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Directional Derivatives

a) " If $f(x,y)$ is a function of 'x' and 'y' and if $\underline{u} = u_1\hat{i} + u_2\hat{j}$ is a unit vector, then the directional derivative of 'f' in the direction of 'u' at (x_0, y_0) is denoted by $D_u f(x_0, y_0)$ and is defined as

$$D_u f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + s u_1, y_0 + s u_2)\right]_{s=0}$$

b) " Similarly if $f(x,y,z)$ is a function of 'x', 'y', and 'z' and if $\underline{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ with point (x_0, y_0, z_0) then

$$D_u f(x_0, y_0, z_0) = \frac{d}{ds}\left[f(x_0 + s u_1, y_0 + s u_2, z_0 + s u_3)\right]_{s=0}$$

classwork

Q7) $f(x,y,z) = \ln(x^2 + 2y^2 + 3z^2)$;
$P(-1, 2, 4)$;
$\vec{u} = -\frac{3}{13}\hat{i} - \frac{4}{13}\hat{j} - \frac{12}{13}\hat{k}$

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As we know that

$$D_u f(-1,2,4) = \frac{d}{ds}\left[f\left(\frac{-1-3s}{13}, \frac{2-4s}{13}, \frac{4-12s}{13}\right)\right]_{s=0}$$

$$= \frac{d}{ds}\left[f\left(\frac{-13-3s}{13}, \frac{26-4s}{13}, \frac{52-12s}{13}\right)\right]_{s=0} \rightarrow (i)$$

Now we take

$$\Rightarrow f\left(\frac{-13-3s}{13}, \frac{26-4s}{13}, \frac{52-12s}{13}\right)$$

$$= \ln\left[\left(\frac{-13-3s}{13}\right)^2 + 2\left(\frac{26-4s}{13}\right)^2 + 3\left(\frac{52-12s}{13}\right)^2\right]_{s=0}$$

$$= \ln\left[\left\{-\left(\frac{13+3s}{13}\right)\right\}^2 + 2\left(\frac{676+16s^2-208s}{169}\right) + 3\left(\frac{2704+144s^2-1248s}{169}\right)\right]_{s=0}$$

$$= \ln\left[\left(\frac{169+9s^2+78s}{169}\right) + \left(\frac{1352+32s^2-416s}{169}\right) + \left(\frac{8112+432s^2-3744s}{169}\right)\right]_{s=0}$$

$$= \ln\left[\frac{169+9s^2+78s+1352+32s^2-416s+8112+432s^2-3744s}{169}\right]_{s=0}$$

$$= \ln\left[\frac{9633+473s^2-4082s}{169}\right]_{s=0} \rightarrow (ii)$$

Substituting (ii) into (i)

$$D_u f(-1,2,4) = \frac{d}{ds}\ln\left[\frac{9633+473s^2-4082s}{169}\right]_{s=0}$$

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$$D_u f(-1,2,4) = \left[\frac{1}{\frac{9633+473s^2-4082s}{169}} \cdot \frac{d}{ds}\left(\frac{9633+473s^2-4082s}{169}\right)\right]_{s=0}$$

$$= \left[\frac{\cancel{169}}{9633+473s^2-4082s} \cdot \frac{1}{\cancel{169}} \cdot (0+946s-4082)\right]_{s=0}$$

$$= \left[\frac{946s-4082}{9633+473s^2-4082s}\right]_{s=0}$$

$$= \frac{946(0)-4082}{9633+473(0)-4082(0)}$$

$$= \frac{-4082}{9633}$$

$$D_u f(-1,2,4) = \frac{-314}{741}$$

Q9) $f(x,y) = 4x^3y^2$; $P(2,1)$

$\vec{a} = 4\hat{i} - 3\hat{j}$

As we know that

$$D_{\vec{a}} f(2,1) = \frac{d}{ds}\left[f(2+4s, 1-3s)\right]_{s=0} \longrightarrow \text{(i)}$$

Now we take

$$f(2+4s, 1-3s) = 4(2+4s)^3(1-3s)^2 \longrightarrow \text{(ii)}$$

Substituting (ii) into (i)

$$D_{\vec{a}} f(2,1) = \frac{d}{ds}\left[4(2+4s)^3(1-3s)^2\right]_{s=0}$$

$$= 4\left[(2+4s)^3 \cdot 2(1-3s)(-3) + (1-3s)^2 \cdot 3(2+4s)^2(4)\right]_{s=0}$$

$$= 4\left[-6(2+4s)^3(1-3s) + 12(2+4s)^2(1-3s)^2\right]_{s=0}$$

$$= \left[-24(2+4s)^3(1-3s) + 48(2+4s)^2(1-3s)^2\right]_{s=0}$$

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$$D_{\vec{a}}f(2,1) = -24(2)^3(1) + 48(2)^2(1)^2$$
$$= -24(8) + 48(4)$$
$$= -192 + 192$$
$$= 0$$

Q13) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$; $P(-2,2)$

$$\vec{a} = -\hat{i} - \hat{j}$$

As we know that

$$D_{\vec{a}}f(-2,2) = \frac{d}{ds}\left[f(-2-s, 2-s)\right]_{s=0} \longrightarrow \text{(i)}$$

Now we take

$$f(-2-s, 2-s) = \tan^{-1}\left(\frac{2-s}{-2-s}\right) \longrightarrow \text{(ii)}$$

Substituting (ii) in (i)

$$D_{\vec{a}}f(-2,2) = \frac{d}{ds}\left[\tan^{-1}\left(\frac{2-s}{-2-s}\right)\right]_{s=0}$$
$$= \left[\frac{1}{1+\left(\frac{2-s}{-2-s}\right)^2} \cdot \frac{d}{ds}\left(\frac{2-s}{-2-s}\right)\right]_{s=0}$$
$$= \left[\frac{1}{1+\left(\frac{2-s}{-2-s}\right)^2} \cdot \frac{(-2-s)(-1)-(2-s)(-1)}{(-2-s)^2}\right]_{s=0}$$
$$= \left[\frac{1}{1+\left(\frac{2-s}{-2-s}\right)^2} \cdot \frac{2+\cancel{s}+2-\cancel{s}}{(-2-s)^2}\right]_{s=0}$$
$$= \left[\frac{1}{1+\left(\frac{2-s}{-2-s}\right)^2} \cdot \frac{4}{(-2-s)^2}\right]_{s=0}$$
$$= \frac{1}{1+\left(\frac{2}{-2}\right)^2} \cdot \frac{4}{(-2-0)^2}$$

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$$D_{\vec{a}}f(-2,2) = \frac{1}{1+(-1)^2} \cdot \frac{4}{4}$$

$$= \frac{1}{1+1} \cdot 1$$

$$= \frac{1}{2}$$

Normalizing the direction vector

Here,

$$\|\vec{a}\| = \sqrt{(-1)^2 + (-1)^2}$$

$$= \sqrt{1+1}$$

$$= \sqrt{2}$$

The directional derivative will now be divided by $\|\vec{a}\|$

$$D_u f(-2,2) = \frac{1/2}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{4}$$

Q15) $f(x,y,z) = xy + z^2$; $P(-3,0,4)$

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

As we know that

$$D_{\vec{a}}f(-3,0,4) = \frac{d}{ds}\left[f(-3+s,\ s,\ 4+s)\right]_{s=0} \longrightarrow (i)$$

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Now we take

$$f(-3+s, s, 4+s) = (-3+s)(s) + (4+s)^2$$
$$= -3s + s^2 + 16 + s^2 + 8s$$
$$= 2s^2 + 5s + 16 \longrightarrow \text{(ii)}$$

Substituting (ii) in (i)

$$D_{\vec{a}} f(-3,0,4) = \frac{d}{ds}(2s^2 + 5s + 16)_{s=0}$$
$$= [2 \cdot 2s + 5(1) + 0]_{s=0}$$
$$= [4s + 5]_{s=0}$$
$$= 4(0) + 5$$
$$= 5$$

Normalizing the direction vector

Here,

$$\|\vec{a}\| = \sqrt{(1)^2 + (1)^2 + (1)^2}$$
$$= \sqrt{1+1+1}$$
$$= \sqrt{3}$$

Dividing $\|\vec{a}\|$ by the direction vector, we get

$$D_{\vec{a}} f(-3,0,4) = \frac{5}{\sqrt{3}}$$

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homework.

Q1) $f(x,y) = (1+xy)^{3/2}$; $P(3,1)$

$\vec{u} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

As we know that

$$D_{\vec{u}}f(3,1) = \frac{d}{ds}\left[f\left(3+\frac{s}{\sqrt{2}}, 1+\frac{s}{\sqrt{2}}\right)\right]_{s=0} \longrightarrow (i)$$

Now we take

$$f\left(3+\frac{s}{\sqrt{2}}, 1+\frac{s}{\sqrt{2}}\right) = \left[1+\left(3+\frac{s}{\sqrt{2}}\right)\left(1+\frac{s}{\sqrt{2}}\right)\right]^{3/2}$$

$$= \left[1+3+\frac{3s}{\sqrt{2}}+\frac{s}{\sqrt{2}}+\frac{s^2}{2}\right]^{3/2}$$

$$= \left[4+\frac{4s}{\sqrt{2}}+\frac{s^2}{2}\right]^{3/2} \longrightarrow (ii)$$

Substituting (ii) in (i).

$$D_{\vec{u}}f(3,1) = \frac{d}{ds}\left[\left(4+\frac{4s}{\sqrt{2}}+\frac{s^2}{2}\right)^{3/2}\right]_{s=0}$$

$$= \left[\frac{3}{2}\left(4+\frac{4s}{\sqrt{2}}+\frac{s^2}{2}\right)^{1/2}\cdot\left(0+\frac{4.1}{\sqrt{2}}+\frac{2s.1}{2}\right)\right]_{s=0}$$

$$= \left[\frac{3}{2}\left(4+\frac{4s}{\sqrt{2}}+\frac{s^2}{2}\right)^{1/2}\cdot\left(\frac{4}{\sqrt{2}}+s\right)\right]_{s=0}$$

$$= \frac{3}{2}\left(4+\frac{4(0)}{\sqrt{2}}+\frac{0}{2}\right)^{1/2}\cdot\left(\frac{4}{\sqrt{2}}+0\right)$$

$$= \frac{3}{2}(4)^{1/2}\cdot\left(\frac{4}{\sqrt{2}}\right)$$

$$= \frac{3.\cancel{2}.4}{\cancel{2}\ \sqrt{2}}$$

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$$D_{\vec{u}}f(3,1) = \frac{12}{\sqrt{2}}$$

$$= \frac{6\times 2}{\sqrt{2}}$$

$$= \frac{6.(\sqrt{2})^2}{\sqrt{2}}$$

$$D_{\vec{u}}f(3,1) = 6\sqrt{2}$$

Normalizing the direction derivative

$$\|\vec{u}\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$= \sqrt{1}$$

$$= 1$$

Now,

$$D_{\vec{u}}f(3,1) = \frac{6\sqrt{2}}{1}$$

$$= 6\sqrt{2}$$

Q3) $f(x,y) = \ln(1+x^2+y)$; $P(0,0)$

$$\vec{u} = -\frac{1}{\sqrt{10}}\hat{i} - \frac{3}{\sqrt{10}}\hat{j}$$

As we know that

$$D_{\vec{u}}f(0,0) = \frac{d}{ds}\left[f\left(\frac{-1}{\sqrt{10}}s, \frac{-3}{\sqrt{10}}s\right)\right]_{s=0} \longrightarrow (i)$$

Now we take

$$f\left(\frac{-1}{\sqrt{10}}s, \frac{-3}{\sqrt{10}}s\right) = \ln\left[1 + \left(\frac{-1}{\sqrt{10}}s\right)^2 + \left(\frac{-3}{\sqrt{10}}s\right)\right]$$

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$$= \ln\left[1 + \frac{s^2}{10} - \frac{3s}{\sqrt{10}}\right] \longrightarrow \text{(ii)}$$

Substituting (ii) into (i)

$$D_{\vec{u}}f(0,0) = \frac{d}{ds}\left[\ln\left(1 + \frac{s^2}{10} - \frac{3s}{\sqrt{10}}\right)\right]_{s=0}$$

$$= \left[\frac{1}{1 + \frac{s^2}{10} - \frac{3s}{\sqrt{10}}} \cdot \left(0 + \frac{1}{\cancel{10}\,5} \cdot \cancel{2}s - \frac{3}{\sqrt{10}} \cdot 1\right)\right]_{s=0}$$

$$= \left[\frac{1}{1 + \frac{s^2}{10} - \frac{3s}{\sqrt{10}}}\left(\frac{s}{5} - \frac{3}{\sqrt{10}}\right)\right]_{s=0}$$

$$= \frac{1}{1+0+0}\left(0 - \frac{3}{\sqrt{10}}\right)$$

$$= 1\left(\frac{-3}{\sqrt{10}}\right)$$

$$D_{\vec{u}}f(0,0) = \frac{-3}{\sqrt{10}}$$

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Q5) $f(x,y,z) = 4x^5y^2z^3$; $P(2,-1,1)$

$$\vec{u} = \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$$

As we know that

$$D_{\vec{u}}f(2,-1,1) = \frac{d}{ds}\left[f\left(2+\frac{s}{3}, -1+\frac{2s}{3}, 1-\frac{2s}{3}\right)\right]_{s=0} \longrightarrow \text{(i)}$$

Now we take

$$\Rightarrow f\left(\frac{6+s}{3}, \frac{-3+2s}{3}, \frac{3-2s}{3}\right)$$

$$= 4\left(\frac{6+s}{3}\right)^5\left(\frac{-3+2s}{3}\right)^2\left(\frac{3-2s}{3}\right)^3 \longrightarrow \text{(ii)}$$

Substituting (ii) into (i)

$$D_{\vec{u}}f(2,-1,1) = \frac{d}{ds}\left[4\left(\frac{6+s}{3}\right)^5\left(\frac{-3+2s}{3}\right)^2\left(\frac{3-2s}{3}\right)^3\right]_{s=0}$$

$$= \left[4\left\{5\left(\frac{6+s}{3}\right)^4 \cdot \frac{1}{3}(0+1) \cdot \left(\frac{-3+2s}{3}\right)^2\left(\frac{3-2s}{3}\right)^3\right\}\right.$$

$$+ \left(\frac{6+s}{3}\right)^5 \cdot 2\left(\frac{-3+2s}{3}\right) \cdot \frac{1}{3}(0+2) \cdot \left(\frac{3-2s}{3}\right)^3$$

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$$= \left(4\left[5\left(\frac{6+s}{3}\right)^4 \cdot \frac{1}{3}(1) \cdot \left(\frac{-3+2s}{3}\right)^2 \cdot \left(\frac{3-2s}{3}\right)^3 + \left(\frac{6+s}{3}\right)^5 \cdot 2\left(\frac{-3+2s}{3}\right) \cdot \frac{1}{3}(2) \cdot \left(\frac{3-2s}{3}\right)^3 + \left(\frac{6+s}{3}\right)^5 \cdot \left(\frac{-3+2s}{3}\right)^2 \cdot 3\left(\frac{3-2s}{3}\right)^2 \cdot \frac{1}{3}(-2)\right]\right)_{s=0}$$

$$= 4\left[5\left(\frac{6}{3}\right)^4 \cdot \frac{1}{3} \cdot \left(\frac{-3}{3}\right)^2 \cdot \left(\frac{3}{3}\right)^3 + \left(\frac{6}{3}\right)^5 \cdot 2\left(\frac{-\cancel{3}}{\cancel{3}}\right) \cdot \frac{2}{3} \cdot \left(\frac{\cancel{3}}{\cancel{3}}\right)^3 + \left(\frac{6}{3}\right)^5 \cdot \left(\frac{-\cancel{3}}{\cancel{3}}\right)^2 \cdot 3\left(\frac{\cancel{3}}{\cancel{3}}\right)^2\left(\frac{-2}{3}\right)\right]$$

$$= 4\left[\frac{5}{3}\left(\frac{1296}{81}\right)(-1)^2(1)^3 + \left(\frac{7776}{243}\right) \cdot 2(-1) \cdot \frac{2}{3} \cdot (1)^3 + \left(\frac{7776}{243}\right)(-1)^2 \cdot \cancel{3}(1)^2\left(\frac{-2}{\cancel{3}}\right)\right]$$

$$= 4\left[\frac{5}{3}(16) + (32)\left(\frac{-4}{3}\right) + (32)(-2)\right]$$

$$= 4\left(\frac{80}{3} - \frac{128}{3} - 64\right)$$

$$= 4\left(\frac{80-128-192}{3}\right)$$

$$= 4\left(\frac{80-320}{3}\right)$$

$$= 4\left(\frac{-240}{3}\right)$$

$$= 4(-80)$$

$$= -320$$

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Q11) $f(x,y) = y^2 \ln x$; $P(1,4)$

$\vec{a} = -3\hat{i} + 3\hat{j}$

As we know that

$$D_{\vec{a}} f(1,4) = \frac{d}{ds}\left[f(1-3s, 4+3s)\right]_{s=0} \longrightarrow \text{(i)}$$

Now we take

$$f(1-3s, 4+3s) = (4+3s)^2 \ln(1-3s) \longrightarrow \text{(ii)}$$

Substituting (ii) into (i)

$$D_{\vec{a}} f(1,4) = \frac{d}{ds}\left[(4+3s)^2 \ln(1-3s)\right]_{s=0}$$

$$= \left[(4+3s)^2 \cdot \frac{1}{1-3s}(-3) + \ln(1-3s) \cdot 2(4+3s)(3)\right]_{s=0}$$

$$= \left[\frac{-3(4+3s)^2}{1-3s} + 6(4+3s)\ln(1-3s)\right]_{s=0}$$

$$= \left[\frac{-3(4+0)^2}{1} + 6(4)\ln(1)\right]$$

$$= -3(4)^2 + 24(0)$$

$$= -3(16)$$

$$= -48$$

Normalizing the direction vector

$$\|\vec{a}\| = \sqrt{(-3)^2 + (3)^2}$$

$$= \sqrt{9+9}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

Now,

$$D_{\vec{a}} f(1,4) = \frac{-\cancel{48}}{\cancel{3}\sqrt{2}}$$

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$$D_{\vec{a}}f(1,4) = \frac{-16}{\sqrt{2}}$$

$$= \frac{-16}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{-\cancel{16}\sqrt{2}}{\cancel{2}}$$

$$D_{\vec{a}}f(1,4) = -8\sqrt{2}$$

Q17) $f(x,y,z) = \dfrac{z-x}{z+y}$; $P(1,0,-3)$

$$\vec{a} = -6\hat{i} + 3\hat{j} - 2\hat{k}$$

As we know that

$$D_{\vec{a}}f(1,0,-3) = \frac{d}{ds}\left[f(1-6s, 3s, -3-2s)\right]_{s=0} \longrightarrow \text{(i)}$$

Now we take

$$f(1-6s, 3s, -3-2s) = \frac{-3-2s-(1-6s)}{-3-2s+3s}$$

$$= \frac{-3-2s-1+6s}{-3+s}$$

$$= \frac{-4+4s}{-3+s} \longrightarrow \text{(ii)}$$

Substituting (ii) into (i)

$$D_{\vec{a}}f(1,0,-3) = \frac{d}{ds}\left[\frac{-4+4s}{-3+s}\right]_{s=0}$$

$$= \left[\frac{(-3+s)(4)-(-4+4s)(1)}{(-3+s)^2}\right]_{s=0}$$

$$= \frac{(-3)(4)-(-4)(1)}{(-3)^2}$$

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$$D_{\vec{a}}f(1,0,-3) = \frac{-12+4}{9}$$

$$= \frac{-8}{9}$$

Normalizing the direction vector

$$\|\vec{a}\| = \sqrt{(-6)^2+(3)^2+(-2)^2}$$

$$= \sqrt{36+9+4}$$

$$= \sqrt{49}$$

$$= 7$$

Now,

$$D_{\vec{a}}f(1,0,-3) = \frac{-8/9}{7}$$

$$= \frac{-8}{9\times 7}$$

$$= \frac{-8}{63}$$

Q26) $f(x,y) = \dfrac{y}{x+y}$

Find a unit vector $\vec{u}$ for which $D_{\vec{u}}f(2,3) = 0$

Here,

$$f(x,y) = \frac{y}{x+y} \quad ; \quad P(2,3)$$

and,

we have to find $\vec{u}$ such that

$$D_{\vec{u}}f(2,3) = 0$$

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Using Gradient Method

As we know that

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$$

Here,

$$\Rightarrow f_x = \frac{\partial}{\partial x}\left(\frac{y}{x+y}\right)$$

$$= \frac{-y}{(x+y)^2}$$

$$f_x(2,3) = \frac{-3}{(2+3)^2}$$

$$= \frac{-3}{(5)^2}$$

$$= \frac{-3}{25}$$

$$\Rightarrow f_y = \frac{\partial}{\partial y}\left(\frac{y}{x+y}\right)$$

$$= \frac{(x+y)(1)-(y)(1)}{(x+y)^2}$$

$$= \frac{x+y-y}{(x+y)^2}$$

$$= \frac{x}{(x+y)^2}$$

$$f_y(2,3) = \frac{2}{(2+3)^2}$$

$$= \frac{2}{25}$$

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Thus, Gradient at $P(2,3)$ is

$$\nabla f(2,3) = \frac{-3}{25}\hat{i} + \frac{2}{25}\hat{j}$$

$$= \left(\frac{-3}{25}, \frac{2}{25}\right)$$

Now,

The directional derivative $D_{\vec{u}}f(2,3)$ is zero only when gradient is orthogonal to $\vec{u}$, such that

$$\nabla f(2,3) \cdot \vec{u} = 0$$

where; $\vec{u} = (\vec{u}_1, \vec{u}_2)$

Thus,

$$\nabla f(2,3) \cdot (\vec{u}_1, \vec{u}_2) = 0$$

$$\left(\frac{-3}{25}, \frac{2}{25}\right) \cdot (\vec{u}_1, \vec{u}_2) = 0$$

$$\frac{-3}{25}\vec{u}_1 + \frac{2}{25}\vec{u}_2 = 0$$

$$-3\vec{u}_1 + 2\vec{u}_2 = 0$$

$$2\vec{u}_2 = 3\vec{u}_1$$

$$\vec{u}_2 = \frac{3\vec{u}_1}{2}$$

Since $\vec{u}$ is a unit vector

So, $(\vec{u}_1)^2 + (\vec{u}_2)^2 = 1$

$$\vec{u}_1^{\,2} + \left(\frac{3\vec{u}_1}{2}\right)^2 = 1$$

$$\vec{u}_1^{\,2} + \frac{9}{4}\vec{u}_1^{\,2} = 1$$

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$$\frac{13}{4}\vec{u}_1^2 = 1$$

$$\vec{u}_1^2 = \frac{4}{13}$$

$$\vec{u}_1 = \pm\frac{2}{\sqrt{13}}$$

Similarly,

$$\vec{u}_2 = \frac{3}{2}\vec{u}_1$$

$$= \frac{3}{\cancel{2}}\left(\pm\frac{\cancel{2}}{\sqrt{13}}\right)$$

$$= \pm\frac{3}{\sqrt{13}}$$

Thus, two possible unit vectors are

$$\vec{u} = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right) \text{ or } \left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$$

Q27) Find the direction derivative of

$$f(x,y,z) = \frac{y}{x+z}$$

at $P(2,1,-1)$ in the direction from P to $Q(-1,2,0)$.

Here,

$$f(x,y,z) = \frac{y}{x+z} \; ; \; P(2,-1,-1)$$

and the direction derivative $\vec{v}$ is given as

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$$\vec{v} = \overrightarrow{PQ}$$
$$= (-1-2,\ 2-1,\ 0+1)$$
$$= (-3,\ 1,\ 1)$$
or, $-3\hat{i} + \hat{j} + \hat{k}$

Using Directional derivative via parameterization method.

As we know that

$$D_{\vec{v}} f(2,1,-1) = \frac{d}{ds}\left[(2-3s,\ 1+s,\ -1+s)\right]_{s=0} \longrightarrow \text{(i)}$$

Now we take

$$f(2-3s,\ 1+s,\ -1+s) = \frac{1+s}{2-3s-1+s}$$
$$= \frac{1+s}{1-2s} \longrightarrow \text{(ii)}$$

Substituting (ii) into (i)

$$D_{\vec{v}} f(2,1,-1) = \frac{d}{ds}\left[\frac{1+s}{1-2s}\right]_{s=0}$$
$$= \left[\frac{(1-2s)(1)-(1+s)(-2)}{(1-2s)^2}\right]_{s=0}$$
$$= \left[\frac{1-\cancel{2s}+2+\cancel{2s}}{(1-2s)^2}\right]_{s=0}$$
$$= \left(\frac{3}{(1-2s)^2}\right)_{s=0}$$
$$= \frac{3}{(1)^2}$$
$$= 3$$

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Normalizing the direction vector

$$\|\vec{v}\| = \sqrt{(-3)^2+(1)^2+(1)^2}$$
$$= \sqrt{9+1+1}$$
$$= \sqrt{11}$$

Now,

$$D_{\vec{v}}f(2,1,-1) = \frac{3}{\sqrt{11}}$$

Examples

E1) $f(x,y) = xy$; $P(1,2)$

$$\vec{u} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

As we know that

$$D_{\vec{u}}f(1,2) = \frac{d}{ds}\left[f\left(1+\frac{\sqrt{3}s}{2}, 2+\frac{s}{2}\right)\right]_{s=0} \rightarrow \text{(i)}$$

Now we take

$$f\left(1+\frac{\sqrt{3}s}{2}, 2+\frac{s}{2}\right) = \left(1+\frac{\sqrt{3}s}{2}\right)\left(2+\frac{s}{2}\right)$$
$$= 2+\frac{s}{2}+\sqrt{3}s+\frac{\sqrt{3}.s^2}{4}$$
$$= \frac{\sqrt{3}s^2}{4}+\frac{s}{2}+\sqrt{3}s+2 \rightarrow \text{(ii)}$$

Substituting (ii) into (i)

$$D_{\vec{u}}f(1,2) = \frac{d}{ds}\left(\frac{\sqrt{3}s^2}{4}+\frac{s}{2}+\sqrt{3}s+2\right)_{s=0}$$

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$$D_{\vec{u}}f(1,2) = \left[\frac{\sqrt{3}}{4}\cdot 2s + \frac{1}{2} + \sqrt{3} + 0\right]_{s=0}$$

$$= \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}\right]_{s=0}$$

$$= 0 + \frac{1}{2} + \sqrt{3}$$

$$= 0.5 + \sqrt{3}$$

$$D_{\vec{u}}f(1,2) = 2.23$$

E2) Find the directional derivative of $f(x,y) = e^{xy}$ at $(-2,0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.

As we know that

$$D_{\vec{u}}f(-2,0) = f_x(-2,0)\,\vec{u_1} + f_y(-2,0)\,\vec{u_2} \longrightarrow (i)$$

Here,

$$\Rightarrow f_x = \frac{\partial}{\partial x}(e^{xy})$$

$$= e^{xy}\cdot y(1)$$

$$= ye^{xy}$$

$$\Rightarrow f_y = \frac{\partial}{\partial y}(e^{xy})$$

$$= e^{xy}\cdot x(1)$$

$$= xe^{xy}$$

Also,

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$$\Rightarrow f_x(-2,0) = (0)\,e^{(-2)(0)} = 0$$

$$\Rightarrow f_y(-2,0) = (-2)\,e^{(-2)(0)}$$

$$= (-2)(1)$$

$$= -2$$

The unit vector $\vec{u}$ that makes an angle of $\pi/3$ with the positive x-axis is

$$\vec{u} = \cos\left(\frac{\pi}{3}\right)\hat{i} + \sin\left(\frac{\pi}{3}\right)\hat{j}$$

$$\vec{u} = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

Now eqn (i) becomes

$$D_{\vec{u}}f(-2,0) = (0)\left(\frac{1}{2}\right) + (-\cancel{2})\left(\frac{\sqrt{3}}{\cancel{2}}\right)$$

$$= 0 - \sqrt{3}$$

$$= -\sqrt{3}$$

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E3) Find the directional derivative of $f(x,y,z) = x^2y - yz^3 + z$ at the point $(1,-2,0)$ in the direction of the vector $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$.

As we know that

$$D_{\vec{a}}f(1,-2,0) = \frac{d}{ds}\left[f(1+2s, -2+s, -2s)\right]_{s=0} \longrightarrow (i)$$

Now we take

$$f(1+2s, -2+s, -2s) = (1+2s)^2(-2+s) - (-2+s)(-2s)^3 + (-2s)$$

$$= (1+4s^2+4s)(-2+s) - (-2+s)(-8s^3) - 2s$$

$$= -2 - 8s^2 - 8s + s + 4s^3 + 4s^2 - (16s^3 - 8s^4) - 2s$$

$$= -2 - 4s^2 - 7s + 4s^3 - 16s^3 + 8s^4 - 2s$$

$$= 8s^4 - 12s^3 - 4s^2 - 9s - 2 \longrightarrow (ii)$$

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Substituting (ii) into (i)

$$D_{\vec{a}}f(1,-2,0) = \frac{d}{ds}\left[(8s^4 - 12s^3 - 4s^2 - 9s - 2)\right]_{s=0}$$

$$= \left[32s^3 - 36s^2 - 8s - 9 - 0\right]_{s=0}$$

$$= 0 - 0 - 0 - 9$$

$$= -9$$

Normalizing the direction vector

$$\|\vec{a}\| = \sqrt{(2)^2 + (1)^2 + (-2)^2}$$

$$= \sqrt{4+1+4}$$

$$= \sqrt{9}$$

$$= 3$$

Now,

$$D_{\vec{a}}f(1,-2,0) = \frac{-9}{3}$$

$$= -3$$

E4) Let $f(x,y) = x^2e^y$. Find the maximum value of a directional derivative at $(-2,0)$, and find the unit vector in the direction in which the maximum value occurs.

As we know that

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j}$$

Here,

$$\Rightarrow f_x = \frac{\partial}{\partial x}(x^2e^y)$$

$$= e^y \cdot 2x$$

$$= 2e^yx$$

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$$f_x(-2,0) = 2e^0(-2)$$
$$= 2(1)(-2)$$
$$= -4$$

Also,

$$\Rightarrow f_y = \frac{\partial}{\partial y}(e^y x^2)$$
$$= x^2 e^y$$
$$f_y(-2,0) = (-2)^2 e^0$$
$$= 4$$

Thus,

$$\nabla f(-2,0) = -4\hat{i} + 4\hat{j}$$

Theorem 13.6.5

"Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$. Assume that f is differentiable at P.

→ If $\nabla f = 0$ at P, then all directional derivatives of f at P are zero.

→ If $\nabla f \neq 0$ at P, then among all possible directional derivative of f at P, the derivative in the direction of ∇f at P has largest value. The value of this largest direction derivative is $\|\nabla f\|$ at P.

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→ If $\nabla f \neq 0$ at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has smallest value. The value of this smallest directional derivate is $-\|\nabla f\|$ at P."

Now using Theorem 13.6.5, the maximum value of directional derivative is

$$\|\nabla f(-2,0)\| = \sqrt{(-4)^2 + (4)^2}$$
$$= \sqrt{16+16}$$
$$= \sqrt{32}$$
$$= 4\sqrt{2}$$

This max. occurs in the direction of $\nabla f(-2,0)$. The unit vector in this direction is

$$\vec{u} = \frac{\nabla f(-2,0)}{\|\nabla f(-2,0)\|}$$
$$= \frac{-4\hat{i} + 4\hat{j}}{4\sqrt{2}}$$
$$= \frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$