

# MTH 311 Homework 8

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## 4.4.9

(a)

Show that a Lipschitz function is uniformly continuous.

*Proof.* Suppose  $f : A \rightarrow \mathbb{R}$  is a Lipschitz function, i.e. there exists  $M > 0$  such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all  $x \neq y \in A$ . We can multiply the above inequality and get  $|f(x) - f(y)| \leq M|x - y|$ . Let  $\epsilon > 0$  be arbitrary, and let  $\delta = \frac{\epsilon}{M}$ . Then if we have  $|x - y| < \frac{\epsilon}{M}$ , we can multiply by  $M$  and get

$$|f(x) - f(y)| \leq M|x - y| < M\delta = M\frac{\epsilon}{M} = \epsilon$$

□

(b)

Is the converse true? No.

*Proof.* Take the function  $f(x) = \sqrt{x}$  on  $[0, 1]$ . We say that it is uniformly continuous because it is continuous on a compact set. Now, since the definition must apply for all  $x, y \in A$  choose  $x = \frac{1}{n} \in [0, 1]$  which works for any  $n \in \mathbb{N}$  and  $y = 0$ . Then

$$\left| \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}} \right| = \sqrt{n}$$

Since  $\sqrt{n}$  is not bounded we say that the function is not Lipschitz.

□

## 4.5.5 b

*Proof.* We continue this proof from where the book left off. We have an interval  $I_1 = [a_1, b_1]$  such that  $f(a_1) < 0$  and  $f(b_1) \geq 0$ . We generalize the process by taking  $z_n = \frac{a_n + b_n}{2}$ , and then if  $f(z_n) > 0$  then  $b_{n+1} = z_n$ . If  $f(z_n) < 0$  then  $a_{n+1} = z_n$ . Otherwise,  $z_n \in [a, b]$  and  $f(z_n) = 0$ , and we are done. By the inductive step, we assume that we have  $I_n = [a_n, b_n]$ , where  $f(a_n) < 0$  and  $f(b_n) > 0$ . Then, we can find a midpoint  $z_n$  and create the interval  $I_{n+1}$  such that  $f(a_{n+1}) < 0$  and  $f(b_{n+1}) > 0$ .

We say that the length of this interval is equal to  $\frac{a-b}{2^n}$ , since each time  $n$  increases by one we cut the interval in half. Then we take the sequence of intervals  $I_n$  for all  $n \in \mathbb{N}$ , and it follows that the length of  $I_n$  approaches 0. Since the length of  $I_n$  approaches zero, and by the Nested Interval Property the infinite intersection of  $I_n$  is non-empty, we say that there is exactly one number  $c$  such that  $\bigcap_{n \in \mathbb{N}} I_n = \{c\}$ .

Suppose that  $f(c) > 0$ . Then there would be some  $n \in \mathbb{N}$  such that  $f(I_n) > 0$  i.e.  $f(a_n) > 0$  and  $f(b_n) > 0$ . This is not possible by the process in which we chose  $f(a_n)$ , therefore  $c \leq 0$ . Now suppose  $f(c) < 0$ . Then, similarly there must be some interval where  $f(b_n) < 0$ , which again is not possible. Thus  $f(c) \geq 0$ . Since we have shown  $f(c)$  not to be strictly positive or strictly negative, it follows that  $f(c) = 0$ .  $\square$

## 5.2.5

(a)

The function  $f$  is continuous at 0 for  $|a| \geq 1$ .

*Proof.* 0 is a limit point of  $[0, \infty)$ . Thus, we can use the functional limit characterization of continuity and say that if  $\lim_{x \rightarrow 0} f(x) = f(0)$  then  $f$  is continuous at 0. Suppose  $|a| < 1$ . Then for all  $a \neq 0$ , this limit is undefined. For  $a = 0$ , the limit is equal to 1  $\neq 0$ . Therefore  $a$  must be greater than or equal to 1. Assume that  $a \geq 1$ , then  $\lim_{x \rightarrow 0} x^a = 0 = f(0)$ . Therefore the function is continuous at 0 assuming  $a \geq 1$ .  $\square$

(b)

$a > 1$ .

## 5.2.7

*Proof.*

$\square$

## 5.3.1 a

*Proof.* Suppose  $f$  is differentiable on  $[a, b]$  and  $f'$  is continuous on  $[a, b]$ . Without loss of generality, let  $x < y \in [a, b]$ . Then choose the interval  $[x, y] \subset [a, b]$ . Then by the mean value theorem there exists  $c \in [x, y]$  such that  $f'(c) = \frac{f(x) - f(y)}{x - y}$ . Since  $f'$  is continuous on the compact set  $[a, b]$  we say that there exists  $M > 0$  such that  $f'(x) \leq M$ . Therefore we have

$$\frac{f(x) - f(y)}{x - y} = f'(c) \leq M$$

And we say that  $f$  is Lipschitz.

$\square$

## 5.3.5 b

