Stochastic Elements of Mathematical Biology - Assignment 1

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Problem 1

We let $S = \{AA, Aa, aa\}$. Then we have $AA \mapsto AA, p = 1, aa \mapsto aa, p = 1$ and our most complicated case:

$$Aa \mapsto \begin{cases} AA, & p = \frac{1}{4} \\ Aa, & p = \frac{1}{2} \\ aa, & p = \frac{1}{4} \end{cases}$$

So then we can write our transition matrix as

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

Where we take alleles in the order listen in S. We know that $P^n = P_n$ (Chapman-Kolmagorov). Then we know as well that we have $P_n = (p_n(i,j))_{i,j \in S}$ by definition. We can immediately determine the first and last row, since at no point in the self polination process can an AA or aa allele deviate from its current form. So we have

$$P_n = \begin{pmatrix} 1 & 0 & 0 \\ p_n(Aa, AA) & p_n(Aa, Aa) & p_n(Aa, aa) \\ 0 & 0 & 1 \end{pmatrix}$$

For the entry $p_n(Aa, Aa)$ we know that if $n = 0$ this value is 1, and if $n = 1$ than it is $\frac{1}{2}$. However, we know that in order to maintain the heterozygous form at the n-th iteration, we must have achieved this $p = \frac{1}{2}$ probability n times. By multiplication of probabilities, we claim that $p_n(Aa, Aa) = \left(\frac{1}{2}\right)^n$. While we could attempt to explicitly compute $p_n(Aa, AA), p_n(Aa, aa)$, instead note that it is equally likely to end up with $Aa \mapsto_n aa$ as $Aa \mapsto_n AA$. Then since these probabilities must add up to 1, we take

$$p_n(Aa, AA) = p_n(Aa, aa) = \frac{1 - \left(\frac{1}{2}\right)^n}{2} = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}$$

Thus,

$$P^n = \begin{pmatrix} 1 & 0 & 0 \\ \\ \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} & \left(\frac{1}{2}\right)^n & \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} \\ \\ 0 & 0 & 1 \end{pmatrix}$$

Then we take

$$\lim_{n \to \infty} P^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2

Now we have $S = \{(AA, AA), (AA, Aa), (AA, aa), (Aa, Aa), (Aa, aa), (aa, aa)\}$. Then we have,

$$
\begin{aligned}
(AA, AA) &\mapsto AA, p = 1 \\
(AA, Aa) &\mapsto AA, p = \frac{1}{2} \quad Aa, p = \frac{1}{2} \\
(AA, aa) &\mapsto Aa, p = 1 \\
(Aa, Aa) &\mapsto AA, p = \frac{1}{4} \quad Aa, p = \frac{1}{2} \quad aa, p = \frac{1}{4} \\
(Aa, aa) &\mapsto Aa, p = \frac{1}{2} \quad aa, p = \frac{1}{2} \\
(aa, aa) &\mapsto aa, p = 1
\end{aligned}
$$

Then for an outcome of the form (x, x), we can simply square its probability. However, for outcomes such as (x, y), we multiply the probability of achieving x with the probability of achieving y, but since we can take either order of x then y or y then x we multiply this product of probabilities by 2. Using these notions, we write our tranistion matrix as follows:

$$
P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
(\frac{1}{2})^2 & 2(\frac{1}{2})^2 & 0 & (\frac{1}{2})^2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
(\frac{1}{4})^2 & 2(\frac{1}{2})(\frac{1}{4}) & 2(\frac{1}{4})^2 & (\frac{1}{2})^2 & 2(\frac{1}{2})(\frac{1}{4}) & (\frac{1}{4})^2 \\
0 & 0 & 0 & (\frac{1}{2})^2 & 2(\frac{1}{2})^2 & (\frac{1}{2})^2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{16} \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

And thus we have our transition matrix for the markov chain.