Exercise 4.4

Question 1:—

$\{(2,1), (3,0)\}$

→ The given vectors $(2,1)$ and $(3,0)$ are linearly independent if the vector equation

$$c_1(2,1) + c_2(3,0) = (0,0) \longrightarrow \text{(i)}$$

has only trivial solution.

→ For these vectors to span $\mathbb{R}^2$, it must be possible to express every vector $\vec{b} = (b_1, b_2)$ in $\mathbb{R}^2$ as

$$c_1(2,1) + c_2(3,0) = (b_1, b_2) \longrightarrow \text{(ii)}$$

equ (i) and (ii) can be rewritten as linear systems

$$\begin{aligned} 2c_1 + 3c_2 &= 0 \\ 1c_1 + 0c_2 &= 0 \end{aligned} \qquad \Big\} \qquad \begin{aligned} 2c_1 + 3c_2 &= b_1 \\ 1c_1 + 0c_2 &= b_2 \end{aligned}$$

Since,

$$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = (2)(0) - (3)(1) = -3$$

As determinant $= -3 \neq 0$:

→ The homogeneous system has only trivial solution so vectors are linearly independent.

→ The nonhomogeneous system is consistent for all reals values b_1 and b_2 ~~so they~~ and span

Question 2 :—

$$\{(3,1,-4),\ (2,5,6),\ (1,4,8)\}$$

⇒ The given vectors $(3,1,-4)$, $(2,5,6)$, $(1,4,8)$ are linearly independent if the vector equation

$$c_1(3,1,-4) + c_2(2,5,6) + c_3(1,4,8) = (0,0,0) \quad \text{(i)}$$

has only trivial solution.

⇒ For these vectors to span $\mathbb{R}^3$, it must be possible to express every vector $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ as

$$c_1(3,1,-4) + c_2(2,5,6) + c_3(1,4,8) = (b_1, b_2, b_3) \quad \text{(ii)}$$

equ (i) and (ii) can be written as linear systems

$$\begin{aligned} 3c_1 + 2c_2 + 1c_3 &= 0 \\ 1c_1 + 5c_2 + 4c_3 &= 0 \\ -4c_1 + 6c_2 + 8c_3 &= 0 \end{aligned}$$

$$\begin{aligned} 3c_1 + 2c_2 + 1c_3 &= b_1 \\ 1c_1 + 5c_2 + 4c_3 &= b_2 \\ 4c_1 + 6c_2 + 8c_3 &= b_3 \end{aligned}$$

The determinant of coefficient matrix of both systems is given as

$$= \begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix}$$

$$= 3\begin{vmatrix} 5 & 4 \\ 6 & 8 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ -4 & 8 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ -4 & 6 \end{vmatrix}$$

$$= 3(40-24) - 2(8+16) + 1(6+20)$$

$$= 3(16) - 2(24) + 1(26)$$

$$= 48 - 48 + 26$$

$$= 26$$

As the determinant $= 26 \neq 0$

→ The homogeneous system has only the trivial solution so the vectors are linearly independent.

→ The non-homogeneous system is consistent for all real values b_1, b_2 and b_3 and span R^3 so they form a basis for R^3.

Question 1 - Remaining

$\mathbb{R}^2$ so they form a basis for $\mathbb{R}^2$.

Question 4.– (Question 3)

$1+x$, $1-x$, $1-x^2$, $1-x^3$

→ The given polynomials $1+x$, $1-x$, $1-x^2$, $1-x^3$ are linearly independent if the equation

$$c_1(1+x) + c_2(1-x) + c_3(1-x^2) + c_4(1-x^3) = 0 \rightarrow \text{(i)}$$

has only the trivial solution.

→ For these polynomials to span P_3, it must be possible to express every polynomial as

$$c_1(1+x) + c_2(1-x) + c_3(1-x^2) + c_4(1-x^3) = a_0 + a_1x + a_2x^2 + a_3x^3 \rightarrow \text{(ii)}$$

For equ (i), we have

$$c_1(1+x) + c_2(1-x) + c_3(1-x^2) + c_4(1-x^3) = 0$$

$$c_1 + c_1x + c_2 - c_2x + c_3 - c_3x^2 + c_4 - c_4x^3 = 0$$

$$(c_1 + c_2 + c_3 + c_4) + (c_1 - c_2)x + (-c_3)x^2 + (-c_4)x^3 = 0$$

The linear system is given as

$$1c_1 + 1c_2 + 1c_3 + 1c_4 = 0$$

$$1c_1 - 1c_2 + 0c_3 + 0c_4 = 0$$

$$0c_1 + 0c_2 - 1c_3 + 0c_4 = 0$$

$$0c_1 + 0c_2 + 0c_3 - 1c_4 = 0$$

From equ (ii), we have

$$c_1(1+x) + c_2(1-x) + c_3(1-x^2) + c_4(1-x^3) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$c_1 + c_1x + c_2 - c_2x + c_3 - c_3x^2 + c_4 - c_4x^3 = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$(c_1 + c_2 + c_3 + c_4) + (c_1 - c_2)x + (-c_3)x^2 + (-c_4)x^3 = a_0 + a_1x + a_2x^2 + a_3x^3$$

The linear system is given as

$$1c_1 + 1c_2 + 1c_3 + 1c_4 = a_0$$

$$1c_1 - 1c_2 + 0c_3 + 0c_4 = a_1$$

$$0c_1 + 0c_2 - 1c_3 + 0c_4 = a_2$$

$$0c_1 + 0c_2 + 0c_3 - 1c_4 = a_3$$

The determinant of coefficient matrix of both systems is given as

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} + 0 + 0$$

$$= 1\left(-1\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + 0 + 0\right) - 1\left(1\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + 0 + 0\right)$$

$$= 1\left(-1\{(-1)(-1) - 0\}\right) - 1\left(1\{(-1)(-1) - 0\}\right)$$

$$= 1\left(-1(1)\right) - 1\left(1(1)\right)$$

$$= -1 - 1 = -2$$

As the determinant $= -2 \neq 0$

→ The homogeneous system has only trivial solution so the polynomials are linearly independent.

→ The non homogeneous system is consistent for all real values a_0, a_1, a_2, a_3 and span P_3 so they form a basis for P_3.

Question 5:— (Question 6)

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

⇒ The given matrices are linearly independent if the equation

$$c_1 \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \text{(i)}$$

has only the trivial solution.

⇒ For these matrices to span M_{22}, it must be possible to express every matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ as

$$c_1 \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$\longrightarrow$ (ii)

Equating corresponding entries on b.s yields linear systems

$$\begin{cases} 3c_1 + 0c_2 + 0c_3 + 1c_4 = 0 \\ 6c_1 - 1c_2 - 8c_3 + 0c_4 = 0 \\ 3c_1 - 1c_2 - 12c_3 - 1c_4 = 0 \\ -6c_1 + 0c_2 - 4c_3 + 2c_4 = 0 \end{cases} \qquad \begin{cases} 3c_1 + 0c_2 + 0c_3 + 1c_4 = 0 \\ 6c_1 - 1c_2 - 8c_3 + 0c_4 = 0 \\ 3c_1 - 1c_2 - 12c_3 - 1c_4 = 0 \\ -6c_1 + 0c_2 - 4c_3 + 2c_4 = 0 \end{cases}$$

The coefficient matrix of both systems has determinant given as

$$= \begin{vmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & -8 & 0 \\ -1 & -12 & -1 \\ 0 & -4 & 2 \end{vmatrix} + 0 + 0 - 1\begin{vmatrix} 6 & -1 & -8 \\ 3 & -1 & -12 \\ -6 & 0 & -4 \end{vmatrix}$$

$$= 3\left(-1\begin{vmatrix} -12 & -1 \\ -4 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -8 & 0 \\ -4 & 2 \end{vmatrix}\right) - 1\left(6\begin{vmatrix} -1 & -12 \\ 0 & -4 \end{vmatrix} + 1\begin{vmatrix} 3 & -12 \\ -6 & -4 \end{vmatrix} - 8\begin{vmatrix} 3 & -1 \\ -6 & 0 \end{vmatrix}\right)$$

$$= 3\left[-1\{-24-4\} + 1(-16-0)\right] - 1\left[6(4-0) + 1(-12-72) - 8(0-6)\right]$$

$$= 3\{-1(-28) + 1(-16)\} - 1\left(6(4) + 1(-84) - 8(-6)\right)$$

$$= 3(28-16) - 1(24-84+48)$$

$= 3(12) - 12$

$= 36 - 12 = 24$

As the determinant $= 24 \neq 0$:

→ The homogeneous system has only the trivial solution so the matrices are linearly independent.

→ The non homogeneous system is consistent for all real values $a_{11}, a_{12}, a_{21}, a_{22}$ and span M_{22} so they form a basis for M_{22}.

Question 7 :—

(a) $\{(2,-3,1), (4,1,1), (0,-7,1)\}$

→ The given vectors are linearly independent if the vector equation

$$c_1(2,-3,1) + c_2(4,1,1) + c_3(0,-7,1) = (0,0,0)$$

has only the trivial solution

This equation can be written as a linear system

$$2c_1 + 4c_2 + 0c_3 = 0$$

$$-3c_1 + 1c_2 - 7c_3 = 0$$

$$1c_1 + 1c_2 + 1c_3 = 0$$

Since the determinant of the coefficient matrix of the system is

$$= \begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -7 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} -3 & -7 \\ 1 & 1 \end{vmatrix}$$

$$= 2\big((1)(1)-(1)(-7)\big) - 4\big((-3)(1)-(1)(-7)\big)$$

$$= 2(1+7) - 4(-3+7)$$

$$= 2(8) - 4(4)$$

$$= 16 - 16$$

$$= 0$$

(d)

As the determinant $= 0$:

→ The homogeneous system has non trivial solutions and the vectors are linearly dependent, thus they do not form a basis for R^3.

Question 10 :—

$\vec{V_1} = \cos^2 x$, $\vec{V_2} = \sin^2 x$, $\vec{V_3} = \cos 2x$

(a) ~~The identity~~

To be a basis, a set of vectors must be linearly independent, if we

can express one vector as a combination of the others, the set is dependent and cannot be a basis.

Since,

$$\cos 2x = \cos^2 x - \sin^2 x$$

or, $$\vec{V_3} = \vec{V_1} - \vec{V_2}$$

Thus, the set $S = \{\vec{V_1}, \vec{V_2}, \vec{V_3}\}$ is linearly dependent, therefore it is not a basis for V.

(b) Since $\vec{V_3} = \cos 2x$ is just a combination of $\vec{V_1}$ and $\vec{V_2}$, so we can remove it without shrinking the space V. Thus, we have the set $\{\vec{V_1}, \vec{V_2}\}$.

Now,

The vectors are linearly independent if the vector equation

$$C_1\vec{V_1} + C_2\vec{V_2} = 0 \rightarrow (i)$$

has only the trivial solution.

~~Let~~

From equ (i), we have

$$C_1 \cos^2 x + C_2 \sin^2 x = 0$$

→ Let $x = 0$:

$$c_1 (1)^2 + c_2 (0)^2 = 0$$

$$\boxed{c_1 = 0}$$

→ Let $x = \pi/2$:

$$c_1 (0)^2 + c_2 (1)^2 = 0$$

$$\boxed{c_2 = 0}$$

Since the only solution is $(0,0)$, these two vectors are linearly independent.

Hence, the basis for V is

$$\{\cos^2 x, \sin^2 x\}$$

Question 8 :-

$1-3x+2x^2$, $1+x+4x^2$, $1-7x$

→ The given vectors $\vec{p_1} = 1-3x+2x^2$, $\vec{p_2} = 1+x+4x^2$, $\vec{p_3} = 1-7x$ are linearly independent if the vector equation

$$c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{0} \rightarrow (i)$$

has only trivial solution.

Now,

From equ (i), we have

$$c_1(1-3x+2x^2) + c_2(1+x+4x^2) + c_3(1-7x) = (0,0,0)$$

$$c_1 - 3c_1x + 2c_1x^2 + c_2 + c_2x + 4c_2x^2 + c_3 - 7c_3x = (0,0,0)$$

$$(c_1+c_2+c_3) + (-3c_1+c_2-7c_3)x + (2c_1+4c_2)x^2 = (0,0,0)$$

This equ can be rewritten as linear system

$$c_1 + c_2 + c_3 = 0$$

$$-3c_1 + c_2 - 7c_3 = 0$$

$$2c_1 + 4c_2 = 0$$

The determinant of the coefficient matrix is

$$= \begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -7 \\ 4 & 0 \end{vmatrix} - 1\begin{vmatrix} -3 & -7 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} -3 & 1 \\ 2 & 4 \end{vmatrix}$$

$= 1(0+28) - 1(0+14) + 1(-12-2)$

$= 28 - 14 - 14$

$= 0$

As the determinant $= 0$;

→ The homogeneous system has non-trivial solutions and the vectors $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ are linearly dependent so they do not form a basis for P_2.

Question 11 :—

(a) $\vec{u_1} = (2,-4)$, $\vec{u_2} = (3,8)$

$\vec{w} = (1,1)$

Expressing $\vec{w}$ as a linear combination of $\vec{u_1}$ and $\vec{u_2}$ we obtain

$$(1,1) = c_1(2,-4) + c_2(3,8)$$

Equating corresponding components on b.s

$$2c_1 + 3c_2 = 1$$

$$-4c_1 + 8c_2 = 1$$

The augmented matrix is given as

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array}\right]$$

The reduced echelon form is

$$\left[\begin{array}{cc|c} 1 & 0 & 5/28 \\ 0 & 1 & 3/14 \end{array}\right]$$

The solution of linear system is

$$c_1 = \frac{5}{28}, \quad c_2 = \frac{3}{14}$$

Therefore, the coordinate vector is

$$(\vec{w})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$$

Question 12 :—

(b) $\vec{u_1} = (1,-1)$, $\vec{u_2} = (1,1)$

$\vec{w} = (0,1)$

Expressing $\vec{w}$ as a linear combination of $\vec{u_1}$ and $\vec{u_2}$, we obtain

$$\vec{w} = c_1\vec{u_1} + c_2\vec{u_2}$$

$$(0,1) = c_1(1,-1) + c_2(1,1)$$

Equating corresponding components on b.s

$$c_1 + c_2 = 0$$

$$-c_1 + c_2 = 1$$

The augmented matrix is given as

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 1 \end{array}\right]$$

The reduced echelon form is

$$\left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{array}\right]$$

The solution of linear system is

$$C_1 = \frac{-1}{2}, \quad C_2 = \frac{1}{2}$$

Therefore, the coordinate vector is

$$(\vec{w})_S = \left(\frac{-1}{2}, \frac{1}{2}\right)$$

Question 13 :-

(a) $\vec{v} = (2, -1, 3)$; $\vec{v_1} = (1, 0, 0)$

$\vec{v_2} = (2, 2, 0)$, $\vec{v_3} = (3, 3, 3)$

Expressing $\vec{v}$ as a linear combination of $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$, we obtain

$$\vec{v} = C_1\vec{v_1} + C_2\vec{v_2} + C_3\vec{v_3}$$

$$(2, -1, 3) = C_1(1, 0, 0) + C_2(2, 2, 0) + C_3(3, 3, 3)$$

Equating corresponding components on b.s

$$c_1 + 2c_2 + 3c_3 = 2$$
$$2c_2 + 3c_3 = -1$$
$$3c_3 = 3$$

It can be solved by back-substitution

$c_3 = 1$, $c_2 = -2$, and $c_1 = 3$

Thus, the coordinate vector is

$$(\vec{w})_S = (3, -2, 1)$$

Question 14 :-

(a) $\vec{p} = 4 - 3x + x^2$;

$\vec{p_1} = 1$, $\vec{p_2} = x$, $\vec{p_3} = x^2$

Expressing $\vec{p}$ as a linear combination of $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ we obtain

$$\vec{p} = c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3}$$
$$4 - 3x + x^2 = c_1(1) + c_2(x) + c_3(x^2)$$
$$4 - 3x + x^2 = c_1 + c_2x + c_3x^2$$

Equating corresponding components on b.s

$$c_1 = 4$$
$$c_2 = -3$$
$$c_3 = 1$$

Thus, the coordinate vector is

$$(\vec{p})_S = (4, -3, 1)$$

Question 15 :—

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

⇒ The given matrices A_1, A_2, A_3 and A_4 are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \vec{0} \longrightarrow (i)$$

has only trivial solution

⇒ For the given matrices to span M_{22}, it must be possible to express every matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ as}$$

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = B \longrightarrow (ii)$$

$$k_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Also equ (i) is given as

$$k_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating corresponding enteries on b.s we get equ (i) and (ii) as

$$k_1 = 0$$
$$k_1 + k_2 = 0$$
$$k_1 + k_2 + k_3 = 0$$
$$k_1 + k_2 + k_3 + k_4 = 0$$

$$k_1 = a$$
$$k_1 + k_2 = b$$
$$k_1 + k_2 + k_3 = c$$
$$k_1 + k_2 + k_3 + k_4 = d$$

The coefficient matrix of both systems is given as

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} - 0 + 0 - 0$$

$$= 1 \left\{ 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 + 0 \right\}$$

$$= 1 \{ 1 (1 - 0) \}$$

$$= 1 (1 (1))$$

$$= 1$$

As, the determinant $= 1 \neq 0$

→ The homogeneous system has only the trivial solution therefore the matrices

A_1, A_2, A_3 and A_4 are linearly independent.

→ The nonhomogeneous system is consistent for all real values a, b, c and d so the matrices span M_{22} and they form a basis for M_{22}.

→ To express $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ as a linear combination of matrices A_1, A_2, A_3 and A_4, we form the nonhomogeneous system with appropriate right hand side values

$$k_1 = 1$$
$$k_1 + k_2 = 0$$
$$k_1 + k_2 + k_3 = 1$$
$$k_1 + k_2 + k_3 + k_4 = 0$$

which can be solved by forward-substitution to obtain

$$k_1 = 1, \quad k_2 = -1, \quad k_3 = 1, \quad k_4 = -1$$

Thus, the coordinate vector is

$$(\vec{A})_S = (1, -1, 1, -1).$$

Question 17 :—

$\vec{p_1} = 1 + x + x^2$, $\vec{p_2} = x + x^2$

$\vec{p_3} = x^2$; $\vec{p} = 7 - x + 2x^2$

→ Vectors $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ are linearly independent if the vector equation

$$c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{0}$$

$$c_1(1+x+x^2) + c_2(x+x^2) + c_3(x^2) = (0,0,0)$$

$$c_1 + c_1x + c_1x^2 + c_2x + c_2x^2 + c_3x^2 = (0,0,0)$$

$$(c_1) + (c_1+c_2)x + (c_1+c_2+c_3)x^2 = (0,0,0) \quad \text{(i)}$$

has only trivial solution

→ For these vectors to span P_2, it must be possible to express every vector $\vec{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{p}$$

$$c_1(1+x+x^2) + c_2(x+x^2) + c_3(x^2) = a_0 + a_1x + a_2x^2$$

$$c_1 + c_1x + c_1x^2 + c_2x + c_2x^2 + c_3x^2 = a_0 + a_1x + a_2x^2$$

$$(c_1) + (c_1+c_2)x + (c_1+c_2+c_3)x^2 = a_0 + a_1x + a_2x^2 \quad \text{(ii)}$$

Equating corresponding components on b.s of equ (i) and (ii)

$c_1 = 0$	$c_1 = a_0$
$c_1 + c_2 = 0$	$c_1 + c_2 = a_1$
$c_1 + c_2 + c_3 = 0$	$c_1 + c_2 + c_3 = a_2$

The determinant ~~mat~~ of the coefficient matrix of both systems is given as

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 + 0$$

$$= 1(1-0)$$

$$= 1(1)$$

$$= 1$$

As the determinant $= 1 \neq 0$

→ The homogeneous system has only the trivial solution therefore, the vectors $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ are linearly independent.

→ The nonhomogeneous system is consistent for all values a_0, a_1 and a_2 therefore, the vectors span P_2 and they form a basis for P_2.

⇒ To express $\vec{p} = 7 - x + 2x^2$ as a linear combination of vectors $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ we form the nonhomogeneous system with appropriate right hand side values.

$$C_1 = 7$$
$$C_1 + C_2 = -1$$
$$C_1 + C_2 + C_3 = 2$$

which can be solved by forward-substitution to obtain

$$C_1 = 7, \quad C_2 = -8, \quad C_3 = 3$$

Thus, the coordinate vector is

$$(\vec{p})_S = (7, -8, 3)$$

Question 19 :—

Concept:

To form a basis for a vector space V, a set of vectors must satisfy both of the following:

1) Linear Independence: No vector in the set can be written as a linear combination of others.
2) Spanning: The set must span the entire vector span, meaning every vector in V can be represented as a linear combination of the basis vectors.

(a) $\vec{u_1} = (1,2)$, $\vec{u_2} = (0,3)$, $\vec{u_3} = (1,5)$ for R^2.

According to the Theorem 4.3.3

"Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent."

Here,

$S = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ and R^2

so, $r > n \Rightarrow 3 > 2$ (True)

Hence, the set contains too many vectors and S is linearly dependent thus, they cannot form a basis.

(b) $\vec{u_1} = (-1,3,2)$, $\vec{u_2} = (6,1,1)$ for R^3.

~~According t~~

Here, the dimension of R^3 is 3. A basis for R^3 must contain exactly 3 vectors to span the entire space. Because this set only contains 2 vectors, it can only span a plane within R^3, not the entire space.

(c) $\vec{p_1} = 1 + x + x^2$, $\vec{p_2} = x$ for P_2.

P_2 is the space of polynomials of degree 2 or less. Its standard basis is $\{1, x, x^2\}$, meaning its dimension is 3. Since the given set only contains 2 polynomials, it is impossible for them to span all of P^2.

(d) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}$

$C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 4 & 2 \end{bmatrix}$

for M_{22}.

→ While the set has 4 matrices and the dimension of M_{22} is 4, notice that every matrix has a zero in upper-right entry.

→ Thus, every linear combination of these matrices will always have a zero in that position. Therefore, they cannot span any matrix in M_{22} that has a non-zero value in the upper right corner.

Question 20 :-

In any vector space a set that contains the zero vector must be linearly dependent. Explain why this is so.

According to the definition of Linear Independence.

> " A set of vectors $\{\vec{V_1}, \vec{V_2}, \ldots, \vec{V_n}\}$ is linearly independent only if the equ:
>
> $$c_1\vec{V_1} + c_2\vec{V_2} + \ldots + c_n\vec{V_n} = \vec{0}$$
>
> can be satisfied only when all ceffici-ents $c_1, c_2, \ldots, c_n$ are zero. If there is any way to solve this equation using at least one non-zero coefficient, the set is linearly dependent. "

Now,

Suppose we have a set S that contains the zero vector. Let's say $\vec{V_1} = 0$. We can set up the linear combination as follows:

$$c_1(0) + c_2\vec{V_2} + \ldots + c_n\vec{V_n} = 0$$

we can satisfy this equation by choosing:

→ $c_1 = 5$ (or any non-zero number)

→ $c_2 = 0, \quad c_3 = 0, \ldots, c_n = 0$

Because $5 \cdot 0 = 0$, the equ holds true.

$$0 + 0 + \ldots + 0 = 0$$

Hence,

Since we found a solution where at least one coefficient (c_1) is not zero, the definition of linear independence is violated. Therefore, any set containing the zero vector is automatically linearly dependent.

Question 21 :—

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{bmatrix}$

The standard basis for R^3 is

$\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

Now, we have

$T_A(1,0,0) = (1,0,-1)$

$T_A(0,1,0) = (1,1,2)$

$T_A(0,0,1) = (1,-3,0)$

The vector equation is given as

$k_1(1,0,-1) + k_2(1,1,2) + k_3(1,-3,0) = (0,0,0)$

can be rewritten as a homogeneous linear system

$$1k_1 + 1k_2 + 1k_3 = 0$$
$$0k_1 + 1k_2 - 3k_3 = 0$$
$$-1k_1 + 2k_2 + 0k_3 = 0$$

The determinant of coefficient matrix is

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= 1(0+6) - 1(0-3) + 1(0+1)$$
$$= 1(6) - 1(-3) + 1(1)$$
$$= 6+3+1$$
$$= 10$$

As, $\det(A) = 10 \neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\vec{e}_1), T_A(\vec{e}_2), T_A(\vec{e}_3)\}$ is linearly independent.

(b) $$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

The standard basis for R^3 is

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

We have,

$$T_A(1,0,0) = (1,0,-1)$$

$T_A(0,1,0) = (1,1,2)$

$T_A(0,0,1) = (2,1,1)$

By inspection,

$$(2,1,1) = (1,1,2) + (1,0,-1)$$

Hence,

we conclude that the set $\{T_A(\vec{e_1}), T_A(\vec{e_2}), T_A(\vec{e_3})\}$ is linearly dependent.

Question 22:—

$\vec{u} = (1,-2,1)$

$S = \{(1,1,0), (0,1,1), (1,1,1)\}$

(a) $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

Finding $T_A(\vec{u})$, we have

$$A\vec{u} = (T_A(\vec{u})) = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(1) + (-1)(-2) + (0)(1) \\ (1)(1) + (1)(-2) + (1)(1) \\ (0)(1) + (-1)(-2) + (2)(1) \end{bmatrix}$$

$$T_A(\vec{u}) = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

Expressing $T_A(\vec{u}) = (4,-2,0)$ as a linear combination of the vectors in S, we obtain

$$(4,-2,0) = c_1(1,1,0) + c_2(0,1,1) + c_3(1,1,1)$$

Equating corresponding components on b.s

$$1c_1 + 0c_2 + 1c_3 = 4$$

$$1c_1 + 1c_2 + 1c_3 = -2$$

$$0c_1 + 1c_2 + 1c_3 = 0$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

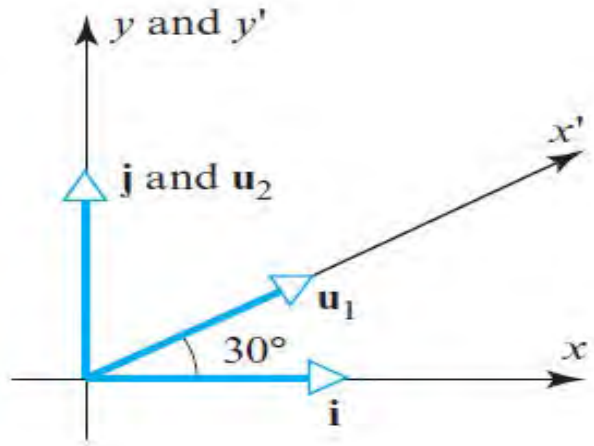
The solution of the linear system is

$$c_1 = -2, \quad c_2 = -6, \quad c_3 = 6$$

The coordinate vector is

$$(T_A(\vec{u}))_S = (-2,-6,6)$$

Question 23 :-



From the provided figure, we can express the new basis vectors $\vec{u_1}$ and $\vec{u_2}$ in terms of the standard xy- system:

→ <u>Vector $\vec{u_1}$</u> : It is a unit vector at a 30° angle to the x-axis.

$$\vec{u_1} = (\cos 30°, \sin 30°) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

→ <u>vector $\vec{u_2}$</u> : It is a unit vector pointing along the positive y-axis. (same as $\hat{j}$)

$$\vec{u_2} = (0, 1)$$

Now,

The relationship b/w xy-coordinates (x, y) and x'y'-coordinates (x', y') is given by:

$$(x, y) = x'\vec{u_1} + y'\vec{u_2}$$

$$(x, y) = x'\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + y'(0, 1)$$

This gives two equations

$\rightarrow \quad x = \frac{\sqrt{3}}{2} x' \Rightarrow x' = \frac{2x}{\sqrt{3}} \rightarrow$ (i)

$\rightarrow \quad y = \frac{1}{2} x' + y' \Rightarrow y' = y - \frac{1}{2} x' \rightarrow$ (ii)

(a) $(\sqrt{3}, 1)$

From equ (i) we have

- $x' = \frac{2}{\sqrt{3}} (\sqrt{3}) = \boxed{2}$

From equ (ii) we have

- $y' = (1) - \frac{1}{2} x'$

$y' = 1 - \frac{1}{\cancel{2}} (\cancel{2})$

$\boxed{y' = 0}$

Hence, we have

$(x', y') = (2, 0)$

(b) $(1, 0)$

From equ (i) we have

$x' = \frac{2}{\sqrt{3}} (1) = \boxed{\frac{2}{\sqrt{3}}}$

From equ (ii) we have

$$y' = y - \frac{1}{2}x'$$

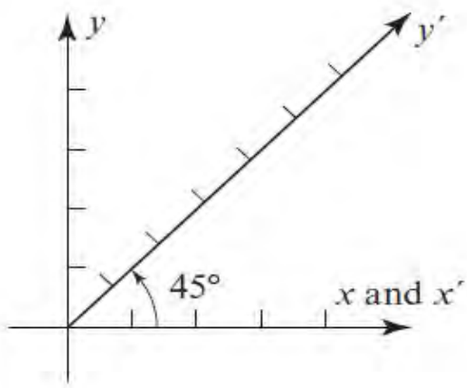
$$y' = 0 - \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{\sqrt{3}}\right)$$

$$\boxed{y' = \frac{-1}{\sqrt{3}}}$$

Hence, we have

$$(x', y') = \left(\frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

Question 24:—



From the figure provided, the x' and y' axes are determined by unit vectors $\vec{u_1}$ and $\vec{u_2}$:

→ $\vec{u_1}$ (along the x'-axis): Since the x-axis and x'-axis are the same, this is the standard unit vector $\hat{i}$.

$$\vec{u_1} = (1, 0)$$

→ $\vec{u_2}$ (along the y'-axis): This is a unit vector at a 45° angle to the +ve x-axis.

$$\vec{u_2} = (\cos 45°, \sin 45°)$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Now,

The relationship b/w the xy-coordinates and $x'y'$-coordinates is given by:

$$(x, y) = x'\vec{u_1} + y'\vec{u_2}$$

$$(x, y) = x'(1, 0) + y'\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

This equation gives two equations

$$x = x' + \frac{y'}{\sqrt{2}} \rightarrow (i)$$

$$y = \frac{y'}{\sqrt{2}}$$

Solving for x' and y', we have

$$y' = y\sqrt{2} \rightarrow (i)$$

$$x' = x - y \quad \because \left(y = \frac{y'}{\sqrt{2}}\right)$$

$\rightarrow$ (ii)