6 (X_1, X_2) – dwuwymiarowa zmienna losowa

$$f(x_1, x_2) = \frac{1}{\pi} \quad \text{dla} \quad 0 < x_1^2 + x_2^2 < 1$$

Gęstości brzegowe x_1, x_2

x_1 – ustalone

$$x_1^2 + x_2^2 < 1$$

$$x_2^2 < 1 - x_1^2$$

$$-\sqrt{1-x_1} < x_2 < \sqrt{1-x_1}$$

$$f_1(x_1) = \int_{-\sqrt{1-x_1}}^{\sqrt{1-x_1}} \frac{1}{\pi}\, dx_2 = \frac{x_2^2}{\pi}\Bigg|_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} = 2 \cdot \frac{\sqrt{1-x_1^2}}{\pi}$$

x_2 – ustalone

$$x_1^2 + x_2^2 < 1$$

$$-\sqrt{1-x_2} < x_1 < \sqrt{1-x_2}$$

$$f_2(x_2) = \int_{-\sqrt{1-x_2}}^{\sqrt{1-x_2}} \frac{1}{\pi}\, dx_1 = 2\,\frac{\sqrt{1-x_1^2}}{\pi}$$

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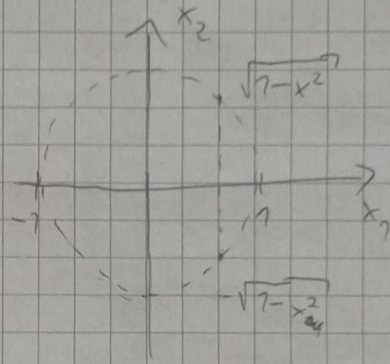
Niezależność:

$$f_1(x_1)\cdot f_2(x_2) = \frac{2\sqrt{1-x_1^2}}{\pi}\cdot\frac{2\sqrt{1-x_2^2}}{\pi} = \frac{4\sqrt{1-x_1^2}\sqrt{1-x_2^2}}{\pi^2} \neq \frac{1}{\pi} = f(x_1,x_2)$$

są zależne

Współczynnik korelacji:

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)\cdot V(Y)}}$$



$$-1 < x_1 < 1$$

$$-\sqrt{1-x_1^2} < x_2 < \sqrt{1-x_1^2}$$

$$cov(X,Y) = E\left((X-\overset{0}{E(X)})(Y-\overset{0}{E(Y)})\right)$$

$$E(X_1) = \int_{-1}^{1}\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} x_1 f(x_1,x_2)\,dx_2\,dx_1 = \int_{-1}^{1} x_1 \overbrace{\frac{2\sqrt{1-x_1^2}}{\pi}}^{0}\,dx_1 = 0$$

$$E(X_2) = \int_{-1}^{1}\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} x_2 f(x_1,x_2)\,dx_2\,dx_1 = \int_{-1}^{1}\left(\frac{x_2}{2\pi}\Big|_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}}\right)dx_1 =$$

$$= \int_{-1}^{1} 0\,dx_1 = 0$$

$$E(X_1^2)\ \iint x_1^2\frac{1}{\pi}\,dx_2\,dx_1 = \int x_1^2\frac{2\sqrt{1-x_1^2}}{\pi}\,dx_1 = \frac{1}{4}$$

$$E(X_2^2)\ \iint x_2^2\frac{1}{\pi}\,dx_2\,dx_1 = \int\frac{2\sqrt{1-x_1^2}\,(1-x_1^2)}{3\pi}\,dx_1 = \frac{1}{4}$$

$$E(X_1X_2) = \iint x_1x_2 f(x_1,x_2)\,dx_2\,dx_1 = \int x_1\int\frac{x_2}{\pi}\,dx_2\,dx_1 = \int x_1\,0\,dx_1 = 0$$

$$\rho(X_1,X_2) = \frac{0}{\sqrt{\frac{1}{16}}} = 0$$

8 $X_1 = Y_1 \cos Y_2$ $\quad 0 < Y_1 < 1$

$X_2 = Y_1 \sin Y_2$ $\quad 0 \leq Y_2 \leq 2\pi$ $\quad f(x_1, x_2) = \frac{1}{\pi}$

Znaleźć gęstość $g(y_1\ y_2)$

$$I = \begin{vmatrix} \cos Y_2 & -Y_1 \sin Y_2 \\ \sin Y_2 & Y_1 \cos Y_2 \end{vmatrix} = Y_1 (\cos^2 Y_2 + \sin^2 Y_2) = Y_1$$

$$g(y_1, y_2) = f(x_1(y_1, y_2), x_2(y_1, y_2)) \cdot |I| = \frac{1}{\pi} \cdot y_1 = \frac{y_1}{\pi}$$

$$g_1(y_1) = \int_0^{2\pi} \frac{y_1}{\pi}\, dy_2 = \frac{y_1}{\pi} \cdot y_2 \Big|_0^{2\pi} = 2y_1$$

$$g_2(y_2) = \int_0^1 \frac{y_1}{\pi}\, dy_1 = \frac{y_1^2}{2\pi} \Big|_0^1 = \frac{1}{2\pi}$$

$$g_1 \cdot g_2 = 2y_1 \cdot \frac{1}{2\pi} = \frac{y_1}{\pi} = g(y_1\ y_2)$$ są niezależne

9 $\quad X = (X_1, X_2, \dots, X_n)^T \qquad Y = (Y_1, \dots, Y_n)^T$

$$Y_1 = \overline{X} \quad , \quad Y_k = X_k - \overline{X} \quad , \quad k = 2, \dots, n$$

$$X_k = Y_k + Y_1 \qquad \text{dla } k = 2, \dots, n$$

$$nY_1 = \sum_{k=1}^{n} X_k = X_1 + X_2 + \dots + X_n = X_1 + (Y_2 + Y_1) + (Y_3 + Y_1) + \dots + (Y_n + Y_1)$$

$$nY_1 = X_1 + (Y_2 + Y_1) + (Y_3 + Y_1) + \dots + (Y_n + Y_1)$$

$$Y_1 - Y_2 - Y_3 - \dots - Y_n = X_1$$

$$D = \begin{vmatrix} 1 & -1 & -1 & -1 & \dots & -1 \\ 1 & 1 & & & & \\ 1 & & 1 & & 0 & \\ 1 & & & 1 & & \\ \vdots & & 0 & & \ddots & \\ 1 & & & & & 1 \end{vmatrix} = n$$

10 $$\sum_{k=1}^{n} (x_k - \mu)^2 = \sum_{k=1}^{n} (x_k - \mu)^2 + n(\bar{x} - \mu)^2$$

$$\sum_{k=1}^{n} (x_k - \mu)^2 = \sum_k \left((x_k - \bar{x}) + (\bar{x} - \mu)\right)^2 =$$

$$= \sum_k \left[(x_k - \bar{x})^2 + (\bar{x} - \mu)^2 + 2(x_k - \bar{x})(\bar{x} - \mu)\right] =$$

$$= \sum_k (x_k - \bar{x})^2 + n(\bar{x} - \mu)^2 + 2(\bar{x} - \mu)\sum_k (x_k - \bar{x}) =$$

$$= \sum_k (x_k - \bar{x})^2 + n(\bar{x} - \mu)^2 + 2(\bar{x} - \mu)\underbrace{\left(\sum_k x_k - n\bar{x}\right)}_{0} =$$

$$= \sum_k (x_k - \bar{x})^2 + n(\bar{x} - \mu)^2$$

11.E1 $\quad X_k \sim N(\mu, \sigma^2)$

$$M = \frac{n}{\sigma^2} \cdot (\bar{X} - \mu)^2 = \left(\frac{\sqrt{n}}{\sigma} \bar{X} - \frac{\sqrt{n}}{\sigma} \mu\right)^2$$

$$M_{X_k}(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

$$M_{\sum_{k=1}^{n} X_k}(t) = \exp\left(n\mu t + \frac{n}{2}\sigma^2 t^2\right)$$

$$M_{\frac{1}{n}\sum_{k=1}^{n} X_k}(t) = M_{\bar{X}} = \exp\left(\mu t + \frac{1}{2n}\sigma^2 t^2\right)$$

$$M_{\frac{\sqrt{n}}{\sigma}\bar{X}} = \exp\left(\mu \frac{\sqrt{n}}{\sigma} t + \frac{1}{2n}\sigma^2 \frac{n}{\sigma^2} t^2\right) = \exp\left(\mu \frac{\sqrt{n}}{\sigma} + \frac{1}{2} t^2\right.$$

$$M_{\frac{\sqrt{n}}{\sigma}\bar{X} - \frac{\sqrt{n}}{\sigma}\mu} = \exp\left(\mu \frac{\sqrt{n}}{\sigma} t + \frac{1}{2}t^2 - \frac{\sqrt{n}}{\sigma}\mu t\right) = e^{\frac{1}{2}t^2} \sim N(0,1)$$

$$\left(\frac{\sqrt{n}}{\sigma}\bar{X} - \frac{\sqrt{n}}{\sigma}\mu\right) \sim N(0,1) \Rightarrow \left(\frac{\sqrt{n}}{\sigma}\bar{X} - \frac{\sqrt{n}}{\sigma}\mu\right)^2 \sim \chi^2(1)$$

12 E1

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k \quad , \quad S^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k-\bar{X})^2 \quad \text{są niezależne}$$

$$n S^2 = \sum_{k=1}^{n}(X_k-\mu)^2 = \sum_{k=1}^{n}(X_k-\bar{X})^2 + n(\bar{X}-\mu)^2 \quad \Big/ \cdot \frac{1}{\sigma^2}$$

pokazać $\frac{nS^2}{\sigma} \sim \chi^2(n-1) \equiv \text{Gamma}\left(\frac{1}{2}, \frac{n-1}{2}\right)$

$$\frac{nS^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{k=1}^{n}(X_k-\mu)^2 = \frac{1}{\sigma^2}\sum_{k=1}^{n}(X_k-\bar{X})^2 + \frac{n}{\sigma^2}(\bar{X}-\mu)^2$$

S^2 $\bar{X}$ są niezależne

$$M_{\frac{1}{\sigma^2}\sum_{k=1}^{n}(X_k-\mu)^2} = M_{\frac{1}{\sigma^2}\sum_{k=1}^{n}(X_k-\bar{X})^2} \, M_{\frac{n}{\sigma^2}(\bar{X}-\mu)^2}$$

$$\Downarrow$$

$$M_{\frac{nS^2}{\sigma^2}} = \frac{M_{\frac{1}{\sigma^2}\sum_{k=1}^{n}(X_k-\mu)^2}}{M_{\frac{n}{\sigma^2}(\bar{X}-\mu)^2}} = \frac{M_Z}{M_Y}$$

$$Z = \frac{1}{\sigma^2}\sum_{k=1}^{n}(X_k-\mu)^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{n}{2}\right) \leftarrow \text{L4 Zad 6}$$

$$Y = \frac{n}{\sigma^2}(\bar{X}-\mu)^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) \leftarrow \text{Z 11}$$

$$M_{\frac{nS^2}{\sigma^2}}(t) = \frac{M_Z(t)}{M_Y(t)} = \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-\frac{1}{2}}} = (1-2t)^{-\frac{n-1}{2}} \sim \text{Gamma}\left(\frac{1}{2}, \frac{n-1}{2}\right)$$

13 E 1

$X_1, X_2 \sim U[1,2]$ $\quad Y_1 = 2X_1 + 2X_2 \quad Y_2 = X_1 X_2$

$E(X) = \frac{a+b}{2}$, $E(X^2) = \frac{b^2+ab+a^2}{3}$

$V(X) = \left(\frac{b-a}{12}\right)^2$ $\quad E(X+Y) = E(X)+E(Y)$

$V(aX) = a^2 V(X)$ $\quad V(X+Y) = V(X)+V(Y) + 2\,cov(X,Y)$

Dla X, Y niezależne $cov(X,Y) = 0$

$V(X) = E(X^2) - E^2(X)$

$E(Y_1) = E(2X_1+2X_2) = E(2X_1)+E(2X_2) = 2(E(X_1)+E(X_2)) = 2\left(\frac{2+1}{2}+\frac{2+1}{2}\right) = 6$

$V(Y_1) = V(2X_1) + V(2X_2) + \underbrace{Cov(2X_1, 2X_2)}_{=0} = 4V(X_1)+4V(X_2) =$

$4\left(2\cdot\left(\frac{2-1}{12}\right)^2\right) = \frac{2}{3}$

$E(Y_2) = E(X_1 X_2) = E(X_1)E(X_2) = \frac{3}{2}\cdot\frac{3}{2} = \frac{9}{4}$

$V(Y_2) = E(Y_2^2) - E^2(Y_2) = E(X_1^2)E(X_2^2) - E^2(X_1)E^2(X_2) =$

$= \left(\frac{4+2+1}{3}\right)^2 - \left(\frac{9}{4}\right)^2 = \frac{55}{144}$

14 E 1

$$\rho(Y_1 Y_2) = \frac{Cov(Y_1, Y_2)}{\sqrt{V(Y_1)\ V(Y_2)}}$$

$$Cov(Y_1, Y_2) = E\Big(Y_1 - E(Y_1)\Big)\Big(Y_2 - E(Y_2)\Big) =$$

$$= E\Big((Y_1 - 6)(Y_2 - \tfrac{9}{4})\Big) = E\Big(Y_1 Y_2 - 6 Y_2 - \tfrac{9}{4} Y_1 + \tfrac{27}{2}\Big) =$$

$$E(Y_1 Y_2) - 6\underbrace{E(Y_2)}_{=\frac{9}{4}} - \frac{9}{4}\underbrace{E(Y_1)}_{=6} + \frac{27}{2} = E(Y_1 Y_2) - \frac{27}{2} =$$

$$E(Y_1 Y_2) = E\Big((2X_1 + 2X_2)\ X_1 X_2 \Big) = E\big(2X_1^2 X_2 + 2X_1 X_2^2\big) =$$

$$= 2E(X_1^2)E(X_2) + 2E(X_1)E(X_2^2) = 2 \cdot \frac{7}{3} \cdot \frac{3}{2} + 2 \cdot \frac{3}{2} \cdot \frac{7}{3} = 14$$

$$= 14 - \frac{27}{2} = \frac{1}{2}$$

$$\rho = \frac{1}{2\sqrt{V(Y_1)\ V(Y_2)}} = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{2}{3} \cdot \frac{55}{144}}} = \frac{3\sqrt{330}}{55}$$