

1

We will use Cauchy's integral formula here. The point $z_0 = 2$ is in the circle of radius 3, so we can use the formula. We have

$$\int_C \frac{2z^2 - z - 2}{z - z_0} dz = f(z_0) = 2\pi i g(z_0),$$

where $g(z) = 2z^2 - z - 2$ for all $z_0 \in D = f(C)$. Then we have $f(2) = 2\pi i(2(2)^2 - 2 - 2) = 8\pi i$.

2

1. We will use Cauchy's integral formula here. Since the specified curve is a square, and it contains $z = \frac{-1}{2}$. Thus we have

$$\int_C \frac{3z}{2z+1} dz = \pi i \frac{1}{2\pi i} \int_C \frac{3z}{z + \frac{1}{2}} dz = \pi i \frac{-3}{2} = \frac{-3}{2},$$

where the formula is used on $3z$.

2. We will use the Cauchy's integral formula for higher derivatives. We have $z = 0$ in the curve, thus we have

$$\int_C \frac{\cosh(z)}{z^{2024}} dz = \frac{2\pi i}{2023!} \frac{d^{2023} \cosh(z)}{dz^{2023}} \Big|_{z=0},$$

which requires us to evaluate repeated derivatives of $\cosh(z)$. We know that $\cosh(z) = \frac{e^z + e^{-z}}{2}$, so $\frac{d^n \cosh(z)}{dz^n} = \frac{e^z + (-1)^n e^{-z}}{2}$. For $n = 2023, z = 0$, we get $\frac{e^0 + (-1)^{2023} e^{-0}}{2} = 0$. Thus we have $\int_C \frac{\cosh(z)}{z^{2024}} dz = 0$.

3

1. We have the annulus $0 < |z - 1| < 2$. The function $f(z) = \frac{z}{(z-1)(z-3)}$ can be written as $f(z) = \frac{A}{(z-1)} + \frac{B}{(z-3)}$, where $A = -0.5$ and $B = 1.5$. The term on the left can be ignored for the time being. We have

$$\begin{aligned} \frac{B}{(z-3)} &= \frac{-B}{2} \cdot \frac{1}{1 - \frac{(z-1)}{2}} \\ &= \frac{-B}{2} \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} \text{ for } 0 < |z-1| < 2. \end{aligned}$$

Since we are in the annulus, we can define the Laurent series $\sum_{n=-\infty}^{\infty} a_n(x-1)^n$ as follows— $a_{-1} = \frac{-1}{2}$, $a_n = \frac{-3}{2^{n+2}}$ for $n \geq 0$, and 0 otherwise.

- 2.

4

Let $D = C_R(0)$ be the open disc of radius R centered around 0. We claim that $f(z) = 2z^2 + z$ is injective on this disc. We must have that for $z_1, z_2 \in D$, we have $f(z_1) = f(z_2) \implies z_1 = z_2$. Then $f(z_1) - f(z_2) = 2z_1^2 + z_1 - 2z_2^2 - z_2 = 2(z_1 - z_2)(2(z_1 + z_2) + 1)$. If this expression is 0, we claim that $2(z_1 + z_2) + 1 \neq 0$ for any $z_1, z_2 \in D$. Clearly $\frac{-1}{2} \notin D$, so $R < \frac{1}{2}$. If we try to solve $z_1 + z_2 = \frac{1}{2}$, then for $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, we have $y_1 = -y_2$ and $x_1 + x_2 = \frac{1}{2}$. Thus see that we need to avoid the point $z = \frac{-1}{4}$ as well. Thus $R < \frac{1}{4}$. If $R > \frac{1}{4}$, it is possible to have points such that the above equations are satisfied. Thus $R = \frac{1}{4}$. On the boundary, any point except $z = \frac{-1}{4}$ works because of the above reason. However, if we choose $z_1 = z_2 = \frac{-1}{4}$, it would be not possible, but note that here $f(z_1) - f(z_2)$ trivially. Thus this works even on the boundary.

5

We decompose f as $u + iv$, where u, v are real valued functions. Then we have

$$(u + iv)^2 = u - iv \implies u^2 - u - v^2 + (2uv + v)i = 0.$$

This implies that $2uv + v = 0$, thus either $v = 0$ on D or $u = \frac{-1}{2}$ on D . Substituting the two in the real part, we have $u^2 = u$ which means $u = 0, 1$ in the first case, and that $v^2 = \frac{3}{4}$ which implies that $v = \pm \frac{\sqrt{3}}{2}$ for the second case. Thus there are four constant functions that satisfy the given condition— $f = 0, 1, \omega, \omega^2$, where ω is a primitive third root of unity.

6

We have f is entire, and its image is in the upper half plane. Then we have $f = u + iv$, where $v \geq 0$. Then $g = \exp(if)$ is also an entire function. We have $|g| = |e^{i(u+iv)}| = |e^{iu}| \cdot |e^{-v}|$, which means that $|g| = |e^{-v}|$, and since $v \geq 0$, we must have that g is a bounded function. This means that g must be constant by Liouville's theorem, which implies that f is constant.

7

8

We are asked to find the maximum modulus of a polynomial on the unit disc D . By the maximum modulus principle, we know that the maximum must be attained on the boundary of the circle, that is the set of points $z \in D$ such that $|z| = 1$. Thus let $z = e^{i\theta}$, and then see that

$$f(e^{i\theta}) = ae^{2i\theta} + 2(|a|^2 - 1)e^{i\theta} - \bar{a} = e^{i\theta} \left(ae^{i\theta} + 2(|a|^2 - 1) - \overline{ae^{i\theta}} \right).$$

Now we can say that

$$|p(z)|^2 = |e^{i\theta}|^2 \left((2|a|^2 - 2)^2 + \left(ae^{i\theta} - \overline{ae^{i\theta}} \right)^2 \right).$$

Now see that the imaginary part is the only part dependent on θ . Let us only look at that. Let $a = re^{i\varphi}$. Here, $r = |a|$. Then we have $g(z) = \left(ae^{i\theta} - \overline{ae^{i\theta}} \right) = 2\Im(re^{i(\theta+\varphi)}) = 2r \sin(\theta + \varphi)$. We can differentiate this with respect to θ to see that $g'(z) = 2r \cos(\theta + \varphi)$. Thus g has a critical point at $\theta = -\varphi$. Plugging this into $|p(z)|^2$, we get

$$|p(z)|^2 = (2(r^2 - 1))^2.$$

This means that $|p(z)| \leq 2(r^2 - 1) \leq 2$, as required.