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Vector Fields

Vector Fields

⇒ A vector field in a plane is a function that associates with each point P in the plane a unique vector $\vec{F}(P)$ parallel to the plane.

⇒ Similarly, a vector field in 3-space is a function that associates with each point P in 3-space a unique vector $\vec{F}(P)$ in the 3-space.

Inverse Square Field

If $\vec{r}$ is a radius vector in 2-space or 3-space, and if c is a constant, then a vector field of the form

$$\vec{F}(\vec{r}) = \frac{c}{\|\vec{r}\|^3}\vec{r}$$

is called an inverse-square field.

Gradient Fields

An important class of vector fields arises from the process of finding gradients. If ϕ is a function of

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three variables, then the gradient of ϕ is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

This formula defines a vector field in 3-space called the gradient field of ϕ.

Conservative Field

A vector field $\vec{F}$ in 2-space or 3-space is said to be conservative in a region if it is the gradient field for some function ϕ in that region, that is, if

$$\vec{F} = \nabla\phi$$

The function ϕ is called a potential function for $\vec{F}$ in the region.

Divergence

If $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$, then we define the divergence of $\vec{F}$, written div $\vec{F}$, to be the function given by

$$\text{div}\,\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

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Curl

If $\vec{F}(x,y,z) = f(x,y,z)\hat{i} + f(x,y,z)\hat{j} + h(x,y,z)\hat{k}$, then we define the curl of $\vec{F}$, written curl $\vec{F}$, to be the vector field given by

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

~~where,~~

$$\cancel{\nabla = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial h}{\partial z}\hat{k}}$$

The Laplacian

The operator that results by taking the dot product of the del operator with itself denoted by ∇^2 and is called the Laplacian operator. This operator has the form

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

when applied to $\phi(x,y,z)$ the Laplacian operator produces the function

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

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If

$$\nabla^2 \phi = 0$$

the laplace equation holds.

Examples

E4) Find the divergence and curl of the vector field

$$\vec{F}(x,y,z) = x^2y\,\hat{i} + 2y^3z\,\hat{j} + 3z\,\hat{k}$$

Here,

$$\text{div } \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$$

$$= 2xy + 6y^2z + 3$$

Also,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y^3z)\right]\hat{i} \quad -$$

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$$\left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(3z)\right]\hat{j} + \left[\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}(x^2y)\right]\hat{k}$$

$$= (0 - 2y^3)\hat{i} - (0-0)\hat{j} + (0 - x^2)\hat{k}$$

$$= -2y^3\hat{i} - x^2\hat{k}$$

Thus,

$$\text{curl } \vec{F} = -2y^3\hat{i} - x^2\hat{k}$$

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Exercise

(18)

Q17) $\vec{F}(x,y,z) = x^2\hat{i} - 2\hat{j} + yz\hat{k}$

As we know that

$$\text{div } \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-2) + \frac{\partial}{\partial z}(yz)$$

$$= 2x + 0 + y$$

$$= 2x + y$$

Also,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2 & yz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(-2)\right]\hat{i}$$

$$\left[\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(x^2)\right]\hat{j} + \left[\frac{\partial}{\partial x}(-2) - \frac{\partial}{\partial y}(x^2)\right]\hat{k}$$

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$$\text{curl } \vec{F} = (z-0)\hat{i} - (0-0)\hat{j} + (0-0)\hat{k}$$
$$= (z)\hat{i} - 0\hat{j} + 0\hat{k}$$
$$= z\hat{i}$$

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Q19) $\vec{F}(x,y,z) = 7y^3z^2\hat{i} - 8x^2z^5\hat{j} - 3xy^4\hat{k}$

As we know that

$$\text{div } \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$
$$= \frac{\partial}{\partial x}(7y^3z^2) + \frac{\partial}{\partial y}(-8x^2z^5) + \frac{\partial}{\partial z}(-3xy^4)$$
$$= 0 + 0 + 0$$
$$= 0$$

Also,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 7y^3z^2 & -8x^2z^5 & -3xy^4 \end{vmatrix}$$
$$= \left[\frac{\partial}{\partial y}(-3xy^4) - \frac{\partial}{\partial z}(-8x^2z^5)\right]\hat{i}$$
$$\left[\frac{\partial}{\partial x}(-3xy^4) - \frac{\partial}{\partial z}(7y^3z^2)\right]\hat{j} +$$
$$\left[\frac{\partial}{\partial x}(-8x^2z^5) - \frac{\partial}{\partial y}(7y^3z^2)\right]\hat{k}$$
$$= (-12xy^3 + 40x^2z^4)\hat{i}$$
$$(-3y^4 - 14y^3z)\hat{j} + (-16xz^5 - 21y^2z^2)\hat{k}$$

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$$\text{curl } \vec{F} = (40x^2z^4 - 12xy^3)\hat{i} - [-(3y^4 + 14y^3z)]\hat{j} + [-(16xz^5 + 21y^2z^2)]\hat{k}$$

$$= (40x^2z^4 - 12xy^3)\hat{i} + (3y^4 + 14y^3z)\hat{j} - (16xz^5 + 21y^2z^2)\hat{k}$$

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Q22) $\vec{F}(x,y,z) = \ln x\,\hat{i} + e^{xyz}\hat{j} + \tan^{-1}(z/x)\hat{k}$

As we know that

$$\text{div } \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial k}$$

$$= \frac{\partial}{\partial x}(\ln x) + \frac{\partial}{\partial y}(e^{xyz}) + \frac{\partial}{\partial z}(\tan^{-1}(z/x))$$

$$= \frac{1}{x} + e^{xyz} \cdot xz + \frac{1}{1+\left(\frac{z}{x}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{1}{x} + xze^{xyz} + \frac{x^2}{x^2} \frac{1}{1+\frac{z^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{1}{x} + xze^{xyz} + \frac{1}{\frac{x^2+z^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{1}{x} + xze^{xyz} + \frac{x^2}{x^2+z^2} \cdot \frac{1}{x}$$

$$= \frac{1}{x} + xze^{xyz} + \frac{x}{x^2+z^2}$$

Also,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \ln x & e^{xyz} & \tan^{-1}\left(\frac{z}{x}\right) \end{vmatrix}$$

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$$\text{curl } \vec{F} = \left[\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{z}{x}\right)\right) - \frac{\partial}{\partial z}\left(e^{xyz}\right)\right]\hat{i}$$

$$- \left[\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{z}{x}\right)\right] - \frac{\partial}{\partial z}\left(\ln x\right)\right]\hat{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(e^{xyz}\right) - \frac{\partial}{\partial y}\left(\ln x\right)\right]\hat{k}$$

$$\text{curl } \vec{F} = \left(0 - xye^{xyz}\right)\hat{i} - \left[\left(\frac{1}{1+\left(\frac{z}{x}\right)^2}\right)\left(\frac{-z}{x^2}\right) - 0\right]\hat{j} + \left(yze^{xyz} - 0\right)\hat{k}$$

$$= -xye^{xyz}\hat{i} + \left(\frac{1}{1+\frac{z^2}{x^2}}\right)\left(\frac{z}{x^2}\right)\hat{j} + yze^{xyz}\hat{k}$$

$$= -xye^{xyz}\hat{i} + \frac{\cancel{x^2}}{x^2+z^2}\left(\frac{z}{\cancel{x^2}}\right)\hat{j} + yze^{xyz}\hat{k}$$

$$= -xye^{xyz}\hat{i} + \frac{z}{x^2+z^2}\hat{j} + yze^{xyz}\hat{k}$$

Find $\nabla \cdot (\vec{F} \times \vec{G})$

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Q 24) $\vec{F}(x,y,z) = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\vec{G}(x,y,z) = xy\hat{j} + yxz\hat{k}$

Here,

$$\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ yz & xz & xy \\ 0 & xy & xyz \end{vmatrix}$$

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$$\vec{F} \times \vec{G} = (x^2yz^2 - x^2y^2)\hat{i} - (xy^2z^2 - 0)\hat{j} + (xy^2z - 0)\hat{k}$$

$$= (x^2yz^2 - x^2y^2)\hat{i} - xy^2z^2\hat{j} + xy^2z\hat{k}$$

Now,

$$\nabla \cdot (\vec{F} \times \vec{G}) = \frac{\partial}{\partial x}(x^2yz^2 - x^2y^2) + \frac{\partial}{\partial y}(-xy^2z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= \cancel{2xyz^2} - 2xy^2 - \cancel{2xyz^2} + xy^2$$

$$= -xy^2$$

Find $\nabla \cdot (\nabla \times \vec{F})$

Q25) $\vec{F}(x,y,z) = \sin x\,\hat{i} + \cos(x-y)\hat{j} + z\hat{k}$

Here,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \cos(x-y) & z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(\cos(x-y))\right]\hat{i} - \left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(\sin x)\right]\hat{j} + \left[\frac{\partial}{\partial x}(\cos(x-y)) - \frac{\partial}{\partial y}(\sin x)\right]\hat{k}$$

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$$\nabla \times \vec{F} = (0-0)\hat{i} - (0-0)\hat{j} + (-\sin(x-y) - 0)\hat{k}$$

$$= -\sin(x-y)\hat{k}$$

Now,

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-\sin(x-y))$$

$$= 0 + 0 + 0$$

$$= 0$$

Find $\nabla \times (\nabla \times \vec{F})$

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Q28) $\vec{F}(x,y,z) = y^2x\hat{i} - 3yz\hat{j} + xy\hat{k}$

Here,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & -3yz & xy \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(-3yz)\right]\hat{i} - \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(xy^2)\right]\hat{j} + \left[\frac{\partial}{\partial x}(-3yz) - \frac{\partial}{\partial y}(xy^2)\right]\hat{k}$$

$$= (x + 3y)\hat{i} - (y-0)\hat{j} + (0 - 2xy)\hat{k}$$

$$\nabla \times \vec{F} = (x+3y)\hat{i} - y\hat{j} - 2xy\hat{k}$$

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Now,

$$\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+3y & -y & -2xy \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-2xy) - \frac{\partial}{\partial z}(-y)\right]\hat{i} - \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial z}(x+3y)\right]\hat{j} + \left[\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(x+3y)\right]\hat{k}$$

$$= (-2x - 0)\hat{i} - (-2y - 0)\hat{j} + (0 - 3)\hat{k}$$

$$\nabla \times (\nabla \times \vec{F}) = -2x\hat{i} + 2y\hat{j} - 3\hat{k}$$

Verify that the radius vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ has the stated property.

Q41) (a) curl $\vec{r} = 0$

As we know that

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right]\hat{i} - \left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right]\hat{j} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\hat{k}$$

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$$\nabla \times \vec{r} = (0-0)\hat{i} - (0-0)\hat{j} + (0-0)\hat{k}$$

$$\nabla \times \vec{r} = 0$$

Thus,

$$\text{curl } \vec{r} = 0$$

(b) $$\nabla \|\vec{r}\| = \frac{\vec{r}}{\|\vec{r}\|}$$

Here,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

so,

$$\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$$

thus,

$$\nabla \|\vec{r}\| = \frac{\partial}{\partial x}\|\vec{r}\|\hat{i} + \frac{\partial}{\partial y}\|\vec{r}\|\hat{j} + \frac{\partial}{\partial z}\|\vec{r}\|\hat{k}$$

$$= \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)\hat{j} + \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)\hat{k}$$

$$= \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}x\,\hat{i} + \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}y\,\hat{j} + \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}}\cdot\cancel{2}z\,\hat{k}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

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$$\nabla \|\vec{r}\| = \frac{x}{\|\vec{r}\|}\hat{i} + \frac{y}{\|\vec{r}\|}\hat{j} + \frac{z}{\|\vec{r}\|}\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\|\vec{r}\|}$$

$$\nabla \|\vec{r}\| = \frac{\vec{r}}{\|\vec{r}\|}$$

Hence Proved.

Q42) a) $\text{div}\, \vec{r} = 3$

Here,

The divergence of $\vec{r}$ is

$$\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

or, $\text{div}\, \vec{r} = 3$

b) $\nabla \frac{1}{\|\vec{r}\|} = \frac{-\vec{r}}{\|\vec{r}\|^3}$

Here,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

So,

$$\frac{1}{\|\vec{r}\|} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= (x^2+y^2+z^2)^{-1/2}$$

Now,

$$\nabla\left(\frac{1}{\|\vec{r}\|}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\|\vec{r}\|}\right)\hat{i}$$

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$$+ \frac{\partial}{\partial y}\left(\frac{1}{\|\vec{r}\|}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{\|\vec{r}\|}\right)\hat{k}$$

$$= \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-1/2}\right]\hat{i} + \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{-1/2}\right]\hat{j}$$

$$+ \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{-1/2}\right]\hat{k}$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot\cancel{2}x\,\hat{i} + \left(-\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot\cancel{2}y\right)\hat{j}$$

$$+ \left(-\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot\cancel{2}z\right)\hat{k}$$

$$= -\frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{i} - \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{j} - \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

$$= -\frac{x}{\|\vec{r}\|^3}\hat{i} - \frac{y}{\|\vec{r}\|^3}\hat{j} - \frac{z}{\|\vec{r}\|^3}\hat{k}$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{\|\vec{r}\|^3}$$

$$= -\frac{\vec{r}}{\|\vec{r}\|^3}$$

or,

$$\nabla\left(\frac{1}{\|\vec{r}\|}\right) = -\frac{\vec{r}}{\|\vec{r}\|^3}$$

Hence Proved