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Drugą część egzaminu

Zadanie 2

$X \sim \chi^2(n)$, $Y \sim \chi^2(k)$ ; $X, Y$ - niezależne

$f(x) \cdot f(y) = f(x, y)$

$Z = X + Y$

$\chi^2(n)$: $f(x) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$ ; $x \in (0, \infty)$

a) przejście od zmiennej $(X, Y)$ do $(Z, V)$

$$f(x,y) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \cdot \frac{1}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} y^{\frac{k}{2}-1} e^{-\frac{y}{2}} =$$

$$= \frac{1}{2^{\frac{n+k}{2}} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k}{2}\right)} \cdot x^{\frac{n}{2}-1} y^{\frac{k}{2}-1} e^{-\frac{(x+y)}{2}}$$

$$\begin{array}{l} Z = X + Y \\ V = Y \end{array} \Rightarrow \begin{array}{l} X = Z - V \\ Y = V \end{array}$$

b) Jacobian

$$J = \begin{vmatrix} \frac{dX}{dZ} & \frac{dX}{dV} \\ \frac{dY}{dZ} & \frac{dY}{dV} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$g(z,v) = f(x(z,v), y(z,v)) \cdot |J| =$$

$$= \frac{1}{2^{\frac{n+k}{2}} \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{k}{2}\right)} (z-v)^{\frac{n}{2}} v^{\frac{k}{2}-1} e^{-\frac{z-v+v}{2}} =$$

$$= \frac{e^{-\frac{z}{2}}}{2^{\frac{n+k}{2}} \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{k}{2}\right)} (z-v)^{\frac{n}{2}-1} v^{\frac{k}{2}-1}$$

c) gęstość brzegowa $g_z(z)$

Ustalamy przedziały całkowania:

$$\begin{cases} 0 < X < \infty \\ 0 < Y < \infty \end{cases} \Rightarrow \begin{cases} 0 < z-v < \infty \\ 0 < v < \infty \end{cases} \Rightarrow \begin{cases} -z < -v < \infty - z \\ 0 < v < \infty \end{cases} \Rightarrow \begin{cases} -\infty + z < v < z \\ 0 < v < \infty \end{cases}$$

$$\Downarrow$$

$$0 < v < z$$

$$g_z(z) = \int_0^z \frac{e^{-\frac{z}{2}}}{2^{\frac{n+k}{2}} \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{k}{2}\right)} (z-v)^{\frac{n}{2}-1} v^{\frac{k}{2}-1} \, dv$$

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$$\frac{e^{-\frac{z}{2}}}{2^{\frac{n+k}{2}} \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{k}{2}\right)} = C \quad ; \quad C - \text{const}$$

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$$g_z(z) = C \int_0^z (z-v)^{\frac{n}{2}-1} v^{\frac{k}{2}-1} \, dv = \left| \begin{matrix} v = zx & \frac{v}{z} = x \\ \frac{dv}{dx} = z & dv = z \cdot dx \end{matrix} \right| =$$

$$= C \int_0^1 (z - zx)^{\frac{n}{2}-1} (zx)^{\frac{k}{2}-1} z \, dx =$$

$$= C \int_0^1 z^{\frac{n}{2}-1} (1-x)^{\frac{n}{2}-1} z^{\frac{k}{2}-1} x^{\frac{k}{2}-1} z \, dx =$$

$$= C \int_0^1 z^{\left[\left(\frac{n}{2}-1\right)+\left(\frac{k}{2}-1\right)+1\right]} (1-x)^{\frac{n}{2}-1} x^{\frac{k}{2}-1} \, dx =$$

$$= C \cdot z^{\frac{n+k}{2}-1} \cdot \int_0^1 x^{\frac{k}{2}-1} (1-x)^{\frac{n}{2}-1} \, dx = C \cdot z^{\frac{n+k}{2}-1} \cdot B\left(\frac{k}{2}, \frac{n}{2}\right) =$$

$$= C \cdot z^{\frac{n+k}{2}-1} \cdot \frac{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{k+n}{2}\right)} =$$

$$= \frac{e^{-\frac{z}{2}}}{2^{\frac{n+k}{2}} \cancel{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{k}{2}\right)}} \cdot \frac{\cancel{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{n}{2}\right)}}{\Gamma\left(\frac{k+n}{2}\right)} \cdot z^{\frac{n+k}{2}-1} =$$

$$= \frac{1}{2^{\frac{n+k}{2}} \Gamma\left(\frac{k+n}{2}\right)} \cdot z^{\frac{n+k}{2}-1} e^{-\frac{z}{2}} = f(z) \text{ dla } \chi^2(k+n)$$

więc $Z \sim \chi^2(n+k)$