Stochastic Elements of Mathematical Biology - Homework 3

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Problem 1

For each $k = 0, 1, 2, \cdots, 9$ in order to compute $p_k(\boldsymbol{i})$ we must first compute the likelihood of each possible gamete column. That is, during the process of segregation with recombination, what is the likelihood that a given gamete column is produced. First note that recombination only affects the probabilities of getting certain gametes from i_4, i_5. This is because for all others, the recombination produces the same two outcomes as without recomination. For example,

$$\begin{pmatrix} A & A \\ B & b \end{pmatrix} \mapsto_r \begin{pmatrix} A & A \\ b & B \end{pmatrix}$$

These clearly will have the same segregated gamete columns, so this process is only relavant to i_4, i_5. Then for each $m \in \{0, 1, \cdots, 9\}$, we add the frequency of this dimension multiplied with the desired gamete column. That is, for $p_0(\boldsymbol{i})$ we can sum $\frac{i_m}{N} \cdot P_m\binom{A}{B}$ over m where $P_m\binom{A}{B}$ is the likelihood that the m-th gamete will produce the desired column. Given this, we write

$$P\begin{pmatrix} A \\ B \end{pmatrix} = \frac{2i_0 + i_1 + i_3 + (1-r)i_4 + ri_5}{2N}$$

$$P\begin{pmatrix} A \\ b \end{pmatrix} = \frac{i_1 + 2i_2 + ri_4 + (1-r)i_5 + 2i_7 + i_8}{2N}$$

$$P\begin{pmatrix} a \\ B \end{pmatrix} = \frac{i_3 + ri_4 + (1-r)i_5 + 2i_7 + i_8}{2N}$$

$$P\begin{pmatrix} a \\ b \end{pmatrix} = \frac{(1-r)i_4 + ri_5 + i_6 + i_8 + 2i_9}{2N}$$

Since the order of columns does not matter, we have 2 options in which we can choose our columns, both with the probability of the likelihood of achivieng this gamete columns multiplied together. That is,

$$p_0(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} A \\ B \end{pmatrix} \cdot P\begin{pmatrix} A \\ B \end{pmatrix} \qquad p_1(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} A \\ B \end{pmatrix} \cdot P\begin{pmatrix} A \\ b \end{pmatrix} \qquad p_2(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} A \\ b \end{pmatrix} \cdot P\begin{pmatrix} A \\ b \end{pmatrix}$$

$$p_3(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} A \\ B \end{pmatrix} \cdot P\begin{pmatrix} a \\ B \end{pmatrix} \qquad p_4(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} A \\ B \end{pmatrix} \cdot P\begin{pmatrix} a \\ b \end{pmatrix} \qquad p_5(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} A \\ b \end{pmatrix} \cdot P\begin{pmatrix} a \\ B \end{pmatrix}$$

$$p_6(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} A \\ b \end{pmatrix} \cdot P\begin{pmatrix} a \\ b \end{pmatrix} \qquad p_7(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} a \\ B \end{pmatrix} \cdot P\begin{pmatrix} a \\ B \end{pmatrix} \qquad p_8(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} a \\ B \end{pmatrix} \cdot P\begin{pmatrix} a \\ b \end{pmatrix}$$

$$p_9(\boldsymbol{i}) = 2 \cdot P\begin{pmatrix} a \\ b \end{pmatrix} \cdot P\begin{pmatrix} a \\ b \end{pmatrix}$$

And these are our probabilities of producing each gamete based on $\boldsymbol{i}$.

Problem 2

To compute the probability of transitioning from $\boldsymbol{i}$ to $\boldsymbol{j}$ is done using a multinomial distribution for our more complex Fisher-Wright model. The probability mass function for this generalization of the binomial distribution is given by

$$\frac{n!}{x_1! \cdot x_2! \cdots x_k!} \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_k^{x_k}$$

Where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$ is the desired outcome state of the probability and $\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{pmatrix}$ is the probability of drawing each corresponding x_i.

So to write the transition probability of $p(\boldsymbol{i}, \boldsymbol{j})$ we must know what $n, \boldsymbol{x}$, and $\boldsymbol{p}$ are. Firstly, we know that $n = N$, secondly, we know that $\boldsymbol{x} = \boldsymbol{j}$. Finally we know that $\boldsymbol{p} = \boldsymbol{p}(i) = (p_0(i), p_1(i), \cdots, p_9(i))$. From here it follows simply that

$$p(\boldsymbol{i}, \boldsymbol{j}) = p(X(t+1) = \boldsymbol{j} | X(t) = \boldsymbol{i}) = \frac{N!}{j_0! \cdots j_9!} p_0^{j_0}(\boldsymbol{i}) \cdots p_9^{j_9}(\boldsymbol{i})$$

And we are done.