Algebraic Topology — Homework 2

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Problem 3.11 (a)

Let X be a Moore space $M(\mathbb{Z}_m, n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m. Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $H_i(\circ;\mathbb{Z})$ for all i but not on $H^{n+1}(\circ;\mathbb{Z})$. Deduce taht the splitting in the universal coefficient theorem for cohomology cannot be natural.

Solution

First we show that the trivial map is induced on the reduced homology with integer coefficients. We say that if $i = n$ then $0 = H_n(S^{n+1}) = H_n(S^{n+1};\mathbb{Z}) \Rightarrow 0 = \tilde{H}_n(S^{n+1})$. Since this is the target of the induced homomorphism from the quotient map, it follows that this map must necessarily be trivial. If $i \neq n$ then we have $0 = H_i(X) = H_i(X;\mathbb{Z}) \Rightarrow 0 = \tilde{H}_i(X;\mathbb{Z})$. Since the domain of our map is trivial, the map must be trivial.
Then we wish to show that we do not have the induced map be trivial on $H^{n+1}(\circ;\mathbb{Z})$. We have the long exact sequence

$$\cdots \to H_{n+1}(S^n) \to H_{n+1}(X) \to H_{n+1}(X/S^n) \to \cdots$$

and its dual

$$\cdots \leftarrow H^{n+1}(S^n;\mathbb{Z}) \leftarrow H^{n+1}(X;\mathbb{Z}) \leftarrow H^{n+1}(X/S^n;\mathbb{Z}) \leftarrow \cdots$$

We know that the $n+1$ cohomology of the n-sphere will be 0. By the universal coefficient theorem we have

$$0 \to Ext(H_n s)$$

Problem 3.1.1

Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori, shown in the text.