Curvature

The curvature k of C at the point P is absolute value of the rate of change of α w.r.t 's'.

$$k = \left| \frac{d\alpha}{ds} \right|$$

Also,

Radius of curvature is given as

$$\rho = \frac{1}{k} = \frac{1}{\left| \frac{d\alpha}{ds} \right|}$$

① For Cartesian Coordinates

$$k = \left| \frac{d\alpha}{ds} \right| = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

② For Parametric Equations

$$k = \frac{|f'(t)\, g''(t) - g'(t)\, f''(t)|}{[(f'(t))^2 + (g'(t))^2]^{3/2}}$$

③ For Polar Coordinates

$$k = \frac{\left| r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right) \right|}{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2 \right]^{3/2}}$$

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1) $y = c\cosh\left(\frac{x}{c}\right)$

$$y = c\cosh\left(\frac{x}{c}\right)$$

$$y_1 = \cancel{c}\cdot\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{\cancel{c}} = \sinh\left(\frac{x}{c}\right)$$

$$y_2 = \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$$

Radius of curvature is given as

$$\rho = \frac{1}{k}$$

$$\rho = \frac{[1+(y_2)^2]^{3/2}}{y_2}$$

$$\rho = \frac{\left[1+\left(\sinh\left(\frac{x}{c}\right)\right)^2\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}}$$

$$\rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}}$$

$$\rho = \frac{\left[\cosh^{\cancel{2}}\left(\frac{x}{c}\right)\right]^{3/\cancel{2}}}{\cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}}$$

$$\rho = \frac{\cosh^2\left(\frac{x}{c}\right)}{\cancel{\cosh\left(\frac{x}{c}\right)}\cdot\frac{1}{c}}$$

$$\rho = \frac{\cosh^2\left(\frac{x}{c}\right)}{1/c}$$

$$\rho = c\cosh^2\left(\frac{x}{c}\right)$$

$$\rho = \frac{c}{c}\cdot c\cosh^2\left(\frac{x}{c}\right)$$

$$\rho = \frac{c^2\cosh^2\left(\frac{x}{c}\right)}{c}$$

$$\boxed{\rho = \frac{y^2}{c}}$$

2) $x = a(\cos t + t\sin t)$, $a > 0$
$y = a(\sin t - t\cos t)$

<u>Solution</u> Let, $f(t) = a(\cos t + t\sin t) \longrightarrow (i)$
$g(t) = a(\sin t - t\cos t) \longrightarrow (ii)$

Differentiating eq: (i) w.r.t 't'

$f'(t) = a(-\sin t + \{(t)(\cos t) + (1)(\sin t)\})$

$f'(t) = a(\cancel{-\sin t} + t\cos t + \cancel{\sin t})$

$f'(t) = a(t\cos t)$

Again Diff b.s w.r.t 't'

$f''(t) = a\{(t)(-\sin t) + (1)(\cos t)\}$

$f''(t) = a(\cos t - t\sin t)$

Also,

Diff : equ (ii) w.r.t 't'

$g'(t) = a\{\cos t - ((t)(-\sin t) + (1)(\cos t))\}$

$g'(t) = a\{\cos t - (\cos t - t\sin t)\}$

$g'(t) = a(\cancel{\cos t} - \cancel{\cos t} + t\sin t)$

$g'(t) = a(t\sin t)$

Again Diff : b.s w.r.t 't'

$g''(t) = a\{(t)(\cos t) + (1)(\sin t)\}$

$g''(t) = a(t\cos t + \sin t)$

Now,

Radius of curvature is given as

$$\rho = \frac{1}{k}$$

$$\rho = \frac{[(f'(t))^2 + (g'(t))^2]^{3/2}}{|f'(t)g''(t) - g'(t)f''(t)|}$$

$$\rho = \frac{[(a(t\cos t))^2 + (a(t\sin t))^2]^{3/2}}{|(a(t\cos t))(a(t\cos t + \sin t)) - (a(t\sin t))(a(\cos t - t\sin t))|}$$

$$\rho = \frac{\left[a^2t^2\cos^2 t + a^2t^2\sin^2 t\right]^{3/2}}{\left|a^2(t\cos t)(t\cos t + \sin t) - a^2(t\sin t)(\cos t - t\sin t)\right|}$$

$$\rho = \frac{\left[a^2t^2(\sin^2 t + \cos^2 t)\right]^{3/2}}{\left|a^2t\{(\cos t)(t\cos t + \sin t) - (\sin t)(\cos t - t\sin t)\}\right|}$$

$$\rho = \frac{\left[a^2t^2(1)\right]^{3/2}}{\left|a^2t(t\cos^2 t + \cancel{\cos t\sin t} - \cancel{\sin t\cos t} + t\sin^2 t)\right|}$$

$$\rho = \frac{(a^2t^2)^{3/2}}{a^2t.t(\sin^2 t + \cos^2 t)}$$

$$\rho = \frac{a^3t^3}{\cancel{a^2}\cancel{t^2}(1)}$$

$$\boxed{\rho = at}$$

3) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $a > 0$

<u>Solution</u>:

Let, $f(t) = a(t - \sin t) \longrightarrow$ (i)

$g(t) = a(1 - \cos t) \longrightarrow$ (ii)

Diff: b.s of equ (i) w.r.t 't'

$f'(t) = a(1 - \cos t)$

Again Diff b.s w.r.t 't'

$f''(t) = a(-(-\sin t))$

$f''(t) = a\sin t$

Also,

Diff: b.s of equ (ii) w.r.t 't'

$g'(t) = a(0 - (-\sin t))$

$g'(t) = a\sin t$

Again Diff: b.s w.r.t 't'

$g''(t) = a\cos t$

Now,

Radius of curvature is given as

$$\rho = \frac{1}{k}$$

$$\rho = \frac{\left[(f'(t))^2 + (g'(t))^2 \right]^{3/2}}{|f'(t)g''(t) - g'(t)f''(t)|}$$

$$\rho = \frac{\left[(a(1-\cos t))^2 + (a\sin t)^2 \right]^{3/2}}{|a(1-\cos t)(a\cos t) - (a\sin t)(a\sin t)|}$$

$$\rho = \frac{\left[a^2(1-\cos t)^2 + a^2\sin^2 t \right]^{3/2}}{|a^2\cos t(1-\cos t) - a^2\sin^2 t|}$$

$$\rho = \frac{\left[a^2(1 - 2\cos t + \cos^2 t) + a^2\sin^2 t \right]^{3/2}}{|a^2\cos t - a^2\cos^2 t - a^2(1-\cos^2 t)|}$$

$$\rho = \frac{\left[a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t \right]^{3/2}}{|a^2\cos t - \cancel{a^2\cos^2 t} - a^2 + \cancel{a^2\cos^2 t}|}$$

$$\rho = \frac{\left[a^2 - 2a^2\cos t + a^2 \right]^{3/2}}{|a^2\cos t - a^2|}$$

$$\rho = \frac{\left[2a^2 - 2a^2\cos t \right]^{3/2}}{|a^2(\cos t - 1)|}$$

$$\rho = \frac{\left[2a^2(1-\cos t) \right]^{3/2}}{|a^2(\cos t - 1)|}$$

$$\rho = \frac{(2)^{3/2}(a^2)^{3/2}(1-\cos t)^{3/2}}{|-a^2(1-\cos t)|}$$

$$\rho = \frac{2\sqrt{2}\, a^{\cancel{3}}\, \cancel{(1-\cos t)} \cdot \sqrt{1-\cos t}}{+\cancel{a^2}\, \cancel{(1-\cos t)}}$$

$$\rho = +2\sqrt{2}\, a \sqrt{2\sin^2\left(\frac{t}{2}\right)}$$

$$\rho = +2\sqrt{2}\, a \cdot \sqrt{2} \cdot \sin\left(\frac{t}{2}\right)$$

$$\rho = +2(2)\, a \sin\left(\frac{t}{2}\right) = +4a\sin\left(\frac{t}{2}\right)$$

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4) $x = a\cos^3\theta$, $y = a\sin^3\theta$

$$x = a\cos^3\theta \longrightarrow (i)$$

Diff: b.s w.r.t 'θ'

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos^3\theta)$$

$$= 3a\cos^2\theta(-\sin\theta)$$

$$= -3a\cos^2\theta\sin\theta$$

Also,

$$y = a\sin^3\theta \longrightarrow (ii)$$

Diff: b.s w.r.t 'θ'

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin^3\theta)$$

$$= 3a\sin^2\theta\cos\theta$$

Now,

Using chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= 3a\sin^2\theta\cancel{\cos\theta}\cdot\frac{-1}{3a\cos^2\theta\cancel{\sin\theta}}$$

$$= -\frac{\sin\theta}{\cos\theta}$$

$$\frac{dy}{dx} = -\tan\theta$$

Diff: b.s w.r.t 'x'

$$\frac{d^2y}{dx^2} = -\sec^2\theta\cdot\frac{d\theta}{dx}$$

$$= -\sec^2\theta\cdot\frac{-1}{3a\cos^2\theta\sin\theta}$$

$$= \frac{\sec^2\theta\cdot\sec^2\theta}{3a\sin\theta}$$

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$$\frac{d^2y}{dx^2} = \frac{\sec^4\theta}{3a\sin\theta}$$

Now,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$\rho = \frac{\left[1 + \tan^2\theta\right]^{3/2}}{\left|\frac{\sec^4\theta}{3a\sin\theta}\right|}$$

$$\rho = \frac{(\sec^{\cancel{2}}\theta)^{3/\cancel{2}}}{\frac{\sec^4\theta}{3a\sin\theta}}$$

$$\rho = \frac{\cancel{\sec^3\theta}}{\frac{\sec^4\theta}{3a\sin\theta}}$$

$$\rho = \frac{3a\sin\theta}{\sec\theta}$$

$$\rho = 3a\sin\theta\cos\theta \longrightarrow \text{(iii)}$$

From equ (i), we have

$$x = a\cos^3\theta$$

$$\frac{x}{a} = \cos^3\theta$$

$$\cos\theta = \left(\frac{x}{a}\right)^{1/3}$$

From equ (ii), we have

$$y = a\sin^3\theta$$

$$\frac{y}{a} = \sin^3\theta$$

$$\sin\theta = \left(\frac{y}{a}\right)^{1/3}$$

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Now equ (iii) becomes

$$\rho = 3a\left(\frac{y}{a}\right)^{1/3} \cdot \left(\frac{x}{a}\right)^{1/3}$$

$$\rho = \frac{3a\, y^{1/3} x^{1/3}}{a^{1/3} \cdot a^{1/3}}$$

$$\rho = \frac{3a x^{1/3} y^{1/3}}{a^{2/3}}$$

$$\rho = 3a^{1-\frac{2}{3}} x^{1/3} y^{1/3}$$

$$\rho = 3a^{1/3} x^{1/3} y^{1/3}$$

$$\boxed{\rho = 3(axy)^{1/3}}$$

5) $r = 2\cos 2\theta$ at $\theta = \frac{\pi}{12}$

$$r = 2\cos 2\theta$$

Diff : b.s w.r.t 'θ'

$$\frac{dr}{d\theta} = \frac{d}{d\theta}(2\cos 2\theta)$$

$$= 2(-\sin 2\theta) \cdot 2$$

$$= -4\sin 2\theta$$

Again Diff : b.s w.r.t 'θ'

$$\frac{d^2r}{d\theta^2} = \frac{d}{d\theta}(-4\sin 2\theta)$$

$$= -4(\cos 2\theta) \cdot 2$$

$$= -8\cos 2\theta$$

Now,

Radius of curvature is given as

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

$$\rho = \frac{\left[(2\cos 2\theta)^2 + (-4\sin 2\theta)^2\right]^{3/2}}{\left|(2\cos 2\theta)^2 + 2(-4\sin 2\theta)^2 - (2\cos 2\theta)(-8\cos 2\theta)\right|}$$

$$\rho = \frac{[4\cos^2 2\theta + 16\sin^2 2\theta]^{3/2}}{|4\cos^2 2\theta + 2(16\sin^2 2\theta) + 16\cos^2 2\theta|}$$

$$\rho = \frac{(4)^{3/2}[\cos^2 2\theta + 4\sin^2 2\theta]^{3/2}}{|4\cos^2 2\theta + 32\sin^2 2\theta + 16\cos^2 2\theta|}$$

$$\rho = \frac{8[\cos^2 2\theta + 4\sin^2 2\theta]^{3/2}}{|20\cos^2 2\theta + 32\sin^2 2\theta|}$$

At $\theta = \frac{\pi}{12}$

$$\rho = \frac{8\left[\cos^2 2\left(\frac{\pi}{12}\right) + 4\sin^2 2\left(\frac{\pi}{12}\right)\right]^{3/2}}{\left|20\cos^2 2\left(\frac{\pi}{12}\right) + 32\sin^2 2\left(\frac{\pi}{12}\right)\right|}$$

$$\rho = \frac{8\left[\left(\frac{\sqrt{3}}{2}\right)^2 + 4\left(\frac{1}{2}\right)^2\right]^{3/2}}{\left|20\left(\frac{\sqrt{3}}{2}\right)^2 + 32\left(\frac{1}{2}\right)^2\right|}$$

$$\rho = \frac{8\left[\frac{3}{4} + 4\left(\frac{1}{4}\right)\right]^{3/2}}{\left|\cancel{20}^{5}\cdot\left(\frac{3}{\cancel{4}}\right) + \cancel{32}^{8}\cdot\left(\frac{1}{\cancel{4}}\right)\right|}$$

$$\rho = \frac{8\left[\frac{3}{4} + 1\right]^{3/2}}{|15 + 8|}$$

$$\rho = \frac{8\,(7/4)^{3/2}}{23}$$

$$\rho = \frac{8\left(\frac{343}{64}\right)^{1/2}}{23}$$

$$\rho = \frac{\cancel{8}\cdot\left(\frac{7\sqrt{7}}{\cancel{8}}\right)}{23}$$

$$\boxed{\rho = \frac{7\sqrt{7}}{23}}$$

6) $r\theta = a$

$r\theta = a$

$r = \frac{a}{\theta}$

$\cos 2\theta)|$

$$r = \frac{a}{\theta}$$

Differentiating b.s w.r.t 'θ'

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{a}{\theta}\right)$$

$$= a\frac{d}{d\theta}(\theta^{-1})$$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{a}{\theta^2}$$

Again Diff b.s w.r.t 'θ'

$$\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right) = \frac{d}{d\theta}(-a\theta^{-2})$$

$$= -a(-2)\theta^{-3}$$

$$\Rightarrow \frac{d^2r}{d\theta^2} = \frac{2a}{\theta^3}$$

Now,

Radius of curvature is given as

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

$$\rho = \frac{\left[\left(\frac{a}{\theta}\right)^2 + \left(\frac{-a}{\theta^2}\right)^2\right]^{3/2}}{\left|\left(\frac{a}{\theta}\right)^2 + 2\left(\frac{-a}{\theta^2}\right)^2 - \left(\frac{a}{\theta}\right)\left(\frac{2a}{\theta^3}\right)\right|}$$

$$\rho = \frac{\left[\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}\right]^{3/2}}{\left|\frac{a^2}{\theta^2} + \cancel{2}\left(\frac{a^2}{\theta^4}\right) - \frac{\cancel{2}a^2}{\theta^4}\right|}$$

$$\rho = \frac{\left[\frac{a^2\theta^2 + a^2}{\theta^4}\right]^{3/2}}{\left|\frac{a^2}{\theta^2}\right|}$$

$$\rho = \frac{(a^2)^{3/2}\left[\frac{(\theta^2+1)^{3/2}}{(\theta^4)^{3/2}}\right]}{a^2/\theta^2}$$

$$\rho = \frac{a^3(\theta^2+1)^{3/2}}{\theta^6 \cdot \frac{a^2}{\theta^2}}$$

$$\rho = \frac{a^{\cancel{2}}(\theta^2+1)^{3/2}}{\theta^4 \cdot \cancel{a^2}}$$

$$\boxed{\rho = \frac{a(\theta^2+1)^{3/2}}{\theta^4}}$$

7) $r^n = a^n \sin n\theta$

$$r^n = a^n \sin n\theta$$

Differentiating b.s w.r.t 'θ'

$$\frac{d}{d\theta}(r^n) = \frac{d}{d\theta}(a^n \sin n\theta)$$

$$\cancel{n} r^{n-1} \cdot \frac{dr}{d\theta} = a^n \cos n\theta \cdot \cancel{n}$$

$$\frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^{n-1}}$$

Multiply and Divide by 'r'

$$\frac{dr}{d\theta} = \frac{r a^n \cos n\theta}{r \cdot r^{n-1}}$$

$$= \frac{r a^n \cos n\theta}{r^n}$$

$$= \frac{r \cancel{a^n} \cos n\theta}{\cancel{a^n} \sin n\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = r \cot n\theta$$

Again Diff: b.s w.r.t 'θ'

$$\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right) = \frac{d}{d\theta}(r \cot n\theta)$$

$$\frac{d^2r}{d\theta^2} = r(-\operatorname{cosec}^2 n\theta \cdot n) + \cot n\theta \cdot \frac{dr}{d\theta}$$

$$\frac{d^2r}{d\theta^2} = -rn \operatorname{cosec}^2 n\theta + \cot n\theta \cdot r \cot n\theta$$

$$\Rightarrow \frac{d^2r}{d\theta^2} = r\cot^2 n\theta - rn\,\mathrm{cosec}^2 n\theta$$

Now,

Radius of curvature is given as

$$\rho = \frac{1}{k}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)\right|}$$

$$\rho = \frac{\left[r^2 + (r\cot n\theta)^2\right]^{3/2}}{\left|r^2 + 2(r\cot n\theta)^2 - r(r\cot^2 n\theta - rn\,\mathrm{cosec}^2 n\theta)\right|}$$

$$\rho = \frac{\left[r^2 + r^2\cot^2 n\theta\right]^{3/2}}{\left|r^2 + 2r^2\cot^2 n\theta - r^2\cot^2 n\theta + r^2 n\,\mathrm{cosec}^2 n\theta\right|}$$

$$\rho = \frac{(r^2)^{3/2}\left[1 + \cot^2 n\theta\right]^{3/2}}{\left|r^2 + r^2\cot^2 n\theta + r^2 n\,\mathrm{cosec}^2 n\theta\right|}$$

$$\rho = \frac{r^{\cancel{3}}(1 + \cot^2 n\theta)^{3/2}}{\left|\cancel{r^2}(1 + \cot^2 n\theta + n\,\mathrm{cosec}^2 n\theta)\right|}$$

$$\rho = \frac{r\,(\mathrm{cosec}^2 n\theta)^{3/2}}{(1 + \cot^2 n\theta) + n\,\mathrm{cosec}^2 n\theta}$$

$$\rho = \frac{r\,\mathrm{cosec}^3 n\theta}{\mathrm{cosec}^2 n\theta + n\,\mathrm{cosec}^2 n\theta}$$

$$\rho = \frac{r\,\mathrm{cosec}^{\cancel{3}} n\theta}{\cancel{\mathrm{cosec}^2 n\theta}\,(1+n)}$$

$$\rho = \frac{r\,\mathrm{cosec}\, n\theta}{1+n}$$

$$\rho = \frac{r}{1+n} \cdot \mathrm{cosec}\, n\theta$$

$$\rho = \frac{r}{(1+n)(\sin n\theta)}$$

$$\rho = \frac{r}{(1+n)\left(\frac{r^n}{a^n}\right)}$$

$$\rho = \frac{r}{1+n} \cdot \frac{a^n}{r^n}$$

$$\boxed{\rho = \frac{a^n}{r^{n-1}(1+n)}}$$

8) $r(1+\cos\theta) = a$

$r(1+\cos\theta) = a$

Diff: w.r.t 'θ'

$$r = \frac{a}{1+\cos\theta}$$

$$\frac{d}{d\theta}(r) = \frac{d}{d\theta}\left(\frac{a}{1+\cos\theta}\right)$$

$$\frac{dr}{d\theta} = \frac{(1+\cos\theta)(0) - (a)(-\sin\theta)}{(1+\cos\theta)^2}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{a\sin\theta}{(1+\cos\theta)^2}$$

Again Diff: b.s w.r.t 'θ'

$$\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right) = \frac{d}{d\theta}\left(\frac{a\sin\theta}{(1+\cos\theta)^2}\right)$$

$$\frac{d^2r}{d\theta^2} = \frac{(1+\cos\theta)^2(a\cos\theta) - (a\sin\theta)(2(1+\cos\theta)(-\sin\theta))}{(1+\cos\theta)^4}$$

$$= \frac{(1+\cos\theta)^2(a\cos\theta) + 2a\sin^2\theta\,(1+\cos\theta)}{(1+\cos\theta)^4}$$

$$= \frac{\cancel{(1+\cos\theta)}\{(1+\cos\theta)(a\cos\theta) + 2a\sin^2\theta\}}{(1+\cos\theta)^4}$$

$$= \frac{(1+\cos\theta)(a\cos\theta) + 2a(1-\cos^2\theta)}{(1+\cos\theta)^3}$$

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$$\frac{d^2r}{d\theta^2} = \frac{(1+\cos\theta)\, a\cos\theta + 2a\,(1+\cos\theta)(1-\cos\theta)}{(1+\cos\theta)^3}$$

$$= \frac{\cancel{(1+\cos\theta)}\{a\cos\theta + 2a\,(1-\cos\theta)\}}{(1+\cos\theta)^{\cancel{3}}}$$

$$= \frac{a\cos\theta + 2a - 2a\cos\theta}{(1+\cos\theta)^2}$$

$$= \frac{2a - a\cos\theta}{(1+\cos\theta)^2}$$

$$\Rightarrow \frac{d^2r}{d\theta^2} = \frac{a\,(2-\cos\theta)}{(1+\cos\theta)^2}$$

Now,

Radius of curvature is given as

$$\rho = \frac{1}{k}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 + r\,\frac{d^2r}{d\theta^2}\right|}$$

$$\rho = \frac{\left[r^2 + \left(\frac{a\sin\theta}{(1+\cos\theta)^2}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{a\sin\theta}{(1+\cos\theta)^2}\right)^2 - r\left(\frac{a(2-\cos\theta)}{(1+\cos\theta)^2}\right)\right|}$$

$$\rho = \frac{\left[\frac{r^2(1+\cos\theta)^4 + a^2\sin^2\theta}{(1+\cos\theta)^4}\right]^{3/2}}{\left|\frac{r^2(1+\cos\theta)^4 + 2a^2\sin^2\theta - ra(2-\cos\theta)(1+\cos\theta)^2}{(1+\cos\theta)^4}\right|}$$

$$\rho = \frac{\frac{\left[r^2(1+\cos\theta)^4 + a^2\sin^2\theta\right]^{3/2}}{(1+\cos\theta)^6}}{\left|\frac{r^2(1+\cos\theta)^4 + 2a^2\sin^2\theta - ra(2-\cos\theta)(1+\cos\theta)^2}{\cancel{(1+\cos\theta)^4}}\right|}$$

$$\rho = \frac{\left[r^2(1+\cos\theta)^4 + a^2\sin^2\theta\right]^{3/2}}{(1+\cos\theta)^2\left[r^2(1+\cos\theta)^4 + 2a^2\sin^2\theta - ra(2-\cos\theta)(1+\cos\theta)^2\right]}$$

Since,

$$r(1+\cos\theta) = a$$

$$1+\cos\theta = \frac{a}{r}$$

$$\cos\theta = \frac{a}{r} - 1$$

$$\cos\theta = \frac{a-r}{r}$$

Using,

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - \left(\frac{a-r}{r}\right)^2$$

$$\sin^2\theta = 1 - \frac{(a-r)^2}{r^2}$$

$$\sin^2\theta = \frac{r^2 - (a^2 - 2ar + r^2)}{r^2}$$

$$\sin^2\theta = \frac{\cancel{r^2} - a^2 + 2ar - \cancel{r^2}}{r^2}$$

$$\sin^2\theta = \frac{2ar - a^2}{r^2}$$

Now,

$$\rho = \frac{\left[r^2\left(\frac{a}{r}\right)^4 + a^2\left(\frac{2ar-a^2}{r^2}\right)\right]^{3/2}}{\left(\frac{a}{r}\right)^2\left[r^2\left(\frac{a}{r}\right)^4 + 2a^2\left(\frac{2ar-a^2}{r^2}\right) - ra(2-\cos\theta)\left(\frac{a}{r}\right)^2\right]}$$

$$\rho = \frac{\left[\cancel{r^2}\left(\frac{a^4}{r^4}\right) + a^2\left(\frac{2ar-a^2}{r^2}\right)\right]^{3/2}}{\frac{a^2}{r^2}\left[r^2 \cdot \frac{a^4}{r^4} + \frac{2a^3}{r^2}(2r-a) - \frac{ra^3}{r^2}(2-\cos\theta)\right]}$$

$$\rho = \frac{\left[\frac{a^4}{r^2} + \frac{a^3}{r^2}(2r-a)\right]^{3/2}}{\frac{a^2}{r^2}\left[\frac{a^4}{r^2} + \frac{2a^3}{r^2}(2r-a) - \frac{ra^3}{r^2}\left\{2 - \frac{a-r}{r}\right\}\right]}$$

$$\rho = \frac{\left[\dfrac{a^4 + a^3(2r-a)}{r^2}\right]^{3/2}}{\dfrac{a^2}{r^2}\cdot\dfrac{a^3}{r^2}\left[a + 2(2r-a) - \cancel{r}\left(\dfrac{2r-a+r}{\cancel{r}}\right)\right]}$$

$$\rho = \frac{(a^3)^{3/2}\dfrac{[\cancel{a} + 2r - \cancel{a}]^{3/2}}{(r^{\cancel{2}})^{3/\cancel{2}}}}{\dfrac{a^5}{r^4}\left[\cancel{a} + 4r - \cancel{2a} - 2r + \cancel{a} - r\right]}$$

$$\rho = \frac{\dfrac{a^{9/2}\,[2r]^{3/2}}{\cancel{r^3}}}{\dfrac{a^5}{\cancel{r^4}}\cdot\cancel{r}}$$

$$\rho = \frac{(2r)^{3/2}}{a^{5-\frac{9}{2}}}$$

$$\rho = \frac{(2r)^{3/2}}{a^{1/2}}$$

$$\rho = \frac{(2)^{3/2}\,(r)^{3/2}}{\sqrt{a}}$$

$$\rho = \frac{\sqrt{8}\,\sqrt{r^3}}{\sqrt{a}}$$

$$\boxed{\rho = \sqrt{\frac{8r^3}{a}}}$$