Algebra 2 Homework 7

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Solution of problem 1*:* □

Solution of problem 2*:* Since $A^k = I$ for some $k \geq 1$, we have that $x^k - 1 = 0$ must be the minimal polynomial. Solving this over $\mathbb{C}$, we have the kth roots of unity, which are exactly k many. The characteristic polynomial must have n roots, all of whom must be kth roots of unity.

For $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, see that $A^k = \begin{pmatrix} 1 & k\alpha \\ 0 & 1 \end{pmatrix}$, for any $k \in \mathbb{N}$. This can easily be checked by induction. See that for $k = p$, we must have I since the field has characteristic p. See that the characteristic of A is just $x^2 - 1 = 0$, which gives us ± 1 as the eigenvalues. We see that the $\ker \dim(A - I) = \{(x, y) \in \mathbb{R}^2 \mid A - I(x, y)^T = 0\} = \dim(\text{ span } (1, 0)) = 1$, while $\ker \dim(A + I) = \{(x, y) \in \mathbb{R}^2 \mid A + I(x, y)^T = 0\} = \dim(\text{ span } (0, 0)) = 0$, which clearly does not add up to 2, since 1 and 2 are not the same number. Thus for $\alpha \neq 0$, this matrix is not diagonalisable. If $\alpha = 0$, in the eigenvalue -1, we would have the null space as $\text{ span } (0, 1)$, which would give us a basis for $\mathbb{R}^2$. □

Solution of problem 3*:* If we see K/F_1, then it is clear that since $K = F_1(\sqrt[8]{2})$, we have a degree 8 extension. To understand $\mathrm{Aut}(K/F_1)$, see that we only need to understand where $\sqrt[8]{2}$ goes. We have eight choices. We can send $\sqrt[8]{2}$ to any □

Solution of problem 4*:* □

Solution of problem 5*:*

1. $[K : F] = n < \infty$, then we say that K is represented as by a finite $F-$basis, by $\{b_1, \dots, b_n\}$. Let $x \in K$, where $x = \sum_{i=1}^n c_i b_i$, where $c_i \in F$ for $i = 1, \dots, n$. Now we have $\alpha \cdot x = \sum_{i=1}^n c_i(\alpha \cdot b_i)$. Since $\alpha \cdot b_i \in K$, $\alpha \cdot b_i = \sum_{j=1}^n a_{ji} b_j$. Then we can put this in matrix form, by $T_\alpha := (a_{ji})$. By this, it is clear that T_α is a linear transformation, since it can be written as a matrix.
2. Let $m(x)$ be the minimal polynomial for α with degree d_1, and $m'(x)$, be the minimal polynomial for A with degree d_2. If $d_1 \geq d_2$, then

$$m(x) = q_1(x)m'(x) + r_1(x),$$

□

Solution of problem 6*:* We have proven in the previous assignment that $f(x) = x^p - x - a$ is irreducible over $\mathbb{F}_p$, and that it is separable. We know that if α is a root, then so is $\alpha + 1$. So it is easy to see that the □

Solution of problem 7: In $E = \mathbb{Q}(\sqrt{1+\sqrt{2}})$,let $x = \sqrt{1+\sqrt{2}}$. Then $x(x^2-1)^2 = 2$. We get that $x^4 - 2x^2 - 1 = 0$. We can explicitly see that none of its roots are rational, so the polynomial we have must be the minimal polynomial. We can calculate the conjugates, which are

$$\pm\sqrt{1+\sqrt{2}}, \pm i\frac{1}{\sqrt{1+\sqrt{2}}}.$$

So it is easy to see that i is missing. Thus, by adjoining i, we can have a normal and separable extension containing $\sqrt{1+\sqrt{2}}$. We propose that $K = \mathbb{Q}(\sqrt{1+\sqrt{2}}, i)$ is the Galois extension. We know K contains all the conjugates, so the Galois field must be sandwiched between K and F. But since $[K : F] = 2$, which is prime, we must have that either the field is equal to E or F, and it cannot be F, since the imaginary conjugates are not in F, the Galois closure must be E. $\square$

Solution of problem 8: $\square$

Solution of problem 9: $\square$