Exercise 4.3

Question 1 :-

(a) $\vec{u_1} = (-1, 2, 4)$ and $\vec{u_2} = (5, -10, -20)$ in R^3.

We can see that each component of $\vec{u_2}$ is exactly -5 times the corresponding component of $\vec{u_1}$.

$$5 = -5(-1)$$
$$-10 = -5(2)$$
$$-20 = -5(4)$$

Since $\vec{u_2} = -5\vec{u_1}$, one vector is a scalar multiple of the other. Hence, two vectors are linearly dependent iff they are scalar multiples of each other.

(b) $\vec{u_1} = (3, -1)$, $\vec{u_2} = (4, 5)$, $\vec{u_3} = (-4, 7)$ in R^2.

<u>According to Theorem 4.3.3</u>

"Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent".

We have,

$$S = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\} \text{ and } R^2$$

so we can see that

$$r > n \quad = \quad 3 > 2$$

Hence, this forms linearly dependent sets of vectors.

Question 2:—

(a) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

For 3 vectors in R^3, we can form a matrix with these vectors as its rows (or colums) and calculate the determinant.

⇒ If determinant $\neq 0$, vectors are linearly Independent

⇒ If determinant $= 0$, vectors are linearly dependent.

Now,

$$A = \begin{bmatrix} -3 & 0 & 4 \\ 5 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -3 & 0 & 4 \\ 5 & -1 & 2 \\ 1 & 1 & 3 \end{vmatrix}$$

$$|A| = -3 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - 0 + 4 \begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix}$$

$$|A| = -3(-3-2) + 4(5+1)$$

$$|A| = -3(-5) + 4(6)$$

$$|A| = 15 + 24 = 39$$

Since $|A| = 39 \neq 0$, the vectors are linearly independent.

(b) Same as Q1 : (b)

Question 3 :-

(a) $(3,8,7,-3)$, $(1,5,3,-1)$
$(2,-1,2,6)$, $(4,2,6,4)$

Concept :

To determine if a set of vectors $\{\vec{V_1}, \vec{V_2}, \vec{V_3}, \vec{V_4}\}$ is linearly independent, we set up the vector eq:

$$a\vec{V_1} + b\vec{V_2} + c\vec{V_3} + d\vec{V_4} = 0$$

- if the only solution is the trivial solution $(a=0, b=0, c=0, d=0)$, the vectors are Linearly Independent.
- if there are non-trivial solutions (where at least one scalar is not zero) the vectors are Linearly dependent.

Solution :

The vector equation is

$$a\vec{V_1} + b\vec{V_2} + c\vec{V_3} + d\vec{V_4} = (0,0,0,0)$$

$$a(3,8,7,-3) + b(1,5,3,-1) + c(2,-1,2,6) + d(4,2,6,4) = (0,0,0,0)$$

The homogeneous linear system is given as

$$3a + 1b + 2c + 4d = 0$$
$$8a + 5b - 1c + 2d = 0$$
$$7a + 3b + 2c + 6d = 0$$
$$-3a - 1b + 6c + 4d = 0$$

The augmented matrix is given as

$$\left[\begin{array}{cccc|c} 3 & 1 & 2 & 4 & 0 \\ 8 & 5 & -1 & 2 & 0 \\ 7 & 3 & 2 & 6 & 0 \\ -3 & -1 & 6 & 4 & 0 \end{array}\right]$$

The reduced row echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, we have

$$a + d = 0$$
$$b - d = 0$$
$$c + d = 0$$

Let, $d = t$

So,

$$a = -t \qquad d = t$$
$$b = t$$
$$c = -t$$

Since, the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) **Same as (a)**

Question 4:—

(a) $2 - x + 4x^2$, $3 + 6x + 2x^2$, $2 + 10x - 4x^2$

The vector equation is

$$a(2 - x + 4x^2) + b(3 + 6x + 2x^2) + c(2 + 10x - 4x^2) = 0$$

$$2a - ax + 4ax^2 + 3b + 6bx + 2bx^2 + 2c + 10cx - 4cx^2 = 0$$

$$(2a + 3b + 2c) + (-a + 6b + 10c)x + (4a + 2b - 4c)x^2 = 0$$

Comparing the coefficients on b.s

we have,

$$2a + 3b + 2c = 0$$
$$-a + 6b + 10c = 0$$
$$4a + 2b - 4c = 0$$

The reduced echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

we have solution as

$$a = 0, \quad b = 0, \quad c = 0$$

Since the system has trivial solution we can say the set of vectors in P_2 is linearly independent

(b) Same as (a)

Question 5 :—

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

The matrix equation is given as

$$a\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1a & 0a \\ 1a & 2a \end{bmatrix} + \begin{bmatrix} 1b & 2b \\ 2b & 1b \end{bmatrix} + \begin{bmatrix} 0c & 1c \\ 2c & 1c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1a+1b+0c & 0a+2b+1c \\ 1a+2b+2c & 2a+1b+1c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding enteries, we get

$$1a + 1b + 0c = 0$$
$$0a + 2b + 1c = 0$$
$$1a + 2b + 2c = 0$$
$$2a + 1b + 1c = 0$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

The reduced row echelon form is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system has only trivial solution $(a = b = c = 0)$. Thus, the given matrices are linearly independent.

(b) **Same as (a)**

Question 6 :—

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

The vector equation is given as

$$a\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + c\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1a & 0a \\ 1a & ka \end{bmatrix} + \begin{bmatrix} -1b & 0b \\ kb & 1b \end{bmatrix} + \begin{bmatrix} 2c & 0c \\ 1c & 3c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1a - 1b + 2c & 0a + 0b + 0c \\ 1a + kb + 1c & ka + 1b + 3c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding enteries, we get

$$1a - 1b + 2c = 0$$

$$0a + 0b + 0c = 0$$

$$1a + kb + 1c = 0$$

$$ka + 1b + 3c = 0$$

Omitting the second equation (which imposes no restrictions), we obtain the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix}$$

After performing elementary row operations

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1+k & -1 \\ 0 & 0 & 4-2k \end{bmatrix}$$

Now,

$$\det(A) = \det(B)$$

$$= 1 \begin{vmatrix} 1+k & -1 \\ 0 & 4-2k \end{vmatrix}$$

$$= 1\{(1+k)(4-2k) - 0\}$$

$$= (1+k)(4-2k)$$

The system has trivial solutions iff

$$\det(A) = (1+k)(4-2k) \neq 0$$

Thus, the given metrices are linearly independent for all k except $k = -1$ and $k = 2$.

Question 7:-

(a) $\vec{V_1} = (2, -2, 0)$

$\vec{V_2} = (6, 1, 4)$

$\vec{V_3} = (2, 0, -4)$

The vector equation is given as

$$a\vec{V_1} + b\vec{V_2} + c\vec{V_3} = 0$$

$$a(2,-2,0) + b(6,1,4) + c(2,0,-4) = (0,0,0)$$

$$(2a, -2a, 0a) + (6b, 1b, 4b) + (2c, 0c, -4c) = (0,0,0)$$

Equating both sides we get

$$2a + 6b + 2c = 0$$

$$-2a + 1b + 0c = 0$$

$$0a + 4b - 4c = 0$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 6 & 2 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right]$$

The row reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the given system has only trivial solution (0,0,0) and thus the vectors are linearly independent and they do not lie in a plane.

Concept:

- If the system has trivial solution, the given vectors are linearly independent, hence they don't lie in a plane
- If the system has non-trivial solutions, the given vectors are linearly dependent, hence they lie in a plane.

(b) Same as (a)

Question 8:—

(a) $\vec{V_1} = (-1, 2, 3)$, $\vec{V_2} = (2, -4, -6)$

$\vec{V_3} = (-3, 6, 0)$

⇒ Check $\vec{V_1}$ and $\vec{V_2}$:

$$\vec{V_1} = (-1, 2, 3)$$
$$\vec{V_2} = (2, -4, -6)$$

Notice that

$$\vec{V_2} = -2\vec{V_1}$$
$$= -2(-1, 2, 3)$$
$$= (2, -4, -6)$$

so, $\vec{V_1}$ and $\vec{V_2}$ are on the same line

⇒ Check $\vec{V_1}$ and $\vec{V_3}$:

$$\vec{V_1} = (-1, 2, 3)$$

$$\vec{V_3} = (-3, 6, 0)$$

The vector equation can be written as

$$a\vec{V_1} + b\vec{V_3} = 0$$

$$a(-1,2,3) + b(-3,6,0) = (0,0,0)$$

Equating each component on b.s

$$-1a - 3b = 0$$

$$2a + 6b = 0$$

$$3a + 0b = 0$$

The augmented matrix is given as

$$\left[\begin{array}{cc|c} -1 & -3 & 0 \\ 2 & 6 & 0 \\ 3 & 0 & 0 \end{array}\right]$$

The row reduced echelon form is

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, $a = 0$, $b = 0$ so the system has trivial solution and vectors are linearly independent Hence they do not lie on the same line.

Thus, from the above analysis it is clear that the three vectors do not lie on the same line (even though the vectors $\vec{V_1}$ and $\vec{V_2}$ are collinear).

(b) $\vec{V_1} = (2,-1,4)$, $\vec{V_2} = (4,2,3)$

$\vec{V_3} = (2,7,-6)$

⇒ <u>Check $\vec{V_1}$ and $\vec{V_2}$</u> :

The vector equation is given as

$$a\vec{V_1} + b\vec{V_2} = 0$$

$$a(2,-1,4) + b(4,2,3) = (0,0,0)$$

Equating each component on b.s

$$2a + 4b = 0$$

$$-1a + 2b = 0$$

$$4a + 3b = 0$$

The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & 2 & 0 \\ 4 & 3 & 0 \end{array}\right]$$

The row reduced echelon form is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

where,

$$a - 2b = 0 \quad \text{and} \quad b = 0$$

so, $a = 0$, $b = 0$

Thus, the system has trivial solution and the two vectors are linearly independent.

→ <u>Check $\vec{V_1}$ and $\vec{V_3}$</u>

The vector equation is given as

$$a\vec{V_1} + b\vec{V_3} = 0$$

$$a(2,-1,4) + b(2,7,-6) = (0,0,0)$$

Equating each component on b.s

$$2a + 2b = 0$$

$$-1a + 7b = 0$$

$$4a - 6b = 0$$

The augmented matrix is given as

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & 7 & 0 \\ 4 & -6 & 0 \end{array}\right]$$

The row reduced echelon form is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so, $a + b = 0 \Rightarrow b = 0$

or $a = 0$ and $b = 0$

Thus, the system has trivial solution and the two vectors are linearly independent.

Hence, from the above analysis, it is clear that the three vectors are linearly independent and they do not lie on the same line.

Question 2 :— Q10

(a) $\vec{V_1} = (0, 3, 1, -1)$,

$\vec{V_2} = (6, 0, 5, 1)$,

$\vec{V_3} = (4, -7, 1, 3)$

The vector equation is given as

$$a\vec{V_1} + b\vec{V_2} + c\vec{V_3} = 0$$

$$a(0,3,1,-1) + b(6,0,5,1) + c(4,-7,1,3) = (0,0,0,0)$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 0 & 6 & 4 & 0 \\ 3 & 0 & -7 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right]$$

The row reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -7/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the general solution of system is

$$a - \frac{7}{3}c = 0 \Rightarrow a - 7/3\,t = 0$$

$$b + \frac{2}{3}c = 0 \Rightarrow b + 2/3\,t = 0$$

where, $c = t$

$a = \frac{7}{3}t$, $b = \frac{-2}{3}t$, $c = t$

(b) Express each vector in part (a) as a linear combination of two other.

From part (a) we have

$$\frac{7}{3}t\vec{V_1} - \frac{2}{3}t\vec{V_2} + t\vec{V_3} = 0$$

→ Let $t = 3/7$

$$\vec{V_1} = \frac{2}{7}\vec{V_2} - \frac{3}{7}\vec{V_3}$$

→ Let $t = -3/2$

$$\vec{V_2} = \frac{7}{2}\vec{V_1} + \frac{3}{2}\vec{V_3}$$

→ Let $t = 1$

$$\vec{V_3} = -\frac{7}{3}\vec{V_1} + \frac{2}{3}\vec{V_2}$$

Question 11 :—

Find value of λ for which

$$\vec{V_1} = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad \vec{V_2} = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right)$$

$$\vec{V_3} = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

are linearly dependent set in R^3.

The vector equation is given as

$$a\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right) + b\left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right) + c\left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right) = (0,0,0)$$

The homogeneous linear system is

$$\lambda a - \frac{1}{2} b - \frac{1}{2} c = 0$$

$$-\frac{1}{2} a + \lambda b - \frac{1}{2} c = 0$$

$$-\frac{1}{2} a - \frac{1}{2} b + \lambda c = 0$$

The determinant of coefficient matrix is

$$\begin{vmatrix} \lambda & -1/2 & -1/2 \\ -1/2 & \lambda & -1/2 \\ -1/2 & -1/2 & \lambda \end{vmatrix} = 0$$

$$\lambda \begin{vmatrix} \lambda & -1/2 \\ -1/2 & \lambda \end{vmatrix} + \frac{1}{2} \begin{vmatrix} -1/2 & -1/2 \\ -1/2 & \lambda \end{vmatrix} - \frac{1}{2} \begin{vmatrix} -1/2 & \lambda \\ -1/2 & -1/2 \end{vmatrix} = 0$$

$$\lambda\left(\lambda^2 - \frac{1}{4}\right) + \frac{1}{2}\left(-\frac{\lambda}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4} + \frac{\lambda}{2}\right) = 0$$

$$\lambda^3 - \frac{\lambda}{4} - \frac{\lambda}{4} - \frac{1}{8} - \frac{1}{8} - \frac{\lambda}{4} = 0$$

$$\lambda^3 - \frac{3\lambda}{4} - \frac{2}{8} = 0$$

$$\lambda^3 - \frac{3\lambda}{4} - \frac{1}{4} = 0$$

$$\left(\lambda + \frac{1}{2}\right)\left(\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}\right) = 0$$

$$\left(\lambda + \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right)(\lambda - 1) = 0$$

Thus, we conclude that the vectors form a linearly dependent set for

$$\lambda = -\frac{1}{2} \quad \text{and} \quad \lambda = 1$$

Question 12 :—

THEOREM 4.3.2

(*a*) *A finite set that contains* **0** *is linearly dependent.*

(*b*) *A set with exactly one vector is linearly independent if and only if that vector is not* **0**.

(*c*) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

Question 14:— Q13

$\vec{u_1} = (1,0,0)$, $\vec{u_2} = (2,-1,1)$

$\vec{u_3} = (0,1,1)$

(a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

The transformed vectors are

$\Rightarrow T_A(\vec{u_1}) = A\vec{u_1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\Rightarrow T_A(\vec{u_2}) = A\vec{u_2} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

$\Rightarrow T_A(\vec{u_3}) = A\vec{u_3} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$

For checking the linear independence we have

$|V| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & -1 & -3 \\ 2 & 2 & 2 \end{vmatrix}$

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$$|V| = 1\begin{vmatrix} -1 & -3 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ -1 & -3 \end{vmatrix}$$

$$|V| = (-2+6) - 1(6-6) + 2(-9+3)$$

$$|V| = (4) - 0 + 2(-6)$$

$$|V| = 4 - 12 = -8$$

Since, $|V| \neq 0$ the vector are linearly independent hence the set is Linearly independent.

The formal test is: Can we find a way to add them up to equal **zero** (for all x) without multiplying them all by zero?

$$c_1(f_1) + c_2(f_2) + c_3(f_3) = 0$$

If we can find non-zero numbers for c, it is **Dependent**.

Question 16 :—

(a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

Here,

$$f_1 = 6$$

$$f_2 = 3\sin^2 x$$

$$f_3 = 2\cos^2 x$$

As we know that

$$\sin^2 x + \cos^2 x = 1$$

Now,

we can write a relationship to get 0

$$c_2(f_2) + c_3(f_3) - c_1(f_1) = 0$$

$$2(3\sin^2 x) + 3(2\cos^2 x) - 1(6) = 0$$

Hence, the given vectors are linearly dependent.

(b) x, $\cos x$

Here,

$$f_1 = x \quad \text{and} \quad f_2 = \cos x$$

we need to prove that

$$C_1(f_1) + C_2(f_2) = 0$$

$$C_1(x) + C_2(\cos x) = 0$$

There is no math formula that can convert x so that adding it to $\cos x$ results in 0. Hence, the given functions are linearly independent.

(c) 1, $\sin x$, $\sin 2x$

Here,

$$f_1 = 1, \quad f_2 = \sin x, \quad f_3 = \sin 2x$$

We need to proove the given equation

$$C_1 f_1 + C_2 f_2 + C_3 f_3 = 0$$

$$C_1(1) + C_2(\sin x) + C_3(\sin 2x) = 0$$

$\Rightarrow$ For $x = 0$

$$C_1 + C_2(0) + C_3(0) = 0$$

$$\boxed{C_1 = 0}$$

$\Rightarrow$ For $x = \pi/2$

$$C_2(\sin 90^\circ) + C_3(\sin 180^\circ) = 0$$

$$C_2(1) + C_3(0) = 0$$

$$\boxed{C_2 = 0}$$

$\Rightarrow$ For $x = \pi/4$

$$c_3 (\sin 2(45^\circ)) = 0$$

$$c_3 (\sin 90^\circ) = 0$$

$$\boxed{c_3 = 0}$$

Since all the constants have become 0. Thus the given functions are linearly independent.

(d) $\cos 2x$, $\sin^2 x$, $\cos^2 x$

Here,

$$f_1 = \cos 2x$$

$$f_2 = \cos^2 x$$

$$f_3 = \sin^2 x$$

We can set an equation like given below

$$c_1 f_1 + c_3 f_3 - c_2 f_2 = 0$$

$$(1)(\cos 2x) + (1)(\sin^2 x) - (1)(\cos^2 x) = 0$$

$$\cos 2x + \sin^2 x - \cos^2 x = 0$$

As we know that

$$\cos 2x = \cos^2 x - \sin^2 x$$

so,

$$\cos^2 x - \sin^2 x + \sin^2 x - \cos^2 x = 0$$

$$0 = 0$$

Hence, the given functions are dependent.

(e) $(3-x)^2,\ x^2-6x,\ 5$

Here,

$$f_1 = (3-x)^2 = 9-6x+x^2$$
$$f_2 = x^2-6x$$
$$f_3 = 5$$

The equation to prove is given as

$$c_1f_1 - c_2f_2 - c_3f_3 = 0$$
$$(1)(9-6x+x^2) - (1)(x^2-6x) - \frac{9}{\cancel{5}}(\cancel{5}) = 0$$
$$9-6x+x^2-(x^2-6x)-9 = 0$$
$$\cancel{9}-\cancel{6x}+\cancel{x^2}-\cancel{x^2}+\cancel{6x}-\cancel{9} = 0$$
$$0 = 0$$

Thus, the given functions are dependent.

(f) $0,\ \cos^3 \pi x,\ \sin^5 3\pi x$

From Theorem 4.3.2 (a), this set is linearly dependent.

THEOREM 4.3.2

(*a*) *A finite set that contains* **0** *is linearly dependent.*

(*b*) *A set with exactly one vector is linearly independent if and only if that vector is not* **0**.

(*c*) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*