General Topology and Fundamental Groups - Homework 3

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Problem 1

(a)

Let $f : X \to Y$ be an open map, and let $A \subset Y$ be arbitrary. Let $f^{-1}(A) \subset C$ where $C \subset X$ is closed. Then there is a closed set $D \subset Y$ such that $A \subset D$ and $f^{-1}(D) \subset C$.

Proof. Let

$$D = Y \setminus f(X \setminus C)$$

Since C is closed, $X \setminus C$ is open. An open map f, will map this complement to an open set, so $f(X \setminus C)$ is open. Thus its complement, D, is closed. Since $f^{-1}(A) \subset C$ it follows by set theory that $A \subset D$. Then we write

$$f^{-1}(D) = f^{-1}(Y) \setminus f^{-1} f(X \setminus C) \subset X \setminus (X \setminus C) = C$$

□

(b)

$f : X \to Y$ is a closed map if and only if $\overline{f(A)} \subset f(\overline{A})$.

Proof. $\boxed{\Rightarrow}$ Assume that f is closed. Then $f(\overline{A})$ is a closed set, which clearly contains $f(A)$ since $A \subset \overline{A}$. Then since the closure is the smallest closed set containing the original, it must be the case that $\overline{f(A)} \subset f(\overline{A})$.

$\boxed{\Leftarrow}$ Suppose $\overline{f(A)} \subset f(\overline{A})$ for every $A \subset X$. Let $C \subset X$ be some arbitrary closed set. Suppose by contradiction that $f(C)$ is not closed. Then by characterization of closed sets we say $f(C) \subsetneq \overline{f(C)}$. However, by assumption, we say $\overline{f(C)} \subset f(\overline{C})$. Since C is closed, $C = \overline{C}$. So

$$f(C) \subsetneq \overline{f(C)} \subset f(\overline{C}) = f(C) \quad \Longrightarrow \quad f(C) \subsetneq f(C) \quad \text{(contradiction)}$$

□

Problem 2

Let $\mathfrak{U} = \{A_i\}, i = 1, 2, \cdots$, be a family of sets in X such that $A_{i+1} \subset A_i$ for all k. Show that if $\bigcap_{i \in \mathbb{N}} \overline{A_i} = \emptyset$, then $\mathfrak{U}$ is locally finite.

Proof. Suppose that that $\bigcap_{i \in \mathbb{N}} \overline{A_i} = \emptyset$ and that $\mathfrak{U}$ is not locally finite. Then there exists some point $x \in X$ such that every neighborhood of x intersects infinitely many sets in $\mathfrak{U}$. Thus for every natural number N there exists another $n > N$ such that $U(x) \cap A_n \neq \emptyset$. Since these sets are nested, it follows that $U(x)$ must intersect every A_m such that $m < n$. Then if $U(x)$ did not intersect every single set A_i then since they are nested, it would only intersect finitely many (contradiction). So then we say that every neighborhood of x intersects each A_i.

It follows that x is a limit point of $\bigcap_{i \in \mathbb{N}} \overline{A_i}$ since every neighborhood of x intersects every A_i. Then since this is an intersection of closed sets, we say that $\bigcap_{i \in \mathbb{N}} \overline{A_i}$ is itself a closed set, and must contain its limit points. In particular $x \in \bigcap_{i \in \mathbb{N}} \overline{A_i}$ so it cannot be empty (contradiction). Therefore, such an x must not exist, and the intersection is locally finite. □

Problem 3

Let $\{B_\alpha\}, \alpha \in \mathcal{A}$ be an open or locally finite, closed cover of Y, and let $f : X \to Y$ be continuous. Suppose that the restrictions $f_{|f^{-1}(B_\alpha)} : f^{-1}(B_\alpha) \to B_\alpha, \alpha \in \mathcal{A}$ are homeomorphisms. Show that f is a homeomorphism.

Proof. $\boxed{\text{Continuity}}$ Let $U \subset Y$ be open. Show that f^{-1}

Continuity is given for f already. For bijectivity let us first show surjectivity. Let $y \in Y$, then it lies in some B_α. Then since we have a homeomorphism from $f^{-1}(B_\alpha) \to B_\alpha$ we know that there exists $x \in X$ such that $f(x) = y$. To show injectivity let $f(x) = f(y)$, then for every B_α that contains $f(x) = f(y)$, we know that this implies $x = y$. Thus we know that f is a bijection.

We wish to show now that f is an open map. We break this into two cases:

$\boxed{\text{Case 1: } \{B_\alpha\} \text{ is an open cover of } Y}$ Let $U \subset X$ be some arbitrary open set. Since $\{B_\alpha\}$ is an open cover of Y and since f is continuous it follows that $\{f^{-1}(B_\alpha)\}$ is an open cover of X. We write $U = X \cap \bigcup_{\alpha \in A} B_\alpha$. Then,

$$U = \bigcup_{\alpha \in A} (f^{-1}(B_\alpha) \cap U) \quad \Longrightarrow \quad f(U) = \bigcup_{\alpha \in A} f(B_\alpha \cap U)$$

Then we know that $f(B_\alpha \cap U)$ is open in Y since $f_{|B_\alpha}$ is a homeomorphism. Therefore f is an open map.

$\boxed{\text{Case 2: } \{B_\alpha\} \text{ is a closed, locally finite cover of } Y}$ Suppose that $V \subset Y$ is some arbitrary closed set. Then we know that $\{f^{-1}(B_\alpha)\}$ is a closed locally-finite cover of X. Then similarly to case 1, except this time with $F \subset A$ being finite.

$$V = \bigcup_{a \in F} (f^{-1}(B_\alpha) \cap V) \quad \Longrightarrow \quad f(V) = \bigcup_{a \in F} f(B_a \cap V)$$

Then, we know that $f(B_\alpha \cap V)$ is a closed set, so a finite union of closed sets is closed therefore $f(V)$ is closed, and f is a closed map.

In either case f is a bijective continuous open and closed map, therefore it is a homeomorphism. $\square$