

Mathematical Statistics - Midterm 2

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Problem 1

Let X be a normally distributed random variable with $\mu = 75, \sigma = 12$.

(1)

Since we have $Z = \frac{X-\mu}{\sigma}$, we say

$$\begin{aligned} P(X < 60) &= P(Z < -\frac{5}{4}) \\ &= P(Z > \frac{5}{4}) && \text{(since } Z \text{ is symmetric)} \\ &= P(Z > 1.25) \\ &= .1056 && \text{(} Z\text{-table from text)} \end{aligned}$$

(2)

To last at least 60 hours, we say that this outcome event is the complement to part (1), so the probability will be $1 - .1056 = .8944$.

(3)

We assume that once the life of a bit is over, that it must be replaced. Then we want to find the proportion of the distribution that lies beyond $X = 90$ or after normalization, $Z = 1.25$. Since we have already computed $P(Z < -1.25)$, this will be the same as $P(Z > 1.25)$ by symmetry, so we say that the answer is .1056.

(4)

For a single bit to fail at or before 48 hours, we compute $P(X < 48) = P(Z < -2.25) = P(Z > 2.25) = .0122$. Then we have a binomial distribution with $n = 10, p = .0122, k = 2$. By looking at the binomial distribution table we get the probability of $1.000 - .996 = .004$.

Problem 2

$$f(x, y) = \begin{cases} 1/\pi, & x^2 + y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(1)

The probability of this event is .5. This is because one can interpret this as integrating over the unit circle only where $y > x$ which will be half of the unit circle above the $y = x$ diagonal. To parameterize this explicitly, one might use polar coordinates and compute some double integral, but we argue that since exactly half of the unit circle lies above this diagonal, we must be left with exactly half of our probability since it is distributed uniformly.

(2)

The perimeter of the circle is described by $x^2 + y^2 = 1 \iff y^2 = 1 - x^2 = (1 + x)(1 - x)$.

$$\begin{aligned} f_x(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{y}{\pi} \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \\ &= \frac{1}{\pi} \left(\sqrt{1-x^2} - (-\sqrt{1-x^2}) \right) \\ &= \frac{2\sqrt{1-x^2}}{\pi} \end{aligned}$$

Similarly, we have the bounds defined by $x = \pm\sqrt{1-y^2}$ on the same function with respect to x , giving us

$$\begin{aligned} f_y(y) &= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx \\ &= \frac{x}{\pi} \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \\ &= \frac{1}{\pi} \left(\sqrt{1-y^2} - (-\sqrt{1-y^2}) \right) \\ &= \frac{2\sqrt{1-y^2}}{\pi} \end{aligned}$$

(3)

We say that X and Y are independent if and only if $f(x, y) = f_x(x)f_y(y)$. Let us verify.

$$\begin{aligned} f_x(x)f_y(y) &= \frac{2\sqrt{1-y^2}}{\pi} \frac{2\sqrt{1-x^2}}{\pi} \\ &= \frac{4\sqrt{(1-y^2)(1-x^2)}}{\pi^2} \\ &= \frac{4\sqrt{1-y^2-x^2+x^2y^2}}{\pi^2} \\ &= \frac{4\sqrt{1-1+x^2y^2}}{\pi^2} \\ &= \frac{4\sqrt{x^2y^2}}{\pi^2} \\ &= \frac{4xy}{\pi^2} \neq \frac{1}{\pi} \end{aligned}$$

Thus they are not independent.

(4)

$$E[XY] = \int \int_{B_1(0)} \frac{xy}{\pi} dx dy = \int_0^1 \int_0^{2\pi} r^2 \sin \theta \cos \theta dr d\theta = 0$$

$$Cov(X, Y) = E[XY] - E[X]E[Y] = -E[X]E[Y] = - \int_{-1}^1 2x\sqrt{1-x^2}\pi dx \int_{-1}^1 2y\sqrt{1-y^2}\pi dy$$

(5)

Problem 3

(1)

To find a constant such that $f(y)$ is a density function, we integrate

$$\begin{aligned}\int_0^\infty cye^{-2y}dy &= c \int_0^\infty ye^{-2y}dy \\ &= c \left(\left[(y) \frac{-e^{-2y}}{2} \right]_0^\infty - \int_0^\infty -\frac{e^{-2y}}{2} (1) dy \right) && \text{(integration by parts)} \\ &= c \left(\left[(y) \frac{-e^{-2y}}{2} \right]_0^\infty + \frac{1}{2} \int_0^\infty e^{-2y} dy \right) \\ &= c \left(\left[(y) \frac{-e^{-2y}}{2} \right]_0^\infty + \frac{1}{2} \left[-\frac{e^{-2y}}{2} \right]_0^\infty \right) \\ &= c \left(0 + \frac{1}{2} \left[-\frac{e^{-2y}}{2} \right]_0^\infty \right) \\ &= c \left(\frac{1}{2} \left[0 + \frac{e^0}{2} \right] \right) = c \frac{1}{2} \cdot \frac{1}{2} = c \frac{1}{4}\end{aligned}$$

We say that $c = 4$.

(2)

Compute $P(Y > t)$ and $P(Y > t + s | Y > t)$.

$$\begin{aligned}P(Y > t) &= \int_t^\infty f(y)dy \\ &= 4 \left(\left[y \frac{e^{-2y}}{-2} \right]_t^\infty + \frac{1}{2} \int_t^\infty -\frac{e^{-2y}}{2} dy \right) && \text{(from part (1))} \\ &= 4 \left(\left[0 - t \frac{e^{-2t}}{-2} + \frac{1}{2} \left[-\frac{e^{-2y}}{2} \right]_t^\infty \right] \right) \\ &= 4 \left(-t \frac{e^{-2t}}{-2} + \frac{1}{2} \left[\frac{e^{-2t}}{-2} \right] \right) \\ &= -2te^{-2t} - e^{-2t} \\ &= (-2t - 1)e^{-2t}\end{aligned}$$

Now for the conditional probability we can compute $\frac{P(Y > t+s)}{P(Y > t)}$. Since $P(A \cap B)$ or the probability that $Y > t + s$ and $Y > t$ is just equal to the probability that $Y > t + s$. Then we get

$$P(Y > t + s | Y > t) = \frac{P(Y > t + s)}{P(Y > t)} = \frac{(-2(t + s) + 1)e^{-2(t+s)}}{(-2t + 1)e^{-2t}}$$

This function does not have the memorylessness property.

(3)

$$E[e^{tY}] = \int_0^\infty e^{ty} 4ye^{-2y} dy = \int_0^\infty e^{(t-2)y} 4y dy = \dots$$

(4)

$$E[e^{t(X+Y)}] = \int_0^\infty e^{tx+ty} 4ye^{-2y} dy = \int_0^\infty e^{tx+ty-2y} 4y dy = \dots$$