Complex Analysis Homework 4

March 27, 2024

Solution of problem 1*:* Since n is odd, $\gcd(2, n) = 1$. Now see that -1 is a primitive 2nd root of unity, while let x be a primitive nth root of unity. Then we claim that $-x$ is our required $2n$th primitive root of unity. See that $(-x)^n = -1$, which means that the order of this term must be strictly greater than n. However, see that $(-x)^{2n} = 1$, which means that the order of $-x$ must divide $2n$, which means that the order must be exactly $2n$. Since $-x$ is clearly in the nth cyclotomic field, we have our solution. $\square$

Solution of problem 2*:* Let us assume that a function f holomorphic on some domain D exists, and we take $a \in D$, and take some $r > 0$ such that the ball of radius r centered at a is in D. Considering the Taylor Series expansion of $f(z)$ at a, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

We want to see if this series converges. See that $a_n = \frac{f^{(n)}(a)}{n!}$. We want to examine $|a_n|$ as $n \to \infty$. See that $|a_n| \geq \frac{n^n n!}{n!} = n^n$. See that $\limsup_{n\to\infty}(a_n)^{\frac{1}{n}} = \lim_{n\to\infty} n$ which diverges to $+\infty$. Then this means that the radius of convergence of this function is 0, which means our function cannot exist.

$\square$

Solution of problem 3*:* $\square$

Solution of problem 4*:* $\square$

Solution of problem 5*:* $\square$

Solution of problem 6*:* $\square$

Solution of problem 7*:* $\square$

Solution of problem 8*:* $\square$