ASSIGNMENT # 6 DUE DATE : 09/11/2023

1. Let R be a PID.

   (i) For any $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \neq \vec{0}$ in R^2, prove that there exists an invertible 2×2 matrix X such that $X\vec{v} = \begin{bmatrix} d \\ 0 \end{bmatrix}$ where d = generator of the ideal (a, b).
   Similarly $\vec{v}^{\,t} X^t = [d \;\; 0]$ (where t = transpose).

   (ii) Let $A = (a_{ij})$ be a nonzero $m\times n$ matrix over R with $a_{11} \neq 0$. Let i be any integer between 2 and m. Prove that there exists an invertible $m\times m$ matrix $\tilde{X}$ such that the $(i, 1)^{th}$ entry of $\tilde{X}A$ is 0 while the entries in the j^{th} row for $j \neq 1, j \neq i$, remain unchanged. A similar result holds for modifying the entries of the first row of A.

   (iii) Prove that for any $m\times n$ matrix A over R, there exists an invertible $m\times m$ matrix X and an invertible $n\times n$ matrix Y such that for $D = XAY^{-1}$, with $D = (d_{ij})$, we have $d_{ij} = 0$ for $i \neq j$. Moreover, we may arrange that $d_{11} | d_{22} | d_{33} | \cdots\cdots$.

2. Let F be a field and G any finite subgroup of the multiplicative group of F^*. Prove that G is cyclic.
$\left(\dim_{\mathbb{F}_p}\left(T_p(F^*)\right) \leq 1\right.$ because $x^p - 1$ has at most p solutions in $F.\left.\right)$

3. Let R be a ring and M an R-module. Consider the natural surjection $R^{\oplus M} \xrightarrow{\pi} M$ with $\pi(e_m) = m$, where $\{e_m\}_{m\in M}$ are the standard basis vectors of $R^{\oplus M}$. Prove that $\ker(\pi)$ is generated by elements of the form $\{e_{rm} - re_m\}$ and $\{e_{m_1+m_2} - e_{m_1} - e_{m_2}\}$ where $m, m_i \in M$ and $r \in R$.

4. Prove the following properties of the $\mathbb{Z}$-module $\mathbb{Q}$.

(i) Any finitely generated submodule of $\mathbb{Q}$ is cyclic.

(ii) Any 2 nonzero submodules of $\mathbb{Q}$ have a nonzero intersection.

(iii) Any homomorphism from $\mathbb{Q}$ to a finitely generated $\mathbb{Z}$-module M is the zero map.

(iv) Any map from $\mathbb{Q}$ to a torsion module M is the zero map.

(v) $\mathbb{Q}$ is not the direct sum of any 2 nonzero modules.

(vi) $\mathbb{Q}$ has no minimal generating set.

5. Let R be a PID. Prove that any submodule M of a free module $R^{\oplus X}$ (for some index set X) is also free.

Hint: Let $\mathbb{T}$ be the set of pairs (B, Y) where $Y \subseteq X$ is a subset such that $M \cap (R^{\oplus Y})$ is free and B is a basis of this module. Define a partial order $\leq$ on $\mathbb{T}$ by $(B_1, Y_1) \leq (B_2, Y_2)$ if $B_1 \subseteq B_2$ and $Y_1 \subseteq Y_2$.

Now check:

(i) $\mathbb{T}$ is nonempty if X is nonempty. (Use the structure theorem for finitely gen. modules).

(ii) $\mathbb{T}$ has a maximal element under $\leq$. (Zorn's lemma)

(iii) If (B, Y) is a maximal element of $\mathbb{T}$, then $Y = X$. (If $Y \subsetneq Y'$ and $Y' \setminus Y$ is finite, then $0 \to M \cap R^{\oplus Y} \to M \cap R^{\oplus Y'} \to M \cap R^{\oplus (Y' \setminus Y)} \to 0$ is split exact by the structure theorem for finitely generated modules.)

6\. (i) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Q}/\mathbb{Z})$ is a cyclic group of order n.

(ii) Deduce that if M is a finite $\mathbb{Z}$-module, then M is isomorphic to the module

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{R}/\mathbb{Z}).$$

This isomorphism is non-canonical.

(iii) Let R be a ring and let M, N be R-modules. Prove that the function

$\delta : M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), N)$ where $\delta(m)$ is the map $\varphi \longmapsto \varphi(m)$ (for $\varphi \in \mathrm{Hom}_R(M,N)$) is an R-linear map of modules.

Moreover, δ commutes with finite direct sums.

(iv) In (iii), let $R = \mathbb{Z}$, $N = \mathbb{Q}/\mathbb{Z}$ (or $\mathbb{R}/\mathbb{Z}$). Prove that the double dualizing map δ is an isomorphism if M is a finite $\mathbb{Z}$-module.

(This is similar to the situation of a finite dimensional vector space V over a field F. We have $V \cong V^{**}$ naturally. There is also an isomorphism $V \simeq V^*$ but there is no canonical choice for such an isomorphism unless some extra data is specified such as a basis of V or a non-degenerate quadratic form on V.)