Real Analysis - Assignmennt 6

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Problem 6.9

If $A \subset B \subset \overline{A} \subset M$, and if A is connected, show that B is connected. In particular, $\overline{A}$ is connected.

Proof. Suppose that $A \subset B \subset \overline{A} \subset M$, and that A is connected. We want to show that there does not exist some disjoint union of open sets such that $U \dot{\cup} V = \overline{A}$ where U and V are non-empty. By contradiction, suppose that such a disconnection does exist. Then it follows that $A \subset U \dot{\cup} V = \overline{A}$. Since A is connected, it must be the case that either $A \cap U = \emptyset$ or $A \cap V = \emptyset$. Otherwise, A yields the disconnection $A = (A \cap U) \dot{\cup} (A \cap V)$. Without loss of generality, choose U and V such that $A \cap U$ is nonempty and $A \cap V$ is empty.

Choose $x \in V \subset \overline{A}$. Since V is open, there exists some ϵ-neighborhood $B_\epsilon(x) \subset V \subset \overline{A}$. However, since x has some ϵ-neighborhood contained in $\overline{A}$, x must lie in the interior of A. Thus, V intersects A, and a contradiction arises. Therefore $\overline{A}$ must be connected and so too must B be connected $\square$

Problem 7.19

Define $c_0 \subset \ell_\infty$ by the set of all sequences converging to 0. Prove that c_0 is complete by showing that c_0 is closed in l_∞.

Proof. Let $f \in \ell_\infty$ be a limit point of c_0. Then let (f_n) be a squence in c_0 converging to $f \in \ell_\infty$. Let $\frac{\epsilon}{2} > 0$ be arbitrary. Then we say

$$|f(k)| \leq |f(k) - f_n(k)| + |f_n(k)|$$

As n grows large, the right hand side is eventually smaller than $\frac{\epsilon}{2} + |fn(k)|$, and as k grows large each $|f_n(k)|$ is eventually less than $\frac{\epsilon}{2}$. At this point we say

$$|f(k)| \leq |f(k) - f_n(k)| + |f_n(k)| < \epsilon$$

Thus $f(k) \to 0$, and we say $f \in c_0$. Since an arbitrary limit point f belongs to c_0, we say that c_0 contains its limit points, and is therefore closed in ℓ_∞. Because this is this case, c_0 is complete. $\square$

Problem 7.35

Prove that a normed vector space is complete if and only if its closed unit ball is complete.

Proof. Let X be a normed vector space. We must show that the bidirectional implication holds in both directions.

$\boxed{\Rightarrow}$ First, assume that X is complete. Then we wish to show that $B = \{x \in X \ : \ ||x|| \leq 1\}$ is complete. We will show that B is closed in X, and therefore is complete. Let $x \in B^c$. We know that $||x|| > 1$, and for any $y \in B, ||y|| \leq 1$. By the reverse triangle inequality, we know that $||x - y|| \geq ||x|| - ||y|| \geq ||x|| - 1 > 0$. Let $\epsilon = \frac{||x||-1}{2}$, and let $y \in B$. For every $y' \in B_\epsilon(x), ||y'|| < \epsilon$.

$$||y' - y|| \geq ||y' - x|| - ||y - x|| > \epsilon - 0 > 0$$

Since $x \in B^c$ has some ϵ-neighborhood contained in B^c, B is closed. Since B is closed in a complete space, the space (B, d) is also complete.

$\boxed{\Leftarrow}$ Now assume that $B = \{x \in X \ : \ ||x|| \leq 1\}$ is a complete subspace of X. We wish to show that the space X is complete. Let (x_n) be a Cauchy sequence in M. Let $\epsilon = \frac{1}{2}$. Then $\exists N \in \mathbb{N}$ such that $\forall n > N, ||x_N - x_n|| < \frac{1}{2}$. Let y_n be a series defined by $y_n = x_n - x_N$. Then $y_N = 0$ and we say that $B_\epsilon(y_N) = B_{\frac{1}{2}}(0) \subset B$. Since x_n is Cauchy, so too is y_n. Then since the limit of x_n must be a limit point of $B_\epsilon(x_N)$, the limit $y = \lim y_n$ must be a limit point of $B_\epsilon(0)$ and therefore must lie in B and be some point $y \in M$. Then since we have closure under vector addition, it follows that $y + x_N \in M$, and that this the limit $x = \lim x_n = y + x_N = \lim y_n + x_N$. $\square$

Problem 7.41

Let M be complete and let $f : M \to M$ be continuous. If f^k is a strict contraction for some integer $k > 1$, show that f has a unique fixed point.

Proof. Since f^k is a strict contraction on a complete space it admits some unique fixed point x. Since $f^k(x) = x$, we can write

$$\begin{aligned} f^k(f(x)) &= (f^k \circ f)(x) \\ &= (f \circ f \circ \cdots \circ f)(x) \\ &= (f \circ f^k)(x) \\ &= f(f^k(x)) \\ &= f(x) \end{aligned}$$

Since $f^k(f(x)) = f(x)$, we say that $f(x)$ is a fixed point for f^k. Then since our fixed point must be unique, we say that it must be the case that $f(x) = x$. Thus x is a fixed point under f. To show that this point x is unique, let $x' \in M$ such that x' is a fixed point under f. Then, $f(x') = x'$. However, this means

$$\begin{aligned} f^k(x') &= f^{k-1}(f(x')) = f^{k-1}(x') \\ &= f^{k-2}(f(x')) = f^{k-2}(x') \\ &\vdots \\ &= f(f(x')) = f(x') = x' \end{aligned}$$

Thus x' is a fixed point under f^k, and since this function has exactly one unique fixed point x, it follows that $x' = x$. Therefore f has a unique fixed point x. $\square$