Differential Geometry - Homework 2

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Problem 1

(a)

$\boxed{\text{Show that all 2-forms in } \mathbb{R}^3 \text{ are decomposable.}}$

Proof. Let $H \in \bigwedge^2(\mathbb{R}^3)$. We assume that this 2-form is defined by some vector field $\vec{H}$, and some surface $S \subset \mathbb{R}^3$. Given our current coordinate system, we say $H = H_x dy \wedge dz + H_y dz \wedge dx + H_z dx \wedge dy$. We want to show that there exist a product of 1-forms that compose H. Let $F, G \in \bigwedge^1(\mathbb{R}^3)$ be 1-forms where $F = F_x dx + F_y dy + F_z dz$ and $G = G_x dx + G_y dy + G_z dz$. Then, taking their wedge product, we have the following:

$$F \wedge G = (F_y G_z - F_z G_y)(dy \wedge dx) - (F_x G_z - F_z G_x)(dx \wedge dz) + (F_x G_y - F_y G_x)(dx \wedge dy)$$

Then, to satisfy $H = F \wedge G$, we produce a system of equations that looks like

$$\begin{aligned} H_x &= F_y G_z - F_z G_y && (1) \\ H_y &= F_z G_x - F_x G_z && (2) \\ H_z &= F_x G_y - F_y G_x && (3) \end{aligned}$$

Notice that this system of equations is equivalent to the system of equations one must solve in order to "undo" the cross product. Given that we have shown this to be possible $\boxed{\text{Homework 1}}$, it follows that such 1-forms F, G exist, and therefore $H = F \wedge G$ is decomposable. □

(b)

$\boxed{\text{Provide an example of a non-decomposable } p\text{-form.}}$

We wish to find some example of a non-decomposable p-form in some vector space V. Let us look at $\mathbb{R}^4$. Call our dimensions x_1, x_2, x_3, x_4. We may have some 2-form,

$$u = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

Suppose that $u = v \wedge w$ for some pair of 1-forms, $v, w \in \bigwedge^1(\mathbb{R}^4)$. Then we say that

$$\begin{aligned} u \wedge u &= (v \wedge w) \wedge (v \wedge w) \\ &= v \wedge (w \wedge v) \wedge w \\ &= v \wedge -(v \wedge w) \wedge w \\ &= (v \wedge -v) \wedge (-w \wedge w) \\ &= (v \wedge v \wedge -1) \wedge (-1 \wedge w \wedge w) \\ &= 0 \wedge 0 \\ &= 0 \end{aligned}$$

However, we can compute this same product $u \wedge u$ as

$$\begin{aligned} u \wedge u &= (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \\ &= 2(dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_2) + 2(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4) \\ &= 0 + 2(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4) \end{aligned}$$

Since $u \wedge u$ cannot be both equal to 0 and not equal to 0 we say that u is not decomposable into a product of 1-forms.

(c)

$\boxed{\text{Is } \gamma \wedge \gamma = 0?}$

It depends on the rank of the form γ, denoted by p. We make the argument using only the anti-symmetric property of the wedge product. Since $a \wedge b = (-1)^{p^2}(b \wedge a)$, if we let $a, b = \gamma$, then

$$\gamma \wedge \gamma = (-1)^{p^2}(\gamma \wedge \gamma)$$

If p is odd, then we are guaranteed that $\gamma \wedge \gamma = 0$. Otherwise, this is not guaranteed. For example, our 2-form example from $\boxed{\text{1b.}}$ serves as an example of differential form that does not equal 0 when wedged with itself.

Problem 2

Let $\alpha = 3dx, \beta = 4dy$.

(a)

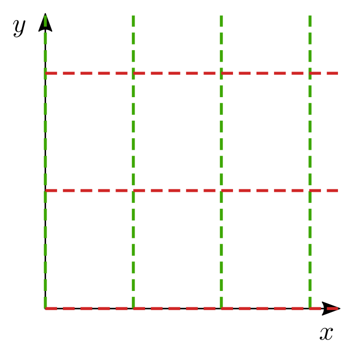

Figure 1: "Stacks" Diagram of α (green) and β (red)

(b)

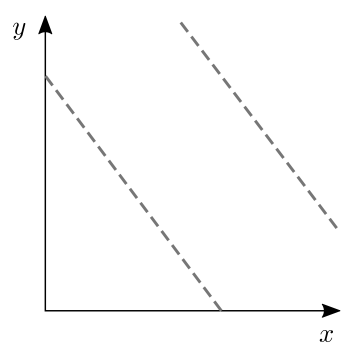

Figure 2: "Stacks" Diagram of $\gamma = \alpha + \beta$

(c)

Let $\vec{v} = \langle 6, 8 \rangle \in \mathbb{R}^2$. From this diagram $\boxed{\text{Figure 3}}$ we can see that $\vec{v}$ crosses exactly 2 of our green stacks, and 2 of our red stacks. So we say that $\alpha(\vec{v}) = 2 = \beta(\vec{v})$. From this second diagram $\boxed{\text{Figure 4}}$ we have $\vec{v}$ crossing exactly 2 of our stacks, which means that $\gamma(\vec{v}) = 2$.

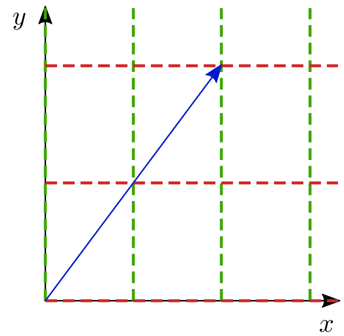

Figure 3: $\vec{v}$ (blue) on α (green) and β (red)

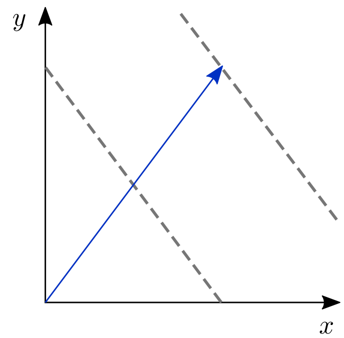

Figure 4: $\vec{v}$ (blue) on γ

(d)

Since $\alpha(\vec{v}) + \beta(\vec{v}) = 2 + 2 = 4 \neq 2 = \gamma(\vec{v})$, we did not obtain $\alpha(\vec{v}) + \beta(\vec{v}) = \gamma(\vec{v})$. It feels like we should have this relationship, but it turns out not to be the case. I believe that this is because if we were to take one "unit" of α, β, and γ as vectors, we do not have $||\alpha|| + ||\beta|| = ||\gamma||$