Gandhar Kulkarni (mmat2304)

1

We will use Cauchy's integral formula here. The point $z_0 = 2$ is in the circle of radius 3, so we can use the formula. We have

$$\int_C \frac{2z^2 - z - 2}{z - z_0} dz = f(z_0) = 2\pi i g(z_0),$$

where $g(z) = 2z^2 - z - 2$ for all $z_0 \in D = \int(C)$. Then we have $f(2) = 2\pi i(2(2)^2 - 2 - 2) = 8\pi i$.

2

1. We will use Cauchy's integral formula here. Since the specified curve is a square, and it contains $z = \frac{-1}{2}$. Thus we have

$$\int_C \frac{3z}{2z+1} dz = \pi i \frac{1}{2\pi i} \int_C \frac{3z}{z + \frac{1}{2}} dz = \pi i \frac{-3}{2} = \frac{-3}{2},$$

where the formula is used on $3z$.

2. We will use the Cauchy's integral formula for higher derivatives. We have $z = 0$ in the curve, thus we have

$$\int_C \frac{\cosh(z)}{z^{2024}} dz = \frac{2\pi i}{2023!} \frac{d^{2023} \cosh(z)}{dz^{2023}} |_{z=0},$$

which requires us to evaluate repeated derivatives of $\cosh(z)$. We know that $\cosh(z) = \frac{e^z + e^{-z}}{2}$, so $\frac{d^n \cosh(z)}{dz^n} = \frac{e^z + (-1)^n e^{-z}}{2}$. For $n = 2023, z = 0$, we get $\frac{e^0 + (-1)^{2023} e^{-0}}{2} = 0$. Thus we have $\int_C \frac{\cosh(z)}{z^{2024}} dz = 0$.

3

1. We have the annulus $0 < |z-1| < 2$. The function $f(z) = \frac{z}{(z-1)(z-3)}$ can be written as $f(z) = \frac{A}{(z-1)} + \frac{B}{(z-3)}$, where $A = -0.5$ and $B = 1.5$. The term on the left can be ignored for the time being. We have

$$\begin{aligned}\frac{B}{(z-3)} &= \frac{-B}{2} \cdot \frac{1}{1 - \frac{(z-1)}{2}} \\ &= \frac{-B}{2} \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} \text{ for } 0 < |z-1| < 2.\end{aligned}$$

Since we are in the annulus, we can define the Laurent series $\sum_{n=-\infty}^{\infty} a_n (x-1)^n$ as follows— $a_{-1} = \frac{-1}{2}$, $a_n = \frac{-3}{2^{n+2}}$ for $n \geq 0$, and 0 otherwise.

2.

4

Let $D = C_R(0)$ be the open disc of radius R centered around 0. We claim that $f(z) = 2z^2 + z$ is injective on this disc. We must have that for $z_1, z_2 \in D$, we have $f(z_1) = f(z_2) \implies z_1 = z_2$. Then $f(z_1) - f(z_2) = 2z_1^2 + z_1 - 2z_2^2 - z_2 = 2(z_1 - z_2)(2(z_1 + z_2) + 1)$. If this expression is 0, we claim that $2(z_1 + z_2) + 1 \neq 0$ for any $z_1, z_2 \in D$. Clearly $\frac{-1}{2} \notin D$, so $R < \frac{1}{2}$. If we try to solve $z_1 + z_2 = \frac{1}{2}$, then for $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, we have $y_1 = -y_2$ and $x_1 + x_2 = \frac{-1}{2}$. Thus see that we need to avoid the point $z = \frac{-1}{4}$ as well. Thus $R < \frac{1}{4}$. If $R > \frac{1}{4}$, it is possible to have points such that the above equations are satisfied. Thus $R = \frac{1}{4}$. On the boundary, any point except $z = \frac{-1}{4}$ works because of the above reason. However, if we choose $z_1 = z_2 = \frac{-1}{4}$, it would be not possible, but note that here $f(z_1) - f(z_2)$ trivially. Thus this works even on the boundary.

5

We decompose f as $u + iv$, where u, v are real valued functions. Then we have

$$(u + iv)^2 = u - iv \implies u^2 - u - v^2 + (2uv + v)i = 0.$$

This implies that $2uv + v = 0$, thus either $v = 0$ on D or $u = \frac{-1}{2}$ on D. Substituting the two in the real part, we have $u^2 = u$ which means $u = 0, 1$ in the first case, and that $v^2 = \frac{3}{4}$ which implies that $v = \pm\frac{\sqrt{3}}{2}$ for the second case. Thus there are four constant functions that satisfy the given condition—$f = 0, 1, \omega, \omega^2$, where ω is a primitive third root of unity.

6

We have f is entire, and its image is in the upper half plane. Then we have $f = u + iv$, where $v \geq 0$. Then $g = \exp(if)$ is also an entire function. We have $|g| = \left|e^{i(u+iv)}\right| = \left|e^{iu}\right| \cdot |e^{-v}|$, which means that $|g| = |e^{-v}|$, and since $v \geq 0$, we must have that g is a bounded function. This means that g must be constant by Liouville's theorem, which implies that f is constant.

7

Let us assume that f, an entire function has non-dense image. Thus there exists $\lambda \in \mathbb{C}, r > 0$ such that $f(\mathbb{C}) \cap B_r(\lambda) + phi$. Then see that $g(z) = \frac{1}{f(z)-\lambda}$ is an entire function, since it cannot be zero. Moreover, since $|f(z) - \lambda| > r \implies |g(z)| < \frac{1}{r}$ which implies that g is constant, hence f is constant. This is a contradiction, hence we must have dense image.

Let us assume g is as given. Let B_M be the ball around the origin of radius M. Then pick an ε neighbourhood of $z \in \mathbb{C}$ that is not in $f(B_M)$ and $\Re(z') > \Im(z')$ for all $z' \in B_\varepsilon(z)$. This cannot be in the image of g, which contradicts the result that the image of a non-constant entire function is dense in $\mathbb{C}$. Thus we must have that g is constant.

8

We are asked to find the maximum modulus of a polynomial on the unit disc D. By the maximum modulus principle, we know that the maximum must be attained on the boundary of the circle, that is the set of points $z \in D$ such that $|z| = 1$. Thus let $z = e^{i\theta}$, and then see that

$$f(e^{i\theta}) = ae^{2i\theta} + 2(|a|^2 - 1)e^{i\theta} - \bar{a} = e^{i\theta}\left(ae^{i\theta} + 2(|a|^2 - 1) - \overline{ae^{i\theta}}\right).$$

Now we can say that

$$|p(z)|^2 = |e^{i\theta}|^2\left(\left(2|a|^2 - 2\right)^2 + \left(ae^{i\theta} - \overline{ae^{i\theta}}\right)^2\right).$$

Now see that the imaginary part is the only part dependent on θ. Let us only look at that. Let $a = re^{i\varphi}$. Here, $r = |a|$. Then we have $g(z) = \left(ae^{i\theta} - \overline{ae^{i\theta}}\right) = 2\Im\left(re^{i(\theta+\varphi)}\right) = 2r\sin(\theta + \varphi)$. We can differentiate this with respect to θ to see that $g'(z) = 2r\cos(\theta + \varphi)$. Thus g has a critical point at $\theta = -\varphi$. Plugging this into $|p(z)|^2$, we get

$$|p(z)|^2 = \big(2(r^2 - 1)\big)^2 .$$

This means that $|p(z)| \leq 2(r^2 - 1) \leq 2$, as required.