Differential Geometry - Homework 6

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Problem 1

a)

We begin with

$$x = r\cos\phi\sin\theta \tag{1}$$
$$y = r\sin\phi\sin\theta \tag{2}$$
$$z = r\cos\theta \tag{3}$$

and the fact that we can express these as differentials,

$$dx = \cos\phi\sin\theta dr + r\cos\phi\cos\theta d\theta - r\sin\phi\sin\theta d\phi \tag{4}$$
$$dy = \sin\phi\sin\theta dr + r\sin\phi\cos\theta d\theta + r\cos\phi\sin\theta d\phi \tag{5}$$
$$dz = \cos\theta dr - r\sin\theta d\theta \tag{6}$$

We can compute the inverse relationships as follows:

$$dr = \cos\phi\sin\theta dx + \sin\phi\sin\theta dy + \cos\theta dz \tag{7}$$
$$d\theta = \frac{1}{r}\cos\phi\cos\theta dx + \frac{1}{r}\sin\phi\cos\theta dy - \frac{1}{r}\sin\theta dz \tag{8}$$
$$d\phi = \frac{-1}{r\sin\theta}\sin\phi dx + \frac{1}{r\sin\theta}\cos\phi dy \tag{9}$$

Then, since there is a correspondence between these differentials, our orthonormal bases' coefficients, and between our $\hat{e}_i$ elements, we write

$$\hat{r} = \cos\phi\sin\theta\hat{x} + \sin\phi\sin\theta\hat{y} + \cos\theta\hat{z} \tag{10}$$
$$\hat{\theta} = \cos\phi\cos\theta\hat{x} + \sin\phi\cos\theta\hat{y} - \sin\theta\hat{z} \tag{11}$$
$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{12}$$

Then we can take the derivative of each of these as a sum of the various partial derivatives; that is, "zapping them with d".

$$d\hat{r} = (-\sin\phi\sin\theta d\phi + \cos\phi\cos\theta d\theta)\hat{x} + (\cos\phi\sin\theta d\phi + \sin\phi\cos\theta d\theta)\hat{y} + (-\sin\theta d\theta)\hat{z} \tag{13}$$
$$d\hat{\theta} = (-\sin\phi\cos\theta d\phi - \cos\phi\sin\theta d\theta)\hat{x} + (\cos\phi\sin\theta d\phi + \sin\phi\cos\theta d\theta)\hat{y} + (-\sin\theta d\theta)\hat{z} \tag{14}$$
$$d\hat{\phi} = (-\sin\phi)\hat{x} + (\cos\phi)\hat{y} \tag{15}$$

We can finally exchange our standard unit basis vectors for spherical ones, which gives us

$$d\hat{r} = \sin\theta d\phi\hat{\phi} + d\theta\hat{\theta} \tag{16}$$
$$d\hat{\theta} = \cos\theta d\phi\hat{\phi} - d\theta\hat{r} \tag{17}$$
$$d\hat{\phi} = -\cos\theta d\phi\hat{\theta} - \sin\theta d\phi\hat{r} \tag{18}$$

The kind of beast that we get for each of these is a vector-valued 1-form, or linear combination of basis vector-valued 1-forms. This is correct because we are taking the derivative of vector valued unctions, or 0-forms in this context.

b)

We say that $\omega_{i\ j} = e_i \cdot de_j$ which is the coefficient of the 2-form $e_i \wedge de_j$. Since we have an orthonormal basis, we say that $\omega_{i\ i} = 0$. Then we need only to compute the remaining 3 combinations. We compute,

$$\begin{aligned}
\omega_{r,\theta}(\hat{r} \wedge \hat{\theta}) &= \hat{r} \wedge d\hat{\theta} && (19)\\
&= \hat{r} \wedge (\cos\theta d\phi\hat{\phi} - d\theta\hat{r}) && (20)\\
&= \cos\theta d\phi(\hat{r} \wedge \hat{\phi}) && (21)\\
\omega_{r,\theta} &= \cos\theta d\phi && (22)\\
& && (23)\\
\omega_{r,\phi}(\hat{r} \wedge \hat{\phi}) &= \hat{r} \wedge d\hat{\phi} && (24)\\
&= \hat{r} \wedge (-\cos\theta d\phi\hat{\theta} - \sin\theta d\phi\hat{r}) && (25)\\
&= -\cos\theta d\phi\hat{\theta}(\hat{r} \wedge \hat{\theta}) && (26)\\
\omega_{r,\phi} &= -\cos\theta d\phi && (27)\\
& && (28)\\
\omega_{\theta,\phi}(\hat{\theta} \wedge \hat{r}) &= \hat{\theta} \wedge d\hat{\phi} && (29)\\
&= \hat{\theta} \wedge (-\cos\theta d\phi\hat{\theta} - \sin\theta d\phi\hat{r}) && (30)\\
&= -\sin\theta d\phi(\hat{\theta} \wedge \hat{r}) && (31)\\
\omega_{\theta,\phi} &= -\sin\theta d\phi && (32)
\end{aligned}$$

c)

Since each of our ω is a scalar function, we should have $d\omega \in \bigwedge^1$, but to take the wedge of two scalar functions, we will have a wedge product that is not vector-valued? Let us try to compute these nontheless.

$$\begin{aligned}
\Omega_{i,i} &= d\omega_{i,i} + \omega_{i,k} \wedge \omega_{k,i} && (33)\\
&= d(0) + \sum_{k=1}^{3} \omega_{i,k} \wedge \omega_{k,i} && (34)\\
&= d(0) + \sum_{k=1}^{3} \omega_{i,k} \wedge \omega_{i,k} && (35)\\
&= 0 + \sum_{k=1}^{3} 0 && (36)\\
&= 0 && (37)\\
& && (38)\\
\Omega_{i,j} &= d\omega_{i,j} + \sum_{k=1}^{3} \omega_{i,k} \wedge \omega_{k,j} && (39)\\
& && (40)
\end{aligned}$$