MTH 430 Metric Spaces and Topology

Final Exam June 10, 2020

Show all your work and justify your answers. And, please, write clearly!

Problem I.

a) Let $\mathbb{Q} \subset \mathbb{R}$ be the set of rational numbers. Find $\overline{\mathbb{Q}}$, $\mathring{\mathbb{Q}}$, $\mathbb{Q}^b$, where the superscript b indicates the frontier of a set.

b) Find all the connected components of $\mathbb{Q}$.

Problem II.

a) Show that $f\colon X \to Y$ is continuous if and only if $f^{-1}(C) \subset X$ is closed for every closed $C \subset Y$.

b) Then show that a function $f\colon X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.

Problem III.

a) Let $g\colon X \to Y$ be continuous, where Y is Hausdorff, and let $q \in Y$. Show that the set $G_q = \{x \in X \mid g(x) = q\}$ is closed in X.

b) Let X denote the space of real numbers with the finite-complement topology, and define $f\colon \mathbb{R}^1 \to X$ by $f(x) = x^3$, where $\mathbb{R}^1$ is equipped with the usual Euclidean topology. Is f continuous? Is it a homeomorphism?

Problem IV.

a) Let X be compact and assume that $\{C_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of closed subsets of X with the property that given any finite set of indices $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathcal{A}$, $n \geq 1$, then $C_{\alpha_1} \cap C_{\alpha_2} \cap \cdots \cap C_{\alpha_n} \neq \emptyset$. Show that $\cap_{\alpha \in \mathcal{A}} C_\alpha \neq \emptyset$.

b) Let $\mathbb{R}^{2*}$ be the 1-point compactification of the plane $\mathbb{R}^2$ as in problem 4 of assignment 4. show that $\mathbb{R}^{2*}$ is homeomorphic with the 2-dimensional unit sphere $S^2 \subset \mathbb{R}^3$.

Problem V. Let X be a topological space, and let $\Delta\colon X \to X \times X$ be the diagonal map $\Delta(x) = (x, x)$. Equip $X \times X$ with the product topology.

a) Show that Δ is continuous.

b) Show that X is Hausdorff if and only if the diagonal $\Delta(X) \subset X \times X$ is closed in $X \times X$.

Problem VI.

a) Show that the product of two path-connected spaces is path-connected.

b) Show that the sphere S^n is path-connected for any $n \geq 1$.

Problem VII.

a) Let $I^2 = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Identify the vertical sides by the relation $(0, y) \sim (1, y)$, $0 \leq y \leq 1$. Show that $I^2/\sim$ is homeomorphic with $S^1 \times I$, where $I = [0, 1]$ denotes the closed unit interval in $\mathbb{R}$.

b) Let $A = [\frac{1}{3}, \frac{2}{3}] \subset I$. Show that I/A is homeomorphic with I.