武汉大学 2011—2012 学年第二学期《高等数学 A2》试题（A 卷）解答

一、解：1、$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x\right)\mathrm{d}x=0, a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x\right)\cos nx\mathrm{d}x=0$，（奇函数在对称区间上积分）

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(x\right)\sin nx\mathrm{d}xa_0=\frac{-2}{n\pi}\int_0^{\pi}x\mathrm{d}\cos nx=\frac{-2}{n\pi}\left[x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx\mathrm{d}x\right]=\left(-1\right)^{n+1}\frac{2}{n}$$

从而 $f\left(x\right)=\sum_{n=1}^{\infty}\left(-1\right)^{n+1}\frac{2}{n}\sin nx, x\neq\left(2k+1\right)\pi$

2、$\iint_{\Sigma}\left(x^2+y^2\right)dxdy=0$； $\iint_{\Sigma}\left(x^2+y^2\right)dS=2\pi$

3、由 $\begin{cases}f_x'=2xy(4-x-y)+xy(-1)=0\\ f_y=x^2(4-x-2y)=0\end{cases}$ 得 D 内的驻点为 $M_0(2,1)$ 且 $f(2,1)=4$，

又 $f(0,y)=0, f(x,0)=0$

而当 $x+y=6, x\geq 0, y\geq 0$ 时，$f(x,y)=2x^3-12x^2 \qquad (0\leq x\leq 6)$

令 $(2x^3-12x^2)'=0$ 得 $x_1=0, x_2=4$

于是相应 $y_1=6, y_2=2$ 且 $f(0,6)=0, f(4,2)=-64$

$\therefore f(x,y)$ 在 D 上的最大值为 $f(2,1)=4$，最小值为 $f(4,2)=-64$

4、设 $s(x)=\sum_{n=1}^{\infty}nx^n \ (-1<x<1)$，则原问题转化为求和函数在 $x=\frac{1}{a}$ 处的值

而 $s(x)=x\sum_{n=1}^{\infty}nx^{n-1}=x\sum_{n=1}^{\infty}(x^n)'=x(\sum_{n=1}^{\infty}x^n)'=x(x\sum_{n=1}^{\infty}x^{n-1})'=x\left(\frac{x}{1-x}\right)'=\frac{x}{(1-x)^2}$

故所求值为 $s\left(\frac{1}{a}\right)=\frac{a}{(a-1)^2}$

5、由于 $dy=f_x'(x,t)dx+f_t'(x,t)dt$， $F_x'dx+F_y'dy+F_t'dt=0$

由上两式消去 dt，即得： $\frac{dy}{dx}=\frac{f_x'\cdot F_t'-f_t'F_x'}{F_t'+f_t'F_y'}$

6、$\int_0^a dx\int_a^x e^{-y^2}dy=-\int_0^a dx\int_x^a e^{-y^2}dy=-\int_0^a dy\int_0^y e^{-y^2}dx$

$$=-\int_0^a ye^{-y^2}dy=\frac{1}{2}\left(e^{-a^2}-1\right).$$

从而 $\lim\limits_{a\to+\infty} I(a) = -\dfrac{1}{2}$

7、**证明：** $\iint\limits_D \dfrac{\ln\left(1+y\right)}{\ln\left(1+x\right)} dxdy$

$$= \frac{1}{2}\iint\limits_D \left[\frac{\ln\left(1+y\right)}{\ln\left(1+x\right)} + \frac{\ln\left(1+x\right)}{\ln\left(1+y\right)}\right] dxdy \geq \iint\limits_D dxdy = 1$$

其中用到了 $\dfrac{1}{2}\left[\dfrac{\ln\left(1+y\right)}{\ln\left(1+x\right)} + \dfrac{\ln\left(1+x\right)}{\ln\left(1+y\right)}\right] = \dfrac{\ln^2\left(1+y\right)+\ln^2\left(1+x\right)}{2\ln\left(1+x\right)\ln\left(1+y\right)} \geq 1$

8、 $\iiint\limits_\Omega (x+y)^2 \mathrm{d}V = \iiint\limits_\Omega (x^2+y^2+2xy)\mathrm{d}V = \iiint\limits_\Omega (x^2+y^2)\mathrm{d}V + \iiint\limits_\Omega 2xy\mathrm{d}V = \iiint\limits_\Omega (x^2+y^2)\mathrm{d}V + 0$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} dr \int_1^2 r^3 dz + \int_0^{2\pi} d\theta \int_{\sqrt{2}}^2 dr \int_{\frac{1}{2}r^2}^2 r^3 dz = \frac{14}{3}\pi$$

二、解： 过已知直线作平面束方程 $10x+2y-2z-27+\lambda\left(x+y-z\right)=0$

即 $\left(10+\lambda\right)x+\left(2+\lambda\right)y-\left(2+\lambda\right)z-27=0$

其法向量为 $\vec{n}=\left\{10+\lambda,\ 2+\lambda,\ -2-\lambda\right\}$.

设所求切平面的切点坐标为 $\left(x_0,\ y_0,\ z_0\right)$，则有

$$\begin{cases} \dfrac{10+\lambda}{6x_0} = \dfrac{2+\lambda}{2y_0} = \dfrac{-2-\lambda}{-2z_0} \\ 3x_0^2+y_0^2-z_0^2=27 \\ \left(10+\lambda\right)x_0+\left(2+\lambda\right)y_0-\left(2+\lambda\right)z_0-27=0 \end{cases},$$

解得 $x_0=3,\ y_0=1,\ z_0=1,\ \lambda=-1$

或 $x_0=-3,\ y_0=-17,\ z_0=-17,\ \lambda=-19$

因此，所求切平面方程为 $9x+3y-3z-27=0$

或 $-9x-17y+17z-27=0$

三、证明：$u_n = 1-\dfrac{\ln n}{\pi}\iint\limits_{x^2+y^2\leq a^2} n^{-x^2-y^2}dxdy = 1-(1-n^{-a^2}) = n^{-a^2}$

所以 $\sum\limits_{n=2}^{\infty} u_n = \sum\limits_{n=2}^{\infty}\dfrac{1}{n^{a^2}}$， $\sum\limits_{n=2}^{\infty}(-1)^n u_n = \sum\limits_{n=2}^{\infty}(-1)^n\dfrac{1}{n^{a^2}}$， 由有关判别法易知。

四、解：方程组$\begin{cases}x^2+y^2+z^2=6\\x+y+z=0\end{cases}$两端对$x$求导，得$\begin{cases}2x+2yy'+2zz'=0\\1+y'+z'=0\end{cases}$

把$\left(1,-2,1\right)$代入得$\begin{cases}1-2y'+z'=0\\1+y'+z'=0\end{cases}$，解得$\begin{cases}y'=0\\z'=-1\end{cases}$，于是在点$\left(1,-2,1\right)$处的切向量为

$\vec{t}=\lambda\left(1,y',z'\right)=\lambda\left(1,0,-1\right)$，单位切向量为$\vec{t}^{\circ}=\pm\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$

由条件得 $\frac{\partial f}{\partial x}=2y,\frac{\partial f}{\partial y}=2x,\frac{\partial f}{\partial z}=-2z$

$\vec{l^0}=\pm\{\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\}=\{\cos\alpha,\cos\beta,\cos\gamma\}\Rightarrow\cos\alpha=\frac{1}{\sqrt{2}},\cos\beta=0,\cos\gamma=-\frac{1}{\sqrt{2}}$

点 A 的梯度方向是$\vec{l}=grad\,f\Big|_A=\{2y,2x,-2z\}\Big|_A=\{-4,2,-2\}$

从而 $\frac{\partial f}{\partial l}=\left[\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma\right]_{A(1,-2,1)}=-2\sqrt{2}+\sqrt{2}=-\sqrt{2}$

而方向导数的最大值是$\left|\frac{\partial f}{\partial l}\right|=\sqrt{2^2+4^2+2^2}=\sqrt{24}=2\sqrt{6}$

五、解：由条件有

$$\frac{\partial}{\partial x}\left[x^2f(x)+y\right]=\frac{\partial}{\partial y}\left[yf^2(x)+x\right]\Rightarrow 2xf+x^2f'=f^2\Rightarrow f'+\frac{2}{x}f=\frac{1}{x^2}f^2$$

设$z=f^{-1}$，则得$z'-\frac{2}{x}z=-\frac{1}{x^2}\Rightarrow f^{-1}=z=\frac{1}{3x}+Cx^2$

代入条件得$C=0\Rightarrow f(x)=3x$，从而原积分变为

$$\int_L(y\,f^2(x)+x)dx+(x^2f(x)+y)dy=\int_L(9x^2y+x)dx+(3x^3+y)dy$$

$$=\int_L 9x^2ydx+3x^3dy=\int_1^2\left[9(3-x)x^2-3x^3\right]dx=\int_1^2\left[27x^2-12x^3\right]dx=18$$

六、解：（1）$\vec{\mathbf{A}}=\mathbf{grad}\,u=2x^3\,\vec{\mathbf{i}}+2y^3\,\vec{\mathbf{j}}+3(z^2-1)\vec{\mathbf{k}}$

$$\iint_S\vec{\mathbf{A}}\cdot\mathbf{dS}=\iint_S 2x^3\,\mathrm{d}y\,\mathrm{d}z+2y^3\,\mathrm{d}z\,\mathrm{d}x+3(z^2-1)\,\mathrm{d}x\,\mathrm{d}y,$$

S: $z=1-x^2-y^2\quad(z\ge 0)$ 不封闭

补充S_1： $z=0\ \ (x^2+y^2\le 1)$下侧， 则 $S+S_1$封闭，取下侧。

$$I=\iint_{S}2x^3\,\mathrm{d}y\,\mathrm{d}z+2y^3\,\mathrm{d}z\,\mathrm{d}x+3(z^2-1)\,\mathrm{d}x\,\mathrm{d}y=\left(\oiint_{S+S_1}-\iint_{S_1}\right)[2x^3\,\mathrm{d}y\,\mathrm{d}z+2y^3\,\mathrm{d}z\,\mathrm{d}x+3(z^2-1)\,\mathrm{d}x\,\mathrm{d}y]$$

由高斯公式，得

$$\oiint_{S+S_1}2x^3\,\mathrm{d}y\,\mathrm{d}z+2y^3\,\mathrm{d}z\,\mathrm{d}x+3(z^2-1)\,\mathrm{d}x\,\mathrm{d}y=\iiint_{\Omega}6(x^2+y^2+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$=6\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho\,\mathrm{d}\rho\int_0^{1-\rho^2}(\rho^2+z)\,\mathrm{d}z=2\pi$$

而 $\iint_{S_1}2x^3\mathrm{d}y\mathrm{d}z+2y^3\mathrm{d}z\mathrm{d}x+3(z^2-1)\mathrm{d}x\mathrm{d}y=-\iint_{x^2+y^2\le 1}(-3)\mathrm{d}x\mathrm{d}y=3\pi$

因此 $I=2\pi-3\pi=-\pi$

（2）$div\vec{\mathbf{A}}=6(x^2+y^2+z)$

$$rot\vec{\mathbf{A}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x^3 & 2y^3 & 3(z^2-1)\end{vmatrix}=0\vec{\mathbf{i}}-0\vec{\mathbf{j}}+0\vec{\mathbf{k}}=\vec{\mathbf{0}}$$

$\mathbf{rot}\vec{\mathbf{A}}=(0,\ 0,\ 0)$.