Measure Theory HW7

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1

If we have $\nu = 0$, then see that $\nu(E) = \int_E 0 d\mu$, thus $\mu(E) = 0 \implies \nu(E) = 0$ trivially, so $\nu << \mu$. Also see that $\nu \perp \mu$, as $X = X \sqcup \phi$, and see that $\mu(E) = \mu(E \cap X)$, while $\nu(E) = \nu(E \cap \phi)$.

Now let us assume that $\nu << \mu$ and $\nu \perp \mu$. Then we have $X = A \sqcup B$, where $\mu(E) = \mu(E \cap A)$, while $\nu(E) = \nu(E \int B)$. Let us pick a measurable set $E \subseteq B$. Then we have $\mu(E) = \mu(E \cap B) = \mu(\phi) = 0$. Since $\nu << \mu$, we have $\nu(E) = 0$. Thus ν is zero on every measurable subset E in B. For a general measurable set E, we have $E = (E \cap A) \sqcup (E \cap B)$. We already know that $\nu(E \cap A) = 0$, now we see that $\nu(E \cap B) = 0$ also. Thus $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = 0$ for all measurable $E \subset X$.

2

If $\nu \perp \mu$, then there exists we have $X = A \sqcup B$, where $\mu(B) = 0$ and $\nu(A) = 0$. Then $\{E_n\}$ is a sequence such that $E_n = B$ for all $n \in \mathbb{N}$. Then see that $\mu(E_n), \nu(X \backslash A) = 0$, as required.

Conversely, we assume that $\{E_n\}$ is a sequence of measurable subsets such that $\mu(E_n) \to \infty$ and $\nu(X \backslash E_n) \to \infty$ as $n \to \infty$. We will use a fun result from real analysis— for any real sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$, we can find a subsequence $\{x_{n_i}\}$ such that $\sum_{i=1}^{\infty} x_{n_i} < \varepsilon$. The proof for this is constructive. Choose $\varepsilon > 0$, then find a $x_{n_1} \leq \frac{\varepsilon}{2}$. Then choose $x_{n_2} \leq \frac{\varepsilon}{2^2}$. In this fashion, we can always find a $x_{n_k} \leq \frac{\varepsilon}{2^{k+1}}$. See that the sum of this sequence is less than ε. Using this, we find a subsequence of $\{E_n\}$ such that $\sum_{i=1}^{\infty} \nu(X \backslash E_{n_i}) < \varepsilon$. This is always true, so without loss of generality we assume that the sequence itself is such that it respects this additional condition. Let $A = \liminf_{n \to \infty} E_k = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k$. Now see that

$$\begin{aligned}
\mu(A) &= \mu\left(\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k\right) \\
&\leq \sum_{n=1}^{\infty} \mu\left(\cap_{k=n}^{\infty} E_k\right).
\end{aligned}$$

See that for a fixed n, let $F_m = \cap_{k=n}^{m} E_k$, so we have $F_{m+1} \subseteq F_m$ thus $\lim_{m \to \infty} \mu(F_m) = \mu\left(\cap_{m=n}^{\infty} F_m\right) = \mu\left(\cap_{k=n}^{\infty} E_k\right)$. However, we have $F_m \subseteq E_m \implies \mu(F_m) \leq \mu(E_m) \implies \lim_{m \to \infty} \mu(F_m) \leq \lim_{m \to \infty} \mu(E_m) \implies \lim_{m \to \infty} \mu(F_m) = 0$, as $\lim_{m \to \infty} \mu(E_m) = 0$. Then we have

$$\sum_{n=1}^{\infty} \mu\left(\cap_{k=n}^{\infty} E_k\right) = \sum_{n=1}^{\infty} 0 = 0.$$

Now consider $B = X \backslash A = \cap_{n=1}\left(\cup_{k=n}^{\infty} D_k\right)$, where $D_k = X \backslash E_k$. Now

$$\begin{aligned}
\nu(B) &= \nu\left(\cap_{n=1}\left(\cup_{k=n}^{\infty} D_k\right)\right) \\
&\leq \nu\left(\cup_{k=n}^{\infty} D_k\right) \leq \sum_{k=n}^{\infty} \nu(D_k) < \varepsilon,
\end{aligned}$$

which means that $\nu(B) = 0$. This means that $\mu \perp \nu$.

3

We can check that $m << m$ trivially, as $m(E) = 0 \implies m(E) = 0$. We can split $\mathbb{R}$ into two disjoint subsets, that is, $\mathbb{R} = \{0\} \sqcup ((\infty, 0) \cup (0, \infty))$. We denote the two sets as A and B. Then observe that

$\delta_0(E) = \delta_0(E \cap A)$, that is, the Dirac measure at 0 only cares if it intersects $\{0\}$, and nothing else. Also, we have $m(E) = m(E \cap B)$, as A is a $m-$null set, hence $m(E) = m(E \cap A \sqcup E \cap B) = m(E \cap A) + m(E \cap B) = m(E \cap B)$, seeing as $E \cap A$ is also a $m-$null set. Thus we have $m \perp \delta_0$. Thus $\nu = m + \delta_0$ is already in the Lebesgue decomposition.

4

- We find the positive and negative parts of f. Note that the roots of this polynomial are $3 + 2\sqrt{2}$ and $3 - \sqrt{2}$. Let us call them α_1 and α_2 for sake of convenience. Then
$$p^+ = \begin{cases} x^2 - 6x + 1 & x \in (-\infty, \alpha_2] \cup [\alpha_1, \infty) \\ 0 & \text{else,} \end{cases}$$
and
$$p^- = \begin{cases} -(x^2 - 6x + 1) & x \in (\alpha_2, \alpha_1) \\ 0 & \text{else.} \end{cases}$$
Then $\nu(E) = \int_E p^+ d\mu - \int_E p^- d\mu = \nu^+ - \nu^-$, where $\nu^+ := \int_E p^+ d\mu$ and $\nu^- := \int_E p^- d\mu$ are two positive measures. Note that it is not possible for both of them to attain ∞ together, since ν^- is a finite measure. Thus it is trivial to see that ν must be a signed measure.

- Let $\mathbb{R} = A \sqcup B$, where $A = (-\infty, \alpha_2] \cup [\alpha_1, \infty)$ and $B = (\alpha_2, \alpha_1)$. See that since both the positive measures have their usual properties, we have that for $E \subseteq A$ measurable, we have $\nu(E) = \nu^+(E) - \nu^-(E) = \nu^+(E) - 0 \geq 0$, and likewise for $E \subseteq B$ measurable, we have $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - \nu^-(E) \leq 0$. Thus the above construction is a Hahn decomposition.

- See that ν^+ lives on A, while ν^- lives on B. That is, $\nu^+(E) = \nu(E \cap A)$, and $\nu^-(E) = -\nu(E \cap B)$. This is easy to see, as $E = (E \cap A) \sqcup (C \cap B)$. Then $\nu(E) = \int_{(E \cap A) \sqcup (E \cap B)} p^+ d\mu - \int_{(E \cap A) \sqcup (E \cap B)} p^- d\mu = \int_{(E \cap A)} p^+ d\mu + \int_{(E \cap B)} p^+ d\mu - \int_{(E \cap A)} p^- d\mu - \int_{(E \cap B)} p^- d\mu$. Since p^+ is 0 on B, and p^- is 0 on A, we have that $\nu^+ \perp \nu^-$. Thus we have the Jordan decomposition.

5

Let us assume that $\mu(E) = 0$. As E is $\mu-$null, then $\mu(E \cap E_n) = 0$ for all n, by monotonicity of the measure. Then we have $\nu(E) = \sum_{n=1}^{N} c_n \mu(E \cap E_n) = 0$. Thus $\nu << \mu$. See that the function $f := \sum_{n=1}^{N} c_n \chi_{E_n}$ is a good candidate for the Radon-Nikodym derivative.

$$\int_E f d\mu = \int_E \sum_{n=1}^{N} c_n \chi_{E_n} d\mu = \sum_{n=1}^{N} c_n \int_X \chi_E \chi_{E_n} d\mu = \sum_{n=1}^{N} c_n \int_X \chi_{E \cap E_n} d\mu = \sum_{n=1}^{N} c_n \mu(E \cap E_n),$$

which is the desired result. Thus $\frac{d\nu}{d\mu} = f$.

6

1. Since $\nu << \mu$, there exists $f \in L^1(\mu)$ such that $\nu(E) = \int_E f d\mu$. We know that $f > 0 \mu-$almost everywhere, then assume that $\nu(E) = 0$. Thus $\int_E f d\mu = 0$. Assume that $\mu(E) > 0$. Then f is greater than zero on all of E, thus $\int_E f d\mu > 0$. However, since $\nu(E) = 0$ this forces $\mu(E)$ to be 0. Thus $\mu << \nu$.

2. For some $g = \chi_E$, we have $\int_X g d\mu = \int_X \chi_E \left(\frac{d\nu}{d\mu}\right) d\mu$ holds. This means that this statement holds for simple functions. From here, we can extend this result to measurable functions as they are the limits of simple functions. This can then be further extended to all $L^1(\nu)$ functions. The exact result holds when we integrate $L^1(\mu)$ functions over μ. Now see that
$$\int_E d\nu = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_E \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\mu} d\nu.$$
Comparing the integrand functions, we have the desired result. Since we know that $\frac{d\nu}{d\mu} > 0$, the reciprocal is always defined.

7

Let $\lambda := \mu + \nu$. Then $f = \frac{d\nu}{d\lambda}$. See that since $\lambda(E) = \int_E 1 d\lambda = \mu(E) + \int_E f d\lambda$. Therefore we have $\mu(E) = \int_E (1-f) d\lambda$. Thus we have $\frac{d\mu}{d\lambda} = 1 - f$. Note that from the previous result, we have $\int_E g d\nu = \int_E g \cdot \frac{d\nu}{d\mu} d\mu$. This works even if we replace ν by μ and μ by λ. Then putting all of this together we have

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int_E \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} d\lambda.$$

Comparing the two integrands gives us $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}$. Putting it all together, we get $f = \frac{d\nu}{d\mu}(1-f)$, which gives us the desired result.

8

In $(\mathbb{N}, \mathbb{P}(\mathbb{N}))$, μ is the counting measure. Note that the empty set is the only $\mu-$null set, since every non-empty set has cardinality more than zero. Then somewhat trivially we have $\mu(E) = 0 \implies E = \phi \implies \nu(E) = 0$. So $\nu << \mu$. We have ν is $\sigma-$finite, thus $\mathbb{N} = \sum_{n=1}^{\infty} \{n\}$, where $\nu(\{n\}) < \infty$. Thus define $f : \mathbb{N} \to \mathbb{R}$, where $f(n) = \nu(\{n\})$. Then we have $\nu(E) = \int_E f d\mu = \sum_{n \in E} f(n)$, is the required function. Thus $f = \frac{d\nu}{d\mu}$.

9

If we assume that $\frac{d\mu}{d\lambda} \cdot \frac{d\nu}{d\lambda} = 0$ λ almost everywhere, we know from the previous assignment that $\mu(E)$ and $\nu(E)$ are mutually singular. To see the converse, let us assume that $\mu \perp \nu$. Then we have $X = A \sqcup B$, where μ lives on A while ν lives on B. Then let $f := \frac{d\mu}{d\lambda}$, and $g := \frac{d\nu}{d\lambda}$. See that

$$\int_E fg d\lambda = \int_{(E \cap A)} fg d\lambda + \int_{(E \cap B)} fg d\lambda.$$

We know that $\int_{(E \cap A)} fg d\lambda = \int_{(E \cap A)} f d\nu$, and $\int_{(E \cap B)} fg d\lambda = \int_{(E \cap B)} g d\mu$. As $E \cap A$ is a $\nu-$null set and $E \cap B$ is a $\mu-$null set, we have $\int_E fg d\lambda = 0 \implies fg = 0$ $\lambda-$almost everywhere.

10

$\nu = \nu^+ - \nu^-$ is a signed measure, where we have $x = P \sqcup N$, that is, $\nu^+(E) = \nu(E \cap P)$, and $\nu^-(E) = \nu(E \cap N)$. Taking $|\nu| = \nu^+ + \nu^-$, see that $\nu^+ << |\nu|$ and $\nu^- << |\nu|$, and thus there must exist $\frac{d\nu^+}{d|\nu|}$ and $\frac{d\nu^-}{d|\nu|}$, the Radon-Nikodym derivatives. Then see that $\frac{d\nu^+}{d|\nu|} = \chi_P$. To see this, see that for E measurable in P,

$$\begin{aligned}\int_E \chi_P d|\nu| &= \int_E \chi_P d\nu^+ + \int_E \chi_P d\nu^- \\ &= \nu^+(E \cap P) + 0 = \nu^+(E),\end{aligned}$$

and similarly for ν^-, $\frac{d\nu^-}{d|\nu|} = \chi_N$.

11

See that

$$\begin{aligned}\left|\int_X f d\nu\right| &\leq \left|\int_X f d\nu^+ - \int_X f d\nu^-\right| \\ &= \left|\int_X f d\nu^+\right| + \left|\int_X f d\nu^-\right| \leq \int_X |f| d\nu^+ + \int_X |f| d\nu^- \\ &= \int_X |f| d|\nu|,\end{aligned}$$

as desired. Now see that $f \leq 1$ implies that $f\chi_E \leq \chi_E$, which means that integrating over X we have $\int_X f\chi_E d\nu \leq \int_X \chi_E d\nu$. Then this gives us

$$\left|\int_X f\chi_E d\nu\right| = \left|\int_E d\nu\right| \leq \int_X |f\chi_E| d|\nu| \leq \int_X \chi_E d|\nu| = |\nu|(E).$$

Therefore we have $|\nu|(E) \geq | \int_E f d\nu |$, for all f such that $f \leq 1$. Then we must have $|\nu|(E) \geq \sup\left\{| \int_E f d\nu |: f \leq 1\right\}$. Now let $f = \chi_{E\cap A} - \chi_{E\cap B}$, where $X = A \sqcup B$, as per the Jordan decomposition. Then we have

$$\begin{aligned}
\int_E f d\nu &= \int_E f d\nu^+ - \int_E f d\nu^- \\
&= \int_E \chi_{E\cap A} d\nu^+ - \int_E \chi_{E\cap B} d\nu^+ + \int_E \chi_{E\cap A} d\nu^- + \int_E \chi_{E\cap B} d\nu^- \\
&= \nu^+(E\cap A) - \nu^+(E\cap B) - \nu^-(E\cap A) + \nu^-(E\cap B).
\end{aligned}$$

Note that $\nu^+(E\cap B) = \nu^-(E\cap A) = 0$. Thus we have $\int_E f d\nu = \nu^+(E\cap A) + \nu^-(E\cap B) = \nu^+(E) + \nu^-(E) = |\nu|(E)$. This means that the right hand side actually attains its supremum within the set of all integrable functions where $f \leq 1$. Thus we have $|\nu|(E) \leq \sup\left\{| \int_E f d\nu |: f \leq 1\right\}$. This gives us the desired equality.

12

We know that the counting measure μ only has one $\nu-$null set, that is ϕ. Let us say that $\mu << \nu$. Then for $\nu(E) = 0$ for some measurable $E \in \mathcal{A}$, we have $\mu(E) = 0$. To take the contrapositive, we see that if $\mu(E) \neq 0$, then $\nu(E) \neq 0$. Since we know that μ has no non-empty null sets, non $\mu-$null sets and non-empty subsets are synonymous. Thus, if $E \neq \phi$, $\nu(E) \neq 0$.

For the Dirac measure at $x_0 \in X$, we know that $\delta_{x_0} << \nu$ means that $\nu(E) = 0$ for some $E \in \mathcal{A}$ implies that $\delta_{x_0}(E) = 0$. The Dirac measure is zero if the measurable subset E does not have x_0. Thus we can take the contrapositive of the absolute continuity to say that if $x_0 \in E$, then we must necessarily have $\nu(E) \neq 0$.