Algebra HW5

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1

If we have $\nu = 0$, then see that $\nu(E) = \int_E 0 d\mu$, thus $\mu(E) = 0 \implies \nu(E) = 0$ trivially, so $\nu << \mu$. Also see that $\nu \perp \mu$, as $X = X \sqcup \phi$, and see that $\mu(E) = \mu(E \cap X)$, while $\nu(E) = \nu(E \int \phi)$.

Now let us assume that $\nu << \mu$ and $\nu \perp \mu$. Then we have $X = A \sqcup B$, where $\mu(E) = \mu(E \cap A)$, while $\nu(E) = \nu(E \int B)$. Let us pick a measurable set $E \subseteq B$. Then we have $\mu(E) = \mu(E \cap B) = \mu(\phi) = 0$. Since $\nu << \mu$, we have $\nu(E) = 0$. Thus ν is zero on every measurable subset E in B. For a general measurable set E, we have $E = (E \cap A) \sqcup (E \cap B)$. We already know that $\nu(E \cap A) = 0$, now we see that $\nu(E \cap B) = 0$ also. Thus $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = 0$ for all measurable $E \subset X$.

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If $\nu \perp \mu$, then there exists we have $X = A \sqcup B$, where $\mu(B) = 0$ and $\nu(A) = 0$. Then $\{E_n\}$ is a sequence such that $E_n = B$ for all $n \in \mathbb{N}$. Then see that $\mu(E_n), \nu(X \backslash A) = 0$, as required.

Conversely, we assume that $\{E_n\}$ is a sequence of measurable subsets such that $\mu(E_n) \to \infty$ and $\nu(X \backslash E_n) \to \infty$ as $n \to \infty$.

3

We can check that $m << m$ trivially, as $m(E) = 0 \implies m(E) = 0$. We can split $\mathbb{R}$ into two disjoint subsets, that is, $\mathbb{R} = \{0\} \sqcup ((\infty, 0) \cup (0, \infty))$. We denote the two sets as A and B. Then observe that $\delta_0(E) = \delta_0(E \cap A)$, that is, the Dirac measure at 0 only cares if it intersects $\{0\}$, and nothing else. Also, we have $m(E) = m(E \cap B)$, as A is a $m-$null set, hence $m(E) = m(E \cap A \sqcup E \cap B) = m(E \cap A) + m(E \cap B) = m(E \cap B)$, seeing as $E \cap A$ is also a $m-$null set. Thus we have $m \perp \delta_0$. Thus $\nu = m + \delta_0$ is already in the Lebesgue decomposition.

4

- We find the positive and negative parts of f. Note that the roots of this polynomial are $3 + 2\sqrt{2}$ and $3 - \sqrt{2}$. Let us call them α_1 and α_2 for sake of convenience. Then

$$p^+ = \begin{cases} x^2 - 6x + 1 & x \in (-\infty, \alpha_2] \cup [\alpha_1, \infty) \\ 0 & \text{else,} \end{cases}$$

  and

$$p^- = \begin{cases} -(x^2 - 6x + 1) & x \in (\alpha_2, \alpha_1) \\ 0 & \text{else.} \end{cases}$$

  Then $\nu(E) = \int_E p^+ d\mu - \int_E p^- d\mu = \nu^+ - \nu^-$, where $\nu^+ := \int_E p^+ d\mu$ and $\nu^- := \int_E p^- d\mu$ are two positive measures. Note that it is not possible for both of them to attain ∞ together, since ν^- is a finite measure. Thus it is trivial to see that ν must be a signed measure.

- Let $\mathbb{R} = A \sqcup B$, where $A = (-\infty, \alpha_2] \cup [\alpha_1, \infty)$ and $B = (\alpha_2, \alpha_1)$. See that since both the positive measures have their usual properties, we have that for $E \subseteq A$ measurable, we have $\nu(E) = \nu^+(E) - \nu^-(E) = \nu^+(E) - 0 \geq 0$, and likewise for $E \subseteq B$ measurable, we have $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - \nu^-(E) \leq 0$. Thus the above construction is a Hahn decomposition.

- See that ν^+ lives on A, while ν^- lives on B. That is, $\nu^+(E) = \nu(E \cap A)$, and $\nu^-(E) = -\nu(E \cap B)$. This is easy to see, as $E = (E \cap A) \sqcup (C \cap B)$. Then $\nu(E) = \int_{(E\cap A)\sqcup(E\cap B)} p^+ d\mu - \int_{(E\cap A)\sqcup(E\cap B)} p^- d\mu = \int_{(E\cap A)} p^+ d\mu + \int_{(E\cap B)} p^+ d\mu - \int_{(E\cap A)} p^- d\mu - \int_{(E\cap B)} p^- d\mu$. Since p^+ is 0 on B, and p^- is 0 on A, we have that $\nu^+ \perp \nu^-$. Thus we have the Jordan decomposition.

5

Let us assume that $\mu(E) = 0$. As E is $\mu-$null, then $\mu(E \cap E_n) = 0$ for all n, by monotonicity of the measure. Then we have $\nu(E) = \sum_{n=1}^{N} c_n \mu(E \cap E_n) = 0$. Thus $\nu << \mu$. See that the function $f := \sum_{n=1}^{N} c_n \chi_{E_n}$ is a good candidate for the Radon-Nikodym derivative.

$$\int_E f d\mu = \int_E \sum_{n=1}^{N} c_n \chi_{E_n} d\mu = \sum_{n=1}^{N} c_n \int_X \chi_E \chi_{E_n} d\mu = \sum_{n=1}^{N} c_n \int_X \chi_{E \cap E_n} d\mu = \sum_{n=1}^{N} c_n \mu(E \cap E_n),$$

which is the desired result. Thus $\frac{d\nu}{d\mu} = f$.

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1. Since $\nu << \mu$, there exists $f \in L^1(\mu)$ such that $\nu(E) = \int_E f d\mu$. We know that $f > 0 \mu-$almost everywhere, then assume that $\nu(E) = 0$. Thus $\int_E f d\mu = 0$. Assume that $\mu(E) > 0$. Then f is greater than zero on all of E, thus $\int_E f d\mu > 0$. However, since $\nu(E) = 0$ this forces $\mu(E)$ to be 0. Thus $\mu << \nu$.

2.

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Let $\theta := \mu + \nu$. Then $f = \frac{d\nu}{d\theta}$. See that since $\theta(E) = \int_E 1 d\theta = \mu(E) + \int_E f d\theta$. Therefore we have $\mu(E) = \int_E (1 - f) d\theta$. Thus we have $\frac{d\mu}{d\theta} = 1 - f$.

8

In $(\mathbb{N}, \mathbb{P}(\mathbb{N}))$, μ is the counting measure. Note that the empty set is the only $\mu-$null set, since every non-empty set has cardinality more than zero. Then somewhat trivially we have $\mu(E) = 0 \implies E = \phi \implies \nu(E) = 0$. So $\nu << \mu$. We have ν is $\sigma-$finite, thus $\mathbb{N} = \sum_{n=1}^{\infty} \{n\}$, where $\nu(\{n\}) < \infty$. Thus define $f : \mathbb{N} \to \mathbb{R}$, where $f(n) = \nu(\{n\})$. Then we have $\nu(E) = \int_E f d\mu = \sum_{n \in E} f(n)$, is the required function. Thus $f = \frac{d\nu}{d\mu}$.

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If we assume that $\frac{d\mu}{d\lambda} \cdot \frac{d\nu}{d\lambda} = 0$ λ almost everywhere, we know from the previous assignment that $\mu(E)$ and $\nu(E)$ are mutually singular. To see the converse, let us assume that $\mu \perp \nu$. Then we have $X = A \sqcup B$, where μ lives on A while ν lives on B. Then let $f := \frac{d\mu}{d\lambda}$, and $g := \frac{d\nu}{d\lambda}$. See that

$$\int_E fg d\lambda = \int_{(E\cap A)} fg d\lambda + \int_{(E\cap B)} fg d\lambda.$$

We know that $\int_{(E\cap A)} fg d\lambda = \int_{(E\cap A)} f d\nu$, and $\int_{(E\cap B)} fg d\lambda = \int_{(E\cap B)} g d\mu$. As $E \cap A$ is a $\nu-$null set and $E \cap B$ is a $\mu-$null set, we have $\int_E fg d\lambda = 0 \implies fg = 0$ $\lambda-$almost everywhere.

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$\nu = \nu^+ - \nu^-$ is a signed measure, where we have $x = P \sqcup N$, that is, $\nu^+(E) = \nu(E \cap P)$, and $\nu^-(E) = \nu(E \cap N)$. Taking $|\nu| = \nu^+ + \nu^-$, see that $\nu^+ << |\nu|$ and $\nu^- << |\nu|$, and thus there must exist $\frac{d\nu^+}{d|\nu|}$ and

$$\frac{d\nu^-}{d|\nu|},$$

the Radon-Nikodym derivatives. Then see that $\frac{d\nu^+}{d|\nu|} = \chi_P$. To see this, see that for E measurable in P,

$$\begin{aligned}\int_E \chi_P d|\nu| &= \int_E \chi_P d\nu^+ + \int_E \chi_P d\nu^- \\ &= \nu^+(E \cap P) + 0 = \nu^+(E),\end{aligned}$$

and similarly for ν^-, $\frac{d\nu^-}{d|\nu|} = \chi_N$.

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See that

$$\begin{aligned}\left|\int_X f d\nu\right| &\leq \left|\int_X f d\nu^+ - \int_X f d\nu^-\right| \\ &= \left|\int_X f d\nu^+\right| + \left|\int_X f d\nu^-\right| \leq \int_X |f| d\nu^+ + \int_X |f| d\nu^- \\ &= \int_X |f| d|\nu|,\end{aligned}$$

as desired. Now see that $f \leq 1$ implies that $f\chi_E \leq \chi_E$, which means that integrating over X we have $\int_X f\chi_E d\nu \leq \int_X \chi_E d\nu$. Then this gives us

$$\mid \int_X f\chi_E d\nu \mid = \mid \int_E d d\nu \mid \leq \int_X |f\chi_E| d|\nu| \leq \int_X \chi_E d|\nu| = |\nu|(E).$$

Therefore we have $|\nu|(E) \geq \mid \int_E f d\nu \mid$, for all f such that $f \leq 1$. Then we must have $|\nu|(E) \geq \sup\left\{\mid \int_E f d\nu \mid : f \leq 1\right\}$. Now let $f = \chi_{E\cap A} - \chi_{E\cap B}$, where $X = A \sqcup B$, as per the Jordan decomposition. Then we have

$$\int_E f d\nu = \int_E f d\nu^+ - \int_E f d\nu^- = \int_E \chi_{E\cap A} d\nu^+ - \int_E \chi_{E\cap B} d\nu^+ + \int_E \chi_{E\cap A} d\nu^- + \int_E \chi_{E\cap B} d\nu^- = \nu^+(E\cap A) - \nu^+(E\cap B) - \nu^-(E$$

Note that $\nu^+(E\cap B) = \nu^-(E\cap A) = 0$. Thus we have $\int_E f d\nu = \nu^+(E\cap A) + \nu^-(E\cap B) = \nu^+(E) + \nu^-(E) = |\nu|(E)$. This means that the right hand side actually attains its supremum within the set of all integrable functions where $f \leq 1$. Thus we have $|\nu|(E) \leq \sup\left\{\mid \int_E f d\nu \mid : f \leq 1\right\}$. This gives us the desired equality.

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We know that the counting measure μ only has one $\nu-$null set, that is ϕ. Let us say that $\mu << \nu$. Then for $\nu(E) = 0$ for some measurable $E \in \mathcal{A}$, we have $\mu(E) = 0$. To take the contrapositive, we see that if $\mu(E) \neq 0$, then $\nu(E) \neq 0$. Since we know that μ has no non-empty null sets, non $\mu-$null sets and non-empty subsets are synonymous. Thus, if $E \neq \phi$, $\nu(E) \neq 0$.

For the Dirac measure at $x_0 \in X$, we know that $\delta_{x_0} << \nu$ means that $\nu(E) = 0$ for some $E \in \mathcal{A}$ implies that $\delta_{x_0}(E) = 0$. The Dirac measure is zero if the measurable subset E does not have x_0. Thus we can take the contrapositive of the absolute continuity to say that if $x_0 \in E$, then we must necessarily have $\nu(E) \neq 0$.