Algebraic Topology - Homework 1

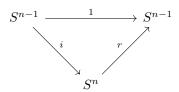
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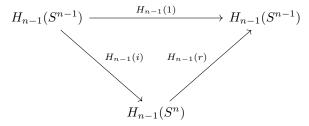
Problem 0.3

Assume, for $n \ge 1$, that $H_i(S^n) = \mathbb{Z}$ if i = 0, n, and that $H_i(S^n) = 0$ otherwise. Using the technique of the proof of Lemma 0.2, prove that the equator of the n-sphere is not a retract.

Proof. Assume that S^{n-1} is the equator of the *n*-sphere. Suppose by contradiction that S^{n-1} is a retract of S^n . Then it follows that there exists some retraction $r: S^n \to S^{n-1}$. Then, with $i: S^{n-1} \to S^n$ being the inclusion map, and with 1 being, of course, the identity map, it follows that we would have a commutative diagram:



To this diagram, we can apply our homology functor, giving us



We know by assumption that $H_{n-1}(S^n)=0$ since $n-1\neq n$ and that $H_n(S^{n-1})=\mathbb{Z}$ since n-1=n-1. This new diagram should continue to commute by the properties of our functor H_{n-1} . Since $H_{n-1}(S^n)=0$, it follows that its image under $H_{n-1}(r)$ must also be zero. That is, $H_{n-1}(S^{n-1})=0$. However, this means that our identity map 1 takes a countable algebra to a trivial one, which is a contradiction, therefore S^{n-1} cannot be a retract of S^n .

Problem 0.5

Problem 0.7

Let $f \in \operatorname{Hom}(A,B)$, and let $g,h \in \operatorname{Hom}(B,A)$ such that $g \circ f = 1_A$ and that $f \circ h = 1_B$. Then g = h.

Proof. Suppose that $h \neq g$,

$$h \neq g$$

$$h \circ f \neq g \circ f$$

$$f \circ h \circ f \neq f \circ g \circ f$$

$$(f \circ h) \circ f \neq f \circ (g \circ f)$$

$$1_B \circ f \neq f \circ 1_A$$

$$f \neq f$$

However, this is a contradiction, and we conclude that h = g.

Problem 0.18

For an abelian group G , let	$tG = \{x \in G : x \text{ has finite order}\}$
denote its torsion subgroup.	

(ii)