Algebraic Topology - Homework 1

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1 Problem 0.3

Assume, for $n \geq 1$, that $H_i(S^n) = \mathbb{Z}$ if $i = 0, n$, and that $H_i(S^n) = 0$ otherwise. Using the technique of the proof of Lemma 0.2, prove that the equator of the n-sphere is not a retract.

Proof. Assume that S^{n-1} is the equator of the n-sphere. Suppose by contradiction that S^{n-1} is a retract of S^n. Then it follows that there exists some retraction $r : S^n \to S^{n-1}$. Then, with $i : S^{n-1} \to S^n$ being the inclusion map, and with 1 being, of course, the identity map, it follows that we would have a commutative diagram:

$$\begin{tikzcd} S^{n-1} \arrow[rr, "1"] \arrow[dr, "i"'] & & S^{n-1} \\ & S^n \arrow[ur, "r"'] & \end{tikzcd}$$

To this diagram, we can apply our homology functor, giving us

$$\begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[rr, "H_{n-1}(1)"] \arrow[dr, "H_{n-1}(i)"'] & & H_{n-1}(S^{n-1}) \\ & H_{n-1}(S^n) \arrow[ur, "H_{n-1}(r)"'] & \end{tikzcd}$$

We know by assumption that $H_{n-1}(S^n) = 0$ since $n - 1 \neq n$ and that $H_n(S^{n-1}) = \mathbb{Z}$ since $n - 1 = n - 1$. This new diagram should continue to commute by the properties of our functor H_{n-1}. Since $H_{n-1}(S^n) = 0$, it follows that its image under $H_{n-1}(r)$ must also be zero. That is, $H_{n-1}(S^{n-1}) = 0$. However, since this diagram commutes, it follows that the identity map takes a countable object to a trivial one (contradiction). Therefore, S^{n-1} is not a retract of S^n. □

2 Problem 0.5

Let $f, g : I \to I \times I$ such that $f(0) = (a, 0), f(1) = (b, 1), g(0) = (0, c), g(1) = (1, d)$. Show that there exists some point (s, t) such that $f(s) = g(t)$.

Proof. Say that its the case that $f(s) \neq g(t)$ for all $(s, t) \in I^2$. Define

$$N(s, t) = \max\{|g_1(t) - f_1(s)|, |g_2(t) - f_2(s)|\}$$

Then we define a function $F : I^2 \to I^2$ to be

$$F(s, t) = \left(\frac{g_1(t) - f_1(s)}{2N(s, t)} + 1, \frac{g_2(t) - f_2(s)}{2N(s, t)} + 1 \right)$$

Then we have that $F(I^2) \subset \partial I^2$. Suppose we have a fixed point (x, y). Then we say that $F(s, t) = (s, t)$ therefore $(s, t) \in \partial I^2$. This means that $s = 0, s = 1, t = 0$ or $t = 1$. Within any one of these cases, the fixed point does not hold so we no fixed point for F (justify). Then since I^2 is homeomorphic to D^2, we say that a contradiction is reached by Brouwer fixed point. □

3 Problem 0.7

Let $f \in \text{Hom}(A, B)$, and let $g, h \in \text{Hom}(B, A)$ such that $g \circ f = 1_A$ and that $f \circ h = 1_B$. Then $g = h$.

Proof. We write the following:

$$\begin{aligned}
f &= f \\
f \circ 1_A &= 1_B \circ f && \text{(identity in } hom(A,A) \text{ and } hom(B,B)) \\
f \circ (g \circ f) &= (f \circ h) \circ f && \text{(by assumption)} \\
(f \circ g \circ f) &= (f \circ h \circ f) && \text{(associativity)} \\
h \circ (f \circ g \circ f) &= h \circ (f \circ h \circ f) && \text{(properties of equivalence relation)} \\
(h \circ f \circ g \circ f) &= (h \circ f \circ h \circ f) && \text{(associativity)} \\
(h \circ f \circ g \circ f) \circ g &= (h \circ f \circ h \circ f) \circ g && \text{(properties of equivalence relation)} \\
(h \circ f) \circ g \circ (f \circ g) &= (h \circ f) \circ h \circ (f \circ g) && \text{(associativity)} \\
1_B \circ g \circ 1_A &= 1_B \circ h \circ 1_A && \text{(by assumption)} \\
g &= h && \text{(identity in } hom(A,A) \text{ and } hom(B,B))
\end{aligned}$$

□

4 Problem 0.18

For an abelian group G, let

$$tG = \{x \in G \ : \ \text{x has finite order}\}$$

denote its torsion subgroup.

(ii)

Assume that t defines a functor and that $t(f) = f\Big|_{tG}$ for every homomorphism f. If f is injective, then $t(f)$ is injective.

Proof. Let G, H be abelian groups, and let $f : G \to H$ be an injection. Then, for any $a, b \in G$ we know that $f(a) = f(b) \implies a = b$. Choose some $x, y \in tG$ such that $tf(x) = tf(y)$, or equivalently, $f\Big|_{tG}(x) = f\Big|_{tG}(y)$. Since both x and y belong to the torsion group, it follows that $f(x) = f(y)$. Then, by the injectivity of f, we know that $x = y$ in G, and since both are in tG, the same is true there. □

(iii)

Give an example of a surjective homomorphism f for which $t(f)$ is not surjective.

Define a function $f : \mathbb{Z} \to \mathbb{Z}_2$ such that

$$f(x) = \begin{cases} [0] & x \text{ is even} \\ [1] & x \text{ is odd} \end{cases}$$

Since $\mathbb{Z}$ contains both odd and even integers, f is clearly surjective. It can also be demonstrated that f is a group homomorphism. But, all non-zero integers have infinite order under addition, so we say that $t\mathbb{Z} = \{0\}$. Since $\mathbb{Z}_2$ is cyclic, we know that $t\mathbb{Z}_2 = \mathbb{Z}_2$. Take the element $[1] \in \mathbb{Z}_2$. Notice that it's pre-image under f is confined to only odd integers, so since 0 is not odd, it can't have a pre-image under $t(f)$. Since there is an element of $t\mathbb{Z}_2$ with no pre-image under $t(f)$, the function is not surjective.

5 Problem 0.19

Let p be a fixed prime in $\mathbb{Z}$. Define a functor $f : \boldsymbol{Ab} \to \boldsymbol{Ab}$ by $F(G) = G/pG$ and $F(f) : x + pG \mapsto f(x) + pH$ (where $f : G \to H$ is a homomorphism).

(i)

Show that if f is a surjection, then $F(f)$ is a surjection.

Proof. Let $y + pH \in H/pH$. Then since $y + pH \in H/pH$ it follows that $y \in H$. Because we know that f is a surjection, it follows that if $y \in H$ there exists some $x \in G$ such that $f(x) = y$. So then take the point $x + pG \in G/pG$, and of course it must be the case that

$$F(f)(x + pG) = f(x) + pH = y + pH$$

And it has been proven that $F(f)$ is surjective. □

(ii)

Give an example of a group homomorphism f that is an injection such that $F(f)$ is not an injection.

Let our prime number p be equal to 2. Then let $G = (\{0, 2\}, + \mod 4), H = \mathbb{Z}_4$. Finally, let f simply be the inclusion map. It follows that f is injective since it is the inclusion map. Then $F(f)(4) = F(f)(2) = 2 + 2H$, and we say that $F(f)$ is not injective. This does not hold, try $p = 3, \{0, 2, 4, 6\}, \mathbb{Z}_8$ and the inclusion map.

6 Problem 0.20

(ii)

Show that there is a contravariant functor between Top and $Ring$, given by $C(X)$.

$\boxed{\text{(i)}}$ Let $X \in objTop$, then $C(X) \in objRing$ by $\boxed{\text{Part (i)}}$.

$\boxed{\text{(ii)}}$ Let $f : X \to Y$. Then let $u \in C(Y)$. We define $f^* : C(Y) \to C(X)$ by

$$f^*(u) = u \circ f$$

Then $f^*(u)$ is a continuous map from X to $\mathbb{R}$. That is, if $u \in C(Y)$ then $f^*(u) \in C(X)$, and f^* acts as a ring homomorphism. To demonstrate this, let $u, v \in C(Y)$ and $h : X \to Y$ where X, Y are topological spaces. Let $x \in X$ be arbitrary. Then we can write

$$\begin{aligned}
h^*(f+g)(x) &= ((f+g) \circ h)(x) && (1)\\
&= (f+g)(h(x)) && (2)\\
&= f(h(x)) + g(h(x)) && (3)\\
&= (f \circ h)(x) + (g \circ h)(x) && (4)\\
&= h^*(f(x)) + h^*(g(x)) && (5)
\end{aligned}$$

Finally we write

$$\begin{aligned}
h^*(fg)(x) &= (fg \circ h)(x) && (6)\\
&= (fg)(h(x)) && (7)\\
&= f(h(x))g(h(x)) && (8)\\
&= (f \circ h)(x) \cdot (g \circ h)(x) && (9)\\
&= h^*(g(x)) \cdot h^*(g(x)) && (10)
\end{aligned}$$

So the ring structure is preserved, adn the property is satisfied.

$\boxed{\text{(iii)}}$ Let $f : X \to Y$ and $g : Y \to Z$ so that both are continuous and $g \circ f$ is defined and a morphism in $Hom(X, Z)$. Let $u \in C(Z)$ be a continuous function on Z. Then we write

$$\begin{aligned}
(g \circ f)^*(u) &= u \circ (g \circ f) && \text{(definition of } (g \circ f)^*\text{)} && (11)\\
&= (u \circ g) \circ f && \text{(associativity of functional composition)} && (12)\\
&= g^*(u) \circ f && \text{(definition of } g^*\text{)} && (13)\\
&= f^*(g^*(u)) && \text{(definition of } f^*\text{)} && (14)\\
&= (f^* \circ g^*)(u) && \text{(definition of composition)} && (15)
\end{aligned}$$

$\boxed{\text{(iv)}}$ Let $X \in objTop$. Then let $u \in C(X)$ be arbitrary. We then can write

$$(1_X)^*(u) = u \circ 1_X \qquad \text{(by definition of } 1_X^*) \qquad (16)$$
$$= u \qquad \text{(identity in } Hom(X,X)) \qquad (17)$$
$$= 1_{C(X)} \circ u \qquad \text{(identity in } Hom(C(X),C(X))) \qquad (18)$$
$$= 1_{C(X)}(u) \qquad \text{(definition of composition)} \qquad (19)$$

Since all of these properties have been shown, we conclude that we have a contravariant functor from the class of topological spaces to the ring of continuous maps.