Gröbner Bases — Homework 4

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Problem 1

Proof. Let $f \in A$. Then we write $f = \sum_{i=1}^{s} u_i X_i$. Then $f \xrightarrow{X_1} f_1$ where $f_1 = f - \frac{u_1 X_1}{lt(X_1)} X_1$. Since $lt(X_i) = X_i$ we can say $f_1 = f - u_1 X_1$. Repeat this process, and then $f_s = f - u_1 X_1 - u_2 X_2 - \cdots - u_s X_s$. However by assumption $f = u_1 X_1 + \cdots + u_2 X_2$. So then we have

$$\begin{aligned} f_s &= \sum_{i=1}^{s} u_i X_i - \sum_{i=1}^{s} u_i X_i \\ &= 0 \end{aligned}$$

Clearly $f \xrightarrow{G} f_s = 0$, so we have one direction complete.

Let $f \xrightarrow{G}_+ 0$ by some reduction. Let each Y_i be some term in f and we say that

$$\begin{aligned} 0 &= f - \sum \frac{Y_i}{lt(X_i)} X_i && \text{where this is a finite sum and all } i \text{ are from 1 to } s \\ &= f - \sum \frac{Y_i}{X_i} X_i \\ &= f - \sum Y_i \end{aligned}$$

Then by definition of reductions we say that $lt(X_i)$ divides Y_i so $Y_i = u_i X_i$. Then we can substitute this for Y_i, giving ,

$$\begin{aligned} 0 &= f - \sum Y_i \\ 0 &= f - \sum u_i X_i \\ \sum u_i X_i &= f \\ \\ &\Rightarrow f \in A \end{aligned}$$

□

Problem 2

First we wish to show that each class in B is independent. Let $X, X' \in \mathbb{T}^n$ such that $lp(g_i) \nmid X$ and $lp(g_i) \nmid X'$ for all i. Let $X + A = X' + A$. Then wish to show that $X = X'$. For the equality of these cosets we can also write

$$\{X + g_i\}_{i=1\cdots t} = \{X' + g_i\}_{i=1\cdots t}.$$

It must be the case that some $X + g_i = X' + g_j$ then. If $i = j$, then $X = X'$ is guaranteed. If $i \neq j$ then we have

$$\begin{aligned} X + g_i &= X' + g_j \\ X - X' &= g_j - g_i \end{aligned}$$

Since $g_j - g_i \in A$ it follows that $X - X' \in A$, and therefore must reduce to 0 by G. This implies that either some $lp(g_k) \in G$ divides $X - X'$, in which case it would have to divide both X and X' leading to a contradiction, or it is the case that $X - X' = 0 \implies X = X'$.

To show that B generates all cosets of A, let $f + A \in R/A$. Then $f \in R$ implies that $f = \sum_{i \in I} c_i X_i$ where c_i belong to our coefficient field and $X_i \in \mathbb{T}^n$. For each power product that can be divided by some $lp(g_i)$, we will say $X_i = Y_i \in \mathbb{T}^n$. Partition our index set I into I_1 and I_2 so that $\{Y_i\} \subset I_2$. Then,

$$\begin{aligned} f &= \sum_{i \in I} c_i X_i \\ &= \sum_{i \in I_1} c_i X_i + \sum_{i \in I_2} c_i Y_i \end{aligned}$$

However each term in the second sum is generated by A. So then it follows that for the coset of f,

$$f + A = \sum_{i \in I_1} c_i X_i + \sum_{i \in I_2} c_i Y_i + A = \sum_{i \in I_1} c_i X_i + A$$

Where each X_i cannot be divided by any g_i, so we say that $f + A \in \langle B \rangle$.