MTH 411 Assignment 1

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Problem 2.5

Show that a set is infinite if and only if it is equivalent to a proper subset of itself.

Proof. First, we show that a countably infinite set is equivalent to a proper subset of itself. Let B be a countable infinite set. Let $x \in B$ and write $B = \{x = b_1, b_2, \cdots, b_n, \cdots\}$. Define $f : B \to B \setminus \{x\}$, where $f(b_i) = b_{i+1}$. To show f is 1-1 let $f(b_i) = f(b_j)$. Rewrite this as $b_{i+1} = b_{j+1}$, and it follows that $i = j$ and $b_i = b_j$ trivially. To show f is onto, let $b_i \in B \setminus \{x\}$. Since $b_1 \notin B \setminus \{x\}, i > 1$ and $i - 1 \in \mathbb{N}$. It follows that b_{i-1} exists and that $f(b_{i-1}) = b_i$, hence f is onto. We now have a 1-1 correspondence, and say that $B \sim B \setminus \{x\}$.

Now we will show that any infinite set has a proper subset which it is equivalent to. We assume that every infinite set has a countable subset (Exercise 2.4). Let A be an infinite set, and let $x \in A$. Let $B \subset A$ be countable, and contain x. Define $f : B \to B \setminus \{x\}$ as before, and $g : A \setminus B \to A \setminus B$ as the identity map. We define $h : A \to A \setminus \{x\}$ as

$$h(a) = \begin{cases} f(a), & a \in B \\ g(a), & a \notin B \end{cases}$$

Since both f and g are 1-1 correspondences, and since B and B^c are disjoint while $B \cup B^c = A$, it follows that h must also be a 1-1 correspondence, therefore $A \sim A \setminus \{x\}$.

Finally, to show the biconditional, we must show that if $A \sim A \setminus \{x\}$ then A is infinite. Suppose that A is finite, then $\exists n \in \mathbb{N}$ for which $|A| = n$. Then $|A \setminus \{x\}|$ must be $n - 1$, and the two cannot be equivalent. Hence, A is infinite. □

Problem 2.19

Show that the set of all functions $f : A \to \{0, 1\}$ is equivalent to $\mathcal{P}(A)$.

Proof. Let A be a set and F be the set of all functions mapping A to $\{0, 1\}$. Define $g : F \to \mathcal{P}(A)$ where $g(f) = \{a \in A | f(a) = 1\}$ for any function $f \in F$. Now we show that g is a 1-1 correspondence.

To show that g is 1-1, let $g(f_x) = g(f_y)$. This is equivalent to saying $\{a \in A | f_x(a) = 1\} = \{a \in A | f_y(a) = 1\}$. Let $a \in A$ be fixed. If $f_x(a) = 0$, then $a \notin g(f_x)$, thus $a \notin g(f_y)$ and it follows that $f_y(a) = 0$. Similarly, if $f_x(a) = 1$ then $f_y(a) = 1$. Since this holds for any $a \in A$, we say $f_x = f_y$.

To show that g is onto, let $B \in \mathcal{P}(A)$. Since F contains all functions mapping A to $\{0, 1\}$, it follows that $f_B \in F$ where

$$f_B(a) = \begin{cases} 0, & a \notin B \\ 1, & a \in B \end{cases}$$

Trivially, $g(f_B) = B$, and g is a 1-1 correspondence. □

Problem 2.24

Show that every point in Δ (The Cantor Set) is the limit of a sequence of distinct endpoints from Δ.

Proof. Let $x \in \Delta$. We want to show that $\lim(x_n) = x$ where $x_n \in \Delta$ for every $n \in \mathbb{N}$.

$\boxed{\text{Case 1: } x \text{ has a finite base 3 expansion}}$ By the construction of the Cantor Set we know that $\exists N \in \mathbb{N}$ such that x is an endpoint for I_N. Choose x_1 to be the other endpoint for the closed interval on which x lies. Then for every $n > N$, choose x_{n-N+1} to be the other endpoint for the interval in I_n on which x lies. Since the length of any interval in I_{n+1} must be $\frac{1}{3}$ the length of any interval in I_n, we know that $|x - x_{n+1}| = \frac{1}{3}|x - x_n|$. Also, since we always remove the middle third, we will choose an endpoint for x_{n+1} that was not available while choosing x_n. This proves that $x_i \neq x_j$ when $i \neq j$.

To show that $\lim(x_n) = x$, let $\epsilon > 0$ be arbitrary. By the Archimedean Property we know $\exists m \in \mathbb{N}$ such that $\frac{1}{3^m} < \epsilon$. Let N be the smallest number such that x is an endpoint in I_N. Since any interval in I_n is of length $\frac{1}{3^{n-1}}$, let $n_0 = \max\{m+1, N\}$. It follows that $|x - x_{n_0}| \leqslant \frac{1}{3^{m+1-1}} < \epsilon$. As we described before, $\forall n > n_0, \;\; |x - x_n| < |x - x_{n_0}|$, so the limit of (x_n) must be x.

$\boxed{\text{Case 2: } x \text{ has an infinite base 3 expansion}}$ Let x_1 be 0. Inductively assume we have some $x_n \in \Delta$ with a finite base 3 expansion that is a truncation of the infinite base 3 expansion of x, such that x_n has m digits. Then define x_{n+1} to be a truncation of the infinite base 3 expansion of x such that there are more than m digits, and such that $x_{n+1} \neq x_n$. These conditions ensure that we do not simply append a 0 to our expansion, resulting in non-distinct members of the sequence. Then it follows that each x_n is distinct, and that $(x_n) \to x$. $\square$