Algebra 2 Homework 5

March 13, 2024

Solution of problem 1*:*

1. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where a_i is the coefficient of the term of degree i, and $a_n \neq 0$. See that the reverse of this polynomial will have degree n, Since $x^n f(1/x) = x^n \left(a_0 + a_1x^{-1} + \cdots + a_nx^{-n}\right)$
2. Let the constant coefficient and leading term both be non-zero (If not, then one could have $x^2 + x$, which is reducible, while its reverse, $x + 1$ is irreducible). It is easy to see that the reverse of the reverse is just the original polynomial. That is, $x^n(x^{-n}f(x)) = f(x)$. Thus we only need to show that if f is reducible, g is reducible. If $f(x) = p(x)q(x)$, where p, q are not units, and $d_p := \deg p(x) > 1$ and $d_q := \deg q(x) > 1$. Since the constant term of f is non-zero, the constant term of p and q must also be non-zero. Replacing x by $\frac{1}{x}$, and multiplying on both sides by x^n, we get

$$x^n f\left(\frac{1}{x}\right) = x^{d_p} p\left(\frac{1}{x}\right) \cdot x^{d_q} q\left(\frac{1}{x}\right) = \ell(x)m(x),$$

which gives us a factorisation for the reverse of f.

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Solution of problem 2*:* We begin by enumerating all irreducible polynomials of degree $1, 2$ and 4. See that x and $x + 1$, the only degree one polynomials, are irreducible. For degree 2, we have four choices. Of these, $x^2, x^2 + x$ and $x^2 + 1$ are reducible. $x^2 + x + 1$ is irreducible since it has no roots, plugging in 0 and 1. For degree 4, we have sixteen choices. Of these, we must weed out the reducible polynomials. We can also calculate the irreducibles of degree 3 easily, since we can use them to find the reducible polynomials of degree 4. We have eight possibilities for polynomials of degree four, we can eliminate six of them easily, with a mixture of clever thinking and brute force ($x^3, x^3+1, x^3+x^2+x+1, x^3+x, x^3+x^2, x^3+x^2+x$ are reducible) we can see that $x^3 + x + 1$ and $x^3 + x^2 + 1$ are irreducible. A clever trick that shall aid us in our effort to weed out reducible polynomials is to notice that if there is a polynomial which has evenly many non-zero terms, than it must be reducible since then we have $1 + \cdots + 1$ even number of times. Therefore an irreducible polynomial must necessarily have odd number of terms with constant term 1. This gives us $x^4 + x^3 + x^2 + x + 1, x^4 + x^3 + 1, x^4 + x^2 + 1$, and $x^4 + x + 1$. In the case of $x^4 + x^2 + 1$, see that it is $(x^2 + x + 1)^2$, so this must be excluded. Thus we only have three irreducible polynomials in $\mathbb{F}_2[x]$. $x(x + 1)(x^2 + x + 1) = (x^2 + x)(x^2 + x + 1) = x^4 + x^2 + x^2 + x = x^4 + x$. Then multiplying the remaining polynomials, we have

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\begin{aligned}
(x^4+x)(x^4+x^3+x^2+x+1)(x^4+x^3+1)(x^4+x+1) &= ((x^4+x^3+1)^2+(x^4+x^3+1)(x^2+x))((x^4+x)^2 \\
&= (x^8+x^6+1+x^6+x^5+x^2+x^5+x^4+x)(x^8+x^4 \\
&= (x^8+x^4+x^2+x+1)(x^8+x) \\
&= x^{16}+x^{12}+x^{10}+x^9+x^8+x^9+x^5+x^3+x^2+x \\
&= x^{16}+x^{12}+x^{10}+x^8+x^5+x^5+x^3+x^2+x
\end{aligned}
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Solution of problem 3*:* □

Solution of problem 4*:* □

Solution of problem 5*:* □

Solution of problem 6*:* □