Gröbner Bases — Homework 3

Philip Warton

January 28, 2022

Problem 1.4.1

Let $f = 3x^4z - 2x^3y^4 + 7x^2y^2z^3 - 8xy^3z^3 \in \mathbb{Q}[x, y, z]$. Determine the leading term, leading coefficient, and leading power product of f with respect to deglex, lex, and degrevlex with $x > y > z$. Repeat the exercise with $x < y < z$.

Solution

$\boxed{x > y > z}$
With respect to deglex, we first note the sum of powers for each term.

$$\begin{aligned} 3x^4z &\to 5 \\ -2x^3y^4 &\to 7 \\ 7x^2y^2z^3 &\to 7 \\ -8xy^3z^3 &\to 7 \end{aligned}$$

So by degree we know that $3x^4z$ is smaller than our other terms. Within the terms that have degree 7, we look at which has the highest power of x. This gives us the following ordering of terms:

$$3x^4z < -8xy^3z^3 < 7x^2y^2z^3 < -2x^3y^4.$$

So, our leading term is $-2x^3y^4$, our leading coefficient is -2, and our leading power product is x^3y^4.

With respect to lex ordering, we look first at our powers of x. This immediately gives us an ordering of our terms,

$$-8xy^3z^3 < 7x^2y^2z^3 < -2x^3y^4 < 3x^4z.$$

Thus our leading term is $3x^4$, our leading coefficient is 3, and our leading power product is x^4z.

With respect to degrevlex, we take the degree order from degrev, but reverse the lexigraphical aspec. This gives us an ordering,

$$3x^4z < -2x^3y^4 < 7x^2y^2z^3 < -8xy^3z^3.$$

Therefore our leading term is $-8xy^3z^3$, our leading coefficient is -8, and our leading power product xy^3z^3.

$\boxed{x < y < z}$
With respect to deglex, we take the same degree order as above. Then we first look at the powers of z. So then $3x^4 < -2x^3y^4$, and both are smaller than the other two. Since the other two both have z^3, we then look at powers of y. This gives us the ordering

$$3x^4z < -2x^3y^4 < 7x^2y^2z^3 < -8xy^3z^3.$$

So we have

$$lt(f) = -8xy^3z^3, \quad lc(f) = -8, \quad lp(f) = xy^3z^3.$$

With respect to lexigraphical ordering, we look at the powers of z. To order the two terms with z^3, we look at powers of y. This gives us

$$-2x^3y^4 < 3x^4z < 7x^2y^2z^3 < -8xy^3z^3.$$

Thus,

$$lt(f) = -8xy^3z^3, \quad lc(f) = -8, \quad lp(f) = xy^3z^3.$$

With respect to degrevlex, we keep the usual degree order. Then we reverse the remaining terms, giving us

$$3x^4z < -8xy^3z^3 < 7x^2y^2z^3 < -2x^3y^4.$$

Therefore

$$lt(f) = -2x^3y^4, \quad lc(f) = -2, \quad lp(f) = x^3y^4.$$

Problem 1.5.5

Let $f, g, h, r, s \in R = K[x_1, \cdots, x_n]$, and let F be a collection of non-zero polynomials in R. Disprove the following (by providing counterexamples).

(a) If $f \xrightarrow{F}_+ r$ and $g \xrightarrow{F}_+ s$, then $f + g \xrightarrow{F}_+ r + s$

(b) If $f \xrightarrow{F}_+ r$ and $g \xrightarrow{F}_+ s$, then $fg \xrightarrow{F}_+ rs$.

Solution

Let $x > y > z$ with degree lexigraphical ordering.

(a)

Let

$$\begin{aligned} F &= \{x + z, y\} \\ f &= x + y \\ g &= -y + z. \end{aligned}$$

Then $f \xrightarrow{F}_+ x, g \xrightarrow{F}_+ z$. So we say that $r + s = x + z$. Then $f + g = x + y - y + z = x + z$. The only reduction that exists is $f + g \xrightarrow{F}_+ 0$. Clearly $x + z \neq 0$.

(b)

Let

$$\begin{aligned} F &= \{x, y\} \\ f &= x + y \\ g &= x - y \end{aligned}$$

By choosing reductions carefully, we say that $f \xrightarrow{y} x$ and $g \xrightarrow{x} -y$. So then $rs = -xy$. Then $fg = (x + y)(x - y) = x^2 - y^2$. Clearly we do not have a way to reduce this to something with an xy term.