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1

To prove the result, we will decompose an arbitrary permutation $\sigma \in S_n$ into transpositions. We will then show that each transposition can be written as a product of transpositions of the form $(ii+1)$. We have $\sigma = \tau_1 \dots \tau_r$, where τ_i is some transposition of the form $(k_1 k + 1 + k_2)_i$. Note that we can assume the first element of τ_i is strictly lesser than the second since if it weren't we could just invert the order without any loss of generality. We state that $(k_1 k_1 + k_2)$ can written as a product of finite transpositions of the form $(tt+1)$. See that $(k_1 k_1 + k_2 - 1) = (k_1 + k_2 - 1 k_1 + k_2)(k_1 k_1 + k_2)(k_1 + k_2 - 1 k_1 + k_2)$. Since we have $(k_1 k_1 + k_2 - 1)$, we have lowered the second entry of the transposition by one. In $k_2 - 1$ steps, we will get $(k_1 k_1 + 1) = (k_1 + 1 k_1 + 2)(k_1 k_1 + 2)(k_1 + 1 k_1 + 2)$, which means that we can stop. We have τ_i as a product of $2(k_2 - 1) + 1$ transpositions of the desired type. We can do this for all transpositions to get our result. Thus we can generate any permutation in S_n by exchanging adjacent elements. The bubble sort algorithm works this way too, which accepts a permutation then returns the list of n numbers. This is essentially the same problem.

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1. For some $1 \leq i \leq n$, $G)i = \{\sigma \in S_n : \sigma(i) = i\} \cong S_{n-1}$. Consider G_i acting on $\{1, 2, \dots, i - 1, i + 1, \dots, n\}$. If $n = 2$, this set will be a singleton, hence trivially transitive. For $n \geq 3$, the set $\{1, 2, \dots, n\} \backslash \{i\}$ has at least two elements. Then $k, \ell \in \{1, 2, \dots, i - 1, i + 1, \dots, n\}$ such that they are distinct (If $k = \ell$, then $i \in G_i$ works). See that $(k\ell) \in G_i$, as it does not affect i. Then we have $(k\ell)k = \ell$, which means that G_i is transitive.
2. For a doubly transitive action of G on X, see that if we choose a proper subset then since

4

Let us denote all elements of Q_8 thus: $1, i, j, k, -1, -i, -j, -k$ are assigned the numbers from 1 to 8. Then see that left multiplication by 1 is the identity permutation on S_8. Left multiplication by i is $(1256)(3478)$, by j is $(1357)(2864)$, and by k is $(1458)(2367)$. Their negatives also have a left regular representation. Now see that i, j can generate Q_8 as a group, then it must stand to reason that their corresponding left regular representations will behave in the same way! Thus, see that $G = \langle (1256)(3478), (1357)(2864) \rangle \cong Q_8$.

5

We know that $|[G : H]| = n$. Consider the group action of left multiplication on left cosets of H. This group action has a permutation representation, let us denote that by π_H. Take $K = \ker \pi_H$, and $|[H : K]| = k$. Then we have $|[G : K]| = |[G : H]||H : K| = nk$. Since H has n many cosets, we must have $\frac{G}{K}$ is isomorphic to some subgroup of S_n. Clearly $K \leq H$, and $K \trianglelefteq G$, and since $nk|n!$, we have

6

We shall prove a result that for $|G| = n$ and p the smallest prime that divides n, then a subgroup of order p must be normal. Let some $H \leq G$, with $|[G : H]| = p$. Consider the group action of left multiplication on left cosets of H. This group action has a permutation representation, let us denote that by π_H. Take

$K = \ker \pi_H$, and $|[H : K]| = k$. Then we have $|[G : K]| = |[G : H]||H : K| = pk$. We know that H has p many left cosets, hence $\frac{G}{K}$ is isomorphic to some subgroup of S_p which is the image of G under π_H. Thus we must have $pk|p! \implies k|(p-1)!$. But since k can only have prime factors greater than or equal to p and $(p-1)!$ has no prime factors greater than p, we must have $k = 1$. Thus $H = K \trianglelefteq G$ is normal.

Let p be the smallest prime dividing n. We know that $p < n$, since n is composite. Then see that there must exist a subgroup of order $\frac{n}{p}$ as given in the problem, hence this subgroup has index $\frac{n}{\frac{n}{p}} = p$, hence this is a normal subgroup. Thus G cannot be simple.

7

We know that $|[G : Z(G)]| = n$. We see that in the class equation we have

$$|G| = |Z(G)| + \sum_{x \notin Z(G)} |[G : C(x)]|.$$

Dividing on both sides by $|Z(G)|$ we have

$$n = 1 + \sum_{x \notin Z(G)} \frac{n}{|C(x)|}.$$

Thus

$$\frac{1}{n} + \sum_{x \notin Z(G)} \frac{1}{|C(x)|} = 1.$$

8

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10