Algebraic Topology I, WS 2021/22
Exercise sheet 1

solutions due: 25.10.21

Exercise 1.1: Let X be a finite CW-complex and Y a space such that for every basepoint $y \in Y$ and every number i that is less than or equal to the dimension of X the set $\pi_i(Y, y)$ is finite. Show that the set $[X, Y]$ of homotopy classes of maps from X to Y is finite.

Exercise 1.2: For $n \geq 1$ consider the maps

$$q_1, q_2 : S^n \vee S^n \to S^n$$

where q_i is the identity on the i-th wedge summand and sends the other wedge summand to the basepoint. A *pinch map* is a continuous map $p : S^n \to S^n \vee S^n$ such that both composites $q_1 \circ p$ and $q_2 \circ p$ are homotopic to the identity map of S^n.

(a) Show that there is a pinch map for every $n \geq 1$.

(b) Show that the effect of a pinch map on singular homology is given by

$$p_*(x) = (i_1)_*(x) + (i_2)_*(x)$$

for all all coefficient groups A and all $x \in H_n(S^n, A)$, where $i_1, i_2 : S^n \to S^n \vee S^n$ are the two wedge summand inclusions.

(c) Suppose now that $n \geq 2$. Use the Hurewicz theorem to show that any two pinch maps are homotopic as based maps.

Exercise 1.3: Let A be a subspace of a space X and let $\pi_2(X, A, x_0)^\dagger$ be the factor group of $\pi_2(X, A, x_0)$ by the normal subgroup generated by all elements of the form

$$(\omega \star f) \cdot f^{-1}$$

for all $\omega \in \pi_1(A, x_0)$ and all $f \in \pi_2(X, A, x_0)$. Show that the factor group $\pi_2(X, A, x_0)^\dagger$ is abelian.