Applied Ordinary Differential Equations - Homework 2

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7.5.11

> Solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.
>
> $$\boldsymbol{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

First off we want to find our eigenvalues. To do this, we just take the determinant of $A - \lambda I$, set it equal to 0 and solve for λ.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} \\ &= (-2-\lambda)(4-\lambda) - (1)(-5) \\ &= \lambda^2 - 2\lambda - 8 + 5 \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda - 3)(\lambda + 1) \end{aligned}$$

So we conclude that $\lambda_1 = 3, \lambda_2 = -1$. So since we have one positive and one negative real eigenvalue, we know that we will have a saddle point style solution. To get the general solution, we'll solve for the eigenvectors. We write

$$A - \lambda_1 I = \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \quad \implies \quad \boldsymbol{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \quad \implies \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This yields the general solution

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Take the fact that $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and it follows that $c_1 = c_2 = \frac{1}{2}$. So we have a solution

$$\boldsymbol{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

This solution will asymptotically approach the span of $(1 \;\; 5)^T$ as $t \to \infty$. The general qualitative properties of the given solution can be seen in Figure 1.

7.5.23

> Consider the system
>
> $$\boldsymbol{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \boldsymbol{x}$$

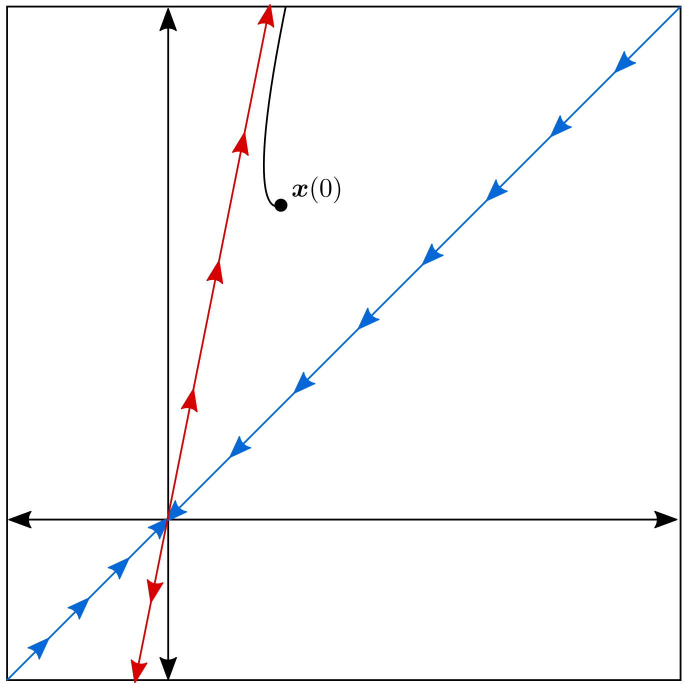


Figure 1: Solution to the diffential equation given in $\boxed{7.5.11}$

(a)

Solve the system for $\alpha = \frac{1}{2}$. Find the eigenvalues of the coefficient matrix, and classify the type of equilibrium point at the origin.

Let $\alpha = \frac{1}{2}$. Then we have a characteristic polynomial given by

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 1 - \frac{1}{2} = \lambda^2 + 2\lambda + \frac{1}{2}$$

This gives us two eigenvalues of $\lambda_1 = -1 + \frac{\sqrt{2}}{2}, \lambda_2 = -1 - \frac{\sqrt{2}}{2}$. Since both eigenvalues are negative, we say that the origin is an unstable 'source' equilibrium point.

(b)

Solve the system for $\alpha = 2$. Find the eigenvalues of the coefficient matrix, and classify the type of equilibrium point at the origin.

Let $\alpha = 2$. Then we have a characteristic polynomial given by

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 1 - 2 = \lambda^2 + 2\lambda - 1$$

This gives us two eigenvalues of $\lambda_1 = -1 + \sqrt{2}, \lambda_2 = -1 - \sqrt{2}$. Since we have $\lambda_1 > 0$ and $\lambda_2 < 0$, we know that we have an unstable saddle point equilibrium.

(c)

We can find the bifurcation point by writing

$$\begin{aligned}\det\begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} &= (-1 - \lambda)^2 - (-1)(-\alpha) \\ &= \lambda^2 + 2\lambda + 1 - \alpha\end{aligned}$$

From here we can determine the eigenvalues as

$$\begin{aligned}\lambda &= \frac{-2 \pm \sqrt{2^2 - 4(1)(1 - \alpha)}}{2} \\ &= -1 \pm \frac{\sqrt{4 - 4 + 4\alpha}}{2} \\ &= -1 \pm \sqrt{\alpha}\end{aligned}$$

The bifurcation point is exactly when one of our eigenvalues is equal to 0 because that is when one of them changes from positive to negative as we vary α. This occurs when $\alpha = 1$ which lies between $\frac{1}{2}$ and 2 as desired.

7.6.5

Compute the general solution in terms of real-valued functions of the following system of differential equations,

$$x' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} x$$

We write

$$\begin{aligned} \det \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} &= (1-\lambda)\left((1-\lambda)(1-\lambda) - (2)(-2)\right) \\ &= (1-\lambda)(\lambda^2 + 2\lambda + 1 + 4) \end{aligned}$$

This gives us eigenvalues of $\lambda_1 = 1, \lambda_2 = 1 - 2i, \lambda_3 = 1 + 2i$. From these values we can determine that the eigenvectors are given by

$$\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$$

Then we can write the general solution,

$$\boldsymbol{x} = c_1 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} e^t + c_2 \boldsymbol{u} + c_3 \boldsymbol{v}$$

Where

$$\begin{aligned} \boldsymbol{u}(t) &= e^t \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(-2t) - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \sin(-2t) \right) \\ \boldsymbol{v}(t) &= e^t \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin(-2t) + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \cos(-2t) \right) \end{aligned}$$

7.6.11

Determine eigenvalues in terms of α, and then find and sketch the bevior around the bifurcation point of this parameter.

$$\boldsymbol{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \boldsymbol{x}$$

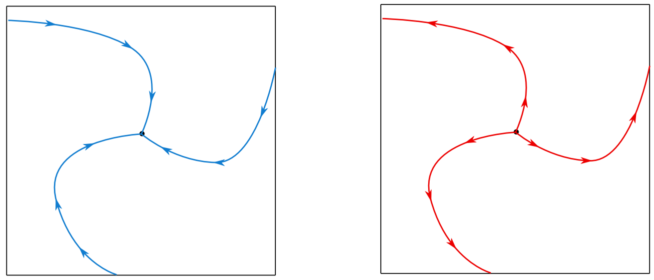
We write

$$\begin{aligned} \det \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} - \lambda \boldsymbol{I} &= 0 \\ (\alpha - \lambda)^2 + (1)(1) &= 0 \\ \lambda^2 - 2\alpha\lambda + \alpha^2 + 1 &= 0 \\ \lambda^2 - 2\alpha\lambda + \alpha^2 &= 1 \\ (\lambda - \alpha)^2 &= -1 \end{aligned}$$

This will give us two eigenvalues given by

$$\lambda = \alpha \pm i$$

The real part of these eigenvalues is positive if and only if α is also positive, giving us a bifurcation point at $\alpha = 0$. Our solutions are stable when $\alpha < 0$, and unstable when $\alpha > 0$. When $\alpha < 0$ our solutions will spiral toward the origin, and when $\alpha > 0$ they will spiral away. This is pictured in our sketches in Figure 2



$\alpha < 0$ $\alpha > 0$

Figure 2: Sketch of phase portraits for given values of α

7.7.12

Show that $\Phi(t)$ does have the principle algebraic properties associated with the exponential.

Proof. We first wanna show that $\Phi(t)\Phi(s) = \Phi(t+s)$. Without loss of generality let s be fixed and t be a variable, then we know that $\Phi(s)$ is some fixed matrix. Now let $Z(t) = \Phi(t)\Phi(s)$. Then we can say that

$$Z(0) = \Phi(0)\Phi(s) = \boldsymbol{I}\Phi(s) = \Phi(s)$$

Now let $Y(t) = \Phi(t+s)$. Then we similarly get

$$Y(0) = \Phi(0+s) = \Phi(s)$$

Given that we can fix any s and let t vary, it follows that this will hold for any initial value given this system of equations. So we conclude that $\Phi(x)\Phi(y) = \Phi(x+y)$.

Now we look at the property stating that $\Phi(t)\Phi(-t) = \boldsymbol{I}$. Now this follows somewhat trivially from our previous result. That is,

$$\Phi(t)\Phi(-t) = \Phi(t-t) = \Phi(0) = \boldsymbol{I}$$

So for any value t we have a matrix that, right-multiplied with $\Phi(t)$, gives us the identity. This gives us the following result

$$\begin{aligned} \Phi(t)\Phi(-t) &= \boldsymbol{I} \\ \Phi^{-1}(t)\Phi(t)\Phi(-t) &= \Phi^{-1}(t)\boldsymbol{I} \\ \Phi(-t) &= \Phi^{-1}(t) \end{aligned}$$

And then finally, it follows obviously that

$$\begin{aligned} \Phi(t-s) &= \Phi(t+(-s)) \\ &= \Phi(t)\Phi(-s) \\ &= \Phi(t)\Phi^{-1}(s) \end{aligned}$$

□