

MTH 311 Homework 8

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4.4.9

(a)

Show that a Lipschitz function is uniformly continuous.

Proof. Suppose $f : A \rightarrow \mathbb{R}$ is a Lipschitz function, i.e. there exists $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x \neq y \in A$. We can multiply the above inequality and get $|f(x) - f(y)| \leq M|x - y|$. Let $\epsilon > 0$ be arbitrary, and let $\delta = \frac{\epsilon}{M}$. Then if we have $|x - y| < \frac{\epsilon}{M}$, we can multiply by M and get

$$|f(x) - f(y)| \leq M|x - y| < M\delta = M\frac{\epsilon}{M} = \epsilon$$

□

(b)

Is the converse true? No.

Proof. Take the function $f(x) = \sqrt{x}$ on $[0, 1]$. We say that it is uniformly continuous because it is continuous on a compact set. Now, since the definition must apply for all $x, y \in A$ choose $x = \frac{1}{n} \in [0, 1]$ which works for any $n \in \mathbb{N}$ and $y = 0$. Then

$$\left| \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}} \right| = \sqrt{n}$$

Since \sqrt{n} is not bounded we say that the function is not Lipschitz.

□

4.5.5 b

Proof. We continue this proof from where the book left off. We have an interval $I_1 = [a_1, b_1]$ such that $f(a_1) < 0$ and $f(b_1) \geq 0$. We generalize the process by taking $z_n = \frac{a_n + b_n}{2}$, and then if $f(z_n) > 0$ then $b_{n+1} = z_n$. If $f(z_n) < 0$ then $a_{n+1} = z_n$. Otherwise, $z_n \in [a, b]$ and $f(z_n) = 0$, and we are done. By the inductive step, we assume that we have $I_n = [a_n, b_n]$, where $f(a_n) < 0$ and $f(b_n) > 0$. Then, we can find a midpoint z_n and create the interval I_{n+1} such that $f(a_{n+1}) < 0$ and $f(b_{n+1}) > 0$.

We say that the length of this interval is equal to $\frac{a-b}{2^n}$, since each time n increases by one we cut the interval in half. Then we take the sequence of intervals I_n for all $n \in \mathbb{N}$, and it follows that the length of I_n approaches 0. Since the length of I_n approaches zero, and by the Nested Interval Property the infinite intersection of I_n is non-empty, we say that there is exactly one number c such that $\bigcap_{n \in \mathbb{N}} I_n = \{c\}$.

Suppose that $f(c) > 0$. Then there would be some $n \in \mathbb{N}$ such that $f(I_n) > 0$ i.e. $f(a_n) > 0$ and $f(b_n) > 0$. This is not possible by the process in which we chose a_n , therefore $f(c) \leq 0$. Now suppose $f(c) < 0$. Then, similarly there must be some interval where $f(b_n) < 0$, which again is not possible. Thus $f(c) \geq 0$. Since we have shown $f(c)$ not to be strictly positive or strictly negative, it follows that $f(c) = 0$. \square

5.2.5

(a)

The function f is continuous at 0 for $a > 0$.

Proof. 0 is a limit point of \mathbb{R} . Thus, we can use the functional limit characterization of continuity and say that if $\lim_{x \rightarrow 0} f(x) = f(0)$ then f is continuous at 0. Suppose $a < 0$. Then, this limit is undefined as it diverges. For $a = 0$, the limit is equal to $1 \neq 0$. Therefore a must be greater than 0. Assume that $a > 0$, then $\lim_{x \rightarrow 0} x^a = 0 = f(0)$. Therefore the function is continuous at 0 assuming $a > 0$. \square

(b)

The function f is differentiable at 0 for all $a > 1$.

Proof. We say that f is differentiable at 0 if $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$ exists. This can be rewritten as

$$\lim_{x \rightarrow 0^-} \frac{0}{x} = \lim_{x \rightarrow 0^+} x^{a-1}$$

Suppose $a < 1$, then the right hand side of this limit diverges. Suppose $a = 1$, then the right hand side is zero, and the left hand side equals 1, therefore the limit does not exist, and f_1 is not differentiable at 0. Thus a must be strictly greater than 1. For any $a > 1$ it is clear that both sides will be equal to zero, and therefore the limit does exist for such values of a . \square

(c)

The function f is twice differentiable at 0 if $a > 2$.

Proof. We say that f is twice differentiable at 0 if f' is differentiable at 0. We know from part b that $f'_a(0) = 0$ for all $a > 1$, so by the power rule, given that $a > 1$, we write

$$f'_a(x) = \begin{cases} ax^{a-1} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

And by the left-right property of functional limits we say that $f'_a(x)$ is differentiable at 0 if

$$\lim_{x \rightarrow 0^-} \frac{0}{x} = \lim_{x \rightarrow 0^+} \frac{ax^{a-1}}{x} = \lim_{x \rightarrow 0^+} ax^{a-2} = 0 \implies a > 2$$

\square

5.2.7

(a)

We choose $a = 3$. Since $a > 2$, we know from recitation that this means g_a will be differentiable at 0, and thus on all of \mathbb{R} . By the power rule and the chain rule, we say that

$$g'_3(x) = x \left(ax \sin \left(\frac{1}{x} \right) - \cos \left(\frac{1}{x} \right) \right)$$

We know that $\cos\left(\frac{1}{x}\right)$ is a function that is not differentiable at 0 as we saw in recitation, and by the Algebraic Differentiability Theorem one can show that the sum of a differentiable function and a non-differentiable function will be non-differentiable. Therefore we say that the function $ax \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ is non-differentiable, and thus $g'_3(x)$ has no derivative at 0. Since it is an infinite discontinuity we say the derivative is unbounded.

(b)

We use our proof from part [a] as a sufficient answer for this part.

(c)

Choose $a = 4$. This guarantees that g' is continuous and differentiable, since we have

$$g'_4(x) = x^2 \left(ax \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \right)$$

Which we can show to be continuous by the squeeze theorem, and differentiable by the algebraic differentiability theorems. However, if we take the second derivative we get

$$\lim_{x \rightarrow 0} x \left(ax \sin\frac{1}{x} - \cos\frac{1}{x} \right)$$

We can show that this limit does not exist via methods from recitation.

5.3.1 a

Proof. Suppose f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$. Without loss of generality, let $x < y \in [a, b]$. Then choose the interval $[x, y] \subset [a, b]$. Then by the mean value theorem there exists $c \in [x, y]$ such that $f'(c) = \frac{f(x) - f(y)}{x - y}$. Since f' is continuous on the compact set $[a, b]$ we say that there exists $M > 0$ such that $f'(x) \leq M$.

Therefore we have

$$\frac{f(x) - f(y)}{x - y} = f'(c) \leq M$$

And we say that f is Lipschitz. □

$$\lim_{x \rightarrow 1} \frac{h(x^2) - h(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{h(x^2) - h(x)}{x - 1} + \lim_{x \rightarrow 1} \frac{h(x) - h(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{h(x^2) - h(x)}{x - 1} + h'(1)$$

5.3.5 b

