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LINE INTEGRAL

"Let $\vec{F}$ be a vector field with continuous components defined along a smooth curve C parameterized by $\vec{r}(t)$, $a \le t \le b$. Then the line integral at $\vec{F}$ along C is"

$$\int_C \vec{F}.\vec{T}\, ds = \int_C \left(\vec{F}.\frac{d\vec{r}}{ds}\right) ds$$

$$= \int_C \vec{F}.d\vec{r} \quad \text{or} \quad \oint_C \vec{F}.d\vec{r}$$

Working Rule:

1) Express the vector field $\vec{F}$ in terms of parameterized curve C as $\vec{F}(\vec{r}(t))$ by substituting components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\vec{r}$.

2) Find the derivative (velocity) vector $\frac{d\vec{r}}{dt}$.

3) Evaluate the line integral

$$\int_C \vec{F}.d\vec{r} = \int_a^b \left(\vec{F}(\vec{r}(t)).\frac{d\vec{r}}{dt}\right) dt$$

TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$

$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$	Parametric vector evaluation
$= \int_a^b \left(Mg'(t) + Nh'(t) + Pk'(t)\right) dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

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16.2

E2) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z) = z\hat{i} + xy\hat{j} - y^2\hat{k}$ along the curve C given by $\vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}$, $0 \le t \le 1$.

The parameterization is

$$\vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$$

$$\vec{F}(\vec{r}(t)) = \sqrt{t}\,\hat{i} + (t^2)(t)\hat{j} - (t)^2\hat{k}$$

$$= \sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}$$

Also,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}\right)$$

$$= 2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}$$

and,

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

$$= \left(\sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}\right)\cdot\left(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right)$$

$$= 2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}$$

Now,

$$\Rightarrow \int_0^1 \left(\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right) dt$$

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$$= \left[\frac{2t^{5/2}}{5/2} + \frac{t^4}{4} - \frac{1}{2}\frac{t^{5/2}}{5/2}\right]_0^1$$

$$= \left[\frac{2 \cdot 2}{5} t^{5/2} + \frac{t^4}{4} - \frac{1}{2} \cdot \frac{2}{5} t^{5/2}\right]_0^1$$

$$= \left[\frac{4}{5} t^{5/2} + \frac{t^4}{4} - \frac{1}{5} t^{5/2}\right]_0^1$$

$$= \left(\frac{4}{5}(1)^{5/2} + \frac{(1)^4}{4} - \frac{1}{5}(1)^{5/2}\right) - 0$$

$$= \frac{4}{5} + \frac{1}{4} - \frac{1}{5}$$

$$= \frac{16 + 5 - 4}{20}$$

$$= \frac{17}{20}$$

E3) Find the line integral $\int_C -y\,dx + z\,dy + 2x\,dz$, where C is the helix

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}, \quad 0 \le t \le 2\pi$$

The parameterization is

$y = \sin t \qquad dy = \cos t\,dt$

$x = \cos t \qquad dx = -\sin t\,dt$

$z = t \qquad dz = 1\,dt$

Thus the line integral becomes

$$= \int_0^{2\pi} -(\sin t)(-\sin t\,dt) + (t)(\cos t\,dt) + 2(\cos t)(dt)$$

$$= \int_0^{2\pi} (\sin^2 t)\,dt + (t\cos t)\,dt + (2\cos t)\,dt$$

$$= \int_0^{2\pi} \sin^2 t\,dt + \int_0^{2\pi} t\cos t\,dt + 2\int_0^{2\pi} \cos t\,dt$$

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$, and $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{4}$$

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$$= \int_0^{2\pi} \left(\frac{1-\cos 2t}{2}\right) dt + \int_0^{2\pi} t\cos t\, dt + 2\left[\sin t\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \left[t(\sin t) - \int_0^{2\pi} \sin t\,(1)\,dt\right]_0^{2\pi} + 2$$

$$= \frac{1}{2}\left[\frac{t - \sin 2t}{2}\right]_0^{2\pi} + \left[t\sin t + \cos t\right]_0^{2\pi} + 2\left[\sin t\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\left(\frac{2\pi - \sin 4\pi}{2}\right) - 0\right] + \left[(2\pi \sin 2\pi + \cos 2\pi) - (0+1)\right]$$

$$+ 2\left[\sin 2\pi - 0\right]$$

$$= \frac{1}{2}(2\pi - 0) + \left[(0+1) - 1\right] + 2(0)$$

$$= \frac{1}{\cancel{2}}(\cancel{2}\pi) + \cancel{1} - \cancel{1}$$

$$= \pi$$

E4) Find the work done by the force field $\vec{F} = (y-x^2)\hat{i} + (z-y^2)\hat{j} + (x-z^2)\hat{k}$ along the curve

$$\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}, \quad 0 \le t \le 1$$

The parameterization is

$$\vec{F} = (y-x^2)\hat{i} + (z-y^2)\hat{j} + (x-z^2)\hat{k}$$

$$\vec{F}(\vec{r}(t)) = (t^2 - t^2)\hat{i} + (t^3 - t^4)\hat{j} + (t - t^6)\hat{k}$$

$$= (t^3 - t^4)\hat{j} + (t - t^6)\hat{k}$$

Also,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(t\hat{i} + t^2\hat{j} + t^3\hat{k})$$

$$= \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

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and,

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

$$= [(t^3-t^4)\hat{j} + (t-t^6)\hat{k}] \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$

$$= (t^3-t^4)(2t) + (t-t^6)(3t^2)$$

$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

Thus, the work done is

$$W = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt$$

$$= \left[\frac{2t^5}{5} - \frac{2t^6}{6} + \frac{3t^4}{4} - \frac{3t^9}{9}\right]_0^1$$

$$= \left(\frac{2}{5}(1)^5 - \frac{1}{3}(1)^6 + \frac{3}{4}(1)^4 - \frac{1}{3}(1)^9\right) - 0$$

$$= \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3}$$

$$= \frac{29}{60}$$

E5) Find the work done by force field $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ in moving an object along the curve C parameterized by $\vec{r}(t) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}$, $0 \le t \le 1$.

The parameterization is

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F}(\vec{r}(t)) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}$$

Also,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}[\cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)]$$

DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{5}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

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$$\frac{d\vec{r}}{dt} = -\pi\sin(\pi t)\hat{i} + 2t\hat{j} + \pi\cos(\pi t)\hat{k}$$

and,

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

$$= [\cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}] \cdot [-\pi\sin(\pi t)\hat{i} + 2t\hat{j} + \pi\cos(\pi t)\hat{k}]$$

$$= \cancel{-\pi\cos(\pi t)\sin(\pi t)} + 2t^3 + \cancel{\pi\sin(\pi t)\cos(\pi t)}$$

$$= 2t^3$$

Thus, the work done is

$$W = \int_0^1 2t^3\, dt$$

$$= 2\left[\frac{t^4}{4}\right]_0^1$$

$$= \frac{1}{2}[(1)^4 - 0]$$

$$= \frac{1}{2}$$

E6) A fluid's velocity field is $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$. Find the flow along the helix $\vec{r}(t) = \cos t\,\hat{i} + (\sin t)\hat{j} + t\hat{k}$, $0 \le t \le \pi/2$

The parameterization is

$$\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$$

$$\vec{F}(\vec{r}(t)) = (\cos t)\hat{i} + (\sin t)\hat{k} + t\hat{j}$$

Also,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}[(\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}]$$

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$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

and,

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

$$= [(\cos t)\hat{i} + t\hat{j} + (\sin t)\hat{k}] \cdot [(-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}]$$

$$= -\sin t\cos t + t\cos t + \sin t$$

Thus, the flow is given as

$$\text{Flow} = \int_0^{\pi/2} (-\sin t\cos t + t\cos t + \sin t)\, dt$$

$$= \int_0^{\pi/2} [\cos t(-\sin t) + t\cos t + \sin t]\, dt$$

$$= \left[\frac{\cos^2 t}{2} + t(\sin t) - (\cos t) + (-\cos t)\right]_0^{\pi/2}$$

$$= \left[\frac{\cos^2 t}{2} + t\sin t + \cancel{\cos t} - \cos t\right]_0^{\pi/2}$$

$$= \left[\frac{\cos^2 t}{2} + t\sin t\right]_0^{\pi/2}$$

$$= \left(\frac{\cos^2\left(\frac{\pi}{2}\right)}{2} + \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)\right) - \left(\frac{\cos^2(0)}{2} + 0\right)$$

$$= 0 + \frac{\pi}{2} - \frac{1}{2}$$

$$= \frac{\pi}{2} - \frac{1}{2}$$

E7) Find the circulation of the field $\vec{F} = (x-y)\hat{i} + x\hat{j}$ around the circle $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$, $0 \le t \le 2\pi$.

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The parameterization is

$$\vec{F} = (x-y)\hat{i} + x\hat{j}$$

$$\vec{F}(\vec{r}(t)) = (\cos t - \sin t)\hat{i} + (\cos t)\hat{j}$$

Also,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(\cos t)\hat{i} + (\sin t)\hat{j}\right]$$

$$= (-\sin t)\hat{i} + (\cos t)\hat{j}$$

and,

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

$$= \left[(\cos t - \sin t)\hat{i} + (\cos t)\hat{j}\right] \cdot \left[-\sin t\,\hat{i} + \cos t\,\hat{j}\right]$$

$$= -\sin t \cos t + \sin t^2 + \cos^2 t$$

$$= -\sin t \cos t + 1$$

Now,

$$\text{Circulation} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^{2\pi} (1 - \sin t \cos t)\, dt$$

$$= \left[t - \frac{\sin^2 t}{2}\right]_0^{2\pi}$$

$$= \left(2\pi - \frac{\sin^2(2\pi)}{2}\right) - \left(0 - \frac{\sin^2(0)}{2}\right)$$

$$= 2\pi - 0 - 0$$

$$= 2\pi$$

Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\,dy - N\,dx \tag{7}$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.

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EB) Find the flux of $\vec{F} = (x-y)\hat{i} + x\hat{j}$ across the circle $x^2+y^2=1$ in the xy- plane.

As we know that
Parametric equ. of circle is

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$$

Here,

$$\text{Flux} = \oint_c M\,dy - N\,dx \longrightarrow (i)$$

and,

$$M = x - y \qquad N = x$$
$$M = \cos t - \sin t \qquad N = \cos t$$

Also,

$$dy = d(\sin t) = \cos t\,dt$$
$$dx = d(\cos t) = -\sin t\,dt$$

Now equ (i) becomes

$$\text{Flux} = \oint_c (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)$$
$$= \int_0^{2\pi} (\cos^2 t - \cancel{\sin t \cos t} + \cancel{\cos t \sin t})\,dt$$
$$= \int_0^{2\pi} \cos^2 t\,dt$$
$$= \int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt$$
$$= \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$
$$= \frac{1}{2}\left[\left(2\pi + \frac{\sin 4\pi}{2}\right) - \left(0 + \frac{\sin 2(0)}{2}\right)\right]$$
$$= \frac{1}{2}\left[(2\pi + 0) - 0\right]$$
$$= \pi$$

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Exercise

Q1) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

As we know that

$$\nabla f(x,y,z) = f_x(x,y,z)\hat{i} + f_y(x,y,z)\hat{j} + f_z(x,y,z)\hat{k}$$

Here,

$$\Rightarrow f_x = \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$= \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \cdot \cancel{2}x$$

$$= \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow f_y = \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$= \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \cdot \cancel{2}y$$

$$= \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow f_z = \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$= \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \cdot (\cancel{2}z)$$

$$= \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Thus, gradient is given as

$$\nabla f(x,y,z) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}\hat{i} + \left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right)\hat{j} +$$

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$$\left(\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)\hat{k}$$

$$\nabla f(x,y,z) = \frac{-x\hat{i} - y\hat{j} - z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

Q3) $g(x,y,z) = e^z - \ln(x^2+y^2)$

As we know that

$$\nabla g(x,y,z) = g_x(x,y,z)\hat{i} + g_y(x,y,z)\hat{j} + g_z(x,y,z)\hat{k}$$

Here,

$$\Rightarrow g_x = \frac{\partial}{\partial x}\left[e^z - \ln(x^2+y^2)\right]$$

$$= 0 - \frac{1}{x^2+y^2}(2x)$$

$$= \frac{-2x}{x^2+y^2}$$

$$\Rightarrow g_y = \frac{\partial}{\partial y}\left[e^z - \ln(x^2+y^2)\right]$$

$$= 0 - \frac{1}{x^2+y^2}(2y)$$

$$= \frac{-2y}{x^2+y^2}$$

$$\Rightarrow g_z = \frac{\partial}{\partial z}\left[e^z - \ln(x^2+y^2)\right]$$

$$= e^z - 0$$

$$= e^z$$

Thus, gradient becomes

$$\nabla g(x,y,z) = \frac{-2x}{x^2+y^2}\hat{i} - \frac{2y}{x^2+y^2}\hat{j} + e^z\hat{k}$$

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Line Integrals of vector fields

9.M

Q7) $\vec{F} = 3y\hat{i} + 2x\hat{j} + 4z\hat{k}$

a) $\vec{r}(t) = t\hat{i} + t\hat{j} + t\hat{k}$,

$0 \le t \le 1$

The parameterization is

$$\vec{F}(\vec{r}(t)) = 3(t)\hat{i} + 2(t)\hat{j} + 4(t)\hat{k}$$
$$= 3t\hat{i} + 2t\hat{j} + 4t\hat{k}$$

Also,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(t\hat{i} + t\hat{j} + t\hat{k})$$
$$= \hat{i} + \hat{j} + \hat{k}$$

and,

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$
$$= (3t\hat{i} + 2t\hat{j} + 4t\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k})$$
$$= 3t + 2t + 4t$$
$$= 9t$$

Now,

$$\int_0^1 \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt = \int_0^1 9t\, dt$$
$$= \left[9 \cdot \frac{t^2}{2}\right]_0^1$$
$$= 9\left[\frac{1}{2} - 0\right]$$
$$= \frac{9}{2}$$

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b) $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^4\hat{k}$,

$0 \leq t \leq 1$

The parameterization is

$\vec{F}(\vec{r}(t)) = 3t^2\hat{i} + 2t\hat{j} + 4t^4\hat{k}$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(t\hat{i} + t^2\hat{j} + t^4\hat{k})$$

$$= \hat{i} + 2t\hat{j} + 4t^3\hat{k}$$

Also,

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}$$

$$= (3t^2\hat{i} + 2t\hat{j} + 4t^4\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 4t^3\hat{k})$$

$$= 3t^2(1) + (2t)(2t) + (4t^4)(4t^3)$$

$$= 3t^2 + 4t^2 + 16t^7$$

$$= 7t^2 + 16t^7$$

Now,

$$\Rightarrow \int_0^1 \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^1 (7t^2 + 16t^7)\, dt$$

$$= \left[\frac{7t^3}{3} + \frac{\cancel{16}^{2}t^8}{\cancel{8}}\right]_0^1$$

$$= \left[\frac{7t^3}{3} + 2t^8\right]_0^1$$

$$= \left(\frac{7(1)^3}{3} + 2(1)^8\right) - 0$$

$$= \frac{7}{3} + 2 = \frac{13}{3}$$

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c) The path $C_3 \cup C_4$ consisting of line segment from $(0,0,0)$ to $(1,1,0)$ followed by the segment from $(1,1,0)$ to $(1,1,1)$.

We can parameterize C_3 as

$$\vec{r_1}(t) = t\hat{i} + t\hat{j} + 0\hat{k},$$
$$0 \le t \le 1$$

The parameterization is

$$\vec{F}(\vec{r_1}(t)) = 3(t)\hat{i} + 2(t)\hat{j} + 4(0)\hat{k}$$
$$= 3t\hat{i} + 2t\hat{j}$$

Also,

$$\frac{d\vec{r_1}}{dt} = \frac{d}{dt}(t\hat{i} + t\hat{j} + 0\hat{k})$$
$$= \hat{i} + \hat{j}$$

and,

$$\Rightarrow \vec{F}(\vec{r_1}(t)) \cdot \frac{d\vec{r_1}}{dt}$$
$$= (3t\hat{i} + 2t\hat{j}) \cdot (\hat{i} + \hat{j})$$
$$= 3t(1) + 2t(1)$$
$$= 3t + 2t$$
$$= 5t$$

Now,

$$\Rightarrow \int_0^1 \vec{F}(\vec{r_1}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$
$$= \int_0^1 5t\, dt$$
$$= \left[\frac{5t^2}{2}\right]_0^1$$
$$= 5/2$$

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We can parameterize C_4 as

For C_4

$$A = (1,1,0)$$
$$B = (1,1,1)$$

The general way to parameterize a straight line segment from point A to B is

$$\vec{r}(t) = A + t(B - A), \quad 0 \le t \le 1$$
$$= (1,1,0) + t[(1,1,1) - (1,1,0)]$$
$$= (1,1,0) + t(0,0,1)$$
$$= (1,1,0) + (0,0,t)$$
$$= (1,1,t)$$

Thus, the parameterization is

$$\vec{r_2}(t) = \hat{i} + \hat{j} + t\hat{k}, \quad 0 \le t \le 1$$

Also,

$$\vec{F}(\vec{r_2}(t)) = 3(1)\hat{i} + 2(1)\hat{j} + 4(t)\hat{k}$$
$$= 3\hat{i} + 2\hat{j} + 4t\hat{k}$$

and,

$$\frac{d\vec{r_2}}{dt} = \frac{d}{dt}(\hat{i} + \hat{j} + t\hat{k})$$
$$= 0\hat{i} + 0\hat{j} + \hat{k}$$

thus,

$$\Rightarrow \int_0^1 \vec{F}(\vec{r_2}(t)) \cdot \frac{d\vec{r_2}}{dt}\, dt$$
$$= \int_0^1 (3\hat{i} + 2\hat{j} + 4t\hat{k}) \cdot (\hat{k})\, dt$$
$$= \int_0^1 4t\, dt$$
$$= \left[\frac{4t^2}{2}\right]_0^1$$

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$$= \left[2t^2\right]_0^1$$

$$= 2(1)^2 - 0$$

$$= 2$$

Now,

Total Integral for $C_3 \cup C_4$

$$\Rightarrow \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}$$

$$= \frac{5}{2} + 2$$

$$= \frac{9}{2}$$

Line Integrals w.r.t x, y, z

Q13) $\int_C (x-y)\,dx$, where $C: x = t,\ y = 2t+1$ for $0 \le t \le 3$.

Here,

$$x = t \qquad dx = dt$$

$$y = 2t+1$$

Thus,

$$\Rightarrow \int_C (x-y)\,dx$$

$$= \int_0^3 [t - (2t+1)]\,dt$$

$$= \int_0^3 (t - 2t - 1)\,dt$$

$$= \int_0^3 (-t-1)\,dt$$

$$= \left[\frac{-t^2}{2} - t\right]_0^3$$

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$$= \left(\frac{-(3)^2 - (3)}{2}\right) - 0$$

$$= \frac{-9-3}{2}$$

$$= \frac{-9-6}{2}$$

$$= \frac{-15}{2}$$

Q17) Along the curve
$\vec{r}(t) = t\hat{i} - \hat{j} + t^2\hat{k}$, $0 \le t \le 1$.

(b,c)

a) $\int_c (x+y-z)\,dx$

Here,

$x = t$ $\qquad dx = dt$

$y = -1$

$z = t^2$

Thus,

$$\Rightarrow \int_c (x+y-z)\,dx$$

$$= \int_0^1 [t - 1 - t^2]\,dt$$

$$= \left[\frac{t^2}{2} - t - \frac{t^3}{3}\right]_0^1$$

$$= \left(\frac{1}{2} - 1 - \frac{1}{3}\right) - 0$$

$$= \frac{3-6-2}{6}$$

$$= \frac{-5}{6}$$

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Work

(19)

Q21) $\vec{F} = 2\hat{i} + x\hat{j} + y\hat{k}$

$\vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}$,

$0 \le t \le 2\pi$

The parameterization is

$\vec{F}(\vec{r}(t)) = t\hat{i} + (\sin t)\hat{j} + (\cos t)\hat{k}$

Also,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}\right]$$

$$= (\cos t)\hat{i} + (-\sin t)\hat{j} + \hat{k}$$

Thus, the work done is

$$\text{Work} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^{2\pi} \left[t\hat{i} + (\sin t)\hat{j} + (\cos t)\hat{k}\right] \cdot \left[(\cos t)\hat{i} - (\sin t)\hat{j} + \hat{k}\right]$$

$$= \int_0^{2\pi} (t\cos t - \sin^2 t + \cos t)\, dt$$

$$= \int_0^{2\pi} t\cos t\, dt - \int_0^{2\pi} \sin^2 t\, dt + \int_0^{2\pi} \cos t\, dt$$

$$= \left[t\sin t - \int_0^{2\pi} \sin t\, dt\right]_0^{2\pi} - \int_0^{2\pi} \frac{1 - \cos 2t}{2}\, dt + \left[\sin t\right]_0^{2\pi}$$

$$= \left[t\sin t + \cos t\right]_0^{2\pi} - \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \left[\sin t\right]_0^{2\pi}$$

$$= \left[2\pi\sin 2\pi + \cos 2\pi\right] - (0 + \cos 0) - \frac{1}{2}\left[2\pi - \frac{\sin 4\pi}{2}\right] + \sin 2\pi$$

$$= (0 + 1) - (0 + 1) - \frac{1}{2}(2\pi - 0) + 0$$

$$= 1 - 1 - \frac{1}{\cancel{2}}(\cancel{2}\pi)$$

$$= 0 - \pi = -\pi$$

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Line Integrals in the plane

Q23) Evaluate $\int_c xy\,dx + (x+y)\,dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

The give line integral is

$$\int_c xy\,dx + (x+y)\,dy \longrightarrow \text{(i)}$$

The curve given is

$$y = x^2 \quad \text{(Parabola)}$$

The parametric equation for the parabola can be written by introducing a parameter t such that

$$x = t$$

$$y = t^2$$

where, $-1 \le t \le 2$

Also,

$$dx = dt$$

$$dy = 2t.\,dt$$

Now equ (i) becomes

$$= \int_{-1}^{2} (t)(t^2)(dt) + (t+t^2)(2t.\,dt)$$

$$= \int_{-1}^{2} \left(t^3 + 2t^2 + 2t^3 \right) dt$$

$$= \int_{-1}^{2} \left(3t^3 + 2t^2 \right) dt$$

$$= \left[\frac{3}{4} t^4 + \frac{2}{3} t^3 \right]_{-1}^{2}$$

$$= \left(\frac{3}{4} (2)^4 + \frac{2}{3} (2)^3 \right) - \left(\frac{3}{4} (-1)^4 + \frac{2}{3} (-1)^3 \right)$$

$$= \left(\frac{3}{\cancel{4}} (\cancel{16}^{4}) + \frac{2}{3} (8) \right) - \left(\frac{3}{4} (1) + \frac{2}{3} (-1) \right)$$

$$= \left(12 + \frac{16}{3} \right) - \left(\frac{3}{4} - \frac{2}{3} \right)$$

$$= \frac{52}{3} - \frac{1}{12}$$

$$= \frac{208 - 1}{12} = \frac{207}{12}$$

$$= \frac{69}{4}$$

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Q25) Evaluate $\int_C \vec{F}\cdot\vec{T}\,ds$ for the vector field $\vec{F} = x^2\hat{i} - y\hat{j}$ along the curve $x = y^2$ from $(4,2)$ to $(1,-1)$.

The given function is

$$\vec{F} = x^2\hat{i} - y\hat{j}$$

The given curve is

$$x = y^2 \quad \text{(Parabola)}$$

The parametric equation for the parabola can be written by introducing a parameter t such that

$$y = t \quad \text{and} \quad x = t^2$$

where, $2 \le t \le -1$.

Thus,

$$\vec{r}(t) = t^2\hat{i} + t\hat{j}$$

Now,

The parameterization is

$$\vec{F}(\vec{r}(t)) = (t^2)^2\hat{i} - (t)\hat{j}$$

$$= t^4\hat{i} - t\hat{j}$$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(t^2\hat{i} + t\hat{j})$$

$$= 2t\hat{i} + \hat{j}$$

Hence,

$$\Rightarrow \int_C \vec{F}\cdot\vec{T}\,ds$$

$$= \int_2^{-1} (t^4\hat{i} - t\hat{j})\cdot(2t\hat{i} + \hat{j})\,dt$$

$$= \int_2^{-1} (2t^5 - t)\,dt$$

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$$= \left[\frac{2}{6}t^6 - \frac{1}{2}t^2\right]_2^{-1}$$

$$= \left(\frac{2}{6}(-1)^6 - \frac{1}{2}(-1)^2\right) - \left(\frac{2}{6}(2)^6 - \frac{1}{2}(2)^2\right)$$

$$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{1}{3}(64) - \frac{1}{2}(4)\right)$$

$$= \frac{2-3}{6} - \left(\frac{64}{3} - 2\right)$$

$$= \frac{-1}{6} - \left(\frac{64-6}{3}\right)$$

$$= \frac{-1}{6} - \frac{58}{3}$$

$$= \frac{-1-116}{6}$$

$$= \frac{-117}{6}$$

$$= \frac{-39}{2}$$

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Work, Circulation and Flux in the plane

Q27) Find the work done by the force $\vec{F} = xy\hat{i} + (y-x)\hat{j}$ over the straight line from (1,1) to (2,3).

First we parameterize the straight line from (1,1) to (2,3)

Here,

Let t be a parameter that varies from 0 to 1. The position vector along the path is

$$\vec{r}(t) = r_0 + t(r_1 - r_0)$$
$$= (1,1) + t((2,3) - (1,1))$$
$$= (1,1) + t(1,2)$$
$$= (1,1) + (t, 2t)$$
$$\vec{r}(t) = (1+t, 1+2t)$$

The parameterization is

$$\vec{F}(\vec{r}(t)) = (1+t)(1+2t)\hat{i} + [1+2t-(1+t)]\hat{j}$$
$$= (1+2t+t+2t^2)\hat{i} + (1+2t-1-t)\hat{j}$$
$$= (2t^2+3t+1)\hat{i} + (t)\hat{j}$$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}[(1+t)\hat{i} + (1+2t)\hat{j}]$$
$$= \hat{i} + 2\hat{j}$$

Thus,

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$$\text{Work} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^1 \left[\left[(2t^2+3t+1)\hat{i} + (t)\hat{j} \right] \cdot (1\hat{i} + 2\hat{j}) \right] dt$$

$$= \int_0^1 (2t^2+3t+1+2t)\, dt$$

$$= \int_0^1 (1+5t+2t^2)\, dt$$

$$= \left[t + \frac{5t^2}{2} + \frac{2t^3}{3} \right]_0^1$$

$$= \left(1 + \frac{5(1)^2}{2} + \frac{2(1)^3}{3} \right) - 0$$

$$= 1 + \frac{5}{2} + \frac{2}{3}$$

$$= \frac{6+15+4}{6}$$

$$= \frac{25}{6}$$

Q29) Find the circulation and flux of the fields

$\vec{F_1} = x\hat{i} + y\hat{j}$ and $\vec{F_2} = -y\hat{i} + x\hat{j}$

around and across each of circles:

a) $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$, $0 \le t \le 2\pi$

b) $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}$, $0 \le t \le 2\pi$

For $\vec{F_1} = x\hat{i} + y\hat{j}$

(a) $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}$, $0 \le t \le 2\pi$

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As we know that

$\Rightarrow$ Circulation $= \cancel{\oint_C M dx + N dy}$ $\oint_C \vec{F} \cdot d\vec{r}$

Here,

$M = x \qquad N = y$

$x = \cos t \qquad dx = -\sin t \, dt$

$y = \sin t \qquad dy = \cos t \, dt$

Also,

$$\vec{F}(\vec{r}(t)) = (\cos t)\hat{i} + (\sin t)\hat{j}$$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(\cos t)\hat{i} + (\sin t)\hat{j}\right]$$

$$= -\sin t\,\hat{i} + \cos t\,\hat{j}$$

Thus,

$$\text{Circulation} = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} [\cos t\,\hat{i} + \sin t\,\hat{j}] \cdot [-\sin t\,\hat{i} + \cos t\,\hat{j}]\,dt$$

$$= \int_0^{2\pi} (-\sin t \cos t + \cos t \sin t)\,dt$$

$$= \int_0^{2\pi} 0 \, dt$$

$$= 0$$

Also,

$$\text{Flux} = \oint_C M dy - N dx$$

$$= \int_0^{2\pi} (x)(\cos t \, dt) - (y)(-\sin t \, dt)$$

$$= \int_0^{2\pi} (\cos t)(\cos t \, dt) - (\sin t)(-\sin t \, dt)$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt$$

$$= \int_0^{2\pi} (1)\,dt$$

$$= [t]_0^{2\pi}$$

$$= 2\pi - 0$$

$$= 2\pi$$

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(b) $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

$\vec{F} = x\hat{i} + y\hat{j}$

The parameterization is

$\vec{F}(\vec{r}(t)) = \cos t\,\hat{i} + 4\sin t\,\hat{j}$

Here,

$M = x = \cos t$

$N = y = 4\sin t$

$x = \cos t \qquad dx = -\sin t\,dt$

$y = 4\sin t \qquad dy = 4\cos t\,dt$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(\cos t)\hat{i} + (4\sin t)\hat{j}\right]$$

$$= (-\sin t)\hat{i} + (4\cos t)\hat{j}$$

Thus,

$$\text{Circulation} = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \left[\cos t\,\hat{i} + 4\sin t\,\hat{j}\right]\cdot\left[(-\sin t)\hat{i} + (4\cos t)\hat{j}\right] dt$$

$$= \int_0^{2\pi} (-\cos t\sin t + 16\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (15\sin t\cos t)\,dt$$

$$= \frac{15}{2}\int_0^{2\pi} (2\sin t\cos t)\,dt$$

$$= \frac{15}{2}\int_0^{2\pi} (\sin 2t)\,dt$$

$$= \frac{15}{2}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi}$$

$$= -\frac{15}{4}\left[(\cos 4\pi) - (\cos 0)\right]$$

$$= -\frac{15}{4}(1-1)$$

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Circulation $= 0$

Also,

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} [(\cos t)(4\cos t\,dt) - (4\sin t)(-\sin t\,dt)]$$

$$= \int_0^{2\pi} 4\cos^2 t\,dt + 4\sin^2 t\,dt$$

$$= \int_0^{2\pi} [4(\cos^2 t + \sin^2 t)]\,dt$$

$$= 4\int_0^{2\pi} (1)\,dt$$

$$= 4[t]_0^{2\pi}$$

$$= 4(2\pi - 0)$$

$$\text{Flux} = 8\pi$$

$$\vec{F_2} = -y\hat{i} + x\hat{j}$$

a) $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \; 0 \le t \le 2\pi$

As we know that

$$\vec{F_2} = -y\hat{i} + x\hat{j}$$

The parameterization is

$$\vec{F}(\vec{r}(t)) = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

Here,

$$M = -y = -\sin t$$

$$N = x = \cos t$$

$$x = \cos t \qquad dx = -\sin t\,dt$$

$$y = \sin t \qquad dy = \cos t\,dt$$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}[(\cos t)\hat{i} + (\sin t)\hat{j}]$$

$$= (-\sin t)\hat{i} + (\cos t)\hat{j}$$

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Thus,

$$\text{Circulation} = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \left[(-\sin t\,\hat{i} + \cos t\,\hat{j}) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j})\right] dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} (1)\, dt$$

$$= [t]_0^{2\pi}$$

$$= 2\pi - 0$$

$$= 2\pi$$

Also,

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} \left[(-\sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)\right]$$

$$= \int_0^{2\pi} (-\sin t \cos t\,dt + \cos t \sin t\,dt)$$

$$= \int_0^{2\pi} (\cancel{-\sin t \cos t} + \cancel{\cos t \sin t})\, dt$$

$$= \int_0^{2\pi} (0)\, dt$$

$$= 0$$

b) $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

$$\vec{F_2} = -y\hat{i} + x\hat{j}$$

The parameterization is

$$\vec{F}(\vec{r}(t)) = -(4\sin t)\hat{i} + (\cos t)\hat{j}$$

Here,

$$M = -y = -\cos t$$

$$N = x = 4\sin t$$

$$x = 4\sin t \qquad dx = 4\cos t\,dt$$

$$y = \cos t \qquad dy = -\sin t\,dt$$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(\cos t\,\hat{i} + 4\sin t\,\hat{j})$$

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$$\frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + 4\cos t\,\hat{j}$$

Thus,

$$\text{Circulation} = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} [-4\sin t\,\hat{i} + \cos t\,\hat{j}]\cdot[-\sin t\,\hat{i} + 4\cos t\,\hat{j}]$$

$$= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)\,dt$$

$$= 4\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= 4\int_0^{2\pi} (1)\,dt$$

$$= 4\,[t]_0^{2\pi}$$

$$= 4\,(2\pi - 0)$$

$$= 8\pi$$

Also,

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} (-\cos t)(-\sin t\,dt) - (4\sin t)(4\cos t\,dt)$$

$$= \int_0^{2\pi} (\cos t\sin t - 16\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (-15\cos t\sin t)\,dt$$

$$= -15\left[\frac{\sin^2 t}{2}\right]_0^{2\pi}$$

$$= -\frac{15}{2}\left[\sin^2 2\pi - \sin^2 0\right]$$

$$= 0$$

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Q33) $\vec{F} = -y\hat{i} + x\hat{j}$

$\vec{\gamma}_1(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}, \; 0 \le t \le \pi$

$\vec{\gamma}_2(t) = t\hat{i}, \; -a \le t \le a$

→ **For $\vec{\gamma}_1(t)$**

$$\vec{F} = -y\hat{i} + x\hat{j}$$

$$\vec{F}(\vec{\gamma}_1(t)) = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

Here,

$$M = x = -a\sin t$$

$$N = y = a\cos t$$

$$x = a\cos t \qquad dx = -a\sin t\,dt$$

$$y = a\sin t \qquad dy = a\cos t\,dt$$

and,

$$\frac{d\vec{\gamma}_1}{dt} = \frac{d}{dt}(a\cos t\,\hat{i} + a\sin t\,\hat{j})$$

$$= -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

Now,

$$\text{Circ}_1 = \int_0^{\pi} \vec{F} \cdot d\vec{\gamma}$$

$$= \int_0^{\pi} [(-a\sin t\,\hat{i} + a\cos t\,\hat{j}) \cdot (-a\sin t\,\hat{i} + a\cos t\,\hat{j})]\,dt$$

$$= \int_0^{\pi} (+a^2\sin^2 t + a^2\cos^2 t)\,dt$$

$$= \int_0^{\pi} a^2(1)\,dt$$

$$= a^2 \int_0^{\pi} (1)\,dt$$

$$= a^2 [t]_0^{\pi}$$

$$= a^2(\pi - 0)$$

$$= \pi a^2$$

Also,

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$$\text{Flux}_1 = \int_0^{\pi} M\,dy - N\,dx$$
$$= \int_0^{\pi} (-a\sin t)(a\cos t\,dt) - (a\cos t)(-a\sin t\,dt)$$
$$= \int_0^{\pi} (-\cancel{a^2\sin t\cos t} + \cancel{a^2\sin t\cos t})\,dt$$
$$= \int_0^{\pi} (0)\,dt$$
$$= 0$$

→ For $\vec{\gamma_2}(t)$

$$\vec{F} = -y\hat{i} + x\hat{j}$$
$$\vec{F}(\vec{\gamma_2}(t)) = -(0)\hat{i} + (t)\hat{j}$$
$$= t\hat{j}$$

Here,

$$M = 0$$
$$N = t$$
$$x = 0 \qquad dx = 0$$
$$y = t \qquad dy = dt$$

and,

$$\frac{d\vec{\gamma_2}}{dt} = \frac{d}{dt}(t\hat{i})$$
$$= \hat{i}$$

Now,

$$\text{Circ}_2 = \int_0^{\pi} \vec{F}\cdot d\vec{\gamma}$$
$$= \int_0^{\pi} (t\hat{j})\cdot(\hat{i})\,dt$$
$$= \int_0^{\pi} (0)\,dt$$
$$= 0$$

Also,

$$\text{Flux}_2 = \int_0^{\pi} M\,dy - N\,dx$$

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$$\text{Flux}_2 = \int_0^{\pi} (0)(dt) - (t)(0)$$
$$= \int_0^{\pi} 0$$
$$= 0$$

Thus,

$$\text{Total Flux} = \text{Flux}_1 + \text{Flux}_2$$
$$= 0 + 0$$
$$= 0$$

and

$$\text{Total Circulation} = \text{Circ}_1 + \text{Circ}_2$$
$$= a^2\pi + 0$$
$$= a^2\pi$$

Flow Integrals in Space

(42)

Q49) $\vec{F} = (x-z)\hat{i} + x\hat{k}$

$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{k}, \quad 0 \le t \le \pi$

The parameterization is

$$\vec{F}(\vec{r}(t)) = [\cos t - \sin t]\hat{i} + (\cos t)\hat{k}$$
$$= (\cos t - \sin t)\hat{i} + (\cos t)\hat{k}$$

and,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(\cos t\,\hat{i} + \sin t\,\hat{k})$$
$$= -\sin t\,\hat{i} + \cos t\,\hat{k}$$

Now,

$$\text{Flow} = \oint_C \vec{F} \cdot d\vec{r}$$
$$= \int_0^{\pi} [(\cos t - \sin t)\hat{i} + \cos t\,\hat{k}] \cdot (-\sin t\,\hat{i} + \cos t\,\hat{k})$$
$$= \int_0^{\pi} (-\sin t \cos t + \sin^2 t + \cos^2 t)\,dt$$

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$$\text{Flow} = \int_0^{\pi} (-\sin t \cos t + 1)\, dt$$

$$= \int_0^{\pi} (1 - \sin t \cos t)\, dt$$

$$= \int_0^{\pi} (1)\, dt - \int_0^{\pi} \sin t \cos t\, dt$$

$$= [t]_0^{\pi} - \left[\frac{\sin^2 t}{2}\right]_0^{\pi}$$

$$= (\pi - 0) - \frac{1}{2}[\sin^2 t]_0^{\pi}$$

$$= \pi - \frac{1}{2}[\sin^2 \pi - \sin^2 0]$$

$$= \pi - \frac{1}{2}(0)$$

$$\text{Flow} = \pi$$

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