Final Exam -Monday June 8, 2020

This final exam is being administered through CANVAS due to the COVID19 public health crisis. As part of the exam you will need to sign a pledge of honor to follow OSU guidelines for academic integrity when taking unproctored final exams. You are allowed to use the textbook and your class notes. If you need to contact me during the exam, I will be available through Zoom from 8:00AM-11:00AM and 1:00PM - 5:00 PM. The meeting ID where you can reach me is 339 644 733.

The final exam consists of 5 problems and one extra credit problem. Copy of the solutions should be uploaded through Canvas as a single document (not separated by problems). You can start working in your exam at any time after 8:00 AM, Monday June 8 but need to finish uploading the solutions by 5:00 PM, Monday June 8. This exam is prepared to be done in 1h and 50' (the usual time allotted for final exams) but you will have a total of 4 hs from the time you start working on the test until you finish uploading your solutions.

Good Luck and keep up the good work!

Problem 1: Determine if each of the following statements are true or false. If True, give a brief justification for it. If False, give an example to illustrate your answer.

Part I: Assume $f_n : [0, 1] \to \mathbb{R}$ is a sequence of integrable functions that converges pointwise to an integrable function f. Then

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$$

Part II: Assume that g is a continuous function on the interval $[a, b]$ and that $f(x) = g(x)$ for all $x \in [a, b] \setminus A$ where is A is a finite set. Then f is integrable on $[a, b]$ and $\int_a^b g = \int_a^b f$.

Problem 2: The following statement of a part of the Fundamental Theorem Calculus contains an error. Give an example to illustrate the error and provide a correction so that the conclusion is valid.

Assume f is an integrable function on $[a, b]$. Then $F(x) = \int_a^x f$ is a differentiable function and $F'(x) = f(x)$.

Problem 3: The purpose of this problem is to obtain an infinite series that evaluates for $b \in [0, 1)$

$$F(b) = \int_0^b f(x)\mathrm{d}x, \quad \text{where} \quad f(x) = \frac{1}{x}\ln\left(\frac{1+x}{1-x}\right)$$

Part I: You can use, without a need for a proof, that the Taylor series (centered at 0) of $h(x) = \ln(1-x)$ is given by

$$\sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1}, \tag{1}$$

Show that the series converges for all $|x| < 1$. [Hint use the Ratio test.]

Part II: You can also assume without proof that the series in (1) converges to $h(x) = \ln(1-x)$ for all $|x| < 1$. Using Part I and algebraic properties of the logarithmic function, show that for all $|x| < 1$,

$$\frac{1}{x} \ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{1}{2k+1} x^{2k}$$

Part III: Show that for any $0 \le b < 1$,

$$F(b) = \int_0^b f(x)\mathrm{d}x = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} b^{2k+1}$$

and that the series converges also for $b = 1$. Give a brief justification for your answer.

Extra Credit Part: Show that the improper integral $\int_0^1 f(x)\mathrm{d}x = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

Problem 4: Let

$$h(x) = \begin{cases} -1 & \text{for } x \le 1, \\ 2 & \text{for } x > 1. \end{cases}$$

and define for any $x \in \mathbb{R}$.

$$H(x) = \int_1^x h$$

Part I: Find an algebraic formula for $H(x)$ for all $x \in \mathbb{R}$.
Part II: Where is H continuous?
Part III: Where is H differentiable?
Part IV: Where is $H' = h$?
Part V: Fix $a \in \mathbb{R}$ and define $G_a(x) = \int_a^x h$. Find $G_a(x)$ in terms of $H(x)$.

Problem 5: For $x \in [0,1]$, consider the sequence of functions

$$f_n(x) = x^n.$$

Part I: For $x \in [0,1]$, find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$. Explain why f is integrable.

Part II: Check that $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$.

Part III: Show that for any $\epsilon > 0$, f_n converges uniformly to f on the interval $[0, 1-\epsilon]$.

Extra Credit Part: Using the result from Part III show that $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$.

Extra Credit Part II: Assume g is integrable on $[0, 1]$ and g is continuous at 0. Define $g_n(x) = g(x^n)$. Show that

$$\lim_{n\to\infty} \int_0^1 g_n = g(0).$$

Extra Credit Problem: Let $0 < a < b$ and let D denote the region in $\mathbb{R}^2$ given by $D = \{t > 0, a \le x \le b\} = (0, \infty) \times [a, b]$. Define $f(t, x) = e^{-xt}, (t, x) \in D$.

Part I: Show that the improper integral $F(x) = \int_0^\infty f(t, x)\mathrm{d}t$ converges uniformly for $x \in [a, b]$ and that $F(x) = 1/x$.

Part II: Let $g(x, t) = \frac{\partial f}{\partial x}(x, t)$. Show that the improper integral $\int_0^\infty g(t, x)\mathrm{d}t$ converges uniformly for $x \in [a, b]$.

Part III: Explain why one can differentiate under the integral sign to obtain

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \int_0^\infty \frac{\partial f}{\partial x}(t, x)\ \mathrm{d}t$$

and thus

$$\frac{1}{x^2} = \int_0^\infty te^{-xt}\mathrm{d}t$$