Algebraic Topology I, WS 2021/22

Exercise sheet 5

solutions due: 22.11.21

Exercise 5.1: We denote by $V_n(\mathbb{R}^k)$ the Stiefel manifold of n-frames in $\mathbb{R}^k$. An element of $V_n(\mathbb{R}^k)$ is an n-tuple $(x_1, \ldots, x_n)$ of orthonormal vectors from $\mathbb{R}^k$. The set $V_n(\mathbb{R}^k)$ is topologized as a subspace of $(\mathbb{R}^k)^n$. We denote by $Gr_n(\mathbb{R}^k)$ the Grassmann manifold of n-dimensional vector subspaces of $\mathbb{R}^k$. The set $Gr_n(\mathbb{R}^k)$ carries the quotient topology with respect to the map $q : V_n(\mathbb{R}^k) \to Gr_n(\mathbb{R}^k)$ that takes an n-frame onto its span. The complex Stiefel and Grassmann manifolds $V_n(\mathbb{C}^k)$ and $Gr_n(\mathbb{C}^k)$ are defined analogously. Show that the following maps are fiber bundles and identify the fibers:

$$\begin{aligned}
q &: V_n(\mathbb{R}^k) \to Gr_n(\mathbb{R}^k) && \text{für } 1 \leq n \leq k, \\
q &: V_n(\mathbb{C}^k) \to Gr_n(\mathbb{C}^k) && \text{für } 1 \leq n \leq k, \\
p &: V_n(\mathbb{R}^k) \to V_m(\mathbb{R}^k) && \text{für } 1 \leq m < n \leq k, \\
p &: V_n(\mathbb{C}^k) \to V_m(\mathbb{C}^k) && \text{für } 1 \leq m < n \leq k,
\end{aligned}$$

here the maps p forget the last $m - n$ vectors: $p(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$. Use the long exact homotopy group sequences to show that $V_n(\mathbb{R}^k)$ is $(k-n-1)$-connected and $V_n(\mathbb{C}^k)$ is $(2k-2n)$-connected. Calculate $\pi_{k-n}(V_n(\mathbb{R}^k))$ and $\pi_{2k-2n+1}(V_n(\mathbb{C}^k))$.

Exercise 5.2: Let G be a topological group that is also a Hausdorff space. Let H be a subgroup of G and G/H the space of right cosets with the quotient topology. Show:

(a) If H is closed in G, then G/H is a Hausdorff space.

(b) Let H be closed in G and suppose that there is a *local section*, i.e., a neighborhood U of $1 \cdot H$ in G/H and a continuous section $\sigma : U \to G$ (i.e., $p \circ \sigma = \mathrm{Id}_U$). Let K be a closed subgroup of H. Then the projection

$$G/K \to G/H \ , \quad gK \mapsto gH$$

is a fibre bundle with fibre H/K. (Hint: show that

$$H/K \times U \ \to \ p^{-1}(U) \ , \quad (hK, x) \mapsto \sigma(x) \cdot hK$$

is a homeomorphism. Use the action of G to produce enough local trivializations.)

Exercise 5.3: Let $p : E \to B$ be a fiber bundle with path connected base, $F = p^{-1}(b)$ the fiber over a point $b \in B$, and $x \in F$. Suppose that the inclusion $F \to E$ is homotopic to a constant map.

Show that the long exact homotopy group sequence degenerates into an isomorphism between $\pi_n(B, b)$ and $\pi_n(E, x) \times \pi_{n-1}(F, x)$ for all $n \geq 1$. Apply this to the Hopf fibrations $\nu : S^7 \to S^4$ and $\sigma : S^{15} \to S^8$ to deduce that the groups $\pi_7(S^4, z)$ and $\pi_{15}(S^8, z)$ each contain a copy of $\mathbb{Z}$ as a direct summand.