MTH 312 Homework 3

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Problem 1 (6.4.2)

(a)

If $\sum_{n=1}^{\infty} g_n$ converges uniformly then (g_n) converges to 0.

Proof. Let $\frac{\epsilon}{2} > 0$ be arbitrary. Then $\exists N \in \mathbb{N}$ such that $\forall n \geqslant N$ and $\forall x \in A$ where A is the domain of g

$$|s_n(x) - s(x)| < \frac{\epsilon}{2}$$
$$\left| \sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{\infty} g_k(x) \right| < \frac{\epsilon}{2}$$
$$\left| \sum_{k=n+1}^{\infty} g_k(x) \right| < \frac{\epsilon}{2}$$

Let $n > N$ and it follows that $n + 1 > N$, so we can say that

$$\left| \sum_{k=n+2}^{\infty} g_k(x) \right| < \frac{\epsilon}{2}$$

Then, we can add these two inequalities giving the result

$$|g_{n+1}(x) - 0| = \left| \sum_{k=n+1}^{\infty} g_k(x) - \sum_{k=n+2}^{\infty} g_k(x) \right| \leqslant \left| \sum_{k=n+1}^{\infty} g_k(x) \right| + \left| - \sum_{k=n+2}^{\infty} g_k(x) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Let $N_\epsilon = N + 1$, and we have $\exists N_\epsilon \in \mathbb{N}$ such that forall $n \geqslant N_\epsilon$

$$|g_n(x) - 0| < \epsilon$$

So we say that the sequence of functions converges uniformly to 0. □

(b)

If $0 \leqslant f_n(x) \leqslant g_n(x)$ and $\sum_{k=1}^{\infty} g_n(x)$ converges uniformly, then $\sum_{k=1}^{\infty} f_n(x)$ converges uniformly.

Proof. I proved it. □

Problem 3 (6.4.7)

Let $f(x) = \sum_{k=1}^{\infty} \dfrac{sin(kx)}{k^3}$.

(a)

Show that $f(x)$ is differentiable and that $f'(x)$ is continuous.

Proof. Let us take the derivative of the inside of the series

$$\frac{d}{dx}\left(\frac{\sin(kx)}{k^3}\right) = \frac{(k)\cos(kx)}{k^3} = \frac{\cos(kx)}{k^2}$$

Then we can look at the series $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$, and notice that it can be bounded by $\frac{1}{k^2}$. Since $|\cos(kx)| \leqslant 1$, it follows that $\left|\frac{\cos(kx)}{k^2}\right| \leqslant \frac{1}{k^2}$. The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, therefore by the Weierstrass M-test the series $\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ converges uniformly on $\mathbb{R}$. Let $x_0 = 0$, and then $\frac{\sin(k(0))}{k^3} = 0 \;\; \forall k \in \mathbb{N}$. So we say that $f(0) = 0$. Since we have converges at some x_0 and uniform convergence of the series of derivatives, we have uniform convergence of $f(x)$. Also, we have $f'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$. Since $\forall k \in \mathbb{N}$ the function $\frac{\cos(kx)}{k^2}$ is continuous, and continuity is preserved when adding two functions, each partial sum is also continuous. Uniform convergence preserves continuity so the series $f'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$ is continuous. □

(b)

Can we check if $f(x)$ is twice differentiable?

Proof. Let $g(x) = f'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$. We will check if term by term differentiation gives us a convergent series. Taking the derivative of the inside of the sum we have

$$\frac{d}{dx}\left(\frac{\cos(kx)}{k^2}\right) = \frac{-\sin(kx)}{k}$$

□