Algebraic Topology - Homework 2

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1.3

> Let $R : S^1 \to S^1$ rotate any given point by x radians, then $R \simeq 1_{S^1}$.

Proof. We define the following function $F : S^1 \times X \to S^1$ and claim that it is a homotopy:

$$F(p,t) = p \cdot e^{i(tx)}$$

Firstly, we can trivially verify that

$$F(p,0) = p \cdot e^0 = p \cdot 1 = p = 1_{S^1}(p)$$
$$F(p,1) = p \cdot e^{ix} = R(p)$$

Since this function simply rotates the point over to x radians as we vary t, it follows that it is continuous and thus a homotopy between R and 1_{S^1}. $\square$

> Every continuous map $f : S^1 \to S^1$ is homotopic to a continuous map $g : S^1 \to S^1$ with $g(1) = 1$.

Proof. Let $f : S^1 \to S^1$ be continuous. Then $f(1) \in S^1$ with some corrosponding argument/angle $x \in [0, 2\pi)$. Let $R_\alpha : S^1 \to S^1$ denote the function given earlier as R with the rotation being given in radians by α. Then,

$$(R_{-x} \circ f)(1) = R_{-x}(f(1)) = R_{-x}(e^{ix}) = e^{i(0)} = 1$$

Let $g = R_{-x} \circ f$ and it follows that since $R \simeq 1_{S^1}$,

$$R_{-x} \circ f \simeq 1_{S^1} \circ f$$
$$g \simeq f$$

where $g(1) = 1$. $\square$

1.5

> Let $X = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ and let Y be a countable discrete space. Then X and Y do not have the same homotopy type.

Proof. Let $f : X \to Y$ be some continuous function. Then there exists some $y \in Y$ such that $0 \in f^{-1}(y)$ (since 0 must of course get mapped to some point in Y). Since Y is a discrete space it follows that $\{y\}$ is an open set. Since f is continuous, $f^{-1}(\{y\})$ is an open set in X containing 0. Assuming that X is equipped with the subspace topology from $\mathbb{R}$ it follows that for any open neighborhood U of 0 the followsing is true:

There exists some $N \in \mathbb{N}$ such that for every $n \geq N$, $\frac{1}{n} \in U$.

By this, it follows that $f^{-1}(\{y\})$ contains infinitely many points from X, and so it must be the case that only finitely many points in X are mapped to points other than y. Let $g : Y \to X$ be continuous. Then we conclude that since $f(X)$ is a finite set, so too is $(g \circ f)(X)$. Somehow XD we conclude that $1_X \not\simeq g \circ f$ for any f, g arbitrarily, and thus the two spaces have two different homotopy types. $\square$

1.7

Let $X = \{x, y\}$ with topology $\{X, \emptyset, \{x\}\}$, then X is contractible.

Proof. Define a function $F : X \times [0,1] \to X$ by

$$F(p,t) = \begin{cases} p, & \text{when } t \leq \frac{1}{2} \\ x, & \text{when } t > \frac{1}{2} \end{cases}$$

We can verify immediately that

$$F(p,0) = p = 1_X(p)$$
$$F(p,1) = x = c_x(p) \qquad \text{(where } c_x \text{ is the constant map to the point } x\text{)}$$

Let us verify that each pre-image of a neighborhood in X is open in X. Firstly $F^{-1}(X) = X \times [0,1]$ since the function is well defined and surjective. Then we know that $F^{-1}(\{x\}) = (\{x\} \times [0,1]) \cup (\{y\} \times (\frac{1}{2}, 1])$ which is open in $X \times [0,1]$. And finally $F^{-1}(\emptyset) = \emptyset$ since the function is well defined. So we conlcude that F is continuous and a homotopy and therefore $1_X \simeq c_x$ so X is contractible. □

1.8

There exists a continuous image of a contractible space that is not contractible.

Proof. Let $f : [0,1] \to S^1$ be given by $f(x) = e^{i(2\pi)x}$. The space $[0,1]$ is contractible trivially, and we claim that $f([0,1]) = S^1$ and is therefore not contractible. Let O be an open set in S^1. Then it is a union of some open intervals along the circle. The pre-image of each interval that does not include 1 will be of the form (a,b) which is clearly open. Otherwise it will be of the form $[0,a) \cup (b,1]$ and will remain open. Thus f is continuous. Let $y \in S^1$ and then it can be written as $y = e^{it}$ where $t \in [0, 2\pi)$. Then it follows that it has some pre-image under f so the function is surjective and $f([0,1]) = S^1$. We assume without proof that S^1 is contractible. □

A retract of a contractible space is contractible

Proof. Let $A \subset X$ be a retract where we have a retraction $r : X \to A$ and $r(a) = a$ for every element $a \in A$. Since X is contractible we can write $1_X \simeq c_{x_0}$ where $x_0 \in X$ and c_{x_0} is the constant map where $c_{x_0}(x) = x_0$. Let $H : X \times [0,1] \to X$ denote this homotopy. That is, $H(\,\cdot\,, 0) = 1_X$ and $H(\,\cdot\,, 1) = c_{x_0}$. Define a function $H_A : A \times [0,1] \to A$ where $H_A = r \circ H$. Since $A \times [0,1] \subset X \times [0,1]$, the function H remains well defined on $A \times [0,1]$. Then it follows that

$$\begin{aligned} H_A(a,0) &= (r \circ H)(a,0) \\ &= r(H(a,0)) \\ &= r(1_X(a)) \\ &= r(a) = a = 1_A(a) \end{aligned}$$

$$\begin{aligned} H_A(a,1) &= (r \circ H)(a,1) \\ &= r(H(a,1)) \\ &= r(c_{x_0}(a)) \\ &= r(x_0) = r_0 = c_{r_0}(a) \end{aligned}$$

Since H_A is a composition of continuous functions, it itself is also continuous, and so it's a homology between the identity and the constant function and obviously A is contracible. □

4.7

Compute $H_n(S^0)$ for all $n \geq 0$.

We invoke Theorem 4.13 (Rotman) which states that we can write

$$H_n(S^0) = \sum_{\lambda} H_n(S^0_\lambda)$$

Where each S^0_λ is a path component of S^0. Since $S^0 = \{-1, 1\}$, we can write

$$\begin{aligned} H_n(S^0) &= \sum_\lambda H_n(S^0_\lambda) \\ &= H_n(\{-1\}) \oplus H_n(\{1\}) \end{aligned}$$

Since each path component is a single-point space, we have

$$H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}, \qquad H_n(S^0) = 0$$

4.8

Compute $H_n(X)$ for all $n \geq 0$ where X is the cantor set.

We invoke Theorem 4.13 (Rotman) once again and conclude that since the cantor set is totally disconnected and so each path component is a singular point. Write $X_\lambda = \{\lambda\}$ as a singleton set in the cantor set. Thus,

$$H_n(X) = \sum_{\lambda \in X} H_n(X_\lambda)$$

And for $n = 0$, we have

$$H_0(X) = \sum_{\lambda \in X} H_0(\{pt\}) = \sum_{\lambda \in X} \mathbb{Z} = \bigoplus_{\lambda \in X} \mathbb{Z}$$

And then trivially for any $n > 0$ we get

$$H_n(X) = \sum_{\lambda \in X} H_n(\{pt\}) = \sum_{\lambda \in X} 0 = 0$$

Other

Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of chain complexes. We can write out the definition of the connecting homomorphisms and verify exactness of the long exact sequence.

Let $f : C' \to C$ and $g : C \to C''$. We define our connecting homomorphism by $h : H_n(C'') \to H_{n-1}(C')$ where $h([a]) = [c]$. But we must define what $[c]$ is in order for this function to be able to exist. Let $n \in \mathbb{N}$ be arbitrary and since C, C', and C'' are all chain complexes we can say let $[a] \in H_n(C''_n)$. That is, let $a \in C''_n$ such that $\partial''_n(a) = 0$, and a is a cycle. Since we have a short exact sequence we know that g is surjective, therefore $\exists b \in C_n$ such that $g_n(b) = a$. Since we know that our diagram of chain homomorphisms on chain complexes commutes it follows that

$$0 = \partial''_n(a) = \partial''_n(g_n(b)) = g_{n-1}(\partial_n(b))$$

Since $\partial_n(b) \in \ker(g_{n-1})$ it follows that $\partial_n(b) \in im(f_{n-1})$. Thus, we have some unique element $c \in C'_{n-1}$ that satisfies $f_{n-1}(c) = \partial_n(b)$. Then by our commutativity we have

$$0 = \partial_{n-1}(\partial_n(b)) = \partial_{n-1}(f_{n-1}(c)) = f_{n-2}(\partial'_{n-1}(c))$$

So then we can say $\partial'_{n-1}(c) \in \ker f_{n-2}$. Since f is always injective, we know that this element is uniquely in the kernel of f_{n-2} and so $\partial'_{n-1}(c)$ must itself be equal to 0. Then it follows trivially that $c \in \ker \partial'_{n-1}$. Thus we have $c \in C'_{n-1}$ as a cycle, so our homomorphism $h([a]) = [c]$ is somewhat plausible.

However for h to be well defined it must be the case that we have only one element $[c]$ which satisfies this property. Suppose that we choose b, b' both to be in the pre-image of a under g_n. Then we take the pre-image of their boundaries under f_{n-1}. That is, let

$$c, c' \in C'_{n-1} \quad | \quad f_n(c) = \partial_n(b), \;\; f_n(c') = \partial_n(b')$$

We must show that c, c' are of the same homology class in $H_{n-1}(C'_{n-1})$. Firstly notice that

$$g_n(b - b') = g_n(b) - g_n(b') = a - a = 0$$

Then since $b - b' \in \ker(g_n) = im(f_n)$ we say there is some unique element $\alpha \in C_n$ such that its image under f_n is $b - b'$. Then we can write

$$\begin{aligned} f_{n-1}(\partial'_n(\alpha)) &= \partial_n(f_n(\alpha)) && \text{(commutativity)} \\ &= \partial_n(b - b') \\ &= \partial_n(b) - \partial_n(b') \\ &= f_{n-1}(c) - f_{n-1}(c') \\ &= f_{n-1}(c - c') \end{aligned}$$

Then since f_{n-1} is injective it follows that $\partial'_n(\alpha) = c - c'$. So then since we have this equality, it follows naturally that c and c' belong to the same homology class in $H_{n-1}(C')$.

We already have short exact sequences from

$$0 \to C'_n \to C_n \to C''_n \to 0$$

So by functorality we also have a short exact sequence,

$$0 \to H_n(C'_k) \to H_n(C_k) \to H_n(C''_k) \to 0$$

What is left to show is that we have ‘exactness’ on our homomorphism $h : H_n(C''_n) \to H_{n-1}(C'_{n-1})$ so that we can follow this chain of homomorphisms all the way down until 0. We must show that $\ker f_* = im(h)$.

$\boxed{\subset}$ Let $[c] \in \ker(f^*_{n-1})$. Then by the funtorality of H_n we say that $[f_{n-1}(c)] = f^*_{n-1}([c]) = 0$. Since $[f_{n-1}(c)] = 0$ we can take some cycle $b \in C_n$ and its boundary will be equal to $f_{n-1}(c)$. That is, $\partial_n(b) = f_{n-1}(c)$. So we write that

$$\partial''_n(g_n(b)) = g_{n-1}(\partial_n(b)) = g_{n-1}(f_{n-1}(c)) = 0$$

So we have that $a = g_n(b) \in \ker \partial''_n$, and thus it is a cycle in C''_n, and forms a homology class $[a] \in H_n(C''_n)$. So it follows that for any $[c] \in \ker(f^*_{n-1})$ we have some $[a] \in H_n(C''_n)$ such that $h([a]) = [c]$. Thus $\ker f^*_{n-1} \subset im(h)$.

$\boxed{\supset}$ Let $[a] \in H_n(C''_n)$, and so $h([a]) \in H_{n-1}(C'_{n-1})$. We wish to show that $h([a]) \in \ker f^*_{n-1}$. So we apply our function h and write

$$\begin{aligned} f^*_{n-1}(h([a])) &= f^*_{n-1}([c]) \\ &= [f_{n-1}(c)] \\ &= [\partial_n(b)] && \text{(commutativity)} \\ &= 0 && \text{(how?)} \end{aligned}$$

So then we have shown that $h([a]) \in \ker f^*_{n-1}$ for any $[a] \in H_n(C''_n)$. Thus it follows that $\ker(f^*_{n-1}) = im(h)$. So it follows that we have a long exact homology sequence

$$0 \to H_n(C'_n) \to H_n(C_n) \to H_n(C''_n) \to_h H_{n-1}(C'_{n-1}) \to H_{n-1}(C_{n-1}) \to \cdots \to H_0(C''_0) \to 0$$