$$1 \quad E(X+Y) = \int_{\mathbb{R}}\int_{\mathbb{R}} (x+y)\, f_{xy}(x,y)\, dx\, dy =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} x\, f_{xy}(x,y)\, dx\, dy + \int_{\mathbb{R}}\int_{\mathbb{R}} y\, f_{xy}(x,y)\, dx\, dy =$$

$$\int_{\mathbb{R}} x f_x(x)\, dx + \int_{\mathbb{R}} y\, f_Y(y)\, dy = E(X) + E(Y)$$

$$2 \quad V(Y) = V(aX+b) = E(aX+b)^2 - E^2(aX+b) =$$

$$= E(a^2X^2 + 2aXb + b^2) - [E(aX+b)][E(aX+b)] =$$

$$= E(a^2X^2 + 2aXb + b^2) - (a\,E(X)+b)^2 =$$

$$= a^2 E(X^2) + \cancel{2abE(X) + b^2} - a^2E(X)^2 - \cancel{2abE(X) - b^2} =$$

$$= a^2 E(X^2) - a^2 E(X)^2 = a^2 \left(E(X^2) - E^2(X) \right) = a^2 V(X)$$

3

$X \sim N(0,1)$

$Y = X^2$

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$f_Y(y) \equiv g(y)$

$F_Y(y) = P(Y < y) = P(X^2 < y) = P(-\sqrt{y} \leq X \leq -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$g(y) \equiv f_Y(y) = (F_Y(y))' = (F_X(\sqrt{y}) - F_X(-\sqrt{y}))' =$

$= f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} =$

$= \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\sqrt{y}^2}{2}} + \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(-\sqrt{y})^2}{2}} \right) =$

$= \frac{1}{2\sqrt{y}} \cdot \frac{2}{\sqrt{2\pi}} \cdot e^{-\frac{y}{2}} = \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y}{2}}$

5

$X \sim \text{Gamma}(b, p)$ wtw $f(x) = \frac{b^p}{\Gamma(p)} x^{p-1} e^{-bx}$

z 3.

$$g(y) = \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y}{2}} =$$

$$= \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{1}{2}y} =$$

$$= y^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{1}{2}y} =$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot y^{\frac{1}{2}-1} e^{-\frac{1}{2}x} = \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \cdot y^{\frac{1}{2}-1} e^{-\frac{1}{2}x}$$

$$\parallel$$

$$\frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{\pi}}$$

$Y \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

6 $X \sim N(0,1)$

$\sigma > 0$

$\mu \in \mathbb{R}$

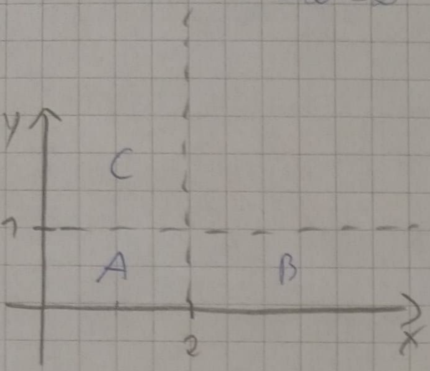
$Y = \sigma X + \mu$

$F_Y(y) = P(Y \leq y) = P(\sigma X + \mu \leq y) = P\left(X \leq \frac{y-\mu}{\sigma}\right) = F_X\left(\frac{y-\mu}{\sigma}\right)$

$f_X(y) = \left(F_X\left(\frac{y-\mu}{\sigma}\right)\right)' = f_X\left(\frac{y-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$

7 (X,Y) $\quad f(x,y) = xy \quad [0,2]\times[0,1]$

$$F_{XY}(s,t) = \int_{-\infty}^{s}\int_{-\infty}^{t} xy\, dy\, dx$$



$$F_{XY}(s,t) = \begin{cases} 0 & \text{dla } s \le 0 \wedge t \le 0 \\ 1 & \text{dla } s \ge 2 \wedge t \ge 2 \\ \frac{s^2t^2}{4} & \text{dla } (s,t) \in A \\ t^2 & \text{dla } (s,t) \in B \end{cases} , \quad \frac{s^2}{4} \quad (s,t) \in C$$

A: $F_{XY}(s,t) = \int_0^s \int_0^t xy\, dy\, dx = \int_0^s \left.\frac{xy^2}{2}\right|_0^t dx = \int_0^s \frac{xt^2}{2}\, dx =$

$$\left.\frac{x^2t^2}{4}\right|_0^s = \frac{s^2t^2}{4}$$

B: $F_{XY}(s,t) = \int_0^2 \int_0^t xy\, dy\, dx =$

$$\int_0^2 \left.\frac{xy^2}{2}\right|_0^t dx = \int_0^2 \frac{xt^2}{2}\, dx = \left.\frac{x^2t^2}{4}\right|_0^2 = t^2$$

C: $F_{XY}(s,t) = \int_0^s \int_0^1 xy\, dy\, dx = \int_0^s \left.\frac{xy^2}{2}\right|_0^1 = \int_0^s \frac{x}{2}\, dx = \left.\frac{x^2}{4}\right|_0^s = \frac{s^2}{4}$

9

(X, Y) – ciągłe

X, Y – niezależne $\Rightarrow E(X)E(Y) = E(X, Y)$

$$\mathrm{Cov}(X, Y) = E\big((X - E(X))(Y - E(Y))\big) = E\big(XY - XE(Y) - YE(X) + E(X)E(Y)\big) =$$

$$= E(X, Y) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y) = 0$$

$$\mathrm{Cov}(X, Y) = E(X, Y) - E(X)E(Y) = 0$$