DAY: DATE:

Double Integral as Volume

Fubini's Theorem (1st Form)

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b, \; c \le y \le d$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx$$

It is called Fubini's Theorem (1st Form).

E1) Calculate $\iint_R f(x,y)\, dA$ for

$f(x,y) = 100 - 6x^2y$

and $R: 0 \le x \le 2, \; -1 \le y \le 1$

As we know that

$$= \iint_R f(x,y)\, dA$$

$$= \iint_R (100 - 6x^2y)\, dA$$

$$= \int_{-1}^{1}\int_0^2 (100 - 6x^2y)\, dx\, dy$$

$$= \int_{-1}^{1} \left[100x - \frac{\cancel{6}^{2}x^3y}{\cancel{3}}\right]_0^2 dy$$

$$= \int_{-1}^{1} \left[100x - 2x^3y\right]_0^2 dy$$

DAY: DATE:

$$= \int_{-1}^{1} [(200 - 2(8)y) - (0-0)] \cdot dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy$$

$$= \left[200y - \frac{16y^2}{2}\right]_{-1}^{1}$$

$$= [200y - 8y^2]_{-1}^{1}$$

$$= [200(1) - 8(1)^2] - [200(-1) - 8(-1)^2]$$

$$= 200 - 8 - (-200 - 8)$$

$$= 192 - (-208)$$

$$= 192 + 208$$

$$= 400$$

E2) Find the volume of region bounded by

$$z = 10 + x^2 + 3y^2$$

$$R: \quad 0 \le x \le 1$$

$$0 \le y \le 2$$

As we know that

$$V = \iint_R f(x,y)\, dA$$

$$V = \iint_R (10 + x^2 + 3y^2)\, dA$$

$$V = \int_0^2 \int_0^1 (10 + x^2 + 3y^2)\, dx\, dy$$

$$V = \int_0^2 \left[10x + \frac{x^3}{3} + 3y^2 x\right]_0^1 dy$$

$$V = \int_0^2 \left(10 + \frac{1}{3} + 3y^2\right) dy$$

$$V = \int_0^2 \left(\frac{30 + 1 + 9y^2}{3}\right) dy$$

$$V = \frac{1}{3} \int_0^2 (31 + 9y^2)\, dy$$

DAY: DATE:

$$V = \frac{1}{3}\left[31y + \frac{\cancel{9}y^3}{\cancel{3}}\right]_0^2$$

$$V = \frac{1}{3}\left(31(2) + 3(2)^3\right)$$

$$V = \frac{1}{3}\left(62 + 3(8)\right)$$

$$V = \frac{1}{3}\left(62 + 24\right)$$

$$V = \frac{86}{3}$$

Exercise

15.1

Evaluating Iterated Integrals.

1 3 5

Q7) $\displaystyle\int_0^1\int_0^1 \frac{y}{1+xy}\,dx\,dy$

$$= \int_0^1 \left[\ln(1+xy)\right]_0^1 dy$$

$$= \int_0^1 \left[\ln(1+y) - \ln(1+0)\right] dy$$

$$= \int_0^1 \left[\ln(1+y) - \ln(1)\right] dy$$

$$= \int_0^1 \left[\ln(1+y) - 0\right] dy$$

$$= \int_0^1 \ln(1+y)\, dy$$

$$= \int_0^1 \ln(1+y) \cdot 1\, dy$$

$$= \left[\ln(1+y) \cdot (y)\right]_0^1 - \int_0^1 (y) \cdot \left(\frac{1}{1+y}\right) dy$$

DAY:_____ DATE:_____

$$= \left[y\ln(1+y)\right]_0^1 - \int_0^1 \frac{y}{1+y}\,dy$$

$$= \left[y\ln(1+y)\right]_0^1 - \int_0^1 \frac{1+y-1}{1+y}\,dy$$

$$= \left[y\ln(1+y)\right]_0^1 - \int_0^1 \left(\frac{1+y}{1+y} - \frac{1}{1+y}\right)dy$$

$$= \left[y\ln(1+y)\right]_0^1 - \int_0^1 \left(1 - \frac{1}{1+y}\right)dy$$

$$= \left[y\ln(1+y)\right]_0^1 - \left[y - \ln(1+y)\right]_0^1$$

$$= \left[y\ln(1+y) - y + \ln(1+y)\right]_0^1$$

$$= (1)\ln(2) - 1 + \ln(2)$$

$$= \ln(2) + \ln(2) - 1$$

$$= 2\ln 2 - 1$$

Q9) $\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$

As we know that

$$= \int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$$

$$= \int_0^{\ln 2}\left[e^{2x+y}\right]_1^{\ln 5}dx$$

$$= \int_0^{\ln 2}\left(e^{2x+\ln 5} - e^{2x+1}\right)dx$$

$$= \int_0^{\ln 2}\left(e^{2x}\cdot e^{\ln 5} - e^{2x+1}\right)dx$$

$$= \int_0^{\ln 2}\left(5e^{2x} - e^{2x+1}\right)dx$$

$$= \left[\frac{5e^{2x}}{2} - \frac{e^{2x+1}}{2}\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left[5e^{2x} - e^{2x+1}\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left[\left(5e^{2\ln 2} - e^{2\ln 2+1}\right) - \left(5e^{0} - e^{1}\right)\right]$$

DAY: DATE:

$$= \frac{1}{2}\left[5(e^{\ln 2})^2 - e^{2\ln 2+1} - 5(1) + e\right]$$

$$= \frac{1}{2}\left[5(2)^2 - e^{2\ln 2}\cdot e - 5 + e\right] \quad \because a^{bc} = (a^b)^c$$

$$= \frac{1}{2}\left(5(4) - (e^{\ln 2})^2 \cdot e - 5 + e\right)$$

$$= \frac{1}{2}\left(20 - (2)^2 \cdot e - 5 + e\right)$$

$$= \frac{1}{2}\left(20 - 4e - 5 + e\right)$$

$$= \frac{1}{2}\left(15 - 3e\right)$$

$$= \frac{15}{2} - \frac{3e}{2}$$

$$= \frac{3}{2}(5 - e)$$

Q11) $\int_{-1}^{2}\int_{0}^{\pi/2} y \sin x \, dx\, dy$

As we know that

$$= \int_{-1}^{2}\int_{0}^{\pi/2} y \sin x \, dx\, dy$$

$$= \int_{-1}^{2} y \int_{0}^{\pi/2} \sin x \, dx\, dy$$

$$= \int_{-1}^{2} y \left[-\cos x\right]_{0}^{\pi/2} dy$$

$$= \int_{-1}^{2} -y \left[\cos x\right]_{0}^{\pi/2} dy$$

$$= \int_{-1}^{2} -y \left[\cos\left(\tfrac{\pi}{2}\right) - \cos 0\right] dy$$

$$= \int_{-1}^{2} -y\,(0-1)\, dy$$

$$= \int_{-1}^{2} y \, dy$$

$$= \left[\frac{y^2}{2}\right]_{-1}^{2}$$

DAY: DATE:

$$= \frac{1}{2}\left[y^2\right]_{-1}^{2}$$

$$= \frac{1}{2}(4-1)$$

$$= \frac{1}{2}(3)$$

$$= \frac{3}{2}$$

Q13) $\int_1^4 \int_1^e \frac{\ln x}{xy}\, dx\, dy$

As we know that

$$= \int_1^4 \int_1^e \frac{\ln x}{xy}\, dx\, dy$$

$$= \int_1^4 \frac{1}{y} \int_1^e (\ln x)\left(\frac{1}{x}\right) dx\, dy$$

$$= \int_1^4 \frac{1}{y}\left[\frac{(\ln x)^2}{2}\right]_1^e dy$$

$$= \int_1^4 \frac{1}{y}\cdot\frac{1}{2}\left[(\ln x)^2\right]_1^e dy$$

$$= \int_1^4 \frac{1}{2y}\left[(\ln e)^2 - (\ln 1)^2\right] dy$$

$$= \int_1^4 \frac{1}{2y}\left[(1)^2 - (0)^2\right] dy$$

$$= \frac{1}{4}\int_1^4 (1)\, dy$$

$$= \frac{1}{4}\left[y\right]_1^4$$

DAY: DATE:

$$= \int_1^4 \frac{1}{2y}(1)\,dy$$

$$= \int_1^4 \frac{1}{2y}\,dy$$

$$= \frac{1}{2}\int_1^4 \frac{1}{y}\,dy$$

$$= \frac{1}{2}\left[\ln y\right]_1^4$$

$$= \frac{1}{2}\left[\ln 4 - \ln 1\right]$$

$$= \frac{1}{2}(\ln 4 - 0)$$

$$= \frac{1}{2}(\ln (2)^2)$$

$$= \frac{1}{\cancel{2}}(\cancel{2}\ln 2)$$

$$= \ln 2$$

Evaluating Double Integrals over Rectangles

15

Q17) $\iint_R xy\cos y\,dA$,

$R: \quad -1 \le x \le 1$

$\quad\quad 0 \le y \le \pi$

As we know that

$$= \iint_R xy\cos y\,dA$$

$$= \int_0^{\pi}\int_{-1}^{1} xy\cos y\,dx\,dy$$

DAY: DATE:

$$= \int_0^{\pi} y\cos y \int_{-1}^{1} x \, dx\,dy$$

$$= \int_0^{\pi} y\cos y \left[\frac{x^2}{2}\right]_{-1}^{1} dy$$

$$= \frac{1}{2}\int_0^{\pi} y\cos y \left[(1)^2 - (-1)^2\right] dy$$

$$= \frac{1}{2}\int_0^{\pi} y\cos y \,(1-1)\, dy$$

$$= \frac{1}{2}\int_0^{\pi} y\cos y \,(0)\, dy$$

$$= \frac{1}{2}(0)$$

$$= 0$$

(19)

Q21) $\iint_R \frac{xy^3}{x^2+1}\, dA$

R: $0 \le x \le 1$

$0 \le y \le 2$

As we know that

$$= \iint_R \frac{xy^3}{x^2+1}\, dA$$

$$= \int_0^2 \int_0^1 \frac{xy^3}{x^2+1}\, dx\,dy$$

$$= \int_0^2 y^3 \int_0^1 \frac{x}{x^2+1}\, dx\,dy$$

$$= \int_0^2 y^3 \cdot \frac{1}{2} \int_0^1 \frac{2x}{x^2+1}\, dx\,dy$$

DAY: DATE:

$$= \int_0^2 \frac{y^3}{2} \left[\ln(x^2+1)\right]_0^1 dy$$

$$= \frac{1}{2}\int_0^2 y^3 \left[\ln(1^2+1) - \ln(0+1)\right] dy$$

$$= \frac{1}{2}\int_0^2 y^3 (\ln 2 - \ln 1)\, dy$$

$$= \frac{1}{2}\int_0^2 y^3 (\ln 2 - 0)\, dy$$

$$= \frac{1}{2}\int_0^2 y^3 \ln 2\, dy$$

$$= \frac{\ln 2}{2}\int_0^2 y^3\, dy$$

$$= \frac{\ln 2}{2}\left[\frac{y^4}{4}\right]_0^2$$

$$= \frac{\ln 2}{2(4)}\left[y^4\right]_0^2$$

$$= \frac{\ln 2}{8}\left[(2)^4 - (0)^2\right]$$

$$= \frac{\ln 2}{\cancel{8}}(\cancel{16}^{2})$$

$$= 2\ln 2$$

Q23) $f(x,y) = \dfrac{1}{xy}$ over the square $1 \le x \le 2$, $0 \le y \le 2$.

As we know that

$$V = \iint_R f(x,y)\, dA$$

DAY: DATE:

$$V = \int_0^2 \int_1^2 \frac{1}{xy} \, dx\,dy$$

$$V = \int_0^2 \frac{1}{y} \int_1^2 \frac{1}{x} \, dx\,dy$$

$$V = \int_0^2 \frac{1}{y} \left[\ln(x)\right]_1^2 dy$$

$$V = \int_0^2 \frac{1}{y} (\ln 2 - \ln 1) \, dy$$

$$V = \ln 2 \int_0^2 \frac{1}{y} \, dy$$

$$V = \ln 2 \left[\ln(y)\right]_0^2$$

$$V = \ln 2 (\ln 2 - \ln 0)$$

$$V = \ln 2 (\ln 2)$$

$$V = (\ln 2)^2$$

Q25) $z = x^2 + y^2$

$R: \; -1 \le x \le 1$

$\quad -1 \le y \le 1$

As we know that

$$V = \iint f(x,y) \, dA$$

$$V = \int_{-2}^{1} \int_{-1}^{R} (x^2 + y^2) \, dx\,dy$$

$$V = \int_{-1}^{1} \left[\frac{x^3}{3} + xy^2\right]_{-1}^{1} dy$$

$$V = \int_{-1}^{1} \left[\left(\frac{1}{3} + y^2\right) - \left(\frac{-1}{3} + (-1)y^2\right)\right] dy$$

$$V = \int_{-1}^{1} \left(\frac{1}{3} + y^2 + \frac{1}{3} + y^2\right) dy$$

DAY: DATE:

$$V = \int_{-1}^{1} \left(\frac{2}{3} + 2y^2 \right) dy$$

$$V = \left[\frac{2}{3}y + \frac{2y^3}{3} \right]_{-1}^{1}$$

$$V = \left(\frac{2}{3}(1) + \frac{2}{3}(1)^3 \right) - \left(\frac{2}{3}(-1) + \frac{2}{3}(-1)^3 \right)$$

$$V = \frac{2}{3} + \frac{2}{3} - \left(\frac{-2}{3} - \frac{2}{3} \right)$$

$$V = \frac{4}{3} - \left(\frac{-4}{3} \right)$$

$$V = \frac{4}{3} + \frac{4}{3}$$

$$V = \frac{8}{3}$$

27

Q29) $z = 2\sin x \cos y$

$R: \; 0 \le x \le \pi/2$

$\quad\;\; 0 \le y \le \pi/4$

As we know that

$$V = \iint_R f(x,y)\, dA$$

$$V = \int_0^{\pi/4} \int_0^{\pi/2} (2\sin x \cos y)\, dx\, dy$$

$$V = \int_0^{\pi/4} 2\cos y \int_0^{\pi/2} \sin x \, dx\, dy$$

$$V = \int_0^{\pi/4} 2\cos y \left[-\cos x \right]_0^{\pi/2} dy$$

$$V = \int_0^{\pi/4} -2\cos y \left[(\cos \pi/2) - (\cos 0) \right] dy$$

$$V = -2 \int_0^{\pi/4} \cos y \, (0 - 1)\, dy$$

$$V = 2 \int_0^{\pi/4} \cos y \, dy$$

$$V = 2 \left[\sin y \right]_0^{\pi/4}$$

DAY: DATE:

$$V = 2\left(\sin(\pi/4) - \sin(0)\right)$$

$$V = 2\left(\frac{1}{\sqrt{2}}\right)$$

$$V = \frac{2}{\sqrt{2}}$$

$$V = \frac{(\sqrt{2})^2}{\sqrt{2}}$$

$$V = \sqrt{2}$$

Q33) $\displaystyle\int_0^2 \int_0^1 \frac{x}{1+xy}\,dx\,dy$

$$= \int_0^2 \int_0^1 \frac{x}{1+xy}\,dx\,dy$$

$$= \int_0^1 \int_0^2 \frac{x}{1+xy}\,dy\,dx$$

$$= \int_0^1 \left[\ln(1+xy)\right]_0^2 dx$$

$$= \int_0^1 \left[\ln(1+2x) - \ln(1+0)\right] dx$$

$$= \int_0^1 \ln(1+2x)\,dx$$

$$= \int_0^1 (\ln(1+2x) \cdot 1)\,dx$$

$$= \ln(1+2x)(x) - \int_0^1 \left(\frac{1}{1+2x}\right)(2)(x)\,dx$$

$$= \ln(1+2x)(x) - \int_0^1 \frac{2x}{1+2x}\,dx$$

$$= 2x\ln(1+2x) - \int_0^1 \frac{1+2x-1}{1+2x}\,dx$$

$$= x\ln(1+2x) - \int_0^1 \left(\frac{1+2x}{1+2x} - \frac{1}{1+2x}\right) dx$$

DAY:

DATE:

$$= x\ln(1+2x) - \int_0^1 \left(1 - \frac{1}{1+2x}\right) dx$$

$$= x\ln(1+2x) - \frac{1}{2}\int_0^1 \left(2 - \frac{2}{1+2x}\right) dx$$

$$= x\ln(1+2x) - \frac{1}{2}\left[2x - \ln(1+2x)\right]_0^1$$

$$= \left[x\ln(1+2x) - \frac{2x}{2} + \frac{\ln(1+2x)}{2}\right]_0^1$$

$$= \left[x\ln(1+2x) - x + \frac{\ln(1+2x)}{2}\right]_0^1$$

$$= \left((1)\ln(1+2) - 1 + \frac{\ln(1+2)}{2}\right) - 0$$

$$= \ln(3) - 1 + \frac{\ln(3)}{2}$$

$$= \frac{2\ln 3 - 2 + \ln 3}{2}$$

$$= \frac{3\ln 3 - 2}{2}$$

$$= \frac{3\ln 3}{2} - \frac{\cancel{2}}{\cancel{2}}$$

$$= \frac{3\ln 3}{2} - 1$$