General Relativity - Homework 3

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April 20, 2021

Take Schwarzschild space to be a 4-dimensional real-valued space with a line element given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)$$

Problem 1

a)

Find the speed of a satellite orbiting a Schwarzschild black hole at constant radius $r = 6m$, as measured by a stationary ("shell") observer at that radius.

Since we have a "shell" observer, we know that we can measure infinitesimal space and time using,

$$\sigma^t = \sqrt{1 - \frac{2m}{r}}dt, \qquad \sigma^\phi = rd\phi$$

So to get the speed S of our satelite, we simply take the ratio $\dfrac{\sigma^\phi}{\sigma^t}$. This gives us the result

$$\begin{aligned} S &= \frac{\sigma^\phi}{\sigma^t} \\ &= \frac{rd\phi}{\sqrt{1 - \frac{2m}{r}}dt} \end{aligned}$$

Then we know that $\dfrac{d\phi}{dt} = \dfrac{\dot{\phi}}{\dot{t}} = \Omega = \sqrt{\dfrac{m}{r^3}}$. So we can We write our generalized speed as

$$\begin{aligned} S &= \frac{r\sqrt{\frac{m}{r^3}}}{\sqrt{1 - \frac{2m}{r}}} \\ &= \frac{r\sqrt{m}}{\sqrt{r^3}\sqrt{1 - \frac{2m}{r}}} \\ &= \frac{r\sqrt{m}}{r\sqrt{r(1 - \frac{2m}{r})}} \\ &= \frac{\sqrt{m}}{\sqrt{r(1 - \frac{2m}{r})}} \\ &= \sqrt{\frac{m}{r(1 - \frac{2m}{r})}} \end{aligned}$$

That the speed is a function of mass and radius makes sense. We plug in $r = 6m$ to get $S = \frac{1}{2}$.

b)

Is a circular orbit at $r = \frac{5}{2}m$ possible?

Let us take our speed computation from part $\boxed{\text{1a}}$ to see if this is the case. We have

$$\begin{aligned} S &= \sqrt{\frac{m}{r(1-\frac{2m}{r})}} \\ &= \sqrt{2} \end{aligned}$$

Then since $\sqrt{2} > 1$ we know that a circular orbit is not possible as a satellite cannot travel faster than lightspeed.

c)

Determine the smallest radius at which a circular orbit is possible, and the (shell) speed of a satellite in such an orbit.

We can immediately assume $r > 2m$ since this is our horizon where the coefficients start becoming non-real. Also, we want to bound our speed $0 < S < 1$. So we write,

$$\begin{aligned} S &< 1 \\ \frac{r\sqrt{\frac{m}{r^3}}}{\sqrt{1-\frac{2m}{r}}} &< 1 \\ r\sqrt{\frac{m}{r^3}} &< \sqrt{1-\frac{2m}{r}} \\ r^2\frac{m}{r^3} &< 1-\frac{2m}{r} \\ \frac{m}{r} &< 1-\frac{2m}{r} \\ \frac{3m}{r} &< 1 \\ 3m &< r \end{aligned}$$

So we must have a radius larger than $3m$ in order for circular orbit to be possible. Exactly at $r = 3m$ our satellite would have to travel at the speed of light, so we have a strict inequality and we say that a "smallest" radius cannot analytically exist assuming that radii can be infinitely divided. We cannot produce a fastest possible speed without having a smallest possible radius.

Problem 2

Imagine a beam of light in orbit around a Schwarzschild black hole at constant radius.

a)

How fast would a shell observer think the beam of light is traveling?

We take the same approach as before, but first we must re-derive our Ω using the proper $\dot{r}$ geodesic. We know that our potential function is now given by

$$V = \frac{1}{2}\left(\frac{\ell^2}{r^2} - \frac{2m\ell^2}{r^3}\right) - \frac{1}{2}$$

This can be rewritten as $V = \frac{\ell^2}{2r^2} - \frac{m\ell^2}{r^3} - \frac{1}{2}$. Then taking the derivative we get

$$V' = \frac{dV}{dr} = -\frac{\ell^2}{r^3} + \frac{3m\ell^2}{r^4} = \frac{\ell^2(3m-r)}{r^4}$$

Then we know that $V' = 0$ if and only if $\ddot{r} = 0$ so we use this fact to say that we must have the following for any circular orbit:

$$\begin{aligned}
0 &= \frac{\ell^2(3m-r)}{r^4} \\
\frac{\ell^2}{r^4} &= \frac{1}{3m-r}
\end{aligned}$$

$$\begin{aligned}
0 &= \dot{r} \\
0 &= e^2 - 1 - 2V \\
1 + 2V &= e^2 \\
1 + \left(1 - \frac{2m}{r}\right)\left(\frac{\ell^2}{r^2}\right) - 1 &= e^2 \\
\left(1 - \frac{2m}{r}\right)\left(\frac{\ell^2}{r^2}\right) &= e^2 \\
\left(1 - \frac{2m}{r}\right)\left(\frac{r^2}{3m-r}\right) &= e^2
\end{aligned}$$

Given these descriptions of e^2 and ℓ^2 in terms of mass and radius, we can now write

$$\begin{aligned}
\Omega^2 &= \frac{\dot{\phi}^2}{\dot{t}^2} \\
&= \frac{\ell^2/r^4}{e^2/\left(1 - \frac{2m}{r}\right)^2} \\
&= \frac{1/(3m-r)}{\left[(1 - \frac{2m}{r})\,(r^2/3m-r)\right]/(1 - \frac{2m}{r})^2} \\
&= \frac{1/(3m-r)}{(r^2/3m-r)\,/(1 - \frac{2m}{r})} \\
&= \frac{(1 - \frac{2m}{r})/(3m-r)}{r^2/(3m-r)} \\
&= \frac{1 - \frac{2m}{r}}{r^2}
\end{aligned}$$

$$\implies \Omega = \frac{\dot{\phi}}{\dot{t}} = \frac{\sqrt{1 - \frac{2m}{r}}}{r}$$

So now we can recompute our speed by

$$S = \frac{r}{\sqrt{1 - \frac{2m}{r}}} \frac{d\phi}{dt} = \frac{r}{\sqrt{1 - \frac{2m}{r}}} \frac{\sqrt{1 - \frac{2m}{r}}}{r} = 1$$

So to the shell observer the beam of light moves at exactly 1, which is the speed of light.

b)

$\boxed{\text{How fast would an observer far away think the beam of light is traveling?}}$

We know that $r = 3m$, and we know that $\frac{d\phi}{dt} = \frac{\sqrt{1 - \frac{2m}{r}}}{r}$. But the rate of change for time will appear different for the shell observer than it will for the far away observer. This is given by the relationship $dt_0 = \sqrt{1 - \frac{2m}{r}}dt_1$. So then we have

$$\begin{aligned}
S = \frac{rd\phi}{\sqrt{1 - \frac{2m}{r}}dt_1} &= \frac{rd\phi}{dt_0} \\
&= \frac{\sqrt{1 - \frac{2m}{r}}}{\sqrt{1 - \frac{2m}{r}}} \frac{rd\phi}{dt_0} \\
&= \sqrt{1 - \frac{2m}{r}} \frac{rd\phi}{\sqrt{1 - \frac{2m}{r}}dt_0} \\
&= \sqrt{1 - \frac{2m}{r}} \sqrt{\frac{m}{r(1 - \frac{2m}{r})}} \\
&= \sqrt{\frac{m}{r}}
\end{aligned}$$

Then we plug in $r = 3m$, which grants us $S = \sqrt{\frac{m}{3m}} = \frac{\sqrt{3}}{3}$.

c)

At what value(s) of r, if any, is such an orbit possible?

As per our computations in part $\boxed{\text{Problem 1 (c)}}$, we say that $r = 3m$ if $S = 1$.