1

$f_1 \dots f_n$ - funkcje liniowo niezależne

(*) $g_1 \dots g_n$ - funkcje liniowo niezależne i $\langle g_i, g_j \rangle = 0 ; i \neq j$

$$\langle f, g \rangle = \sum_{k=0}^{N} f(x_k)\, g(x_k)$$

$$\begin{cases} g_0 = f_0 \\ g_k = f_k - \sum_{i=0}^{k-1} \frac{\langle f_k, g_i \rangle}{\langle g_i, g_i \rangle} \cdot g_i \end{cases}$$

D-d

Baza: $k=0$ $\quad g_0 = f_0$ $\quad$ (*) zachodzi

$k=1$

$$g_1 = f_1 - \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} \cdot g_0 \qquad \langle g_0, g_1 \rangle \overset{?}{=} 0$$

$$\langle g_0, g_1 \rangle = \langle g_0, f_1 - \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} \cdot g_0 \rangle = \langle g_0, f_1 \rangle - \langle g_0, \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} \cdot g_0 \rangle =$$

$$= \langle g_0, f_1 \rangle - \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} \cdot \langle g_0, g_0 \rangle = \langle g_0, f_1 \rangle - \langle f_1, g_0 \rangle = 0$$

$$\cancel{\frac{\langle g_0, f_1 \rangle \langle g_0, g_0 \rangle - \langle f_1, g_0 \rangle \langle g_0, g_0 \rangle}{\langle g_0, g}}$$

Krok Załóżmy że mamy $g_0 \dots g_{k-1}$ ortogonalne

Pokażmy że g_k ortogonalne z g_i $i < k$

$$g_k = f_k - \sum_{i=0}^{k-1} \frac{\langle f_k, g_i \rangle}{\langle g_i, g_i \rangle} \cdot g_i$$

$$\langle g_i, g_k \rangle = \left\langle g_i, f_k - \sum_{i=0}^{k-1} \frac{\langle f_k, g_i \rangle}{\langle g_i, g_i \rangle} \cdot g_i \right\rangle =$$

$$= \langle g_i, f_k \rangle - \sum_{i=0}^{k-1} \frac{\langle f_k, g_i \rangle}{\langle g_i, g_i \rangle} \langle g_i, g_i \rangle =$$

z zał. $\langle g_i, g_j \rangle = 0$ dla $i \neq j$; $i, j < k$

$$= \langle g_i, f_k \rangle - \frac{\langle f_k, g_i \rangle}{\cancel{\langle g_i, g_i \rangle}} \cancel{\langle g_i, g_i \rangle} = \langle g_i, f_k \rangle - \langle f_k, g_i \rangle = 0$$

ckd

2

$$w = \alpha_0 P_0 + \alpha_1 P_1 + \dots + \alpha_{k-1} P_{k-1}$$

$$\langle w, P_k \rangle = \langle \alpha_0 P_0 + \alpha_1 P_1 + \dots + \alpha_{k-1} P_{k-1}, P_k \rangle =$$

$$= \langle \alpha_0 P_0, P_k \rangle + \langle \alpha_1 P_1, P_k \rangle + \dots + \langle \alpha_{k-1} P_{k-1}, P_k \rangle =$$

$$= \alpha_0 \underbrace{\langle P_0, P_k \rangle}_{0} + \alpha_1 \underbrace{\langle P_1, P_k \rangle}_{0} + \dots + \alpha_{k-1} \underbrace{\langle P_{k-1}, P_k \rangle}_{0} = 0$$

Bo P_i ortogonalne

4

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - c_1 \\ P_k(x) = (x - c_k) P_{k-1}(x) - d_k P_{k-2}(x) \end{cases}$$

$$c_k = \frac{\langle x P_{k-1}, P_{k-1} \rangle}{\langle P_{k-1}, P_{k-1} \rangle}$$

$$d_k = \frac{\langle P_{k-1}, P_{k-1} \rangle}{\langle P_{k-2}, P_{k-2} \rangle}$$

c_1 : $\langle P_0, P_0 \rangle \to 0$

$\langle x P_0, P_0 \rangle \to N$ obliczeń

$\}$ N obliczeń i 1 mnożenie

$P_1(x)$: $c_1 + 1$ obl

$$\langle P_k, P_k \rangle = \sum_{k=0}^{N} P_k^2(x_i) \to N+1 \text{ obl}, \; N+1 \text{ mnoż}, \; +1 \text{ na dzielenie} \to d_k$$

$$\langle x P_k, P_k \rangle = \sum_{k=0}^{N} x_i \underbrace{P_k^2(x_i)}_{\text{mamy}} \to N \text{ obl}, \; N \text{ mnożeń}, \; +1 \text{ na dzielenie} \to c_k$$

$P_k(x_0) \dots P_k(x_N)$ $2N+1$ obl, $2N+1$ mn

Do obliczenia P_k potrzebujemy $4N$ działań

Więc do obliczenia $P_0(x), P_1(x), \dots, P_N(x)$ potrzebujemy $4N^2$ działań

5
$$(1)\quad \begin{cases} Q_0(x) = 1, \quad Q_1(x) = x - c_1 & Q_1(x) = x - c_1 Q_0 \\ Q_k = \cancel{\alpha_k}(x - c_k) Q_{k-1}(x) - d_k Q_{k-2}(x) & (k = 2, 3, \dots) \end{cases}$$

$$\sum_{k=0}^{m} a_k Q_k(x) = \sum_{k=0}^{m} \left(B_k - (x - c_{k+1}) B_{k+1} + d_{k+2} B_{k+2} \right) Q_k =$$

$$= \sum_{k=0}^{m} B_k Q_k - \sum_{k=0}^{m-1} (x - c_{k+1}) B_{k+1} Q_k + \sum_{k=0}^{m-2} d_{k+2} B_{k+2} Q_k =$$

$$= B_0 \overset{=1}{Q_0} + B_1 Q_1 - (x - c_1) B_1 Q_0 + \sum_{k=2}^{m} B_k Q_k - \sum_{k=1}^{m-1} (x - c_{k+1}) B_{k+1} Q_k + \sum_{k=0}^{m-2} d_{k+2} B_{k+2} Q_k =$$

$$= B_0 + B_1 \underbrace{\left(Q_1 - \overbrace{(x - c_1) Q_0}^{Q_1} \right)}_{=0} + \sum_{k=2}^{m} B_k Q_k - (x - c_k) B_k Q_{k-1} + d_k B_k Q_{k-2} =$$

$$= B_0 + B_k \sum_{k=2}^{m} \underbrace{Q_k - (x - c_k) B_k Q_{k-1} + d_k B_k Q_{k-2}}_{=0 \text{ z } (1)} = B_0$$

$a_0 = a_1 = \dots = a_{m-1} = 0 \qquad a_m = 1$

$-10, -5, 0, 5, 10$

6

I Gram - Schmitt

$$f_0(x)=1, \quad f_1(x)=x, \quad f_2(x)=x^2$$

$$\begin{cases} g_0(x) = f_0(x) \\ g_k(x) = f_k(x) - \sum_{i=0}^{k-1} \frac{\langle f_k, g_i \rangle}{\langle g_i, g_i \rangle} \cdot g_i \end{cases}$$

$$g_0(x) = 1$$

$$g_1(x) = x - \frac{-10-5+0+5+10}{1+1+1+1+1} = x$$

$$g_2(x) = x^2 - \left(\frac{(-10)^2+(-5)^2+0^2+5^2+10^2}{5} + \frac{(-10)^3+(-5)^3+0^3+(5)^3+10^3}{(-10)^2+(-5)^2+0^2+5^2+10^2} \cdot x \right) =$$

$$x^2 - \frac{250}{5} = x^2 - 50 = g_2(x)$$

II ciąg P_0, P_1, P_2

$$\begin{cases} P_0 = 1 \\ P_1 = x - c_1 \\ P_k = (x - c_k) P_{k-1} - d_k P_{k-2}, \; k = 2, 3, \ldots \end{cases}$$

$$c_k = \frac{\langle x P_{k-1}, P_{k-1} \rangle}{\langle P_{k-1}, P_{k-1} \rangle}$$

$$d_k = \frac{\langle P_{k-1}, P_{k-1} \rangle}{\langle P_{k-2}, P_{k-2} \rangle}$$

$$P_0 = 1$$

$$c_1 = \frac{-10-5+0+5+10}{5} = 0$$

$$P_1 = x$$

$$c_2 = \frac{(-10)^3+(-5)^3+0^3+5^3+10^3}{(-10)^2+(-5)^2+0^2+5^2+10^2} = 0$$

$$d_2 = \frac{250}{5} = 50$$

$$P_2 = x \cdot x - 50 = x^2 - 50$$

7 $\quad x: \quad -10, -5, 0, 5, 10$

$h(x): \quad 3, -5, -1, -5, 3$

$P_0 = 1 \quad P_1 = x$

$P_2 = x^2 - 50$

$$\varphi_2^*(x) = \sum_{k=0}^{n} a_k f_k \qquad a_k = \frac{\langle h, P_k \rangle}{\langle P_k, P_k \rangle}$$

$$a_0 = \frac{3-5-1-5+3}{5} = -1$$

$$a_1 = \frac{(3\cdot(-10)) + ((-5)\cdot(-5)) + ((-5)\cdot 5) + (3\cdot 10)}{(-10)^2 + (-5)^2 + 5^2 + 10^2} = 0$$

$$a_2 = \frac{(3\cdot 50) + ((-5)\cdot(-25)) + 50 + ((-5)\cdot(-25)) + (3\cdot 50)}{50^2 + (-25)^2 + (-50)^2 + (-25)^2 + 50^2} = \frac{12}{175}$$

$$\varphi_2^*(x) = (-1)\cdot 1 + 0\cdot x + \frac{12}{175}(x^2-50) = -1 + \frac{12}{175}(x^2-50) = \frac{12}{175}x^2 - \frac{13}{7}$$