Mathematical Statistics - Assignment 8

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Problem 5.9

Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = \begin{cases} k(1-y_2) & 0 \leqslant y_1 \leqslant y_2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

(a)

Find the value of k that makes this a probability density function.

We want to find some value of k such that the double integral on $\mathbb{R}^2$ evaluates to 1. We write:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 \\ &= \int_0^1 \int_0^{y_2} k(1-y_2) dy_1 dy_2 \\ &= \int_0^1 \left[ky_1(1-y_2)\right]_0^{y_2} dy_2 \\ &= \int_0^1 ky_2 - ky_2^2 dy_2 \\ &= \left[\frac{ky_2^2}{2} - \frac{ky_2^3}{3}\right]_0^1 \\ &= \frac{k}{2} - \frac{k}{3} \\ \implies 6 &= 3k - 2k \\ &= k \end{aligned}$$

(b)

Find $P(Y_1 \leqslant \frac{3}{4}, Y_2 \geqslant \frac{1}{2})$.

We can rewrite this as $P(Y_2 \geqslant \frac{1}{2}) - P(Y_1 > \frac{3}{4})$. Then this becomes an integral that we can compute:

$$
\begin{aligned}
P(Y_2 \geqslant \frac{1}{2}) - P(Y_1 > \frac{3}{4}) &= \int_{\frac{1}{2}}^{1}\int_{0}^{y_2} f(y_1,y_2)dy_1dy_2 - \int_{\frac{3}{4}}^{1}\int_{y_1}^{1} f(y_1,y_2)dy_2dy_1 \\
&= \int_{\frac{1}{2}}^{1}\int_{0}^{y_2} 6(1-y_2)dy_1dy_2 - \int_{\frac{3}{4}}^{1}\int_{y_1}^{1} 6(1-y_2)dy_2dy_1 \\
&= \int_{\frac{1}{2}}^{1} [6y_1(1-y_2)]_0^{y_2}\, dy_2 - \int_{\frac{3}{4}}^{1}\int_{y_1}^{1} 6(1-y_2)dy_2dy_1 \\
&= \int_{\frac{1}{2}}^{1} 6y_2 - 6y_2^2 dy_2 - \int_{\frac{3}{4}}^{1}\int_{y_1}^{1} 6(1-y_2)dy_2dy_1 \\
&= \left[3y_2^2 - 2y_2^3\right]_{\frac{1}{2}}^{1} - \int_{\frac{3}{4}}^{1}\int_{y_1}^{1} 6(1-y_2)dy_2dy_1 \\
&= [3-2] - [\frac{3}{4} - \frac{1}{4}] - \int_{\frac{3}{4}}^{1}\int_{y_1}^{1} 6(1-y_2)dy_2dy_1 \\
&= \frac{1}{2} - \int_{\frac{3}{4}}^{1}\int_{y_1}^{1} 6(1-y_2)dy_2dy_1 \\
&= \frac{1}{2} - \int_{\frac{3}{4}}^{1} 6y_2 - 3y_2^2\Big|_{y_1}^{1} dy_1 \\
&= \frac{1}{2} - \int_{\frac{3}{4}}^{1} [6-3] - [6y_1 - 3y_1^2]dy_1 \\
&= \frac{1}{2} - \int_{\frac{3}{4}}^{1} 3 - 6y_1 + 3y_1^2 dy_1 \\
&= \frac{1}{2} - \left[3y_1 - 3y_1^2 + y_1^3\right]_{\frac{3}{4}}^{1} \\
&= \frac{1}{2} - \left([3-3+1] - \left[\frac{9}{4} - \frac{27}{16} + \frac{27}{64}\right]\right) \\
&= \frac{1}{2} - \frac{1}{64} = \frac{31}{64}
\end{aligned}
$$

Problem 5.16

$$
f(y_1,y_2) = \begin{cases} y_1 + y_2 & y_1, y_2 \in [0,1] \\ 0 & \text{otherwise} \end{cases}
$$

(a)

$$
\begin{aligned}
P(Y_1 < 1/2, Y_2 > 1/4) &= \int_{\frac{1}{4}}^{1}\int_{0}^{\frac{1}{2}} y_1 + y_2 dy_1dy_2 \\
&= \int_{\frac{1}{4}}^{1} \frac{y_1^2}{2} + y_2y_1\Big|_0^{\frac{1}{2}} dy_2 \\
&= \int_{\frac{1}{4}}^{1} \frac{1}{8} + \frac{y_2}{2} dy_2 \\
&= \frac{y_2}{8} + \frac{y_2^2}{4}\Big|_{\frac{1}{4}}^{1} \\
&= \frac{1}{8} + \frac{1}{4} - [\frac{1}{32} + \frac{1}{64}] \\
&= \frac{3}{8} - [\frac{3}{64}] = \frac{21}{64}
\end{aligned}
$$

(b)

$$\begin{aligned}
P(Y_1+Y_2\leqslant 1) &= P(Y_1\leqslant 1-Y_2)\\
&= \int_0^1\int_0^{1-y_2} y_1+y_2 dy_1 dy_2\\
&= \int_0^1 \frac{y_1^2}{2}+y_1y_2\Big|_0^{1-y_2} dy_2\\
&= \int_0^1 \frac{(1-y_2)^2}{2}+(1-y_2)y_2 dy_2\\
&= \int_0^1 \frac{1-2y_2+y_2^2}{2}+y_2-y_2^2 dy_2\\
&= \int_0^1 \frac{1}{2}-y_2+\frac{1}{2}y_2^2+y_2-y_2^2 dy_2\\
&= \int_0^1 \frac{1}{2}-\frac{1}{2}y_2^2 dy_2\\
&= \frac{1}{2}\int_0^1 1-y_2^2 dy_2\\
&= \frac{1}{2}\left[y_2-\frac{y_2^3}{3}\right]_0^1\\
&= \frac{1}{2}\frac{2}{3}=\frac{1}{3}
\end{aligned}$$

Problem 5.24

$$f(y_1,y_2)=\begin{cases}1 & y_1,y_2\in[0,1]\\ 0 & \text{otherwise}\end{cases}$$

(a)

Find the marginal denisty functions for Y_1, Y_2.

$$f_1(y_1)=\begin{cases}\int_0^1 1dy_2=1 & 0\leqslant y_2\leqslant 1\\ 0 & \text{otherwise}\end{cases}$$

$$f_2(y_2)=\begin{cases}\int_0^1 1dy_1=1 & 0\leqslant y_1\leqslant 1\\ 0 & \text{otherwise}\end{cases}$$

(b)

$$\begin{aligned}
P(.3<Y_1<.5) &= \int_{.3}^{.5} f_1(y_1)dy_1\\
&= \int_{.3}^{.5} 1dy_1\\
&= .5-.3=.2
\end{aligned}$$

$$\begin{aligned}
P(.3<Y_2<.5) &= \int_{.3}^{.5} f_2(y_2)dy_2\\
&= \int_{.3}^{.5} 1dy_1\\
&= .5-.3=.2
\end{aligned}$$

(c)

We know that

$$f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)} = \frac{1}{1} = 1$$

So this is defined everywhere for every $y_2 \in [0,1]$. Otherwise, f_2 is 0 which is clearly undefined.

(d)

From part c, we know that for any $y_2 \in [0,1]$ that $f(y_1|y_2) = 1$ for all $y_1 \in [0,1]$.

(e)

$$P(.3 < Y_1 < .5|Y_2 = .3) = \int_{.3}^{.5} 1 dy_1 = .5 - .3 = .2$$

(f)

$$P(.3 < Y_1 < .5|Y_2 = .5) = \int_{.3}^{.5} 1 dy_1 = .5 - .3 = .2$$

(g)

The answer does not change at all, this is probably because of the fact that the density function is trivial, and the variables are independent.

Problem 5.38

The variable Y_1 is uniform over $0 \leqslant y_1 \leqslant 1$. The variable Y_2 is uniform over $0 \leqslant y_2 \leqslant y_1$.

(a)

$$f(y_1,y_2) = \begin{cases} 1 & 0 \leqslant y_2 \leqslant y_1 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$f_1(y_1) = \int_0^{y_1} 1 dy_2 = y_1, \forall y_1 \in [0,1]$.

$$\begin{aligned} P(Y_2 > 1/4|Y_1 = 1/2) &= \int_{1/4}^{1} f(y_2|y_1 = 1/2) dy_2 \\ &= \int_{1/4}^{1} \frac{1}{(1/2)} dy_2 = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

Problem 5.48

No, these variables are not independent. The probability $P(Y_1 = 1) = .24$ whereas $P(Y_1 = 1|Y_2 = 1) = \frac{.02}{.16} = .125$.

Problem 5.68

(a)

We say $g(y_1) = \binom{2}{y_1}.2^{y_1}.8^{2-y_1}, h(y_2) = \binom{1}{y_2}.3^{y_2}.7^{1-y_2}$. Then since they are independent the joint density function will be the product of these functions.

$$f(y_1,y_2) = \left(\binom{2}{y_1}.2^{y_1}.8^{2-y_1}\right)\left(\binom{1}{y_2}.3^{y_2}.7^{1-y_2}\right)$$

(b)

$$\begin{aligned} P(Y_1 + Y_2 \leqslant 1) &= f(0,0) + f(0,1) + f(1,0) \\ &= (.8^2)(.7) + (.8^2)(.3) + 2(.2)(.8)(.7) \\ &= .864 \end{aligned}$$

Problem 5.72

(a)

$$E(Y_1) = np = 2\frac{1}{3} = \frac{2}{3}$$

(b)

$$V(Y_1) = np(1-p) = 2\frac{2}{3}\frac{1}{3} = \frac{4}{9}$$

(c)

$$E(Y_1 - Y_2) = E(Y_1) - E(Y_2) = \frac{2}{3} - \frac{2}{3} = 0$$

Problem 5.84

(a)

$$\begin{aligned} E(Y_1) &= \frac{p}{q} \\ E(Y_2) &= \frac{p}{q} \\ E(Y_1 - Y_2) &= E(Y_1) - E(Y_2) = 0 \end{aligned}$$

(b)

$$\begin{aligned} E(Y_1^2) &= p\left(\frac{2}{q^2} - \frac{1}{q}\right) \\ E(Y_2^2) &= p\left(\frac{2}{q^2} - \frac{1}{q}\right) \\ E(Y_1 Y_2) &= E(Y_1)E(Y_2) = \frac{p^2}{q^2} \end{aligned}$$

(c)

$$E(Y_1 - Y_2)^2 = 0$$

$$\begin{aligned} V(Y_1 - Y_2) &= E((Y_1 - Y_2)^2) - 0 \\ &= E(Y_1^2 - 2Y_1Y_2 + Y_2^2) \\ &= E(Y_1^2) + E(Y_2^2) - 2E(Y_1Y_2) \\ &= 2p\left(\frac{2}{q^2} - \frac{1}{q}\right) - 2\frac{p^2}{q^2} \\ &= \frac{2p}{q}\left(\frac{2}{q} - 1 - \frac{p}{q}\right) \end{aligned}$$

Problem 5.94

(a)

$$
\begin{aligned}
Cov(U_1, U_2) &= E(U_1U_2) - E(U_1)E(U_2) \\
&= E([Y_1 + Y_2][Y_1 - Y_2]) - E(Y_1 + Y_2)E(Y_1 - Y_2) \\
&= E(Y_1^2 - Y_2^2) - [E(Y_1) + E(Y_2)][E(Y_1) - E(Y_2)] \\
&= E(Y_1^2) - E(Y_2^2) - E(Y_1)^2 + E(Y_2)^2 \\
&= V(Y_1) - V(Y_2)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\rho &= \frac{Cov(U_1, U_2)}{\sqrt{V(U_1)}\sqrt{V(U_2)}} \\
&= \frac{V(Y_1) - V(Y_2)}{\sqrt{V(Y_1 + Y_2)V(Y_1 - Y_2)}} \\
&= \frac{V(Y_1) - V(Y_2)}{\sqrt{(V(Y_1) + V(Y_2) + 2Cov(Y_1, Y_2))(V(Y_1) + V(Y_2) - 2Cov(Y_1, Y_2))}} \\
&= \frac{V(Y_1) - V(Y_2)}{\sqrt{(V(Y_1) + V(Y_2))(V(Y_1) + V(Y_2))}} \\
&= \frac{V(Y_1) - V(Y_2)}{V(Y_1) + V(Y_2)}
\end{aligned}
$$

(c)

It is possible that this coviariance is equal to 0. This is the case when both variables Y_1 and Y_2 have the same variance.

Problem 5.100

(a)

We have $E(Y_1) = E(Z) = 0$. Then we have $E(Y_2) = E(Z^2) = V(Z) + E(Z)^2 = \sigma^2 + 0^2 = 1$.

(b)

Now we have $E(Y_1Y_2) = E(ZZ^2) = E(Z^3) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^3 e^{-\frac{x^2}{2}} dx = 0$.

(c)

$$
Cov(Y_1, Y_2) = Cov(Z, Z^2) = E(ZZ^2) - E(Z)E(Z^2) = 0 - 0(1) = 0
$$

(d)

So we konw that $P(Y_2 > 1|Y_1 > 1) = P(Z^2 > 1|Z > 1) = 1$. However since both, $P(Z^2 > 1)$ and $P(Z > 1)$ are stricly smaller than 1 (in fact most of the normal probability lies between -1 and 1), we know that $P(Z^2 > 1)P(Z > 1)$ must also lie in $(0, 1)$ which does not contain 1, therefore $P(Z^2 > 1|Z > 1) \neq P(Z^2 > 1)P(Z > 1)$. The two are not independent.