Advanced Multivariable Calculus - Homework 1

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Problem 1.64

a)

Show that if U is a unit vector and if $U \cdot V = ||U||\ ||V||$ then $V = (U \cdot V)U$.

Proof. Let U be a unit vector and suppose that $U \cdot V = ||U||\ ||V||$. Then

$$||U||\ ||V|| = (1)||V|| = ||V|| = U \cdot V$$

Then we compute the orthogonal projection of U onto V,

$$proj_V U = \frac{U \cdot V}{V \cdot V} V = \frac{||V||}{||V||^2} V = \frac{V}{||V||}$$

This is clearly a unit vector, which means that U had no change in magnitude under this projection, and must of course be parallel to V. So if $U \cdot V$ is positive and U is a unit vector parallel to V, it follows trivially that

$$(U \cdot V)U = ||V||U = V$$

$\square$

b)

Show that $|U \cdot V| = ||U||\ ||V||$ implies that U is a multiple of V.

Proof. If $U = 0$ then trivially it is equal to $0V$. If $||U|| = 1$ then refer to $\boxed{\text{1.64 (a)}}$. Otherwise we have $\frac{U}{||U||}$ is a unit vector thus

$$\begin{aligned} V &= \left(\frac{U}{||U||} \cdot V \right) \frac{U}{||U||} \\ &= \left(\frac{U}{||U||^2} \cdot V \right) U \end{aligned}$$

And since the term in parenthesis is a real number it follows that U is a multiple of V. $\square$

Problem 2.25

$F : \mathbb{R}^n \to \mathbb{R}^2$ is continuous. $g : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

a)

Show that $g \circ F$ is continuous.

Proof. Let $\epsilon > 0$ be arbitrary. By the continuity of g we know that there exists $\alpha > 0$ such that $||\vec{x} - \vec{y}|| < \alpha \implies |g(\vec{x}) - g(\vec{y})| < \epsilon$. Then by the continuity of F we know that with $\alpha > 0$ there exists some $\delta > 0$ such that $||\vec{a} - \vec{b}|| < \delta \implies ||F(\vec{a}) - F(\vec{b})|| < \alpha$. So it follows that

$$||\vec{x} - \vec{y}|| < \delta \implies ||F(\vec{x}) - F(\vec{y})|| < \alpha \implies |g(F(\vec{x})) - g(F(\vec{y}))| < \epsilon$$

$\square$

b)

Show that the funciton $g(x, y) = x + y$ is continuous.

Proof. Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Then let $\epsilon > 0$, and let $\delta = \epsilon/2$. Then if $||(x, y) - (a, b)|| < \delta$, we say

$$\begin{aligned} |g(x, y) - g(a, b)| &= |(x + y) - (a + b)| \\ &= |(x - a) + (y - b)| \\ &\leqslant |x - a| + |y - b| \\ &< \delta + \delta = 2\delta = \epsilon \end{aligned}$$

□

c)

Show that if $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are both continuous that $f_1 + f_2 : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Proof. Since f_1, f_2 are continuous it follows that $F(x, y) = (f_1(x, y), f_2(x, y))$ is continous. Then we know that the function $g(x, y) = x + y$ is continuous from $\boxed{\text{2.25b}}$. Thus from $\boxed{\text{2.25a}}$ we know that $g \circ F = f_1 + f_2$ is continuous. □

2.35

Show that the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\} \subset \mathbb{R}^3$ is open.

Proof. Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that $x_0^2 + y_0^2 < 1$. Then we have $1 - x_0^2 - y_0^2 > 0$. Choose $\epsilon = (1 - x_0^2 - y_0^2)/2$, and by the triangle inequality it follows that if $(x, y, z) \in B_\epsilon(x_0, y_0, z_0)$ then $x^2 + y^2 < 1$. Thus $B_\epsilon(x_0, y_0, z_0) \subset \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and we say that the set must be open. □

2.36

Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$. Then $(0, 0) \in \partial S$.

Proof. Let $\epsilon > 0$ be arbitrary. Trivially, $B_\epsilon(0, 0) \cap S^c \neq \emptyset$ since $(0, 0) \in B_\epsilon(0, 0)$. Then choose $x \in \mathbb{R} : 0 < x < \epsilon$, we have $||(0, x) - (0, 0)|| = x < \epsilon$, therefore $(0, x) \in B_\epsilon(0, 0)$ and we say $B_\epsilon(0, 0)$ has a non-empty intersection with S. Since any neighborhood of $(0, 0)$ intersects both S and its complement we say that $(0, 0) \in \partial S$. □

2.41

b)

Use the Cauchy-Shwarz inequality to show that $g(X, Y) = X \cdot Y$ mapping $\mathbb{R}^{2n}$ to $\mathbb{R}$ is continuous.

Proof. Choose $\begin{bmatrix} X \\ Y \end{bmatrix} \in \mathbb{R}^{2n}$ arbitrarily. Let $\begin{bmatrix} X_n \\ Y_N \end{bmatrix}$ be a sequence converging to $\begin{bmatrix} X \\ Y \end{bmatrix}$. Then it follows that $(X_n) \to X$ and $(Y_n) \to Y$ trivially. We wish to show that $(X_n \cdot Y_n) \to X \cdot Y$, or alternatively that $|X \cdot Y - X_n \cdot Y_n| \to 0$. We write

$$\begin{aligned} |X \cdot Y - X_n \cdot Y_n| &= |X \cdot Y - X_n \cdot Y + X_n \cdot Y - X_n \cdot Y_n| \\ &\leqslant |X \cdot Y - X_n \cdot Y| + |X_n \cdot Y - X_n \cdot Y_n| \\ &= |(X - X_n) \cdot Y| + |X_n \cdot (Y - Y_n)| \\ &\leqslant ||X - X_n||\ ||Y|| + ||Y - Y_n||\ ||X_n|| \end{aligned}$$

Then we know that the norm of Y is fixed, and that the norm of X_n must converge to that of X. So then we have $||X - X_n||\ ||Y|| \to 0||Y|| = 0$ and $||Y - Y_n||\ ||X_n|| \to 0||X|| = 0$, so we say that $X_n \cdot Y_n \to X \cdot Y$ and the dot product is continuous. □

2.44

Let $F : \mathbb{R}^n \to \mathbb{R}^m$. If for every sequence $(X_n) \to X$, $F(X_n) \to F(X)$ then F is continuous.

Proof. Suppose that F is not continuous at $A \in \mathbb{R}^n$. Then $\exists \epsilon > 0$ such that $\forall \delta > 0,\ \ ||A - B|| < \delta \not\Rightarrow ||F(A) - F(B)|| \leqslant \epsilon$. That means that it is not the case that for all B where $||A - B|| < \delta$ we have $|F(A) - F(B)|| \leqslant \epsilon$. Clearly it must be the case that there exists some B within the δ-ball of A such that $||F(A) - F(B)|| > \epsilon$.

By the Archimedean property we of course have for every $k \in \mathbb{N}$ there is some $\delta > 0$ such that $\frac{1}{k} > \delta$. Thus it follows that for the same epsilon which we assume to exist in the previous paragraph that for all $k \in \mathbb{N}$ there exists some $\delta < 1/k$ so we have a point B within a δ-neighborhood of A where $||F(A) - F(B)|| > \epsilon$. Call this point $B = X_k$, and we have a point X_k within a $1/k$-neighborhood of A where $||F(A) - F(X_k)|| > \epsilon$.

Take these points X_k to be a sequence. Clearly we have $(X_k) \to A$ since for every $\epsilon > 0$ choose $k \in \mathbb{N}$ such that $1/k < \epsilon$ and there is some index in the sequence where every point X_k lies within this radius. However, as shown in the previous paragraph, for every $k \in \mathbb{N}$ we have $||F(A) - F(X_k)|| > \epsilon$ for some $\epsilon > 0$ that we know to exist. So it cannot be the case that $F(X_n) \to F(A)$.

Having shown that if F is not continuous at A then there exists a sequence $X_k \to A$ such that $F(X_k) \not\to F(A)$, it follows that the contra-positive holds as well. That is, if we have $X_k \to A \Rightarrow F(X_k) \to F(A)$, then F is continuous at A. If this holds for every A in the domain of F then we say F is continuous. $\square$