Systems of ODE's - Homework 2

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Problem 2.7

Consider the 2×2 matrix

$$A = \begin{bmatrix} a & 1 \\ 0 & 1 \end{bmatrix}$$

Find the value a_0 of the parameter a for which A has repeated real eigenvalues. What happens to the eigenvectors of the matrix a approaches a_0?

Generally, we are looking for solutions to the equation $\det(A - \lambda I) = 0$. So we write the following

$$\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & 1 \\ 0 & 1 - \lambda \end{bmatrix} = (a - \lambda)(1 - \lambda)$$

This clearly has solutions at a and 1, which means that it will have repeated eigenvalues exactly when $a = a_0 = 1$. As the variable a approaches 1 the second eigenvector rotates towards the $\langle 1, 0 \rangle$ direction, so that both are parallel.

Problem 3.4a

Consider the harmonic oscillator system

$$X' = \begin{bmatrix} 0 & 1 \\ -b & -k \end{bmatrix} X$$

where $b \geqslant 0, k \geqslant 0$, and the mass $m = 1$. For which values of k, b does this system have complex eigenvalues? Repeated eigenvalues? Real and distinct eigenvalues?

The eigenvalues are determined by the solutions to the following equation.

$$\det \begin{bmatrix} 0 - \lambda & 1 \\ -b & -k - \lambda \end{bmatrix} = (-\lambda \cdot -k - \lambda) - (1 \cdot -b) = \lambda^2 + k\lambda + b = 0$$

By the quadratic formula, this has solutions at,

$$\lambda = \frac{-k \pm \sqrt{k^2 - 4(b)}}{2}$$

There will be complex eigenvalues whenever $k^2 - 4b < 0$. There will be repeated eigenvalues whenever $k^2 - b = 0$, and finally there will be real and distinct eigenvalues when $k^2 - 4b > 0 \iff k > 2\sqrt{b}$.

Problem 3.12

Prove that $\alpha e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta e^{\lambda t} \begin{bmatrix} t \\ 1 \end{bmatrix}$ is the general solution of $X' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} X$.

Proof. Write $X' = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}, X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Then we say that $y'(t) = \lambda y(t) \implies y(t) = \beta e^{\lambda t}$ for some $\beta \in \mathbb{R}$. Now if $\beta = 0$, we say that $y(t) = 0$, and from there we have $x(t) = \alpha e^{\lambda t}$ for some $\alpha \in \mathbb{R}$ However if we have $\beta \neq 0$, then we must solve the system $x'(t) = \lambda x(t) + \beta e^{\lambda t}$.

$$x'(t) - \lambda x(t) = \beta e^{\lambda t}$$
$$C\lambda e^{\lambda t} - C\lambda e^{\lambda t} = \beta e^{\lambda t}$$

Using the undetermined coefficients method we get the result $x'(t) = \alpha e^{\lambda t} + Cte^{\lambda t}$. From there we have the final result $X = \alpha e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta e^{\lambda t} \begin{bmatrix} t \\ 1 \end{bmatrix}$ $\square$