Solution for Exercise sheet 4

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Exercise 4.1

(a) We can consider G as a small category in this way: the set of object of the category G is $\{*\}$, consisting of a single point, and the set of morphisms is just the underlying set of G, i.e. elements of the group G are in one-to-one correspondence with endomorphisms of $*$ in the category G. The unit 1 of the group G corresponds to the identity in $\mathrm{End}_G(*)$. Any two morphisms in G are composable and the composition of two morphisms is just the product of the two elements in G. In this way $BG = NG$ and the result follows directly from Exercise 3.3(a).

(b) We show that $(g, (0, 1))$ and $(1, 1)$ represent the same point in $|BG|$. Let $\alpha\colon [0] \to [1]$ be a map sending 0 to 1. Then in ∇^1, we have

$$(0, 1) = e_1 = e_{\alpha(0)} = \alpha_*(1),$$

where the 1 in the last term is the point in ∇^0. This implies that in $|BG|$ we have

$$(g, (0, 1)) = (g, \alpha_*(1)) = (\alpha^*(g), 1) = (1, 1).$$

The last equation follows because $\alpha^*(g) \in |BG|_0 = \{1\}$. This proves our claim at the beginning. By letting α map 0 to 0, the argument also shows that $(g, (1, 0))$ and $(1, 1)$ represent the same point.

(c) It suffices to verify that for two fixed $g, h \in G$ we have $\omega(g) \cdot \omega(h) = \omega(gh)$. Their formulas are given as follows

$$\begin{aligned} \omega(g)\cdot\omega(h)\colon [0,1] &\to |BG| \\ t &\mapsto \begin{cases} (g, (2t, 1-2t)), & \text{if } t \le \frac{1}{2} \\ (h, (2t-1, 2-2t)), & \text{otherwise.} \end{cases} \end{aligned} \qquad \begin{aligned} \omega(gh)\colon [0,1] &\to |BG| \\ t &\mapsto (gh, (t, 1-t)). \end{aligned}$$

Following the hint, we consider the composition H of maps

$$\begin{aligned} H(s,t)\colon [0,1]\times[0,1] &\to (BG)_2 \times \nabla^2 \to |BG| \\ (s,t) &\mapsto \begin{cases} ((g,h), (st, (2-2s)t, 1-(2-s)t)), & \text{if } t \le \frac{1}{2} \\ ((g,h), (st + (1-s)(2t-1), (1-s)(2-2t), s(1-t))), & \text{otherwise.} \end{cases} \end{aligned}$$

From this formula it is clear that H is continuous. Moreover, H satisfies

$$H(0,t) = \begin{cases} ((g,h), d_{0*}(2t, 1-2t)), & \text{if } t \le \frac{1}{2} \\ ((g,h), d_{2*}(2t-1, 2-2t), & \text{otherwise.} \end{cases} = \begin{cases} (g, (2t, 1-2t)), & \text{if } t \le \frac{1}{2} \\ (h, (2t-1, 2-2t)), & \text{otherwise.} \end{cases},$$

$$H(1,t) = ((g,h), d_{1*}(t, 1-t)) = (gh, (t, 1-t)) \text{ and}$$
$$H(s,0) = ((g,h), (0,0,1)) = (1,1) = ((g,h), (1,0,0)) = H(s,1).$$

(The (1,1) in the third equation is the basepoint of $|BG|$.) These together show that H is a homotopy between $\omega(g)\cdot\omega(h)$ and $\omega(gh)$ and this finishes the proof.