Osculating circle

Centre of curvature

The coordinates (h,k) of the centre of curvature are given as

$$h = x - y' \frac{(1+(y')^2}{y''}$$

$$k = y + \frac{1+(y')^2}{y''}$$

Equation of Osculating circle is

$$(x-h)^2 + (y-k)^2 = \rho^2$$

1) Find an osculating circle's equation to the curve $y = \ln x$ at the point $(1,0)$.

$$y = \ln x$$

Diff: b.s w.r.t 'x'

$$\frac{d}{dx}(y) = \frac{d}{dx}(\ln x)$$

$$\Rightarrow y' = \frac{1}{x} \Rightarrow y'_{(1,0)} = 1$$

Again Diff: b.s w.r.t 'x'

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\Rightarrow y'' = -\frac{1}{x^2}$$

$$y''_{(1,0)} = -\frac{1}{1^2} = -1$$

Radius of curvature is given as

$$\rho = \frac{[1 + (y')^2]^{3/2}}{|y''|}$$

$$\rho_{(1,0)} = \frac{[1 + 1]^{3/2}}{|-1|}$$

$$\rho_{(1,0)} = \frac{(2)^{3/2}}{1}$$

$$\rho_{(1,0)} = 2\sqrt{2}$$

Now,

Coordinates of centre of curvature are

$$h = x - y' \frac{1 + (y')^2}{y''}$$

$$h_{(1,0)} = 1 - 1 \frac{(1 + 1)}{-1}$$

$$h_{(1,0)} = 1 - 1 \frac{(2)}{-1}$$

$$h_{(1,0)} = 1 + 2 = 3$$

$$k = y + \frac{1 + (y')^2}{y''}$$

$$k_{(1,0)} = 0 + \frac{1 + 1}{-1}$$

$$k_{(1,0)} = -2$$

Now,

Equation of osculating circle is

$$(x - h)^2 + (y - k)^2 = \rho^2$$

$$(x - 3)^2 + (y + 2)^2 = (2\sqrt{2})^2$$

$$\boxed{(x - 3)^2 + (y + 2)^2 = 8}$$

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2) Find an equation of the osculating circle to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at the point $(-2, 0)$.

Solution:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

Diff: b.s w.r.t 'x'

$$\frac{2x}{4} - \frac{2y}{9} \cdot \frac{dy}{dx} = 0$$

$$\frac{2x}{4} = \frac{2y}{9} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cancel{2}x}{4} \times \frac{9}{\cancel{2}y}$$

$$\Rightarrow \quad y' = \frac{9x}{4y}$$

Again Diff: w.r.t 'x'

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{9x}{4y}\right)$$

$$y'' = \frac{9}{4}\left[\frac{(y)(1) - (x)y'}{y^2}\right]$$

$$y'' = \frac{9}{4}\left[\frac{y - (x)(9x/4y)}{y^2}\right]$$

$$y'' = \frac{9}{4}\left[\frac{y - \frac{9x^2}{4y}}{y^2}\right]$$

$$y'' = \frac{9}{4}\left[\frac{\frac{4y^2 - 9x^2}{4y}}{y^2}\right]$$

$$\Rightarrow \quad y'' = \frac{9}{4}\left[\frac{4y^2 - 9x^2}{4y^3}\right]$$

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Radius of curvature is given as

$$\rho = \frac{[1 + (y')^2]^{3/2}}{|y''|}$$

$$\rho = \frac{\left[1 + \left(\frac{9x}{4y}\right)^2\right]^{3/2}}{\left|\frac{9}{4}\left[\frac{4y^2 - 9x^2}{4y^3}\right]\right|}$$

$$\rho = \frac{\left[1 + \frac{81x^2}{16y^2}\right]^{3/2}}{\left|\frac{9}{4}\left[\frac{4y^2 - 9x^2}{4y^3}\right]\right|}$$

$$\rho = \left[\frac{16y^2 + 81x^2}{16y^2}\right]^{3/2} \times \frac{4(4y^3)}{9(4y^2 - 9x^2)}$$

$$\rho = \frac{(16y^2 + 81x^2)^{3/2}}{\cancel{64y^2}_{4}} \times \frac{\cancel{16y^3}}{9(4y^2 - 9x^2)}$$

$$\rho = \frac{(16y^2 + 81x^2)^{3/2}}{36(4y^2 - 9x^2)}$$

$$\rho_{(-2,0)} = \frac{(16(0) + 81(-2)^2)^{3/2}}{36[4(0) - 9(-2)^2]}$$

$$\rho_{(-2,0)} = \frac{[81(4)]^{3/2}}{|36(-9(4))|}$$

$$\rho_{(-2,0)} = \frac{(324)^{3/2}}{|-1296|}$$

$$\rho^2_{(-2,0)} = \frac{(324)^3}{(1296)^2}$$

$$\rho^2_{(-2,0)} = \frac{(324)^{\cancel{3}}}{(4)^2 \times \cancel{(324)^2}}$$

$$\rho^2_{(-2,0)} = \frac{324}{16}$$

$$\rho^2_{(-2,0)} = \frac{81}{4}$$

Coordinates of centre of curvature

$$\mathbf{h} = x - y' \frac{1 + (y')^2}{y''}$$

$$h = x - \frac{9x}{4y} \frac{1 + \left(\frac{9x}{4y}\right)^2}{\frac{9}{4}\left[\frac{4y^2 - 9x^2}{4y^3}\right]}$$

$$h = x - \frac{x}{y}\left[\left(1 + \frac{81x^2}{16y^2}\right) \times \frac{4y^3}{4y^2 - 9x^2}\right]$$

$$h = x - x\frac{(16y^2 + 81x^2)}{{}^{4}\cancel{16}\cancel{y^2}} \times \frac{\cancel{4}y^{\cancel{3}}}{4y^2 - 9x^2}$$

$$h = x - x\frac{(16y^2 + 81x^2)}{4(4y^2 - 9x^2)}$$

$$h_{(-2,0)} = -2 - \frac{(-\cancel{2})(16(0) + 81(4))}{{}^{2}\cancel{4}(4(0) - 9(4))}$$

$$h_{(-2,0)} = -2 + \frac{81(\cancel{4})}{\cancel{2}(-\cancel{36})^{18}}$$

$$h_{(-2,0)} = -2 + \frac{\cancel{81}^{9}}{-\cancel{18}_{2}}$$

$$h_{(-2,0)} = -2 - \frac{9}{2}$$

$$h_{(-2,0)} = \frac{-13}{2}$$

$$\mathbf{k} = y + \frac{1 + (y')^2}{y''}$$

$$k = y + \frac{1 + \left(\frac{9x}{4y}\right)^2}{\frac{9}{4}\left[\frac{4y^2 - 9x^2}{4y^3}\right]}$$

$$k = y + \left[1 + \frac{81x^2}{16y^2} \times \frac{4(4y^3)}{9(4y^2 - 9x^2)}\right]$$

$$k = y + \left[\frac{16y^2 + 81x^2}{\cancel{16}\cancel{y^2}} \times \frac{\cancel{16}y^3}{9(4y^2 - 9x^2)}\right]$$

$$k = y + \frac{y\,(16y^2 + 81x^2)}{9\,(4y^2 - 9x^2)}$$

$$k_{(-2,0)} = 0 + \frac{0}{9(4(0) - 9(-2)^2)}$$

$$k_{(-2,0)} = 0$$

Now,

Equation of osculating circle is

$$(x-h)^2 + (y-k)^2 = \rho^2$$

$$(x + 13/2)^2 + (y-0)^2 = \frac{81}{4}$$

$$\boxed{\left(x + \frac{13}{2}\right)^2 + y^2 = \frac{81}{4}}$$

3) Show that the evolute of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

Solution:

$$x = a\cos\theta$$

Diff: b.s w.r.t 'θ'

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta)$$

$$\frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta$$

Diff: b.s w.r.t 'θ'

$$\frac{d}{d\theta}(y) = \frac{d}{d\theta}(b\sin\theta)$$

$$\frac{dy}{d\theta} = b\cos\theta$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$y' = b\cos\theta \cdot \frac{-1}{a\sin\theta}$$

$$y' = \frac{-b}{a}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad y' = \frac{-b}{a}\cot\theta$$

Again Diff: b.s w.r.t 'θ'

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{-b}{a}\cot\theta\right)$$

$$y'' = \frac{-b}{a}\left(-\operatorname{cosec}^2\theta \cdot \frac{d\theta}{dx}\right)$$

$$y'' = \frac{b}{a}\operatorname{cosec}^2\theta \cdot \frac{-1}{a\sin\theta}$$

$$y'' = \frac{b}{a} \cdot \frac{1}{\sin^2\theta} \cdot \frac{-1}{a\sin\theta}$$

$$\Rightarrow \quad y'' = \frac{-b}{a^2\sin^3\theta}$$

Coordinates of centre of curvature

$$h = x - y' \frac{1 + (y')^2}{y''}$$

$$h = x - \left(\frac{-b}{a}\cot\theta\right) \frac{1 + \left(\frac{-b}{a}\cot\theta\right)^2}{\frac{-b}{a^2\sin^3\theta}}$$

$$h = x + \frac{b}{a}\cot\theta \frac{1 + \left(\frac{b^2\cot^2\theta}{a^2}\right)}{\frac{-b}{a^2\sin^3\theta}}$$

$$h = a\cos\theta + \frac{\cancel{b}}{\cancel{a}}\cot\theta \left(\frac{1 + \left(\frac{b^2\cot^2\theta}{a^2}\right)}{\frac{-\cancel{b}}{a^2\sin^3\theta}}\right)$$

$$h = a\cos\theta + \cot\theta\left[\left(\frac{1 + b^2\cot^2\theta}{a^2}\right) \times \left(\frac{-a^2\sin^3\theta}{a}\right)\right]$$

$$h = a\cos\theta + \cot\theta\left[\frac{(a^2 + b^2\cot^2\theta)\sin^3\theta}{-a}\right]$$

$$h = a\cos\theta + \left[\frac{a^2\sin^3\theta\cot\theta + b^2\sin^3\theta\cot^3\theta}{-a}\right]$$

$$h = a\cos\theta - \frac{a^2\sin^3\theta\left(\frac{\cos\theta}{\sin\theta}\right) - b^2\sin^3\theta\left(\frac{\cos^3\theta}{\sin^3\theta}\right)}{a}$$

$$h = a\cos\theta - \frac{a^2\sin^2\theta\cos\theta - b^2\cos^3\theta}{a}$$

$$h = \frac{a^2\cos\theta - a^2\sin^2\theta\cos\theta - b^2\cos^3\theta}{a}$$

$$ah = a^2\cos\theta\,(1 - \sin^2\theta) - b^2\cos^3\theta$$

$$ah = a^2\cos\theta\,(\cos^2\theta) - b^2\cos^3\theta$$

$$ah = a^2\cos^3\theta - b^2\cos^3\theta$$

$$ah = \cos^3\theta\,(a^2 - b^2) \longrightarrow \text{(i)}$$

$$k = y + \frac{1 + (y')^2}{y''}$$

$$k = y + \frac{1 + \left(\frac{-b}{a}\cot\theta\right)^2}{\frac{-b}{a^2\sin^3\theta}}$$

$$k = y + \left[\left(\frac{1 + b^2\cot^2\theta}{a^2}\right) \times \frac{a^2\sin^3\theta}{-b}\right]$$

$$k = b\sin\theta + \left[\frac{a^2 + b^2\cot^2\theta}{a^2} \times \frac{a^2\sin^3\theta}{-b}\right]$$

$$k = b\sin\theta + \frac{\sin^3\theta\,(a^2 + b^2\cot^2\theta)}{-b}$$

$$k = b\sin\theta - \frac{\sin^3\theta\,(a^2 + b^2\cot^2\theta)}{b}$$

$$k = \frac{b^2\sin\theta - a^2\sin^3\theta - b^2\sin^3\theta\cot^2\theta}{b}$$

$$bk = b^2\sin\theta - a^2\sin^3\theta - b^2\sin^3\theta\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$bk = b^2\sin\theta - a^2\sin^3\theta - b^2\sin\theta\,(1-\sin^2\theta)$$

$$bk = \cancel{b^2\sin\theta} - a^2\sin^3\theta - \cancel{b^2\sin\theta} + b^2\sin^3\theta$$

$$bk = b^2\sin^3\theta - a^2\sin^3\theta$$

$$bk = -\sin^3\theta\,(a^2-b^2) \longrightarrow \text{(ii)}$$

To evolute eliminate θ from (i) and (ii)
Take power $2/3$ on b.s of (i) and (ii)
then adding both

$$(ah)^{2/3} + (bk)^{2/3} = (\cos^2\theta)^{2/3}(a^2-b^2)^{2/3} + (-\sin^2\theta)^{2/3}(a^2-b^2)^{2/3}$$

$$(ah)^{2/3} + (bk)^{2/3} = \cos^2\theta\,(a^2-b^2)^{2/3} + \sin^2\theta\,(a^2-b^2)^{2/3}$$

$$(ah)^{2/3} + (bk)^{2/3} = (a^2-b^2)^{2/3}(\sin^2\theta + \cos^2\theta)$$

$$(ah)^{2/3} + (bk)^{2/3} = (a^2-b^2)^{2/3}(1)$$

$$(ah)^{2/3} + (bk)^{2/3} = (a^2-b^2)^{2/3}$$

Put $h = x$ and $k = y$

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$$

Hence Proved

Find the centre of curvature for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Show that its evolute is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

Solution:

$$x = a\sec\theta$$

Diff: b.s w.r.t 'θ'

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\sec\theta)$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$y = b\tan\theta$$

Diff: b.s w.r.t 'θ'

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\tan\theta)$$

$$\frac{dy}{d\theta} = b\sec^2\theta$$

Using chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = b\sec^2\theta\cdot\frac{1}{a\sec\theta\tan\theta}$$

$$y' = b\cdot\frac{1}{\cancel{\cos\theta}}\cdot\frac{\cancel{\cos\theta}}{a\sin\theta}$$

$$y' = \frac{b}{a\sin\theta}$$

$$\Rightarrow y' = \frac{b}{a}\operatorname{cosec}\theta$$

Again Diff: b.s w.r.t 'x'

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{b}{a}\operatorname{cosec}\theta\right)$$

$$y'' = \frac{b}{a}\left(-\operatorname{cosec}\theta\cot\theta\cdot\frac{d\theta}{dx}\right)$$

$$y'' = \frac{b}{a}\left(-\operatorname{cosec}\theta\cot\theta\cdot\frac{1}{a\sec\theta\tan\theta}\right)$$

$$y'' = -\frac{b}{a}\left[\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta\cdot\cos\theta}{a\cdot\sin\theta}\right]$$

$$\Rightarrow y'' = -\frac{b}{a^2}\left(\frac{\cos^3\theta}{\sin^3\theta}\right)$$

Coordinates of centre of curvature

$$\mathbf{h} = x - y' \frac{1+(y')^2}{y''}$$

$$h = a\sec\theta - \frac{b}{a}\operatorname{cosec}\theta \cdot \frac{1+\left(\frac{b}{a}\operatorname{cosec}\theta\right)^2}{-\frac{b}{a^2}\left(\frac{\cos^3\theta}{\sin^2\theta}\right)}$$

$$h = \frac{a}{\cos\theta} + \frac{\cancel{b}}{\cancel{a}}\operatorname{cosec}\theta \cdot \frac{1+\left(\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{+\frac{\cancel{b}}{a^{\cancel{2}}}\left(\frac{\cos^3\theta}{\sin^2\theta}\right)}$$

$$h = \frac{a}{\cos\theta} + \frac{1}{\sin\theta}\left[\frac{a^2+b^2\operatorname{cosec}^2\theta}{a^2} \div \frac{\cos^3\theta}{a\sin^3\theta}\right]$$

$$h = \frac{a}{\cos\theta} + \frac{1}{\cancel{\sin\theta}}\left[\frac{(a^2+b^2\operatorname{cosec}^2\theta)}{a^{\cancel{2}}} \times \frac{\cancel{a}\sin^{\cancel{3}}\theta}{\cos^3\theta}\right]$$

$$h = \frac{a}{\cos\theta} + \left[\frac{\sin^2\theta a^2 + b^2\operatorname{cosec}^2\theta\sin^2\theta}{a\cos^3\theta}\right]$$

$$h = \frac{a}{\cos\theta} + \left[\frac{\sin^2\theta a^2 + b^2\left(\frac{1}{\cancel{\sin^2\theta}}\right)\cancel{\sin^2\theta}}{a\cos^3\theta}\right]$$

$$h = \frac{a}{\cos\theta} + \frac{a^2\sin^2\theta + b^2}{a\cos^3\theta}$$

$$h = \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta}$$

$$h = \frac{a^2(\sin^2\theta + \cos^2\theta) + b^2}{a\cos^3\theta}$$

$$h = \frac{a^2+b^2}{a\cos^3\theta}$$

$$ah = \frac{a^2+b^2}{\cos^3\theta} \longrightarrow (i)$$

$$\mathbf{k} = y + \frac{1+(y')^2}{y''}$$

$$k = b\tan\theta + \frac{1+\left(\frac{b}{a}\operatorname{cosec}\theta\right)^2}{-\frac{b}{a^2}\left(\frac{\cos^3\theta}{\sin^2\theta}\right)}$$

$$k = b\tan\theta + \left[\left(1 + \frac{b^2}{a^2}\operatorname{cosec}^2\theta\right) \div \left(\frac{-b\cos^2\theta}{a^2\sin^2\theta}\right)\right]$$

$$k = b\tan\theta + \left[\left(\frac{a^2 + b^2\operatorname{cosec}^2\theta}{a^2}\right) \times \left(\frac{a^2\sin^2\theta}{-b\cos^2\theta}\right)\right]$$

$$k = b\tan\theta - \left[\frac{a^2\sin^2\theta + b^2\operatorname{cosec}^2\theta \cdot \sin^2\theta}{b\cos^3\theta}\right]$$

$$k = b\tan\theta - \frac{a^2\sin^3\theta - b^2 \cdot \sin^2\theta \cdot \frac{1}{\sin^2\theta}}{b\cos^3\theta}$$

$$k = \frac{b\sin\theta}{\cos\theta} - \frac{a^2\sin^3\theta - b^2\sin\theta}{b\cos^3\theta}$$

$$k = \frac{b^2\sin\theta\cos^2\theta - a^2\sin^3\theta - b^2\sin\theta}{b\cos^3\theta}$$

$$k = \frac{b^2\sin\theta\,(\cos^2\theta - 1) - a^2\sin^3\theta}{b\cos^3\theta}$$

$$k = \frac{b^2\sin\theta\,(-\sin^2\theta) - a^2\sin^3\theta}{b\cos^3\theta}$$

$$k = \frac{-b^2\sin^3\theta - a^2\sin^3\theta}{b\cos^3\theta}$$

$$k = -\frac{\sin^3\theta\,(a^2+b^2)}{\cos^3\theta\; b}$$

$$bk = -\tan^3\theta\,(a^2+b^2) \longrightarrow \text{(ii)}$$

For evolute eliminate θ from (i) and (ii) and adding both

Taking 2/3 power of (i) and (ii)

$$(ah)^{2/3} + (bk)^{2/3} = \frac{(a^2+b^2)^{2/3}}{(\cos^2\theta)^{2/3}} - (-\tan^3\theta)^{2/3}(a^2+b^2)^{2/3}$$

$$(ah)^{2/3} - (bk)^{2/3} = \frac{(a^2+b^2)^{2/3}}{\cos^2\theta} - \tan^2\theta\,(a^2+b^2)^{2/3}$$

$$(ah)^{2/3} - (bk)^{2/3} = \frac{(a^2+b^2)^{2/3}}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}(a^2+b^2)^{2/3}$$

$$(ah)^{2/3} - (bk)^{2/3} = \frac{(a^2+b^2)^{2/3} - \sin^2\theta\,(a^2+b^2)^{2/3}}{\cos^2\theta}$$

$$= \frac{(a^2+b^2)^{2/3}\,(1-\sin^2\theta)}{\cos^2\theta}$$

$$(ah)^{2/3} - (bk)^{2/3} = \frac{(a^2+b^2)^{2/3}\,\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}$$

Put $h = x$ and $k = y$

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

Hence Proved

5) Prove that the evolute of the hyperbola $2xy = a^2$ is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$$

Solution:

$$2xy = a^2$$

$$y = \frac{a^2}{2x}$$

Diff. b.s w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{a^2}{2x}\right)$$

$$\Rightarrow y' = \frac{-a^2}{2x^2}$$

Again Diff. b.s w.r.t 'x'

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{-a^2}{2x^2}\right)$$

$$\Rightarrow y'' = \frac{-a^2}{\cancel{2}} \cdot \frac{-\cancel{2}}{x^3} = \frac{a^2}{x^3}$$

Coordinates of centre of curvature

$$h = x - y' \frac{1 + (y')^2}{y''}$$

$$h = x - \left(-\frac{a^2}{2x^2}\right) \frac{1 + \left(\frac{-a^2}{2x^2}\right)^2}{a^2/x^3}$$

$$h = x + \frac{a^2}{2x^2}\left[\left(1 + \frac{a^4}{4x^4}\right) \div \frac{a^2}{x^3}\right]$$

$$h = x + \frac{a^2}{2x^2}\left[\frac{4x^4 + a^4}{4x^4} \times \frac{x^3}{a^2}\right]$$

$$h = x + \frac{\cancel{a^2}}{2x^2}\left[\frac{4x^4 + a^4}{4\cancel{x^3}x}\right]$$

$$h = x + \frac{4x^4 + a^4}{8x^3}$$

$$h = \frac{8x^4 + 4x^4 + a^4}{8x^3}$$

$$h = \frac{12x^4 + a^4}{8x^3} \longrightarrow (i)$$

$$k = y + \frac{1 + (y')^2}{y''}$$

$$k = y + \frac{1 + \left(\frac{-a^2}{2x^2}\right)^2}{a^2/x^3}$$

$$k = y + \left[\left(1 + \frac{a^4}{4x^4}\right) \times \frac{x^3}{a^2}\right]$$

$$k = y + \left[\frac{4x^4 + a^4}{4x^4} \times \frac{\cancel{x^3}}{a^2}\right]$$

$$k = \frac{a^2}{2x} + \frac{4x^4 + a^4}{4a^2x}$$

$$k = \frac{2a^2(a^2) + 4x^4 + a^4}{4a^2x}$$

$$k = \frac{2a^4 + 4x^4 + a^4}{4a^2x}$$

$$k = \frac{3a^4 + 4x^4}{4a^2x} \longrightarrow \text{(ii)}$$

For evolute we eliminate x from equ (i) and (ii)

Adding (i) and (ii)

$$h+k = \frac{12x^4 + a^4}{8x^3} + \frac{3a^4 + 4x^4}{4a^2x}$$

$$h+k = \frac{a^2(12x^4 + a^4) + (3a^4 + 4x^4)2x^2}{8a^2x^3}$$

$$h+k = \frac{12a^2x^4 + a^6 + 6x^2a^4 + 8x^6}{8a^2x^3}$$

$$h+k = \frac{(2x^2 + a^2)^3}{8a^2x^3}$$

$$(h+k)^{2/3} = \frac{(2x^2 + a^2)^2}{(8a^2x^3)^{2/3}}$$

$$(h+k)^{2/3} = \frac{(2x^2 + a^2)^2}{4x^2a^{4/3}} \longrightarrow \text{(iii)}$$

Subtracting (ii) from (i)

$$h-k = \frac{12x^4 + a^4}{8x^3} - \frac{3a^4 + 4x^4}{4a^2x}$$

$$h-k = \frac{12x^4a^2 + a^6 - 6x^2a^4 - 8x^6}{8a^2x^3}$$

$$h-k = \frac{(a^2 - 2x^2)^3}{8a^2x^3}$$

$$(h-k)^{2/3} = \frac{(a - 2x^2)^2}{4x^2a^{4/3}} \longrightarrow \text{(iv)}$$

Subtracting (iv) from (iii), we get

$$(h+k)^{2/3} - (h-k)^{2/3} = \frac{(2x^2+a^2)^2}{4x^2a^{4/3}} - \frac{(a^2-2x^2)^2}{4x^2a^{4/3}}$$

$$(h+k)^{2/3} - (h-k)^{2/3} = \frac{(2x^2+a^2)^2 - (a^2-2x^2)^2}{4x^2a^{4/3}}$$

$$(h+k)^{2/3} - (h-k)^{2/3} = \frac{4x^4+a^4+4a^2x^2-(a^4+4x^4-4a^2x^2)}{4x^2a^{4/3}}$$

$$= \frac{\cancel{4x^4}+\cancel{a^4}+4a^2x^2-\cancel{a^4}-\cancel{4x^4}+4a^2x^2}{4x^2a^{4/3}}$$

$$= \frac{{}^{2}\cancel{8}a^2\cancel{x^2}}{\cancel{4}\cancel{x^2}a^{4/3}}$$

$$= \frac{2a^2}{a^{4/3}}$$

$$(h+k)^{2/3} - (h-k)^{2/3} = 2a^{2/3}$$

Put $h = x$ and $y = k$

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$$

Hence Proved