MTH 311 Homework 1

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1.3.1

a.)

Write a formal definition for infimum:

A real number t is a greatest lower bound for a set $A \subseteq \mathbb{R}$ if it meets the following conditions:

a) t is a lower bound for A

b) If b is any lower bound for A, then $b \leqslant t$

b.)

Given a set $A \subseteq \mathbb{R}$, and a lower bound t, $t = \inf A$ if and only if for every choice of $\epsilon > 0, \exists a \in A$ such that $t + \epsilon > a$.

Proof. Assume $t \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$. We want to show that $t = \inf A$ if and only if for every choice of $\epsilon > 0, \exists a \in A$ such that $t + \epsilon > a$. Let us show that the implication holds in each direction.

”$\Rightarrow$” Assume that $t = \inf A$. We have $t + \epsilon > t$ for all $\epsilon > 0$, which by our definition from $\boxed{\text{1.3.1 a.)}}$ means that $t + \epsilon$ cannot be a lower bound for A, since any lower bound b has the property $b \leqslant t < t + \epsilon$. Therefore, by definition of lower bound, $\exists a \in A : t + \epsilon > a$.

”$\Leftarrow$” Assume now that $\forall \epsilon > 0, \exists a \in A$ such that $t + \epsilon > a$. This means that $t + \epsilon$ is not a lower bound for A for all $\epsilon > 0$, by defition of lower bound. Since $t + \epsilon$ is not a lower bound for A with any $\epsilon > 0$ chosen arbitrarily, it must be the case that any lower bound $b \in \mathbb{R}$ for A satisfies the following: $b = t + x \; \exists x \leqslant 0$. This implies $b \leqslant t$ for any lower bound b. And therefore $t = \inf A$.

$\square$

1.3.3

a.)

Let $A \neq \emptyset$ and bounded below, and define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. $\sup B = \inf A$.

Proof. Let s be the infimum of A. We know that $s \geqslant b$ where b is any lower bound for A. Therefore $s \geqslant b\ \forall b \in B$, so we have that s is an upper bound for B. Let t be an upper bound for B chosen arbitrarily. If there exists some t such that $t < s$, then we would have $b \leqslant t < s\ \forall b \in B$, therefore $s \notin B$ and s would not be an upper bound for A (contradiction). To avoid this contradiction we must say that for any upper bound t for B, $t \geqslant s$. Having shown that s is both an upper bound for B and that for any other upper bound t, $s \leqslant t$, it can be said that $s = \sup B$.

□

b.)

There is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness because one could always choose the set B of lower bounds, and by finding the least upper bound for B, you find the greatest lower bound for a bounded below set A.

1.3.5

a.)

Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ and define the set $cA = \{ca : a \in A\}$. Want to show that $c \sup A = \sup cA$, given $c > 0$.

Proof. Let $s = \sup A$. We have $s \geqslant a\ \forall a \in A$. Multiplying both sides by $c > 0$ we get $cs \geqslant ca\ \forall a \in A$. By definition of upper bound we have cs is an upper bound for cA. Since $cs = c \sup A$, we have that $c \sup A$ is an upper bound for cA.

Let b be an upper bound for cA chosen arbitrarily. By definition we have $b \geqslant ac\ \forall a \in A$. Dividing by c we get $\frac{b}{c} \geqslant a\ \forall a \in A$. Then $\frac{b}{c}$ is an upper bound for A. Since $s = \sup A$ and $\frac{b}{c}$ is an upper bound for A, we have $\frac{b}{c} \geqslant s$ by definition of least upper bound. We can multiply both sides by c and get $b \geqslant cs$ which is equivalent to $b \geqslant c \sup A$. Thus any upper bound b for cA is greater or equal to $c \sup A$. Since $c \sup A$ is an upper bound for cA, and $c \sup A \leqslant b$ where b is an upper bound for cA, by definition of least upper bound we have $c \sup A = \sup cA$.

□

b.)

Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ and define the set $cA = \{ca : a \in A\}$. Postulate: $c \sup A = \inf cA\ \forall c < 0$.