Equilibria in Non-linear Systems - Notes

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We want to look at systems of ODE's that are non-linear, but that resemble linearized ones near equilibria. Consider the syste:

$$\begin{aligned} x' &= x + y^2 \\ y' &= -y \end{aligned}$$

Note that when y is small, y^2 is much smaller. So it follows that as we make y very small the system $x' = x + y^2$ will vaguely 'converge' or resemble $x' = x$. So instead we look at the system

$$\begin{aligned} x' &= x \\ y' &= -y \end{aligned}$$

Then we know that the bottom equation is solved by

$$y(t) = y_0 e^{-t}$$

So we then plug this in to our non-linear system, giving us

$$x' = x + y_0^2 e^{-2t}$$

Recall from good old diffEQ's that we can plug in a "guess" solution of ce^{-2t} and yield a particular solution to this first order nonautonomous equation. Namely,

$$x(t) = -\frac{1}{3} y_0^2 e^{-2t}$$

So we get a general solution of

$$\begin{aligned} x(t) &= \left(x_0 + \frac{1}{3} y_0^2\right) e^t - \frac{1}{3} y_0^2 e^{-2t} \\ y(t) &= y_0 e^{-t} \end{aligned}$$

$\boxed{\text{Polar Coordinates Example}}$ Take the following system,

$$\begin{aligned} x' &= \frac{1}{2}x - y - \frac{1}{2}(x^3 + y^2 x) \\ y' &= x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2 y) \end{aligned}$$

Then if you simply drop the non-linear terms you get the system,

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

This clearly gives us a spiral source set of solutions, which can be checked by taking the eigenvalues. But now we want to solve the non-linear system. Oh shit.