Differential Geometry - Homework 3a

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Problem 1

(a)

Determine the Hodge dual operator $*$ on all forms (expressed in spherical coordinates) by computing its action on basis forms at each rank.

We think of the Hodge operator as a sort of "completion" upon a differential form. This is described by the property that states

$$\alpha \wedge *\alpha = g(\alpha, \alpha)\omega$$

In order to compute this on our basis forms, we must know both how the metric tensor will operate on our spherical bases, and some reasonable orientation ω for us to work in. We take the orientation $\omega = r^2 \sin\theta dr \wedge d\theta \wedge d\phi$ as described in the problem statement. Observing the basis elements $\{dr, r\sin\theta d\theta, rd\phi\}$, can we check that they are orthonormal?

If we assume that they are, then we know that $g(\alpha, \alpha)$ should be equal to 1 in most cases. So, we can begin to compute the "Hodge Complement" of our basis elements. We begin with the 0-form:

$$*1 = \omega = r^2 \sin\theta dr \wedge d\theta \wedge d\phi$$

Then we move on to 1-forms,

$$\begin{aligned} dr \wedge *dr &= g(dr, dr) r^2 \sin\theta dr \wedge d\theta \wedge d\phi = dr \wedge (r^2 \sin\theta d\theta \wedge d\phi) \\ \implies *dr &= r^2 \sin\theta d\theta \wedge d\phi \end{aligned}$$

$$\begin{aligned} r\sin\theta d\theta \wedge *r\sin\theta d\theta &= g(r\sin\theta d\theta, r\sin\theta d\theta) r^2 \sin\theta dr \wedge d\theta \wedge d\phi \\ \implies *r\sin\theta d\theta &= -r^2 \sin\theta dr \wedge d\phi \\ \implies *d\theta &= -r dr \wedge d\phi \end{aligned}$$

$$\begin{aligned} rd\phi \wedge *rd\phi &= g(rd\phi, rd\phi) r^2 \sin\theta dr \wedge d\theta \wedge d\phi \\ \implies *rd\phi &= r^2 \sin\theta dr \wedge d\theta \\ \implies *d\phi &= r\sin\theta dr \wedge d\theta \end{aligned}$$

Next, we must compute the dual for each 2-form,

$$\begin{aligned}(dr \wedge r\sin\theta d\theta) \wedge *(dr \wedge r\sin\theta d\theta) &= g((dr \wedge r\sin\theta d\theta), (dr \wedge r\sin\theta d\theta))r^2 \sin\theta dr \wedge d\theta \wedge d\phi \\ \implies *(dr \wedge r\sin\theta d\theta) &= r d\phi \\ \implies *(dr \wedge d\theta) &= \frac{1}{\sin\theta} d\phi\end{aligned}$$

$$\begin{aligned}(dr \wedge r d\phi) \wedge *(dr \wedge r d\phi) &= g((dr \wedge r d\phi), (dr \wedge r d\phi))r^2 \sin\theta dr \wedge d\theta \wedge d\phi \\ \implies *(dr \wedge r d\phi) &= -r\sin\theta d\theta \\ \implies *(dr \wedge d\phi) &= -\sin\theta d\theta\end{aligned}$$

$$\begin{aligned}(r\sin\theta d\theta \wedge r d\phi) \wedge *(r\sin\theta d\theta \wedge r d\phi) &= g((r\sin\theta d\theta \wedge r d\phi), (r\sin\theta d\theta \wedge r d\phi))r^2 \sin\theta dr \wedge d\theta \wedge d\phi \\ \implies *(r\sin\theta d\theta \wedge r d\phi) &= dr \\ \implies *(d\theta \wedge d\phi) &= \frac{1}{r^2 \sin\theta} dr\end{aligned}$$

Finally, for the 3-form, the task should be somewhat trivial.

$$\begin{aligned}(dr \wedge r\sin\theta d\theta \wedge r d\phi) \wedge *(dr \wedge r\sin\theta d\theta \wedge r d\phi) &= g((dr \wedge r\sin\theta d\theta \wedge r d\phi), (dr \wedge r\sin\theta d\theta \wedge r d\phi))r^2 \sin\theta dr \wedge d\theta \wedge d\phi \\ \implies *(dr \wedge r\sin\theta d\theta \wedge r d\phi) &= 1 \\ \implies *(dr \wedge d\theta \wedge d\phi) &= \frac{1}{r^2 \sin\theta}\end{aligned}$$

What is lacking here is the justification for the metric tensor on our bases with themselves being positive normal.

(b)

Compute the dot and cross products of 2 generic "vector fields" (really 1-forms) in sphericalcoordinates using the expressions:

$$\alpha \cdot \beta = *(\alpha \wedge *\beta)$$

$$\alpha \times \beta = *(\alpha \wedge \beta)$$

Let us begin with the dot product. First, note that α, β are both 1-forms. This means that their "Hodge Complements" will be 2-forms, since we are in a 3-dimensional space. So then we can compute the rank of our dot product using only these facts. The term $\alpha \wedge *\beta$ is the wedge product of a 2-form and a 1-form, which will of course result in some 3-form. Then the "Hodge Complement" of a 3-form in a 3-dimensional space is some constant $x \in \mathbb{R}$. We can write $\alpha = a_1 dr + a_2 d\theta + a_3 d\phi$, and similarly $\beta = b_1 dr + b_2 d\theta + b_3 d\phi$ (this is with respect to a coordinate basis rather than an orthonormal one). Then we say

$$*\beta = b_1(*dr) + b_2(*d\theta) + b_3(*d\phi)$$

So then

$$\begin{aligned}\alpha \wedge *\beta = & [a_1 dr \wedge b_1(*dr)] + [a_1 dr \wedge b_2(*d\theta)] + [a_1 dr \wedge b_3(*d\phi)] \\ & + [a_2 d\theta \wedge b_1(*dr)] + [a_2 d\theta \wedge b_2(*d\theta)] + [a_2 d\theta \wedge b_3(*d\phi)] \\ & + [a_3 d\phi \wedge b_1(*dr)] + [a_3 d\phi \wedge b_2(*d\theta)] + [a_3 d\phi \wedge b_3(*d\phi)] \\ \\ & = (a_1 b_1) dr \wedge d\phi \wedge d\theta + 0 + 0 \\ & + 0 + (a_2 b_2) dr \wedge d\phi \wedge d\theta + 0 \\ & + 0 + 0 + (a_3 b_3) dr \wedge d\phi \wedge d\theta \\ \\ & = (a_1 b_1 + a_2 b_2 + a_3 b_3) dr \wedge d\phi \wedge d\theta\end{aligned}$$

And from there it follows quite easily that

$$*(\alpha \wedge *\beta) = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{r^2 \sin\theta}$$

Now, we compute the cross product. The wedge product of two 1-forms α, β provides a 2-form. Then its dual should be a 1-form in a 3-dimensional space. We write

$$\alpha \wedge \beta = (a_1 b_2 - a_2 b_1) dr \wedge d\theta - (a_1 b_3 - a_3 b_1) dr \wedge d\phi + (a_2 b_3 - a_3 b_2) d\theta \wedge d\phi$$

$$\begin{aligned} \implies *(\alpha \wedge \beta) &= (a_1 b_2 - a_2 b_1)(*(dr \wedge d\theta)) + (a_1 b_3 - a_3 b_1)(*(dr \wedge d\phi)) + (a_2 b_3 - a_3 b_2)(*(d\theta \wedge d\phi)) \\ &= \frac{a_1 b_2 - a_2 b_1}{\sin\theta} d\phi - \sin\theta (a_1 b_3 - a_3 b_1) d\theta + \frac{a_2 b_3 - a_3 b_2}{r^2 \sin\theta} dr \\ &= \left(\frac{1}{r^2 \sin\theta} \right) \left[(a_2 b_3 - a_3 b_2) dr - r^2 \sin^2\theta (a_1 b_3 - a_3 b_1) d\theta + r^2 (a_1 b_2 - a_2 b_1) d\phi \right] \end{aligned}$$

Thus we have the cross product.