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1.

2. This statement is false. For sake of contradiction, let $(X, ||\cdot||)$ be a normed linear space such that the induced metric is the discrete metric. Then for $x, y \in X, x \neq y$ we must have $||x-y|| = 1$. Note that $2x \neq 2y$, so we must have $||2x - 2y|| = 2||x-y|| = 2$, but by the discrete metric the answer should still be 1! Thus there can be no such norm.

2

We wish to show that the function $||\cdot||$ on X satisfies the triangle inequality iff the closed unit ball is convex. Assume that the function $||\cdot||$ is indeed a norm. Then let $x, y \in D$, the closed unit ball. Then $||x||, ||y|| \leq 1$. Now we have for $\alpha \in [0,1]$ $z = \alpha x + (1-\alpha)y$. See that

$$||z|| = ||\alpha x + (1-\alpha)y|| \leq \alpha||x|| + (1-\alpha)||y|| \leq \alpha \cdot 1 + (1-\alpha)\cdot 1 \leq 1,$$

thus we have $z \in D$.

Take two elements $x, y \in X$ both non-zero, since if either were zero the inequality would be trivial. Then

$$||x+y|| = (||x|| + ||y||) \cdot \left|\left|\alpha\frac{x}{||x||} + (1-\alpha)\frac{y}{||y||}\right|\right|,$$

where $\alpha = \frac{||x||}{||x||+||y||}$. Note that $\frac{x}{||x||} = \frac{y}{||y||} = 1$, thus we can use the convexity condition to see that $\frac{||x+y||}{||x||+||y||} \leq 1$, which is the triangle inequality.

3

1. Pick a $f \in C([a,b])$. Then $|f| \leq M = \sup\{|f(x)| : x \in [a,b]\}$. Now see that

$$\int_a^b |f(t)|^p dt \leq (b-a)M^p \geq 0.$$

Thus $||f||_p \geq 0$. For $f = 0$, we have $M = 0$, so $\int_a^b |0|^p dt = 0$. If $\int_a^b |f(t)|^p dt = 0$, then see that $0 \leq (b-a)M^p \geq 0$. Thus we must have

$$(b-a)M^p = 0 \implies M = 0 \implies f = 0.$$

To see the next axiom, for $\alpha \in \mathbb{K}$, we have $\int_a^b |\alpha f(t)|^p dt = \int_a^b |\alpha|^p |f(t)|^p dt$. Then

$$||\alpha f||_p = \left(|\alpha|^p \int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} = |\alpha|||f||_p.$$

Now let $f, g \in C[a,b]$. Then we have to prove Minkowski's inequality to show the triangle inequality.

$$\begin{aligned}
||f+g||_p^p &= \int_a^b |f+g|^p dx \\
&= \int_a^b |f+g| \cdot |f+g|^{p-1} dx \\
&\leq \int_a^b |f||f+g|^{p-1} dx + \int_a^b |g||f+g|^{p-1} dx \\
&\leq \left(\int_a^b |f|^p dx + \int_a^b |g|^p dx \right) \left(\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} dx \right)^{1-\frac{1}{p}} && \text{(Hölder's inequality)} \\
&= (||f||_p + ||g||_p) \frac{||f+g||_p^p}{||f+g||_p},
\end{aligned}$$

which yields the required result.

2.

4

1. $||f_1 - F||_\infty$ is to be found, where F is the subspace of constant functions. Unfolding the term, we get
$$||f_1 - F||_\infty = \inf_{c \in \mathbb{R}} \{ \sup_{t \in [0,1]} |t - c| \},$$
which for $c \in [-1, 0] \cup [1, 2]$ is 1, while in $(0, 1)$ it decreases to $\frac{1}{2}$ then goes back up to 1. Thus, we must have $||f_1 - F||_\infty = \frac{1}{2}$.

2. We want to now see the distance between $f_2 = t^2$ and G, the space of all polynomials with degree at most 1. Then for some polynomial $-ax - b \in G$, we want to see $\sup_{t\in[0,1]}\{|t^2 + ax + b|\}$

5

Let Y and X/Y be Banach spaces. Then take (x_n) to be a Cauchy sequence in X. That is, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m, n \geq N$ we have

$$||x_m - x_n|| < \varepsilon.$$

By the canonical projection to X/Y we can see that the sequence $(x_n + Y)$ is also Cauchy, since we have

$$||x_m - x_n + Y|| \leq ||x_m - x_n + 0|| < \varepsilon.$$

Since X/Y is Banach, we have $(x_n + Y) \to (x_0 + Y)$. Now let $y_n :=$

Let X and X/Y be Banach spaces. Then take (y_n), a Cauchy sequence in Y. Thus for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$, we have $||y_m - y_n|| < \varepsilon$. As a Cauchy sequence in X, this must converge to some element $y_0 \in X$. We now need to show that $y_0 \in Y$. But since Y is closed and y_0 is a limit point, we must have $y_0 \in Y$.

Let X and Y be Banach spaces.

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Assume that X is a Banach space. Let (x_n) be an absolutely convergent series in X, that is, $\sum_{i=1}^n ||x_i||$ converges to $c \in \mathbb{R}$ as $n \to \infty$. Thus for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$,

$$|||x_n|| - \alpha| < \varepsilon.$$

Now we have

For the converse, take a Cauchy sequence (x_n). Pick $\varepsilon > 0$. Pick $N(n) \in \mathbb{N}$ such that for $m, n \geq N$, we have $||x_m - x_n|| < \frac{\varepsilon}{2^n}$ Now see that $\sum_{i=1}^n ||x_{i+1} - x_i||$

7

Let ℓ^p be the space of all $p-$power summable sequences. Then $||(x_n)|| = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. Now we want to show that the space $K = \{(x_n) \in \ell^p : x_i = 0 \forall i > n, n \in \mathbb{N}\}$ is dense in ℓ^p. Take any $(x_n)_{n\in\mathbb{N}}^{(0)} \in \ell^p$. There is a sequence of elements in ℓ^p $(x_n)^{(m)} \subseteq K$ such that $x_n^{(m)} = x_n$ if $n \leq m$, and 0 otherwise. See that $||(x_n)^{(m)} - (x_n)^{(0)}||_p = \left(\sum_{i=m+1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$. Since $||(x_n)^{(m)}||_p \to ||(x_n)^{(0)}||_p$ as $m \to \infty$, we must have for a choice of $\varepsilon > 0$ there being $N \in \mathbb{N}$ such that for $m \geq N$, $|||(x_n)^{(m)}||_p - ||(x_n)^{(0)}||_p| < \varepsilon$. Multiplying on both sides by $\frac{||(x_n)^{(m)}||_p^p - ||(x_n)^{(0)}||_p^p}{||(x_n)^{(m)}||_p - ||(x_n)^{(0)}||_p}$, and after a change in the value of ε, we get

$$||(x_n)^{(m)}||_p^p - ||(x_n)^{(0)}||_p^p < \varepsilon.$$

Thus we have shown that K is dense in ℓ^p. For a fixed m, and a fixed n, look at $x_n^{(m)}$. This is a real number or a complex number, which can be approximated by a sequence of rationals (or elements of $\mathbb{Q}(i)$) such that $x_n^{(m)}$ is its limit. Now we have a sequence (t_k), where $t_k \to x_n^{(m)}$ as $k \to \infty$.

Now let us see that $K' = \{(x_n) \in K : x_n \in \mathbb{Q} \text{ or} \mathbb{Q}(i)\} \subset K$ is countable, and as we saw above must be dense in ℓ^p. Thus it is separable.

8

We first show that ℓ^∞ is not separable. Take the space K of all sequences in ℓ^∞ such that their entries are either 0 or 1. Then for $x \neq y \in K$, we have $||x_y||_\infty = 1$, since at least one entry is different between the two. Then we have uncountably points. consider a ball of radius $\frac{1}{2}$ centred at $x \in K$, for all such points in K. We know what uncountably many open sets, all of which are disjoint from each other. Let S be a possibly dense set. Then we have that each open ball contains at least one point of S. This then means that there must be uncountably many elements, so ℓ^∞.

If there was a Schauder basis for ℓ^∞, then for all $x \in \ell^\infty$ we would have a sequence $(a_1, a_2, \dots)$ described using the basis. We can approximate each term by a sequence of elements in $\mathbb{Q}$ or $\mathbb{Q}(i)$, which would then mean that we would have all the rational points of ℓ^∞ as a dense subset, which contradicts the fact that the space is not separable. Thus there can be no Schauder basis.

9

We want to see if this function is continuous. Pick a $x_0 \in \mathbb{R}$. Now for all $\varepsilon > 0$ have $f(z_0+h) - f(z_0) = f(h)$.

See that $f(2) = 2f(1)$, $f(0) = 2f(0) \implies f(0) = 0$, and that $f(0) = 0 = f(1) + f(-1) \implies f(-1) = -f(1)$. Thus $f(n) = nf(1)$ for all $n \in \mathbb{Z}$. We know that $f(1) = f\left(n\frac{1}{n}\right) \implies f\left(\frac{1}{n}\right) = \frac{1}{n}f(1)$. Thus we have $f(q) = qf(1)$ for all $q \in \mathbb{Q}$. Since any real number can be approximated by a sequence of rationals, we can easily see that for $q_n \to r$ and $n \to \infty$, we have $f(q_n) = q_n f(1) \to rf(1)$. Thus we can extend this to all real numbers. Now we have $|f(z_0 + h) - f(z_0)| = |hf(1)| < \varepsilon$, for a choice of $h = \frac{\varepsilon}{|f(1)|}$.

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Let $P([0,1])$ be the space of all real polynomials defined on $[0,1]$ be a real vector space. Let the norm of a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n \in P([0,1])$ be given thus: $||f|| = |a_0| + |a_1| + \cdots + |a_n|$. Now see that the operator $I : P([0,1]) \to P([0,1])$ such that $x^t \mapsto \frac{x^{t+1}}{t+1}$, we then see that

$$||I|| = \sup_{0 \neq x \in P([0,1])} \frac{||If||}{||f||} = \frac{a_0x + \dots \frac{a_n}{n+1}a_{n+1}}{a_0 + \dots + a_nx^n} = \frac{|a_0| + \left|\frac{a_1}{2}\right| + \dots + \left|\frac{a_n}{n+1}\right|}{|a_0| + \dots + |a_n|} \leq 1.$$

Also see that this supremum is indeed attained since $I(a_0) = a_0x$, and in this case $\frac{||I(a_0)||}{||a_0||} = 1$. Thus we have $||I|| = 1$. We wish to find the inverse of this operator, see that the differential operator D such that $x^t \mapsto tx^{t-1}$, is the required inverse. However, see that for $f(x) = x^n$, we have $Df = I^{-1}f = nx^{n-1}$. Now we have

$$||D|| = \sup_{0 \neq x \in P([0,1])} \frac{||Df||}{||f||} \geq \frac{||Dx^n||}{||x^n||} = \frac{n}{1}.$$

Thus we have that our operator D is unbounded, since for any chosen $N \in \mathbb{N}$ we can choose x^{N+1}, such that $\frac{||Dx^{N+1}||}{||x^{N+1}||}$ is larger.

11

Since the linear functional $f : X \to \mathbb{R}$ is unbounded, we have a sequence (x_n) such that $f(x_n) > n||x_n||$. We can say without loss of generality, and by discarding non-zero elements, we have a sequence (x_n) of norm 1. Pick $x \in X$. Then see that the sequence $z_n = x - \frac{f(x)}{f(x_n)}x_n$, which is clearly in K, the kernel of f. Also it can be seen that as $n \to \infty$, we have $||z_n|| \leq$

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