Lecture 4

Chapter 1. The Foundations

- Predicates and Quantifiers
- Nested Quantifiers

Previously…

- In predicate logic, a ***predicate*** is modeled as a ***proposional function P(·)*** from subjects to propositions.
  - $P(x)$: "x is a prime number" (x: any subject)
  - $P(3)$: "3 is a prime number." (proposition!)
- Propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take
  - $P(x,y,z)$: "***x*** gave ***y*** the grade ***z***"
  - P(Mike,Mary,A): "**Mike** gave **Mary** the grade **A**."

Universe of Discourse (U.D.)

- The power of distinguishing subjects from predicates is that it lets you state things about *many* objects at once.
- e.g., let $P(x)$ = "$x + 1 > x$". We can then say, "For ***any*** number x, $P(x)$ is true" instead of $(\mathbf{0} + 1 > \mathbf{0}) \wedge (\mathbf{1} + 1 > \mathbf{1}) \wedge (\mathbf{2} + 1 > \mathbf{2}) \wedge \ldots$
- The collection of values that a variable x can take is called x's ***universe of discourse*** or the ***domain of discourse*** (often just referred to as the ***domain***).

Quantifier Expressions

- ***Quantifiers*** provide a notation that allows us to *quantify* (count) *how many* objects in the universe of discourse satisfy the given predicate.
- "∀" is the FOR∀LL or ***universal*** quantifier.
  $\forall x\, P(x)$ means *for all* x in the domain, $P(x)$.
- "∃" is the ∃XISTS or ***existential*** quantifier.
  $\exists x\, P(x)$ means there *exists* an x in the domain (that is, 1 or more) such that $P(x)$.

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The Universal Quantifier $\forall$

- ***$\forall x\, P(x)$: For all x in the domain, P(x).***
- $\forall x\, P(x)$ is
  - *true* if $P(x)$ is true <u>for every</u> x in D (*D: domain of discourse*)
  - *false* if $P(x)$ is false for <u>at least one</u> x in D
    - For every real number x, $x^2 \geq 0$ TRUE
    - For every real number x, $x_2 - 1 > 0$ FALSE
- A ***counterexample*** to the statement $\forall x\, P(x)$ is a value x in the domain D that makes $P(x)$ false
- What is the truth value of $\forall x\, P(x)$ when the domain is empty? TRUE

The Universal Quantifier $\forall$

- If all the elements in the domain can be listed as x_1, $x_2,\ldots, x_n$ then, $\forall x\, P(x)$ is the same as the conjunction:

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

- Example: Let the domain of x be parking spaces at UH. Let $P(x)$ be the statement "x is full." Then the ***universal quantification*** of $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at UH are full."
  - or "Every parking space at UH is full."
  - or "For each parking space at UH, that space is full."

The Existential Quantifier $\exists$

- ***$\exists x\, P(x)$: There exists an x in the domain (that is, 1 or more) such that $P(x)$.***
- $\exists x\, P(x)$ is
  - *true* if $P(x)$ is true for at least one x in the domain
  - *false* if $P(x)$ is false for every x in the domain
- What is the truth value of $\exists x\, P(x)$ when the domain is empty? FALSE
- If all the elements in the domain can be listed as $x_1, x_2, \ldots, x_n$ then, $\exists x\, P(x)$ is the same as the disjunction:

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

The Existential Quantifier ∃

- Example:
  Let the domain of *x* be parking spaces at UH.
  Let $P(x)$ be the statement "*x* is full."
  Then the ***existential quantification*** of $P(x)$,
  $\exists x\, P(x)$, is the *proposition*:
  - "Some parking spaces at UH are full."
  - or "There is a parking space at UH that is full."
  - or "At least one parking space at UH is full."

Free and Bound Variables

- An expression like $P(x)$ is said to have a ***free variable x*** (meaning, x is undefined).

- A quantifier (either ∀ or ∃) *operates* on an expression having one or more free variables, and ***binds*** one or more of those variables, to produce an expression having one or more ***bound variables***.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\, P(x,y)$ has 1 free variable y and one bound variable. x [Which is which?]
- "$P(x)$, where $x = 3$" is another way to bind x.
- An expression with <u>zero</u> free variables is a bona-fide (actual) proposition.
- An expression with <u>one or more</u> free variables is not a proposition:

  e.g. $\forall x\, P(x,y) = Q(y)$

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x{>}0\ P(x)$ is shorthand for
    "For all x that are greater than zero, $P(x)$."
    $= \forall x\ (x > 0 \rightarrow P(x))$
  - $\exists x{>}0\ P(x)$ is shorthand for
    "There is an x greater than zero such that $P(x)$."
    $= \exists x\ (x > 0 \wedge P(x))$

Translating from English

- Express the statement "*Every student in this class has studied calculus*" using predicates and quantifiers.
  - Let $C(x)$ be the statement: "x has studied calculus."
  - If <u>domain for x consists of the students in this class</u>, then
  - it can be translated as $\forall x\, C(x)$

or

  - If <u>domain for x consists of all people</u>
  - Let $S(x)$ be the predicate: "x is in this class"
  - Translation: $\forall x\, (S(x) \rightarrow C(x))$

Translating from English

- Express the statement "*Some students in this class has visited Mexico*" using predicates and quantifiers.
  - Let $M(x)$ be the statement: "x has visited Mexico"
  - If <u>domain for x consists of the students in this class</u>, then
  - it can be translated as $\exists x\, M(x)$

  or
  - If <u>domain for x consists of all people</u>
  - Let $S(x)$ be the statement: "x is in this class"
  - Then, the translation is $\exists x\, (S(x) \land M(x))$

Translating from English

- Express the statement "*Every student in this class has visited either Canada or Mexico*" using predicates and quantifiers.
  - Let $C(x)$ be the statement: "x has visited Canada" and $M(x)$ be the statement: "x has visited Mexico"
  - If domain for x consists of the students in this class, then
  - it can be translated as $\forall x\ (C(x) \vee M(x))$

Negations of Quantifiers

- $\forall x\, P(x)$: "Every student in the class has taken a course in calculus" ($P(x)$: "x has taken a course in calculus")
  - "Not every student in the class … calculus"

    $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

- Consider $\exists x\, P(x)$: "There is a student in the class who has taken a course in calculus"
  - "There is no student in the class who has taken a course in calculus"

    $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

Negations of Quantifiers

- Definitions of quantifiers: If the domain = {*a*, *b*, *c*,…}
  - $\forall x\, P(x) \equiv P(a) \land P(b) \land P(c) \land \cdots$
  - $\exists x\, P(x) \equiv P(a) \lor P(b) \lor P(c) \lor \cdots$
- From those, we can prove the laws:
  - $$\begin{aligned}\neg\forall x\, P(x) &\equiv \neg(P(a) \land P(b) \land P(c) \land \cdots) \\ &\equiv \neg P(a) \lor \neg P(b) \lor \neg P(c) \lor \cdots \\ &\equiv \exists x\, \neg P(x)\end{aligned}$$
  - $$\begin{aligned}\neg\exists x\, P(x) &\equiv \neg(P(a) \lor P(b) \lor P(c) \lor \cdots) \\ &\equiv \neg P(a) \land \neg P(b) \land \neg P(c) \land \cdots \\ &\equiv \forall x\, \neg P(x)\end{aligned}$$
- Which *propositional* equivalence law was used to prove this?

DeMorgan's

Negations of Quantifiers

Theorem:

- **Generalized De Morgan's laws for logic**

1. $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

2. $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

Negations: Examples

- What are the negations of the statements $\forall x\,(x^2 > x)$ and $\exists x\,(x^2 = 2)$?
  - $\neg\forall x\,(x^2 > x) \equiv \exists x\,\neg(x_2 > x) \equiv \exists x\,(x^2 \leq x)$
  - $\neg\exists x\,(x_2 = 2) \equiv \forall x\,\neg(x_2 = 2) \equiv \forall x\,(x^2 \neq 2)$
- Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.
  - $\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x\,\neg(P(x) \rightarrow Q(x))$

$$\equiv \exists x\,\neg(\neg P(x) \lor Q(x))$$

$$\equiv \exists x\,(P(x) \land \neg Q(x))$$

Summary

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Nesting of Quantifiers

- Example:

  Let the domain of x and y be people.

  Let $L(x,y)$ = "x likes y" (A statement with 2 free variables – not a proposition)

  - Then $\exists y\, L(x,y)$ = "There is someone whom x likes." (A statement with 1 free variable x – not a proposition)

  - Then $\forall x\, (\exists y\, L(x,y))$ =
    "Everyone has someone whom they like."
    (A Proposition with 0 free variables.)

Nested Quantifiers

- Nested quantifiers are quantifiers that occur within the scope of other quantifiers.
- The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Nested Quantifiers

- Let the domain of x and y is **R**, and $P(x,y)$: $xy = 0$.

  Find the truth value of the following propositions.
  - $\forall x\, \forall y\, P(x, y)$ (F)
  - $\forall x\, \exists y\, P(x, y)$ (T)
  - $\exists x\, \forall y\, P(x, y)$ (T)
  - $\exists x\, \exists y\, P(x, y)$ (T)

R: set of real numbers

- $\forall x\, \exists y\, P(x,y) \not\equiv \exists y\, \forall x\, P(x,y)$
  - For every x, there exists y such that $x + y = 0$. (T)
  - There exists y such that, for every x, $x + y = 0$. (F)

Nested Quantifiers: Example

- Let the domain = {1, 2, 3}. Find an expression equivalent to $\forall x \; \exists y \; P(x,y)$ where the variables are bound by substitution instead:
  - Expand from inside out or outside in.
  - Outside in:

$$\forall x \, \exists y \, P(x,y)$$

$$\equiv \exists y \, P(1,y) \wedge \exists y \, P(2,y) \wedge \exists y \, P(3,y)$$

$$\equiv [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge$$
$$[P(2,1) \vee P(2,2) \vee P(2,3)] \wedge$$
$$[P(3,1) \vee P(3,2) \vee P(3,3)]$$

Quantifier Exercise

- If $R(x,y)$="x relies upon y," express the following in unambiguous English when the domain is all people

$\forall x(\exists y\, R(x,y))$ = Everyone has *someone* to rely on.

$\exists y(\forall x\, R(x,y))$ = There's a poor overburdened soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y\, R(x,y))$ = There's some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x\, R(x,y))$ = Everyone has *someone* who relies upon them.

$\forall x(\forall y\, R(x,y))$ = *Everyone* relies upon *everybody*, (including themselves)!

Negating Nested Quantifiers

- Successively apply the rules for negating statements involving a single quantifier
- Example: Express the negation of the statement $\forall x\, \exists y\, (P(x,y) \land \exists z\, R(x,y,z))$ so that all negation symbols immediately precede predicates.
  - $\neg \forall x\, \exists y\, (P(x,y) \land \exists z\, R(x,y,z))$

    $\equiv \exists x\, \neg \exists y\, (P(x,y) \land \exists z\, R(x,y,z))$

    $\equiv \exists x\, \forall y\, \neg (P(x,y) \land \exists z\, R(x,y,z))$

    $\equiv \exists x\, \forall y\, (\neg P(x,y) \lor \neg \exists z\, R(x,y,z))$

Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$

  $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$

- $\forall x\ \mathbf{(}P(x) \land Q(x)\mathbf{)} \equiv \mathbf{(}\forall x\ P(x)\mathbf{)} \land \mathbf{(}\forall x\ Q(x)\mathbf{)}$

  $\exists x\ \mathbf{(}P(x) \lor Q(x)\mathbf{)} \equiv \mathbf{(}\exists x\ P(x)\mathbf{)} \lor \mathbf{(}\exists x\ Q(x)\mathbf{)}$

- Exercise:

  See if you can prove these yourself.

Notational Conventions

- Quantifiers have higher precedence than all logical operators from propositional logic:

$$(\forall x\, P(x)) \wedge Q(x)$$

- Consecutive quantifiers of the same type can be combined:

$\forall x\ \forall y\ \forall z\ P(x,y,z) \equiv \forall x,y,z\ P(x,y,z)$ or even $\forall xyz\ P(x,y,z)$