Intro to Mathematical Statistics - Final Exam

Philip Warton

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Problem 1

(a)

The variable X can be represented by a binomial distribution such that $X \sim B(n = 100, p = .02)$. We say $P(X \leqslant 2) = P(X = 0) + P(X = 1) + P(X = 2) = \binom{100}{0}.02^0.98^{100-0} + \binom{100}{1}.02^1.98^{100-1} + \binom{100}{2}.02^2.98^{100-2} \approx .6767$.

(b)

To use Poisson, first note that $\lambda = E[X] = np = 100 * .02 = 2$. Then we write

$$P(X \leqslant 2) = P(X = 0) + P(X = 1) + P(X = 2) = e^{-2} \sum_{0 \leqslant i \leqslant 2} \frac{2^i}{i!} = 5e^{-2} \approx .6767$$

Problem 2

(a)

$$\begin{aligned}
\int_0^1 \int_0^{1-y} k\,dx\,dy &= k \int_0^1 \int_0^{1-y} (1)\,dx\,dy \\
&= k \int_0^1 [1-y] - [0]dy \\
&= k \left[y - \frac{y^2}{2} \right]_0^1 \\
&= k(1 - \frac{1}{2}) \\
&= k\frac{1}{2}
\end{aligned}$$

Since we need to have a total probability of 1, $k = 2$.

(b)

We say that for $x \in [0, 1]$,

$$\begin{aligned}
f_X(x) &= \int_{1-x}^1 2dy \\
&= 2[1 - (1 - x)] \\
&= 2x
\end{aligned}$$

Similarly for $y \in [0, 1]$

$$\begin{aligned}
f_Y(y) &= \int_0^{1-y} 2dx \\
&= 2(1 - y)
\end{aligned}$$

(c)

$P(X > .5 \text{ and } Y < .5)$ Simply change the bounds of our integral so that the requirements are matched.

$$\begin{aligned}
\int_0^{.5}\int_{.5}^{1-y} 2dxdy &= \int_0^{.5} 2-2y-1dy \\
&= \int_0^{.5} 1-2ydy \\
&= y-y^2\Big|_0^{.5} \\
&= [.5-.5^2]-[0-0] \\
&= .25
\end{aligned}$$

(d)

To find $Cov(X,Y)$ we must find $E[XY], E[X], E[Y]$ and then we write

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

To compute the first we take the integral

$$\begin{aligned}
\int_0^1\int_0^{1-y}(xy)2dxdy &= \int_0^1 \left[x^2y\right]_0^{1-y} dy \\
&= \int_0^1 (1-y)^2 ydy \\
&= \int_0^1 (1-2y+y^2)ydy \\
&= \left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1 \\
&= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}
\end{aligned}$$

Then we compute $E[X]$ and $E[Y]$.

$$E[X] = \int_0^1 x(2x)dx = 2x^3/3\Big|_0^1 = 2/3$$

Then

$$E[Y] = \int_0^1 (y)2(1-y)dy = \int_0^1 2y-2y^2dy = y^2 - \frac{2y^3}{3}\Big|_0^1 = 1-\frac{2}{3} = \frac{1}{3}$$

Finally we can write the covariance as

$$Cov(X,Y) = \frac{1}{12} - \frac{2}{3}\frac{1}{3} = \frac{1}{12} - \frac{2}{9}$$

I would think that this value should come to 0, rather than some negative value since it appears that X, Y should be independent.

Problem 3

(a)

Let $X = Z_1^2$, we can compute the moment generating function of X by

$$\begin{aligned} m_X(t) &= E[e^{tZ^2}] \\ &= \int_{\mathbb{R}} e^{tz^2}\varphi(z)dz \\ &= \int_{\mathbb{R}} e^{tz^2}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\ &= \int_{\mathbb{R}} e^{z^2(t-\frac{1}{2})}\frac{1}{\sqrt{2\pi}}dz \\ &= \int_{\mathbb{R}} e^{-z^2(\frac{1}{2}-t)}\frac{1}{\sqrt{2\pi}}dz \end{aligned}$$

Let $v = z\sqrt{1-2t}$ and $dv = dz\sqrt{1-2t}$. So then we have

$$\begin{aligned} \int_{\mathbb{R}} e^{-z^2(\frac{1}{2}-t)}\frac{1}{\sqrt{2\pi}}dz &= \int_{\mathbb{R}} e^{-v^2/2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1-2t}}dv \\ &= \frac{1}{\sqrt{1-2t}}\int_{\mathbb{R}}\varphi(v)dv \\ &= (1-2t)^{-1/2} \end{aligned}$$

Then, notice that this is the moment-generating function of a χ^2 distribution with $k = 1$ degrees of freedom. Since its mgf is unique, we say the densities are the same. Thus,

$$f(x) = \frac{1}{\sqrt{2}\Gamma(1/2)}x^{1/2-1}e^{-x/2} = \frac{1}{\sqrt{2\pi}}x^{1/2-1}e^{-x/2}$$

(b)

We can differentiate the mgf to get expectation and variance.

$$E[X] = m'(0) = -\frac{1}{2}(1-2t)^{-3/2}(-2)\Big|_{t=0} = (1-2t)^{-3/2}\Big|_{t=0} = 1^{-3/2} = 1$$

$$E[X^2] = m''(0) = -\frac{3}{2}(1-2t)^{-5/2}(-2)\Big|_{t=0} = 3(1-2t)^{-5/2}\Big|_{t=0} = 3$$

Then the variance will be $E[X^2] - E[X]^2 = 3 - 1^2 = 2$.

(c)

Since Y is also a squared standard normal random variable, it will have the same mgf. Then since the two are independent we say $m_{X+Y}(t) = m_X(t)m_Y(t) = \frac{1}{\sqrt{1-2t}}^2 = \frac{1}{1-2t}$. This is identical to the χ^2 moment-generating function with degrees of freedom $k = 2$.

Problem 4

(a)

We have the probability of having a success or failure by some $n-th$ trial as $.5^{n-1}.5$ then the probability of achieving the same flip on the next toss is $.5$. This will be a geometric random variable shifted over by one, so

$$P(X = k) = P(Geom(.5) = k+1) = .5^k.5 = .5^{k+1} \quad \forall k = 2, 3, 4, \cdots$$

(b)

Since it is simply a shifted geometric distribution, we say $\mu = E[Geom(.5) + 1] = 2 + 1 = 3$. Then the variance will be unchanged, so $Var(X) = \frac{.5}{.5^2} = \frac{.5}{.25} = 2$.

(c)

We compute the mgf. We have

$$\begin{aligned}
E[e^t X] &= \sum_{2\leqslant k<\infty} e^{tk}.5^{k+1} \\
&= \sum_{2\leqslant k<\infty} \frac{1}{e^t}(e^t.5)^{k+1} \\
&= \frac{1}{e^t}\sum_{2\leqslant k<\infty} (e^t.5)^{k+1} \\
&= \frac{1}{e^t}\sum_{3\leqslant k<\infty} (e^t.5)^{k} \\
&= \frac{1}{e^t}\left[\frac{1}{1-e^t.5} - (e^t.5)^0 - (e^t.5)^1 - (e^t.t)^2\right]
\end{aligned}$$

Problem 5

(a)

We know that $m_X(t) = \sum_{k\in\mathbb{N}} e^t P(X=k)$. That is,

$$\begin{aligned}
\sum_{k\in\mathbb{N}} e^{tk}P(X=k) &= c\left(\frac{1}{8}e^{-t} + \frac{1}{4} + \frac{1}{4}e^t + \frac{3}{8}e^{2t}\right) \\
&= c\sum_{k\in\{-1,0,1,2\}} e^{tk}P(X=k)
\end{aligned}$$

Then from this we can deduce that $P(X=k)$ must be

$$P(X=k) = \begin{cases} \frac{c}{8} & k=-1 \\ \frac{c}{4} & k=0 \\ \frac{c}{4} & k=1 \\ \frac{3c}{8} & k=2 \end{cases}$$

Since the sum of these probabilities is c, we say that $c=1$ and

$$P(X=k) = \begin{cases} \frac{1}{8} & k=-1 \\ \frac{1}{4} & k=0 \\ \frac{1}{4} & k=1 \\ \frac{3}{8} & k=2 \end{cases}$$

(b)

We can compute the first moment by taking the derivative of $m_X(t)$ giving us

$$m'(t) = (-1)\frac{1}{8}e^{-t} + 0 + \frac{1}{4}e^t + (2)\frac{3}{8}e^{2t}$$

Then we can say that $m'(0) = \frac{7}{8}$. We compute the second moment by

$$m''(t) = \frac{1}{8}e^{-t} + 0 + \frac{1}{4}e^t + (4)\frac{3}{8}e^{2t}$$

Then $m''(0) = \frac{13}{8}$.

(c)

The probability of X being stricly between 0 and 2 non-inclusive will be $P(X=1) = \frac{1}{4}$.