

Applied Ordinary Differential Equations Notes

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We begin with the following form of a second order autonomous ODE:

$$ay''(t) + by'(t) + cy(t) = f(t)$$

This equation can be transformed to a first order system of ODE's, if we define two equations to be

$$\begin{aligned}x_1(t) &= y(t) \\x_2(t) &= y'(t)\end{aligned}$$

Then the equation can be rewritten in terms of x_1, x_2 as

$$ax_2'(t) + bx_2(t) + cx_1(t) = f(t)$$

Which immediately gives us this first order system of ODE's:

$$\begin{aligned}x_1'(t) &= x_2(t) \\x_2'(t) &= -\frac{c}{a}x_1(t) - \frac{b}{a}x_2(t) + \frac{f(t)}{a}\end{aligned}$$

For notational purposes, write $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$. Then we can write many first order systems of ODE's as

$$\begin{aligned}A\vec{x}(t) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\&= \begin{pmatrix} a_{11}x_1(t) + a_{12}x_2(t) \\ a_{21}x_1(t) + a_{22}x_2(t) \end{pmatrix}\end{aligned}$$

For our previous example we would have $A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$ ignoring the $f(t)$ term.

Now we study autonomous first order systems of ODE's. That is, $\vec{x}'(t) = A\vec{x}(t)$. These have solutions that can be represented as

$$\vec{x}(t) = e^{At}\vec{x}_0, \quad e^{At} = \sum_{j=0}^{\infty} \frac{1}{j!} A^j t^j$$

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2.1 Matrix Exponential

(Taylor Series) We define the Taylor series at 0 as

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$$

For example, $e^x = \sum_{j=1}^{\infty} \frac{x^j}{j!}$. Now we are prepared for the matrix exponential, which we define as follows:

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

When $A \mapsto e^A$, we have a group homomorphism from the set of $n \times n$ matrices with addition over to $n \times n$ invertible matrices with multiplication. Take the initial value problem

$$\begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

The unique solution is $e^{At}\vec{x}_0$. Let $X(t) = e^{At}$. This is the matrix of fundamental solutions.

$$X(0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

Then $X'(t) = AX(t)$, $X(0) = I$. Therefore we get $\vec{x}(0) = I\vec{x}_0 = \vec{x}_0$.

Suppose A is diagonalizable,

$$A = U\Lambda U^{-1}, \quad \text{where } \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad U = (\vec{u}_1, \dots, \vec{u}_n)$$

Then we know that

$$X(t) = e^{At} \tag{1}$$

$$= \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} \tag{2}$$

$$= \sum_{j=0}^{\infty} \frac{(U\Lambda U^{-1})^j t^j}{j!} \tag{3}$$

$$= \sum_{j=0}^{\infty} \frac{U\Lambda^j U^{-1} t^j}{j!} \tag{4}$$

In the case of diagonal matrices, we get a linear composition.

$$\begin{aligned} \vec{x}(t) &= Ue^{\Lambda t}U^{-1}\vec{x}_0 \\ &= c_1 e^{\lambda_1 t} \vec{u}_1 + \cdots c_n e^{\lambda_n t} \vec{u}_n \\ &= c_1 \vec{\Psi}_1(t) + \cdots + c_n \vec{\Psi}_n(t) \end{aligned}$$

When we have a diagonalizable matrix we have a nice formula for the solution. Now we will move on to an example:

Consider the following system:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

We know the solution to this is just $\vec{x}(t) = e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Now we write

$$e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t} = \sum_{j=0}^{\infty} \frac{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^j t^j}{j!}$$

Now we know that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$. This means that we can actually write out what our matrix exponential is explicitly in

terms of series in each component. We do this as follows:

$$e \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^t = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{t^j}{j!} & \sum_{j=1}^{\infty} \frac{j t^j}{j!} \\ 0 & \sum_{j=0}^{\infty} \frac{t^j}{j!} \end{pmatrix} = \begin{pmatrix} e^t & t e^t \\ 0 & e^t \end{pmatrix}$$

The Wronskian again:

$$W(t) = \det(X(t)) \quad W'(t) = \frac{d}{dt} \det(e^{At}) = \frac{d}{dt} e^{\text{tr}(At)} = \text{tr}(A) e^{\text{tr}(A)t} = \text{tr}(A) W(t)$$