Differential Geometry - Homework 3b

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Problem 1

(a)

Determine the Hodge dual operator $*$ on all forms by computing its action on basis forms at each rank.

Lets begin with the 0-form.

$$\begin{aligned} 1 \wedge *1 &= g(1,1) dx \wedge dy \wedge dz \wedge dt \\ \implies *1 &= dx \wedge dy \wedge dz \wedge dt \end{aligned}$$

Now we do each of the 1-forms. We can determine the sign, using the inner product and counting how many times we must commute wedges to get our LHS wedge product factored out of the RHS.

$$\begin{aligned} dx \wedge *dx &= g(dx, dx) dx \wedge dy \wedge dz \wedge dt \\ *dx &= dy \wedge dz \wedge dt \\ *dy &= -dx \wedge dz \wedge dt \\ *dz &= dx \wedge dy \wedge dt \\ *dt &= dx \wedge dy \wedge dz \end{aligned}$$

Before moving onto 2-forms, lets compute the inner products we will need:

$$\begin{aligned} g(dx \wedge dy, dx \wedge dy) &= \det \begin{bmatrix} g(dx,dx) & g(dx,dy) \\ g(dy,dx) & g(dy,dy) \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 1 \end{aligned}$$

$$\begin{aligned} g(dx \wedge dz, dx \wedge dz) &= \det \begin{bmatrix} g(dx,dx) & g(dx,dz) \\ g(dz,dx) & g(dz,dz) \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 1 \end{aligned}$$

We can see that clearly any pair involving dx, dy, dz will have an inner product of 1. Any pair including dt will have an inner product of -1. Now we move on to computing the Hodge operator on 2-forms:

$$\begin{aligned} *(dx \wedge dy) &= dz \wedge dt \\ *(dx \wedge dz) &= -dy \wedge dt \\ *(dx \wedge dt) &= -dy \wedge dz \\ *(dy \wedge dz) &= dx \wedge dt \\ *(dy \wedge dt) &= dx \wedge dz \\ *(dz \wedge dt) &= -dx \wedge dy \end{aligned}$$

Before attending to the 3-forms, we must compute the inner product of each triple. We use the following

$$g(dx \wedge dy \wedge dz, dx \wedge dy \wedge dz) = g((dx \wedge dy) \wedge dz, (dx \wedge dy) \wedge dz) = \det \begin{bmatrix} g((dx \wedge dy), (dx \wedge dy)) & g((dx \wedge dy), dz) \\ g(dz, (dx \wedge dy)) & g(dz, dz) \end{bmatrix}$$
$$= \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

We will see that any inner product involving dt will gather a minus sign, while each other inner product will not. So we compute the hodge operator on 3-forms as follows:

$$*(dx \wedge dy \wedge dz) = dt$$
$$*(dx \wedge dy \wedge dt) = dz$$
$$*(dx \wedge dz \wedge dt) = -dy$$
$$*(dy \wedge dz \wedge dt) = dx$$

Finally for our 4-form, we compute $g(\omega, \omega) = -1$ since it will be the product of $g(dx \wedge dy \wedge dz, dx \wedge dy \wedge dz)$ and $g(dt, dt)$ which will of course be -1. So we say that

$$*(dx \wedge dy \wedge dz \wedge dt) = -1$$

Or equivalently

$$*\omega = -1$$

(b)

If we change our orientation so that $\omega = dt \wedge dx \wedge dy \wedge dz$, by the anti-commutativity of the wedge product, we can notice that

$$dx \wedge dy \wedge dz \wedge dt = -dt \wedge dx \wedge dy \wedge dz$$

So using the fact that $\alpha \wedge *\alpha = g(\alpha, \alpha)\omega$ it follows that if our bases are the same and the sign of ω is flipped, we can simply change the sign on our Hodge operator computations.