General Topology and Fundamental Groups - Homework 2

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Problem 1

(a)

Let $\tau = \{A \subset \mathbb{R} | \mathbb{R} \setminus A \text{ is countable or } A = \emptyset\}$. Show that τ forms a topology on $\mathbb{R}$.

Proof. Clearly we have $\emptyset \in tau$, and since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is countable, we also have $\mathbb{R} \in \tau$. Then let $\bigcup_{\alpha \in A} U_\alpha$ be an arbitrary union of sets in τ. The complement of the union will be the intersection of the complements, so we say

$$\mathbb{R} \setminus \left(\bigcup_{\alpha \in A} U_\alpha \right) = \bigcap_{\alpha \in A} (\mathbb{R} \setminus U_\alpha)$$

Of course, an intersection of countable sets is countable, so it follows that $\bigcup_{\alpha \in A} U_\alpha \in \tau$. Finally let $\bigcap_{\beta \in B} U_\beta$ be a finite intersection of sets in τ. Then the complement of the intersection will be a union of the complements, so we can write

$$\mathbb{R} \setminus \left(\bigcap_{\beta \in B} U_\beta \right) = \bigcup_{\beta \in B} (\mathbb{R} \setminus U_\beta)$$

Recall that B is finite, then we have a finite union of countable sets, which therefore must be countable. Since the complement of finite intersections is countable, we say $\bigcap_{\beta \in B} U_\beta \in \tau$. Having satisfied each axiom, τ is a topology on $\mathbb{R}$. $\square$

(b)

Let $A \subset \mathbb{R}$ be uncountable. Find $\overline{A}$ in this topology.

We know that $\overline{A}$ is the smallest closed set containing A. By definition of our topology, a set B is closed if either B is countable or $B = \mathbb{R}$. Since it is impossible for any countable set B to contain an uncountable set A, $\mathbb{R} \supset A$ is the only closed set containing A. Therefore $\overline{A} = \mathbb{R}$. As an immediate corollary, in this topology, any uncountable set is dense in $(\mathbb{R}, \tau)$.

(c)

Show that no sequence of distinct real numbers $\{a_n\} \subset \mathbb{R}$ converges in the topology τ.

Proof. Let $\{a_n\} \subset \mathbb{R}$ be a sequence in this topological space. Denote some limit candidate by $a \in \mathbb{R}$. Choose the set defined by $U = \mathbb{R} \setminus \left(\bigcup_{n \in \mathbb{N}} \{a_n\} \setminus \{a\}\right)$. Its complement, $\bigcup_{n \in \mathbb{N}} \{a_n\} \setminus \{a\}$ is countable, so of course U is an open neighborhood of a. However, notice that the every element of the sequence except for a itself is excluded from U. Since we assume that our sequence is not eventually constant, it follows that it cannot be eventually in U. Therefore $\exists U(x)$ such that $\forall N \in \mathbb{N}, \exists n \geqslant N : a_n \notin U$, and we say that this open neighborhood U exists for any sequence, so no sequence of distinct numbers converges. $\square$

(d)

Let $\iota : \mathbb{R} \to \mathbb{R}$ be the indentity map. Is ι continuous when mapping from the standard topology to τ, or vice versa?

In the case where we are mapping from $(\mathbb{R}, \tau) \to (\mathbb{R}, d)$ where d is the standard metric topology on $\mathbb{R}$, ι is not continuous. Choose $(0, 1) \subset \mathbb{R}$. It is open in the standard topology but since its complement is not countable, it is not open in τ. For the other direction, take the set of all irrational numbers. This set is open in τ, but is not open in the standard topology. So for neither direction is ι continuous.

Problem 2

(a)

Suppose that $G \subset X$ is a G_δ set. Show that there is a sequence of sets $G_1 \supset G_2 \supset \cdots$ of open sets such that $G = \bigcap_{k\in\mathbb{N}} G_k$.

Proof. Suppose G is G_δ set. Then G is a countable intersection of open sets. Then we say $G = \bigcap_{n\in\mathbb{N}} G_n$. Define $G_k = \bigcup_{n=1}^{k} G_n$, then for all $k \in \mathbb{N}, G_k \supset G_{k+1}$. The limit as $k \to \infty$ of G_k will clearly be $\bigcap_{n\in\mathbb{N}} G_n$. Since finite intersections of open sets are open, G_k is open for every $k \in \mathbb{N}$. □

Suppose that $F \subset X$ is a F_σ set. Show that there is a sequence of sets $F_1 \subset F_2 \subset \cdots$ such that $F = \bigcup_{k\in\mathbb{N}} F_k$.

Proof. F is an F_σ set. Then we say $F = \bigcup_{n\in\mathbb{N}} F_n$ where F_n is closed for every $n \in \mathbb{N}$. Define $F_k = \bigcup_{n=1}^{k} F_n$, then $F_k \subset F_{k+1} \forall k$. □

(b)

Let $Z \subset Y \subset X$ with their subspace topologies. Suppose that Z is G_δ in Y, and Y is G_δ in X. Show Z is G_δ in X.

Proof. $Z = \bigcap_{n\in\mathbb{N}} V_n$ where V_n is open in Y, and $Y = \bigcap_{m\in\mathbb{N}} O_n$ where O_n is open in X. Since V_n is open in Y with the subspace topology, we say $V_n = U_n \cap Y$ where U_n is open in X. Finally,

$$\begin{aligned} Z &= \bigcap_{n\in\mathbb{N}} V_n \\ &= \bigcap_{n\in\mathbb{N}} (U_n \cap Y) \\ &= \left(\bigcap_{n\in\mathbb{N}} U_n\right) \cap Y \\ &= \left(\bigcap_{n\in\mathbb{N}} U_n\right) \cap \left(\bigcap_{m\in\mathbb{N}} O_m\right) \end{aligned}$$

Thus Z is a countable intersection of open sets in X. □

(c)

Suppose that $Z \subset Y$ is G_δ in Y. Show that there is a G_δ set $W \subset X$ such that $Z = W \cap Y$.

Proof. We say that V_n are open in Y, and $V_n = Y \cap U_n$ where U_n is open in X. Then $Z = \bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} (U_n \cap Y) = (\bigcap_{n\in\mathbb{N}} U_n) \cap Y$. Let $W = \bigcap_{n\in\mathbb{N}} U_n$ and clearly W is a G_δ in X □

Problem 3

(a)

Show that any closed set in a metric space is G_δ.

Proof. Let (X, d) be a metric space and let $A \subset X$ be a closed set. Define the open set $A_n = \bigcup_{a\in A} B_{1/n}(a)$. Then we say that $\bigcap_{n\in\mathbb{N}} A_n = A$. Since every point in A is a limit point (A being closed), it follows that $a \in \bigcap_{n\in\mathbb{N}} A_n$. Then for every $x \notin A$, there is some neighborhood of x not intersecting A (since the complement of A is open). Thus it follows that $x \notin \bigcap_{n\in\mathbb{N}} A_n$, and $A = \bigcap_{n\in\mathbb{N}} A_n$. □

(b)

Show that $\mathbb{Q}$ is not G_δ in $\mathbb{R}$.

Proof. Suppose by contradiction that $\mathbb{Q}$ is G_δ. Then we must have $\mathbb{Q} = \bigcap_{n\in\mathbb{N}} U_n$. It must be the case that for every n, $U_n \supset \mathbb{Q}$. Then for each n, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbb{Q}) \subset U_n$. Let $\epsilon > 0$ be arbitrary, then $\mathbb{Q} \cap B_\epsilon(\sqrt{2}) \neq \emptyset$. This means that $\sqrt{2} \in B_\epsilon(\mathbb{Q})$, and therefore we have $\sqrt{2} \in \bigcap_{n\in\mathbb{N}} U_n = \mathbb{Q}$ (contradiction). □

Problem 4

(a)

Show that a smallest σ-algebra exists, $\mathcal{B}$.

Proof. Let $\{\mathcal{M}\rangle_{i\in I}$ be the collection of all σ-algebras on X. We claim that the smallest σ-algebra can be constructed by taking the intersection of this collection. Clearly this will be smaller than any σ-algebra, but we must demonstrate that this intersection is itself a σ-algebra.

$\boxed{\text{(i)}}$ For every $i \in I$, $X \in \mathcal{M}_i$ therefore $X \in \bigcap_{i\in I} \mathcal{M}_i$.

$\boxed{\text{(ii)}}$ Suppose $A \in \bigcap_{i\in I} \mathcal{M}_i$, then for all i, $A \in \mathcal{M}_i$, then of course $A^c \in \mathcal{M}_i$, so it follows that $A^c \in \bigcap_{i\in I} \mathcal{M}_i$.

$\boxed{\text{(iii)}}$ Suppose for some indexed collection of subsets $A_k \in \bigcap_{i\in I} \mathcal{M}_i$ for every $k \in \mathbb{N}$. Then for every $i \in I$, we have $A_k \in \mathcal{M}_i \implies \bigcap_{k\in\mathbb{N}} A_k \in \mathcal{M}_i$. Therefore $\bigcap_{k\in\mathbb{N}} A_k \in \bigcap_{i\in I} \mathcal{M}_i$.

Since each property of the σ-algebra is satisfied, we say that $\bigcap_{i\in I} \mathcal{M}_i$ is a σ-algebra. $\square$

(b)

Find a borel set in $\mathbb{R}$ that is neither F_σ or G_δ.

Proof. Let the set

$$A = ((-\infty, 0] \cap \mathbb{Q}) \cup ((0,1) \cap \mathbb{I}) \cup ([1,\infty) \cap \mathbb{Q})$$

Then we we say that $\square$