Algebraic Topology I, WS 2021/22
Exercise sheet 4

solutions due: 15.11.21

Exercise 4.1: (Bar construction) Let G be a group. For $n \geq 0$, let $(BG)_n = G^n$ be the cartesian product of n copies of the underlying set of G (for $n = 0$ we interpret this as $(BG)_0 = \{1\}$). For $n \geq 1$ and $0 \leq i \leq n$ define $d_i : (BG)_n \to (BG)_{n-1}$ by

$$d_i(g_n, \dots g_1) \; = \begin{cases} (g_n, \dots, g_2) & \text{for } i = 0, \\ (g_n, \dots, g_i, g_{i+1} \cdot g_i, g_{i-1}, \dots, g_1) & \text{for } 0 < i < n, \\ (g_{n-1}, \dots, g_1) & \text{for } i = n. \end{cases}$$

(For $n = 1$ we interpret this as $d_0(g) = 1 = d_1(g)$.) For $n \geq 1$ and $0 \leq i \leq n-1$ define $s_i : (BG)_{n-1} \to (BG)_n$ by

$$s_i(g_{n-1}, \dots, g_1) \; = \; (g_{n-1}, \dots, g_{i+1}, 1, g_i, \dots, g_1) \; .$$

(a) Show that BG extends to a simplicial set. Identify BG as the nerve, in the sense of Exercise 3.3, of a suitable category.

(b) In the geometric realization $|BG|$ we take the class of $(1, 1) \in (BG)_0 \times \nabla^0$ as the basepoint and call it '1'. Every element $g \in G = (BG)_1$ yields a continuous map

$$\{g\} \times \Delta^1 \xrightarrow{\text{inclusion}} \bigcup_{n \geq 0} (BG)_n \times \Delta^n \xrightarrow{\text{projection}} |BG| \; .$$

Show that this map takes the two boudary points of $\{g\} \times \nabla^1$ to the basepoint of $|BG|$.

(c) We identify the interval $[0, 1]$ with $\{g\} \times \nabla^1$ via the homeomorphism sending t to $(g, (t, 1-t))$. By part (b) the composition

$$[0, 1] \xrightarrow{\cong} \{g\} \times \Delta^1 \to |BG|$$

is a loop at the basepoint $1 \in |BG|$. We let $\omega(g)$ denote the homotopy class of this loop in the fundamental group $\pi_1(|BG|, 1)$. Show that

$$\omega \; : \; G \; \to \; \pi_1(|BG|, 1)$$

is a group homomorphism.
(Hint: for the proof of $\omega(g) \cdot \omega(h) = \omega(g \cdot h)$ use the map

$$\Delta^2 \; \to \; \bigcup_{n \geq 0} (BG)_n \times \Delta^n \xrightarrow{\text{projection}} |BG|$$

parametrized by the 2-simplex $(g, h) \in (BG)_2$.

Background: the homomorphism $\omega : G \to \pi_1 (|BG|, 1)$ is even an isomorphism of groups. Moreover, $|BG|$ is a path-connected CW-complex all of whose higher homotopy groups are trivial. These properties characterize the space $|BG|$ up to homotopy equivalence; $|BG|$ is called a *classifying space* for the group G.

Exercise 4.2: Show that every compact simply-connected 3-manifold without boundary is homotopy equivalent to S^3. (Hint: you might want to use Poincaré duality and the Hurewicz theorem.)

Exercise 4.3: Let F be a topological space, $f : F \to F$ a continuous self map and

$$T_f \;=\; F \times [0,1] \,/\, (x,0) \sim (f(x),1)$$

the mapping torus of f. The map

$$\exp \circ \operatorname{pr}_2 \;:\; F \times [0,1] \to S^1 \;, \quad (x,t) \longmapsto e^{2\pi i t}$$

factors over a continuous map $p : T(f) \to S^1$. Show:

(a) If $f : F \to F$ is a homeomorphism, then the projection $p : T_f \to S^1$ is a fibre bundle.

(b) Every fiber bundle over S^1 is homeomorphic to the mapping torus of some self-homeomorphism of the fiber.