Solution for Exercise sheet 1

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Exercise 1.1 For a fixed integer j, let $\{e_i^j\}_i$ denote the set of j-cells of X, let $\Phi_i^j \colon D^j \to X^j$ be the characteristic map of e_i^j. Then a map $f \colon X \to Y$ is the union of maps

$$f_{ij} := f|_{\overline{e_i^j}} \circ \Phi_i^j \colon D^j \to Y.$$

Viewing D^j as the upper half of S^j, we can define maps

$$\begin{aligned} \widetilde{f_{ij}} \colon S^j &\to Y \\ (x_0, \ldots, x_n) &\mapsto \begin{cases} f_{ij}(x_0, \ldots, x_n), & \text{if } x_n \geq 0 \\ f_{ij}(x_0, \ldots, -x_n), & \text{if } x_n < 0 \end{cases} \end{aligned}$$

This map is continuous and can be view as an element of $\pi_j(Y, y)$ for some $y_{ij} \in Y$. We then show that two maps f, $g \colon X \to Y$ are homotopic if and only if for all i, j, $\widetilde{f_{ij}}$ and $\widetilde{g_{ij}}$ are homotopic and the following condition is satisfied:

> Let $\widetilde{H_{ij}} \colon S^j \times [0,1] \to Y$ be the continuous map such that $\widetilde{H_{ij}}(x,0) = \widetilde{f_{ij}}(x)$ and $\widetilde{H_{ij}}(x,1) = \widetilde{g_{ij}}(x)$. If there exist some i, j, i', j' and $x \in D^j, y \in D^{j'}$ such that $\Phi_i^j(x) = \Phi_{i'}^{j'}(y)$, then $\widetilde{H_{ij}}(x,t) = \widetilde{H_{i'j'}}(y,t)$ for all $t \in [0,1]$.

For one thing, let $H \colon X \times [0,1] \to Y$ be a continuous map such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. By restricting H to $\overline{e_i^j}$ and then use the method of defining $\widetilde{f_{ij}}$ from f_{ij}, we can get the required maps $\widetilde{H_{ij}}$.

For another, let $\widetilde{H_{ij}} \colon S^j \times [0,1] \to Y$ be continuous maps with $\widetilde{H_{ij}}(x,0) = \widetilde{f_{ij}}(x)$ and $\widetilde{H_{ij}}(x,1) = \widetilde{g_{ij}}(x)$ for all i and j and satisfying the condition above. Define

$$\begin{aligned} H \colon X \times [0,1] &\to Y \\ (x,t) &\mapsto \widetilde{H_{ij}}(x,t), \text{ for } x \in \Phi_i^j(D^j) \end{aligned}$$

Then the condition ensures that H is well-defined. This H is also continuous, so this proves our claim.

Note that a homotopy class of $\widetilde{f_{ij}}$ is an element of $\pi_j(Y, y_{ij})$ for some $y_{ij} \in Y$. So the homotopy class of $\widetilde{f_{ij}}$ has only finitely many possibilities, and so does the homotopy class of f.

Exercise 1.2

(a) To give the map explicitly we use coordinates. Set

$$\begin{aligned} S^n &:= \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\} \\ S^n \vee S^n &:= \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid (x_0 \pm \frac{1}{2})^2 + \cdots + x_n^2 = \frac{1}{4}\} \end{aligned}$$

The pinch map then can be given by

$$p(x_0, \ldots, x_n) = \left(x_0, \sqrt{\frac{|x_0|}{1+|x_0|}} x_1, \ldots, \sqrt{\frac{|x_0|}{1+|x_0|}} x_n \right),$$

which is clearly continuous. In this coordinate system q_1 can be given by

$$q_1\colon S^n \vee S^n \Rightarrow S^n$$
$$(x_0, \ldots, x_n) \mapsto \begin{cases} (2x_0+1,\, 2x_1,\, \ldots,\, 2x_n), & \text{if } x_0 \le 0 \\ (1,\, 0,\, \ldots,\, 0), & \text{if } x_0 > 0 \end{cases}$$

Both maps and their composition are illustrated for $n = 1$ in the figure below. A homotopy

$$H\colon S^n \times [0,1] \to S^n$$

from $\mathrm{id}|_{S^n}$ to $q_1 \circ p$ can be described as follows. Let $x \in S^n$ and C be the geodesic from $(-1, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$ and passing through x. Let f be a homeomorphism from $[0,1]$ to C that sends 0 to $(-1, 0, \ldots, 0)$ and 1 to $(1, 0, \ldots, 0)$. Let

$$\tilde{H}\colon [0,1] \times [0,1] \to [0,1]$$

be a homotopy of $[0,1]$ such that

$$H(-,0) = \mathrm{id},\ H([0,\tfrac{1}{2}],1) = [0,1] \text{ and that}$$
$$H(0,t) = 0,\ H(1,t) = 1 \text{ for all } t \in [0,1].$$

Then

$$H(x,t) = f \circ \tilde{H}(f^{-1}(x),t)$$

is the required homotopy. The case for $q_2 \circ p$ is similar.

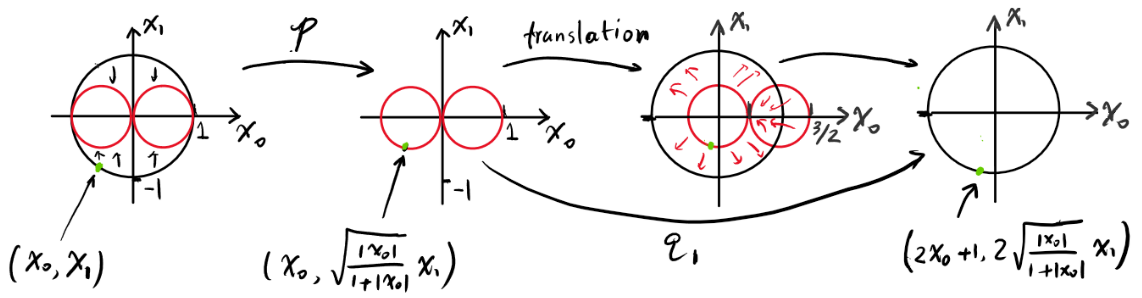

Figure 1: maps p and q_1, $n = 1$

(b) By part (a) we have

$$(q_1 \circ p)_* = (q_2 \circ p)_* = \mathrm{id} \text{ on } H_n(S^n, A).$$

Thus

$$\begin{aligned}(i_1)_* + (i_2)_* &= (i_1)_* \circ (q_1)_* \circ p_* + (i_2)_* \circ (q_2)_* \circ p_* \\ &= ((i_1 \circ q_1)_* + (i_2 \circ q_2)_*) \circ p_* \\ &= (\mathrm{id})_* \circ p_* \quad \text{(This step is explained below.)} \\ &= p_*.\end{aligned}$$

Let us explain why $(i_1 \circ q_1)_* + (i_2 \circ q_2)_* = \mathrm{id}_*$ on $H_n(S^n \vee S^n, A)$. Using the coordinate expression of $S^n \vee S^n$ as above, we define

$$B_1 := \{(x_0, \ldots, x_n) \in S^n \vee S^n \mid x_0 \le \frac{1}{4}\} \text{ and } B_2 := \{(x_0, \ldots, x_n) \in S^n \vee S^n \mid x_0 \ge -\frac{1}{4}\}.$$

Then we have the Mayer-Vietoris sequence (note that $B_1 \cup B_2 = S^n \vee S^n$)

$$\cdots \to H_n(B_1 \cap B_2, A) \to H_n(B_1, A) \oplus H_n(B_2, A) \xrightarrow{j_*} H_n(S^n \vee S^n) \to H_{n-1}(B_1 \cap B_2, A) \to \cdots$$

and $j_* = (i_1)_* + (i_2)_*$. It is clear that

$$((q_1)_* + (q_1)_*) \circ ((i_1)_* + (i_2)_*) = \mathrm{id}$$

So

$$((i_1)_* + (i_2)_*) \circ ((q_1)_* + (q_1)_*) \circ ((i_1)_* + (i_2)_*) = (i_1)_* + (i_2)_*.$$

Since $B_1 \cap B_2$ is contractible, we know that j_* is an isomorphism and composing with j_*^{-1} on both sides gives us the required result.

(c) Suppose that $p_1, p_2 \colon S^n \to S^n \vee S^n$ are two pinch maps. They can be seen as elements of $\pi_n(S^n \vee S^n, x_0)$. Since $S^n \vee S^n$ is $(n-1)$-connected, we have the Hurewicz map

$$h \colon \pi_n(S^n \vee S^n, x_0) \xrightarrow{\sim} H_n(S^n \vee S^n, \mathbb{Z}).$$

By part (b), we know that $h(p_1) = h(p_2)$. Since h is an isomorphism, it follows that p_1 and p_2 are homotopic as based maps.

Exercise 1.3 Let

$$\partial \colon \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

be the group homomorphism given by restricting a map $(I^2, \partial I^2, J^1) \to (X, A, x_0)$ to I, which can be viewed as $I \times \{0\} \subset \partial I^2$. Take h_1, h_2 in $\pi_2(X, A, x_0)$, then we can construct a homotopy from $h_1 h_2 h_1^{-1}$ to $(\partial h_1) \star h_2$ as illustrated in the following picture.

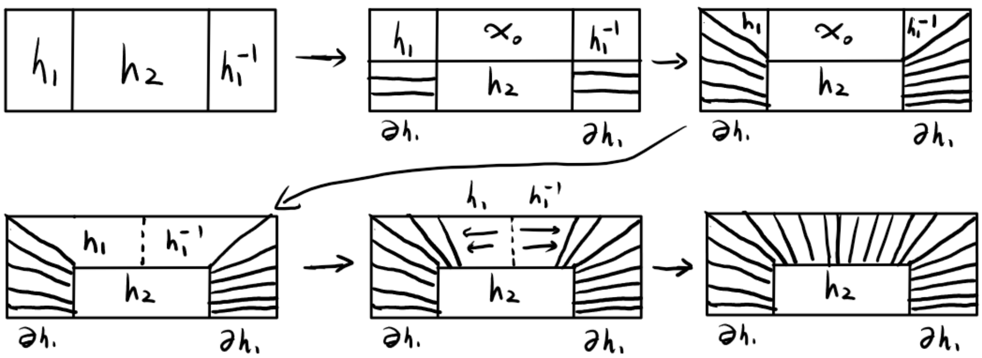

Figure 2: homotopy from $h_1 h_2 h_1^{-1}$ to $(\partial h_1) \star h_2$

Here in the fourth picture the dotted line is mapped to x_0. In the fifth picture the images of points in the interior of the dotted line are moving from x_0 to the corresponding point in ∂h_1. From this we know that in $\pi_2(X, A, x_0)^\dagger$,

$$h_1 h_2 h_1^{-1} = (\partial h_1) \star h_2 = h_2.$$