Solution for Exercise sheet 9

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Exercise 9.1 Following the hint, we can attach k-cells ($k \geq n+2$) to X to get a relative CW-complex (Y, X) such that $\pi_i(Y, x) \cong \pi_i(X, x)$ for $1 \leq i \leq n$ and $\pi_i(Y, x) = 0$ for $i \geq n+1$. By checking definitions we get the following commutative diagram (The coefficient $\mathbb{Z}$ in homology groups will be omitted for short.)

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+2}(Y,X,x) \arrow[r] \arrow[d, "h_1"] & \pi_{n+1}(X,x) \arrow[r] \arrow[d, "h_2"] & \pi_{n+1}(Y,x) \arrow[r] \arrow[d, "h_3"] & \pi_{n+1}(Y,X,x) \arrow[r] \arrow[d, "h_4"] & \cdots \\
\cdots \arrow[r] & H_{n+2}(Y,X) \arrow[r] & H_{n+1}(X) \arrow[r] & H_{n+1}(Y) \arrow[r] & H_{n+1}(Y,X) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical maps are all Hurewicz maps. By our construction and celluar approximation theorem we get $\pi_i(Y, X, x) = 0$ for $1 \leq i \leq n+1$, so h_4 is injective. Exercise 8.3 shows that $H_{n+1}(Y, X) = 0$, so h_3 is surjective. Since $n \geq 2$ we can use the relative Hurewicz theorem to conclude that h_1 is surjective. So by the five lemma h_2 is surjective.

Exercise 9.2

(a)

Well-definedness Two polynomials $f, g \in \mathbb{C}[x]$ represent a same element in $\mathbb{C}P^\infty$ if and only if $f = \lambda g$ for some $\lambda \in \mathbb{C}$. So for $\lambda_1, \lambda_2 \in \mathbb{C}$ we have

$$\mu(\mathbb{C}\lambda_1 f, \mathbb{C}\lambda_2 g) = \mathbb{C}(\lambda_1\lambda_2 fg) = \mathbb{C}(fg) = \mu(\mathbb{C}f, \mathbb{C}g).$$

Continuity If we set $\mathbb{C}_0^\infty := \{(x_i)_{i \in \mathbb{Z}_{>0}} \mid x_i \in \mathbb{C}$ for all i and only finitely many terms are nonzero$\}$ and give it the usual metric topology, then we have the commutative diagram

$$\begin{tikzcd}
\mathbb{C}_0^\infty \times \mathbb{C}_0^\infty \arrow[r] \arrow[d] & \mathbb{C}_0^\infty \arrow[d] \\
\mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "\mu"] & \mathbb{C}P^\infty
\end{tikzcd} .$$

The upper map is continuous because every component of it is a polynomial of finitely many variables. The two vertical maps are quotient maps, hence continuous. So μ, as their composition, is also continuous.

Associativity, Commutative and Unital All obvious by the corresponding properties of $\mathbb{C}[x]$.

(b) We need to show that the diagram on the left below commutes. Let $\alpha, \beta \colon (I^2, \partial I^2) \to (\mathbb{C}P^\infty, \mathbb{C}\cdot 1)$ be based maps representing elements in $\pi_2(\mathbb{C}P^\infty, \mathbb{C}\cdot 1)$. Choose $\varphi \colon \pi_2(\mathbb{C}P^\infty, \mathbb{C}\cdot 1)) \xrightarrow{\sim} \mathbb{Z}$. Then it is shown on the right below how an element is mapped in two ways.

$$\begin{tikzcd}
\pi_2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty, (\mathbb{C}\cdot 1, \mathbb{C}\cdot 1)) \arrow[r, "\mu_*"] \arrow[d] & \pi_2(\mathbb{C}P^\infty, \mathbb{C}\cdot 1) \arrow[dd, "\varphi"] \\
\pi_2(\mathbb{C}P^\infty, \mathbb{C}\cdot 1) \times \pi_2(\mathbb{C}P^\infty, \mathbb{C}\cdot 1) \arrow[d, "\varphi\times\varphi"] & \\
\mathbb{Z} \times \mathbb{Z} \arrow[r, "+"] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(\alpha,\beta)]} \arrow[rr, mapsto] \arrow[d, mapsto] & & {[\mu\circ(\alpha,\beta)]} \arrow[dd, mapsto] \\
{([\alpha],[\beta])} \arrow[d, mapsto] & & \\
{(\varphi([\alpha]),\varphi([\beta]))} \arrow[r, mapsto] & {\varphi([\alpha]+[\beta])} \arrow[r, equal] & {\varphi([\mu\circ(\alpha,\beta)])}
\end{tikzcd}$$

The equality $\varphi([\alpha]+[\beta]) = \varphi([\mu\circ(\alpha,\beta)])$ can be seen as follows: choose a representative $\gamma\colon (I^2,\partial I^2)\to(\mathbb{C}P^\infty,\mathbb{C}\cdot 1)$ for $[\alpha]+[\beta]$, say

$$\gamma(x,y) = \begin{cases}\alpha(2x,y), & \text{if } 0\le x\le \frac{1}{2}\\ \beta(2x-1,y), & \text{if } \frac{1}{2}< x\le 1,\end{cases} \quad \text{for } (x,y)\in I^2.$$

We can construct linear homotopies between each coordinate of γ and $\mu\circ(\alpha,\beta)$, which is constant on ∂I^2. Then by putting them together we can get a homotopy between γ and $\mu\circ(\alpha,\beta)$.

(d) By definition $[f]+[g] = [\mu\circ(f,g)]$ is sent to

$$\begin{aligned}[(f,g)^*\mu^*(\gamma)] &= [(f,g)^*(p_1^*(\gamma)\otimes p_2^*(\gamma))]\\ &= [(f,g)^*p_1^*(\gamma)\otimes(f,g)^*p_2^*(\gamma)]\\ &= [f^*(\gamma)\otimes g^*(\gamma)].\end{aligned}$$

(e) The well-definedness and continuity of i can be proved in a similar way as in (a). For every $[f]\in\pi_2(\mathbb{C}P^\infty,\mathbb{C}\cdot 1)$ we have

$$[f]+i_*([f]) = [\mu(f,\bar{f})] = [\mathbb{C}\cdot 1].$$

so i induces the additive inverse. To construct an isomorphism we denote by E_γ the total space of γ and consider the following diagram which commutes without the dashed arrow.

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left, "z\mapsto(\mathbb{C}\cdot f(z),f(z))"] \arrow[ddr, bend right, "z\mapsto g(z)"'] & & \\
& \bar{\gamma} \arrow[r] \arrow[d] & E_\gamma \arrow[d, "\gamma"] \\
& \mathbb{C}P^\infty \arrow[r, "i"] & \mathbb{C}P^\infty
\end{tikzcd}$$

We want to show that there is a unique $\varphi\colon Z\to\bar{\gamma}$ such that the whole diagram commutes. This is clear because the commutativity forces φ to map any $z\in Z$ to the pair $(\mathbb{C}f(z),\overline{g(z)})\in\mathbb{C}P^\infty\times\mathbb{C}[x]$, which is well-defined by the commutativity from our assumption. So $\bar{\gamma}$ is the fiber product $E_\gamma\times_{\mathbb{C}P^\infty}\mathbb{C}P^\infty$ and hence there is an isomorphism $i^*(\gamma)\cong\bar{\gamma}$.