commandline leftmargin=10pt, rightmargin=10pt, innerleftmargin=15pt, middlelinecolor=black!50!white, middlelinewidth=2pt, frametitlerule=false, backgroundcolor=black!5!white, frametitle=Command Line, frametitlefont=white, frametitlebackgroundcolor=black!50!white, nobreak,

file innertopmargin=1.6nerbottommargin=0.8topline=false, bottomline=false, leftline=false, rightline=false, leftmargin=2cm, rightmargin=2cm, singleextra=[fill=black!10!white](P)++(0,-1.2em)rectangle(P-|O); [anchor=north west] at(P-|O); (O-|P)++(-3em,0)–++(3em,3em)–(P)–(P-|O)–(O)–cycle; (O-|P)++(-3em,0)–++(0,3em)–++(3em,0); , nobreak,

question innertopmargin=1.2nerbottommargin=0.8roundcorner=5pt, nobreak, singleextra=(P-|O)node[xshift=1em,anchor=west,fill=white,draw,rounded corners=5pt]Question ; ,

warning topline=false, bottomline=false, leftline=false, rightline=false, nobreak, singleextra=(P-|O)++(-0.5em,0)node(tmp1); (P-|O)++(0.5em,0)node(tmp2); [black,rotate around=45:(P-|O)](tmp1)rectangle(tmp2); at(P-|O)white!; [very thick](P-|O)++(0,-1em)–(O);

infotopline=false, bottomline=false, leftline=false, rightline=false, nobreak, singleextra=[black](P-|O)circle[radius=0.4em]; at(P-|O)white𝐢; [very thick](P-|O)++(0,-0.8em)–(O);

Homework 4 Measure Theor

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1

Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Suppose

$$\mu(\{x \in X : |f(x)| \geq \varepsilon\}) = 0,$$

for all $\varepsilon > 0$. Prove that $f = 0$ a.e.

Solution: For all $n \in \mathbb{N}$, we can say that $\mu(\{x \in X : |f(x)| \geq 1/n\}) = 0$. Define $\{x \in X : |f(x)| \geq 1/n\}$ as N_n. See that $N_n \subseteq N_{n+1}$. Consider $\mu(\cap_{n=1}^{\infty} N_n) = \mu(\lim_{n\to\infty} N_n) = \lim_{n\to\infty} \mu(N_n) = 0$. See that $\lim_{n\to\infty} N_n = \{x \in X : |f(x)| > 0\}$. This is precisely the definition of $f = 0$ a.e., proving the statement.

2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that the set

$$\{x \in \mathbb{R} : a \leq f(x) \leq b\},$$

is measurable for any $a < b$. Prove that f is measurable.

Solution: Consider the collection $\{[a - 1/n, b + n]\}_{n\in\mathbb{N}}$. Thus union of this collection is (a, ∞). Let $C_n := f^{-1}([a - 1/n, b + n])$. See that C_n is measurable for all n. Since measurable sets are closed under countable union, we have $\cup_{n=1}^{\infty} C_n = f^{-1}([a - 1/n, b + n]) = f^{-1}((a, \infty))$ is also measurable. Our choice of a was arbitrary, which implies that f is a measurable function.

3

Suppose that $f : [a, b] \to \mathbb{R}$ is a function such that

$$\{x \in [a, b] : f(x) = c\},$$

is measurable for each $c \in \mathbb{R}$. Is f necessarily measurable?

Solution: We know that $f^{-1}(\{c\})$ is measurable for all $c \in \mathbb{R}$. See that the function $f : [0, 1] \to \mathbb{R}$ where $f(x) = x$ if $x \in V$, where V is the Vitali set, and $x + 1$ otherwise. See that this function is actually one-one, so the pre-image has only one point, which is measurable. However, $f^{-1}([0, 1] \cap f([0, 1]))$ is V, a non-measurable set. Thus f needn't be measurable as a function even though the fibre of each point is measurable as a set.

4

5

6

Prove that an increasing function $f : [a, b] \to \mathbb{R}$ is measurable.

Solution: We know from the previous homework that monotone functions are measurable, since we can explicitly find $f^{-1}((a, \infty))$, where f is a monotone function. Since $f : [a, b] \to \mathbb{R}$ in this case is given to be increasing, it is also monotone. Therefore it must also be measurable.

$e^{i\pi}$

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8

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then $g \circ f$ is Lebesgue measurable. True or false?

Solution: Let $f : [0,2] \to [0,1]$ be the inverse of the function $K : [0,1] \to [0,2]$, which is given by $K(x) = \Lambda(x) + x$, where Λ is the Cantor function. It is strictly increasing, hence one-one. It is also surjective onto $[0,2]$ as it takes every value in that range due to the continuity of Λ and x. Thus K is biijective, and the function $f; = K^{-1}$ is also bijective and continuous, thus a measurable function.

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