MTH 430 Homework 2

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Problem 1

$X = \mathbb{R} \cup \{p\}$
$\beta = \{\text{open intervals in } \mathbb{R} \text{ and neighborhoods of p}\}$
$V(p) = (a, 0) \cup \{p\} \cup (0, b) : a < 0 < b$

(a)

Show that β is a basis for a topology.

Proof. To prove that β is a basis, there are two requirements.

$$\begin{array}{rl} \text{(i)} & \forall x \in X \quad \exists B \in \beta : x \in B \\ \text{(ii)} & \forall B_1, B_2 \in \beta, \forall x \in B_1 \cap B_2 \ \exists B \in \beta : x \in B \subset B_1 \cap B_2 \end{array}$$

We want to show that $\forall sx \in X, \exists B \in \beta : x \in B$. Let $x \in \mathbb{R}$ be arbitrary. If $x = p$, then any neighborhood of p is automatically in β. Otherwise, $x \neq p \implies x \in \mathbb{R}$. Choose a real open interval $(a, b) \subset \mathbb{R} : a < x < b$.

Now let $B_1, B_2 \in \beta$. We want to show that $\forall x \in B_1 \cap B_2, \ \exists B \in \beta$ such that $x \in B \subset B_1 \cap B_2$. Let $x \in B_1 \cap B_2$. If $x = p$, then $p \in B_1$ and $p \in B_2$, and we can write

$$B_1 = (a_1, 0) \cup \{p\} \cup (0, b_1) \qquad B_2 = (a_2, 0) \cup \{p\} \cup (0, b_2)$$

Then let $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$ and we have a neighborhood around p, $B = (a, 0) \cup \{p\} \cup (0, b)$ where $B \subset B_1 \cap B_2$.

If $x \neq p$, then $x \in \mathbb{R}$. Denote our intervals as

$$B_1 = (a_1, b_1) \qquad B_2 = (a_2, b_2)$$

Take a and b as done previously and we have $a \geqslant a_1, a_2$ and $b \leqslant b_1, b_2$, which gives us $(a, b) = B \subset B_1$ and $B \subset B_2$ so then $B \subset B_1 \cap B_2$. $\square$

(b)

Show that $\forall U, V \subset X$ such that $0 \in U, p \in V$ and both sets are open, that $U \cap V \neq \emptyset$.

Proof. Let $U \subset X$ be an open set containing 0. Let $V \subset X$ be an open set containing p. We say $\exists U' \subset U : 0 \in U'$ where U' is of the form

$$U' = (-a, a) = (-a, 0) \cup \{0\} \cup (0, a) : a > 0$$

Similarly there exists $V' \subset V$ such that

$$V' = (-b, 0) \cup \{p\} \cup (0, b) : b > 0$$

If such subsets do not exist, then either U or V is not an open set containing the point 0 or p respectively. Let $c = \min\{a, b\}$ then $V' \cap U' = (-c, 0) \cup (0, c)$ and then $\emptyset \neq V' \cap U' \subset V \cap U$. $\square$

(c)

Show that $\mathbb{Q} \subset \mathbb{R}$ is dense in X.

Proof. We say that $\mathbb{Q}$ is dense in X if $\overline{\mathbb{Q}} = X$. Note that $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}'$ where $\mathbb{Q}' = \{\text{all limit points of } Q\}$. We know that each irrational is a limit point of $\mathbb{Q}$, and so it is in the closure of $\mathbb{Q}$. This leaves us with only the point p. Let $V(p)$ be a neighborhood of p arbitrarily. We say $V(p) = (a, 0) \cup \{p\} \cup (0, b)$. Since there exists a rational number between any two real numbers, $\exists c \in \mathbb{Q} : a < c < 0$. Therefore $V(p) \cap \mathbb{Q} \neq \emptyset$ and so the set $\mathbb{Q}$ is dense in X. □

(d)

Let $f : X \to \mathbb{R}$ be a function such that $f(x) = x$ if $x \in \mathbb{R}$, otherwise $f(x) = 0$. Show that f is continuous on X.

Proof. Let $O \subset \mathbb{R}$ be and open set in $\mathbb{R}$. We want to show that $f^{-1}(O)$ is open in X.

$\boxed{\text{Case 1: } 0 \in O}$ If $0 \in O$ then $f^{-1}(O) = O \cup \{p\}$. Since O is an open set containing 0, there exists $a \in mathbbR$ and $A_1, A_2 \subset O$ such that

$$A_1 = (-a, 0), \qquad A_2 = (0, a)$$

Then since $O = O \cup A_1 \cup A_2$, we can rewrite

$$f^{-1}(O) = O \cup \{p\} = (O \cup A_1 \cup A_2) \cup \{p\} = O \cup (A_1 \cup \{p\} \cup A_2)$$

Then we have the union of the open set O and the a-neighborhood of p. Since the union of two open sets is open, $f^{-1}(O)$ is open in X.

$\boxed{\text{Case 2: } 0 \notin O}$ If $0 \notin O$, then it follows that $f^{-1}(O) = O$, and is open in X. □

Problem 2

(a)

Let T_1, T_2 be topologies on X, show that the intersection $T_1 \cap T_2$ is also a topology on X.

Proof. We want to show that $\emptyset, X \in T_1 \cap T_2$. Since T_1, T_2 are both topologies $\emptyset \in T_1$ and $\emptyset \in T_2$, so $\emptyset \in T_1 \cap T_2$. Similarly, since both are topologies on X, the set X must be in both, and we say $X \in T_1 \cap T_2$.s

Now we want to show that $\bigcup_{\alpha \in A} O_\alpha \in T_1 \cap T_2$. We know that for each open set in our intersection, $O_\alpha \in T_1$ and $O_\alpha \in T_2$. Since T_1 is a topology on X, we know that any union of sets in T_1 will also be in T_1, so $\bigcup_{\alpha \in A} O_\alpha \in T_1$. Similarly, all open sets in our intersection are sets in T_2, and we can say that $\bigcup_{\alpha \in A} O_\alpha \in T_2$. If this arbitrary union is a set in both T_1 and T_2 it must be the case that $\bigcup_{\alpha \in A} O_\alpha \in T_1 \cap T_2$.

Finally we must show that any finite intersection of sets in $T_1 \cap T_2$ is also a set in $T_1 \cap T_2$. The argument is very similar to that regarding unions. For every $\alpha \in A$, $O_\alpha \in T_1$. Since T_1 is a topology on X, any finite intersection of sets in T_1 is also in T_1, so $\bigcap_{k=1}^n O_k \in T_1$. Similarly we say $\bigcap_{k=1}^n O_k \in T_2$. With our finite intersection being a set in both T_1 and T_2, it must be in the intersection $T_1 \cap T_2$. □

(b)

This argument is similar to that of the construction of the closure of a set. Take the collection of all topologies that contain the collection β, if we take the intersection of all of these, then we have a smallest topology. This is the smallest topology because for every toplogy τ_0 containing $\beta, \tau \subset \tau_0$. As we have shown in part (a), the intersection of two topologies on X is also a topology on X, and for this reason the arugment holds.

(c)

Proof. Let $A = \{\text{every finite intersection of sets in } \beta\text{, and every set in } \beta\}$. Then let $\tau = \{\text{every union of sets in } A\} \cup \{\emptyset\}$. From part (b) we know that this is the smallest topology on X containing β if $\tau =$ the intersection of all topologies containing β. If we can show that for every topology containing β, τ is a subset of it, then it is the smallest one.

$\boxed{\tau \text{ is a topology on } X}$ By construction we have $\emptyset$ and $\bigcup_{i\in I} B_i = X \in \tau$. Since our topology is constructed only from unions of sets in A, it follows that unions of sets in τ will simply be unions of sets in A which are in τ by construction. For intersections, let $G, H \in \tau$. Then $\exists G_0 \subset A$ and $H_0 \subset A$ such that

$$G = \left\{\bigcup G_0\right\} \qquad H = \left\{\bigcup H_0\right\}$$

Then their intersection can be written as

$$G \cap H = \left\{\bigcup G_0\right\} \cap \left\{\bigcup H_0\right\} = \bigcup (G_0 \cap H_0)$$

Recall that $G_0, H_0 \subset A$, which means that they are finite intersections of sets in β. For this reason, their intersection is also a finite intersection and $G_0 \cap H_0 = \{B_{1_g} \cap B_{2_g} \cap \cdots \cap B_{n_g}\} \cap \{B_{1_h} \cap B_{2_h} \cap \cdots \cap B_{m_h}\}$. Thus $G_0 \cap H_0 \subset A$, and so the union of elements in this intersection will be in the topology by construction. It follows that by induction any finite intersection will be in τ.

$\boxed{\tau \text{ is the smallest topology containing } \beta}$ Since topologies must be closed under finite intersections, the smallest one must contain every set in A. Since topologies must also be closed under infinite unions, any topology containing β must contain at least every element in τ, and we conclude that for any topology τ_0 such that $B \in \beta \Rightarrow B \in \tau_0$, $\tau \subset \tau_0$. $\square$

Problem 3

(a)

Let $X = \mathbb{R}$ with τ consisting of all subsets $B \subset \mathbb{R}$ such that B^c contains finitely many elements or $B^c = \mathbb{R}$. Show that τ is a topology on X.

Proof. First we want to show that the empty set and the entire set are in the topology. Since $\emptyset \subset X$, and $\emptyset^c = \mathbb{R}$, $\emptyset \in \tau$. For X we have $X \subset X$, and also $X^c = \emptyset$, and we say that the empty set has finitely many elements therefore $X \in \tau$.

Now we wish to prove that any union of sets in τ is also a set in τ. For any set $B \in \tau$, we say there is some natural number n for which $B^c = \{x_1, x_2, \cdots, x_n\}$. Take the complement of the union $\left(\bigcup_{i\in I} B_i\right)^c$ and if it has finitely many elements, the union must be a set in τ. Let $x \in \left(\bigcup_{i\in I} B_i\right)^c$. Then $\forall i \in I, x \notin B_i$ which is equivalent to saying $\forall i \in I, x \in B_i^c$. Then since x is in the complement of each B_i, $x \in \bigcap_{i\in I} B_i^c$. This gives us the result $\left(\bigcup_{i\in I} B_i\right)^c \subset \bigcap_{i\in I} B_i^c$ (both sets are in fact equal but we need not prove this here). Since each B_i^c has finitely many terms, it follows logically that $\bigcap_{i\in I} B_i^c$ must have at most finitely many terms. Since $\left(\bigcup_{i\in I} B_i\right)^c \subset \bigcap_{i\in I} B_i^c$ and the superset has finitely many terms, the subset must also have finitely many terms. With the complement having finitely many terms, $\bigcup_{i\in I} B_i \in \tau$.

Finally we must show that any finite intersection of sets in τ is also in τ. Take the complement of the union $(\bigcap_{k=1}^n B_k)^c$. Let $x \in (\bigcap_{k=1}^n B_k)^c$. Then x is not in every B_k, which means that $\exists k : x \in B_k^c$. Since there is some set where x is in the complement, we say that $x \in \bigcup_{k=1}^n B_k^c$. This gives us the result $(\bigcap_{k=1}^n B_k)^c \subset \bigcup_{k=1}^n B_k^c$. Now we have a finite union from 1 to n of sets with finitely many elements (the complements of $B \in \tau$). It follows that this union must have finitely many elements, and since it is a superset of the complement of $\bigcap_{k=1}^n B_k$, the complement of said set has finitely many elements so the set is in τ. $\square$

(b)

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Show that τ is a topology on X.

Proof. We know that $\emptyset, X \in \tau$ by construction. Let us check if the union of each pair of nonempty sets in τ is also in τ.

$$\begin{aligned}
\{c\} \cup \{a, c\} &= \{a, c\} \\
\{c\} \cup \{b, c\} &= \{b, c\} \\
\{c\} \cup \{a, b, c\} &= \{a, b, c\} \\
\{a, c\} \cup \{b, c\} &= \{a, b, c\} \\
\{a, c\} \cup \{a, b, c\} &= \{a, b, c\} \\
\{b, c\} \cup \{a, b, c\} &= \{a, b, c\}
\end{aligned}$$

Any finite union can be broken up by associativity to unions of pairs of sets and every pair of sets in τ has a union in τ therefore any finite union of sets in τ is a set in τ.

We make a similar argument for a finite intersection of sets in τ, and we look at the intersection of every pair of non-empty sets.

$$\begin{aligned}
\{c\} \cap \{a, c\} &= \{c\} \\
\{c\} \cap \{b, c\} &= \{c\} \\
\{c\} \cap \{a, b, c\} &= \{c\} \\
\{a, c\} \cap \{b, c\} &= \{c\} \\
\{a, c\} \cap \{a, b, c\} &= \{a, c\} \\
\{b, c\} \cap \{a, b, c\} &= \{b, c\}
\end{aligned}$$

And since all pairs result in sets that belong to τ we say that any finite intersection of sets in τ belongs to τ. No infinite cases exist since our topology is a finite set. $\square$

Problem 4

Show that $\overline{A} = A^\circ \cup A^b$.

Proof. $\boxed{A^\circ \cup A^b \subset \overline{A}}$ If a point p is in $A^\circ \subset A \subset \overline{A}$, then it is in $\overline{A}$. Otherwise, suppose $p \in A^b$. Assume by contradiction that the closed set $\overline{A}$ did not contain p. Then the complement of $\overline{A}$ would both contain p. Since $(\overline{A})^c$ is the complement of a closed set, it is open. Since $A \subset \overline{A}$ we say that $(\overline{A})^c \cap A = \emptyset$. Therefore there exists a nieghborhood of p that does not intersect A, and p is not a boundary point (contradiciton). So $p \in A^b \implies p \in \overline{A}$.

$\boxed{\overline{A} \subset A^\circ \cup A^b}$ Let $p \in \overline{A}$. Suppose by contradiction that $p \notin A^\circ \cup A^b$. Then both $p \notin A$ and p is not a limit point of A. Therefore there exists a neighborhood of p, $O(p)$ that does not intersect A. The complement of $O(p)$ would then be a closed set containing A that did not contain $p \implies p \notin \overline{A}$ (contradiction). $\square$

Problem 5

Show that $Cl(Int(Cl(Int(A)))) = Cl(Int(A))$.

Proof. Note that $Cl(A)$ is closed set, so it must be equal to its closure, i.e $Cl(A) = Cl(Cl(A))$. Similarly an interior $Int(A)$ is an open set, and therefore must be equal to its interior which means $Int(A) = Int(Int(A))$. Also recall that $Int(A) \subset A \subset Cl(A)$.

$\boxed{\subset}$ Let A'' be a set. We can say

$$Int(A'') \subset A''$$
$$Cl(Int(A'')) \subset Cl(A'')$$

Now let $A'' = Cl(A')$ and it follows that

$$Cl(Int(Cl(A'))) \subset Cl(Cl(A')) = Cl(A')$$

Finally let $A' = Int(A)$ and we get

$$Cl(Int(Cl(Int(A)))) \subset Cl(Int(A))$$

$\boxed{\supset}$

$$A' \subset Cl(A')$$
$$\implies Int(A') \subset Int(Cl(A'))$$

Let $A' = Int(A)$, then

$$\begin{aligned} Int(Int(A)) = Int(A) &\subset Int(Cl(Int(A))) \\ Cl(Int(A)) &\subset Cl(Int(Cl(Int(A)))) \end{aligned}$$

$\square$