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1. Let $\mathcal{K}$ be a conjugacy class, let $x \in G$ be a representative. Then $\mathcal{K} = \{gxg^{-1} : x \in G\}$, and
$$\sigma(\mathcal{K}) = \{\sigma(gxg^{-1}) = \sigma(g)\sigma(x)\sigma(g)^{-1} : g \in G\}.$$
Since σ is an automorphism, it is a bijection. Then $\sigma(g)$ spans all of G. Then we have $\sigma(\mathcal{K}) = \sigma(\mathrm{Cl}(x)) = \mathrm{Cl}(\sigma(x))$. Thus this is also a conjugacy class.

2. See that in S_n $\mathcal{K} = \mathrm{Cl}((12))$, that is, all elements that switch two and only two elements. However, in general, the product of multiple transpositions also has order 2, but they are not transpositions. For example, in S_n, where $n \geq 4$, $(12)(34)$ has order 2, but are not $2-$cycles. For $n = 2$, there are no order 2 permutations that aren't $2-$cycles, so this is vacuously true for $n = 2$. For $n = 3$, the same principle applies since it is not possible to find two disjoint $2-$cycles. We have $|\mathcal{K}| = \binom{n}{2}$, while $|\mathcal{K}'| = \binom{n}{2}\binom{n-2}{2} + \cdots + \binom{n}{2}\ldots\binom{n-2\lfloor \frac{n}{2} \rfloor}{2}$, which is greater than $|\mathcal{K}|$ for $n \geq 2$. Thus $|\mathcal{K}| \neq |\mathcal{K}'|$.

3.

2

1. Let H be a non-abelian simple group. Consider $DH \trianglelefteq H$. Then since H is simple, we must have $DH = H$ or $DH = 0$. If $DH = 0$, then H would be abelian, which is not possible, thus H is perfect.

2. We know that $DH = H, DK = K$. Thus any term in H and K can be seen as an element of the type $h_1h_2h_1^{-1}h_2^{-1}$ for H and respectively for K. Thus we consider any word in $\langle H, K\rangle$. Then it is of the form $h_1k_1 \ldots h_nk_n$, where $h_1, \ldots, h_n \in H, k_1, \ldots, k_n \in K$. We know that we can write any $h \in H$ and $k \in K$ can be written as an element of the commutator. The commutator is generated by such elements. Thus we have $\langle H, K\rangle = D\langle H, K\rangle$.

3. We propose that $D(g^{-1}Hg) = g^{-1}DHg$. Take any element of $D(g^{-1}Hg)$, which is of the form $(g^{-1}h_1g)(g^{-1}h_2g)(g^{-1}h_1^{-1}g)(g^{-1}h_2^{-1}g) = g^{-1}(h_1h_2h_1^{-1}h_2^{-1})g \in g^{-1}DHg$. These operations are all if and only if statements, hence $D(g^{-1}Hg) = g^{-1}DHg$. If $H = DH$, $D(g^{-1}Hg) = g^{-1}Hg$.

4. If G is simple, then either $DG = 0$ or $DG = G$. Then for abelian simple groups the maximal perfect subgroup is 0, and in the non-abelian case it is G itself. Both are clearly normal in G.

3

Since K is cyclic, it is generated by some element, say $x \in K$, and we know that the map φ_1 and φ_2 are decided entirely by where it sends the element x. We know that there exists $\sigma \in \mathrm{Aut}(H)$ such that $\sigma\varphi_1(K)\sigma^{-1} = \varphi_2(K)$, so we have $\sigma\varphi_1(x)\sigma^{-1} = \varphi_2(x)^l$, for some $l \in \mathbb{Z}$. Then $k = x^t \in K$, we have $\sigma\varphi_1(k)\sigma^{-1} = (\sigma\varphi_1(x)\sigma^{-1})^t = \varphi_2(x)^a$, where $t \in \mathbb{Z}$.

Take the map $\psi : H \rtimes_{\varphi_1} K \to H \rtimes_{\varphi_2} K$ where $\psi(h, k) = (\sigma(h), k^a)$. To see that this is a homomorphism, we have

$$\begin{aligned}\psi((h_1, k_1) \circ_1 (h_2, k_2)) &= \psi((h_1\varphi_1(k_1)(h_2), k_1k_2)) \\ &= (\sigma(h_1)\sigma(\varphi_1(k_1)(h_2)), (k_1k_2)^a).\end{aligned}$$

Also, see that

$$\begin{aligned}\psi((h_1,k_1)) \circ_2 \psi((h_2,k_2)) &= (\sigma(h_1),k_1^a) \circ_2 (\sigma(h_2),k_2^a)\\ &= (\sigma(h_1)\varphi_2(k_1)^a(\sigma(h_2)),(k_1k_2)^a)\\ &= (\sigma(h_1)\sigma\varphi_1(k_1)\sigma^{-1}\sigma(h_2),(k_1k_2)^a)\\ &= (\sigma(h_1)\sigma(\varphi_1(k_1)(h_2)),(k_1k_2)^a).\end{aligned}$$

This shows that we have a homomorphism. Define $\theta : H \rtimes_{\varphi_2} K \to H \rtimes_{\varphi_1} K$, such that $\theta(h,k) = (\sigma^{-1}(h), k^{a'})$, for some $a' \in \mathbb{Z}$. If K is finite then there exists some

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We have $|G| = 75$. We clearly have a $5-$Sylow subgroup of order 25, as well as a $3-$Sylow group of order 3. Since the number of $5-$Sylow subgroups must be $5k+1$ and $5k+1|3$, we are forced to have a normal subgroup of order 25. Then let $H \trianglelefteq G$ where $|H| = 25$. Let K be some $3-$Sylow subgroup of order 3. Then by Lagrange's theorem, $|H \cap K| = 1$, so we can construct a semi-direct product of the groups.

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Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ be such that $\alpha_i = 1, 2$ for all $1 \leq i \leq r$ and $p_i \nmid p_j^{\alpha_j} - 1$ for all i, j. Assume WLOG that $p_1 < \cdots < p_r$. Then we have that the number of p_r-Sylow subgroups is of the form p_rk+1, and since $p_rk+1|p_1^{\alpha_1} \dots p_{r-1}^{\alpha_{r-1}}$, we have $p_rk+1|p_i^{\alpha_i}$, for $1 \leq i \leq r-1$. Since p_r is the largest prime, we

The converse can be shown by proving the contrapositive. See that if we have $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, and $\alpha_i > 2$ for some i, or $p_i | p_j^{\alpha_j} - 1$ for some i, j, then we are done. To check the first case, take D_8, which has order $8 = 2^3$. This group has centre $\{1, r^2\}$, which implies that this group is not abelian.

Now for the other case, take the group of order 75 as we showed in the previous question was non-abelian. In that case, we had $3|25-1$. Thus we have that a n