# MTH 342 Homework 2

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## 1.

Let $U, V$ be subspaces of vector space $W$.

### a.)

Show $U + V$ is a subspace of $W$.

*Proof.* We must begin by showing $U + V \subseteq W$, and then show closure under vector addition, closure under scalar multiplication, and the presence of the additive identity $\mathbf{0}$.

Let $u \in U$ and $v \in V$, chosen arbitrarily. Since $U \subseteq W$ and $u \in U$, we have $u \in W$. Similarly, we can say $v \in W$. Because $W$ is a vector space, we know that $W$ is closed under vector addition, and therefore with $u, v \in W$, and therefore $u + v \in W$. Now we have $U + V \subseteq W$.

Let $x, y \in U + V$. By definition of $U + V$ we know $\exists u_1, u_2 \in U,\ \exists v_1, v_2 \in V : x = u_1 + v_1$ and $y = u_2 + v_2$. We can write

$$\begin{aligned} x + y &= (u_1 + v_1) + (u_2 + v_2) \\ &= u_1 + v_1 + u_2 + v_2 && \text{(by associative addition in } W\text{)} \\ &= u_1 + u_2 + v_1 + v_2 && \text{(by commutative addition in } W\text{)} \\ &= (u_1 + u_2) + (v_1 + v_2) \end{aligned}$$

Withh $u_1, u_2 \in U$, and $U$ being a vector space, we know that $(u_1 + u_2) \in U$. Similarly, we know $(v_1 + v_2) \in V$. Hence, we have $x + y$ is the sum of a vector in $U$ and a vector in $V$, and $U + V$ is closed under vector addition.

Next, we must show closure under scalar multiplication. Let $c$ be a scalar in the field $F$. Let $t$ be a vector in $U + V$. By the definition of $U + V$, we know $\exists u \in U,\ \exists v \in V : t = u + v$. Multiplying $t$ by our scalar $c$ we get

$$\begin{aligned} c(t) &= c(u + v) \\ &= c(u) + c(v) && \text{(by distributive scaling in } W\text{)} \end{aligned}$$

Since $U$ is a vector space and therefore closed under scalar multiplication, and $u \in U$, we know $c(u) \in U$. Similarly, we know that $c(v) \in V$. Thus, $c(t)$ can be expressed as the sum of a vector in $U$ with a vector in $V$, and $U + V$ is closed under scalar multiplication.

Finally, we need the additive identity, $\mathbf{0}_{U+V} \in U + V$. Suppose $u, \mathbf{0}_U \in U$. Since $u \in U \subseteq W$, $u \in W$. With $u \in W$ and $u + \mathbf{0}_U = u$, we know that $\mathbf{0}_U = \mathbf{0}_W$ by the uniqueness of the additive identity in $W$. Using the same methods it can be shown that $\mathbf{0}_V = \mathbf{0}_W = \mathbf{0}_U$. Since $\mathbf{0}_U + \mathbf{0}_V = \mathbf{0}_W$ is the sum of a vector in $U$ and a vector in $V$, we have $\mathbf{0}_W \in U + V$. $\mathbf{0}_W$ is the additive identity for all vectors in $W$ including any subset of $W$.

□

**b.)**

Let $w \in W \setminus V$. Show $w + v \notin V \ \ \forall v \in V$.

*Proof.* Suppose by contradiction that $w + v \in V$. Let $u \in V : u = v + w$. By the existence of an additive inverse we have $v + (-v) = 0$. Therefore we can write

$$\begin{aligned} u &= v + w \\ u + (-v) &= v + w + (-v) \\ &= v + (-v) + w \\ &= (v + (-v)) + w \\ &= \mathbf{0} + w \\ &= w \end{aligned}$$

Since both $u, (-v) \in V$, we know $u + (-v) = w \in V$. Therfore $w \notin W \setminus V$ (contradiction). Thus $w + v \notin V$. $\square$

**c.)**

Show that $U \cup V$ is a subspace of $W$ if and only if either $U \subset V$ or $V \subset U$.

*Proof.* We must show that the implication holds in both directions.

"$\Rightarrow$" Assume that $U \cup V$ is a subspace of $W$. We want to show that $U \subset V$ or $V \subset U$. Pick $u \in U$ and $v \in V$ arbitrarily. We have $u, v \in U \cup V$, and since we assume $U \cup V$ to be a vector space, we know $u + v \in U \cup V$. Therefore it must be the case that either $u + v \in U$ or $u + v \in V$. If $u + v \in U$, then we have $v \in V \implies v \in U$, and $V \subset U$. In the other case, it can similarly be shown that $U \subset V$. This shows that the forwards implication holds.

"$\Leftarrow$" Assume that either $U \subset V$ or $V \subset U$. If $U \subset V$ then $U \cup V = V$ and is therefore a subspace of $W$. The same holds if $V \subset U$. Thus the implication holds in the backwards direction. $\square$

**d.)**

Show that $U + V$ is a direct sum if and only if $U \cap V = \{\mathbf{0}\}$.

*Proof.* To show the double implication we must show that the implication holds going both forwards and backwards.

"$\Rightarrow$" Assume $U + V$ is a direct sum, and therefore $x \in U + V$ has one unique pair of $u, v$ where $u \in U$ and $v \in V$ such that $x = u + v$. Suppose $y \in U$ and $y \in V$. Since $y \in U$ and $U$ is a subspace we also have $-y \in U$. With $-y \in U$ and $y \in V$ we know that $y + (-y) \in U + V$. Since every vector has one unique set of parts, $\mathbf{0_U} \in U, \mathbf{0_V} \in V$, and $\mathbf{0} \in U + V$, it must be the case that $y + (-y) = \mathbf{0_V} + \mathbf{0_U} \Rightarrow y = \mathbf{0}$. Therefore, any vector $y \in U \cap V$ is equal to $\mathbf{0}$, thus $U \cap V = \{\mathbf{0}\}$.

"$\Leftarrow$" Assume $U \cap V = \{\mathbf{0}\}$. There are three cases for vectors in $U + V$. Either $t \in U \setminus V$, $t \in V \setminus U$, or $t \notin U \cup V$. Suppose we take a vector $u \in U$. To write this as a vector in $U + V$ we must write it as $u + \mathbf{0}$, and thus there is only one solution. Now choose a vector $v \in V$, and it must be written as $\mathbf{0} + v$. Finally let $w \in U + V$ where $w \notin U \cup V$. Suppose $w$ can be written as $u_1 + v_1$ or as $u_2 + v_2$ where $u_1, u_2 \in U$ and $v_1, v_2 \in V$. This gives us $u_1 + v_1 = u_2 + v_2$. By reordering this can be written as $u_1 - u_2 = v_2 - v_1$ Since $u_1 - u_2 \in U$ and $v_2 - v_1 \in V$, we can write $u_1 - u_2 = v_2 - v_1 \in U \cap V$. With $U \cap V = \{\mathbf{0}\}$ this means that $u_1 = u_2$ and $v_1 = v_2$ and therefore $w \in U + V$ has a unique solution. $\square$

## 2.

Let A be the set of vectors $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ x \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ x \\ x^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} \right\}$

*Proof.* We want to show that this is a basis for polynomial space $P_3(F)$. To do this we must show that span$(A) = P_3(F)$ and that $A$ is linearly independent. Pick 4 constants arbitrarily $c_1, c_2, c_3, c_4 \in F$ where $F$ is our field. Any linear combination of the basis can be written as:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & x & x & x \\ 0 & 0 & x^2 & x^2 \\ 0 & 0 & 0 & x^3 \end{bmatrix} \times \begin{bmatrix} c_1 \\ c2 \\ c3 \\ c4 \end{bmatrix} = (c_1 + c_2 + c_3 + c_4) + x(c_2 + c_3 + c_4) + x^2(c_3 + c_4) + x^3(c_4)$$

We can say that this set of of vectors spans $P_3(F)$ if we show that each of the standard basis vectors of $P_3(F)$ can be written as a linear combination of $A$. For the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ we can write $c_1 = 1$ and $c_2, c_3, c_4 = 0$. For the vector $\begin{bmatrix} 0 \\ x \\ 0 \\ 0 \end{bmatrix}$ we can write $c_1 = -1, c_2 = 1, c_3 = 0, c_4 = 0$. Now for the vector $\begin{bmatrix} 0 \\ 0 \\ x^2 \\ 0 \end{bmatrix}$ we write $c_1 = 0, c_2 = -1, c_3 = 1, c_4 = 0$.

Finally for $\begin{bmatrix} 0 \\ 0 \\ 0 \\ x^3 \end{bmatrix}$ we have $c_1 = 0, c_2 = 0, c_3 = -1, c_4 = 1$. Since we can write all the standard basis vectors of $P_3(F)$ as linear combinations of vectors in $A$, we have span$(A) = P_3(F)$. To demonstrate linear independence, we wish to show $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & x & x & x \\ 0 & 0 & x^2 & x^2 \\ 0 & 0 & 0 & x^3 \end{bmatrix} \times \begin{bmatrix} c_1 \\ c2 \\ c3 \\ c4 \end{bmatrix} = \mathbf{0} \Rightarrow c_1, c_2, c_3, c_4 = 0$. Suppose $x = -1$, we get

$$\begin{aligned} (c_1 + c_2 + c_3 + c_4) + (-1)(c_2 + c_3 + c_4) + (1)(c_3 + c_4) + (-1)(c_4) &= 0 \\ (c_1 + c_2 + c_3 + c_4 - c_2 - c_3 - c_4) + (c_3 + c_4 - c_4) &= 0 \\ (c_1) + (c_3) &= 0 \\ \Rightarrow c_1 &= -c_3 \end{aligned}$$

Now if we choose $x = 0$ we get $(c_1 + c_2 + c_3 + c_4) = 0$, and since we have show that $c_1 = -c_3$ this can be written as $c_2 + c_4 = 0$, therefore $c_2 = -c_4$. Now suppose $x = 1$, this tells us

$$\begin{aligned} c_1 + 2c_2 + 3c_3 + 4c_4 &= 0 \\ c_1 + 2c_2 + 3(-c_1) + 4(-c_2) &= 0 \\ -2c_1 - 4c_2 &= 0 \end{aligned}$$

By replacing $c_3$ and $c_4$ we get the following:

$$\begin{aligned} c_1 + 2c_2 + 3(-c_1) + 4(-c_2) &= 0 \\ -2c_1 - 2c_2 &= 0 \\ c_1 &= c_2 \end{aligned}$$

If we have $c_1 = c_2 = -c_3 = -c_4$ then they must all be equal to zero for the previous equations to hold. Replace all coefficients with $c_1$ or $-c_1$ and observe $c_1 + 2c_1 - 3c_1 - 4c_1 = -4c_1 = 0 \Rightarrow c_1 = 0$.

□

## 3.

Let $U$ and $V$ be vector spaces over a field $F$. Let $f : U \to V$ be a linear map.

### a.)

Let $u_1, u_2, ..., u_k \in U$. Show that if $f(u_1), f(u_2), ..., f(u_k)$ are linearly independent, then their inverse images are also linearly independent.

*Proof.* Let $f(u_1), f(u_2), ..., f(u_k)$ be a set of independent vectors. By contradiction, suppose $u_1, u_2, ..., u_k$ are linearly dependent. Then there exist scalars $c_1, c_2, ..., c_k$ not all equal to 0 such that $c_1u_1 + c_2u_2 + ... + c_ku_k = \mathbf{0}$. By taking the transform of this vector we get the following:

$$\begin{aligned} f(c_1u_1 + c_2u_2 + ... + c_ku_k) &= f(\mathbf{0}) \\ f(c_1u_1 + c_2u_2 + ... + c_ku_k) &= \mathbf{0} \\ f(c_1u_1) + f(c_2u_2) + ... + f(c_ku_k) &= \mathbf{0} \\ c_1(f(u_1)) + c_2(f(u_2)) + ... + c_k(f(u_k)) &= \mathbf{0} \end{aligned}$$

This having non-trivial solutions would imply that $f(u_1), f(u_2), ..., f(u_k)$ are not linearly independent (contradiction). Therefore $u_1, u_2, ..., u_k$ are linearly independent.

□

### b.)

The function $f$ is monomorphic if and only if $f = \{\mathbf{0}\}$.

*Proof.* Let us show that the implication holds in both directions.

"$\Rightarrow$" Assume $f$ is monomorphic. Choose a vector $n \in U$ where $n \in \text{null}(f)$. If we take the image of $n$ we get $f(n) = \mathbf{0}$ by definition of null space. By taking the image of $\mathbf{0} \in U$ we get $f(\mathbf{0}) = \mathbf{0}$. Since $f$ is a monomorphism we have $f(n) = f(\mathbf{0}) \Rightarrow n = \mathbf{0}$. With $n = \mathbf{0}$ for any $n \in \text{null}(U)$, we have shown that $\text{null}(f) = \{\mathbf{0}\}$.

"$\Leftarrow$" Let us prove the contrapositive of this implication. Suppose $f$ is not monomorphic, we want to show that $\text{null}(f) \neq \{\mathbf{0}\}$. Since $f$ is not monomorphic, then there exists $x, y \in U$ such that $f(x) = f(y)$ and $x \neq y$. We can write that $x - y \neq \mathbf{0}$, and then we have a non-zero vector, $x - y$, where $f(x - y) = f(x) - f(y) = 0$. Therefore $x - y \neq \mathbf{0} \in \text{null}(f)$.

□

### c.)

Suppose $f$ is monomorphic. We want to show that linear independence is preserved under the transformation.

*Proof.* Let $X$ be a set of linearly independet vectors where $X = \{x_1, x_2, ..., x_k\}$. We can write $c_1x_1 + c_2x_2 + ... + c_kx_k = \mathbf{0} \Rightarrow c_1, c_2, ..., c_k = 0$. Since we have $\mathbf{0} = c_1x_1 + c_2x_2 + ... + c_kx_k$ we know that $f(\mathbf{0}) = f(c_1x_1 + c_2x_2 + ... + c_kx_k)$. With the zero vector being preserved under linear transformations, we can say $\mathbf{0} = f(c_1x_1 + c_2x_2 + ... + c_kx_k)$. We can rewrite this as follows:

$$\begin{aligned} \mathbf{0} &= f(c_1x_1 + c_2x_2 + ... + c_kx_k) \\ &= f(c_1x_1) + f(c_2x_2) + ... + f(c_kx_k) \\ &= c_1f(x_1) + c_2f(x_2) + ... + c_kf(x_k) \end{aligned}$$

Since this final statement is equivalent to $c_1x_1 + c_2x_2 + ... + c_kx_k = \mathbf{0}$, both imply that $c_1, c_2, ..., c_k = 0$.

□

## 4.

Suppose we have a vector space $V$ over a field $F$ and a linear transform $f : V \to F$. Let $v \in V \setminus \text{null}(f)$. We can show that $V = \text{null}(f) + \text{span}\{v\}$ and is a direct sum.

This is a direct sum.

*Proof.* Since $\text{null}(f)$ is a vector space and $v \notin \text{null}(f)$, we know that $cv \notin \text{null}(f)$ for any $c \in F$ where $c \neq 0$. Therefore, $\text{null}(f) \cap \text{span}\{v\} = \{\mathbf{0}\}$, and thus the sum is direct (by the property proven in $\boxed{\text{1d.)}}$. □