MTH 311 HW 6

Philip Warton

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Theorem 1. *A point x is a limit point of a set A if and only if $x = lim(a_n)$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

3.2.5

A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also in F.

Proof. "$\Rightarrow$" Assume F is closed. Then if $x \in \mathbb{R}$ is a limit point for F, $x \in F$. Let (a_n) be a Cauchy sequence contained in F. By the Cauchy Critereon (a_n) converges to some limit $a \in \mathbb{R}$.
$\boxed{\text{Case 1: Limit Point}}$ If there exists a subsequence of $(a_n), (a_{n_k})$ such that $a_{n_k} \neq x$ for all $k \in \mathbb{N}$, then we can say that x is a limit point of F by $\boxed{\text{Theorem 1}}$. Since F is closed, this means that $x \in F$.
$\boxed{\text{Case 2: Isolated Point}}$ If such a subsequence does not exist, it follows that there exists a constant subsequence where $a_{n_k} = x$ for all $k \in \mathbb{N}$. Therefore, this sequence trivially converges to x and since our terms a_{n_k} belong to F, so too does x.

"$\Leftarrow$" Assume every Cauchy sequence if F has a limit that is also in F. Suppose x is a limit point of F. By $\boxed{\text{Theorem 1}}$, $x = lim\ a_n$ for some sequence (a_n) contained in F where $a_n \neq x$ for all $n \in \mathbb{N}$. By the Cauchy Critereon (a_n) is a Cauchy sequence. By assumption, $lim\ a_n \in F$. □

3.2.7

Given $A \subseteq \mathbb{R}$, let L be the set of all limit points of A.

(a)

Show L is closed.

Proof. Suppose $x \in \mathbb{R}$ is a limit point for L. Then, every ϵ-neighborhood $V_\epsilon(x)$ of x intersects L at some point that is not x. Let $\frac{\epsilon}{2} > 0$, then there exists $l \in L$ such that $0 < |x - l| < \frac{\epsilon}{2}$. Since $l \in L$, we know l is a limit point for A. Therefore there exists $a \in A$ such that $0 < |l - a| < |x - l| < \frac{\epsilon}{2}$. It follows that

$$\begin{aligned} &0 < |x - l| + |l - a| < \epsilon \\ &0 < |x - l + l - a| \leqslant |x - l| + |l - a| < \epsilon \\ &0 < |x - a| < \epsilon \end{aligned}$$

Since $|l - a| < |l - x|$, $x \neq a$. Therefore x is a limit point for A, and hencely is contained in L.

□

(b)

If x is a limit point of $A \cup L$, then x is either a limit point of A, or it is a limit point of L. If x is a limit point for A, then we are done. If x is a limit point for L, then by $\boxed{\text{(a)}}$ $x \in L$ and therefore is a limit point of A.

Proof. If L is the set of limit points of A, then it is immediately clear that $\overline{A}$ contains the limit points of A. Taking the union of $A \cup L$ produces a closed set that contains all limit points of A. Any closed set containing A must contain L as well. Therefore $\overline{A} = A \cup L$ is the smallest closed set containing A.

□

Theorem 2. *A point $s \in \mathbb{R}$ is the supremem of a set A if for any $\epsilon > 0$, there exists some element $a \in A$ such that $s - \epsilon < a$. Similarly a point $t \in \mathbb{R}$ is the infimum of a set A if for any $\epsilon > 0$, there exists some element $a \in A$ such that $s + \epsilon > a$.*

3.3.1

Show that if K is compact and nonempty, the sup$K$ and inf$K$ both exist and are elements of K.

Proof. Since K is compact, by the Heine-Borel Theorem K is both closed and bounded. Since K is bounded and $K \neq \emptyset$, it has an infimum and a supremem both in $\mathbb{R}$ by the Axiom of Completeness. Denote $s = sup(K)$ and $t = inf(K)$. The set K is closed, and thus it contains its limit points. Let $\epsilon > 0$. By $\boxed{\text{Theorem 2}}$, we know that there exists $k_1 = s - \epsilon \in K$, and thus $k_1 \in V_\epsilon(s)$. Similarly, there exists $k_2 \neq t \in K$ such that $k_2 \in V_\epsilon(t)$. Therefore s and t are limit points of the closed set K, and we conclude that $s, t \in K$.

□

3.3.3

Prove the converse fo Theorem 3.3.4 by showing that if a set $K \subseteq \mathbb{R}$ is closed and bounded, then it is compact.

Proof. Let K be a set that is closed and bounded. Let (k_n) be an arbitrary sequence in K. Then, by the Bolzano Weierstrass Theorem, there exists some subsequence (k_{n_m}) that converges to a limit $x \in \mathbb{R}$. By the $\boxed{\text{Theorem 1}}$, since $\lim(k_{n_m}) = x$, x is a limit point of K. Since K is closed, it follows that x is contained in K. Therefore every sequence in K has a subsequence that converges to a limit that is also in K. By definition, K is compact.

□