Algebra 2 Homework 7

March 18, 2024

Solution of problem 1*:* □

Solution of problem 2*:* Since $A^k = I$ for some $k \geq 1$, we have that $x^k - 1 = 0$ must be the minimal polynomial. Solving this over $\mathbb{C}$, we have the kth roots of unity, which are exactly k many. The characteristic polynomial must have n roots, all of whom must be kth roots of unity.

For $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, see that $A^k = \begin{pmatrix} 1 & k\alpha \\ 0 & 1 \end{pmatrix}$, for any $k \in \mathbb{N}$. This can easily be checked by induction. See that for $k = p$, we must have I since the field has characteristic p. See that the characteristic of A is just $x^2 - 1 = 0$, which gives us ± 1 as the eigenvalues. We see that the $\ker \dim(A - I) = \{(x, y) \in \mathbb{R}^2 \mid A - I(x, y)^T = 0\} = \dim(\text{ span } (1, 0)) = 1$, while $\ker \dim(A + I) = \{(x, y) \in \mathbb{R}^2 \mid A + I(x, y)^T = 0\} = \dim(\text{ span } (0, 0)) = 0$, which clearly does not add up to 2, since 1 and 2 are not the same number. Thus for $\alpha \neq 0$, this matrix is not diagonalisable. If $\alpha = 0$, in the eigenvalue -1, we would have the null space as $\text{ span } (0, 1)$, which would give us a basis for $\mathbb{R}^2$. □

Solution of problem 3*:* If we see K/F_1, then it is clear that since $K = F_1(\sqrt[8]{2})$, we have a degree 8 extension. To understand $\text{Aut}(K/F_1)$, see that we only need to understand where $\sqrt[8]{2}$ goes. We have eight choices. We can send $\sqrt[8]{2}$ to any □

Solution of problem 4*:* We can see that $x^4 - 14x^2 + 0 = 0$ is solved as a quadratic in x^2, that is, $x^2 = 7 \pm 2\sqrt{10}$. We can write $7 - 2\sqrt{10} = 7 - 2\sqrt{10}\frac{7+2\sqrt{10}}{7+2\sqrt{10}} = \frac{9}{7+2\sqrt{10}}$. Then we can see that $x = \pm\sqrt{7 + 2\sqrt{10}}, \pm\frac{3}{\sqrt{7+2\sqrt{10}}}$. All of these roots lie in $K = \mathbb{Q}(\sqrt{7 + 2\sqrt{10}})$.

We propose that this field is Galois. The minimal polynomial for $\sqrt{7 + 2\sqrt{10}}$ is all in K, thus it is normal, since it contains all of its conjugates. It is also clear that none of the roots are repeated, so the extension is separable. Thus K is the Galois splitting field. □

Solution of problem 5*:*

1. $[K : F] = n < \infty$, then we say that K is represented as by a finite $F-$basis, by $\{b_1, \dots, b_n\}$. Let $x \in K$, where $x = \sum_{i=1}^n c_i b_i$, where $c_i \in F$ for $i = 1, \dots, n$. Now we have $\alpha \cdot x = \sum_{i=1}^n c_i(\alpha \cdot b_i)$. Since $\alpha \cdot b_i \in K$, $\alpha \cdot b_i = \sum_{j=1}^n a_{ji} b_j$. Then we can put this in matrix form, by $T_\alpha := (a_{ji})$. By this, it is clear that T_α is a linear transformation, since it can be written as a matrix.
2. Let $m(x)$ be the minimal polynomial for α with degree d_1, and $m'(x)$, be the minimal polynomial for A with degree d_2. If $d_1 \geq d_2$, then

$$m(x) = q_1(x)m'(x) + r_1(x),$$

□

Solution of problem 6*:* We have proven in the previous assignment that $f(x) = x^p - x - a$ is irreducible over $\mathbb{F}_p$, and that it is separable. We know that if α is a root, then so is $\alpha + 1$. So it is easy to see that the □

Solution of problem 7*:* In $E = \mathbb{Q}(\sqrt{1+\sqrt{2}})$,let $x = \sqrt{1+\sqrt{2}}$. Then $x(x^2-1)^2 = 2$. We get that $x^4 - 2x^2 - 1 = 0$. We can explicitly see that none of its roots are rational, so the polynomial we have must be the minimal polynomial. We can calculate the conjugates, which are

$$\pm\sqrt{1+\sqrt{2}}, \pm i\frac{1}{\sqrt{1+\sqrt{2}}}.$$

So it is easy to see that i is missing. Thus, by adjoining i, we can have a normal and separable extension containing $\sqrt{1+\sqrt{2}}$. We propose that $K = \mathbb{Q}(\sqrt{1+\sqrt{2}}, i)$ is the Galois extension. We know K contains all the conjugates, so the Galois field must be sandwiched between K and F. But since $[K : F] = 2$, which is prime, we must have that either the field is equal to E or F, and it cannot be F, since the imaginary conjugates are not in F, the Galois closure must be E. □

Solution of problem 8*:* □

Solution of problem 9*:* □