Systems of ODEs - Homework 1

Philip Warton

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Problem 1.11

(a)

Find a general solution to the following differential equaiton: $x' = x^2$.

We can solve this by doing a speration of variables and then some simple calculus.

$$\begin{aligned}
x' &= x^2 \\
\frac{dx}{dt} &= x^2 \\
x^{-2}dx &= dt \\
\int x^{-2}dx &= \int dt \\
(-1)x^{-1} + c_1 &= t + c_2 \\
x^{-1} &= c - t \\
x &= (c-t)^{-1}
\end{aligned}$$

(b)

The domain of the above solution is $\mathbb{R} \setminus \{c\}$ for every constant $c \in \mathbb{R}$.

(c)

Give a differential equation such that $x(0) = 0$ is a solution defined only on $-1 < t < 1$.

$$x' = \sqrt{1 - x^2} - 1$$

Problem 2.11

Let $V = (v_1, v_2), W = (w_1, w_2)$ be vectors in $\mathbb{R}^2$. Prove that V and W are linearly independent if and only if $\det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = 0$.

Proof. $\boxed{\Rightarrow}$ Assume that V and W are linearly independent. Then V and W are not co-linear. That is, $\forall x \in \mathbb{R} xV \neq W$ or equivalently $xv_1 \neq w_1$ and $xv_2 \neq w_2$. So $\det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} = v_1w_2 - v_2w_1$. Suppose, by contradiction, that $v_1w_2 - v_2w_1 = 0$. We know that neither vector can be the 0 vector, otherwise, scale the other by a 0 coefficient. So we must have $v_1w_2 = v_2w_1$ non-trivially. This can be rewritten as $v_1\frac{w_2}{v_2} = w_1$ (contradiciton).

$\boxed{\Leftarrow}$ Assume that V and W are not linearly independent. Then $\exists x \in \mathbb{R}$ such that $xV = W$ (if V is the 0 vector, without loss of generality, swap V and W). Then we know that $\det[X\ X] = 0$ for any vector $X \in \mathbb{R}^2$, so it follows that

$$\det[V\ W] = v_1w_2 - v_2w_1 = xv_1w_2 - xv_2w_1 = \det[xV\ W] = \det[W\ W] = 0$$

$\square$

Problem 2.14

Prove that two eigenvectors from distinct real eigenvalues of a 2×2 matrix are always linearly independent.

Proof. Let $A \in M_{2\times 2}(\mathbb{R})$ be a matrix such that $\exists \lambda_1 \neq \lambda_2 \in \mathbb{R}$ eigenvalues of A, and let $\vec{v}_1, \vec{v}_2$ be the associated eigenvectors. We want to show that $x_1\vec{v}_1 + x_2\vec{v}_2 = 0$ has only the trivial solution, $x_1, x_2 = 0$. This equation scaled by λ_1 is $x_1\lambda_1\vec{v}_1 + x_2\lambda_1\vec{v}_2 = 0$. Another equation can be produced by multiplying our matrix A to both sides of the equation, giving us $Ax_1\vec{v}_1 + Ax_2\vec{v}_2 = x_1A\vec{v}_1 + x_2A\vec{v}_2 = x_1\lambda_1\vec{v}_1 + x_2\lambda_2\vec{v}_2 = 0$. We then have the following:

$$\begin{aligned} x_1\lambda_1\vec{v}_1 + x_2\lambda_1\vec{v}_2 &= x_1\lambda_1\vec{v}_1 + x_2\lambda_1\vec{v}_2 \\ x_2\lambda_1\vec{v}_2 &= x_2\lambda_2\vec{v}_2 \\ \implies x_2 &= 0 \end{aligned}$$

Then it follows that x_1 must also be equal to 0 since we have real non-trivial eigenvectors so we say that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent. $\square$