

# The last Home-work

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## 1

1. Since  $R$  is a PID, we know that  $(a, b) = (d)$ , for some  $d, a, b \in R$ . Then we have  $d = am + bn$ , for some  $m, n \in R$ . Now we have a vector  $v = [a, b]^T \in R^2 \setminus \{0\}$ . Then we show that there exists a  $2 \times 2$  matrix that does what we want by constructing one. Let the desired matrix be given by  $X = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$ . Now we have  $Xv = [x_{11}a + x_{21}b, x_{12}a + x_{22}b]^T = [d, 0]^T$ . Comparing terms, we have  $x_{12}a + x_{22}b = 0$ . Then we have  $x_{12}a = -x_{22}b$ , which implies that  $x_{12} \mid b$ , and  $x_{22} \mid a$ . It is easy to see that  $x_{12} = -a/d$  and  $x_{22} = b/d$  does the trick. For  $x_{11}a + x_{21}b = d$ , see that  $x_{11} = m$  and  $x_{21} = n$  are good choices, since their linear combination produces  $d$ . Thus see that

$$X = \begin{pmatrix} m & n \\ -b/d & a/d \end{pmatrix}$$

is a matrix that achieves the intended result.

2. The above result shall be of much use to us. We see that we want to send  $a_{11}$  to  $d$ , that is the gcd of  $a_{11}$  and  $a_{i1}$ , and  $a_{i1}$  to 0. Let  $m, n \in R$  such that  $d = a_{11}m + a_{i1}n$ . Define the matrix  $\tilde{X} = (x_{kl})$  thus—  $x_{11} = m, x_{1i} = n, x_{i1} = -a_{i1}/d, x_{ii} = a_{11}/d$ . Also we have  $x_{kk} = 1$  if  $k \neq 1, i$ . All other elements are 0. Then we have  $A' = \tilde{X}A = (a'_{kl})$ , where  $a'_{11} = a_{11}m + a_{i1}n, a'_{i1} = (-a_{i1})a_{11} + (a_{11})a_{i1} = 0$ , and  $a'_{kl} = a_{kl}$  for  $k \neq 1, i$ . We need to see that this here matrix is invertible. For  $\tilde{X}$  a  $m \times m$  matrix, we want  $\det \tilde{X}$ . We expand the determinant along the first row. Then we have  $\det \tilde{X} = m \det \tilde{X}[1|1] + (-1)^{i+1}((-1)^{i+1} \det \tilde{X}[1|i])$ .  $\tilde{X}[1|1]$  is a diagonal matrix with  $a_{11}$  on the  $a_{ii}$ th entry, and 1 otherwise on the diagonal. Thus  $\det \tilde{X}[1|1] = a_{11}$ .  $\tilde{X}[1|i]$  is a matrix with  $x_{i1}$  at the  $(i-1, 1)$ th entry, with every element below and above it zero. We take the determinant along this column, we have  $(-1)^{i-1+1}x_{i1} \cdot \det I_{m-2} = (-1)^i x_{i1}$ . Thus see that  $\det \tilde{X} = ma_{11} + (-1)^{2i+1}(-a_{i1}) = 1$ , means that  $\tilde{X}$  is invertible.
3. The above result and the result above that shall be of much use to us. If  $A = 0$ , then there is nothing to do. We then have  $A \neq 0$ . Without loss of generality, we take  $a_{11} \neq 0$ . This is because we can shift the row with a non-zero element to the top, then send the column with that element to the first column. Now using the above result, there is a  $\tilde{X}_1$  such that  $a_{21} = 0$ . The value of  $a_{11}$  changes. Now we have  $\tilde{X}_2$  that sends  $a_{31}$  to 0. We repeat this process till  $a_{i1} = 0$  for all  $i > 1$ . Now we have the first column all zero except for  $a_{11}$ . Let  $X_1 := \tilde{X}_{i-1} \dots \tilde{X}_1$ . Let us denote  $X_1 A$  by  $A'$ . Then consider  $A'^T$ . The first row now becomes the first column, and we can do the same thing that we did earlier, to reduce all elements below  $a_{11}$  in  $A'^T$  to 0. Let that operation be given by the matrix  $Y_1$ . Naturally, this matrix is the product of matrices obtained from  $j-1$  operations as given in the previous part. Then we take the transpose of the matrix  $Y_1 A'^T$  to have

$$A'' = (Y_1 A'^T)^T = (Y_1 (X_1 A)^T)^T = X_1^T A Y_1^T.$$

The matrix we have obtained has no non-zero elements below  $a_{11}$  or to its right.

Now note that we can modify our previous result to the second row. Earlier, we reduced all the leading terms of rows other than the first row to zero, then we did the same with columns. Here we reduce the second terms of the  $i$ th rows for  $i > 2$ , then do the same for the columns. We can find a  $X_2$  and  $Y_2$  both invertible such that  $X_2 A'' Y_2$  has all elements of the type  $a_{2j}$  and  $a_{i2}$  zero, for  $i, j \neq 2$ . Now we have a matrix where  $a_{11}$  and  $a_{22}$  may or may not be zero, but all elements sharing the same row or column with them is zero. We continue this process for the entire matrix, which gives us at every stage two invertible matrices that do the above reduction. To be precise,

we have  $X_1, \dots, X_t, Y_1, \dots, Y_t$  where  $t = \min(m, n)$ . We say  $X = X_1 \dots X_t$ , and  $Y = Y_t^{-1} \dots Y_1^{-1}$ . Then putting all of these results together, we get  $D = XAY^{-1}$ .

We do not know a priori if  $a_{11}|a_{22}| \dots |a_{tt}$ , but we can ensure this. We first make sure that  $a_{11}|a_{22}$ , then the general case is easy to see. We execute the elementary column operation  $C_1 \mapsto C_1 + C_2$ . Now using the previous result we can change  $a_{11}$  to  $\gcd(a_{11}, a_{22})$  and  $a_{21}$  goes to 0. All other terms remain unchanged. We know that  $d|a_{22}$ . We repeat this procedure for  $d_{ii}$  and  $d_{(i+1)(i+1)}$ , to get the desired result.

## 2

We are dealing with abelian groups, that is, we are dealing with  $\mathbb{Z}$ -modules. Applying the structure theorem on  $G$ , we must have

$$G \cong \bigoplus_{i=1}^k \frac{\mathbb{Z}}{(a_i)} \oplus \mathbb{Z}^d.$$

Since  $G$  is finite, we must have  $d = 0$ . By the Chinese remainder theorem, we can split the module into a direct sum of modules of the form  $\frac{\mathbb{Z}}{(p^r)}$ , where  $p$  is a prime and  $r > 0$ . We want to show that if a prime appears, it appears only once. Let us assume that there are two submodules of  $G$  of the form  $\frac{\mathbb{Z}}{(p^r)}$  and  $\frac{\mathbb{Z}}{(p^s)}$ . Then this means that we have more than  $p$  number of elements of order  $p$  in  $G$  (This is because elements of both the submodules have an element of order  $p$ . They generate a cyclic subgroup all of which have order  $p$ , but they have a trivial intersection). However, see that there are only at most  $p$  solutions of  $x^p - 1$  in  $F$ , which means that we have found more than  $p$  roots to this equation in a finite subgroup! This is clearly a contradiction, which means that each prime appears only once. Now note that since our module decomposition is such that  $a_1 | \dots | a_k$ , if any prime appears in the decomposition for some  $a_i$ , it has to appear in the decomposition for every  $a_j$ , for  $i < j \leq k$ . This then means that no primes can appear until  $a_k$ , since otherwise they would appear more than once. Thus we have  $a_1 = a_2 = \dots = a_{k-1} = 1$ , which means that  $G \cong \frac{\mathbb{Z}}{(a_k)}$ , which is a cyclic group.

## 3

We have  $M$  a  $R$ -module which has itself as a generating set. Then  $\pi : R^{\oplus M} \rightarrow M$  is the surjective map sending  $e_m$  to  $m$ . We see that  $\pi(e_{rm} - re_m) = \pi(e_{rm}) - r\pi(e_m) = rm - rm = 0$ . Also,  $\pi(e_{m_1+m_2} - e_{m_1} - e_{m_2}) = \pi(e_{m_1+m_2}) - \pi(e_{m_1}) - \pi(e_{m_2}) = (m_1 + m_2) - m_1 - m_2 = 0$ . Therefore we have  $e_{rm} - re_m \in \ker \pi$ , and  $e_{m_1+m_2} - e_{m_1} - e_{m_2} \in \ker \pi$ . Thus we have

$$(e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2}) \subseteq \ker \pi.$$

To see the other inclusion, let there be some  $re_m \in \ker \pi$ . Then we have  $\pi(re_m) = rm = 0$ . See that we can write  $re_m$  as

$$re_m = -((e_{rm} - re_m) + (e_{0+0} - e_0 - e_0)),$$

as  $e_{rm} = e_0$ . Thus for any general element  $\sum re_m \in \ker \pi$ , we can write the term as a linear combination of terms in  $(e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2})$ . Thus

$$(e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2}) = \ker \pi,$$

as required.

## 4

1. By the structure theorem for finitely generated modules, we have  $N \cong \frac{\mathbb{Z}}{(a_1)} \oplus \dots \oplus \frac{\mathbb{Z}}{(a_k)} \oplus \mathbb{Z}^d$ , where  $a_1 | a_2 | \dots | a_k$  and  $N$  is a finitely generated submodule in  $\mathbb{Q}$ . Since no element in  $\mathbb{Q}$  is a torsion element, we must have  $T(N) = \{0\}$ . Thus  $N \cong \mathbb{Z}^d$ . Let  $d > 1$ , say  $d = 2$ . Then we have a map  $f : \mathbb{Z}^2 \rightarrow N$  from  $\mathbb{Z}^2$  to  $N$ , where  $f(1, 0) = \frac{p_1}{q_1}$  and  $f(0, 1) = \frac{p_2}{q_2}$ , for some  $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$ . Then we have  $f(q_1 p_2, -q_2 p_1) = 0$ , which contradicts the linear independence of  $\mathbb{Z}$ . Then  $d = 0, 1$ , which means  $N$  is either zero or a cyclic module.

2. We have  $N_1, N_2$ , two non-zero submodules of  $\mathbb{Q}$ . We have  $\frac{p_1}{q_2} \in N_1, \frac{p_2}{q_2} \in N_2$ . See that  $\frac{p_1}{q_1}(q_1 p_2) = \frac{p_2}{q_2}(q_2 p_1)$ , thus  $p_1 p_2 \in N_1 \cap N_2$ . Thus for any non-zero submodule we can find a non-zero element they have in common.
3. Let  $f : \mathbb{Q} \rightarrow M \cong \bigoplus_{i=1}^k \frac{\mathbb{Z}}{(a_i)} \oplus \mathbb{Z}^d$  be a  $\mathbb{Z}$ -linear map. By the structure theorem, we can write  $M$  as given. Then we have  $f(1) = (m_1, \dots, m_k, n_1, \dots, n_d)$ , where  $m_1 \in \frac{\mathbb{Z}}{(a_1)}, \dots, m_k \in \frac{\mathbb{Z}}{(a_k)}$ , and  $n_1, \dots, n_d \in \mathbb{Z}^d$ . We dissect this function per component. Let us consider  $g : \mathbb{Q} \rightarrow \mathbb{Z}$ . We can use the well ordered principle to see that there is some  $r \in \mathbb{Q}$  such that  $g(r)$  is the lowest positive integer. Let us assume that this function is not uniformly zero, then see that there is some rational number  $r'$  such that  $g(r')$  is not zero. It could be positive, in which case we know that  $\mathbb{Z}^+ \cap g(\mathbb{Q})$  has a supremum; if it is negative,  $g(-r')$  has to be positive. Thus we can pick the smallest positive number in the range of  $g(\mathbb{Q})$ . We then see that  $g(\frac{r}{2}) = m$ , so we have  $g(r) = g(\frac{r}{2}) + g(\frac{r}{2}) \implies g(r) = 2g(\frac{r}{2}) > g(\frac{r}{2}) > 0$ , which contradicts the minimality. Thus this map must be uniformly zero. Then we must have that  $f : \mathbb{Q} \rightarrow \bigoplus_{i=1}^k \frac{\mathbb{Z}}{(a_i)}$ . Now we try to see that behaviour of  $f_i : \mathbb{Q} \rightarrow \frac{\mathbb{Z}}{(a_i)}$ , where  $f_i = f|_{\frac{\mathbb{Z}}{(a_i)}}$ . Let  $f_i(\frac{1}{k}) = n$ , for some  $k \in \frac{\mathbb{Z}}{(a_i)} \setminus \{0\}, n \in \frac{\mathbb{Z}}{(a_i)}$ . Then we have  $f_i(\frac{a_i}{a_i k}) = n \implies a_i f_i(\frac{1}{a_i k}) = n$ . However, the term on the left must be zero in  $\frac{\mathbb{Z}}{(a_i)}$ , which means that  $f_i(\frac{1}{a_i k}) = 0$ . However, since our choice of  $k$  was arbitrary, we must have  $f_i = 0$  uniformly.

Putting all of the above together, any map from  $\mathbb{Q} \rightarrow M$  must be the zero map.

- 4.
5. Let us assume that it is possible to have two non-zero  $\mathbb{Z}$ -modules  $M, N$  such that  $M \oplus N = \mathbb{Q}$ , then we must have  $M \cap N = \phi$ . However, we showed earlier that any two non-zero submodules of  $\mathbb{Q}$  must have non-empty intersection. Thus we cannot have  $\mathbb{Q}$  splitting into such a direct sum.
6. We want to show that any generating set can be reduced to fewer elements. We know that a finite number of elements clearly will not suffice, as  $\mathbb{Q}$  is not cyclic. Then the generating set must necessarily be infinite. For sake of contradiction, let  $S$  be a minimal set that generates  $\mathbb{Q}$ . Then take some  $a \in S$ . Then we have  $\frac{a}{2} = ak_0 + \sum_{i=1}^n a_i k_i$ , where  $k_i \in \mathbb{Z}, a_i \in S$ . Then  $a = a(2k_0) + \sum_{i=1}^n a_i(2k_i)$ , thus

$$am = \sum_{i=1}^n a_i(2k_i),$$

where  $m \in \mathbb{Z}$  is non-zero, as  $m = 1 - 2k_0$ .  $\frac{a}{m}$  can also be written as a linear combination of elements in  $S$ , so we have  $a = a(mr_0) + \sum_{i=1}^l b_i(mr_i)$ , thus we have

$$a = \sum_{i=1}^n a_i r_0(2k_i) + \sum_{i=1}^l b_i(mr_i),$$

which means that  $a$  can be discarded safely since  $a$  itself can be expressed as a combination of elements in  $S \setminus \{a\}$ . This contradicts the minimality of  $S$ , which is the required result.

## 5

$R^{\oplus X}$  is a free module, for some indexing set  $X$ .  $M$  is some submodule of  $R^{\oplus X}$ . We find a subset  $Y \subseteq X$  such that  $M \cap R^{\oplus Y}$  is free and  $B$  is a basis for this free module. Let  $(B, Y)$  be such a pair with the given partial order.  $\mathbb{T}$  is the poset of all such submodules in  $R^{\oplus X}$ .

1.  $X$  is non-empty. Then we can have  $M$  non-empty (if it is empty, there is nothing to prove). We take  $M$  non-empty, so we can pick a finite subset of  $M$ , elements  $m_1, \dots, m_k$ . Then let  $S = \{m_1, \dots, m_k\}$ . See that  $R^{\oplus S}$  is a finitely generated module over a PID, and thus so must be  $R^{\oplus S} \cap M$ , which is non-empty (since our chosen generators were taken from  $M$ ). Then this is a submodule of  $R^{\oplus S}$ , which means it is also finitely generated. We can use the structure theorem to claim that

$R^{\oplus S} \cap M \cong \bigoplus_{i=1}^l \frac{R}{(a_i)} \oplus R^d$ . However, since  $R^{\oplus X}$  and  $R^{\oplus S}$  are torsion free,  $M$  is also torsion free and thus so is  $R^{\oplus S} \cap M$ . Thus

$$R^{\oplus S} \cap M \cong R^d,$$

for some  $d \in \mathbb{N}$ . Thus we have  $R^{\oplus S} \cap M$  free, with it having some basis. Therefore we have that  $\mathbb{T}$  is non-empty.

2. We consider the chain  $\{K_i\}$  where  $K_i = (B_i, Y_i)$ . Let  $B_\infty = \bigcup_{i=1}^\infty B_i$ . Then we propose that  $Y_\infty = \bigcup_{i=1}^\infty Y_i$  is a basis for  $R^{\oplus B_\infty} \cap M$ , which should make it a free module. Take any element  $m \in R^{\oplus B_\infty} \cap M$ . Then it can be written as  $m = \sum r_m e_m$ , where  $e_m$  is a basis element in some  $B_i$ . Since this summation is finite, there is a maximum  $i$  such that  $B_i$  contains all the  $e'_m$ s. Then we know that  $B_i \subseteq B_\infty$ , which means that  $B_\infty$  is a generating set for  $R^{\oplus B_\infty} \cap M$ . Now to check linear independence, see that as any element  $m \in R^{\oplus B_\infty} \cap M$  has basis elements in one basis set  $B_i$  as we previously saw, it inherits its linear independence from  $B_i$ . Thus we have that  $(B_\infty, Y_\infty)$  is a maximal element for the chain  $\{K_i\}$ . By Zorn's lemma, we have that  $\mathbb{T}$  must have a maximal element.
3. Let us assume that  $(B, Y)$  is a maximal element of  $\mathbb{T}$ . If we have that there is a  $Y \subsetneq Y'$ , where  $Y' \setminus Y$  is finite, then we have that  $i : M \cap R^{\oplus Y} \hookrightarrow M \cap R^{\oplus Y'}$  where the cokernel is a submodule of the cokernel of  $i' : R^{\oplus Y} \hookrightarrow R^{\oplus Y'}$  as the former map is the restriction map of the latter. Then the cokernel is a submodule of the cokernel of the latter map, which is  $\frac{R^{\oplus Y'}}{R^{\oplus Y}}$ , which makes the cokernel a free module as well by the structure theorem for finitely generated modules on a PID. See that  $i$  splits  $M \cap R^{\oplus Y'}$ , as the surjective map  $\pi : M \cap R^{\oplus Y'} \rightarrow \text{coker } i$  is such that  $\pi \circ i = 1_{\text{coker } i}$ , which means that  $i$  splits. This means that we have  $M \cap R^{\oplus Y'}$  is a free module, which contradicts the maximality of  $(B, Y)$ . Thus either  $Y = X$ , or  $Y \setminus X$  has infinitely many elements. If we assume the latter, then we have that any finite subset of  $Y \setminus X$  can be included into  $Y$ , thus we must have inductively that all of  $X$  must be included in  $Y$ , which completes the argument.

## 6

1. Let  $f \in \text{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}}{(n)}, \frac{\mathbb{Q}}{\mathbb{Z}}\right)$ . Then we have an abelian group, as  $\text{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}}{(n)}, \frac{\mathbb{Q}}{\mathbb{Z}}\right)$  is a  $\mathbb{Z}$ -module ( $\mathbb{Z}$ -modules and abelian groups are the same). Elements in  $\frac{\mathbb{Q}}{\mathbb{Z}}$  are precisely the elements of  $\mathbb{Q} \cap [0, 1)$ . Then due to the  $\mathbb{Z}$ -linearity of  $f$ , we only need to ask where 1 is sent to. Let us say that  $f(1) = \frac{p}{q}$ . We see that  $n \cdot f(1) = f(n) = f(0) = 0$ , thus  $n \cdot \frac{p}{q} = 0$ . This then means that  $\frac{np}{q} \in \mathbb{Z} \implies q|n$ . This means that  $f(1) = \frac{p}{n}$ , for some  $p < n$ , which means that we have  $n$  choices at most. Now note that any map of the form  $f_i : \frac{\mathbb{Z}}{(n)} \rightarrow \frac{\mathbb{Q}}{\mathbb{Z}}$  where  $f_i(k) = \frac{ik}{n}$ , where  $i = 0, 1, \dots, n-1$  is in  $\text{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}}{(n)}, \frac{\mathbb{Q}}{\mathbb{Z}}\right)$ , which means that it must have at least  $n$  elements. Since the only number that is at least  $n$  but is at most  $n$  is  $n$ , we are done.
2. Since  $M$  is a finite  $\mathbb{Z}$ -module, it is finitely generated, hence it must be of the form  $\bigoplus_{i=1}^k \frac{\mathbb{Z}}{(a_i)} \oplus \mathbb{Z}^d$ , for some  $d \geq 0$ . Note that since the module is finite, we must have  $d = 0$ . Then we have  $M = \bigoplus_{i=1}^k \frac{\mathbb{Z}}{(a_i)}$ , where  $(a_1) \supseteq (a_2) \supseteq \dots \supseteq (a_k)$ . Let us denote  $M_i := \frac{\mathbb{Z}}{(a_i)}$ , so we have  $M = \bigoplus_{i=1}^k M_i$ . See that  $\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$  is a cyclic group of order  $a_i$  as given in the previous result. Moreover, we know explicitly what those maps are. Also see that in  $\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{R}}{\mathbb{Z}})$ , the map is determined solely by where 1 is taken to. If it is taken to an irrational number, then this map cannot be a cyclic group, as it will not have a period. Therefore these maps must be rational, and hence we defer to the rational case to see that the module  $\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{R}}{\mathbb{Z}})$  must have the same maps as  $\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$ , giving us the isomorphism.

We will state a few facts that will make proving that  $M \cong \text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}})$  easier. Note that  $\text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \bigoplus_{i=1}^k \text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$ , and that for modules  $M_1, \dots, M_k$  and  $N_1, \dots, N_k$  where  $M_i \cong N_i$  for all  $1 \leq i \leq k$ , we have  $\bigoplus_{i=1}^k M_i \cong \bigoplus_{i=1}^k N_i$ . By the universal property of direct sums, we know that the composition of isomorphic maps shall also be an isomorphic map.

We want to show that  $M_i \cong \text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$ , the required result should follow directly. We know that  $M_i = \{0, 1, \dots, a_i - 1\}$ , and  $\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}}) = \{\varphi_0^i, \dots, \varphi_{a_i-1}^i\}$ , where  $\varphi_m^i(1) = \frac{m}{a_i}$ . Now define

the map  $\sigma : M_i \rightarrow \text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$ , where  $\sigma(m_i) = \varphi_{m_i}^i$ . Our map is a  $\mathbb{Z}$ -module homomorphism. We define  $\theta : \text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}}) \rightarrow M_i$ , where  $\theta(\varphi_{m_i}^i) = m_i$ . Then see that  $\sigma \circ \theta = \iota_{\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})}$ , and  $\theta \circ \sigma = \iota_{M_i}$ . Thus we see that  $M_i \cong \text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$ .

Since  $\text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \bigoplus_{i=1}^k \text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$ , we have

$$M \cong \bigoplus_{i=1}^k M_i \cong \bigoplus_{i=1}^k \text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}}),$$

as required.

3. We have the map  $\delta : M \rightarrow \text{Hom}_R(\text{Hom}_R(M, N), N)$  where  $\delta(m) = (\varphi \mapsto \varphi(m))$ . Let us denote by  $f_m \in \text{Hom}_R(\text{Hom}_R(M, N), N)$ , where  $f_m(\varphi) = \varphi(m)$ . So  $\delta(m) = f_m$ . We want to see if  $\delta$  is a  $R$ -linear map.

We will check the behaviour of  $\delta$  with respect to addition and scalar multiplication. We have  $\delta(m+n) = f_{m+n}$ , which means that  $f_{m+n}(\varphi) = \varphi(m+n) = \varphi(m) + \varphi(n)$ . We know that  $f_m(\varphi) = \varphi(m)$  and  $f_n(\varphi) = \varphi(n)$ , so we have  $f_{m+n} = f_m + f_n$ . Also we know that  $\delta(m) = f_m$  and  $\delta(n) = f_n$ , so putting all this together we have  $\delta(m+n) = \delta(m) + \delta(n)$ . For scalar multiples, see that  $\delta(rm) = f_{rm}$ , where  $f_{rm}(\varphi) = \varphi(rm) = r\varphi(m)$ . See that  $f_m(\varphi) = \varphi(m)$ , so  $r\varphi(m) = rf_m$ . We know that  $r\delta(m) = rf_m$ , which is the required result. Thus we have that  $\delta$  is a  $R$ -linear map.

To see that  $\delta$  commutes with finite direct sums, we note that since finite direct sums in the first slot and  $\text{Hom}$  commute, we can see that

$$\text{Hom}_R(\text{Hom}_R(\bigoplus_{i=1}^k M_i, N), N) \cong \bigoplus_{i=1}^k \text{Hom}_R(\text{Hom}_R(M_i, N), N).$$

We specify an isomorphism between the two using the terminology defined earlier. For the ease of typing and the eyes of the reader, let

$$H_1 = \text{Hom}_R(\text{Hom}_R(\bigoplus_{i=1}^k M_i, N), N),$$

and

$$H_2 = \bigoplus_{i=1}^k \text{Hom}_R(\text{Hom}_R(M_i, N), N).$$

We have  $\delta_{\bigoplus_{i=1}^k M_i} : \bigoplus_{i=1}^k M_i \rightarrow H_1$ , where  $\delta_{\bigoplus_{i=1}^k M_i}(m_1, \dots, m_k) = (\varphi \mapsto \varphi(m_1, \dots, m_k))$ . Denote the map on the right by  $f_{m_1, \dots, m_k}$ . We have  $\delta_{M_i} : M_i \rightarrow \text{Hom}_R(\text{Hom}_R(M_i, N), N)$ , where  $\delta_{M_i}(m_i) = (\varphi \mapsto \varphi(m_i))$ . We denote the map on the right by  $f_{m_i}^i$ . We extend this map to  $H_2$ , where  $\delta|_{M_i} = \delta_{M_i}$ . Here, we abuse notation since more accurately  $\delta_{M_i}$  acts on  $M_k$ , while  $\delta$  acts on  $\bigoplus_{i=1}^k M_i$ . This product of maps is denoted by  $\bigoplus_{i=1}^k \delta_{M_i}$ .

Now we wish to show that the given diagram in the problem commutes. Define  $\psi_1 : H_1 \rightarrow H_2$ , and  $\psi_2 : H_2 \rightarrow H_1$ , two module maps. We know a map between the two exists such that their composition is the identity maps for  $H_1$  and  $H_2$ . Then we have  $\psi_1 \circ \delta_{\bigoplus_{i=1}^k M_i} : \bigoplus_{i=1}^k M_i \rightarrow H_2$ , where  $\psi_1 \circ \delta_{\bigoplus_{i=1}^k M_i}(m_1, \dots, m_k) = \psi_1(f_{(m_1, \dots, m_k)}) = (f_{m_1}^1, \dots, f_{m_k}^k)$ . Similarly, we have the map  $\psi_2 \circ \bigoplus_{i=1}^k \delta_{M_i} : \bigoplus_{i=1}^k M_i \rightarrow H_1$  where  $\psi_2 \circ \bigoplus_{i=1}^k \delta_{M_i}(m_1, \dots, m_k) = \psi_2(f_{m_1}^1, \dots, f_{m_k}^k) = f_{(m_1, \dots, m_k)}$ . See that this diagram commutes, which is the desired result.

4. We will use the previous two results liberally to get our result. We know  $M \cong \text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}})$ . Thus  $\text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}}), \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong M$ . Now we need to check that  $\delta_{\bigoplus_{i=1}^k M_i}$  is an isomorphism. Previously we established a correspondence between the elements of each  $M_i$  with its  $\text{Hom}$  module, we extend it to all of  $M$ . A general element of  $M$  is  $(m_1, \dots, m_k)$ , and a general map from  $M$  to  $\frac{\mathbb{Q}}{\mathbb{Z}}$  is  $(\phi_{m_1}^1, \dots, \phi_{m_k}^k)$ , where  $\phi_{m_i}^i : M_i \rightarrow \frac{\mathbb{Q}}{\mathbb{Z}}$  is the map where  $\phi_{m_i}^i(1) = \frac{m_i}{a_i}$ .

Now see that  $\delta_{\bigoplus_{i=1}^k M_i}(m_1, \dots, m_k) = f_{m_1, \dots, m_k}$ . This map takes any map in  $\text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}})$  and sends it to that map evaluated at  $(m_1, \dots, m_k)$ . That is,  $\delta_{\bigoplus_{i=1}^k M_i}(m_1, \dots, m_k) = (f_{m_1}, \dots, f_{m_k})$ , which is the map where each component sends any map in  $\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$  to that map evaluated at  $m_i$ , for  $i = 1, \dots, k$ . We wish to have a map from  $\text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}}), \frac{\mathbb{Q}}{\mathbb{Z}})$  to  $M$ , such that their composition is the identity map. Any element of  $\text{Hom}_{\mathbb{Z}}(M_i, \frac{\mathbb{Q}}{\mathbb{Z}})$  is of the form  $\phi_{\ell}^i$ , where  $\ell = 0, 1, \dots, a_i - 1$ . As mentioned before, it can be mapped to  $\ell \in M_i$ . Then we specify a map from  $M_i$  to  $\frac{\mathbb{Q}}{\mathbb{Z}}$ , which sends  $\ell$  to  $\phi^i(\ell)$ . Thus any element of  $\text{Hom}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}(M, \frac{\mathbb{Q}}{\mathbb{Z}}), \frac{\mathbb{Q}}{\mathbb{Z}})$  can be represented by  $(\phi_{\ell_1}^1, \dots, \phi_{\ell_k}^k)$ . We naturally see that we can send this to  $(\ell_1, \dots, \ell_k) \in M$ , which is the required inverse map.