Systems of ODE's - Homework 5

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Problem 8.1 (iii)

Take the following system:

$$x' = x + y^2$$
$$y' = 2y$$

a)

Find all of the equilibrium points and describe the behavior of the associated linearized system.

So we want solutions such that both $x' = y' = 0$. If we want $y' = 0$ then we must have $2y = 0 \implies y = 0$. Then once we have $y = 0$ it follows that for $x' = x + y^2 = 0$ to also hold we must have $x' = x + y^2 = x + 0^2 = 0$ therefore $x = 0$. So the one and only equilibrium point is at $(0, 0)$. Having found our equilibrium point, now we wish to linearize the system, and observe the behavoir of this system. Let

$$f(x, y) = x + y^2$$
$$g(x, y) = 2y$$

Then we perform a (trivial) change of variables, where (a, b) is an equilibrium point, we let

$$u = x - a$$
$$v = y - b$$

So in our case $(u, v) = (x, y)$. Then we compute

$$u' = f_x(0, 0)u + f_y(0, 0)v$$
$$v' = g_x(0, 0)u + g_y(0, 0)v$$

$$\iff u' = u$$
$$v' = 2v$$

This gives us the linearized system $U' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} U$. Clearly we have positive real and distinct eigenvalues 1 and 2. Or we can compute our trace and determinant to be $T = 1 + 2 = 3$ and $D = 1 \cdot 2 = 2$. Then $\Delta = T^2 - 4D = 9 - 8 = 1$. There will obviously be a nodal source behavoir of this system if we were to draw a phase portrait.

b)

For the non-linear system, notice that we can take the equilibrium solutions $2y = 0$ and $x + y^2 = 0$. We get one solution $\langle x(t), y(t) \rangle = \langle Ce^{-t}, 0 \rangle$. Then we also get another solution that follows the parabola $x + y^2 = 0$ that must be oriented towards the origin. We can draw parallels between this and the nodal source linear equilibrium solutions, which lie upon the x and y axis. Then, we can imagine bending the y-axis into this parabola $x + y^2 = 0$, and this will give us a vague idea of the behavior of the non-linearized system.

c)

Yes, the linearized system does resemble the non-linearized system near the origin. Notice that the parabola $x + y^2 = 0$ is tangential to the y-axis as the distance from the origin approaches 0. Then also the solution where $2y = 0$ is indentical to the solution to the linearized system everywhere including near the origin.

Problem 8.5

a)

We have the system of differential equations,

$$\begin{aligned} x' &= x^2 + y \\ y' &= x - a + a \end{aligned}$$

We first wish to find the equilibrium points. That is, when both x' and y' are equal to 0. To do this first we can write

$$y' = x - y + a = 0 \quad \iff \quad y = x + a$$

Then, we can put this into our equation for x', giving us

$$\begin{aligned} x' &= x^2 + y \\ &= x^2 + x + a \end{aligned}$$

Since this is simply a quadratic function, it has roots given by the quadratic formula. It follows that we have equilibrium points at

$$\left(\frac{-1 \pm \sqrt{1-4a}}{2}, \frac{-1 \pm \sqrt{1-4a}}{2} + a \right) = \{p_1, p_2\} \subset \mathbb{R}^2$$

Both equilibrium only exist when $a < \frac{1}{4}$.We can linearize the system by doing an exchange of variables giving us

$$U' = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix} U$$

Simply plug in x_1, x_2 to compute the system for each respective equilibrium point.

b)

Now we will compute the trace, determinant, and big delta of this linear system at both equilibrium points p_1, p_2. We write

$$\begin{aligned} T &= 2x - 1 \\ D &= -2x - 1 \\ \Delta &= 4x^2 + 4x + 5 \end{aligned}$$

So for $x = \frac{-1+\sqrt{1-4a}}{2}$ (i.e. for p_1), we have

$$\begin{aligned} T &= -2 + \sqrt{1-4a} \\ D &= -\sqrt{1-4a} \\ \Delta &= 5 - 4a \end{aligned}$$

So at p_1 we are guaranteed a negative determinant, which means that around this point there will be saddle behavior.

Then if $x = \frac{-1-\sqrt{1-4a}}{2}$, or at p_2,

$$\begin{aligned} T &= -2 - \sqrt{1-4a} \\ D &= \sqrt{1-4a} \\ \Delta &= 5 - 4a \end{aligned}$$

Here our determinant is positive, so we are in the upper half of the trace-determinant plane. Then, our trace is guaranteed to be negative, so we are in the upper left quadrant of the trace -determinant plane. Then if $a < \frac{5}{4}$ (which is guaranteed when p_2 exists) then we are guaranteed $\Delta > 0$, that is, we have real and distinct eigenvalues, thus we have nodal source resembling behavior around p_2.

Problem 8.8

We consider the system given by

$$\begin{aligned} r' &= r - r^3 \\ \theta' &= \sin^2\theta + a \end{aligned}$$

Clearly we have an equilibrium point at the origin. Nearby to this point we have a spiral source for any initial value with $0 < r < 1$. For $a \leqslant -1$, for any initial value not at the origin, we have $r \to 1$ as $t \to \infty$, meaning this system resembles the one presented by the hopf bifurcation. For any solution with an initial value of $r = 1$, we have a solution that simply travels clockwise along the unit circle. If we observe $a \in (-1, 0)$, we notice that two equilibrium points appear in our phase portrait, at $(1, 3\pi/4), (1, 7\pi/4)$ that locally exibit nodal source behavior.