Question 6 :-

$$x = at \ , \quad y = bt \ , \quad z = ct$$

$\vec{V_1} = (x_1, y_1, z_1)$ and $\vec{V_2} = (x_2, y_2, z_2)$

If the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are both on L, then there must exist real numbers t_1 and t_2 such that

$$x_1 = at_1 \ , \quad y_1 = bt_1 \ , \quad z_1 = ct_1$$

$$x_2 = at_2 \ , \quad y_2 = bt_2 \ , \quad z_2 = ct_2$$

$\Rightarrow$ L is closed under addition since

$$= (x_1, y_1, z_1) + (x_2, y_2, z_2)$$

$$= (at_1, bt_1, ct_1) + (at_2, bt_2, ct_2)$$

$$= (at_1 + at_2 \ , \ bt_1 + bt_2 \ , \ ct_1 + ct_2)$$

$$= (a(t_1 + t_2), \ b(t_1 + t_2), \ c(t_1 + t_2))$$

$\Rightarrow$ L is also closed under scalar multiplication

because

$$= k\,(x_1, y_1, z_1)$$
$$= k\,(at_1, bt_1, ct_1)$$
$$= (kat_1, kbt_1, kct_1)$$
$$= (a(kt_1), b(kt_1), c(kt_1))$$

Hence,

L is a subspace of R^3.

Exercise 4.2

Question 7 — (8)

$\vec{u} = (0,-2,2)$ $\vec{v} = (1,3,-1)$

(b) $(0,4,5)$

For $(0,4,5)$ to be a linear combination of the vectors $\vec{u}$ and $\vec{v}$, there must exist scalars a and b such that:

$$a\vec{u} + b\vec{v} = (x,y,z)$$

$$a(0,-2,2) + b(1,3,-1) = (0,4,5)$$

Equating corresponding components on b.s

$$0a + b = 0$$

$$-2a + 3b = 4$$

$$2a - b = 5$$

The augmented matrix is given as

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array}\right]$$

The reduced echelon form after performing row operation is

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The Last row corresponds to the equation

$$0a + 0b = 1 \quad \text{(Impossible)}$$

Thus, the linear is inconsistent and therefore $(0,4,5)$ is not a linear combination of $\vec{u}$ and $\vec{v}$.

(a) $(2,2,2)$

For $(2,2,2)$ to be a linear combination of the vectors $\vec{u}$ and $\vec{v}$; there must exist scalars a and b such that:

$$a\vec{u} + b\vec{v} = (x,y,z)$$

$$a(0,-2,2) + b(1,3,-1) = (2,2,2)$$

Equating corresponding components on b.s

$$0a + 1b = 2$$

$$-2a + 3b = 2$$

$$2a - 1b = 2$$

The augmented matrix is given as

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The corresponding equations are given as

$$1a + 0b = 2 \Rightarrow a = 2$$

$$0a + 1b = 2 \Rightarrow b = 2$$

$$0 = 0 \quad \text{(consistent)}$$

Since the linear system is consistent with a unique solution $(a = 2, b = 2)$, $(2, 2, 2)$ is a linear combination of $\vec{u}$ and $\vec{v}$.

Question 8 :-

$\vec{u} = (2, 1, 4)$ $\quad$ $\vec{w} = (3, 2, 5)$

$\vec{v} = (1, -1, 3)$

(a) $(-9, -7, -15)$

For $(-9, -7, -15)$ to be a linear combination of the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, there must exist scalars a, b, c such that

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (-9,-7,-15)$$

Equating corresponding components on b.s

$$2a + 1b + 3c = -9$$
$$1a - 1b + 2c = -7$$
$$4a + 3b + 5c = -15$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The solution to this system is given as

$a = -2, \quad b = 1, \quad c = -2$

Thus,

$$(-9, -7, -15) = -2\vec{u} + 1\vec{v} - 2\vec{w}$$

(b) (6, 11, 6)

For (6, 11, 6) to be a linear combination of vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, there must exist scalars a, b and c such that

$$a\vec{u} + b\vec{v} + c\vec{w} = (6, 11, 6)$$

$$a(2,1,4) + b(1,-1,3) + c(3,2,5) = (6,11,6)$$

Equating corresponding components on b.s

$$2a + 1b + 3c = 6$$
$$1a - 1b + 2c = 11$$
$$4a + 3b + 5c = 6$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Exercise 4.2

Question 2:—

Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$

For the given matrix M to be a linear combination of A, B and C, there must exist scalars a, b and c such that

$$aA + bB + cC = M$$

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

Equating corresponding entries on both sides, we get the linear system

$$4a + 1b + 0c = 6$$

$$0a - 1b + 2c = -8$$

$$-2a + 2b + 1c = -1$$

$$-2a + 3b + 4c = -8$$

The augmented matrix of the given linear system is given as

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right]$$

After applying row operations we get the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus,

$$a = 1 \quad , \quad b = 2 \quad , \quad c = -3$$

Hence, the linear system is consistent.

Therefore, $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is a linear combination of A, B and C because

$$1A + 2B - 3C = M$$

$$\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

In Linear algebra, the zero matrix is always a linear combination of any set of matrices. This is called the trivial combination, where all scalars are zero.

$$0.A + 0.B + 0.C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

For the given matrix M to be a linear combination of A, B and C there must exist scalars a, b and c such that

$$aA + bB + cC = M$$

$$a\begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$$

Equating corresponding entries on b.s, we get

$$4a + 1b + 0c = -1$$

$$0a - 1b + 2c = 5$$

$$-2a + 2b + 1c = 7$$

$$-2a + 3b + 4c = 1$$

The augmented matrix of the given linear system is given as

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ -2 & 2 & 1 & 7 \\ -2 & 3 & 4 & 1 \end{array}\right]$$

After applying the row operations, we get the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row corresponds to the equation

$$0 = 1$$

which is contradictory.

Thus,

we conclude that the given matrix M is not a linear combination of A, B and C.

Question 10 :—

$\vec{p}_1 = 2 + x + 4x^2$

$\vec{p}_2 = 1 - x + 3x^2$

$\vec{p}_3 = 3 + 2x + 5x^2$

(a) $-9 - 7x - 15x^2$

For $-9 - 7x - 15x^2$ to be a linear combination of vectors $\vec{p}_1$, $\vec{p}_2$ and $\vec{p}_3$ there must exist scalars a, b and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2)$$
$$= -9 - 7x - 15x^2$$

$$2a + ax + 4ax^2 + b - bx + 3bx^2 + 3c + 2cx + 5cx^2 = -9 - 7x - 15x^2$$

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = -9 - 7x - 15x^2$$

Grouping the terms corresponding to the powers of x

$$(2a + b + 3c) = -9$$

$$(a - b + 2c)x = -7x$$

$$(4a + 3b + 5c)x^2 = -15x^2$$

This results in the linear system

$$2a + 1b + 3c = -9$$

$$1a - 1b + 2c = -7$$

$$4a + 3b + 5c = -15$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This is the only one solution to system

$$a = -2, \quad b = 1, \quad c = -2$$

Therefore,

$$-9 - 7x - 15x^2 = -2\vec{p}_1 + 1\vec{p}_2 - 2\vec{p}_3$$

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Question 11 :-

(a) $\vec{V_1} = (2,2,2)$, $\vec{V_2} = (0,0,3)$

$\vec{V_3} = (0,1,1)$

The given vectors span R^3 if an arbitrary vector $\vec{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2,2,2) + k_2(0,0,3) + k_3(0,1,1)$$

Equating corresponding components on b-s yeilds the linear system

$$2k_1 + 0k_2 + 0k_3 = b_1$$
$$2k_1 + 0k_2 + 1k_3 = b_2$$
$$2k_1 + 3k_2 + 1k_3 = b_3$$

By inspection, regardless of the right hand side values b_1, b_2, b_3 the first equation can be solved for k_1.

The second equation can be used to obtain k_3 and the third would yeild k_2.

$\Rightarrow$ Thus, we conclude that $\vec{V_1}$, $\vec{V_2}$ and $\vec{V_3}$ span R^3.

(b) $\vec{V_1} = (2,-1,3)$, $\vec{V_2} = (4,1,2)$
$\vec{V_3} = (8,-1,8)$

The given vectors span R^3 if an arbitrary vector $\vec{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2,-1,3) + k_2(4,1,2) + k_3(8,-1,8)$$

Equating corresponding components on b.s yeilds the linear system

$$2k_1 + 4k_2 + 8k_3 = b_1$$
$$-1k_1 + 1k_2 - 1k_3 = b_2$$
$$3k_1 + 2k_2 + 8k_3 = b_3$$

The determinent of the coefficient matrix is given as

$$\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$$

Therefore, the system cannot be consistent for all right hand side vectors $\vec{b}$. We conclude that $\vec{V_1}$, $\vec{V_2}$ and $\vec{V_3}$ do not span R^3.

The solution to this system is given as

$a = 4, \quad b = -5, \quad c = 1$

Thus,

$$(6, 11, 6) = 4\vec{u} - 5\vec{v} + 1\vec{w}$$

Question 12 :—

$\vec{V_1} = (2, 1, 0, 3)$

$\vec{V_2} = (3, -1, 5, 2)$

$\vec{V_3} = (-1, 0, 2, 1)$

(a) $(2, 3, -7, 3)$

In order for vectors $(2, 3, -7, 3)$ to be in span $\{\vec{V_1}, \vec{V_2}, \vec{V_3}\}$, there must be exist scalars a, b and c such that

$$a\vec{V_1} + b\vec{V_2} + c\vec{V_3} = (2, 3, -7, 3)$$

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (2,3,-7,3)$$

Equating corresponding components on b.s

$2a + 3b - 1c = 2$

$1a - 1b + 0c = 3$

$0a + 5b + 2c = -7$

$3a + 2b + 1c = 3$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent because it has only one solution $(a = 2,\ b = -1,\ c = -1)$

Thus, $(2, 3, -7, 3)$ is in span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

(d) $(-4, 6, -13, 4)$

In order for vector $(-4, 6, -13, 4)$ to be in span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ there must exist scalars a, b and c such that

$$a\vec{v_1} + b\vec{v_2} + c\vec{v_3} = (-4, 6, -13, 4)$$

$$a(2,1,0,3) + b(3,-1,5,2) + c(-1,0,2,1) = (-4,6,-13,4)$$

Equating corresponding components on b.s

$$2a + 3b - 1c = -4$$

$$1a - 1b + 0c = 6$$

$$0a + 5b + 2c = -13$$

$$3a + 2b + 1c = 4$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{array}\right]$$

The reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This system is consistent because it has only one solution $(a = 3, b = -3, c = 1)$.

Thus,

$(-4, 6, -13, 4)$ is in span $\{\vec{V_1}, \vec{V_2}, \vec{V_3}\}$.

(c) $(1, 1, 1, 1)$

In order for the vector $(1, 1, 1, 1)$ to be in span $\{\vec{V_1}, \vec{V_2}, \vec{V_3}\}$ there must exist scalars a, b and c such that

$$a\vec{V_1} + b\vec{V_2} + c\vec{V_3} = (1, 1, 1, 1)$$

$$a(2, 1, 0, 3) + b(3, -1, 5, 2) + c(-1, 0, 2, 1) = (1, 1, 1, 1)$$

Equating corresponding components on b.s

$$2a + 3b - 1c = 1$$
$$1a - 1b + 0c = 1$$
$$0a + 5b + 2c = 1$$
$$3a + 2b + 1c = 1$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This system is inconsistent therefore $(1,1,1,1)$ is not in span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

Question 13 :—

$\vec{p_1} = 1 - x + 2x^2$

$\vec{p_2} = 3 + x$

$\vec{p_3} = 5 - x + 4x^2$

$\vec{p_4} = -2 - 2x + 2x^2$

The given polynomials span P_2 if an arbitrary polynomial in P_2, $\vec{p} = a_0 + a_1x + a_2x^2$ can be expressed as a linear combination

$$a_0 + a_1x + a_2x^2 = k_1\vec{p_1} + k_2\vec{p_2} + k_3\vec{p_3} + k_4\vec{p_4}$$

$$a_0 + a_1x + a_2x^2 = k_1(1 - x + 2x^2) + k_2(3 + x) + k_3(5 - x + 4x^2) + k_4(-2 - 2x + 2x^2)$$

Grouping terms according to pow (x)

$$a_0 + a_1x + a_2x^2 = k_1 - k_1x + 2k_1x^2 + 3k_2 + k_2x + 5k_3 - k_3x + 4k_3x^2 - 2k_4 - 2k_4x + 2k_4x^2$$

$$a_0 + a_1x + a_2x^2 = (k_1 + 3k_2 + 5k_3 - 2k_4) + (-k_1 + k_2 - k_3 - 2k_4)x + (2k_1 + 4 + 2k_4)x^2$$

The linear system is given as

$$1k_1 + 3k_2 + 5k_3 - 2k_4 = a_0$$

$$-k_1 + k_2 - k_3 - 2k_4 = a_1$$

$$2k_1 + 0k_2 + 4k_3 + 2k_4 = a_2$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -2 & a_0 \\ -1 & 1 & -1 & -2 & a_1 \\ 2 & 0 & 4 & 2 & a_2 \end{array}\right]$$

The reduced echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & \frac{1}{4}a_0 - \frac{3}{4}a_1 \\ 0 & 1 & 1 & -1 & \frac{1}{4}a_0 + \frac{1}{4}a_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \end{array}\right]$$

The given system is inconsistent because

$$-\frac{1}{2}a_0 + \frac{3}{2}a_1 + a_2 \neq 0$$

Since, the polynomial $\vec{p} = a_0 + a_1 x + a_2 x^2$ cannot be expressed as a linear combination of $\vec{p_1}$, $\vec{p_2}$, $\vec{p_3}$ and $\vec{p_4}$.

Thus, we conclude that the polynomials $\vec{p_1}$, $\vec{p_2}$, $\vec{p_3}$ and $\vec{p_4}$ do not span P_2.

Question 14 :—

$\vec{f} = \cos^2 x$ and $\vec{g} = \sin^2 x$

(a) $\cos 2x$

For the given functions $\vec{f}$ and $\vec{g}$ to span $\cos 2x$ there must be a linear combination b/w $\vec{f}$, $\vec{g}$ and $\cos 2x$

According to trignometric identity

$$\cos 2x = \cos^2 x - \sin^2 x$$

Thus, the function $\cos 2x$ is in span $\{\vec{f}, \vec{g}\}$.

(b) $3 + x^2$

In order for $3+x^2$ to be in span $\{\vec{f}, \vec{g}\}$, there must exist scalars a and b such that

$$a\cos^2 x + b\sin^2 x = 3 + x^2$$

⇒ When $x = 0$

$$a\cos^2(0) + b\sin^2(0) = 3 + (0)^2$$

$$a(1) + b(0) = 3$$

$$\boxed{a = 3}$$

⇒ when $x = \pi$

$$a\cos^2(\pi) + b\sin^2(\pi) = 3 + (\pi)^2$$

$$a(-1)^2 + b(0)^2 = 3 + \pi^2$$

$$\boxed{a = 3 + \pi^2}$$

Thus, there is a contradiction.

Hence, we conclude that $3+x^2$ is not in span $\{\vec{f}, \vec{g}\}$.

(c) 1

It follows from the trignometric identity

$$\cos^2 x + \sin^2 x = 1$$

that 1 is in span $\{\vec{f}, \vec{g}\}$.

(d) $\sin x$

In order for $\sin x$ to be in span $\{\vec{f}, \vec{g}\}$, there must exist scalars a and b such

$$a\cos^2 x + b\sin^2 x = \sin x$$

→ when $x = \pi/2$

$$a\cos^2\left(\frac{\pi}{2}\right) + b\sin^2\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)$$

$$a(0) + b(1) = 1$$

$$\boxed{b = 1}$$

→ when $x = -\pi/2$

$$a\cos^2\left(-\frac{\pi}{2}\right) + b\sin^2\left(\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right)$$

$$a(0) + b(-1)^2 = -1$$

$$\boxed{b = -1}$$

Thus, there is a contradiction.

Hence, we conclude that $\sin x$ is not in span $\{\vec{f}, \vec{g}\}$.

(e) 0

Since,

$$0\cos^2 x + 0\sin^2 x = 0$$

holds for all real x values, we conclude that 0 is in span $\{\vec{f}, \vec{g}\}$.

To solve this, we perform row reduction on the augmented matrix $[A|0]$. The number of free variables determines the geometry:

- **0 Free Variables:** Trivial solution only ($x = 0, y = 0, z = 0$). The space is the **origin**.
- **1 Free Variable:** The solution depends on one parameter (t). The space is a **line**.
- **2 Free Variables:** The solution depends on two parameters (s, t). The space is a **plane**.

Question 15 :-

(a) $A = \begin{bmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{bmatrix}$

~~Performing row operations~~

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right]$$

$R_1 = (-1)R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right]$$

$R_2 - 3R_1, \quad R_3 - 2R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right]$$

$R_3 + R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The linear equations are

$$x - y - z = 0$$
$$2y + 3z = 0$$

Let, $t = z$

Thus,

$$x - y - t = 0$$
$$2y + 3t = 0 \Rightarrow y = -3/2\, t$$
$$\Rightarrow x = -1/2\, t$$
$$\Rightarrow z = t$$

Hence, the solution space is a line through the origin.

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 0 & 8 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -2 & 5 & 0 \end{array}\right] \quad R_2 - 2R_1, \quad R_3 - R_1$$

$R_3 + 2R_2$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

The linear equations are

$$x + 2y + 3z = 0$$
$$y - 3z = 0$$
$$-z = 0 \Rightarrow z = 0$$

Thus,

$$x + 2y + 3z = 0$$
$$y - 3(0) = 0 \Rightarrow y = 0$$

and,

$$x + 2(0) + 3(0) = 0 \Rightarrow x = 0$$

Hence,

The solution space is the origin only.

(c) $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right]$$

$R_2 - 2R_1 \quad , \quad R_3 - 3R_1$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The linear equations are

$$x - 3y + z = 0$$

Thus, the solution space is a plane through the origin.