

Algebra 2 Homework 2

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See that for all $i \in \{1, 2, 3, 4, 5\}$ we have t'_i is a product of disjoint transpositions. We know that the order of t'_i is the least common multiple of all the cycles, which is clearly 2 in this case. Then we have $t'^2_i = 1$.

Now see that $t'_1 t'_2 = (1, 2)(3, 4)(5, 6) \circ (1, 4)(2, 5)(3, 6) = (1, 3, 5)(2, 6, 4)$, $t'_2 t'_3 = (1, 4)(2, 5)(3, 6) \circ (1, 3)(2, 4)(5, 6) = (1, 6, 2)(3, 4, 5)$, $t'_3 t'_4 = (1, 3)(2, 4)(5, 6) \circ (1, 2)(3, 6)(4, 5) = (1, 4, 6)(2, 3, 5)$, and $t'_4 t'_5 = (1, 2)(3, 6)(4, 5) \circ (1, 4)(2, 3)(5, 6) = (1, 5, 3)(2, 6, 4)$. These are all products of disjoint 3-cycles, thus they have order 3.

It is also interesting to see that $t'_1 t'_3 = (1, 2)(3, 4)(5, 6) \circ (1, 3)(2, 4)(5, 6) = (1, 4)(2, 3)$, $t'_1 t'_4 = (1, 2)(3, 4)(5, 6) \circ (1, 2)(3, 6)(4, 5) = (3, 5)(4, 6)$, $t'_1 t'_5 = (1, 2)(3, 4)(5, 6) \circ (1, 4)(2, 3)(5, 6) = (1, 3)(2, 4)$, $t'_2 t'_4 = (1, 4)(2, 5)(3, 6) \circ (1, 2)(3, 6)(4, 5) = (1, 5)(2, 4)$, $t'_2 t'_5 = (1, 4)(2, 5)(3, 6) \circ (1, 4)(2, 3)(5, 6) = (2, 6)(3, 5)$, and $t'_3 t'_5 = (1, 3)(2, 4)(5, 6) \circ (1, 4)(2, 3)(5, 6) = (1, 2)(3, 4)$, which is a product of disjoint 2-cycles.

We know from a previous assignment that S_n can be generated by elements of the f

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Let $\alpha \in \mathbb{Q}$. It satisfies a monic polynomial $f(x) \in \mathbb{Z}[x]$. Then we have

$$\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_0 = 0.$$

Let $\alpha = \frac{p}{q}$, with $(p, q) = 1$. Then we have

$$p^n + c_{n-1}p^{n-1}q + \cdots + c_0q^n = 0.$$

Reducing this equation modulo q , we have

$$p^n \equiv 0 \pmod{q}.$$

Since $(p, q) = 1$, we have $q = 1$. Thus $\alpha \in \mathbb{Z}$.

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We have $f(x) = x^5 - ax - 1$. For $a = 0$, $f(x) = (x - 1)(x^4 + x^3 + x^2 + x + 1)$, which is a non-trivial reduction. For $a = 2$, we have $f(-1) = 0$, so this is also reducible. For $a = -1$, we have $f(x) = x^5 - x - 1$ which can be factored as given by the problem. Now assume $a \neq -1, 0, 2$.

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See that $x^2 - 4x + 1$ is a polynomial that $2 + \sqrt{3}$ satisfies. The minimal polynomial if it is any smaller would have degree 1. But since $2 + \sqrt{3}$ is not rational, the degree of its minimal polynomial must be at least 2. Thus we have that $2 + \sqrt{3}$ has exactly degree 2 over \mathbb{Q} .

Consider the number field $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, which is a degree 3 extension. It is a \mathbb{Q} vector space, so it clearly contains the element $1 + \sqrt[3]{2} + \sqrt[3]{4}$. Therefore it must have degree at most 3 in \mathbb{Q} . Since it is not rational, its degree over \mathbb{Q} must be at least 2. Let us see if any quadratic polynomial can satisfy it. Let

$x^2 + ax + b \in \mathbb{Q}[x]$ be some rational polynomial. Assume $\alpha = \sqrt[3]{2}$. Assume that $(1 + \alpha + \alpha^2)$ satisfies this quadratic polynomial, so we must have

$$\begin{aligned}(1 + \alpha + \alpha^2)^2 + a(1 + \alpha + \alpha^2) + b &= 3\alpha^2 + 4\alpha + 5 + a + a\alpha + a\alpha^2 + b \\ &= (3 + a)\alpha^2 + (4 + a)\alpha + (5 + a + b).\end{aligned}$$

If this is to be 0, then we must have $a = -3$ and $a = -4$, which is absurd.

Therefore $1 + \sqrt[3]{2} + \sqrt[3]{4}$ must be of degree 3.

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If we can find a $a + bi \in \mathbb{Q}(i)$, such that $x^3 - q$ vanishes for $q \in \{2, 3\} \in \mathbb{Q}$, then we can reduce the polynomial. If this is possible, then we must have

$$(a + bi)^3 = (a^3 - 3ab^2) + i(3a^2b - b^3) \in \mathbb{Q},$$

which forces $3a^2b = b^3$. If $b = 0$, then it is equivalent to asking if a rational root for q exists, which is not true. Thus we must have $b \neq 0$. Then we have $3a^2 = b^2$, which has no rational solution since $\sqrt{3}$ is not rational. Therefore we must have that $x^3 - 2$ and $x^3 - 3$ are both irreducible, since they have no solutions on $\mathbb{Q}(i)$.

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