Advanced Multivariable Calculus - Notes

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1 Introduction to $\mathbb{R}^n$

Definition 1.1 (n-dimensional Vector)**.** An n-dimensional vector is an ordered tuple

$$\vec{x} = (x_1, x_2, \cdots, x_n)$$

With $x_i \in \mathbb{R}$ for every $i \in \{1, 2, \cdots, n\}$.

Then it is simple to say that $\mathbb{R}^n$ is the set of all n-dimensional vectors. Given some scalar constant $c \in \mathbb{R}$ and a vector $\vec{x} \in \mathbb{R}^n$ we write

$$c\vec{x} = (cx_1, cx_2, \cdots, cx_n)$$

We also define a norm on $\mathbb{R}^n$ which is given by $||\vec{x}|| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.

Definition 1.2 (Dot Product)**.** The dot product maps from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. Given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ we define their dot product to be $\vec{x} \cdot \vec{y} = \sum_{i=1}^n x_i y_i$.

Then we have $||\vec{x}|| = \sqrt{\vec{x} \cdot \vec{x}}$. The angle between two vectors is given by

$$\theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{||\vec{x}||\ ||\vec{y}||}\right)$$

Theorem 1.1 (Cauchy Shwarz Inequality)**.** *Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then we say*

$$|\vec{x} \cdot \vec{y}| \leqslant ||\vec{x}||\ ||\vec{y}||$$

2 Functions

Definition 2.1 (Function)**.** For $m, n \in \mathbb{N}, D \subset \mathbb{R}^n$, a function $F : D \to \mathbb{R}^m$ assigns to each $\vec{x} \in D$ a unique point $\vec{y} \in \mathbb{R}^m$. We write $F(\vec{x}) = \vec{y}$. For each $\vec{x} \in D$, we can write

$$\vec{y} = F(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \cdots, f_n(\vec{x}))$$

Where $f_j : D \to \mathbb{R} \quad \forall j, 1 \leqslant j \leqslant m$.

We can call f_j the j-th component function of F. Of course D is the domain of F and $F(D)$ is the image of F. Now there are lots of great examples of functions that map $\mathbb{R}^m \to \mathbb{R}^n$. But we care mostly about the following property:

Definition 2.2 (Continuity)**.** Let $D \subset \mathbb{R}^n$. Then $f : D \to \mathbb{R}^m$ is continuous at $x \in D$ if given any $\epsilon > 0$ there exists some $\delta > 0$ such that $||\vec{x} - \vec{y}|| < \delta$ implies $||f(\vec{x}) - f(\vec{y})|| < \epsilon$.

3 Integration

3.1 Partitions on $\mathbb{R}^n$

Let $I = I_1 \times I_2 \times \cdots \times I_k$ be a generalized rectangle, so $I_\ell = [a_\ell, b_\ell], 1 \leqslant \ell \leqslant k$. For each ℓ between 1 and k, let P_ℓ be a partition of I_ℓ. The collection of generalized rectangles

$$\{J = J_1 \times \cdots \times J_\ell \times \cdots \times J_k \mid J_\ell \text{ is an interval in } P_\ell\}$$

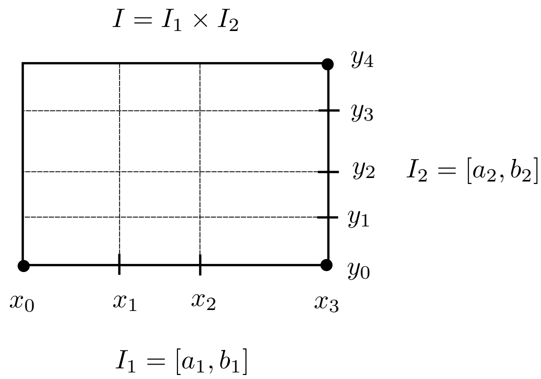

Figure 1: Generalized Rectangle Partition

Is a partition of I, and is denoted $P = (P_1, P_2, \cdots, P_m)$. For example, we can take this rectangle with the following partition: $P_1 = \{x_0, x_1, x_2, x_3\}, P_2 = \{y_0, y_1, y_2, y_3, y_4\}$ $\boxed{\text{Figure 1}}$.

The volume $Vol(I) = \sum_{J \in P} Vol(J)$. Let I be a generalized rectangle. Let P be a partition of I. Let $f : I \to \mathbb{R}$ be bounded. For J a generalized sub-rectangle in P, let

$$M(f, J) = \sup\{f(\boldsymbol{x}) : \boldsymbol{x} \in J\}$$

$$m(f, J) = \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \in J\}$$

The upper sum of f with respect to P is

$$U(f, P) = \sum_{J \in P} M(f, J) Vol(J)$$

And the lower sum is

$$L(f, P) = \sum_{J \in P} m(f, J) Vol(J)$$

We wish to find an upper bound for $L(f, P)$ and a lower bound for $U(f, P)$, so we use the following lemma.

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be a bounded function on a generalized rectangle I. Suppose*

$$m \leqslant f(\boldsymbol{x}) \leqslant M \quad \forall \boldsymbol{x} \in I$$

Then for every partition P of I,

$$m \cdot Vol(I) \leqslant L(f, P) \leqslant U(f, P) \leqslant M \cdot Vol(I)$$

Now we can write the following definition to start thinking about integration in $\mathbb{R}^n$.

Definition 3.1. Let $f : I \to \mathbb{R}$ be bounded, and I a generalized rectangle in $\mathbb{R}^n$. The lower integral of f on I is

$$\underline{\int_I} f = \sup\{L(f, P) : P \text{ partition of } I\}$$

The upper integral of f on I is

$$\overline{\int_I} f = \inf\{U(f, P) : P \text{ partition of } I\}$$

So then we also need to get a definition for integrable functions, which we say is the following:

Definition 3.2. Let $f : I \to \mathbb{R}$ be bounded where I is a generalized rectangle. Then f is integrable on I if

$$\underline{\int_I} f = \overline{\int_I} f$$

In which case we write

$$\int_I f = \underline{\int_I} f = \overline{\int_I} f$$

For an example, let I be a generalized rectangle. Define a function $f : I \to \mathbb{R}$ as follows:

$$f(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \text{ has a rational coordinate} \\ 0, & \text{otherwise} \end{cases}$$

Then by the density of rational and irrational numbers in $\mathbb{R}$, for any partition P of I, it must be the case that

$$L(f, P) = \sum_{J \in P} m(f, J)Vol(J) = 0$$

and

$$U(f, P) = \sum_{J \in P} M(f, J)Vol(J) = \sum_{J \in P} Vol(J) = Vol(I) \neq 0$$