Functional Analysis

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Solution of problem 1*:* We need to check that rules of inner products hold—

1. For $A = B$, we have $\langle A, A\rangle = tr(AA^*) = \sum_{i,j} |a_{ij}|^2 \geq 0$, where a_{ij} denotes the elements of A. Moreover, $||A|| = 0 \implies |a_{ij}| = 0$ for all $1 \leq i, j \leq n \implies A = 0$.
2. $\langle B, A\rangle = tr(BA^*) = tr(A\overline{B}^T)$. See that $A\overline{B}^T(c_{ij})$ is such that $c_{ij} = \sum_{i=1}^n a_{i1}\overline{b_{j1}}$. See that $\overline{c_{ij}} = \sum_{i=1}^n \overline{a_{i1}}b_{j1}$, gives us $\sum_{1\leq i,j\leq n} a_{ij}\overline{b_{ij}}$. Note that replacing A and B just gives us the conjugate, which is the desired result, that
$$\langle B, A\rangle = \overline{\langle A, B\rangle}.$$
3. We have $\langle A + B, C\rangle = tr((A + B)C^*)$. We know that
$$tr((A+B)C^*) = \sum_{1\leq i,j\leq n} (a_{ij}+b_{ij})\overline{c_{ij}} = \sum_{1\leq i,j\leq n} a_{ij}\overline{c_{ij}}+\sum_{1\leq i,j\leq n} b_{ij}\overline{c_{ij}} = tr(AC^*)+tr(BC^*).$$
4. We have
$$\langle(\alpha A), B\rangle = tr(\alpha AB^*) = \sum_{1\leq i,j\leq n} \alpha a_{ij}\overline{b_{ij}} = \alpha \sum_{1\leq i,j\leq n} a_{ij}\overline{b_{ij}} = \alpha \langle A, B\rangle .$$

Therefore we have defined an inner product. Now fix $\varepsilon > 0$. Then we take a Cauchy sequence of matrices (A_n). There exists $N \in \mathbb{N}$ such that
$$\left(\sum_{i,j} \left|a_{ij}^{(n)} - a_{ij}^{(m)}\right|^2\right)^{1/2} < \varepsilon,$$
for $n, m \geq N$.

Thus we have that
$$\left|a_{ij}^{(n)} - a_{ij}^{(m)}\right| < \left(\sum_{i,j} \left|a_{ij}^{(n)} - a_{ij}^{(m)}\right|^2\right)^{1/2} < \varepsilon.$$

Thus we know that $a_{ij}^{(n)} \to a_{ij}$ in $\mathbb{C}$. We claim that $A = (a_{ij})$ is the desired limit.

We have
$$||A_n - A||^2 = \sum_{i,j} \left|a_{ij}^{(n)} - a_{ij}^{(m)}\right|^2 < \varepsilon,$$

which gives us the required answer. To solve the second part, see that since we can apply the Cauchy Schwarz inequality on inner product spaces, we have

$$|\langle A, B\rangle|^2 \leq ||A||^2 \cdot ||B||^2,$$

which gives us the required answer. $\square$

Solution of problem 2*:* We calculate $||x-y||^2 + ||x-z||^2 - ||x-u||^2$. Then see that

$$\begin{aligned}
t\,||x-y||^2 + (1-t)\,||x-z||^2 - ||x-u||^2 &= t(||x||^2 + ||y||^2 - \langle x, y\rangle - \langle y, x\rangle) \\
&+ (1-t)(||x||^2 + ||z||^2 - \langle x, z\rangle - \langle z, x\rangle) \\
&- (||x||^2 + ||u||^2 - \langle x, u\rangle - \langle u, x\rangle) \\
&= ||x||^2 - \langle x, u\rangle - \langle u, x\rangle + t\,||y||^2 + (1-t)\,||z||^2 \\
&- ||x||^2 - ||u||^2 + \langle x, u\rangle + \langle u, x\rangle \\
&= t\,||y||^2 + (1-t)\,||z||^2 - ||u||^2 \\
&= t\,||y||^2 + (1-t)\,||z||^2 - (\langle ty + (1-t)z, ty + (1-t)z\rangle) \\
&= t\,||y||^2 + (1-t)\,||z||^2 \\
&- \left(t^2\,||y||^2 + t(1-t)\,\langle y, z\rangle + t(1-y)\,\langle z, y\rangle + (1-t)^2\,||z||^2\right) \\
&= t(1-t)\,||y-z||^2.
\end{aligned}$$

The second result follows easily by setting $t = \frac{1}{2}$, which gives us $u = \frac{1}{2}(y+z)$. $\square$

Solution of problem 3*:* For any $z \in Y$, we want to show that $\Re\langle x-y, y-z\rangle \geq 0$. We know that for any $a, b \in H$ we have $||a+b||^2 = ||a||^2 + ||b||^2 + 2\Re\langle a, b\rangle$. Using this, we have

$$\Re\langle x-y, y-z\rangle = \frac{1}{2}\left(||x-z||^2 - ||x-y||^2 - ||y-z||^2\right).$$

We now just have to show that

$$||x-z||^2 \geq ||x-y||^2 + ||y-z||^2.$$

Let us set $u := ty + (1-t)z$ by Apollonius' identity, for some $t \in [0,1]$, then we have

$$t\,||x-y||^2 + (1-t)\,||x-z||^2 = ||x-u||^2 + t(1-t)\,||y-z||^2 \geq ||x-y||^2 + t(1-t)\,||y-z||^2.$$

Thus we have

$$||x-z||^2 \geq ||x-y||^2 + t\,||y-z||^2.$$

Putting $t \to 1$ gives us the desired result.

Conversely, since $\Re\langle x-y, y-z\rangle \geq 0$, we have

$$||x-z||^2 \geq ||x-y||^2 + ||y-z||^2,$$

implying that $||x-z||^2 \geq ||x-y||^2$ for all $z \in Y$. Thus $\inf_{z\in Y} ||x-z|| \geq ||x-y||$. For the other side, since $y \in Y$, we have

$$\inf_{z\in Y} ||x-z|| \leq ||x-y||,$$

which gives us the desired inequality.

$\square$

Solution of problem 4*:* We propose that $\mathbb{R}^\infty$ is an IPS that does not satisfy the projection theorem. This is an IPS as a subspace of ℓ^2. Let $a^{(n)} = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, \ldots\right)$. Then see that

$$\begin{aligned}\left|\left|a^{(n)} - a^{(m)}\right|\right| &= \sum_{k=1}^{\infty} \left|a_k^{(n)} - a_k^{(m)}\right|^2 \\ &= \sum_{k=m+1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{n} \frac{1}{k^2},\end{aligned}$$

which is a convergent sequence, hence the above is bounded above, meaning that this sequence is Cauchy. However, in ℓ^2 this sequence converges to a point $\alpha = \left(1, \frac{1}{2}, \ldots\right)$ outside $\mathbb{R}^\infty$, thus this space is not complete.

Define the linear functional $f : \mathbb{R}^\infty \to \mathbb{K}$ where $(x_n) \mapsto \langle (x_n), \alpha \rangle$.

Let $W = \ker f$. Our claim is that W is a proper subspace. $(1, 0, \ldots)$ is not in W, so that is done. We need to find $W^\perp$. Let $x \in W^\perp$, and $\beta^{(n)} = (1, \ldots, -n, 0, \ldots)$ which is in W. Then we have $x \perp W \implies x \perp \beta_n$. Then

$$\sum_{k=1}^{\infty} x_k \beta_k^{(n)} = x_1 - n x_n = 0,$$

which means that x has infinitely many non-zero terms. Thus $W^\perp = 0$, and this contradicts the statement of the projection theorem. □

Solution of problem 5*:* Since A_1 is bounded, all subsets are bounded. We can pick any $a_n \in A_n$ such that it's norm is minimum. Then we claim that (a_n) is a Cauchy sequence, and hence convergent. If we do show that it is Cauchy, then we have that the limit is contained within $\cap_{n=1}^\infty A_n$, which would complete the proof.

Now we have that $\forall x \in A_1, ||x|| \leq K$, for some $K > 0$. Since the norm is a monotone convergent sequence, the norm converges. By the parallelogram law,

$$||x_n - x_m||^2 = 2\,||x_n||^2 + 2\,||x_m||^2 - ||x_n + x_m||^2,$$

for $n \geq m$. Since $x_m \in A_m$ and $A_n \subseteq A_m$, have $x_n, x_m \in A_m$. Then from the convexity of A_m we have $\frac{1}{2}(x_n + x_m) \in A_m$. Thus $\frac{1}{2}\,||x_n + x_m|| \geq ||x_m||$. Thus, we have

$$||x_n - x_m||^2 \leq 2\,||x_n||^2 + 2\,||x_m||^2 - 4\,||x_m||^2 = 2(||x_n||^2 - ||x_m||^2).$$

Since $||x_n||$ is convergent, we must have that (x_n) is Cauchy, which completes the proof. □

Solution of problem 6*:* We want to construct an isometric isomorphism between H, a separable Hilbert space and ℓ^2, the sequence of square summable sequences over a linear field. We have H is separable, hence there exists a countable dense subset. This, in fact gives us an orthonormal Schauder basis $\{b_n\}_{n\in\mathbb{N}}$. Let the standard orthonormal basis for ℓ^2 be given by $\{e_n\}_{n\in\mathbb{N}}$. Define $T : H \to \ell^2$ be such that

$$T(\sum_{n=1}^{\infty} a_n b_n) = \sum_{n=1}^{\infty} a_n e_n.$$

For $\mathbf{a} = \{k_n\}, \mathbf{b} = \{l_n\} \in H$, we have

$$\langle T\mathbf{a}, T\mathbf{b}\rangle = \langle \sum_{n=1}^{\infty} k_n e_n, \mathbf{b}\rangle = \sum_{n=1}^{\infty} k_n \langle e_n, \mathbf{b}\rangle.$$

We can see that

$$\langle e_n, \sum_{m=1}^{\infty} l_n e_n\rangle = \sum_{m=1}^{\infty} \overline{l_m}\langle e_n, e_m\rangle = \bar{l}_n.$$

Thus we have $\langle T\mathbf{a}, T\mathbf{b}\rangle = \sum_{n=1}^{\infty} k_n \bar{l}_n = \langle \mathbf{a}, \mathbf{b}\rangle$. Thus our map is an isometry. It is clearly one-one. It is also onto, as the pre-image of any $\sum_{n=1}^{\infty} c_n e_n$ is $\sum_{n=1}^{\infty} c_n b_n$. Therefore we have an isomorphism of Hilbert spaces. □

Solution of problem 7*:* If V is a finite-dimensional vector space, then any total orthonormal set must be finite as there can be at most some finite number of linearly independent elements. Since a total orthonormal set must span the entire space, we have a Hamel basis since any element can be written as a finite linear combination of elements from the total orthonormal set.

Conversely, let V be a vector space such that every total orthonormal set is a Hamel basis. Take B to be a total orthonormal set, which is a Hamel basis by assumption. Let this be finite. Then we have $\{\bar{e}_n\} \subseteq B$. Now consider the series $\sum_n \frac{\bar{e}_n}{2^n}$. This converges in V, as $\sum_n \frac{1}{n}$ converges, hence let $x = \sum_n \frac{\bar{e}_n}{2^n}$. Since B is Hamel basis, we have $x \sum \alpha_i e_i$, which is a finite summation of terms $e_n \in B$ from the orthonormal set. However, from equating the two we see that clearly x has infinitely many non-zero coefficients, which contradicts that there is an infinite total orthonormal set. □

Solution of problem 8*:* Let C be a closed convex non-empty subset of H, a real Hilbert space. By Riesz Representation Theorem, we know that there exists $y \in H$ such that $f(x) = \langle x, y\rangle$. Then

$$\begin{aligned} g(x) &= \langle x, x\rangle - \langle x, y\rangle \\ &= \langle x, x - y\rangle \\ &= \frac{1}{2}(||x||^2 + ||x - y||^2 - ||y||^2) \end{aligned}$$

$$\geq \frac{1}{2}(||x_0||^2 + ||x_1 - y||^2 - ||y||^2),$$

where $x_0 \in C$ such that it has minimum norm, and $x_1 \in C$ is the point which is the best approximation of y to C.

We know that $\delta = \inf_{x\in C} g(x)$ exists. Let $\{x_n\}$ be a sequence in H such that $\lim_{n\to\infty} g(x_n) = \delta$. To see that this sequence is Cauchy, see that for $\varepsilon > 0$, we have

$$||x_n - x_m||^2 = 2\,||x_n||^2 + 2\,||x_m||^2 - ||x_n - x_m||^2 .$$

Since $\frac{1}{2}(x_n + x_m) \in C$, we have $g((x_n + x_m)/2) \geq \delta$. Thus $||x_n + x_m||^2 \geq 4\delta + 2\,\langle x_n, y\rangle + 2\,\langle x_m, y\rangle$. Then we use the above to see that

$$||x_n - x_m||^2 \leq 2(g(x_n)g(x_m)) - 4\delta.$$

Thus, for a large enough $n, m \in \mathbb{N}$, we get that $||x_n - x_m||$ is arbitrarily small, and hence $\{x_n\}$ is a Cauchy sequence. Clearly, this must converge in a Hilbert space, to some x_0. Since this is a closed convex set, this minimum is unique. $\square$

Solution of problem 9*:* We are given $T : \mathbb{K}^n \to \mathbb{K}^m$, where

$$(Tx)(i) = \sum_{j=1}^{n} k_{ij} x_j,$$

where $i = 1, 2, \ldots, m$. Let a_i denote the ith row of T. Then we have $\langle Tx, y \rangle = \sum_{j=1}^{m} (Tx)(i) y_j$. Expanding the entire thing, we have

$$\langle Tx, y \rangle = \sum_{1 \leq i \leq m, 1 \leq j \leq n} k_{ij} x_j \bar{y_i}.$$

We can write this as

$$\sum_{j=1}^{n} x_j \overline{\overline{k_{1i}} y_1 + \cdots + \overline{k_{mi}} y_m} = \langle x, \overline{T}^T y \rangle!$$

Therefore from uniqueness of adjoint we must have $T^* = \overline{T}^T$.

$\square$

Solution of problem 10*:* See that for any operator we have

$$|\langle Tx, x \rangle| \leq ||Tx|| \cdot ||x|| \leq ||T||,$$

taking $||x|| = 1$. Since the left of the inequality depends on x while the right is independent, we have $\sup_{||x||=1} \langle Tx, x \rangle \leq ||T||$. For the other direction, let $\alpha := \sup\{|\langle Tx, x \rangle| \mid ||x|| = 1\}$. We want to show that for $||x|| = ||\langle Tx, y \rangle|| \leq \alpha$. Since T is self-adjoint, we have $\langle Tx, y \rangle \in \mathbb{R}$. Then we have

$$\langle Tx, y \rangle = \frac{(\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle)}{4}.$$

But then

$$|\langle Tx, y \rangle| \leq \alpha \frac{||x+y||^2 + ||x-y||^2}{4} = \alpha,$$

by the parallelogram identity. $\square$