Indian Statistical Institute

Measure Theory (HW - 5) Due date: 09.10.2023 Instructor : Jaydeb Sarkar

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NOTE: (i) m = the Lebesgue measure on $\mathbb{R}$. (ii) m^* = the Lebesgue outer measure on $\mathbb{R}$. (iii) $\mathcal{M}(\mathbb{R})$ = the Lebesgue measurable subsets of $\mathbb{R}$. (iv) $(X, \mathcal{A}, \mu)$ is a measure space.

(1) Prove the monotone convergence theorem using Fatou's lemma.

(2) Let $f \in L^+$. Prove that

$$\lim_{n\to\infty} \int_{\{x: f(x) \geq n\}} f = 0.$$

(3) Let $\{f_n\} \subseteq L^+$, f_n decreases pointwise to f, and suppose $\int f_1 < \infty$. Prove that

$$\lim \int f_n = \int f.$$

(4) Let $f \in L^+$, and suppose $\int_X f d\mu < \infty$. Prove that for $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$ and

$$\int_A f d\mu > \int_X f d\mu - \epsilon.$$

(5) Let $f \in L^+$. Prove that there exists a sequence of bounded measurable functions $\{f_n\}$ such that

$$\lim \int f_n = \int f.$$

(6) Give an example of a measurable function f for which f^+ is integrable and f^- is not.

(7) Show that the inequality in Fatou's Lemma is strict for the sequence

$$f_n(x) = (n+1)x^n \qquad (n \geq 1, x \in [0,1]).$$

(8) Prove or disprove: If $\{f_n\}$ is a sequence of integrable functions on X and $f_n \to f$ pointwise, then f is integrable.

(9) Let $\{f_n\}$ be a sequence of functions on $[0,1]$ defined by

$$f_{2n+1}(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{3}] \\ 1 & \text{otherwise,} \end{cases} \text{ and } f_{2n}(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{3}] \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \geq 1$. Prove that

$$\int_0^1 \liminf f_n < \liminf \int_0^1 f_n < \limsup \int_0^1 f_n < \int_0^1 \limsup f_n.$$

(10) (Chebychesv's Inequality) Let $f : X \to [0, \infty]$ be a nonnegative measurable function. Suppose $\alpha > 0$. Prove that

$$\mu(\{x \in X : f(x) \geq \alpha\}) \leq \frac{1}{\alpha} \int_X f \, d\mu.$$

(11) Let $f : X \to [0, \infty]$ be a measurable function. Prove that $f \in L^1(\mu)$ if and only if

$$\sum_{n \geq 1} \mu(X_n) < \infty,$$

where

$$X_n = \{x \in X : f(x) \geq n\} \qquad (n \geq 1).$$

(12) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on X that converges pointwise on X to an integrable function f. If

$$f_n \leq f \text{ on } X,$$

for all n, show that

$$\lim \int f_n = \int f.$$