Algebra HW5

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1. See that $bx - a \in \ker \pi$, since $b \cdot \left(\frac{a}{b}\right) - a = 0$. Therefore $(bx - a) \subseteq \ker \pi$. Now consider $f(x) \in R[x]$ where $f(x) \in \ker \pi$. We consider the polynomials $f(x)$ and $x - \frac{a}{b}$ as elements of $Q[x]$ the ring of polynomials with coefficients from the fraction field of R. Then this is a PID, which is why we can apply the division algorithm to see that $f(x) = q(x)\left(x - \frac{a}{b}\right) + c$, where $c \in Q, q(x) \in Q[x]$. Setting $x = \frac{a}{b}$ we get $c = 0$. Thus we have $f(x) = q(x)\left(x - \frac{a}{b}\right)$. We rewrite all polynomials are primitive polynomials in $R[x]$; thus we have $f(x) = a_1 \cdot f_0(x)$, $q(x) = a_2 \cdot q_0(x)$, and $x - \frac{a}{b} = b^{-1} \cdot (bx - a)$. Then we have $a_1 \cdot f_0(x) = (a_2 b^{-1}) q_0(x)(bx - a)$. We multiply on both sides by some $k \in R$ such that $ka_1 b \in R$ and $ka_2 \in R$ and the two are coprime. The constant cannot divide the polynomials as they are all primitive, hence we must have $ka_1 b | ka_2$, and by Gauss' lemma we can say that $f_0(x) | q_0(x)(bx - a)$, that is, $f \in (bx - a)$. Thus we have $\ker \pi = (bx - a)$.
2. Note that $(1 + \sqrt{-3}) \cdot (1 - \sqrt{-3}) = 2 \cdot 2 = 4$, which means that R is not a UFD. Therefore the above result needn't apply. To show that the above result strictly does not apply, we need to find $f \in \ker \pi$ such that $f \notin (2x - (1 + \sqrt{-3}))$. See that $f(x) = x^2 - x + 1$ does the trick well. It is in fact the minimal polynomial, but it is not in $(2x - (1 + \sqrt{-3}))$. It is easy to prove, as the ideal of leading coefficients $R \cap (2x - (1 + \sqrt{-3})) = (2)$, and this clearly does not include the leading coefficient of $f(x)$. Thus $f \in \ker \pi \backslash (2x - (1 + \sqrt{-3}))$.

   The underlying reason for why this fails stems from the fact that the ring of integers of the number field $\mathbb{Q}(\sqrt{-3})$ is $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right] \supsetneq \mathbb{Z}[\sqrt{-3}]$; that is, the ring of integers is strictly larger than R as given in this problem. This has to do with the fact that $-3 \cong 1 \mod 4$, which introduces interesting additional algebraic integers into the number field.

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Let $R = \frac{F[x,y]}{xy}$, and $I = (\bar{x}, \bar{y})$. Then

$$\frac{R}{I} \equiv \frac{\frac{F[x,y]}{(xy)}}{(\bar{x}, \bar{y})} \equiv \frac{F[x,y]}{(xy, x, y)} \equiv \frac{F[x]}{0 \cdot x, x} \equiv F,$$

which is a domain. Thus I is prime. To see that it is not principal, we assume for the sake of contradiction that $I = (f_0)$, where $f_0 \in F[x,y]$. Note that once seen modulo (xy), we have $\bar{f_0} = f_1(x) + f_2(y)$, where $f_1 \in F[x], f_2 \in F[y]$. If we say that $(f_0) = (\bar{x}, \bar{y})$, then we have $f_0 | x$ and $f_0 | y$. $f_0(x,y) = f_1(x) + f_2(y)$ must have degree less than or equal to 1, with no term of y, hence $f_2(y) = c_2$ and $f_1(x) = c_1 + dx$. Set $c = c_1 + c_2$, then see that we must have $x = t(c + dx)$ for some $t \in F[x,y]$. Comparing degrees, we must have $t \in F \backslash \{0\}$. Comparing the two sides, see that $c = 0, d = 1/t$ which is the only possibility. However, $x \nmid y$, so such a f_0 cannot exist. Thus I is not principal.

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Let

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 5 & 5 & 4 & 4 \\ 6 & 7 & 7 & 8 \\ 10 & 10 & 9 & 9 \end{pmatrix}.$$

Through a myriad set of row and column operations shall we reduce our matrix A to form that will generate an alike cokernel. We shall use R_1, R_2, and R_3 to denote the rows, while C_1, C_2, and C_3 shall denote the columns of A. First we execute $C_2 \mapsto C_2 - C_1, C_4 \mapsto C_4 - C_3$. Then we execute $C_4 \mapsto C_4 - C_2$ to get

$$A' = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 5 & 0 & 4 & 0 \\ 6 & 1 & 7 & 0 \\ 10 & 0 & 9 & 0 \end{pmatrix}.$$

Execute $C_2 \mapsto C_2 - C_1, C_3 \mapsto C_3 - 2C_1$ to get

$$A'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & -5 & -6 & 0 \\ 6 & -5 & -5 & 0 \\ 10 & -10 & -11 & 0 \end{pmatrix}.$$

Execute $R_2 \mapsto R_2 - R_1, R_3 \mapsto R_3 - 6R_1$, and $R_4 \mapsto R_4 - 10R_1$. After this, execute $R_3 \mapsto R_3 - R_2$ and $R_4 \mapsto R_4 - 2R_2$ to get

$$A''' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Execute $R_2 \mapsto -R_2$, and $R_4 \mapsto R_4 - R_3$. After this execute $R_2 \mapsto R_2 - 6R_3$ to get

$$A'''' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We are well aware that the cokernel is left unchanged due to our row and column operations. Thus it can be seen that the image of this matrix is $A''''(x_1x_2x_3x_4)^T = (x_15x_2x_30)$. Therefore the image is $\mathbb{Z} \oplus 5\mathbb{Z} \oplus \mathbb{Z} \oplus 0$. Then $\text{coker}A = \frac{Z^4}{\mathbb{Z}\oplus 5\mathbb{Z} \oplus \mathbb{Z} \oplus 0} = \frac{\mathbb{Z}}{5\mathbb{Z}}$.

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1. Since $f \circ g = 0$, we have $f(g(p)) = 0 \forall p \in P$. Then we have $g(p) \in \ker f \forall p \in P \implies g(P) \subseteq \ker f$. Thus we can define $h : P \to \ker f$ as $h(p) = g(p)$. If another $h' : P \to \ker f$ exists such that $g = i \circ h'$, then we have $g = i \circ h = i \circ h'$. Since i is injective, for all $p \in P$ we have $i(h(p)) = i(h'(p)) \implies h(p) = h'(p)$, thus we have $h = h'$, proving the uniqueness of h.

2. Let $h(\bar{n}) = g \circ \pi^{-1}(\bar{n})$, for $\bar{n} \in \text{coker}f$. We propose that this is the desired map. We need to see that this map is well defined. $\pi^{-1}(\bar{n}) = n + f(M)$, for some $n \in N$. We need to see that the choice of representative does not matter. We can see that since g is $R-$linear we have $g(n + f(M)) = g(n) + g \circ f(M) = g(n) \in P$. Thus this map is well defined. To see that this map is unique, for another such map $h' : \text{coker}f \to P$ such that $g = h \circ \pi$, we have $g = h \circ \pi = h' \circ \pi$, which implies that $h = h'$ is surjective, where right cancellation is possible. Thus this map is unique.

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We can see that $(0) \subseteq \ker f \subseteq \ker f^2 \subseteq \dots$ which is an ascending chain of submodules of M. This clearly must stabilise as the Noetherian condition is equivalent to the ascending chain condition. That is, for some $n \in \mathbb{N}$ we have $\ker f^n = \ker f^{n+1} = \ker f^{n+2} = \dots$. Now see that for some $m \in \ker f$ we have $f(m) = 0$. Since f is surjective, we can find a $m' \in M$ such that $f(m') = m$. Repeating this process, see that there must exist some $m_n \in M$ such that $f^n(m_n) = m$. Applying f on both sides, we have $f^{n+1}(m_n) = f(m) = 0$. Thus $m_n \in \ker f^{n+1} = \ker f^n$, we must have $f^n(m_n) = m = 0$. Thus m must necessarily be zero, meaning that a surjective endomorphism on a Noetherian module must necessarily be injective, and thus an isomorphism.