Solution for Exercise sheet 8

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Exercise 8.1

Exercise 8.2

(a) Denote by $\mathrm{tel}_n X_n$ the mapping telescope of the sequence $\{f_m\}$. Then Exercise 2.3(b) shows that we have for every m an isomorphism

$$\pi_m(\mathrm{tel}_n X_n, i_0(x_0)) \cong \mathrm{colim}_k \pi_m(X_k, x_k),$$

where $x_k \in X_k$ are base points and $i_0\colon X_0 \hookrightarrow \mathrm{tel}_n X_n$ is inclusion. By definition of Eilenberg-Maclane spaces the right hand side is 0 when $m \neq n$ and G_∞ when $m = n$. So $\mathrm{tel}_n X_n$ is an Eilenberg-Maclane space of type $K(G_\infty, n)$.

(b) The first part follows directly from (a) and the only nontrivial homotopy group of the telescope is its fundamental group, which is the colimit of the following diagram.

$$G_0 \xrightarrow{\times n} G_1 \xrightarrow{\times n} \cdots \xrightarrow{\times n} G_m \xrightarrow{\times n} \cdots$$

where all groups $G_i = \mathbb{Z}$.

We claim that the colimit is $\mathbb{Z}[n^{-1}]$ with morphisms $G_i \to \mathbb{Z}[n^{-1}]$ given by $x \mapsto x/n^i$. We verify this by checking the universal property of colimits. Suppose that we have another abelian group A and morphisms $\beta_i\colon G_i \to A$, $i \geq 1$, such that for every $i < j$ we have

$$\beta_i(x) = \beta_j(n^{j-i}x) \tag{1}$$

for all $x \in \mathbb{Z}$. Then there is a unique group homomorphism $\gamma\colon \mathbb{Z}[n^{-1}] \to A$ such that $\beta_i(x) = \gamma(x/n^i)$ for all i. The value of the map is uniquely determined by this condition, so uniqueness is clear. This γ is well-defined because of (1). It is a group homomorphism because every β_i is. This shows that the required colimit, hence the required fundamental group is $\mathbb{Z}[n^{-1}]$.