Math 343 Final Exam - Spring 2020

Date: June 11, 2020 **Time:** 6.00pm **Venue:** Virtual (Zoom)

First Name	Last Name	OSU Student ID #

Instructions:

- The length of time for this exam is **110 minutes** (That is 1hour 50minutes)
- Put your name and work on this paper. Please do not add additional pages. Adding pages would make it difficult to grade.
- Show your work and explain your reasoning when needed. **Please write neatly (so that it is easy to read).**
- If you could put down your answers on this paper (maintaining the format), that would be great, this would help a lot in grading, albeit if you don't have a printer or an option of writing electronically, you can do otherwise.
- This exam is worth 30% of your total grade. Be sure to show **ALL** your work, as partial credit will be given. Full credit will not be given for answers which are not accompanied with a mathematical justification.
- Be sure to show **ALL** your work, as partial credit will be given. Full credit will not be given for answers which are not accompanied with a mathematical justification.
- Index card or cheat sheet **<u>is not allowed</u>** for this exam

Good Luck!

MULTIPLE CHOICE - TOTAL OF 10POINTS

(1) (10 points: 1 points each) Choose the option that best answers the question. Note that if a statement is true for a particular case but not others, you should answer false.

(a) Every subset of a ring is a subring

◯ (a) True

◯ (b) False

(b) The multiplicative inverse of 5 in $\mathbb{Z}_7$ is

◯ (a) 1

◯ (b) 3

◯ (c) 5

◯ (d) 2

◯ (e) 0

(c) The order of $\langle 5 \rangle \in \mathbb{Z}_{12}$ is

◯ (a) 5

◯ (b) 8

◯ (c) 12

◯ (d) 2

◯ (e) 10

(d) Homomorphism must be a bijection

◯ (a) True

◯ (b) False

(e) A group G can have more than 1 identity element depending on the binary operations.

◯ (a) True

◯ (b) False

(f) If $g(x)$ and $h(x)$ are factors of the polynomial $f(x)$ over a field, then the

$$\deg(f(x)) < \deg(g(x)) + \deg(h(x))$$

◯ (a) True

◯ (b) False

(g) If the order of a group is 6, then the index is also 6.

◯ (a) True

◯ (b) False

(h) Ideals of a ring are always prime.

◯ (a) True

◯ (b) False

(i) A zero divisor is a nonzero element a of a commutative ring such that there is a nonzero element $b \in R$ and $ab = 0$

◯ (a) True

◯ (b) False

(j) Rings are subsets of Groups

◯ (a) True

◯ (b) False

Groups and Subgroups

(2) (5 Points) Let $M_2(\mathbb{Z})$ be the set of all 2×2 matrices with integer entries. $M_2(\mathbb{Z})$ is a group under addition. Prove or disprove

Answer:

(3) (5points) Let G consist of the 2×2 matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta \in \mathbb{R}$. Prove that G is a subgroup of $SL_2(\mathbb{R})$.
(**Remark**: $SL_2(\mathbb{R})$ is a set of matrices whose determinant is equal to 1)

Answer:

(4) (a) (5 Points) Is $\mathbb{Z}_6$ a cyclic group? What are the generators?
(**Note**: Saying Yes or No will earn you only 1pt)

Answer:

(b) (5 Points) Let H be a cyclic group and $a \in H$ be a generator for H defined as $h = a^m$. Prove that "Every cyclic group is abelian".

Answer:

(5) (5 Points) Let $H = \{id, (12)\}$ be a subgroup of S_3. Find the left and right cosets of H and determine if H is a normal subgroup

Answer:

Ring and Subrings

(6) (a) $M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a, b, c, d \in \mathbb{R} \right\}$ is a ring with matrix addition and multiplication. Prove or disprove. (5 Points)

Answer:

(b) (5 Points) The set of non-negative integers ($\mathbb{Z}^+$) is a ring under addition and multiplication. Prove or disprove.

Answer:

(7) (5 Pts) Let $T := \left\{ A \in M_2(\mathbb{R}) \mid A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},\ a, b, d \in \mathbb{R} \right\}$ be the set of all upper triangular matrices in $M_2(\mathbb{R})$. Is T a subring of $M_2(\mathbb{R})$?

Answer:

Fields and Polynomials

(8) Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$ for each of the following pairs of polynomials in question (8a) and (8b).

(a) (5 Points) $a(x) = x^4 + x^2 + 3x + 4$ and $b(x) = x^2 + 2x + 3$ in $\mathbb{Z}_5[x]$

Answer:

(b) (5 Points) $a(x) = x^5 + x^3 + x + 1$ and $b(x) = x^3 + x^2 + 1$ in $\mathbb{Z}_2[x]$

Answer:

(c) (5 Points) Determine whether $f(x) = x^3 + 2x + 1$ is irreducible in $\mathbb{Z}_3(x)$

Answer:

Theorems

(9) (a) (5points) First isomorphism theorem for groups.

Answer:

(b) Let $\mathbb{R}$ denote the group of real numbers with addition and $\mathbb{R}^*$ denote the group of non-zero real numbers with multiplication. For each part, determine whether the mapping given is a group homomorphism. Justify your answers briefly.

(i) (5points) Define $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ by $\phi(x) = 3x$ for all $x \in \mathbb{R}$

Answer:

(ii) (5points) Define $\phi : \mathbb{R}^* \longrightarrow \mathbb{R}^*$ by $\phi(x) = 3x$ for all $x \in \mathbb{R}$

Answer:

Computation

(10) (2 Pts each)

Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

(a) Put α and β in cycle notations

Answer:

(b) Compute α^{-1} in cycle notation

Answer:

(c) Compute $\beta\alpha$ in cycle notation

Answer:

(d) Compute $\alpha\beta$ in cycle notation

Answer:

(e) Is $\alpha\beta = \beta\alpha$?

Answer: