# Functional Analysis Homework 2

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## 1

We need to check that rules of inner products hold—

1. For $A = B$, we have $\langle A, A \rangle = tr(AA^*) = \sum_{i,j} |a_{ij}|^2 \geq 0$, where $a_{ij}$ denotes the elements of $A$. Moreover, $||A|| = 0 \implies |a_{ij}| = 0$ for all $1 \leq i, j \leq n \implies A = 0$.
2. $\langle B, A \rangle = tr(BA^*) = tr()$

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We are given $T : \mathbb{K}^n \to \mathbb{K}^m$, where

$$(Tx)(i) = \sum_{j=1}^{n} k_{ij} x_j,$$

where $i = 1, 2, \ldots, m$. Let $a_i$ denote the $i$th row of $T$. Then we have $\langle Tx, y \rangle = \sum_{j=1}^{m} (Tx)(i) y_j$. Expanding the entire thing, we have

$$\langle Tx, y \rangle = \sum_{1 \leq i \leq m, 1 \leq j \leq n} k_{ij} x_j \bar{y_i}.$$

We can write this as

$$\sum_{j=1}^{n} x_j \overline{\overline{k_{1i}} y_1 + \cdots + \overline{k_{mi}} y_m} = \langle x, \overline{T}^T y \rangle!$$

Therefore from uniqueness of adjoint we must have $T^* = \overline{T}^T$.

## 10

See that for any operator we have

$$|\langle Tx, x \rangle| \leq ||Tx|| \cdot ||x|| \leq ||T||,$$

taking $||x|| = 1$. Since the left of the inequality depends on $x$ while the right is independent, we have $\sup_{||x||=1} \langle Tx, x \rangle \leq ||T||$.