Part a : show $x_n \in (0,1)$, $n = 0, 1, \cdots$, for all $r \in (0,1)$

$$x_{n+1} = r\sin(\pi x_n), \quad x_0 = 0.5$$

base case: $x_0 = 0.5 \in (0,1)$

Inductive step $(n \to n+1)$:

assume $x_n \in (0,1) \implies \pi x_n \in (0, \pi)$

$\implies \sin(\pi x_n) \in (0,1]$

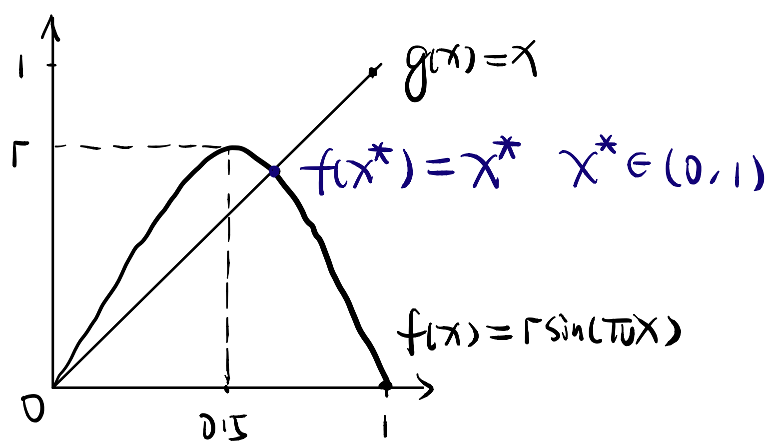
$\implies x_{n+1} = r\sin(\pi x_n) \in (0, r)$

if $\sin(\pi x_n) = 1$ (max at $x_n = 0.5$) $\implies x_{n+1} = r < 1$

$r \in (0,1) \implies x_{n+1} \in (0,1)$

$\therefore x_n \in (0,1)$ for all $n \in \mathbb{N}$

Part b:



part c : show that iteration $x_{n+1} = f(x_n) = r\sin(\pi x_n)$

converges to 0 when $r < \frac{1}{\pi}$

The fixed point $x^* = 0$ since $\sin(0) = 0$

To determine stability, first compute derivative

$$f'(x) = r\pi \cos(\pi x)$$

At $x^* = 0$ $\quad f'(0) = r\pi \cos(0) = r\pi$

For fixed point stable, $|f'(0)| < 1$

$$r\pi < 1 \Rightarrow r < \frac{1}{\pi}$$

$\therefore$ when $r < \frac{1}{\pi}$, x_n sequence converges to 0