Gröbner Bases — Homework 1

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Problem 1

Let $S = \{x - y, z\} \subseteq \mathbb{R}[x, y, z]$. Describe the variety $V(S)$ both using set notation and geometrically.

Solution

We know that the variety $V(S)$ is the set of points such that every function in S is equal to 0 at said point. That is, in set notation,

$$\begin{aligned} V(S) &= \{(x, y, z) \subset \mathbb{R}^3 \mid x - y = 0 = z\} \\ &= \{(x, y, z) \subset \mathbb{R}^3 \mid x = y \text{ and } z = 0\}. \end{aligned}$$

We can also just write this as all points in $\mathbb{R}^3$ of the form $(\alpha, \alpha, 0)$ where $\alpha \in \mathbb{R}$. Geometrically, this consists of a straight line passing through the origin that lies in the x-y plane that is represented by the equation $x = y$.

Problem 2

Let K be a field. Recall that for a set $S \subseteq K^n$, we define

$$I(S) = \{f \in K[x_1, \cdots, x_n] \mid f(\vec{a}) = 0 \text{ for all } \vec{a} \in S\}$$

Prove that $I(S)$ is an ideal of the ring $K[x_1, \cdots, x_n]$.

Solution

Proof. First, we will show that $I(S) \subset K^n[x]$ is a subring, and then we will show that it is an ideal. To show that $I(S)$ is a subring, first one should notice that the additive identity, 0 is trivially an element of $I(S)$ since clearly the 0 function will map everything to 0, making it a memeber of $I(S)$. Let $f \in I(S)$ arbitrarily, then the typical additive inverse will still evaluate to 0, so we say that $-f \in I(S)$. Let $f, g \in I(S)$ arbitrarily, then we say that for any $\vec{a} \in S$ $f(\vec{a}) = 0 = g(\vec{a})$. So then $(f + g)(\vec{a}) = f(\vec{a}) + g(\vec{a}) = 0 + 0 = 0$ so we say that $I(S)$ is closed under addition. Recall that a subring I of a ring R is an ideal if for every $x \in R, y \in I$, we can say $xy \in I$. In our case specifically, let $S \subseteq K^n$, let

$$f \in K[x_1, \cdots, x_n],$$

and let

$$g \in I(S).$$

Choose any $\vec{a} \in S$ to be arbitrary. Then

$$\begin{aligned} (fg)(\vec{a}) &= f(\vec{a})g(\vec{a}) \\ &= f(\vec{a}) \cdot 0 \\ &= 0 \end{aligned}$$

Since $(fg)(\vec{a}) = 0$ for any $\vec{a} \in S$, it follows that $fg \in I(S)$ and it must be true that $I(S)$ is an ideal. $\square$