Solution for Exercise sheet 6

Yikai Teng, You Zhou

Exercise session: Thu. 8-10

Exercise 6.1 We first show that $m = n+k$. Every point $b \in S^n$ has a small neighborhood U that is homeomorphic to $\mathbb{R}^n$. Since p is a fiber bundle, we may assume that $p^{-1}(U) \cong U \times S^k \cong \mathbb{R}^n \times S^k$. A point $x \in \mathbb{R}^n \times S^k$ then has a neighborhood homeomorphic to $\mathbb{R}^n \times \mathbb{R}^k$ and such that its inverse image under the homeomorphism $p^{-1}(U) \xrightarrow{\cong} \mathbb{R}^n \times S^k$ is homeomorphic to $\mathbb{R}^m$. This shows that $m = n+k$.

We then show that $k = n-1$. If $m = 1$, then this follows from $m = n+k$ and $n \geq 1$. If $m = 2$, then either $n = 2, k = 0$ or $n = k = 1$. In the first case, p is a two-fold covering map. But this is impossible since $\pi_1(S^2) = 0$ and has no index-2 subgroup. In the second case, we have the long exact sequence

$$\cdots \to \pi_i(S^k) \to \pi_i(S^m) \to \pi_i(S^n) \to \pi_{i-1}(S^k) \to \cdots .$$

Since $\pi_2(S^1) = 0$, taking $n = 2$ gives us an exact sequence $0 \to \mathbb{Z} \to 0$, which is also impossible. When $m > 2$, still consider the exact sequence above. Then we have exact sequences

$$0 \to \pi_i(S^n) \to \pi_{i-1}(S^k) \to 0$$

for all $1 < i < m$. From this we get n cannot be 1 since otherwise we would have $\pi_{k+1}(S^1) \to \pi_k(S^k) \to 0$ exact. So $\pi_1(S^n) = 0$ and there is a common i that is the minimal i such that $\pi_i(S^n)$ and $\pi_{i-1}(S^k)$ are not trivial. This shows that $k = n-1$.

Exercise 6.3 First note that there is a homeomorphism

$$\Omega X \times \Omega X \cong X^{(S^1 \vee S^1, y_0)},$$

where y_0 is the point of intersection of the two circles. The map from the left side to the right side can be given by

$$\varphi\colon (f_1, f_2) \mapsto (x \mapsto f_i(x), \text{ if } x \text{ is in the } i\text{-th wedge summand})$$

and map in the opposite direction is

$$\psi\colon g \mapsto (g \circ i_1, g \circ i_2),$$

where i_1, i_2 are two wedge summand inclusions. By definition they are inverse to each other and are both continuous. (The continuity can be easily checked on subbasis.)

Let $\widetilde{\nabla^2}$ be ∇^2 with the three vertices glued together and $q\colon \nabla^2 \to \widetilde{\nabla^2}$ be the quotient map, then $q^*\colon X^{\widetilde{\nabla^2}} \to X^{\nabla^2}$ is a homeomorphism onto its image, which is just E. So it suffices to show that the map

$$\Phi\colon X^{\widetilde{\nabla^2}} \to X^{S^1 \vee S^1}, \quad f \mapsto (f \circ i') \vee (f \circ j')$$

is a homotopy equivalence, where

$$i', j'\colon S^1 \to \nabla^2$$

is defined by

$$i'(e^{2\pi i t}) = (t, 1-t, 0) \quad \text{respectively} \quad j'(e^{2\pi i t}) = (0, t, 1-t)$$

and

$$(f \circ i') \vee (f \circ j')(x) = \begin{cases} f \circ i'(x), & \text{if } x \text{ is in the first wedge summand} \\ f \circ j'(x), & \text{if } x \text{ is in the second wedge summand.} \end{cases}.$$

To prove this, we will first construct a homotopy equivalence between $\widetilde{\nabla^2}$ and $S^1 \vee S^1$ whose induced map on mapping spaces is Φ and then show a general property that under certain assumptions, a homotopy equivalence

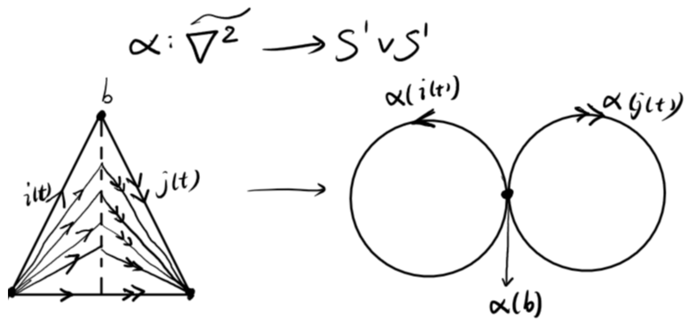

Figure 1: The map α

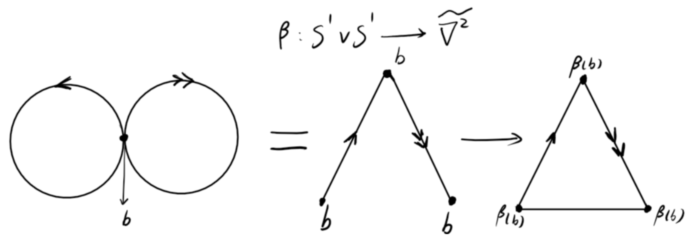

Figure 2: The map β

map induces homotopy equivalence on mapping spaces. For the first step, to avoid messy notations and formulas, I express the maps $\alpha\colon \widetilde{\nabla^2} \to S^1 \vee S^1$ and $\beta\colon S^1 \vee S^1 \to \widetilde{\nabla^2}$ in figures 1 and 2 below. Then by definition $\alpha \circ \beta = \mathrm{id}_{S^1 \vee S^1}$. Moreover, a homotopy equivalence from $\mathrm{id}_{\widetilde{\nabla^2}}$ to $\beta \circ \alpha$ can also be constructed as illustrated in figure 3 below.

So we have constructed our expected homotopy equivalence. By definition its induced map $\beta^*\colon X^{\widetilde{\nabla^2}} \to X^{S^1 \vee S^1}$ is just Φ.

Now we do the second step. Suppose that X, Y and Z are topological spaces, $f\colon X \to Y$ is a homotopy equivalence and that $g\colon Y \to X$ is the homotopy inverse of f. We may assume that X and Y are both Hausdorff since $\widetilde{\nabla^2}$ and $S^1 \vee S^1$ have this property. We have a continuous map

$$H\colon Y \times [0,1] \to Y$$

such that $H(y,0) = f \circ g(y)$ and $H(y,1) = y$ for all $y \in Y$. Since Y is Hausdorff,

$$H^*\colon Z^Y \times [0,1] \to Z^Y, \quad (h(\cdot), t) \mapsto (h(H(\cdot, t)))$$

is a continuous map that gives a homotopy between the identity and $(f \circ g)^*$. Similarly, we can show that $(g \circ f)^*$ is also homotopy equivalent to the identity. Thus f and g really induce homotopy equivalence between mapping spaces Z^X and Z^Y. Combining these two steps then finishes our proof.

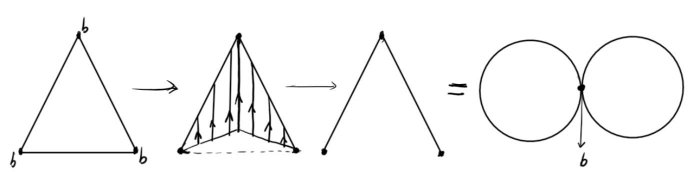

Figure 3: Homotopy between $\widetilde{\nabla^2}$ and $S^1 \vee S^1$