Stuff

Philip Warton

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1 Chapter 6 Solutions

Moment Generating Function Method

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Let $Y_1, Y_2, \cdots, Y_n$ be independent and identiacally distributed random variables such that for $0 < p < 1, P(Y_i = 1) = p$ and $P(Y_i = 0) = q = 1 - p$.

(a) Find the moment generating function for the Bernoulli random variable Y_1. The moment generating function can be found by computing $E[e^{tY_1}]$. We do so now:

$$E[e^{tY_1}] = e^{t(0)}q + e^{t(1)}p = q + e^t p$$

(b) Find the moment-generating function for $W = Y_1 + Y_2 + \cdots + Y_n$. We use the fact that they are independent to say that

$$m_W(t) = m_{Y_1}(t) \cdot m_{Y_2}(t) \cdot \cdots \cdot m_{Y_n}(t) = (q + e^t p)^n$$

(c) This is the same moment generating function as that of the binomial distribution with n Bernoulli trials. Thus the distribution of $W \sim Binom(n, p)$.

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Let Y_1 and Y_2 be independent random variables with moment-generating functions $m_{Y_1}(t)$ and $m_{Y_2}(t)$ respectively. If a_1 and a_2 are constants and $U = a_1 Y_1 + a_1 Y_2$ show that the moment-generating function for U is $m_U(t) = m_{Y_1}(a_1 t) \times m_{Y_2}(a_2 t)$.

$$E[e^{tU}] = E[e^{t(a_1Y_1 + a_2Y_2)}] = E[e^{ta_1Y_1} e^{ta_2Y_2}] = E[e^{ta_1Y_1}]E[e^{ta_1Y_1}] = m_{Y_1}(a_1 t) \cdot m_{Y_2}(a_2 t)$$

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Suppose Y_1 and Y_2 are indpenedent, standard normal random variables. Find the density function of $U = Y_1^2 + Y_2^2$.

We can compute the moment generating function as

$$m_U(t) = m_{Y_1^2}(t) m_{Y_2^2}(t)$$

Now lets compute the moment-generating function of a standard normal random variable squared.

$$\begin{aligned}
m_{Z^2}(t) &= E[e^{tZ^2}] \\
&= \int_{\mathbb{R}} e^{tz^2} \varphi(z) dz \\
&= \int_{\mathbb{R}} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \int_{\mathbb{R}} e^{z^2(t - \frac{1}{2})} \frac{1}{\sqrt{2\pi}} dz \\
&= \int_{\mathbb{R}} e^{-z^2(\frac{1}{2} - t)} \frac{1}{\sqrt{2\pi}} dz
\end{aligned}$$

Let $v = z\sqrt{1-2t}$ and $dv = dz\sqrt{1-2t}$. So then we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{-z^2(\frac{1}{2}-t)} \frac{1}{\sqrt{2\pi}} dz &= \int_{\mathbb{R}} e^{-v^2/2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-2t}} dv \\
&= \frac{1}{\sqrt{1-2t}} \int_{\mathbb{R}} \varphi(v) dv \\
&= (1-2t)^{-1/2}
\end{aligned}$$

Having computed this, we can now say

$$m_U(t) = (1-2t)^{-\frac{1}{2}} \cdot (1-2t)^{-\frac{1}{2}} = \frac{1}{1-2t}$$

This is the moment generating function for a χ^2 distribution with $k = 2$ degrees of freedom.