Advanced Multivariable Calculus - Homework 2

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Preamble

> Suppose that $f : \mathbb{R}^k \to \mathbb{R}^m$ and that $g : \mathbb{R}^m \to \mathbb{R}^n$, both continuously. Then, it follows that $g \circ f : \mathbb{R}^k \to \mathbb{R}^n$ is continuous.

Proof. Let $\epsilon > 0$ be arbitrary. We wish to show that $\exists \delta > 0$ such that $||X - Y|| < \delta \Rightarrow ||g \circ f(X) - g \circ f(Y)|| < \epsilon$. We know that $\exists \delta_g > 0$ such that $||f(X) - f(Y)|| < \delta_g$ implies $||g \circ f(X) - g \circ f(Y)|| < \epsilon$ by the continuity of g. Then by the continuity of f, take δ_g as the "ϵ" for the function f, and we know that $\exists \delta > 0$ such that $||X - Y|| < \delta \Rightarrow ||f(X) - f(Y)|| < \delta_g$. Of course, we then have the implications,

$$||X - Y|| < \delta \quad \Longrightarrow \quad ||f(X) - f(Y)|| < \delta_g \quad \Longrightarrow \quad ||g \circ f(X) - g \circ f(Y)|| < \epsilon$$

Therefore $g \circ f$ is continuous. □

> Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then we have the following:
>
> (i) $f(X) + g(X)$ is continuous
>
> (ii) $f(X)g(X)$ is continuous
>
> (iii) $\dfrac{f(X)}{g(X)}$ is continuous when $g(X) \neq 0$

Proof. $\boxed{\text{(i)}}$ Since $G(x, y) = x + y$ is continuous, and the function $F(X) = (f(X), g(X))$ is continuous, it follows that $G \circ F$ is continuous. For more details see $\boxed{\text{Homework 1}}$.

$\boxed{\text{(ii)}}$ We argue that $G(x, y) = xy$ is continuous, and thus by the same logic $f(X)g(X)$ will be continuous as well. To show this, let $\epsilon > 0$ be arbitrary. Then we have

$$\begin{aligned}
|xy - x_0y_0| &= |xy - x_0y + x_0y - x_0y_0| \\
&\leqslant |xy - x_0y| + |x_0y - x_0y_0| \\
&= |(x - x_0)y| + |x_0(y - y_0)| \\
&= |x - x_0||y| + |x_0||y - y_0| \\
&< \epsilon(|y| + |x_0|) \\
&< \epsilon(|y| + |x + \epsilon|)
\end{aligned}$$

This can be made arbitrarily small since $(x, y) \in \mathbb{R}^2$ is a fixed value. Thusly, $f(X)g(X) = G \circ F(X)$ and is continuous.

$\boxed{\text{(iii)}}$ Assume that $g(X) \neq 0$. Then it follows that $\frac{1}{g(X)}$ is continuous. Then by $\boxed{\text{(ii)}}$ we have $f(X)\frac{1}{g(X)}$ is continuous, so it follows that the quotient of them is continuous. □

Problem 1

> Let $C \subset \mathbb{R}^n$, and assume that whenever $\{x_n\}$ is a sequence in C with $x_n \to x$, it follows that $x \in C$. Show that C is closed.

Proof. Let $x \notin C$. We want to show that there is some ϵ-ball around x such that $B_\epsilon(x) \subset \mathbb{R}^n \setminus C$. Suppose that this is not the case. Then for every $\epsilon > 0$ there is a point in $B_\epsilon(x) \cap C$. We can take $\epsilon_k = \frac{1}{k}$, and then let x_k belong to that intersection. Clearly there is a sequence of points $\{x_k\}$ such that $x_k \to x$, and we have $x \in C$ (contradiction). So it must be the case that there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset \mathbb{R}^n \setminus C$. Thus the complement is open, so C is closed. □

Problem 2

> Let $O \subset \mathbb{R}^n$ be open. Assume $F : O \to \mathbb{R}^m$ is a function such that if $V \subset \mathbb{R}^m$ is open then so too is $F^{-1}(V) \subset \mathbb{R}^n$. Prove that F is continuous on O.

Proof. Let $\epsilon > 0$ be arbitrary. We want to show that $\exists \delta > 0$ such that $||X - Y|| < \delta \Rightarrow ||F(X) - F(Y)|| < \epsilon$. We know that $B_\epsilon(F(X))$ is an open set in $\mathbb{R}^m$. Thus we know that $F^{-1}(B_\epsilon(F(X)))$ is open in $\mathbb{R}^n$. Trivially it must contain X, since it is the pre-image of a set containing $F(X)$. Then, we know that there is some δ-neighborhood of X contained in $F^{-1}(B_\epsilon(F(X)))$ since it is an open set. So it follows that

$$Y \in B_\delta(X) \subset F^{-1}(B_\epsilon(F(X))) \implies F(Y) \in F(B_\delta(X)) \subset B_\epsilon(F(X))$$

Or alternatively,

$$||X - Y|| < \delta \implies ||F(X) - F(Y)|| < \epsilon$$

And we conclude that F must be continuous. $\square$

Problem 3

> Prove that for any subsets A and B of $\mathbb{R}^n$, if $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proof. Let $A \subset B \subset \mathbb{R}^n$. Let $x \in \overline{A}$ be arbitrary. We wish to show that $x \in \overline{B}$. We know that since $x \in \overline{A}$ that every ϵ-neighborhood of x must intersect A. Then since $B_\epsilon(x) \cap A$ is non-empty it follows that $B_\epsilon \cap B$ is also non-empty. Thus x is a limit point of B and belongs in its closure. $\square$

Problem 4

> Show that the function
>
> $$f(x, y, z) = \frac{\sin(x^2 + y^2)}{e^{z+y}}$$
>
> is continuous at all points $(x, y, z) \in \mathbb{R}^3$.

Proof. We know that $x \mapsto x^2$ is continuous, so it follows that $(x, y, z) \mapsto x^2$ is also continuous. The same is true for $(x, y, z) \mapsto y^2$, and also for $(x, y, z) \mapsto e^{-x}$ and $(x, y, z) \mapsto e^{-y}$. We know also that $\sin(h)$ is continuous given that h is continuous. Given these facts, we use the algebraic properties of functional continuity to assert that f must of course be a continuous function. $\square$

Problem 5

> Let $f : C \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous and let C be closed and bounded. The function f is uniformly continuous.

Proof. Suppose by contradiction that f is not uniformly continuous. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, there exist two points $X, Y \in \mathbb{R}^n$ where $||X - Y|| < \delta$ and $||f(X) - f(Y)|| \geqslant \epsilon$. Let $k \in \mathbb{N}$ be arbitrary. Choose $\delta_k = \frac{1}{k}$, and since it is true for every $\delta > 0$, we know that there are two points X_k, Y_k such that $||X_k - Y_k|| < \frac{1}{k}$ and $||f(X_k) - f(Y_k)|| \geqslant \epsilon$.

Since C is a closed and bounded set in $\mathbb{R}^n$ it is compact (Heine-Borel). Therefore every sequence has a convergent subsequence that converges to a point $X \in C$. Since f is continuous on C we know that

$$\lim_{k_i \to \infty} X_{k_i} \to X \implies \lim_{k_i \to \infty} f(X_{k_i}) \to f(X)$$

Take $\epsilon > 0$ to be arbitrary, and let $\alpha = \frac{\epsilon}{2}$. Then we know that $\frac{1}{k_i} < \alpha$ after some index in the sequence. So it follows that

$$\begin{aligned} ||X - Y_{k_i}|| &\leqslant ||X - X_{k_i}|| + ||X_{k_i} - Y_{k_i}|| \\ &< \alpha + \frac{1}{k_i} \\ &< \alpha + \alpha = \epsilon \end{aligned}$$

Thus we say that $Y_{k_i} \to X$.

Then from our conlcusion before we must have $Y_{k_i} \to X \implies f(Y_{k_i}) \to f(X)$. But if both $f(X_{k_i})$ and $f(Y_{k_i})$ converge to the point $f(X)$, it follows that their difference must converge to 0. Simply choose $\epsilon > 0$ arbitrary, and make both $||f(X_{k_i}) - f(X)|| < \epsilon/2$ and $||f(Y_{k_i}) - f(X)|| < \epsilon/2$ and then we have

$$||f(X_{k_i}) - f(Y_{k-i})|| = ||f(X_{k_i}) - f(X) + f(X) - f(Y_{k_i})|| \leqslant ||f(X_{k_i}) - f(X)|| + ||f(Y_{k_i}) - f(X)|| < \epsilon$$

However this lies in direct contradiction to the fact that there exists $\epsilon > 0$ such that $||f(X_{k_i}) - f(Y_{k_i})|| \geqslant \epsilon$. Thus our assumption that f is not uniformly continuous must be false, and f is uniformly continuous on C. $\square$

Problem 6

Provide an example and show that it holds.

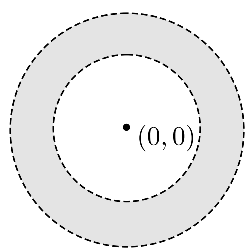
i)

> A function $f : \mathbb{R}^2 \to \mathbb{R}$ and an open set V in $\mathbb{R}$ such that $f^{-1}(V)$ is not open in $\mathbb{R}^2$.

Take the function

$$f(x, y) = \begin{cases} ||(x, y)||, & (x, y) \neq 0 \\ 1, & \text{otherwise} \end{cases}$$

Then take the interval $(\frac{1}{2}, \frac{2}{3}) \subset \mathbb{R}$ which is clearly open. Then its preimage is all (x, y) such that $1/2 < ||(x, y)|| < 3/2$ and $(x, y) = 0$. Any open ball of $(0, 0)$ will contain points such that $||(x, y)|| < 1/2$ and will intersect $f^{-1}(\frac{1}{2}, \frac{2}{3})^c$.



$\{(x, y) \in \mathbb{R}^2 | \frac{1}{2} < ||(x, y)|| < \frac{2}{3}\}$

Figure 1: $f^{-1}(\frac{1}{2}, \frac{2}{3})$

Since clearly no neighborhood of $(0, 0)$ is contained in the set, it cannot be open.

ii)

> A bounded $A \subset \mathbb{R}$ and a function $f : A \to \mathbb{R}$ which is continuous on A but not uniformly continuous on A.

Take the function $f = x^{-1}$ and let $A = (0, 1)$. Then we know that f is continuous on A, but it is not uniformly continuous due to its asymptote at $x = 0$.

Let $\epsilon > 0$ be arbitrary. Choose $\delta < \min\{\frac{x^2\epsilon}{2}, \frac{x}{2}\}$. Then if $|x - y| < \delta$ we have

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{y}\right| &= \left|\frac{x - y}{xy}\right| \\ &= \frac{|x - y|}{xy} \\ &\leqslant \frac{2|x - y|}{x^2} \\ &< 2\delta/x^2 \\ &< \epsilon \end{aligned}$$

So it follows that f is continuous on $(0, 1)$.

Let $\epsilon = 1$. We argue that for all $\delta > 0$ there exists $x \in (0,1)$ such that there exists $y \in (x-\delta, x+\delta)$ where $\left|\frac{1}{x} - \frac{1}{y}\right| \geqslant 1$. Let $x = \min\{\frac{1}{2}, \delta\}$. Then for every $x > y > 0$ we know that $y \in (x-\delta, x+\delta)$. Choose $0 < y = \frac{x}{2} < x$. Then we have

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} = \frac{\frac{x}{2}}{\frac{x^2}{2}} = \frac{1}{x} > 1$$