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Triple Integrals

Triple Integrals in Rectangular Coordinates

The valume of a closed, bounded region D in space is

$$V = \iiint_D dV$$

or,

$$V = \iiint f(x,y,z)\, dz\,dy\,dx$$

Examples

E3) Evaluate the integral

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\,dy\,dx$$

Here,

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\,dy\,dx$$

$$V = \int_0^1 \int_x^1 \left[z \right]_0^{y-x} dy\,dx$$

$$V = \int_0^1 \int_x^1 (y - x - 0)\, dy\,dx$$

$$V = \int_0^1 \int_x^1 (y - x)\, dy\,dx$$

$$V = \int_0^1 \left[y^2/2 - yx \right]_x^1 dx$$

$$V = \int_0^1 \left[\left(\frac{1}{2} - x \right) - \left(\frac{x^2}{2} - x^2 \right) \right] dx$$

$$V = \int_0^1 \left(\frac{1}{2} - x - \frac{x^2}{2} + x^2 \right) dx$$

$$V = \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2} \right) dx$$

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$$V = \int_0^1 \left(\frac{1-2x+x^2}{2}\right) dx$$

$$V = \frac{1}{2}\int_0^1 (1-2x+x^2)\,dx$$

$$V = \frac{1}{2}\left[x - \frac{\cancel{2}x^2}{\cancel{2}} + \frac{x^3}{3}\right]_0^1$$

$$V = \frac{1}{2}\left[x - x^2 + \frac{x^3}{3}\right]_0^1$$

$$V = \frac{1}{2}\left(\cancel{1} - \cancel{1} + \frac{1}{3}\right)$$

$$V = \frac{1}{6}$$

15.5

Evaluating Triple Iterated Integrals

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Q8) $$\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$$

$$= \int_0^{\sqrt{2}}\int_0^{3y} \left[z\right]_{x^2+3y^2}^{8-x^2-y^2} dx\,dy$$

$$= \int_0^{\sqrt{2}}\int_0^{3y} \left[(8-x^2-y^2) - (x^2+3y^2)\right] dx\,dy$$

$$= \int_0^{\sqrt{2}}\int_0^{3y} (8-x^2-y^2-x^2-3y^2)\,dx\,dy$$

$$= \int_0^{\sqrt{2}}\int_0^{3y} (8-2x^2-4y^2)\,dx\,dy$$

$$= \int_0^{\sqrt{2}}\int_0^{3y} (8-2x^2-4y^2)\,dx\,dy$$

$$= \int_0^{\sqrt{2}} \left[8x - \frac{2x^3}{3} - 4y^2x\right]_0^{3y} dy$$

$$= \int_0^{\sqrt{2}} \left(8(3y) - \frac{2}{3}(3y)^3 - 4y^2(3y) \right) dy$$

$$= \int_0^{\sqrt{2}} \left(24y - \frac{2}{3}(27y^3) - 12y^3 \right) dy$$

$$= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3)\, dy$$

$$= \left[\frac{24y^2}{2} - \frac{\cancel{18}^{9}\, y^4}{\cancel{4}_{2}} - \frac{\cancel{12}^{3}\, y^4}{\cancel{4}_{2}} \right]_0^{\sqrt{2}}$$

$$= \left[12y^2 - \frac{9y^4}{2} - 3y^4 \right]_0^{\sqrt{2}}$$

$$= \left[12y^2 - \frac{15y^4}{2} \right]_0^{\sqrt{2}}$$

$$= \left(12(\sqrt{2})^2 - \frac{15(\sqrt{2})^4}{2} \right) - 0$$

$$= 12(2) - \frac{15(4)}{2}$$

$$= 24 - 30$$

$$= -6$$

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Q9) $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz} \, dx\,dy\,dz$

$= \int_1^e \int_1^{e^2} \frac{1}{yz} \int_1^{e^3} \frac{1}{x} \, dx\,dy\,dz$

$= \int_1^e \int_1^{e^2} \frac{1}{yz} \left[\ln x\right]_1^{e^3} dy\,dz$

$= \int_1^e \int_1^{e^2} \frac{1}{yz} \left[\ln(e^3) - \ln(1)\right] dy\,dz$

$= \int_1^e \int_1^{e^2} \frac{1}{yz} \left(3(1) - 0\right) dy\,dz$

$= \int_1^e \frac{3}{z} \int_1^{e^2} \frac{1}{y} \, dy\,dz$

$= \int_1^e \frac{3}{z} \left[\ln y\right]_1^{e^2} dz$

$= \int_1^e \frac{3}{z} \left[\ln(e^2) - \ln(1)\right] dz$

$= \int_1^e \frac{3}{z} \left(2(1) - 0\right) dz$

$= 6 \int_1^e \frac{1}{z} \, dz$

$= 6 \left[\ln z\right]_1^e$

$= 6 \left[\ln(e) - \ln(1)\right]$

$= 6 (1 - 0)$

$= 6$

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Q11) $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z \, dx\,dy\,dz$

$= \int_0^{\pi/6}\int_0^1 y\sin z \int_{-2}^3 (1)\, dx\, dy\, dz$

$= \int_0^{\pi/6}\int_0^1 y\sin z \, [x]_{-2}^3 \, dy\, dz$

$= \int_0^{\pi/6}\int_0^1 y\sin z \, [3-(-2)]\, dy\, dz$

$= \int_0^{\pi/6}\int_0^1 y\sin z \, (3+2)\, dy\, dz$

$= \int_0^{\pi/6}\int_0^1 y\sin z \, (5)\, dy\, dz$

$= \int_0^{\pi/6} 5\sin z \int_0^1 (y)\, dy\, dz$

$= \int_0^{\pi/6} 5\sin z \left[\frac{y^2}{2}\right]_0^1 dz$

$= \int_0^{\pi/6} \frac{5}{2}\sin z \, [y^2]_0^1 \, dz$

$= \frac{5}{2}\int_0^{\pi/6} \sin z \, (1^2-0)\, dz$

$= \frac{5}{2}\int_0^{\pi/6} \sin z \, dz$

$= \frac{5}{2}\left[-\cos z\right]_0^{\pi/6}$

$= -\frac{5}{2}\left[\cos\left(\frac{\pi}{6}\right) - \cos(0)\right]$

$= -\frac{5}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$

$= -\frac{5}{2}\left(\frac{\sqrt{3}-2}{2}\right)$

$= \frac{5(2-\sqrt{3})}{4}$

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Q13) $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

$= \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} (1)\, dz\,dy\,dx$

$= \int_0^3 \int_0^{\sqrt{9-x^2}} \left[z \right]_0^{\sqrt{9-x^2}} dy\,dx$

$= \int_0^3 \int_0^{\sqrt{9-x^2}} \left(\sqrt{9-x^2} - 0 \right) dy\,dx$

$= \int_0^3 \int_0^{\sqrt{9-x^2}} \left(\sqrt{9-x^2} \right) dy\,dx$

$= \int_0^3 \sqrt{9-x^2} \int_0^{\sqrt{9-x^2}} (1)\, dy\,dx$

$= \int_0^3 \sqrt{9-x^2} \left[y \right]_0^{\sqrt{9-x^2}} dx$

$= \int_0^3 \sqrt{9-x^2} \left(\sqrt{9-x^2} - 0 \right) dx$

$= \int_0^3 \left(\sqrt{9-x^2} \cdot \sqrt{9-x^2} \right) dx$

$= \int_0^3 \left[\left(\sqrt{9-x^2} \right)^2 \right] dx$

$= \int_0^3 (9 - x^2)\, dx$

$= \left[\frac{9x - x^3}{3} \right]_0^3$

$= \left[\frac{27x - x^3}{3} \right]_0^3$

$= \frac{1}{3} \left[27x - x^3 \right]_0^3$

$= \frac{1}{3} \left[27(3) - (3)^3 \right]$

$= \frac{1}{3} (81 - 27)$

$= \frac{1}{3} (54)$

$= 18$

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Q14) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\,dx\,dy$

$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} (1)\,dz\,dx\,dy$

$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left[z \right]_0^{2x+y} dx\,dy$

$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y-0)\,dx\,dy$

$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y)\,dx\,dy$

$= \int_0^2 \left[\frac{\cancel{2}x^2}{\cancel{2}} + yx \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$

$= \int_0^2 \left[x^2 + yx \right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$

$= \int_0^2 \left[\left\{ (\sqrt{4-y^2})^2 + y(\sqrt{4-y^2}) \right\} - \left\{ (-\sqrt{4-y^2})^2 + y(-\sqrt{4-y^2}) \right\} \right] dy$

$= \int_0^2 \left[\left\{ 4-y^2 + y\sqrt{4-y^2} \right\} - \left\{ 4-y^2 - y\sqrt{4-y^2} \right\} \right] dy$

$= \int_0^2 \left(\cancel{4}-\cancel{y^2} + y\sqrt{4-y^2} - \cancel{4} + \cancel{y^2} + y\sqrt{4-y^2} \right) dy$

$= \int_0^2 \left(2y\sqrt{4-y^2} \right) dy$

$= -\int_0^2 (4-y^2)^{1/2} (-2y)\,dy$

$= -\left[\frac{(4-y^2)^{3/2}}{3/2} \right]_0^2$

$= \frac{-2}{3} \left[(4-y^2)^{3/2} \right]_0^2$

$= \frac{-2}{3} \left[(4-(2)^2)^{3/2} - (4-(0)^2)^{3/2} \right]$

$= \frac{-2}{3} \left[(0)^{3/2} - (4)^{3/2} \right]$

$= \frac{-2}{3} \left[-(2^{\cancel{2}})^{3/\cancel{2}} \right]$

$= \frac{+2}{3} \left[+8 \right]$

$= \frac{16}{3}$

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Q18) $\int_0^1 \int_1^{\sqrt{e}} \int_1^e s e^s \ln r \frac{(\ln t)^2}{t} \, dt \, dr \, ds$

Here,

$$= \int_0^1 \int_1^{\sqrt{e}} s e^s \ln r \int_1^e \frac{(\ln t)^2}{t} \, dt \, dr \, ds$$

$$= \int_0^1 \int_1^{\sqrt{e}} s e^s \ln r \int_1^e (\ln t)^2 \cdot \frac{1}{t} \, dt \, dr \, ds$$

$$= \int_0^1 \int_1^{\sqrt{e}} s e^s \ln r \left[\frac{(\ln t)^3}{3} \right]_1^e dr \, ds$$

$$= \frac{1}{3} \int_0^1 \int_1^{\sqrt{e}} s e^s \ln r \left[(\ln t)^3 \right]_1^e dr \, ds$$

$$= \frac{1}{3} \int_0^1 \int_1^{\sqrt{e}} s e^s \ln r \left[\{ (\ln e)^3 - (\ln 1)^3 \} \right] dr \, ds$$

$$= \frac{1}{3} \int_0^1 \int_1^{\sqrt{e}} s e^s \ln r \, (1-0) \, dr \, ds$$

$$= \frac{1}{3} \int_0^1 s e^s \int_1^{\sqrt{e}} \ln r \, dr \, ds$$

Using By parts integration

Here,

$$\int_1^{\sqrt{e}} \ln r \, dr$$

$$= \int_1^{\sqrt{e}} (\ln r)(1) \, dr$$

$$= (\ln r)(r) - \int_1^{\sqrt{e}} r \cdot \frac{1}{r} \, dr$$

$$= r \ln r - \int_1^{\sqrt{e}} (1) \, dr$$

$$= r \ln r - r$$

So,

$$= \frac{1}{3} \int_0^1 s e^s \left[r \ln r - r \right]_1^{\sqrt{e}} ds$$

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$$= \frac{1}{3}\int_0^1 se^s \left[\left\{\sqrt{e}\,\ln(\sqrt{e}) - \sqrt{e}\right\} - \left\{(1)\ln(1) - 1\right\}\right] ds$$

$$= \frac{1}{3}\int_0^1 se^s \left[\left\{(\sqrt{e})\tfrac{1}{2}(1) - \sqrt{e}\right\} - \{0 - 1\}\right] ds$$

$$= \frac{1}{3}\int_0^1 \left[\left(\frac{\sqrt{e}}{2} - \sqrt{e}\right) + 1\right] se^s\, ds$$

$$= \frac{1}{3}\int_0^1 se^s \left(\frac{1-\sqrt{e}}{2}\right) ds$$

$$= \frac{1}{3}\left(\frac{1-\sqrt{e}}{2}\right)\int_0^1 se^s\, ds$$

(Using by parts integration

Here,

$$\int se^s\, ds$$

$$= (s)(e^s) - \int (e^s)(1)\, ds$$

$$= se^s - \int e^s\, ds$$

$$= se^s - e^s$$

So,

$$= \frac{1}{3}\left(\frac{1-\sqrt{e}}{2}\right)\left[se^s - e^s\right]_0^1$$

$$= \frac{1}{3}\left(\frac{1-\sqrt{e}}{2}\right)\left[(1)e^1 - e^1 - \{(0)e^0 - e^0\}\right]$$

$$= \frac{1}{3}\left(\frac{1-\sqrt{e}}{2}\right)\left[0 - (0-1)\right]$$

$$= \frac{1}{3}\left(\frac{1-\sqrt{e}}{2}\right)(1)$$

$$= \frac{1}{3}\left(\frac{1-\sqrt{e}}{2}\right)$$

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Q19) $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x \, dx\,dt\,dv$

Here,

$$= \int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x \, dx\,dt\,dv$$

$$= \int_0^{\pi/4}\int_0^{\ln\sec v} \left[e^x\right]_{-\infty}^{2t} dt\,dv$$

$$= \int_0^{\pi/4}\int_0^{\ln\sec v} \left(e^{2t} - e^{-\infty}\right) dt\,dv$$

$$= \int_0^{\pi/4}\int_0^{\ln\sec v} \left(e^{2t} - \lim_{x\to-\infty} e^x\right) dt\,dv$$

$$= \int_0^{\pi/4}\int_0^{\ln\sec v} \left(e^{2t} - 0\right) dt\,dv$$

$$= \int_0^{\pi/4}\int_0^{\ln\sec v} \left(e^{2t}\right) dt\,dv$$

$$= \int_0^{\pi/4} \left[\frac{e^{2t}}{2}\right]_0^{\ln\sec v} dv$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[e^{2t}\right]_0^{\ln\sec v} dv$$

$$= \frac{1}{2}\int_0^{\pi/4} \left[e^{2\ln\sec v} - e^0\right] dv$$

$$= \frac{1}{2}\int_0^{\pi/4} \left((\sec v)^2 - 1\right) dv$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\sec^2 v - 1\right) dv$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\tan^2 v\right) dv$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\sec^2 v - 1\right) dv$$

$$= \frac{1}{2}\left[\tan v - v\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\tan 0 - 0\right)\right]$$

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$$= \frac{1}{2}\left(\frac{1-\pi}{4}\right)$$

$$= \frac{1}{2}\left(\frac{4-\pi}{4}\right)$$

$$= \frac{4-\pi}{8}$$

$$= \frac{1}{2} - \frac{\pi}{8}$$

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