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## 1

We are given $f(z) = z^2 - z\bar{z}^2 - 2|z|^2$. Since $z = x + iy$, we can expand it in $f$ to get

$$f(x, y) = -2(x^2 + y^2) + i4xy.$$

Thus $u = -2(x^2+y^2), v = 4xy$. Then we have $u_x = -4x, u_y = -4y, v_x = 4y, v_y = 4x$. If $f$ is holomorphic, then we must have $u_x = v_y \implies -4x = 4x \implies x = 0$. Also we must have $u_y = -v_x \implies -4y = -4y \implies y \in \mathbb{R}$. Thus $f$ satisfies the Cauchy Riemann equations on $\{0\} \times \mathbb{R}$, which is not a domain since it is not open. Thus it is complex differentiable at each point of the type $(0, y)$ where $y \in \mathbb{R}$, but not holomorphic at any point in $\mathbb{C}$ since the points at which it satisfies the Cauchy Riemann equations is not open in $\mathbb{C}$.

## 2

Let us assume that there exists a holomorphic function on a domain $D$ such that its image lies entirely on a vertical line, say $x = \frac{1}{2}$. Thus for $f = u + iv$, we must have that $u = \frac{1}{2}$, a constant. Then $u_x = u_y = 0$, and by the Cauchy-Riemann equations, we have $v_y = u_x = 0 = u_y = -v_x$. Thus we have $v$ constant as well, which means that $f$ must be a constant.

## 3

## 4

Let us assume that the image of $f$ lies on a line passing through the origin. We have $f(t) = \gamma_1(t) + i\gamma_2(t)$, and $\exists \alpha \in \mathbb{R} f([0, 1]) \subseteq ((x, \alpha x), x \in \mathbb{R})$ or $f([0, 1]) \subseteq i\mathbb{R}$. In the second case, we have $\gamma_1(t) = 0$. Then $|\int_0^1 f(t)dt| = |i \int_0^1 \gamma_2(t)dt| = |\int_0^1 \gamma_2(t)dt|$. Also $\int_0^1 |f(t)|dt = \int_0^1 |\gamma_2(t)|dt$.

For the first case, we have $\gamma_1(t) = \alpha\gamma_2(t)$. Then $|\int_0^1 f(t)dt| = |(1+i\alpha) \int_0^1 \gamma_1(t)dt| = \sqrt{1 + \alpha^2}|\int_0^1 \gamma_1(t)dt|$. Also see that $\int_0^1 |f(t)|dt = |\int_0^1 |\gamma_1(t) + i\alpha\gamma_1(t)|dt = \sqrt{1 + \alpha^2} \int_0^1 |\gamma_1(t)|dt$.

## 5

1. The curve can be parametrised by $\gamma : [0, 1] \to \mathbb{C}$. We have $\gamma(t) = \omega(1 - t) + \omega^2 t$. Expanding this, we can see that $\gamma(t) = \frac{-1+\sqrt{3}i(1-2t)}{2}$. See that $\gamma'(t) = -\sqrt{3}i$, then we have

$$\begin{aligned}
\int_\gamma |z^2|dz &= \int_0^1 |\frac{-1 + \sqrt{3}i(1 - 2t)}{2}|^2|(-\sqrt{3}i)|dt \\
&= \frac{\sqrt{3}}{4} \int_0^1 (\frac{-1 + \sqrt{3}i(1 - 2t)}{2})^2 dt \\
&= \frac{\sqrt{3}}{4} \int_0^1 (1 - 3(1 - 2t)^2 - 2\sqrt{3}(1 - 2t))dt \\
&= \frac{\sqrt{3}}{4} \int_0^1
\end{aligned}$$

2. The curve can be represented as $\gamma : \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \to \mathbb{C}$. We have $\gamma(t) = e^{it}$. For this we have $\gamma'(t) = ie^{it}$, so $|\gamma'(t)| = 1$. Then the integral is

$$\begin{aligned}\int_\gamma |z^2| dz &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} |e^{it}|^2 |1| dt \\ &= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} dt \\ &= \frac{2\pi}{3}.\end{aligned}$$

3. Here, $\gamma$ has four parts labelled 1 to 4. The curves are separately parametrised by the same parameter $t$, in a classic fashion of abuse of notation. These are the four curves: $\left(\frac{1}{2}, t - \frac{1}{2}\right), \left(\frac{1}{2} - t, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2} - t\right)$, and $\left(t - \frac{1}{2}, -\frac{1}{2}\right)$. Their respective absolute values of derivatives are 1 throughout. The integral is calculated thus:

$$\begin{aligned}\int_\gamma |z|^2 dz = &\int_0^1 \left(\frac{1}{2}\right)^2 + \left(t - \frac{1}{2}\right)^2 dt + \int_0^1 \left(\frac{1}{2} - t\right)^2 + \left(\frac{1}{2}\right)^2 dt + \\ &\int_0^1 \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2} - t\right)^2 dtz + \int_0^1 \left(t - \frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 dt \\ &= 4\left(\frac{1}{4} + \frac{1}{12}\right) = \frac{4}{3}.\end{aligned}$$

# 6

Let $C := \{e^{it} : t \in \left[0, \frac{pi}{2}\right]\}$, which parametrizes the curve. Then

$$\begin{aligned}\int_C \overline{\mathrm{Log}(z)} dz &= \int_0^{\frac{\pi}{2}} \overline{\mathrm{Log}(e^{it})} |ie^{it}| dt \\ &= \int_0^{\frac{\pi}{2}} \overline{\log(1) + it} dt \\ &= \int_0^{\frac{\pi}{2}} -it dt \\ &= -i\frac{\pi^2}{4}.\end{aligned}$$

# 7

# 8

# 9

We will try to describe the function $\log \log z$, the objective is to make sure that the argument of the function lies in $(-\pi, \pi]$. Let $z = x + iy$. Then we have

$$\begin{aligned}\log \log(x + iy) &= \log\left(\log|z| + i2n\pi \arg(z)\right) \\ &= \log\left(\frac{1}{2}\log(x^2 + y^2) + i2n\pi \tan^{-1}\left(\frac{y}{x}\right)\right) \\ &= \frac{1}{2}\log\left(\frac{1}{4}\log\left(x^2 + y^2\right)^2 + 4n\pi^2 n^2 \tan^{-1}\left(\frac{y}{x}\right)^2\right) \\ &+ i\left(\frac{4\pi n}{\log(x^2 + y^2)} \tan^{-1}\left(\frac{y}{x}\right)\right),\end{aligned}$$

where $n \in \mathbb{Z}$.

## 10

We know that $|dz| = -iR\frac{dz}{z}$, then our integral is $I = \int_{|z|=R} \frac{-iRdz}{|z-a|^2}$. If $|a| > R$, then our function $\frac{1}{|z-a|^2}$ is holomorphic in the interior of the circle $|z| = R$, so the integral must be 0. In the case where $|a| < R$, we can use Cauchy's