

# Algebra HW5

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## 1

If we have  $\nu = 0$ , then see that  $\nu(E) = \int_E 0 d\mu$ , thus  $\mu(E) = 0 \implies \nu(E) = 0$  trivially, so  $\nu \ll \mu$ . Also see that  $\nu \perp \mu$ , as  $X = X \sqcup \phi$ , and see that  $\mu(E) = \mu(E \cap X)$ , while  $\nu(E) = \nu(E \cap \phi)$ .

Now let us assume that  $\nu \ll \mu$  and  $\nu \perp \mu$ . Then we have  $X = A \sqcup B$ , where  $\mu(E) = \mu(E \cap A)$ , while  $\nu(E) = \nu(E \cap B)$ . Let us pick a measurable set  $E \subseteq B$ . Then we have  $\mu(E) = \mu(E \cap B) = \mu(\phi) = 0$ . Since  $\nu \ll \mu$ , we have  $\nu(E) = 0$ . Thus  $\nu$  is zero on every measurable subset  $E$  in  $B$ . For a general measurable set  $E$ , we have  $E = (E \cap A) \sqcup (E \cap B)$ . We already know that  $\nu(E \cap A) = 0$ , now we see that  $\nu(E \cap B) = 0$  also. Thus  $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = 0$  for all measurable  $E \subset X$ .

## 2

## 3

We can check that  $m \ll m$  trivially, as  $m(E) = 0 \implies m(E) = 0$ . We can split  $\mathbb{R}$  into two disjoint subsets, that is,  $\mathbb{R} = \{0\} \sqcup ((-\infty, 0) \cup (0, \infty))$ . We denote the two sets as  $A$  and  $B$ . Then observe that  $\delta_0(E) = \delta_0(E \cap A)$ , that is, the Dirac measure at 0 only cares if it intersects  $\{0\}$ , and nothing else. Also, we have  $m(E) = m(E \cap B)$ , as  $A$  is a  $m$ -null set, hence  $m(E) = m(E \cap A \sqcup E \cap B) = m(E \cap A) + m(E \cap B) = m(E \cap B)$ , seeing as  $E \cap A$  is also a  $m$ -null set. Thus we have  $m \perp \delta_0$ . Thus  $\nu = m + \delta_0$  is already in the Lebesgue decomposition.

## 4

- We find the positive and negative parts of  $f$ . Note that the roots of this polynomial are  $3 + 2\sqrt{2}$  and  $3 - \sqrt{2}$ . Let us call them  $\alpha_1$  and  $\alpha_2$  for sake of convenience. Then

$$p^+ = \begin{cases} x^2 - 6x + 1 & x \in (-\infty, \alpha_2] \cup [\alpha_1, \infty) \\ 0 & \text{else,} \end{cases}$$

and

$$p^- = \begin{cases} -(x^2 - 6x + 1) & x \in (\alpha_2, \alpha_1) \\ 0 & \text{else.} \end{cases}$$

Then  $\nu(E) = \int_E p^+ d\mu - \int_E p^- d\mu = \nu^+ - \nu^-$ , where  $\nu^+ := \int_E p^+ d\mu$  and  $\nu^- := \int_E p^- d\mu$  are two positive measures. Note that it is not possible for both of them to attain  $\infty$  together, since  $\nu^-$  is a finite measure. Thus it is trivial to see that  $\nu$  must be a signed measure.

- Let  $\mathbb{R} = A \sqcup B$ , where  $A = (-\infty, \alpha_2] \cup [\alpha_1, \infty)$  and  $B = (\alpha_2, \alpha_1)$ . See that since both the positive measures have their usual properties, we have that for  $E \subseteq A$  measurable, we have  $\nu(E) = \nu^+(E) - \nu^-(E) = \nu^+(E) - 0 \geq 0$ , and likewise for  $E \subseteq B$  measurable, we have  $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - \nu^-(E) \leq 0$ . Thus the above construction is a Hahn decomposition.
- See that  $\nu^+$  lives on  $A$ , while  $\nu^-$  lives on  $B$ . That is,  $\nu^+(E) = \nu(E \cap A)$ , and  $\nu^-(E) = -\nu(E \cap B)$ . This is easy to see, as  $E = (E \cap A) \sqcup (E \cap B)$ . Then  $\nu(E) = \int_{(E \cap A) \sqcup (E \cap B)} p^+ d\mu - \int_{(E \cap A) \sqcup (E \cap B)} p^- d\mu = \int_{(E \cap A)} p^+ d\mu + \int_{(E \cap B)} p^+ d\mu - \int_{(E \cap A)} p^- d\mu - \int_{(E \cap B)} p^- d\mu$ . Since  $p^+$  is 0 on  $B$ , and  $p^-$  is 0 on  $A$ , we have that  $\nu^+ \perp \nu^-$ . Thus we have the Jordan decomposition.

## 5

Let us assume that  $\mu(E) = 0$ . As  $E$  is  $\mu$ -null, then  $\mu(E \cap E_n) = 0$  for all  $n$ , by monotonicity of the measure. Then we have  $\nu(E) = \sum_{n=1}^N c_n \mu(E \cap E_n) = 0$ . Thus  $\nu \ll \mu$ . See that the function  $f := \sum_{n=1}^N c_n \chi_{E_n}$  is a good candidate for the Radon-Nikodym derivative.

$$\int_E f d\mu = \int_E \sum_{n=1}^N c_n \chi_{E_n} d\mu = \sum_{n=1}^N c_n \int_X \chi_E \chi_{E_n} d\mu = \sum_{n=1}^N c_n \int_X \chi_{E \cap E_n} d\mu = \sum_{n=1}^N c_n \mu(E \cap E_n),$$

which is the desired result. Thus  $\frac{d\nu}{d\mu} = f$ .

## 6

1. Since  $\nu \ll \mu$ , there exists  $f \in L^1(\mu)$  such that  $\nu(E) = \int_E f d\mu$ . We know that  $f > 0$   $\mu$ -almost everywhere, then assume that  $\nu(E) = 0$ . Thus  $\int_E f d\mu = 0$ . Assume that  $\mu(E) > 0$ . Then  $f$  is greater than zero on all of  $E$ , thus  $\int_E f d\mu > 0$ . However, since  $\nu(E) = 0$  this forces  $\mu(E)$  to be 0. Thus  $\mu \ll \nu$ .

2.

## 7

Let  $\theta := \mu + \nu$ . Then  $f = \frac{d\nu}{d\theta}$ . See that since  $\theta(E) = \int_E 1 d\theta = \mu(E) + \int_E f d\theta$ . Therefore we have  $\mu(E) = \int_E (1 - f) d\theta$ . Thus we have  $\frac{d\mu}{d\theta} = 1 - f$ .

## 8

In  $(\mathbb{N}, \mathbb{P}(\mathbb{N}))$ ,  $\mu$  is the counting measure. Note that the empty set is the only  $\mu$ -null set, since every non-empty set has cardinality more than zero. Then somewhat trivially we have  $\mu(E) = 0 \implies E = \emptyset \implies \nu(E) = 0$ . So  $\nu \ll \mu$ . We have  $\nu$  is  $\sigma$ -finite, thus  $\mathbb{N} = \sum_{n=1}^{\infty} \{n\}$ , where  $\nu(\{n\}) < \infty$ . Thus define  $f : \mathbb{N} \rightarrow \mathbb{R}$ , where  $f(n) = \nu(\{n\})$ . Then we have  $\nu(E) = \int_E f d\mu = \sum_{n \in E} f(n)$ , is the required function. Thus  $f = \frac{d\nu}{d\mu}$ .

## 9

If we assume that  $\frac{d\mu}{d\lambda} \cdot \frac{d\nu}{d\lambda} = 0$   $\lambda$  almost everywhere, we know from the previous assignment that  $\mu(E)$  and  $\nu(E)$  are mutually singular. To see the converse, let us assume that  $\mu \perp \nu$ . Then we have  $X = A \sqcup B$ , where  $\mu$  lives on  $A$  while  $\nu$  lives on  $B$ . Then let  $f := \frac{d\mu}{d\lambda}$ , and  $g := \frac{d\nu}{d\lambda}$ . See that

$$\int_E f g d\lambda = \int_{(E \cap A)} f g d\lambda + \int_{(E \cap B)} f g d\lambda.$$

We know that  $\int_{(E \cap A)} f g d\lambda = \int_{(E \cap A)} f d\nu$ , and  $\int_{(E \cap B)} f g d\lambda = \int_{(E \cap B)} g d\mu$ . As  $E \cap A$  is a  $\nu$ -null set and  $E \cap B$  is a  $\mu$ -null set, we have  $\int_E f g d\lambda = 0 \implies f g = 0$   $\lambda$ -almost everywhere.

## 10

$\nu = \nu^+ - \nu^-$  is a signed measure, where we have  $x = P \sqcup N$ , that is,  $\nu^+(E) = \nu(E \cap P)$ , and  $\nu^-(E) = \nu(E \cap N)$ . Taking  $|\nu| = \nu^+ + \nu^-$ , see that  $\nu^+ \ll |\nu|$  and  $\nu^- \ll |\nu|$ , and thus there must exist  $\frac{d\nu^+}{d|\nu|}$  and

$$\frac{d\nu^-}{d|\nu|},$$

the Radon-Nikodym derivatives. Then see that  $\frac{d\nu^+}{d|\nu|} = \chi_P$ . To see this, see that for  $E$  measurable in  $P$ ,

$$\begin{aligned}\int_E \chi_P d|\nu| &= \int_E \chi_P d\nu^+ + \int_E \chi_P d\nu^- \\ &= \nu^+(E \cap P) + 0 = \nu^+(E),\end{aligned}$$

and similarly for  $\nu^-$ ,  $\frac{d\nu^-}{d|\nu|} = \chi_N$ .

## 11

See that

$$\begin{aligned}\left| \int_X f d\nu \right| &\leq \left| \int_X f d\nu^+ - \int_X f d\nu^- \right| \\ &= \left| \int_X f d\nu^+ \right| + \left| \int_X f d\nu^- \right| \leq \int_X |f| d\nu^+ + \int_X |f| d\nu^- \\ &= \int_X |f| d|\nu|,\end{aligned}$$

as desired.

## 12

We know that the counting measure  $\mu$  only has one  $\nu$ -null set, that is  $\phi$ . Let us say that  $\mu \ll \nu$ . Then for  $\nu(E) = 0$  for some measurable  $E \in \mathcal{A}$ , we have  $\mu(E) = 0$ . To take the contrapositive, we see that if  $\mu(E) \neq 0$ , then  $\nu(E) \neq 0$ . Since we know that  $\mu$  has no non-empty null sets, non  $\mu$ -null sets and non-empty subsets are synonymous. Thus, if  $E \neq \phi$ ,  $\nu(E) \neq 0$ .

For the Dirac measure at  $x_0 \in X$ , we know that  $\delta_{x_0} \ll \nu$  means that  $\nu(E) = 0$  for some  $E \in \mathcal{A}$  implies that  $\delta_{x_0}(E) = 0$ . The Dirac measure is zero if the measurable subset  $E$  does not have  $x_0$ . Thus we can take the contrapositive of the absolute continuity to say that if  $x_0 \in E$ , then we must necessarily have  $\nu(E) \neq 0$ .