Measure Theory Homework 5

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We have a sequence of non-negative measurable functions $\{f_n\}$ such that $0 \leq f_1 \leq \dots f$, where f is a non-negative measurable function that is defined as $\lim_{n\to\infty} f_n$. Then we know that $f_n \leq f \implies \int_X f_n \leq \int_X f$ for all $n \in \mathbb{N}$. This means that $\lim_{n\to\infty} \int_X f_n \leq \int_X f$. Note that from Fatou's lemma we can say that

$$\int_X \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_X f_n.$$

Since $\{f_n\}$ is increasing, we can say that $\liminf_{n\to\infty} f_n = f$. Therefore we have

$$\int_X \liminf_{n\to\infty} f_n = \int_X f \leq \liminf_{n\to\infty} \int_X f_n.$$

We know that integration preserves order; that is, $f_n \leq f_{n+1} \implies \int_X f_n \leq \int_X f_{n+1}$ for all n. This means that $\liminf_{n\to\infty} \int_X f_n = \lim_{n\to\infty} \int_X f_n$. Therefore we have $\int_X f \leq \lim_{n\to\infty} \int_X f_n$, which implies that $\lim_{n\to\infty} f_n = \int_X f$.

2

Let $X_n = \{x \in X : f(x) \geq n\}$. Then

$$\int_{X_n} f = \nu_f(X_n),$$

where ν_f is the measure function determined by f. Note that $X_1 \supseteq X_2 \supseteq \dots$. This means that $\nu_f(X_1) \geq \nu_f(X_2) \geq \dots$. This is clearly a decreasing sequence. Since $f \in L^1(\mu)$, $\nu_f(X_1) < \infty$, $\nu_f(X_n) < \infty$ for all n. Moreover, $A_\infty\{x \in X : f(x) = \infty\}$, and $\mu(A_\infty) = 0$. Thus $\nu_f(A_\infty) = 0$. We can use continuity from above to see that $\nu_f(\cap_{n=1}^\infty X_n) = \lim_{n\to\infty} \nu_f(X_n) = \nu_f(A_\infty) = 0$. This gives us our result.

3

Since $f_n \to f$, and $f_n \geq f_{n+1} \geq f$, we have $\int f_n \geq \int f$. Also note that $\int f_1 < \infty$ means that $\int f_n < \infty$ and $\int f < \infty$. Thus $\left\{\int f_n\right\}$ is a bounded monotone sequence, hence it must be convergent. Since $\int f_n \geq \int f$, we have $\lim_{n\to\infty} \int f_n \geq \int f$. To get the second inequality, see that by Fatou's lemma, $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$. Since $\{f_n\}$ is convergent, $\liminf_{n\to\infty} f_n = \lim_{n\to\infty} f_n = f$. Since $\left\{\int f_n\right\}$ is convergent, we have $\liminf_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n$. Putting this together, we have $\int f \leq \lim_{n\to\infty} \int f_n$. This gives us our result, that is $\lim_{n\to\infty} f_n = \int f$.

4

Let $\varepsilon > 0$. Choose a measurable partition $A_1, \dots, A_m$ of X where

$$\int_X f < \varepsilon + \mathcal{L}(f, P).$$

Denote by E the union of those A_j's such that $\inf_{A_j} g > 0$. Then we have $\mu(E) < \infty$ as otherwise we could have $\mathcal{L}(f, P) = \infty$, contradicting that $\int_X f < \infty$. Now see that

$$\int_{X\setminus E} f = \int_X f - \int_X \chi_E f < (\varepsilon + \mathcal{L}(f, P)) - \mathcal{L}(\chi_E f, P) < \varepsilon,$$

which gives us the desired result.

5

Let $f \in L^+$. We know that a sequence of simple functions $\{f_n\}$ (that are bounded) that approximate f. Moreover, $|f_n| \le |f_{n+1}| \le |f|$. Simple functions are measurable, and we have $f_n \to f$. Note that we can have $f_n \in L^+$, without loss of generality as we can replace the function f_n with $|f_n|$. Using the monotone convergence theorem we can say that $\lim_{n\to\infty} \int f_n = \int f$.

6

Define $f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \begin{cases} \frac{1}{x} & x < 0 \\ \frac{1}{1+x^2} & x \ge 0. \end{cases}$$

This function is measurable, as $f^{-1}(a,\infty)$ is $\left[0, \sqrt{\frac{1}{a}-1}\right)$ for $0 \le a < 1$, ϕ for $a > 1$, $[0,\infty)$ for $a = 0$, and $\left(-\infty, \frac{1}{a}\right) \cup [0,\infty)$ for $a < 0$. These are all measurable, however, see that $f^+ = f|_{x\ge 0}$ is integrable, while $f^- = -f|_{x<0}$ is not integrable as the integral does not exist. To show this, see that the sequence of simple functions $\{\varphi_n\}$, where $\varphi_n(x) = \frac{1}{i}$ for $x \in [i, i+1), i \le n$, and 0 for $x > n$. $\int_{-\infty}^0 \varphi_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. See that $\varphi_n \le f^-$, which means that the integral for f^- cannot exist.

7

Since $f_n = (n+1)x^n$, we see that $\int_0^1 f_n = [x^{n+1}]_{x=0}^{x=1} = 1$, we have $\liminf_{n\to\infty} \int_0^1 f_n = 1$. To find $\liminf_{n\to\infty} f_n(x)$, see that for $0 < x < 1$, $f_n(x) = (n+1)x^n$ will go to zero as $n \to \infty$. For $x = 0$, $\liminf_{n\to\infty} f_n(0) = 0$. For $x = 1$, $\liminf_{n\to\infty} f_n(1) = n+1 \to \infty$. Then we have $\liminf_{n\to\infty} f_n(x)$ is zero for all $x \in [0,1]$ except $x = 1$, where it is $+\infty$. This is zero almost everywhere, hence $\int_0^1 \liminf_{n\to\infty} f_n = 0$. Thus in this case $\int_0^1 \liminf_{n\to\infty} f_n \lneq \liminf_{n\to\infty} \int_0^1 f_n$; that is, Fatou's lemma applies strictly.

8

Define $\{f_n\}$ as a sequence of simple (hence integrable) functions on $\mathbb{R}$, and $f(x)$ is $\frac{1}{x}$ for $x > 0$ and 0 otherwise. We define $f_n(x)$ as

$$\begin{aligned} f_n(x) &= \frac{2^n}{i}, \quad x \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right), 0 \le i \le n2^n \\ &= 0 \text{ otherwise.} \end{aligned}$$

See that

$$\int_{\mathbb{R}} f_n = \int_0^\infty f_n = \sum_{i=0}^{n2^n} \frac{2^n}{i} \cdot \frac{1}{2^n} < \infty.$$

Also note that $f_n \le f_{n+1} \le f$, and see that this sequence converges pointwise to f. However, see that f is not integrable, which means that there could be a sequence of integral functions that converges pointwise to a limit but that limit need not be integrable.

9

We will fix $x \in [0,1]$ and observe its behaviour as n varies. For $x \in \left[0, \frac{1}{3}\right]$ we have $f_n(x) = 0, 1, 0, 1, \ldots$ for $n \ge 1$. For $x \in \left[\frac{1}{3}, 1\right]$, $f_n(x) = 1, 0, 1, 0, \ldots$. Thus $\liminf_{n\to\infty} f_n(x) = 0$, while $\limsup_{n\to\infty} f_n(x) = 1$. See that $\int_0^1 f_{2n} = \frac{1}{3}$, while $\int_0^1 f_{2n+1} = \frac{2}{3}$. Then $\int_0^1 f_n \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$. We can then infer that $\limsup_{n\to\infty} \int_0^1 f_n = \frac{2}{3}$, and $\limsup_{n\to\infty} \int_0^1 f_n = \frac{1}{3}$. Putting all of this together gives us our required result.

10

Let $X_\alpha := \{x \in X : f(x) \geq \alpha\}$. Then define the function $g(x) = \alpha \cdot \chi_{X_\alpha}$. Note that $f(x) \geq g(x)$ for all $x \in X$, and $f \in L^+$, hence

$$\int_X f \geq \int_X g = \int_{X_n} \alpha + \int_{X \setminus X_n} 0 = \alpha \cdot \mu(X_\alpha),$$

which is the desired inequality.

11

Let $A_n := \{x \in X : f(x) \in [n, n+1)\}$, for $n \geq 1$. See that $X_n = \sqcup_{i=n}^{\infty} A_i$, for $n \geq 1$. Define $A_0 = \{x \in X : f(x) = 0\}$, and $A_\infty = \{x \in X : f(x) = \infty\}$. Since $f \in L^1(\mu)$, we must have $\mu(A_\infty) = 0$, as otherwise the integral would not be finite. Thus see that

$$\begin{aligned}
\int_X f &= \int_{A_0} 0 + \int_{A_\infty} \infty + \sum_{i=1}^{\infty} \int_{A_i} f \\
&\geq 0 + 0 + \sum_{i=1}^{\infty} i \cdot \mu(A_i) = (\mu(A_1) + \mu(A_2) + \dots) + (\mu(A_2) + \mu(A_3) + \dots) + \dots \\
&= \sum_{n=1}^{\infty} \mu(X_n) < \infty.
\end{aligned}$$

Thus $f \in L^1(\mu) \implies \sum_{n=1}^{\infty} \mu(X_n) < \infty$. To see the converse, we will first show that $\mu(A_\infty) = 0$. Since $\sum_{n=1}^{\infty} \mu(X_n) < \infty$, we must have $\mu(X_n) \to 0$ and $n \to \infty$. Then we have $\mu(\cap_{n=1}^{\infty} X_n) = \lim_{n\to\infty} \mu(X_n) = \mu(A_\infty) = 0$. Then see that for $K \in \mathbb{N}, 4 \int_{A_0} 0 + \int_{A_\infty} \infty + \sum_{i=1}^{K} \int_{A_i} f \leq 0 + 0 + \sum_{i=1}^{K} (i+1)\mu(A_{i+1}) \leq \sum_{i=1}^{\infty} (i+1)\mu(A_{i+1}) \leq \sum_{i=1}^{\infty} \mu(X_i) + \mu(X_1) < \infty$. Since the left hand side holds for all $K \in \mathbb{N}$, we see that $\int_X f < \infty$, which means that $f \in L^1(\mu)$.

12

We know that $\{f_n\} \subseteq L^+$, and that $f_n \leq f$. From monotonicity of integral, we have that $\int f_n \leq \int f$, which means that $\lim_{n\to\infty} \int f_n \leq \int f$. Now see that since $f_n \to f$, $\liminf_{n\to\infty} f_n = \lim_{n\to\infty} f_n = f$. We also know that since $\{f_n\}$ is convergent, $\left\{\int f_n\right\}$ is also convergent. Therefore by Fatou's lemma see that $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$. Thus we have $\int f \leq \lim_{n\to\infty} \int f_n$. This is the inequality in the other direction, which implies that $\lim_{n\to\infty} \int f_n = \int f$.