The last Home-work

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1. Since R is a PID, we know that $(a,b) = (d)$, for some $d, a, b \in R$. Then we have $d = am + bn$, for some $m, n \in R$. Now we have a vector $v = [a, b]^T \in R^2 \setminus \{0\}$. Then we show that there exists a $2x2$ matrix that does what we want by constructing one. Let the desired matrix by be given by $X = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$. Now we have $Xv = [x_{11}a + x_{21}b, x_{12}a + x_{22}b]^T = [d, 0]^T$. Comparing terms, we have $x_{12}a + x_{22}b = 0$. Then we have $x_{12}a = -x_{22}b$, which implies that $x_{12} | -b$, and $x_{22} | a$. It is easy to see that $x_{12} = -a/d$ and $x_{22} = b/d$ does the trick. For $x_{11}a + x_{21}b = d$, see that $x_{11} = m$ and $x_{21} = n$ are good choices, since their linear combination produces d. Thus see that
$$X = \begin{pmatrix} m & n \\ -b/d & a/d \end{pmatrix}$$
is a matrix that achieves the intended result.

2. The above result shall be of much use to us. We see that we want to send a_{11} to d, that is the gcd of a_{11} and a_{i1}, and a_{i1} to 0. Let $m, n \in R$ such that $d = a_{11}m + a_{i1}n$. Define the matrix $\tilde{X} = (x_{kl})$ thus— $x_{11} = m, x_{1i} = n, x_{i1} = -a_{i1}/d, x_{ii} = a_{11}/d$. Also we have $x_{kk} = 1$ if $k \neq 1, i$. All other elements are 0. Then we have $A' = \tilde{X}A = (a'_{kl})$, where $a'_{11} = a_{11}m + a_{i1}n$, $a'_{i1} = (-a_{i1})a_{11} + (a_{11})a_{i1} = 0$, and $a'_{kl} = a_{kl}$ for $k \neq 1, i$. We need to see that this here matrix is invertible. For $\tilde{X}$ a $m \times m$ matrix, we want $\det \tilde{X}$. We expand the determinant along the first row. Then we have $\det \tilde{X} = m \det \tilde{X}[1|1] + (-1)^{i+1}((-1)^{i+1} \det \tilde{X}[1|i])$. $\tilde{X}[1|1]$ is a diagonal matrix with a_{11} on the a_{ii}th entry, and 1 otherwise on the diagonal. Thus $\det \tilde{X}[1|1] = a_{11}$. $\tilde{X}[1|i])$ is a matrix with x_{i1} at the $(i-1, 1)$th entry, with every element below and above it zero. We take the determinant along this column, we have $(-1)^{i-1+1}x_{i1} \cdot \det I_{m-2} = (-1)^i x_{i1}$. Thus see that $\det \tilde{X} = ma_{11} + (-1)^{2i+1}(-a_{i1}) = 1$, means that $\tilde{X}$ is invertible.

3. The above result and the result above that shall be of much use to us. If $A = 0$, then there is nothing to do. We then have $A \neq 0$. Without loss of generality, we take $a_{11} \neq 0$. This is because we can shift the row with a non-zero element to the top, then send the column with that element to the first column. Now using the above result, there is a $\tilde{X}_1$ such that $a_{21} = 0$. The value of a_{11}

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We have M a $R-$module which has itself as a generating set. Then $\pi : R^{\oplus M} \twoheadrightarrow M$ is the surjective map sending e_m to m. We see that $\pi(e_{rm} - re_m) = \pi(e_{rm}) - r\pi(e_m) = rm - rm = 0$. Also, $\pi(e_{m_1+m_2} - e_{m_1} - e_{m_2}) = \pi(e_{m_1+m_2}) - \pi(e_{m_1}) - \pi(e_{m_2}) = (m_1 + m_2) - m_1 - m_2 = 0$. Therefore we have $e_{rm} - re_m \in \ker \pi$, and $e_{m_1+m_2} - e_{m_1} - e_{m_2} \in \ker \pi$. Thus we have
$$(e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2}) \subseteq \ker \pi.$$
Let us have $\sum_{m \in M} r_m e_m \in R^{\oplus M}$. Note that there are only finitely many terms in the summation. See that
$$\pi(\sum_{m \in M} r_m e_m) = \sum_{m \in M} r_m \pi(e_m) = \sum_{m \in M} r_m m = \pi(\sum_{m \in M} e_{r_m m}).$$

This means that $\pi(\sum_{m\in M} r_m e_m - \sum_{m\in M} e_{r_m m}) = 0$. This, in turn implies that $\sum_{m\in M} r_m e_m - e_{r_m m} \in \ker \pi$. We also see that

$$\pi(\sum_{m\in M} r_m e_m) = \sum_{m\in M} r_m \pi(e_m) = \sum_{m\in M} r_m m = \pi(e_{\sum_{m\in M} r_m m}).$$

This means that $\sum_{m\in M} r_m e_m - e_{\sum_{m\in M} r_m m} \in \ker \pi$. Given an element in $R^{\oplus M}$, we can choose which summands to clump and which to leave unchanged. Either ways, we see that we get a linear combination of $re_m - e_{rm}$ and $e_{m_1+m_2} - e_{m_1} - e_{m_2}$, which implies that $\ker \pi \subseteq (e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2})$. This gives us the desired equality.

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1. We take the module $\mathbb{Z}[q_1, q_2]$, where $q_1, q_2 \in \mathbb{Q}$.

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