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1

1. Let $\mathcal{K}$ be a conjugacy class, let $x \in G$ be a representative. Then $\mathcal{K} = \{gxg^{-1} : x \in G\}$, and
$$\sigma(\mathcal{K}) = \{\sigma(gxg^{-1}) = \sigma(g)\sigma(x)\sigma(g)^{-1} : g \in G\}.$$
Since σ is an automorphism, it is a bijection. Then $\sigma(g)$ spans all of G. Then we have $\sigma(\mathcal{K}) = \sigma(\mathrm{Cl}(x)) = \mathrm{Cl}(\sigma(x))$. Thus this is also a conjugacy class.

2. See that in S_n $\mathcal{K} = \mathrm{Cl}((12))$, that is, all elements that switch two and only two elements. However, in general, the product of multiple transpositions also has order 2, but they are not transpositions. For example, in S_n, where $n \geq 4$, $(12)(34)$ has order 2, but are not $2-$cycles. For $n = 2$, there are no order 2 permutations that aren't $2-$cycles, so this is vacuously true for $n = 2$. For $n = 3$, the same principle applies since it is not possible to find two disjoint $2-$cycles. We have $|\mathcal{K}| = \binom{n}{2}$, while $|\mathcal{K}'| = \binom{n}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\ldots\binom{n-2\lfloor\frac{n}{2}\rfloor}{2}$, which is greater than $|\mathcal{K}|$ for $n \geq 2, n \neq 6$. Thus $|\mathcal{K}| \neq |\mathcal{K}'|$.

   See that for any transposition (ab), it swaps two elements. When an automorphism σ acts on (ab), we know that $a \mapsto a'$, $b \mapsto b'$, where $a' \neq b'$, since automorphisms are bijective. Then $\sigma(ab)$ is also a transposition.

3. We have $\sigma((12)) = (ab_2)$. See that $\sigma((12k)) = \sigma((1k)(12)) = \sigma((1k))\sigma((12)) = \sigma((1k))(ab_2)$. If $\sigma((1k))$ is disjoint with (ab_2), then the resultant element is of order 2. Thus $\sigma((1k))$ must overlap with (ab_2) for k. On the other hand, we have
$$\sigma((1k)(ij)) = \sigma((1k)(1j)(1i)) = \sigma((1k))(ab_i)(ab_j)(ab_i) = \sigma((1k))(b_ib_j).$$
Now, if $\sigma((1k))$ has elements in common with (b_ib_j) then we get a $3-$cycle, which cannot happen. Thus all elements of $\sigma((1k))$ overlap with (12) have the same element a. Thus $\sigma((1k)) = (ab_k)$.

4. Any permutation in S_n can be written as $(a_1 \ldots a_k) = (1a_k)\ldots(1a_1)$. Then any automorphism is decided by where it sends its generators, which is $(1k)$. First to choose a we have n choices. We have $n-1$ choices for b_2, and so on. Thus we have at most $n!$ automorphisms of n. However, we know that S_n has trivial centre which means that
$$\mathrm{Inn}(S_n) \cong \frac{S_n}{\{e\}} \cong S_n,$$
and $\mathrm{Inn}(S_n) \leq \mathrm{Aut}(S_n)$, which means that we must have at least $n!$ automorphisms! This means that $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n) = S_n$.

2

1. Let H be a non-abelian simple group. Consider $DH \trianglelefteq H$. Then since H is simple, we must have $DH = H$ or $DH = 0$. If $DH = 0$, then H would be abelian, which is not possible, thus H is perfect.

2. We know that $DH = H, DK = K$. Thus any term in H and K can be seen as an element of the type $h_1h_2h_1^{-1}h_2^{-1}$ for H and respectively for K. Thus we consider any word in $\langle H, K\rangle$. Then it is of the form $h_1k_1\ldots h_nk_n$, where $h_1, \ldots, h_n \in H, k_1, \ldots, k_n \in K$. We know that we can write any $h \in H$ and $k \in K$ can be written as an element of the commutator. The commutator is generated by such elements. Thus we have $\langle H, K\rangle = D\langle H, K\rangle$.

3. We propose that $D(g^{-1}Hg) = g^{-1}DHg$. Take any element of $D(g^{-1}Hg)$, which is of the form $(g^{-1}h_1g)(g^{-1}h_2g)(g^{-1}h_1^{-1}g)(g^{-1}h_2^{-1}g) = g^{-1}(h_1h_2h_1^{-1}h_2^{-1})g \in g^{-1}DHg$. These operations are all if and only if statements, hence $D(g^{-1}Hg) = g^{-1}DHg$. If $H = DH$, $D(g^{-1}Hg) = g^{-1}Hg$.

4. If G is simple, then either $DG = 0$ or $DG = G$. Then for abelian simple groups the maximal perfect subgroup is 0, and in the non-abelian case it is G itself. Both are clearly normal in G.

3

Since K is cyclic, it is generated by some element, say $x \in K$, and we know that the map φ_1 and φ_2 are decided entirely by where it sends the element x. We know that there exists $\sigma \in \text{Aut}(H)$ such that $\sigma\varphi_1(K)\sigma^{-1} = \varphi_2(K)$, so we have $\sigma\varphi_1(x)\sigma^{-1} = \varphi_2(x)^l$, for some $l \in \mathbb{Z}$. Then $k = x^t \in K$, we have $\sigma\varphi_1(k)\sigma^{-1} = (\sigma\varphi_1(x)\sigma^{-1})^t = \varphi_2(x)^a$, where $t \in \mathbb{Z}$.

Take the map $\psi : H \rtimes_{\varphi_1} K \to H \rtimes_{\varphi_2} K$ where $\psi(h,k) = (\sigma(h), k^a)$. To see that this is a homomorphism, we have

$$\begin{aligned}\psi((h_1,k_1) \circ_1 (h_2,k_2)) &= \psi((h_1\varphi_1(k_1)(h_2), k_1k_2)) \\ &= (\sigma(h_1)\sigma(\varphi_1(k_1)(h_2)), (k_1k_2)^a).\end{aligned}$$

Also, see that

$$\begin{aligned}\psi((h_1,k_1)) \circ_2 \psi((h_2,k_2)) &= (\sigma(h_1), k_1^a) \circ_2 (\sigma(h_2), k_2^a) \\ &= (\sigma(h_1)\varphi_2(k_1)^a(\sigma(h_2)), (k_1k_2)^a) \\ &= (\sigma(h_1)\sigma\varphi_1(k_1)\sigma^{-1}\sigma(h_2), (k_1k_2)^a) \\ &= (\sigma(h_1)\sigma(\varphi_1(k_1)(h_2)), (k_1k_2)^a).\end{aligned}$$

This shows that we have a homomorphism. We need to find the kernel of ψ. See that if $\sigma(h) = e_H$, then we must have $h = e_H$, since $\sigma \in \text{Aut}(H)$, thus it is bijective and preserves the identity. If $k^a = e_K$, then applying φ_2 on both sides, we have $\varphi_2(k^a) = \sigma\varphi_1(k)\sigma^{-1} = e_K \implies \varphi_1(k) = e_K$. Since φ_1 is injective, we must have $k = e_K$. Thus ψ is injective since the kernel is trivial.

We know that x is a generator of K, so $\varphi_1(x)$ generates $\varphi_1(K)$, and hence $\varphi_2(x)^a$ generates $\varphi_2(K)$. Thus there is some power $a' \in \mathbb{Z}$ such that $(x^a)^{a'} = e$ and $\sigma^{-1}\varphi_2(x)\sigma = \varphi_1(x^{a'})$. Then see that for any $(h,k) \in H \rtimes_{\varphi_2} K$, we have $(\sigma^{-1}(h), k^{a'})$ as the corresponding pre-image, which means that ψ is onto, as required.

4

We have $|G| = 75$. We clearly have a $5-$Sylow subgroup of order 25, as well as a $3-$Sylow group of order 3. Since the number of $5-$Sylow subgroups must be $5k+1$ and $5k+1|3$, we are forced to have a normal subgroup of order 25. Then let $H \trianglelefteq G$ where $|H| = 25$. Let K be some $3-$Sylow subgroup of order 3. Then by Lagrange's theorem, $|H \cap K| = 1$, so we can construct a semi-direct product of the groups.

We have that the number of $3-$Sylow subgroups of G are $n_3 = 3k+1$, where $3k+1|25$. The only choices are $n_3 = 1$, which would give us an abelian group, so assume that $n_3 = 25$, the only other choice. These subgroups are all conjugate to each other.

We have H of order 25, hence it is abelian as it is the square of a prime. Then it is either $\mathbb{Z}^{25}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. Since the units of $\mathbb{Z}^{25}$ have order $25 - 5 = 20$, we cannot have any non trivial map from K to H. Therefore we must have $H = \mathbb{Z}_5 \times \mathbb{Z}_5$. The automorphisms of this group are basically the invertible maps on $\mathbb{Z}_5 \times \mathbb{Z}_5$, which means $\text{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5) = \text{GL}_2(\mathbb{Z}_5)$. This group has order $(p^2-1)(p^2-p) = 480$.

See that $|K|||H|$, but only once. Using the previous result we want to show that $H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$, for some $\varphi_1, \varphi_2 : K \to \text{Aut}(H)$. See that any map from $k \to \text{Aut}(H)$ is determined solely by where it sends $a \neq e_K$. Then we have 480 choices, with one choices being where a is sent to the trivial automorphism. This corresponds to the direct product of H and K.

For $\varphi_1, \varphi_2 : K \to \text{Aut}(H)$, both non-trivial, we have $\varphi_1(K)$, and $\varphi_2(K)$ with three elements each. Hence there can only be one non-abelian group. We pick a, the generator of K to be sent to some non-trivial automorphism of H. That gives us $H \rtimes_\varphi K$, where $\varphi(a) \in \text{GL}_2(\mathbb{Z}_5)\backslash\{0\}$.

5

Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be such that $\alpha_i = 1, 2$ for all $1 \leq i \leq r$ and $p_i \nmid p_j^{\alpha_j} - 1$ for all i, j. Assume WLOG that $p_1 < \cdots < p_r$. We will proceed by induction on r, the number of prime factor For $r = 1$, we have order p and p^2, which are always abelian. For $r = 2$, we have four possibilities— $p_1p_2, p_1p_2^2, p_1^2p_2, p_1^2p_2^2$. The number of $p-$Sylow groups in any case must divide p_j^{α} for the other prime, and be 1 modulo p_i which is only possible if the number is 1, since $p_i \nmid p_j^{\alpha_j} - 1$. So we have that all these groups are the direct product of two abelian groups, which is thus abelian.

For $r \geq 3$, we consider the result that G is solvable. To see this, we use our assumption that all proper subgroups of G are abelian. Thus any normal series we have should have abelian quotient. Then DG, the commutator subgroup of G is abelian. We suppose that $|DG|$ isn't the power of a prime. Then we have $DG = H \times K$, where H, K are non-trivial with coprime order. Then we have G/H is abelian since it is a proper quotient of G, which contradicts the fact that DG is the smallest normal subgroup of G such that the quotient is abelian. Thus DG is a $p-$group. We pick a p_i-Sylow subgroup (for some p_i), say P, that contains DG. It contains DG, so P is normal.

For $p_j \neq p_i$, and P_j is a p_j-Sylow subgroup of G, then $DG \cdot P_j$ is an abelian normal subgroup of G (Since it is a subgroup containing DG). We have $\gcd(|DG|, |P_j|) = 1$, thus $P_j \trianglelefteq DG \cdot P_j$. Then we have that the p_j-Sylow subgroup is normal when $p_j \neq p_i$, and we already know that the p_i-Sylow subgroup is normal. Then we have $G = P_1 \times \cdots \times P_r$, where these are all normal. This is a product of groups of the order p_i or p_i^2, which is abelian, giving us the result.

The converse can be shown by proving the contrapositive. See that if we have $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, and $\alpha_i > 2$ for some i, or $p_i | p_j^{\alpha_j} - 1$ for some i, j, then we are done.

To check the first case, take G, which has order p^3. Let $H = \mathbb{Z}_{p^2}$ and $K = \mathbb{Z}_p$. Then we can find a non-trivial map φ from K to $\mathrm{Aut}(H)$ since $p | (p^2 - 1)(p^2 - p)$. Thus $H \rtimes_\varphi K$ will be a non-abelian group of order p^3.

Let us assume then, that $\alpha_i = 1$ or 2, but that $p_i | p_j - 1$, where $\alpha_j = 1$. Then there is a non trivial map $\varphi : K \to \mathrm{Aut}(H)$, where K is the group of order p_i and the group of automorphisms of H, the group of order p_j. Then we have $\mathbb{Z}_{\frac{n}{p_i p_j}} \times (H \rtimes_\varphi K)$ is a non-abelian group.

Now for $p_i | p_j^2 - 1$,, take the group of order 75 as we showed in the previous question was non-abelian. Generalising this, we have a non-trivial map $\varphi : K \to \mathrm{Aut}(H)$, where K is the group of order p_i and the group of automorphisms of H, the group of order p_j^2. The group of automorphisms will be $\mathrm{GL}_2(\mathbb{Z}_{p_j})$, of order $(p_j^2 - 1)(p_j^2 - p_j)$, which p_i clearly divides. Then we have $\mathbb{Z}_{\frac{n}{p_i p_j^2}} \times (H \rtimes_\varphi K)$, where φ is the non-trivial map.

Thus if n is abelian, the aforementioned condition must apply.