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1

We are given $f(z) = z^2 - z\bar{z}^2 - 2|z|^2$. Since $z = x + iy$, we can expand it in f to get

$$f(x, y) = -2(x^2 + y^2) + i4xy.$$

Thus $u = -2(x^2+y^2), v = 4xy$. Then we have $u_x = -4x, u_y = -4y, v_x = 4y, v_y = 4x$. If f is holomorphic, then we must have $u_x = v_y \implies -4x = 4x \implies x = 0$. Also we must have $u_y = -v_x \implies -4y = -4y \implies y \in \mathbb{R}$. Thus f satisfies the Cauchy Riemann equations on $\{0\} \times \mathbb{R}$, which is not a domain since it is not open. Thus it is complex differentiable at each point of the type $(0, y)$ where $y \in \mathbb{R}$, but not holomorphic at any point in $\mathbb{C}$ since the points at which it satisfies the Cauchy Riemann equations is not open in $\mathbb{C}$.

2

Let us assume that there exists a holomorphic function on a domain D such that its image lies entirely on a vertical line, say $x = \frac{1}{2}$. Thus for $f = u + iv$, we must have that $u = \frac{1}{2}$, a constant. Then $u_x = u_y = 0$, and by the Cauchy-Riemann equations, we have $v_y = u_x = 0 = u_y = -v_x$. Thus we have v constant as well, which means that f must be a constant.

3

Note that $f(z) = \exp(-z^{-2})$. We know that the exponential function is entire, so any value of $-z^{-2}$ is permissible. However, $-z^{-2}$ is analytic on $\mathbb{C}\backslash\{0\}$. Thus f is analytic on $\mathbb{C}\backslash\{0\}$.

Since the above holds, the Cauchy-Riemann equations are satisfied on this domain. Now see that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\exp(-h^{-2})}{h}.$$

For $h = h_1 \in \mathbb{R}$, we have

$$\lim_{h\to 0} \frac{\exp(-h^{-2})}{h} \leq \frac{1}{h} \to 0.$$

However, for $h = ih_2$, we have

$$\lim_{h\to 0} \frac{\exp(h^{-2})}{h},$$

which is unbounded as $h \to 0$. Thus no limit exists and f is not differentiable at 0.

4

Let us assume that the image of f lies on a line passing through the origin. We have $f(t) = f_1(t)+if_2(t)$, and $\exists \alpha \in \mathbb{R} f([0, 1]) \subseteq ((x, \alpha x), x \in \mathbb{R})$ or $f([0, 1]) \subseteq i\mathbb{R}$. In the second case, we have $f_1(t) = 0$. Then $|\int_0^1 f(t)dt| = |i\int_0^1 f_2(t)dt| = |\int_0^1 f_2(t)dt|$. Also $\int_0^1 |f(t)|dt = \int_0^1 |f_2(t)|dt$. Since f lies on a ray, the integral $|\int_0^1 f_2(t)dt| = \int_0^1 |f_2(t)|dt$ as f_2 is only positive or only negative.

For the first case, we have $f_1(t) = \alpha f_2(t)$. Then $|\int_0^1 f(t)dt| = |(1+i\alpha)\int_0^1 f_1(t)dt| = \sqrt{1 + \alpha^2}|\int_0^1 f_1(t)dt|$. Also see that $\int_0^1 |f(t)|dt = |\int_0^1 |f_1(t)+i\alpha f_1(t)|dt = \sqrt{1 + \alpha^2}\int_0^1 |f_1(t)|dt$. Since f lies on a ray, the integral $|\int_0^1 f_1(t)dt| = \int_0^1 |f_1(t)|dt$ as f_1 is only positive or only negative.

We are given $\left|\int_0^1 f(t)dt\right| = \int_0^1 |f(t)|dt$. Then let $\alpha = \int_0^1 f(t)dt$ and let $\beta = \frac{|\alpha|}{\alpha}$. See that $|\alpha| = \int_0^1 \beta f(t)dt \implies |\alpha| = \Re\left(\int_0^1 \beta f(t)dt\right) = \int_0^1 \Re(\beta f(t))dt = \int_0^1 |\beta f(t)|dt$. Thus we have

$$\int_0^1 |\beta f(t)| - \Re(\beta f(t))dt = 0.$$

But since the integrand must always be greater than zero, we must have $|\beta f(t)| = \Re(\beta f(t))$. Thus $\Im(\beta f(t)) = 0 \implies f_1 = cf_2$, for some $c \in \mathbb{R}$.

5

1. The curve can be parametrised by $\gamma : [0,1] \to \mathbb{C}$. We have $\gamma(t) = \omega(1-t) + \omega^2 t$. Expanding this, we can see that $\gamma(t) = \frac{-1+\sqrt{3}i(1-2t)}{2}$. See that $\gamma'(t) = -\sqrt{3}i$, then we have

$$\begin{aligned}\int_\gamma |z^2|dz &= \int_0^1 \left|\frac{-1+\sqrt{3}i(1-2t)}{2}\right|^2 (-\sqrt{3}i)dt \\ &= \frac{-\sqrt{3}i}{4}\int_0^1 (-1+\sqrt{3}i(1-2t))^2 dt \\ &= \frac{-\sqrt{3}i}{4}\int_0^1 \left(1+3(1-2t)^2\right)dt \\ &= \frac{-\sqrt{3}i}{4}\left[t + \frac{-1}{2}(1-2t)^3\right]_0^1 \\ &= \frac{-\sqrt{3}i}{4}(1+1/2-(-1/2)) = \frac{-\sqrt{3}i}{2}.\end{aligned}$$

2. The curve can be represented as $\gamma : \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \to \mathbb{C}$. We have $\gamma(t) = e^{it}$. For this we have $\gamma'(t) = ie^{it}$, then the integral is

$$\begin{aligned}\int_\gamma |z^2|dz &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} |e^{it}|^2 ie^{it}dt \\ &= i\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} e^{it}dt \\ &= \left[e^{it}\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = -\sqrt{3}i.\end{aligned}$$

3. Here, γ has four parts labelled 1 to 4. The curves are separately parametrised by the same parameter t, in a classic fashion of abuse of notation. These are the four curves: $\left(\frac{1}{2}, t-\frac{1}{2}\right), \left(\frac{1}{2}-t, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}-t\right)$, and $\left(t-\frac{1}{2}, -\frac{1}{2}\right)$. Their derivatives are $i, -1, -i, 1$. The integral is calculated thus:

$$\begin{aligned}\int_\gamma |z|^2 dz = i\int_0^1 \left(\frac{1}{2}\right)^2 + \left(t-\frac{1}{2}\right)^2 dt + (-1)\int_0^1 \left(\frac{1}{2}-t\right)^2 + \left(\frac{1}{2}\right)^2 dt + \\ (-i)\int_0^1 \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}-t\right)^2 dtz + 1\int_0^1 \left(t-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 dt \\ = 0.\end{aligned}$$

In the first two parts, the path integrals differed based on path taken. So even though the previous integral was zero, there is no primitive for f.

6

Let $C := \{e^{it} : t \in \left[0, \frac{pi}{2}\right]\}$, which parametrizes the curve. Then

$$\begin{aligned}
\int_C \overline{\operatorname{Log}(z)} dz &= \int_0^{\frac{\pi}{2}} \overline{\operatorname{Log}(e^{it})} i e^{it} dt \\
&= \int_0^{\frac{\pi}{2}} \overline{\log(1) + it} i e^{it} dt \\
&= \int_0^{\frac{\pi}{2}} t e^{it} dt \\
&= [-ite^{it} + e^{it}]_0^{\frac{\pi}{2}} = (-\frac{\pi}{2} + i) - 1 \\
&= -\left(\frac{\pi}{2} + 1\right) + i.
\end{aligned}$$

7

We let $z = Re^{it}$, and see that $z' = iRe^{it}$. Then substituting into the integral, we get

$$\int_{\gamma_R} \frac{e^{iz}}{z} dz = \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it} dt.$$

We just consider the integral $\int_0^\pi \frac{e^{iRe^{it}}}{Re^{it}} Re^{it} dt$. We have $\int_0^\pi \frac{e^{iRe^{it}}}{Re^{it}} Re^{it} dt = \int_0^\pi e^{iRe^{it}} dt = \int_0^\pi e^{i(R\cos t + iR\sin t)} dt = \int_0^\pi e^{iR\cos t} e^{-R\sin t} dt$. Now $\left|\int_0^\pi e^{iR\cos t} e^{-R\sin t} dt\right| \leq \int_0^\pi e^{-R\sin t} dt$. Since $\sin(\pi - t) = \sin t$, then the function is symmetric about $\pi/2$. Then we have $\int_0^\pi e^{-R\sin t} dt = 2\int_0^{\pi/2} e^{-R\sin t} dt$. We have $\sin t \geq \frac{2t}{\pi}$. Thus we have $e^{-R\sin t} \leq e^{\frac{-2tR}{\pi}} \implies \int_0^{\pi/2} e^{-R\sin t} \leq \int_0^{\pi/2} e^{\frac{-2tR}{\pi}} = \frac{\pi(e^{-R}-1)}{2R} \to 0$, which is the desired result.

8

We need to look at $\sqrt{1+z} + \sqrt{1-z}$. The square root function is defined for all complex numbers except the points $(-\infty, 0]$, that is, non-positive real numbers. Then $\sqrt{1+z}$ must avoid $(-\infty, -1]$, and $\sqrt{1-z}$ must avoid $[1, \infty)$. Thus $\sqrt{1+z} + \sqrt{1-z}$ is defined on $\mathbb{C} \backslash (-\infty, -1] \cup [1, \infty)$.

9

We have $\log\log z$. We know that log is defined on $\mathbb{C}\backslash(-\infty, 0]$. Then $\log\log z$ needs to avoid $(-\infty, 0]$. For this, $\log z$ must avoid $[0, 1)$. Therefore z must avoid $[1, e)$. Therefore $\log\log z$ is defined on $\mathbb{C}\backslash\{[1, e)\}$.

10

We proceed case-wise. For $a = 0$, we have

$$\int_{|z|=R} \frac{|dz|}{|z|^2} = \int_{|z|=R} \frac{-idz}{Rz} = \frac{-i}{R} \int_{|z|=R} \frac{dz}{z} = \frac{2\pi}{R}.$$

Now by rotating a through some angle θ we can bring it to the real line. See that for $z' = e^{i\theta} z$, we get $|dz'| = |e^{i\theta} dz| = |dz|$. Also, $|z' - a|^2 = |e^{i\theta}|^2 |z - ae^{ß\theta}|^2$, so we assume that $a \in \mathbb{R}, a > 0$. Note that on the circle $|z| = R$, we have $\bar{z} = \frac{R^2}{z}$. Now see that

$$\begin{aligned}
\int_{|z|=R} \frac{|dz|}{|z|^2} = \int_{|z|=R} \frac{-iRdz/z}{(z-a)(\bar{z}-a)} &= -iR \int_{|z|=R} \frac{dz}{(z-a)(R^2 - az)} \\
&= \frac{iR}{a^2 - a^2} \left(\int_{|z|=R} \frac{dz}{z-a} + a \int_{|z|=R} \frac{dz}{R^2 - az} \right).
\end{aligned}$$

Now, if $a < R$, then the first integral in the bracket is $2\pi i$ while the second is 0 and the reverse when $a > R$.

Thus we have

$$\frac{2\pi R}{R^2 - |a|^2}$$

for $|a| < R$ and

$$\frac{2\pi R}{|a|^2 - R^2}$$

for $|a| > R$. This also takes care of the case where $a = 0$.