Real Analysis - Assignment 8

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December 8, 2020

A sequence of real valued functions $f_n : X \to \mathbb{R}$ is uniformly continuous if and only if it is uniformly Cauchy.

Proof. We must show the bi-conditional by showing that the implication holds in both directions.

$\boxed{\Rightarrow}$ Assume that f_n is uniformly convergent. Then $||f_n - f||_\infty \to 0$. Equivalently, we say that $\sup_{x\in X} |f_n(x) - f(x)| \to 0$. Thus we say that for every $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geqslant N \sup_{x\in X} |f_n(x) - f(x)| < \epsilon$.

Choose some $\epsilon' > 0$ arbitrarily. Then $\exists N_{\epsilon'/2} \in \mathbb{N}$ such that $\forall n \geqslant N_{\epsilon'/2},\ |||f_n - f||_\infty < \epsilon'/2$. Then choose $m, n \geqslant N_{\epsilon'/2}$ and it follows that

$$\sup_{x\in X} |f_n(x) - f_m(x)| = \sup_{x\in X} |f_n(x) - f(x) + f(x) - f_m(x)| \leqslant \sup_{x\in X} |f_n(x) - f(x)| + \sup_{x\in X} |f_m(x) - f(x)| \leqslant 2\epsilon'/2 = \epsilon'$$

$\boxed{\Leftarrow}$ Assume that f_n is uniformly Cauchy. Then it must be the case that f_n is pointwise Cauchy, and therefore pointwise convergent. Thus $f_n \to f$ pointwise. Suppose that this convergence is not uniform. Then $\exists\epsilon > 0$ such that $||f_n - f||_\infty \geqslant \epsilon\ \ \forall n$. Choose some $\epsilon > \delta > 0$ arbitrarily. Then $\exists x \in X$ such that $|f_n(x) - f(x)| > \epsilon - \delta > 0 \forall n$. Therefore f_n is not pointwise convergent at some x (contradiction). Finally f_n must be uniformly convergent. □

Let $(X, d), (Y, \rho)$ be metric spaces. Let $f, f_n : X \to Y$ and let f_n converge uniformly to f on X. Show that $D(f) \subset \bigcup_{n=1}^\infty D(f_n)$.

Proof. Let $x \in X$ such that $x \notin D(f_n)$ for every natural number n. If x is some isolated point, then $f(x)$ must be continuous trivially. Otherwise, we say that $x_n \to x \implies f_k(x_n) \to f_k(x)$ for every natural number k. Choose $\frac{\epsilon}{3} > 0$ to be arbitrary. Then $\exists N_1$ such that $\forall n \geqslant N_1$, $\rho(f_k(x_n), f_k(x)) < \frac{\epsilon}{3}$ since f_k is continuous at x. Since $f_k \to f$ uniformly, $\exists N_2$ such that $\forall k \geqslant N_2$, $\rho(f_k(x), f(x)) < \frac{\epsilon}{3}$ for every point $x \in X$. Choose $N = \max\{N_1, N_2\}$. Then it follows that for $k, n \geqslant N$

$$\rho(f(x_n), f(x)) \leqslant \rho(f(x_n), f_k(x_n)) + \rho(f_k(x_n), f(x)) \leqslant \rho(f(x_n), f_k(x_n)) + \rho(f_k(x_n), f_k(x)) + \rho(f_k(x), f(x)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Therefore $x_n \to x \implies f(x_n) \to f(x)$ so f is continuous at x. Thus $x \notin \bigcup_{\mathbb{N}} D(f_n)$ implies $x \notin D(f)$. □

Let $f, f_n \in C[0, 1]$ where f_n converges uniformly to f. Then $\int_0^{1-1/n} f_n \to \int_0^1 f$.

Proof.

$$\left| \int_0^{1-1/n} f_n - \int_0^1 f \right| = \left| \int_0^1 f_n - f - \int_{1-1/n}^1 f_n \right| \leqslant$$

Yikes idk how to do this □