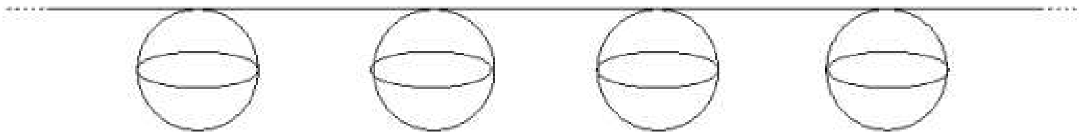
Algebraic Topology I, WS 2021/22
Exercise sheet 2

solutions due: 03.11.21
(note: November 1 is a public holiday)

Exercise 2.1: This exercise shows, among other things, that the higher homotopy groups of a finite CW-complex need not be finitely generated (in contrast to the integral homology groups).

Let $X = S^1 \vee S^2$ be the onepoint union of a circle and 2-sphere.

(a) Find a universal cover of X and calculate $\pi_1(X, x_0)$ as the deck transformation group of the universal cover.



(b) Use the Hurewicz theorem for the universal cover to calculate the group $\pi_2(X, x_0)$.

(c) Define a family of continuous maps $S^2 \to X$ whose homotopy classes form a basis of $\pi_2(X, x_0)$. Determine the action of the fundamental group $\pi_1(X, x_0)$ on $\pi_2(X, x_0)$ in terms of your basis.

Exercise 2.2: Let X be an acyclic CW-complex, i.e., all reduced integral homology groups of X are trivial. Show that the suspension of X is contractible.

Exercise 2.3: Let

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \dots$$

be a sequence of topological spaces and continuous maps. The *mapping telescope* of the sequence is the space

$$\mathrm{tel}_n X_n = \left\{ \coprod_{n \geq 0} X_n \times [0,1] \right\} / \sim ,$$

where $\sim$ denotes the equivalence relation generated by

$$(x, 1) \sim (f_n(x), 0)$$

for all $n \geq 0$ and all $x \in X_n$. We define continuous maps

$$i_k : X_k \to \mathrm{tel}_n X_n$$

be letting $i_k(x)$ be the equivalence class of $(x, 0)$.

(a) Show that the maps

$$(i_k)_* : H_*(X_k, A) \to H_*(\mathrm{tel}_n X_n, A)$$

induce an isomorphism

$$\mathrm{colim}_k H_*(X_k, A) \to H_*(\mathrm{tel}_n X_n, A)$$

for every abelian coefficient group A.

(b) Let $x_0 \in X_0$ be a basepoint and define $x_n \in X_n$ inductively by $x_n = f_{n-1}(x_{n-1})$. Use the maps

$$(i_k)_* \ : \ \pi_m(X_k, x_k) \ \rightarrow \ \pi_m(\mathrm{tel}_n X_n, i_k(x_k))$$

to define an isomorphism

$$\mathrm{colim}_k \ \pi_m(X_k, x_k) \ \rightarrow \ \pi_m(\mathrm{tel}_n X_n, i_0(x_0)) \ .$$

Some care has to be taken here with basepoints ...