Applied Ordinary Differential Equations — Homework 5

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5.2

5.2.5

Do the following:

a. Seek the power series relationship around the point x_0, and figure out the recurrence relation on a_n.

b. Find the first four non-zero terms in the power series' for two solutions y_1, y_2.

c. Show that y_1, y_2 form a fundamental set of solutions by computing the wronskian $W[y_1, y_2](x_0)$.

d. If possible, find the general term in each solution.

$$y'' + k^2x^2y = 0, \quad x_0 = 0, k \text{ is constant}$$

a.

To find the power series solution around the point x_0, we must observe that $P(x) = 1, Q(x) = 0, R(X) = k^2x^2$. This equation does not change when we divide by $P(X)$ since it is equal to 1, so we can also write $p(x) = 0, q(x) = k^2x^2$. So we write,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Then if we substitute this into our original equation we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + k^2x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

We can shift the index by 2 and then we have

$$\begin{aligned} \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + k^2x^2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + k^2x^2 a_n x^n &= 0 \\ \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + k^2x^2a_n\right)x^n &= 0 \end{aligned}$$

So for this to be true for every x, it follows that

$$\begin{aligned} (n+2)(n+1)a_{n+2} + k^2x^2a_n &= 0 \\ (n+2)(n+1)a_{n+2} &= -k^2x^2a_n \\ a_{n+2} &= \frac{-k^2x^2}{(n+2)(n+1)} \cdot a_n \end{aligned}$$

And thus we have our recurrence relation.

b.

5.2.18

5.3

5.3.8

5.3.17

5.3.18