General Topology and Fundamental Groups - Homework 4

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Problem 1

Let (X, d) be a metric space and let $A \subset X$. Define the distance $d(x, A)$ of a point $x \in X$ from A by $d(x, A) = \inf\{d(x, a) | a \in A\}$.

a)

Fix $A \subset X$. The function $d(\cdot, A) : X \to \mathbb{R}, x \to d(x, A)$ is continuous.

Proof. Since we have two metric spaces, we can use the ϵ-δ definition of continuity. That is, a function $f : X \to Y$ is continuous if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon \tag{1}$$

Before our notation becomes confusing we will define our distance in the space X to be denoted by $d(x, y)$ for two points $x, y \in X$. Then we will denote our function described in the problem statement as $f : X \to \mathbb{R}$ where

$$f(x) = d(x, A) \tag{2}$$

Then for our metric on $\mathbb{R}$ we say that $\rho(x, y) = |x - y| \;\; \forall x, y \in \mathbb{R}$. Having made our notation follow our definition of continuity clearly we may now continue.

Let $\epsilon > 0$ be arbitrary. Then choose $\delta = \epsilon$. Choose any point $z \in A$. If $d(x, y) < \delta$, then by the reverse triangle inequality in metric spaces, we have

$$|d(x, A) - d(y, A)| \leqslant |d(x, z) - d(y, z)| < d(x, y) < \delta = \epsilon \tag{3}$$

Therefore the function is continuous. $\square$

b)

The distance $d(x, A) = 0$ if and only if $x \in \overline{A}$.

Proof. $\boxed{\Rightarrow}$ Assume that $d(x, A) = 0$. Then let $\epsilon > 0$ be arbitrary. There exists some $a \in A$ such that $d(x, a) < \epsilon$. Since x is clearly a limit point of A, it follows that $x \in \overline{A}$.

$\boxed{\Leftarrow}$ Assume that $x \in \overline{A}$. Then for every $\epsilon > 0, \;\; B_\epsilon(x) \cap A \neq \emptyset$. So it follows that there exists $a \in A$ such that $d(x, a) < \epsilon$ for every positive radius ϵ. Therefore the infimum of all strictly positive radii must be 0, and we say that $d(x, A) = \inf\{d(x, a) | a \in A\} = 0$. $\square$

c)

Let $A, B \subset X$ be subsets of X with disjoint closures. The distance function yields some Urysohn function for A and B directly.

Define a function $f : X \to \{0, 1\}$ as follows

$$f(x) = \begin{cases} 0, & d(x, A) = 0 \\ 1, & \text{otherwise} \end{cases} \tag{4}$$

Problem 2

Let X be a topological space and let $D \subset \mathbb{R}$ be dense. Suppose that there is an open $X_r \subset X$ for each $r \in D$ such that $\overline{X}_r \subset X_{\tilde{r}}$ when $r < \tilde{r}$, and $\bigcup_{r \in D} X_r = X$. Define $f : X \to \mathbb{R}$ by $f(x) = \inf\{r \in D | x \in X_r\}$. The function f is continuous.

Proof. Choose some open interval $(a, b) \in \mathbb{R}$. Sets of such a form are a basis for the standard topology on $\mathbb{R}$. Therefore if we can show that $f^{-1}((a, b))$ is an open set in X, then we say that f is continuous. Define the set $X_s = \bigcup_{r<s} X_r$, which allows us to take such sets on any point in $\mathbb{R}$. So we write the following,

$$\begin{aligned}
f^{-1}((a, b)) &= \{x \in X | f(x) \in (a, b)\} &(5)\\
&= \{x \in X | \inf\{r \in D | x \in X_r\} \in (a, b)\} &(6)\\
&= \{x \in X | a < \inf\{r \in D | x \in X_r\} < b\} &(7)\\
&= \{x \in X | a < \inf\{r \in D | x \in X_r\}\} \cap \{x \in X | b > \inf\{r \in D | x \in X_r\}\} &(8)\\
&= \{x \in X | a \geqslant \inf\{r \in D | x \in X_r\}\}^c \cap \{x \in X | b > \inf\{r \in D | x \in X_r\}\} &(9)\\
&= \left(\overline{X_a}\right)^c \cap X_b &(10)
\end{aligned}$$

This last line is justified by the fact that these sets are nested, and that the closure of the nested set still lies within any containing open set. So since the the closure of X_a is closed, its complement is open, and clearly X_b, a union of open sets, is also open. Then we have the intersection of two open sets which is of course open. □

Problem 4

Define a strong limit point to be a limit point x of a set A such that for any neighborhood U of x has infinitely many points in its intersection with A. Define countably compact to mean that any countable open cover of a topological space X yields some finite sub-cover.

A singleton set in a T_1 space is closed.

Proof. Let X be a T_1 topological space. Let $x \in X$. For every $y \neq x$, there exists some neighborhood $U(y)$ not containing x. Take the union

$$\bigcup_{y \in X \setminus \{x\}} U(y) = X \setminus \{x\} \tag{11}$$

As this is a union of open sets, it is open, therefore the singleton set $\{x\}$ is closed. □

a)

Assume that X is a T_1 space. Any limit point of a set is a strong limit point of that set.

Proof. Let $A \subset X$ with some limit point $x \in X$. Let U be some arbitrary open neighborhood of x. Then suppose by contradiction that $U \cap A$ is finite. Write

$$\begin{aligned}
U \cap A \setminus \{x\} &= \{a_1, a_2, \cdots, a_k\} \setminus \{x\} &(12)\\
&= \bigcup_{1 \leqslant i \leqslant k} \{a_i\} \setminus \{x\} &(13)
\end{aligned}$$

Since singleton sets are closed, a finite union of singleton sets will be closed. Therefore can take the complement of this set and it will be open. That is,

$$X \setminus \left(\bigcup_{1 \leqslant i \leqslant k} \{a_i\} \setminus \{x\} \right) \tag{14}$$

However, this is an open neighborhood of x that does not intersect A anywhere other than at the point x itself, therefore x is not a limit point of A (contradiction). □

b)

Assume that X is T_1 and assume that any infinite set in X has an accumulation point. Then X is coutably compact.

Proof. Let $\bigcup_{n\in\mathbb{N}} \mathcal{O}_n = X$ be some countable cover of X. We want to show that there exists some finite subcover of X. Suppose by contradiction that no finite subcover exists. That is, for every reordering of our countable cover,

$$\bigcup_{i=1}^{k} \mathcal{O}_i \neq X \tag{15}$$

Then for every $i \in \mathbb{N}$ there exists some point $x_i \in X \setminus \bigcup_{i=1}^{k} \mathcal{O}_i$. Thus the set $\{x_k\}_{k\in\mathbb{N}}$ is infinite. For any $i \in \mathbb{N}$, the intersection $\{x_k\}_{k\in\mathbb{N}} \cap \mathcal{O}_i$ is finite by definition (it will have exactly $\max\{0, i-k\}$ elements in fact). Then since the set $\{x_k\}_{k\in\mathbb{N}}$ is infinite, it will of course have some limit point $x \in X$. Then since we have an open cover of X, we know that for some $n \in \mathbb{N}$ $x \in \mathcal{O}_n$. Since x is a limit point of $\{x_k\}_{k\in\mathbb{N}}$ and $\mathcal{O}_n$ is a neighborhood of x, it must be the case that $\mathcal{O}_n \cap \{x_k\}_{k\in\mathbb{N}}$ is an infinite set (contradiction). □

Problem 5

Let (X, τ_1) be a compact Hausdorff space.

a)

Suppose X is also compact in topology τ_2 and assume that τ_2 is finer than τ_1. Then $\tau_1 = \tau_2$.

We use the following facts without proof: A closed subset of a compact space is compact in the subspace topology. A compact subset of a compact Hausdorff space is closed.

Proof. Let C be closed in (X, τ_2). Then, arbitrarily choose an open cover of C in the subspace topology consisting only of sets from τ_1. Then clearly it must yield some finite sub-cover. That is,

$$\bigcup_{\alpha\in A} U_\alpha \cap C = C \quad \Longrightarrow \quad \bigcup_{\alpha\in F} U_\alpha \cap C = C \tag{16}$$

It follows then that C is compact in the subspace topology of $(X \cap C, \tau_1)$. Since C is compact in this subspace, it is the case that C is a closed set in (X, τ_1). Since any closed set in τ_2 is closed in τ_1, we have $\tau_2 \subset \tau_1$. Then since τ_2 is finer than τ_1, we have $\tau_1 = \tau_2$. □

b)

Assume only that τ_2 is Hausdorff and finer than τ_1. Then still $\tau_1 = \tau_2$.

Proof. Let f be the identity map. Then $f : (X, \tau_2) \to (X, \tau_1)$ is continuous, by τ_2 being finer than τ_1. That is for every open set $O \in \tau_1, O \in \tau_2$. Then clearly the identity map is a bijection. So it follows that this function f must be a homeomorphism since it is mapping from a compact Hausdorff space to a Hausdorff space. Therefore (X, τ_2) is a compact space. From there, the conditions of 5a are met so we say that $\tau_1 = \tau_2$. □