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Graduation Project

Oscillation Theory for Second Order Differential Equations

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1 Introduction

A Sturm-Liouville differential equation is said to be oscillatory at ∞, if there exists a nontrivial solution of this equation which having infinitly many zeros in every interval of the form $[\alpha, \infty)$. Otherwise, it said to be nonoscillatory at ∞. As we shall see that disconjugacy and admissible functions are some important concepts related with this theory. If no solution of a differential equation has more than one zero in interval I then it is said to be disconjugate on I. If a function and its first derivative are continuous on some interval, and the function attains the value zero at both ends of the interval then it is said to be admissible.

Let us begin with the simple wave equaiton

$$y''(x) + k^2 y(x) = 0. \tag{1}$$

This equation has the solutions,

$$C(\sin k)(x - x_0)$$

or

$$C(\cos k)(x - x_0),$$

and clearly these solutions are oscillatory in $\mathbb{R}$, where k is a positive constant and C is an arbitrary constant. It can be seen that the zeros of linearly independent solutions of the equation (1) are interlaced. Equation (1) can be considered as the special case of the following equation,

$$y''(x) + q(x)y(x) = 0, \tag{2}$$

where $q(x)$ is continuous on some interval. In general as we all see that, if $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of (1) and x_1 and x_2 are consecutive zeros of $y_1(x)$, then $y_2(x)$ has one and only one zero in (x_1, x_2).

It is known that a solution $y(x)$ of the equation (2) is continuous on the given interval whenever $q(x)$ is at least continuous on that interval. According to the theory of ordinary differential equations a solution $y(x)$ of the equation (2) satisfying the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$ should be a trivial solution. Therefore, nothing can be said about the zeros of such trivial solutions. So, the zeros of nontrivial solutions are needed to the examined.

It is better to remind that a point $x = x_1$ is called a zero of the solution $y(x)$ of (2) if $y(x_1) = 0$, and according to previous explanation $y'(x_1)$ should be different from zero since otherwise it becomes a trivial solution. From the theory of calculus, it is known that if $y'(x_1) = \alpha > 0$ then there exists an interval $(x_1 - h_1, x_1 + h_2)$ in the domain of $y(x)$ such that, $y(x)$ is negative (positive) on $(x_1 - h_1, x_1)$ and positive (negative) on $(x_1, x_1 + h_2)$, where h_1 and h_2 are some positive constants. The points x_1 and x_2 are called consecutive zeros of the solution $y(x)$ of (2) if $y(x_1) = y(x_2) = 0$ and there is no other zero between x_1 and x_2, where $x_1 < x_2$.

We shall note that some Sturm-Liouville equations may have the solutions so that these solutions may not have any zeros on the given interval. Indeed, the following second order Sturm-Liouville equation

$$y''(x) - k^2 y(x) = 0$$

has the solutions

$$Ce^{|k|x}$$

and

$$Ce^{-|k|x},$$

where C and k are some arbitrary real numbers, and as can be seen that these solutions do not have any zero on $\mathbb{R}$, unless $C = 0$.

2 Sturm's Seperation Theorem

The first fundamental theorem on comparison of the zeros of the Sturm-Liouville equation has been given by Sturm. First of all we shall share the following seperation theorem.

Theorem 2.1 *Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the equation (2). If x_1 and x_2 are consecutive zeros of $y_1(x)$ then $y_2(x)$ has one and only one zero in (x_1, x_2).*

Proof: Assume that $y_1(x) > 0$ for $x_1 < x < x_2$ such that $y_1'(x_1) > 0$. According to the assumption x_1 and x_2 are consecutive zeros of $y(x)$ so that $y_1'(x_2) < 0$. Therefore, $y_1(x)$ is increasing at $x = x_1$ and $y_1(x)$ is decreasing at $x = x_2$. It is known that the Wronskian of $y_1(x)$ and $y_2(x)$ is a non-zero constant. Here the Wronskian $W[y_1, y_2]$ of the solutions y_1 and y_2 is defined by

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

So, at $x = x_1$ we have

$$y_1(x_1)y_2'(x_1) - y_1'(x_1)y_2(x_1) = -y_1'(x_1)y_2(x_1).$$

Similarly for $x = x_2$,

$$y_1(x_2)y_2'(x_2) - y_1'(x_2)y_2(x_2) = -y_1'(x_2)y_2(x_2)$$

is obtained. So

$$y_2(x_1)y_1'(x_1) = y_2(x_2)y_1'(x_2),$$

and hence,

$$(sgn)y_2(x_1) = -(sgn)y_2(x_2).$$

This implies that $y_2(x)$ has at least one zero in (x_1, x_2).

If there were two zeros ξ_1 and ξ_2 of $y_2(x)$ between (x_1, x_2), $y_1(x)$ would have at least one zero in (ξ_1, ξ_2). This contradicts the assumpsion of x_1, x_2 being consecutive zeros of $y_1(x)$. This completes the proof. ■

3 Comparison Theorems

We shall introduce some results on the zeros of solutions of some Sturm-Liouville equations.

3.1 Sturm's Comparison Theorem

Following theorem is due to Sturm [2].

Theorem 3.1.1 *Consider two equations,*

$$u''(x) + F(x)u(x) = 0, \tag{3}$$

$$v''(x) + G(x)v(x) = 0, \tag{4}$$

where $F(x), G(x)$ are positive, continuous and $G(x) \geq F(x)$ in (a, b). Let the solution $u(x)$ of the equation (3) have two consecutive zeros, x_1 and x_2, $a < x_1 < x_2 < b$, and $v(x)$ be solution of the equation (4) with a zero at x_1. Then $v(x)$ has at least a zero x_3 in (x_1, x_2).

Proof: Firstly, let us multiply the equation (3) by $v(x)$ and multiply the equation (4) by $u(x)$. Then we shall subtract the first equation from the second equation so that

$$v''(x)u(x) - u''(x)v(x) + G(x)v(x)u(x) - F(x)v(x)u(x) = 0. \tag{5}$$

Equation (5) can also be written as

$$\tfrac{d}{dx}W[u, v] + (G(x) - F(x))u(x)v(x) = 0. \tag{6}$$

Now, integrating the equation (6) on (x_1, x_2) one has

$$W[u, v](x)\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} (G(x) - F(x))u(x)v(x)dx = 0,$$

or

$$\begin{aligned} u(x_2)v'(x_2) - v(x_2)u'(x_2) - u(x_1)v'(x_1) + v(x_1)u'(x_1) \\ + \int_{x_1}^{x_2} (G(x) - F(x))u(x)v(x)dx = 0. \end{aligned} \tag{7}$$

Hence

$$-v(x_2)u'(x_2) + \int_{x_1}^{x_2} (G(x) - F(x))u(x)v(x)dx = 0.$$

If $v'(x_1) > 0$ and $v(x)$ were positive in interval (x_1, x_2) then the integral would be positive and it can be inferred that

$$-v(x_2)u'(x_2) \geq 0.$$

This expression shows that $v(x)$ cannot keep a constant sign throughout the interval (x_1, x_2). ■

Theorem 3.1.2 *Let $u(x)$ and $v(x)$ be the solutions of (3) and (4), respectively, such that*

$$u(x_1) = v(x_1) = 0 \quad , \quad u'(x_1) = v'(x_1) > 0.$$

Suppose that $u(x)$ is increasing in $[x_1, x_2]$ and reaches a maximum at $x = x_2$. Then $v(x)$ reaches a maximum at some point x_3 with $x_1 < x_3 < x_2$.

Proof: Assume that $u(x)$ reaches a maximum point at $x = x_2$. This means that $u'(x_2) = 0$. Then (7) can be written as

$$u(x_2)v'(x_2) + \int_{x_1}^{x_2} (G(x) - F(x))u(x)v(x)dx = 0.$$

This equation shows that $v'(x_2) < 0$ if $v(x) > 0$ on (x_1, x_2). So there should be a maximum of $v(x)$ at x_3 on $x_1 < x < x_2$. This completes the proof. ■

Remark: It is possible to consider much more general version of the Sturm-Liouville equations rather than the form given in (2) as follows. However in many applications the sign of the leading function of the Sturm-Liouville equation is -1. So we shall consider the following equations

$$-(p_1(x)u'(x))' + p_0(x)u(x) = 0, \tag{8}$$

$$-(q_1(x)v'(x))' + q_0(x)v(x) = 0, \tag{9}$$

where $p_1 > 0$, $q_1 > 0$ and p_0, q_0 are some real-valued functions on an interval (x_1, x_2).

3.2 Sturm-Picone Comparison Theorem

The restriction $p_1(x) = q_1(x)$ has been removed by M.Picone in 1909.

Theorem 3.2.1 *Suppose that $p_1(x) \geq q_1(x)$ and $p_0(x) \geq q_0(x)$ on an interval $[x_1, x_2]$. If x_1 and x_2 are consecutive zeros of $u(x)$ of (8) then there should* *exists at least one zero of every solution $v(x)$ of (9).*

Proof: Consider that $v(x) \neq 0$ for $x \in [x_1, x_2]$. Following the equation is known as Picone's identity;

$$\begin{aligned}
\tfrac{d}{dx}\left[\frac{u}{v}(vp_1u' - uq_1v')\right] =& \tfrac{d}{dx}\left[up_1u' - \frac{u^2q_1v'}{v}\right] \qquad (10)\\
=&(u')^2p_1 + (p_1u')'u - \left[\frac{2uu'v - v'u^2}{v^2}(v'q_1) + \frac{u^2}{v}(q_1v')'\right]\\
=&(u')^2p_1 + p_0u^2 + \frac{q_1}{v^2}[uv' - vu']^2 - q_1(u')^2 - u^2q_0\\
=&(u')^2(p_1 - q_1) + u^2(p_0 - q_0) + \frac{q_1}{v^2}[uv' - vu']^2.
\end{aligned}$$

Integrating both sides of (10) on $[x_1, x_2]$ we have,

$$\begin{aligned}
&\frac{u}{v}\left[vp_1u' - uq_1v'\right]\Bigg|_{x_1}^{x_2} = 0\\
&= \int_{x_1}^{x_2}\left((p_1 - q_1)(u')^2 + (p_0 - q_0)u^2 + \frac{q_1}{v^2}[uv' - vu']^2\right)dx\\
&\geq \int_{x_1}^{x_2}\left((p_1 - q_1)(u')^2 + (p_0 - q_0)u^2\right)dx
\end{aligned}$$

which is a contradiction unless $p_1 \equiv q_1$, $p_0 \equiv q_0$ and $\frac{uv'-vu'}{v^2} = (\frac{-u}{v})' = 0$. However this implies that $\frac{u}{v}$ = constant. Consequently, $v(x_1) = v(x_2) = 0$. This is a contradiction. ■

3.3 Leighton's Comparison Theorem

Theorem 3.3.1 *If $\int_\alpha^\infty \frac{1}{p_1}dx = \infty$ and $\int_\alpha^\infty p_0 dx = -\infty$ then the Sturm-Liouville equation (8) is oscillatory at ∞, where α is a fixed constant.*

Proof: Suppose that $h = -\frac{p_1u'}{u}$. Then it is obtained that

$$\begin{aligned}
h' = -\left(\frac{(p_1u')'u - u'(p_1u')}{u^2}\right) =& \frac{-p_0u^2 + u'^2p_1}{u^2}\\
=& -p_0 + h^2\frac{1}{p_1} = h'.
\end{aligned}$$

If $u > 0$ on $[\beta, \infty)$, $\alpha \leq \beta \leq \infty$, the equation (8) is nonoscillatory at ∞. Integrating h' on $[\beta, \infty)$ it is obtained that

$$h(x) - h(\beta) = -\int_\beta^x p_0(t)dt + \int_\beta^x \frac{h^2(t)}{p_1(t)}dt,$$

and

$$h(x) = h(\beta) - \int_\beta^x p_0(t)dt + \int_\beta^x \frac{h^2(t)}{p_1(t)}dt.$$

Adopting the notation

$$g(x) = \int_\beta^x \frac{h^2(t)}{p_1(t)}dt,$$

one gets that $h(x) > g(x)$ on (β, ∞). Taking the derivative of $g(x)$ it is obtained that

$$g'(x) \geq \frac{h^2(x)}{p_1(x)}.$$

It can be written as

$$\frac{h^2(x)}{p_1(x)} > \frac{g^2(x)}{p_1(x)},$$

or

$$\frac{g'(x)}{g^2(x)} > \frac{1}{p_1(x)}. \tag{11}$$

Integrating both sides of (11) on $[\gamma, \infty)$, one obtained that

$$\int_\gamma^\infty \frac{g'(x)}{g^2(x)}dx > \int_\gamma^\infty \frac{1}{p_1(x)}dx$$

which gives

$$\int_\gamma^\infty u^{-2}du = -u^{-1}\Big|_\gamma^\infty = -\frac{1}{g}\Big|_\gamma^\infty > \int_\gamma^\infty \frac{1}{p_1}dx,$$

and

$$\int_\gamma^\infty \frac{1}{p_1}dx < \frac{1}{g(\gamma)} < \infty.$$

This is a contradiction. ■

The proof of the following theorem can be obtained from the proof of Theorem 3.3.1.

Theorem 3.3.2 *If $\int_\alpha^\infty \frac{1}{p_1(x)}dx < \infty$ and $\int_\alpha^\infty |p_0(x)|dx < \infty$ then (8) is nonoscillatory at ∞.*

3.4 Levin's Comparison Theorem

Levin considers the following equations

$$u'' + p_0(x)u = 0, \tag{12}$$
$$v'' + q_0(x)v = 0, \tag{13}$$

respectively, where $x \in [\alpha, \beta]$, p_0, q_0 are some continuous functions on $[\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$.

The method used by Levin converts the differential equations (12) and (13) into the Riccati equations

$$w'(x) = w^2(x) + p_0(x), \tag{14}$$
$$h'(x) = h^2(x) + q_0(x), \tag{15}$$

where $x \in [\alpha, \beta]$. Indeed, substituting $w(x) = -\frac{u'(x)}{u(x)}$ and $h(x) = -\frac{v'(x)}{v(x)}$ into the equations (14) and (15), respectively, then the equations (12) and (13) are obtained.

Theorem 3.4.1 *Let $u(x)$ and $v(x)$ be nontrivial solutions of (12) and (13), respectively, such that $u(x)$ does not vanish on $[\alpha, \beta]$, $v(\alpha) \neq 0$ and the inequality*

$$-\frac{u'(\alpha)}{u(\alpha)} + \int_\alpha^x p_0(t)dt > \left| -\frac{v'(\alpha)}{v(\alpha)} + \int_\alpha^x q_0(t)dt \right| \tag{16}$$

holds for all $x \in [\alpha, \beta]$. Then $v(x)$ does not vanish on $[\alpha, \beta]$ and the inequality

$$-\frac{u'(x)}{u(x)} > \left| -\frac{v'(x)}{v(x)} \right| \tag{17}$$

holds, where $x \in [\alpha, \beta]$.

Proof: If $u(x)$ does not vanish then $w(x) = -\frac{u'(x)}{u(x)}$ is continuous on $[\alpha, \beta]$. Integrating the equation (14) on $[\alpha, x]$ one gets that

$$w(x) - w(\alpha) = \int_\alpha^x w^2(t)dt + \int_\alpha^x p_0(t)dt$$

or

$$w(x) = w(\alpha) + \int_\alpha^x w^2(t)dt + \int_\alpha^x p_0(t)dt. \tag{18}$$

By the hypotesis (16), it can be written as

$$w(x) \geq -\frac{u'(\alpha)}{u(\alpha)} + \int_\alpha^x p_0(t)dt > 0.$$

Since $v(\alpha) \neq 0$ then $h(x) = -\frac{v'(x)}{v(x)}$ is continuous on $[\alpha, \gamma]$, where $\alpha < \gamma \leq \beta$. Integrating the equation (15) on $[\alpha, \gamma]$ where $x = \gamma$ we get

$$h(x) - h(\alpha) = \int_\alpha^x h^2(t)dt + \int_\alpha^x q_0(t)dt,$$

or

$$h(x) = h(\alpha) + \int_\alpha^x h^2(t)dt + \int_\alpha^x q_0(t)dt. \tag{19}$$

From (19) and we have

$$\begin{aligned} h(x) &\geq -\frac{v'(\alpha)}{v(\alpha)} + \int_\alpha^x q_0(t)dt \\ &> -w(\alpha) - \int_\alpha^x p_0(t)dt \geq -w(x), \end{aligned}$$

where $x \in [\alpha, \gamma]$. Consequently, $w(x) > -h(x)$. In order to show that

$$|h(x)| < w(x) \tag{20}$$

on $\alpha \leq x \leq \gamma$, it is sufficient to show that $w(x) > h(x)$ on $\alpha \leq x \leq \gamma$.

If we suppose to the contrary, there exists an x_0 on the interval $[\alpha, \gamma]$ such that the inequality $w(x_0) \le h(x_0)$ holds. Then since $w(\alpha) > |h(\alpha)|$ from (17) (with $x = \alpha$) and $w(x)$, $h(x)$ are continuous on the interval $[\alpha, \gamma]$, there exists some point x_1 in $(\alpha, x_0]$ such that $w(x_1) = h(x_1)$ and $h(x) < w(x)$ for $\alpha \le x < x_1$. Since $w(x) > -h(x)$ was established previously, it follows that $|h(x)| < w(x)$ for $\alpha \le x < x_1$. Consequently,

$$\int_{\alpha}^{x_1} h^2(t)dt < \int_{\alpha}^{x_1} w^2(t)dt.$$

Hence, it is obtained that

$$\begin{aligned} h(x_1) &= h(\alpha) + \int_{\alpha}^{x_1} h^2(t)dt + \int_{\alpha}^{x_1} q_0(t)dt \\ &< w(\alpha) + \int_{\alpha}^{x_1} w^2(t)dt + \int_{\alpha}^{x_1} p_0(t)dt = w(x_1) \end{aligned}$$

from equations (18) and (19). However, this is a contradiction. ■

Theorem 3.4.2 *Let $u(x)$ and $v(x)$ be two solutons of (12) and (13), respectively, such that $u(x)$ does not vanish on $[\alpha, \beta]$, $v(\beta) \ne 0$ and the inequality*

$$\frac{u'(\beta)}{u(\beta)} + \int_{x}^{\beta} p_0(t)dt > \left| \frac{v'(\beta)}{v(\beta)} + \int_{x}^{\beta} q_0(t)dt \right| \tag{21}$$

holds for all $x \in [\alpha, \beta]$. Then $v(x)$ does not vanish on $[\alpha, \beta]$, and the inequality

$$\frac{u'(x)}{u(x)} > \left| \frac{v'(x)}{v(x)} \right|, \qquad \alpha \le x \le \beta. \tag{22}$$

holds.

Proof: If $u(x)$ does not vanish, $w(x) = -\frac{u'(x)}{u(x)}$ is continuous on $[\alpha, \beta]$ and integrating the equation (14) and (15) on the interval $[x, \beta]$. We obtain that

$$w(\beta) - w(x) = \int_{x}^{\beta} w^2(t)dt + \int_{x}^{\beta} p_0(t)dt$$

and

$$h(\beta) - h(x) = \int_{x}^{\beta} h^2(t)dt + \int_{x}^{\beta} q_0(t)dt.$$

So it is found that

$$
\begin{aligned}
-w(x) &\geq -w(\beta) + \int_x^\beta p_0(t)dt \\
&> -h(x) \geq -h(\beta) + \int_x^\beta q_0(t)dt.
\end{aligned}
$$

Consequently, $-w(x) > |h(x)|$ on $\xi \leq x \leq \beta$. Let us suppose the contrary. Then there exists an x_0 on $[\xi, \beta]$ such that $-w(x_0) \leq -h(x_0)$. Then since $-w(\beta) > |h(\beta)|$ from hypotesis (22) (with $x = \beta$) and since $w(x)$ and $h(x)$ are continuous on $[\xi, \beta]$, there exists x_1 in $[x_0, \beta)$ such that

$$-w(x_1) = -h(x_1)$$

and

$$-w(x) > -h(x)$$

for $x_1 < x \leq \beta$. Since $-w(x) > -h(x)$ we also have

$$
\begin{aligned}
-h(x_1) &= -h(\beta) + \int_{x_1}^\beta h^2(t)dt + \int_{x_1}^\beta q_0(t)dt \\
&< -w(\beta) + \int_{x_1}^\beta w^2(t)dt + \int_{x_1}^\beta p_0(t)dt = -w(x_1).
\end{aligned}
$$

However this is a contradiction. ■

Theorem 3.4.3 *Suppose that there exists a nontrivial solution $v(x)$ of the equation (13) satisfying the conditions $v(\alpha) = v(\beta) = v'(\gamma) = 0$, where $\alpha < \gamma < \beta$. If the inequalities*

$$\int_x^\gamma p_0(t)dt \geq \left| \int_x^\gamma q_0(t)dt \right|, \quad \int_\gamma^x p_0(t)dt \geq \left| \int_\gamma^x q_0(t)dt \right| \tag{23}$$

hold for all x on $[\alpha, \gamma]$ and $[\gamma, \beta]$, then every solution u of the equation (12) has at least one zero on $[\alpha, \beta]$.

Proof: Let u be a nontrivial solution of (12) satisfying $u'(\gamma) = 0$. Levin asserted that u has at least one zero in each interval $[\alpha, \gamma)$ and $(\gamma, \beta]$. First, observe that $u(\gamma) \neq 0$. Otherwise, u would be a trivial solution of (12). If u had no zero in $(\gamma, \beta]$ then u would have no zero on $[\gamma, \beta]$. Theorem 3.4.1

supports the hypothesis (21). Thus v would have no zero on $[\gamma, \beta]$. This contradicts the hypothesis $v(\beta) = 0$. Likewise, if u had no zero on $[\alpha, \gamma)$, Theorem 3.4.1 gives the contradiction that v has no zero on $[\alpha, \gamma]$. Indeed, Levin has shown that u has at least two zeros on $[\alpha, \beta]$, and hence every solution of (12) has at least one zero on $[\alpha, \beta]$ by Sturm Seperation Theorem. ■

4 Some Criteria

In this section we will introduce some criteria on disconjugacy, existence of zeros of solutions and oscillation.

4.1 Kreith's Criteria

Kreith introduced the following theorem on disconjugacy.

Theorem 4.1.1 *If the differential equation (9) is disconjugate on $[\alpha, \beta]$, then*

$$\int_{\alpha}^{\beta} (q_1 \eta'^2 + q_0 \eta^2) dx > 0, \tag{24}$$

for all admissable $\eta(x) \not\equiv 0$.

Proof: If the equation (9) is disconjugate on $[\alpha, \beta]$, then there exists a solution $v(x)$ of (9) which is different from zero on $[\alpha, \beta]$. Using The Picone identity with $p_1 \equiv q_1$ the following can be written

$$\frac{d}{dx}\left[\frac{\eta}{v}\left(\eta' q_1 v - \eta q_1 v'\right)\right] = \frac{d}{dx}\left(\eta' \eta q_1 - \frac{\eta^2 q_1 v'}{v}\right). \tag{25}$$

Therefore, proceeding the steps in equation (25) it is obtained that

$$\begin{aligned} &(\eta' q_1)'\eta + \eta'^2 q_1 - \left[(q_1 v')' \frac{\eta^2}{v} + (q_1 v')\left(\frac{2\eta\eta' v - \eta^2 v'}{v^2}\right)\right] \\ =&(\eta' q_1)'\eta + \eta'^2 q_1 - q_0 \eta^2 - \frac{2 q_1 v' \eta \eta'}{v} + \frac{q_1 \eta^2 v'^2}{v^2} \\ =&(\eta' q_1)'\eta - q_0 \eta^2 + q_1 \left(\eta' - \frac{\eta v'}{v}\right)^2 = (\eta' q_1)'\eta + q_1 \eta'^2 - \frac{d}{dx}\left(\frac{q_1 \eta^2 v'}{v}\right). \end{aligned}$$

Cancelling $(\eta' q_1)'\eta$ and multiplying both sides of the equation by -1 we get

$$q_1\eta'^2 + q_0\eta^2 = q_1\left(\eta' - \frac{\eta v'}{v}\right)^2 + \frac{d}{dx}\left(\frac{q_1\eta^2 v'}{v}\right). \tag{26}$$

Integrating both sides of the equation (26) on $[\alpha, \beta]$ it is found that

$$\int_\alpha^\beta (q_1\eta'^2 + q_0\eta^2)dx = \int_\alpha^\beta q_1\left(\eta' - \frac{\eta v'}{v}\right)^2 + \left(\frac{q_1\eta^2 v'}{v}\right)\Bigg|_\alpha^\beta.$$

Since $\eta(\alpha) = \eta(\beta) = 0$,

$$\left(\frac{q_1\eta^2 v'}{v}\right)\Bigg|_\alpha^\beta = 0.$$

So, we get

$$\int_\alpha^\beta (q_1\eta'^2 + q_0\eta^2)dx = \int_\alpha^\beta q_1\left(\eta' - \frac{\eta v'}{v}\right)^2 \geq 0$$

with equality if and only if

$$\begin{aligned} \eta' - \frac{\eta v'}{v} &= \frac{\eta' v - v'\eta}{v} \\ &= v\left(\frac{\eta'}{v} - \frac{v'\eta}{v^2}\right) \\ &= 0 \end{aligned}$$

holds for the solution $v > 0$. Then $(\frac{\eta}{v})' = 0$ such that $\frac{\eta}{v} = C$, C is a arbitrary constant. This gives that $\eta = Cv$. This is a contradiction. Consequently, equation (24) is established for all admissible $\eta(x) \not\equiv 0$ on the interval $[\alpha, \beta]$. ■

Corollary 4.1.1 *If there exists an admissible $\eta(x) \not\equiv 0$ such that*

$$\int_\alpha^\beta (q_1\eta'^2 + q_0\eta^2)dx \leq 0,$$

then every solution of the equation (9) has a zero in $[\alpha, \beta)$.

4.2 Leighton's Criterion

Leighton has noted that Sturm-Liouville Comparision Theorem is a special case of Corollary 4.1.1. Indeed, if the following inequality holds

$$\int_{\alpha}^{\beta}[(p_1-q_1)\eta'^2dx+(p_0-q_0)\eta^2]dx \le \int_{\alpha}^{\beta}(q_1\eta'^2+q_0\eta^2)dx \tag{27}$$

the following is obtained.

Corollary 4.2.1 *If there exists an admissible η such that the inequality (27) is satisfied, then every solutions of (9) has a zero in $[\alpha,\beta)$.*

4.3 Wintner's Criteria

A.Wintner introduced some criteria on the oscillatory property of the equation

$$y''+q(x)y=0 \tag{28}$$

where $q(x)$ is real valued, continuous function for large positive x, where $x_0 \le x < \infty$ and $x_0 \in \mathbb{R}$.

Theorem 4.3.1 *The equation (28) is oscillatory at ∞ if*

$$Q(x)=\int^{x} q(s)ds$$

satisfies

$$\lim_{x\to\infty}\frac{\int^{x} Q(s)ds}{x}=\infty. \tag{29}$$

Proof: Let us assume the contrary. So that the equation (28) is nonoscillatory. If $y(x)$ is a nontrivial solution of (28), then for large x $y(x)$ is positive. Since $y(x)$ does not vanish on some interval around infinity, it has a logaritmic derivative so that $h=\frac{y'}{y}$. It satisfies the Riccati equation

$$h'=-q-h^2. \tag{30}$$

Indeed one has

$$\begin{aligned} h' &= \frac{y''y - y'^2}{y^2} \\ &= \frac{y''y}{y^2} - \frac{y'^2}{y^2} \\ &= \frac{-qy^2}{y^2} - \frac{y'^2}{y^2} = -q - h^2. \end{aligned}$$

The equation (30) implies the inequality $h' \leq -q$ or $-h' \geq q$. Integrating the inequality $q \leq -h'$ twice on some intervals whose upper bounds are x, it is obtained that

$$\int^x q(s)ds \leq -h(x) + c_1$$

and

$$\int^x Q(s)ds \leq -\int^x h(s)ds + c_1x + \tilde{c_2}$$

where $c_1, \tilde{c_2}$ are some constants. Therefore,

$$\int^x Q(s)ds \leq -\ln|y(x)| + c_1x + c_2$$

or

$$\int^x Q(s)ds - c_1x - c_2 \leq -\ln|y(x)|. \tag{31}$$

Multiplying both sides of the inequality (31) by $\frac{1}{x}$, we get

$$\frac{\int^x Q(s)ds}{x} - c_1 - \frac{c_2}{x} \leq \frac{-\ln|y(x)|}{x}.$$

If hypotesis (29) holds, then

$$\lim_{x\to\infty} \ln|y(x)| = -\infty. \tag{32}$$

Hence, this gives a contradiction. ■

Theorem 4.3.2 *The equation (12) is nonoscillatory at* ∞ *if and only if there exists a function* v *so that* v *and* v' *are continuous satisfying the inequality*

$$v'(x) + v^2(x) \le -c(x) \tag{33}$$

for sufficiently large x.

Proof: If equation (12) is nonoscilatory at ∞ and $u(x)$ is nontrivial solution of the equation (12), there exists some number x_0 such that $u(x)$ has no zero on $[x_0, \infty)$ and $v = \frac{u'}{u}$ satisfies the Riccati equation

$$v'(x) + v^2(x) = -c(x).$$

Conversely, if there exists a fuction v satisfying

$$-C(x) \equiv v'(x) + v^2(x) \le -c(x), x \ge x_0.$$

Then

$$u(x) = \exp\left[\int_{x_0}^{x} v(t)dt\right]$$

satisfies $u'' + C(x)u = 0$. Indeed

$$\begin{aligned} u'(x) =& v(x)\exp\left[\int_{x_0}^{x} v(t)dt\right], \\ u''(x) =& v'(x)\exp\left[\int_{x_0}^{x} v(t)dt\right] + v^2(x)\exp\left[\int_{x_0}^{x} v(t)dt\right] \\ & -[v'(x) + v^2(x)]\exp\left[\int_{x_0}^{x} v(t)dt\right] = 0. \end{aligned}$$

Since $c(x) \le C(x)$ on $x_0 \le x < \infty$, it follows from Sturm's Theorem 2.1 that no solution of $u'' + c(x)u = 0$ can have more than one zero on $[x_0, \infty)$. ■

4.4 Hartman's Criterion

Theorem 4.4.1 *Every nonoscillatory equation (12) has a solution* u *such that*

$$\int^{\infty} u^{-2}(t)dt < \infty \tag{34}$$

is finite and a nontrivial solution v such that

$$\int^{\infty} v^{-2}(t)dt = \infty. \tag{35}$$

Proof: Let w be a solution of the equation (12) such that $w(x) > 0$ for $x > x_0$, and Hartman defines a second solution by

$$u(x) = w(x) \int_{x_0}^{x} w^{-2}(t)dt, x > x_0. \tag{36}$$

Taking the derivative of the equation (36) twice, it is obtained

$$u'(x) = w'(x) \int_{x_0}^{x} w^{-2}(t)dt + w^{-1}(x),$$

and

$$u''(x) = w''(x) \int_{x_0}^{x} w^{-2}(t)dt. \tag{37}$$

Adding $p_0(x)u(x)$ to both sides of the equation (37) we get

$$0 = w''(x) \int_{x_0}^{x} w^{-2}(t)dt + p_0(x)w(x) \int_{x_0}^{x} w^{-2}(t)dt.$$

or

$$0 = (w''(x) + p_0(x)w(x)) \int_{x_0}^{x} w^{-2}(t)dt,$$

where $w > 0$ on $[x_0, x]$. Then

$$w''(x) + p_0(x)w(x) = 0.$$

The Wronskian of u and w is

$$\begin{aligned} u(x)w'(x) - w(x)u'(x) &= w'(x)w(x) \int_{x_0}^{x} w^{-2}(t)dt \\ &\quad - w(x)\left[w'(x) \int_{x_0}^{x} w^{-2}(t)dt + w^{-1}(x)\right] \\ &= -1. \end{aligned}$$

Since $u(x) > 0$ for $x > x_0$, it follows that the ratio $w(x)/u(x)$ is differentiable for $x > x_0$ and

$$\left(\frac{w}{u}\right)' = \frac{w'u - u'w}{u^2} = \frac{-1}{u^2}.$$

Hence for $x_0 < x_1 \leq x < \infty$, integrating $(\frac{w}{u})'$ on $[x_1, x]$ one gets that

$$\frac{w(x)}{u(x)} = \frac{w(x_1)}{u(x_1)} - \int_{x_1}^{x} \frac{dt}{u^2(t)}.$$

If

$$\lim_{x\to\infty} \int_{x_1}^{x} \frac{dt}{u^2(t)} = \infty$$

then $w(x)/u(x)$ would approach $-\infty$, contrary to fact that both $u(x)$ and $w(x)$ are positive for $x \geq x_1$.

To prove the second assertion of the Theorem 4.4.1, define another solution by

$$v(x) = u(x) \int_{x}^{\infty} \frac{dt}{u^2(t)}, x > x_0.$$

Then $v(x)/u(x) \to 0$ as $x \to \infty$ since u^{-2} is integrable on (x_0, ∞). From the identity

$$\begin{aligned}\left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \\ &= \frac{u'u \int_x^\infty \frac{dt}{u^2} - uu' \int_x^\infty \frac{dt}{u^2} + u^2 \frac{1}{u^2}}{v^2} \\ &= \frac{1}{v^2}\end{aligned}$$

it is obtained that

$$\frac{u(x)}{v(x)} = \frac{u(x_1)}{v(x_1)} + \int_{x_1}^{x} \frac{dt}{v^2(t)}, x_0 \leq x_1 \leq x,$$

and hence

$$\lim_{x\to\infty} \int_{x_1}^{x} \frac{dt}{v^2(t)} = \infty.$$

This completes the proof. ■

5 Conclusion

In this project our main aim is to clarify the very well known results on oscillation and nonoscillation (disconjugacy) of some second-order Sturm-Liouville equations defined on some finite and infinite intervals. As is known such equations arise, for instance, in the study of harmonic standing wave, vibrations of elastic bars and vibrating membranes. Therefore this project may help the students in understanding some real-world problems in detail.

6 Referances

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