Algebra 2 Homework 5

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Solution of problem 1:

- BIT $\implies$ OMT: We first need to show that $\pi : X \to X/M$ is an open map, where M is a closed subspace. Let U be open in X. Then we want to understand $q(U)$. Let $x \in U$. Then there must be $r > 0$ such that $B(x, r) \subsetneq U$. Let $x' + M$ be such that $||(x + M) - (x' + M)|| < r$, that is, $\inf_{m \in M} ||(x - x') + m|| < r$. We must obtain some $m \in M$, where $||(x - x') + m|| < r$, since the inequality is strict. But now $x' - m \in B(x, r) \subset U$, and so $q(x' - m) \in q(U)$, and note that $q(x' - m) = x' + M$, which lies in $B(x' + M, r)$, a ball in $q(U)$. Thus we have an open map.
  Now we assume that bounded inverse theorem. Let $T : X \to Y$ be a bounded surjective map. Then let $M = \ker T$. Then we have the map $\bar{T} : X/M \to Y$ where $x + M \mapsto T(x)$. This map is clearly well-defined since if $x + M = x' + M$ then $x - x' \in M$. Then $T(x) = T(x')$. This is clearly a bijection by the first isomorphism theorem. Now by the Bounded Inverse Theorem, we have that $S = \bar{T}^{-1} : Y \to X/M$ exists and is a bounded and linear map. Hence it is also continuous.
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□

Solution of problem 2: Let $T : C^1[-1, 1] \to \mathbb{R}$, where $T(f) = f'$. This is clearly linear. Note that $||f|| := \sup_{x \in [-1,1]} |f(x)|$. Then see that $||T|| = \sup_{||f||=1} |Tf|$, which is unbounded. Then note for (u_n), a sequence of differentiable functions that converge uniformly to u, and $Tu_n = u_n'$ converges to f, then we have $Tu = u' = f$. Thus this a discontinuous linear operator than has closed graph. □

Solution of problem 3: □

Solution of problem 4:

1. Let X be a Banach space, and $p : X \to [0, \infty)$ is a semi-norm (a norm, but without the rule that $p(x) = 0 \implies x = 0$). If we take any absolutely convergent series $\sum_{n=1}^{\infty} x_n \in X$, we have
$$p\left(\sum_{n=1}^{\infty} x_n\right) \leq \sum_{n=1}^{\infty} p(x_n) \in [0, \infty],$$
then p is continuous. To prove this, we let $A_n = p^{-1}([0, n])$ and $F_n = \overline{A_n}$. See that A_n and F_n are symmetric convex sets since p is a seminorm. We have $X = \cup_{n=1}^{\infty} F_n$, and by Baire's theorem there must be some N such that F_N has non-empty interior. Therefore, there exist $x_0 \in X$, and $R > 0$ such that $B_R(x_0) \subset F_n$. By symmetry of

$F_N, B_R(-x_0) = -B_R(x_0) \subset F_N$. If $||x|| < R$, then $x + x_0 \in B_R(x_0), x - x_0 \in B_R(-x_0)$, so we have $x \pm x_0 \in F_N$. Since F_N is convex, we have $\frac{1}{2}(x_0 + (-x_0)) = 0 \in F_N$. Then we have $B_R(0) \subset F_N$. We want to show that $B_R(0) \subset A_N$. Suppose $||x|| < r < R$. Fix $0 < q < 1 - \frac{r}{R}$, so that $\frac{1}{1-q} \cdot \frac{r}{R} < 1$. Then $y = \frac{R}{r} x \in B_R(0) \subset F_N = \overline{A_N}$. Thus there is $y_0 \in A_N$ such that $||y - y_0|| < qR$, so $q^{-1}(y - y_0) \in B_R$. Choose a $y_1 \in A_N$ such that $\left|\left|q^{-1}(y - y_0) - y_1\right|\right| < qR$, so $||y - y_0 - qy_1|| < q^2 R$. By induction we have (y_n) such that

$$\left|\left| y - \sum_{k=0}^{n} q^k y_k \right|\right| < q^n R,$$

for all $n \geq 0$, thus we have $y = \sum_{k=0}^{\infty} q^k y_k$. We see that $||y_k|| \leq R + qR$ for all k, so y as a series exists since the constructed series is absolutely convergent. Now, using the subadditivity that was given in the hypothesis, we have

$$p(y) = p\left(\sum_{k=0}^{\infty} q^k y_k\right) \leq \sum_{k=0}^{\infty} q^k p(y_k) \leq \frac{1}{1-q} N,$$

and hence $p(x) \leq \frac{N(1+\varepsilon)}{R} ||x||$, which proves the continuity.

2. (a) $T :\to Y$ is a bounded linear operator. Then suppose the $T(U)$, where U is the open unit ball, is open. In that case, let V be some open neighbourhood of X. Then for $x \in V$, we have that some ball of radius r centered at x is in V. We can then see that $T(rU + x) \subset V$, which means that we only need to see that $T(U)$ is open.

   Define $p(y) := \inf\{||x|| \mid Tx = y\}$. We need to show that this is a seminorm with countable subadditivity. Let $\alpha \neq 0$ be a scalar. Then we have $\{x \mid x \in X, Tx = \alpha y\} = \{\alpha x \mid x \in X, Tx = y\}$, and taking infimums, we have $p(\alpha y) = |\alpha| p(y)$. For $\alpha = 0$, this can be easily checked. Let $\sum_n y_n$ be a convergent series. We need to show that $p(\sum_n y_n) \leq \sum_n p(y_n)$, so we assume $\sum_n p(y_n)$ is finite, since if it was infinite there would be nothing to prove. Fixing some $\varepsilon > 0$, we take a sequence (x_n) in X such that $Tx_n = y_n$, and $||x_n|| < p(y_n) + 2^{-n}\varepsilon$. Then we have $\sum_n ||x_n|| < \sum_n p(y_n) + \varepsilon$, which is finite. Since in Banach spaces absolutely convergent series are also convergent, we have $\sum_n x_n$ converges. Then $T(\sum_n x_n) = \sum_n Tx_n = \sum_n y_n$, so

   $$p\left(\sum_n y_n\right) \leq \left|\left|\sum_n x_n\right|\right| \leq \sum_n ||x_n|| < \sum_n p(x_n) + \varepsilon.$$

   Therefore subadditivity is confirmed, so by Zabreiko's lemma, we have

   $$T(U) = \{y : y \in Y, Tx = y \text{ for some } x \in U\} = \{y : y \in Y, p(y) = 1\},$$

   which is open. This proves the open mapping theorem.

   (b) If we have a one-one onto linear mapping from a topological space to another is a homeomorphism if and only if it is continuous and open. Using the open mapping theorem, we have that this map is open, and continuous. Thus T^{-1} exists and must be bounded, as it is a homeomorphism too. This proves the bounded inverse theorem.

(c) Let $\mathcal{F}$ be a non-empty family of bounded linear operators from a Banach space X to a normed space Y, where $\sup\{||Tx|| \mid T \in \mathcal{F}\}$. Now, let $p(x) := \sup\{||Tx|| \mid T \in \mathcal{F}\}$. See that $p(\alpha x) = |\alpha|\, p(x)$ from definition. For $\sum_n x_n$ a convergent series, we have
$$\left|\left|T\left(\sum_n x_n\right)\right|\right| = \left|\left|\sum_n x_n\right|\right| \leq \sum_n ||Tx_n|| \leq \sum_n p(x_n),$$
which implies that $p\left(\sum_n x_n\right) \leq \sum_n p(x_n)$. In particular, we have $p(x_1 + x_2) \leq p(x_1) + p(x_2)$. Now, since p is continuous, we have $\delta > 0$ such that $p(x) \leq 1$ for $||x|| \leq \delta$. Whenever $x \in X, ||x|| = 1$ we have $p(x) \leq \delta^{-1}$, which means that $||T|| \leq \delta^{-1}$ for each $T \in \mathcal{F}$.

(d) Let, $T : X \to Y$. Now pick $p(x) = ||Tx||$. If p was continuous, then there would be a neighbourhood U of 0 such that the set $p(U)$ is bounded, which implies that $T(U)$ is bounded, and this implies continuity of T. p is a semi-norm, so we need to check its continuity. We only need to check that this has countable subadditivity. Take $\sum_n x_n$, a convergent series, then we can assume that $\sum_n ||Tx_n||$ is finite without loss of generality. Now see that if $\sum_n ||Tx_n||$ is convergent, then so is $\sum_n Tx_n$ is convergent in Y, as it is complete. Since $\sum_{k=1}^n x_k \to \sum_n x_n$, then we have $T\left(\sum_{k=1}^n x_k\right) \to T\left(\sum_n x_n\right)$. Then, from hypothesis we have $T\left(\sum_n x_n\right) = \sum_n Tx_n$. Taking norm, we have the norm subadditivity. Since p is continuous by Zabreiko's lemma, we have proven the closed graph theorem.

□

Solution of problem 5*:* We know that $x_n \xrightarrow{w} x$, so for any $f \in X^*$, we have $f(x_n) \to f(x)$. Using the Hahn-Banach theorem, we can find a linear functional f where $||f|| = 1$, and $f(x) = ||x||$. Then we have
$$||x|| = \lim_{n\to\infty} |f(x_n)| \leq \liminf_{n\to\infty} ||f||\, ||x_n|| = \liminf_{n\to\infty} ||x_n||,$$
as desired. □

Solution of problem 6*:* X is a normed linear space. We say that (x_n), a sequence in X is *weakly Cauchy* if the sequence (fx_n) converges for all $f \in X^*$. X is *weakly complete* if all weakly Cauchy sequences converge weakly.

Let X be reflexive. Let (x_n) be a weakly Cauchy sequence in X. Pick $f \in X^*$. Then since $(f(x_n))$ is a Cauchy sequence in $\mathbb{C}$, we have that $f(x_n) \to \alpha(f)$, where $\alpha \in X^{**}$. We do not know what element in X this element corresponds to, but we know that since X is reflexive we can think of it as an element of the bidual acting on f. For any $f \in X^*$ we have $E_{x_n}(f) = f(x_n) \to \alpha(f)$. We define α as the element of X^{**}, as the limit of $(f(x_n))$ as $n \to \infty$. Now see that $|\alpha(f)|$ is bounded as for each $f \in X^*$, we have that pointwise the set $(f(x_n))$ is bounded for each $f \in X^*$. Then by the Uniform Bounded Principle, we have (x_n) must be bounded in X^{**}. Since $||x_n||_{X^{**}} = ||x_n||$, we know that (x_n) is bounded in X by $M > 0$. Then,
$$|f(x_n)| \leq M\, ||f|| \implies \alpha(f) \leq M\, ||f||.$$

Since X is reflexive, $\alpha \in X$. Then by definition for each $f \in X^*$ we have $f(x_n) \to \alpha(f) = f(\alpha)$, which confirms weak converges. □