General Relativity - Homework 6

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May 19, 2021

Problem 1

a)

We must manipulate our function algebraically as follows,

$$\begin{aligned} 8mr &= h^2 + 16m^2 \\ 8mr - 16m^2 &= h^2 \\ \pm\sqrt{8mr - 16m^2} &= h \end{aligned}$$

Plotting this function with m fixed to be some constant (in our case $m = 1$, but the overall shape is invariant), we get

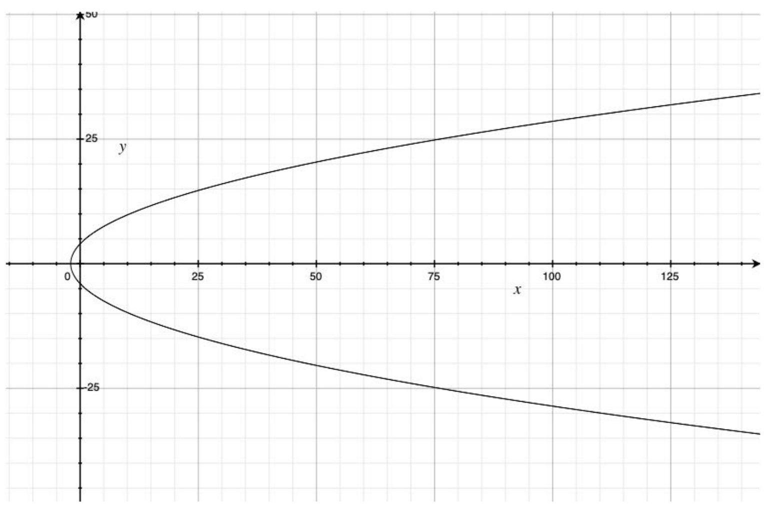

Figure 1: $h = \pm\sqrt{8mr - 16m^2}$

b)

To compute the arclength in terms of r and dr, we wish to use the Euclidean line element. We say that

$$ds^2 = dr^2 + dh^2$$

We must take the derivative of h with respect to r.

$$\frac{d}{dr}\left(\pm\sqrt{8mr - 16m^2}\right) = \frac{\pm 8m}{2\sqrt{8mr - 16m^2}}$$

Then we want to use this to compute our arclength.

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+f'^2(r)}dr \\
&= \int_a^b \sqrt{1+\left(\frac{\pm 8m}{2\sqrt{8mr-16m^2}}\right)^2} dr \\
&= \int_a^b \sqrt{1+\frac{2m^2}{mr-2m^2}} dr \\
&= \int_a^b \sqrt{\frac{mr}{mr-2m^2}} dr \\
&= \int_a^b \sqrt{\frac{r}{r-2m}} dr \\
&= \int_a^b \frac{dr}{\sqrt{1-\frac{2m}{r}}}
\end{aligned}
$$

And we say that this is the arclength of our parabola based on a, b and m. However, we notice that this term within the integral looks highly similar to σ^r from Schwarzchild geometry.

c)

To rotate around the h-axis, consider a new coordinate, ϕ. If we take our line element, and write dh in terms of r and dr we may be able to more easily understand the geometry of what we are doing. Since $h = \pm\sqrt{8mr - 16m^2}$ we can write

$$
\begin{aligned}
h &= \sqrt{8mr-16m^2} \\
dh &= d\left(\sqrt{8mr-16m^2}\right) \\
&= 0 \cdot dm + \frac{dr}{\sqrt{\frac{r}{2m}-1}}
\end{aligned}
$$

Then our line element becomes

$$
\begin{aligned}
ds^2 &= dr^2 + dh^2 \\
&= dr^2 + \left(\frac{dr}{\sqrt{\frac{r}{2m}-1}}\right)^2 \\
&= dr^2\left(1+\frac{1}{\frac{r}{2m}-1}\right) \\
&= dr^2\left(\frac{\frac{r}{2m}-1+1}{\frac{r}{2m}-1}\right) \\
&= dr^2\left(\frac{\frac{r}{2m}}{\frac{r}{2m}-1}\right) \\
&= dr^2\left(\frac{1}{1-\frac{2m}{r}}\right) \\
&= \frac{dr^2}{1-\frac{2m}{r}}
\end{aligned}
$$

Knowing that if we fix some radius, are arclength traveling along ϕ should mimic a circular arclength (given that we are rotating our whole parabola in a circle), we add a ϕ piece to our line element yielding

$$
ds^2 = \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\phi
$$

Then if $dr = 0$ and we have a fixed radius, we should get arclength that is equal to a circle. Notice that this is equal to the line element of the Schwarzchild geometry given that $dt = 0$ and that $\theta = \frac{\pi}{2}$.

d)

Given our line element ds^2 we say that our basis forms are given by

$$\sigma^r = \frac{dr}{\sqrt{1-\frac{2m}{r}}}$$
$$\sigma^\phi = r d\phi$$

So then by the torsion free condition we say that

$$0 = d\sigma^r + \omega^r{}_\phi \wedge \sigma^\phi$$
$$0 = d\left(\frac{dr}{\sqrt{1-\frac{2m}{r}}}\right) + \omega^r{}_\phi \wedge r d\phi$$
$$0 = \omega^r{}_\phi \wedge r d\phi$$

By the torsion free condition we say that

$$0 = d\sigma^\phi + \omega^\phi{}_r \wedge \sigma^r$$
$$0 = d(r d\phi) + \omega^\phi{}_r \wedge \left(\frac{dr}{\sqrt{1-\frac{2m}{r}}}\right)$$
$$0 = dr \wedge d\phi + r \wedge d^2\phi + \omega^\phi{}_r \wedge \left(\frac{dr}{\sqrt{1-\frac{2m}{r}}}\right)$$
$$0 = dr \wedge d\phi + \omega^\phi{}_r \wedge \left(\frac{dr}{\sqrt{1-\frac{2m}{r}}}\right)$$
$$0 = \sqrt{1-\frac{2m}{r}} dr \wedge d\phi + \omega^\phi{}_r \wedge dr$$
$$\sqrt{1-\frac{2m}{r}} d\phi \wedge dr = \omega^\phi{}_r \wedge dr$$
$$\sqrt{1-\frac{2m}{r}} d\phi = \omega^\phi{}_r$$

Then by metric compatibility we say that $\omega^r{}_\phi = -\sqrt{1-\frac{2m}{r}} d\phi$. Finally we can write

$$\Omega^r{}_\phi = d\omega^r{}_\phi = K\omega$$
$$= d\left(-\sqrt{1-\frac{2m}{r}} d\phi\right)$$
$$K\omega = \frac{-m}{r^2} \cdot \frac{1}{\sqrt{1-\frac{2m}{r}}} dr \wedge d\phi$$

Then since ω is our orientation which given the line element can be written

$$\omega = \frac{dr}{\sqrt{1-\frac{2m}{r}}} \wedge r d\phi$$

We write that

$$\begin{aligned}
K\left(\frac{dr}{\sqrt{1-\frac{2m}{r}}}\wedge r d\phi\right) &= \frac{-m}{r^2}\cdot\frac{1}{\sqrt{1-\frac{2m}{r}}}dr\wedge d\phi \\
K\left(\frac{r}{\sqrt{1-\frac{2m}{r}}}\right)dr\wedge d\phi &= \frac{-m}{r^2}\cdot\frac{1}{\sqrt{1-\frac{2m}{r}}}dr\wedge d\phi \\
K\left(\frac{r}{\sqrt{1-\frac{2m}{r}}}\right) &= \frac{-m}{r^2}\cdot\frac{1}{\sqrt{1-\frac{2m}{r}}} \\
Kr &= \frac{-m}{r^2} \\
K &= \frac{-m}{r^3}
\end{aligned}$$

And we say that this is the curvature of our surface devised in $\boxed{\text{1c}}$.

Problem 2

a)

We seek to describe $\vec{e}_i \cdot \vec{\nabla} f$ in terms of partial derivatives. Now we can say that

$$\begin{aligned}
\vec{e}_i \cdot \vec{\nabla} f = \vec{e}_i \cdot \begin{pmatrix} \frac{\partial}{\partial x_1}\vec{e}_1 \\ \vdots \\ \frac{\partial}{\partial x_n}\vec{e}_n \end{pmatrix} \\
= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\vec{e}_i \cdot \vec{e}_j \\
= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} g_{ij}
\end{aligned}$$

Which is the sum of the Jacobian J_g where g is the metric on the i-th row.

b)

We wish to express $g^{ij} = g(dx^i, dx^j)$ in terms of of components g_{ij}. Let $\vec{u}_i$ be a vector such that $\vec{u}_i \cdot d\vec{r} = dx^i$. This vector must exist by definition of our basis forms. Now we get the following system of equations

$$\begin{aligned}
dx^i = \sum_{j=1}^{n} dx^j(\vec{u}_i \cdot \vec{e}_j) \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix} \\
= \begin{pmatrix} \sum_{j=1}^{n} dx^j(\vec{u}_1 \cdot \vec{e}_j) \\ \vdots \\ \sum_{j=1}^{n} dx^j(\vec{u}_n \cdot \vec{e}_j) \end{pmatrix} \\
= \begin{pmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_n \end{pmatrix} \cdot \begin{pmatrix} dx^1\vec{e}_1 \\ \vdots \\ dx^n\vec{e}_n \end{pmatrix}
\end{aligned}$$

We know that along the diagonal of g we will get 1's, and elsewhere we will get 0's, given we have an orthonormal basis, so it follows that $g^{ij} = g(dx^i, dx^j) = \vec{u}_i \cdot \vec{u}_j$. Thus we have

$$\sum_{j=1}^{n} g^{ij} g_{jk} = \delta^i{}_k$$

Where ${\delta^i}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. So it follows that we can write this as a matrix product

$$g^\circ \times g_\circ = I$$

And it follows that g_{ij} works as the corresponding element to the inverse matrix of g^{ij}.