Algebraic Topology - Homework 1

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Problem 0.3

Assume, for $n \geq 1$, that $H_i(S^n) = \mathbb{Z}$ if $i = 0, n$, and that $H_i(S^n) = 0$ otherwise. Using the technique of the proof of Lemma 0.2, prove that the equator of the n-sphere is not a retract.

Proof. Assume that S^{n-1} is the equator of the n-sphere. Suppose by contradiction that S^{n-1} is a retract of S^n. Then it follows that there exists some retraction $r : S^n \to S^{n-1}$. Then, with $i : S^{n-1} \to S^n$ being the inclusion map, and with 1 being, of course, the identity map, it follows that we would have a commutative diagram:

$$\begin{tikzcd} S^{n-1} \arrow[rr, "1"] \arrow[dr, "i"'] & & S^{n-1} \\ & S^n \arrow[ur, "r"'] & \end{tikzcd}$$

To this diagram, we can apply our homology functor, giving us

$$\begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[rr, "H_{n-1}(1)"] \arrow[dr, "H_{n-1}(i)"'] & & H_{n-1}(S^{n-1}) \\ & H_{n-1}(S^n) \arrow[ur, "H_{n-1}(r)"'] & \end{tikzcd}$$

We know by assumption that $H_{n-1}(S^n) = 0$ since $n - 1 \neq n$ and that $H_n(S^{n-1}) = \mathbb{Z}$ since $n - 1 = n - 1$. This new diagram should continue to commute by the properties of our functor H_{n-1}. Since $H_{n-1}(S^n) = 0$, it follows that its image under $H_{n-1}(r)$ must also be zero. That is, $H_{n-1}(S^{n-1}) = 0$. However, this means that our identity map 1 takes a countable algebra to a trivial one, which is a contradiction. therefore S^{n-1} cannot be a retract of S^n. □

Problem 0.5

Problem 0.7

Let $f \in \text{Hom}(A, B)$, and let $g, h \in \text{Hom}(B, A)$ such that $g \circ f = 1_A$ and that $f \circ h = 1_B$. Then $g = h$.

Proof. Suppose that $h \neq g$,

$$\begin{gathered} h \neq g \\ h \circ f \neq g \circ f \\ f \circ h \circ f \neq f \circ g \circ f \\ (f \circ h) \circ f \neq f \circ (g \circ f) \\ 1_B \circ f \neq f \circ 1_A \\ f \neq f \end{gathered}$$

However, this is a contradiction, and we conclude that $h = g$. □

Problem 0.18

For an abelian group G, let
$$tG = \{x \in G \ : \ \text{x has finite order}\}$$
denote its torsion subgroup.

(ii)