ASSIGNMENT # 5   DUE DATE 31/10/2023

1. (a) Let R be a UFD. Let a, b be nonzero elements such that a, b have no common factor. Prove that the kernel of the natural surjection $R[x] \twoheadrightarrow R[\frac{a}{b}]$ sending x to $\frac{a}{b}$ is generated by $(bx-a)$.
(Hint: Use Gauss Lemma).

   (b) Let $R = \mathbb{Z}[\sqrt{-3}]$. Let $a = 1+\sqrt{-3}$, $b = 2$. Explain why the kernel of the natural map $R[x] \twoheadrightarrow R[\frac{a}{b}]$, $x \mapsto \frac{a}{b}$, is not generated by $(bx-a)$.

2. (a) Let $R = F[x, y, z]$ where F is a field. Prove that the ideal $I = (x, y, z)$ cannot be generated by less than 3 elements.

   (b) Prove that the result is also true for the ring $R = A[x, y, z]$ where A is any commutative ring.

3. Let $R = F[x, y]/(xy)$. Prove that the ideal $(\bar{x}, \bar{y})$ in R is not principal. Prove that every other prime ideal in R is principal. Here F is a field.

4. Compute the cokernel of the natural $\mathbb{Z}^4 \to \mathbb{Z}^4$ given by the matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 5 & 5 & 4 & 4 \\ 6 & 7 & 7 & 8 \\ 10 & 10 & 9 & 9 \end{bmatrix}$$

(You need not follow the algorithm given in class.)

5. Universal property of kernels and cokernels:

(a) Let $f: M \to N$ be a map of R-modules and let $i: \ker(f) \hookrightarrow M$ be the inclusion map. For any map $g: P \to M$ such that $f \circ g = 0$, prove that there is a unique map $h: P \to \ker(f)$ such that $g = i \circ h$.

(b) Let $f: M \to N$ be a map of R-modules and let $\pi: N \twoheadrightarrow \operatorname{Coker}(f)$ be the natural surjection. For any map $g: N \to P$ such that $g \circ f = 0$ prove that there exists a unique map $h: \operatorname{Coker}(f) \to P$ such that $g = h \circ \pi$.

6. Let M be a noetherian R-module and let $f: M \to M$ be a surjective map. Prove that f is an isomorphism. (Look at $0 \subseteq \ker(f) \subseteq \ker(f^2) \subseteq \ker(f^3) \subseteq \cdots$).