MTH 411 Post Midterm Notes

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1 Midterm Solutions and Review

1.1 Let (M, d) be a metric space with the discrete metric. Show that any convergent sequence is eventually constant.

Proof. Let (x_n) be a convergent sequence in the space. Choose $\epsilon = 1$. Our sequence will eventually be in the epsilon ball of its limit, and therefore it will be eventually constant. $\square$

1.2 The set $A = \{y \in M : d(x, y) \leqslant r\}$ is called the closed ball with radius r about x.

1.2.1 Show that A is closed.

Proof. Assume that (y_n) is a convergent sequence in A. We will show that its limit is in A. Let $\epsilon > 0$ be arbitrary. Then,

$$d(x, y) \leqslant d(x, y_n) + d(y_n, y) \leqslant r + \epsilon$$

Since this is true for any $\epsilon > 0$ we say that $d(x, y) \leqslant r$, and $y \in A$. $\square$

1.2.2 Give an example where A is not the closure of the open ball.

Choose the space of integers, with an open ball radius 1 around 0. Then $B_1(0) = \{0\}$ is already closed, and is a proper subset of A.

1.3 If $x_n \to x$ in a metric space, show that $d(x_n, y) \to d(x, y)$.

Proof. By the reverse triangle inequality and the squeeze theorem, the result follows trivially. $\square$

1.4 Show that the collection of polynomials with integer coefficients is countable.

Proof. Let $\mathcal{P}$ be the set of all polynomials with integer coefficients, $\mathcal{P}_n$ be the set of polynomials $p(x) = \sum_{k=0}^{n} a_k x^k$ with integer coefficients and degree at most n. Then

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$$

To show that $\mathcal{P}_n$ are countable, map $\mathcal{P}_{n-1}$ onto Z^n with the bijection:

$$f(z_1, z_2, \cdots, z_3) = \sum_{k=1}^{n} z_k x^k$$

Then we assume that $\mathbb{Q}^n$ is countable, and $\mathbb{Z}^n \subset \mathbb{Q}^n$ and we say that $\mathcal{P}$ must be countable. $\square$

2 Continuity

3 Homeomorphisms

4 Connectedness

A space M is said to be disconnected if $M = A \dot{\cup} B$. That is to say, if M can be written as a disjoint union of open sets. Such a construction is called a disconnection of M, and M is connected if it yields no disconnection.

Theorem 4.1. *M is connected if and only if M contains no nontrivial clopen sets.*

5 Completeness

Definition 5.1 (Totally Bounded)**.** We define total boundedness to be the following: a set A in a metric space (M, d) is totally bounded $\Leftrightarrow$

$$\forall \epsilon > 0, \exists n \in \mathbb{N}, x_1, \cdots, x_n \in M : A \subset \bigcup_{j=1}^{n} B_\epsilon(x_j)$$

If we look at $B_1(0) \in l_1$, we find that although this set is bounded, it is not totally bounded.

Theorem 5.1. *We can characterize total boundedness by:* $\forall \epsilon > 0 \exists n \in \mathbb{N}, A_1, \cdots, A_n \subset A$ *such that* $diam(A_j) < \epsilon, j = 1, \cdots, n$ *and* $A \subset \bigcup_{j=1}^{n} A_j$.

The property of total boundedness can be considered as a generalization of compactness.

Definition 5.2 (Bounded)**.** We say that a set $A \subset M$ is bounded if there exists some ball of finite radius such that A is contained in this ball.

Lemma 5.1. *Let* (x_n) *be a sequence in* (M, d) *and* $A = \{x_n | n \in \mathbb{N}\}$ *its range.*

(i) *if* (x_n) *is Cauchy, then* A *is totally bounded*
(ii) *if* A *is totally bounded, then* x_n *has a Cauchy subsequence*

Proof. $\boxed{\text{(i)}}$ Let $\epsilon > 0$ be arbitrary. Since (x_n) is Cauchy, we say that for some $N \in \mathbb{N}$, for every $m, n \geq N, d(x_m, x_n) < \epsilon$. So we say that $\bigcup_{n=1}^{N} B_\epsilon(x_n) \supset A$ and is a finite union of open balls, and is therefore open.

$\boxed{\text{(ii)}}$ If A is finite, then every sequence $(x_n) \in A$ has a constant subsequence. Otherwise, A will be infinite. □

Definition 5.3. A metric space (M, d) is complete if every Cauchy sequence in M converges to a point in M.

Of course the set of real numbers will be complete, however the set of rational numbers will not be complete. The Lebesgue space ℓ_2 is complete. To prove this is fairly difficult.

Theorem 5.2. *For any metric space* M*, the following are equivalent*

(i) M *is complete*
(ii) *The Nested Set Property holds*
(iii) *The Bolzano Weirstrass Property holds. That is, every totally bounded set has a limit point*

This is another way to characterize completeness, this time for a normed vector space.

Theorem 5.3. *A normed vector space* V *is complete if and only If*

$$\sum_{n=1}^{\infty} ||x_n|| < \infty \Rightarrow \sum_{n=1}^{\infty} x_n \text{ converges in } V$$

Every absolutely summable series in V *is summable.*

Proof. $\boxed{\Rightarrow}$ Assume V is complete, and let $(x_n) \subset V$ be such that $\sum_{n=1}^{\infty} ||x_n|| < \infty$. Let S_n be the sequence of partial sums. We wish to show that S_n is a cauchy sequence.

$$||S_n - S_m|| = ||\sum k = m + 1^n x_k|| \leq \sum_{k=m+1}^{n} ||x_k|| \to 0$$

Thus (S_n) is a Cauchy sequence in V. Since V is complete (S_n) converges to $S = \sum_{k=1}^{\infty} x_k$.

$\boxed{\Leftarrow}$ Now assume that $\sum ||x_n|| < \infty \Rightarrow \sum x_n$ converges in v and let (x_n) be a Cauchy sequence in V. For $k = 1, 2, \cdots$ let N_k

be such that $\forall n > m \geq N_k : d(x_n, x_m) < 2^{-k}$. Then let $m = N_k \Rightarrow x_n \in B_{2^{-k}}(x_{N_k}) \forall n > N_k$. Consider the subsequence $y_k = x_{N_k}, k \in \mathbb{N}$. Then $y_{k+1} = x_{N_{k+1}} \in B_{2^{-k}}(x_{N_k}) = B_{2^{-k}}(y_k)$. And $||y_{k+1} - y_k|| < 2^{-k}$. Hence $\sum_{k=1}^{\infty} ||y_{k+1} - y_k||$ converges and therefore also $\sum_{k=1}^{\infty} y_{k+1} - y_k$ converges. The partial sums for this series are $S_n = \sum_{k=1}^{n} y_{k+1} - y_k = y_{nn} - y_1$. Therefore the sequence $(y_k) = (x_{N_k})$ converges. Thus there exists some $x \in M : x = \lim_{k\to\infty} x_{N_k}$ and (x_n) is Cauchy. □

Note: Banach Space is a complete normed vector space V.

Definition 5.4. A function $f : (M, d) \to (N, s)$ is called Lipschitz if there is a constant $k < \infty$ such that $s(f(x), f(y)) \leq kd(x, y)$ for every $x, y \in M$.

Immediately it should be clear that a Lipschitz mapping will be continuous.

Proof. Let $x_n \to x$ in M. Then $d(x, x_n) \to 0$. So $s(f(x), f(x_n)) < kd(x, x_n) \to 0$. Thus $s(f(x), f(x_n)) \to 0$ and f is continuous. □

Definition 5.5. A map $f : M \to M$ on a metric space (M, d) is called a contraction if there is $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$.

Since a contraction is Lipschitz with $k = \alpha$ it is continuous.

Definition 5.6. Let $f : M \to M$. Any $x \in M$ such that $f(x) = x$ is called a fixed point of f.

Theorem 5.4. *(Contraction Mapping Theorem, Banach Fixed Point Theorem) Let (M, d) be a complete metric space and let $f : M \to M$ be a contraction. Then, f has a unique fixed point. For any $x_0 \in M$, the iteration $x_{n+1} = f(x_n)$ converges to x. One has $d(x_n, x) \leq d(x_1, x_0)\frac{\alpha^n}{1-\alpha}$.*

Definition 5.7. Let $f'(x) = f(x), f^{n+1}(x) = f(f^n(x))$, i.e. f^n is the n-fold composition of f with itself.

Proof. The sequence x_n can be written as $x_n = f^n(x_0)$. Let $x_0 \in M$ be arbitrary.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\ &\leq \alpha d(x_n, x_{n-1}) = \alpha d(f(x_{n-1}), f(x_{n-2})) \\ &\leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n d(x_1, x_0) = c\alpha^n \qquad\qquad c = d(x_1, x_0) \end{aligned}$$

□

6 Compactness

Definition 6.1. A metric space (M, d) is said to be compact if it is both complete and totally bounded.

Theorem 6.1. *(M, d) is compact if and only if every seuqence has a Cauchy subsequence that converges to a point in M.*

Theorem 6.2. *The image of a compact set under a continuous function is compact in metric spaces.*

Theorem 6.3. *Let $(V, ||.||)$ and $(W, |||.|||)$ be normed vector spaces and let $T : V \to W$ be a linear map. Then the following are equivalent:*

(i)T is Lipschitz
(ii)T is uniformly continuous
(iii)T is everywhere continuous
(iv)T is continuous at $0 \in V$
(v)there is a constant $C < \infty$ such that $|||T(x)||| \leqslant C||x||$ for all $x \in V$