# Algebra 2 Homework 5

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*Solution of problem* 1: • BIT $\implies$ OMT: We first need to show that $\pi : X \to X/M$ is an open map, where $M$ is a closed subspace. Let $U$ be open in $X$. Then we want to understand $q(U)$. Let $x \in U$. Then there must be $r > 0$ such that $B(x,r) \subsetneq U$. Let $x' + M$ be such that $||(x + M) - (x' + M)|| < r$, that is, $\inf_{m \in M} ||(x - x') + m|| < r$. We must obtain some $m \in M$, where $||(x - x') + m|| < r$, since the inequality is strict. But now $x' - m \in B(x,r) \subset U$, and so $q(x' - m) \in q(U)$, and note that $q(x' - m) = x' + M$, which lies in $B(x' + M, r)$, a ball in $q(U)$. Thus we have an open map.

Now we assume that bounded inverse theorem. Let $T : X \to Y$ be a bounded surjective map. Then let $M = \ker T$. Then we have the map $\bar{T} : X/M \to Y$ where $x + M \mapsto T(x)$. This map is clearly well-defined since if $x + M = x' + M$ then $x - x' \in M$. Then $T(x) = T(x')$. This is clearly a bijection by the first isomorphism theorem. Now by the Bounded Inverse Theorem, we have that $S = \bar{T}^{-1} : Y \to X/M$ exists and is a bounded and linear map. Hence it is also continuous.

• CGT $\implies$ OMT: Let $T : X \to Y$ is continuous and onto. Then the map $\pi : X \to X/\ker T$ is open, onto, and continuous. Then the map $\overline{T} := \pi \circ T$ is a continuous linear bijection. The graph of $\overline{T}$ must be closed. The map that switches the two coordinates gives us the graph of $\overline{T}^{-1}$ which must also be closed, since the map switching coordinates is a homeomorphism. By closed map theorem, $\overline{T}^{-1}$ is continuous. For any open $U \subseteq X$, $T(U)$ must be open since it is the pre-image of $\pi(U)$, which is open, and $\overline{T}^{-1}$ is continuous. This proves that the map $T$ is open.

• CGT $\implies$ UBP: Consider the map $\Phi : X \to \ell^\infty(\mathcal{F})$, where $x \mapsto (f(x))_{f \in \mathcal{F}}$. Since $f$ is a continuous linear functional, $\Phi$ has a closed graph.

Thus it is continuous, thus bounded so

$$\sup_{||x|| \leq 1} ||\Phi(x)|| = \sup_{f \in \mathcal{F}} ||f||$$

must be finite. Thus CGT implies UBP.

$\square$

*Solution of problem* 2: Let $T : C^1[-1,1] \to \mathbb{R}$, where $T(f) = f'$. This is clearly linear. Note that $||f|| := \sup_{x \in [-1,1]} |f(x)|$. Then see that $||T|| = \sup_{||f|| = 1} |Tf|$, which is unbounded. Then note for $(u_n)$, a sequence of differentiable functions that converge uniformly to $u$, and $Tu_n = u_n'$ converges to $f$, then we have $Tu = u' = f$. Thus this a discontinuous linear operator than has closed graph. $\square$

*Solution of problem* 3: Take $\ell_{00}^1$. This is the subspace of $\ell^1$ which have only finitely many non-zero terms. Consider $\Lambda_n(x) = \sum_{k=0}^n k \cdot x_k$. Then $\{\Lambda_n : n \in \mathbb{N}\}$, is a finite set for all $x \in \ell_{00}^1$, but since $||Lambda_n|| = n$, this is not uniformly bounded. This is easy to see, since $\Lambda_n(e_n) = n$, hence it See that the uniform boundedness principle does not apply here that $\ell_{00}^1$ is not complete, since the sequence $\{y_n\}$ where $y_n = (1, 1/2, \ldots, 1/n, 0, \ldots)$, does not converge in the space. $\qquad\square$

*Solution of problem* 4:     1. Let $X$ be a Banach space, and $p : X \to [0, \infty)$ is a semi-norm (a norm, but without the rule that $p(x) = 0 \implies x = 0$). If we take any absolutely convergent series $\sum_{n=1}^\infty x_n \in X$, we have

$$p\left(\sum_{n=1}^\infty x_n\right) \le \sum_{n=1}^\infty p(x_n) \in [0, \infty],$$

then $p$ is continuous. To prove this, we let $A_n = p^{-1}([0, n])$ and $F_n = \overline{A_n}$. See that $A_n$ and $F_n$ are symmetric convex sets since $p$ is a seminorm. We have $X = \cup_{n=1}^\infty F_n$, and by Baire's theorem there must be some $N$ such that $F_N$ has non-empty interior. Therefore, there exist $x_0 \in X$, and $R > 0$ such that $B_R(x_0) \subset F_n$. By symmetry of $F_N$, $B_R(-x_0) = -B_R(x_0) \subset F_N$. If $||x|| < R$, then $x + x_0 \in B_R(x_0), x - x_0 \in B_R(-x_0)$, so we have $x \pm x_0 \in F_N$. Since $F_N$ is convex, we have $\frac{1}{2}(x_0 + (-x_0)) = 0 \in F_N$. Then we have $B_R(0) \subset F_N$. We want to show that $B_R(0) \subset A_N$. Suppose $||x|| < r < R$. Fix $0 < q < 1 - \frac{r}{R}$, so that $\frac{1}{1-q} \cdot \frac{r}{R} < 1$. Then $y = \frac{R}{r} x \in B_R(0) \subset F_N = \overline{A_N}$. Thus there is $y_0 \in A_N$ such that $||y - y_0|| < qR$, so $q^{-1}(y - y_0) \in B_R$. Choose a $y_1 \in A_N$ such that $||q^{-1}(y - y_0) - y_1|| < qR$, so $||y - y_0 - qy_1|| < q^2 R$. By induction we have $(y_n)$ such that

$$\left\| y - \sum_{k=0}^n q^k y_k \right\| < q^n R,$$

for all $n \ge 0$, thus we have $y = \sum_{k=0}^\infty q^k y_k$. We see that $||y_k|| \le R + qR$ for all $k$, so $y$ as a series exists since the constructed series is absolutely convergent. Now, using the subadditivity that was given in the hypothesis, we have

$$p(y) = p\left(\sum_{k=0}^\infty q^k y_k\right) \le \sum_{k=0}^\infty q^k p(y_k) \le \frac{1}{1-q} N,$$

and hence $p(x) \le \frac{N(1+\varepsilon)}{R} ||x||$, which proves the continuity.

2. (a) $T :\to Y$ is a bounded linear operator. Then suppose the $T(U)$, where $U$ is the open unit ball, is open. In that case, let $V$ be some open neighbourhood of $X$. Then for $x \in V$, we have that some ball of radius $r$ centered at $x$ is in $V$. We can then see that $T(rU + x) \subset V$, which means that we only need to see that $T(U)$ is open.

Define $p(y) := \inf\{||x|| \mid Tx = y\}$. We need to show that this is a seminorm with countable subadditivity. Let $\alpha \ne 0$ be a scalar. Then we have $\{x \mid x \in X, Tx = \alpha y\} = \{\alpha x \mid x \in X, Tx = y\}$, and taking infimums, we have $p(\alpha y) = |\alpha| p(y)$. For $\alpha = 0$, this can be easily checked. Let $\sum_n y_n$ be a convergent series. We need to show that $p\left(\sum_n y_n\right) \le \sum_n p(y_n)$, so we assume $\sum_n p(y_n)$ is finite,

since if it was infinite there would be nothing to prove. Fixing some $\varepsilon > 0$, we take a sequence $(x_n)$ in $X$ such that $Tx_n = y_n$, and $||x_n|| < p(y_n) + 2^{-n}\varepsilon$. Then we have $\sum_n ||x_n|| < \sum_n p(y_n) + \varepsilon$, which is finite. Since in Banach spaces absolutely convergent series are also convergent, we have $\sum_n x_n$ converges. Then $T(\sum_n x_n) = \sum_n Tx_n = \sum_n y_n$, so

$$p\left(\sum_n y_n\right) \leq \left|\left|\sum_n x_n\right|\right| \leq \sum_n ||x_n|| < \sum_n p(x_n) + \varepsilon.$$

Therefore subadditivity is confirmed, so by Zabreiko's lemma, we have

$$T(U) = \{y : y \in Y, Tx = y \text{ for some } x \in U\} = \{y : y \in Y, p(y) = 1\},$$

which is open. This proves the open mapping theorem.

(b) If we have a one-one onto linear mapping from a topological space to another is a homeomorphism if and only if it is continuous and open. Using the open mapping theorem, we have that this map is open, and continuous. Thus $T^{-1}$ exists and must be bounded, as it is a homeomorphism too. This proves the bounded inverse theorem.

(c) Let $\mathcal{F}$ be a non-empty family of bounded linear operators from a Banach space $X$ to a normed space $Y$, where $\sup\{||Tx|| \mid T \in \mathcal{F}\}$. Now, let $p(x) := \sup\{||Tx|| \mid T \in \mathcal{F}\}$. See that $p(\alpha x) = |\alpha| p(x)$ from definition. For $\sum_n x_n$ a convergent series, we have

$$\left|\left|T\left(\sum_n x_n\right)\right|\right| = \left|\left|\sum_n x_n\right|\right| \leq \sum_n ||Tx_n|| \leq \sum_n p(x_n),$$

which implies that $p\left(\sum_n x_n\right) \leq \sum_n p(x_n)$. In particular, we have $p(x_1 + x_2) \leq p(x_1) + p(x_2)$. Now, since $p$ is continuous, we have $\delta > 0$ such that $p(x) \leq 1$ for $||x|| \leq \delta$. Whenever $x \in X, ||x|| = 1$ we have $p(x) \leq \delta^{-1}$, which means that $||T|| \leq \delta^{-1}$ for each $T \in \mathcal{F}$.

(d) Let, $T : X \to Y$. Now pick $p(x) = ||Tx||$. If $p$ was continuous, then there would be a neighbourhood $U$ of $0$ such that the set $p(U)$ is bounded, which implies that $T(U)$ is bounded, and this implies continuity of $T$. $p$ is a semi-norm, so we need to check its continuity. We only need to check that this has countable subadditivity. Take $\sum_n x_n$, a convergent series, then we can assume that $\sum_n ||Tx_n||$ is finite without loss of generality. Now see that if $\sum_n ||Tx_n||$ is convergent, then so is $\sum_n Tx_n$ is convergent in $Y$, as it is complete. Since $\sum_{k=1}^n x_k \to \sum_n x_n$, then we have $T\left(\sum_{k=1}^n x_k\right) \to T\left(\sum_n x_n\right)$. Then, from hypothesis we have $T\left(\sum_n x_n\right) = \sum_n Tx_n$. Taking norm, we have the norm subadditivity. Since $p$ is continuous by Zabreiko's lemma, we have proven the closed graph theorem.

$\square$

*Solution of problem* 5: We know that $x_n \xrightarrow{w} x$, so for any $f \in X^*$, we have $f(x_n) \to f(x)$. Using the Hahn-Banach theorem, we can find a linear functional $f$ where $||f|| = 1$, and $f(x) = ||x||$. Then we have

$$||x|| = \lim_{n\to\infty} |f(x_n)| \leq \liminf_{n\to\infty} ||f|| \, ||x_n|| = \liminf_{n\to\infty} ||x_n||,$$

as desired.

$\square$

*Solution of problem* 6*:* $X$ is a normed linear space. We say that $(x_n)$, a sequence in $X$ is *weakly Cauchy* if the sequence $(fx_n)$ converges for all $f \in X^*$. $X$ is *weakly complete* if all weakly Cauchy sequences converge weakly.

Let $X$ be reflexive. Let $(x_n)$ be a weakly Cauchy sequence in $X$. Pick $f \in X^*$. Then since $(f(x_n))$ is a Cauchy sequence in $\mathbb{C}$, we have that $f(x_n) \to \alpha(f)$, where $\alpha \in X^{**}$. We do not know what element in $X$ this element corresponds to, but we know that since $X$ is reflexive we can think of it as an element of the bidual acting on $f$. For any $f \in X^*$ we have $E_{x_n}(f) = f(x_n) \to \alpha(f)$. We define $\alpha$ as the element of $X^{**}$, as the limit of $(f(x_n))$ as $n \to \infty$. Now see that $|\alpha(f)|$ is bounded as for each $f \in X^*$, we have that pointwise the set $(f(x_n))$ is bounded for each $f \in X^*$. Then by the Uniform Bounded Principle, we have $(x_n)$ must be bounded in $X^{**}$. Since $||x_n||_{X^{**}} = ||x_n||$, we know that $(x_n)$ is bounded in $X$ by $M > 0$. Then,

$$|f(x_n)| \leq M\, ||f|| \implies \alpha(f) \leq M\, ||f||.$$

Since $X$ is reflexive, $\alpha \in X$. Then by definition for each $f \in X^*$ we have $f(x_n) \to \alpha(f) = f(\alpha)$, which confirms weak converges. $\qquad\square$