Algebra 2 Homework 5

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Solution of problem 1*:* If we have $[K_1K_2 : F] = [K_1 : F][K_2 : F]$, then for $K_1 = F(\{\alpha_m\}_{m\in I})$, and $K_2 = F(\{\beta_n\}_{n\in I'})$. Then we have $K_1K_2 = F(\{\alpha_m\beta_n\}_{m\in I, n\in I'})$. We have that $K_1 \otimes_F K_2$ is the $F-$module generated by the generating elements $\alpha_m \otimes_F \beta_n$. We have an obvious $F-$module homomorphism from $K_1 \otimes_F K_2$ to K_1K_2 where $\alpha_m \otimes_F \beta_n \mapsto \alpha_m\beta_n$. This clearly is an isomorphism of modules, and since K_1K_2 is a field, so is $K_1 \otimes_F K_2$.

Conversely, we have $K_1 \otimes_F K_2$ is a field. Then we have $[K_1 \otimes_F K_2 : F] = [K_1 : F][K_2 : F]$, since as $F-$modules this must happen. Now define $\varphi : K_1 \times K_2 \to K_1K_2$ where $(a, b) \mapsto ab$. This is a map that distributes over addition and scalar multiplication over both a and b. Then by the universal property of tensor products we have a unique map $\Omega : K_1 \otimes_F K_2 \to K_1K_2$ where $\Omega(a \otimes b) = ab$. This homomorphism preserves multiplication, and it is necessarily injective. Moreover, for any $ab \in K_1K_2$, we have a corresponding $a \otimes b \in K_1 \otimes_F K_2$. Thus we have an isomorphism of fields.

Thus we have $[K_1K_2 : F] = [K_1 \otimes_F K_2 : F] = [K_1 : F][K_2 : F]$. □

Solution of problem 2*:* We have a quadratic equation in x^2, which gives us $x^2 = \pm\omega$. Solving this further, we get $x^4 + x^2 + 1 = (x - \omega)(x + \omega)(x - i\omega)(x + i\omega)$. Then clearly $\mathbb{Q}(i, \omega)$ contains the splitting field. Also, adjoining all the roots to $\mathbb{Q}$ gives us $\mathbb{Q}(\omega, -\omega, i\omega, -i\omega)$ which certainly contains $\mathbb{Q}(i, \omega)$. Thus the splitting field is $\mathbb{Q}(i, \omega)$. □

Solution of problem 3*:* The polynomial $x^6 - 4$ splits into linear factors in $\mathbb{C}$, where the roots $\pm\zeta_3\alpha$, where $\zeta_3 \in \{1, \omega, \omega^2\}$, and $\alpha = \sqrt[3]{2}$. We propose that the splitting field is $\mathbb{Q}(\alpha, \omega)$. This clearly contains all the roots of this polynomial, thus it contains the splitting field. Also, we get the splitting field by adjoining all the roots to $\mathbb{Q}$, which clearly contains $\mathbb{Q}(\alpha, \omega)$, thus it is the splitting field. □

Solution of problem 4*:* Let us assume K is a splitting field over F for some polynomial $f(x) \in F[x]$. Then we take some irreducible polynomial $p(x) \in F[x]$ such that some root of p is in K. Let $K = K(\alpha)$. We also have another root β. We know that since $F(\alpha) \cong F(\beta)$ we can extend this isomorphism to their splitting fields, that is $K(\alpha) \cong K(\beta)$. Then we have that K/F and $K(\beta)/F$ have the same degree. Thus $[K(\alpha) : K(\beta)][K(\beta) : F] = [K(\alpha) : F] \implies [K(\alpha) : K(\beta)] = 1$. Thus $K(\beta) \cong K(\alpha) \cong K$.

Conversely, assume that any irreducible polynomial over F either contains all its roots in K or none of its roots in K. We can assume $K = F(\alpha_1, \ldots, \alpha_n)$. Take $m(x)$, the minimal polynomial of K over F. This is irreducible, and it is clearly all in K. Then we must have the splitting field of $m(x)$ is contained in K. However, the splitting field must also contain all the roots of $m(x)$, and thus must also contain K. Thus we have K is a splitting field. □