MTH 430 Homework 1

Philip Warton

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Problem 1

Let $f : X \to Y$ be a function.

(a)

Show that for all $A_1, A_2 \subset X, \quad f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Proof. Let $A_1, A_2 \subset X$.

$\boxed{\subset}$ Let $y \in Y$ such that $y \in f(A_1 \cup A_2)$. Then $\exists x \in A_1 \cup A_2$ such that $f(x) = y$. If $x \in A_1$, then $f(x) = y \in f(A_1) \subset f(A_1) \cup f(A_2)$. If $x \notin A_1$ then $x \in A_2$, and similarly it follows that $y \in f(A_1) \cup f(A_2)$.

$\boxed{\supset}$ Now, let $y \in Y$ such that $y \in f(A_1) \cup f(A_2)$. Then either $y \in f(A_1)$ or $y \in f(A_2)$. If $y \in f(A_1)$ then $\exists a_1 \in A_1$ such that $f(a_1) = y$. Thus, $a_1 \in A_1 \cup A_2$ and $f(a_1) = y \in f(A_1 \cup A_2)$ Otherwise, $y \in f(A_2)$, and then $\exists a_2 \in A_2 : f(a_2) = y$, and thus $f(a_2) = y \in f(A_1 \cup A_2)$. $\square$

(b)

Show that for all $A_1, A_2 \subset X, \quad f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

Proof. Let $A_1, A_2 \subset X$ be arbitrary. Let $y \in Y$ such that $y \in f(A_1 \cap A_2)$. Then, there exists $a \in A_1 \cap A_2$ such that $f(a) = y$. Since $a \in A_1$, $f(a) \in f(A_1)$, and similarly $f(a) \in f(A_2)$. Thus $f(a) = y \in f(A_1) \cap f(A_2)$. $\square$

Problem 2

(a)

Show that for all $B_1, B_2 \in Y, \quad f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

Proof. Let $B_1, B_2 \subset Y$.

$\boxed{\subset}$ Let $x \in X$ such that $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$. If $f(x) \in B_1$ then $x \in f^{-1}(B_1) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$. Otherwise, $f(x) \in B_2$ thus $x \in f^{-1}(B_2) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$.

$\boxed{\supset}$ Let $x \in X$ such that $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. If $x \in f^{-1}(B_1)$, then $f(x) \in B_1 \subset B_1 \cup B_2$, thus $x \in f^{-1}(B_1 \cup B_2)$. Otherwise, $x \in f^{-1}(B_2)$, and it follows that $f(x) \in B_2 \subset B_1 \cup B_2$ so $x \in f^{-1}(B_1 \cup B_2)$. $\square$

(b)

Show that for all $B_1, B_2 \in Y, \quad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

Proof. Let $B_1, B_2 \subset Y$.

$\boxed{\subset}$ Let $x \in X$ such that $x \in f^{-1}(B_1 \cap B_2)$. Then $f(x) \in B_1 \cap B_2$, thus $f(x) \in B_1$ and $f(x) \in B_2$. Since $f(x) \in B_1, x \in f^{-1}(B_1)$, and similarly $x \in f^{-1}(B_2)$. Therefore $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.

$\boxed{\supset}$ Let $x \in X$ such that $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Since $x \in f^{-1}(B_1), f(x) \in B_1$, and since $x \in f^{-1}(B_2), f(x) \in B_2$. Since $f(x) \in B_1$ and $f(x) \in B_2$ and thus $f(x) \in B_1 \cap B_2$, it follows that $x \in f^{-1}(B_1 \cap B_2)$. □

Problem 3

(b)

We wish to show that (a) and (b) are equivalent.

Proof. Let $A \subset X$ be arbitrary.

"$\Rightarrow$" Assume f is injective. Let $a \in A$, then $f(a) \in f(A)$. Since $f(a) \in f(A)$, by definition $a \in f^{-1}(f(A))$. Thus $\forall a \in A, a \in f^{-1}(f(A))$ and we say $A \subset f^{-1}(f(A))$. Now let $a \in f^{-1}(f(A))$ be arbitrary, then $f(a) \in f(A)$. Since $f(a) \in f(A)$, then $\exists a_0 \in A$ such that $f(a_0) = f(a)$. We know that f is injective therefore $a = a_0 \in A$. Thus $f^{-1}(f(A)) \subset A$, and $f^{-1}(f(A)) \supset A$, so $f^{-1}(f(A)) = A$.

"$\Leftarrow$" Assume that $f^{-1}(f(A)) = A \quad \forall A \subset X$. Let $a, b \in X$ such that $f(a) = f(b)$ and let $A = \{a\}$. Then $f(A) = \{f(a)\}$ and since $f(a) = f(b)$ it follows that $f(b) \in f^{-1}(f(A))$. Therefore by our assumption that $f^{-1}(f(A)) = A$, we have $b \in A$, and thus $b = a$. □

(c)

We wish to show that (a) and (c) are equivalent.

Proof. A function f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$.

"$\Rightarrow$" Assume that f is injective. We wish to show that $f(A \cap B) = f(A) \cap f(B)$. Let $y \in f(A \cap B)$, then $\exists x \in A \cap B$ such that $f(x) = y$. Since $x \in A, f(x) = y \in f(A)$. Similarly $y \in f(B)$, thus $y \in f(A) \cap f(B)$, and we say $f(A \cap B) \subset f(A) \cap f(B)$.

Now let $y \in f(A) \cap f(B)$, then $\exists x_1 \in A : f(x_1) = y$. Similarly $\exists x_2 \in B : f(x_2) = y$. Since f is an injection we can say $x_1 = x_2 = x$. Thus $x \in A$ and $x \in B$ so $x \in A \cap B$ and it follows that $y = f(x) \in f(A \cap B)$.

"$\Leftarrow$" Assume that $f(A \cap B) = f(A) \cap f(B) \quad \forall A, B \subset X$. Let $a, b \in X$ such that $f(a) = y = f(b)$. Let $A = \{a\}$ and $B = \{b\}$, then $f(A) = \{y\} = f(B) = f(A) \cap f(B) = f(A \cap B)$. Since $y \in f(A \cap B)$, then there must exist some $x \in A \cap B$ such that $f(x) = y$. Therefore $x \in \{a\} \cap \{b\}$ and $a = x = b$. □

(d)

We wish to show that (c) and (d) are equivalent.

Proof. We want to show $f(A \cap B) = f(A) \cap f(B)$ if and only if $A \cap B = \emptyset \Rightarrow f(A) \cap f(B) = \emptyset$.

"$\Rightarrow$" Assume that for all $A, B \subset X$ that $f(A \cap B) = f(A) \cap f(B)$. Let $A, B \subset X$ such that $A \cap B = \emptyset$. Then $f(A \cap B) = \emptyset = f(A) \cap f(B)$, and the desired implication holds.

"$\Leftarrow$" Assume that for all $A, B \subset X$ that $A \cap B = \emptyset \Rightarrow f(A) \cap f(B) = \emptyset$. Let $a \in \{a\} = A \subset X$ and $b \in \{b\} = B \subset X$, and suppose $a \neq b$. Then $A \cap B = \emptyset = f(A) \cap f(B)$, which means that $f(a) \neq f(b)$. Since $a \neq b \Rightarrow f(a) \neq f(b)$, it follows that f is injective, which is equivalent to $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subset X$. $\square$

(e)

We wish to show that f is injective if and only if $\forall B \subset A \subset X, f(A \setminus B) = f(A) \setminus f(B)$.

Proof. "$\Rightarrow$" Assume that f is injective. Let $B \subset A \subset X$ be arbitrary. We want to show that $f(A \setminus B) = f(A) \setminus f(B)$.

$\boxed{\subset}$ Let $y \in Y : y \in f(A \setminus B)$. Since f is injective, there exists a unique $x \in A \setminus B$ such that $f(x) = y$. Since $x \in A, f(x) = y \in f(A)$. We know that for all $b \in B, f(b) \neq y$, because our unique solution $x \notin B$. Since $\nexists b \in B : f(b) = y, y \notin f(B)$. Then with $y \in f(A)$ and $y \notin f(B), y \in f(A) \setminus f(B)$.

$\boxed{\supset}$ Let $y \in Y$ such that $y \in f(A) \setminus f(B)$. Then $\nexists b \in B : f(b) = y$, and $\exists x \in A : f(x) = y$. It follows that if $x \in A$ and $f(x) = y$ that $x \in A \setminus B$. Therefore $f(x) = y \in f(A \setminus B)$.

"$\Leftarrow$" Assume that for all $B \subset A \subset X \quad f(A \setminus B) = f(A) \setminus f(B)$. Let $a, b \in X$ such that $f(a) = y = f(b)$. We want to show that $a = b$. Suppose by contradiction that $a \neq b$. Let $B = \{b\} \subset A = \{a, b\} \subset X$. Then $A \setminus B = \{a\}$, and then $f(A \setminus B) = \{f(a)\} = \{y\}$. However, we also know that $f(A) = \{y\}$ and $f(B) = \{y\}$ so then $f(A) \setminus f(B) = \emptyset$. By assumption $f(A \setminus B) = f(A) \setminus f(B)$, therefore $\{y\} = \emptyset$ (contradiction). It must then be the case that $a = b$. $\square$

Problem 4

Proof. To show that this set, denote τ is a topology on X, we must check 3 things.

$\boxed{(i) : \emptyset \text{ and } x \in \tau}$ By construction we know that $\emptyset \in \tau$. Since $\forall p \in X, \exists B \in \beta : p \in B$, the union of all $B \in \beta$ must be equal to X, thus $X \in \beta$.

$\boxed{(ii) : \forall T_1, T_2 \in \tau, T_1 \cup T_2 \in \tau}$ Let $T_1, T_2 \in \tau$ be arbitrary. If $T_1 = \emptyset$ or $T_2 = \emptyset$, the $T_1 \cup T_2 \in \tau$ trivially. Otherwise, we can denote

$$T_1 = B_{1_1} \cup B_{1_2} \cup \cdots, \qquad T_2 = B_{2_1} \cup B_{2_2} \cup \cdots$$

Then $T_1 \cup T_2 = B_{1_1} \cup B_{2_1} \cup B_{2_1} \cup B_{2_2} \cup \cdots$ which will be an element of τ.

$\boxed{(iii) : \text{Any intersection of a finite subcollection of members of } \tau \text{ is in } \tau}$ We want to show that $\forall T_1, T_2, \cdots T_k \in \tau, \quad T_1 \cap T_2 \cap \cdots \cap T_k \in \tau$. If $T_1 \cap T_2 \in \tau$ it follows that any finite intersection $\bigcap_{i=1}^{k} T_i \in \tau$. So let $T_1, T_2 \in \tau$ be arbitrary, we wish to show that $T_1 \cap T_2 \in \tau$. If either $T_1 = \emptyset$ or $T_2 = \emptyset$, the intersection is empty and thus in τ. Otherwise, we can denote

$$\begin{aligned} T_1 \cap T_2 &= (B_{1_1} \cup B_{1_2} \cup B_{1_3} \cdots) \cap (B_{2_1} \cup B_{2_2} \cup B_{2_3} \cdots) \\ &= \bigcup_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} (B_{1_i} \cap B_{2_j}) \end{aligned}$$

Thus if for all $B_1, B_2 \in \beta, B_1 \cap B_2 \in \tau$, then by (ii) their union will be in τ, and we are done. Let $B_1, B_2 \in \beta$ arbitrarily. Then, since for all $p \in B_1 \cap B_2$, there exists $B_p \subset B_1 \cap B_2 : p \in B_p$, it follows that

$$\bigcup_{p \in B_1 \cap B_2} B_p \quad = \quad B_1 \cap B_2$$

Since this is a union of elements of β, we say that $B_1 \cap B_2 \in \tau$, thus any $T_1 \cap T_2 \in \tau$. It follows from logic before that there intersection any finite subcollection of in τ will be a member of τ. $\square$

Problem 5

(a)

The function f is continuous. Let O be an arbitrary open set in $\mathbb{R}_\tau : O = \{[a,b)|a < b\}$. Then we write

$$\begin{aligned} f^{-1}(O) &= \{x \in \mathbb{R}|f(x) \in O\} \\ &= \{x \in \mathbb{R}|a \leqslant f(x) < b\} \\ &= \{x \in \mathbb{R}|a \leqslant 2x - 5 < b\} \\ &= \left\{x \in \mathbb{R}\middle|\frac{a+5}{2} \leqslant x < \frac{a+5}{2}\right\} \end{aligned}$$

Choose $a_0 = \dfrac{a+5}{2}$ and $b_0 = \dfrac{a+5}{2}$, and then $f^{-1}(O)$ is in $\mathbb{R}_\tau$, thus f is continuous.

(b)

The function $f(x) = -x$ is not continuous.

$\boxed{\text{Counterexample}}$ Let $O = \{[0,1)\}$. Then, $f^{-1}(O) = \{x \in \mathbb{R}|x \in O\} = \{(-1,0]\}$. Since the interval is open on the left and closed on the right, it is not in $\mathbb{R}_\tau$.

(c)

The function $f(x) = x^2$ is not continuous.

$\boxed{\text{Counterexample}}$ Let $O = \{[0,1)\}$. Then, $f^{-1}(O) = \{x \in \mathbb{R}|x \in O\} = \{x \in \mathbb{R}|x^2 \in O\}$. For $x^2 \in O$, we must have $0 \leqslant x^2 < 1$, which holds for any x in $(-1,1)$. Since this interval is open on the left, it is not in $\mathbb{R}_\tau$, therefore f is not continuous.