Algebraic Topology I, WS 2021/22

Exercise sheet 3

solutions due: 08.11.21

Exercise 3.1: Let X be a simplicial set, T a topological space and $f : X \to \mathcal{S}(T)$ a morphism of simplicial sets, where $\mathcal{S}(-)$ denotes the singular simplicial set.

(a) Consider the continuous map

$$\begin{array}{rcl} \dot{\bigcup\limits_{n\geq 0}} X_n \times \nabla^n & \longrightarrow & T \ , \\ X_n \times \nabla^n \ni (x,t) & \longmapsto & f_n(x)(t) \ . \end{array}$$

Show that this induces a continuous map $\hat{f} : |X| \to T$ defined on the geometric realization.

(b) Show that for every simplicial set X and every topological space T the assignment

$$\mathrm{Hom}_{\text{simpl. sets}}(X, \mathcal{S}(T)) \longrightarrow \mathrm{Hom}_{\text{top. spaces}}(|X|, T) \ , \qquad f \longmapsto \hat{f}$$

is bijective.

(c) Show that the bijection of part (b) is natural in both variables.

In other words: this exercise shows that the geometric realization functor

$$|-| : (\text{simplicial sets}) \to (\text{topoplogical spaces})$$

and the singular simplicial set functor

$$\mathcal{S} : (\text{topological spaces}) \to (\text{simplicial sets})$$

are an *adjoint functor pair.*

Exercise 3.2: Show that for every $n \geq 0$ the map

$$(|p_1|, |p_2|) \ : \ |\Delta[n] \times \Delta[1]| \ \to \ |\Delta[n]| \times |\Delta[1]|$$

is a homeomorphism, where $p_1 : \Delta[n] \times \Delta[1] \to \Delta[n]$ and $p_2 : \Delta[n] \times \Delta[1] \to \Delta[1]$ are the projections to the two factors.

Exercise 3.3: Let I be a small category (i.e., the objects form a set, as opposed to a class). The *nerve* of I is the simplicial set NI given by

$$\begin{aligned} (NI)_n &= \text{set of all composable } n\text{-tuples of morphisms in } I \\ &= \{(f_n, \ldots, f_1) \mid \text{source}(f_{i+1}) = \text{target}(f_i) \text{ for } 1 \leq i \leq n-1\} \end{aligned}$$

Also, by convention, $(NI)_0$ is the set of objects of I.

(a) Show that there is a unique way to extend this data to a simplicial set in such a way that the face and degeneracy morphisms are given by the following formulas. For $n \geq 1$ and $0 \leq i \leq n$, the i-th boundary map $d_i : (NI)_n \to (NI)_{n-1}$ is defined by

$$d_i(f_n, \dots, f_1) \;=\; \begin{cases} (f_n, \dots, f_2) & \text{for } i = 0, \\ (f_n, \dots, f_{i+2}, f_{i+1} \circ f_i, f_{i-1}, \dots, f_1) & \text{for } 0 < i < n, \\ (f_{n-1}, \dots, f_1) & \text{for } i = n. \end{cases}$$

For $n = 1$, $d_0(f)$ and $d_1(f)$ are to be interpreted as the target, respectively source, of f. For $n \geq 0$ and $0 \leq i \leq n$, the degeneracy map $s_i : (NI)_n \to (NI)_{n+1}$ is given by

$$s_i(f_n, \dots, f_1) \;=\; (f_n, \dots, f_{i+1}, \mathrm{Id}, f_i, \dots, f_1) \ .$$

(b) Let J be another small category and $F : I \to J$ a functor. We define maps $(NF)_n : (NI)_n \to (NJ)_n$ by

$$(NF)(f_n, \dots, f_1) \;=\; (F(f_n), \dots, F(f_1)) \ .$$

Shows that these maps form a morphism $NF : NI \to NJ$ of simplicial sets.

(c) Show that the nerve construction preserves products, i.e., that the morphisms

$$N(I \times J) \xrightarrow{(N\mathrm{proj}_I,\ N\mathrm{proj}_J)} NI \times NJ$$

is an isomorphism of simplicial sets.

(d) Let $[1]$ denote the category with two objects 0 and 1 and three morphisms Id_0, Id_1 and $f : 0 \to 1$. Let $\tau : F \to G$ be a natural transformation of functors from I to J. Define a Funktor $H : I \times [1] \to J$ satisfying $H(-,0) = f$, $H(-,1) = g$ and $H(i,f) = \tau_i : fi \to gi$. Show that the natural transformation τ yields a simplicial homotopy between the morphisms $NF, NG : NI \to NJ$.

(e) Show that the nerves NI and NJ of two equivalent small categories are homotopy equivalent simplicial sets.