The last Home-work

Gandhar Kulkarni (mmat2304)

1

1. Since R is a PID, we know that $(a,b) = (d)$, for some $d, a, b \in R$. Then we have $d = am + bn$, for some $m, n \in R$. Now we have a vector $v = [a,b]^T \in R^2 \backslash \{0\}$. Then we show that there exists a $2x2$ matrix that does what we want by constructing one. Let the desired matrix by be given by $X = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$. Now we have $Xv = [x_{11}a + x_{21}b, x_{12}a + x_{22}b]^T = [d, 0]^T$. Comparing terms, we have $x_{12}a + x_{22}b = 0$. Then we have $x_{12}a = -x_{22}b$, which implies that $x_{12}| - b$, and $x_{22}|a$. It is easy to see that $x_{12} = -a$ and $x_{22} = b$ does the trick. For $x_{11}a + x_{21}b = d$, see that $x_{11} = m$ and $x_{21} = n$ are good choices, since their linear combination produces d. Thus see that
$$X = \begin{pmatrix} m & n \\ -b & a \end{pmatrix}$$
is a matrix that achieves the intended result.

2

3

We have M a $R-$module which has itself as a generating set. Then $\pi : R^{\oplus M} \twoheadrightarrow M$ is the surjective map sending e_m to m. We see that $\pi(e_{rm} - re_m) = \pi(e_{rm}) - r\pi(e_m) = rm - rm = 0$. Also, $\pi(e_{m_1+m_2} - e_{m_1} - e_{m_2}) = \pi(e_{m_1+m_2}) - \pi(e_{m_1}) - \pi(e_{m_2}) = (m_1 + m_2) - m_1 - m_2 = 0$. Therefore we have $e_{rm} - re_m \in \ker \pi$, and $e_{m_1+m_2} - e_{m_1} - e_{m_2} \in \ker \pi$. Thus we have
$$(e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2}) \subseteq \ker \pi.$$
Let us have $\sum_{m\in M} r_m e_m \in R^{\oplus M}$. Note that there are only finitely many terms in the summation. See that
$$\pi(\sum_{m\in M} r_m e_m) = \sum_{m\in M} r_m \pi(e_m) = \sum_{m\in M} r_m m = \pi(\sum_{m\in M} e_{r_m m}).$$
This means that $\pi(\sum_{m\in M} r_m e_m - \sum_{m\in M} e_{r_m m}) = 0$. This, in turn implies that $\sum_{m\in M} r_m e_m - e_{r_m m} \in \ker \pi$. We also see that
$$\pi(\sum_{m\in M} r_m e_m) = \sum_{m\in M} r_m \pi(e_m) = \sum_{m\in M} r_m m = \pi(e_{\sum_{m\in M} r_m m}).$$
This means that $\sum_{m\in M} r_m e_m - e_{\sum_{m\in M} r_m m} \in \ker \pi$. Given an element in $R^{\oplus M}$, we can choose which summands to clump and which to leave unchanged. Either ways, we see that we get a linear combination of $re_m - e_{rm}$ and $e_{m_1+m_2} - e_{m_1} - e_{m_2}$, which implies that $\ker \pi \subseteq (e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2})$. This gives us the desired equality.

4

1. We take the module $\mathbb{Z}[q_1, q_2]$, where $q_1, q_2 \in \mathbb{Q}$.

5

6