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$f(x,y) = c(x+y)e^{-x-y}$ $; x, y > 0$

a) $f(x,y)$ gustoće $X\,Y$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dx\,dy = 1$$

$$\int_0^\infty\int_0^\infty c(x+y)e^{-x-y}\,dx\,dy = c\int_0^\infty\int_0^\infty (x+y)\,e^{-x}e^{-y}\,dx\,dy$$

$$\int_0^\infty (x+y)e^{-x}e^{-y}\,dx = e^{-y}\int_0^\infty (x+y)e^{-x}\,dx = \begin{vmatrix} u=x+y & v'=e^{-x} \\ u'=1 & v=-e^{-x}\end{vmatrix} =$$

$$= e^{-y}\left(\left[(x+y)(-e^{-x})\right]\Big|_0^\infty - \int_0^\infty -e^{-x}\,dx\right) = e^{-y}\left(0+y+1\right) = e^{-y}(y+1)$$

$$= c\int_0^\infty e^{-y}(y+1)\,dy =$$

$$\int_0^\infty e^{-y}(y+1)\,dy = \begin{vmatrix} u=y+1 & v'=e^{-y} \\ u'=1 & v=-e^{-y}\end{vmatrix} = \left[(y+1)(-e^{-y})\right]\Big|_0^\infty + \int_0^\infty e^{-y} =$$

$= 1+1 = 2$

$\Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$

b) Nezavisne $X, Y$ ?

$$f_1(x) = \int_0^\infty \frac{1}{2}(x+y) \cdot e^{-(x+y)} \, dy = \frac{1}{2} e^{-x} \int_0^\infty (x+y) e^{-y} \, dy =$$

$$= \frac{1}{2} e^{-x} (x+1)$$

$$f_2(y) = \int_0^\infty \frac{1}{2}(x+y) \cdot e^{-(x+y)} \, dx = \frac{1}{2} e^{-y} (y+1)$$

$$f_1 \cdot f_2 = \frac{1}{4} e^{-(x+y)} (x+1)(y+1)$$

$$f(x,y) = \frac{1}{2}(x+y) \, e^{-(x+y)}$$

$$f_1 \cdot f_2 \overset{?}{=} f(x,y)$$

$$\frac{1}{2}\cancel{\frac{1}{4}} \cancel{e^{-(x+y)}} (x+1)(y+1) \overset{?}{=} \cancel{\frac{1}{2}} (x+y) \cancel{e^{-(x+y)}}$$

$$(x+1)(y+1) \overset{?}{=} 2(x+y) \qquad x=3, y=3$$

$$4 \cdot 4 \neq 2 \cdot 6$$

Nisu nezavisne

c) Policzyć $m_{10}$, $m_{01}$

$$m_{pq} = \int_R \int_R x^p y^q f(x,y)\,dx\,dy$$

$$m_{10} = \int_R \int_R \frac{1}{2} x (x+y) e^{-x} e^{-y}\,dx\,dy = \frac{1}{2} \int_0^\infty \int_0^\infty x(x+y) e^{-x} e^{-y}\,dy\,dx =$$

$$\int_0^\infty x(x+y) e^{-x} e^{-y}\,dy = x e^{-x} \int_0^\infty (x+y) e^{-y}\,dy = x e^{-x} (1+x)$$

$$= \frac{1}{2} \int_0^\infty x e^{-x} (1+x)\,dx =$$

$$\cancel{\int_0^\infty x e^{-x} (1+x)\,dx = \begin{vmatrix} u = 1+x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{vmatrix} = \left[(-e^{-x})(1+x)\right]_0^\infty}$$

$$\int_0^\infty e^{-x} (x+x^2)\,dx = \begin{vmatrix} u = x+x^2 & v' = e^{-x} \\ u' = 1+2x & v = -e^{-x} \end{vmatrix} = \left[(x+x^2)(-e^{-x})\right]_0^\infty - \int_0^\infty -e^{-x}(1+2x)\,dx =$$

$$= 0 + \int_0^\infty e^{-x}(1+2x)\,dx = \begin{vmatrix} u = 1+2x & v' = e^{-x} \\ u' = 2 & v = -e^{-x} \end{vmatrix} = \left[(-e^{-x})(1+2x)\right]_0^\infty - \int_0^\infty 2(-e^{-x})\,dx =$$

$$= 1 + 2\int_0^\infty e^{-x}\,dx = 3$$

$$= \frac{1}{2} \cdot 3 = \frac{3}{2}$$

$m_{01}$ analogicznie

2 $f_{XY}(x,y) = cxy + x + y$ $\qquad f_{XY} \geq 0$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_{XY}\, dx\, dy = 1$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\, dx\, dy = \int_1^2 \int_0^3 (cxy + x + y)\, dx\, dy =$$

$$\int_0^3 (cxy + x + y)\, dx = \int_0^3 cxy\, dx + \int_0^3 x\, dx + \int_0^3 y\, dx = cy\int_0^3 x\, dx + \int_0^3 x\, dx + y\int_0^3 1\, dx =$$

$$cy\left[\frac{x^2}{2}\right]\Big|_0^3 + \frac{x^2}{2}\Big|_0^3 + y\, x\Big|_0^3 = cy\frac{9}{2} + \frac{9}{2} + 3y$$

$$= \int_1^2 \frac{9}{2}cy + \frac{9}{2} + 3y\, dy = \frac{9}{2}c\int_1^2 y\, dy + \frac{9}{2}\int 1\, dy + 3\int y\, dy =$$

$$\frac{9}{4}c\, y^2\Big|_1^2 + \frac{9}{2}\, y\Big|_1^2 + \frac{3}{2}\, y^2\Big|_1^2 = c\frac{9}{4}\cdot 3 + \frac{9}{2} + \frac{9}{2} = \frac{27}{4}c + 9 = 1$$

$$\frac{27}{4}c = -8$$

$$\boxed{c = -\frac{32}{27}}$$

ALE

$$f_{XY}(3,2) = -\frac{32}{27}\cdot 3\cdot 2 + 3 + 2 = -\frac{32}{27}\cdot 6 + 6 < 0$$

3. $f_{XY}(x,y) = -xy + x \quad ; \quad x \in \langle 0;2\rangle, \; y \in \langle 0;1\rangle$

$$f_1(x) = \int_0^1 (-xy+x)\,dy = -x\int_0^1 y\,dy + x\int_0^1 1\,dy = -x\,\frac{1}{2}y^2\Big|_0^1 + x\,y\Big|_0^1 =$$

$$= -x\,\frac{1}{2} + x = \frac{x}{2}$$

$$f_2(y) = \int_0^2 (-xy+x)\,dx = -y\int_0^2 x + \int_0^2 x\,dx = -\frac{1}{2}y\,x^2\Big|_0^2 + \frac{1}{2}x^2\Big|_0^2 =$$

$$= -2y + 2$$

$$f_1 \cdot f_2 = \frac{x}{2}\cdot(-2y+2) = -xy + x = f_{XY}(x,y)$$

X, Y Niezależne

Pułapka

$$4.\; P(1 \le X \le 3, \; 0 \le Y \le \tfrac{1}{2}) = \int_1^3 \int_0^{\frac{1}{2}} (-xy+x)\,dy\,dx \quad ; \quad 0 \le x \le 2$$

$$\int_0^{\frac{1}{2}} (-xy+x)\,dy = -x\int_0^{\frac{1}{2}} y\,dy + x\int_0^{\frac{1}{2}} 1\,dy = -\frac{1}{2}x\,y^2\Big|_0^{\frac{1}{2}} + x\,y\Big|_0^{\frac{1}{2}} = -\frac{1}{8}x + \frac{1}{2}x = \frac{3}{8}x$$

$$\int_1^2 \frac{3}{8}x\,dx = \frac{3}{8}\cdot\frac{1}{2}\,x^2\Big|_1^2 = \frac{3}{16}\cdot 3 = \frac{9}{16}$$

5. $X \sim U[0,1]$
$Y = X^n$

Pokazać $f_Y(y) = \dfrac{y^{\frac{1}{n}-1}}{n}$ $\quad 0 \leq Y \leq 1$

$F_Y(y) = P(Y < y) = P(X^n < y) = P(X < y^{\frac{1}{n}}) = F_X(y^{\frac{1}{n}})$

$(F_X)' = f(x)$

$f_Y(y) = (F_X(y^{\frac{1}{n}}))' = \dfrac{y^{\frac{1}{n}-1}}{n}$

6. $X \sim U[-1;1]$ $\quad f(x) = \frac{1}{2}$
$Y = |X|$

$F_Y(y) = P(Y < y) = P(|X| < y) = P(-y < x < y) = P(x < y) - P(x < -y) =$
$= P(x < t) - P(x < -y) = F(y) - F(-y) = \int_0^y \frac{1}{2}\,dy - \int_0^{-y} \frac{1}{2}\,dy = \frac{1}{2}y + \frac{1}{2}y = y$

$f(x) = F_Y(y)' = 1$

8 $Y = X^2 \qquad X \in \mathbb{R}$

Pokazać: $f_Y(y) = \dfrac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \quad ; \; y > 0$

$F_Y(t) = P(Y \leq t) = P(X^2 \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) =$

$= F_X(\sqrt{t}) - F_X(-\sqrt{t}) = \int_0^{\sqrt{t}} f(x)\,dx - \int_0^{-\sqrt{t}} f(x)\,dx =$

$= \sqrt{t}\, f_X(\sqrt{t}) + \sqrt{t}\, f_X(-\sqrt{t})$

$f_X(x) = \left(\sqrt{t}\, f_X(\sqrt{t} + \sqrt{t}\, f_X(-\sqrt{t})\right)' = \dfrac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$

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$X \sim U[a,b] \qquad V(x) = E(x^2) - E^2(x)$

$$f_X(x) = \frac{1}{b-a}$$

$$E(x) = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\,\frac{1}{2}\,x^2\Big|_a^b =$$

$$= \frac{1}{b-a}\,\frac{1}{2}\left(b^2 - a^2\right) = \frac{b+a}{2} \Rightarrow E^2(x) = \frac{(b+a)^2}{4}$$

$$E(x^2) = \int_a^b x^2\,\frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\,\frac{1}{3}\,x^3\Big|_a^b =$$

$$= \frac{1}{b-a}\,\frac{1}{3}\left(b^3 - a^3\right) = \frac{1}{\cancel{b-a}}\,\frac{1}{3}\,\cancel{(a-b)}\left(a^2 + ab + b^2\right) =$$

$$\frac{a^2 + ab + b^2}{3}$$

$$V(x) = \frac{a^2 + ab + b^2}{3} - \frac{b^2 + 2ab + a^2}{4} = \frac{4a^2 + 4ab + 4b^2 - 3b^2 - 6ab - 3a^2}{12} =$$

$$= \frac{a^2 - 2ab + b^2}{12} = \frac{(a-b)^2}{12}$$