MTH 411 Post Midterm Notes

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1 Midterm Solutions and Review

1.1 Let (M, d) be a metric space with the discrete metric. Show that any convergent sequence is eventually constant.

Proof. Let (x_n) be a convergent sequence in the space. Choose $\epsilon = 1$. Our sequence will eventually be in the epsilon ball of its limit, and therefore it will be eventually constant. □

1.2 The set $A = \{y \in M : d(x, y) \leqslant r\}$ is called the closed ball with radius r about x.

1.2.1 Show that A is closed.

Proof. Assume that (y_n) is a convergent sequence in A. We will show that its limit is in A. Let $\epsilon > 0$ be arbitrary. Then,

$$d(x, y) \leqslant d(x, y_n) + d(y_n, y) \leqslant r + \epsilon$$

Since this is true for any $\epsilon > 0$ we say that $d(x, y) \leqslant r$, and $y \in A$. □

1.2.2 Give an example where A is not the closure of the open ball.

Choose the space of integers, with an open ball radius 1 around 0. Then $B_1(0) = \{0\}$ is already closed, and is a proper subset of A.

1.3 If $x_n \to x$ in a metric space, show that $d(x_n, y) \to d(x, y)$.

Proof. By the reverse triangle inequality and the squeeze theorem, the result follows trivially. □

1.4 Show that the collection of polynomials with integer coefficients is countable.

Proof. Let $\mathcal{P}$ be the set of all polynomials with integer coefficients, $\mathcal{P}_n$ be the set of polynomials $p(x) = \sum_{k=0}^{n} a_k x^k$ with integer coefficients and degree at most n. Then

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$$

To show that $\mathcal{P}_n$ are countable, map $\mathcal{P}_{n-1}$ onto Z^n with the bijection:

$$f(z_1, z_2, \cdots, z_3) = \sum_{k=1}^{n} z_k x^k$$

Then we assume that $\mathbb{Q}^n$ is countable, and $\mathbb{Z}^n \subset \mathbb{Q}^n$ and we say that $\mathcal{P}$ must be countable. □

2 Continuity

3 Homeomorphisms

4 Completeness

Definition 4.1 (Totally Bounded)**.** We define total boundedness to be the following: a set A in a metric space (M, d) is totally bounded $\Leftrightarrow$

$$\forall \epsilon > 0, \exists n \in \mathbb{N}, x_1, \cdots, x_n \in M : A \subset \bigcup_{j=1}^{n} B_\epsilon(x_j)$$

If we look at $B_1(0) \in l_1$, we find that although this set is bounded, it is not totally bounded.

Theorem 4.1. *We can characterize total boundedness by:* $\forall \epsilon > 0 \exists n \in \mathbb{N}, A_1, \cdots, A_n \subset A$ *such that diam*$(A_j) < \epsilon, j = 1, \cdots, n$ *and* $A \subset \bigcup_{j=1}^{n} A_j$.

The property of total boundedness can be considered as a generalization of compactness.

Definition 4.2 (Bounded)**.** We say that a set $A \subset M$ is bounded if there exists some ball of finite radius such that A is contained in this ball.

Lemma 4.1. *Let* (x_n) *be a sequence in* (M, d) *and* $A = \{x_n | n \in \mathbb{N}\}$ *its range.*

(i) *if* (x_n) *is Cauchy, then* A *is totally bounded*

(ii) *if* A *is totally bounded, then* x_n *has a Cauchy subsequence*

Proof. $\boxed{\text{(i)}}$ Let $\epsilon > 0$ be arbitrary. Since (x_n) is Cauchy, we say that for some $N \in \mathbb{N}$, for every $m, n \geq N, d(x_m, x_n) < \epsilon$. So we say that $\bigcup_{n=1}^{N} B_\epsilon(x_n) \supset A$ and is a finite union of open balls, and is therefore open.

$\boxed{\text{(ii)}}$ If A is finite, then every sequence $(x_n) \in A$ has a constant subsequence. Otherwise, A will be infinite. $\square$

Definition 4.3. A metric space (M, d) is complete if every Cauchy sequence in M converges to a point in M.

Of course the set of real numbers will be complete, however the set of rational numbers will not be complete. The Lebesgue space ℓ_2 is complete. To prove this is fairly difficult.