Intro to Differential Geoemetry - Final Exam

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Problem 1

a)

Express the triple product $T(\vec{u}, \vec{v}, \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ in differential forms.

If $\vec{a}$ is a vector $\vec{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$ we say that it is equivalent to the differential form $a = a_x dx + a_y dy + a_z dz \in \bigwedge^1(\mathbb{R}^3)$. Then we say that

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\star(u \wedge v)) \cdot \vec{w} = \star((\star(u \wedge v)) \wedge \star w)$$

b)

Show that $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$.

Proof. We begin by rewriting $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{w} \cdot (\vec{u} \times \vec{v})$. This is the case because the dot product in $\mathbb{R}^3$ is of course symmetric. This can now be written in differnetial froms as

$$\star(w \wedge \star(\star(u \wedge v)))$$

Then, since $\star \star \alpha = (-1)^{k(n-k)} \alpha = \alpha$, we write

$$\begin{aligned}
(\vec{u} \times \vec{v}) \cdot \vec{w} &= \star(w \wedge (u \wedge v)) \\
&= \star(w \wedge u \wedge v) && \text{(by associativity of the wedge)} \\
&= \star(u \wedge v \wedge w) && \text{(by antisymmetry of the wedge)} \\
&= \star(u \wedge (v \wedge w)) \\
&= \star(u \wedge \star(\star(v \wedge w))) \\
&= \vec{u} \cdot (\vec{v} \times \vec{w})
\end{aligned}$$

□

Problem 2

a)

β_1

The form is closed. That is,

$$\begin{aligned}
d\beta_1 &= d(xdy \wedge dz - ydz \wedge dx) \\
&= (1)dx \wedge dy \wedge dz - (1)dy \wedge dz \wedge dx \\
&= dx \wedge dy \wedge dz - dx \wedge dy \wedge dz && \text{(antisymmetry)} \\
&= 0
\end{aligned}$$

β_2

The form is not closed. That is,

$$\begin{aligned}
d\beta_2 &= d(xdy \wedge dz + ydz \wedge dx) \\
&= (1)dx \wedge dy \wedge dz + (1)dy \wedge dz \wedge dx \\
&= dx \wedge dy \wedge dz + dx \wedge dy \wedge dz && \text{(antisymmetry)} \\
&= 2dx \wedge dy \wedge dz
\end{aligned}$$

b)

β_1

This form may be exact since $d\beta_1 = 0$, otherwise we would violate the notion of $d^2 = 0$. We know that to produce a $xdy \wedge dz$ term we must have either a $yxdz$ or $-zxdy$ term, and similarly to produce a $ydz \wedge dx$ term we must have either a $zydx$ or $-xydz$ term. The combination that produces the necessary cancellations is

$$\alpha_1 = -zxdy - zydx + Cdz$$

Where $C \in \mathbb{R}$ is a constant. Taking the derivative of this will produce β_1.

$$\begin{aligned} d\alpha_1 &= d(-zxdy) - d(zydx) + dCdz \\ &= [xdy \wedge dz - zdx \wedge dy] - [-zdx \wedge dy + ydz \wedge dx] + [0] \\ &= xdy \wedge dz - ydz \wedge dx \\ &= \beta_1 \end{aligned}$$

β_2

This form cannot be exact. Suppose that $\exists \alpha_2$ such that $d\alpha_2 = \beta_2$. Then $d(d\alpha_2) = d^2\alpha_2 = d\beta_2 \neq 0$ (contradiction).

Problem 3

a)

Let our space be $\bigwedge^p(M)$ where M is a 3-dimensional coordinate space with coordinates t, ψ, ϕ. Our orthonormal basis will be determined by the coefficients of our $d\vec{r}$ vector which can be accurately derived by observing the line element,

$$\begin{aligned} ds^2 &= -dt^2 + r^2(d\psi^2 + \sinh^2\psi d\phi^2) \\ \implies d\vec{r} &= -\hat{t} + r\hat{\psi} + r\sinh\psi\hat{\phi} \end{aligned}$$

So we write our orothonormal basis as $\{dt, rd\psi, r\sinh\psi d\phi\}$. Hence the orientation is the wedge product of these. To compute all $\star$ operations in this space, let us begin with the obvious 0-form,

$$\star 1 = \omega$$

Then before moving on let us express the metric tensor as a matrix of orthonormal basis, based upon the signs observed in the line element. That is,

$$g^{ij} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where $i, j = dt, rd\psi, r\sinh\psi d\phi$. Then we can compute the hodge duals of our basis 1-forms,

$$\begin{aligned} dt \wedge \star dt &= g^{dt,dt} r^2 \sinh\psi dt \wedge d\psi \wedge d\phi \\ \star dt &= (-1) r^2 \sinh\psi d\psi \wedge d\phi \end{aligned}$$

$$\begin{aligned} rd\psi \wedge \star rd\psi &= g^{rd\psi,rd\psi} r^2 \sinh\psi dt \wedge d\psi \wedge d\phi \\ \star rd\psi &= -r\sinh\psi dt \wedge d\phi \end{aligned}$$

$$\begin{aligned} r\sinh\psi d\phi \wedge \star r\sinh\psi d\phi &= g^{r\sinh\psi d\phi, r\sinh\psi d\phi} r^2 \sinh\psi dt \wedge \psi \wedge d\phi \\ \star r\sinh\psi d\phi &= rdt \wedge d\psi \end{aligned}$$

Now can move on to our hodge duals of basis 2-forms, which we can derive from our hodge duals of basis 1-forms, knowing that $(-1)^{k(n-k)} = 1$ since $k(n-k)$ is even as long as n is odd (in 3 dimensions $n = 3$). So it follows that $\star\star\alpha = \alpha$. So immediately we have

$$\begin{aligned} \star r\sinh\psi d\phi &= rdt \wedge d\psi & &\implies \star\, rdt \wedge d\psi = r\sinh\psi d\phi \\ \star rd\psi &= -r\sinh\psi dt \wedge d\phi & &\implies \star\, r\sinh\psi dt \wedge d\phi = -rd\psi \\ \star dt &= (-1)r^2\sinh\psi d\psi \wedge d\phi & &\implies \star\, r^2 \sinh\psi d\psi \wedge d\phi = -dt \end{aligned}$$

Then finally for our basis 3-form, we say that $\star\omega = \star\star 1 = 1$.

b)

We compute the Laplacian of a function $f, \Delta f$ by the relation $\Delta f = \star d \star df$. We write,

$$\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial \psi}d\psi + \frac{\partial f}{\partial \phi}d\phi \\
\star df &= \frac{\partial f}{\partial t}\star dt + \frac{\partial f}{\partial \psi}\star d\psi + \frac{\partial f}{\partial \phi}\star d\phi \\
d\star df &= \frac{\partial^2 f}{\partial t^2}dt\wedge\star dt + \frac{\partial^2 f}{\partial \psi^2}d\psi\wedge\star d\psi + \frac{\partial^2 f}{\partial \phi^2}d\phi\wedge\star d\phi \\
d\star df &= (-f_{tt} + f_{\psi\psi} + f_{\phi\phi})\omega \\
\star d\star df &= -f_{tt} + f_{\psi\psi} + f_{\phi\phi} \\
\Delta f &= -f_{tt} + f_{\psi\psi} + f_{\phi\phi}
\end{aligned}$$

Problem 4

We have the line element

$$ds^2 = a^2\left(\frac{dX^2 + dY^2}{Y^2}\right)$$

Which yields the $d\vec{r}$ vector as

$$d\vec{r} = \frac{a}{Y}dX\hat{X} + \frac{a}{Y}dY\hat{Y}$$

We take our orthonormal basis to be,

$$\left\{\sigma^X = \frac{a}{Y}dX, \quad \sigma^Y = \frac{a}{Y}dY\right\}$$

With $\omega = \frac{a^2}{Y^2}dX\wedge dY$. By the structure equations we can derive the following,

$$\begin{aligned}
d\sigma^X &= (0)dX\wedge dX - \frac{a}{Y^2}dY\wedge dX = -\frac{a}{Y^2}dY\wedge dX \\
d\sigma^Y &= (0)dX\wedge dY - \frac{a}{Y^2}dY\wedge dX = 0
\end{aligned}$$

So then from $0 = d\sigma^X + \omega^X_{\ Y}\wedge\sigma^Y$ it follows that

$$\begin{aligned}
-\frac{a}{Y^2}dY\wedge dX &= -\omega^X_{\ Y}\wedge\frac{a}{Y}dY \\
\frac{1}{Y}dY\wedge dX &= \omega^X_{\ Y}\wedge dY \\
\frac{1}{Y}dX &= \omega^X_{\ Y}
\end{aligned}$$

And then taking the derivative we have,

$$d\omega^X_{\ Y} = \frac{-1}{Y^2}dX\wedge dY = K\frac{a^2}{Y^2}dX\wedge dY$$

Which immediately yields $K = -1/a^2$. This result seems correct because with an extremely large scaling factor a, our curvature will become much smaller, but we will always remain in negative curvature since it is a hyperbolic surface.

Problem 5

a)

We want to compute the integral over the interior of S^3. First let us derive our orientation and orthonormal basis from the line element. We get $\{dr, rd\psi, r\sin\psi d\theta, r\sin\psi\sin\theta d\phi\}$. Then we can write

$$\omega = r^3\sin^2\psi\sin\theta dr\wedge d\psi\wedge d\theta\wedge d\phi$$

Now notice that if we take the 3-form $\alpha = \frac{1}{4}r^4\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi$, its derivative is equal to ω. So then, by Stokes theorem, we have

$$\int_{B^4}\omega = \int_{B^4}d\alpha = \int_{S^3}\alpha$$

Where B^4 is the closed 4-ball and S^3 is its boundary. So we can parameterize this integral as

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\frac{1}{4}a^4\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi\\
=&\frac{1}{4}a^4\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi\\
=&\left(\frac{\pi}{2}\right)\left(\frac{1}{4}a^4\right)\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta\wedge d\phi\\
=&(2)\left(\frac{\pi}{2}\right)\left(\frac{1}{4}a^4\right)\int_0^{2\pi}d\phi\\
=&(2\pi)(2)\left(\frac{\pi}{2}\right)\left(\frac{1}{4}a^4\right)\\
=&\frac{\pi^2a^4}{2}
\end{aligned}$$

b)

Now we want to compute

$$\int_{S^3} r\star dr$$

First let's compute $r\star dr$, which we can do by using the properties of the hodge dual. Namely,

$$\begin{aligned}
dr\wedge\star dr &= g(dr,dr)\omega\\
&= dr\wedge r^3\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi\\
\star dr &= r^3\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi\\
r\star dr &= r^4\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi
\end{aligned}$$

Then of course we can integrate over the 3-sphere just as before, and one may notice that after factoring out the first coefficient term, the integral becomes the same,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}r^4\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi\\
=&r^4\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\sin^2\psi\sin\theta d\psi\wedge d\theta\wedge d\phi\\
=&\left(\frac{\pi}{2}\right)\left(r^4\right)\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta\wedge d\phi\\
=&(2)\left(\frac{\pi}{2}\right)\left(r^4\right)\int_0^{2\pi}d\phi\\
=&(2\pi)(2)\left(\frac{\pi}{2}\right)\left(r^4\right)\\
=&2\pi^2r^4
\end{aligned}$$

Problem 6

Intuitively, one might try to first normalize the basis elements and hope they become orthogonal. But this does not yield a correct answer. If we choose a basis of

$$\{\cosh X dT, \cosh X dX\}$$

We are still left with non-orthogonal, but now normal, basis elements. The goal is to find a way to eliminate the $-\dfrac{\sinh X}{\cosh^2 X}$ term from the metric $g(d\alpha, d\beta)$ where α,β are our basis elements. In lieu of finding this, we resort to Graham Schmidtt orthogonalization,

$$\begin{aligned}
u_1 &= dT\\
u_2 &= dX - proj_{dT}dX\\
&= dX - \frac{g(dT,dX)}{g(dT,dT)}dT\\
&= dX - \sinh X dT
\end{aligned}$$

Then let $e_1 = u_1/\sqrt{g(u_1,u_1)} = \cosh X dT$ and $e_2 = u_2/\sqrt{g(u_2,u_2)} = dX - \sinh X dT/\sqrt{g(u_2,u_2)}$. We can compute this by using properties of inner products we know to be true. Computing this gets us,

$$\begin{aligned}
g(u_2,u_2) &= g(dX - \sinh X dT, dX - \sinh X dT) \\
&= g(dX, dX - \sinh X dT) - \sinh X g(dT, dX - \sinh X dT) \\
&= g(dX,dX) - \sinh X g(dX,dT) - \sinh X g(dT,dX) + \sinh^2 X g(dT,dT) \\
&= \frac{1}{\cosh^2 X} - 2\sinh X \frac{-\sinh X}{\cosh^2 X} + \sinh^2 X \frac{-1}{\cosh^2 X} \\
&= \frac{1 + 2\sinh^2 X - \sinh^2 X}{\cosh^2 X} \\
&= \frac{1+\sinh^2 X}{\cosh^2 X}
\end{aligned}$$

$$\sqrt{g(u_2,u_2)} = \frac{\sqrt{1+\sinh^2 X}}{\cosh X}\}$$

So finally, we have an orthonormal basis given by,

$$\left\{ \cosh X dT, \frac{(\cosh X)(dX - \sinh X dT)}{\sqrt{1+\sinh^2 X}} \right\}$$

b)

We compute $g(e_1,e_1), g(e_2,e_2)$ in order to find the signature. Firstly,

$$\begin{aligned}
g(\cosh X dT, \cosh X dT) &= \cosh^2 X g(dT,dT) = -1 \\
g\left((\frac{(\cosh X)(dX - \sinh X dT)}{\sqrt{1+\sinh^2 X}}, \frac{(\cosh X)(dX - \sinh X dT)}{\sqrt{1+\sinh^2 X}} \right) &= \frac{\cosh^2 X}{1+\sinh^2 X} g(dX - \sinh X dT, dX - \sinh X dT) \\
&= \frac{\cosh^2 X}{1+\sinh^2 X}\left(\frac{1+\sinh^2 X}{\cosh^2 X}\right) \\
&= 1
\end{aligned}$$

The signature is given by the number of basis elements whose inner product is negative, so for ours we have a signature of $s = 1$. Or if you wanted to write it as a plus/minus ordered pair, we would write $(1,1)$ as in 1 positive, 1 negative respectively. Or we could even write $(-,+)$, or write out the metric as a matrix

$$g = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

As we did for a previous problem.

c)

The line element is given by

$$ds^2 = g_{ij} dx^i dx^j$$

So we write this out as

$$ds^2 = \frac{-1}{\cosh^2 X} dT^2 + 2\left(\frac{-\sinh X}{\cosh^2 X}\right) dX dT + \frac{1}{\cosh^2 X} dX^2$$