1 $X \sim \text{Geom}(p)$

$$P(X \geq k) = (1-p)^{k-1} p = q^{k-1} p \qquad \{ 1-p = q$$

$$M_X(t) = \sum_k e^{t x_k} \cdot p_k$$

$$M_X(t) = \sum_k e^{t x_k} q^{k-1} p = \frac{p}{q} \sum_k e^{t k} q^k = \frac{p}{q} \sum_k (e^t q)^k = \frac{p}{q} \frac{e^t q}{1 - e^t q} =$$

$$= \frac{p e^t}{1 - q e^t}$$

2 $$M_x(t) = \frac{pe^t}{1-qe^t}$$

$$M_x'(0) = E(X)$$

$$M_x'(t) = \left(\frac{pe^t}{1-qe^t}\right)' = p\left(\frac{e^t(1-qe^t) - e^t(-qe^t)}{(1-qe^t)^2}\right) =$$

$$= p\frac{e^t - qe^{2t} + qe^{2t}}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$E(X) = M_x'(0) = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$V(X) = E(X^2) - E^2(X)$$

$$M_x''(0) = E(X^2)$$

$$M''_x(t) = p\left(\frac{e^t}{(1-qe^t)^2}\right)' = p\,\frac{e^t(1-qe^t)^2 - e^t\,2(1-qe^t)(-qe^t)}{(1-qe^t)^4} =$$

$$= p\,\frac{e^t(1-qe^t)^2 + 2e^t(1-qe^t)(qe^t)}{(1-qe^t)^4} = p\,\frac{(1-qe^t)\left[e^t(1-qe^t) + 2e^t qe^t\right]}{(1-qe^t)^4} =$$

$$= p\,\frac{e^t(1-qe^t) + 2e^{2t}q}{(1-qe^t)^3} = p\,\frac{e^t - qe^{2t} + 2e^{2t}q}{(1-qe^t)^3} = p\,\frac{e^t + qe^{2t}}{(1-qe^t)^3} = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

$$M''_x(0) = \frac{p(1+q)}{(1-q)^3} = \frac{p(1+q)}{(1-1+p)^3} = \frac{1+q}{p^2} = \frac{2-p}{p^2}$$

$$V(X) = E(X^2) - E^2(X) = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$$

3 $X \sim N(\mu, \sigma^2)$ $\qquad f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R}$

Pokazać $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$$M_X(t) = \int_R e^{tx} \cdot f(x)\,dx = \int_R e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \left\{ \begin{array}{l} x = \sigma z + \mu \\ \frac{x-\mu}{\sigma} = z \\ \frac{1}{\sigma}dx = dz \\ dx = \sigma dz \end{array} \right\} =$$

$$= \int_R e^{t(\sigma z + \mu)} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{1}{2}z^2}\,dz = \int_R e^{t\sigma z} \cdot e^{t\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz =$$

$$= e^{t\mu} \int_R e^{t\sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \overset{\text{z wykładu}}{=} e^{t\mu}\, e^{\frac{1}{2}t^2\sigma^2} = e^{t\mu + \frac{1}{2}t^2\sigma^2}$$

5 $X_1 \dots X_n$ – niezależne $\qquad M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$

$X_k \sim N(\mu, \sigma^2)$

$\bar{X}$ – śr $X_1 \dots X_n$ $\qquad \bar{X} = \frac{1}{n} X_1 + \frac{1}{n} X_2 + \frac{1}{n} X_3 + \dots + \frac{1}{n} X_n$

$M_{\bar{X}}(t) = M_{X_1}\left(\frac{1}{n} t_1\right) \cdot M_{X_2}\left(\frac{1}{n} t_2\right) \cdot \dots M_{X_n}\left(\frac{1}{n}(t_n)\right)$

$M_{\bar{X}}(t) = \left(M_{\frac{1}{n} X_i}(t)\right)^n = \left(M_{X_i}\left(\frac{1}{n} t\right)\right)^n = \left(e^{\frac{1}{n}\mu t + \frac{1}{n}\frac{t^2\sigma^2}{2}}\right)^n =$

$= e^{\mu t + \frac{t^2\sigma^2}{2n}} = e^{\mu t + \frac{t^2 \frac{\sigma^2}{n}}{2}}$

$M_{\bar{X}} \sim N(\mu, \sigma^2)$

5

$X \sim N(\mu, \sigma^2)$ $\qquad Y = \left(\frac{X-\mu}{\sigma}\right)^2 = Z^2 \qquad Z = \frac{X-\mu}{\sigma}$

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t) = M_{\frac{X}{\sigma}-\frac{\mu}{\sigma}}(t) = e^{-t\frac{\mu}{\sigma}} M_X\left(\frac{t}{\sigma}\right) =$$

$$= e^{-t\frac{\mu}{\sigma}} e^{t\frac{\mu}{\sigma} + \frac{1}{2}t^2} = e^{\frac{1}{2}t^2} \qquad Z \sim N(0,1)$$

$Y = Z^2$

Z poprzedniej listy

$Y \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

6 $Y_k = \left(\frac{X_k - \mu}{\sigma}\right)^2$

Z 5. wiemy że $Y \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$$Z_n = \sum_{k=1}^{n} Y_k$$

[redacted]

$$M_{Z_n}(t) = M_{\left(\frac{X_k - \mu}{\sigma}\right)^2}(t)^n = \left(\left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{\frac{1}{2}}\right)^n = \left(\frac{1}{1-2t}\right)^{\frac{n}{2}}$$

$$Z_n \sim \text{Gamma}\left(\frac{1}{2}, \frac{n}{2}\right)$$

7

$$Z = \sum_{k=1}^{n} X_k \qquad M_{X_k}(t) = \left(\frac{b}{b-t}\right)^{p_k}$$

$$X_k \sim \text{Gamma}(b, p_k) \qquad k = 1, \dots, n$$

$$MGF_{Z}(t) = \prod^{n} \left(\frac{b}{b-t}\right)^{p_k} = \left(\frac{b}{b-t}\right)^{p_1} \cdot \left(\frac{b}{b-t}\right)^{p_2} \cdot \ldots \cdot \left(\frac{b}{b-t}\right)^{p_n} = \left(\frac{b}{b-t}\right)^{\sum_{i=1}^{n} p_i}$$

$$Z = \text{Gamma}\left(b, \sum_{k=1}^{n} p_k\right)$$

8

$$X_k \sim B(m_k, p) \qquad k = 1, \dots, n$$

$$M_{X_k}(t) = (pe^t + q)^{m_k} \qquad q = 1-p$$

FGM:

$$M_Z(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t) = (pe^t+q)^{m_1} (pe^t+q)^{m_2} \cdot \ldots \cdot (pe^t+q)^{m_n}$$

$$= (pe^t + q)^{\sum m_k}$$

$$MGF_Z(t) = (pe^t + 1 - p)^{\sum_{k=1}^{n} m_k}$$

$$Z \sim B\left(\sum m_k, p\right)$$

10

a) $M_u(t) = \frac{2}{2-3t}$

$E_x(t) = M_u'(0)$

$M_u'(t) = \frac{6}{(2-3t)^2}$ $\quad$ $M_u'(0) = \frac{6}{4}$

b) $V_u(X) = E(x^2) - (E^2(x))$

$E(x^2) = M_u''(0)$

$E^2(x) = \frac{36}{16} = \frac{9}{4}$

$M_u''(t) = \left(\frac{6}{(2-3t)^2}\right)' = \frac{36}{(2-3t)^3}$ $\quad$ $M_u''(0) = \frac{36}{8} = \frac{18}{4} = \frac{9}{2}$

$V_u(x) = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}$

c) $\alpha = 0{,}9u$

$M_{\alpha x}(t) = M_x(\alpha t) = \frac{2}{2-3\cdot 0{,}9t} = \frac{2}{2-2{,}7t}$