MTH 351 HW 5

Philip Warton

February 21, 2020

1.

Consider the function $f(x) = \frac{1}{4}(5 - x^2)$.

a.

We want to solve for all fixed points of f. We wish to find all x where $f(x) = x$. We have,

$$\frac{1}{4}(5 - x^2) = x.$$

By arithmatic rearangement this is equivalent to the equation

$$0 = x^2 + 4x - 5.$$

Thus, we can factor the right hand side giving us,

$$0 = (x - 1)(x + 5)$$

This gives us solutions at $x = 1$ and $x = -5$.

b.

For an iterative formula, we write

$$\frac{1}{4}(5 - {x_n}^2) = x_{n+1}.$$

With $x_0 = 0.8$, we can plug this into our iterative formula on a calculator, which gives us $lim(x_n) = 1$.

c.

To verify this, let $L = lim(x_n) = lim(x_{n+1})$. Then,

$$\frac{1}{4}(5 - L^2) = L$$

By arithmetic from $\boxed{\text{1a.}}$, we have $L = 1$ and $L = -5$.

d.

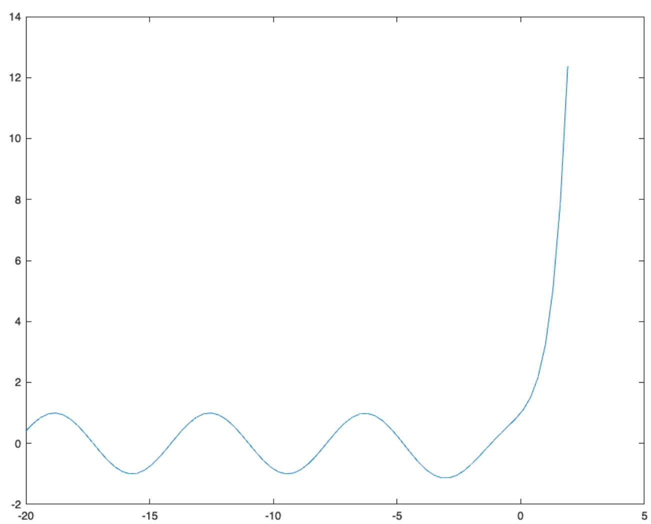
To find the order of convergence, let $e_n = |x_n - \alpha|$ and $e_{n+1} = |x_{n+1} - \alpha|$. Subtracting 1 from both sides of our iterative formula, we have

$$\begin{aligned} x_{n+1} - 1 &= \frac{1}{4}(5 - x_n^2) - 1 \\ e_{n+1} &= \frac{-1}{4}(x_n^2 - 1) \\ &= \frac{-1}{4}(e_n^2 - 2x_n) && \text{(for large enough } n, x = 1 \text{ thus)} \\ e_{n+1} &= \frac{2 - e_n^2}{4} \end{aligned}$$

And we have an order of convergence of 2 for f.

2.

a.



b.

To find the largest root for $f(x) = xe^x + cos(x)$ using Newton's method, let $x_0 = 0$. We know that $f'(x) = xe^x + e^x - sin(x)$. So our iteration method is

$$x_{n+1} = x_n e^{x_n} + cos(x_n) - \frac{x_n e^{x_n} + cos(x_n)}{x_n e^{x_n} + e^{x_n} - sin(x_n)}$$

The iteration will stop when $|x_n - x_{n+1}| < 10^{-4}$. Using the online Geogebra app for Newton's root-finding method, we get that after 3 iterations we can stop, with the largest root being approximately at $x = -1.20106$.

c.

To convert this problem to a fixed-point problem, we must begin with the equation that has solutions at every root,

$$\begin{aligned} 0 &= xe^x + cos(x) \\ -x &= xe^x + cos(x) - x \\ x &= x - xe^x - cos(x) \end{aligned}$$

Now we have a fixed point problem with solutions at the roots of f. We have the iteration formula

$$x_{n+1} = x_n - x_n e^{x_n} - cos(x_n)$$

Let $x_0 = 0$, and we will stop iteration when $|x_n - x_{n+1}| < 10^{-4}$. After 6 iterations, we get the largest root to be at $x = -1.2011$, and there may be rounding error due to my use of the Geogebra cobweb diagram applet.

3.

Let $f(x) = x^2$.

a.

See hand-drawn paper attached

b.

For the iteration formula of Newton's method, we have $x_{n+1} = x_n + \dfrac{f(x)}{f'(x)}$. We can write

$$x_{n+1} = x_n + \frac{x_n^2}{2x_n}$$

c.

To find the limit of x_n, let $L = lim(x_n) = lim(x_{n+1})$. Then we have

$$\begin{aligned} L &= L - \frac{L^2}{L} \\ &= L - L \\ &= 0 \end{aligned}$$

For the order of convergence, let $e_{n+1} = x_{n+1} - 0$ and $e_n = x_n - 0$. Then,

$$\begin{aligned} x_{n+1} &= x_n + \frac{x_n^2}{2x_n} \\ e_{n+1} &= x_n + \frac{x_n}{2} \\ &= \frac{x_n}{2} \\ &= \frac{e_n}{2} \end{aligned}$$

Thus, we have a linear order of convergence with a rate of $\dfrac{1}{2}$.

4.

Find a polynomial passing through$(-1, 1), (0, -1), (1, 0), (2, 2)$, denoted by $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$.

a.

Let $L_1(x) = \dfrac{(x-0)(x-1)(x-2)}{(-1)(-2)(-3)} = \dfrac{x^3-3x^2+2x}{-6}$, and let $L_2(x) = \dfrac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} = \dfrac{x^3-2x^2-x+2}{2}$. Then let $L_3(x) = \dfrac{(x+1)(x)(x-2)}{(2)(1)(-1)} = \dfrac{x^3-x^2-2x}{-2}$. Finally, let $L_4(x) = \dfrac{(x+1)(x)(x-1)}{(3)(2)(1)} = \dfrac{x^3-x}{6}$. Then we have

$$P(x) = y_1L_1(x) + y_2L_2(x) + y_3L_3(x) + y_4L_4(x)$$

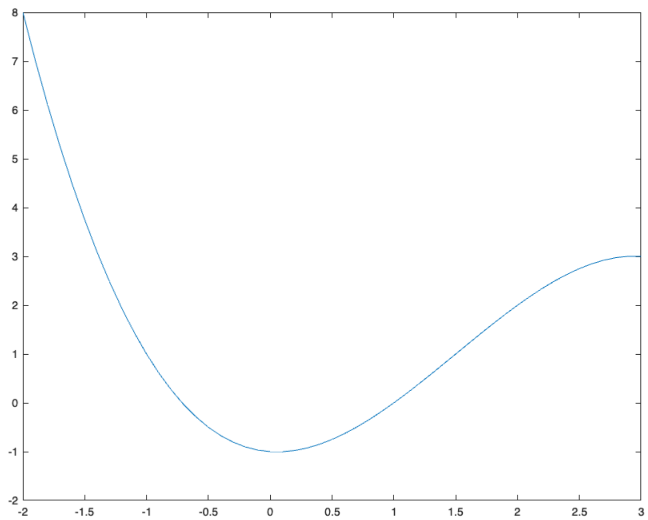
Given that we know y_1, y_2, y_3, y_4 and $L_1(x), L_2(x), L_3(x), L_4(x)$, we have

$$P(x) = (1)\frac{x^3-3x^2+2x}{-6} + (-1)\frac{x^3-2x^2-x+2}{2} + (0)\frac{x^3-x^2-2x}{-2} + (2)\frac{x^3-x}{6}$$

This can be simplified to

$$P(x) = \frac{-1}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{6}x - 1$$

b.



c.

Let $x = 1.5$. Then $P(1.5) = \dfrac{-1}{3}\left(\dfrac{27}{8}\right) + \dfrac{3}{2}\left(\dfrac{9}{4}\right) - \dfrac{1}{6}\left(\dfrac{3}{2}\right) - 1$, which, when simplified, equals 1. For the slope at 1.5, write

$$P'(x) = -x^2 + 3x - \frac{1}{6}$$

Then we have

$$P'(1.5) = -\frac{9}{4} + \frac{9}{2} - \frac{1}{6} = \frac{25}{12}$$