Measure Theory HW6

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We wish to show that $h \in L^1(\mu \times \nu)$. See that since $f \in L^1(\mu)$ and $g \in L^1(\nu)$. Then see that $|h(x,y)| = |f(x)| \cdot |g(y)|$. Since these measure space are $\sigma-$finite and the functions $|f|$ and $|g|$ are positive measurable, we can apply the Fubini- Tolleni theorem. Integrating over $X \times Y$, we get

$$\begin{aligned}
\int_{X \times Y} |h(x,y)| d(\mu \times \nu) &= \int_{X \times Y} |f(x)| \cdot |g(y)| d(\mu \times \nu) \\
&= \int_Y \left(\int_X |h^y(x)| d\mu(x) \right) d\nu(y) = \int_Y |g(y)| \cdot \left(\int_X |f(x)| d\mu(x) \right) d\nu(y) \\
&= \int_Y |g(y)| d\nu(y) \cdot \left(\int_X |f(x)| d\mu(x) \right) < \infty.
\end{aligned}$$

Thus $h \in L^1(\mu \times \nu)$. To calculate the integral, see that

$$\begin{aligned}
\int_{X \times Y} h(x,y) d(\mu \times \nu) &= \int_{X \times Y} f(x) \cdot g(y) d(\mu \times \nu) \\
&= \int_Y \left(\int_X h^y(x) d\mu(x) \right) d\nu(y) = \int_Y g(y) \cdot \left(\int_X f(x) d\mu(x) \right) d\nu(y) \\
&= \int_Y g(y) d\nu(y) \cdot \left(\int_X f(x) d\mu(x) \right),
\end{aligned}$$

which is the desired result.

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The Fubini-Tolleni theorem requires the two spaces X and Y to both be $\sigma-$finite with respect to both μ and ν respectively. In this case see that the counting measure over $\mathbb{N}$ is indeed $\sigma-$finite, as $\mathbb{N} = \cup_{n=1}^{\infty}\{n\}$, where $\mu(\{n\}) = 1 < \infty$. Thus for $X = Y = \mathbb{N}$, and $\Sigma_1 = \Sigma_2 = P(\mathbb{N})$, and $\mu = \nu = m$, where $m(A)$ denotes the cardinality of the set A if it is finite and $+\infty$ otherwise. Note that $\Sigma_1 \otimes \Sigma_2 = P(\mathbb{N}^2)$, since $\Sigma_1 \otimes \Sigma_2 \subseteq P(\mathbb{N}^2)$, and for any $A \times B \in P(\mathbb{N}^2)$, we have $A \times B = \cup_{x \times y \in A \times B}\{x\} \times \{y\} \in \Sigma_1 \otimes \Sigma_2$, which gives us the other inequality.

We wish to see what sorts of functions over $\mathbb{N}$ are measurable. All functions are clearly $\Sigma_1 \otimes \Sigma_2$ measurable, as the entire power set constitutes the $\sigma-$algebra. See that functions can be indexed by two natural numbers, hence they can be described as $a_{m,n}$ Note that $(a_{m_0})_n = a_{m_0,n}$ fixes the first variable at some $m_0 \in \mathbb{N}$, and $a_m^{n_0} = a_{m,n_0}$ fixes the second variable at some $n_0 \in \mathbb{N}$. We also need to understand what integration looks like. Integration in this case is just summation, as we can see that integrating over a point gives us the value of the function at that point. Thus $\int_{\mathbb{N}}\{a_n\}dm(n) = \sum_{n=1}^{\infty} a_n$ and $\int_{\mathbb{N}^2}\{a_{m,n}\}dm(m,n) = \sum_{(m,n)\in\mathbb{N}^2} a_{m,n}$. We take a positive measurable function $\{a_{m,n}\}$, and the Fubini-Tolleni theorem tells us that

1. $$m \mapsto \sum_{n=1}^{\infty} (a_m)_n$$

   is $P(\mathbb{N})$ is measurable,

and

$$n \mapsto \sum_{m=1}^{\infty} a_m^n$$

is $P(\mathbb{N})$ is measurable;

2.

$$\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} dm(m,n) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_m^n\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}(a_m)_n\right).$$

The above statement means in the case of $\mathbb{N}^2$ is that we can switch the limits of double summations without issue.

Fubini's theorem can also be stated since our product measure consists of two $\sigma-$finite measures. We need to consider functions $\{a_{m,n}\} \in L^1(m^2)$, where we have $\sum_{(m,n)\in\mathbb{N}^2} |a_{m,n}| < \infty$. Then Fubini's theorem says that:

1. $(a_m)_n \in L^1(m)$, $m-$almost everywhere, and $a_n^m \in L^1(m)$, $m-$almost everywhere.

2.

$$m \mapsto \sum_{n=1}^{\infty}(a_m)_n$$

and

$$n \mapsto \sum_{m=1}^{\infty} a_m^n$$

are $L^1(m)$.

In the case of the counting measure it means that if the double summation is absolutely convergent, then so are the summations of all sections.

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We need to find a function $g(x)$ that dominates $f_n = f(nx)$ for all $n \in \mathbb{N}$. Then see that

$$\mid \frac{\sin(n^2x^2)}{nx} \mid \leq \frac{1}{nx} \leq \frac{1}{x}.$$

Also see that $\frac{cnx}{1+nx} \leq c$. See that this holds for all $x \in [0,\infty)$

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1. We know that $\mu(E) = 0$. Also $f = (\Re(f)^+ - \Re(f)^-) + i(\Im(f)^+ - \Im(f)^-)$, which are all positive measurable functions. Then $\nu(E) = \int_E \Re(f)^+ d\mu \leq +\infty \cdot \mu(E) \leq 0$. Since measure cannot be negative, we have $\nu(E) = 0$. Use this same strategy for the other three functions $\Re(f)^-, \Im(f)^+$, and $\Im(f)^-$.

2. Let $\{E_n\}_{n\in\mathbb{N}}$ be a countable disjoint collection of measurable subsets. Then define g as $g = \sum_{n=1}^{\infty} f\chi_{E_n}$. Then $f-g$ is zero on $\coprod_{n=1}^{\infty} E_n$. Then we must have

$$\int_{\coprod_{n=1}^{\infty} E_n} (f-g)d\mu = 0 \implies$$

$$\begin{aligned}
\nu(\coprod_{n=1}^{\infty} E_n) = \int_{\coprod_{n=1}^{\infty} E_n} f d\mu &= \int_{\coprod_{n=1}^{\infty} E_n} \sum_{n=1}^{\infty} f\chi_{E_n} \\
&= \sum_{n=1}^{\infty} \int_{\coprod_{n=1}^{\infty} E_n} f\chi_{E_n} = \sum_{n=1}^{\infty} \int_{E_n} f d\mu \\
&= \sum_{n=1}^{\infty} \nu(E_n),
\end{aligned}$$

which means that $\nu(\coprod_{n=1}^{\infty} E_n) = \int_{\coprod_{n=1}^{\infty} E_n} f d\mu$.

3. Let $\varepsilon > 0$. We wish to find a $\delta > 0$ such that $\mu(E) < \delta \implies |\nu(E)| < \varepsilon$.

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From the solution of problem 1, the answer to the problem follows directly.

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Note that $|f(m,n)| = 1$ for $m = n$ and $m = n+1$. Then

$$\int_{\mathbb{N}^2} |f(m,n)| dm(m,n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |f(m,n)| = \sum_{p=1} |f(p,p)| + \sum_{q=1}^{\infty} |f(q+1,q)| = \infty.$$

Now we fix each variable and successively integrate. $\sum_{m=1}^{\infty} f(m,n_0) = f(n_0,n_0) + f(n_0+1,n_0) = 1 + (-1) = 0$. Thus $\sum_{n=1}^{\infty} 0 = 0$. If we fix m, and $m > 1$, then we have $\sum_{n=1}^{\infty} f(m,n) = f(m,m) + f(m,m-1) = 1 + (-1) = 0$. If $m = 1$, there is no n such that $m = n+1$, so $\sum_{n=1} f(1,n) = f(1,1) = 1$. Thus $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = 1 \neq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = 0$.

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Since $f(x)\cdot g(x) = 0$, we cannot have $f(x)$ and $g(x)$ nonzero for the same value of x almost everywhere. We divide our domain X into four disjoint parts $X = X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$, where X_1 is the set of all $x \in X$ where f is nonzero but g is zero; X_2 is the set of all $x \in X$ such that g is nonzero but f is zero; X_3 is the set of all $x \in X$ such that f and g are both zero, and X_4 is the set of all $x \in X$ such that both f and g are nonzero. We know that $\mu'(X_4) = 0$. Let $A = (X_1 \sqcup X_3 \sqcup X_4)$ and $B = X_2$. Clearly the two are disjoint and their union is X, by construction. We argue that this should show that $\mu' \perp \nu$. We try to evaluate $\mu'(E_B)$, for E_B a measurable subset in B. See that

$$\mu'(E_B) = \mu'(E_B|\cap(X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4)) = \int_{X_1\cap E_B} f d\mu + \int_{X_2\cap E_B} f d\mu + \int_{X_3\cap E_B} f d\mu + \int_{X_4\cap E_B} f d\mu.$$

We know that $E_B \cap X_1 = E_B \cap X_3 = E_B \cap X_4 = \phi$. Thus we have $\mu'(E_B) = \int_{E_B\cap X_2} f d\mu$. However, we know that f is zero on X_2, hence $\mu'(E_B) = 0$.

Similarly, see that $\nu(E_A)$ should also be zero for E_A a measurable set on A. So see that

$$\nu(E_A) = \nu(E_A|\cap(X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4)) = \int_{X_1\cap E_A} g d\mu + \int_{X_2\cap E_A} g d\mu + \int_{X_3\cap E_A} g d\mu + \int_{X_4\cap E_A} g d\mu.$$

We have $E_A \cap X_2 = \phi$, and we have that g is zero on X_1 and X_3. Thus $\int_{E_A \cap X_1} g d\mu = \int_{E_A \cap X_3} g d\mu = 0$. Note that on X_4 the function may be non-zero, but $X_4 \cap E$ is a null set as it is a subset of X_4, a $\mu-$null set. Thus $\int_{E_A \cap X_4} g d\mu = 0$. (This exact question has been solved in 6). Thus we have $\nu(E_A) = 0$. Therefore $\mu' \perp \nu$.

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