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1

We are given $f(z) = z^2 - z\bar{z}^2 - 2|z|^2$. Since $z = x + iy$, we can expand it in f to get

$$f(x, y) = -2(x^2 + y^2) + i4xy.$$

Thus $u = -2(x^2+y^2), v = 4xy$. Then we have $u_x = -4x, u_y = -4y, v_x = 4y, v_y = 4x$. If f is holomorphic, then we must have $u_x = v_y \implies -4x = 4x \implies x = 0$. Also we must have $u_y = -v_x \implies -4y = -4y \implies y \in \mathbb{R}$. Thus f satisfies the Cauchy Riemann equations on $\{0\} \times \mathbb{R}$, which is not a domain since it is not open. Thus it is complex differentiable at each point of the type $(0, y)$ where $y \in \mathbb{R}$, but not holomorphic at any point in $\mathbb{C}$ since the points at which it satisfies the Cauchy Riemann equations is not open in $\mathbb{C}$.

2

Let us assume that there exists a holomorphic function on a domain D such that its image lies entirely on a vertical line, say $x = \frac{1}{2}$. Thus for $f = u + iv$, we must have that $u = \frac{1}{2}$, a constant. Then $u_x = u_y = 0$, and by the Cauchy-Riemann equations, we have $v_y = u_x = 0 = u_y = -v_x$. Thus we have v constant as well, which means that f must be a constant.

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Let $C := \{e^{it} : t \in \left[0, \frac{pi}{2}\right]\}$, which parametrizes the curve. Then

$$\begin{aligned}
\int_C \overline{\mathrm{Log}(z)} dz &= \int_0^{\frac{\pi}{2}} \overline{\mathrm{Log}(e^{it})} |ie^{it}| dt \\
&= \int_0^{\frac{\pi}{2}} \overline{\log(1) + it} dt \\
&= \int_0^{\frac{\pi}{2}} -it dt \\
&= -i\frac{\pi^2}{4}.
\end{aligned}$$

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