Real Analysis - Assignment 9

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> A finite dimensional normed vector space is complete.

Proof. Let $(V, ||\cdot||)$ be a normed n-dimensional vector space. Then we say that any vector $v \in V$ consists of n real or complex numbers. Let $(v_k) \subset V$ be an arbitrary sequence in our space, then it follows that we can write this sequnce out as

$$v_1 = \begin{bmatrix} x_{1_1} \\ x_{2_1} \\ \vdots \\ x_{n_1} \end{bmatrix}, v_2 = \begin{bmatrix} x_{1_2} \\ x_{2_2} \\ \vdots \\ x_{n_2} \end{bmatrix}, \cdots$$

Then for any sequence $(x_{i_k}) \subset F$ where F is either the real or complex numbers, F is complete and therefore there exists some subsequence that converges to $x_i \in F$. We say that this subsequence relies on some $K_1 \subset \mathbb{N}$. Take the intersection of all the K_i sets, and you have a Cauchy subsequence of (v_k) that converges to some vector $v \in V$. $\square$

Since a linear subspace of a normed vector space is also a normed vector space, thus it is complete.

> Show that $\mathcal{P}_n$ is closed in $C[a,b]$. Then show that $P = \bigcup_{i\in\mathbb{N}} \mathcal{P}_n \neq C[a,b]$.

Proof. We know that $\mathcal{P}_n$ is a finite dimensional linear subspace of a normed vector space. Thus it is complete, and then it contains its limit points and is therefore closed. $\square$

Proof. Take some function with an infinite polynomial expansion, such as e^x. Then $\forall i \in \mathbb{N}$ we say that $e^x \notin \mathcal{P}_n$ thus there exists some continuous function on $[a,b]$ that is not contained in the set $\mathcal{P}$. $\square$

> Let p_n be a polynomial of degree m_n. The suppose that $p_n \to f$ in $C[a,b]$. That is, it converges uniformly to f on this interval where f is not a polynomial. Show that $m_n \to \infty$.

Proof. Suppose that m_n does not diverge to infinity. Then it must be eventually bounded by some integer $k \in \mathbb{N}$. So some polynomial of degree k will be a function such that $\forall \epsilon > 0, ||p_n - f|| < \epsilon$. Then it follows that f must be a polynomial of degree k (contradiction). $\square$

> Show that the set of all polynomials $\mathcal{P}$ is first category.

Proof. We know that $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$. Then we wish to show that for any $n \in \mathbb{N}, int(cl(\mathcal{P}_n)) = \emptyset$. We know already that $\mathcal{P}_n$ is closed so $cl(\mathcal{P}_n) = \mathcal{P}_n$. Let $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$ arbitrarily. Choose some $p' \in \mathcal{P}_{n+1}$ with all n coefficients identical. This function can be made arbitrarily close to p, so we say that no neighborhood of p lies in $\mathcal{P}_n$. Thus $int(cl(\mathcal{P}_n))$ is empty, and we say $\mathcal{P}_n$ is nowhere dense. Thus $\mathcal{P}$ is a first category set. $\square$

> Suppose that $f : [1,\infty) \to \mathbb{R}$ is continuous and that $\lim_{x\to\infty} f(x)$ exists. For $\epsilon > 0$ there is a polynomial p such that $|f(x) - p(1/x)| < \epsilon$ for all $x \geqslant 1$.

Proof. Let $g : [0,1] \to \mathbb{R}$ where $g(x) = f(x^{-1})$ for $x \in (0,1]$ and $g(x) = \lim_{x\to\infty} f(x)$ at $x = 0$. Then by the Weierstrass Approximation Theorem it follows that for any arbitrary $\epsilon > 0$ there exists some polynomial $p \in \mathcal{P}$ such that $||p - g||_\infty < \epsilon$. Then it follows that

$$|f(x) - p(x^{-1})| < \epsilon$$

$\square$