Algebraic Topology 1: Exercise 3

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Exercise 3.1

(a) Recall the definition of geometric realization: For a simplicial set X, its geometric realization $|X| = \coprod X_n \times \nabla^n / \sim$, equipped with quotient topology inherited from $\coprod X_n \times \nabla^n$. Thus by the properties of quotient topology, a continuous map $\coprod X_n \times \nabla^n \longrightarrow T$ induces a continuous map $|X| \to T$ if and only if it factors through the quotient map:

$$\begin{tikzcd} \coprod X_n \times \nabla^n \arrow[r] \arrow[d, "q"] & T \\ \coprod X_n \times \nabla^n / \sim \arrow[ur, dotted] & \end{tikzcd}$$

Thus we only need to check that equivalent elements under $\sim$ are sent to the same target.

Recall that the equivalence relation $\sim$ is spanned by $(x, \alpha_*(t)) \sim (\alpha^*(x), t)$ for all $x \in X_m, t \in \nabla^n, \alpha : [n] \to [m]$. Thus we only need to ensure that $f_m(x)(\alpha_*(t)) = f_n(\alpha^*(x))(t)$. However, this equation follows directly from the following commutative diagram, which depicts the definition of $f : X \to \mathcal{S}(T)$ being a morphism. (recall that α^* on the right is just precompose with α_*).

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d, "\alpha^*"] & \mathcal{S}(T)_m \arrow[d, "-\circ\alpha_*"] \\ X_n \arrow[r, "f_n"] & \mathcal{S}(T)_n \end{tikzcd}$$

Thus we conclude that each morphism $f : X \to \mathcal{S}(T)$ induces a map $\hat{f} : |X| \to T$.

(b) For notation reasons, we start by denoting the map described in the previous subproblem as $\phi : Hom_{sim}(X, \mathcal{S}(T)) \to Hom_{TOP}(|X|, T)$ by sending $f \mapsto \phi(f) = \hat{f}$. We want to show that ϕ is bijiective. We proceed by constructing an explicit inverse $\psi : Hom_{TOP}(|X|, T) \to Hom_{sim}(X, \mathcal{S}(T))$ described as follows:

Given a function $g : \coprod(X_n \times \nabla^n)/ \sim \longrightarrow T$, $\psi(g)$ is a morphism $X \to \mathcal{S}(T)$ such that $\psi(g)_n$ sends $x_n \in X_n$ to the map $g(x_n, -) \in \mathcal{S}_n(T)$.

Now we need to check that ϕ and ψ are indeed inverse to each other.

($\phi \circ \psi$): Given a map $g : \coprod(X_n \times \nabla^n)/\sim\longrightarrow T$, by definition $\psi(g)_n : X_n \to \mathcal{S}(T)$ with $x \mapsto g(x, -)$. Then by composing ϕ, we end up with a map $\phi(\psi(g)) : \coprod(X_n \times \nabla^n)/\sim\longrightarrow T$ sending $(x, t) \mapsto \psi(g)_n(x)(t) = g(x, t)$. Thus this composition is identity.

($\psi \circ \phi$): Given a morphism $f : X \to \mathcal{S}(T)$, we know by definition that $\phi(f) : \coprod(X_n \times \nabla^n)/\sim\longrightarrow T$ with $(x, t) \mapsto f_n(x)(t)$. Thus by composing ψ, we have the map $\psi(\phi(f)) : X \to \mathcal{S}(T)$ such that $\psi(\phi(f))_n : X_n \to \mathcal{S}(T)_n$ by sending $x \mapsto \phi(f)(x, -) = f_n(x)(-)$, thus is the same map as f. Thus this compossition is also identity.

Finally we conclude that the map ϕ is a bijection.

(c) We want to show that ϕ is natural in both diagram, i.e. we need to show that for all morphisms $\alpha : Y \to X$ between simplicial sets, all continuous maps between topological spaces $h : T \to T'$, the following diagram commutes:

$$\begin{tikzcd}
Hom_{\mathcal{S}}(X, \mathcal{S}(T)) \arrow[r, "\phi"] \arrow[d] & Hom_{TOP}(|X|, T) \arrow[d] \\
Hom_{\mathcal{S}}(Y, \mathcal{S}(T')) \arrow[r, "\phi"] & Hom_{TOP}(|Y|, T')
\end{tikzcd}$$

We prove commutativity by directly chasing the diagram, starting from the morphism $f : X \to \mathcal{S}(T)$.

(Go down then go right): We first go down. The morpshism f is mapped to a morphism $Y \to \mathcal{S}(T')$, sending $y_n \in Y_n$ to $h_* \circ f_n \circ \alpha_n(y, n) \in \mathcal{S}(T')_n$. This means if we apply ϕ (go right), we get a map $\coprod Y_n \times \nabla^n/\sim\longrightarrow T'$, sending $(y_n, t_n) \mapsto h(f_n(\alpha_n(y_n))(t_n))$.

(Go right then go down): We first apply ϕ, which means we get a map $\coprod X_n \times \nabla^n/\sim\longrightarrow T$ sending $(x_n, t_n) \mapsto f_n(x_n)(t_n)$. Then we go down, and we again get the map $\coprod Y_n \times \nabla^n/\sim\longrightarrow T'$, sending $(y_n, t_n) \mapsto h(f_n(\alpha_n(y_n))(t_n))$.

Thus we conclude that the above diagram commutes, and thus ϕ is natural with respect to both variables, and $| = |$ and $\mathcal{S}$ are adjoint functor pair.

Exercise 3.2 For any topological space T, we have the following calculation based on the ajunction we have from problem 1).

$$
\begin{aligned}
Hom_{TOP}(|\Delta[n]\times\Delta[1]|,T) &= Hom_{\mathcal{S}}(\Delta[n]\times\Delta[1],\mathcal{S}(T))\\
&= Hom_{\mathcal{S}}(\Delta[n],\mathcal{S}(T))\times Hom_{\mathcal{S}}(\Delta[1],\mathcal{S}(T))\\
&= Hom_{TOP}(|\Delta[n]|,T)\times Hom_{TOP}(|\Delta[n]|,T)\\
&= Hom_{TOP}(|\Delta[n]|\times|\Delta[1]|,T)
\end{aligned}
$$

Finally we apply Yoneda Lemma and we have $|\Delta[n]\times\Delta[1]|\cong|\Delta[n]|\times|\Delta[1]|$.

Exercise 3.3

(a) Consider the following composition of functors (which we also denote by NI)

$$NI\colon \Delta^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathcal{C}\mathrm{at}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(-,I)} (\mathrm{Sets}).$$

We first explain the notations involved. Here $\mathcal{C}$at is the category of all small categories. For every object $[n]$ of Δ, the functor $\mathcal{C}$ sends $[n]$ to a category $\mathcal{C}_{[n]} := \mathcal{C}([n])$ defined as

- The set of objects is $\mathrm{Ob}(\mathcal{C}_{[n]}) = [n] = \{0,1,\ldots,n\}$;
- $\mathrm{Hom}_{\mathcal{C}_{[n]}}(x,y) = \begin{cases}\{*\}, & \text{if } x\le y\\ \emptyset, & \text{otherwise.}\end{cases}$ In other words, for $x,y\in[n]$, there is at most one morphism from x to y and the morphism exists if and only if $x\le y$.

Every morphism $\alpha\colon[n]\to[m]$ is sent by $\mathcal{C}$ to a functor $\mathcal{C}_\alpha$ from $\mathcal{C}_{[m]}$ to $\mathcal{C}_{[n]}$ defined as

- For every $x\in[n], \mathcal{C}_\alpha(x)=\alpha(x)$
- A morphism $x\to y$ in $\mathcal{C}_{[n]}$ is sent to the unique morphism $\alpha(x)\to\alpha(y)$, which exists since α is weakly monotone.

The second functor $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(-,I)$ sends a small category $\mathcal{D}$ to the set of all functors from $\mathcal{D}$ to I, which is indeed a set since I is small. Its action on morphisms between small categories is natural.

Next we verify that this NI coincides with the NI defined in the problem for simplices by showing that giving a functor in $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n]},I)$ is equivalent to giving a composable n-tuple of morphisms in I. Note that every morphism in $\mathcal{C}_{[n]}$ can be written as composition of morphisms of the form $(j-1)\to j$ with $j\in\{1,\ldots,n\}$. Thus giving images of all morphisms of $\mathcal{C}_{[n]}$ is equivalent to only giving images of morphisms of form $(j-1)\to j$. Now If we have a functor F from $\mathcal{C}_{[n]}$ to I, then we have for every $x\in[n]$ an object $F(x)$ of I and for every morphism $(j-1)\to j$ in $\mathcal{C}_{[n]}$ a morphism $F_j\colon F(j-1)\to F(j)$ in I. Note that by definition the target of F_{j-1} equals the source of F_j, so $(F_n,\ldots,F_1)$ is a composable n-tuple of morphisms in I. Conversely, given a composable n-tuple of morphisms in I, say $(F_n,\ldots,F_1)$ as described in the problem, we may define $F\colon\mathcal{C}_{[n]}\to I$ by

- $F(j) := \begin{cases}\text{source}\,(F_{j+1}), & \text{if } 0\le j<n\\ \text{target}\,(F_n), & \text{if } j=n.\end{cases}$

- $F((j-1) \to j) := F_j$.

The functor is well-defined by composability of $(F_n, \dots, F_1)$ and it clearly preserves identity and composition of morphisms. This finishes this verification. Moreove r, the above argument shows that a functor in $\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n]}, I)$ is determined by the image of all morphisms $(j-1) \to j$ under it.

Finally we verify that the action of $d_i\colon [n-1] \to [n]$ on $(NI)_n$ is as required. The case for s_i will be similar and thus omitted. First, d_i induces a functor $\mathcal{C}_{d_i}\colon \mathcal{C}_{[n-1]} \to \mathcal{C}_{[n]}$, whose action on objects and morphisms are explained in the first paragraph. This $\mathcal{C}_{d_i}$ induces morphism of sets

$$\begin{gathered}\mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n]}, I) \to \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[n-1]}, I)\\ \begin{pmatrix}\text{images of } (j-1 \to j)\\ 1 \le j \le n\end{pmatrix} \mapsto \begin{pmatrix}\text{images of } d_i(j-1) \to d_i(j)\\ 1 \le j \le n-1\end{pmatrix}.\end{gathered}$$

Therefore, when $j < i$, the image of f_j is still f_j. When $j = i$, the image of f_j is $f_{j+1} \circ f_j$ since $d_i(i-1) \to d_i(i)$ is composition of $(i-1) \to i$ and $i \to (i+1)$. When $j > i$, the image of f_j is f_{j+1}, as the morphism $d_i(j-1) \to d_i(j))$ becomes $j \to j+1$. This finishes the verification and hence the proof.

(b) For every $[n] \in \Delta$, the definition of $(NF)_n$ is already given in the problem. So we only need to verify that for every morphism $\alpha\colon [n] \to [m]$, the diagram

$$\begin{tikzcd}(NI)_m \arrow[r, "(NF)_m"] \arrow[d, "\alpha^*"] & (NJ)_m \arrow[d, "\alpha^*"]\\ (NI)_n \arrow[r, "(NF)_n"] & (NJ)_n\end{tikzcd}$$

commutes. For this, let $(f_m, \dots, f_1) \in (NI)_m$. By the equivalence

$$\{\text{elements in } (NI)_m\} \leftrightarrow \{\text{functors in } \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[m]}, I) \leftrightarrow \begin{Bmatrix}\text{image of morphisms } (j-1) \to j\\ \text{under a functor in } \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[m]}, I)\end{Bmatrix} \tag{1}$$

established in part (a), we may write $f_j = G((j-1) \to j)$ for all $j \in [m]$ and some functor $G \in \mathrm{Hom}_{\mathcal{C}\mathrm{at}}(\mathcal{C}_{[m]}, I)$. Then by definition of maps in the diagram the image of f_j under $\alpha^* \circ (NF)_m$ and $(NF)_n \circ \alpha^*$ are both $F \circ G(\alpha(j-1) \to \alpha(j))$.

(c) The inverse morphism $\psi\colon NI \times NJ \to N(I \times J)$ can be constructed as

$$\begin{aligned}(NI \times NJ)_n = (NI)_n \times (NJ)_n &\xrightarrow{\psi_n} N(I \times J)_n\\ (f_n, \dots, f_1) \times (g_n, \dots, g_1) &\mapsto (f_n \times g_n, \dots, f_1 \times g_1)\end{aligned}$$

By writing elements of $(NI)_n$ as images of morphisms in $\mathcal{C}_{[n]}$ under some functor from $\mathcal{C}_{[n]}$ to I, as we did in the previous parts, we can show that for every $\alpha \in \mathrm{Mor}_\Delta([n],[m])$ the diagram

$$\begin{tikzcd}
(NI)_m \times (NJ)_m \arrow[r, "\psi_m"] \arrow[d, "\alpha^*"'] & N(I \times J)_m \arrow[d, "\alpha^*"] \\
(NI)_n \times (NJ)_n \arrow[r, "\psi_n"] & N(I \times J)_n
\end{tikzcd}$$

commutes. Thus ψ is indeed a morphism between simplicial sets. By computing directly the action on elements, we can show that both the compositions

$$(NI \times NJ)_n = (NI)_n \times (NJ)_n \xrightarrow{\psi_n} N(I \times J)_n \xrightarrow{(N_{\mathrm{proj}_I}, N_{\mathrm{proj}_J})_n} (NI)_n \times (NJ)_n = (NI \times NJ)_n$$
$$\text{and } N(I \times J)_n \xrightarrow{(N_{\mathrm{proj}_I}, N_{\mathrm{proj}_J})_n} (NI)_n \times (NJ)_n = (NI \times NJ)_n \xrightarrow{\psi_n} N(I \times J)_n$$

are identities. So $(N_{\mathrm{proj}_I}, N_{\mathrm{proj}_J})$ is an isomorphism between simplicial sets.

(d) We claim that there is an isomorphism $\varphi \colon \Delta[1] \to N[1]$ between simplicial sets, which will be constructed later. If this holds, we have the following diagram (identify NI with $NI \times \Delta[0]$)

$$\begin{tikzcd}
NI \arrow[r, "1\times d^1"] \arrow[dddr, bend right, "NF"'] & NI \times \Delta[1] \arrow[d, "\cong"] & NI \arrow[l, "1\times d^0"'] \arrow[dddl, bend left, "NG"] \\
& NI \times N[1] \arrow[d, "\cong"] & \\
& N(I \times [1]) \arrow[d, "NH"] & \\
& NJ &
\end{tikzcd}$$

where NH is the morphism constructed from H as in part (b). The commutativity of this diagram can be showed by routine verifications on simplices, using the definition of each maps and the equivalence (1). And the existence of such a commutative diagram is just the definition of a simplicial homotopy between NF and NG.

Now we show our claim by briefly explaining the construction of the isomorphism $\varphi \colon \Delta[1] \to N[1]$. For every $n \in \mathbb{N}$, define

$$\varphi_n \colon (\Delta[1])_n \to (N[1])_n$$
$$(g \colon [n] \to [1]) \mapsto (g_n, \ldots, g_1), \text{ with } g_j = \begin{cases} \mathrm{Id}_0, & \text{if } g(j) = g(j-1) = 0 \\ f \colon 0 \to 1, & \text{if } g(j) = 1, g(j-1) = 0 \\ \mathrm{Id}_1, & \text{if } g(j) = g(j-1) = 1. \end{cases}$$

For every $\alpha\colon [n] \to [m]$, The diagram

$$\begin{tikzcd}
(\Delta[1])_m \arrow[r, "\alpha^*"] \arrow[d, "\varphi_m"] & (\Delta[1])_n \arrow[d, "\varphi_n"] \\
(N[1])_m \arrow[r, "\alpha^*"] & (N[1])_n
\end{tikzcd}$$

commutes because if we choose some $(g\colon [m] \to [1]) \in (\Delta[1])_m$, then by computing with the definition of maps and the equivalence (1), we find that

$$\alpha^* \circ \varphi_m(g) = \varphi_n \circ \alpha^*(g) = (g_n, \dots, g_1)$$
$$\text{with } g_j = \begin{cases} \mathrm{Id}_0, & \text{if } g \circ \alpha(j) = g \circ \alpha(j-1) = 0 \\ f\colon 0 \to 1, & \text{if } g \circ \alpha(j) = 1, g \circ \alpha(j-1) = 0 \\ \mathrm{Id}_1, & \text{if } g \circ \alpha(j) = g \circ \alpha(j-1) = 1. \end{cases}$$

So φ is a morphism. Now we describe its inverse $\psi\colon N[1] \to \Delta[1]$. Given some composable n-tuple $(g_n, \dots, g_1) \in (N[1])_n$, there are three possibilities (by composability)

- $g_n = \cdots = g_1 = \mathrm{Id}_0$
- There is one j such that $g_n = \cdots = g_{j+1} = \mathrm{Id}_1$, $g_j = f$ and $g_{j-1} = \cdots = g_1 = \mathrm{Id}_0$
- $g_n = \cdots = g_1 = \mathrm{Id}_1$

In the first case, we let $\psi_n(g_n, \dots, g_1)$ sends $[n]$ to 0. In the second case, $\psi_n(g_n, \dots, g_1)$ send $1, \dots, j-1$ to 0 and $j, \dots, n$ to 1. In the third case, $\psi_n(g_n, \dots, g_1)$ sends $[n]$ to 1. It is then straightforward to verify that ψ is indeed a morphism and that both $(\psi \circ \varphi)_n$ and $(\varphi \circ \psi)_n$ are identities for all $n \in \mathbb{N}$.

(e) Let $F\colon I \to J$ and $G\colon J \to I$ be functors such that there exist natural transformations $\tau\colon GF \to \mathrm{Id}_I$ and $\tau'\colon FG \to \mathrm{Id}_J$. The process in part (d) shows that τ yields a simplicial homotopy between $N(GF) = NG \circ NF$ and $N(\mathrm{Id}_I) = \mathrm{Id}_{NI}$. Similarly, τ' yields a simplicial homotopy between $N(FG) = NF \circ NG$ and $N(\mathrm{Id}_J) = \mathrm{Id}_{NJ}$. Thus NI and NJ are homotopy equivalent simplicial sets.