Functional Analysis

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Solution of problem 1*:* We need to check that rules of inner products hold—

1. For $A = B$, we have $\langle A, A\rangle = tr(AA^*) = \sum_{i,j} |a_{ij}|^2 \geq 0$, where a_{ij} denotes the elements of A. Moreover, $||A|| = 0 \implies |a_{ij}| = 0$ for all $1 \leq i, j \leq n \implies A = 0$.
2. $\langle B, A\rangle = tr(BA^*) = tr(A\overline{B}^T)$. See that $A\overline{B}^T(c_{ij})$ is such that $c_{ij} = \sum_{i=1}^n a_{i1}\overline{b_{j1}}$. See that $\overline{c_{ij}} = \sum_{i=1}^n \overline{a_{i1}}b_{j1}$, gives us $\sum_{1\leq i,j\leq n} a_{ij}\overline{b_{ij}}$. Note that replacing A and B just gives us the conjugate, which is the desired result, that
$$\langle B, A\rangle = \overline{\langle A, B\rangle}.$$
3. We have $\langle A + B, C\rangle = tr((A + B)C^*)$. We know that
$$tr((A+B)C^*) = \sum_{1\leq i,j\leq n} (a_{ij}+b_{ij})\overline{c_{ij}} = \sum_{1\leq i,j\leq n} a_{ij}\overline{c_{ij}} + \sum_{1\leq i,j\leq n} b_{ij}\overline{c_{ij}} = tr(AC^*)+tr(BC^*).$$
4. We have
$$\langle(\alpha A), B\rangle = tr(\alpha AB^*) = \sum_{1\leq i,j\leq n} \alpha a_{ij}\overline{b_{ij}} = \alpha \sum_{1\leq i,j\leq n} a_{ij}\overline{b_{ij}} = \alpha \langle A, B\rangle .$$

Therefore we have defined an inner product. To solve the second part, see that since we can apply the Cauchy Schwarz inequality on inner product spaces, we have
$$|\langle A, B\rangle|^2 \leq ||A||^2 \cdot ||B||^2,$$
which gives us the required answer. □

Solution of problem 2*:* We calculate $||x - y||^2 + ||x - z||^2 - ||x - u||^2$. Then see that
$$\begin{aligned}
t\,||x - y||^2 + (1 - t)\,||x - z||^2 - ||x - u||^2 &= t(||x||^2 + ||y||^2 - \langle x, y\rangle - \langle y, x\rangle) \\
&\quad + (1 - t)(||x||^2 + ||z||^2 - \langle x, z\rangle - \langle z, x\rangle) \\
&\quad - (||x||^2 + ||u||^2 - \langle x, u\rangle - \langle u, x\rangle) \\
&= ||x||^2 - \langle x, u\rangle - \langle u, x\rangle + t\,||y||^2 + (1 - t)\,||z||^2 \\
&\quad - ||x||^2 - ||u||^2 + \langle x, u\rangle + \langle u, x\rangle \\
&= t\,||y||^2 + (1 - t)\,||z||^2 - ||u||^2 \\
&= t\,||y||^2 + (1 - t)\,||z||^2 - (\langle ty + (1 - t)z, ty + (1 - t)z\rangle) \\
&= t\,||y||^2 + (1 - t)\,||z||^2 \\
&\quad - \left(t^2\,||y||^2 + t(1 - t)\,\langle y, z\rangle + t(1 - y)\,\langle z, y\rangle + (1 - t)^2\,||z||^2\right) \\
&= t(1 - t)\,||y - z||^2 .
\end{aligned}$$

The second result follows easily by setting $t = \frac{1}{2}$, which gives us $u = \frac{1}{2}(y + z)$. $\square$

Solution of problem 3*:* We assume that there is $y \in Y$ such that $||x - y|| = d(x, Y)$. Then we have $x - y \perp Y$. Thus $\Re\langle x - y, y\rangle = 0$. This implies the other side trivially.

For the converse, we assume that

$$\Re\langle x - y, z\rangle \leq \Re\langle x - y, y\rangle.$$

$\square$

Solution of problem 4*:* $\square$

Solution of problem 5*:* $\square$

Solution of problem 6*:* We want to construct an isometric isomorphism between H, a separable Hilbert space and ℓ^2, the sequence of square summable sequences over a linear field. We have H is separable, hence there exists a countable dense subset. This, in fact gives us an orthonormal Schauder basis $\{b_n\}_{n\in\mathbb{N}}$. Let the standard orthonormal basis for ℓ^2 be given by $\{e_n\}_{n\in\mathbb{N}}$. Define $T : H \to \ell^2$ be such that

$$T(\sum_{n=1}^{\infty} a_n b_n) = \sum_{n=1}^{\infty} a_n e_n.$$

For $\mathbf{a} = \{k_n\}, \mathbf{b} = \{l_n\} \in H$, we have

$$\langle T\mathbf{a}, T\mathbf{b}\rangle = \langle \sum_{n=1}^{\infty} k_n e_n, \mathbf{b}\rangle = \sum_{n=1}^{\infty} k_n \langle e_n, \mathbf{b}\rangle.$$

We can see that

$$\langle e_n, \sum_{m=1}^{\infty} l_n e_n\rangle = \sum_{m=1}^{\infty} \overline{l_m}\langle e_n, e_m\rangle = \bar{l}_n.$$

Thus we have $\langle T\mathbf{a}, T\mathbf{b}\rangle = \sum_{n=1}^{\infty} k_n \bar{l}_n = \langle \mathbf{a}, \mathbf{b}\rangle$. Thus our map is an isometry. It is clearly one-one. It is also onto, as the pre-image of any $\sum_{n=1}^{\infty} c_n e_n$ is $\sum_{n=1}^{\infty} c_n b_n$. Therefore we have an isomorphism of Hilbert spaces. $\square$

Solution of problem 7*:* If V is a finite-dimensional vector space, then any total orthonormal set must be finite as there can be at most some finite number of linearly independent elements. Since a total orthonormal set must span the entire space, we have a Hamel basis since any element can be written as a finite linear combination of elements from the total orthonormal set.

Conversely, let V be a vector space such that every total orthonormal set is a Hamel basis. Let us assume that V is infinite dimensional. Let us take some total orthonormal set $\mathcal{B}$ in V. Then we know that $\overline{\text{span } \mathcal{B}} = H$, and that this is a Hamel basis.

Take the unit ball, then we know that $\mathcal{B}$ lies on it. If $\mathcal{B}$ was finite, there would be nothing to prove, so assume that it is infinite.

$\square$

Solution of problem 8*:* $\square$

Solution of problem 9*:* We are given $T : \mathbb{K}^n \to \mathbb{K}^m$, where

$$(Tx)(i) = \sum_{j=1}^{n} k_{ij} x_j,$$

where $i = 1, 2, \ldots, m$. Let a_i denote the ith row of T. Then we have $\langle Tx, y\rangle = \sum_{j=1}^{m}(Tx)(i)y_j$. Expanding the entire thing, we have

$$\langle Tx, y\rangle = \sum_{1\le i\le m, 1\le j\le n} k_{ij} x_j \bar{y}_i.$$

We can write this as

$$\sum_{j=1}^{n} x_j \overline{\overline{k_{1i}} y_1 + \cdots + \overline{k_{mi}} y_m} = \langle x, \overline{T}^T y\rangle!$$

Therefore from uniqueness of adjoint we must have $T^* = \overline{T}^T$.

$\square$

Solution of problem 10*:* See that for any operator we have

$$|\langle Tx, x\rangle| \le ||Tx|| \cdot ||x|| \le ||T||,$$

taking $||x|| = 1$. Since the left of the inequality depends on x while the right is independent, we have $\sup_{||x||=1}\langle Tx, x\rangle \le ||T||$. For the other direction, let $\alpha := \sup\{|\langle Tx, x\rangle| \mid ||x|| = 1\}$. We want to show that for $||x|| = ||\langle Tx, y\rangle|| \le \alpha$. Since T is self-adjoint, we have $\langle Tx, y\rangle \in \mathbb{R}$. Then we have

$$\langle Tx, y\rangle = \frac{(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle)}{4}.$$

But then

$$|\langle Tx, y\rangle| \le \alpha \frac{||x+y||^2 + ||x-y||^2}{4} = \alpha,$$

by the parallelogram identity. $\square$