MTH 342 Homework 6

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February 28, 2020

1

Let V be an inner-product space and $||\cdot||$ be the norm where $||v|| = \sqrt{(v,v)}$.

a

Show that $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$.

Proof. Let $u, v \in V$ arbitrarily. Then $||u+v|| = \sqrt{(u+v, u+v)}$. Similarly $||u-v|| = \sqrt{(u-v, u-v)}$. Taking the square of eache we get

$$||u+v||^2 = (u+v, u+v) \qquad ||u-v||^2 = (u-v, u-v)$$

Then we have $||u+v||^2 + ||u-v||^2 = (u+v, u+v) + (u-v, u-v)$. By the linearity on the first argument and conjugate semi-linearity on the second argument we have

$$\begin{aligned}
||u+v||^2 + ||u-v||^2 &= (u, u+v) + (v, u+v) + (u, u-v) + (-v, u-v) \\
&= (u,u) + (u,v) + (v,u) + (v,v) + (u,u) + (u,-v) + (-v,u) + (-v,-v) \\
&= (u,u) + (u,v) + (v,u) + (v,v) + (u,u) - (u,v) - (v,u) + (v,v) \\
&= 2(u,u) + (u,v) - (u,v) + (v,u) - (v,u) + 2(v,v) \\
&= 2((u,u) + (v,v)) \\
&= 2(||u||^2 + ||v||^2)
\end{aligned}$$

$\square$

b

Show that $|(u,v)| \leqslant ||u||\,||v||$.

Proof. Let $u, v \in V$ be arbitrary. Then $||u||\,||v|| = \sqrt{(u,u)}\sqrt{(v,v)}$. We can write this as $\sqrt{(u,u)(v,v)}$. By our properties of linearity, we can write

$$\begin{aligned}
||u||\,||v|| &= \sqrt{\overline{vu}(u,1)(v,1)} \\
&= \sqrt{(u,v)(v,u)} \\
&= \sqrt{(u,v)\overline{(u,v)}} && \text{Let our field } F \text{ not be complex, then} \\
&= \sqrt{(u,v)^2} \\
&= |(u,v)|
\end{aligned}$$

$\square$

2

Show that the taxicab norm given by $||x|| = |x_1| + |x_2| + \cdots + |x_n|$ is a norm on $\mathbb{R}^2$.

Proof. We want to show the three norm space axioms

$$\begin{array}{rl} \text{(i)} & \text{Triangle Inequality} \\ \text{(ii)} & ||cx|| = |c|\ ||x|| \\ \text{(iii)} & \text{Positive definiteness} \end{array}$$

Let $x, y \in \mathbb{R}^2$ denoted $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then $||x+y|| = |x_1 + y_1| + |x_2 + y_2|$, and $||x|| + ||y|| = |x_1| + |x_2| + |y_1| + |y_2|$. By the triangle inequality in the reals, we get $|x_1+y_1|+|x_2+y_2| \leqslant |x_1|+|x_2|+|y_1|+|y_2|$, and thus $||x+y|| \leqslant ||x|| + ||y||$. Therefore (i) holds.

For (ii), let $c \in \mathbb{R}$. Then

$$\begin{aligned} ||cx|| &= |cx_1| + |cx_2| \\ &= |c||x_1| + |c||x_2| \\ &= |c|(|x_1| + |x_2|) \\ &= |c|\ ||x|| \end{aligned}$$

Now suppose $||x|| = 0$. Then $|x_1| + |x_2| = 0$, and it follows trivially that $x_1 = x_2 = 0$, thus $x = \mathbf{0}$, and we have shown (iii). This shows that all three axioms of normed space hold, and therefore the taxicab norm is a norm.

□

3

Let V be an inner product space. Let U be a subspace of V.

a

Show that $U^\perp$ is a subspace of V.

Proof. To show a vector $v \in U^\perp$, we must show that $(v, u) = 0$ for all $u \in U$.

Since the inner product of $(\mathbf{0}, v) = 0$ for all $v \in V$, we know that $(\mathbf{0}, u) = 0$ for all $u \in U$ and therefore $\mathbf{0} \in U^\perp$.

We want to show that we have closure under vector addition. Let $u \in U$ be arbitrary and $v_1, v_2 \in U^\perp$ aribitrarily. Since $(v_1, u) = 0$ and $(v_2, u) = 0$, it follows that $(v_1 + v_2, u) = 0 + 0 = 0$, and therefore $v_1 + v_2 \in U^\perp$ and we have closure under vector addition.

To show closure under scaling, let $c \in F$ where F is a field, and let $u \in U$ and $v \in U^\perp$. Since $(v, u) = 0$, it follows that $(cv, u) = c(v, u) = 0$. Therefore $U^\perp$ is closed under scaling. Since we have shown the existence of the additive identity, closure under vector addition, and closure under scaling, $U^\perp$ is a vector subspace.

□

b

Proof. First we must show this sum to be a direct sum. Let $v \in U \cap U^\perp$. We want to show that $v = \mathbf{0}$. Since $v \in U^\perp$, it follows that $(u, v) = 0$ for all $u \in U$. Since $v \in U$, then it must true when $u = v$, and therefore $(v, v) = 0$, which means that $v = \mathbf{0}$. Hence, $U \cap U^\perp = \{\mathbf{0}\}$ and thus this is a direct sum.

Since $U \subset V$ and $U^\perp \subset V$, we know $U \oplus U^\perp \subset V$. To show $U \oplus U^\perp = V$, we want to show that $V \subset U \oplus U^\perp$. Let $v \in V$.

We claim there exists $u \in U$ and $u' \in U^\perp$ such that $u + u' = v$. Let $T : V \to U$ where $T(x) = proj_{(U)}(x)$ for all $x \in V$. Then $v = (v - T(v)) + T(v)$. We can show that $v - T(v) \in U^\perp$ and that $T(v) \in U$. Therefore, let $u = T(v)$ and $u' = v - T(v)$, and it follows that $V \subset U \oplus U^\perp$.

□

c

Proof. We want to show $(U^\perp)^\perp = U$.

Let $x \in (U^\perp)^\perp$, and let $y \in U^\perp$. From $\boxed{\text{b}}$, we have $x = u + v$ for some $u \in U$ and for some $v \in U^\perp$. Then, since $(x, y) = 0$, we have $(u + v, y) = 0$. This can be rewritten $(u, y) + (v, y) = 0$.

Then since $u \in U$ and $y \in U^\perp$, we know that $(u, y) = 0$. Therefore we have $0 + (v, y) = 0$, and since $v, y \in U^\perp$, they cannot be perpendicular. This means that $v = \mathbf{0}$. Thus we have $x = u \in U$. □

4

Proof. We know that

$$(x, y) = \left\langle \sum_{k=1}^{n} \alpha_k v_k, \sum_{k=1}^{n} \beta_k v_k \right\rangle$$

We can then write this out as

$$\langle \alpha_1 v_1 + \cdots + \alpha_n v_n, \beta_1 v_1 + \cdots + \beta_n v_n \rangle$$

If we use the properties of linearity and split this into all of its different parts, we get the sum of the inner product of each combination, i.e.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle \alpha_i v_i, \beta_j v_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i, v_j \rangle$$

However, since $v_i \perp v_j$ if $i \neq j$, this is equivalent to

$$\sum_{k=1}^{n} \alpha_k \overline{\beta_k} \langle v_k, v_k \rangle = \sum_{k=1}^{n} \alpha_k \overline{\beta_k}$$

□

5

To find p, we must compute $proj_x x^2$. Taking this projection we get

$$p = proj_x x^2 = \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x$$

$$= \frac{\int_0^1 x^3 dx}{\int_0^1 x^2 dx} x$$

$$= \frac{3}{4} x$$

Therefore we can write

$$\begin{aligned}
||x^2 - p|| &= \int_0^1 \left(x^2 - \frac{3}{4}x\right)^2 dx \\
&= \int_0^1 x^4 - \frac{3}{2}x^3 + \frac{9}{16}x^2 dx \\
&= \frac{1}{5}x^5 - \frac{3}{8}x^4 + \frac{9}{48}x^3 \Big|_0^1 \\
&= \frac{1}{5} - \frac{3}{8} + \frac{9}{48} && = \frac{1}{80}
\end{aligned}$$