Algebra 2 HW1

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1

Assume for the sake of contradiction that there exists an isomorphism $\varphi : \mathbb{C}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$. Then we must have

$$\varphi(i^4) = \varphi(i)^4 = 1.$$

Thus we must have $\varphi(i) = \pm 1$, since $\varphi(i) \in \mathbb{R}\backslash\{0\}$. If $\varphi(i) = 1$, then φ is not one-one. If $\varphi(i) = -1$, then $\varphi(i^2) = -1^2 = 1$, which also means that φ is not one-one. Thus no such isomorphism exists.

2

1. To characterise a linear transformation, it is enough to understand its action on the basis elements, that is $(1,0)^T$ and $(0,1)^T$. Looking at the point on the unit circle that has an angle θ to the x-axis, we can see that it has the coordinate $(\cos\theta, \sin\theta)$. Similarly, we want to see the coordinates of the point that has an angle of $\frac{\pi}{2} + \theta$ to the x-axis. Its coordinates are $(-\sin\theta, \cos\theta)$. Putting it together, we get the required rotation matrix that describes the linear transformation.
2. To confirm that $\varphi : D_{2n} \to GL_2(\mathbb{R})$ is a homomorphism, we need to confirm that $\varphi(r)^n = \varphi(s)^2 = I_2$, and that $\phi(r)\phi$

3

$$D_{2n} = \left\{r^i s^j : 0 \le i \le n-1, 0 \le j \le 1, rs = sr^{n-1}\right\}.$$

For any two elements $r^{i_1}s^{j_1}, r^{i_2}s^{j_2} \in D_{2n}$. Let $r^{i_1}s^{j_1} \in Z(D_{2n})$ commute with $r^{i_2}s^{j_2} \in D_{2n}$. Then

$$r^{i_1}s^{j_1} \cdot r^{i_2}s^{j_2} = r^{i_2}s^{j_2} \cdot r^{i_1}s^{j_1}.$$

Working this out, we get

$$r^{n+i_1+(-1)^{j_1}i_2}s^{j_1+j_2} = r^{n+i_2+(-1)^{j_2}i_1}s^{j_2+j_1},$$

which implies that

1. If n is odd,

4

Let $x \in G$ be such that $xZ(G)$ generates $G/Z(G)$. Thus any term in $G/Z(G)$ is of the form $x^a Z(G)$ for some $a \in \mathbb{Z}$. Consider the canonical quotient map $\pi : G \twoheadrightarrow G/Z(G)$ where $\pi(g) = gZ(G)$. Its kernel is $Z(G)$, so we have $G \cong Z(G) \times G/Z(G)$. Thus we can write $g \in G$ as (z, x^a), such that $g = x^a z$. Now take $g_1, g_2 \in G$, and consider $g_1 \cdot g_2 = x^{a_1}z_1 \cdot x^{a_2}z_2 = g_2 = x^{a_1}x^{a_2}z_1z_2 = g_2 \cdot g_1$, as the order of multiplication of z_1 and z_2 can be switched as it is in the centre. Thus we have G is abelian.

5

Let $n = |G|$, $k = |[G : N]|$.

6

$G = MN$, where $M, N \trianglelefteq G$. Define the map $f : G \to (G/M) \times (G/N)$, where $f(g) = (gM, gN)$. To see that this map is well-defined, see that for $g = g'$ in G, we have $gM = g'M$ and $gN = g'N$ as the canonical projections from G to G/M and G/N are well-defined. From these two maps it can be seen that the map f also respects the group operation, hence this is also a homomorphism. Note that for all $g \in G$, $g = mn$, for $m \in M, n \in N$. Then an arbitrary element of $(G/M) \times (G/N)$ is of the form (nM, mN). Thus we can see that this corresponds to an element $mn \in G$, which can cover all of G. Thus f is surjective. To compute the kernel of f, see that $gM = e_{G/M} \implies g \in M$, and $gN = e_{G/N} \implies g \in N$. Thus $g \in M \cap N$, thus $\ker f = M \cap N$. Using the first isomorphism theorem gives us our result.