THEOREM 1.6.2 *If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

Exercise 1.6

Solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

Question 1:—

$$x_1 + x_2 = 2$$
$$5x_1 + 6x_2 = 9$$

The given system can be written in matrix form as

$$Ax = b \longrightarrow (i)$$

where,

$$A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

We begin by inverting coefficient matrix A

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -5 & 1 \end{array}\right] \quad -5R_1 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -5 & 1 \end{array}\right] \quad (-1)R_2 + R_1$$

Since,

$$A^{-1} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$$

From (i), we have

$$Ax = b$$

$$x = A^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence,

$$x_1 = 3, \quad x_2 = -1$$

Question 3:—

$$x_1 + 3x_2 + x_3 = 4$$

$$2x_1 + 2x_2 + x_3 = -1$$

$$2x_1 + 3x_2 + x_3 = 3$$

The given system can be written in matrix form as

$$Ax = b \longrightarrow (i)$$

where,

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

We begin by inverting the coefficient matrix A

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-2R_1 + R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-2R_1 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$(-1)R_2 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$R_2 \longleftrightarrow R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & -4 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$4R_2 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -2 & -3 & 4 \end{array}\right]$$

$$(-1)R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right]$$

$$(-1)R_3 + R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & -3 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right]$$

$$-3R_2 + R_1$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right]$$

From (i), we have

$$Ax = b$$

$$x = A^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$$

Hence,

$$x_1 = -1, \quad x_2 = 4, \quad x_3 = -7$$

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Question 5 :—

$$x + y + z = 5$$
$$x + y - 4z = 10$$
$$-4x + y + z = 0$$

The given system can be written in matrix form

$$Ax = b \longrightarrow (i)$$

where,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

We begin by inverting matrix A

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$(-1)R_1 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$4R_1 + R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right]$$

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$R_2 \longleftrightarrow R_3$

$$\overset{R}{\sim}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right]$$

$\left(\frac{1}{5}\right)R_2$ and $\left(\frac{-1}{5}\right)R_3$

$$\overset{R}{\sim}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$(-1)R_3 + R_1$

$$\overset{R}{\sim}\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 4/5 & 1/5 & 0 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$(-1)R_3 + R_2$

$$\overset{R}{\sim}\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 4/5 & 1/5 & 0 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

$(-1)R_2 + R_1$

$$\overset{R}{\sim}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right]$$

Since,

$$A^{-1} = \begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix}$$

Now equ (i) becomes

$$Ax = b$$

$$x = A^{-1}b$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & 1/5 \\ 3/5 & 1/5 & -1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

Hence,

$$x = 1, \quad y = 5, \quad z = -1$$

Question 8 :—

$$x_1 + 2x_2 + 3x_3 = b_1$$

$$2x_1 + 5x_2 + 5x_3 = b_2$$

$$3x_1 + 5x_2 + 8x_3 = b_3$$

The given system can be written in matrix form

$$Ax = b \longrightarrow (i)$$

where,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We begin by inverting matrix A

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$-2R_1 + R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$-3R_1 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$R_2 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -5 & 1 & 1 \end{array}\right]$$

$$\left(\frac{-1}{2}\right)R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right]$$

$$R_2 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right]$$

$(-3)R_3 + R_1$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -13/2 & 3/2 & 3/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right]$$

$-2R_2 + R_1$

$$\underset{\sim}{R}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -15/2 & 1/2 & 5/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right]$$

Since,

$$A^{-1} = \begin{bmatrix} -15/2 & 1/2 & 5/2 \\ 1/2 & 1/2 & -1/2 \\ 5/2 & -1/2 & -1/2 \end{bmatrix}$$

Now equ (i) becomes

$$Ax = b$$

$$x = A^{-1}b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15/2 & 1/2 & 5/2 \\ 1/2 & 1/2 & -1/2 \\ 5/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2}b_1 + \frac{1}{2}b_2 + \frac{5}{2}b_3 \\ \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3 \\ \frac{5}{2}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3 \end{bmatrix}$$

where,

$$x_1 = -\frac{15}{2}b_1 + \frac{1}{2}b_2 + \frac{5}{2}b_3 \qquad x_3 = \frac{5}{2}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3$$

$$x_2 = \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3$$