Mathematical Statistics - Assignment 6

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Problem 4.8

Suppose that Y has a density function

$$f(y) = \begin{cases} ky(1-y), & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a)

Find some $k \in \mathbb{R}$ such that $f(y)$ is a probability density function. Notice that $\forall y \in [0,1], \quad y(1-y) > 0$. This indicates that $k \geq 0$. The integral $\int_{-\infty}^{0} f(y)dy + \int_{1}^{\infty} f(y)dy = 0$ since we know that the function is constantly zero on these open intervals. Then $P(-\infty, \infty) = P(0 \leq X \leq 1) = \int_0^1 f(y)dy$. Thus we want to find some k such that $\int_0^1 ky(y-1)dy = 1$. Let us use rules of algebra and integration to derive the following

$$\begin{aligned} 1 &= \int_0^1 ky(1-y)dy \\ &= k\int_0^1 y - y^2 dy \\ &= k\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 \\ &= \frac{k}{6} \end{aligned}$$

Since we must have $\frac{k}{6} = 1$, of course our answer will be $k = 6$.

(b)-(e)

Compute various probabilities using this value of k and the normalized density function $f(y)$. First we must find $P(.4 \leq Y \leq 1)$.

$$\begin{aligned} P(.4 \leq Y \leq 1) &= \int_{0.4}^1 6y(1-y)dy \\ &= 6\int_{.4}^1 y - y^2 dy \\ &= 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{.4}^1 \\ &= 6\left[\frac{1}{6} - \left(\frac{2}{25} - \frac{2^3}{(3)5^3}\right)\right] \\ &= .648 \end{aligned}$$

The next probability to compute is $P(.4 \leq Y < 1)$. However, this probability will be the same as the one we just computed because $P(.4 \leq Y \leq 1) = P(.4 \leq Y < 1) + P(Y = 1)$. Since $P(Y = 1) = \int_1^1 f(y)dy = 0$, both probabilities must be equal.

To compute $P(Y \leq .4 | Y \leq .8)$ by definition the probability equals

$$\frac{P(Y \leq .4 \text{ and } Y \leq .8)}{P(Y \leq .8)}$$

The probability in the numberator will simply be $P(Y \leq .4)$ since this event is a subset of $Y \leq .8$. Since the probability $P(.4 \leq Y \leq 1)$ is already known, $P(Y \leq .4) = 1 - P(Y > .4) = 1 - P(.4 \leq Y \leq 1) = .352$. For the denominator, the integral must be computed.

$$6 \int_0^{.8} f(y)dy = 6 \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^{.8} = 6 \left[\frac{4^2}{(2)5^2} - \frac{4^3}{(3)5^3} \right] = .896$$

Knowing both the numerator and denominator we say $P(Y \leq .4 | Y \leq .8) = \frac{.352}{.896} = .393$.

Problem 4.14

Problem 4.18

Problem 4.28

Problem 4.32

Problem 4.40

Problem 4.48

Problem 4.50