MTH 483 Homework 2

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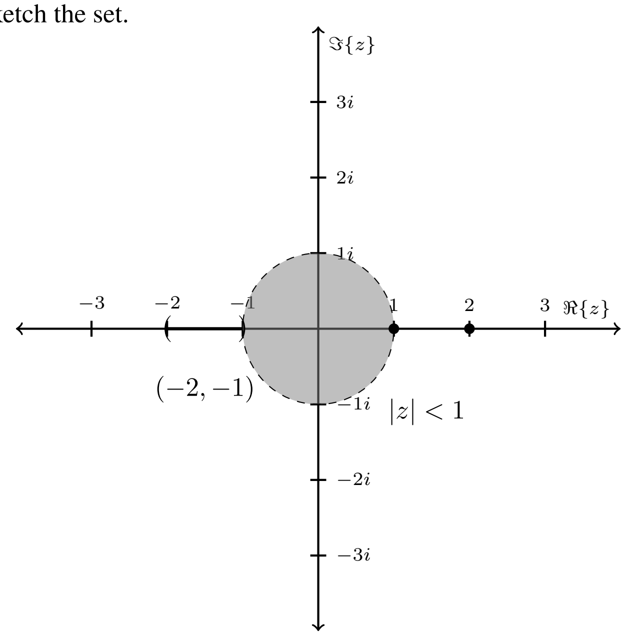
July 11, 2020

Problem 1.29

Let $G = (-2,-1) \cup B_1(0) \cup \{1\} \cup \{2\}$.

(a)

Sketch the set.



(b)

What are the interior points? We claim that $G^{\circ} = B_1(0)$. For any other point in G, every open disk must intersect $\neg G$.

(c)

What are the boundary points? For every point in $[-2,-1]$, we have a boundary point. Then for every point z such that $|z| = 1$, this will be a boundary point. Both 1 and 2 will be boundary points as well. So we say $G^b = [-2,-1] \cup S^1 \cup \{1\} \cup \{2\}$.

(d)

Our only isolated point in this set will be 2.

Problem 1.33

Construct a path for the following.

(a)

A circle radius 1 centered at $1+i$, oriented counter clock-wise. Let $f : [0,1] \to \mathbb{C}$ be a path where $f(t) = e^{i2\pi t}+1+i$.

(b)

The line segment from $-1-i$ to $2i$. Let $g : [0,1] \to \mathbb{C}$ be a path where $g(t) = (t-1)+(3t-1)i$.

(c)

The semi-circle radius 34 centered at 0, above the real axis, oriented clock-wise. Let $h : [0,1] \to \mathbb{C}$ where $h(t) = 34e^{i(\pi-\pi t)}$.

Problem 2.2

Compute the limit.

(a)

Compute $\lim_{z\to i} \dfrac{iz^3-1}{z+i}$. To do this we can simply plug in $z = i$ since this is a continuous function defined everywhere besides $z = -i$. So we have

$$\begin{aligned}
\lim_{z\to i} \frac{iz^3-1}{z+i} &= \frac{(i)i^3-1}{i+i} \\
&= \frac{i^4-1}{2i} \\
&= \frac{1-1}{2i} \\
&= 0
\end{aligned}$$

(b)

Compute $\lim_{z\to 1-i}[x+(2x+y)i]$. Let us approach this in the real and imaginary directions. For the real direction, let $y = -1$ be fixed, then

$$\begin{aligned}
\lim_{z\to 1-i}[x+(2x+y)i] &= \lim_{x\to 1}[x+(2x-1)i] \\
&= 1+\left(\lim_{x\to 1} 2xi\right)-i \\
&= 1+2i-i \\
&= 1+i
\end{aligned}$$

Now we fix $x = 1$ and then we say

$$\begin{aligned}
\lim_{z\to 1-i}[x+(2x+y)i] &= \lim_{y\to -1}[1+(2(1)+y)i] \\
&= 1+\left(\lim_{y\to -1} 2+y\right)i \\
&= 1+i
\end{aligned}$$

Problem 2.12

Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(z) = \dfrac{1}{z}$. Show that $f'(z) = -\dfrac{1}{z^2}$.

Proof. Let $z \in \mathbb{C} \setminus \{0\}$. Then

$$\begin{aligned}
f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{z+h} - \frac{1}{z}}{h} \\
&= \lim_{h\to 0} \frac{\frac{z-(z+h)}{z(z+h)}}{h} \\
&= \lim_{h\to 0} \frac{z - z - h}{hz(z+h)} \\
&= \lim_{h\to 0} \frac{-h}{hz(z+h)} \\
&= \lim_{h\to 0} \frac{-1}{z(z+h)} \\
&= -\frac{1}{z^2}
\end{aligned}$$

□

Problem 2.15

Let $T(z) = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$. Compute $T'(z)$ and find which z make $T'(z) = 0$. Let us use the limit to find the derivative.

$$\begin{aligned}
T'(z_0) &= \lim_{z\to z_0} \frac{T(z) - T(z_0)}{z - z_0} \\
&= \lim_{z\to z_0} \frac{\frac{az+b}{cz+d} - \frac{az_0+b}{cz_0+d}}{z - z_0} \\
&= \lim_{z\to z_0} \frac{\frac{(az+b)(cz_0+d)-(az_0+b)(cz+d)}{(cz+d)(cz_0+d)}}{z - z_0} \\
&= \lim_{z\to z_0} \frac{[aczz_0 + adz + bcz_0 + bd] - [aczz_0 + adz_0 + bcz + bd]}{(z-z_0)(cz+d)(cz_0+d)} \\
&= \lim_{z\to z_0} \frac{adz - adz_0 - (bcz - bcz_0)}{(z-z_0)(cz+d)(cz_0+d)} \\
&= \lim_{z\to z_0} \frac{ad(z-z_0) - bc(z-z_0)}{(z-z_0)(cz+d)(cz_0+d)} \\
&= \lim_{z\to z_0} \frac{(z-z_0)(ad-bc)}{(z-z_0)(cz+d)(cz_0+d)} \\
&= \lim_{z\to z_0} \frac{ad-bc}{(cz+d)(cz_0+d)} \\
&= \frac{ad-bc}{(cz_0+d)^2}
\end{aligned}$$

There is no $z \in \mathbb{C}$ such that $T'(z) = 0$. By definition of $T(z)$ the numerator of $T'(z)$ is non-zero.

Problem 2.18

Where are the following functions differentiable, holomorphic? What is their derivative on such sets?

(a)

Let $f(z) = e^{-x}e^{-iy}$. Let us begin by rewriting this function

$$\begin{aligned} f(z) &= e^{-x}e^{-iy} \\ &= e^{-x-iy} \\ &= e^{(-1)x+iy} \\ &= e^{z^{-1}} \\ &= (e^z)^{-1} \\ &= \frac{1}{e^z} \end{aligned}$$

Now we us the quotient rule to compute the derivative

$$\begin{aligned} f'(z) &= \frac{(e^z)(1') - (1)(e^z)}{(e^z)^2} \\ &= -\frac{e^z}{e^{2z}} \\ &= -\frac{1}{e^z} \end{aligned}$$

Since e^z cannot equal zero, this derivative is defined on all of $\mathbb{C}$. And thus it will be holomorphic on any open subset of $\mathbb{C}$.

(b)

Let $f(z) = 2x + ixy^2$. This function is differentiable nowhere. Suppose that there is a point $z = x_0 + y_0 i \in \mathbb{C}$ such that $f(z)$ is differentiable. Then $f_x(z)$ must equal $-if_y(z)$. So we take

$$\begin{aligned} f_x(z) &= \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(2x + ixy_0^2) - (2x_0 + ix_0y_0^2)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{x(2 + iy_0^2) - x_0(2 + iy_0^2)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(x - x_0)(2 + iy_0^2)}{x - x_0} \\ &= 2 + iy_0^2 \end{aligned}$$

And then we do the same for $f_y(z)$.

$$\begin{aligned}
f_y(z) &= \lim_{y\to y_0} \frac{f(x_0,y)-f(x_0,y_0)}{y-y_0} \\
&= \lim_{y\to y_0} \frac{(2x_0+ix_0y^2)-(2x_0+ix_0y_0^2)}{y-y_0} \\
&= \lim_{y\to y_0} \frac{ix_0y^2-ix_0y_0^2}{y-y_0} \\
&= \lim_{y\to y_0} \frac{(y^2-y_0^2)(ix_0)}{y-y_0} \\
&= \lim_{y\to y_0} \frac{(y-y_0)(y+y_0)(ix_0)}{y-y_0} \\
&= \lim_{y\to y_0} (y+y_0)ix_0 \\
&= 2y_0x_0i
\end{aligned}$$

So we must find $x_0, y_0 \in \mathbb{R}$ such that $f_x(z) = -if_y(z)$. We rewrite this as $2 + iy_0^2 = -i(2y_0x_0i)$. By simplifying the right hand side we get $2 + iy_0^2 = 2x_0y_0$. Suppose $y_0 = 0$, then we have $2 = 0$, so $y_0 \neq 0$. Then when $y_0 \neq 0$ we have a complex number on the left hand side, and a real number on the right (contradiction). Hence there are no solutions to the equation, and f must be differentiable nowhere.

(h)

Let $f(z) = zIm(z)$. The function f is holomorphic nowhere and differentiable at 0. By the Cauchy-Riemann equations we can rule out $\mathbb{C} \setminus \{0\}$. For $z = x_0 + y_0i$ we get $f_x(z) = y_0$, and $(-i)f_y(z) = (-i)(x_0 + 2y_0i)$. If f is differentiable at z these will be equal, and we will have $(-i)(x_0 + 2y_0i) = -ix_0 + y_0 = y_0$. In order to have both sides be real valued, $x_0 = 0$, and then it follows that $y_0 = 0$. This means that f is not differentiable for any point other than 0. At $z = 0$ the limit exists, and is $f'(0) = 0$. Since $\{0\}$ has no non-empty open subsets, f is holomorphic nowhere.

(i)

Let $f(z) = \frac{ix+1}{y}$. This function is differentiable nowhere. Taking the partial derivatives with respect to x and y we get $f_x(z) = \frac{i}{y_0}$ and $f_y(z) = -\frac{ix_0+1}{y_0^2}$. For the equation $f_x(z) = -if_y(z)$ to hold it follows that $y_0 = ix_0 + 1$. This only holds for $z = 0$, however the derivative is not defined at 0 since $f_x(z) = \frac{i}{y_0}$ is not defined when $y_0 = 0$. Thus the function is not differentiable anywhere.

Problem 2.20

Let $f(z)$ be holomorphic on a region $G \subset \mathbb{C}$, and be real valued. Show that f is constant on G.

Proof. Let $z \in G$, then $f'(z)$ exists and also is equal to $f_x(z) = -if_y(z)$. Since $f_x(z) = \lim_{x\to x_0} \frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}$, we know that both the numerator and denominator are real valued, therefore $f'(z)$ is real valued. Similarly, $f_y(z)$ will be real valued by the same logic. In order to satisfy $f_x(z) = f'(z) = -if_y(z)$ while both $f_x(z)$ and $f_y(z)$ are real valued, it must be the case that $f'(z) = 0$, therefore f is constant on G. $\square$