MTH 342 Homework 1

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1.

a.)

Proof. Let $u_1, u_2, v \in V$, we want to show that $u_1 + v = u_2 + v \Rightarrow u_1 = u_2$. By the axiom that states an additive inverse must exist, given $v \in V$, we have $-v \in V$ such that $v + (-v) = 0$. Therefore we can can add $-v$ to both sides of our equation, giving us $(u_1 + v) + (-v) = (u_2 + v) + (-v)$. From there, we can use the property of additive associativity and say that $u_1 + (v + (-v)) = u_2 + (v + (-v))$. By definition of additive inverse, this is equivalent to stating that $u_1 + \mathbf{0} = u_2 + \mathbf{0}$. By the axiom of the additive identity, we can rewrite the former statement as $u_1 = u_2$, showing that the cancellation law holds.

□

b.)

Proof. Suppose that a and b are neutral elements of V. Want to show that $a = b$. By definition of the neutral element we have $a + v = v \; \forall v \in V$, and $b + v = v \; \forall v \in V$. Thus, we have $a + v = b + v$, since both are equal to v. Then by the cancellation law shown in $\boxed{\text{1a.}}$, we have $a = b$.

□

c.)

Proof. Let $0 \in F$ and $v \in V$, and denote the zero vector by $\mathbf{0}$. Want to show that $0v = \mathbf{0}$. Rewriting $0 \in F$ as $0 + 0$ we can write $0v = (0 + 0)v$. Then, by distributivity of multiplication we have $0v = 0v + 0v$. Also, by the additive identity, we have $0v = 0v + \mathbf{0}$. Since both are equal to $0v$, we can write that $0v + 0v = 0v + \mathbf{0}$. Then, by commutativity of addition this can be written as $0v + 0v = \mathbf{0} + 0v$. Now we invoke $\boxed{\text{1a.}}$ once again which implies that $0v = \mathbf{0}$.

□

d.)

Proof. Suppose $v, w \in V$ such that $v + w = \mathbf{0}$. We want to show that $w = (-1)v$. Let us take the additive inverse denoted by $(-v)$ and add it to both sides, giving us $v + w + (-v) = \mathbf{0} + (-v)$. By reordering and invoking the axiom of associativity, this can be written as $(v + (-v)) + w = \mathbf{0} + (-v)$. By definition of the additive inverse, this is equivalent to $\mathbf{0} + w = \mathbf{0} + (-v)$. Invoking $\boxed{\text{1a.}}$ we get $w = (-v)$.

$\square$

2.

Proof. We want to show that $V = \mathbb{C}$ is a vector space over $F = \mathbb{C}$, when scalar multiplication is defined as $z * v = \overline{z}v \; \forall z \in F, \; \forall v \in V$.

Since we know $\mathbb{C}^n = V$ to be a vector space under normal rules, one can assume that with no changes to how $\mathbb{C}$ operates under vector addition that the axioms for addition are already satisfied.

We must now show that scalar multiplication is associative within our new scaling operation. Let $z_1, z_2 \in \mathbb{C} = F$, and let $v \in \mathbb{C} = V$. Let us write the term $z_1 * (z_2 * v)$ and show that it is equal to $(z_1 z_2) * v$. By our new multiplication operation we have

$$\begin{aligned} z_1 * (z_2 * v) &= z_1 * (\overline{z_2}v) \\ &= \overline{z_1}(\overline{z_2}v) \\ &= (\overline{z_1 z_2})v \\ &= (z_1 z_2) * v \end{aligned}$$

For the multiplicative identity, we still have $1 \in \mathbb{C} = F$, since it has no complex part. We can show this by writing $1 = 1 + 0i = 1 - 0i = \overline{1}$. Therefore, presence of a multiplicative identity is not changed by our scalar multiplication defintion.

To show distributivity, we must consider two types. For the first, let $z \in \mathbb{C} = F$ and $v_1, v_2 \in \mathbb{C} = V$. Then, $z * (v_1 + v_2) = \overline{z}(v_1 + v_2)$. This can be rewritten as $\overline{z}v_1 + \overline{z}v_2 = z*v_1 + z*v_2$. For the second kind of distributivity, now let $z_1, z_2 \in \mathbb{C} = F$ and $v \in \mathbb{C} = V$. We can write the following

$$\begin{aligned} (z_1 + z_2) * v &= (\overline{z_1 + z_2})v \\ &= (\overline{z_1} + \overline{z_2})v \\ &= \overline{z_1}v + \overline{z_2}v \\ &= z_1 * v + z_2 * v. \end{aligned}$$

Therefore we have shown that even within the redefined scalar multiplication operation $V = \mathbb{C}$ is still a vector space over $F = \mathbb{C}$.

$\square$

3.

Let F be a field and $V = \{A \in M_{2\times 2}(F) : A + A^T = 0\}$. z

a.)

Proof. Want to show that V is a vector space over F. Firstly, we must note that $V \subseteq U$ where $U = \{M_{2\times 2}(F)\}$. Therefore we must only show that the properties of subspaces hold for V to show that it is a vector space. Let us show that V is closed under vector addition. Let $v, w \in V$, want to show $v + w \in V$. Let A be a matrix chosen arbitrarily, denoted by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in F$. By adding together A and A^T we get
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \mathbf{0} \Rightarrow a, d = 0$ and $b = -c$

Therefore, any matrix in the space V will be of the form $\begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} : f \in F$. Denote $v = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, and denote $w = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$. Given that matrix addition operates entrywise, we can write $v + w = \begin{bmatrix} 0 & a+b \\ -a+-b & 0 \end{bmatrix}$. Factoring out the -1 from the bottom left entry, we get $\begin{bmatrix} 0 & a+b \\ -(a+b) & 0 \end{bmatrix}$. Which is of the desired form for an matrix chosen arbitrarily in V. Therefore $v + w \in V \ \forall v, w \in V$. Let $v = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ where $a \in F$ and suppose we have $f \in F$. By scaling v by a factor of f, we get $\begin{bmatrix} 0 & af \\ -af & 0 \end{bmatrix}$. Since the matrix is of the form we desire with $af \in F$ we have shown U is closed under scaling. Therefore U is a vector space.

$\square$

b.)

For this space we have dimension $= 1$. This is because all $v \in V$ are scalar multiples of the matrix shown in $\boxed{\text{3a.}}$. We can write the basis for this space as $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

4.

Let $V = \{f : \{(1, 3) \cap \mathbb{Q}\} \to \mathbb{Q}\}$. Let our field $F = \mathbb{Q}$.

a.)

(i)

Proof. Suppose we have a function $f(x) = \frac{x}{x-2}$. We want to show that $f(x) \notin V \ \forall x \in \mathbb{Q}$ by counter-example. Let $x = 2$, we have $f(2) = \frac{2}{2-2} = \frac{2}{0}$. Since our denominator cannot be zero we have $f(2) \notin \mathbb{Q}$.

$\square$

(ii)

Proof. Suppose we have the function $g(x) = \sqrt{x}$. We want to show that $g(x) \notin V\ \forall x \in \mathbb{Q}$ by counter-example. Let $x = 2$ again, and we have $g(2) = \sqrt{2}$. Since $\sqrt{2}$ is irrational we have $g(2) \notin \mathbb{Q}$.

$\square$

b.)

Proof. We want to show that constants $a, b, c \in \mathbb{Q}$ must be zero in order to satisfy the equation $af_1(x) + bf_2(x) + cf_3(x) = 0$. Let us choose 3 points $x_1 = \frac{1}{2}, x_2 = 2, x_3 = \frac{3}{2}$. We can then create a system of equations with a corrosponding coefficient matrix $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 2 \\ 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{2}{3} \end{bmatrix}$. By row reducing, we get the 3×3 identity matrix, which shows that these functions are linearly independent.

$\square$