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1.

2. This statement is false. For sake of contradiction, let $(X, ||\cdot||)$ be a normed linear space such that the induced metric is the discrete metric. Then for $x, y \in X, x \neq y$ we must have $||x-y|| = 1$. Note that $2x \neq 2y$, so we must have $||2x-2y|| = 2||x-y|| = 2$, but by the discrete metric the answer should still be 1! Thus there can be no such norm.

2

We wish to show that the function $||\cdot||$ on X satisfies the triangle inequality iff the closed unit ball is convex. Assume that the function $||\cdot||$ is indeed a norm. Then let $x, y \in D$, the closed unit ball. Then $||x||, ||y|| \leq 1$. Now we have for $\alpha \in [0,1]$ $z = \alpha x + (1-\alpha)y$. See that

$$||z|| = ||\alpha x + (1-\alpha)y|| \leq \alpha||x|| + (1-\alpha)||y|| \leq \alpha \cdot 1 + (1-\alpha)\cdot 1 \leq 1,$$

thus we have $z \in D$.

Take two elements $x, y \in X$ both non-zero, since if either were zero the inequality would be trivial. Then

$$||x+y|| = (||x|| + ||y||) \cdot \left|\left|\alpha \frac{x}{||x||} + (1-\alpha)\frac{y}{||y||}\right|\right|,$$

where $\alpha = \frac{||x||}{||x||+||y||}$. Note that $\frac{x}{||x||} = \frac{y}{||y||} = 1$, thus we can use the convexity condition to see that $\frac{||x+y||}{||x||+||y||} \leq 1$, which is the triangle inequality.

3

1. Pick a $f \in C([a,b])$. Then $|f| \leq M = \sup\{|f(x)| : x \in [a,b]\}$. Now see that

$$\int_a^b |f(t)|^p dt \leq (b-a)M^p \geq 0.$$

Thus $||f||_p \geq 0$. For $f = 0$, we have $M = 0$, so $\int_a^b |0|^p dt = 0$. If $\int_a^b |f(t)|^p dt = 0$, then see that $0 \leq (b-a)M^p \geq 0$. Thus we must have

$$(b-a)M^p = 0 \implies M = 0 \implies f = 0.$$

To see the next axiom, for $\alpha \in \mathbb{K}$, we have $\int_a^b |\alpha f(t)|^p dt = \int_a^b |\alpha|^p |f(t)|^p dt$. Then

$$||\alpha f||_p = \left(|\alpha|^p \int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} = |\alpha| ||f||_p.$$

Now let $f, g \in C[a,b]$.

4

1. $||f_1 - F||_\infty$ is to be found, where F is the subspace of constant functions. Unfolding the term, we get
$$||f_1 - F||_\infty = \inf_{c\in\mathbb{R}}\{\sup_{t\in[0,1]} |t-c|\},$$
which for $c \in [-1,0] \cup [1,2]$ is 1, while in $(0,1)$ it decreases to $\frac{1}{2}$ then goes back up to 1. Thus, we must have $||f_1 - F||_\infty = \frac{1}{2}$.

2. We want to now see the distance between $f_2 = t^2$ and G, the space of all polynomials with degree at most 1. Then for some polynomial $-ax - b \in G$, we want to see $\sup_{t\in[0,1]}\{|t^2 + ax + b|\}$

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6

Assume that X is a Banach space. Let (x_n) be an absolutely convergent sequence in X, that is, $||x_n|| \to \alpha$, as $n \to \infty$. Thus for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$,
$$|||x_n|| - \alpha| < \varepsilon.$$
Now we have

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