Real Analysis - Final Exam

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Problem 1

Let $(M,d),(N,\rho)$ be metric spaces, and let $f : M \to N$ be continuous and onto. If (M,d) is seperable then (N,ρ) is seperable.

Proof. Suppose (M,d) is seperable. Then there exists some countable dense subset of M, $\{x_n\}_{n\in\mathbb{N}}$. Then for any non-empty open set in M, we have natural number n such that x_n belongs to this open set. We claim that $f(\{x_n\}_{n\in\mathbb{N}})$ is a countable dense subset of N. Being the image of a countable set, it is obviously countable. Suppose that this set is not dense. Then there exists some non-empty open set $V \subset N$ such that it is disjoint with $f(\{x_n\}_{n\in\mathbb{N}})$. Note that since f is continuous and onto, the pre-image of V, $U = f^{-1}(V)$ is a non-empty open set in M. Then, since f is onto we say $A \subset f^{-1}(f(A))$, and we have

$$\begin{aligned}
\{x_n\}_{n\in\mathbb{N}} \cap U &\subset f^{-1}(f(\{x_n\}_{n\in\mathbb{N}})) \cap f^{-1}(V) \\
&= f^{-1}(f(\{x_n\}_{n\in\mathbb{N}}) \cap V) \\
&= f^{-1}(\emptyset) \\
&= \emptyset
\end{aligned}$$

However, this means that $\{x_n\}_{n\in\mathbb{N}}$ is not dense in M (contradiction). Therefore it must be the case that $f(\{x_n\}_{n\in\mathbb{N}})$ is dense in N. Thus (N,ρ) is seperable. □

Problem 2

Let $g : [0,1] \to \mathbb{R}$ be continuous. There exists a unique continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) + \int_0^x f(t)\sin(\pi t/4)dt = g(x) \quad \forall x \in [0,1]$$

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be a function in $C^{2\pi}$ that equals $g(0)$ on $[-\pi, 0)$, $g(1)$ on $(1,\pi]$ and equals $g(x)$ on $[0,1]$. Then we say that for every ϵ there exists a trigonometric polynomial such that $||T(x) - g(x)||_\infty < \epsilon$ (Weierstrass's Second Theorem). Perhaps it is possible to find some trigonometric polynomial that is orthoganal to $\sin(\pi x/4)$, that is, $\int_{-\pi}^{\pi} f(t)\sin(\pi t/4)dt = 0$. Then it may be possible to some limit of such polynomials to provide some function that is orthogonal to $\sin(t\pi/4)$ and will have the property of

$$f_n(x) + \int_0^x f_n(t)\sin(\pi t/4)dt = f_n(x) \to g(x)$$

I do not know how to prove this. □

Problem 3

Let $\mathcal{F} \subset C[0,1]$ where $\mathcal{F} = \{p \in \mathcal{P} : \max_{x\in[0,1]} |p(x)| \leqslant 2\}$. The closed unit ball $B = \{f \in C[0,1] : ||f||_\infty \leqslant 1\}$ is contained in the closure of $\mathcal{F}$.

Proof. Let $f \in B$ be arbitrary. Then $||f||_\infty \leqslant 1$ by assumption. By the Weierstrass Approximation Theorem, for every $\epsilon > 0$ there exists some polynomial p such that $||f - p||_\infty < \epsilon$. Let $\epsilon < 1$, there will exist some polynomial p such that $||f-p||_\infty < \epsilon$. In other words

$$\max_{x\in[0,1]} |f(x) - p(x)| < \epsilon$$

Since f is bounded by 1, it follows that p must be bounded by $1 + \epsilon < 2$. Thus for every $\epsilon < 1$ we say that $p \in \mathcal{F}$, thus as ϵ approaches 0 we can take a sequence of polynomials in $\mathcal{F}$ and they will converge to f. Finally, f is a limit point of $\mathcal{F}$ for any $f \in B$, and thus $B \subset \overline{\mathcal{F}}$. □

Problem 4

We define the colleciton of functions $\mathcal{F} \subset C[0,1]$ as

$$\mathcal{F} = \{\sin(nx) \mid n \in \mathbb{N}\}$$

(a)

$\boxed{\mathcal{F} \text{ is uniformly bounded.}}$

Proof. We know that $|\sin(x)|$ is bounded by 1 on all of $\mathbb{R}$. Then for any natural number n, $nx \in \mathbb{R}$. So it follows that $|\sin(nx)| \leqslant 1$ for every natural number n, for every $x \in [0,1]$. Thus we say that this collection of functions is uniformly bounded ($||\sin(nx)||_\infty \leqslant 1 \ \forall n \in \mathbb{N}$). □

(b)

$\boxed{\mathcal{F} \text{ is not equicontinuous.}}$

Proof. Let $x = 0$, and let $\epsilon = \frac{1}{2}$. Then choose any stricly positive δ. By the Archimedean Property there exists some natural number n such that $\frac{2\pi}{n} < \delta$. It follows then that since $\sin(x)$ is 2π periodic that $\sin(nx)$ is $\frac{2\pi}{n}$ periodic. Since $\sin(x)$ achieves its maximum 1 within this period then there must exist some $x \in [0, \frac{2\pi}{n}]$ such that $\sin(nx) = 1$. Then since $\sin(n(0)) = 0$ and there exists $y \in [0, \frac{2\pi}{n}] \subset [0, \delta)$ such that $\sin(ny) = 1$, we have $|\sin(nx) - \sin(ny)| = |0 - 1| > \frac{1}{2} = \epsilon$. If we choose $\epsilon = \frac{1}{2}$, then for every $\delta > 0$ there is some $n \in \mathbb{N}$ such that $\sin(nx)$ is not uniformly continuous by this δ, and we say that the collection $\mathcal{F}$ is not equicontinuous. □

(c)

$\boxed{\mathcal{F} \text{ is not compact in } C[0,1].}$

Proof. Observe the sequence $(\sin(x), \sin(2x), \sin(3x), \sin(4x), \cdots)$ we claim that there is no Cauchy subsequence, and that therefore the collection is not compact. Choose any $m \in \mathbb{N}$ arbitrarily. Then we say that $\sin(mx)$ is non-negative on $[0, \frac{\pi}{m}]$. Then choose any $n \geqslant 2m$, and we say that $\sin(nx)$ will be equal to -1 exactly at $x = \frac{3\pi}{2n} = \frac{3\pi}{4m} \in [0, \frac{\pi}{m}]$. It follows that since we have a non-negative function and a function that achieves -1 on this interval that

$$||\sin(mx) - \sin(nx)||_\infty \geqslant 1$$

Since any subsequence that is not eventually constant must contain some $n \geqslant 2m$ for any $m \in \mathbb{N}$ arbitrarily, there does not exist any Cauchy subsequence, thus $\mathcal{F}$ is not totally bounded, and is not compact. □

Problem 5

Let $a_n = \frac{1}{2^n}$ and consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ where

$$f_n(x) = \begin{cases} \frac{1}{a_{n+2}^2}(x - a_{n+1})(a_n - x) & x \in [a_{n+1}, a_n] \\ 0 & \text{otherwise} \end{cases}$$

(a)

$\boxed{\forall n \in \mathbb{N} \ \ \max_{x \in [0,1]} |f_n(x)| = 1}$

Proof. We can first restrict our domain to $[a_{n+1}, a_n]$ since any non-zero value will immediately have an absolute value greater than 0. Since $f_n(x)$ is a product of 3 positive terms in this domain, we say that $|f(x)| = f(x)$. Then by the given hint, we say that

$$\max_{x \in [a_{n+1}, a_n]} |f_n(x)| = (a_{n+2})^{-2} \left(\frac{a_{n+1} - a_n}{2}\right)^2 = (a_{n+2})^{-2}(a_1)^2(a_{n+1} - a_n)^2$$

Then we can write this out using the definition of a_n, giving us

$$\begin{aligned}\left(\frac{1}{2^{n+2}}\right)^{-2}\left(\frac{1}{2^1}\right)^2\left(\left(\frac{1}{2^{n+2}}\right)^2-2\left(\frac{1}{2^{n+1}}\right)\left(\frac{1}{2^n}\right)+\left(\frac{1}{2^n}\right)^2\right) &= \left(\frac{1}{2^{n+2}}\right)^{-2}\left(\frac{1}{2^1}\right)^2\left(\left(\frac{1}{2^{n+1}}\right)^2-\left(\frac{1}{2^n}\right)\left(\frac{1}{2^n}\right)+\left(\frac{1}{2^n}\right)^2\right)\\ &= \left(\frac{1}{2^{n+2}}\right)^{-2}\left(\frac{1}{2^1}\right)^2\left(\left(\frac{1}{2^{n+1}}\right)^2-\left(\frac{1}{2^n}\right)^2+\left(\frac{1}{2^n}\right)^2\right)\\ &= \left(\frac{1}{2^{n+2}}\right)^{-2}\left(\frac{1}{2^1}\right)^2\left(\frac{1}{2^{n+1}}\right)^2\\ &= \left(\frac{1}{2^{n+2}}\right)^{-2}\left(\frac{1}{2^1}\frac{1}{2^{n+1}}\right)^2\\ &= \left(\frac{1}{2^{n+2}}\right)^{-2}\left(\frac{1}{2^{n+2}}\right)^2\\ &= 1\end{aligned}$$

Thus for any $n \in \mathbb{N}, \max_{x\in[0,1]} |f_n(x)| = 1$. □

(b)

$\boxed{\text{The pointwise limit of } f_n(x) \text{ is 0 for all } x.}$

Proof. If $x = 0$, then for every $n \in \mathbb{N}, 0 < \frac{1}{2^{n+1}}$, so we say $f_n(x) = 0$ for all n, thus the constant sequence $(f_n(0)) = (0) \to 0$. Now let $x \in (0, 1]$ be fixed. Then by the convergence of the geometric series, we know that $\exists n \in \mathbb{N}$ such that $\frac{1}{2^n} < x$. Thus this sequence is also eventually the constant sequence (0) which converges to 0. Therefore at any point $x \in [0, 1]$ the point-wise limit is 0. □

(c)

I invoke my proof of the following property in order to answer the question:

$\boxed{\text{A sequence of real valued functions } f_n : X \to \mathbb{R} \text{ is uniformly continuous if and only if it is uniformly Cauchy.}}$

Proof. We must show the bi-conditional by showing that the implication holds in both directions.

$\boxed{\Rightarrow}$ Assume that f_n is uniformly convergent. Then $||f_n - f||_\infty \to 0$. Equivalently, we say that $\sup_{x\in X} |f_n(x) - f(x)| \to 0$. Thus we say that for every $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geqslant N \sup_{x\in X} |f_n(x) - f(x)| < \epsilon$.

Choose some $\epsilon' > 0$ arbitrarily. Then $\exists N_{\epsilon'/2} \in \mathbb{N}$ such that $\forall n \geqslant N_{\epsilon'/2},\ |||f_n - f||_\infty < \epsilon'/2$. Then choose $m, n \geqslant N_{\epsilon'/2}$ and it follows that

$$\sup_{x\in X} |f_n(x) - f_m(x)| = \sup_{x\in X} |f_n(x) - f(x) + f(x) - f_m(x)| \leqslant \sup_{x\in X} |f_n(x) - f(x)| + \sup_{x\in X} |f_m(x) - f(x)| \leqslant 2\epsilon'/2 = \epsilon'$$

$\boxed{\Leftarrow}$ Assume that f_n is uniformly Cauchy. Then it must be the case that f_n is pointwise Cauchy, and therefore pointwise convergent. Thus $f_n \to f$ pointwise. Suppose that this convergence is not uniform. Then $\exists\epsilon > 0$ such that $||f_n - f||_\infty \geqslant \epsilon\ \ \forall n$. Choose some $\epsilon > \delta > 0$ arbitrarily. Then $\exists x \in X$ such that $|f_n(x) - f(x)| > \epsilon - \delta > 0 \forall n$. Therefore f_n is not pointwise convergent at some x (contradiction). Finally f_n must be uniformly convergent. □

$\boxed{f_n \text{ does not converge uniformly.}}$

Proof. Firstly $f_n \in C[0, 1]$ for every $n \in \mathbb{N}$. This is the case because the piece on (a_{n+1}, a_n) can be expressed as a finite polynomial which is of course continuous. Then outside of this interval we have the continuous constant function 0. Finally at $x = a_{n+1}$ and $x = a_n$ both functions have a limit point at these values of x and the value of their limits is the same, that is

$$\lim_{x\to a_n} \frac{1}{a_{n+2}^2}(x - a_{n+1})(a_n - x) = 0, \qquad \lim_{x\to a_n} 0 = 0$$

And similarly,

$$\lim_{x\to a_{n+1}} \frac{1}{a_{n+2}^2}(x - a_{n+1})(a_n - x) = 0, \qquad \lim_{x\to a_{n+1}} 0 = 0$$

Thus $f_n \in C[0, 1]$.

Having established this, suppose it does converge uniformly, then it must be uniformly Cauchy in $C[0,1]$ (by Uniform Convergence Def. and Uniform Convergence Uniform Cauchy Equivalence). However, choose any distinct $m, n \in \mathbb{N}$, and it is guaranteed that $||f_m - f_n||_\infty \geqslant 1$. This is the case because we know that f_n achieves its absolute maximum at the midpoint of a_n and a_{n+1}. Since the sequence (a_n) is monotone decreasing it must be the case that this midpoint does not lie in $[a_{m+1}, a_m]$ (that is, these intervals must overlap only at endpoints, so no midpoint will lie in two). At the point $x = \frac{a_{n+1}+a_n}{2}$, $|f_n(x) - f_m(x)| = 1$ so it follows that $||f_n - f_m||_\infty \geqslant 1$. Therefore the sequence is not uniformly Cauchy, and thus not uniformly continuous. $\square$

(d)

$\boxed{\text{Let } g_n = f_{2n}\text{. For any } m, n \in \mathbb{N},\ ||g_m - g_n||_\infty = 1.}$

Proof. We have already demonstrated most of the steps in order to prove that this is true. Note that with our new series, we choose only every even $n \in \mathbb{N}$. This means that the intervals $[a_{n+1}, a_n]$ and $[a_{m+1}, a_m]$ will always be disjoint for any distinct natural numbers m, n, as they can never share an endpoint now. Then it follows that

$$||g_m - g_n||_\infty = \max\left\{\max_{x\in[a_{n+1},a_n]} |g_m(x) - g_n(x)|, \max_{x\in[a_{m+1},a_m]} |g_m(x) - g_n(x)|, \max_{x\notin[a_{n+1},a_n]\cup[a_{m+1},a_m]} |g_m(x) - g_n(x)|\right\}$$

This is equal to $\max\{1,1,0\}$ since the intervals are disjoint, and $g_n(x)$ achieves its maximum of 1 within the its interval $[a_{n+1}, a_n]$, while g_n will be the constant function 0 there. Therefore $||g_m - g_n||_\infty = 1$. $\square$

(e)

$\boxed{\text{The set } \mathcal{F} = \{f_n \mid n \in \mathbb{N}\} \text{ is not totally bounded in } C[0,1]}$

Proof. Choose the sequence $(f_1, f_2, f_3, f_4, \cdots)$. As we established in part $\boxed{\text{(c)}}$ for any two natural numbers n, m $||f_n - f_m||_\infty \geqslant 1$. This means it is impossible to take any non-constant Cauchy subsequence of our sequence $(f_1, f_2, \cdots)$. Since a set is totally bounded if and only if every sequence yields some Cauchy sub-sequence, it must be the case that $\mathcal{F}$ is not totally bounded. $\square$