Measure Theory HW8

Gandhar Kulkarni (mmat2304)

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We will use $i = 1, 2$ to denote the measures. Let $E_{n,i} = \{x : \frac{d\nu_i}{d\mu_i} < -\frac{1}{n}\}$, and see that

$$\nu_i(E_{n,i}) = \int_E d\nu_i = \int_{E_{n,i}} -\frac{1}{n} \int_{E_{n,i}} d\mu_i = -\frac{1}{n}\mu_i(E_{n,i}).$$

Since ν_i is a positive measure, $\mu_i(E_{n,i}) = 0 = \nu_i(E_{n,i})$ for all n. We have $E = \cup_{n=1}^{\infty} E_{n,i}$, that is, $E = \{x : \frac{d\nu_i}{d\mu_i} < 0\}$. By continuity from below, $\mu(E) = \lim_{n\to\infty} \mu(E_{n,i}) = 0$. Then $\frac{d\nu_i}{d\mu_i} \geq 0$ almost everywhere. Then by Tonelli's theorem,

$$\begin{aligned}
(\nu_1 \times \nu_2)(E) &= \int_E d(\nu_1 \times \nu_2) = \int \chi_E d(\nu_1 \times \nu_2) \\
&= \int\int \chi_E d\nu_1 d\nu_2 = \int \left(\int \chi_E \frac{d\nu_1}{d\mu_1} d\mu_1 \right) \frac{d\nu_2}{d\mu_2} d\mu_2 \\
&= \int\int \chi_E \frac{d\nu_1}{d\mu_1} \frac{d\nu_2}{d\mu_2} d\mu_1 d\mu_2 \\
&= \int_E \frac{d\nu_1}{d\mu_1} \frac{d\nu_2}{d\mu_2} d(\mu_1 \times \mu_2).
\end{aligned}$$

This gives us the desired result.

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$f \in BV$ means that $V_a^b(f) \leq M$ for some $M \in \mathbb{N}$. Pick a partition $\mathcal{P}$ of $[a, b]$, then we have $\sum_{\mathcal{P}} |f(x_{i+1}) - f(x_i)| \geq \sum_{\mathcal{P}} ||(x_{i+1})| - |f(x_i)||$. Thus we have

$$V(|f|, \mathcal{P}) \leq V(f, \mathcal{P}).$$

Taking the limit over all partitions, we have $V(|f|) \leq V(f) < \infty$.

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1. Since $V(x)$ is an increasing function, it is differentiable almost everywhere. Choose a partition $\mathcal{P}_1$ for $[a, x]$, and a partition $\mathcal{P}_2$ for $[a, x+h]$. Then we have a partition finer than both of them, say $\mathcal{P}$. Then we can write $V(f, \mathcal{P})(a, x+h) = V(f, \mathcal{P})(a, x) + V(f, \mathcal{P})(x, x+h)$. Thus $V(f, \mathcal{P})(a, x+h) - V(f, \mathcal{P})(a, x) = V(f, \mathcal{P})(x, x+h)$. Then see that
$$\frac{V(f, \mathcal{P})(a, x+h) - V(f, \mathcal{P})(a, x)}{h} = \frac{V(f, \mathcal{P})(x, x+h)}{h}.$$
See that
$$\frac{V(f, \mathcal{P})(x, x+h)}{h} \geq \frac{|\sum_{i=1}^{N} f(x_{i+1} - x_i)|}{h} = \frac{|f(x+h) - f(x)|}{h}.$$
Since the term on the right is dependent on h, and not the partition, we let it go to zero, which gives us $|f'(x)|$. The term on the left can be refined with respect to partitions, which gives us $\frac{V(x+h)-V(x)}{h}$. Since the derivative of V exists almost everywhere, we have the desired inequality.

2. From monotonicity of the integral, we have $\int_a^b |f'| \leq \int_a^b V' \leq V(b) - V(a) = V(b)$, as required.

3. If f is AC, then we have $f(x) - f(a) = \int_a^x g$, for some L^1 function g. Then see that for some partition $\mathcal{P}$, we have $V(f, \mathcal{P}) = \sum_{i=1}^n \left| \int_{x_i}^{x_{i+1}} g \right| \leq \int_{x_i}^{x_{i+1}} |g| = \int_a^b |g|$. Thus taking the limit over all partitions, we have $V(b) \leq \int_a^b |f'|$, which combined with the opposite inequality shown gives us the required result.

   For the converse, we assume $\int_a^b |f'| = V(b)$. The term on the right is also $\int_a^b V'$, so V must be absolutely continuous as it is monotonous and of bounded variation. See that then $\int_a^b V' - |f'| = 0$. Since we know $|f'| \leq V'$ almost everywhere, we must have $V' = |f'|$ almost everywhere. Choose a partition $\mathcal{P}$ such that $\sum_{\mathcal{P}} |x_{i+1} - x_i| < \delta$, we have $\sum_{\mathcal{P}} |V(x_{i+1}) - V(x_i)| < \varepsilon$, for some $\varepsilon > 0$ from the absolute continuity of V. Then at any subinterval of the partition P_k we have $V(f, P_i) = |f(x_{i+1}) - f(x_i)| \leq V(x_{i+1}) - V(x_i) = |V(x_{i+1}) - V(x)|$. Then summing over all subintervals we have our desired result.

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Define h_n, h where $h_n = \sum_{i=1}^n |f_i|, h = \sum_{i=1}^\infty |f_i|$. We have an increasing sequence of positive functions that converge to h, so we have by the monotone convergence theorem,

$$\int h^p = \lim_{n\to\infty} \int h_n^p.$$

By Minkowski's inequality we have

$$||h_n||_p \leq \sum_{i=1}^n ||g_n||_p \leq M,$$

where $M = \sum_{i=1}^\infty ||g_n||_p$. Then we have h is L^p and it is finite a.e. We also have that $\sum_{i=1}^n g_i$ is convergent to some f such that $|f| \leq h$, so we have

$$\left| f - \sum_{i=1}^n g_i \right|^p = \left(|f| + \sum_{i=1}^n |g_i| \right) \leq (2h)^p,$$

so by the dominated convergence theorem we have $\int (f - \sum_{i=1}^n g_i)^p \to \infty$ as $n \to \infty$. Thus we have $\sum_{i=1}^\infty g_i$ converging to f in L^p, as desired.

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1. Let us have $f \in L^{p_2}$. Then let $r = \frac{p_2}{p_1} > 1$. We can see that the Hölder dual of r is $r' = \frac{p_2}{p_2 - p_1}$. We apply Hölder's inequality on $|f|^{p_1}$ and 1, using r and r', we get

$$\begin{aligned} \int |f|^{p_1} d\mu &\leq \left(\int |f^{p_1}| d\mu \right)^{\frac{1}{r}} \cdot \left(\int 1^{r'} d\mu \right)^{\frac{1}{r'}} \\ &= \mu(X)^{\frac{p_2 - p_1}{p_2}} \left(\int |f|^{p_2} d\mu \right)^{\frac{p_1}{p_2}}. \end{aligned}$$

Now raising both sides to $\frac{1}{p_1}$, we have

$$(|f|^{p_1} d\mu)^{\frac{1}{p_1}} \leq \mu(X)^{\frac{p_2-p_1}{p_2}} \left(\int |f|^{p_2} d\mu \right)^{\frac{1}{p_2}}.$$

Since $\mu(X) < \infty$, $f \in L^{p_1}$.

2. Using the above inequality and putting $\mu(X) \leq 1$ gives us the required result.

3. In the measure space $[1, \infty)$ we have $f(x) = \frac{1}{x}$, which is a L^2 function, as $\int_1^\infty \frac{1}{x^2} dx = 1$, but $\int_1^\infty \frac{1}{x} dx = \infty$, thus we have a L^2 function that is not L^1.

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We integrate e^{-px} as see when it has a finite value. See that

$$\int_0^\infty e^{px} dx = \lim_{n\to\infty} \left[\frac{e^{-px}}{-p} \right]_0^n = \lim_{n\to\infty} \frac{1 - e^{np}}{p} = \frac{1}{p},$$

which implies that $e^{-x} \in L^p$ for $p \in [1, \infty]$.

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We want to evaluate $\sum_{i=1}^\infty \frac{1}{(\sqrt{n}\log n)^p}$. See that this is a decreasing sequence (let $a_n = \frac{1}{(\sqrt{n}\log n)^p}$). Then $\sum a_n$ converges iff $\sum 2^n a_{2^n}$ converges, by the Cauchy condensation test. We can reduce this to $\sum 2^n \frac{1}{(\sqrt{2^n}\log(2^n))^p} = \sum 2^{k-\frac{kp}{2}} \frac{1}{(n^p(\log 2)^p}$.

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