

Algebra HW5

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1

If we have $\nu = 0$, then see that $\nu(E) = \int_E 0 d\mu$, thus $\mu(E) = 0 \implies \nu(E) = 0$ trivially, so $\nu \ll \mu$. Also see that $\nu \perp \mu$, as $X = X \sqcup \phi$, and see that $\mu(E) = \mu(E \cap X)$, while $\nu(E) = \nu(E \cap \phi)$.

Now let us assume that $\nu \ll \mu$ and $\nu \perp \mu$. Then we have $X = A \sqcup B$, where $\mu(E) = \mu(E \cap A)$, while $\nu(E) = \nu(E \cap B)$. Let us pick a measurable set $E \subseteq B$. Then we have $\mu(E) = \mu(E \cap B) = \mu(\phi) = 0$. Since $\nu \ll \mu$, we have $\nu(E) = 0$. Thus ν is zero on every measurable subset E in B . For a general measurable set E , we have $E = (E \cap A) \sqcup (E \cap B)$. We already know that $\nu(E \cap A) = 0$, now we see that $\nu(E \cap B) = 0$ also. Thus $\nu(E) = \nu(E \cap A) + \nu(E \cap B) = 0$ for all measurable $E \subset X$.

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If $\nu \perp \mu$, then there exists we have $X = A \sqcup B$, where $\mu(B) = 0$ and $\nu(A) = 0$. Then $\{E_n\}$ is a sequence such that $E_n = B$ for all $n \in \mathbb{N}$. Then see that $\mu(E_n), \nu(X \setminus A) = 0$, as required.

Conversely, we assume that $\{E_n\}$ is a sequence of measurable subsets such that $\mu(E_n) \rightarrow \infty$ and $\nu(X \setminus E_n) \rightarrow \infty$ as $n \rightarrow \infty$.

3

We can check that $m \ll m$ trivially, as $m(E) = 0 \implies m(E) = 0$. We can split \mathbb{R} into two disjoint subsets, that is, $\mathbb{R} = \{0\} \sqcup ((-\infty, 0) \cup (0, \infty))$. We denote the two sets as A and B . Then observe that $\delta_0(E) = \delta_0(E \cap A)$, that is, the Dirac measure at 0 only cares if it intersects $\{0\}$, and nothing else. Also, we have $m(E) = m(E \cap B)$, as A is a m -null set, hence $m(E) = m(E \cap A \sqcup E \cap B) = m(E \cap A) + m(E \cap B) = m(E \cap B)$, seeing as $E \cap A$ is also a m -null set. Thus we have $m \perp \delta_0$. Thus $\nu = m + \delta_0$ is already in the Lebesgue decomposition.

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- We find the positive and negative parts of f . Note that the roots of this polynomial are $3 + 2\sqrt{2}$ and $3 - \sqrt{2}$. Let us call them α_1 and α_2 for sake of convenience. Then

$$p^+ = \begin{cases} x^2 - 6x + 1 & x \in (-\infty, \alpha_2] \cup [\alpha_1, \infty) \\ 0 & \text{else,} \end{cases}$$

and

$$p^- = \begin{cases} -(x^2 - 6x + 1) & x \in (\alpha_2, \alpha_1) \\ 0 & \text{else.} \end{cases}$$

Then $\nu(E) = \int_E p^+ d\mu - \int_E p^- d\mu = \nu^+ - \nu^-$, where $\nu^+ := \int_E p^+ d\mu$ and $\nu^- := \int_E p^- d\mu$ are two positive measures. Note that it is not possible for both of them to attain ∞ together, since ν^- is a finite measure. Thus it is trivial to see that ν must be a signed measure.

- Let $\mathbb{R} = A \sqcup B$, where $A = (-\infty, \alpha_2] \cup [\alpha_1, \infty)$ and $B = (\alpha_2, \alpha_1)$. See that since both the positive measures have their usual properties, we have that for $E \subseteq A$ measurable, we have $\nu(E) = \nu^+(E) - \nu^-(E) = \nu^+(E) - 0 \geq 0$, and likewise for $E \subseteq B$ measurable, we have $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - \nu^-(E) \leq 0$. Thus the above construction is a Hahn decomposition.

- See that ν^+ lives on A , while ν^- lives on B . That is, $\nu^+(E) = \nu(E \cap A)$, and $\nu^-(E) = -\nu(E \cap B)$. This is easy to see, as $E = (E \cap A) \sqcup (E \cap B)$. Then $\nu(E) = \int_{(E \cap A) \sqcup (E \cap B)} p^+ d\mu - \int_{(E \cap A) \sqcup (E \cap B)} p^- d\mu = \int_{(E \cap A)} p^+ d\mu + \int_{(E \cap B)} p^+ d\mu - \int_{(E \cap A)} p^- d\mu - \int_{(E \cap B)} p^- d\mu$. Since p^+ is 0 on B , and p^- is 0 on A , we have that $\nu^+ \perp \nu^-$. Thus we have the Jordan decomposition.

5

Let us assume that $\mu(E) = 0$. As E is μ -null, then $\mu(E \cap E_n) = 0$ for all n , by monotonicity of the measure. Then we have $\nu(E) = \sum_{n=1}^N c_n \mu(E \cap E_n) = 0$. Thus $\nu \ll \mu$. See that the function $f := \sum_{n=1}^N c_n \chi_{E_n}$ is a good candidate for the Radon-Nikodym derivative.

$$\int_E f d\mu = \int_E \sum_{n=1}^N c_n \chi_{E_n} d\mu = \sum_{n=1}^N c_n \int_X \chi_E \chi_{E_n} d\mu = \sum_{n=1}^N c_n \int_X \chi_{E \cap E_n} d\mu = \sum_{n=1}^N c_n \mu(E \cap E_n),$$

which is the desired result. Thus $\frac{d\nu}{d\mu} = f$.

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1. Since $\nu \ll \mu$, there exists $f \in L^1(\mu)$ such that $\nu(E) = \int_E f d\mu$. We know that $f > 0$ μ -almost everywhere, then assume that $\nu(E) = 0$. Thus $\int_E f d\mu = 0$. Assume that $\mu(E) > 0$. Then f is greater than zero on all of E , thus $\int_E f d\mu > 0$. However, since $\nu(E) = 0$ this forces $\mu(E)$ to be 0. Thus $\mu \ll \nu$.

2.

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Let $\theta := \mu + \nu$. Then $f = \frac{d\nu}{d\theta}$. See that since $\theta(E) = \int_E 1 d\theta = \mu(E) + \int_E f d\theta$. Therefore we have $\mu(E) = \int_E (1 - f) d\theta$. Thus we have $\frac{d\mu}{d\theta} = 1 - f$.

8

In $(\mathbb{N}, \mathbb{P}(\mathbb{N}))$, μ is the counting measure. Note that the empty set is the only μ -null set, since every non-empty set has cardinality more than zero. Then somewhat trivially we have $\mu(E) = 0 \implies E = \emptyset \implies \nu(E) = 0$. So $\nu \ll \mu$. We have ν is σ -finite, thus $\mathbb{N} = \sum_{n=1}^{\infty} \{n\}$, where $\nu(\{n\}) < \infty$. Thus define $f : \mathbb{N} \rightarrow \mathbb{R}$, where $f(n) = \nu(\{n\})$. Then we have $\nu(E) = \int_E f d\mu = \sum_{n \in E} f(n)$, is the required function. Thus $f = \frac{d\nu}{d\mu}$.

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If we assume that $\frac{d\mu}{d\lambda} \cdot \frac{d\nu}{d\lambda} = 0$ λ almost everywhere, we know from the previous assignment that $\mu(E)$ and $\nu(E)$ are mutually singular. To see the converse, let us assume that $\mu \perp \nu$. Then we have $X = A \sqcup B$, where μ lives on A while ν lives on B . Then let $f := \frac{d\mu}{d\lambda}$, and $g := \frac{d\nu}{d\lambda}$. See that

$$\int_E f g d\lambda = \int_{(E \cap A)} f g d\lambda + \int_{(E \cap B)} f g d\lambda.$$

We know that $\int_{(E \cap A)} f g d\lambda = \int_{(E \cap A)} f d\nu$, and $\int_{(E \cap B)} f g d\lambda = \int_{(E \cap B)} g d\mu$. As $E \cap A$ is a ν -null set and $E \cap B$ is a μ -null set, we have $\int_E f g d\lambda = 0 \implies f g = 0$ λ -almost everywhere.

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$\nu = \nu^+ - \nu^-$ is a signed measure, where we have $x = P \sqcup N$, that is, $\nu^+(E) = \nu(E \cap P)$, and $\nu^-(E) = \nu(E \cap N)$. Taking $|\nu| = \nu^+ + \nu^-$, see that $\nu^+ \ll |\nu|$ and $\nu^- \ll |\nu|$, and thus there must exist $\frac{d\nu^+}{d|\nu|}$ and

$$\frac{d\nu^-}{d|\nu|},$$

the Radon-Nikodym derivatives. Then see that $\frac{d\nu^+}{d|\nu|} = \chi_P$. To see this, see that for E measurable in P ,

$$\begin{aligned} \int_E \chi_P d|\nu| &= \int_E \chi_P d\nu^+ + \int_E \chi_P d\nu^- \\ &= \nu^+(E \cap P) + 0 = \nu^+(E), \end{aligned}$$

and similarly for ν^- , $\frac{d\nu^-}{d|\nu|} = \chi_N$.

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See that

$$\begin{aligned} \left| \int_X f d\nu \right| &\leq \left| \int_X f d\nu^+ - \int_X f d\nu^- \right| \\ &= \left| \int_X f d\nu^+ \right| + \left| \int_X f d\nu^- \right| \leq \int_X |f| d\nu^+ + \int_X |f| d\nu^- \\ &= \int_X |f| d|\nu|, \end{aligned}$$

as desired.

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We know that the counting measure μ only has one ν -null set, that is ϕ . Let us say that $\mu \ll \nu$. Then for $\nu(E) = 0$ for some measurable $E \in \mathcal{A}$, we have $\mu(E) = 0$. To take the contrapositive, we see that if $\mu(E) \neq 0$, then $\nu(E) \neq 0$. Since we know that μ has no non-empty null sets, non μ -null sets and non-empty subsets are synonymous. Thus, if $E \neq \phi$, $\nu(E) \neq 0$.

For the Dirac measure at $x_0 \in X$, we know that $\delta_{x_0} \ll \nu$ means that $\nu(E) = 0$ for some $E \in \mathcal{A}$ implies that $\delta_{x_0}(E) = 0$. The Dirac measure is zero if the measurable subset E does not have x_0 . Thus we can take the contrapositive of the absolute continuity to say that if $x_0 \in E$, then we must necessarily have $\nu(E) \neq 0$.