Differential Geometry - Homework 7

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Problem 1

a)

From $\boxed{\text{Homework 6}}$, we write the following elements in $\mathbb{R}^3$,

$$\begin{aligned} d\hat{r} &= \sin\theta d\phi\hat{\phi} + d\theta\hat{\theta} \\ d\hat{\theta} &= \cos\theta d\phi\hat{\phi} - d\theta\hat{r} \\ d\hat{\phi} &= -\cos\theta d\phi\hat{\theta} - \sin\theta d\phi\hat{r} \end{aligned}$$

If we drop the $\hat{r}$ components, then we get the resulting elments,

$$\begin{aligned} d\hat{\theta} &= \cos\theta d\phi\hat{\phi} \\ d\hat{\phi} &= -\cos\theta d\phi\hat{\theta} \end{aligned}$$

Then, to get our connection components, we know that

$$\omega_{ij} = g_{ik}\omega^k{}_j = \omega^i{}_j$$

Since $d\hat{e}_j = \omega^i{}_j\hat{e}_i$, it follows that

$$\begin{aligned} d\hat{\theta} &= \omega^\phi{}_\theta d\hat{\phi} \\ \implies \omega^\phi{}_\theta &= \cos\theta d\phi \end{aligned}$$

$$\begin{aligned} d\hat{\phi} &= \omega^\theta{}_\phi d\hat{\theta} \\ \implies \omega^\theta{}_\phi &= -\cos\theta d\phi \end{aligned}$$

So writing all our connective componenets into a matrix we can write this out as

$$\begin{bmatrix} \omega_{\theta\theta} & \omega_{\theta\phi} \\ \omega_{\phi\theta} & \omega_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \omega^\theta{}_\theta & \omega^\theta{}_\phi \\ \omega^\phi{}_\theta & \omega^\phi{}_\phi \end{bmatrix} = \begin{bmatrix} 0 & -\cos\theta d\phi \\ \cos\theta d\phi & 0 \end{bmatrix}$$

b)

Then we compute $\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$ for this part. First note that since we have only two coordinates, any term of the form

$$\omega^i{}_k \wedge \omega^k{}_j$$

Is guaranteed to have one of them be $\omega^k{}_k = 0$, and therefore the whole term goes to 0. Thus we have $\Omega^i{}_j = d\omega^i{}_j$. From this we get

$$\begin{aligned} \Omega_{\theta\phi} &= d\omega^\theta{}_\phi = d(-cos\theta d\phi) = \sin\phi d\phi \wedge d\theta \\ \Omega_{\phi\theta} &= d\omega^\phi{}_\theta = d(cos\theta d\phi) = -\sin\phi d\phi \wedge d\theta \end{aligned}$$

Note that $\Omega_{ii} = d\omega^i{}_i = d(0) = 0$ is guaranteed quite easily. Now we simply write these Ω as a matrix,

$$\begin{bmatrix} \Omega_{\theta\theta} & \Omega_{\theta\phi} \\ \Omega_{\phi\theta} & \Omega_{\phi\phi} \end{bmatrix} = \begin{bmatrix} 0 & \sin\phi d\phi \wedge d\theta \\ -\sin\phi d\phi \wedge d\theta & 0 \end{bmatrix}$$