MTH 311 Homework 3

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2.2.2

c.)

Show that $\lim \frac{sin(n^2)}{\sqrt[3]{n}} = 0$.

Proof. Choose $\epsilon > 0$, arbitrarily. Let $N_\epsilon > \frac{1}{\epsilon^3}$. We can therefore write that for all $n \geqslant N_\epsilon$ we have $n > \frac{1}{\epsilon^3}$ which is equivalent to stating that $\epsilon^3 n > 1$. By properties of the sine function we also know that $|sin(n^2)| \leqslant 1$ and therefore $|sin(n^2)|^3 \leqslant 1^3 = 1$. From there, we have by ordering that $|sin(n^2)|^3 < \epsilon^3 n$. We can divide this expression by n to get the inequality $\frac{|sin(n^2)|^3}{n} < \epsilon^3$. Since both sides are positive, this is equivalent to $\frac{|sin(n^2)|}{\sqrt[3]{n}} < \epsilon$. With both the numerator and denominator positive, we write $\left|\frac{sin(n^2)}{\sqrt[3]{n}}\right| < \epsilon$. Trivially subtracting a zero we get $\left|\frac{sin(n^2)}{\sqrt[3]{n}} - 0\right| < \epsilon$. By definition of convergence we have $\lim \frac{sin(n^2)}{\sqrt[3]{n}} = 0$.

□

2.2.5

a.)

Let $[[x]]$ be the floor function, where $[[x]]$ is equal to the greatest integer less than or equal to x. Show that $\lim \left[\left[\frac{5}{n}\right]\right] = 0$.

Proof. Let $\epsilon > 0$, chosen arbitrarily. Choose $N_\epsilon > \frac{5}{\epsilon}$. From there we have for all $n \geqslant N_\epsilon$ that $n > \frac{5}{\epsilon}$, which is equivalent to stating $\epsilon > \frac{5}{n}$. With n positive, we have $\left[\left[\frac{5}{n}\right]\right] \leqslant \frac{5}{n} < \epsilon$, and with all terms being always positive this is equivalent to $\left|\left[\left[\frac{5}{n}\right]\right] - 0\right| < \epsilon$. Therefore, we have $\lim \left[\left[\frac{5}{n}\right]\right] = 0$.

□

2.2.7

a.)

Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

The sequence $(-1)^n$ is not eventually in the set $\{1\}$. In other words, for all $N \in \mathbb{N}$, $\exists n \geqslant N : (-1)^n \notin \{1\}$.

Proof. Let $N \in \mathbb{N}$, arbitrarily. There are two cases, $(-1)^N = (-1)$, and $(-1)^N = 1$.

$\boxed{\text{Case 1: } (-1)^N = (-1)}$ Let $n = N$. We have $(-1)^n = (-1)^N = (-1)$, and $(-1) \notin \{1\}$.

$\boxed{\text{Case 2: } (-1)^N = 1}$ Let $n = N + 1$. From there we can write $(-1)^n = (-1)^{N+1}$, equivalent to $(-1)^N(-1)^1$. Thus, we have $1(-1) = -1 \notin \{1\}$.

□

The sequence $(-1)^n$ is frequently in the set $\{1\}$ (ie. $\forall N \in \mathbb{N},\ \exists n \geqslant N : (-1)^n \in \{1\}$).

Proof. Let $N \in \mathbb{N}$, arbitrarily. There are once again two cases, $(-1)^N = (-1)$, and $(-1)^N = 1$.

$\boxed{\text{Case 1: } (-1)^N = (-1)}$ Let $n = N + 1$. We have $(-1)^n = (-1)^{N+1}$ which is equal to $(-1)^N(-1) = 1$. Since $1 \in \{1\}$, we have $(-1)^n \in 1$ for some $n \geqslant N$.

$\boxed{\text{Case 2: } (-1)^N = 1}$ Let $n = N$. From there we can write $(-1)^n = 1 \in \{1\}$.

□

b.)

Eventually is stronger than frequently. We can state that eventually implies frequently, but frequently does not imply eventually.

Suppose we have a sequence a_n eventually in the set S. Eventaully implies frequently because if there exists N_e where $\forall n_e \geqslant N_e$, $a_{n_e} \in S$, then for every $N_f \in \mathbb{N}$, choose $n_f \geqslant \max\{N_e, N_f\}$. Then we have $a_{n_f} \in S$ since $n_f \geqslant N_e$.

This does not hold in the other direction, since frequently only defines that there must exist one index greater than any N that is in the proposed set. Therefore it cannot be said that all indecies greater than some N will be in the set.

c.)

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_e(a)$ of a, (a_n) is eventually in $V_e(a)$.

d.)

Suppose an infinite number of terms of a sequence x_n are equal to 2. This does not imply that x_n is eventually in the interval $(1.9, 2.1)$. The counterexample would be suppose $x_n = \{1, 2, 1, 2, 1, 2, ...\}$.

However, having an infinite number of terms be equal to 2 does imply that x_n would be frequently in the aforementioned interval. Suppose that this implication was not true, then there would be some N after which all x_n would not be in the interval, and therefore not equal to two. Thus there would be a finite number of terms equal to 2, which is a contradiction.

2.3.9

Suppose a_n bounded and $\lim b_n = 0$. Show that $\lim a_n b_n = 0$.

Proof. Since a_n is bounded we know there exists $M > 0$ such that $|a_n| \leqslant M$ for all $n \in \mathbb{N}$. Also since b_n converges to 0 we have $\forall \epsilon > 0\ \exists N_{\epsilon B} : \forall n \geqslant N_{\epsilon B},\ \ |b_n - 0| < \epsilon$.

Let $\epsilon_j > 0$ be arbitrary. Choose $N_{\epsilon AB} = N_{\epsilon B}$. We know that for all $n \geqslant N_{\epsilon AB}$, the following hold:

$$
\begin{aligned}
|a_n| &\leqslant M \\
|b_n - 0| &< \epsilon_j \qquad \text{(with all terms positive we can multiply these inequalities resulting in strict inequality)} \\
\implies |a_n||b_n - 0| &< M(\epsilon_j) \\
|a_n||b_n| = |a_n b_n - 0| &< M(\epsilon_j)
\end{aligned}
$$

Since we chose ϵ_j arbitrarily, we can write this inequality for any other $\epsilon_k > 0$ by letting $\epsilon_k = M\epsilon_j$. Therefore we can say that for all $\epsilon_k > 0$, there exists $N_{\epsilon AB}$ such that for all $n \geqslant N_{\epsilon AB}$, $|a_n b_n - 0| < \epsilon_k$. Thus $\lim a_n b_n = 0$.

□