The last Home-work

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1. Since R is a PID, we know that $(a, b) = (d)$, for some $d, a, b \in R$. Then we have $d = am + bn$, for some $m, n \in R$. Now we have a vector $v = [a, b]^T \in R^2 \backslash \{0\}$. Then we show that there exists a $2x2$ matrix that does what we want by constructing one. Let the desired matrix by be given by $X = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$. Now we have $Xv = [x_{11}a + x_{21}b, x_{12}a + x_{22}b]^T = [d, 0]^T$. Comparing terms, we have $x_{12}a + x_{22}b = 0$. Then we have $x_{12}a = -x_{22}b$, which implies that $x_{12}| - b$, and $x_{22}|a$. It is easy to see that $x_{12} = -a/d$ and $x_{22} = b/d$ does the trick. For $x_{11}a + x_{21}b = d$, see that $x_{11} = m$ and $x_{21} = n$ are good choices, since their linear combination produces d. Thus see that

$$X = \begin{pmatrix} m & n \\ -b/d & a/d \end{pmatrix}$$

is a matrix that achieves the intended result.

2. The above result shall be of much use to us. We see that we want to send a_{11} to d, that is the gcd of a_{11} and a_{i1}, and a_{i1} to 0. Let $m, n \in R$ such that $d = a_{11}m + a_{i1}n$. Define the matrix $\tilde{X} = (x_{kl})$ thus— $x_{11} = m, x_{1i} = n, x_{i1} = -a_{i1}/d, x_{ii} = a_{11}/d$. Also we have $x_{kk} = 1$ if $k \neq 1, i$. All other elements are 0. Then we have $A' = \tilde{X}A = (a'_{kl})$, where $a'_{11} = a_{11}m + a_{i1}n$, $a'_{i1} = (-a_{i1})a_{11} + (a_{11})a_{i1} = 0$, and $a'_{kl} = a_{kl}$ for $k \neq 1, i$. We need to see that this here matrix is invertible. For $\tilde{X}$ a $m \times m$ matrix, we want $\det \tilde{X}$. We expand the determinant along the first row. Then we have $\det \tilde{X} = m \det \tilde{X}[1|1] + (-1)^{i+1}((-1)^{i+1} \det \tilde{X}[1|i])$. $\tilde{X}[1|1]$ is a diagonal matrix with a_{11} on the a_{ii}th entry, and 1 otherwise on the diagonal. Thus $\det \tilde{X}[1|1] = a_{11}$. $\tilde{X}[1|i]$) is a matrix with x_{i1} at the $(i-1, 1)$th entry, with every element below and above it zero. We take the determinant along this column, we have $(-1)^{i-1+1}x_{i1} \cdot \det I_{m-2} = (-1)^i x_{i1}$. Thus see that $\det \tilde{X} = ma_{11} + (-1)^{2i+1}(-a_{i1}) = 1$, means that $\tilde{X}$ is invertible.

3. The above result and the result above that shall be of much use to us. If $A = 0$, then there is nothing to do. We then have $A \neq 0$. Without loss of generality, we take $a_{11} \neq 0$. This is because we can shift the row with a non-zero element to the top, then send the column with that element to the first column. Now using the above result, there is a $\tilde{X}_1$ such that $a_{21} = 0$. The value of a_{11} changes. Now we have $\tilde{X}_2$ that sends a_{31} to 0. We repeat this process till $a_{i1} = 0$ for all $i > 1$. Now we have the first column all zero except for a_{11}. Let $X_1 := \tilde{X}_{i-1} \dots \tilde{X}_1$. Let us denote X_1A by A'. Then consider A'^T. The first row now becomes the first column, and we can do the same thing that we did earlier, to reduce all elements below a_{11} in A'^T to 0. Let that operation be given by the matrix Y_1. Naturally, this matrix is the product of matrices obtained from $j - 1$ operations as given in the previous part. Then we take the transpose of the matrix $Y_1A'^T$ to have

$$A'' = (Y_1A'^T)^T = (Y_1(X_1A)^T)^T = X_1^T AY_1^T.$$

The matrix we have obtained has no non-zero elements below a_{11} or to its right.

Now note that we can modify our previous result to the second row. Earlier, we reduced all the leading terms of rows other than the first row to zero, then we did the same with columns. Here we reduce the second terms of the ith rows for $i > 2$, then do the same for the columns. We can find a X_2 and Y_2 both invertible such that $X_2A''Y_2$ has all elements of the type a_{2j} and a_{i2} zero, for $i, j \neq 2$. Now we have a matrix where a_{11} and a_{22} may or may not be zero, but all elements sharing the same row or column with them is zero. We continue this process for the entire matrix, which gives us at every stage two invertible matrices that do the above reduction. To be precise, we have $X_1, \dots, X_t, Y_1, \dots, Y_t$ where $t = \min(m, n)$. We say $X = X_1 \dots X_t$, and $Y = Y_t^{-1} \dots Y_q^{-1}$. Then putting all of these results together, we get $D = XAY^{-1}$.

We do not know a priori if $a_{11}|a_{22}| \dots |a_{tt}$, but we can ensure this. We first make sure that $a_{11}|a_{22}$, then the general case is easy to see. We execute the elementary column operation $C_1 \mapsto C_1 + C_2$. Now using the previous result we can change a_{11} to $\gcd(a_{11}, a_{22},)$ and a_{21} goes to 0. All other terms remain unchanged. We know that $d|a_{22}$. We repeat this procedure for d_{ii} and $d_{(i+1)(i+1)}$, to get the desired result.

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We have M a $R-$module which has itself as a generating set. Then $\pi : R^{\oplus M} \twoheadrightarrow M$ is the surjective map sending e_m to m. We see that $\pi(e_{rm} - re_m) = \pi(e_{rm}) - r\pi(e_m) = rm - rm = 0$. Also, $\pi(e_{m_1+m_2} - e_{m_1} - e_{m_2}) = \pi(e_{m_1+m_2}) - \pi(e_{m_1}) - \pi(e_{m_2}) = (m_1 + m_2) - m_1 - m_2 = 0$. Therefore we have $e_{rm} - re_m \in \ker \pi$, and $e_{m_1+m_2} - e_{m_1} - e_{m_2} \in \ker \pi$. Thus we have

$$(e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2}) \subseteq \ker \pi.$$

Let us have $\sum_{m \in M} r_m e_m \in R^{\oplus M}$. Note that there are only finitely many terms in the summation. See that

$$\pi(\sum_{m\in M} r_m e_m) = \sum_{m\in M} r_m \pi(e_m) = \sum_{m\in M} r_m m = \pi(\sum_{m\in M} e_{r_m m}).$$

This means that $\pi(\sum_{m\in M} r_m e_m - \sum_{m\in M} e_{r_m m}) = 0$. This, in turn implies that $\sum_{m\in M} r_m e_m - e_{r_m m} \in \ker \pi$. We also see that

$$\pi(\sum_{m\in M} r_m e_m) = \sum_{m\in M} r_m \pi(e_m) = \sum_{m\in M} r_m m = \pi(e_{\sum_{m\in M} r_m m}).$$

This means that $\sum_{m\in M} r_m e_m - e_{\sum_{m\in M} r_m m} \in \ker \pi$. Given an element in $R^{\oplus M}$, we can choose which summands to clump and which to leave unchanged. Either ways, we see that we get a linear combination of $re_m - e_{rm}$ and $e_{m_1+m_2} - e_{m_1} - e_{m_2}$, which implies that $\ker \pi \subseteq (e_{rm} - re_m, e_{m_1+m_2} - e_{m_1} - e_{m_2})$. This gives us the desired equality.

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1. We take the module $\mathbb{Z}[q_1, q_2]$, where $q_1, q_2 \in \mathbb{Q}$.

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$R^{\oplus X}$ is a free module, for some indexing set X. M is some submodule of $R^{\oplus X}$. We find a subset $Y \subseteq X$ such that $M \cap R^{\oplus Y}$ is free and B is a basis for this free module. Let (B, Y) be such a pair with the given partial order. $\mathbb{T}$ is the poset of all such submodules in $R^{\oplus X}$.

1. X is non-empty. Then we can pick a singleton subset $\{x\} \subseteq X$. $R^{\oplus Y}$ must be a finitely generated module (hence the free module generated by a singleton must be R), and hence so must $M \cap R$, as this is merely an ideal in R, which is an ideal generated by one element. Thus the ideal is isomorphic to R as a module. Thus $R \cong Ra$, where $(a) = I$. Thus this is an element of $\mathbb{T}$, which means that $\mathbb{T}$, where this above example corresponds to the element $(R, \{1\})$.

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1. Let $f \in \mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}}{(n)}, \frac{\mathbb{Q}}{\mathbb{Z}}\right)$. Then we have an abelian group, as $\mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}}{(n)}, \frac{\mathbb{Q}}{\mathbb{Z}}\right)$ is a $\mathbb{Z}-$module ($\mathbb{Z}-$modules and abelian groups are the same). Elements in $\frac{\mathbb{Q}}{\mathbb{Z}}$ are precisely the elements of $\mathbb{Q} \cap [0, 1)$. Then due to the $\mathbb{Z}-$linearity of f, we only need to ask where 1 is sent to. Let us say that $f(1) = \frac{p}{q}$.

We see that $n \cdot f(1) = f(n) = f(0) = 0$, thus $n \cdot \frac{p}{q} = 0$. This then means that $\frac{np}{q} \in \mathbb{Z} \implies n|q$. Also see that $qf(1) = p \in \mathbb{Z}$. Thus $f(q) = 0$, which means that $q \in \ker f$.