Chapter 1. The Foundations

2. Propositional Equivalences
3. Predicates and Quantifiers

1.2 Propositional Equivalence

- A ***tautology*** is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are!
  - *e.g.* $p \lor \neg p$ ("Today the sun will shine or today the sun will not shine.") [What is its truth table?]
- A ***contradiction*** is a compound proposition that is **false** no matter what!
  - *e.g.* $p \land \neg p$ ("Today is Wednesday and today is not Wednesday.") [Truth table?]
- A ***contingency*** is a compound proposition that is neither a tautology nor a contradiction.
  - *e.g.* $(p \lor q) \to \neg r$

Logical Equivalence

- Compound proposition p is ***logically equivalent*** to compound proposition q, written $p \equiv q$ or $p \Leftrightarrow q$, **iff** the compound proposition $p \leftrightarrow q$ is a tautology.

- Compound propositions p and q are logically equivalent to each other **iff** p and q contain the same truth values as each other in <u>all</u> corresponding rows of their truth tables.

Proving Equivalence via Truth Tables

- Prove that $\neg(p \wedge q) \equiv \neg p \vee \neg q$. (De Morgan's law)

p q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q)$
T T	T F F			F	F
T F	F F T			T	T
F	F T F			T	T
T	F T T			T	T
T					
F					

- Show that

Check out the solution in the textbook!

  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's law)
  - $p \rightarrow q \equiv \neg p \vee q$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (distributive law)

Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.

- They provide a pattern or template that can be used to match part of a much more complicated proposition and to find an equivalence for it and possibly simplify it.

Equivalence Laws

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- *Identity*: $p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
- *Domination*: $p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
- *Idempotent*: $p \vee p \equiv p$ $p \wedge p \equiv p$
- *Double negation:* $\neg\neg p \equiv p$
- *Commutative:* $p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
- *Associative:* $(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

More Equivalence Laws

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- *Distributive*: $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
  $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
- *De Morgan's*:
  $\neg(p \land q) \equiv \neg p \lor \neg q$
  $\neg(p \lor q) \equiv \neg p \land \neg q$
- *Absorption*
  $p \lor (p \land q) \equiv p$ $\quad$ $p \land (p \lor q) \equiv p$
- *Trivial tautology/contradiction*:
  $p \lor \neg p \equiv \mathbf{T}$ $\quad$ $p \land \neg p \equiv \mathbf{F}$

See Table 6, 7, and 8 of Section 1.2

Defining Operators via Equivalences

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Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

  $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$
- Implies: $p \rightarrow q \equiv \neg p \vee q$
- Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

  $p \leftrightarrow q \equiv \neg(p \oplus q)$

This way we can "normalize" propositions

An Example Problem

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- Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$\neg(p \rightarrow q)$ [Expand definition of $\rightarrow$]

$\equiv \neg(\neg p \vee q)$ [DeMorgan's Law]

$\equiv \neg(\neg p) \wedge \neg q$ [Double Negation]

$\equiv p \wedge \neg q$

Another Example Problem

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- Check using a symbolic derivation whether

$$\boxed{(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r}$$

$(p \wedge \neg q) \rightarrow (p \oplus r)$ [Expand definition of $\rightarrow$]

$\equiv \neg(p \wedge \neg q) \vee (p \oplus r)$ [Expand definition of $\oplus$]

$\equiv \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$

[DeMorgan's Law]

$\equiv (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$

cont.

Example Continued...

$$(p \land \neg q) \to (p \oplus r) \equiv \neg p \lor q \lor \neg r$$

$(\neg p \lor q) \lor ((p \lor r) \land \neg(p \land r))$ [$\lor$ Commutative]

$\equiv (q \lor \neg p) \lor ((p \lor r) \land \neg(p \land r))$ [$\lor$ Associative]

$\equiv q \lor (\neg p \lor ((p \lor r) \land \neg(p \land r)))$ [Distribute $\lor$ over $\land$]

$\equiv q \lor (((\neg p \lor (p \lor r)) \land (\neg p \lor \neg(p \land r)))$ [$\lor$ Assoc.]

$\equiv q \lor (((\neg p \lor p) \lor r) \land (\neg p \lor \neg(p \land r)))$[Trivial taut.]

$\equiv q \lor ((\mathbf{T} \lor r) \land (\neg p \lor \neg(p \land r)))$ [Domination]

$\equiv q \lor (\mathbf{T} \land (\neg p \lor \neg(p \land r)))$ [Identity]

$\equiv q \lor (\neg p \lor \neg(p \land r))$

cont.

End of Long Example

$$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r$$

$q \vee (\neg p \vee \neg(p \wedge r))$ [DeMorgan's Law]

$\equiv q \vee (\neg p \vee (\neg p \vee \neg r))$ [∨ Associative]

$\equiv q \vee ((\neg p \vee \neg p) \vee \neg r)$ [Idempotent]

$\equiv q \vee (\neg p \vee \neg r)$ [Associative]

$\equiv (q \vee \neg p) \vee \neg r$ [∨ Commutative]

$\equiv \neg p \vee q \vee \neg r$ ■

Review: Propositional Logic (1.1-1.2)

- Atomic propositions: p, q, r, …
- Boolean operators: $\neg \wedge \vee \oplus \rightarrow \leftrightarrow$
- Compound propositions: $(p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \Leftrightarrow \equiv \neg(p \rightarrow q)$
- Proving equivalences using:
  - Truth tables
  - Symbolic derivations (series of logical equivalences) $p \equiv q \equiv r \equiv \cdots$

1.3 Predicate Logic

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- Consider the sentence

  "For every x, $x > 0$"

  If this were a true statement about the positive integers, it could not be adequately symbolized using only statement letters, parentheses and logical connectives.

  The sentence contains two new features: a ***predicate*** *and a* ***quantifier***

Subjects and Predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the ***subject*** – the *object* or *entity* that the sentence is about.
  - The phrase "is sleeping" denotes the ***predicate*** – a property that the subject of the statement can have.
- In predicate logic, a ***predicate*** is modeled as a ***proposional function*** $P(\cdot)$ from subjects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any subject).
  - P(The cat) = "*The cat* is sleeping" (proposition!)

More About Predicates

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- Convention: Lowercase variables *x*, *y*, *z*... denote subjects; uppercase variables *P*, *Q*, *R*… denote propositional functions (or predicates).

- Keep in mind that the *result of applying* a predicate *P* to a value of subject *x* is the *proposition*. But the predicate *P*, or the statement *P*(*x*) **itself** (*e.g. P* = "is sleeping" or *P*(*x*) = "*x* is sleeping") is **not** a proposition.
  - *e.g.* if *P*(*x*) = "*x* is a prime number",
    P(3) is the *proposition* "3 is a prime number."

Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.
  - *e.g.*:
    let $P(x,y,z)$ = "x gave y the grade z"
    then if
    x = "Mike", y = "Mary", z = "A",
    then
    $P(x,y,z)$ = "**Mike** gave **Mary** the grade **A**."

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Examples

- Let $P(x)$: $x > 3$. Then
  - $P(4)$ is TRUE/FALSE $\boxed{4 > 3}$
  - $P(2)$ is TRUE/FALSE $\boxed{2 > 3}$
- Let $Q(x, y)$: x is the capital of y. Then
  - Q(Washington D.C., U.S.A.) is TRUE
  - Q(Hilo, Hawaii) is FALSE
  - Q(Massachusetts, Boston) is FALSE
  - Q(Denver, Colorado) is TRUE
  - Q(New York, New York) is FALSE
- Read EXAMPLE 6 (pp.33)
  - If **x > 0** then x:= x + 1 (in a computer program)