Mathematical Statistics - Assignment 6

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Problem 4.8

Suppose that Y has a density function

$$f(y) = \begin{cases} ky(1-y), & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a)

Find some $k \in \mathbb{R}$ such that $f(y)$ is a probability density function. Notice that $\forall y \in [0,1], \quad y(1-y) > 0$. This indicates that $k \ge 0$. The integral $\int_{-\infty}^{0} f(y)dy + \int_{1}^{\infty} f(y)dy = 0$ since we know that the function is constantly zero on these open intervals. Then $P(-\infty, \infty) = P(0 \le X \le 1) = \int_0^1 f(y)dy$. Thus we want to find some k such that $\int_0^1 ky(y-1)dy = 1$. Let us use rules of algebra and integration to derive the following

$$\begin{aligned} 1 &= \int_0^1 ky(1-y)dy \\ &= k\int_0^1 y - y^2 dy \\ &= k\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 \\ &= \frac{k}{6} \end{aligned}$$

Since we must have $\frac{k}{6} = 1$, of course our answer will be $k = 6$.

(b)-(e)

Compute various probabilities using this value of k and the normalized density function $f(y)$. First we must find $P(.4 \le Y \le 1)$.

$$\begin{aligned} P(.4 \le Y \le 1) &= \int_{0.4}^1 6y(1-y)dy \\ &= 6\int_{.4}^1 y - y^2 dy \\ &= 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{.4}^1 \\ &= 6\left[\frac{1}{6} - \left(\frac{2}{25} - \frac{2^3}{(3)5^3}\right)\right] \\ &= .648 \end{aligned}$$

The next probability to compute is $P(.4 \le Y < 1)$. However, this probability will be the same as the one we just computed because $P(.4 \le Y \le 1) = P(.4 \le Y < 1) + P(Y = 1)$. Since $P(Y = 1) = \int_1^1 f(y)dy = 0$, both probabilities must be equal.

To compute $P(Y \le .4 | Y \le .8)$ by definition the probability equals

$$\frac{P(Y \le .4 \text{ and } Y \le .8)}{P(Y \le .8)}$$

The probability in the numberator will simply be $P(Y \leq .4)$ since this event is a subset of $Y \leq .8$. For the denominator, the integral must be computed.

$$6\int_0^{.8} f(y)dy = 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{.8} = 6\left[\frac{4^2}{(2)5^2} - \frac{4^3}{(3)5^3}\right] = .896$$

Knowing both the numerator and denominator we say $P(Y \leq .4|Y \leq .8) = \frac{.352}{.896} = .393$.

Problem 4.14

We have the following probability density function

$$f(y) = \begin{cases} y, & 0 < y < 1 \\ 2 - y, & 1 \leq y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a)

We sketch the function $f(y)$.

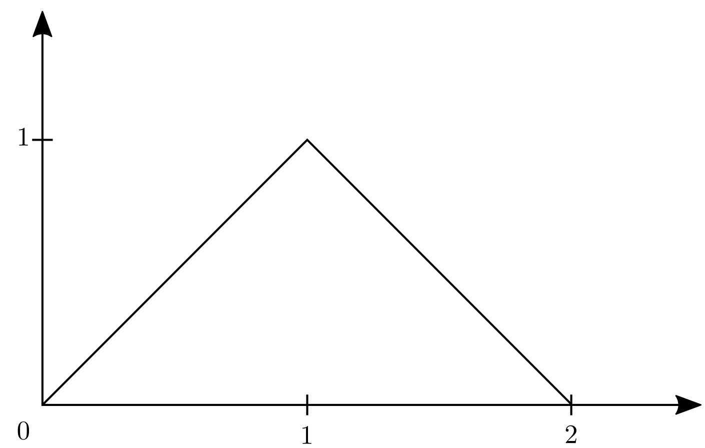

Figure 1: Probability Density Function $f(y)$

(b)

$$F(y) = \begin{cases} \int_{-\infty}^{y} 0dt = 0, & y \leq 0 \\ \int_{-\infty}^{0} 0dt + \int_0^y tdt = \frac{t^2}{2}\Big|_0^y = \frac{y^2}{2}, & 0 < y < 1 \\ \int_{-\infty}^{0} 0dt + \int_0^y tdt + \int_1^y 2 - tdt = \frac{1}{2} + \left[2t - \frac{t^2}{2}\right]_1^y = \frac{1}{2} + \left[2y - \frac{y^2}{2} - (2 - \frac{1}{2})\right] = -\frac{y^2}{2} + 2y - 1, & 1 \leq y \leq 2 \\ 1, & y > 2 \end{cases}$$

(c)

To find the probability that the station pumps between 8000 and 12000 gallons of gas, we say $P(.8 < Y < 1.2) = F(1.2) - F(.8) = -\frac{1.2^2}{2} + 2(1.2) - 1 - [\frac{.8^2}{2}] = .36$

(d)

$P(Y > 1.5|Y > 1) = \frac{P(Y>1.5)}{P(Y>1)}$. We must compute each of these in order to find the conditional probability.

$$P(Y > 1.5) = (.5)\int_{1.5}^{\infty} f(y)dy = (.5)\frac{(.5)^2}{2} = (.5)\frac{.25}{2} = (.5).128 = .064$$
$$P(Y > 1) = (.5)\int_{1}^{\infty} f(y)dy = (.5)\frac{1}{2} = .25$$

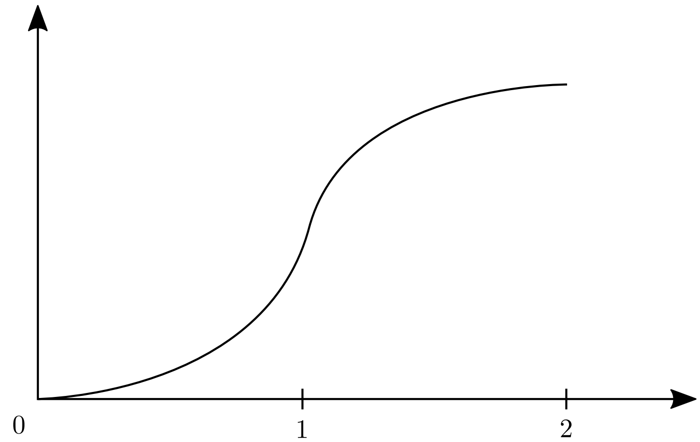

Figure 2: The Anti-Derivative $F(y)$

This is clear by the geometric argument where we are simply taking the area of square right triangles. Finally our conditional probability is going to be $\frac{.064}{.25} = .256$.

Problem 4.18

For this problem we have the density function

$$f(y) = \begin{cases} .2, & -1 < y \le 0 \\ .2 + cy, & 0 < y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a)

Find the value of c such that f is a proper probability density function. We know that the integral up to 0 will be .2 (area of a rectangle), which means the integral from 0 onward must be .8. We say that $.2 + \frac{c}{2} = .8 \Leftrightarrow c = 1.2$

(b)

$$F(y) = \begin{cases} 0, & y \le -1 \\ .2y|_{-1}^{y} = .2y + .2, & -1 < y \le 0 \\ .2(0) + .2 + \left[.2y + \frac{1.4y^2}{2}\right]_0^y = .2 + .2y + \frac{1.2y^2}{2}, & 0 < y \le 1 \\ 1, & y > 1 \end{cases}$$

(c)

Graph $f(y), F(y)$. See Figure 3.

(d)

$F(-1) = 0, F(0) = .2, F(1) = 1$

(e)

$P(0 \le Y \le .5) = F(.5) - F(0) = .2 + .2(.5) + \frac{1.2(.5)^2}{2} - [.2] = .25$

(f)

$P(Y > .5|Y > .1) = \frac{P(Y>.5)}{P(Y>.1)} = \frac{1-P(Y<.5)}{1-P(Y<.1)} = \frac{1-.45}{1-.2+.2(.1)+\frac{1.2(.1)^2}{2}} = .67.$

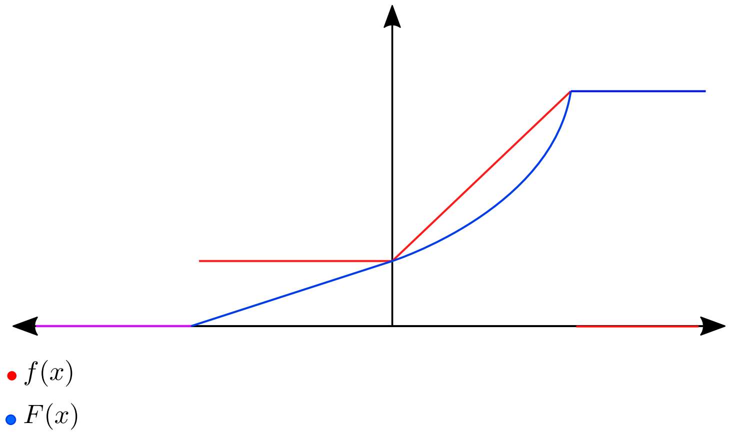

Figure 3: Graphs of $f(x)$ and $F(x)$

Problem 4.28

$$f(y) = \begin{cases} cy^2(1-y)^4, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a)

Find c such that $f(y)$ is a probability density function.

$$\int_0^1 cy^2(1-y)^4 dy = \frac{c}{105} \implies c = 105$$

(b)

Find $E(Y)$.

$$E(Y) = \int_{-\infty}^{\infty} yf(y)dy = 105 \int_0^1 y^3(1-y)^4 dy = \frac{3}{8} = .375$$

Problem 4.32

$$f(y) = \begin{cases} \frac{3}{64}y^2(4-y), & 0 \leq y \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

(a)

Find $E(Y)$ and $V(Y)$.

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} yf(y)dy \\
&= \int_{0}^{4} yf(y)dy \\
&= \int_{0}^{4} y\frac{3y^2(4-y)}{64}dy \\
&= \int_{0}^{4} \frac{3y^3(4-y)}{64}dy \\
&= \int_{0}^{4} \frac{12y^3-3y^4}{64}dy \\
&= \frac{12y^4}{4\cdot 64} - \frac{3y^5}{5\cdot 64}\Big|_0^4 \\
&= \frac{12}{5} = 2.4
\end{aligned}$$

$$\begin{aligned}
V(Y) &= E(Y^2) - E(Y)^2 \\
&= \int_{0}^{4} \frac{12y^4-3y^5}{64}dy - 2.4^2 \\
&= .64
\end{aligned}$$

(b)

If Y is our weekly CPU time in hours, and the cost is \$200 dollars per hour, we cna find the expected value and variance of CPU cost by multiplying $V(Y)$ and $E(Y)$ by 200. The expected weekly cost is $200 \cdot 2.4 = 480$ dollars. The variance will be 128 dollars.

(c)

We expect this to occur somewhat often. For this to occur we must use at least 3 hours of CPU. $P(Y > 3) = \int_3^4 \frac{3}{64}y^2(4-y)dy = \frac{3}{64}\int_3^4(4y^2-y^3)dy = \frac{67}{256} = .263$. This means our probability of this occurring will be more than $\frac{1}{4}$, which depending on how often is defined is a fairly reasonable chance.

Problem 4.40

Here we have a continuous uniform distribution, which must have a constant function between the endpoints A, B and must integrate to 1 so we say, $P(x_i \in (A,B)$ for some $i \in \{1,2,3\}) = 1 - P(x_i \notin (A,B) \forall i = 1,2,3) = 1 - \left(\frac{1}{2}\right)^3 = 1 - \frac{1}{8} = \frac{7}{8} = .875$

Problem 4.48

(a)

Here we have another continuous uniform distribution on the interval $[0, 500]$. It must be the case that $f(y)$ is constant and that $\int_0^{500} f(y)dy = 1$ so $f(y) = \frac{1}{500}$. $\frac{1}{500}\int_{475}^{500} 1dy = \frac{1}{500}[500-475] = \frac{25}{500} = .05$

(b)

We have the same probability except we are integrating over $(0, 25)$ but since the length is still 25, we get the same value. .05

(c)

$P(Y > 250) = \int_{250}^{500} \frac{1}{500}dy = \frac{250}{500} = .5$.

Problem 4.50

We have success from 12:00-1:00, failure from 1:00-3:00, success from 3:00-4:00, and failure from 4:00-5:00. Our total probability is 1 distributed evenly over the 5 hour period. Since $\frac{2}{5}$ of the possible time lies within the windows of success, we say the probability of success is $\frac{2}{5} = .4$.