I

a) $4\cos^2 x - 3 =$

$= 4\cos^2 x - 3(\sin^2 x + \cos^2 x) =$

$= 4\cos^2 x - 3\sin^2 x - 3\cos^2 x =$

$= \cos^2 x - \cancel{3}\sin^2 x - 2\sin x =$

$\cos^2 x - \sin^2 x = \cancel{\sin 2x}\ \cos 2x$

$= (\cos 2x - 2\sin x) \cdot \frac{\cos x}{\cos x} =$

$2\sin x \cos x = \sin 2x$

$= \frac{\cos x \cos 2x - 2\sin x \cos x \sin x}{\cos x} =$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{\cos x \cos 2x - \sin x \sin 2x}{\cos x} =$

$= \frac{\cos 3x}{\cos x}$

b) $x^{-3}\left(\frac{\pi}{2} - x - \operatorname{arcctg}(x)\right) =$

$= x^{-3}(\operatorname{arctg}(x) - x)$

$\operatorname{arctg}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$x^{-3}(\operatorname{arctg}(x) - x) = -\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \frac{x^6}{9}$

II

$x_1 x_2 = \frac{a}{c}$

$x_2 = \frac{a}{c x_1}$

$\boxed{III}$

$$x = \left(r + \sqrt{q^3+r^2}\right)^{\frac{1}{3}} + \left(r - \sqrt{q^3+r^2}\right)^{\frac{1}{3}}$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x = \frac{2r}{\left(r + \sqrt{q^3+r^2}\right)^{2/3} + \left(r - \sqrt{q^3+r^2}\right)^{2/3} + q}$$

$$(a+b)(a-b) = a^2 - b^2$$

$$x = \frac{2r}{\left(r + \sqrt{q^3+r^2}\right)^{2/3} + \left(\frac{1}{r + \sqrt{q^3+r^2}}\right)^{2/3} \cdot q^2 + q}$$

L 3.4

$$\left|\frac{f(x+h)-f(x)}{f(x)}\right| \qquad \left|\frac{x-(x+h)}{x}\right| = \left|\frac{h}{x}\right| \qquad h \text{ - małe}$$

względna zmiana wyniku przy zmianie argumentu

względna zmiana argumentu

$$\left|\frac{f(x+h)-f(x)}{f(x)}\right| = \left|\frac{f(x+h)-f(x)}{h}\right| \cdot \left|\frac{h}{f(x)}\right| \approx f'(x)\left|\frac{h}{f(x)}\right| =$$

$$= f'(x) \cdot \left|\frac{h}{x}\,\frac{x}{f(x)}\right| = \left|\frac{f'(x)\,x}{f(x)}\right| \cdot \left|\frac{h}{x}\right|$$

wskaźnik uwarunkowania: $\left|\frac{f'(x)\cdot x}{f(x)}\right|$

V

$$\frac{f'(x)\cdot x}{f(x)}$$

a) $f(x) = x^3 + 2020$ $\qquad f'(x) = 3x^2$

$$W(f(x)) = \left|\frac{3x^2 \cdot x}{x^3+2020}\right| = \left|\frac{3}{1+\frac{2020}{x^3}}\right|$$

$$\lim_{x \to -\sqrt[3]{2020}} f(x) = \infty$$

b) $f(x) = x^{-1}\ln(x)$ $\qquad f'(x) = x^{-2}(1-\ln(x))$

$$WU(f(x)) = \left|\frac{x^{-2}(1-\ln(x))\cdot x}{x^{-1}\ln(x)}\right| = \left|\frac{1-\ln(x)}{\ln(x)}\right|$$

$$\lim_{x\to 1}\left|\frac{1}{\ln(x)} - 1\right| = \infty$$

c) $f(x) = \cos(5x)$ $\qquad f'(x) = -5\sin(5x)$

$$WU(f(x)) = \left|\frac{-5x\sin(5x)}{\cos 5x}\right| = |5x\,tg(5x)|$$

$$tg\,\frac{\pi}{2} + k\pi = \infty \implies \frac{\pi}{2} + k\pi = 5x \qquad k \in \mathbb{Z}$$

$$\frac{\pi}{10} + \frac{k\pi}{5} = x$$

$$\lim_{x\to \frac{\pi}{10}+\frac{k\pi}{5}} |5x\,tg(5x)| = \infty$$

d) $f(x) = (\sqrt{x^4+2020} + x)^{-1}$ $\quad f'(x) = -\dfrac{2x^3 + \sqrt{x^4+2020}}{(\sqrt{x^4+2020}+x)^2 \cdot \sqrt{x^4+2020}}$

$$WU(f(x)) = \left| \frac{(2x^4 + x\sqrt{x^4+2020})(\sqrt{x^4+2020}+x)}{(\sqrt{x^4+2020}+x)^2 \cdot \sqrt{x^4+2020}} \right| =$$

$$= \left| \frac{2x^4 + x\sqrt{x^4+2020}}{(\sqrt{x^4+2020}+x)\sqrt{x^4+2020}} \right| = \left| \frac{2x^4 + x\sqrt{x^4+2020}}{x^4+2020 + x\sqrt{x^4+2020}} \right|$$

Dobrze uwarunkowane

VII

$$w(x) = x + x^{-1}$$

$$\tilde{w}(x) = \left(x + \frac{1}{x}(1+\varepsilon_1)\right)(1+\varepsilon_2) = x + x\,\varepsilon_2 + \frac{(1+\varepsilon_1)(1+\varepsilon_2)}{x} =$$

$$= x(1+\varepsilon_2) + \frac{1}{x(1+\varepsilon')} \qquad \varepsilon' = \frac{1}{(1+\varepsilon_1)(1+\varepsilon_2)}$$

Dokładny wynik dla lekko zaburzonych danych

VIII

$$I = \left(\left(\left(x_1(1+\varepsilon_1)\right)\cdot x_2\right)(1+\varepsilon_2)\cdot x_3\right)(1+\varepsilon_3)\cdot \ldots \cdot x_n(1+\varepsilon_n) = \quad \{ \varepsilon_1 = 0$$

$$= x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n \cdot (1+\varepsilon_1)\cdot(1+\varepsilon_2)\cdot(1+\varepsilon_3)\cdot\ldots\cdot(1+\varepsilon_n) =$$

$$= \prod_{i=1}^{n} x_i (1+\varepsilon_i)$$

$$I = \prod_{i=1}^{n} x_i \cdot \prod_{i=1}^{n} (1+\varepsilon_i) = \left(\prod_{i=1}^{n} x_i\right)(1+E)$$

$$|\varepsilon_i| \leq 2^{-t} \Rightarrow |E| \leq n \cdot 2^{-t}$$

Algorytm numerycznie poprawny

x_k nie są maszynowe:

$$rd(x_k) = x_k(1+\varepsilon_k) \qquad |\varepsilon_k| \leq 2^{-t}$$

$$I = x_1(1+\varepsilon_1)(1+\alpha_1)\cdot x_2(1+\varepsilon_2)(1+\alpha_2)\cdot\ldots\cdot x_n(1+\varepsilon_n)(1+\alpha_n) \quad \{ \varepsilon_1 = 0$$

$$I = \prod_{i=1}^{n} x_i (1+\varepsilon_i)(1+\alpha_i)$$

$$I = \prod_{i=1}^{n} x_i \cdot \prod_{i=1}^{n} (1+\varepsilon_i) \cdot \prod_{i=1}^{n} (1+\alpha_i) = \left(\prod_{i=1}^{n} x_i\right)(1+E)$$

$$|\varepsilon_i| \leq 2^{-t}, \quad |\alpha_i| \leq 2^{-t}$$

$$\Downarrow$$

$$|E| \leq 2n\, 2^{-t} = n\, 2^{-t+1}$$

Algorytm numerycznie poprawny