Solution for Exercise sheet 10

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Exercise session: Thu. 8-10

Exercise 10.1 This follows from taking A as a point in X and $(Y,B) = (X,A)$ in the following lemma.

Lemma. *Let (X,A) be a CW pair and let (Y,B) be any pair with $B \neq \emptyset$. For each n such that $X \setminus A$ has cells of dimension n, assume that $\pi_n(Y,B,y_0) = 0$ for all $y_0 \in B$. Then every map $f\colon (X,A) \to (Y,B)$ is homotopic rel A to a map $X \to B$.*

Proof. Denote by X^n the n-skeleton of the pair (X,A). We prove by induction. First, $f|_{X^0}$ maps A into B, so nothing needs to be done.

Now suppose that $k \geq 1$ and that f has already been homotoped rel A to another map $\bar{f}\colon (X,A) \to (Y,B)$ such that $\bar{f}|_{X^{k-1}}$ maps X^{k-1} into B. Let Φ be the characteristic map of a cell e^k of $X \setminus A$. The condition $\pi_k(Y,B,y_0) = 0$ implies that $\bar{f} \circ \Phi\colon (D^k, \partial D^k) \to (Y,B)$ can be homotoped rel ∂D^k to some map $D^k \to B$. Thus $\bar{f}|_{e^k}\colon (e^k, \Phi(\partial D^k)) \to (Y,B)$ can be homotoped rel $\Phi(\partial D^k)$ to some map $e^k \to B$. The condition "rel $\Phi(\partial D^k)$" allows us to extend this homotopy to a homotopy from $\bar{f}\colon (X^{k-1} \cup e^k, X^{k-1}) \to (Y,B)$ to some $X^{k-1} \cup e^k \to B$ relative to X^{k-1}. This process can be carried out simultaneous on all k-cells of $X \setminus A$. Thus we successfully homotoped $\bar{f}|_{X^k}$ rel A to a map $X^k \to B$. By the homotopy extension property, this homotopy can be extend to one defined on all of X.

Now if (X,A) is finite-dimensional, using the induction steps finitely many times then shows that f is homotopic rel A to some map $X \to B$. In general we can do the homotopy constructed in the k-th induction step during the t-interval $[1 - 1/2^{k-1}, 1 - 1/2^k]$ to combine all the homotopies constructed together to a new homotopy f_t. Since every point in X lies in a finite-dimensional cell, by our construction $f_1(x)$ is well-definied for all $x \in X$ and $f_1(x) \in B$. □

Exercise 10.2 We prove $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iv)$.

$(iv) \Rightarrow (iii)$ Since $[K,f]$ is bijective, we get some $[\lambda] \in [K,X]$ such that $[f \circ \lambda] = [\beta] \in [K,Y]$. So $[\beta|_L] = [f \circ (\lambda|_L)] = [f \circ \alpha]$. Since $[L,f]$ is bijective by (iv), we get that $[\lambda|_L] = [\alpha]$. By the homotopy extension property, $[\lambda] = [\lambda'] \in [K,X]$ for some λ' such that $\lambda'|_L = \alpha$ and $[f \circ \lambda'] = [\beta] \in [K,Y]$.

$(iii) \Rightarrow (ii)$ Just take $(K,L) = (D^n, \partial D^n)$.

$(ii) \Rightarrow (i)$ It suffices to prove that $\pi_n(f) = 0$. Every element in $\pi_n(f)$ is represented by (α, β) satisfying the commutative diagram

$$\begin{tikzcd} \partial D^n \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^n \arrow[r, "\beta"] & Y \end{tikzcd}.$$

By (ii) there is some $\lambda\colon D^n \to X$ such that $\lambda|_{\partial D^n} = \alpha$ and $f \circ \lambda$ is homotopic relative to ∂D^n to β. So (α, β) and $(\lambda|_{\partial D^n}, f \circ \lambda)$ represent the same element in $\pi_n(f)$. But D^n is contractible, so λ is always null-homotopic. So $(\lambda|_{\partial D^n}, f \circ \lambda)$ always represents the zero element in $\pi_n(f)$. This shows that $\pi_n(f) = 0$.

$(i) \Rightarrow (iv)$ Surjectivity: Let $h\colon K \to Y$ and M_f be the mapping cylinder of f. Since M_f is a deformation retraction of Y and f is weak homotopy equivalence, we have $\pi_n(M_f, X, x_0) = 0$ for all $x_0 \in X$ and $n \geq 1$. Let $i\colon X \hookrightarrow M_f$, $j\colon Y \hookrightarrow M_f$ and $p\colon M_f \to Y$ be natural inclusions or projections. The lemma above shows that (take $(X,A) = (K, \emptyset)$ and $(Y,B) = (M_f, X)$) there is a map $g\colon K \to X$ such that $i \circ g$ is homotopic with $j \circ h$. So in $[K,Y]$ we have

$$[f \circ g] = [p \circ i \circ g] = [p \circ j \circ h] = [h].$$

This proves the surjectivity.

Injectivity: Suppose that $g_1, g_2 \colon K \to X$ such that $[f \circ g_1] = [f \circ g_2]$. Then $[p \circ i \circ g_1] = [p \circ i \circ g_2]$. Since p is deformation retraction, we have $[i \circ g_1] = [i \circ g_2]$. Let $G \colon K \times [0,1] \to M_f$ be a homotopy such that $G(x,0) = i \circ g_0$ and that $G(x,1) = i \circ g_1$. Using the lemma above (take $(X,A) = (K \times [0,1], K \times \{0,1\})$ and $(Y,B) = (M_f, X)$) gives a homotopy $G' \colon K \times [0,1] \to X$ such that $G'(x,0) = g_0$ and $G'(x,1) = g_1$. This proves the injectivity.

Exercise 10.3

(i) We only need to prove that $H^m(K(A,n), B) = 0$. By the universal coefficient theorem we have

$$H^m(K(A,n), B) \cong \operatorname{Ext}(H_{m-1}(K(A,n), \mathbb{Z}), B) \oplus \hom(H_m(K(A,n), \mathbb{Z}), B).$$

The Hurewicz theorem tells that $\tilde{H}_m(K(A,n), \mathbb{Z}) = 0$ for $0 \le m \le n-1$. Combining this and the fact that $\operatorname{Ext}(\mathbb{Z}, B) = 0$ gives the result.

(ii)

(iii)