Homework 4 Measure Theory

Gandhar Kulkarni (mmat2304)

1

Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Suppose

$$\mu(\{x \in X : |f(x)| \geq \varepsilon\}) = 0,$$

for all $\varepsilon > 0$. Prove that $f = 0$ a.e.

Solution: For all $n \in \mathbb{N}$, we can say that $\mu(\{x \in X : |f(x)| \geq 1/n\}) = 0$. Define $\{x \in X : |f(x)| \geq 1/n\}$ as N_n. See that $N_n \subseteq N_{n+1}$. Consider $\mu(\cap_{n=1}^{\infty} N_n) = \mu(\lim_{n\to\infty} N_n) = \lim_{n\to\infty} \mu(N_n) = 0$. See that $\lim_{n\to\infty} N_n = \{x \in X : |f(x)| > 0\}$. This is precisely the definition of $f = 0$ a.e., proving the statement.

2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that the set

$$\{x \in \mathbb{R} : a \leq f(x) \leq b\},$$

is measurable for any $a < b$. Prove that f is measurable.

Solution: Consider the collection $\{[a - 1/n, b + n]\}_{n\in\mathbb{N}}$. Thus union of this collection is (a, ∞). Let $C_n := f^{-1}([a - 1/n, b + n])$. See that C_n is measurable for all n. Since measurable sets are closed under countable union, we have $\cup_{n=1}^{\infty} C_n = f^{-1}([a - 1/n, b + n]) = f^{-1}((a, \infty))$ is also measurable. Our choice of a was arbitrary, which implies that f is a measurable function.

3

Suppose that $f : [a, b] \to \mathbb{R}$ is a function such that

$$\{x \in [a, b] : f(x) = c\},$$

is measurable for each $c \in \mathbb{R}$. Is f necessarily measurable?

Solution: We know that $f^{-1}(\{c\})$ is measurable for all $c \in \mathbb{R}$. See that the function $f : [0, 1] \to \mathbb{R}$ where $f(x) = x$ if $x \in V$, where V is the Vitali set, and $x + 1$ otherwise. See that this function is actually one-one, so the pre-image has only one point, which is measurable. However, $f^{-1}([0, 1] \cap f([0, 1]))$ is V, a non-measurable set. Thus f needn't be measurable as a function even though the fibre of each point is measurable as a set.

4

Show that Egorov's theorem is no longer true if the condition that the sequence of functions be measurable is dropped.

Solution: Take the measure space $(\mathbb{Z}, \mathcal{M}, \mu)$ where $\mathcal{M} = \{\phi, \mathbb{Z}, E, \mathbb{Z}\backslash E\}$, where E is the set of even integers. The measure μ is zero for ϕ, and $\mathbb{Z}\backslash E$, and one otherwise. This is a finite measure space. Let $\{f_n\}$ be such that $f_n(k) = \frac{k}{n}$. This sequence has pointwise convergence for all $k \in \mathbb{Z}$ to the function $f = 0$. Each f_n is an injective function, thus $f^{-1}(\{\frac{k}{n}\}) = \{k\}$. $\{\frac{k}{n}\}$ is measurable in $B_{\mathbb{R}}$, but $\{k\}$ is not measurable in $\mathcal{M}$. So none of the functions f_n are measurable.

To show the Egorov's theorem cannot hold in this case, we need to find a subset in $\mathbb{Z}$ such that it does not have arbitrarily low measure. Thus there exists an $\varepsilon > 0$ such that for all measurable subsets

S of $\mathbb{Z}$ $\{f_n\}$ does not converge uniformly, or a subset exists where the function converges uniformly but $\mu(S^c) \geq \varepsilon$. For $\varepsilon = \frac{1}{2}$, see that if $S = \phi$, then $\mu(S^c) = 1$, so this is not a valid candidate. If $S = E$, then $\sup_{k\in E}(\mid f_n(x) - f(x) \mid) = \infty$. Therefore the function does not converge uniformly. For $S = \mathbb{Z}\backslash E$, $\mu(E) = 1$, so we cannot choose this set. Finally, for $S = \mathbb{S}$, then $\sup_{k\in\mathbb{Z}}(\mid f_n(x) - f(x) \mid) = \infty$. Therefore the function does not converge uniformly.

See that this example gives a reason as to why Egorov's theorem will fail provided the functions are not measurable.

5

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose $f = g$ a.e. Prove that $f = g$ everywhere.

Solution: Let $h = f - g$, which is continuous due to the continuity of f and g. Let Z be the set of points where $h(x) = 0$. We know that $\mu(Z^c) = 0$. Then $Z^c = \{x \in \mathbb{R} : h(x) > 0 \text{ or} h(x) < 0\} = h^{-1}((\infty, 0)) \cup h^{-1}((0, \infty))$. By the continuity of h, $h^{-1}((\infty, 0))$ and $h^{-1}((0, \infty))$ are both open sets. If Z^c is non-empty, then there exists a $a \in Z^c$. As Z^c is open, there is an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq Z^c$. However, this implies that $\mu(Z^c) > 2\varepsilon > 0$, which is a contradiction. Thus Z^c must be empty, implying that $f = g$ for all $x \in \mathbb{R}$.

6

Prove that an increasing function $f : [a, b] \to \mathbb{R}$ is measurable.

Solution: We know from the previous homework that monotone functions are measurable, since we can explicitly find $f^{-1}((a, \infty))$, where f is a monotone function. Since $f : [a, b] \to \mathbb{R}$ in this case is given to be increasing, it is also monotone. Therefore it must also be measurable.

7

Let $(X, \mathcal{S}, \mu)$, and let $f_n, f : X \to \overline{\mathbb{R}}, n \geq 1$ be measurable functions. Suppose $f_n \xrightarrow{p} f$ a.e. Prove that there exist measurable functions $\{g_n\}_{n\geq 1}$ such that $f_n = g_n$ a.e. for all $n \geq 1$ and $f = \lim_{n\to\infty} g_n$ everywhere.
Solution: Let E be the set of all points where $f_n \to f$. Since this happens almost everywhere, $\mu(E^c) = 0$. Then define g_n thus— $g(x) = \begin{cases} f_n(x) & x \in E \\ f(x) & x \notin E. \end{cases}$ Thus f_n and g_n both differ only on a set of measure zero, hence they are equal a.e. g_n is also a measurable function for all n, as $g_n(B) = (f_n^{-1}(B) \cap E) \cup (f^{-1}(B) \cap E^c)$, which is measurable. Thus by Egorov's theorem, the sequence $\{g_n\}$ converges a.e., with the additional condition that f is exactly equal to $\lim_{n\to\infty} g_n$.

8

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then $g \circ f$ is Lebesgue measurable. True or false?

Solution: Let $f : [0, 2] \to [0, 1]$ be the inverse of the function $K : [0, 1] \to [0, 2]$, which is given by $K(x) = \Lambda(x) + x$, where Λ is the Cantor function. It is strictly increasing, hence one-one. It is also surjective onto $[0, 2]$ as it takes every value in that range due to the continuity of Λ and x. Thus K is bijective, and the function $f := K^{-1}$ is also bijective and continuous, thus a measurable function. Let C be the Cantor set. Then we know that $K(C)$ is a set of measure 1, since $K([0, 1]\backslash C) + K(C) = [0, 2] \implies m(K([0, 1]\backslash C)) + m(K(C)) = 2$. Then $[0, 1]\backslash C$ is open, as C is closed. Then $[0, 1]\backslash C = \coprod_{j=1}^{\infty} I_j$, where I_j is an open interval, (a_j, b_j). See that Λ is constant on $[0, 1]\backslash C$, and that $m(\coprod_{j=1}^{\infty} I_j) = 1$, since $m(C) = 0$.

Then

$$\begin{aligned} m(K([0,1]\backslash C)) &= m(K(\coprod_{j=1}^{\infty} I_j) \\ &= \sum_{j=1}^{\infty} m(K(I_j)) = \sum_{i=1}^{\infty} (m(\Lambda(b_j) - \Lambda(a_j)) + m(b_j - a_j)) \\ &= \sum_{j=1}^{\infty} m(I_j) = 1. \end{aligned}$$

Thus $m(K(C)) = 2 - 1$, hence has non-zero measure. This implies that there exists a non-measurable set contained in $K(C)$, denoted by A. Then $f(A)$ is a subset of C, a null set. Therefore by the completeness of measure, $f(A)$ must be measurable.

Let $g = \chi_B$. We know that B is Lebesgue measurable, thus g is also measurable. Now consider $g \circ f$. See that

$$\begin{aligned} (g \circ f)^{-1}((\frac{1}{2}, \infty)) &= K \circ g^{-1}((\frac{1}{2}, \infty)) = \{x \in [0,2] : \chi_B(K(x)) \in (\frac{1}{2}, \infty)\} \\ &= \{x \in [0,2] : K(x) \in B\} \\ &= K(B) = A. \end{aligned}$$

As A is not Lebesgue measurable, $g \circ f$ is not a Lebesgue measurable function.

9

Let $E \in \mathcal{M}(\mathbb{R})$ and let $\{f_n\}$ be a sequence of real valued measurable functions on E. Prove that the set of points at which this sequence converges is measurable.

Solution: Let C be the set of all points such that $f_n(x)$ converges. For a fixed $x \in E$, we have a sequence of real numbers. This sequence is convergent if and only if the sequence is Cauchy. The claim is that

$$C = \cap_{K=1}^{\infty} \left(\cup_{N=1}^{\infty} \left(\cap_{n \geq N} \left(\cap_{m \geq N} \left((f_n - f_m)^{-1} \left(-\frac{1}{K}, \frac{1}{K} \right) \right) \right) \right) \right).$$

Let D denote the RHS of the above equation. Denote by D_K the set $\cup_{N=1}^{\infty} \left(\cap_{n \geq N} \left(\cap_{m \geq N} \left((f_n - f_m)^{-1} \left(-\frac{1}{K}, \frac{1}{K} \right) \right) \right) \right)$. Denote by $D_{K,N}$ the set $\cap_{n \geq N} \left(\cap_{m \geq N} \left((f_n - f_m)^{-1} \left(-\frac{1}{K}, \frac{1}{K} \right) \right) \right)$. Fix $x \in D$. Then $x \in D_K$ for all $K \in \mathbb{N}$. See that for all $K \in \mathbb{N}, x \in \cup_{N=1}^{\infty} D_{K,N}$. We can see that for all $K \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $x \in D_{K,N}$. $D_{K,N} = \cap_{n \geq N} \left(\cap_{m \geq N} \left((f_n - f_m)^{-1} \left(-\frac{1}{K}, \frac{1}{K} \right) \right) \right)$. For $x \in D_{K,N}$, for all m, n we have $-\frac{1}{K} < f_n(x) - f_m(x) < \frac{1}{K}$. Now we can see that this is true for all m, n. We now know that for all $\varepsilon > 0$ we can find a $K \in \mathbb{N}$ such that the above condition holds. This is the exact criterion for convergence, thus $x \in C$. Thus $D \subseteq C$. By a similar argument, $D \supseteq C$. Thus $C = D$. Thus, each f_n is a measurable function. C is a countable union of countable intersection of measurable sets, implying that C must be measurable.