1

a) Wiemy:

$\Omega \in \Sigma$

$A \in \Sigma \Rightarrow A^c = (\Omega \setminus A) \in \Sigma$

$A = \Omega$

$\Omega \in \Sigma \Rightarrow (\Omega \setminus \Omega) = \emptyset \in \Sigma$

b) $A_i \in \Sigma,\ (i = 1, 2, \ldots) \Rightarrow \bigcup_i A_i \in \Sigma$

$A_k \in \Sigma \Rightarrow \Omega \setminus A_k \in \Sigma \Rightarrow \bigcup_k \Omega \setminus A_k \in \Sigma$

$\bigcup_k \Omega \setminus A_k \in \Sigma \Rightarrow \Omega \setminus \left(\bigcup_k \Omega \setminus A_k\right) \in \Sigma$

$$\boxed{\Omega \setminus \bigcap_k A_k = \left(\bigcap_k A_k\right)^c = \bigcup_k A_k^c = \bigcup_k \Omega \setminus A_k}$$

$\bigcup_k \Omega \setminus A_k = \Omega \setminus \bigcap_k A_k$

$\Omega \setminus \left(\bigcup_k \Omega \setminus A_k\right) \in \Sigma \Rightarrow \underbrace{\Omega \setminus \left(\Omega \setminus \bigcap_k A_k\right)}_{= \bigcap_k A_k} \in \Sigma \Rightarrow \bigcap_k A_k \in \Sigma$

2 $\Omega = \{a, b, c\}$

a) $\Sigma_1 = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$

$\Sigma_2 = \{\emptyset, \Omega, \{b\}, \{a, c\}\}$

$\Sigma_3 = \{\emptyset, \Omega, \{c\}, \{a, b\}\}$

$\Sigma_4 = \{\emptyset, \Omega\}$

b)

$\Omega = \{a, b, c\}$

$\Sigma = \{\emptyset, \Omega, \{a\}, \{b, c\}\}$

$X: a \to 0, b \to 1, c \to 1$

$Y: a \to 1, b \to 0, c \to 1$

$X^{-1}((-\infty, 0]) = \{a\} \in \Sigma$

$Y^{-1}((-\infty, 0]) = \{b\} \notin \Sigma$

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3 $\Omega = \{1, 2, 3, 4, 5\}$, $S = \{1, 4\}$

$\Sigma = \{\emptyset, \Omega, \underbrace{\{1, 4\}}_{S}, \underbrace{\{2, 3, 5\}}_{S^c}\}$

4

x_i	2	3	4	5
p_i	0,2	0,4	0,1	0,3

Dystrybuanta:

x	$(-\infty; 2]$	$(2; 3]$	$(3; 4]$	$(4; 5]$	$(5; +\infty)$
$F(x)$	0	0,2	0,6	0,7	1

Wartość oczekiwana

$$E(x) = \sum_i x_i \cdot p_i$$

$$E(x) = 0{,}4 + 1{,}2 + 0{,}4 + 1{,}5 = 3{,}5$$

5

x	$(-\infty; -2]$	$(-2; 3]$	$(3; 5]$	$(5; \infty)$
$F(x)$	0	0,2	0,7	1

x_i	-2	3	5
p_i	0,2	0,5	0,3

6 $E(aX+b) = \sum_k (ax_k+b)p_k = \sum_k ax_k p_k + \sum b\, p_k =$

$= a\sum_k x_k p_k + b \cdot \underbrace{\sum p_k}_{=1} = \cancel{a\, E_X}\; a \cdot E(X) + b$

7 $E(aX+b) = \int_{-\infty}^{\infty} (aX+b) \cdot f(x)\, dx = \int_{-\infty}^{\infty} (aX f(x) + b f(x))\, dx =$

$a\int_{-\infty}^{\infty} X f(x)\, dx + b \underbrace{\int_{-\infty}^{\infty} f(x)\, dx}_{=1} = a\, E(X) + b$

8

a) $B(p,q+1) = B(p,q)\dfrac{q}{p+q}$

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$$

$$B(p,q+1) = \int_0^1 t^{p-1}(1-t)^{q}\,dt = \left|\begin{matrix} (1-t)^{q} & t^{p-1} \\ -q(1-t)^{q-1} & \dfrac{t^{p}}{p} \end{matrix}\right| =$$

$$= \underbrace{\left[\frac{t^{p}}{p}(1-t)^{q}\right]_0^1}_{=0} - \int_0^1 -q(1-t)^{q-1}\frac{t^{p}}{p}\,dt = \frac{q}{p}\int_0^1 (1-t)^{q-1}t^{p}\,dt =$$

$$\left\{ t^{p} = t^{p} + t^{p-1} - t^{p-1} = t^{p-1} - t^{p-1}(1-t) \right\}$$

$$= \frac{q}{p}\int_0^1 (1-t)^{q-1}\left(t^{p-1} - t^{p-1}(1-t)\right)dt = \frac{q}{p}\int_0^1 (1-t)^{q-1}t^{p-1} - (1-t)^{q}t^{p-1}\,dt =$$

$$= \frac{q}{p}\left[\int_0^1 (1-t)^{q-1}t^{p-1}\,dt - \int_0^1 (1-t)^{q}t^{p-1}\,dt\right] = \frac{q}{p}B(p,q) - \frac{q}{p}B(p,q+1)$$

$$\frac{q}{p}B(p,q) - \frac{q}{p}B(p,q+1) = B(p,q+1)$$

$$\frac{q}{p}B(p,q) = \frac{p\,B(p,q+1) + q\,B(p,q+1)}{p}$$

$$q\,B(p,q) = (p+q)B(p,q+1)$$

$$B(p,q+1) = \frac{q}{p+q}B(p,q)$$

b) $B(p,q) = B(p,q+1) + B(p+1,q)$

$$B(p,q+1) = B(p,q)\frac{q}{p+q} \qquad \text{z } a)$$

$$B(p,q) = B(q,p)$$

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \left| \begin{matrix} 1-t=x & & t=1-x \\ dt=-dx & & \end{matrix} \right| =$$

$$= \int_0^1 (1-x)^{p-1} x^{q-1}\,dx = B(q,p)$$

$$B(p+1,q) = B(q,p+1) = B(p,q)\frac{p}{p+q}$$

$$B(p,q+1) + B(p+1,q) = \frac{q}{p+q}B(p,q) + \frac{p}{p+q}B(p,q) = \frac{p+q}{p+q}B(p,q) = B(p,q)$$

9 Pokazać: $\Gamma(p)\Gamma(q) = \Gamma(p+q)\, B(p,q)$

$$\Updownarrow$$

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q)$$

Indukcja po q

$q = 1$

$L = \dfrac{1!\cdot(p-1)!}{p!} = \dfrac{1}{p}$ $\qquad P = B(p,1) = \displaystyle\int_0^1 t^{p-1}(1-t)^0\,dt = \int_0^1 t^{p-1}\,dt =$

$\dfrac{t^p}{p}\Big|_0^1 = \dfrac{1}{p} = L$

dla $q=n$ działa $\qquad \dfrac{\Gamma(p)\Gamma(n)}{\Gamma(p+n)} = B(p,n)$

$q = n+1$

$L = \dfrac{(p-1)!\,n!}{(p+n)!}$

$$P = \int_0^1 t^{p-1}(1-t)^n\,dt = \begin{vmatrix} (1-t)^n & t^{p-1} \\ -n(1-t)^{n-1} & \frac{t^p}{p} \end{vmatrix} =$$

$$= \underbrace{\left[(1-t)^n \frac{t^p}{p}\right]\Big|_0^1}_{=0} + \frac{n}{p}\int_0^1 (1-t)^{n-1} t^p\,dt \overset{\text{zał}}{=} \frac{n}{p}\,\frac{p!\,(n-1)!}{(p+\underset{n+1}{\cancel{x}}-1)!} =$$

$$= \frac{(p-1)!\,n!}{(p+n)!} = L$$