1

$$w(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n =$$

$$(\ldots(((a_n)x + a_{n-1})x + a_{n-2})x + \ldots)x + a_1)x + a_0 \qquad \alpha_n = 0$$

$$w'(x) = (a_n(1+\alpha_n)\cdot x(1+\beta_n) + a_{n-1})(1+\alpha_{n-1})\cdot x(1+\beta_{n-1}) + \ldots)x(1+\beta_2) + a_1(1+\alpha_1)x(1+\beta_1)) + a_0(1+\alpha_0)$$

$$w'(x) = a_n x^n \prod_{i=0}^{n}(1+\alpha_i)\prod_{i=1}^{n}(1+\beta_i) + a_{n-1}x^{n-1}\prod_{i=0}^{n-1}(1+\alpha_i)\prod_{i=1}^{n-1}(1+\beta_i) + \ldots + a_0\prod_{i=0}^{0}(1+\alpha_i)\prod_{i=1}^{0}(1+\beta_i)$$

$$w'(x) = \sum_{k=0}^{n} a_k x^k \prod_{i=0}^{n-k}(1+\alpha_i)\prod_{i=1}^{n-k}(1+\beta_i)$$

$$|\alpha_i|, |\beta_i| \leq 2^{-t}$$

$$(1+E_k) = \prod_{i=0}^{n-k}(1+\alpha_i)\prod_{i=1}^{n-k}(1+\beta_i) \qquad |E_k| \lesssim (2k+1)\cdot 2^{-t}$$

$$\tilde{a}_k = a_k(1+E_k)$$

$$w'(x) = \tilde{a}_n x^n + \tilde{a}_{n-1}x^{n-1} + \ldots + \tilde{a}_1 x^1 + \tilde{a}_0$$

Dokładny wynik dla nieco zaburzonych danych

4

a) $T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x)$

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 2^2 x^3 - 3x$

$T_4(x) = 2x(2^2x^3 - 3x) - 2x^2 + 1 = 2^3 x^4 - 8x^2 + 1$

$T_5(x) = 2x(2^3x^4 - 8x^2 + 1) - 2^2x^3 - 3x = 2^4x^5 - 16x^3 + 2x - 2^2x^3 + 3x = 2^4x^5 - 20x^3 + 5x$

$T_6(x) = 2x(2^4x^5 - 20x^3 + 5x) - 2^3x^4 + 8x^2 - 1 = 2^5x^6 - 40x^4 + 10x^2 - 2^3x^4 + 8x^2 - 1 = 2^5x^6 - 48x^4 + 18x^2 - 1$

b) a_n – współczynnik przy x^n

$a_n = 2^{n-1}$

$a_{n-1} = 0$

$T_1(x) = 2^{1-1} \cdot x + 0 \cdot x^0$

$T_2(x) = 2^{2-1} \cdot x^2 + 0 \cdot x^1$

$\cancel{T_{n+2}(x) = 2x(T_{n-1}(x)) - T_{n-2}(x) = 2x(2^{n-1}}$

$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x) = 2x(2^{n-1}x^n + 0 \cdot x^{n+1} + \mathcal{E}) - (2^{n-2}x^{n-1} + 0x^{n-2} + \mathcal{E}') =$

$= 2^n x^{n+1} + E = 2^n x^{n+1} + 0 \cdot x^n + E$

c)

I $|T_n(x)| \leqslant 1$

$T_n(x) = \cos(n \arccos x) \qquad |\cos\alpha| \leqslant 1$

II

$|T_n(x)| = 1 \qquad |\cos\alpha| = 1$ wtw $\alpha = k\pi$; $k \in \mathbb{Z}$

$n \arccos x = k\pi$

$\arccos x = \frac{k\pi}{n}$

$x = \cos\left(\frac{k\pi}{n}\right)$

$0 \leqslant \frac{k\pi}{n} \leqslant \pi$

$0 \leqslant k \leqslant n$

III

$T_{n+1} = \cos((n+1)\arccos x) \qquad T_{n+1} = 0 \qquad \cos\alpha = 0$ wtw $\alpha = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

$(n+1)\arccos x = \frac{\pi}{2} + k\pi$

$\arccos x = \frac{2k+1}{2n+2}\pi$

$x = \cos\left(\frac{2k+1}{2n+2}\pi\right)$

$0 \leqslant \frac{2k+1}{2n+2}\pi \leqslant \pi$

$0 \leqslant 2k+1 \leqslant 2n+2$

$-\frac{1}{2} \leqslant k \leqslant n + \frac{1}{2}$

$k \in [-\frac{1}{2}; n+\frac{1}{2}]$

6

$$\lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$$

1° $x_k \in \{x_0, x_1, \dots, x_n\}$ $k \neq i$

$$\lambda_i(x_k) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x_k-x_j}{x_i-x_j} = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x_i-x_j}{x_i-x_j} = 1$$

2° $x_k \in \{x_0, \dots, x_n\}$ $k \neq i$

$$\lambda_i(x_k) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x_k-x_j}{x_i-x_j} = \frac{(x_k-x_0)(x_k-x_1)\dots\overset{=0}{(x_k-x_k)}\dots(x_k-x_n)}{m} = 0$$

$x_i \in \{x_0, x_1, \dots, x_n\}$

$$L_n(x_i) = y_0\lambda_0(x_i) + y_1\lambda_1(x_i) + \dots + y_n\lambda_n(x_i)$$

$$L_n(x_i) = y_i$$

$P(x)$, $Q(x)$ – wielomiany stopnia n interpolujące tę samą funkcję

$x_i \in \{x_0, \dots, x_n\}$ $P(x_i) = Q(x_i)$

Weźmy $R(x) = P(x) - Q(x)$

$R(x_i) = P(x_i) - Q(x_i) = 0$ czyli $R(x)$ ma $n+1$ miejsc zerowych

czyli $R(x) = 0$

$P(x) - Q(x) = 0 \Rightarrow P(x) = Q(x)$

7.

k	0	1	2	3
x_k	-3	-2	0	4
$y_k = f(x_k)$	0	2	6	-10

$$L(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = (x+2)(x)(x-4) \cdot \frac{1}{(-1)\cdot(-3)\cdot(-7)} = -\frac{x(x+2)(x-4)}{21}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = (x+3)(x)(x-4) \cdot \frac{1}{1\cdot(-2)(-6)} = \frac{x(x+3)(x-4)}{12}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = (x+3)(x+2)(x-4) \cdot \frac{1}{3\cdot 2\cdot(-4)} = -\frac{(x+2)(x+3)(x-4)}{24}$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = (x+3)(x+2)\cancel{(x-4)}x \cdot \frac{1}{7\cdot 6\cdot 4} = \frac{x(x+2)(x+3)}{168}$$

$$L(x) = 2 \cdot \frac{x(x+3)(x-4)}{12} - 6 \cdot \frac{(x+2)(x+3)(x-4)}{24} - 10 \cdot \frac{x(x+2)(x+3)}{168}$$

$$L(x) = \frac{x(x+3)(x-4)}{6} - \frac{(x+2)(x+3)(x-4)}{4} - 5 \cdot \frac{x(x+2)(x+3)}{84} =$$

$$L(x) = \frac{x^3 - x^2 - 12x}{6} - \frac{x^3 + x^2 - 14x - 24}{4} - 5 \cdot \frac{x^3 + 5x^2 + 6x}{84} =$$

$$= \frac{14x^3 - 14x^2 - 168x - 21x^3 - 21x^2 + 294x + 504 - 5x^3 - 25x^2 - 30x}{84}$$

$$= \frac{-12x^3 - 60x^2 + 96x + 504}{84} =$$

$$L(x) = -\frac{1}{7}x^3 - \frac{5}{7}x^2 + \frac{8}{7}x + 6$$

8

a) Interpolując wielomian piątego stopnia wielomianem 5 stopnia otrzymamy ten sam wielomian ponieważ interpolacja jest jednoznaczna. Czyli $f(x) = L(x)$

b)

k	0	1	2
x_k	-1	0	1
y_k	-2369	-1791	2741

$$\lambda_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x)(x-1)}{(-1)\cdot(-1)} = x^2 - x$$

$$\lambda_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{1\cdot(-1)} = -x^2+1$$

$$\lambda_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{2\cdot 1} = x^2+x$$

$$L(x) = -2369x^2 + 2369x + 1791x^2 - 1791 + 2741x^2 + 2741x =$$

$$= 2163x^2 + 5110x - 1791$$

9

a) Vezmy $f(x)=1$

$$1=f(x)=L_n(f)=\sum_{k=0}^{n} f(x_k)\cdot\lambda_k(x)=\sum_{k=0}^{n}\lambda_k(x)$$

b)

$$f(x)=\begin{cases}1 \text{ ak } j=0\\ x^j \text{ ak } j\neq 0\end{cases}$$

1° $j=0$

$$f(x)=\sum_{n=0}^{n}\lambda_k(x)\cdot x_k^j$$

$$1=\sum_{n=0}^{n}\lambda_k(0)\cdot 1=\sum_{n=0}^{n}\lambda_k(0)$$

2° $j\neq 0$

$$x^j=\sum_{n=0}^{n}\lambda_k(x)\,x_k^j$$

$$0=\sum_{n=0}^{n}\lambda_k(0)\,x_k^j$$