General Relativity - Homework 7

Philip Warton

May 28, 2021

1 Traces

We assume the following three equations:

$$\vec{G} = G^i{}_j \sigma^j \hat{e}_i$$
$$\vec{R} = R^i{}_j \sigma^j \hat{e}_i$$
$$G^i{}_j = R^i{}_j - \frac{1}{2}\delta^i{}_j R$$

Then to compute the $trace(G)$ we write

$$\begin{aligned} trace(G) &= \sum_{i \in I} G^i{}_i \\ &= \sum_{i \in I} R^i{}_i - \frac{1}{2}\delta^i{}_i R \\ &= \sum_{i \in I} R^i{}_i - \frac{1}{2}\sum_{i \in I} \delta^i{}_i R \\ &= R - \frac{|I|R}{2} \end{aligned}$$

This is the case because δ^a_b is the Kroenecker delta we say that $\delta^i{}_i = 0 \;\; \forall i \in I$. Let $n = |I|$ be our number of dimensions and we get

$$trace(G) = R\left(1 - \frac{n}{2}\right)$$

2 Robertson-Walker Geometry

Firstly, the connection forms are written as

$$\begin{aligned} \Omega^t{}_r &= \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^r \\ \Omega^t{}_\theta &= \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\theta \\ \Omega^t{}_\phi &= \frac{\ddot{a}}{a}\sigma^t \wedge \sigma^\phi \\ \Omega^r{}_\theta &= \frac{\dot{a}^2 + k}{a^2}\sigma^r \wedge \sigma^\theta \\ \Omega^r{}_\phi &= \frac{\dot{a}^2 + k}{a^2}\sigma^r \wedge \sigma^\phi \\ \Omega^\theta{}_\phi &= \frac{\dot{a}^2 + k}{a^2}\sigma^\theta \wedge \sigma^\phi \end{aligned}$$

Then we use the following relationship to compute our Riemannian curvature tensors

$$\Omega^i{}_j = \frac{1}{2}R^i{}_{jkl}\sigma^k \wedge \sigma^l$$

So we can write

$$\Omega^t{}_r = \frac{1}{2}\left(\sum_{k\in I}\sum_{l\in I} R^t{}_{rkl}\sigma^k \wedge \sigma^l\right)$$

However, since we know that we only have $\sigma^t \wedge \sigma^r$ on the left hand side we can eliminate any terms that do not have $\sigma^t \wedge \sigma^r$ on the right hand side. Thus

$$\begin{aligned}
\Omega^t{}_r &= \frac{1}{2}\left(\sum_{k\in I}\sum_{l\in I} R^t{}_{rkl}\sigma^k \wedge \sigma^l\right)\\
\frac{\ddot{a}}{a}\sigma^t \wedge \sigma^r &= \frac{1}{2}\left(R^t{}_{rtr}\sigma^t \wedge \sigma^r + R^t{}_{rrt}\sigma^r \wedge \sigma^t\right)\\
&= \frac{1}{2}\left(R^t{}_{rtr}\sigma^t \wedge \sigma^r - R^t{}_{rrt}\sigma^t \wedge \sigma^r\right)\\
&= \frac{1}{2}\left(R^t{}_{rtr}\sigma^t \wedge \sigma^r + R^t{}_{rtr}\sigma^t \wedge \sigma^r\right)\\
&= \frac{1}{2}\left(2R^t{}_{rtr}\sigma^t \wedge \sigma^r\right)\\
&= R^t{}_{rtr}\sigma^t \wedge \sigma^r
\end{aligned}$$

So thus it follows that $R^t{}_{rtr} = \frac{\ddot{a}}{a}$. Also notice that this implies that all other Riemann tensors must be equal to 0 by this argument. Similarly,

$$\begin{aligned}
R^t{}_{rtr} = R^t{}_{\theta t\theta} = R^t{}_{\phi t\phi} &= \frac{\ddot{a}}{a}\\
R^r{}_{\theta r\theta} = R^r{}_{\phi r\phi} = R^\theta{}_{\phi\theta\phi} &= \frac{\dot{a}+k}{a^2}
\end{aligned}$$

So then we can write our diagonal Ricci curvature in terms of Riemannian curvature,

$$R_{ii} = \sum_{k\in I} R^k{}_{iki}$$

So we state that

$$\begin{aligned}
R_{tt} &= \sum_{k\in I} R^k{}_{tkt}\\
&= \sum_{k\in I} -R^k{}_{ttk}\\
&= \sum_{k\in I} -R^t{}_{ktk}\\
&= -R^t{}_{ttt} - R^t{}_{rtr} - R^t{}_{\theta t\theta} - R^t{}_{\phi t\phi}\\
&= 0 - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a}\\
&= \frac{-3\ddot{a}}{a}
\end{aligned}$$

For our space-like variables, x, we write

$$\begin{aligned}
R_{xx} &= \sum_{k\in I} R^k{}_{xkx}\\
&= R^t{}_{xtx} + \sum_{k\in I\setminus\{t\}} R^k{}_{xkx}\\
&= \frac{\ddot{a}}{a} + \sum_{k\in I\setminus\{t,x\}} R^k{}_{xkx}\\
&= \frac{\ddot{a}}{a} + \sum_{k\in I\setminus\{t,x\}} R^x{}_{kxk}
\end{aligned}$$

So by the Riemann curvatures computed earlier we get

$$R_{rr} = R_{\theta\theta} = R_{\phi\phi} = \frac{\ddot{a}}{a} + \frac{2(\dot{a}+k)}{a^2}$$

Computing the Ricci scalar, the sum of these diagonal components is

$$\begin{aligned}
R &= -R_{tt} + R_{rr} + R_{\theta\theta} + R_{\phi\phi} \\
&= \frac{3\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + \frac{2(\dot{a}+k)}{a^2}\right) \\
&= \frac{6\ddot{a}}{a} + \frac{6(\dot{a}+k)}{a^2} \\
&= \frac{6\ddot{a}a + 6\dot{a} + 6k}{a^2}
\end{aligned}$$

Then we know simply that for $x \neq t$,

$$R^t{}_t = -R_{tt} = \frac{3\ddot{a}}{a} \qquad R^x{}_x = R_{xx} = \frac{\ddot{a}}{a} + \frac{2(\dot{a}+k)}{a^2}$$

To compute the diagonal components $G^i{}_i$ we simply use the Ricci terms that we have already have computed,

$$\begin{aligned}
G^t{}_t &= R^t{}_t - \frac{1}{2}\delta^t{}_t R \\
&= \frac{3\ddot{a}}{a} - \frac{6\ddot{a}a + 6\dot{a} + 6k}{2a^2} \\
&= \frac{-6\dot{a} - 6k}{2a^2} \\
&= \frac{-3(\dot{a}+k)}{a^2} \\
G^x{}_x &= R^x{}_x - \frac{1}{2}\delta^x{}_x R \\
&= \frac{\ddot{a}}{a} + \frac{2(\dot{a}+k)}{a^2} - \frac{6\ddot{a}a + 6\dot{a} + 6k}{2a^2} \\
&= \frac{2\ddot{a}a}{2a^2} + \frac{4(\dot{a}+k)}{2a^2} - \frac{6\ddot{a}a + 6\dot{a} + 6k}{2a^2} \\
&= \frac{2\ddot{a}a}{2a^2} + \frac{4\dot{a} + 4k}{2a^2} - \frac{6\ddot{a}a + 6\dot{a} + 6k}{2a^2} \\
&= \frac{-4\ddot{a}a - 2\dot{a} - 2k}{2a^2} \\
&= -\frac{2\ddot{a}a + \dot{a} + k}{2a^2}
\end{aligned}$$

Where $x = r, \theta, \phi$. Since the Kroenecker delta is only non-zero on the diagonal, and also our Ricci components are only non-zero as computed, we say that no other $G^i{}_j$ is non-trivial.