Algebraic Topology I, WS 2021/22

Exercise sheet 6

solutions due: 29.11.21

Exercise 6.1: Let $p : S^m \to S^n$ be a fiber bundle with $m, n \geq 1$ whose fiber is homeomorphic to the sphere S^k. Show that then $k = n - 1$ and $m = 2n - 1$.

Exercise 6.2: Show by examples that various hypotheses in the exponential law are really necessary.

(a) Find spaces X (not loc. cpt.) and Z such that the evaluation map

$$\mathrm{ev} \; : \; Z^X \times X \to Z \; , \quad (f, x) \longmapsto f(x)$$

is not continuous. $Z = \{0,1\}$ open $= \{\{0\}, \emptyset, Z\}$ $X = \mathbb{Q}$ not loc.cpt. , $Y = Z^X$

(b) Find spaces X, Y and Z such that the exponential map

$$\Phi \; : \; Z^{X \times Y} \to (Z^X)^Y$$

$\Phi^{-1}(\mathrm{id}) = \mathrm{ev}$, not conti. by (*).

is not surjective.

(c) Find spaces X, Y and Z such that X is locally compact but the exponential map Φ is not a homeomorphism.

Exercise 6.3: Let X be a topological space with basepoint x_0. Let

$$E \; = \; \{ f \in X^{\nabla^2} \mid f(1,0,0) = f(0,1,0) = f(0,0,1) = x_0 \}$$

be the space of continuous maps from the 2-simplex to X that takes all three vertices to the basepoint. We define continuous maps

$$i, j \; : \; [0,1] \to \nabla^2$$

by

$$i(t) \; = \; (t, 1-t, 0) \quad \text{respectively} \quad j(t) \; = \; (0, t, 1-t) \; .$$

Show that the map

$$\Phi \; : \; E \to \Omega X \times \Omega X \; , \quad f \longmapsto (f \circ i, f \circ j)$$

is a homotopy equivalence.

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Exercise sheet 7

solutions due: 06.12.21

Exercise 7.1: Let X be a topological space and Z a Hausdorff space. Let $f : X \to Z$ be a continuous map and $\{f_n\}_{n\geq 1}$ a sequence in Z^X. Suppose that f is a limit of the sequence $\{f_n\}$ in the compact-open topology on Z^X. Show that then the sequence $\{f_n\}$ converges pointwise to f. Show by example that the converse need not hold.

Exercise 7.2: Let $p : E \to B$ be a Serre fibration whose total space is contractible. Show that the fiber of p over a point $b \in B$ is weakly homotopy equivalent to the loop space of B, based at b.

Exercise 7.3: Given a permutation $\sigma \in \Sigma_3$ of the set $\{1,2,3\}$, define the homeomorphism

$$\sigma_* : \nabla^2 \to \nabla^2 \qquad \text{by} \qquad \sigma_*(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) .$$

For each of the six elements of the group Σ_3, identify the composite

$$\pi_1(X) \times \pi_1(X) \cong \pi_0(\Omega X \times \Omega X) \xrightarrow{\pi_0(\Phi)^{-1}} \pi_0(E) \xrightarrow{\pi_0(\sigma^*)} \pi_0(E) \xrightarrow{\pi_0(\Phi)} \pi_0(\Omega X \times \Omega X) \cong \pi_1(X) \times \pi_1(X)$$

explicitly in terms of group theoretic operations. Here $\Phi : E \to \Omega X \times \Omega X$ is the homotopy equivalence introduced in Exercise 6.3, and $\sigma^* : E \to E$ is the homeomorphism obtained by restricting $(\sigma_*)^* : X^{\nabla^2} \to X^{\nabla^2}$ to E.

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Exercise sheet 8

solutions due: 13.12.21

Exercise 8.1: Let

$\omega_{n+1}(0)$ $\quad \omega_n(0)$ $\quad \omega_{n-1}(0)$

$\omega_n(1)$ $\quad \omega_{n-1}(1)$

$$\cdots \longrightarrow P_{n+1} \xrightarrow{p_n} P_n \xrightarrow{p_{n-1}} P_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_0} P_0$$

be a sequence of topological spaces and continuous maps. The *homotopy limit* of the sequence is the space

$$\operatorname{holim}_n P_n \;=\; \left\{(\omega_n)_{n\geq 0} \in \prod\nolimits_{n\geq 0} P_n^{[0,1]} \mid \omega_n(1) = p_n(\omega_{n+1}(0)) \text{ for all } n \geq 0\right\} \,.$$

The homotopy limit has the subspace topology of the product topology. We define continuous maps

$$q_k \;:\; \operatorname{holim}_n P_n \;\to\; P_k \qquad \text{by} \qquad q_k((\omega_n)_{n\geq 0}) \;=\; \omega_k(0)\,.$$

Let $\omega = (\omega_n)$ be a basepoint in $\operatorname{holim}_n P_n$ and let $m \geq 1$.

(a) We define a homomorphism Ψ_n as the composite

$$\pi_m(P_{n+1}, \omega_{n+1}(0))) \xrightarrow{(p_n)_*} \pi_m(P_n, p_n(\omega_{n+1}(0))) \xrightarrow{(\omega_n)_*} \pi_m(P_n, \omega_n(0))\,.$$

Here $(\omega_n)_*$ is the isomorphism given by conjugation with the path ω_n. Show that the homomorphisms

$$\pi_m(q_k) \;:\; \pi_m(\operatorname{holim}_n P_n, \omega) \;\to\; \pi_m(P_k, \omega_k(0))$$

satisfy

$$\Psi_k \circ \pi_m(q_{k+1}) \;=\; \pi_m(q_k)$$

and hence they assemble into a homomorphism

$$\pi_m(\operatorname{holim}_n P_n, \omega) \;\to\; \lim_n \pi_m(P_n, \omega_n(0))\,, \tag{1}$$

where the limit in the right hand side is taken over the homomorphisms Ψ_n.

(b) Show the homomorphism (1) defined in (a) is surjective.

(c) Suppose that there is an $N \geq 0$ such that for all $n \geq N$ the map

$$\Psi_n \;:\; \pi_{m+1}(P_{n+1}, \omega_{n+1}(0)) \;\to\; \pi_{m+1}(P_n, \omega_n(0))$$

is surjective (note the index $m+1$ instead of m). Show that then the homomorphism (1) is bijective.

Let $f: (I^m, \partial I^m) \to (\operatorname{holim}_n P_n, \omega)$ s.t. $[f_k] = 0$, $\forall k$

Choose based htpy $H^{N-1}: I^m \times I \to P_{N-1}$ s.t. $H^{N-1} = p_{N-1} f_N$ on $I^m \times 1$, $\text{const}_{\omega_{N-1}(1)}$ on $I^m \times 0$

Extend H^{N-1} to $\overline{H^{N-1}}: I^m \times I \times I \to P_{N-1}$ s.t. $\overline{H^{N-1}} = H^{N-1}$ on $I^m \times I \times 1$, const on $\partial I^m \times I \times I \cup I^m \times \partial I \times I$

Do same thing for $p_{N-2} \circ \overline{H^{N-2}} \simeq f_{N-2}$ $\rightsquigarrow \overline{H^k}$ $(0 \leq k \leq N-1)$

Find a lift $\overline{f}^{N+1}: (I^m \times I, \partial(I^m \times I)) \to (P_{N+1}, \omega_{N+1}(0))$

Exercise 8.2:

(a) Let

$$G_0 \xrightarrow{\alpha_0} G_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{m-1}} G_m \xrightarrow{\alpha_m} \dots$$

be a sequence of groups and group homomorphism. Consider $n \geq 1$; if $n \geq 2$, assume that all groups G_m are abelian. Let X_m be an Eilenberg-MacLane space of type $K(G_m, n)$ and $f_m : X_m \to X_{m+1}$ a continuous based map that realizes α_m on π_n. Show that the mapping telescope of the sequence $\{f_m\}$ is an Eilenberg-MacLane space of type $K(G_\infty, n)$, where G_∞ is a colimit of the original sequence of group homomorphims.

(b) Let $f : S^1 \to S^1$ be the standard degree n map on the circle defined by $f(z) = z^n$. Show that the mapping telescope of the sequence

$$S^1 \xrightarrow{f} S^1 \xrightarrow{f} \dots \xrightarrow{f} S^1 \xrightarrow{f} \dots$$

is an Eilenberg-MacLane space and describe its fundamental group.

Exercise 8.3: Let A be an abelian group and $n \geq 2$. Show that the homology group

$$H_{n+1}(K(A, n), \mathbb{Z})$$

is trivial. (Hint: construct an Eilenberg-MacLane space from a 'Moore space', i.e., a space with only one non-vanishing reduced integral homology group, namely A concentrated in dimension n.)

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Exercise sheet 9
solutions due: 20.12.21

Exercise 9.1: Let X be an $(n-1)$-connected space for $n \geq 2$. Show that the Hurewicz homomorphism

$$\pi_{n+1}(X,x) \to H_{n+1}(X,\mathbb{Z})$$

is surjective in dimension $n+1$. (Hint: build an Eilenberg-MacLane space from X and use a previous exercise.)

$\mathbb{C}P^\infty = \operatorname{colim}_n \mathbb{C}P^n$

Exercise 9.2: Let $\mathbb{C}P^\infty$ denote the space of complex lines in $\mathbb{C}[x]$, the complex polynomial algebra in one variable x.

(a) Show that the map

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty\ , \quad \mu(\mathbb{C}f, \mathbb{C}g) = \mathbb{C}(f \cdot g)$$

need $\mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{n+m}$ conti. (follows from bilinearity of

is well-defined, continuous, associative, commutative and unital, where $f \cdot g$ is the product in the polynomial ring.

(b) Show that the map μ induces the addition on the homotopy group $\pi_2(\mathbb{C}P^\infty, \mathbb{C}\cdot 1)$

(c) Let γ denote the tautological complex line bundle over $\mathbb{C}P^\infty$, with total space consisting of all pairs $(L,x) \in \mathbb{C}P^\infty \times \mathbb{C}[x]$ such that $x \in L$. Construct an isomorphism of line bundles over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ between $\mu^*(\gamma)$ and $p_1^*(\gamma) \otimes_\mathbb{C} p_2^*(\gamma)$, where $p_1, p_2 : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ are the projections. $\mathbb{C}(f\cdot g) \mapsto \mathbb{C}f \otimes_\mathbb{C} \mathbb{C}g$

?

(d) Show that for every paracompact space X the map $\lambda(f\cdot g) \mapsto \lambda(f \otimes_\mathbb{C} g)$

Fact: over paracpt spaces, pullback of bundles along homotopic maps gives isomorphic bundles.

$$[X, \mathbb{C}P^\infty] \to \mathrm{Pic}_\mathbb{C}(X)\ , \quad [f] \longmapsto [f^*(\gamma)]$$

to the set $\mathrm{Pic}_\mathbb{C}(X)$ of isomorphism classes of complex line bundle over X is a group homomorphism for the addition on the source induced by μ, and for the tensor product of line bundles on the target.

(e) Define $i : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ by $i(\mathbb{C}f) = \mathbb{C}\bar{f}$, where $\bar{f}$ is obtained from f by conjugating all coefficients of the polynomial f. Show that i is well-defined, continuous, and that it induces the additive inverse map of $\pi_2(\mathbb{C}P^\infty, \mathbb{C}\cdot 1)$. Construct an isomorphism of line bundles over $\mathbb{C}P^\infty$ between $i^*(\gamma)$ and $\bar{\gamma}$, the complex conjugate of the tautological line bundle.

$\alpha : (S^2, *) \to (\mathbb{C}P^\infty, \mathbb{C}\cdot 1)$

then $[\alpha] + [i \circ \alpha] = [\alpha \cdot \bar{\alpha}]$. Observe $\alpha\cdot\bar{\alpha} : S^2 \to \mathbb{R}P^\infty$

But $\pi_2(\mathbb{R}P^\infty) = 0$ $(\mathbb{Z}/2\mathbb{Z} \to S^\infty \to \mathbb{R}P^\infty)$

$\mathbb{R}P^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$, so $[\alpha\cdot\bar{\alpha}] = 0$

Def. Pointed space X is an H-space if $\exists$ pointed map $X \times X \xrightarrow{\mu} X$ s.t. $\mu(-,x) \simeq \mathrm{id}_X \simeq \mu(x,-)$.

Lem. X: H-space, then $+$ on $\pi_n(X,x)$ is induced by μ.
(proof by Eckmann-Hilton argument)

$n=0$: ✓

$n \geq 1$: Notice $H_k(X)=0, \forall k \geq n+1$

n-conn. $\Rightarrow \tilde{H}_k(X)=0, \forall k \leq n$

$\pi_k(X)=H_k(X)=0 \ \forall k \geq n+1$ by Hurewicz & induction

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Exercise sheet 10

solutions due: 10.01.22

Exercise 10.1: Show that every CW-complex that is n-connected and n-dimensional is contractible.

Exercise 10.2: Show that for every continuous map $f : X \to Y$, the following conditions are equivalent.

$3 \Rightarrow 2$ obvious

$2 \Rightarrow 1$ $\pi_n(f) = [0^*, X] = \{0\}$

(i) The map f is a weak homotopy equivalence.

(ii) For all $n \geq 0$, all continuous maps $\alpha : \partial D^n \to X$ and $\beta : D^n \to Y$ such that $\beta|_{\partial D^n} = f \circ \alpha$, there is a continuous map $\lambda : D^n \to X$ such that $\lambda|_{\partial D^n} = \alpha$ and such that $f \circ \lambda : D^n \to Y$ is homotopic, relative to ∂D^n, to β.

$\times I$ is left adjoint $\Rightarrow$ preserves pushout (colim) [See Mapping spaces]

(iii) For every CW-complex K and every subcomplex L of K, all continuous maps $\alpha : L \to X$ and $\beta : K \to Y$ such that $\beta|_L = f \circ \alpha$, there is a continuous map $\lambda : K \to X$ such that $\lambda|_L = \alpha$ and such that $f \circ \lambda : K \to Y$ is homotopic, relative to L, to β.

(iv) For every CW-complex K, the induced map

$$[K, f] : [K, X] \to [K, Y]$$

of homotopy classes of continuous maps is bijective.

Exercise 10.3: Let A and B be abelian groups, and $n \geq 1$.

$H^n(K(A,n), A) \cong \mathrm{Hom}(A, A)$

$\iota_{K(A,n)} \leftrightarrow id_A$

fundamental class

(i) Show that for $1 \leq m < n$ the only natural transformation $H^n(X; A) \to H^m(X; B)$ is the zero transformation.

(ii) Every group homomorphism $\varphi : A \to B$ gives rise to a coefficient homomorphism

$$\varphi_* : H^n(X; A) \to H^n(X; B)$$

where X is any space. Show that this assignment defines an isomorphism of groups

$$\mathrm{Hom}(A, B) \to \mathrm{Nat}(H^n(-; A), H^n(-; B))$$

to the group of cohomology operations of type (A, n, B, n).

(iii) In an earlier exercise we constructed from a short exact sequence of abelian groups

$$0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0$$

a Bockstein homomorphism

$$\beta(i, p) : H^n(X; A) \to H^{n+1}(X; B)$$

where X is any space. The Bockstein homomorphism only depends on the class of the extension in $\mathrm{Ext}(A, B)$.

Show that this assignment defines an isomorphism of groups

$$\beta : \mathrm{Ext}(A, B) \to \mathrm{Nat}(H^n(-; A), H^{n+1}(-; B))$$

to the group of cohomology operations of type $(A, n, B, n+1)$.

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Exercise sheet 11

solutions due: 17.01.22

Exercise 11.1:

(i) A simplicial set X is *m-dimensional* if all simplices of X_n for $n > m$ are degenerate. Show that the product of an m-dimensional simplicial set and an n-dimensional simplicial set is $(m+n)$-dimensional.

(ii) Identify the non-degenerate simplices of the simplicial set $\Delta^n \times \Delta^1$. How many non-degenerate $(n+1)$-simplices does $\Delta^n \times \Delta^1$ have?

Remark: We showed in an earlier exercise that the geometric realization of the simplicial set $\Delta^n \times \Delta^1$ is homeomophic to $|\Delta^n| \times |\Delta^1|$ and hence to $\nabla^n \times \nabla^1$. Part (ii) shows that for $n \geq 1$, the preferred CW-structure on the realization of $\Delta^n \times \Delta^1$ is *not* the product CW-structure for the preferred CW-structures on $|\Delta^n|$ and $|\Delta^1|$.

$\mathcal{N}([n]) \times \mathcal{N}([m]) \cong \mathcal{N}([n] \times [m])$

Exercise 11.2:

(i) Let K be a compact space. Show that for every space X the map

$$\eta : X \longrightarrow (X \times K)^K, \quad \eta(x)(k) = (x, k)$$

$x \longmapsto \mathrm{const}_x \times \mathrm{id}_K$

adjoint of $X \times K \xrightarrow{id} X \times K$

is continuous.

(ii) Show that for every compact space K, the functor $- \times K$ from the category of topological spaces to itself is left adjoint to the functor sending a space Y to Y^K with the compact-open topology. K c,H $\Rightarrow$ $Z^{X \times K} \longrightarrow (Z^K)^X$ bij.

(iii) Show that for every simplicial set A the map

$|A \times \Delta^1| \cong |\operatorname{colim}_{\Delta/A} \Delta^n \times \Delta^1| \cong \operatorname{colim}_{\Delta/A} |\Delta^n \times \Delta^1|$

$\cong \operatorname{colim}_{\Delta/A} |\Delta^n| \times |\Delta^1|$

$$(|p_1|, |p_2|) : |A \times \Delta^1| \longrightarrow |A| \times |\Delta^1|$$

is a homeomorphism, where $p_1 : A \times \Delta^1 \to A$ and $p_2 : A \times \Delta^1 \to \Delta^1$ are the projections to the two factors. (Hint: use the simplicial skeleton filtration for A and the fact that $- \times |\Delta^1|$ preserves colimits to reduced the claim to the special case $A = \Delta^n$ which was shown earlier.)

Exercise 11.3: Let G be a group, and let X be a G-simplicial set. Suppose that the G-action on the set X_0 of vertices is free. $x \in X_n \mapsto \beta^* x$; $\beta : [0] \to [n]$, $0 \mapsto 0$

(i) Show that the action of G on the set X_n of n-simplices is free for every $n \geq 0$.

partially ordered set with non-deg. spaces & strongly monotone maps

(ii) Show that the action of G on the geometric realization $|X|$ is free and properly discontinuous, i.e., every point $x \in |X|$ has a neighborhood U in $|X|$ such that $U \cap (g \cdot U) = \emptyset$ for all $g \in G$ with $g \neq 1$.

$sd(\Delta^n) := N(P\Delta^n)$

$sd(X) = \operatorname{colim}_{\Delta/X} (sd\,\Delta^n)$

$|sd\,X| \cong |X|$

barycentric subdivision $\Rightarrow$ take all simplicies containing the pt.

$U = \bigcup_{\Delta^k \in sd(X),\ x \in \mathrm{im}(|\Delta^k| \to |sd(X)| \cong |X|)} \mathring{\nabla}^k$

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Exercise sheet 12

This special exercise sheet covers material from the entire semester, and is intended as an opportunity to recapitulate the material. Solutions to this exercise sheet should *not* be handed in, and they will *not* be corrected. Still, the last tutorial meetings in the week January 24-28, 2022, will be devoted to discussing some of these exercises.

Exercise 1. Let $f\colon S^n \times S^n \to S^{2n}$ be the map which collapses $S^n \vee S^n$ to a point. Show that f induces the trivial map on all homotopy groups but f is not nullhomotopic.

Exercise 2. Let X be a connected CW complex with $\pi_i(X,x) = 0$ for $1 < i < n$ for some $n \geq 2$. Let h denote the Hurewicz map. Show that $H_n(X;\mathbb{Z})/h(\pi_n(X,x))$ is isomorphic to $H_n(K(\pi_1(X,x),1);\mathbb{Z})$.

Exercise 3. A space Y is a homotopy retract of a space X if there are continuous maps $Y \xrightarrow{i} X \xrightarrow{r} Y$ such that $r \circ i \simeq \mathrm{id}_Y$. Show that any simply-connected CW complex X which is a homotopy retract of a wedge of spheres $\bigvee_i S^{n_i}$ is again homotopy equivalent to a wedge of spheres.

Hint: First show the following: If the Hurewicz map is surjective for a space Y, then the same is true for any homotopy retract X of Y. Use this to construct a map from a wedge of spheres to X which induces isomorpisms on all homology groups.

Exercise 4. Let X be a path-connected space. Show that the suspension map

$$\pi_1(X,x) \longrightarrow \pi_2(SX,\tilde{x})$$

exhibits $\pi_2(SX,\tilde{x})$ as the abelianisation of $\pi_1(X)$.

Exercise 5. Show that $K(\mathbb{Z},n)$ has no model that is a finite CW-complex for even $n \geq 2$. Also show that $K(\mathbb{Z}/2,n)$ has no model that is a finite CW-complex for $n \geq 1$.

Hint: Think about cohomology operations and cup powers.

Exercise 6. Recall that two spaces are called weakly equivalent if there is a zig-zag of weak homotopy equivalences relating them.

1. Let $f\colon Y_1 \to Y_2$ be a continuous map and $\alpha_i\colon X_i \to Y_i$ be CW-approximations for $i = 1,2$. Show that there is a continuous map $g\colon X_1 \to X_2$ such that $\alpha_2 g \simeq f\alpha_1$.
2. Show that two spaces are weakly equivalent if and only if they admit a common CW-approximation.
3. Show that two CW-complexes are weakly equivalent if and only if they are homotopy equivalent.

Exercise 7. Let $f\colon E \to B$ be a Serre fibration where B is path connected. Show that if f is injective, then f is a weak homotopy equivalence.

Exercise 8. Let M be a connected surface (possibly non-compact or with non-empty boundary) with infinite fundamental group. Show that M is aspherical (i.e. $\pi_k(M, x) = 0$ for all $k \geq 2$).

Exercise 9. Let M be a closed connected n-manifold that is homotopy equivalent to a n-dimensional CW-complex. Show that the (homological) degree induces an isomorphism

$$[M, S^n] \to \mathbb{Z}$$

if M is oriented. Also show that the mod 2-degree induces an isomorphism

$$[M, S^n] \to \mathbb{Z}/2$$

if M is not orientable. Try to construct a representative of every homotopy class.

Hint: Build an Eilenberg-MacLane space from S^n.