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1. This statement is true. Take any vector space V , which necessarily has a Hamel basis, trivially the entire space is a basis. Then for a choice of basis (b_n) we have for all $x \in V$, $x = \sum k_i b_i$, where the sum is finite. Then define $\|x\| := \max\{|k_i|\}$. See that this fulfills the definition of a norm.
2. This statement is false. For sake of contradiction, let $(X, \|\cdot\|)$ be a normed linear space such that the induced metric is the discrete metric. Then for $x, y \in X, x \neq y$ we must have $\|x - y\| = 1$. Note that $2x \neq 2y$, so we must have $\|2x - 2y\| = 2\|x - y\| = 2$, but by the discrete metric the answer should still be 1! Thus there can be no such norm.

2

We wish to show that the function $\|\cdot\|$ on X satisfies the triangle inequality iff the closed unit ball is convex. Assume that the function $\|\cdot\|$ is indeed a norm. Then let $x, y \in D$, the closed unit ball. Then $\|x\|, \|y\| \leq 1$. Now we have for $\alpha \in [0, 1]$ $z = \alpha x + (1 - \alpha)y$. See that

$$\|z\| = \|\alpha x + (1 - \alpha)y\| \leq \alpha\|x\| + (1 - \alpha)\|y\| \leq \alpha \cdot 1 + (1 - \alpha) \cdot 1 \leq 1,$$

thus we have $z \in D$.

Take two elements $x, y \in X$ both non-zero, since if either were zero the inequality would be trivial. Then

$$\|x + y\| = (\|x\| + \|y\|) \cdot \left\| \alpha \frac{x}{\|x\|} + (1 - \alpha) \frac{y}{\|y\|} \right\|,$$

where $\alpha = \frac{\|x\|}{\|x\| + \|y\|}$. Note that $\frac{x}{\|x\|} = \frac{y}{\|y\|} = 1$, thus we can use the convexity condition to see that $\frac{\|x+y\|}{\|x\| + \|y\|} \leq 1$, which is the triangle inequality.

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1. Pick a $f \in C([a, b])$. Then $|f| \leq M = \sup\{|f(x)| : x \in [a, b]\}$. Now see that

$$\int_a^b |f(t)|^p dt \leq (b - a)M^p \geq 0.$$

Thus $\|f\|_p \geq 0$. For $f = 0$, we have $M = 0$, so $\int_a^b |0|^p dt = 0$. If $\int_a^b |f(t)|^p dt = 0$, then see that $0 \leq (b - a)M^p \geq 0$. Thus we must have

$$(b - a)M^p = 0 \implies M = 0 \implies f = 0.$$

To see the next axiom, for $\alpha \in \mathbb{K}$, we have $\int_a^b |\alpha f(t)|^p dt = \int_a^b |\alpha|^p |f(t)|^p dt$. Then

$$\|\alpha f\|_p = \left(|\alpha|^p \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} = |\alpha| \|f\|_p.$$

Now let $f, g \in C[a, b]$. Then we have to prove Minkowski's inequality to show the triangle inequality.

$$\begin{aligned}
\|f + g\|_p^p &= \int_a^b |f + g|^p dx \\
&= \int_a^b |f + g| \cdot |f + g|^{p-1} dx \\
&\leq \int_a^b |f| |f + g|^{p-1} dx + \int_a^b |g| |f + g|^{p-1} dx \\
&\leq \left(\int_a^b |f|^p dx + \int_a^b |g|^p dx \right) \left(\int_a^b |f + g|^{(p-1) \frac{p}{p-1}} dx \right)^{1-\frac{1}{p}} \quad (\text{Hölder's inequality}) \\
&= (\|f\|_p + \|g\|_p) \frac{\|f + g\|_p^p}{\|f + g\|_p},
\end{aligned}$$

which yields the required result.

2.

4

1. $\|f_1 - F\|_\infty$ is to be found, where F is the subspace of constant functions. Unfolding the term, we get

$$\|f_1 - F\|_\infty = \inf_{c \in \mathbb{R}} \left\{ \sup_{t \in [0, 1]} |t - c| \right\},$$

which for $c \in [-1, 0] \cup [1, 2]$ is 1, while in $(0, 1)$ it decreases to $\frac{1}{2}$ then goes back up to 1. Thus, we must have $\|f_1 - F\|_\infty = \frac{1}{2}$.

2. We want to now see the distance between $f_2 = t^2$ and G , the space of all polynomials with degree at most 1. Then for some polynomial $-ax - b \in G$, we want to see $\sup_{t \in [0, 1]} \{ |t^2 + ax + b| \}$

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Let Y and X/Y be Banach spaces. Then take (x_n) to be a Cauchy sequence in X . That is, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m, n \geq N$ we have

$$\|x_m - x_n\| < \varepsilon.$$

By the canonical projection to X/Y we can see that the sequence $(x_n + Y)$ is also Cauchy, since we have

$$\|x_m - x_n + Y\| \leq \|x_m - x_n + 0\| < \varepsilon.$$

Since X/Y is Banach, we have $(x_n + Y) \rightarrow (x_0 + Y)$. Now let $y_n :=$

Let X and X/Y be Banach spaces. Then take (y_n) , a Cauchy sequence in Y . Thus for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$, we have $\|y_m - y_n\| < \varepsilon$. As a Cauchy sequence in X , this must converge to some element $y_0 \in X$. We now need to show that $y_0 \in Y$. But since Y is closed and y_0 is a limit point, we must have $y_0 \in Y$.

Let X and Y be Banach spaces. Take (x_n) as a sequence in X such that $(x_n + Y)$ is Cauchy. Then pick $\varepsilon > 0$. Then there is a $N \in \mathbb{N}$ such that for $m, n \geq N$ $\|(x_m - x_n) + Y\| < \varepsilon$. See that $\|x_m - x_n\| \leq \|(x_m - x_n) + y\| + \|y\|$. Since the expression on the left is independent of y , then we can set $\|(x_m - x_n) + y\|$ to $\|(x_m - x_n) + Y\|$, and $\|y\|$ to 0, as $0 \in Y$. Then we have $\|x_m - x_n\| < \varepsilon$, which means that (x_n) is Cauchy. Then since X is Banach, we have $x_n \rightarrow x$. Then we propose that $x + Y$ is the limit of $(x_n + Y)$. See that $\|(x_n + Y) - (x + Y)\| = \|(x_n - x) + Y\| = \inf_{y \in Y} \{ \|(x_n - x) + y\| \}$. Then, for any choice of $y \in Y$, we have

$$\|(x_n - x) + Y\| \leq \|(x_n - x) + y\|.$$

Then for $y = 0$, we have

$$\|(x_n - x) + Y\| \leq \|x_n - x\| < \varepsilon,$$

which implies that X/Y is Banach.

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Assume that X is a Banach space. Let (x_n) be an absolutely convergent series in X , that is, $\sum_{i=1}^n \|x_i\|$ converges to $c \in \mathbb{R}$ as $n \rightarrow \infty$. Thus for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$,

$$|||x_n|| - \alpha| < \varepsilon.$$

Now we have

For the converse, take a Cauchy sequence (x_n) . Pick $\varepsilon > 0$. Pick $N(n) \in \mathbb{N}$ such that for $m, n \geq N$, we have $\|x_m - x_n\| < \frac{\varepsilon}{2^n}$. Now see that $\sum_{i=1}^n \|x_{i+1} - x_i\|$

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Let ℓ^p be the space of all p -power summable sequences. Then $\|(x_n)\| = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. Now we want to show that the space $K = \{(x_n) \in \ell^p : x_i = 0 \forall i > n, n \in \mathbb{N}\}$ is dense in ℓ^p . Take any $(x_n)_{n \in \mathbb{N}}^{(0)} \in \ell^p$. There is a sequence of elements in ℓ^p $(x_n)^{(m)} \subseteq K$ such that $x_n^{(m)} = x_n$ if $n \leq m$, and 0 otherwise. See that $\|(x_n)^{(m)} - (x_n)^{(0)}\|_p = (\sum_{i=m+1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. Since $\|(x_n)^{(m)}\|_p \rightarrow \|(x_n)^{(0)}\|_p$ as $m \rightarrow \infty$, we must have for a choice of $\varepsilon > 0$ there being $N \in \mathbb{N}$ such that for $m \geq N$, $|\|(x_n)^{(m)}\|_p - \|(x_n)^{(0)}\|_p| < \varepsilon$. Multiplying on both sides by $\frac{\|(x_n)^{(m)}\|_p^p - \|(x_n)^{(0)}\|_p^p}{\|(x_n)^{(m)}\|_p - \|(x_n)^{(0)}\|_p}$, and after a change in the value of ε , we get

$$|\|(x_n)^{(m)}\|_p^p - \|(x_n)^{(0)}\|_p^p| < \varepsilon.$$

Thus we have shown that K is dense in ℓ^p . For a fixed m , and a fixed n , look at $x_n^{(m)}$. This is a real number or a complex number, which can be approximated by a sequence of rationals (or elements of $\mathbb{Q}(i)$) such that $x_n^{(m)}$ is its limit. Now we have a sequence (t_k) , where $t_k \rightarrow x_n^{(m)}$ as $k \rightarrow \infty$.

Now let us see that $K' = \{(x_n) \in K : x_n \in \mathbb{Q} \text{ or } \mathbb{Q}(i)\} \subset K$ is countable, and as we saw above must be dense in ℓ^p . Thus it is separable.

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We first show that ℓ^∞ is not separable. Take the space K of all sequences in ℓ^∞ such that their entries are either 0 or 1. Then for $x \neq y \in K$, we have $\|x - y\|_\infty = 1$, since at least one entry is different between the two. Then we have uncountably points. consider a ball of radius $\frac{1}{2}$ centred at $x \in K$, for all such points in K . We know what uncountably many open sets, all of which are disjoint from each other. Let S be a possibly dense set. Then we have that each open ball contains at least one point of S . This then means that there must be uncountably many elements, so ℓ^∞ .

If there was a Schauder basis for ℓ^∞ , then for all $x \in \ell^\infty$ we would have a sequence (a_1, a_2, \dots) described using the basis. We can approximate each term by a sequence of elements in \mathbb{Q} or $\mathbb{Q}(i)$, which would then mean that we would have all the rational points of ℓ^∞ as a dense subset, which contradicts the fact that the space is not separable. Thus there can be no Schauder basis.

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We want to see if this function is continuous. Pick a $x_0 \in \mathbb{R}$. Now for all $\varepsilon > 0$ have $f(z_0 + h) - f(z_0) = f(h)$.

See that $f(2) = 2f(1)$, $f(0) = 2f(0) \implies f(0) = 0$, and that $f(0) = 0 = f(1) + f(-1) \implies f(-1) = -f(1)$. Thus $f(n) = nf(1)$ for all $n \in \mathbb{Z}$. We know that $f(1) = f(\frac{1}{n}) \implies f(\frac{1}{n}) = \frac{1}{n}f(1)$. Thus we have $f(q) = qf(1)$ for all $q \in \mathbb{Q}$. Since any real number can be approximated by a sequence of rationals, we can easily see that for $q_n \rightarrow r$ and $n \rightarrow \infty$, we have $f(q_n) = q_n f(1) \rightarrow r f(1)$. Thus we can extend this to all real numbers. Now we have $|f(z_0 + h) - f(z_0)| = |hf(1)| < \varepsilon$, for a choice of $h = \frac{\varepsilon}{|f(1)|}$.

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Let $P([0, 1])$ be the space of all real polynomials defined on $[0, 1]$ be a real vector space. Let the norm of a polynomial $f(x) = a_0 + a_1x + \dots + a_nx^n \in P([0, 1])$ be given thus: $\|f\| = |a_0| + |a_1| + \dots + |a_n|$. Now

see that the operator $I : P([0, 1]) \rightarrow P([0, 1])$ such that $x^t \mapsto \frac{x^{t+1}}{t+1}$, we then see that

$$\|I\| = \sup_{0 \neq x \in P([0, 1])} \frac{\|If\|}{\|f\|} = \frac{a_0x + \dots + \frac{a_n}{n+1}a_{n+1}}{a_0 + \dots + a_nx^n} = \frac{|a_0| + \left|\frac{a_1}{2}\right| + \dots + \left|\frac{a_n}{n+1}\right|}{|a_0| + \dots + |a_n|} \leq 1.$$

Also see that this supremum is indeed attained since $I(a_0) = a_0x$, and in this case $\frac{\|I(a_0)\|}{\|a_0\|} = 1$. Thus we have $\|I\| = 1$. We wish to find the inverse of this operator, see that the differential operator D such that $x^t \mapsto tx^{t-1}$, is the required inverse. However, see that for $f(x) = x^n$, we have $Df = I^{-1}f = nx^{n-1}$. Now we have

$$\|D\| = \sup_{0 \neq x \in P([0, 1])} \frac{\|Df\|}{\|f\|} \geq \frac{\|Dx^n\|}{\|x^n\|} = \frac{n}{1}.$$

Thus we have that our operator D is unbounded, since for any chosen $N \in \mathbb{N}$ we can choose x^{N+1} , such that $\frac{\|Dx^{N+1}\|}{\|x^{N+1}\|}$ is larger.

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Since the linear functional $f : X \rightarrow \mathbb{R}$ is unbounded, we have a sequence (x_n) such that $f(x_n) > n\|x_n\|$. We can say without loss of generality, and by discarding non-zero elements, we have a sequence (x_n) of norm 1. Pick $x \in X$. Then see that the sequence $z_n = x - \frac{f(x)}{f(x_n)}x_n$, which is clearly in K , the kernel of f . Also it can be seen that as $n \rightarrow \infty$, we have $\|z_n\| \leq$

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