Applied Ordinary Differential Equations - Homework 2

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7.5.11

Solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\boldsymbol{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

First off we want to find our eigenvalues. To do this, we just take the determinant of $A - \lambda I$, set it equal to 0 and solve for λ.

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{pmatrix} \\
&= (-2-\lambda)(4-\lambda) - (1)(-5) \\
&= \lambda^2 - 2\lambda - 8 + 5 \\
&= \lambda^2 - 2\lambda - 3 \\
&= (\lambda - 3)(\lambda + 1)
\end{aligned}$$

So we conclude that $\lambda_1 = 3, \lambda_2 = -1$. So since we have one positive and one negative real eigenvalue, we know that we will have a saddle point style solution. To get the general solution, we'll solve for the eigenvectors. We write

$$A - \lambda_1 I = \begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{u} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This yields the general solution

$$\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Take the fact that $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and it follows that $c_1 = c_2 = \frac{1}{2}$. So we have a solution

$$\boldsymbol{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

This solution will asymptotically approach the span of $(1 \;\; 5)^T$ as $t \to \infty$. The general qualitative properties of the given solution can be seen in Figure 1.

Problem 7.5.23

Consider the system

$$\boldsymbol{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \boldsymbol{x}$$

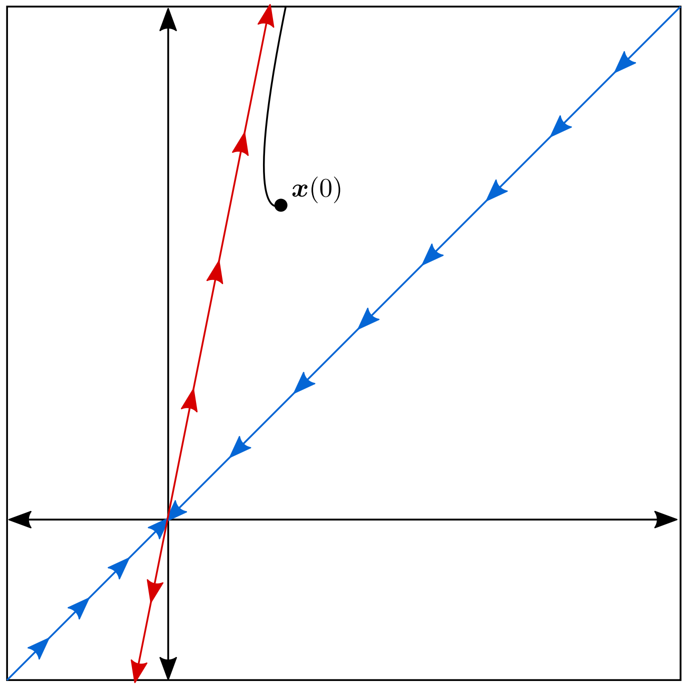

Figure 1: Solution to the diffential equation given in $\boxed{7.5.11}$

(a)

Solve the system for $\alpha = \frac{1}{2}$. Find the eigenvalues of the coefficient matrix, and classify the type of equilibrium point at the origin.

Let $\alpha = \frac{1}{2}$. Then we have a characteristic polynomial given by

$$\det(A - \lambda I) = \lambda^2 + 2\lambda + 1 - \frac{1}{2} = \lambda^2 + 2\lambda + \frac{1}{2}$$

This gives us two eigenvalues of $\lambda_1 = -1 + \frac{\sqrt{2}}{2}, \lambda_2 = -1 - \frac{\sqrt{2}}{2}$. Since both eigenvalues are negative, we say that the origin is an unstable 'source' equilibrium point.

(b)

Solve the system for $\alpha = 2$. Find the eigenvalues of the coefficient matrix, and classify the type of equilibrium point at the origin.