Indian Statistical Institute, Bangalore
M. Math.
First Year, Second Semester
Functional Analysis

Home Assignment II — Due Date: 17 February 2024

1. Let $\mathcal{M}_n(\mathbb{C})$ be the linear space of all $n \times n$ complex matrices, and we define the map $\langle \cdot, \cdot \rangle : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ by $\langle A, B \rangle = tr(AB^*)$, where B^* is the adjoint of the matrix B and tr is the trace. Prove that $(\mathcal{M}_n(\mathbb{C}), \langle \cdot, \cdot \rangle)$ is a Hilbert space, and deduce that for two complex matrices A and B of order n we have
$$|tr(AB^*)|^2 \leq tr(AA^*)\ tr(BB^*).$$
2. Prove that for all x, y, z in an inner product space X we have the Appolonius identity:
$$t\|x-y\|^2 + (1-t)\|x-z\|^2 = \|x-u\|^2 + t(1-t)\|y-z\|^2,$$
where $u = ty + (1-t)z$, $t \in \mathbb{R}$. Deduce that
$$\|x-y\|^2 + \|x-z\|^2 = 2\left(\|x - \frac{1}{2}(y+z)\|^2 + \|\frac{1}{2}(y-z)\|^2\right).$$
Note: It generalizes the theorem with this name in plane geometry: if ABC is a triangle, and D is the mid-point of the side BC, then
$$(AB)^2 + (AC)^2 = 2[(AD)^2 + (BD)^2].$$
3. Let Y be a nonempty closed convex subset of a Hilbert space H and $x \in H$. Show that an element $y \in Y$ is a best approximation to x from Y if and only if
$$Re\langle x-y, z \rangle \leq Re\langle x-y, y \rangle$$
for all $z \in Y$.
4. Is the projection theorem valid in general inner product spaces? Justify your answer.
5. Prove that in a Hilbert space any decreasing sequence of nonempty closed convex bounded sets has a nonempty intersection. Note: This is a Cantor type of Theorem in Hilbert spaces.
6. Show that every separable infinite-dimensional Hilbert space is isomorphic to l^2.
7. Prove that a Hilbert space is finite dimensional if and only if every total orthonormal set in it is a Hamel basis.
8. Let H be a real Hilbert space and f be a bounded linear functional on H. Prove that the function $g(x) = \|x\|^2 - f(x)$ is lower bounded and attains its infimum at a unique point on any nonempty closed convex subset of H.
9. Find T^* if $T : \mathbb{K}^n \to \mathbb{K}^m$ is the linear operator defined by
$$(Tx)(i) = \sum_{j=1}^{n} k_{ij}x(j), \quad i = 1, 2, \ldots, m,\ x \in \mathbb{K}^n.$$
10. If T is a self-adjoint operator on a Hilbert space H, then prove that
$$\|T\| = \sup_{\|x\|=1} \langle Tx, x \rangle.$$

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