MTH 342 Homework 7

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1

Let $E = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_2 = x_1 + x_3 + x_4 \}$.

(a)

Find the projection of u onto E where $u = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$.

Proof. We can write our space E as $\left\{ \begin{bmatrix} x_1 \\ x_1 + x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_1, x_3, x_4 \in \mathbb{R} \right\}$. Then since there are no restrictions on x_1, x_2, x_3, x_4, we can write $E = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Denote these vectors as e_1, e_2, e_3. To find an orthogal basis, $E' = \{e'_1, e'_2, e'_3\}$ we begin by letting $e'_1 = e_1$. Then, let

$$\begin{aligned} e'_2 &= e_2 - proj_{e'_1} e_2 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{(e_2, e'_1)}{(e'_1, e'_1)} e'_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Then, let $e_3' = e_3 - proj_{\{e_1', e_2'\}} e_3$. We can write

$$
\begin{aligned}
e_3' &= e_3 - proj_{e_1'} e_3 - proj_{e_2'} e_3 \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(e_3, e_1')}{(e_1', e_1')} e_1' - \frac{(e_3, e_2')}{(e_2', e_2')} e_2' \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}
\end{aligned}
$$

Then we can write $E' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right\}$, an orthogonal basis for E. Then, to compute the projection of u onto E, we can write $proj_E u = proj_{e_1'} u + proj_{e_2'} u + proj_{e_3'} u$. Then we have

$$
\begin{aligned}
proj_E u &= \frac{1}{2} e_1' - e_2' + \frac{3}{2} e_3' \\
&= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$\square$

(b)

Since we have E' is an orthogonal basis, all we need to do is normalize it. We can simply divide these vectors by their magnitudes, and we get $E_n' = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{6}}{9} \\ \frac{\sqrt{6}}{9} \\ -\frac{\sqrt{6}}{9} \\ \frac{\sqrt{6}}{3} \end{bmatrix} \right\}$.

2

Let E be the space $E = \{A \in M_{2\times 2} : AC = CA\}$ where $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the projection of $u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ onto E.

First, we must find an orthogonal basis for E. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $a, b, c, d \in \mathbb{R}$ arbitrarily. Then,

$$AC = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

and

$$CA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

And it follows that $AC = CA \Rightarrow b = c, a = d$. Then we write $E = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Since we have no restrictions on a and b, we have a basis for E, $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Since their inner product is 0, they are orthogonal. Denote the vectors in B as e_1, e_2 respectively. We now have

$$proj_E(u) = proj_{e_1}(u) + proj_{e_1}(u)$$

Using the inner product characterization of projection, this is equivalent to

$$\begin{aligned} proj_E(u) &= \frac{(u, e_1)}{(e_1, e_1)} e_1 + \frac{(u, e_2)}{(e_2, e_2)} e_2 \\ &= \frac{5}{2} e_1 + \frac{5}{2} e_2 \\ &= \frac{5}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

And we are done.

3

To minimize this difference, we must project e^x onto polynomial space of degree 3. Then we choose the shifted Legendre Polynomials as our basis for $P_3(\mathbb{R})$ where $B = \left\{1, 2x-1, 6x^2-6x+1, 20x^3-30x^2+12x-1\right\}$. Denote these vectors as $p_0, ..., p_3$ respectively. Then,

$$proj_{P_3(\mathbb{R})}(e^x) = proj_{p_0}(e^x) + proj_{p_1}(e^x) + proj_{p_2}(e^x) + proj_{p_3}(e^x)$$

Then, we must take each projection individually. For p_0, we get

$$proj_{p_0} e^x = \int_0^1 e^x dx = e - 1$$

Then for p_1, we get

$$proj_{p_1} e^x = \frac{\int_0^1 (e^x)(2x-1)dx}{\int_0^1 (2x-1)^2 dx}(2x-1) = (9-3e)(2x-1)$$

$$proj_{p_2} e^x = \frac{\int_0^1 (e^x)(6x^2-6x+1)dx}{\int_0^1 (6x^2-6x+1)^2 dx}(6x^2-6x+1) = (35e-95)(6x^2-6x+1)$$

$$proj_{p_3} e^x = \frac{\int_0^1 (e^x)(20x^3-30x^2+12x-1)dx}{\int_0^1 (20x^3-30x^2+12x-1)^2 dx}(20x^3-30x^2+12x-1) = (1351 = 497e)(20x^3-30x^2+12x-1)$$

Then we can add up all of these projections to get $proj_{P_3(\mathbb{R})}$,

$$proj_{P_3(\mathbb{R})} = (e-1) + (9-3e)(2x-1) + (35e-95)(6x^2-6x+1) + (1351-497e)(20x^3-30x^2+12x-1)$$

4

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$.

(a)

Show that $Ax = b$ has no solutions where $x \in \mathbb{R}^2$.

Proof. Suppose it did have some solution $x = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Then

$$Ax = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+2b \\ b \\ 2a+b \\ 2a-b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} = b$$

This gives us $b = -1$, from the second row. And since $a + 2b = 1$, we also have $a = 3$. Looking at the third row we have $6 - 1 = 5 = 0$ (contradiction), therefore no solution may exist. □

(b)

Let us take $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ to be our basis. To get an orthogonal bases, let $b_1' = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$. Then let $b_2' = b_2 - proj_{b_1'} b_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{9} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix}$. Then, we have an orthogonal bases $B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix} \right\}$. Then, since this is an orthogonal basis, we can write

$$\begin{aligned} proj_{B'} b &= proj_{b_1'} b + proj_{b_2'} b \\ &= \frac{(b, b_1')}{(b_1', b_1')} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} + \frac{(b, b_2')}{(b_2', b_2')} \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix} \\ &= \frac{7}{9} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} + \frac{-32}{531} \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix} \end{aligned}$$

Then, we must find the solution to $Ax = proj_{B'} b$. We write

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{7}{9} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} + \frac{-32}{531} \begin{bmatrix} 16 \\ 9 \\ 5 \\ -13 \end{bmatrix}$$

We can determine from the second row of Ax that $b = -\frac{32}{59}$. It then follows that a must be equal to $\frac{53}{59}$ from the other equations.

5

Let $u_1' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$u_2' = u_2 - proj_{u_1'} u_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u_3' = u_3 - proj_{u_2'} u_3 - proj_{u_1'} u_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{7}\frac{1}{4} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{112} \begin{bmatrix} 25 & -75 & -3 \\ 0 & 25 & 112 \\ 0 & 0 & 25 \end{bmatrix}$$

We write our orthogonal basis as $U' = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \frac{1}{4} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \frac{1}{112} \begin{bmatrix} 25 & -75 & -3 \\ 0 & 25 & 112 \\ 0 & 0 & 25 \end{bmatrix} \right\}$