

Measure Theory HW6

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1

We wish to show that $h \in L^1(\mu \times \nu)$. See that since $f \in L^1(\mu)$ and $g \in L^1(\nu)$. Then see that $|h(x, y)| = |f(x)| \cdot |g(y)|$. Since these measure space are σ -finite and the functions $|f|$ and $|g|$ are positive measurable, we can apply the Fubini- Tolleni theorem. Integrating over $X \times Y$, we get

$$\begin{aligned} \int_{X \times Y} |h(x, y)| d(\mu \times \nu) &= \int_{X \times Y} |f(x)| \cdot |g(y)| d(\mu \times \nu) \\ &= \int_Y \left(\int_X |h^y(x)| d\mu(x) \right) d\nu(y) = \int_Y |g(y)| \cdot \left(\int_X |f(x)| d\mu(x) \right) d\nu(y) \\ &= \int_Y |g(y)| d\nu(y) \cdot \left(\int_X |f(x)| d\mu(x) \right) < \infty. \end{aligned}$$

Thus $h \in L^1(\mu \times \nu)$. To calculate the integral, see that

$$\begin{aligned} \int_{X \times Y} h(x, y) d(\mu \times \nu) &= \int_{X \times Y} f(x) \cdot g(y) d(\mu \times \nu) \\ &= \int_Y \left(\int_X h^y(x) d\mu(x) \right) d\nu(y) = \int_Y g(y) \cdot \left(\int_X f(x) d\mu(x) \right) d\nu(y) \\ &= \int_Y g(y) d\nu(y) \cdot \left(\int_X f(x) d\mu(x) \right), \end{aligned}$$

which is the desired result.

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The Fubini-Tolleni theorem requires the two spaces X and Y to both be σ -finite with respect to both μ and ν respectively. In this case see that the counting measure over \mathbb{N} is indeed σ -finite, as $\mathbb{N} = \cup_{n=1}^{\infty} \{n\}$, where $\mu(\{n\}) = 1 < \infty$. Thus for $X = Y = \mathbb{N}$, and $\Sigma_1 = \Sigma_2 = P(\mathbb{N})$, and $\mu = \nu = m$, where $m(A)$ denotes the cardinality of the set A if it is finite and $+\infty$ otherwise. Note that $\Sigma_1 \otimes \Sigma_2 = P(\mathbb{N}^2)$, since $\Sigma_1 \otimes \Sigma_2 \subseteq P(\mathbb{N}^2)$, and for any $A \times B \in P(\mathbb{N}^2)$, we have $A \times B = \cup_{x \times y \in A \times B} \{x\} \times \{y\} \in \Sigma_1 \otimes \Sigma_2$, which gives us the other inequality.

We wish to see what sorts of functions over \mathbb{N} are measurable. All functions are clearly $\Sigma_1 \otimes \Sigma_2$ measurable, as the entire power set constitutes the σ -algebra. See that functions can be indexed by two natural numbers, hence they can be described as $a_{m,n}$. Note that $(a_{m_0})_n = a_{m_0,n}$ fixes the first variable at some $m_0 \in \mathbb{N}$, and $a_m^{n_0} = a_{m,n_0}$ fixes the second variable at some $n_0 \in \mathbb{N}$. We also need to understand what integration looks like. Integration in this case is just summation, as we can see that integrating over a point gives us the value of the function at that point. Thus $\int_{\mathbb{N}} \{a_n\} dm(n) = \sum_{n=1}^{\infty} a_n$ and $\int_{\mathbb{N}^2} \{a_{m,n}\} dm(m, n) = \sum_{(m,n) \in \mathbb{N}^2} a_{m,n}$. We take a positive measurable function $\{a_{m,n}\}$, and the Fubini-Tolleni theorem tells us that

1.

$$m \mapsto \sum_{n=1}^{\infty} (a_m)_n$$

is $P(\mathbb{N})$ is measurable,

and

$$n \mapsto \sum_{m=1}^{\infty} a_m^n$$

is $P(\mathbb{N})$ is measurable;

2.

$$\sum_{(m,n) \in \mathbb{N}^2} a_{m,n} dm(m,n) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_m^n \right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} (a_m)_n \right).$$

The above statement means in the case of \mathbb{N}^2 is that we can switch the limits of double summations without issue.

Fubini's theorem can also be stated since our product measure consists of two σ -finite measures. We need to consider functions $\{a_{m,n}\} \in L^1(m^2)$, where we have $\sum_{(m,n) \in \mathbb{N}^2} |a_{m,n}| < \infty$. Then Fubini's theorem says that:

1. $(a_m)_n \in L^1(m)$, m -almost everywhere, and $a_n^m \in L^1(m)$, m -almost everywhere.

2.

$$m \mapsto \sum_{n=1}^{\infty} (a_m)_n$$

and

$$n \mapsto \sum_{m=1}^{\infty} a_m^n$$

are $L^1(m)$.

In the case of the counting measure it means that if the double summation is absolutely convergent, then so are the summations of all sections.

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We need to find a function $g(x)$ that dominates $f_n = f(nx)$ for all $n \in \mathbb{N}$. Then see that

$$\left| \frac{\sin(n^2 x^2)}{nx} \right| \leq \frac{1}{nx} \leq \frac{1}{x}.$$

Also see that $\frac{cnx}{1+nx} \leq c$. See that this holds for all $x \in [0, \infty)$

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1. We know that $\mu(E) = 0$. Also $f = (\Re(f)^+ - \Re(f)^-) + i(\Im(f)^+ - \Im(f)^-)$, which are all positive measurable functions. Then $\nu(E) = \int_E \Re(f)^+ d\mu \leq +\infty \cdot \mu(E) \leq 0$. Since measure cannot be negative, we have $\nu(E) = 0$. Use this same strategy for the other three functions $\Re(f)^-, \Im(f)^+, \text{ and } \Im(f)^-$.

2. Let $\{E_n\}_{n \in \mathbb{N}}$ be a countable disjoint collection of measurable subsets. Then define g as $g = \sum_{n=1}^{\infty} f \chi_{E_n}$. Then $f - g$ is zero on $\coprod_{n=1}^{\infty} E_n$. Then we must have

$$\begin{aligned} \int_{\coprod_{n=1}^{\infty} E_n} (f - g) d\mu &= 0 \implies \\ \nu\left(\coprod_{n=1}^{\infty} E_n\right) &= \int_{\coprod_{n=1}^{\infty} E_n} f d\mu = \int_{\coprod_{n=1}^{\infty} E_n} \sum_{n=1}^{\infty} f \chi_{E_n} \\ &= \sum_{n=1}^{\infty} \int_{\coprod_{n=1}^{\infty} E_n} f \chi_{E_n} = \sum_{n=1}^{\infty} \int_{E_n} f d\mu \\ &= \sum_{n=1}^{\infty} \nu(E_n), \end{aligned}$$

which means that $\nu(\coprod_{n=1}^{\infty} E_n) = \int_{\coprod_{n=1}^{\infty} E_n} f d\mu$.

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From the solution of problem 1, the answer to the problem follows directly.

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Note that $|f(m, n)| = 1$ for $m = n$ and $m = n + 1$. Then

$$\int_{\mathbb{N}^2} |f(m, n)| dm(m, n) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(m, n)| = \sum_{p=1}^{\infty} |f(p, p)| + \sum_{q=1}^{\infty} |f(q+1, q)| = \infty.$$

Now we fix each variable and successively integrate. $\sum_{m=1}^{\infty} f(m, n_0) = f(n_0, n_0) + f(n_0 + 1, n_0) = 1 + (-1) = 0$. Thus $\sum_{n=1}^{\infty} 0 = 0$. If we fix m , and $m > 1$, then we have $\sum_{n=1}^{\infty} f(m, n) = f(m, m) + f(m, m-1) = 1 + (-1) = 0$. If $m = 1$, there is no n such that $m = n + 1$, so $\sum_{n=1}^{\infty} f(1, n) = f(1, 1) = 1$. Thus $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) = 1 \neq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m, n) = 0$.

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