(1) Let (X, Σ_1, μ) and (Y, Σ_2, ν) be two σ-finite measure spaces and let $f \in L^1(\mu)$ and $g \in L^1(\nu)$. Define

$$h(x,y) = f(x)g(y) \qquad (x \in X, y \in Y).$$

Prove that $h \in L^1(\mu \times \nu)$ and

$$\int_{X \times Y} h \, d(\mu \times \nu) = \Big(\int_X f d\mu\Big)\Big(\int_Y g d\nu\Big).$$

(2) Let $X = Y = \mathbb{N}$, $\Sigma_1 = \Sigma_2 = P(\mathbb{N})$, and let $\mu = \nu =$ the counting measure. Restate Fubini's and Tonelli's Theorems in this setting.

(3) Let $c \in \mathbb{R}$. Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = \frac{\sin(x^2)}{x} + \frac{cx}{1+x}.$$

Suppose $a > 0$. Prove that

$$\lim_{n \to \infty} \int_0^a f(nx) dx = ac.$$

(4) Let $X = Y = [0,1]$, $\Sigma_1 = \Sigma_2 = \mathcal{B}_{[0,1]}$, $\mu =$ the Lebesgue measure and $\nu =$ the counting measure. Consider the diagonal

$$D = \{(x,x) : x \in [0,1]\},$$

in $X \times Y$. Prove that $\int_{X \times Y} \chi_D d(\mu \times \nu)$, $\int\int \chi_D d\mu d\nu$ and $\int\int \chi_D d\nu d\mu$ are all unequal.

(5) Give an example of a nonempty set X and a monotone class $\mathcal{C}$ of subsets of X which contains X and $\emptyset$ and which is not a σ-algebra.

(6) Let (X, Σ, μ) be a measure space, and let $f : X \to \mathbb{C}$ be μ-integrable function. Define

$$\nu(E) = \int_E f d\mu \qquad (E \in \Sigma).$$

Prove that: (i) For $E \in \Sigma$, if $\mu(E) = 0$ then $\nu(E) = 0$. (ii)

$$\nu\Big(\coprod_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \nu(E_n).$$

(iii) For $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) < \delta \Rightarrow |\nu(E)| < \epsilon.$$

(7) Let $\sum_{m,n} a_{m,n}$ be a double series whose terms are nonnegative. Use the Fubini-Tonelli theorem to prove that

$$\sum_m \sum_n a_{m,n} = \sum_n \sum_n a_{m,n}.$$

(8) Let $X = Y = \mathbb{N}$, $\Sigma_1 = \Sigma_2 = P(\mathbb{N})$, and let $\mu = \nu =$ counting measure. Define

$$f(m,n) = \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } m = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\int_{X\times Y} |f| d(\mu\times\nu) = \infty,$$

and $\int\int f d\mu d\nu$ and $\int\int f d\nu d\mu$ exist and are unequal.

(9) Let $(X, \mathcal{A}, \nu)$ be a signed measure, and let $\{E_n\}_n^\infty \subset \mathcal{A}$. (a) If

$$E_n \subseteq E_{n+1} \qquad (n \geq 1),$$

then prove that

$$\nu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \nu(E_n).$$

(b) If $\nu(E_1)$ is finite and

$$E_{n+1} \subseteq E_n \qquad (n \geq 1),$$

then prove that

$$\nu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \nu(E_n).$$

(10) Let (X, Σ, μ) be a measure space and let $f, g : X \to [0, \infty)$ be Σ-measurable functions such that

$$f(x)g(x) = 0,$$

μ-a.e. on X. Suppose

$$\mu(E) = \int_E f d\mu \qquad (E \in \Sigma).$$

Prove that $\mu \perp \nu$, where

$$\nu(E) = \int_E g d\mu \qquad (E \in \Sigma).$$

(11) Let $(X, \mathcal{A}, \nu)$ be a signed measure, and let $A \in \mathcal{A}$. (a) If A is a positive set and $B \subseteq A$, $B \in \mathcal{A}$, then prove that B is also a positive set. (b) If $\{P_n\}_{n=1}^\infty \subseteq \mathcal{A}$ is a sequence of positive sets, then prove that $\bigcup_{n=1}^\infty P_n$ is also a positive set.