Task 1: $X^T X \hat{a} = X^T \hat{f}$

$f(t) = a + b\sin(t) + c\cos(t)$

data points:

t	F
0	5.0
1	5.3
2	2.5
3	-0.5
4	-1.4
5	1.0
6	4.3

For each data point (t_i, F_i), $a + b\sin(t_i) + c\cos(t_i) = F_i$

$a + b\sin(0) + c\cos(0) = 5.0 \longrightarrow a + 0\cdot b + 1\cdot c = 5.0$

$a + b\sin(1) + c\cos(1) = 5.3$

$a + b\sin(2) + c\cos(2) = 2.5$

$a + b\sin(3) + c\cos(3) = -0.5$

$a + b\sin(4) + c\cos(4) = -1.4$

$a + b\sin(5) + c\cos(5) = 1.0$

$a + b\sin(6) + c\cos(6) = 4.3$

Write matrix form: $X\hat{a} = \hat{f}$

X design matrix

$\hat{a} = [a, b, c]^T$ vectors of unknowns

$\hat{f} = [F_1, F_2, \cdots, F_7]^T$ vectors of observed values

construct X and $\hat{f}$

$$X = \begin{bmatrix} 1 & \sin(0) & \cos(0) \\ 1 & \sin(1) & \cos(1) \\ 1 & \sin(2) & \cos(2) \\ 1 & \sin(3) & \cos(3) \\ 1 & \sin(4) & \cos(4) \\ 1 & \sin(5) & \cos(5) \\ 1 & \sin(6) & \cos(6) \end{bmatrix} \qquad \hat{f} = \begin{bmatrix} -5.0 \\ 5.3 \\ 2.5 \\ -0.5 \\ -1.4 \\ 1.0 \\ 4.3 \end{bmatrix}$$

Derive equation $X\hat{a} = \hat{f}$

$X \in \mathbb{R}^{7\times 3}$ $\quad \hat{a} = [a, b, c]^T \quad \hat{f} = [F_1, F_2, \cdots, F_7]^T$

residual vector: $\hat{r} = X\hat{a} - f$

Cost function : $W = \|\hat{r}\|^2 = (X\hat{a} - \hat{f})^T (X\hat{a} - \hat{f})$

$$= \hat{a}^T X^T X \hat{a} - 2\hat{a}^T X^T \hat{f} + \hat{f}^T \hat{f}$$

$$\nabla_{\hat{a}} (\hat{a}^T A \hat{a}) = 2A\hat{a} = 2X^T X \hat{a}$$

$$\nabla_{\hat{a}} (-2\hat{a}^T X^T \hat{f}) = -2X^T \hat{f}$$

$$\nabla_{\hat{a}} (\hat{f}^T \hat{f}) = 0$$

$$\nabla_{\hat{a}} W = 2X^T X \hat{a} - 2X^T \hat{f}$$

minimize W $\quad \nabla W = 0 \quad 2X^T X \hat{a} - 2X^T \hat{f} = 0$

$$\boxed{X^T X \hat{a} = X^T \hat{f}}$$

Task 3: $W = \sum_{i=1}^{7} [f(t_i) - F_i]^2$

$$f(t_i) = a + b\sin(t_i) + c\cos(t_i)$$

① $\frac{\partial W}{\partial a}$: $\quad W = \sum_{i=1}^{7} [(a + b\sin(t_i) + c\cos(t_i)) - F_i]^2$

$$\frac{\partial W}{\partial a} = 2\sum_{i=1}^{7} [(a + b\sin(t_i) + c\cos(t_i)) - F_i] \cdot \frac{\partial}{\partial a}(a + b\sin(t_i) + c\cos(t_i))$$

$$\frac{\partial W}{\partial a} = 2\sum_{i=1}^{7} [(a + b\sin(t_i) + c\cos(t_i)) - F_i] \cdot 1$$

$$= 2\sum_{i=1}^{7} [a + b\sin(t_i) + c\cos(t_i) - F_i]$$

② $\frac{\partial W}{\partial b} = 2\sum_{i=1}^{7} [(a + b\sin(t_i) + c\cos(t_i)) - F_i] \cdot \frac{\partial}{\partial b}(a + b\sin(t_i) + c\cos(t_i))$

$$= 2\sum_{i=1}^{7} [a + b\sin(t_i) + c\cos(t_i) - F_i] \cdot \sin(t_i)$$

③ $$\frac{\partial W}{\partial b} = 2\sum_{i=1}^{7}\left[\left(a+b\sin(t_i)+c\cos(t_i)\right)-F_i\right]\cdot\frac{\partial}{\partial c}\left(a+b\sin(t_i)+c\cos(t_i)\right)$$

$$= 2\sum_{i=1}^{7}\left[a+b\sin(t_i)+c\cos(t_i)-F_i\right]\cdot\cos(t_i)$$