Exercise 1.2

Question 3

(b)
$$
\begin{array}{c}
\begin{array}{ccccc} w & x & y & z & \end{array} \\
\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}
\end{array}
$$

The linear system is given as

$$w + 8y - 5z = 6$$
$$x + 4y - 9z = 3$$
$$y + z = 2$$

It can also be written as

$$w = 6 + 5z - 8y \rightarrow \text{(i)}$$
$$x = 9z - 4y + 3 \rightarrow \text{(ii)}$$
$$y = 2 - z \rightarrow \text{(iii)}$$

Putting $y = 2-z$ in equ (i) & (ii)

$$w = 5z - 8(2-z) + 6$$

$$x = 9z - 4(2-z) + 3$$

$$y = 2-z$$

Let $z = t$, we have

$$w = 5t - 8(2-t) + 6$$

$$x = 9t - 4(2-t) + 3$$

$$y = 2-t$$

So,

$$w = 5t - 16 + 8t + 6$$

$$x = 9t - 8 + 4t + 3$$

$$y = 2-t$$

Also,

$$\left\{\begin{array}{l} w = -10 + 13t \\ x = -5 + 13t \\ y = 2-t \end{array}\right\}$$

where t is an arbitrary value.

Thus, the original linear system has infinitely many solutions.

(c)
$$\begin{array}{c} \begin{matrix} v & w & x & y & z & \end{matrix} \\ \begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The linear system is given as

$$v + 7w - 2x - 8z = -3$$
$$x + y + 6z = 5$$
$$y + 3z = 9$$

It can also be written as

$$v = -3 - 7w + 2x + 8z \rightarrow \text{(i)}$$
$$x = 5 - y - 6z \rightarrow \text{(ii)}$$
$$y = 9 - 3z \rightarrow \text{(iii)}$$

Putting $y = 9 - 3z$ in equ (ii)

$$v = -3 - 7w + 2x + 8z$$
$$x = 5 - (9 - 3z) - 6z$$
$$y = 9 - 3z$$

Also,

$$v = -3 - 7w + 2x + 8z$$
$$x = 5 - 9 + 3z - 6z = -4 - 3z$$
$$y = 9 - 3z$$

Putting $x = -4 - 3z$ in equ (i)

$$v = -3 - 7w + 2(-4 - 3z) + 8z$$
$$= -3 - 7w - 8 - 6z + 8z$$
$$= -11 - 7w + 2z$$

Thus,

$$v = -11 - 7w + 2z$$
$$x = -4 - 3z$$
$$y = 9 - 3z$$

Let $w = s$ and $z = t$

$$\begin{cases} v = -11 - 7s + 2t \\ x = -4 - 3t \\ y = 9 - 3t \end{cases}$$

where, s and t are arbitrary values

Therefore, the original linear system has infinitely many solutions.

Question 4

(a)
$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix} \end{array}$$

The linear system is given as

$$\begin{cases} x = -3 \\ y = 0 \\ z = 7 \end{cases}$$

Thus, the original linear system has a unique solution.

(d) $\begin{array}{cccc} x & y & z & \\ \end{array}$
$$\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The linear system is given as

$$\left\{ \begin{aligned} x - 3y &= 0 \\ z &= 0 \\ 0x + 0y + 0z &= 1 \end{aligned} \right\}$$

The system is inconsistent since the third equation is contradictory.

Gaussian Elimination

Question 5:

$$x_1 + x_2 + 2x_3 = 8$$
$$-x_1 - 2x_2 + 3x_3 = 1$$
$$3x_1 - 7x_2 + 4x_3 = 10$$

The augmented matrix for system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$R_1 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

$$-3R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$(-1)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$10R_2 + R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$(-1/52)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$x_1 + x_2 + 2x_3 = 8$$

$$x_2 - 5x_3 = -9$$

$$x_3 = 2$$

It can also be written as

$$x_1 = 8 - x_2 - 2x_3 \rightarrow \text{(i)}$$
$$x_2 = 5x_3 - 9 \rightarrow \text{(ii)}$$
$$x_3 = 2$$

Putting $x_3 = 2$ in (ii) and (i)

$$x_1 = 8 - x_2 - 2(2) = 4 - x_2 \rightarrow \text{(iii)}$$
$$x_2 = 5(2) - 9 = 1$$
$$x_3 = 2$$

Putting $x_2 = 1$ in (iii)

$$\left\{\begin{aligned} x_3 &= 2 \\ x_2 &= 1 \\ x_1 &= 4 - 1 = 3 \end{aligned}\right\}$$

Hence, the linear has a unique solution:
$(3, 1, 2)$

Question 7:

$$x - y + 2z - w = -1$$
$$2x + y - 2z - 2w = -2$$
$$-x + 2y - 4z + w = 1$$
$$3x - 3w = -3$$

The augmented matrix for system is

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

$$R_1 + R_3$$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

$$-2R_1 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

$$-3R_1 + R_4$$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$\left(\frac{1}{3}\right) R_2$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$(-1) R_2 + R_3$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$(-3) R_2 + R_4$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system of equations corresponding to this augment matrix in row echelon form is

$$x - y + 2z - w = -1$$
$$y - 2z = 0$$
$$0 = 0$$
$$0 = 0$$

It can also be written as

$$x = y - 2z + w - 1 \longrightarrow \text{(i)}$$

$$y = 2z$$

Putting $y = 2z$ in equ (i)

$$x = 2z - 2z + w - 1$$

$$= w - 1$$

Thus,

$$x = w - 1$$

$$y = 2z$$

Let, $w = s$ and $z = t$

$$\begin{Bmatrix} x = s - 1 \\ y = 2t \end{Bmatrix}$$

where, s and t are arbitrary values.
Hence, the linear system has infinitely many solutions.

Gauss - Jordan Elimination

Question 10:

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$-2x_1 + 5x_2 + 2x_3 = 1$$

$$8x_1 + x_2 + 4x_3 = -1$$

The augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

$\left(\frac{1}{2}\right) R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

$2R_1 + R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 4 & 4 & -1 \end{array}\right]$$

$-8R_1 + R_3$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right]$$

$\left(\frac{1}{7}\right) R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & -7 & -4 & -1 \end{array}\right]$$

$7R_2 + R_3$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(-1) R_2 + R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system of equations corresponding to this augmented matrix in row reduced echelon form is

$$x_1 + \frac{3}{7}x_3 = -1/7$$

$$x_2 + 4/7\, x_3 = 1/7$$

It can also be written as

$$x_1 = \frac{-1}{7} - \frac{3}{7}x_3$$

$$x_2 = \frac{1}{7} - \frac{4}{7}x_3$$

Let, $t = x_3$ we have

$$\left\{ \begin{array}{l} x_1 = \dfrac{-1}{7} - \dfrac{3}{7}t \\[2ex] x_2 = \dfrac{1}{7} - \dfrac{4}{7}t \end{array} \right\}$$

where, t is an arbitrary value.

Hence, the linear system has infinitely many solutions.

Question 12:

$$-2b + 3c = 1$$
$$3a + 6b - 3c = -2$$
$$6a + 6b + 3c = 5$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$\underset{\sim}{R_{12}} \left[\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$(1/3)\ R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$-6R_1 + R_3$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$(-1/2)\ R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$6R_2 + R_3$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

$(1/6)\, R_3$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$(1/2)\, R_3 + R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\left(\frac{2}{3}\right) R_3 + R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$-2R_2 + R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The equations corresponding to this augmented matrix in row reduced echelon form is

$$x_1 + 2x_3 = 0$$
$$(-3/2)\, x_3 = 0$$
$$0 = 1$$

The system is inconsistent because the third equation is contradictory.

Solve the linear system by any method.

Question 15:

$$2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 = 0$$
$$x_2 + x_3 = 0$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_{12}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$-2R_1 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left(-\tfrac{1}{3}\right) R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$(-1) R_2 + R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left(\tfrac{1}{2}\right) R_3$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 + R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$-2R_2 + R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The equations corresponding to this augmented matrix in row reduced echelon form are:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0$$

or,
$$(0, 0, 0)$$

Question 18:

$$v + 3w - 2x = 0$$
$$2u + v - 4w + 3x = 0$$
$$2u + 3v + 2w - x = 0$$
$$-4u - 3v + 5w - 4x = 0$$

The augmented matrix for this system is

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array}\right]$$

$$\underset{\sim}{R_{12}} \left[\begin{array}{cccc|c} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array}\right]$$

$$\left(\tfrac{1}{2}\right) R_1$$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array}\right]$$

$$-2R_1 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array}\right]$$

$$+4R_1 + R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ 0 & -1 & -3 & 2 & 0 \end{array}\right]$$

$$-2R_2 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 2 & 0 \end{array}\right]$$

$$R_2 + R_4$$

$$\underset{\sim}{R}\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\frac{-1}{2}\right) R_2 + R_1$$

$$\underset{\sim}{R} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The linear equations corresponding to this augmented matrix in row reduced echelon form are:

$$x_1 - \frac{7}{2}x_3 + \frac{5}{2}x_4 = 0$$

$$x_2 + 3x_3 - 2x_4 = 0$$

It can also be written as

$$x_1 = 7/2\, x_3 - 5/2\, x_4$$

$$x_2 = 2x_4 - 3x_3$$

Let, $x_3 = s$ and $x_4 = t$

$$\left\{ \begin{array}{l} x_1 = \frac{7}{2}s - \frac{5}{2}t \\ \\ x_2 = 2t - 3s \end{array} \right\}$$

where, s and t are arbitrary values.