n be even, then

$$a_n = \frac{(-1)^{\frac{n}{2}}}{(n)^2(n-2)^2\cdots(2)^2}.$$

And since all the terms in the denominator are even, we can write

$$a_n = \frac{(-1)^k a_0}{(2k)^2(2(k-1))^2\cdots(2)^2} = \frac{(-1)^k a_0}{2^2(k)^2 2^2(k-1)^2\cdots{}_2^2\,(2)^2} = \frac{(-1)^k a_0}{2^{2k}(k!)^2} \quad \left(\text{where } k = \frac{n}{2}\right).$$

Let a_1 be zero and it follows that we have a solution

$$\begin{aligned} a_0x^0 + \sum_{n=2}^{\infty} \frac{(-1)^k a_0 x^n}{2^{2^k}(k!)^2} &= 0 \\ x^0 + \sum_{n=2}^{\infty} \frac{(-1)^k x^n}{2^{2^k}(k!)^2} &= 0 \\ J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} &= 0 \end{aligned}$$

$\square$

d.

Proof. We need only check that the series converges for all x. The ratio test will be useful in testing the convergence of this series. So we will first simplify the fraction $a_{n+1}x^{n+1}/a_nx^n$.

$$\begin{aligned} \frac{\frac{(-1)^{k+1}x^{2k+1}}{2^{2(k+1)}(k+1)!^2}}{\frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}} &= \frac{(-1)^{k+1}x^{2k+1}}{2^{2(k+1)}(k+1)!^2} \cdot \frac{2^{2k}(k!)^2}{(-1)^k x^{2k}} \\ &= \frac{(-1)(-1)^k x x^{2k}}{2^2 2^{2k}(k+1)^2(k!)^2} \cdot \frac{2^{2k}(k!)^2}{(-1)^k x^{2k}} \\ &= \frac{(-1)x}{2^2(k+1)^2} \end{aligned}$$

Now we take the limit as $k \to \infty$ and we get

$$\begin{aligned} \lim_{k\to\infty} \frac{(-1)x}{2^2(k+1)^2} &= \frac{(-1)x}{2^2} \lim_{k\to\infty} \frac{1}{(k+1)^2} \\ &= \frac{(-1)x}{2^2} \cdot 0 \\ &= 0 \quad \forall x. \end{aligned}$$

This means that we can conclude that the series converges for all x. $\square$

Section 5.5 – Problem 12

The Bessel equation of order one is

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

a. Show that $x = 0$ is a regular singular point.

b. Show that the roots of the indical equation are $r_1 = 1$ and $r_2 = -1$.

c. Show that one solution for $x > 0$ is

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!n!2^{2n}}$$

The function J_1 is known as the Bessel function of the first kind of order 1.

d. Show that the series for $J_1(x)$ converges for all x.

Solution

a.

Firstly, it must be identified that $P(x) = x^2, Q(x) = x, R(x) = x^2 - 1$. The point $x_0 = 0$ must be a singular point since

$$P(x_0) = P(0) = 0^2 = 0$$

To check that it is regular, it must be the case that the following limits are both finite.

1. $\lim_{x\to 0} x\frac{Q(x)}{P(x)}$
2. $\lim_{x\to 0} x^2\frac{R(x)}{P(x)}$

The first can be evaulated simply.

$$\begin{aligned}\lim_{x\to 0} x\frac{Q(x)}{P(x)} &= \lim_{x\to 0} x\frac{x}{x^2} \\ &= \lim_{x\to 0} 1 \\ &= 1\end{aligned}$$

Similarly we can also evaluate the second limit somewhat trivially.

$$\begin{aligned}\lim_{x\to 0} x^2\frac{R(x)}{P(x)} &= \lim_{x\to 0} x^2\frac{x^2 - 1}{x^2} \\ &= \lim_{x\to 0} x^2 - 1 \\ &= -1\end{aligned}$$

Since both limits are finite, we conclude the point $x = 0$ is a regular singular point.

b.

Proof. We assume that we have some solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}$$

This can be differentiated twice to find y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^{r+n} \quad\longrightarrow\quad y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1} \quad\longrightarrow\quad y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}$$

These power series representations can be substituted in to our original order 1 Bessel equation.

$$\begin{aligned}
x^2\left[\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}\right] + x\left[\sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1}\right] + (x^2-1)\left[\sum_{n=0}^{\infty} a_n x^{r+n}\right] &= 0\\
\left[\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n}\right] + \left[\sum_{n=0}^{\infty} a_n(r+n)x^{r+n}\right] + \left[\sum_{n=0}^{\infty} a_n x^{r+n+2} - a_n x^{r+n}\right] &= 0\\
\left[\sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n}\right] + \left[\sum_{n=0}^{\infty} a_n(r+n)x^{r+n}\right] + \left[\sum_{n=0}^{\infty} a_n x^{r+n+2}\right] - \left[\sum_{n=0}^{\infty} a_n x^{r+n}\right] &= 0\\
\sum_{n=0}^{\infty} a_n x^{r+n}\left((r+n)(r+n-1)+(r+n)-1\right) + \sum_{n=2}^{\infty} a_{n-2}x^{r+n} &= 0\\
a_0x^r\left(r(r-1)+r-1\right) + a_1x^{r+1}\left((r+1)(r)+r\right) + \sum_{n=2}^{\infty}\left[x^{r+n}\left(a_n\left((r+n)(r+n-1)+(r+n)-1\right)+a_{n-2}\right)\right] &= 0
\end{aligned}$$

Since $a_0 \neq 0$ and our x^r term must be non-trivial, the only way for this equation to hold is if our indical equation,

$$r(r-1)+r-1,$$

is equal to 0. So we solve for roots on this equation.

$$\begin{aligned}
r(r-1)+r-1 &= 0\\
r^2-r+r-1 &= 0\\
r^2-1 &= 0\\
r^2 &= 1\\
r &= \pm 1
\end{aligned}$$

Therefore, we have two roots to the indical equation $r_1 = 1, r_2 = -1$. □

c.

Proof. For the remaining terms, we must ensure that they are 0 as well. That is, that

$$a_n((r+n)(r+n-1)+(r+n)-1)+a_{n-2} = 0$$

From this, we can find a recurrence relation.

$$\begin{aligned}
a_n((r+n)(r+n-1)+(r+n)-1) &= -a_{n-2}\\
a_n &= \frac{-a_{n-2}}{(r+n)(r+n-1)+(r+n)-1}
\end{aligned}$$

Let us fix $r = r_1 = 1$, then this can be simplified to

$$\begin{aligned}
a_n &= \frac{-a_{n-2}}{(1+n)(1+n-1)+(1+n)-1}\\
a_n &= \frac{-1}{(1+n)n+n}\cdot a_{n-2}\\
a_n &= \frac{-1}{n(n+2)}\cdot a_{n-2}
\end{aligned}$$

For $n = 2, 4, 6, \cdots$ we can write

$$\begin{aligned}
a_2 &= \frac{(-1)}{2(4)}\cdot a_0\\
a_4 &= \frac{(-1)^2}{2(4)(4)(6)}\cdot a_0\\
a_6 &= \frac{(-1)^3}{2(4)(4)(6)(6)(8)}\cdot a_0
\end{aligned}$$

So we can write the following general relation for even n.

$$\begin{aligned} a_n &= \frac{(-1)^{n/2}}{(n+2)\cdot 2(n)2(n-2)\cdots 2(4)\cdot 2}\cdot a_0 \\ &= \frac{(-1)^{n/2}}{[(n+2)(n)(n-2)\cdots(4)][(n)(n-2)\cdots(4)(2)]}\cdot a_0 \end{aligned}$$

Since n is even, let $k = \frac{n}{2}$. Now a 2 can be factored out fom both terms in the denominator.

$$\begin{aligned} a_n &= \frac{(-1)^{n/2}}{[(n+2)(n)(n-2)\cdots(4)][(n)(n-2)\cdots(4)(2)]}\cdot a_0 \\ &= \frac{(-1)^k}{2^k[(k+1)(k)(k-1)\cdots(2)]2^k[(k)(k-1)\cdots(2)(1)]}\cdot a_0 \\ &= \frac{(-1)^k}{2^{2k}(k+1)!k!}\cdot a_0 \end{aligned}$$

And this begins to take the desired form. Let $a_1 = 0$ and the odd terms will vanish, so what we are left with is

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{1+2k}}{2^{2k}(k+1)!k!}\cdot a_0 = \frac{x}{2}\cdot\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{(k+1)!k!2^{2k}}\cdot a_0.$$

So we can ommit the a_0 term, which finally gives us

$$J_1(x) = \frac{x}{2}\cdot\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{(k+1)!k!2^{2k}}.$$

□

d.

Proof. To show that the series converges, it is sufficient to show that

$$\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{(k+1)!k!2^{2k}}$$

converges. By the ratio test, if $\lim_{k\to\infty}|a_{k+1}/a_k| < 1$ then the series converges.

$$\begin{aligned} \frac{\frac{(-1)^{k+1}x^{2(k+1)}}{(k+2)!(k+1)!2^{2(k+1)}}}{\frac{(-1)^k x^{2k}}{(k+1)!k!2^{2k}}} &= \frac{(-1)^{k+1}x^{2(k+1)}}{(k+2)!(k+1)!2^{2(k+1)}}\cdot\frac{(k+1)!k!2^{2k}}{(-1)^k x^{2k}} \\ &= \frac{(-1)(-1)^k x^2 x^{2k}}{(k+2)(k+1)2^2(k+1)!k!2^{2k}}\cdot\frac{(k+1)!k!2^{2k}}{(-1)^k x^{2k}} \\ &= \frac{-x^2}{2^2(k+2)(k+1)} \\ &= \frac{-x^2}{2^2}\cdot\frac{1}{k^2+3k+2} \end{aligned}$$

So if we take the limit as $k\to\infty$, then we get

$$\begin{aligned} \lim_{k\to\infty}\left|\frac{-x^2}{2^2}\cdot\frac{1}{k^2+3k+2}\right| &= \frac{x^2}{4}\lim_{k\to\infty}\left|\frac{1}{k^2+3k+2}\right| \\ &= \frac{x^2}{4}\cdot 0 \\ &= 0 \end{aligned}$$

So we conclude that the series does converge for all x. □

Section 5.7 – Problem 8

Consider the Bessel equation of order ν.

$$x^2y'' + xy' + (x^2 - \nu^2) = 0$$

where ν is real and positive.

a. Show that $x = 0$ is a regular singular point and that the roots of the indical equation are ν and $-\nu$.

b. Corresponding to the larger root ν, show that one solution is

$$y_1(x) = x^\nu \left(1 - \frac{1}{1!(1+\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1+\nu)(2+\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty} \frac{(-1)^m}{m!(1+\nu)\cdots(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right)$$

c. If 2ν is not an integer, show that a second solution is

$$y_1(x) = x^{-\nu} \left(1 - \frac{1}{1!(1-\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1-\nu)(2-\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty} \frac{(-1)^m}{m!(1-\nu)\cdots(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right)$$

d. Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution, provided only that is not an integer.

Solution

a.

Proof. Firstly, it must be identified that $P(x) = x^2, Q(x) = x, R(x) = x^2 - \nu^2$. The point $x_0 = 0$ must be a singular point since

$$P(x_0) = P(0) = 0^2 = 0$$

To check that it is regular, it must be the case that the following limits are both finite.

1. $\lim_{x\to 0} x\frac{Q(x)}{P(x)}$

2. $\lim_{x\to 0} x^2\frac{R(x)}{P(x)}$

The first can be evaulated simply.

$$\begin{aligned}\lim_{x\to 0} x\frac{Q(x)}{P(x)} &= \lim_{x\to 0} x\frac{x}{x^2} \\ &= \lim_{x\to 0} 1 \\ &= 1\end{aligned}$$

Similarly we can also evaluate the second limit somewhat trivially.

$$\begin{aligned}\lim_{x\to 0} x^2\frac{R(x)}{P(x)} &= \lim_{x\to 0} x^2\frac{x^2-\nu^2}{x^2} \\ &= \lim_{x\to 0} x^2 - \nu^2 \\ &= -\nu^2\end{aligned}$$

Since both limits are finite, we conclude the point $x = 0$ is a regular singular point. We assume that we have some solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}$$

This can be differentiated twice to find y' and y''.

$$y = \sum_{n=0}^{\infty} a_n x^{r+n} \quad\longrightarrow\quad y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1} \quad\longrightarrow\quad y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}$$

These power series representations can be substituted in to our original order 1 Bessel equation.

$$x^2\left[\sum_{n=0}^{\infty}a_n(r+n)(r+n-1)x^{r+n-2}\right]+x\left[\sum_{n=0}^{\infty}a_n(r+n)x^{r+n-1}\right]+(x^2-\nu^2)\left[\sum_{n=0}^{\infty}a_nx^{r+n}\right]=0$$

$$\left[\sum_{n=0}^{\infty}a_n(r+n)(r+n-1)x^{r+n}\right]+\left[\sum_{n=0}^{\infty}a_n(r+n)x^{r+n}\right]+\left[\sum_{n=0}^{\infty}a_n(x^2-\nu^2)x^{r+n}\right]=0$$

$$\left[\sum_{n=0}^{\infty}a_n(r+n)(r+n-1)x^{r+n}\right]+\left[\sum_{n=0}^{\infty}a_n(r+n)x^{r+n}\right]+\left[\sum_{n=0}^{\infty}a_nx^{r+n+2}\right]-\left[\sum_{n=0}^{\infty}a_n\nu^2x^{r+n}\right]=0$$

$$\left[\sum_{n=0}^{\infty}a_n(r+n)(r+n-1)x^{r+n}+a_n(r+n)x^{r+n}-a_n\nu^2x^{r+n}\right]+\left[\sum_{n=0}^{\infty}a_nx^{r+n+2}\right]=0$$

$$\left[\sum_{n=0}^{\infty}a_nx^{r+n}\left((r+n)(r+n-1)+(r+n)-\nu^2\right)\right]+\left[\sum_{n=2}^{\infty}a_{n-2}x^{r+n}\right]=0$$

$$a_0x^r\left((r)(r-1)+r-\nu^2\right)+a_1x^{r+1}\left((r+1)(r)+(r+1)-\nu^2\right)+\left[\sum_{n=2}^{\infty}a_nx^{r+n}\left((r+n)(r+n-1)+(r+n)-\nu^2\right)+a_{n-2}x^{r+n}\right]=0$$

So we have our indical equation,

$$r(r-1)+r-\nu^2=r^2-\nu^2=(r+\nu)(r-\nu).$$

Clearly, this has two roots that are equal to ν and $-\nu$. $\square$