

## 1

1. This statement is true. Take any vector space  $V$ , which necessarily has a Hamel basis, trivially the entire space is a basis. Then for a choice of basis  $(b_n)$  we have for all  $x \in V$ ,  $x = \sum k_i b_i$ , where the sum is finite. Then define  $\|x\| := \max\{|k_i|\}$ . See that this fulfills the definition of a norm.
2. This statement is false. For sake of contradiction, let  $(X, \|\cdot\|)$  be a normed linear space such that the induced metric is the discrete metric. Then for  $x, y \in X, x \neq y$  we must have  $\|x - y\| = 1$ . Note that  $2x \neq 2y$ , so we must have  $\|2x - 2y\| = 2\|x - y\| = 2$ , but by the discrete metric the answer should still be 1! Thus there can be no such norm.

## 2

We wish to show that the function  $\|\cdot\|$  on  $X$  satisfies the triangle inequality iff the closed unit ball is convex. Assume that the function  $\|\cdot\|$  is indeed a norm. Then let  $x, y \in D$ , the closed unit ball. Then  $\|x\|, \|y\| \leq 1$ . Now we have for  $\alpha \in [0, 1]$   $z = \alpha x + (1 - \alpha)y$ . See that

$$\|z\| = \|\alpha x + (1 - \alpha)y\| \leq \alpha\|x\| + (1 - \alpha)\|y\| \leq \alpha \cdot 1 + (1 - \alpha) \cdot 1 \leq 1,$$

thus we have  $z \in D$ .

Take two elements  $x, y \in X$  both non-zero, since if either were zero the inequality would be trivial. Then

$$\|x + y\| = (\|x\| + \|y\|) \cdot \left\| \alpha \frac{x}{\|x\|} + (1 - \alpha) \frac{y}{\|y\|} \right\|,$$

where  $\alpha = \frac{\|x\|}{\|x\| + \|y\|}$ . Note that  $\frac{x}{\|x\|} = \frac{y}{\|y\|} = 1$ , thus we can use the convexity condition to see that  $\frac{\|x+y\|}{\|x\| + \|y\|} \leq 1$ , which is the triangle inequality.

## 3

1. Pick a  $f \in C([a, b])$ . Then  $|f| \leq M = \sup\{|f(x)| : x \in [a, b]\}$ . Now see that

$$\int_a^b |f(t)|^p dt \leq (b - a)M^p \geq 0.$$

Thus  $\|f\|_p \geq 0$ . For  $f = 0$ , we have  $M = 0$ , so  $\int_a^b |0|^p dt = 0$ . If  $\int_a^b |f(t)|^p dt = 0$ , then see that  $0 \leq (b - a)M^p \geq 0$ . Thus we must have

$$(b - a)M^p = 0 \implies M = 0 \implies f = 0.$$

To see the next axiom, for  $\alpha \in \mathbb{K}$ , we have  $\int_a^b |\alpha f(t)|^p dt = \int_a^b |\alpha|^p |f(t)|^p dt$ . Then

$$\|\alpha f\|_p = \left( |\alpha|^p \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} = |\alpha| \|f\|_p.$$

Now let  $f, g \in C[a, b]$ . Then we have to prove Minkowski's inequality to show the triangle inequality.

$$\begin{aligned}
\|f + g\|_p^p &= \int_a^b |f + g|^p dx \\
&= \int_a^b |f + g| \cdot |f + g|^{p-1} dx \\
&\leq \int_a^b |f| |f + g|^{p-1} dx + \int_a^b |g| |f + g|^{p-1} dx \\
&\leq \left( \int_a^b |f|^p dx + \int_a^b |g|^p dx \right) \left( \int_a^b |f + g|^{(p-1) \frac{p}{p-1}} dx \right)^{1-\frac{1}{p}} \quad (\text{Hölder's inequality}) \\
&= (\|f\|_p + \|g\|_p) \frac{\|f + g\|_p^p}{\|f + g\|_p},
\end{aligned}$$

which yields the required result.

2.

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1.  $\|f_1 - F\|_\infty$  is to be found, where  $F$  is the subspace of constant functions. Unfolding the term, we get

$$\|f_1 - F\|_\infty = \inf_{c \in \mathbb{R}} \{ \sup_{t \in [0,1]} |t - c| \},$$

which for  $c \in [-1, 0] \cup [1, 2]$  is 1, while in  $(0, 1)$  it decreases to  $\frac{1}{2}$  then goes back up to 1. Thus, we must have  $\|f_1 - F\|_\infty = \frac{1}{2}$ .

2. We want to now see the distance between  $f_2 = t^2$  and  $G$ , the space of all polynomials with degree at most 1. Then for some polynomial  $-ax - b \in G$ , we want to see  $\sup_{t \in [0,1]} \{ |t^2 + ax + b| \}$

5

Let  $Y$  and  $X/Y$  be Banach spaces. Then take  $(x_n)$  to be a Cauchy sequence in  $X$ . That is, for all  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for  $m, n \geq N$  we have

$$\|x_m - x_n\| < \varepsilon.$$

By the canonical projection to  $X/Y$  we can see that the sequence  $(x_n + Y)$  is also Cauchy, since we have

$$\|x_m - x_n + Y\| \leq \|x_m - x_n + 0\| < \varepsilon.$$

Since  $X/Y$  is Banach, we have  $(x_n + Y) \rightarrow (x_0 + Y)$ . Now let  $y_n :=$

Let  $X$  and  $X/Y$  be Banach spaces. Then take  $(y_n)$ , a Cauchy sequence in  $Y$ . Thus for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for  $m, n \geq N$ , we have  $\|y_m - y_n\| < \varepsilon$ . As a Cauchy sequence in  $X$ , this must converge to some element  $y_0 \in X$ . We now need to show that  $y_0 \in Y$ . But since  $Y$  is closed and  $y_0$  is a limit point, we must have  $y_0 \in Y$ .

Let  $X$  and  $Y$  be Banach spaces.

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Assume that  $X$  is a Banach space. Let  $(x_n)$  be an absolutely convergent series in  $X$ , that is,  $\sum_{i=1}^n \|x_i\|$  converges to  $c \in \mathbb{R}$  as  $n \rightarrow \infty$ . Thus for all  $\varepsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|\|x_n\| - \alpha| < \varepsilon.$$

Now we have

For the converse, take a Cauchy sequence  $(x_n)$ . Pick  $\varepsilon > 0$ . Pick  $N(n) \in \mathbb{N}$  such that for  $m, n \geq N$ , we have  $\|x_m - x_n\| < \frac{\varepsilon}{2^n}$ . Now see that  $\sum_{i=1}^n \|x_{i+1} - x_i\|$

## 7

Let  $\ell^p$  be the space of all  $p$ -power summable sequences. Then  $\|(x_n)\| = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$ . Now we want to show that the space  $K = \{(x_n) \in \ell^p : x_i = 0 \forall i > n, n \in \mathbb{N}\}$  is dense in  $\ell^p$ . Take any  $(x_n)_{n \in \mathbb{N}}^{(0)} \in \ell^p$ . There is a sequence of elements in  $\ell^p$   $(x_n)^{(m)} \subseteq K$  such that  $x_n^{(m)} = x_n$  if  $n \leq m$ , and 0 otherwise. See that  $\|(x_n)^{(m)} - (x_n)^{(0)}\|_p = (\sum_{i=m+1}^{\infty} |x_i|^p)^{\frac{1}{p}}$ . Since  $\|(x_n)^{(m)}\|_p \rightarrow \|(x_n)^{(0)}\|_p$  as  $m \rightarrow \infty$ , we must have for a choice of  $\varepsilon > 0$  there being  $N \in \mathbb{N}$  such that for  $m \geq N$ ,  $|\|(x_n)^{(m)}\|_p - \|(x_n)^{(0)}\|_p| < \varepsilon$ . Multiplying on both sides by  $\frac{\|(x_n)^{(m)}\|_p - \|(x_n)^{(0)}\|_p}{\|(x_n)^{(m)}\|_p - \|(x_n)^{(0)}\|_p}$ , and after a change in the value of  $\varepsilon$ , we get

$$\|(x_n)^{(m)}\|_p^p - \|(x_n)^{(0)}\|_p^p < \varepsilon.$$

Thus we have shown that  $K$  is dense in  $\ell^p$ . For a fixed  $m$ , and a fixed  $n$ , look at  $x_n^{(m)}$ . This is a real number or a complex number, which can be approximated by a sequence of rationals (or elements of  $\mathbb{Q}(i)$ ) such that  $x_n^{(m)}$  is its limit. Now we have a sequence  $(t_k)$ , where  $t_k \rightarrow x_n^{(m)}$  as  $k \rightarrow \infty$ .

Now let us see that  $K' = \{(x_n) \in K : x_n \in \mathbb{Q} \text{ or } \mathbb{Q}(i)\} \subset K$  is countable, and as we saw above must be dense in  $\ell^p$ . Thus it is separable.

## 8

We first show that  $\ell^\infty$  is not separable. Take the space  $K$  of all sequences in  $\ell^\infty$  such that their entries are either 0 or 1. Then for  $x \neq y \in K$ , we have  $\|x - y\|_\infty = 1$ , since at least one entry is different between the two. Then we have uncountably points. consider a ball of radius  $\frac{1}{2}$  centred at  $x \in K$ , for all such points in  $K$ . We know what uncountably many open sets, all of which are disjoint from each other. Let  $S$  be a possibly dense set. Then we have that each open ball contains at least one point of  $S$ . This then means that there must be uncountably many elements, so  $\ell^\infty$ .

If there was a Schauder basis for  $\ell^\infty$ , then for all  $x \in \ell^\infty$  we would have a sequence  $(a_1, a_2, \dots)$  described using the basis. We can approximate each term by a sequence of elements in  $\mathbb{Q}$  or  $\mathbb{Q}(i)$ , which would then mean that we would have all the rational points of  $\ell^\infty$  as a dense subset, which contradicts the fact that the space is not separable. Thus there can be no Schauder basis.

## 9

We want to see if this function is continuous. Pick a  $x_0 \in \mathbb{R}$ . Now for all  $\varepsilon > 0$  have  $f(z_0 + h) - f(z_0) = f(h)$ .

See that  $f(2) = 2f(1)$ ,  $f(0) = 2f(0) \implies f(0) = 0$ , and that  $f(0) = 0 = f(1) + f(-1) \implies f(-1) = -f(1)$ . Thus  $f(n) = nf(1)$  for all  $n \in \mathbb{Z}$ . We know that  $f(1) = f(n \frac{1}{n}) \implies f(\frac{1}{n}) = \frac{1}{n}f(1)$ . Thus we have  $f(q) = qf(1)$  for all  $q \in \mathbb{Q}$ . Since any real number can be approximated by a sequence of rationals, we can easily see that for  $q_n \rightarrow r$  and  $n \rightarrow \infty$ , we have  $f(q_n) = q_n f(1) \rightarrow r f(1)$ . Thus we can extend this to all real numbers. Now we have  $|f(z_0 + h) - f(z_0)| = |h f(1)| < \varepsilon$ , for a choice of  $h = \frac{\varepsilon}{|f(1)|}$ .

## 10

Let  $P([0, 1])$  be the space of all real polynomials defined on  $[0, 1]$  be a real vector space. Let the norm of a polynomial  $f(x) = a_0 + a_1x + \dots + a_nx^n \in P([0, 1])$  be given thus:  $\|f\| = |a_0| + |a_1| + \dots + |a_n|$ . Now see that the operator  $I : P([0, 1]) \rightarrow P([0, 1])$  such that  $x^t \mapsto \frac{x^{t+1}}{t+1}$ , we then see that

$$\|I\| = \sup_{0 \neq x \in P([0, 1])} \frac{\|If\|}{\|f\|} = \frac{a_0x + \dots + \frac{a_n}{n+1}a_{n+1}}{a_0 + \dots + a_nx^n} = \frac{|a_0| + |\frac{a_1}{2}| + \dots + |\frac{a_n}{n+1}|}{|a_0| + \dots + |a_n|} \leq 1.$$

Also see that this supremum is indeed attained since  $I(a_0) = a_0x$ , and in this case  $\frac{\|I(a_0)\|}{\|a_0\|} = 1$ . Thus we have  $\|I\| = 1$ . We wish to find the inverse of this operator, see that the differential operator  $D$  such that  $x^t \mapsto tx^{t-1}$ , is the required inverse. However, see that for  $f(x) = x^n$ , we have  $Df = I^{-1}f = nx^{n-1}$ . Now we have

$$\|D\| = \sup_{0 \neq x \in P([0, 1])} \frac{\|Df\|}{\|f\|} \geq \frac{\|Dx^n\|}{\|x^n\|} = \frac{n}{1}.$$

Thus we have that our operator  $D$  is unbounded, since for any chosen  $N \in \mathbb{N}$  we can choose  $x^{N+1}$ , such that  $\frac{\|Dx^{N+1}\|}{\|x^{N+1}\|}$  is larger.

## 11

Since the linear functional  $f : X \rightarrow \mathbb{R}$  is unbounded, we have a sequence  $(x_n)$  such that  $f(x_n) > n\|x_n\|$ . We can say without loss of generality, and by discarding non-zero elements, we have a sequence  $(x_n)$  of norm 1. Pick  $x \in X$ . Then see that the sequence  $z_n = x - \frac{f(x)}{f(x_n)}x_n$ , which is clearly in  $K$ , the kernel of  $f$ . Also it can be seen that as  $n \rightarrow \infty$ , we have  $\|z_n\| \leq$

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