3.3

(a) Consider the following composition of functors (which we also denote by NI)

$$NI\colon \Delta^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathcal{C}\mathrm{at}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(-,I)} (\mathrm{Sets}).$$

We first explain the notations involved. Here $\mathcal{C}at$ is the category of all small categories. For every object $[n]$ of Δ, the functor $\mathcal{C}$ sends $[n]$ to a category $\mathcal{C}_{[n]} := \mathcal{C}([n])$ defined as

- The set of objects is $\mathrm{Ob}(\mathcal{C}_{[n]}) = [n] = \{0, 1, \ldots, n\}$;
- $\mathrm{Hom}_{\mathcal{C}_{[n]}}(x, y) = \begin{cases} \{*\}, & \text{if } x \le y \\ \emptyset, & \text{otherwise.} \end{cases}$ In other words, for $x, y \in [n]$, there is at most one morphism from x to y and the morphism exists if and only if $x \le y$.

Every morphism $\alpha\colon [n] \to [m]$ is sent by $\mathcal{C}$ to a functor $\mathcal{C}_\alpha$ from $\mathcal{C}_{[m]}$ to $\mathcal{C}_{[n]}$ defined as

- For every $x \in [n], \mathcal{C}_\alpha(x) = \alpha(x)$
- A morphism $x \to y$ in $\mathcal{C}_{[n]}$ is sent to the unique morphism $\alpha(x) \to \alpha(y)$, which exists since α is weakly monotone.

The second functor $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(-, I)$ sends a small category $\mathcal{D}$ to the set of all functors from $\mathcal{D}$ to I, which is indeed a set since I is small. Its action on morphisms between small categories is natural.

Next we verify that this NI coincides with the NI defined in the problem for simplices by showing that giving a functor in $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n]}, I)$ is equivalent to giving a composable n-tuple of morphisms in I. Note that every morphism in $\mathcal{C}_{[n]}$ can be written as composition of morphisms of the form $(j-1) \to j$ with $j \in \{1, \ldots, n\}$. Thus giving images of all morphisms of $\mathcal{C}_{[n]}$ is equivalent to only giving images of morphisms of form $(j-1) \to j$. Now If we have a functor F from $\mathcal{C}_{[n]}$ to I, then we have for every $x \in [n]$ an object $F(x)$ of I and for every morphism $(j-1) \to j$ in $\mathcal{C}_{[n]}$ a morphism $F_j\colon F(j-1) \to F(j)$ in I. Note that by definition the target of F_{j-1} equals the source of F_j, so $(F_n, \ldots, F_1)$ is a composable n-tuple of morphisms in I. Conversely, given a composable n-tuple of morphisms in I, say $(F_n, \ldots, F_1)$ as described in the problem, we may define $F\colon \mathcal{C}_{[n]} \to I$ by

- $F(j) := \begin{cases} \mathrm{source}\,(F_{j+1}), & \text{if } 0 \le j < n \\ \mathrm{target}\,(F_n), & \text{if } j = n. \end{cases}$
- $F((j-1) \to j) := F_j$.

The functor is well-defined by composability of $(F_n, \ldots, F_1)$ and it clearly preserves identity and composition of morphisms. This finishes this verification. Moreove r, the above argument shows that a functor in $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n]}, I)$ is determined by the image of all morphisms $(j-1) \to j$ under it.

Finally we verify that the action of $d_i\colon [n-1] \to [n]$ on $(NI)_n$ is as required. The case for s_i will be similar and thus omitted. First, d_i induces a functor $\mathcal{C}_{d_i}\colon \mathcal{C}_{[n-1]} \to \mathcal{C}_{[n]}$, whose action on objects and morphisms are explained in the first paragraph. This $\mathcal{C}_{d_i}$ induces morphism of sets

$$\begin{gathered}\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n]}, I) \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n-1]}, I) \\ \begin{pmatrix} \text{images of } (j-1 \to j) \\ 1 \le j \le n \end{pmatrix} \mapsto \begin{pmatrix} \text{images of } d_i(j-1) \to d_i(j) \\ 1 \le j \le n-1 \end{pmatrix}.\end{gathered}$$

Therefore, when $j < i$, the image of f_j is still f_j. When $j = i$, the image of f_j is $f_{j+1} \circ f_j$ since $d_i(i-1) \to d_i(i)$ is composition of $(i-1) \to i$ and $i \to (i+1)$. When $j > i$, the image of f_j is f_{j+1}, as the morphism $d_i(j-1) \to d_i(j))$ becomes $j \to j+1$. This finishes the verification and hence the proof.

(b) For every $[n] \in \Delta$, the definition of $(NF)_n$ is already given in the problem. So we only need to verify that for every morphism $\alpha\colon [n] \to [m]$, the diagram

$$\begin{tikzcd}
(NI)_m \arrow[r, "(NF)_m"] \arrow[d, "\alpha^*"] & (NJ)_m \arrow[d, "\alpha^*"] \\
(NI)_n \arrow[r, "(NF)_n"] & (NJ)_n
\end{tikzcd}$$

commutes. For this, let $(f_m, \ldots, f_1) \in (NI)_m$. By the equivalence

$$\{\text{elements in } (NI)_m\} \leftrightarrow \{\text{functors in } \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[m]}, I) \leftrightarrow \left\{ \begin{array}{c} \text{image of morphisms } (j-1) \to j \\ \text{under a functor in } \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[m]}, I) \end{array} \right\} \tag{1}$$

established in part (a), we may write $f_j = G((j-1) \to j)$ for all $j \in [m]$ and some functor $G \in \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[m]}, I)$. Then by definition of maps in the diagram the image of f_j under $\alpha^* \circ (NF)_m$ and $(NF)_n \circ \alpha^*$ are both $F \circ G(\alpha(j-1) \to \alpha(j))$.

(c) The inverse morphism $\psi\colon NI \times NJ \to N(I \times J)$ can be constructed as

$$\begin{aligned}
(NI \times NJ)_n = (NI)_n \times (NJ)_n &\xrightarrow{\psi_n} N(I \times J)_n \\
(f_n, \ldots, f_1) \times (g_n, \ldots, g_1) &\mapsto (f_n \times g_n, \ldots, f_1 \times g_1)
\end{aligned}$$

By writing elements of $(NI)_n$ as images of morphisms in $\mathcal{C}_{[n]}$ under some functor from $\mathcal{C}_{[n]}$ to I, as we did in the previous parts, we can show that for every $\alpha \in \mathrm{Mor}_\Delta([n], [m])$ the diagram

$$\begin{tikzcd}
(NI)_m \times (NJ)_m \arrow[r, "\psi_m"] \arrow[d, "\alpha^*"'] & N(I \times J)_m \arrow[d, "\alpha^*"] \\
(NI)_n \times (NJ)_n \arrow[r, "\psi_n"] & N(I \times J)_n
\end{tikzcd}$$

commutes. Thus ψ is indeed a morphism between simplicial sets. By computing directly the action on elements, we can show that both the compositions

$$(NI \times NJ)_n = (NI)_n \times (NJ)_n \xrightarrow{\psi_n} N(I \times J)_n \xrightarrow{(N_{\mathrm{proj}_I}, N_{\mathrm{proj}_J})_n} (NI)_n \times (NJ)_n = (NI \times NJ)_n$$
$$\text{and } N(I \times J)_n \xrightarrow{(N_{\mathrm{proj}_I}, N_{\mathrm{proj}_J})_n} (NI)_n \times (NJ)_n = (NI \times NJ)_n \xrightarrow{\psi_n} N(I \times J)_n$$

are identities. So $(N_{\mathrm{proj}_I}, N_{\mathrm{proj}_J})$ is an isomorphism between simplicial sets.

(d) We claim that there is an isomorphism $\varphi\colon \Delta[1] \to N[1]$ between simplicial sets, which will be constructed later. If this holds, we have the following diagram (identify NI with $NI \times \Delta[0]$)

$$\begin{tikzcd}
NI \arrow[r, "1 \times d^1"] \arrow[dddr, bend right, "NF"'] & NI \times \Delta[1] \arrow[d, "\cong"] & NI \arrow[l, "1 \times d^0"'] \arrow[dddl, bend left, "NG"] \\
& NI \times N[1] \arrow[d, "\cong"] & \\
& N(I \times [1]) \arrow[d, "NH"] & \\
& NJ &
\end{tikzcd}$$

where NH is the morphism constructed from H as in part (b). The commutativity of this diagram can be showed by routine verifications on simplices, using the definition of each maps and the equivalence (1). And the existence of such a commutative diagram is just the definition of a simplicial homotopy between NF and NG.

Now we show our claim by briefly explaining the construction of the isomorphism $\varphi\colon \Delta[1] \to N[1]$. For every $n \in \mathbb{N}$, define

$$\begin{aligned} \varphi_n\colon (\Delta[1])_n &\to (N[1])_n \\ (g\colon [n] \to [1]) &\mapsto (g_n, \dots, g_1), \text{ with } g_j = \begin{cases} \mathrm{Id}_0, & \text{if } g(j) = g(j-1) = 0 \\ f\colon 0 \to 1, & \text{if } g(j) = 1, g(j-1) = 0 \\ \mathrm{Id}_1, & \text{if } g(j) = g(j-1) = 1. \end{cases} \end{aligned}$$

For every $\alpha\colon [n] \to [m]$, The diagram

$$\begin{tikzcd} (\Delta[1])_m \arrow[r, "\alpha^*"] \arrow[d, "\varphi_m"] & (\Delta[1])_n \arrow[d, "\varphi_n"] \\ (N[1])_m \arrow[r, "\alpha^*"] & (N[1])_n \end{tikzcd}$$

commutes because if we choose some $(g\colon [m] \to [1]) \in (\Delta[1])_m$, then by computing with the definition of maps and the equivalence (1), we find that

$$\alpha^* \circ \varphi_m(g) = \varphi_n \circ \alpha^*(g) = (g_n, \dots, g_1)$$
$$\text{with } g_j = \begin{cases} \mathrm{Id}_0, & \text{if } g \circ \alpha(j) = g \circ \alpha(j-1) = 0 \\ f\colon 0 \to 1, & \text{if } g \circ \alpha(j) = 1, g \circ \alpha(j-1) = 0 \\ \mathrm{Id}_1, & \text{if } g \circ \alpha(j) = g \circ \alpha(j-1) = 1. \end{cases}$$

So φ is a morphism. Now we describe its inverse $\psi\colon N[1] \to \Delta[1]$. Given some composable n-tuple $(g_n, \dots, g_1) \in (N[1])_n$, there are three possibilities (by composability)

- $g_n = \cdots = g_1 = \mathrm{Id}_0$
- There is one j such that $g_n = \cdots = g_{j+1} = \mathrm{Id}_1$, $g_j = f$ and $g_{j-1} = \cdots = g_1 = \mathrm{Id}_0$
- $g_n = \cdots = g_1 = \mathrm{Id}_1$

In the first case, we let $\psi_n(g_n, \dots, g_1)$ sends $[n]$ to 0. In the second case, $\psi_n(g_n, \dots, g_1)$ send $1, \dots, j-1$ to 0 and $j, \dots, n$ to 1. In the third case, $\psi_n(g_n, \dots, g_1)$ sends $[n]$ to 1. It is then straightforward to verify that ψ is indeed a morphism and that both $(\psi \circ \varphi)_n$ and $(\varphi \circ \psi)_n$ are identities for all $n \in \mathbb{N}$.

(e) Let $F\colon I \to J$ and $G\colon J \to I$ be functors such that there exist natural transformations $\tau\colon GF \to \mathrm{Id}_I$ and $\tau'\colon FG \to \mathrm{Id}_J$. The process in part (d) shows that τ yields a simplicial homotopy between $N(GF) = NG \circ NF$ and $N(\mathrm{Id}_I) = \mathrm{Id}_{NI}$. Similarly, τ' yields a simplicial homotopy between $N(FG) = NF \circ NG$ and $N(\mathrm{Id}_J) = \mathrm{Id}_{NJ}$. Thus NI and NJ are homotopy equivalent simplicial sets.