Mth 342 Homework 4

Philip Warton

February 14, 2020

1.

a.

To get the transformation matrix $[T]_B$ let us take the image of each basis vector and write it in the coordinates of B. We have

$$T(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_B$$
$$T(x) = 2x = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_B$$
$$T(x^2) = 8x^2 + 4x = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}_B$$

Putting these together in a matrix, we get

$$[T]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

b.

To get the characteristic polynomial we take the determinant of the matrix minus some multiple of the identity λI_3.

$$\begin{aligned} det([T]_B - \lambda I_3) &= det(\begin{bmatrix} 0-\lambda & 0 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 8-\lambda \end{bmatrix}) \\ &= -\lambda(2-\lambda)(8-\lambda) \\ \Longrightarrow & \lambda_1 = 0, \\ & \lambda_2 = 2, \\ & \lambda_3 = 8 \end{aligned}$$

Thus our eigenvalues are λ_1, λ_2, and λ_3.

c.

Let us take the null space of our matrix $[T]_B - \lambda I_3$ for each value of lambda. For $\lambda_1 = 0$ we get

$$null\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix}\right) = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$$

With $\lambda_2 = 2$ we have

$$null\left(\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}\right) = \left\{ x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$$

And finally with $\lambda_3 = 8$ we get

$$null\left(\begin{bmatrix} -8 & 0 & 0 \\ 0 & -6 & 4 \\ 0 & 0 & 0 \end{bmatrix}\right) = \left\{ x \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} : x \in \mathbb{R} \right\}$$

2.

a.

We want to show $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of $S \circ T \circ S^{-1}$.

Proof. We wish to show that such an implication holds in both directions.

"$\Rightarrow$" Assume that $\lambda \in \mathbb{R}$ is an eigenvalue for T. By definition, this means that $\exists v \in V : T(v) = \lambda v$. Let $w \in V$ such that $w = S(v)$. By taking S^{-1} of both sides we have $S^{-1}w = v$. From there we can write

$$\begin{aligned} T(S^{-1}(w)) &= \lambda S^{-1}(w) && \text{(since } v = S^{-1}w) \\ S(T(S^{-1}(w))) &= S(\lambda S^{-1}(w)) \\ &= \lambda S(S^{-1}(w)) \\ &= \lambda w \end{aligned}$$

Thus λ is an eigenvalue for $S \circ T \circ S^{-1}$.

"$\Leftarrow$" Assume that λ is an eigenvalue for $S \circ T \circ S^{-1}$. We know that $\exists w \in V : S \circ T \circ S^{-1}(w) = \lambda w$. Let $v \in V$ such that $v = S^{-1}(w)$. We can write

$$\begin{aligned} S(T(S^{-1}(w))) &= \lambda w \\ S^{-1}(S(T(S^{-1}(w)))) &= S^{-1}(\lambda w) \\ T(S^{-1}(w)) &= \lambda S^{-1}(w) \\ T(v) &= \lambda v \end{aligned}$$

And thusforth λ is an eigenvalue for T. We have now shown that $\lambda \in \mathbb{R}$ is an eigenvalue of T if and only if λ is an eigenvalue of $S \circ T \circ S^{-1}$.

□

b.

Given a specific $\lambda \in \mathbb{R}$, if $v \in V$ is an eigenvector for T, then $S(v)$ is an eigenvector for STS^{-1}.

3.

a.

True. Two similar matricies have the same eigenvalues. As we just showed in $\boxed{\text{2a.}}$ if λ is an eigenvalue for B, then it is also an eigenvalue for PBP^{-1}. Since $A = PBP^{-1}$, it is an eigenvalue for A.

b.

False. There exists two matricies with the same eigenvalues that are not similar. Suppose we have

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then both have $\lambda = 1$ as their only eigenvalues, but if we take

$$\begin{aligned} A &= PIP^{-1} \\ A &= PP^{-1} \\ A &= I \end{aligned}$$

This is a contradiction, therefore the equation is false, and the matricies are not similar.

4.

We want to show that for $A, B \in M_{n\times n}(F)$, $trace(AB) = trace(BA)$.

Proof. Suppose $A, B \in M_{n\times n}(F)$. Let $AB = C$ and $BA = C'$. By definition of matrix multiplication we have

$$c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki} \quad \text{and} \quad c'_{ii} = \sum_{k=1}^{n} b_{ik} a_{ki}$$

Therefore the trace of C is

$$\begin{aligned} trace(C) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} b_{ki} \\ &= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ki} a_{ik} \right) \\ &= \sum_{k=1}^{n} c'_{kk} \\ &= trace(C') \end{aligned}$$

□

5.

We have

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

To get the eigenvalues, we will use matlab with the following command:

```
>> eig(A)
```

This gives us eigenvalues $\lambda_1 = 0$ with algebraic mutliplicity 3 and $\lambda_2 = 30$. We can use another matlab command to get the null space of $A - 0I$ and $A - 30I$.

```
>> null(A)

ans =
        0.1144    -0.9765          0
       -0.2553     0.0383    -0.8944
        0.8130     0.1977          0
       -0.5107     0.0767     0.4472

>> null(A - (30 * eye(4)))

ans =
        0.1826
        0.3651
        0.5477
        0.7303
```

Thus we can write that

$$E_{\lambda=0} = \left\{ \begin{bmatrix} 0.1144 \\ -0.2553 \\ 0.8130 \\ -0.5107 \end{bmatrix}, \begin{bmatrix} -0.9765 \\ 0.0383 \\ 0.1977 \\ 0.0767 \end{bmatrix}, \begin{bmatrix} 0 \\ -0.8944 \\ 0 \\ 0.4472 \end{bmatrix} \right\}$$

and

$$E_{\lambda=30} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Since $dim(E_{\lambda=0}) = 3$ and $dim(E_{\lambda=30}) = 1$, A is diagonalizable. We have

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 \end{bmatrix}$$

and the diagonalizable matrix

$$Q = \begin{bmatrix} -2 & -3 & -4 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$