Solution for Exercise sheet 2

Yikai Teng, You Zhou

Exercise session: Thu. 8-10

Exercise 2.1

(a) Let $\{S_n^2\}_{n\in\mathbb{Z}}$ be a sequence of spheres. Fix x_n in each sphere S_n^2 and consider

$$Y := \mathbb{R} \coprod \left(\coprod_{n\in\mathbb{Z}} S_n^2 \right).$$

Then the universal cover, as shown in the question sheet, is

$$\widetilde{X} = Y/\sim .$$

The equivalence relation $\sim$ on Y is defined as

$$\forall x, y \in Y, p \sim q \Leftrightarrow \exists n \in \mathbb{Z} \text{ s.t. } x = n, y = x_n \text{ or } x = x_n, y = n.$$

To describe the covering map $p\colon \widetilde{X} \to X$, we will denote points in S^1 by $\{e^{i\theta} : \theta \in \mathbb{R}\}$ and points in every S^2 by $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. In particular, in X we let $e^{-i\pi/2}$ and $(0,0,1)$ be identified and in Y we let $(0,0,1) \in S_n^2$ be identified with $n \in \mathbb{R}$. Then p maps $\theta \in \mathbb{R}$ to $e^{2\pi i\theta} \in S^1$, and $(x,y,z) \in S_n^2$ to the "same" point $(x,y,z) \in S^2$. In particular, all x_n are mapped to the "intersection" of S^1 and S^2 in X.

Next we compute the deck transformation group G of $\widetilde{X}$. Suppose $\varphi \in G$ and $\tilde{x} \in \widetilde{X}$. Since $\varphi(\tilde{x})$ and $\tilde{x}$ are mapped by p to the same point in X, the only possible choices of $\varphi(\tilde{x})$ is

$$\begin{cases} \tilde{x} + n \text{ for some } n \in \mathbb{Z}, & \text{if } \tilde{x} \in \mathbb{R} \\ (x,y,z) \in S_n^2 \text{ for some } n \in \mathbb{Z}, & \text{if } \tilde{x} = (x,y,z) \in S_m^2 \text{ for some } m. \end{cases}$$

The continuity of φ forces φ to be a transformation by n units, with n some integer. This shows that $G \cong \mathbb{Z}$.

Finally, the covering space theory tells us that since $\widetilde{X}$ is simply-connected, we have $\pi_1(X, x_0) \cong G \cong \mathbb{Z}$.

(b) Since the universal cover is simply connected, the Hurewicz theorem tells us that

$$\pi_2(\widetilde{X}, \tilde{x_0}) \cong H_2(\widetilde{X}, \mathbb{Z}).$$

Since covering maps induces isomorphisms on n-th homotopy groups for every $n \geq 2$, it suffices to compute $H_2(\widetilde{X}, \mathbb{Z})$.

Let A and B denote $\mathbb{R}$ and $\coprod_{n\in\mathbb{Z}} S_n^2$, as subsets of $\widetilde{X}$, respectively. Then $A \cap B = \coprod_{n\in\mathbb{Z}}\{\text{point}\}$. and the interiors of A and B covers $\widetilde{X}$. By the Mayer-Vietoris sequence

$$\cdots \to H_2(A \cap B, \mathbb{Z}) \to H_2(A, \mathbb{Z}) \oplus H_2(B, \mathbb{Z}) \to H_2(\widetilde{X}, \mathbb{Z}) \to H_1(A \cap B, \mathbb{Z}) \to \cdots$$

we have $H_2(A, \mathbb{Z}) \oplus H_2(B, \mathbb{Z}) \cong H_2(\widetilde{X}, \mathbb{Z})$. Since $\mathbb{R}$ can be seen as a CW-complex with no n-cells for $n \geq 2$, we have $H_2(A, \mathbb{Z}) = 0$. Thus

$$H_2(\widetilde{X}, \mathbb{Z}) \cong H_2\left(\coprod_{n\in\mathbb{Z}} S_n^2, \mathbb{Z} \right) \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$$

(c) We first construct a basis for $\pi_2(\widetilde{X}, x_0)$. Let $f_n\colon (I^2, \partial I^2) \to (\widetilde{X}, \tilde{x_0})$ be the map illustrated below.

Exercise 2.2 Let ΣX denote the suspension of X and denote by x_1 and x_2 the points $X \times \{0\}$ and $X \times \{1\}$ in ΣX, respectively. Let

$$A := \Sigma X \setminus \{x_1\} \text{ and } B := \Sigma X \setminus \{x_2\},$$

then the interior of A and B covers X. By Mayer-Vietoris sequence we have the exact sequence

$$\cdots \to \widetilde{H}_n(A \cap B) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(\Sigma X) \to \widetilde{H}_{n-1}(A \cap B) \to \cdots$$

Note that both A and B are contractible and that $A \cap B$ and X are homotopy equivalent. Thus all the reduced integral homology groups of ΣX are trivial, which implies that X is path-connected and that the n-th integral homology group of ΣX is trivial for all $n \geq 1$. So the Hurewicz theorem shows that $\pi_n(X, x_0) = 0$ for all $n \geq 1$. Since X is a CW-complex, Whitehead's theorem shows that ΣX is contractible.

Exercise 2.3