General Topology and the Fundamental Group - Homework 3

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June 10, 2021

Problem 1

Let G be a topological group.

a)

> Suppose that $H \subset G$ is an open subgroup. Show that H is closed, and conclude that if G is connected that $H = G$.

Proof. $\boxed{H \text{ is closed}}$ We wish to show that $G \setminus H$ is open in order to show that H is closed. Let $g \in G \setminus H$ be arbitrary. Let $U = gH = \{gh \mid h \in H\}$ be an open neighborhood of g. We know this to be true since H must contain the identity element, so $g \in U$ and also we know that m is an open map, therefore gH is open since H is open. Now we wish to show that gH is contained in $G \setminus H$. Suppose by contradiction that there exists some element $h \in H \cap gH$. So of course $h \in H$, and since $h \in gH$ we say $\exists h_0 \in H$ such that $h = gh_0$. If we multiply both sides of the equality on the right by h_0^{-1} then we get $hh_0^{-1} = g$. Since h, h_0^{-1} both belong to H and H is a group, their product g also must belong to H. This a contradiction, since by assumption $g \in G \setminus H$ so $g \in H$ cannot also be the case.

$\boxed{G \text{ is connected} \implies H = G}$ We will do a proof by contrapositive. Suppose that $H \neq G$. Then we can construct a non-trivial disconnection $H \cup (G \setminus H)$. Clearly the union is disjoint and covers G by definition. Since H is both open and closed it follows that the same must be true for its complement $G \setminus H$. Then since $H \neq G$ we know that $G \setminus H$ is non-empty. H must also be non-empty as it must contain at least the identity element. $\square$

b)

> Let $K_1, K_2 \subset G$ be compact sets. Show that their product K_1K_2 is compact.

Proof. First, we write $K_1K_2 = m(K_1, K_2)$. Since K_1 and K_2 are compact we know that $K_1 \times K_2$ is compact in the product toplogy. Then since $K_1 \times K_2 \subset G \times G$ is compact, we know that its continuous image must be compact. So since m is continuous, $K_1K_2 = m(K_1, K_2)$ is compact. $\square$

c)

> Show that $m : G \times G \to G$ is an open map.

Proof. Let $U \times V \subset G \times G$ be an arbitrary basis element of the topology on $G \times G$. Then both U and V are clearly open by properties of the product toplogy. Then we write

$$m(U \times V) = UV = \{uv \in G \mid u \in U, v \in V\} = \bigcup_{u \in U} L_u(V)$$

Since V is open, for any $u \in G$, $L_u(V)$ is also open, and we conclude that $m(U, V)$ is open, therefore m is an open map. $\square$

Problem 2

Let G be a topological group.

a)

Show that the connected component G_0 is a normal subgroup of G and that any connected component is of the form gG_0 for some $g \in G$.

Proof. By definition, we know that our identity element $e \in G_0$. We wish to show that $g(G_0)g^{-1} \subset G_0$ for every $g \in G$. Since L_g and $R_{g^{-1}}$ are both continuous functions, it follows that $g(G_0)g^{-1} = R_{g^{-1}}(L_g(G_0))$ is a connected set, since the image of a connected set is connected. Then also $geg^{-1} \in g(G_0)g^{-1}$ where e is the identity element. Then since $geg^{-1} = gg^{-1} = e$, it follows that $g(G_0)g^{-1}$ is a connected set containing e. Since G_0 is the connected component containing e, it contains any connected set containing e as well, so $g(G_0)g^{-1} \subset G_0$. Thus G_0 is a normal subgroup.

Now to show that any connected componenet is of the form gG_0 for some $g \in G$, let $H \subset G$ be a connected component. Let $h \in H$ be some element, since we assume H is non-empty. Then $L_{h^{-1}}(H)$ is an open connected set containing the identity element e. Suppose by contradiction that $h^{-1}H \neq G_0$ and there exists some element $a \in G_0$ that is not in $h^{-1}H$. However ha is an element of hG_0 which is of course a connected set containing h. But since $a \notin h^{-1}H$, we know that $ha \notin hh^{-1}H = H$. This means that H is not a connected componenet (contradiction). Therefore $h^{-1}H = G_0$, so H can be written in the form

$$H = hG_0$$

□

b)

Show that $G \setminus G_0$ is a totally disconnected group.

Proof. Let $[g] \in G \setminus G_0$. Let $H \subset G$ be a connected componenet other than G_0. Let $a, b \in H$ be two elements. Then since $H = hG_0$ by part (a), it follows that $h^{-1}a, h^{-1}b \in G_0$. So then it follows that $[h^{-1}a] = [h^{-1}b] \Rightarrow [a] = [b]$. Since any two elements of H are equivalent, $H \subset [h]$. Let $g \in G$ be an element such that $g \simeq h$. Then it must be the case that $h^{-1}g \simeq h^{-1}h = e$. Therefore $h^{-1}g \in G_0$, which implies that $g \in hG_0 = H$, which implies that $H = [h]$. Since every point in the space belongs to a connected componenet, and its equivalence class will be equal to its connected component, it follows that each equivalence class in G/G_0 is a connected componenet, and that G/G_0 is a totally disconnected group. □

Problem 3

Let $Gl(n, \mathbb{C})$ be the set of invertible $n \times n$ matrices over the complex numbers. Let $SO(n)$ be the set of orthogonal $n \times n$ matrices.

a)

Show that $Gl(n, \mathbb{C})$ is path connected.

Proof. Let $A \in Gl(n, \mathbb{C})$. Then write its Jordan normal form as

$$J = P^{-1}AP$$

Since our matrices are invertible, our eigenvalues for A, $\lambda_1, \cdots, \lambda_n$ and for B, $\Lambda_1, \cdots, \Lambda_n$ are all non-zero. Since they all belong to the complex plane with 0 removed, we know that λ_i and $z = 1$ are path connected without crossing 0. This is because the set of complex numbers without 0 is a path connected space, simply go around the origin. So let γ_i connect λ_i with Λ_i such that $\gamma_i(t) \neq 0 \; \forall t$. Then accross the superdiagonal simply linearly move from one to zero given by $s(t) = 1 - t$ with $t \in [0, 1]$. Then write our path,

$$\Gamma_{J_1,I}(t) = \begin{pmatrix} \gamma_1 & 0 & & \\ & \gamma_2 & s & \\ & & \ddots & s \\ & & & \gamma_n \end{pmatrix}$$

Where for every sub-path along the superdiagonal we choose between $\{0, s\}$ appropriately (that is, if there is a 1 there for J_1 choose s, otherwise choose 0). Then we have $\Gamma_{J,I}(0) = J_1, \Gamma_{J,I}(1) = I$, and $\Gamma_{J,I}(t) \in Gl(n, \mathbb{C}) \; \forall t \in [0, 1]$. This last fact follows from the fact that our matrices remain as upper triangular matrices with no zeroes on the diagonal. We know that P, P^{-1} are group homeomorphisms, so it follows that

$$\Gamma = P^{-1}(\Gamma_{J,I})P$$

Then we know that $\Gamma(0) = A$ and $\Gamma(1) = I$, so any matrix is path connected to the identity matrix, therefore $Gl(n, \mathbb{C})$ is path connected. □

b)

Show that $SO(n)$ is path connected.

Proof. $\boxed{\text{i}}$ Let $A \in SO(n)$ and $V \subset \mathbb{R}^n$ be a linear subspace. Suppose $L_A(V) = V$ and that $L_A(V^\perp) \neq V^\perp$. Since L_A is an isomorphism on $\mathbb{R}^n$, it follows that there exists some $\boldsymbol{v} \in V^\perp$ such that $L_A(\boldsymbol{v}) \notin V^\perp$. We know that there must exist also some $\boldsymbol{u} \in V$ such that $\boldsymbol{u} \perp \boldsymbol{v}$. Then we also have $L_A(\boldsymbol{u}) \perp L_A(\boldsymbol{v})$ since dot-product is preserved. However, this implies that $L_A(\boldsymbol{v}) \in V^\perp$ (contradiction). So it must be that if $L_A(V) = V$, then $L_A(V^\perp) = V^\perp$.

$\boxed{\text{ii}}$ Now we wanna show that all real eigenvalues λ of A have the property $|\lambda| = 1$. Let λ be some real eigenvalue of A. There exists some corresponding vector $\boldsymbol{v}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Since A is orthogonal, we know that $A^{-1} = A^T$. So then,

$$||\lambda \boldsymbol{v}|| = ||A\boldsymbol{v}|| = (A\boldsymbol{v})^T(A\boldsymbol{v}) = \boldsymbol{v}^T A^T A \boldsymbol{v} = \boldsymbol{v}^T \boldsymbol{v} = ||\boldsymbol{v}||$$

Knowing that $||\lambda\boldsymbol{v}|| = ||\boldsymbol{v}||$ we can write the following:

$$\begin{aligned} ||\lambda \boldsymbol{v}|| &= ||\boldsymbol{v}|| \\ |\lambda| \, ||\boldsymbol{v}|| &= ||\boldsymbol{v}|| \\ \Rightarrow |\lambda| &= 1 \end{aligned}$$

Now we've gotta show that $\mathbb{R}^n = V \oplus V^\perp$ where V admits some basis $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ such that $L_A(\boldsymbol{v}_i) = \pm\boldsymbol{v}_i$ and $L_A(\boldsymbol{u}) \neq \boldsymbol{u}$ for every $\boldsymbol{u} \notin V$. We will do this by induction. Let $n = 1$. On the real line, the matrix whose transpose is it's own inverse is $(1) = (1)^{-1} = (1)^T$. Then clearly $V = \text{span}(1)$, and $V^\perp = \{0\}$ is trivial so $\mathbb{R} = V \oplus V^\perp$. Assume by induction that the statement is true for some integer n, and we will show it holds for $n+1$. Let us assume that the problem is non-trivial and that A possesses at least one real eigenvector $\boldsymbol{x}$. Then let $U = \{\boldsymbol{u} \in \mathbb{R}^{n+1} \mid \boldsymbol{u} \perp \boldsymbol{x}\}$. Clearly U is a subspace of $\mathbb{R}^{n+1}$ with 1 less dimension, so it follows that since $\lambda\boldsymbol{x} = \pm\boldsymbol{x}$, we have $U = \mathbb{R}^n$. Then by induction $U = V \oplus V^\perp$ in $\mathbb{R}^{n-1}$. Since $U \perp \boldsymbol{x}$ we know that $V^\perp \perp \boldsymbol{x}$, and it follows that its orthogonal complement in $\mathbb{R}^{n+1}$ will be equal to

$$\text{span}\{V, \boldsymbol{x}\}$$

From here it follows that $\text{span}\{V, \boldsymbol{x}\} \oplus V^{\perp_n}$ is our direct sum as desired, with $\text{span}\{V, \boldsymbol{x}\}$ having the desired basis, that being the basis of V appended with $\boldsymbol{x}$. Thus, by induction the property is true for all $n \in \mathbb{N}$. Since L_A is an orthogonal matrix, it is an orthogonal transformation, and in particular since by $\boxed{\text{i}}$ we know that $L_A(V^\perp) = V^\perp$ so it follows that under such a restriction, $L_A\big|_{V^\perp}$ remains an orthogonal transformation.

$\boxed{\text{iii}}$ By the same argument from $\boxed{\text{ii}}$, we know that $|\lambda| = 1$ even if it is complex. Let $\boldsymbol{v} = \boldsymbol{w}_1 + i\boldsymbol{w}_2$ be a complex eigenvector. Write $\lambda = e^{i\alpha} = a + bi$. Suppose that $\boldsymbol{w}_1 = c\boldsymbol{w}_2$. Then we write $L_A(\boldsymbol{v}) = (a - b(c))\boldsymbol{w}_1 + i((c)a + b)\boldsymbol{w}_1$. So we say that

$$L_A(\boldsymbol{w}_1) = (a - b(c))\boldsymbol{w}_1 \qquad L_A(\boldsymbol{w}_2) = ((c)a + b)\boldsymbol{w}_1 \qquad (c)L_A(\boldsymbol{w}_1) = L_A(\boldsymbol{w}_2)$$

So it follows that $L_A(\boldsymbol{w}_1) = L_A(\boldsymbol{w}_2)/c = \left(a + \frac{b}{c}\right)\boldsymbol{w}_1$. From these equations we write

$$\begin{aligned} (a - b(c))\boldsymbol{w}_1 &= \left(a + \frac{b}{c}\right)\boldsymbol{w}_1 \\ a - b(c) &= a + \frac{b}{c} \\ -b(c) &= \frac{b}{c} \\ -b(c^2) &= b \\ -c^2 &= 1 \\ c^2 &= -1 \end{aligned}$$

This clearly implies that the scalar c must be equal to i, making $\boldsymbol{v} = \boldsymbol{w}_1 + \frac{i}{i}\boldsymbol{w}_1$ which is not complex (contradiction). So it must be the case that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are independent. Since the vectors are independent the images under L_A will also be independent since dot product is preserved. So it follows that their span will be preserved as well. So it follows that in the subspace of $\text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ since both angle and span are preserved, we have a two dimension rotation matrix representing $L_A|_W$,

$$L_A|_W = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$\boxed{\text{iv}}$ We will use induction to show that $\mathbb{R}^n$ admits a basis such that $L_A(\boldsymbol{v}_i) = \pm\boldsymbol{v}_i$ or L_A is a rotation matrix of the above form. For $n = 1$ we claerly have this since $O(1) = 1$. For the inductive step, by $\boxed{\text{ii}}$ we have the direct sum $V \oplus V^\perp$ for which V already has the desired property. By $\boxed{\text{iii}}$, and since the dimension of $V^{\perp}$ is less than that of $\mathbb{R}^n$ we can also assume that, being a subspace isomorphic to $\mathbb{R}^k$ where $k < n$, it too must yield a basis with the desired property. Thus $\mathbb{R}^n$ is a direct sum of subspaces with the desired basis vector properties, so the space itself must be so as well.

$\boxed{\text{v}}$ Since $\det(A) = 1$ it follows that the number of -1's along the diagonal must be even. So it follows that the rotation matrix can take the W subspaces to the identity matrix without causing the determinant to equal 0, so we end up with a path to the identity matrix as follows:

$$\begin{pmatrix} 1 & f & \cdots & f \\ 0 & 1 & \cdots & f \\ 0 & \cos \pi t & -\sin \pi t & \\ 0 & \sin \pi t & \cos \pi t & \end{pmatrix}$$

Or something of this form, which will be A at $t = 0$, and I at $t = 1$. $\square$

Problem 4

Compute $\pi_1(S^1 \vee S^1)$.

Let $X = \{z \in \mathbb{Z} \mid z = e^{2\pi i t} \text{ or } z = e^{it} + 1, t \in \mathbb{R}\}$ with the subspace topology inherited from the complex plane. Then let

$$A = \left\{ z \in \mathbb{Z} \mid z = e^{2\pi i t} \text{ or } z = e^{2\pi i s}, t \in \mathbb{R}, s \in \left(-\frac{1}{2}, \frac{1}{2} \right) \right\} \qquad B = \left\{ z \in \mathbb{Z} \mid z = e^{2\pi i t} \text{ or } z = e^{2\pi i s}, t \in \left(-\frac{1}{2}, \frac{1}{2} \right), s \in \mathbb{R} \right\}$$

Since the additional half circle is contractible, we say that $\pi_1(A, 1) \cong \pi_1(S^1, 1)$, and that $\pi_1(B, 1) \cong \pi_1(S^1 + 1, 1)$. Then we know that both A and B are open in X, since their complements are the closed left or right half circles. We can write $X = A \cup B = S^1 \vee S^1$. Then the point $z = 1 = x_0$ lies within the intersection $A \cap B$. By $\boxed{\text{Theorem 59.1 (Munkres)}}$, it follows that $\pi_1(X, 1)$ is generated by path homotopy classes in A and B. That is, we can concatenate any finite number of loops in either to create a loop in X. For two circles touching at one point, this means all paths are generated by going around in some kind of ordered set of loops (clockwise/counterclockwise, multiple times/trivially no times) in either the first or second circle. Since the fundamental group for S^1 is isomorphic to $\mathbb{Z}$, it follows that $\pi_1(S^1 \vee S^1)$ will be isomorphic to $\mathbb{Z} \times \mathbb{Z}$.