Task 1: 2D poisson equation

$$-\nabla^2 u = f(x,y) \qquad \Omega = [0,1] \times [0,1]$$

$$u|_{\partial\Omega} = 0 \qquad h = \frac{1}{N+1}$$

Finite difference approximation, 5-point stencil

interior point (x_i, y_i)

Laplacian: $-\nabla^2 u(x_i, y_i) \approx \frac{1}{h^2}(4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1})$

following stencil pattern: $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ for example.

And then map 2D grid into 1D vector of unknowns

$$u = [u_{1,1}, u_{2,1}, \cdots u_{N,1}, u_{1,2}, \cdots u_{N,N}]$$

Each entry in $A \in R^{N^2 \times N^2} \longrightarrow$ interaction of a grid point with itself and 4 neighbors.

The diagonal entry: $\frac{4}{h^2}$

four off-diagonal entries: $-\frac{1}{h^2}$ (left, right, top, bottom neighbors).

if a neighbor is outside the domain (boundary), it's not included due to Dirichlet conditions.

Block structure of A

$$A = \frac{1}{h^2}\begin{bmatrix} T & -I & 0 & \cdots & 0 \\ -I & T & -I & \cdots & 0 \\ 0 & -I & T & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & -I \\ 0 & \cdots & 0 & -I & T \end{bmatrix}_{N^2 \times N^2}$$

$$T = \begin{bmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \cdots & 0 \\ 0 & -1 & 4 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{bmatrix} \quad \text{tridiagonal}(-1, 4, -1)$$

Task 2 prove: A symmetric positive definite.

① prove symmetric:

$$-\nabla^2 u(x_i, y_j) \approx \frac{1}{h^2}(-u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} + 4u_{i,j})$$

$u_{i,j} = \frac{4}{h^2}$ each neighbor $= -\frac{1}{h^2}$

$\therefore$ matrix entries satisfy $A_{ij} = A_{ji} = -\frac{1}{h^2}$

$A_{ii} = \frac{4}{h^2}$

other entries are 0

$\therefore A = A^T$ A is symmetri

$$A = \frac{1}{h^2}\begin{bmatrix} T & -I & 0 & \cdots & 0 \\ -I & T & -I & \cdots & 0 \\ 0 & -I & T & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & -I \\ 0 & \cdots & 0 & -I & T \end{bmatrix}$$

$$A^T = \frac{1}{h^2}\begin{bmatrix} T & -I & 0 & \cdots & 0 \\ -I & T & -I & \cdots & 0 \\ 0 & -I & T & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & -I \\ 0 & \cdots & 0 & -I & T \end{bmatrix}^T$$

$$= \frac{1}{h^2}\begin{bmatrix} T & -I & 0 & \cdots & 0 \\ -I & T & -I & \cdots & 0 \\ 0 & -I & T & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & -I \\ 0 & \cdots & 0 & -I & T \end{bmatrix} = A$$

$\therefore$ A is symmetric $\because A^T = A$

② prove positive definite.

$$x^T A x > 0 \text{ for all } x \neq 0$$

let $x = [x_1, \cdots, x_{N^2}]^T$ reshape into grid values $x_{i,j}$

$$x^T A x = \frac{1}{h^2}\sum_{i,j}[4x_{i,j}^2 - x_{i,j}x_{i+1,j} - x_{i,j}x_{i-1,j} - x_{i,j}x_{i,j+1} - x_{i,j}x_{i,j-1}]$$

$$x_{i,j}^2 + x_{i+1,j}^2 - 2x_{i,j}x_{i+1,j} = (x_{i,j} - x_{i+1,j})^2$$

$$x_{i,j}^2 + x_{i,j+1}^2 - 2x_{i,j}x_{i,j+1} = (x_{i,j} - x_{i,j+1})^2$$

$$x^T A x = \frac{1}{h^2} \sum_{i,j} \left[(x_{i,j} - x_{i+1,j})^2 + (x_{i,j} - x_{i,j+1})^2 \right] \geq 0$$

Now suppose $x^T A x = 0$

$(x_{i,j} - x_{i+1,j}) = 0$ $(x_{i,j} - x_{i,j+1}) = 0$ for all i,j

$x_{i,j}$ constant for all grid points, but $x_{i,j} = 0$ on boundary

so constant must be 0 $\therefore x = 0$

$x^T A x > 0$ for all $x \neq 0$

so A is symmetric and positive definite.