Gandhar Kulkarni (mmat2304)

1

1. This statement is true. Take any vector space V, which necessarily has a Hamel basis, trivially the entire space is a basis. Then for a choice of basis (b_n) we have for all $x \in V$, $x = \sum k_i b_i$, where the sum is finite. Then define $||x|| := \max\{|k_i|\}$. See that this fulfills the definition of a norm.

2. This statement is false. For sake of contradiction, let $(X, ||\cdot||)$ be a normed linear space such that the induced metric is the discrete metric. Then for $x, y \in X, x \neq y$ we must have $||x - y|| = 1$. Note that $2x \neq 2y$, so we must have $||2x - 2y|| = 2||x - y|| = 2$, but by the discrete metric the answer should still be 1! Thus there can be no such norm.

2

We wish to show that the function $||\cdot||$ on X satisfies the triangle inequality iff the closed unit ball is convex. Assume that the function $||\cdot||$ is indeed a norm. Then let $x, y \in D$, the closed unit ball. Then $||x||, ||y|| \leq 1$. Now we have for $\alpha \in [0, 1]$ $z = \alpha x + (1 - \alpha)y$. See that

$$||z|| = ||\alpha x + (1 - \alpha)y|| \leq \alpha||x|| + (1 - \alpha)||y|| \leq \alpha \cdot 1 + (1 - \alpha) \cdot 1 \leq 1,$$

thus we have $z \in D$.

Take two elements $x, y \in X$ both non-zero, since if either were zero the inequality would be trivial. Then

$$||x + y|| = (||x|| + ||y||) \cdot \left|\left|\alpha \frac{x}{||x||} + (1 - \alpha)\frac{y}{||y||}\right|\right|,$$

where $\alpha = \frac{||x||}{||x||+||y||}$. Note that $\frac{x}{||x||} = \frac{y}{||y||} = 1$, thus we can use the convexity condition to see that $\frac{||x+y||}{||x||+||y||} \leq 1$, which is the triangle inequality.

3

1. Pick a $f \in C([a, b])$. Then $|f| \leq M = \sup\{|f(x)| : x \in [a, b]\}$. Now see that

$$\int_a^b |f(t)|^p dt \leq (b - a)M^p \geq 0.$$

Thus $||f||_p \geq 0$. For $f = 0$, we have $M = 0$, so $\int_a^b |0|^p dt = 0$. If $\int_a^b |f(t)|^p dt = 0$, then see that $0 \leq (b - a)M^p \geq 0$. Thus we must have

$$(b - a)M^p = 0 \implies M = 0 \implies f = 0.$$

To see the next axiom, for $\alpha \in \mathbb{K}$, we have $\int_a^b |\alpha f(t)|^p dt = \int_a^b |\alpha|^p |f(t)|^p dt$. Then

$$||\alpha f||_p = \left(|\alpha|^p \int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} = |\alpha|||f||_p.$$

Now let $f, g \in C[a,b]$. Then we have to prove Minkowski's inequality to show the triangle inequality.

$$\begin{aligned}
||f+g||_p^p &= \int_a^b |f+g|^p dx \\
&= \int_a^b |f+g| \cdot |f+g|^{p-1} dx \\
&\leq \int_a^b |f||f+g|^{p-1} dx + \int_a^b |g||f+g|^{p-1} dx \\
&\leq \left(\int_a^b |f|^p dx + \int_a^b |g|^p dx \right) \left(\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} dx \right)^{1-\frac{1}{p}} \quad \text{(Hölder's inequality)} \\
&= (||f||_p + ||g||_p) \frac{||f+g||_p^p}{||f+g||_p},
\end{aligned}$$

which yields the required result.

2. We consider (f_n), which is a Cauchy sequence of L^p functions. Take a subsequence of functions (f_{n_i}) where $||f_{n_i} - f_{n_{i+1}}||_p < 2^{-i}$. Now define $g_n(x) = \sum_{i=1}^n |f_{n_i} - f_{n_{i+1}}|$ and $g(x) = \sum_{i=1}^\infty |f_{n_i} - f_{n_{i+1}}|$. Using the triangle triangle inequality, we have $||g_n||_p \leq 1$. By Fatou's lemma, we have

$$\int_a^b \liminf_n g_n \leq \liminf_n \int_a^b g_n.$$

We must have g_n converges almost everywhere on X. We let $f(x) = \liminf_n f_{n_i}(x)$ for almost $x \in X$, where the function 0 wherever the pointwise limit does not exist. Then we claim that this is the limit. For $\varepsilon > 0$, take N such that $||f_m - f_n||_p < \varepsilon$. By Fatou's lemma, we have

$$\int_a^b |f - f_n|^p \leq \liminf_n \int_a^b |f_{n_i} - f_n|^p \leq \varepsilon^p,$$

which means $f \in L^p$ and $(f_n) \to f$.

4

1. $||f_1 - F||_\infty$ is to be found, where F is the subspace of constant functions. Unfolding the term, we get

$$||f_1 - F||_\infty = \inf_{c \in \mathbb{R}} \{ \sup_{t \in [0,1]} |t - c| \},$$

which for $c \in [-1, 0] \cup [1, 2]$ is 1, while in $(0, 1)$ it decreases to $\frac{1}{2}$ then goes back up to 1. Thus, we must have $||f_1 - F||_\infty = \frac{1}{2}$.

2. We want to now see the distance between $f_2 = t^2$ and G, the space of all polynomials with degree at most 1. Then for some polynomial $-ax - b \in G$, we want to see $\inf_{a,b \in \mathbb{R}} \sup_{t \in [0,1]} \{|t^2 + ax + b|\}$. See that we have $\sup_{t \in [0,1]} |t^2 - t + 1/8| = \frac{1}{8}$. This is an upper bound. See that by varying a, we translate it along the $x-$axis. We can vary b, which gives us translation in the $y-$axis. Note that to minimise the maximum value of $|t^2 + at + b|$ we must have the value of the polynomial at $t = 0, t = 1, t = \frac{-a}{2}$ be the same. Then we take the polynomial to be symmetric about $t = \frac{1}{2}$. For this, $a = -1$. Now we have $|t^2 - t + b|$, which attains a critical value at $t = \frac{1}{2}$, that is $\left| b - \frac{1}{4} \right|$. Thus we have $|b| = \left| b - \frac{1}{4} \right|$. A solution is $b = \frac{1}{8}$. See that the this then must indeed be $||t^2 - G||$.

5

Let Y and X/Y be Banach spaces. Then take (x_n) to be a Cauchy sequence in X. That is, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m, n \geq N$ we have

$$||x_m - x_n|| < \varepsilon.$$

Consider the canonical projection $p : X \twoheadrightarrow X/Y$. Its kernel is Y. The sequence (x_n) sent to X/Y is also Cauchy, as $||(x_m - x_n) + Y|| \leq ||x_m - x_n|| < \varepsilon$. This must converge to some coset K in X/Y, since X/Y is Banach. Now pick an element y_n such that $||(x_m - x_n) + y_n|| < \varepsilon$. Then this sequence (y_n) is Cauchy since $||y_m - y_n|| < ||(x_n - y_n) - (x_m - y_m)|| < \varepsilon$. Then we have that $y_n \to y$, where $y \in Y$. Now we propose that there is an element in X that is $K + y$, where K is of the form $x + Y$. Then we say that $x \in x + Y + y$ is the limit of (x_n). Note that Y contains $-y$, so the element in $x + Y$ that corresponds to $-y \in Y$ is our choice of X. See that $x_n + Y \to x + Y$, and $y_n \to y$, by setting $-y$ instead of Y and adding the two convergent sequences in X we get our desired result.

Let X and X/Y be Banach spaces. Then take (y_n), a Cauchy sequence in Y. Thus for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n \geq N$, we have $||y_m - y_n|| < \varepsilon$. As a Cauchy sequence in X, this must converge to some element $y_0 \in X$. We now need to show that $y_0 \in Y$. But since Y is closed and y_0 is a limit point, we must have $y_0 \in Y$.

Let X and Y be Banach spaces. Take (x_n) as a sequence in X such that $(x_n + Y)$ is Cauchy. Then pick $\varepsilon > 0$. Then there is a $N \in \mathbb{N}$ such that for $m, n \geq N$ $||(x_m - x_n) + Y|| < \varepsilon$. See that $||x_m - x_n|| \leq ||(x_m - x_n) + y|| + ||y||$. Since the expression on the left is independent of y, then we can set $||(x_m - x_n) + y||$ to $||(x_m - x_n) + Y||$, and $||y||$ to 0, as $0 \in Y$. Then we have $||x_m - x_n|| < \varepsilon$, which means that (x_n) is Cauchy. Then since X is Banach, we have $x_n \to x$. Then we propose that $x + Y$ is the limit of $(x_n + Y)$. See that $||(x_n + Y) - (x + Y)|| = ||(x_n - x) + Y|| = \inf_{y \in Y}\{||(x_n - x) + y||\}$. Then, for any choice of $y \in Y$, we have

$$||(x_n - x) + Y|| \leq ||(x_n - x) + y||.$$

Then for $y = 0$, we have

$$||(x_n - x) + Y|| \leq ||x_n - x|| < \varepsilon,$$

which implies that X/Y is Banach.

6

Assume that X is a Banach space. Let (x_n) be an absolutely convergent series in X, that is, $\sum_{i=1}^{n} ||x_i||$ converges to $c \in \mathbb{R}$ as $n \to \infty$. Thus $s_n = \sum_{i=1}^{n} x_i$, so for any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for $n \geq N$, we have $\sum_{i=N}^{\infty} ||x_i|| < \varepsilon$. Now we have $||s_n - s_m|| \leq \sum_{i=n+1}^{m} ||x_i|| < \varepsilon$. Since (s_n) is Cauchy, it must converge to some value $s \in X$. Thus this sequence converges.

For the converse, take a Cauchy sequence (x_n). For $k \in \mathbb{N}$ pick $n_k \in \mathbb{N}$ such that for $m, n \geq n_k$, we have $||x_m - x_n|| < 2^{-k}$. In particular, $||x_{n_{k+1}} - x_{n_k}|| < 2^{-k}$. Let $y_1 = x_{n_1}$, and $y_n = x_{n_{k+1}} - x_{n_k}$. From this, it follows that $\sum ||y_n|| \leq ||x_{n_1}|| + 1$, thus this sequence is absolutely convergent. From this, see that this series must converge. Thus $\sum_{i=1}^{\infty} y_n = \lim_{n\to\infty} x_n$ is defined, hence the space X is Banach.

7

Let ℓ^p be the space of all $p-$power summable sequences. Then $||(x_n)|| = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. Now we want to show that the space $K = \{(x_n) \in \ell^p : x_i = 0 \forall i > n, n \in \mathbb{N}\}$ is dense in ℓ^p. Take any $(x_n)_{n\in\mathbb{N}}^{(0)} \in \ell^p$. There is a sequence of elements in ℓ^p $(x_n)^{(m)} \subseteq K$ such that $x_n^{(m)} = x_n$ if $n \leq m$, and 0 otherwise. See that $||(x_n)^{(m)} - (x_n)^{(0)}||_p = \left(\sum_{i=m+1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$. Since $||(x_n)^{(m)}||_p \to ||(x_n)^{(0)}||_p$ as $m \to \infty$, we must have for a choice of $\varepsilon > 0$ there being $N \in \mathbb{N}$ such that for $m \geq N$, $|||(x_n)^{(m)}||_p - ||(x_n)^{(0)}||_p| < \varepsilon$. Multiplying on both sides by $\frac{||(x_n)^{(m)}||_p^p - ||(x_n)^{(0)}||_p^p}{||(x_n)^{(m)}||_p - ||(x_n)^{(0)}||_p}$, and after a change in the value of ε, we get

$$||(x_n)^{(m)}||_p^p - ||(x_n)^{(0)}||_p^p < \varepsilon.$$

Thus we have shown that K is dense in ℓ^p. For a fixed m, and a fixed n, look at $x_n^{(m)}$. This is a real number or a complex number, which can be approximated by a sequence of rationals (or elements of $\mathbb{Q}(i)$) such that $x_n^{(m)}$ is its limit. Now we have a sequence (t_k), where $t_k \to x_n^{(m)}$ as $k \to \infty$.

Now let us see that $K' = \{(x_n) \in K : x_n \in \mathbb{Q} \text{ or} \mathbb{Q}(i)\} \subset K$ is countable, and as we saw above must be dense in ℓ^p. Thus it is separable.

8

We first show that ℓ^∞ is not separable. Take the space K of all sequences in ℓ^∞ such that their entries are either 0 or 1. Then for $x \neq y \in K$, we have $||x_y||_\infty = 1$, since at least one entry is different between the two. Then we have uncountably points. consider a ball of radius $\frac{1}{2}$ centred at $x \in K$, for all such points in K. We know what uncountably many open sets, all of which are disjoint from each other. Let S be a possibly dense set. Then we have that each open ball contains at least one point of S. This then means that there must be uncountably many elements, so ℓ^∞.

If there was a Schauder basis for ℓ^∞, then for all $x \in \ell^\infty$ we would have a sequence $(a_1, a_2, \dots)$ described using the basis. We can approximate each term by a sequence of elements in $\mathbb{Q}$ or $\mathbb{Q}(i)$, which would then mean that we would have all the rational points of ℓ^∞ as a dense subset, which contradicts the fact that the space is not separable. Thus there can be no Schauder basis.

9

We want to see if this function is continuous. Pick a $x_0 \in \mathbb{R}$. Now for all $\varepsilon > 0$ have $f(z_0+h)-f(z_0) = f(h)$.

See that $f(2) = 2f(1)$, $f(0) = 2f(0) \implies f(0) = 0$, and that $f(0) = 0 = f(1) + f(-1) \implies f(-1) = -f(1)$. Thus $f(n) = nf(1)$ for all $n \in \mathbb{Z}$. We know that $f(1) = f\left(n\frac{1}{n}\right) \implies f\left(\frac{1}{n}\right) = \frac{1}{n}f(1)$. Thus we have $f(q) = qf(1)$ for all $q \in \mathbb{Q}$. Since any real number can be approximated by a sequence of rationals, we can easily see that for $q_n \to r$ and $n \to \infty$, we have $f(q_n) = q_n f(1) \to rf(1)$. Thus we can extend this to all real numbers. Now we have $|f(z_0 + h) - f(z_0)| = |hf(1)| < \varepsilon$, for a choice of $h = \frac{\varepsilon}{|f(1)|}$.

10

Let $P([0,1])$ be the space of all real polynomials defined on $[0,1]$ be a real vector space. Let the norm of a polynomial $f(x) = a_0 + a_1 x + \dots + a_n x^n \in P([0,1])$ be given thus: $||f|| = |a_0| + |a_1| + \dots + |a_n|$. Now see that the operator $I : P([0,1]) \to P([0,1])$ such that $x^t \mapsto \frac{x^{t+1}}{t+1}$, we then see that

$$||I|| = \sup_{0 \neq x \in P([0,1])} \frac{||If||}{||f||} = \frac{a_0 x + \dots \frac{a_n}{n+1} a_{n+1}}{a_0 + \dots + a_n x^n} = \frac{|a_0| + \left|\frac{a_1}{2}\right| + \dots + \left|\frac{a_n}{n+1}\right|}{|a_0| + \dots + |a_n|} \leq 1.$$

Also see that this supremum is indeed attained since $I(a_0) = a_0 x$, and in this case $\frac{||I(a_0)||}{||a_0||} = 1$. Thus we have $||I|| = 1$. We wish to find the inverse of this operator, see that the differential operator D such that $x^t \mapsto tx^{t-1}$, is the required inverse. However, see that for $f(x) = x^n$, we have $Df = I^{-1}f = nx^{n-1}$. Now we have

$$||D|| = \sup_{0 \neq x \in P([0,1])} \frac{||Df||}{||f||} \geq \frac{||Dx^n||}{||x^n||} = \frac{n}{1}.$$

Thus we have that our operator D is unbounded, since for any chosen $N \in \mathbb{N}$ we can choose x^{N+1}, such that $\frac{||Dx^{N+1}||}{||x^{N+1}||}$ is larger.

11

Since the linear functional $f : X \to \mathbb{R}$ is unbounded, we have a sequence (x_n) such that $f(x_n) > n||x_n||$. We can say without loss of generality, and by discarding non-zero elements, we have a sequence (x_n) of norm 1. Pick $x \in X$. Then see that the sequence $z_n = x - \frac{f(x)}{f(x_n)} x_n$, which is clearly in K, the kernel of f. Also it can be seen that as $n \to \infty$, we have $||z_n|| \leq$

12

Let X be the subspace of all functions in $C[0,1]$ such that $f(0) = 0$. Let Y be the closed subspace of functions in X such that $\int_0^1 f(t)dt = 0$. Then take any function $f \in X$ such that $||f|| = 1$. Consider the map $F : X \to \mathbb{R}$ where

$$F(f) = \int_0^1 f(t)dt.$$

Then $Y = \ker F$, and we have $||F|| = \sup_{||f||=1} F(f) \leq 1 \cdot ||f|| = 1$. Thus $||F|| = 1$. Now see that for $f \notin Y$ we have $X = Y \oplus \operatorname{span}\{f\}$. Thus for $g \in X$ we have $g = f - ty$, where $t \in \mathbb{R}$ and $y \in Y$. Thus $||F|| = \sup_{y \in Y, t \in \mathbb{R}} \frac{|F(f-ty)|}{||f-ty||}$. Since $F(y) = 0$ we have

$$||F|| = \sup_{y \in Y, t \in \mathbb{R}} \frac{|F(f)|}{||f - ty||} = \sup_{g \in Y} \frac{|F(f)|}{||f - g||} = \frac{|F(x)|}{d(f, Y)}.$$

Thus we have for $||F|| = 1$,

$$d(f, Y) = |F(f)|.$$

Note that $x = 1$ and $x = -1$ are the functions for which $|F(f)| = 1$, but these functions are not in X. Thus we must have $d(f, Y) < 1$, which means Riesz's lemma cannot hold for $r = 1$ here.