Differential Geoemetry - Homework 1

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Let $\vec{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z} \in \mathbb{R}^3$. Determine two vectors $\vec{v}$ and $\vec{w}$ such that $\vec{u} = \vec{v} \times \vec{w}$.

We know that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$, the cross product is defined as

$$\vec{v} \times \vec{w} = det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = \hat{x}(v_y w_z - v_z w_y) - \hat{y}(v_x w_z - v_z w_x) + \hat{z}(v_x w_y - v_y w_x)$$

And, we know also that the cross product is orthogonal to both of the input vectors.

In the case that $\vec{u} = 0$, simply choose $\vec{v} = 0 = \vec{w}$ (this is only one of many possible solutions). Otherwise $\vec{u} \neq 0$. In this scenario, we know that for any two vectors that lie on the plane $0 = u_x x + u_y y + u_z z$ (that is, a plane whose normal vector is $\vec{u}$), both will be orthogonal to $\vec{u}$. We choose $\vec{v_0}, \vec{w_0}$ to be two orthogonal unit vectors that lie on this plane. Then, let $\vec{v} = \vec{v_0}$, and let $\vec{w} = ||\vec{u}||\vec{w_0}$. As a result, we will have either $\vec{u} = \vec{v} \times \vec{w}$ or $-\vec{u} = \vec{v} \times \vec{w}$. This is because of the property of cross products where $||a \times b|| = ||a||\,||b||\,|\sin\theta|$ where θ is the angle between a and b. Since we chose two vectors orthogonal to each other $\sin\theta = 1$. In the case where $-\vec{u} = \vec{v} \times \vec{w}$, without loss of generality, relabel $\vec{v}$ and $\vec{w}$ so that they are swapped, and then we will have $\vec{u} = \vec{v} \times \vec{w}$ by the anti-commutative property of cross products.