Solution for Exercise sheet 5

Yikai Teng, You Zhou

Exercise session: Thu. 8-10

Exercise 5.1 We show that these are fibre bundles by using Exercise 5.2(b). We first consider the real case. Note that $O(k)$ acts on both $V_n(\mathbb{R}^k)$ and $Gr_n(\mathbb{R}^k)$ transitively with an isotropy group $O(k-n)$ and $O(n) \times O(k-n)$, respectively. So $V_n(\mathbb{R}^k) \cong O(k)/O(k-n)$ and $Gr_n(\mathbb{R}^k) \cong O(k)/O(n) \times O(k-n)$. From this we see that q and p are just the projections expressed in Exercise 5.2(b). Moreover, if U is a neighborhood of $I \cdot O(k-n)$ in $V_n(\mathbb{R}^k)$, then we can form a continuous section $U \cdot O(k-n) \to O(k)$ by the Gram-Schmidt method. If V is a neighborhood of $I \cdot O(n) \times O(k-n)$, then we can form a continuous section $V \to O(k)$ by just putting two matrices on the diagonal. Then the conclusion of Exercise 5.2(b) shows that q and p are fibre bundles and that their fibres are $O(n)$ and $V_{n-m}(\mathbb{R}^{k-m})$, respectively. It can be proved similarly that in the complex cases q and p are still fibre bundles but their fibres are $U(n)$ and $V_{n-m}(\mathbb{C}^{k-m})$, respectively.

Next we consider the long exact homotopy group sequences. Noting that $V_1(\mathbb{R}^n) = S^{n-1}$ and that $V_1(\mathbb{C}^n) = S^{2n-1}$, we have the following two exact sequences (corresponding to $m = 1$ in the definition of p)

$$\cdots \to \pi_i(S^{k-n}) \to \pi_i(V_n(\mathbb{R}^k)) \to \pi_i(V_{n-1}(\mathbb{R}^k)) \to \pi_{i-1}(S^{k-n}) \to \cdots \tag{1}$$

$$\cdots \to \pi_i(S^{2k-2n+1}) \to \pi_i(V_n(\mathbb{C}^k)) \to \pi_i(V_{n-1}(\mathbb{C}^k)) \to \pi_{i-1}(S^{2k-2n+1}) \to \cdots \tag{2}$$

Using induction on n and these two sequences give immediately that $\pi_i(V_n(\mathbb{R}^k)) = 0$ when $i \leq k-n-1$ and that $\pi_i(V_n(\mathbb{C}^k)) = 0$ when $i \leq 2k-2n$. So $V_n(\mathbb{R}^k)$ is $(k-n-1)$-connected and $V_n(\mathbb{C}^k)$ is $(2k-2n)$-connected.

Finally we compute the homotopy groups. Taking $m = 1$ in the definition of p then gives us

$$\cdots \to \pi_{k-n+1}(S^{k-1}) \to \pi_{k-n}(V_{n-1}(\mathbb{R}^{k-1})) \to \pi_{k-n}(V_n(\mathbb{R}^k)) \to \pi_{k-n}(S^{n-1}) \to \cdots \tag{3}$$

$$\cdots \to \pi_{2k-2n+2}(S^{2k-1}) \to \pi_{2k-2n+1}(V_{n-1}(\mathbb{C}^{k-1})) \to \pi_{2k-2n+1}(V_n(\mathbb{C}^k)) \to \pi_{2k-2n+1}(S^{2k-1}) \to \cdots \tag{4}$$

So when $n > 1$ we have

$$\pi_{2k-2n+1}(V_n(\mathbb{C}^k)) \cong \pi_{2k-2n+1}(V_{n-1}(\mathbb{C}^{k-1})) \cong \cdots \cong \pi_{2k-2n+1}(V_1(\mathbb{C}^{k-n+1})) = \mathbb{Z}.$$

When $n > 2$ we have

$$\pi_{k-n}(V_n(\mathbb{R}^k)) \cong \pi_{k-n+1}(V_{n-1}(\mathbb{R}^{k-1})) \cong \cdots \cong \pi_{k-n}(V_2(\mathbb{R}^{k-n+2})).$$

Using the CW structure on $V_n(\mathbb{R}^k)$ (explained in P302 of Hatcher's *Algebraic Topology*) we get

$$H_{k-n}(V_2(\mathbb{R}^{k-n+2})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } k-n \text{ odd} \\ \mathbb{Z}, & \text{if } k-n \text{ even.} \end{cases}$$

This holds also for $n = 2$. Since by the definition of p we have $n > 1$, by Hurewicz's theorem we get

$$\pi_{k-n}(V_2(\mathbb{R}^{k-n+2})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } k-n \text{ odd} \\ \mathbb{Z}, & \text{if } k-n \text{ even.} \end{cases}$$

Exercise 5.2

(a) We need the following lemma.

Lemma. *If G is a topological group and $g \in G$, and U is a neighborhood of g, then there is a neighborhood V of e such that $V = V^{-1}$ and that $VgV^{-1} \subset U$.*

Proof of the Lemma. Consider the following two functions

$$f\colon G\times G\to G \qquad\qquad h\colon G\times G\to G$$
$$(v_1,v_2)\mapsto v_1gv_2^{-1} \qquad\qquad (v_1,v_2)\mapsto v_2gv_1^{-1}$$

Then f and h are both continuous since G is a topological group. Set

$$W_1 := f^{-1}(U)\cap(f^{-1}(U))^{-1},\ W_2 := h^{-1}(U)\cap(h^{-1}(U))^{-1},\ W := W_1\cap W_2.$$

(For a subset $Z\subset G\times G$, we define here $Z^{-1} := \{(z_1^{-1},z_2^{-1})\mid(z_1,z_2)\in Z\}$.) Then W is an open neighborhood of (e,e) and $W=W^{-1}$ since both W_1 and W_2 satisfy these two conditions. Moreover, let $p_1,p_2\colon G\times G\to G$ be projection to the first and second coordinate, respectively, then by our construction $p_1(W)=p_2(W)$ and it is an open neighborhood of e. In addition, $V:=p_1(W)$ satisfies that $V=V^{-1}$ and that $VgV^1=f(W)\subset f(f^{-1}(U))=U$. So V is our required neighborhood. $\square$

Now come back to this problem. Let xH and yH be different points in G/H. Then $y^{-1}x\notin H$. Since $G\setminus H$ is an open neighborhood of $y^{-1}x$, by the lemma there is an open neighborhood U of e such that $U=U^{-1}$ and that $Uy^{-1}xU^{-1}\cap H=\emptyset$. This implies that $y^{-1}xU\cap UH=\emptyset$. Thus xUH and yUH are neighborhoods of xH and yH, respectively, and they do not intersect.

(b) Following the hint, we first show that

$$\varphi\colon H/K\times U\to p^{-1}(U),\quad (hK,xH)\mapsto\sigma(xH)\cdot hK$$

is a homeomorphism by finding its inverse. Suppose that $xK\subset p^{-1}(U)$. Then $p(xK)=xH\in U$. From the following commutative diagram

$$\begin{tikzcd} G \arrow[r] & G/K \arrow[d, "p"] \\ U \arrow[u, "\sigma"] \arrow[r, hook] & G/H \end{tikzcd}$$

(the top map is quotient map and the bottom is inclusion) we get that $xH=p(\sigma(xH)K)=\sigma(xH)\cdot H$. So $(\sigma(xH))^{-1}x\in H$ and we can define

$$\psi\colon p^{-1}(U)\to H/K\times U,\quad xK\mapsto((\sigma(xH))^{-1}xK,xH)$$

This ψ is continuous by continuity of multiplication and inverse and the continuity of σ. It is also not hard to verify by expanding the definitions that $\psi\circ\varphi=\mathrm{id}$ and $\varphi\circ\psi=\mathrm{id}$. So we have shown that φ is a homeomorphism.

Now for every $gH\in G/H, gU$ is a neighborhood of gH and we can prove in a similar way that the map

$$H/K\times gU\to p^{-1}(gU),\quad (hK,gxH)\mapsto\sigma(gxH)\cdot hK$$

is a homeomorphism. This finishes the proof.