Functional Analysis

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Solution of problem 1*:* We consider the operator $(\lambda^{-1}A - I)^{-1}$, for $\lambda \in \mathbb{C}\backslash\{0\}$. This is equal to $\sum_{n=0}^{\infty} A^n(\lambda^{-1})^n$ where the radius of convergence R is

$$\frac{1}{R} = \liminf_{n\to\infty} ||A^n||^{1/n} .$$

Thus we must have $|\lambda| > \liminf_{n\to\infty} ||A^n||^{1/n}$.

See that for $n > 1$, we have

$$\begin{aligned} A^n - \lambda^n I &= (A - \lambda)(A^{n-1} + \cdots + \lambda^{n-1} I)(A - \lambda I) \\ &= (A - \lambda I)(A^{n-1} + \cdots + \lambda^{n-1} I). \end{aligned}$$

If $A^n - \lambda^n I$ were invertible, then $A - \lambda I$ would have a left and a right inverse, hence would be invertible. Now if $A - \lambda I$ were not invertible, $A^n - \lambda^n I$ would not be invertible. Thus, if $\lambda \in \sigma(A)$, we would have $\lambda^n \in \sigma(A^n)$. Thus $|\lambda|^n \leq ||A^n||$, which gives us the other inequality. □

Solution of problem 2*:*

1. We shall prove the contrapositive. Let us take $\lambda \in \rho(S + T)$. Then $(T + S - \lambda I)$ has a bounded inverse. If we assume that $S - \lambda I$ is injective, then we can show that $(I - (T + S - \lambda I)^{-1}T)$ is injective. This is because if $S - \lambda I = (T + S - \lambda I)(I - (T + S - \lambda I)^{-1}T)$, which proves it. Thus we know that λ cannot lie in the point spectrum, if at all it lies in the spectrum. This is the required result.
2. We have that $S + T \in \mathcal{B}(X)$, and $-T \in \mathcal{K}(X)$. Then replacing S by $S + T$ and $S + T$ by $(S + T) - T$ we get that $\sigma(S + T)\backslash\sigma_p(S + T) \subseteq \sigma(S)$. Taking the union on both sides by the point spectrum of $S + T$ gives us the required answer.

□

Solution of problem 3*:*

1.
2. If $\lambda \in \sigma_p(ST)$, then we have that for some $v \neq 0 \in X$, $STv = \lambda v$. Then see that $y = Tv$ is an eigenvector for TS, such that $\lambda \in \sigma_p(TS)$.
3. Consider $D(x_1, \dots) = \left(x_1, \dots, \frac{x_n}{n}, \dots\right)$, and R is the right shift operator. Then see that $RD(x_1, \dots) = (0, x_1, \frac{x_2}{2}, \dots)$, whose

□

Solution of problem 4*:* If $p(0) = 0$ then take a bounded sequence of values (x_n). Now see that $p(T)$ is of the form $T(p'(T))$, where p' is some polynomial. Now we have that (Tx_n) has a convergent subsequence, so see that $p(T)(x_n) = (Tx_n)(p'(T)(x_n))$. If we consider

(x_{n_k}), the subsequence such that Tx_{n_k} converges, we have that $(Tx_{n_k})(p'(T)(x_{n_k}))$ must also converge, since (x_n) is bounded, and T is bounded, and p' is a polynomial, we have that $(p'(T)(x_{n_k}))$ is bounded, hence $p(T)$ is compact.

Conversely, if $p(T)$ is compact, and assume that $p(0) \neq 0$ then we have that $p(T) - p(0)I$ must be compact, since it is zero for $T = 0$. Then we have that $p(T) - p(0)I - p(T)$ must be compact, which implies that I must be a compact operator, which is impossible. Thus $p(0) = 0$. □

Solution of problem 5*:* Let T be a compact operator. Take $\{u_n\}$ be an orthonormal basis. It is bounded, thus $\{Tu_n\}$ has a convergent subsequence. Using the next to next problem (all but the last part) we can see that $Tx = \sum_{i=1}^{\infty} c_i \langle x, e_i \rangle e_i$, where $c_n := \langle Te_i, e_i \rangle$. This is a bounded operator, since T is compact and it sends the unit ball (hence the orthonormal basis in particular) to a paracompact, hence bounded set. Moreover, $\{Tu_i\}$ has a convergent subsequence. □

Solution of problem 6*:*

1. Using Parseval's identity, we have $\sum_{\alpha\in\Lambda} ||Tu_\alpha||^2 = \sum_{\alpha,\beta\in\Lambda} |\langle Tu_\alpha, u_\beta \rangle|^2 = \sum_{\alpha,\beta\in\Lambda} |\langle u_\alpha, T^* u_\beta \rangle|^2 = \sum_{\alpha\in\Lambda} ||T^* u_\alpha||^2$.
2. Take another orthonormal basis $\{v_\alpha\}$. Since we have
$$\sum_{\alpha\in\Lambda} ||Tu_\alpha||^2 = \sum_{\alpha\in\Lambda} ||T^* u_\alpha||^2 = \sum_{\alpha,\beta\in\Lambda} |\langle v_\alpha, T^* u_\beta \rangle|^2 = \sum_{\alpha,\beta\in\Lambda} |\langle Tv_\alpha, u_\beta \rangle|^2 = \sum_{\alpha\in\Lambda} ||Tv_\alpha||^2 ,$$
which is the required result.
3. We will approximate T using a sequence of finite rank operators. Let $\{T_n\}$ be such that $T_n(e_i) = Te_i$ for $i < n$, and 0 otherwise. These are clearly finite rank operators, and see that
$$||T_n - T|| \leq \sum_{i=1}^{\infty} ||(T_n - T)u_i||^2 = \sqrt{\left(\sum_{i>n} ||Te_i||^2 \right)} \to 0,$$
for $i \to \infty$.

□

Solution of problem 7*:*

1. Let us see that for n such that $|\langle x, u_n \rangle| < 1$, we have $|\langle x, u_n \rangle|^2 < 1$. We have
$$\begin{aligned} ||Tx|| &\leq \sum_{n=1}^{\infty} |k_n| \, |\langle x, u_n \rangle| \\ &\leq M \sum_{n=1}^{\infty} |\langle x, u_n \rangle| \leq M \, ||x|| , \end{aligned}$$
since we can replace all those n such that $|\langle x, u_n \rangle| < 1$ by 0, and consider the norm of that. Thus this is a bounded operator.
2. If T is compact, then we must have that Conversely, if $|k_n| \to 0$, then we can approximate this operator by finite rank operator, and it is easy to see that the tail that $T_n - T$ gives must converge. Thus it is compact.

3.

□

Solution of problem 8*:* If $\theta_1 \neq \theta_2$, then if $f(\theta_1) = f(\theta_2)$ we have

$$\sum_{k=0}^{n} c_k \left(e^{ik\theta_1} - e^{ik\theta_2}\right) = 0,$$

and if we pick $f \in \mathcal{A}$ such that $c_k \neq 0$ for at least one nonzero value of k, then by linear independence of $e^{ik\theta}$, we must have $\theta_1 = \theta_2$, a contradiction. Thus $\mathcal{A}$ separates points on K.

We can also show that for all θ, there is a $f \in \mathcal{A}$ such that $f(\theta) \neq 0$. Clearly, any non-zero constant function defined on $\mathcal{A}$ does not disappear.

See that $\iota(e^{i\theta}) = e^{-i\theta}$ is a continuous function. However, no element of $\mathcal{A}$ can approximate it since all functions on it can only have positive k. □