2013-2014 高数 B2 期中考试试题解

一、（40 分）试解下列各题：

1.（8 分）设$(\vec{a}\times\vec{b})\cdot\vec{c}=2$，求$[(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})]\cdot(\vec{c}+\vec{a})$。

解：$[(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})]\cdot(\vec{c}+\vec{a})=[\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}]\cdot(\vec{c}+\vec{a})$

$=(\vec{a}\times\vec{b})\cdot\vec{c}+(\vec{b}\times\vec{c})\cdot\vec{a}=2(\vec{a}\times\vec{b})\cdot\vec{c}=4$。

2.（8 分）求过直线$\dfrac{x-2}{5}=\dfrac{y+1}{2}=\dfrac{z-2}{4}$且与平面$x+4y-3z+1=0$垂直的平面方程。

解：直线$\dfrac{x-2}{5}=\dfrac{y+1}{2}=\dfrac{z-2}{4}$的一般方程$\begin{cases}\dfrac{x-2}{5}=\dfrac{y+1}{2}\\ \dfrac{y+1}{2}=\dfrac{z-2}{4}\end{cases}$即

$\begin{cases}2x-5y-9=0\\ 2y-z+4=0\end{cases}$。用平面束方法。设所求平面为

$$\pi_\lambda:2x-5y-9+\lambda(2y-z+4)=0$$

即

$$\pi_\lambda:2x+(2\lambda-5)y-\lambda z+4\lambda-9=0$$

$$\vec{n}_\lambda=\{2,2\lambda-5,-\lambda\}$$

平面$x+4y-3z+1=0$的法向量$\vec{n}=\{1,4,-3\}$。$\vec{n}_\lambda\perp\vec{n},\vec{n}_\lambda\cdot\vec{n}=0$，

$2+4(2\lambda-5)+3\lambda=0$解得$\lambda=\dfrac{18}{11}$。所求平面为

$$\pi_{\frac{18}{11}}:2x-5y-9+\frac{18}{11}(2y-z+4)=0$$

即

$$\pi_{\frac{18}{11}}:22x-19y-18z-27=0$$

3.（8 分）求由曲线$\begin{cases}3x^2+2y^2=12\\ z=0\end{cases}$绕$y$轴转一周得到的旋转曲面在点$(0,\sqrt{3},\sqrt{2})$处的指向外侧的单位法向量。

解：旋转曲面 $F = 3x^2 + 3z^2 + 2y^2 - 12 = 0$。在点 $\left(0,\sqrt{3},\sqrt{2}\right)$ 处的指向外侧的法向量 $\vec{n} = \left\{F_x, F_y, F_z\right\}\Big|_{\left(0,\sqrt{3},\sqrt{2}\right)} = \left\{6x,4y,6z\right\}\Big|_{\left(0,\sqrt{3},\sqrt{2}\right)} = \left\{0,4\sqrt{3},6\sqrt{2}\right\}$。单位化得到指向外侧的单位法向量

$$\vec{e}_{\vec{n}} = \left\{0, \frac{\sqrt{10}}{5}, \frac{\sqrt{15}}{5}\right\}$$

4.（8 分）设直线 $L:\begin{cases} x + y + b = 0 \\ x + ay - z - 3 = 0 \end{cases}$ 在平面 π 上，且平面 π 又与曲面 $z = x^2 + y^2$ 相切于点 $P\left(1,-2,5\right)$，求 a, b。

解：曲面 $z = x^2 + y^2$ 为 $F = x^2 + y^2 - z = 0$。π 的法向量 $\vec{n} = \left\{F_x, F_y, F_z\right\}\Big|_P = \left\{2x,2y,-1\right\}\Big|_P = \left\{2,-4,-1\right\}$。用平面束方法。设 π 为

$$\pi : x + y + b + \lambda(x + ay - z - 3) = 0$$

法向量 $\vec{n}_\lambda = \left\{\lambda + 1, a\lambda + 1, -\lambda\right\}$。 $\vec{n}_\lambda // \vec{n}, \dfrac{\lambda + 1}{2} = \dfrac{a\lambda + 1}{-4} = \lambda$。解得 $\lambda = 1, a = -5$。把 $P\left(1,-2,5\right)$ 代入 $\pi : 2x - 4y - z + b - 3 = 0$ 得 $b = -2$。

5.（8 分）设变换 $\begin{cases} u = x + a\sqrt{y} \\ v = x + 2\sqrt{y} \end{cases}$ 把方程 $\dfrac{\partial^2 z}{\partial x^2} - y\dfrac{\partial^2 z}{\partial y^2} - \dfrac{1}{2}\dfrac{\partial z}{\partial y} = 0$ 化为 $\dfrac{\partial^2 z}{\partial u \partial v} = 0$，试确定 a。

解：按照函数图

$$\begin{tikzcd}[row sep=small] & & x \\ & u \arrow[ur] \arrow[r] & y \\ z \arrow[ur] \arrow[dr] & & \\ & v \arrow[r] \arrow[dr] & x \\ & & y \end{tikzcd}$$

求导。

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial u}{\partial x}\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial x}\right) + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\left(\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\right) + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}\frac{a}{2\sqrt{y}}+\frac{\partial z}{\partial v}\frac{1}{\sqrt{y}},$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$=\frac{\partial u}{\partial y}\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial v}{\partial y}\right)+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial v}{\partial y}\left(\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}+\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial y}\right)+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$=\frac{a}{2\sqrt{y}}\left(\frac{\partial^2 z}{\partial u^2}\frac{a}{2\sqrt{y}}+\frac{\partial^2 z}{\partial u\partial v}\frac{1}{\sqrt{y}}\right)-\frac{\partial z}{\partial u}\frac{a}{4y^{\frac{3}{2}}}+\frac{1}{\sqrt{y}}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{\sqrt{y}}+\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2\sqrt{y}}\right)-\frac{\partial z}{\partial v}\frac{1}{2y^{\frac{3}{2}}}$$

$$=\frac{\partial^2 z}{\partial u^2}\frac{a^2}{4y}+\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2y}-\frac{\partial z}{\partial u}\frac{a}{4y^{\frac{3}{2}}}+\frac{\partial^2 z}{\partial v^2}\frac{1}{y}+\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2y}-\frac{\partial z}{\partial v}\frac{1}{2y^{\frac{3}{2}}}$$

$$\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial y^2}-\frac{1}{2}\frac{\partial z}{\partial y}$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial^2 z}{\partial u^2}\frac{a^2}{4}-\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2}+\frac{\partial z}{\partial u}\frac{a}{4y^{\frac{1}{2}}}-\frac{\partial^2 z}{\partial v^2}-\frac{\partial^2 z}{\partial u\partial v}\frac{a}{2}+\frac{\partial z}{\partial v}\frac{1}{2y^{\frac{1}{2}}}$$

$$-\frac{\partial z}{\partial u}\frac{a}{4\sqrt{y}}-\frac{\partial z}{\partial v}\frac{1}{2\sqrt{y}}$$

$$=\left(1-\frac{a^2}{4}\right)\frac{\partial^2 z}{\partial u^2}+(2-a)\frac{\partial^2 z}{\partial u\partial v}$$

令$1-\frac{a^2}{4}=0, 2-a\neq 0$解得$a=-2$。

二、（12 分）证明函数$f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$在点$(0,0)$连续且偏导数存在，但在此点不可微。

解：$0\leq\left|\frac{x^2y}{x^2+y^2}\right|\leq|x|$又$\lim\limits_{\substack{x\to 0\\ y\to 0}}|x|=0$，所以$\lim\limits_{\substack{x\to 0\\ y\to 0}}\left|\frac{x^2y}{x^2+y^2}\right|=0$，从而

$$\lim_{\substack{x\to 0\\ y\to 0}}f(x,y)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{x^2+y^2}=0=f(0,0)$$

所以$f(x,y)$在点$(0,0)$连续。

$\varphi(x)=f(x,0)\equiv 0, \psi(y)=f(0,y)\equiv 0$，

$f_x(0,0)=\varphi'(0)=0, f_y(0,0)=\psi'(0)=0$都存在。

$$\lim_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)-[f_x(0,0)x+f_y(0,0)y]}{\sqrt{x^2+y^2}}=\lim_{\substack{x\to0\\y\to0}}\frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}$$

当 $y=kx(k\neq0)$ 时，$\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}=\dfrac{k}{(1+k^2)^{\frac{3}{2}}}$ 与 k 有关，$\lim\limits_{\substack{x\to0\\y\to0}}\dfrac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}$ 不存在。故 $f(x,y)$ 在点 $(0,0)$ 不可微。

三、（12 分）设 $z=f(2x-y)+g(x,xy)$，其中函数 $f(t)$ 二阶可导，$g(u,v)$ 有连续二阶偏导数，求 $\dfrac{\partial^2 z}{\partial x\partial y}$。

解：$\dfrac{\partial z}{\partial x}=2f'(2x-y)+g_1(x,xy)+yg_2(x,xy)$，

$$\frac{\partial^2 z}{\partial x\partial y}=-2f''(2x-y)+xg_{12}(x,xy)+g_2(x,xy)+xyg_{22}(x,xy)。$$

四、（，所以 12 分）设 $u=f(x,y,z),\varphi(x^2,e^y,z)=0,y=\sin x$，其中 f,φ 都具有一阶连续偏导数且 $\dfrac{\partial\varphi}{\partial z}\neq0$，求 $\dfrac{du}{dx}$。

解：$\varphi(x^2,e^{\sin x},z)=0$ 两边对 x 求导

$$2x\varphi_1(x^2,e^{\sin x},z)+e^{\sin x}\cos x\varphi_2(x^2,e^{\sin x},z)+\varphi_3(x^2,e^{\sin x},z)\frac{dz}{dx}=0，$$

$$\frac{dz}{dx}=-\frac{2x\varphi_1(x^2,e^{\sin x},z)+e^{\sin x}\cos x\varphi_2(x^2,e^{\sin x},z)}{\varphi_3(x^2,e^{\sin x},z)}。$$

$$\frac{du}{dx}=f_1+\cos xf_2-\frac{2x\varphi_1(x^2,e^{\sin x},z)+e^{\sin x}\cos x\varphi_2(x^2,e^{\sin x},z)}{\varphi_3(x^2,e^{\sin x},z)}f_3。$$

五、（12 分）设 $f(x,y,z)=x^3-xy^2-z$。

（1）求 $f(x,y,z)$ 在点 $P_0(1,1,0)$ 处的梯度及方向导数的最大值；

（2）问：$f(x,y,z)$ 在哪些点的梯度垂直于 x 轴？

解：（1）

$$\overrightarrow{grad}f(P_0)=\{f_x(P_0),f_y(P_0),f_z(P_0)\}=\{3x^2-y^2,-2xy,-1\}\big|_{P_0}=\{2,-2,-1\}。$$

在点 $P_0(1,1,0)$ 处方向导数的最大值 $= \left|\overrightarrow{gradf}(P_0)\right| = 3$。

（2） $\overrightarrow{gradf}(x,y,z) = \{f_x, f_y, f_z\} = \{3x^2 - y^2, -2xy, -1\}$ 在两条相交直线 $3x^2 - y^2 = 0$ 上点的梯度垂直于 x 轴。

六、（12 分）在椭球面 $4x^2 + y^2 + z^2 = 4$ 的第一卦限部分上求一点，使得椭球面在该点的切平面、椭球面及三个坐标平面所围成在第一卦限部分的立体的体积最小。

解：$F = 4x^2 + y^2 + z^2 - 4 = 0$ 在点 (x, y, z) 的切平面

$$4x(X - x) + y(Y - y) + z(Z - z) = 0$$

化简为

$$4xX + yY + zZ = 4$$

三个截距分别是 $\frac{1}{x}, \frac{4}{y}, \frac{4}{z}$。四面体的体积 $V_4 = \frac{8}{3xyz}$。

椭球第一卦限部分的体积 V_1 显然是常数。所给立体的体积 $=V_4 - V_1$。因此归结为条件极值问题

$$\begin{cases} f = \dfrac{1}{xyz} \\ 4x^2 + y^2 + z^2 - 4 = 0 \end{cases} \quad (x > 0, y > 0, z > 0)$$

$L = \frac{1}{xyz} + \lambda(4x^2 + y^2 + z^2 - 4)$，令

$$\begin{cases} L_x = -\dfrac{1}{x^2yz} + 8\lambda x = 0 \\ L_y = -\dfrac{1}{xy^2z} + 2\lambda y = 0 \\ L_z = -\dfrac{1}{xyz^2} + 2\lambda z = 0 \\ L_\lambda = 4x^2 + y^2 + z^2 - 4 = 0 \end{cases}$$

由前三个方程有 $\lambda \neq 0, 4x^2 = y^2 = z^2$。代入最后方程解得 $x = \frac{\sqrt{3}}{3}$，$y = z = \frac{2\sqrt{3}}{3}$。

根据问题的实际，这就是所给立体体积最小的点。