Measure Theory HW6

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1

We wish to show that $h \in L^1(\mu \times \nu)$. See that since $f \in L^1(\mu)$ and $g \in L^1(\nu)$. Then see that $|h(x,y)| = |f(x)| \cdot |g(y)|$. Since these measure space are $\sigma-$finite and the functions $|f|$ and $|g|$ are positive measurable, we can apply the Fubini- Tolleni theorem. Integrating over $X \times Y$, we get

$$\begin{aligned}
\int_{X\times Y} |h(x,y)| d(\mu \times \nu) &= \int_{X\times Y} |f(x)| \cdot |g(y)| d(\mu \times \nu) \\
&= \int_Y \left(\int_X |h^y(x)| d\mu(x) \right) d\nu(y) = \int_Y |g(y)| \cdot \left(\int_X |f(x)| d\mu(x) \right) d\nu(y) \\
&= \int_Y |g(y)| d\nu(y) \cdot \left(\int_X |f(x)| d\mu(x) \right) < \infty.
\end{aligned}$$

Thus $h \in L^1(\mu \times \nu)$. To calculate the integral, see that

$$\begin{aligned}
\int_{X\times Y} h(x,y) d(\mu \times \nu) &= \int_{X\times Y} f(x) \cdot g(y) d(\mu \times \nu) \\
&= \int_Y \left(\int_X h^y(x) d\mu(x) \right) d\nu(y) = \int_Y g(y) \cdot \left(\int_X f(x) d\mu(x) \right) d\nu(y) \\
&= \int_Y g(y) d\nu(y) \cdot \left(\int_X f(x) d\mu(x) \right),
\end{aligned}$$

which is the desired result.

2

The Fubini-Tolleni theorem requires the two spaces X and Y to both be $\sigma-$finite with respect to both μ and ν respectively. In this case see that the counting measure over $\mathbb{N}$ is indeed $\sigma-$finite, as $\mathbb{N} = \cup_{n=1}^{\infty} \{n\}$, where $\mu(\{n\}) = 1 < \infty$. Thus for $X = Y = \mathbb{N}$, and $\Sigma_1 = \Sigma_2 = P(\mathbb{N})$, and $\mu = \nu = m$, where $m(A)$ denotes the cardinality of the set A if it is finite and $+\infty$ otherwise. Note that $\Sigma_1 \otimes \Sigma_2 = P(\mathbb{N}^2)$, since $\Sigma_1 \otimes \Sigma_2 \subseteq P(\mathbb{N}^2)$, and for any $A \times B \in P(\mathbb{N}^2)$, we have $A \times B = \cup_{x \times y \in A \times B} \{x\} \times \{y\} \in \Sigma_1 \otimes \Sigma_2$, which gives us the other inequality.

We wish to see what sorts of functions over $\mathbb{N}$ are measurable. All functions are clearly $\Sigma_1 \otimes \Sigma_2$ measurable, as the entire power set constitutes the $\sigma-$algebra. See that functions can be indexed by two natural numbers, hence they can be described as $a_{m,n}$ Note that $(a_{m_0})_n = a_{m_0,n}$ fixes the first variable at some $m_0 \in \mathbb{N}$, and $a_m^{n_0} = a_{m,n_0}$ fixes the second variable at some $n_0 \in \mathbb{N}$. We also need to understand what integration looks like. Integration in this case is just summation, as we can see that integrating over a point gives us the value of the function at that point. Thus $\int_{\mathbb{N}} \{a_n\} dm(n) = \sum_{n=1}^{\infty} a_n$ and $\int_{\mathbb{N}^2} \{a_{m,n}\} dm(m,n) = \sum_{(m,n)\in\mathbb{N}^2} a_{m,n}$. We take a positive measurable function $\{a_{m,n}\}$, and the Fubini-Tolleni theorem tells us that

1. $$m \mapsto \sum_{n=1}^{\infty} (a_m)_n$$

   is $P(\mathbb{N})$ is measurable,

and

$$n \mapsto \sum_{m=1}^{\infty} a_m^n$$

is $P(\mathbb{N})$ is measurable;

2.

$$\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} dm(m,n) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_m^n\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} (a_m)_n\right).$$

The above statement means in the case of $\mathbb{N}^2$ is that we can switch the limits of double summations without issue.

Fubini's theorem can also be stated since our product measure consists of two $\sigma-$finite measures. We need to consider functions $\{a_{m,n}\} \in L^1(m^2)$, where we have $\sum_{(m,n)\in\mathbb{N}^2} |a_{m,n}| < \infty$. Then Fubini's theorem says that:

1. $(a_m)_n \in L^1(m)$, $m-$almost everywhere, and $a_n^m \in L^1(m)$, $m-$almost everywhere.

2.

$$m \mapsto \sum_{n=1}^{\infty} (a_m)_n$$

and

$$n \mapsto \sum_{m=1}^{\infty} a_m^n$$

are $L^1(m)$.

In the case of the counting measure it means that if the double summation is absolutely convergent, then so are the summations of all sections.

3

We need to find a function $g(x)$ that dominates $f_n = f(nx)$ for all $n \in \mathbb{N}$. Then see that for $x \in \left[0, \frac{1}{n}\right)$ we have $nx < 1$, which means that $\frac{\sin(n^2x^2)}{nx} < \frac{n^2x^2}{nx} < nx < 1$. For $x \in \left[\frac{1}{n}, \infty\right)$ see that $\frac{\sin(n^2x^2)}{nx} \leq \frac{1}{nx} \leq 1$. Also see that $\frac{cnx}{1+nx} \leq c$. Thus we using DCT we can see that for $a \leq 1$ we have $f_n < 1 + c$, and for $a > 1$ we have $f_n < 1 + c$ either way. Thus we can say that $\lim_{n\to\infty} \int_0^a f_n = \int_0^a \lim_{n\to\infty} f_n = \int_0^a 0 + c = ac$.

4

To find the value of $\int_0^1 (\int_0^1 \chi_D d\mu) d\nu$, see that we need to fix the second coordinate. Thus for a fixed $y \in [0,1]$ we have $\int_0^1 \chi_D(x,y) d\mu(x) = 0$, as this function is zero almost everywhere with respect to the Lebesgue measure. Thus we have $\int_0^1 0 d\nu(y) = 0$. Thus we have $\int_0^1 (\int_0^1 \chi_D d\mu) d\nu = 0$. To find the value of $\int_0^1 (\int_0^1 \chi_D d\nu) d\mu$, see that we need to fix the first coordinate. Thus for a fixed $x \in [0,1]$ we have $\int_0^1 \chi_D(x,y) d\nu(y) = 0$, as this function is 1 at $y = x$, on a set of cardinality 1. Thus we have $\int_0^1 1 d\mu(x) = 1$. Thus we have $\int_0^1 (\int_0^1 \chi_D d\nu) d\mu = 1$.

To find the double integral, we want to find the measure of D with respect to the product measure $\mu \times \nu$. Using the definition of the product measure, we know that $(\mu \times \nu)(D) = \inf\{\sum_{i=1}^{\infty} (\mu \times \nu)(B_i) : D \subseteq \cup_{i=1}^{\infty} B_i\}$. Note that for a box $B_i = L_i \times H_i$ we have $(\mu \times \nu)(B_i) = \mu(L_i)\nu(H_i)$. Note that (H_i) has to necessarily have non-zero $\mu(H_i)$ for some i, since if for all $i \in \mathbb{N}$ H_i had zero measure, it would only cover at most countable many points of $[0,1]$ which clearly cannot cover the entire space. Thus for some i $\mu(H_i) > 0$, which then means that $\nu(H_i) = \infty$. For this same i, $\mu(L_i) > 0$, thus we have $(\mu \times \nu)(B_i) = \infty$. Thus for an arbitrary covering we have $\sum_{i=1}^{\infty} (\mu \times \nu)(B_i) \geq (\mu \times \nu)(D)$. But since the term on the left is always infinite, we have $\int_{X\times Y} \chi_D d(\mu \times \nu) = \infty$.

5

We pick $X = \{1, 2, 3\}$, and $\mathcal{C} = \{\phi, \{1\}, \{2\}, \{3\}, X\}$. This is a monotone class, as it can be checked. However, $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{C}$, which means that $\mathcal{C}$ is a monotone class but not a $\sigma-$algebra.

6

1. We know that $\mu(E) = 0$. Also $f = (\Re(f)^+ - \Re(f)^-) + i(\Im(f)^+ - \Im(f)^-)$, which are all positive measurable functions. Then $\nu(E) = \int_E \Re(f)^+ d\mu \leq +\infty \cdot \mu(E) \leq 0$. Since measure cannot be negative, we have $\nu(E) = 0$. Use this same strategy for the other three functions $\Re(f)^-, \Im(f)^+$, and $\Im(f)^-$.

2. Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable disjoint collection of measurable subsets. Then define g as $g = \sum_{n=1}^{\infty} f\chi_{E_n}$. Then $f - g$ is zero on $\coprod_{n=1}^{\infty} E_n$. Then we must have

$$\int_{\coprod_{n=1}^{\infty} E_n} (f - g)d\mu = 0 \implies$$

$$\begin{aligned}\nu(\coprod_{n=1}^{\infty} E_n) = \int_{\coprod_{n=1}^{\infty} E_n} f d\mu &= \int_{\coprod_{n=1}^{\infty} E_n} \sum_{n=1}^{\infty} f\chi_{E_n} \\ &= \sum_{n=1}^{\infty} \int_{\coprod_{n=1}^{\infty} E_n} f\chi_{E_n} = \sum_{n=1}^{\infty} \int_{E_n} f d\mu \\ &= \sum_{n=1}^{\infty} \nu(E_n),\end{aligned}$$

   which means that $\nu(\coprod_{n=1}^{\infty} E_n) = \int_{\coprod_{n=1}^{\infty} E_n} f d\mu$.

3. Let $\varepsilon > 0$. We wish to find a $\delta > 0$ such that $\mu(E) < \delta \implies |\nu(E)| < \varepsilon$. See that $\nu(E) = \int_E (\Re(f)^+ - \Re(f)^-) + i(\Im(f)^+ - \Im(f)^-) d\mu = (I_1 - I_2) + i(I_3 - I_4)$, where $I_1 = \int_E \Re(f)^+ d\mu, I_2 = \int_E \Re(f)^- d\mu, I_3 = \int_E \Im(f)^+ d\mu, I_4 = \int_E \Im(f)^- d\mu$. Then see that $|\mu(E)| = \sqrt{(I_1 - I_2)^2 + (I_3 - I_4)^2} \leq \sqrt{I_1^2 + I_2^2 + I_3^2 + I_4^2}$. Take $\Re(f)^+$ integrated on X. Since f is integrable, $I_1 < \infty$. Then we find a simple measurable function h such that $0 \leq h \leq \Re(f)^+$ $\int_X \Re f^+ d\mu - \int_E h d\mu < \varepsilon/4$. Let H be the maximum of $h(x)$ on X. Choose $\delta_1 > 0$ such that $H\delta < \varepsilon/4$. Now see that for $\mu(E) < \delta$,

$$\int_E \Re(f)^+ d\mu = \int_E (\Re(f)^+ - h)d\mu + \int_E h d\mu \leq \varepsilon/4 + \varepsilon/4 < \varepsilon/2.$$

   Thus $I_1 < \varepsilon/2$. We can do the same to get I_2, I_3, I_4 in the same fashion. In all cases we have a value of δ_i, $i = 1, 2, 3, 4$. We pick $\delta = \min\{\delta_i\}$, then we are done as $|\nu(E)| \leq \sqrt{I_1^2 + I_2^2 + I_3^2 + I_4^2} < \varepsilon$.

7

Using the same product measure as the one in 2, we take a positive measurable function $\{a_{m,n}\}$, and the Fubini-Tolleni theorem tells us that

1. $$m \mapsto \sum_{n=1}^{\infty} (a_m)_n$$

   is $P(\mathbb{N})$ is measurable,

   and

$$n \mapsto \sum_{m=1}^{\infty} a_m^n$$

   is $P(\mathbb{N})$ is measurable;

2.

$$\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} dm(m,n) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_m^n\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}(a_m)_n\right).$$

Therefore we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}$.

8

Note that $|f(m,n)| = 1$ for $m = n$ and $m = n+1$. Then

$$\int_{\mathbb{N}^2} |f(m,n)| dm(m,n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |f(m,n)| = \sum_{p=1}^{\infty} |f(p,p)| + \sum_{q=1}^{\infty} |f(q+1,q)| = \infty.$$

Now we fix each variable and successively integrate. $\sum_{m=1}^{\infty} f(m,n_0) = f(n_0,n_0) + f(n_0+1,n_0) = 1 + (-1) = 0$. Thus $\sum_{n=1}^{\infty} 0 = 0$. If we fix m, and $m > 1$, then we have $\sum_{n=1}^{\infty} f(m,n) = f(m,m) + f(m,m-1) = 1 + (-1) = 0$. If $m = 1$, there is no n such that $m = n+1$, so $\sum_{n=1} f(1,n) = f(1,1) = 1$. Thus $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = 1 \neq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = 0$.

9

1. We have an increasing sequence of subsets $\{E_n\}$. We wish to show that continuity from below above to signed measures as well. By the Jordan decomposition theorem, we have $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are two positive measures, $\nu^+ \perp \nu^-$. Note that ν^+ lives on P, ν^- lives on N, where $P \cup N = X$, $P, N \in \mathcal{A}$, and $P \cap N = \phi$.

   We can see that $E_n = P_n \sqcup N_n$, where $P_n := (E_n \cap P)$ and $N_n := (E_n \cap N)$. Thus we have $E = \cup_{n=1}^{\infty} E_n = (\cup_{n=1}^{\infty} P_n) \cup (\cup_{n=1}^{\infty} N_n)$. Note that

   $$\nu(E) = \nu^+((\cup_{n=1}^{\infty} P_n) \cup (\cup_{n=1}^{\infty} N_n) - \nu^-((\cup_{n=1}^{\infty} P_n) \cup (\cup_{n=1}^{\infty} N_n) = \nu^+(\cup_{n=1}^{\infty} P_n) - \nu^-(\cup_{n=1}^{\infty} N_n).$$

   We will set $P_0 = \cup_{n=1}^{\infty} P_n$ and $N_0 = \cup_{n=1}^{\infty} N_n$. We will rewrite P_0 and N_0 as a disjoint union $\bigsqcup_{n=1}^{\infty} P_n'$ and $\bigsqcup_{n=1}^{\infty} Q_n'$ where $P_1' = P_1, Q_1' = Q_1$ and $P_n' = P_n \backslash (\cup_{i=1}^{n-1} P_i)$ and $Q_n' = Q_n \backslash (\cup_{i=1}^{n-1} Q_i)$. We denote by $P_n'' = \cup_{i=1}^{n} P_n'$ and $Q_n'' = \cup_{i=1}^{n} Q_n'$. Then using continuity from below we have $\nu^+(\cup_{n=1}^{\infty} P_n'') = \lim_{n\to\infty} \nu^+(P_n'') = \nu^+(P_0)$ and similarly $\nu^-(\cup_{n=1}^{\infty} Q_n'') = \nu^-(N_0)$. Thus we have $\nu(E) = \nu^+(P_0) - \nu^-(N_0) = \lim_{n\to\infty} \nu(E_n)$.

2. We know that $\nu(E_1)$ is finite. Thus we must have both ν^+ and ν^- are finite, as at most one of the two positive measures constituting the signed measure can be infinite. Thus we define P_n'' and Q_n'' in the same way as in the previous question, except that this sequence is decreasing. Using continuity from above for positive measures gives us the desired result.

10

Since $f(x) \cdot g(x) = 0$, we cannot have $f(x)$ and $g(x)$ nonzero for the same value of x almost everywhere. We divide our domain X into four disjoint parts $X = X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$, where X_1 is the set of all $x \in X$ where f is nonzero but g is zero; X_2 is the set of all $x \in X$ such that g is nonzero but f is zero; X_3 is the set of all $x \in X$ such that f and g are both zero, and X_4 is the set of all $x \in X$ such that both f and g are nonzero. We know that $\mu'(X_4) = 0$. Let $A = (X_1 \sqcup X_3 \sqcup X_4)$ and $B = X_2$. Clearly the two are disjoint and their union is X, by construction. We argue that this should show that $\mu' \perp \nu$. We try to evaluate $\mu'(E_B)$, for E_B a measurable subset in B. See that

$$\mu'(E_B) = \mu'(E_B | \cap (X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4)) = \int_{X_1\cap E_B} f d\mu + \int_{X_2\cap E_B} f d\mu + \int_{X_3\cap E_B} f d\mu + \int_{X_4\cap E_B} f d\mu.$$

We know that $E_B \cap X_1 = E_B \cap X_3 = E_B \cap X_4 = \phi$. Thus we have $\mu'(E_B) = \int_{E_B\cap X_2} f d\mu$. However, we know that f is zero on X_2, hence $\mu'(E_B) = 0$.

Similarly, see that $\nu(E_A)$ should also be zero for E_A a measurable set on A. So see that

$$\nu(E_A) = \nu(E_A| \cap (X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4)) = \int_{X_1 \cap E_A} g d\mu + \int_{X_2 \cap E_A} g d\mu + \int_{X_3 \cap E_A} g d\mu + \int_{X_4 \cap E_A} g d\mu.$$

We have $E_A \cap X_2 = \phi$, and we have that g is zero on X_1 and X_3. Thus $\int_{E_A \cap X_1} g d\mu = \int_{E_A \cap X_3} g d\mu = 0$. Note that on X_4 the function may be non-zero, but $X_4 \cap E$ is a null set as it is a subset of X_4, a $\mu-$null set. Thus $\int_{E_A \cap X_4} g d\mu = 0$. (This exact question has been solved in 6). Thus we have $\nu(E_A) = 0$. Therefore $\mu' \perp \nu$.

11

1. Since A is a positive set, we have $\nu(E) \geq 0 \; \forall E \subset A$. We fix $B \subseteq A \in \mathcal{A}$. We know that $\nu(B) = 0$. Also note that for any set $C \in \mathcal{A}$ where $C \subseteq B \subseteq A$, we can infer that $C \subseteq A \implies \nu(C) \geq 0$ from the positivity of A. Since our choice of subset C was arbitrary, we must have that B is a positive set. Hence every subset of a positive set is also a positive set.

2. Let $P := \cup_{n=1}^{\infty} P_n$. Let $E \subset P$. We can rewrite P as a disjoint union $\bigsqcup_{n=1}^{\infty} Q_n$, where $Q_1 = P_1$, and $Q_n = P_n \backslash (\cup_{i=1}^{n-1} P_i)$. Note that $Q_n \subset P_n$, hence from the previous section we can see that Q_n is also a positive set for all n. We now consider $E \cap Q_n$, which is clearly a subset of Q_n, hence $\nu(E \cap Q_n) \geq 0$. We now have

$$\nu(E \cap P) = \nu \left(E \cap \bigsqcup_{n=1}^{\infty} Q_n \right) = \sum_{n=1}^{\infty} \nu \left(E \cap Q_n \right) \geq 0,$$

which means that for any measurable subset E of P we have positive measure.