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1. Let H be a non-abelian simple group. Consider $DH \trianglelefteq H$. Then since H is simple, we must have $DH = H$ or $DH = 0$. If $DH = 0$, then H would be abelian, which is not possible, thus H is perfect.
2. We need to calculate $D(\langle H, K \rangle)$. Any element of $\langle H, K \rangle$ can be realised as a word $w = h_1 k_1 \dots h_n k_n$ for some $n \in \mathbb{N}$ and $h_1, \dots, h_n \in H, k_1, \dots, k_n \in K$. We also assume that the terms are reduced. We consider two such words w_1 and w_2 , where $w_1 = h_1 k_1 \dots h_n k_n, w_2 = h_1 k_1 \dots h_{n'} k_{n'}$. Then any term in $D\langle H, K \rangle$ is of the form $w_1 w_2 w_1^{-1} w_2^{-1} = (h_1 k_1 \dots h_n k_n)(h_1 k_1 \dots h_{n'} k_{n'})(k_n^{-1} h_n^{-1} \dots k_1^{-1} h_1^{-1})(k_{n'}^{-1} h_{n'}^{-1} \dots k_1'^{-1} h_1'^{-1})$, which is another word in $\langle H, K \rangle$. Now take any word $w = h_1 k_1 \dots h_n k_n$ in $\langle H, K \rangle$.
3. We propose that $D(g^{-1}Hg) = g^{-1}DHg$. Take any element of $D(g^{-1}Hg)$, which is of the form $(g^{-1}h_1g)(g^{-1}h_2g)(g^{-1}h_1^{-1}g)(g^{-1}h_2^{-1}g) = g^{-1}(h_1h_2h_1^{-1}h_2^{-1})g \in g^{-1}DHg$. These operations are all if and only if statements, hence $D(g^{-1}Hg) = g^{-1}DHg$. If $H = DH$, $D(g^{-1}Hg) = g^{-1}Hg$.
4. If G is simple, then either $DG = 0$ or $DG = G$. Then for abelian simple groups the maximal perfect subgroup is 0, and in the non-abelian case it is G itself. Both are clearly normal in G .

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We have $|G| = 75$. We clearly have a 5-Sylow subgroup of order 25, as well as a 3-Sylow group of order 3. Since the number of 5-Sylow subgroups must be $5k + 1$ and $5k + 1 \mid 3$, we are forced to have a normal subgroup of order 25. Then let $H \trianglelefteq G$ where $|H| = 25$. Let K be some 3-Sylow subgroup of order 3. Then by Lagrange's theorem, $|H \cap K| = 1$, so we can construct a semi-direct product of the groups.

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