MTH 430 Homework 1

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Problem 1

Let $f : X \to Y$ be a function.

(a)

Show that for all $A_1, A_2 \subset X, \quad f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Proof. Let $A_1, A_2 \subset X$. Then $A_1 \cup A_2 \subset X$.

Let $y \in Y$ such that $y \in f(A_1 \cup A_2)$. Then $f^{-1}(y) \in A_1 \cup A_2$ and either $f^{-1}(y) \in A_1$ or $f^{-1}(y) \in A_2$. If $f^{-1}(y) \in A_1$, then $y \in f(A_1)$ and thus $y \in f(A_1) \cup f(A_2)$. Similarly if $f^{-1}(y) \in A_2$, it follows that $y \in f(A_1) \cup f(A_2)$. Therefore $f(A_1 \cup A_2) \subset f(A_1) \cup f(A_2)$.

Now, let $y \in Y$ such that $y \in f(A_1) \cup f(A_2)$. Then either $y \in f(A_1)$ or $y \in f(A_2)$. If $y \in f(A_1)$ then $f^{-1}(y) \in A_1 \subset A_1 \cup A_2$ thus $y \in f(A_1 \cup A_2)$. Similarly if $y \in f(A_2)$ it follows that $y \in f(A_1 \cup A_2)$. $\square$

(b)

Show that for all $A_1, A_2 \subset X, \quad f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

Proof. Let $A_1, A_2 \subset X$. Then $A_1 \cap A_2 \subset X$.

Let $y \in Y$ such that $y \in f(A_1 \cap A_2)$. Since $y \in f(A_1 \cap A_2)$, $f^{-1}(y) \in A_1 \cap A_2$. Thus $f^{-1}(y) \in A_1$ and $y \in f(A_1)$. Similarly, $f^{-1}(y) \in A_1$ and $y \in f(A_2)$. Therefore $y \in f(A_1) \cap f(A_2)$ and $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$. $\square$