Algebra 2 Homework 9

April 18, 2024

Solution of problem 1*:* We have $\alpha + \beta = \theta$, since $\beta = \gamma$ and $\alpha + \gamma = \theta$ due to the exterior angle. Now see that $\theta + \alpha + (\pi - 2\beta) = \pi$, and we also have $(\pi - 2\beta)\theta + \alpha = \pi \implies 2\alpha = \beta$. Thus $\alpha + \beta = \theta \implies \alpha = \theta/3$. Now we have □

Solution of problem 2*:* We can see that $x^3x^2 - 2x - 1$ cannot be reduced for $\mathbb{Q}$ as the thing is irreducible in $\mathbb{Z}_3$. Then see that the α that satisfies this polynomial gives us a degree 3 extension of $\mathbb{Q}$. This cannot be constructed by straight edge and compass, as the degree is not a power of 2. Thus $\cos\left(\frac{2\pi}{7}\right)$ cannot be constructed.

See that for a regular $7-$gon we will have exterior angle $\cos\left(\frac{2\pi}{7}\right)$ which we cannot construct, hence this polygon is not possible. □

Solution of problem 3*:* Since $b \in \mathbb{R}$ is constructible, we have $b \in K_n = \mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_n})$, where $a_i > 0, a_i \in K_{i-1} = \mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_{i-1}})$. Then we have that by the fundamental theorem of Galois theory, we must have that the Galois group of K_i is $\oplus_{k=1}^{i}\mathbb{Z}_2$. This is a finite $p-$group, which is clearly a solvable group.

Now let the roots of the minimal polynomial for b over $\mathbb{Q}$ be $b_1 = b, \ldots, b_n$. Since $\mathbb{Q}(b) \cong \mathbb{Q}(b_i)$ for $1 \leq i \leq n$, then $K = \mathbb{Q}(b_1, \ldots, b_n)$ is Galois. As $\mathbb{Q}(b)$ is a radical extension, so is $\mathbb{Q}(b_i)$. Thus we can get $\mathbb{Q}(b_i)$ by successively adjoining square roots of elements from the smaller field. We can see that K can be obtained by adjoining all these numbers, which is also a radical extension. Thus its corresponding Galois group must be solvable. □

Solution of problem 4*:* If we have K/F is inseparable, then for all $\alpha \in K \backslash F$, there is some $m \in \mathbb{N}$ such that $\alpha^{p^m} \in F$. Let L be the algebraic closure of F. Then F □

Solution of problem 5*:* We know that if $A_{F/K}$ is the $[F : K] \times [F : K]$ matrix that represents the $K-$linear operator of multiplication by x, then $T_{F/K}(x) = \operatorname{tr}(A_{F/K})$, and $N_{F/K}(x) = \det A_{F/K}$. We want to show that for $F \subset K \subset L$, we want to see that trace and norm are multiplicative.

1. If L/F is not separable, K/F or L/K is inseparable. In either case, the trace would turn out to be zero, hence vacuously true. To see this, we have that if we assume K/F is inseparable, then the field has characteristic $p > 0$ and the minimal polynomial must necessarily have degree p^m, since otherwise it would not have repeated roots. Then see that the coefficient for x^{p^m-1} must be zero since the derivative of the polynomial needs to have a common factor with the minimal polynomial. Thus trace must be zero. The same applies in the other case.

Assume now that the extension is inseparable. Then choose K' as Galois containing L. We have $G = \text{Gal}(K/F), H' = \text{Gal}(L/F), H = \text{Gal}(K'/K)$. Now, we know that trace is just sum of embeddings. Then $\text{tr}_{L/F}(x) = \sum_{\sigma\in G/H'} \sigma(x)$, where G/H' is the set of left cosets. Now we have that $\text{tr}_{L/K}(x) = \sum_{\sigma\in H/H'} \sigma(x)$, so putting the two together, we have

$$\text{tr}_{L/K}(\text{tr}_{K/F}(x)) = \sum_{\sigma\in G/H} \sigma\left(\sum_{\sigma'\in H/H'} \sigma'(\sigma(x))\right),$$

which means that the automorphism $\sigma' \circ \sigma$ ranges over all of G/H'. This is the same as the formula for RHS, the desired result.

2.

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