

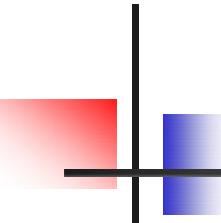
Lecture 4

Chapter 1. The Foundations

- Predicates and Quantifiers
- Nested Quantifiers

Previously...

- In predicate logic, a ***predicate*** is modeled as a ***propositional function* $P(\cdot)$** from subjects to propositions.
 - $P(x)$: “ x is a prime number” (x : any subject)
 - $P(3)$: “3 is a prime number.” (proposition!)
- Propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take
 - $P(x,y,z)$: “***x*** gave ***y*** the grade ***z***”
 - $P(\text{Mike},\text{Mary},\text{A})$: “**Mike** gave **Mary** the grade **A**.”



Universe of Discourse (U.D.)

- The power of distinguishing subjects from predicates is that it lets you state things about *many* objects at once.
- e.g., let $P(x) = "x + 1 > x"$. We can then say, “For **any** number x , $P(x)$ is true” instead of $(0 + 1 > 0) \wedge (1 + 1 > 1) \wedge (2 + 1 > 2) \wedge \dots$
- The collection of values that a variable x can take is called x ’s ***universe of discourse*** or the ***domain of discourse*** (often just referred to as the ***domain***).



Quantifier Expressions

- **Quantifiers** provide a notation that allows us to *quantify* (count) *how many* objects in the universe of discourse satisfy the given predicate.
- “**forall**” is the FORALL or *universal* quantifier.
 $\forall x P(x)$ means for all x in the domain, $P(x)$.
- “**exists**” is the EXISTS or *existential* quantifier.
 $\exists x P(x)$ means there exists an x in the domain (that is, 1 or more) such that $P(x)$.



The Universal Quantifier

- $\forall x P(x)$: *For all x in the domain, $P(x)$.*
- $\forall x P(x)$ is
 - *true* if $P(x)$ is true for every x in D (D : domain of discourse)
 - *false* if $P(x)$ is false for at least one x in D
 - For every real number x , $x^2 \geq 0$ TRUE
 - For every real number x , $x^2 - 1 > 0$ FALSE
- A **counterexample** to the statement $\forall x P(x)$ is a value x in the domain D that makes $P(x)$ false
- What is the truth value of $\forall x P(x)$ when the domain is empty? TRUE

The Universal Quantifier

- If all the elements in the domain can be listed as x_1, x_2, \dots, x_n then, $\forall x P(x)$ is the same as the conjunction:

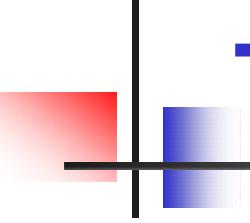
$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

- Example: Let the domain of x be parking spaces at UH. Let $P(x)$ be the statement “ x is full.” Then the ***universal quantification*** of $P(x)$, $\forall x P(x)$, is the *proposition*:
 - “All parking spaces at UH are full.”
 - or “Every parking space at UH is full.”
 - or “For each parking space at UH, that space is full.”

The Existential Quantifier \exists

- $\exists x P(x)$: *There exists an x in the domain (that is, 1 or more) such that $P(x)$.*
- $\exists x P(x)$ is
 - *true* if $P(x)$ is true for at least one x in the domain
 - *false* if $P(x)$ is false for every x in the domain
- What is the truth value of $\exists x P(x)$ when the domain is empty? FALSE
- If all the elements in the domain can be listed as x_1, x_2, \dots, x_n then, $\exists x P(x)$ is the same as the disjunction:

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$



The Existential Quantifier \exists

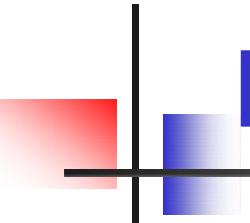
- Example:

Let the domain of x be parking spaces at UH.

Let $P(x)$ be the statement “ x is full.”

Then the ***existential quantification*** of $P(x)$,
 $\exists x P(x)$, is the *proposition*:

- “Some parking spaces at UH are full.”
- or “There is a parking space at UH that is full.”
- or “At least one parking space at UH is full.”



Free and Bound Variables

- An expression like $P(x)$ is said to have a ***free variable*** x (meaning, x is undefined).
- A quantifier (either \forall or \exists) *operates* on an expression having one or more free variables, and ***binds*** one or more of those variables, to produce an expression having one or more ***bound variables***.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y .
- $\forall x P(x,y)$ has 1 free variable  and one bound variable.  [Which is which?]
- “ $P(x)$, where $x = 3$ ” is another way to bind x .
- An expression with zero free variables is a bona-fide (actual) proposition.
- An expression with one or more free variables is not a proposition:

e.g. $\forall x P(x,y) = Q(y)$

- Sometimes the universe of discourse is restricted within the quantification, e.g.,
 - $\forall x > 0 P(x)$ is shorthand for
“For all x that are greater than zero, $P(x)$.
= $\forall x (x > 0 \rightarrow P(x))$
 - $\exists x > 0 P(x)$ is shorthand for
“There is an x greater than zero such that $P(x)$.
= $\exists x (x > 0 \wedge P(x))$

Translating from English

- Express the statement “*Every student in this class has studied calculus*” using predicates and quantifiers.
 - Let $C(x)$ be the statement: “*x has studied calculus.*”
 - If domain for x consists of the students in this class, then
 - it can be translated as $\forall x C(x)$

or

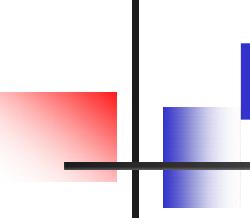
- If domain for x consists of all people
- Let $S(x)$ be the predicate: “*x is in this class*”
- Translation: $\forall x (S(x) \rightarrow C(x))$

Translating from English

- Express the statement “*Some students in this class has visited Mexico*” using predicates and quantifiers.
 - Let $M(x)$ be the statement: “*x has visited Mexico*”
 - If domain for x consists of the students in this class, then
 - it can be translated as $\exists x M(x)$
or
 - If domain for x consists of all people
 - Let $S(x)$ be the statement: “*x is in this class*”
 - Then, the translation is $\exists x (S(x) \wedge M(x))$

Translating from English

- Express the statement “*Every student in this class has visited either Canada or Mexico*” using predicates and quantifiers.
 - Let $C(x)$ be the statement: “ x has visited Canada” and $M(x)$ be the statement: “ x has visited Mexico”
 - If domain for x consists of the students in this class, then
 - it can be translated as $\forall x (C(x) \vee M(x))$



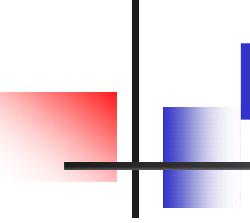
Negations of Quantifiers

- $\forall x P(x)$: “Every student in the class has taken a course in calculus” ($P(x)$: “ x has taken a course in calculus”)
 - “Not every student in the class ... calculus”
 $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- Consider $\exists x P(x)$: “There is a student in the class who has taken a course in calculus”
 - “There is no student in the class who has taken a course in calculus”
 $\neg \exists x P(x) \equiv \forall x \neg P(x)$

Negations of Quantifiers

- Definitions of quantifiers: If the domain = { a, b, c, \dots }
 - $\forall x P(x) \equiv P(a) \wedge P(b) \wedge P(c) \wedge \dots$
 - $\exists x P(x) \equiv P(a) \vee P(b) \vee P(c) \vee \dots$
- From those, we can prove the laws:
 - $\neg \forall x P(x) \equiv \neg(P(a) \wedge P(b) \wedge P(c) \wedge \dots)$
 $\equiv \neg P(a) \vee \neg P(b) \vee \neg P(c) \vee \dots$
 $\equiv \exists x \neg P(x)$
 - $\neg \exists x P(x) \equiv \neg(P(a) \vee P(b) \vee P(c) \vee \dots)$
 $\equiv \neg P(a) \wedge \neg P(b) \wedge \neg P(c) \wedge \dots$
 $\equiv \forall x \neg P(x)$
- Which *propositional* equivalence law was used to prove this?

DeMorgan's
Demorgan's



Negations of Quantifiers

Theorem:

- **Generalized De Morgan's laws for logic**

$$1. \neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$2. \neg \exists x P(x) \equiv \forall x \neg P(x)$$

Negations: Examples

- What are the negations of the statements $\forall x (x^2 > x)$ and $\exists x (x^2 = 2)$?
 - $\neg \forall x (x^2 > x) \equiv \exists x \neg(x^2 > x) \equiv \exists x (x^2 \leq x)$
 - $\neg \exists x (x^2 = 2) \equiv \forall x \neg(x^2 = 2) \equiv \forall x (x^2 \neq 2)$
- Show that $\neg \forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.
 - $\neg \forall x(P(x) \rightarrow Q(x)) \equiv \exists x \neg(P(x) \rightarrow Q(x))$
 $\equiv \exists x \neg(\neg P(x) \vee Q(x))$
 $\equiv \exists x (P(x) \wedge \neg Q(x))$

Summary

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TABLE 1 Quantifiers.

<i>Statement</i>	<i>When True?</i>	<i>When False?</i>
$\forall x P(x)$	$P(x)$ is true for every x .	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x .

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TABLE 2 De Morgan's Laws for Quantifiers.

<i>Negation</i>	<i>Equivalent Statement</i>	<i>When Is Negation True?</i>	<i>When False?</i>
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x , $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x .

Nesting of Quantifiers

■■ Example:

Let the domain of x and y be people.

Let $L(x,y)$ = “ x likes y ” (A statement with 2 free variables – not a proposition)

■■ Then $\exists y L(x,y)$ = “There is someone whom x likes.” (A statement with 1 free variable x – not a proposition)

■■ Then $\forall x (\exists y L(x,y))$ =
“Everyone has someone whom they like.”

(A ~~Proposition~~ with ~~free~~ variables.)

Nested Quantifiers

- Nested quantifiers are quantifiers that occur within the scope of other quantifiers.
- The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

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TABLE 1 Quantifications of Two Variables.

<i>Statement</i>	<i>When True?</i>	<i>When False?</i>
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y .	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y .
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y .	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y .

Nested Quantifiers

- Let the domain of x and y is \mathbb{R} , and $P(x,y)$: $xy = 0$.

Find the truth value of the following propositions.

$$\exists x \forall y P(x, y) \quad (\text{T})$$

$$\forall x \exists y P(x, y) \quad (\text{T})$$

$$\exists x \forall y P(x, y) \quad (\text{T})$$

$$\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y)$$

$$\exists x \forall y P(x, y), \text{ for every } x, \text{ there exists } y \text{ such that } x + y = 0 \quad (\text{T})$$

$$\exists x \forall y P(x, y), \text{ There exists } y \text{ such that, for every } x, x + y = 0.$$

$$(\text{F})$$

\mathbb{R} : set of real numbers

Nested Quantifiers: Example

■■ Let the domain = {1, 2, 3}. Find an expression equivalent to $\forall x \exists y P(x,y)$ where the variables are bound by substitution instead:

- Expand from inside out or outside in.
- Outside in:

$$\forall x \exists y P(x,y)$$

$$\equiv \exists y P(1,y) \wedge \exists y P(2,y) \wedge \exists y P(3,y)$$

$$\equiv [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge$$

$$[P(2,1) \vee P(2,2) \vee P(2,3)] \wedge$$

$$[P(3,1) \vee P(3,2) \vee P(3,3)]$$

Quantifier Exercise

- If $R(x,y)$ = “ x relies upon y ,” express the following in unambiguous English when the domain is all people

$$\forall x(\exists y R(x,y)) =$$

Everyone has *someone* to rely on.

$$\exists y(\forall x R(x,y)) =$$

There's a poor overburdened soul whom *everyone* relies upon (including himself)!

$$\exists x(\forall y R(x,y)) =$$

There's some needy person who relies upon *everybody* (including

$$\forall y(\exists x R(x,y)) =$$

himself). Everyone has *someone* who relies upon them.

$$\forall x(\forall y R(x,y)) =$$

Everyone relies upon *everybody*, (including themselves)!

Negating Nested Quantifiers

- Successively apply the rules for negating statements involving a single quantifier
- Example: Express the negation of the statement

$\forall x \exists y (P(x,y) \wedge \exists z R(x,y,z))$ so that all
negation symbols immediately precede
predicates.

$$\begin{aligned} & \neg \forall x \exists y (P(x,y) \wedge \exists z R(x,y,z)) \\ & \equiv \exists x \neg \exists y (P(x,y) \wedge \exists z R(x,y,z)) \\ & \equiv \exists x \forall y \neg (P(x,y) \wedge \exists z R(x,y,z)) \\ & \equiv \exists x \forall y (\neg P(x,y) \vee \neg \exists z R(x,y,z)) \end{aligned}$$

Equivalence Laws

■■ $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

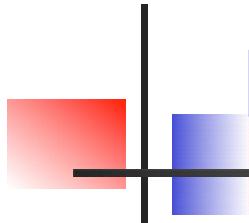
$$\exists x \exists y P(x,y) \equiv \exists y \exists x P(x,y)$$

■■ $\forall x (P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$

$$\exists x (P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$$

■■ Exercise:

See if you can prove these yourself.



Notational Conventions

- Quantifiers have higher precedence than all logical operators from propositional logic:

$$(\forall x P(x)) \wedge Q(x)$$

- Consecutive quantifiers of the same type can be combined:

$$\forall x \forall y \forall z P(x,y,z) \equiv \forall x,y,z$$

$$P(x,y,z) \text{ or even } \forall xyz P(x,y,z)$$