DEFINITION 1 The ***dimension*** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

▶ EXAMPLE 1 Dimensions of Some Familiar Vector Spaces

$$\dim(R^n) = n \qquad \text{[The standard basis has } n \text{ vectors.]}$$

$$\dim(P_n) = n + 1 \qquad \text{[The standard basis has } n + 1 \text{ vectors.]}$$

$$\dim(M_{mn}) = mn \qquad \text{[The standard basis has } mn \text{ vectors.]}$$

Exercise 4.5

Question 1 :— Basis & Dimension

$$x_1 + x_2 - x_3 = 0$$
$$-2x_1 - x_2 + 2x_3 = 0$$
$$-x_1 + x_3 = 0$$

The augmented matrix of the given linear system is given as

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is given as

$$x_1 - x_3 = 0$$
$$x_2 = 0$$

Now,

Let $x_3 = t$

so,

$$x_1 - t = 0 \Rightarrow x_1 = t$$

Thus, we have

$$(x_1, x_2, x_3) = (t, 0, t) = t(1, 0, 1)$$

Therefore, the solution space is spanned by a vector $\vec{v_1} = (1, 0, 1)$. This vector is non-zero, therefore it forms a linearly independent set.

Hence,

We conclude that $\vec{v_1}$ forms a basis for the solution space and that the dimension of the solution space is 1.

Question 3:— Basis & Dimension

$$2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 5x_3 = 0$$
$$x_2 + x_3 = 0$$

The augmented matrix is given as

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The reduced echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The only solution is given as

$(x_1, x_2, x_3) = (0, 0, 0)$

Hence,

the solution space has no basis and its dimension is 0.

Question 4 :— Basis & Dimension

$$x_1 - 4x_2 + 3x_3 - x_4 = 0$$
$$2x_1 - 8x_2 + 6x_3 - 2x_4 = 0$$

The augmented matrix is given as

$$\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{array}\right]$$

The reduced echelon form is given as

$$\left[\begin{array}{cccc|c} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is

$$x_1 - 4x_2 + 3x_3 - x_4 = 0$$

Here,

Let $x_2 = r$; $x_3 = s$; $x_4 = t$

so,

$$x_1 - 4r + 3s - t = 0$$
$$x_1 = 4r - 3s + t$$

In vector form we have

$$(x_1, x_2, x_3, x_4) = (4r - 3s + t, r, s, t)$$
$$= r(4, 1, 0, 0) + s(-3, 0, 1, 1) + t(1, 0, 0, 1)$$

Therefore,

the solution space is spanned by $\vec{V_1} = (4, 1, 0, 0)$, $\vec{V_2} = (-3, 0, 1, 1)$, and $\vec{V_3} = (1, 0, 0, 1)$.

By inspection,

these vectors are linearly independent since $r\vec{V_1} + s\vec{V_2} + t\vec{V_3} = \vec{0}$ implies $r = s = t = 0$.

Hence,

the vectors $\vec{V_1}, \vec{V_2}$ and $\vec{V_3}$ form a basis for the solution space and the dimension of solution space is 3.

Question 7:— Basis & Dimension

(a) The plane $3x - 2y + 5z = 0$

Finding the general solution, we have

$$3x - 2y + 5z = 0$$
$$3x = 2y - 5z$$

$$x = \frac{2}{3}y - \frac{5}{3}z$$

Let $y = s$ and $z = t$

Thus, In vector form we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3}s - \frac{5}{3}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

<u>Basis</u> : $\left\{\left(\frac{2}{3}, 1, 0\right), \left(\frac{-5}{3}, 0, 1\right)\right\}$

<u>Dimension</u>: 2 (s, t)

(b) The plane $x - y = 0$

Finding the general solution, we have

$$x - y = 0$$

$$x = y$$

Let $y = s$ and $z = t$

Thus, In vector form we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y = s \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

<u>Basis</u> : $\{(1, 1, 0), (0, 0, 1)\}$

<u>Dimension</u>: 2 $(s t)$

(c) The line $x = 2t$, $y = -t$, $z = 4t$

This is already given in terms of one parameter t, so the vector form is given as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ 4t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Hence,

Basis: $\{(2, -1, 4)\}$

Dimension: 1

(d) All vectors of the form (a, b, c) where $b = a + c$.

The vector we have is given as

$$(a, b, c) = (a, a + c, c)$$

The general solution is given as

$$b = a + c$$

Let $a = s$, $c = t$

so, $b = s + t$

Thus, in vector form we have

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} s \\ s+t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence,

Basis: $\{(1,1,0), (0,1,1)\}$

Dimension: 2

Question 8:— Basis & Dimension

(a) All vectors of the form $(a,b,c,0)$.

The vector is given as

$(a, b, c, 0)$

There are three independent variables so we can rewrite the vector as

$$(a,b,c,0) = a(1,0,0,0) + b(0,1,0,0) + c(0,0,1,0)$$

Hence,

Basis: $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$

Dimension: Since there are three vectors in the basis so the dimension is 3.

(b) All vectors of the form (a,b,c,d) where $d = a+b$ and $c = a-b$.

Substituting the constraints for c and d into the vector (a,b,c,d):

$(a, b, a-b, a+b)$

There are two independent variables so we can rewrite the vector as

$$(a, b, a-b, a+b) = a(1,0,1,1) + b(0,1,-1,1)$$

Hence,

Basis : $\{(1,0,1,1), (0,1,-1,1)\}$

Dimension : 2

(c) All vectors of the form (a,b,c,d) where $a=b=c=d$.

Substituting the constraints into (a,b,c,d)

$$(a, a, a, a)$$

There is only one independent variable

$$(a,a,a,a) = a(1,1,1,1)$$

Hence,

Basis : $\{(1,1,1,1)\}$

Dimension : 1

Question 9.— Dimension

(a) The vector space of all diagonal n×n matrices.

A diagonal matrix is defined by the condition that all entries off main diagonal are zero.

- → Free variables: The only non-zero entries are on the main diagonal: $a_{11}, a_{22}, \ldots, a_{nn}$.
- → Count: For an $n \times n$ matrix, there are exactly n positions on the main diagonal.
- → Dimension: Since there are n independent parameters, the dimension is n.

(b) The vector space of all symmetric n×n matrices.

A symmetric matrix satisfies the condition $A = A^T$, which implies that $a_{ij} = a_{ji}$.

- → Free variables:
  1) n entries of the main diagonal.
  2) All entries above the diagonal

(upper triangle). Once these are choosen, the entries below the diagonal are fixed by symmetry $(a_{ji} = a_{ij})$

→ Calculation:

- Total entries in $n \times n$ matrix $= n^2$
- Entries on diagonal $= n$
- Entries not on diagonal $= n^2 - n$
- Entries in upper triangle $= (n^2-n)/2$
- Total Dimension = Diagonal + Upper triangle

$$n + \frac{n^2-n}{2} = \frac{2n+n^2-n}{2} = \frac{n^2+n}{2}$$

→ Dimension: $\frac{n(n+1)}{2}$

(c) The vector space of all upper triangular $n \times n$ matrices

An upper triangular matrix has all zero entries below the main diagonal

→ Free Variables: We are free to choose any entry on the diagonal or above it.

→ Calculation:

- Row 1 has n non-zero entries.

- Row 2 has $n-1$ non-zero entries.
- ....
- Row n has 1 non-zero entry.
- Total $= n + (n-1) + \dots + 1 = \sum_{k=1}^{n} k$

→ Dimension: $\frac{n(n+1)}{2}$

Question 10 :- Dimension

The vector space P_3 consists of all polynomials of degree 3 or less. The standard general form is:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

but applying the given constraint $a_0 = 0$.
so we have,

$$p(x) = 0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$p(x) = a_1 x + a_2 x^2 + a_3 x^3$$

Now,
The given subspace can be expressed as span (S) where $S = \{x, x^2, x^3\}$ is a set of linearly independent vectors in P_3.
Therefore S forms a basis for subspace with dimension $= 3$.