Algebraic Topology - Homework 2

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1.3

Let $R : S^1 \to S^1$ rotate any given point by x radians, then $R \simeq 1_{S^1}$.

Proof. We define the following function $F : S^1 \times X \to S^1$ and claim that it is a homotopy:

$$F(p,t) = p \cdot e^{i(tx)}$$

Firstly, we can trivially verify that

$$F(p,0) = p \cdot e^0 = p \cdot 1 = p = 1_{S^1}(p)$$
$$F(p,1) = p \cdot e^{ix} = R(p)$$

Since this function simply rotates the point over to x radians as we vary t, it follows that it is continuous and thus a homotopy between R and 1_{S^1}. □

Every continuous map $f : S^1 \to S^1$ is homotopic to a continuous map $g : S^1 \to S^1$ with $g(1) = 1$.

Proof. Let $f : S^1 \to S^1$ be continuous. Then $f(1) \in S^1$ with some corrosponding argument/angle $x \in [0, 2\pi)$. Let $R_\alpha : S^1 \to S^1$ denote the function given earlier as R with the rotation being given in radians by α. Then,

$$(R_{-x} \circ f)(1) = R_{-x}(f(1)) = R_{-x}(e^{ix}) = e^{i(0)} = 1$$

Let $g = R_{-x} \circ f$ and it follows that since $R \simeq 1_{S^1}$,

$$R_{-x} \circ f \simeq 1_{S^1} \circ f$$
$$g \simeq f$$

where $g(1) = 1$. □

1.5

Let $X = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ and let Y be a countable discrete space. Then X and Y do not have the same homotopy type.

Proof. Let $f : X \to Y$ be some continuous function. Then there exists some $y \in Y$ such that $0 \in f^{-1}(y)$ (since 0 must of course get mapped to some point in Y). Since Y is a discrete space it follows that $\{y\}$ is an open set. Since f is continuous, $f^{-1}(\{y\})$ is an open set in X containing 0. Assuming that X is equipped with the subspace topology from $\mathbb{R}$ it follows that for any open neighborhood U of 0 the followsing is true:

There exists some $N \in \mathbb{N}$ such that for every $n \geq N$, $\frac{1}{n} \in U$.

By this, it follows that $f^{-1}(\{y\})$ contains infinitely many points from X, and so it must be the case that only finitely many points in X are mapped to points other than y. Let $g : Y \to X$ be continuous. Then we conclude that since $f(X)$ is a finite set, so too is $(g \circ f)(X)$. Somehow XD we conclude that $1_X \not\simeq g \circ f$ for any f, g arbitrarily, and thus the two spaces have two different homotopy types. □

1.7

Let $X = \{x, y\}$ with topology $\{X, \emptyset, \{x\}\}$, then X is contractible.

Proof. Define a function $F : X \times [0,1] \to X$ by

$$F(p,t) = \begin{cases} p, & \text{when } t \leq \frac{1}{2} \\ x, & \text{when } t > \frac{1}{2} \end{cases}$$

We can verify immediately that

$$F(p,0) = p = 1_X(p)$$
$$F(p,1) = x = e_x(p) \qquad \text{(where } e_x \text{ is the constant map to the point } x\text{)}$$

Let us verify that each pre-image of a neighborhood in X is open in X. Firstly $F^{-1}(X) = X \times [0,1]$ since the function is well defined and surjective. Then we know that $F^{-1}(\{x\}) = (\{x\} \times [0,1]) \cup (\{y\} \times (\frac{1}{2}, 1])$ which is open in $X \times [0,1]$. And finally $F^{-1}(\emptyset) = \emptyset$ since the function is well defined. So we conlcude that F is a homotopy and therefore $1_X \simeq e_x$ and therefore X is contractible. □

1.8

There exists a continuous image of a contractible space that is not contractible.

Proof. Let $f : [0,1] \to S^1$ be given by $f(x) = e^{i(2\pi)x}$. The space $[0,1]$ is contractible trivially, and we claim that $f([0,1]) = S^1$ and is therefore not contractible. Let O be an open set in S^1. Then it is a union of some open intervals along the circle. The pre-image of each interval that does not include 1 will be of the form (a, b) which is clearly open. Otherwise it will be of the form $[0, a) \cup (b, 1]$ and will remain open. Thus f is continuous. Let $y \in S^1$ and then it can be written as $y = e^{it}$ where $t \in [0, 2\pi)$. Then it follows that it has some pre-image under f so the function is surjective and $f([0,1]) = S^1$. We assume without proof that S^1 is contractible. □

A retract of a contractible space is contractible

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