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1

2

See that for all $i \in \{1,2,3,4,5\}$ we have t'_i is a product of disjoint transpositions. We know that the order of t'_i is the least common multiple of all the cycles, which is clearly 2 in this case. Then we have $t'^2_i = 1$.

Now see that $t'_1t'_2 = (1,2)(3,4)(5,6) \circ (1,4)(2,5)(3,6) = (1,3,5)(2,6,4)$, $t'_2t'_3 = (1,4)(2,5)(3,6) \circ (1,3)(2,4)(5,6) = (1,6,2)(3,4,5)$, $t'_3t'_4 = (1,3)(2,4)(5,6) \circ (1,2)(3,6)(4,5) = (1,4,6)(2,3,5)$, and $t'_4t'_5 = (1,2)(3,6)(4,5) \circ (1,4)(2,3)(5,6) = (1,5,3)(2,6,4)$. These are all products of disjoint $3-$cycles, thus they have order 3.

It is also interesting to see that $t'_1t'_3 = (1,2)(3,4)(5,6)\circ(1,3)(2,4)(5,6) = (1,4)(2,3), t'_1t'_4 = (1,2)(3,4)(5,6)\circ(1,2)(3,6)(4,5) = (3,5)(4,6), t'_1t'_5 = (1,2)(3,4)(5,6)\circ(1,4)(2,3)(5,6) = (1,3)(2,4), t'_2t'_4 = (1,4)(2,5)(3,6)\circ(1,2)(3,6)(4,5) = (1,5)(2,4), t'_2t'_5 = (1,4)(2,5)(3,6)\circ(1,4)(2,3)(5,6) = (2,6)(3,5)$, and $t'_3t'_5 = (1,3)(2,4)(5,6)\circ(1,4)(2,3)(5,6) = (12,)(3,4)$, which is a product of disjoint $2-$cycles.

We know from a previous assignment that S_n can be generated by elements of the form $(n, n+1)$. Then see that $(1,2) =$

3

Let $\alpha \in \mathbb{Q}$. It satisfies a monic polynomial $f(x) \in \mathbb{Z}[x]$. Then we have

$$\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_0 = 0.$$

Let $\alpha = \frac{p}{q}$, with $(p,q) = 1$. Then we have

$$p^n + c_{n-1}p^{n-1}q + \cdots + c_0q^n = 0.$$

Reducing this equation modulo q, we have

$$p^n \equiv 0 \mod q.$$

Since $(p,q) = 1$, we have $q = 1$. Thus $\alpha \in \mathbb{Z}$.

4

We have $f(x) = x^5 - ax - 1$. For $a = 0$, $f(x) = (x-1)(x^4 + x^3 + x^2 + x + 1)$, which is a non-trivial reduction. For $a = 2$, we have $f(-1) = 0$, so this is also reducible. For $a = -1$, we have $f(x) = x^5 - x - 1$ which can be factored as given by the problem. Now assume $a \neq -1, 0, 2$.

5

See that $x^2 - 4x + 1$ is a polynomial that $2 + \sqrt{3}$ satisfies. The minimal polynomial if it is any smaller would have degree 1. But since $2 + \sqrt{3}$ is not rational, the degree of its minimal polynomial must be at least 2. Thus we have that $2 + \sqrt{3}$ has exactly degree 2 over $\mathbb{Q}$.

Consider the number field $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, which is a degree 3 extension. It is a $\mathbb{Q}$ vector space, so it clearly contains the element $1 + \sqrt[3]{2} + \sqrt[3]{4}$. Therefore it must have degree at most 3 in $\mathbb{Q}$. Since it is not rational, its degree over $\mathbb{Q}$ must be at least 2. Let us see if any quadratic polynomial can satisfy it. Let

$x^2 + ax + b \in \mathbb{Q}[x]$ be some rational polynomial. Assume $\alpha = \sqrt[3]{2}$. Assume that $(1 + \alpha + \alpha^2)$ satisfies this quadratic polynomial, so we must have

$$\begin{aligned}(1 + \alpha + \alpha^2)^2 + a(1 + \alpha + \alpha^2) + b &= 3\alpha^2 + 4\alpha + 5 + a + a\alpha + a\alpha^2 + b \\ &= (3 + a)\alpha^2 + (4 + a)\alpha + (5 + a + b).\end{aligned}$$

If this is to be 0, then we must have $a = -3$ and $a = -4$, which is absurd.

Therefore $1 + \sqrt[3]{2} + \sqrt[3]{4}$ must be of degree 3.

6

If we can find a $a + bi \in \mathbb{Q}(i)$, such that $x^3 - q$ vanishes for $q \in \{2, 3\} \in \mathbb{Q}$, then we can reduce the polynomial. If this is possible, then we must have

$$(a + bi)^3 = (a^3 - 3ab^2) + i(3a^2b - b^3) \in \mathbb{Q},$$

which forces $3a^2b = b^3$. If $b = 0$, then it is equivalent to asking if a rational root for q exists, which is not true. Thus we must have $b \neq 0$. Then we have $3a^2 = b^2$, which has no rational solution since $\sqrt{3}$ is not rational. Therefore we must have that $x^3 - 2$ and $x^3 - 3$ are both irreducible, since they have no solutions on $\mathbb{Q}(i)$.

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