1

$X_1, X_2, \dots, X_n \sim e^{\lambda}$ niezależne

$Y_i = X_1 + X_2 + \dots + X_i$ dla $i = 1, 2, \dots, n$

$f_{X_i}(x) = \lambda e^{-\lambda x}$

Z niezależności:

$$f_{X_1, X_2, \dots X_n}(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} \lambda e^{-\lambda x} = \lambda e^{-\lambda \sum_{i=1}^{n} x_i}$$

$Y_1 = X_1 \Rightarrow X_1 = Y_1$

$Y_2 = X_1 + X_2 \Rightarrow X_2 = Y_2 - Y_1$

$Y_3 = X_1 + X_2 + X_3 \Rightarrow X_3 = Y_3 - Y_2 + Y_1 - Y_1 = Y_3 - Y_2$

$\vdots$

$Y_i = \sum_{k=1}^{i} X_k \Rightarrow X_i = Y_i - Y_{i-1}$

$$J = \begin{vmatrix} \frac{dx_1}{dy_1} & \cdots & \frac{dx_1}{dy_n} \\ \vdots & \ddots & \\ \frac{dx_n}{dy_1} & \cdots & \frac{dx_n}{dy_n} \end{vmatrix} = \begin{vmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & -1 & 1 & \\ 0 & & \ddots & \ddots \\ & & -1 & 1 \end{vmatrix} = 1$$

$$f_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) = f_{x_1 \dots}(x_1(Y_1 \dots), x_2(Y_1 \dots), \dots) \cdot |J| =$$

$$= \lambda^n e^{-\lambda \left(Y_1 + \sum_{i=2}^{n} (Y_i - Y_{i-1})\right)} = \underbrace{\lambda^n e^{-\lambda y_n}}$$

2

$0 < y_1 < y_2 < \dots < y_n$

$f_{Y_1 \dots}(y_1 \dots) = \lambda^n e^{-\lambda y_n}$

$f_{Y_n}(y_n) = \int_0^{y_n} \int_0^{y_{n-1}} \dots \int_0^{y_2} f_{Y_1 \dots}(y_1 \dots)\, dy_1\, dy_2 \dots dy_{n-1} =$

$= \lambda^n e^{-\lambda y_n} \int_0^{y_n} \int_0^{y_{n-1}} \dots \int_0^{y_2} 1\, dy_1\, dy_2 \dots dy_n =$

$= \lambda^n e^{-\lambda y_n} \int_0^{y_n} \int_0^{y_{n-1}} \dots \int_0^{y_3} y_2\, dy_2\, dy_3 \dots dy_n =$

$= \lambda^n e^{-\lambda y_n} \int_0^{y_n} \dots \int_0^{y_4} \frac{y_3^2}{2}\, dy_3\, dy_4 \dots dy_n =$

$= \lambda^n e^{-\lambda y_n} \int_0^{y_n} \dots \int_0^{y_5} \frac{y_4^3}{2 \cdot 3}\, dy_4 \dots dy_n =$

$= \lambda^n e^{-\lambda y_n} \frac{y_n^{n-1}}{(n-1)!}$

6

X, Y, Z - niezależne $\sim U[0,1]$

$P(X \geqslant YZ)$

$Z \in [0,1]$ $Y \in [0,1]$ $X \in [ZY, 1]$

$f(x,y,z) = 1$ bo niezależne

$$P(X \geqslant YZ) = \int_0^1 \int_0^1 \int_0^1 f(x,y,z)\, dx\, dy\, dz =$$

$$= \int_0^1 \int_0^1 (1-yz)\, dy\, dz = \int_0^1 1 - \frac{z}{2}\, dz = \left[z - \frac{z^2}{4}\right]\Big|_0^1 = 1 - \frac{1}{4} = \underline{\frac{3}{4}}$$

7

X_1, X_2, X_3 - niezależne $\sim e^{\lambda}$

$$(Y_1, Y_2, Y_3) = [(X_1 + X_2), (X_1 + X_3), (X_2 + X_3)]$$

$Y_1 = X_1 + X_2$

$$X_1 = Y_1 - X_2 = Y_1 - Y_3 + X_3 = Y_1 - Y_3 + Y_2 - X_1 = X_1$$

$$\Downarrow$$

$$X_1 = \frac{Y_1 + Y_2 - Y_3}{2}$$

$Y_2 = X_1 + X_3$

$$X_3 = Y_2 - X_1 = Y_2 - Y_1 + X_2 = Y_2 - Y_1 + Y_3 - X_3 \Rightarrow X_3 = \frac{-Y_1 + Y_2 + Y_3}{2}$$

$Y_3 = X_2 + X_3$

$$X_2 = Y_3 - X_3 = Y_3 - Y_2 + X_1 = Y_3 - Y_2 + Y_1 - X_2 \Rightarrow X_2 = \frac{Y_1 - Y_2 + Y_3}{2}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix} = -\frac{1}{2}$$

$$f_{Y_1 Y_2 Y_3}(y_1, y_2, y_3) = f_{X_1 X_2 X_3}(x_1(Y_1 Y_2 Y_3), x_2(Y), x_3(Y..)) \cdot |J| =$$

$$= \frac{1}{2} \cdot \lambda^3 e^{-\lambda\left(\frac{y_1 + y_2 + y_3}{2}\right)}$$

9

$Z = \frac{X}{Y}$ $\qquad f(x,y) = 1 \quad 0 \leq x, y \leq 1$

$V = Y$

$$\begin{cases} X = ZY = ZV \\ Y \quad\;\; = V \end{cases}$$

$$J = \begin{vmatrix} v & z \\ 0 & 1 \end{vmatrix} = v$$

$$\begin{cases} 0 \leq x \leq 1 \\ 0 < Y \leq 1 \end{cases} \Leftrightarrow \begin{cases} 0 \leq \frac{X}{Y} \leq \frac{1}{Y} \\ 0 < V \leq 1 \end{cases} \Leftrightarrow \begin{cases} 0 \leq Z \leq \frac{1}{V} \\ 0 < V \leq 1 \end{cases}$$

$$\begin{cases} 0 \leq Z \\ V < \frac{1}{Z} \\ 0 < V \leq 1 \end{cases}$$

$1°$ $0 \leq z \leq 1 \wedge v \in (0, 1]$

$2°$ $1 < Z \wedge v \in (0, \frac{1}{z}]$

$$g_Z(z) = \begin{cases} \int_0^1 v \, dv = \frac{1}{2} & , z \in [0, 1] \\ \int_0^{\frac{1}{z}} v \, dv = \frac{1}{2z^2} & , z \in (1, \infty) \end{cases}$$