1 Geom (p) , parametr p

$$f(p, x) = (1-p)^{x-1} p$$

$$L(p, x_1 \ldots x_n) = \prod_{i=1}^{n} f(p, x_i) = \prod_i (1-p)^{x_i - 1} p =$$

$$= (1-p)^{\sum x_i - n} p^n$$

$$l = \ln L(p, x_{\ldots})$$

$$l = \ln\left((1-p)^{\sum x_i - n} p^n \right) = \ln (1-p)^{\sum x_i - n} \ln p^n =$$

$$= \left(\sum x_i - n\right) \ln(1-p) + n \cdot \ln p$$

$$\frac{\partial l}{\partial p} = -\frac{\sum x_i - n}{1-p} + \frac{n}{p} = 0$$

$$\frac{\sum x_i - n}{1-p} = \frac{n}{p}$$

$$p \sum x_i \cancel{- np} = n \cancel{- np}$$

$$p = \frac{n}{\sum x_i}$$

$$\hat{p} = \frac{n}{n \bar{x}} = \frac{1}{\bar{x}}$$

2

$f(x; a, k)$ k-znane, a- parametr

$$f(x; a, k) = \frac{k a^k}{x^{k+1}}$$

$$L(a,k) = f(x_1, x_2, \dots, x_n; a, k) = \prod_{i=1}^{n} \frac{k a^k}{x_i^{k+1}} =$$

$$= \left(k^n a^{kn}\right) \prod_{i=1}^{n} x_i^{-(k+1)}$$

$$\ln L(a,k) = n \ln k + \underbrace{nk \ln a}_{} - (k+1) \ln \left(\prod_i x_i\right)$$

Żeby $L(a,k)$ było maksymalne $\log a$ musi być maksymalny

$$x_i \in [a, +\infty) \Rightarrow x_i \geq a$$

Więc max a to min x_i

$$\hat{a} = \min(x_i)$$

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$$f(x; a, k) = \frac{k a^k}{x^{k+1}}$$

a - znane

k - param

$$L(x; a, k) = \prod_i \frac{k a^k}{x_i^{k+1}} = k^n a^{nk} \prod x_i^{-(k+1)}$$

$$\ln(L(x; a, k)) = n \ln(k) + nk \ln(a) - (k+1) \sum_i \ln(x_i)$$

$$\frac{\partial L(a, k)}{\partial k} = \frac{n}{k} + n \ln(a) - \sum_i \ln(x_i) = 0$$

$$\hat{k} = \frac{n}{\sum_i \ln(x_i) - n \ln(a)}$$

4

$$f(x;\lambda) = \lambda \exp(-\lambda x) \qquad ; x \in (0;\infty) \qquad \lambda - \text{param}$$

$$L(\lambda) = \prod_i \left(\lambda e^{-\lambda x_i}\right) = \lambda^n \cdot e^{-\lambda \sum_i x_i}$$

$$l = \ln(L(\lambda)) = \ln\left(\lambda^n \cdot e^{-\lambda \sum_i x_i}\right) = n \ln(\lambda) - \lambda \sum_i x_i$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_i x_i = \frac{n}{\lambda} - \bar{x} n = 0$$

$$\frac{n}{\lambda} = \bar{x} n$$

$$\hat{\lambda} = \frac{1}{\bar{x}}$$

5

$$f(x;k,\lambda) = \frac{k}{\lambda} \cdot \left(\frac{x}{\lambda}\right)^{k-1} \exp\left(-\left(\frac{x}{\lambda}\right)^{k}\right) \quad ; \quad x \in [0, \infty)$$

k – znane

λ – parametr

$$L(\lambda) = \prod_{i=1}^{n} \frac{k}{\lambda} \cdot \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^{k}} = \prod_{i=1}^{n} \frac{k}{\lambda^{k}} x^{k-1} e^{-\left(\frac{x}{\lambda}\right)^{k}} =$$

$$= \left(\frac{k}{\lambda^{k}}\right)^{n} \prod_{i} x_i^{k-1} \cdot e^{-\sum_i \left(\frac{x_i}{\lambda}\right)^{k}}$$

$$l = \ln(L(\lambda)) = \ln \left(\frac{k}{\lambda^{k}}\right)^{n} \prod_{i} x_i^{k-1} e^{-\sum_i \left(\frac{x_i}{\lambda}\right)^{k}} =$$

$$= n \ln k - nk \ln \lambda - \sum_i \left(\frac{x_i}{\lambda}\right)^{k} + (k-1)\sum_i \ln x_i$$

$$\frac{\partial l}{\partial \lambda} = -\frac{nk}{\lambda} + k \sum_i \frac{x_i^{k}}{\lambda^{k+1}} = 0$$

$$\frac{nk}{\lambda} = k \sum_i \frac{x_i^{k}}{\lambda^{k+1}}$$

$$\frac{1}{n\lambda^{k}} \sum_i x_i^{k} = 1$$

$$\lambda = \frac{\left(\sum x_i^{k}\right)^{\frac{1}{k}}}{n^{\frac{1}{k}}}$$

6

$$f(a,b,c) = \sum_i (a + bx_i + cx_i^2 - y_i)^2$$

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} = 0$$

$$\frac{\partial f}{\partial a} = 2\sum_i (a + bx_i + cx_i^2 - y_i) = 0$$

$$na + b\sum x_i + c\sum x_i^2 = \sum y_i$$

$$\frac{\partial f}{\partial b} = 2\sum_i (a + bx_i + cx_i^2 - y_i) \cdot x_i = 0$$

$$a\sum x_i + b\sum x_i^2 + c\sum x_i^3 = \sum y_i x_i$$

$$\frac{\partial f}{\partial c} = 2\sum_i (a + bx_i + cx_i^2 - y_i) \cdot x^2 = 0$$

$$a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 = \sum y_i x_i^2$$

Wtedy:

$$\begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix} = \begin{bmatrix} an + b\sum x_i + c\sum x_i^2 \\ a\sum x_i + b\sum x_i^2 + c\sum x_i^3 \\ a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 \end{bmatrix} = \begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

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$$f(a,b,c) = \sum_{i=1}^{n} (a + bx_i + cy_i - z_i)^2$$

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} = 0$$

$$\begin{cases} 2\sum_i (a + bx_i + cy_i - z_i) = 0 \\ 2\sum_i (a + bx_i + cy_i - z_i)\, x_i = 0 \\ 2\sum_i (a + bx_i + cy_i - z_i)\, y_i = 0 \end{cases}$$

$$\begin{cases} na + b\sum x_i + c\sum y_i = \sum z_i \\ a\sum x_i + b\sum x_i^2 + c\sum y_i x_i = \sum z_i x_i \\ a\sum y_i + b\sum x_i y_i + c\sum y_i^2 = \sum z_i y_i \end{cases}$$

$$\begin{bmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \\ \sum y_i & \sum x_i y_i & \sum y_i^2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum z_i \\ \sum x_i z_i \\ \sum y_i z_i \end{bmatrix}$$

10 $X \sim U[0,3]$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_0^3 \frac{1}{3} e^{tx}\, dx =$$

$$= \frac{1}{3t} e^{tx}\Big|_0^3 = \frac{1}{3t}(e^{3t}-1)$$

$$M'_X(t) = \frac{e^{3t}(3t-1)+1}{3t^2}$$

$$M'_X(0) = \lim_{t\to 0} \frac{e^{3t}(3t-1)+1}{3t^2} = \lim_{t\to 0} \frac{3e^{3t}}{2} = \frac{3}{2} = E(X)$$