Solution for Exercise sheet 2

Yikai Teng, You Zhou

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Exercise 2.1

(a) Let $\{S_n^2\}_{n\in\mathbb{Z}}$ be a sequence of spheres. Fix x_n in each sphere S_n^2 and consider

$$Y := \mathbb{R} \coprod \left(\coprod_{n\in\mathbb{Z}} S_n^2 \right).$$

Then the universal cover, as shown in the question sheet, is

$$\widetilde{X} = Y/\sim.$$

The equivalence relation $\sim$ on Y is defined as

$$\forall x, y \in Y, p \sim q \Leftrightarrow \exists n \in \mathbb{Z} \text{ s.t. } x = n, y = x_n \text{ or } x = x_n, y = n.$$

To describe the covering map $p\colon \widetilde{X} \to X$, we will denote points in S^1 by $\{e^{i\theta} : \theta \in \mathbb{R}\}$ and points in every S^2 by $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. In particular, in X we let $e^{-i\pi/2}$ and $(0,0,1)$ be identified and in Y we let $(0,0,1) \in S_n^2$ be identified with $n \in \mathbb{R}$. Then p maps $\theta \in \mathbb{R}$ to $e^{2\pi i\theta} \in S^1$, and $(x,y,z) \in S_n^2$ to the "same" point $(x,y,z) \in S^2$. In particular, all x_n are mapped to the "intersection" of S^1 and S^2 in X.

Next we compute the deck transformation group G of $\widetilde{X}$. Suppose $\varphi \in G$ and $\tilde{x} \in \widetilde{X}$. Since $\varphi(\tilde{x})$ and $\tilde{x}$ are mapped by p to the same point in X, the only possible choices of $\varphi(\tilde{x})$ is

$$\begin{cases} \tilde{x} + n \text{ for some } n \in \mathbb{Z}, & \text{if } \tilde{x} \in \mathbb{R} \\ (x,y,z) \in S_n^2 \text{ for some } n \in \mathbb{Z}, & \text{if } \tilde{x} = (x,y,z) \in S_m^2 \text{ for some } m. \end{cases}$$

The continuity of φ forces φ to be a transformation by n units, with n some integer. This shows that $G \cong \mathbb{Z}$.

Finally, the covering space theory tells us that since $\widetilde{X}$ is simply-connected, we have $\pi_1(X, x_0) \cong G \cong \mathbb{Z}$.

(b) Since the universal cover is simply connected, the Hurewicz theorem tells us that

$$\pi_2(\widetilde{X}, \tilde{x_0}) \cong H_2(\widetilde{X}, \mathbb{Z}).$$

Since covering maps induces isomorphisms on n-th homotopy groups for every $n \geq 2$, it suffices to compute $H_2(\widetilde{X}, \mathbb{Z})$.

Let A and B denote $\mathbb{R}$ and $\coprod_{n\in\mathbb{Z}} S_n^2$, as subsets of $\widetilde{X}$, respectively. Then $A \cap B = \coprod_{n\in\mathbb{Z}}\{\text{point}\}$. and the interiors of A and B covers $\widetilde{X}$. By the Mayer-Vietoris sequence

$$\cdots \to H_2(A\cap B, \mathbb{Z}) \to H_2(A, \mathbb{Z}) \oplus H_2(B, \mathbb{Z}) \to H_2(\widetilde{X}, \mathbb{Z}) \to H_1(A \cap B, \mathbb{Z}) \to \cdots$$

we have $H_2(A, \mathbb{Z}) \oplus H_2(B, \mathbb{Z}) \cong H_2(\widetilde{X}, \mathbb{Z})$. Since $\mathbb{R}$ can be seen as a CW-complex with no n-cells for $n \geq 2$, we have $H_2(A, \mathbb{Z}) = 0$. Thus

$$H_2(\widetilde{X}, \mathbb{Z}) \cong H_2\left(\coprod_{n\in\mathbb{Z}} S_n^2, \mathbb{Z} \right) \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}$$

(c) We first construct a basis for $\pi_2(\widetilde{X}, x_0)$. Let $f_n : (I^2, \partial I^2) \to (\widetilde{X}, \tilde{x_0})$ be the map illustrated in figure 1 below. So f_n can be viewed as an element in $\pi_2(\widetilde{X}, x_0)$. Addition of f_n and f_m and the fact

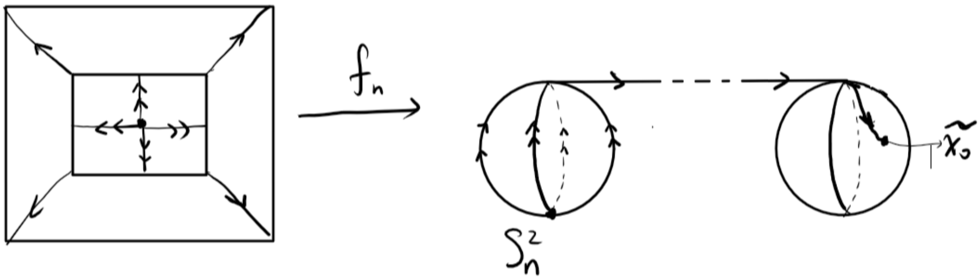

Figure 1: Definition of f_n

that $f_n + f_m = f_m + f_n$ are illustrated in the picture below. We claim that f_n and f_m are not homotopy

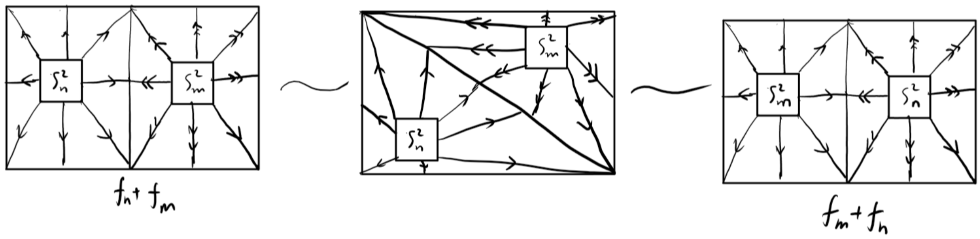

Figure 2: Commutativity of the addition

equivalent when $n \neq m$. Then we have an isomorphism

$$\pi_2(\widetilde{X}, x_0) \xrightarrow{\cong} \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}$$

that sends f_n to $(\ldots, 0, 0, 1, 0, 0, \ldots)$, where 1 is in the n-th position. Then $p \circ f_n$ is a basis of $\pi_2(X, x_0)$, where p is the covering map described in part a.

Now we prove the claim above. Suppose not and let $H : I^2 \times I \to \widetilde{X}$ be a continuous map such that $H(x, 0) = f_n(x)$ and $H(x, 1) = f_m(x)$. Then $H^{-1}(S^2_n)$ is a closed subset of $I^2 \times I$ containing the "inner square" of the domain of f_n (the smaller square drawn in figure 1). By continuity the boundary of $H^{-1}(S^2_n)$ must be mapped to the "point of intersection" of S^2_n and the line $\mathbb{R}$, denoted by y_n for short. For every $x \in S^2_n$, the line $x \times [0, 1]$ must have intersection with $\partial H^{-1}(S^2_n)$, i.e. there is some $t_x \in [0, 1]$ such that $H(x, t_x) = y_n$. But this would imply that S^2 can be retracted to a point, which is impossible. The idea is illustrated in figure 3 below.

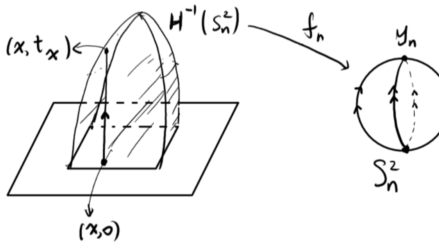

Figure 3: Study of f_n

Finally, by the isomorphisms established above, describing the action of $\pi_1(X, x_0)$ on $\pi_2(X, x_0)$ is equivalent to describing the action of $\mathbb{Z}$ on $\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}$. The latter is not hard. It gives the required action to let $p \in \mathbb{Z}$ send $(\ldots, 0, 0, 1, 0, 0, \ldots)$, where 1 is in the n-th position, to $(\ldots, 0, 0, 1, 0, 0, \ldots)$, where 1 is in the $(n+p)$-th position.

Exercise 2.2 Let ΣX denote the suspension of X and denote by x_1 and x_2 the points $X \times \{0\}$ and $X \times \{1\}$ in ΣX, respectively. Let

$$A := \Sigma X \setminus \{x_1\} \text{ and } B := \Sigma X \setminus \{x_2\},$$

then the interior of A and B covers X. By Mayer-Vietoris sequence we have the exact sequence

$$\cdots \to \widetilde{H}_n(A \cap B) \to \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \to \widetilde{H}_n(\Sigma X) \to \widetilde{H}_{n-1}(A \cap B) \to \cdots$$

Note that both A and B are contractible and that $A \cap B$ and X are homotopy equivalent. Thus all the reduced integral homology groups of ΣX are trivial, which implies that X is path-connected and that the n-th integral homology group of ΣX is trivial for all $n \geq 1$. So the Hurewicz theorem shows that $\pi_n(X, x_0) = 0$ for all $n \geq 1$. Since X is a CW-complex, Whitehead's theorem shows that ΣX is contractible.

Exercise 2.3