Gröbner Bases — Homework 7

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Problem 1

Let $R = \mathbb{Q}[x, y]$, and set $f = x^4y + x^2y^3 - 3x^3y - 3xy^3$, $g = x^5y - 3x^4y - x^2y^2 + 3xy^2$. Use our Gröbner basis elimination order strategy from class to compute the following. (Note: you can check your work using the gcd command in sagemath, but please explain the whole method and do all the relevant computations in sagemath for your hand in work.)

1. $\text{lcm}(f, g)$
2. $\text{gcd}(f, g)$

Solution

First, we will compute the reduced Gröbner basis for the ideal $\langle wf, (1-w)g\rangle$ with respect to the elimination order of x, y being smaller than w. We will use sagemath in order to do this. We get

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REDUCED_GROEBNER_BASIS:
w*x^4*y - 3*w*x^3*y + w*x^2*y^3 - 3*w*x*y^3
w*x^3*y^2 - w*x^2*y^5 - 3*w*x^2*y^2 + 3*w*x*y^5 + x^6*y - 3*x^5*y - x^3*y^2 + 3*x^2*y^2
w*x^2*y^6 + w*x^2*y^2 - 3*w*x*y^6 - 3*w*x*y^2 - x^6*y^2 + 3*x^5*y^2 + x^5*y - 3*x^4*y + x^3*y^3 -
x^7*y - 3*x^6*y + x^5*y^3 - 3*x^4*y^3 - x^4*y^2 + 3*x^3*y^2 - x^2*y^4 + 3*x*y^4
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The last polynomial is our lcm since there is no w variable in this polynomial. That is,

$$lcm(f, g) = x^7y - 3x^6y + x^5y^3 - 3x^4y^3 - x^4y^2 + 3x^3y^2 - x^2y^4 + 3xy^4.$$

Then by our theorem, we know that $fg = lcm(f, g)gcd(f, g)$. We will use sagemath to compute $\frac{fg}{lcm(f,g)} = gcd(f, g)$. We get as our result,

$$gcd(f, g) = x(x - 3)y.$$

Problem 2

Let $R = \mathbb{Q}[x, y]$, and set

$$\begin{aligned} f_1 &= x^2 \\ f_2 &= x + y \\ g_1 &= x(x+y)^2 \\ g_2 &= y. \end{aligned}$$

Let $A = \langle f_1, f_2\rangle$ and let $B = \langle g_1, g_2\rangle$. Use our Gröbner basis elimination order strategy from class to compute the ideal quotient $B : A$. (Note: again you can check your work using the quotient(A) command in sagemath, but please explain the whole method and do all the relevant computations in sagemath for your hand in work.)

Solution

To solve this we can first use Lemma 2.3.10 and Lemma 2.3.11 which grants us that

$$B : A = (B : \langle f_1 \rangle) \cap (B : \langle f_2 \rangle) = \left(\frac{1}{f_1} (B \cap \langle f_1 \rangle) \right) \cap \left(\frac{1}{f_2} (B \cap \langle f_2 \rangle) \right).$$

Then we can use lemma 2.3.7 to get the result that

$$B \cap \langle f_i \rangle = \langle lcm(B, f_i) \rangle = \langle lcm(g_1, g_2, f_i) \rangle.$$

First, we can simply find the Groöbner basis, G_1, of $\langle wg_1, wg_2, (1-w)f_1 \rangle$ and G_2 of $\langle wg_1, wg_2, (1-w)f_2 \rangle$ using sagemath. This grants us

$$G_1 = \{x^3, x^2y\}, \quad G_2 = \{x^3 + y^3, xy + y^2\}.$$

So then dividing by f_1 and f_2 respectively, we say

$$\frac{1}{f_1}(B \cap \langle f_1 \rangle) = \langle x, y \rangle, \qquad \frac{1}{f_2}(B \cap \langle f_2 \rangle) = \langle x^2 - yx + y^2, y \rangle.$$

So now to intersect these two, we use sagemath to compute the Gröbner basis of $\langle wx, wy, (1-w)(x^2 - yx + y^2), (1-w)y \rangle$, which is given by

$$\langle x, y^2 \rangle.$$

So, in all, $B : A = \langle x, y^2 \rangle$.