Lecture

Chapter 2. Basic Structures

2.3 Functions

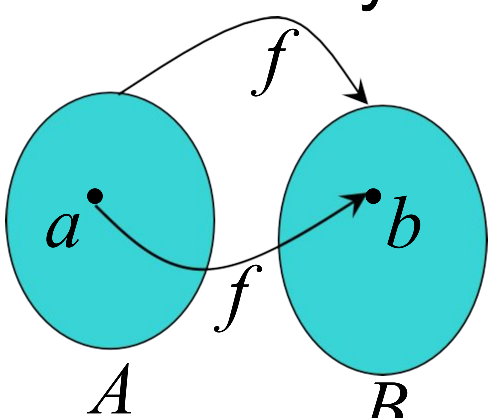
2.3 Functions

- From calculus, you are familiar with the concept of a real-valued function f, which assigns to each number $x \in \mathbf{R}$ a value $y = f(x)$, where $y \in \mathbf{R}$.

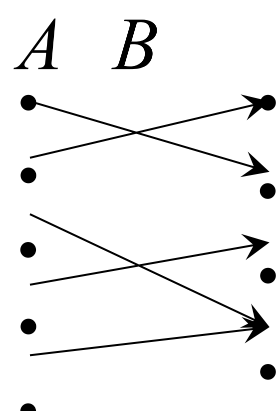
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *map*.)

Function: Formal Definition

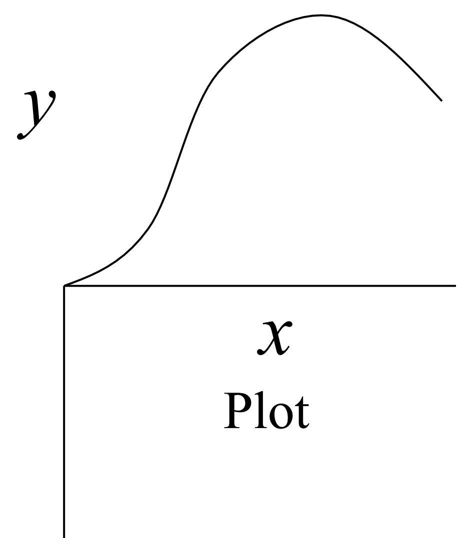
- For any sets A and B, we say that a ***function*** **(*or "mapping"*) *f from A to B* (*f* : *A* $\rightarrow$ *B*)** is a particular assignment of **exactly one element** $f(x) \in B$ to **each element** $x \in A$.
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

- If it is written that $f : A \to B$, and $f(a) = b$ (where $a \in A$ and $b \in B$), then we say:
  - A is the ***domain*** of f
  - B is the ***codomain*** of f
  - b is the ***image*** of a under f
    - a can not have more than 1 image
  - a is a ***pre-image*** of b under f
    - b may have more than 1 pre-image
  - The ***range*** $R \subseteq B$ of f is $R = \{b \mid \exists a\ f(a) = b\}$

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

- Suppose I declare that: "*f* is a function mapping students in this class to the set of grades {A, B, C, D, F}."

- At this point, you know *f* 's codomain is: _{A, B, C, D, F}_, and its range is _unknown!_

- Suppose the grades turn out all As and Bs.

- Then the range of *f* is _{A, B}_, but its ... still {A, B, C, D,

Function Operators

- $+$, $\times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply two *real-valued functions* f,g: $\mathbf{R} \to \mathbf{R}$:
  - $(f+g)$: $\mathbf{R} \to \mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - (fg): $\mathbf{R} \to \mathbf{R}$, where $(fg)(x) = f(x)g(x)$
- <u>Example 6</u>:
  Let f and g be functions from **R** to **R** such that $f(x) = x^2$ and $g(x) = x - x^2$. What are the functions $f + g$ and fg?

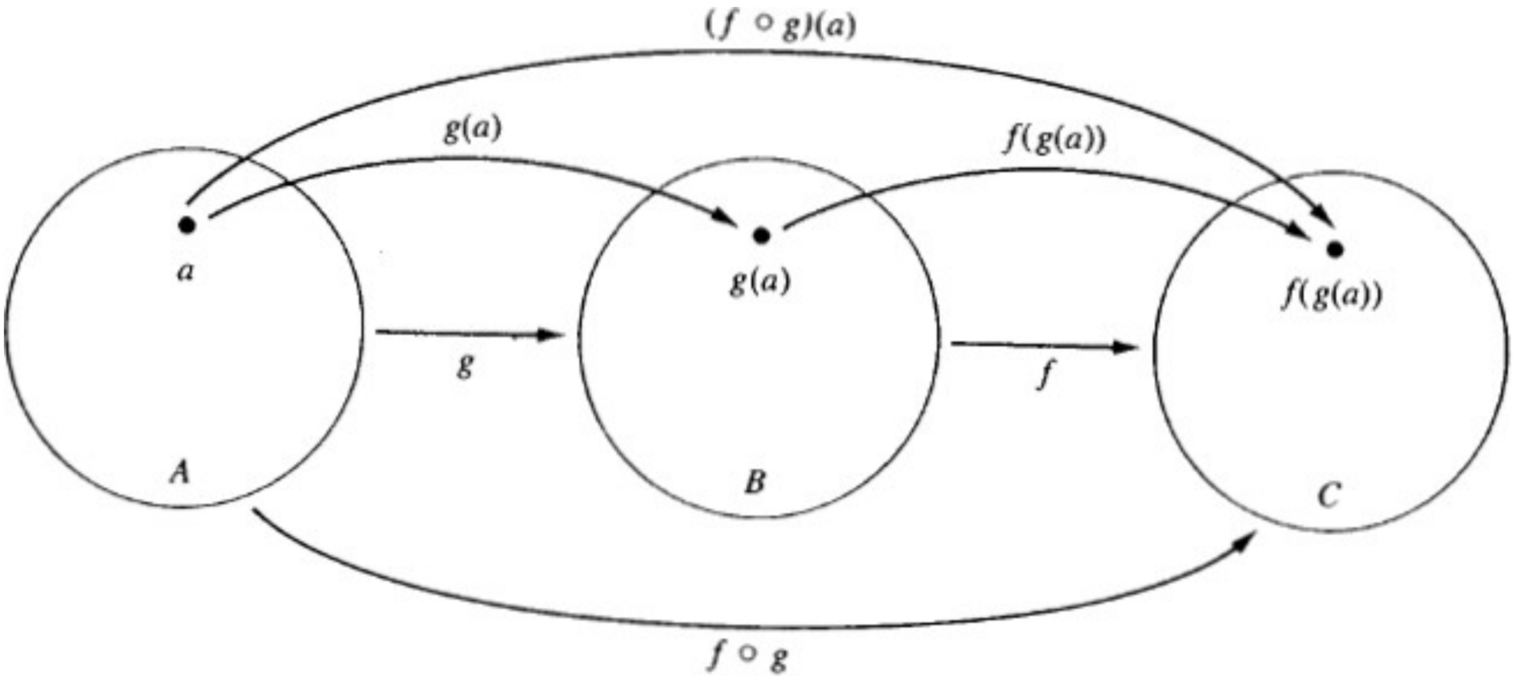
Function Composition Operator

Note the match here. It's necessary!

- For functions $g: A \to B$ and $f: B \to C$, there is a special operator called ***compose*** ("∘").
  - It <u>composes</u> (creates) a new function from f and g by applying f to the result of applying g.
  - We say $(f \circ g): A \to C$, where $(f \circ g)(a) = f(g(a))$.
  - Note: $f \circ g$ cannot be defined unless range of g is a subset of the domain of f.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.

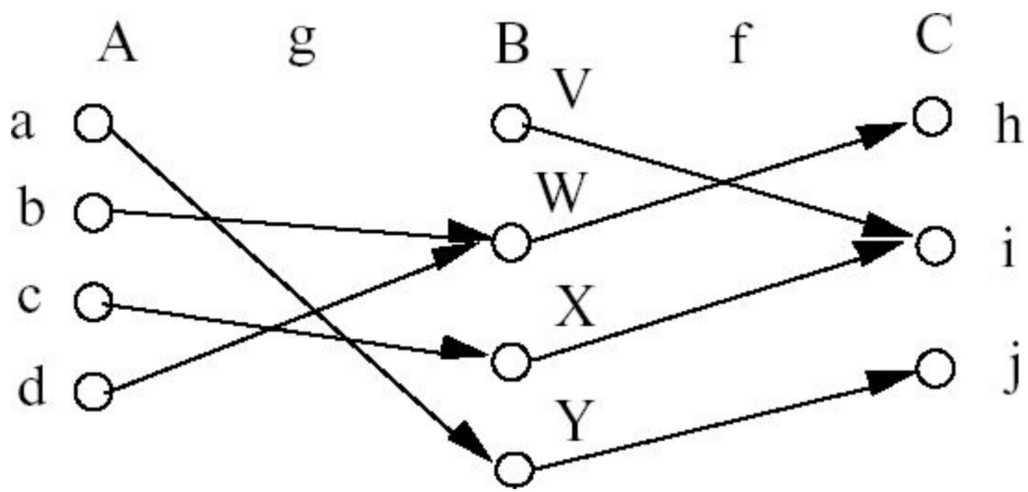
Function Composition Illustration

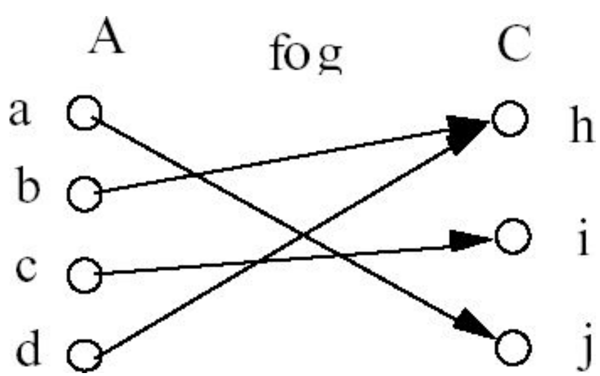
- $g: A \rightarrow B, \quad f: B \rightarrow C$



Function Composition: Example

- $g: A \to B, \quad f: B \to C$





Function Composition: Example

- Example 20: Let g: $\{a, b, c\} \rightarrow \{a, b, c\}$ such that

$$g(a) = b,\ g(b) = c,\ g(c) = a.$$

Let f : $\{a, b, c\} \rightarrow \{1, 2, 3\}$ such that

$$f(a) = 3,\ f(b) = 2,\ f(c) = 1.$$

What is the composition of f and g, and what is the composition of g and f ?

- $f\circ g$: {… …t

  $(f\circ g)($… … $= 3.$

- $g\circ f$ is not defined (why?)

Function Composition: Example

- If $f(x) = x^2$ and $g(x) = 2x + 1$, then what is the composition of f and g, and what is the composition of g and f ?
  - $(f\circ g)(x) = f(g(x))$
    $= f(2x+1)$
    $= (2x+1)^2$
  - $(g\circ f)(x) = g(f(x))$
    $= g(x^2)$
    $= 2x^2 + 1$

  Note that $f\circ g \neq g\circ f$. $(4x^2+4x+1 \neq 2x^2+1)$

Images of Sets under Function niverssi

- Given $f : A \rightarrow B$, and $S \subseteq A$,
- The ***image*** of S under f is simply the set of all images (under f) of the elements of S.

$$f(S) = \{f(t) \mid t \in S\}$$
$$= \{b \mid \exists t \in S: f(t) = b\}.$$

- Note the range of f can be defined as simply the image (under f) of f's domain.

One-to-One Functions

- A function f is ***one-to-one*** (1–1), or ***injective***, or an ***injection***, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f (i.e. every element of its range has *only* 1 pre-image).
  - Formally, given $f : A \rightarrow B$,

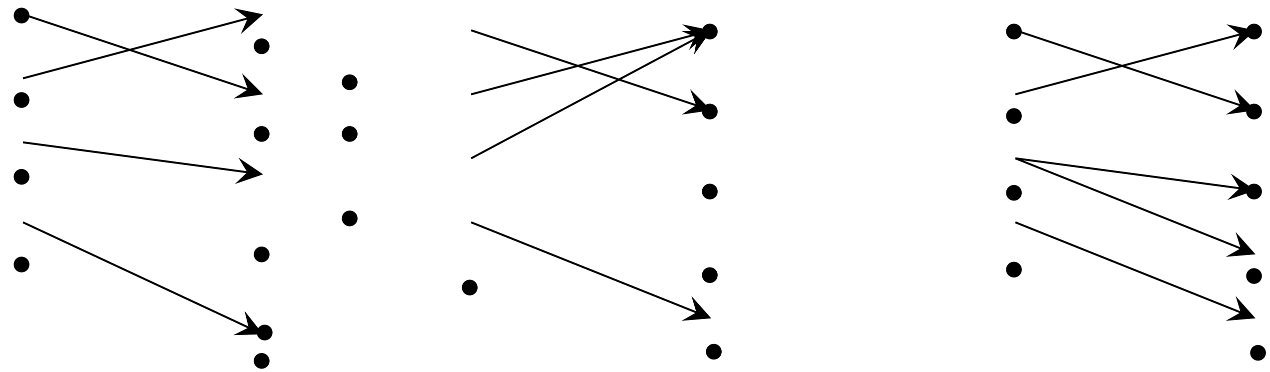
    "f is injective": $\forall a,b\ (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a,b\ (a \neq b \rightarrow f(a) \neq f(b))$
- Only <u>one</u> element of the domain is mapped <u>to</u> any given <u>one</u> element of the range.
  - Domain & range have the same cardinality.

    May Be Larger

    What about codomain?

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one Not one-to-one Not even a function!

- Example 8:

  Is the function $f : \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ one-to-one?

- Example 9:

  Let $f : \mathbf{Z} \to \mathbf{Z}$ such that $f(x) = x^2$. Is f one-to-one?

Sufficient Conditions for 1-1ness

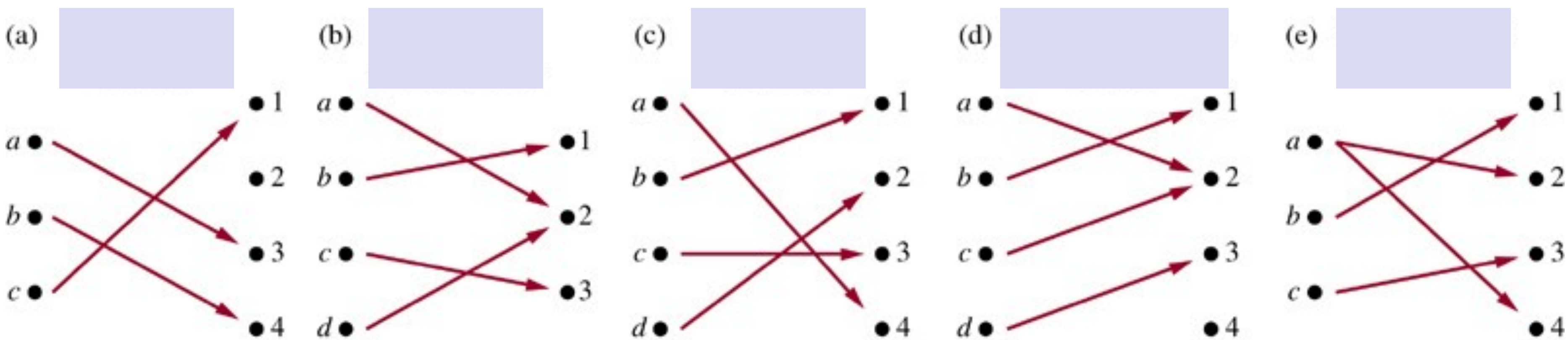
- For functions f over numbers, we say:
  - f is ***strictly*** (or ***monotonically***) ***increasing*** iff $x > y \rightarrow f(x) > f(y)$ for all x, y in domain;
  - f is ***strictly*** (or ***monotonically***) ***decreasing*** iff $x > y \rightarrow f(x) < f(y)$ for all x, y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one.

Onto (Surjective) Functions

- A function $f : A \rightarrow B$ is ***onto*** or ***surjective*** or a ***surjection*** iff <u>for every element $b \in B$</u> there is an element $a \in A$ with $f(a) = b$ ($\forall b \in B, \exists a \in A: f(a) = b$) (i.e. its range is equal to its codomain).
- Think: An *onto* function maps the set A <u>onto</u> (over, covering) the *entirety* of the set B, not just over a piece of it.
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are, or are not, *onto* their codomains:



- <u>Example13</u>: Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Bijections and Inverse Function.

- A function *f* is said to be *a* ***one-to-one correspondence***, or *a* ***bijection***, or *reversible*, or *invertible*, iff it is both one-to-one and onto.

- Let $f : A \to B$ be a bijection.

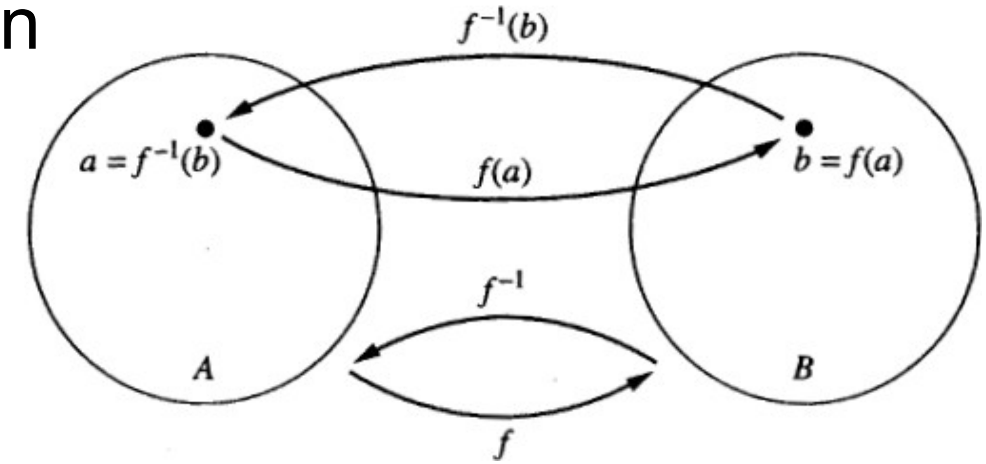
  The ***inverse function*** of *f* is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

  The inverse function of *f* is denoted by $f^{-1} : B \to A$.

  Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Function Illustration

- Let $f: A \to B$ be a bijection



- Example 16: Let $f : \{a, b, c\} \to \{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

- Example 18: Let f be the

  $f(x) = x^2$. Is f invertible?

  No. f is not a one-to-one function. So it's not invertible.