MTH 342 HW 5

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1.

We want to determine if f is diagonalizable, then find a basis that diagonalizes the matrix if possible. Let $A \in M_{2\times 2}(\mathbb{R})$ such that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, $f(A) = \begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix}$. To find out if f is diagonalizable, we wish to find the sum of dimensions of eigenspaces. Let $f(A) = \lambda A$. Then,

$$\begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix} = \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since $\lambda a = b - c$, we have $-\lambda a = c - b = \lambda d$.
$\boxed{\text{Case 1: } \lambda = 0}$
If $\lambda = 0$, then $b = c$, and $a = d$, so we have $E_0 = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$, since this set is clearly linearly independent, and $b = c$ and $a = d$.
$\boxed{\text{Case 2: } \lambda \neq 0}$
Since $\lambda a = -\lambda d$, we know $a = -d$. Similarly, $b = -c$. In this case we write

$$f(A) = \begin{bmatrix} 2b & 2a \\ -2a & -2b \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

Thus, $2b = \lambda a$ and $2a = \lambda b$. From this we have

$$\begin{aligned} \frac{2b}{\lambda} &= a = \frac{\lambda b}{2} \\ \Rightarrow \quad \frac{4b}{2\lambda} &= \frac{\lambda^2 b}{2\lambda} \\ \Rightarrow \quad 4 &= \lambda^2 \end{aligned}$$

Therefore $\lambda = \pm 2$. If $\lambda = 2$, then $a = b$. Otherwise $a = -b$. Thus,

$$E_2 = span\left\{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right\}, \quad \text{and} \quad E_{(-2)} = span\left\{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right\}$$

We know $rank(E_0) + rank(E_2) + rank(E_{(-2)}) = 2 + 1 + 1 = 4$, hence f is diagonalizable. Let

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right\}$$

, then $[f]_B$ will be the diagonal matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

2.

0.1 $(x, y) = x_1y_1 + 2x_2y_2$

Linearity on 1^{st}

Let $x, y, z \in \mathbb{R}^2$. Then

$$\begin{aligned}(x + y, z) &= (x_1 + y_1)z_1 + 2(x_2 + y_2)z_2 \\ &= x_1z_1 + y_1z_1 + 2x_2z_2 + 2y_2z_2 \\ &= (x_1z_1 + 2x_2z_2) + (y_1z_1 + 2y_2z_2) \\ &= (x, z) + (y, z)\end{aligned}$$

Let $x, y \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}(cx, y) &= (cx_1)y_1 + 2(cx_2)y_2 \\ &= c(x_1y_1) + c(2x_2y_2) \\ &= c(x_1y_1 + 2x_2y_2) \\ &= c(x, y)\end{aligned}$$

Therefore we have linearity on the first argument.

Conjugate Symmetry

Let $x, y \in \mathbb{R}^2$. Then

$$\begin{aligned}(x, y) &= x_1y_1 + 2x_2y_2 \\ &= y_1x_1 + 2y_2x_2 \\ &= \overline{y_1x_1 + 2y_2x_2} \\ &= \overline{(y, x)}\end{aligned}$$

Thus we have conjugate symmetry.

Positive Definiteness

Let $x \in \mathbb{R}^2$. Then

$$(x, x) = x_1^2 + 2x_2^2 \geqslant 0$$

Now let $y \in \mathbb{R}^2$ and suppose $(y, y) = 0$. Then

$$0 = y_1^2 + 2y_2^2 \implies y = \mathbf{0}$$

Therefore we have positive definiteness

0.2 $(x, y) = x_1x_2 + y_1y_2$

Let $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then,

$$\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 1(-1) + 1(-1) = -2 < 0$$

Through this counter-example we have shown that the positive definite axiom is violated.

0.3 $(x,y) = (x_1+x_2)(y_1+y_2)$

Let $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Then,

$$\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = (1-1)(1-1) = 0$$

Therefore $(x,x) = 0$ does not imply that $x = \mathbf{0}$.

3.

Let $u_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Define $(.,.)$ such that it satisfies $(u_1,u_2) = i$, $(u_1,u_1) = 3$, and $(u_2,u_2) = 1$. Compute $\left(\begin{bmatrix} i+1 \\ 2i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$. We can write

$$\begin{aligned}
\left(\begin{bmatrix} i+1 \\ 2i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}\right) &= \Big(2i(u_1) + 3(u_2) + i(u_2), i(u_1) + 2(u_2)\Big) \\
&= \Big(2i(u_1), i(u_1) + 2(u_2)\Big) + \Big(3(u_2), i(u_1) + 2(u_2)\Big) + \Big(i(u_2), i(u_1) + 2(u_2)\Big) \\
&= \Big(2i(u_1), i(u_1)\Big) + \Big(2i(u_1), 2(u_2)\Big) + \Big(3(u_2), i(u_1)\Big) + \Big(3(u_2), 2(u_2)\Big) + \Big(i(u_2), i(u_1)\Big) + \Big(i(u_2), 2(u_2)\Big) \\
&= -2i^2\Big(u_1, u_1\Big) + 4i\Big(u_1, u_2\Big) - 3i\Big(u_2, u_1\Big) + 6\Big(u_2, u_2\Big) - i^2\Big(u_2, u_1\Big) + 2i\Big(u_2, u_2\Big) \\
&= 2\Big(u_1, u_1\Big) + 4i\Big(u_1, u_2\Big) - 3i\overline{\Big(u_1, u_2\Big)} + 6\Big(u_2, u_2\Big) + \overline{\Big(u_1, u_2\Big)} + 2i\Big(u_2, u_2\Big) \\
&= 2(3) + 4i(i) - 3 + 6 - i + 2i(1) \\
&= 5 + i
\end{aligned}$$

Thus we have $\left(\begin{bmatrix} i+1 \\ 2i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}\right) = 5 + i$.

4.

Define a product as $(u,v) = c_1\overline{d_1} + c_2\overline{d_2} + ... + c_n\overline{d_n}$ where $[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and $[v]_B = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$. We want to show that this product (denote $(\star)$) is an inner product space. We wish to show three different things:

(i) linearity on the first argument

(ii) conjugate symmetry

(iii) positive definite

To show (i), let $u, v \in V$ and $z \in F$ where F is a field. Then

$$\begin{aligned}
((z)u, v) &= zc_1\overline{d_1} + zc_2\overline{d_2} + \cdots + zc_n\overline{d_n} \\
&= z(c_1\overline{d_1} + c_2\overline{d_2} + \cdots + c_n\overline{d_n}) \\
&= z(u,v)
\end{aligned}$$

Now let $u, v, w \in V$. Then, $(u+v), w) = (c_1+d_1)\overline{e_1} + (c_2+d_2)\overline{e_2} + \cdots + (c_n+d_n)\overline{e_n}$ where $w = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$. It follows that

$$\begin{aligned}(u+v, w) &= c_1\overline{e_1} + d_1\overline{e_1} + c_2\overline{e_2} + d_2\overline{e_2} + \cdots + c_n\overline{e_n} + d_n\overline{e_n} \\ &= (c_1\overline{e_1} + c_2\overline{e_2} + \vdots + c_n\overline{e_n}) + (d_1\overline{e_1} + d_2\overline{e_2} + \vdots + d_n\overline{e_n}) \\ &= (u, w) + (v, w)\end{aligned}$$

Therefore we have (i). For (ii), let $u, v \in V$. Then $(u, v) = c_1\overline{d_1} + c_2\overline{d_2} + \cdots + c_n\overline{d_n}$. We can write this as $(u, v) = \overline{\overline{c_1}d_1 + \overline{c_2}d_2 + \cdots + \overline{c_n}d_n}$. Thus

$$(u, v) = \overline{\overline{c_1}d_1 + \overline{c_2}d_2 + \cdots + \overline{c_n}d_n} = \overline{(v, u)}$$

Therefore we have (ii). To show (iii) for $(\star)$, let $u \in V$. Then

$$(u, u) = c_1\overline{c_1} + c_2\overline{c_2} + \cdots + c_n\overline{c_n}$$

Since $z\overline{z} \geqslant 0\ \forall z \in \mathbb{C}$, we have $(u, u) \geqslant 0$. Let $u \in V$ such that $(u, u) = 0$. We have

$$c_1\overline{c_1} + c_2\overline{c_2} + \cdots + c_n\overline{c_n} = 0$$

Since all terms $c_k\overline{c_k}$ are non-negative, it follows trivially that $u = \mathbf{0}$.

5.

Let $x, y \in \mathbb{R}^2$ such that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Define the inner product as $(u, v) = 2x_1y_1 + x_2y_2$. Then,

$$\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = 2(1)(-1) + (2)(1) = 0$$

Therefore, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are perpendicular. To show this to be an inner product see the proof $\boxed{2.1}$ as the two products are nearly identical.