(1) Let μ_i, ν_i be σ-finite positive measures on X, $i = 1, 2$. Suppose $\nu_1 << \mu_1$ and $\nu_2 << \mu_2$. Prove that

$$\nu_1 \times \nu_2 << \mu_1 \times \mu_2,$$

and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_1)}(x, y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y) \qquad a.e.$$

(2) Prove that if $f \in BV[a, b]$, then $|f| \in BV[a, b]$, and $V_a^b(|f|) \leq V_a^b(f)$.

(3) Suppose $f : [a, b] \to \mathbb{R}$ is differentiable at every point in $[a, b]$. Prove that f is absolutely continuous on $[a, b]$ if and only if $f \in BV[a, b]$.

(4) Let $f \in BV[a, b]$. Define

$$V(x) = V_a^x(f) \qquad (x \in [a, b]).$$

Prove that:
(i) $|f'(x)| \leq V'(x)$ a.e. $x \in [a, b]$.
(ii) $\int_a^b |f'| \leq V_a^b(f)$.
(iii) $\int_a^b |f'| = V_a^b(f)$ if and only if f is absolutely continuous.

(5) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \leq p < \infty$. Suppose $\{f_n\}_n \subseteq L^p$. (i) If

$$\sum_n \|f_n\|_p < \infty,$$

then prove that there exists $f \in L^p$ such that

$$f(x) = \sum_n f_n(x) \quad (x \in X \text{ a.e.})$$

and

$$f = \sum_n f_n.$$

(ii) Prove that if $f_n \to f$ in L^p, then $\{f_n\}_n$ has a subsequence which converges pointwise a.e. to f.

(6) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{f_n\}_n \subseteq L^p$, $1 \leq p \leq \infty$. Suppose $f_n \to f$ in L^p and $f_n \to \tilde{f}$ pointwise a.e. for some $f, \tilde{f} \in L^p$. Does that mean $f = \tilde{f}$ a.e?

(7) Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{h \to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dm(x) = 0.$$

Prove that $f = 0$ a.e.

(8) Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that every L^∞ Cauchy sequence of measurable functions converges uniformly almost everywhere.

(9) Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove that L^∞ norm convergence implies pointwise convergence. What about L^p, $1 \leq p < \infty$?

(10) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $1 \leq p_1 < p_2 < \infty$. Prove the following assertions:
(i) If $\mu(X) < \infty$, then $L^{p_2} \subset L^{p_1}$.
(ii) If $\mu(X) = 1$, then $\|f\|_{p_1} \leq \|f\|_{p_2}$ for all $f \in L^{p_2}$.

(iii) Give an example to show that if $\mu(X) = \infty$, then the conclusion in (i) need not be true.

(11) Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = e^{-x} \qquad (x \in \mathbb{R}).$$

For what values of p is f an L^p function?

(12) For what values of p is

$$\left\{ \frac{1}{\sqrt{n} \log n} \right\} \in l^p?$$

(13) Prove that $L^\infty(\mathbb{R})$ is not separable. Here, $\mathbb{R}$ is equipped with the Lebesgue measure.