Advanced Multivariable Calculus - Assignment 3

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Problem 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ where

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) = 0 \\ 0, & \text{otherwise} \end{cases}$$

a)

Compute the partial derivatives of f at $(x,y) = 0$.

For the partial derivative with respect to x at 0, we write,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h(0)}{h^2+0^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= 0 \end{aligned}$$

And since the function is identical if we switch variables x and y, it follows that $\frac{\partial f}{\partial y}(0,0) = 0$ as well.

b)

Show that f is not continuous at 0 and therefore not differentiable at 0.

Proof. Let $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n})$. Then it follows that $(x_n, y_n) \to 0$ as $n \to \infty$. But, if we take the limit of $f(x_n, y_n)$, we find that

$$\begin{aligned} \lim_{n\to\infty} f(x_n, y_n) &= \lim_{n\to\infty} \frac{(\frac{1}{n})(\frac{1}{n})}{(\frac{1}{n})^2 + (\frac{1}{n})^2} \\ &= \lim_{n\to\infty} \frac{(\frac{1}{n})^2}{2(\frac{1}{n})^2} \\ &= \lim_{n\to\infty} \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Since $(x_n, y_n) \to 0$ but $f(x_n, y_n) \not\to f(0)$, we say that f cannot be continuous at 0. Any function from $\mathbb{R}^2 \to \mathbb{R}$ that is differentiable at a point A, is continuous at A, so by the contrapositive of that statement it follows that f cannot be differentiable at 0. □

Problem 2

Consider the partial differential equation

$$u_t + 3u_x = 0$$

for a differentiable function $u(x,t)$.

a)

Suppose f is a differentiable function of one variable. Show that $u(x,t) = f(x-3t)$ satisfies the partial differential equation.

Let $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(x,t) = x - 3t$. Then we can write $f(x-3t) = (f \circ g)(x,t)$. Then, we compute the partial derivatives of u using the chain rule

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}(f(g(x,t))) = \frac{df}{dy} \cdot \frac{\partial g}{\partial x} \\ &= \frac{df}{dy}(1) = \frac{df}{dy} \end{aligned}$$

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}(f(g(x,t))) = \frac{df}{dy} \cdot \frac{\partial g}{\partial t} \\ &= \frac{df}{dy}(-3) \end{aligned}$$

Then it follows that

$$u_t + 3u_x = \frac{df}{dy}(-3) + (3)\frac{df}{dy} = 0$$

So we say that the partial differential equation is satisfied.

b)

Let $V = \frac{1}{\sqrt{10}}(3,1)$. Show that if $u(x,t)$ satisfies the differential equation then the directional derivative $D_v u = 0$.

Proof. Let $u(x,t)$ be some function such that $u_t + 3u_x = 0$. We wish to show that $D_v u = 0$. So by definition of the directional derivative,

$$\begin{aligned} D_v u(x,t) &= \nabla u(x,t) \cdot V \\ &= \langle u_x, u_t \rangle \cdot \frac{1}{\sqrt{10}} \langle 3, 1 \rangle \\ &= (3u_x + u_t)\frac{1}{\sqrt{10}} \\ &= 0 \end{aligned}$$

□

c)

Show that a line that passes through (x,t) and is parallel to V passes through $(x-3t, 0)$.

We know that there is only one line parallel to V passing through (x,t) by one of Euclid's geometric postulates. So if we take the line passing through both (x,t) and $(x-3t,0)$ and show it is parallel to V, we have shown the desired statement. The slope of a line passing through both points is equal to

$$\frac{t-0}{x-(x-3t)} = \frac{t}{3t} = \frac{1}{3}$$

Then since V is a scalar multiple of $\langle 3, 1 \rangle$ it follows that it is parallel to this line.

d)

Show that every solution $u(x,t)$ has the property $u(x,t) = u(x-3t, 0)$.

Proof. We know that for any solution $u(x,t)$ that its directional derivative with respect to V is 0. Because of this, u is constant on any line parallel to V. Then since for any (x,t) we know that the line parallel to V passing through it also passes through $(x-3t,0)$. So u is constant on this line therefore $u(x,t) = u(x-3t,0)$. □

Problem 3

Let $G \subset \mathbb{R}^2$ be an open set, and assume $f : G \to \mathbb{R}$ is differentiable on G. Let C be a smooth curve in G, given by $\gamma : (a, b) \to G$. Assume f is constant on C. Prove that $\nabla f(\gamma(t)) \cdot \gamma'(t) = 0$.

Proof. Since f is constant on C, we write $f(\gamma(t)) = c \in \mathbb{R}$ for every $t \in (a, b)$. By the chain rule we write

$$\begin{aligned}\nabla f(\gamma(t)) \cdot \gamma'(t) &= \frac{d}{dt} f(\gamma(t)) \\ &= \frac{d}{dt} c \\ &= 0\end{aligned}$$

□

Problem 4

Let $f : [a, b] \to \mathbb{R}^2$ continuously on $[a, b]$ such that f is differentiable on (a, b). Then there exists some $t \in (a, b)$ such that

$$||f(b) - f(a)|| \leqslant ||f'(t)||(b - a)$$

Proof. Let $z = f(b) - f(a)$ and let $\varphi = z \cdot f(t)$. Then we have $\varphi : (a, b) \to \mathbb{R}$. By the ordinary mean value theorem on real valued functions, we say that $\exists t \in (a, b)$ such that

$$\begin{aligned}\varphi(b) - \varphi(a) &= \varphi'(t)(b - a) \\ z \cdot f(b) - z \cdot f(a) &= z \cdot f'(t)(b - a) \\ z \cdot (f(b) - f(a)) &= z \cdot f'(t)(b - a) \\ (f(b) - f(a)) \cdot (f(b) - f(a)) &= (f(b) - f(a)) \cdot f'(t)(b - a) \\ ||f(b) - f(a)||^2 &= [(f_1(b) - f_1(a))f_1'(c) + (f_2(b) - f_2(a))f_2'(c)](b - a)\end{aligned}$$

Then by the mean value theorem in $\mathbb{R}$, we know that $f_i(b) - f_i(b) = f_i'(t)(b - a)$, but multiplying both sides by $f_i'(t)$ we get $f_i(b)f_i'(t) - f_i(a)f_i'(t) = f_i'(t)^2(b - a)$. Therefore we can write

$$\begin{aligned}||f(b) - f(a)||^2 &= [f_1'(c)^2(b - a) + f_2'(c)^2(b - a)](b - a) \\ ||f(b) - f(a)||^2 &= [f_1'(c)^2 + f_2'(c)^2](b - a)^2 \\ ||f(b) - f(a)||^2 &= ||f'(c)||^2(b - a)^2\end{aligned}$$

Since all terms are positive, we can take the square root of each side without consequences, showing that $||f(b)-f(a)|| = ||f'(t)||(b-a)$, and so the inclusive inequality will hold as well. □

Problem 5

Fix $r > 0$ and let $B_r(0)$ be the open ball in $\mathbb{R}^2$. Assume $f : B_r(0) \to \mathbb{R}$ is differentiable on $B_r(0)$ and assume there exists a positivie real number M such that $||\nabla f(x)|| \leqslant M$ for all $x \in B_r(0)$. Prove that for any $\boldsymbol{a}, \boldsymbol{b} \in B_r(0)$,

$$|f(\boldsymbol{b}) - f(\boldsymbol{a})| \leqslant M||\boldsymbol{b} - \boldsymbol{a}||$$

Proof. Let $\gamma : [0, 1] \to \mathbb{R}^2$ where $\gamma(t) = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$. Then we write $\boldsymbol{a} = \gamma(0), \boldsymbol{b} = \gamma(1)$. Then we can rewrite

$$|f(\boldsymbol{b}) - f(\boldsymbol{a})| = |(f \circ \gamma)(1) - (f \circ \gamma)(0)|$$

By the mean value theorem, we know that $\exists c \in (0, 1)$ such that

$$(f \circ \gamma)(1) - (f \circ \gamma)(0) = (f \circ \gamma)'(c)$$

Then by the chain rule, we write

$$(f \circ \gamma)'(c) = \nabla f(\gamma(c)) \cdot \gamma'(c)$$

Then since $\gamma(t) = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$, we know that $\gamma'(t) = \boldsymbol{b} - \boldsymbol{a}$ for every t. So then we have

$$\begin{aligned}\nabla f(\gamma(c)) \cdot \gamma'(c) &= \nabla f(\gamma(c)) \cdot (\boldsymbol{b} - \boldsymbol{a}) \\ &\leqslant ||\nabla f(\gamma(c))||\,||\boldsymbol{b} - \boldsymbol{a}|| \\ &\leqslant M||\boldsymbol{b} - \boldsymbol{a}||\end{aligned}$$

And transitively we have shown that $|f(\boldsymbol{b}) - f(\boldsymbol{a})| \leqslant M||\boldsymbol{b} - \boldsymbol{a}||$. □

Proof. Assume by contradiction that there exists some $n \in \{2, 3, 4, \cdots\}$ such that n is neither prime or a product of two or more primes. By the well-ordering principle there must exist some such n that is the smallest element of $\{2, 3, 4, \cdots\}$ such that it is neither prime nor the product of two or more primes. Call this smallest such element n. Then since n is not prime it must be the case that $\exists a, b \in \{2, 3, \cdots, n-1\}$ such that $n = a \cdot b$. However since n is the smallest natural number larger than 1 that is neither prime or a product of primes, it must be the case that both a and b are prime or a product of two or more primes. So it follows that n is either prime or a product of two or more primes (contradiction). Therefore for every $n \in \mathbb{N}$ such that $n \geqslant 2$ we conclude that n is prime or a product of two or more primes. □