Algebraic Topology 1: Exercise 1

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Exercise 1.1 Note that X is a finite CW complex, so X has a finite number of path components. Also Y has a finite number of path components, since $\pi_0(Y)$ is assumed finite. If there are only finite number of homotopy classes of maps from each path component of X to each path component of Y, $[X, Y]$ is clearly finite. Thus it is safe to assume both X and Y are path connected.

Also note that since every path connected CW complex is homotopy equivalent to a CW complex with only one 0-cell (by finding and collapsing a maximal tree of the 1-skeleton), thus we can also assume X has only one 0-cell.

Then we can proceed with induction not worrying about path-connectedness. Since X is a finite CW complex, it is a union of finite number of cells of ascending index.

We first check the base case , where X in this case only have the 0-skeleton. i.e. a single point. Then obviously $[X, Y]$ is finite.

Thus the problem can be reduced into the following: If $\tilde{X}$ is obtained from X attaching an n-cell via the attaching map $\phi : S^{n-1} \to X$, and we have $[X, Y]$ finite, then $[\tilde{X}, Y]$ is also finite.

We consider the map $[\tilde{X}, Y] \to [X, Y]$ by restriction. Since $[X, Y]$ is assumed finite, all we need to show is all its fibers are finite, i.e. given any map $f : X \to Y$, we need to show the set $F_f := \{[\tilde{f} : \tilde{X} \to Y] : [\tilde{f}|_X] = [f] \in [X, Y]\}$ is finite.

We pick a basepoint y_0 in the image of the attaching map $\phi(S^{n-1})$, then we claim that the group $\pi_n(Y, y_0)$ acts on F_f transitively. If so, by the orbit-stabilizer theorem, we have $|F_f| \le |\pi_n(Y, y_0)|$, thus is finite.

All that is left is to describe the group action and justify the above claim.

As indicated in figure 1 below, S^n can be seen as the pointed suspension of S^{n-1} or $\tilde{X}/X$, and the thickened line is quotiented to the basepoint, and is mapped to y_0 under f. On the other hand, $\tilde{X}$ can be seen as the mapping cone of the attaching map ϕ. Again the thickened line is quotiented to the basepoint, and is mapped to y_0 under f. The action then is done by identifying the top vertex of the cone and the lower vertex of the suspension. The action is transitive indicated by figure 2 below.

Thus we have fully proved the problem. To conclude, we start by justifying both X and Y can be assumed path connected and then we proceed with induction on cells. Within each inductive step, we start by fixing a homotopy class of maps on the base space (which only have finite number

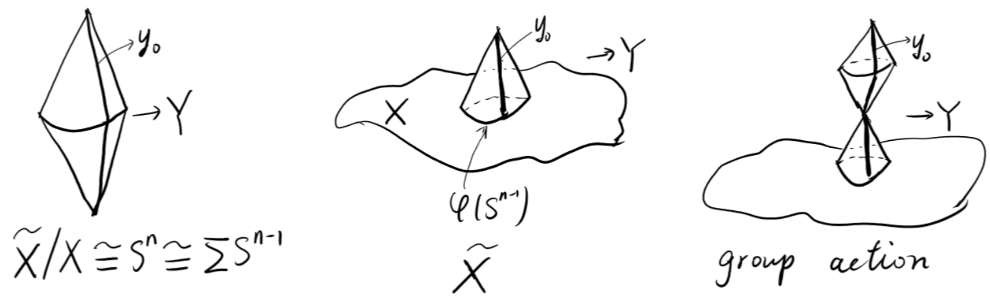

Figure 1: Group action of $\pi_n(Y, y_0)$ on F_f

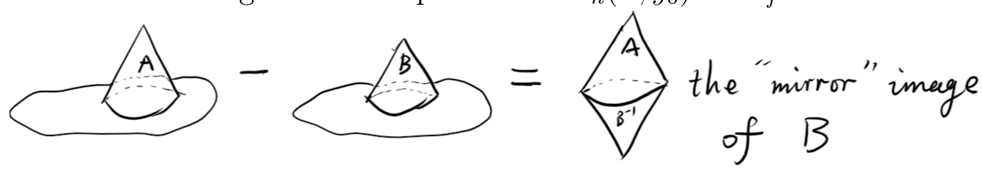

Figure 2: The action is transitive

of choices) ans we try to extend it to the attaching cell. It turns out that there are only finite number of ways to extend it. Thus we conclude that $[X, Y]$ is a finite set.

Exercise 1.2

(a) To give the map explicitly we use coordinates. Set

$$S^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}$$
$$S^n \vee S^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid (x_0 \pm \frac{1}{2})^2 + \cdots + x_n^2 = \frac{1}{4}\}$$

The pinch map then can be given by

$$p(x_0, \ldots, x_n) = \left(x_0, \sqrt{\frac{|x_0|}{1 + |x_0|}} x_1, \ldots, \sqrt{\frac{|x_0|}{1 + |x_0|}} x_n \right),$$

which is clearly continuous. In this coordinate system q_1 can be given by

$$q_1 \colon S^n \vee S^n \Rightarrow S^n$$
$$(x_0, \ldots, x_n) \mapsto \begin{cases} (2x_0 + 1, 2x_1, \ldots, 2x_n), & \text{if } x_0 \leq 0 \\ (1, 0, \ldots, 0), & \text{if } x_0 > 0 \end{cases}$$

Both maps and their composition are illustrated for $n = 1$ in the figure below. A homotopy

$$H \colon S^n \times [0,1] \to S^n$$

from $\mathrm{id}|_{S^n}$ to $q_1 \circ p$ can be described as follows. Let $x \in S^n \setminus \{(\pm 1, 0, \ldots, 0)\}$ and C be the geodesic from $(-1, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$ and passing through x. Let f be a homeomorphism from $[0,1]$

to C that sends 0 to $(-1, 0, \ldots, 0)$, $1/2$ to x and 1 to $(1, 0, \ldots, 0)$. Let

$$\tilde{H}\colon [0,1] \times [0,1] \to [0,1]$$

be a homotopy of $[0,1]$ given by

$$\tilde{H}(x,t) = \begin{cases} (1+t)x, & \text{if } x < \dfrac{1}{2} \\ 1-(1-t)(1-x), & \text{if } x > \dfrac{1}{2} \end{cases}$$

Intuitively, $\tilde{H}$ extends $[0, 1/2]$ to $[0,1]$ and fixes 0 and 1. Then

$$H(x,t) = f \circ \tilde{H}(f^{-1}(x), t)$$

is the required homotopy. The case for $q_2 \circ p$ is similar.

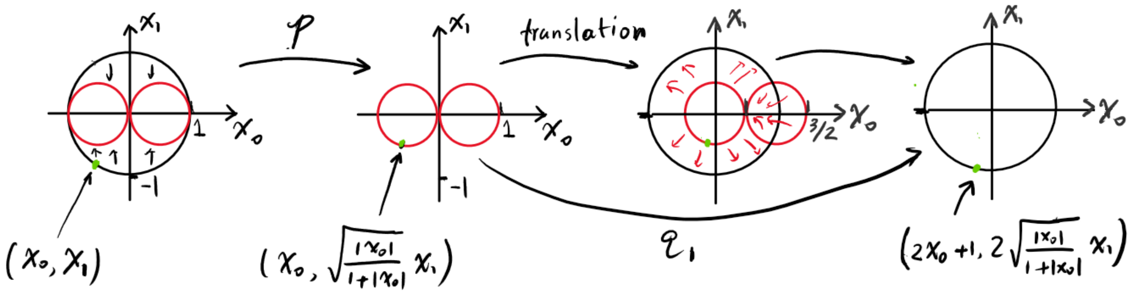

Figure 3: maps p and q_1, $n = 1$

(b) By part (a) we have

$$(q_1 \circ p)_* = (q_2 \circ p)_* = \text{id on } H_n(S^n, A).$$

Thus

$$\begin{aligned} (i_1)_* + (i_2)_* &= (i_1)_* \circ (q_1)_* \circ p_* + (i_2)_* \circ (q_2)_* \circ p_* \\ &= ((i_1 \circ q_1)_* + (i_2 \circ q_2)_*) \circ p_* \\ &= (\text{id})_* \circ p_* \quad \text{(This step is explained below.)} \\ &= p_*. \end{aligned}$$

Let us explain why $(i_1 \circ q_1)_* + (i_2 \circ q_2)_* = \text{id}_*$ on $H_n(S^n \vee S^n, A)$. Using the coordinate expression of $S^n \vee S^n$ as above, we define

$$B_1 := \{(x_0, \ldots, x_n) \in S^n \vee S^n \mid x_0 \leq \frac{1}{4}\} \text{ and } B_2 := \{(x_0, \ldots, x_n) \in S^n \vee S^n \mid x_0 \geq -\frac{1}{4}\}.$$

Then we have the Mayer-Vietoris sequence (note that $B_1 \cup B_2 = S^n \vee S^n$)

$$\cdots \to H_n(B_1 \cap B_2, A) \to H_n(B_1, A) \oplus H_n(B_2, A) \xrightarrow{j_*} H_n(S^n \vee S^n) \to H_{n-1}(B_1 \cap B_2, A) \to \cdots$$

and $j_* = (i_1)_* + (i_2)_*$. It is clear that

$$((q_1)_* + (q_1)_*) \circ ((i_1)_* + (i_2)_*) = \mathrm{id}$$

So

$$((i_1)_* + (i_2)_*) \circ ((q_1)_* + (q_1)_*) \circ ((i_1)_* + (i_2)_*) = (i_1)_* + (i_2)_*.$$

Since $B_1 \cap B_2$ is contractible, we know that j_* is an isomorphism and composing with j_*^{-1} on both sides gives us the required result.

(c) Suppose that p_1, $p_2 \colon S^n \to S^n \vee S^n$ are two pinch maps. They can be seen as elements of $\pi_n(S^n \vee S^n, x_0)$. Since $S^n \vee S^n$ is $(n-1)$-connected, we have the Hurewicz map

$$h \colon \pi_n(S^n \vee S^n, x_0) \xrightarrow{\sim} H_n(S^n \vee S^n, \mathbb{Z}).$$

By part (b), we know that $h(p_1) = h(p_2)$. Since h is an isomorphism, it follows that p_1 and p_2 are homotopic as based maps.

Exercise 1.3 Let

$$\partial \colon \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

be the group homomorphism given by restricting a map $(I^2, \partial I^2, J^1) \to (X, A, x_0)$ to I, which can be viewed as $I \times \{0\} \subset \partial I^2$. Take h_1, h_2 in $\pi_2(X, A, x_0)$, then we can construct a homotopy from $h_1 h_2 h_1^{-1}$ to $(\partial h_1) \star h_2$ as illustrated in the following picture.

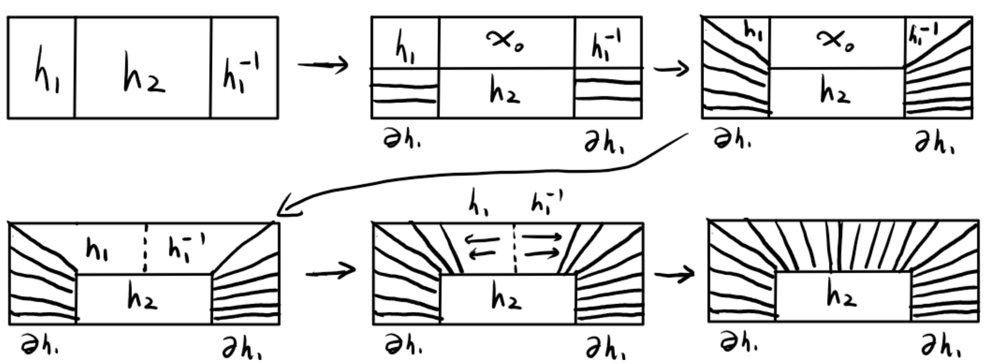

Figure 4: homotopy from $h_1 h_2 h_1^{-1}$ to $(\partial h_1) \star h_2$

Here in the fourth picture the dotted line is mapped to x_0. In the fifth picture the images of points in the interior of the dotted line are moving from x_0 to the corresponding point in ∂h_1. From this we know that in $\pi_2(X, A, x_0)^\dagger$,

$$h_1 h_2 h_1^{-1} = (\partial h_1) \star h_2 = h_2.$$