General Topology and Fundamental Group - Homework 1

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Problem 1

Let $[a,b] \subset \mathbb{R}$. Write $k_{a,b} : I \to [a,b]$ for the linear bijection of the form $y = kx + l, k, l \in \mathbb{R}$. Let $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$. Then let $\alpha_i = \alpha \circ k_{a_{i-1},a_i}, \forall i \in \{1, 2, \cdots, n\}$ where $\alpha : [a,b] \to X$ is continuous. Show that $[\alpha] = [\alpha_1] * [\alpha_2] * \cdots * [\alpha_n]$.

Proof. We immediately begin by writing

$$\begin{aligned}
[\alpha_1] * [\alpha_2] * \cdots * [\alpha_n] &= [\alpha_1 * \alpha_2 * \cdots * \alpha_n] \\
&= [(\alpha \circ k_{a_0,a_1}) * (\alpha \circ k_{a_1,a_2}) * \cdots * (\alpha \circ k_{a_{n-1},a_n})] \\
&= [\alpha \circ (k_{a_0,a_1} * k_{a_1,a_2} * \cdots * k_{a_{n-1},a_n})]
\end{aligned}$$

Then we argue that $k_{a_0,a_1} * k_{a_1,a_2} * \cdots * k_{a_{n-1},a_n} \cong_p id_I$. This is the case because we concatenate linear paths from $0 = a_0$ to a_1, a_1 to a_2, and so on... so it clearly follows that this produces a path in the unit interval that travels from 0 to 1 moving only in the positive direction. This will be path homotopic to id_I.

From there we have

$$[\alpha \circ (k_{a_0,a_1} * k_{a_1,a_2} * \cdots * k_{a_{n-1},a_n})] = [\alpha] \circ [k_{a_0,a_1} * k_{a_1,a_2} * \cdots * k_{a_{n-1},a_n}] = [\alpha] \circ [id_I] = [\alpha \circ id_I] = [\alpha]$$

□

Problem 2

Let $\alpha : I \to S^2$ be a path.

a)

Suppose α is not surjective. Show that if $\alpha(0) \neq \alpha(1)$, then α is path homotopic to to an injective $\beta : I \to S^2$.

Proof. If α is injective then trivially $\alpha \cong_p \alpha$. Otherwise α is not injective, that is, it must intersect itself at some point on the sphere. We know it is not closed, so it connects two distinct points on the sphere. Since α is not surjective, we know that there exists some point $p \in S^2$ not contained in $\alpha(I)$. Thus we can take $S^2 \setminus \{p\}$, and take the stereographic projection $spr_p : S^2 \setminus \{p\} \to \mathbb{R}^2$ from this point, and have a homeomorphism.

Then we have a path between two points in $\mathbb{R}^2$, which is of course homotopic to the straight line between $spr_p(\alpha(0))$ and $spr_p(\alpha(1))$, since $\mathbb{R}^2$ is simply connected. Since this straight line path γ is injective, let

$$\beta = spr_p^{-1} \circ \gamma$$

and it follows that β is an injective path homotopic to α on S^2. □

b)

Suppose α is not surjective. Show that if $\alpha(0) = \alpha(1)$, then α is path homotopic to the constant map $e_{\alpha(0)}$.

Proof. Since α is not surjective, $\exists p \in S^2 \setminus \alpha(I)$, so we take the stereographic projection $spr_p : S^2 \setminus \{p\} \to \mathbb{R}^2$ once again, which we know to be a homeomorphism. Then since $\mathbb{R}^2$ is simply connected, there exists some path homotopy P that brings $spr_p \circ \alpha$ to the trivial path at its starting / ending point. It follows that $spr_p^{-1} \circ P$ will be a path homotopy between α and $e_{\alpha(0)}$. □

Problem 3

Let $\alpha : I \to S^2$ be injective. Show that $int(\alpha(I)) = \emptyset$.

Proof. Suppose that $\alpha(I)$ has a non-trivial interior. Then $B_\epsilon(p) \subset int(\alpha(I))$ by assumption. Let $A = \alpha^{-1}(B_\epsilon(p))$. Of course $A \subset I$ and also $\alpha|_A : A \to B_\epsilon(p)$ is a homeomorphism. If A is not of the form $[a,b] : 0 < a < b < 1$, then it is disconnected, and a contradiction arises. If A is of the form $[a,b]$, simply remove one point $x \in (a,b)$ and we have

$$\alpha|_{A \setminus \{x\}} : A \setminus \{x\} \to B_\epsilon(p) \setminus \alpha(x)$$

However, this is clearly a homeomorphism between a disconnected and connected space (contradiction). So it must be the case that $int(\alpha(I)) = \emptyset$. □

Problem 4

A continuous surjective map $\alpha : I \to I \times I$ produces a space filling curve.

a)

Show that a space filling curve $\alpha : I \to S^2$ must exist.

Let α be a space filling curve mapping I to $I \times I$. We know that S^2 can be described in spherical coordinates as $\{(\rho, \phi, \theta) \in \mathbb{R}^3 : \rho = 1\}$. If we map $I \times I$ to S^2 by $f(x,y) = (1, \pi x, 2\pi y)$ then we have an open (but not continuous) mapping from $I \times I$ to S^2. This makes sense considering the relationship of S^1 and I where the periodic-ness is preventing continuity. However, take $f \circ \alpha$, and this will be a surjective path mapping I to S^2. That is, a space filling curve.

b)

Given a curve $\alpha : I \to S^2$, show that there are $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n$ such that $\alpha([a_{i-1}, a_i]) \neq S^2, i = 1, 2, \cdots, n$.

Proof. Suppose that for every partition of I there exists some $i \in \{1, 2, \cdots, n-1, n\}$ such that $\alpha([a_{i-1}, a_i]) = S^2$. Then take a sequence of partitions $P_k = [0, \frac{1}{k}, \frac{2}{k}, \cdots, \frac{k-1}{k}, 1]$. It must be the case that for each $k \in \mathbb{N}$ some sub-interval maps to all of S^2. Then we can take α_k to be the map of this sub-interval to S^2. If we take the limit of these partitions it must be the case that we map an arbitrarily small closed interval (a point) to all of S^2, which is clearly a contradiction. □

c)

Show that any path $\alpha : I \to S^2$ is path homotopic to a path $\beta : I \to S^2$ with $\beta(I) \neq S^2$.

Proof. If α is not a surjection then let $\beta = \alpha$ thus $\alpha \cong_p \beta$ such that $\beta(I) \neq S^2$. If α is a surjection then it is a space filling curve. Let $p \in S^2$ be equal to neither endpoints of α. Take the partition that we know to exist from part (b). Then we know that $[\alpha] = [\alpha_1] * [\alpha_2] * \cdots * [\alpha_k]$ from Problem 1. Each of these will not be a surjection. For each of these they will be homotopic to a path not containing p. Then call this path β and $\alpha \cong_p \beta$ such that $\beta(I) \neq S^2$. □

d)

Conclude that S^2 is a simply connected space.

Suppose $\exists p \in \alpha(I) \cap \beta(I)$. Then take $spr_p : S^2 \to \mathbb{R}^2$ and since $\mathbb{R}^2$ is simply connected it follows that the two paths are homotopic. If such a point p does not exist then we must take another approach. If both are not surjective, then they will be homotopic to some path that does not include p, and the same proof will still hold. Otherwise, α must be a space-filling curve. Take the partition that we know to exist from part (b). Then we know that $[\alpha] = [\alpha_1] * [\alpha_2] * \cdots * [\alpha_k]$ from Problem 1. Each of these will not be a surjection. For each of these they will be homotopic to a path not containing p, so we call this modified version path not containing p, α'. Then $\alpha \cong_p \alpha' \cong_p \beta$ by our initial method. For part (d), if both are space filling curves, simply do the same process to β, and apply the logic again. Then it follows that any two paths with the same endpoints are path homotopic, and the space is simply connected.