Solution for Exercise sheet 11

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Exercise 11.1

(i) We make two claims: Let $i \leq m$ and $j \leq n$, $\alpha\colon [k] \to [i]$ and $\beta\colon [k] \to [j]$ be two surjective weakly monotone maps. Then

1. If $k > m+n$, then α and β can always both factor through one weakly monotone surjection $\gamma\colon [k] \to [m+n]$.
2. If $k = m+n$, then there exist i, j and α, β as above such that for every $p < m+n$ and every $\gamma\colon [k] \to [p]$, it is impossible that α and β both factor through γ.

We show that it suffices to prove these two claims. Suppose that our claims hold and X and Y be m and n-dimensional simplicial sets, respectively. On one hand, for every $k > m+n$ and $(x, y) \in X_k \times Y_k$, we can write $x = \alpha^*(x')$ and $y = \beta^*(y')$, where $\alpha\colon [k] \to [i]$ and $\beta\colon [k] \to [j]$ are weakly monotone surjections with $i \leq m$ and $j \leq n$. By the first claim, we can write $\alpha = \alpha' \circ \gamma$ and $\beta = \beta' \circ \gamma$ with $\gamma\colon [k] \to [m+n]$ weakly monotone and surjective. Then $(x, y) = \gamma^*(\alpha'^*(x'), \beta'^*(y'))$ is degenerate. On the other hand, let α and β be as in the second claim. Consider $(\alpha^*(x), \beta^*(y)) \in X_{m+n} \times Y_{m+n}$ with $x = X_m$ and $y \in Y_n$ both non-degenerate. If $(x, y) = (\gamma^*(x'), \gamma^*(y'))$, then by the uniqueness of minimal representative of a simplex proven in the lecture, we must have factorizations $\alpha = \alpha' \circ \gamma$ and $\beta = \beta' \circ \gamma$, which is impossible by our second claim.

Now we prove the claims, which are purely combinatoric problems. Consider the following model: Put $k+1$ balls in one row. There are k spaces between adjacent balls. Giving a weakly monotone surjection $\alpha\colon [k] \to [i]$ is equivalent to putting i bars in the spaces to separate the balls into $i+1$ groups. Surjection means that we cannot put several bars in one space. (Then numbers in $[k]$ corresponding to balls in the l-th group are mapped by α to l.) For the first claim, to say that $\alpha\colon [k] \to [i]$ and $\beta\colon [k] \to [j]$ both factor through one weakly monotone surjection $\gamma\colon [k] \to [m+n]$ is equivalent to say that after putting $i+j$ bars in the spaces, we can still add some bars (this "some" can be 0) in some spaces to separate the k balls into $m+n+1$ groups. Since $k > m+n$ and $i+j \leq m+n$, this is always achievable. This proves the first claim.

For the second claim, let $i = m$ and $j = n$, $\alpha\colon [m+n] \to [m]$ be the unique surjection such that $\alpha(m) = m$ and $\beta\colon [m+n] \to [n]$ be the unique surjection such that $\beta(m) = 0$. In our model, given such α and β is equivalent to having exactly one bar in each of the $m+n$ spaces between $m+n+1$ balls. It is then impossible to add any bar to separate the balls into p groups for $p < m+n$. So this proves the second claim.

(ii) When $n = 0$ we can show by hand that the only non-degenerate element of $\Delta^0_1 \times \Delta^1_1$ is $(\mathrm{id}_{[0]}, \mathrm{id}_{[1]})$. So then assume $n \geq 1$. For $m \leq n+1$, giving a non-degenerate pair $(f, g) \in \Delta^n_m \times \Delta^1_m$ is equivalent to using n red bars and 1 blue bar to fill in the m spaces between $m+1$ adjacent balls such that in each space there is at least one bar. This means that a non-degenerate m-simplex of $\Delta^n \times \Delta^1$ is a pair $(f, g)\colon [m] \to [n] \times [1]$ such that for every $k \in \{0, 1 \ldots, m\}$ exactly one of the following three conditions hold (for simplicity assume that $f(-1) = g(-1) = -1$ and $f(n+1) = g(2) = n+1$)

1. $f(k-1) < f(k) < f(k+1)$ (i.e. the k-th ball is between 2 red bars)
2. $f(k) < f(k+1)$ and $g(k-1) < g(k)$ (i.e. the k-th ball is on the left of a red bar and right of the blue bar)
3. $f(k-1) < f(k)$ and $g(k) < g(k+1)$ (i.e. the k-th ball is on the right of a red bar and left of the blue bar)

In particular, when $m = n+1$, the place of the blue bar uniquely determines the places for the red bars and there are $n+1$ available places for the blue bar, so there are $n+1$ non-degenerate $(n+1)$-simplicies.