Conservative Fields

Component test for conservative fields

1) When $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ then $\vec{F}$ is said to be conservative if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

2) When $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j} + P(x,y)\hat{k}$ then $\vec{F}$ is said to be conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

E3) Show that $\vec{F} = (e^x\cos y + yz)\hat{i} + (xz - e^x\sin y)\hat{j} + (xy + z)\hat{k}$ is conservative over its natural domain and find a potential function for it.

Here,

$$M = e^x\cos y + yz$$

$$N = xz - e^x\sin y$$

$$P = xy + z$$

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Now,

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xy+z)$$

$$= x$$

$$\Rightarrow \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(xz - e^x \sin y)$$

$$= x$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$$

Also,

$$\Rightarrow \frac{\partial M}{\partial z} = \frac{\partial}{\partial z}(e^x \cos y + yz)$$

$$= y(1)$$

$$= y$$

$$\Rightarrow \frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(xy+z)$$

$$= y(1)$$

$$= y$$

$$\therefore \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$

Also,

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xz - e^x \sin y)$$

$$= z(1) - \sin y\,(e^x)$$

$$= z - e^x \sin y$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y + yz)$$

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$$\frac{\partial M}{\partial y} = e^x(-\sin y) + z(1)$$

$$= z - e^x \sin y$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Hence, the given function is conservative.

Exercise

16.3

Testing for Conservative Fields

Q1) $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

Here,

$$M = yz$$
$$N = xz$$
$$P = xy$$

Now,

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(xy) = x$$

$$\Rightarrow \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(xz) = x$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$$

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Also,

$$\Rightarrow \frac{\partial M}{\partial z} = \frac{\partial (yz)}{\partial z} = y$$

$$\Rightarrow \frac{\partial P}{\partial x} = \frac{\partial (xy)}{\partial x} = y$$

$$\therefore \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$

Also,

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial (xz)}{\partial x} = z$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial (yz)}{\partial y} = z$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Hence, the given function is conservative.

Q3) $\vec{F} = y\hat{i} + (x+z)\hat{j} - y\hat{k}$

Here,

$M = y$

$N = x+z$

$P = -y$

Now,

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial (-y)}{\partial y} = -1$$

$$\Rightarrow \frac{\partial N}{\partial z} = \frac{\partial (x+z)}{\partial z} = 1$$

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$$\therefore \frac{\partial P}{\partial y} \neq \frac{\partial N}{\partial z}$$

Hence,

the given function is not conservative.

Q5) $\vec{F} = (z+y)\hat{i} + z\hat{j} + (y+x)\hat{k}$

Here,

$$M = z+y$$
$$N = z$$
$$P = y+x$$

Now,

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(z+y) = 1$$

$$\Rightarrow \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(\cancel{y+x}\ z) = 1$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$$

Also,

$$\Rightarrow \frac{\partial M}{\partial z} = \frac{\partial}{\partial z}(z+y) = 1$$

$$\Rightarrow \frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(y+x) = 1$$

$$\therefore \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$

Also,

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(z) = 0$$

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$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(z+y) = 1$$

$$\therefore \frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$$

Hence, the given function is not conservative.