MTH 311 Homework 3

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February 9, 2020

2.2.2

c.)

Show that $\lim \dfrac{sin(n^2)}{\sqrt[3]{n}} = 0$.

Proof. Choose $\epsilon > 0$, arbitrarily. Let $N_\epsilon > \dfrac{1}{\epsilon^3}$. We can therefore write that for all $n \geqslant N_\epsilon$ we have $n > \dfrac{1}{\epsilon^3}$ which is equivalent to stating that $\epsilon^3 n > 1$. By properties of the sine function we also know that $|sin(n^2)| \leqslant 1$ and therefore $|sin(n^2)|^3 \leqslant 1^3 = 1$. From there, we have by ordering that $|sin(n^2)|^3 < \epsilon^3 n$. We can divide this expression by n to get the inequality $\dfrac{|sin(n^2)|^3}{n} < \epsilon^3$. Since both sides are positive, this is equivalent to $\dfrac{|sin(n^2)|}{\sqrt[3]{n}} < \epsilon$. With both the numerator and denominator positive, we write $\left|\dfrac{sin(n^2)}{\sqrt[3]{n}}\right| < \epsilon$. Trivially subtracting a zero we get $\left|\dfrac{sin(n^2)}{\sqrt[3]{n}} - 0\right| < \epsilon$. By definition of convergence we have $\lim \dfrac{sin(n^2)}{\sqrt[3]{n}} = 0$.

□

2.5.1

a.)

By the Bolzano Weirstrass Theorem, every bounded sequence has a convergent subsequence. Therefore, if a sequence has as a bounded subsequence, it also contains a convergent subsequence.

b.)

We want a sequence with subsequences converging to 0 and 1, that does not contain 0 or 1. Let our sequence $a_n = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, ...\}$.

c.)

We want a sequence that contains subsequences that converge to $\dfrac{1}{n}$ $\forall n \in \mathbb{N}$.

2.5.3

Problem: Show that grouping terms of a convergent series results in a series that stil converges.

Proof. Assume that $\sum_{n=1}^{\infty} a_n$ converges to L. We want to show that any regrouping of terms results in a series that converges to L. Denote the grouping as $\sum_{n=1}^{\infty} a_n = (a_1 + a_2 + a_{n_1}) + (a_{n_1+1} + ... + a_{n_2}) + (a_{n_2+1} + ... + a_{n_3}) + ...$ We write the sequence of partial sums as $s_m = a_1 + a_2 + ... + a_m$. Let p_k be a sequence such that $p_k = s_{n_k}$ where n_k

is derived from our grouping. Since the series converges L, by the definition of series', the sequence of partial sums converges to L. Since p_k is a subsequence of s_m, p_k converges to L, therefore grouping terms does not interfere with convergent series'. $\square$

This proof does not apply to series' that do not converge, as it is reliant on the property that states: any subsequence of a convergent sequence converges.