Applied Ordinary Differential Equations Notes

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We begin with the following form of a second order autonomous ODE:

$$ay''(t) + by'(t) + cy(t) = f(t)$$

This equation can be transformed to a first order system of ODE's, if we define two equations to be

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{aligned}$$

Then the euqation can be rewritten in terms of x_1, x_2 as

$$ax_2'(t) + bx_2(t) + cx_1(t) = f(t)$$

Which immediately gives us this first order system of ODE's:

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -\frac{c}{a}x_1(t) - \frac{b}{a}x_2(t) + \frac{f(t)}{a} \end{aligned}$$

For notational purposes, write $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \vec{x'}(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$. Then we can write many first order systems of ODE's as

$$\begin{aligned} A\vec{x}(t) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t) \end{pmatrix} \end{aligned}$$

For our previous example we would have $A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$ ignoring the $f(t)$ term.

Now we study autonomous first order systems of ODE's. That is,$\vec{x'}(t) = A\vec{x}(t)$. These have solutions that can be represented as

$$\vec{x}(t) = e^{At}\vec{x_0}, \qquad e^{At} = \sum_{j=0}^{\infty} \frac{1}{j!}A^j t^j$$

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2.1 Matrix Exponential

(Taylor Series) We define the Taylor series at 0 as

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}x^j$$

For example, $e^x = \sum_{j=1}^{\infty} \frac{x^j}{j!}$. Now we are prepared for the matrix exponential, which we define as follows:

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

When $A \mapsto e^A$, we have a group homomorphism from the set of $n \times n$ matrices with addition over to $n \times n$ invertible matrices with multiplication. Take the initial value problem

$$\begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

The unique solution is $e^{At}\vec{x}_0$. Let $X(t) = e^{At}$. This is the matrix of fundamental solutions.

$$X(0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

Then $X'(t) = AX(t), X(0) = I$. Therefore we get $\vec{x}(0) = I\vec{x}_0 = \vec{x}_0$.

Suppose A is diagonalizable,

$$A = U\Lambda U^{-1}, \qquad \text{where } \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad U = (\vec{u}_1, \cdots, \vec{u}_n)$$

Then we know that

$$\begin{aligned} X(t) &= e^{At} && (1) \\ &= \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} && (2) \\ &= \sum_{j=0}^{\infty} \frac{(U\Lambda U^{-1})^j t^j}{j!} && (3) \\ &= \sum_{j=0}^{\infty} \frac{U\Lambda^j U^{-1} t^j}{j!} && (4) \end{aligned}$$

In the case of diagonal matrices, we get a linear composition.

$$\begin{aligned} \vec{x}(t) &= Ue^{\Lambda t}U^{-1}\vec{x}_0 \\ &= c_1 e^{\lambda_1 t}\vec{u}_1 + \cdots c_n e^{\lambda_n t}\vec{u}_n \\ &= c_1 \vec{\Psi}_1(t) + \cdots + c_n \vec{\Psi}_n(t) \end{aligned}$$

When we have a diagonalizable matrix we have a nice formula for the solution. Now we will move on to an example:

Consider the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

We know the solution to this is just $\vec{x}(t) = e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Now we write

$$e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t} = \sum_{j=0}^{\infty} \frac{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^j t^j}{j!}$$

Now we know that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$. This means that we can actually write out what our matrix exponential is explicitly in terms of series in each component. We do this as follows:

$$e^{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}t} = \begin{pmatrix}\sum_{j=0}^{\infty}\frac{t^j}{j!} & \sum_{j=1}^{\infty}\frac{jt^j}{j!}\\ 0 & \sum_{j=0}^{\infty}\frac{t^j}{j!}\end{pmatrix} = \begin{pmatrix}e^t & te^t\\ 0 & e^t\end{pmatrix}$$

The Wronskian again:

$$W(t) = \det(X(t)) \quad W'(t) = \frac{d}{dt}\det(e^{At}) = \frac{d}{dt}e^{\text{tr}(At)} = \text{tr}(A)e^{\text{tr}(A)t} = \text{tr}(A)W(t)$$

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With a system that has only complex eigenvalues we have trajectories that are ellipses, with fixed semiaxes and fixed ratios $\frac{\rho_2}{\rho_1} = c$.

Let $\vec{x}^0$ be an isolated critical point. WLOG takes $\vec{x}^0 = \vec{0}(\vec{x} = \vec{x} - \vec{x}^0)$.

Proof. $x' = f(x)$ is nearly linear at isolated critical point 0 if $f(x) = Ax + g(x)$ such that $\det(A) \neq 0$ and $\frac{||g||}{||f||} = 0$? idk what he wrote □

Example: Let us have the given (damped pendulum) system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y$$

Then let

$$\boldsymbol{x} = \begin{pmatrix}x\\ y\end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix}0 & 1\\ -\omega^2 & -\gamma\end{pmatrix}\boldsymbol{x} - \omega^2\begin{pmatrix}0\\ \sin(x) - x\end{pmatrix}$$

So we have

$$-\omega^2 \sin x = -\omega^2 x + \frac{\omega^2 x^3}{3!} - \cdots$$

But we still need to show that $\frac{||\boldsymbol{g}(\boldsymbol{x})||}{||\boldsymbol{x}||} \to \boldsymbol{0}$ as $\boldsymbol{x} \to \boldsymbol{0}$.

$$\begin{aligned}||\boldsymbol{g}(\boldsymbol{x})|| &= \omega^2|\sin(\boldsymbol{x}) - \boldsymbol{x}|\\ &\approx \frac{\omega^2}{3!}|x^3|\end{aligned}$$

$$\begin{aligned}\implies \frac{||\boldsymbol{g}(\boldsymbol{x})||}{||\boldsymbol{x}||} &\approx \frac{1}{r}\frac{\omega^2}{3!}|r^3\cos^3\theta|\\ &= \frac{r^2\omega^2|\cos^3\theta|}{3!}\\ &\leq \frac{r^2\omega^2}{3!} \to \boldsymbol{0}\end{aligned}$$

Non-linear stability and pendulum problemns

$$x' = f(x) = A(x - x^0) + g(x), \quad \lim_{x\to x^0}\frac{||g(x)||}{||x - x^0||} = 0$$

Where A is a 2x2 real matrix (non-singular), the matrix has two eigenvalues r_1, r_2. Stability of locally linear system at x^0 (critical point).

$$\begin{aligned}
&r_2 > r_1 > 0 && \text{(unstable node)}\\
&r_1 < r_2 < 0 && \text{(stable node)}\\
&r_2 < 0 < r_1 && \text{(saddle point)}\\
&r_1 = r_2 > 0 && \text{(unstable node or spiral point)}\\
&r_1 = r_2 < 0 && \text{(stable node or spiral point)}\\
&r_1, r_2 = \lambda \pm i\mu && \text{(unstable/stable spiral, or circles)}
\end{aligned}$$