DIFFERENTIAL EQUATION

① $Mdx + Ndy = 0$ is exact

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$$

or $\underbrace{\int M dx}_{y \text{ is const.}} + \underbrace{\int N dy}_{\text{all } f^n \text{ of } x = 0} = c$

② Integrating factor if not exact

- $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = g(x)$, $IF = e^{\int g(x) dx}$

- $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = h(y)$, $IF = e^{-\int h(y) dy}$

③ By inspection

- $d(xy) = xdy + ydx$
- $d\left(\frac{x}{y}\right) = \dfrac{ydx - xdy}{y^2}$
- $d\left(\frac{y}{x}\right) = \dfrac{xdy - ydx}{x^2}$
- $d\left(\tan^{-1}\frac{y}{x}\right) = \dfrac{ydx - xdy}{x^2+y^2}$
- $d(x^2+y^2) = 2xdx + 2ydy$
- $d\left(\ln\frac{x}{y}\right) = \dfrac{ydx - xdy}{xy}$
- $d\left(\ln(x^2+y^2)\right) = \dfrac{2xdx + 2ydy}{x^2+y^2}$

④ Linear Differential Eq^n of 1^{st} Order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$IF = e^{\int P(x) dx}$$

$$y \, IF = \int Q(x) \, IF \, dx + c$$

⑤ Bernoulli's Differential Eq^n

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

put $y^{1-n} = v$

SECOND ORDER DIFFERENTIAL EQUATION

⑥ Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

LD if $W = 0$, otherwise LI

⑦ for HDE

$$y_g = c_1 y_1 + c_2 y_2$$

⑧ Use of a known sol^n to find another

$$y'' + P(x)y' + Q(x)y = 0$$

$y_1(x) \rightarrow$ given

$y_2(x) = v(x) . y_1(x)$

where $v(x) = \int \frac{1}{(y_1(x))^2} . e^{-\int P(x)dx} \, dx$

⑨ Homogeneous Eq^n with Constant Coefficients

$$y'' + py' + qy = 0$$

Let $y = e^{mx}$ be a sol^n

$$m^2 + pm + q = 0$$

$$m = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

Case i m_1 & m_2 are real & distinct

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case ii m_1 & m_2 are real & equal

$$y(x) = (c_1 + c_2 x)e^{mx}$$

Case iii m_1 & m_2 are imaginary

$$m = a \pm ib$$

$$y(x) = e^{ax}[c_1 \cos bx + c_2 \sin bx]$$

(10) Cauchy - Euler Differential Eqⁿ

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0$$

put $x = e^z$

$Z = \ln x$

$$xy' = \frac{dy}{dz}$$

$$x^2 y'' = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

(11) Legendre's Differential Eqⁿ

$$a_2 (ax+b)^2 y'' + a_1 (ax+b) y' + a_0 y = 0$$

$$(ax+b) y' = a\frac{dy}{dz}$$

$$(ax+b)^2 y'' = a^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

(12) Method of Undetermined Coefficients

$$y'' + py' + qy = r(x)$$

Case i $r(x) = ke^{ax}$

Condⁿ	Particular Solⁿ
$m_1 \neq a$ & $m_2 \neq a$	$y_p(x)$ or $Y(x) = Ae^{ax}$
$m_1 = a$ or $m_2 = a$	$y_p(x) = Axe^{ax}$
$m_1 = a$ & $m_2 = a$	$y_p(x) = Ax^2 e^{ax}$

Case ii $r(x) = k_1 \cos(bx) + k_2 \sin(bx)$

$y_p(x) = A\cos(bx) + B\sin(bx)$

if $y_g(x) = c_1 \cos(bx) + c_2 \sin(bx$

then $y_p(x) = x[A\cos(bx) + B\sin(bx)]$

Case iii $r(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

when 'y' term is present

$\Rightarrow y_p(x) = A_0 + A_1 x + \cdots + A_n x^n$

when 'y' term is not present

$\Rightarrow y_p(x) = x[A_0 + A_1 x + \cdots + A_n x^n]$

(13) Variation of Parameters Method

$y'' + py' + qy = r(x)$

for $r = 0 \Rightarrow y'' + py' + qy = 0$

$y_g = c_1 y_1(x) + c_2 y_2(x)$

$y_p = v_1(x) y_1(x) + v_2(x) y_2(x)$

where $v_1 = -\int \frac{y_2(x)\, r(x)}{W(y_1, y_2)}\, dx$

$v_2 = \int \frac{y_1(x)\, r(x)}{W(y_1, y_2)}\, dx$

LAPLACE TRANSFORM

① $L[f(x)] = \int_0^\infty e^{-px} f(x)\,dx$

② $L[1] = \frac{1}{p}$

$L[k] = \frac{k}{p}$

$L[x^n] = \frac{n!}{p^{n+1}}$

$L[e^{ax}] = \frac{1}{p-a}$

$L[\sin ax] = \frac{a}{p^2+a^2}$

$L[\cos ax] = \frac{p}{p^2+a^2}$

$L[\sinh x] = \frac{a}{p^2-a^2}$

$L[\cosh x] = \frac{p}{p^2-a^2}$

③ Gamma f^n

$n! = \Gamma(n+1) = n\Gamma(n)$

$\Gamma(1/2) = \sqrt{\pi}$

④ I^{st} Shifting Property

$L[f(x)] = F(p)$

$\Rightarrow L[e^{ax} f(x)] = F(p-a)$

⑤ Differentiation of LT

$L[f(x)] = F(p)$

$\Rightarrow L[x f(x)] = \frac{-d[F(p)]}{dp}$

$L[x^n f(x)] = (-1)^n \frac{d^n}{dp^n}[F(p)]$

⑥ Integration of LT

$L[f(x)] = F(p)$

$\Rightarrow L\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)\,dp$

⑦ LT of Derivatives

$L[f(x)] = F(p)$

$L[f'(x)] = pL[f(x)] - f(0)$

$L[f''(x)] = p^2 L[f(x)] - pf(0) - f'(0)$

$\therefore L[f^n(x)] = p^n L[f(x)] - p^{n-1} f(0) - p^{n-2} f'(0) \ldots\ldots - f^{n-1}(0)$

⑧ LT of Integrals

$L[f(x)] = F(p)$

$L\left[\int_0^x f(x)\,dx\right] = \dfrac{F(p)}{p}$

$L\left[\underbrace{\int_0^x \int_0^x \cdots \int_0^x}_{n \text{ times}} f(x)\,dx\right] = \dfrac{F(p)}{p^n}$

⑨ Laplace Inverse Transformation

- $L^{-1}\left(\frac{1}{p}\right) = 1$
- $L^{-1}\left(\frac{1}{p^n}\right) = \dfrac{x^{n-1}}{(n-1)!}$
- $L^{-1}\left[\frac{1}{p-a}\right] = e^{ax}$
- $L^{-1}\left[\frac{a}{p^2+a^2}\right] = \sin ax$
- $L^{-1}\left[\frac{p}{p^2+a^2}\right] = \cos ax$
- $L^{-1}\left[\frac{a}{p^2-a^2}\right] = \sinh ax$
- $L^{-1}\left[\frac{p}{p^2-a^2}\right] = \cosh ax$

⑩ $L[f(x)] = F(p)$

$L^{-1}[F(p)] = f(x)$

$\Rightarrow L^{-1}[F(p-a)] = e^{ax} f(x)$

⑪ $L^{-1}[F(p)] = f(x)$

$\Rightarrow L^{-1}[F^n(p)] = (-1)^n x^n f(x)$

⑫ $L^{-1}[F(p)] = f(x)$

$\Rightarrow L^{-1}\left[\int_p^\infty F(p)\,dp\right] = \dfrac{f(x)}{x}$

13. Convolution Theorem

$$L[(f*g)(x)] = L[f(x)].L[g(x)]$$
$$= F(p).G(p)$$

where $F(p) = L[f(x)]$
$G(p) = L[g(x)]$

$$L^{-1}[F(p).G(p)] = (f*g)(x)$$

$f(x) = L^{-1}[F(p)]$
$g(x) = L^{-1}[G(p)]$

$$(f*g)(x) = \int_0^x f(t)\, g(x-t)\, dt$$

14. 2^{nd} Shifting Property

$$L[f(x)\, u(x-a)] = e^{-ap} L[f(x+a)]$$

15. Piecewise to Unit Step function

$$g(x) = \begin{cases} g_1(x), & 0 \le x < a \\ g_2(x), & \text{Otherwise} \end{cases}$$

$$g(x) = g_1(x) + [g_2(x) - g_1(x)]\, u(x-a)$$

16. Inverse of 2^{nd} Shifting Property

$$L^{-1}[e^{-ap} F(p)] = f(x-a)\, u(x-a)$$

where $f(x) = L^{-1}[F(p)]$

17. Dirac Delta f^n

$$L[\delta(t-a)] = e^{-ap}$$

18. Existence of LT

- piecewise continuous f^n
- exponential order

i.e. $|f(x)| \le Me^{cx} \ \forall\ x > x_1,\ M > 0$

FOURIER SERIES

① Period of $f(ax)$ is p/a
Period of $f(x/a)$ is ap } if Period of $f(x)$ is 'p'

② FS Expansion on $[-l, l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

③ $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd f^n i.e. $f(-x) = -f(x)$

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even f^n i.e. $f(-x) = f(x)$

④ FS of odd f^n

$a_0 = 0 = a_n$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

FS of even f^n

$b_n = 0$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_{0}^{l} f(x)\,dx$$

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

⑤ Convergence of FS

- Series converges to $f(x)$ at pt. of continuity.
- At pt. of discontinuity, series converges to $\frac{1}{2}[LHL + RHL]$ at that pt.

⑥ Fourier Half-Range Series

- Fourier Cosine Series

$f(x)$ on $[0, l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

- Fourier Sine Series

$f(x)$ on $[0, l]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

⑦ 1-D Heat Equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad -①$$

3 Condⁿ ⟶ 2 boundary condⁿ

$u(0,t) = g_1(t)$ Say $= 0$

$u(l,t) = g_2(t) = 0$

↓

1 initial condⁿ

$u(x,0) = f(x)$

Let, $u(x,t) = X(x)\,T(t) \quad -②$

eqⁿ ② in eqⁿ ①

$XT' = kX''T$

$$\frac{X''}{X} = \frac{T'}{kT} = \omega \quad -③$$

$$X'' - \omega X = 0 \quad - \text{(4a)}$$

$$T' - \omega k T = 0 \quad - \text{(4b)}$$

from boundary cond^n

$$u(0,t) = X(0)T(t) = g_1(t) = 0 \text{ say}$$

$$u(l,t) = X(l)T(t) = g_2(t) = 0$$

AE: $m^2 - \omega = 0$

$$m = \pm\sqrt{\omega}$$

Case i : $\omega = 0$, $m = 0, 0$ (real & equal roots)

$$X = (c_1 + x c_2)e^{0x} = c_1 + x c_2$$

$$X(0) = 0 \Rightarrow c_1 = 0$$

$$X(l) = 0 \Rightarrow c_1 + l c_2 = 0 \Rightarrow c_2 = 0$$

$\therefore$ trivial sol^n as $X = 0$

Case ii : $\omega > 0$, $m = \pm\sqrt{\omega}$ (real & distinct roots)

$$X = c_1 e^{\sqrt{\omega}x} + c_2 e^{-\sqrt{\omega}x}$$

$$X(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$X(l) = 0 \Rightarrow c_1 e^{\sqrt{\omega}l} + c_2 e^{-\sqrt{\omega}l} = 0$$

$$\Rightarrow c_1 e^{2\sqrt{\omega}l} + c_2 = 0$$

$$\Rightarrow c_1(e^{2\sqrt{\omega}l} - 1) = 0$$

$$\therefore c_1 = 0 = c_2$$

$\therefore$ trivial sol^n as $X = 0$

Case iii : $\omega < 0$, $m = \pm\sqrt{\omega}$

put $\omega = -\lambda$, $\lambda > 0$

$$m = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i$$

$$X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$X(0) = 0 \Rightarrow c_1 = 0$$

$$X(l) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda}l) = 0$$

$$\sin(\sqrt{\lambda}l) = \sin(n\pi)$$

$$\sqrt{\lambda} = \frac{n\pi}{l}$$

$$\lambda = \frac{n^2\pi^2}{l^2}$$

$X = c_2 \sin\left(\frac{n\pi x}{l}\right)$

$X_n = C_n \sin\left(\frac{n\pi x}{l}\right)$

lly,

$T' = \omega k T$

put $\omega = -\lambda, \lambda > 0, \lambda = \frac{n^2\pi^2}{l^2}$

$\frac{dT}{T} = \omega k \, dt$

$\ln T = \omega k t = -\lambda k t + c_3$

$T = e^{\omega k t + c_3} = e^{-\lambda k t + c_3}$

$T = B e^{-\lambda k t}$

$T_n = B_n e^{-\frac{n^2\pi^2 k t}{l^2}}$

$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{l^2} k t}$

or $u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2}{l^2} k t}$

from initial cond^n

$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$

$a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

(B) 1-D Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = k \frac{\partial^2 u}{\partial x^2} \quad - (1)$$

4 Cond^n $\longrightarrow$ 2 initial cond^n

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

$\downarrow$

2 ~~initial~~ boundary cond^n

$u(0,t) = 0$

$u(l,t) = 0$

Let $u(x,t) = X(x)T(t)$ — ②

eqⁿ ② in eqⁿ ①

$$T''X = kX''T$$

$$\frac{X''}{X} = \frac{T''}{kT} = \omega \quad - ③$$

$X'' - \omega X$ — (4a)

$T'' - \omega kT$ — (4b)

from boundary condⁿ

$u(0,t) = 0 = X(0)T(t) \Rightarrow X(0) = 0$

$u(l,t) = 0 = X(l)T(t) \Rightarrow X(l) = 0$

Case i $\omega = 0 \rightarrow$ trivial solⁿ, $X = 0$

Case ii $\omega > 0 \rightarrow$ trivial solⁿ, $X = 0$

Case iii $\omega < 0$, put $\omega = -\lambda$, $\lambda > 0$

AE: $m^2 + \lambda = 0$

$m = \pm\sqrt{\lambda}\, i$

$$X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(l) = 0 \Rightarrow c_2 \sin(\sqrt{\lambda}l) = 0$

$\sqrt{\lambda}l = n\pi$

$\sqrt{\lambda} = \frac{n\pi}{l}$

$$\therefore X_n = C_n \sin\left(\frac{n\pi x}{l}\right)$$

$\frac{T''}{kT} = \omega = -\lambda$

$\frac{T''}{T} + \lambda kT = 0$

AE: $m^2 + \lambda k = 0$

$\lambda = \pm\sqrt{\lambda k}\, i$

$\lambda = \frac{n^2\pi^2}{l^2}$

$$\therefore\ T_n = A_n \cos\left(\frac{n\pi}{\ell}\sqrt{k}\,t\right) + b_n \sin\left(\frac{n\pi}{\ell}\sqrt{k}\,t\right)$$

$$u_n(x,t) = C_n \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \cos\left(\frac{n\pi}{\ell}\sqrt{k}\,t\right) + b_n \sin\left(\frac{n\pi}{\ell}\sqrt{k}\,t\right)\right]$$

$$\text{or}\quad u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi}{\ell}\sqrt{k}\,t\right) + B_n \sin\left(\frac{n\pi}{\ell}\sqrt{k}\,t\right)\right]\sin\left(\frac{n\pi x}{\ell}\right)$$

use initial condⁿ to find A_n & B_n

- $u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$

$$A_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

- $\frac{\partial u}{\partial t}(x,0) = g(x)$

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi}{\ell}\sqrt{k}\,\sin\left(\frac{n\pi x}{\ell}\right)$$

$$B_n = \frac{2\cancel{\ell}}{\cancel{\ell}\, n\pi\sqrt{k}}\int_0^{\ell} g(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

LINEAR ALGEBRA

① Row-Reduced Echelon Form

- 1^{st} non-zero element of each non-zero row is 1.
- if a column contains first non-zero entry of any row, every other element in that column is zero.
- zero-rows occur below all the non-zero rows.

② Rank of Matrix

no. of non-zero rows in RREF

If det. of $A_{n\times n} \neq 0$, then Rank of $A = n$

③ System of Linear Equations

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Augmented Matrix

$$[A \,\vdots\, B]$$

④

Non-homogeneous System $(AX = B)$

- Consistent — if $\text{Rank}(A) = \text{Rank}([A \,\vdots\, B])$
  - Unique Solution if $\text{Rank}(A) = n$
  - Many Infinite solutions if $\text{Rank}(A) < n$
- Inconsistent — if $\text{Rank}(A) \neq \text{Rank}([A \,\vdots\, B])$
  - No Solution

★ n = no. of unknowns

⑤ Homogeneous System ($AX = 0$)

Consistent (always)

- if $\text{Rank}(A) = n$ → Unique Solution
- if $\text{Rank}(A) < n$ → Many infinite solutions

No. of free variables = $n - \text{Rank}(A)$

⑥ Eigen Values & Eigen Vectors

$AX = \lambda X$ → eigen vector of A associated with eigen value

λ ↓ eigen values of A

$|A - \lambda I| = 0$

⑦ Vector Space

Non-empty set V is Vector Space is it satisfies –

- vector addition
  - $u, v \in V$
  - $u + v \in V$
  - commutative
  - associative
  - identity
  - inverse
- scalar multiplication
  - $\alpha \in \mathbb{R}$
  - $u \in V$
  - $\alpha u \in V$
  - distributive
  - associative
  - identity
  - inverse

⑧ Subspace

Non-empty subset S of a vector space V if it follows –
vector addition & scalar multiplication

$u, v \in S$

$\alpha, \beta \in \mathbb{R}$

$\Rightarrow \alpha u + \beta v \in S$

9 Span of a Set

Linear Combination

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

$$u_1, u_2, \cdots u_n \in V$$

Span of a subset S of vector space V is set of all finite LC of S.

$$[S] = \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \; ; \; u_i \in S \; ; \; \alpha_i \in \mathbb{R}\}$$

10

$$S = \{u_1, u_2, \cdots u_n\}$$

if $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$

where atleast one $\alpha_i \neq 0$

then Linearly Dependent

otherwise Linearly Independent

if we get $AX = 0$

then $|A| \neq 0 \Rightarrow LI$

$|A| = 0 \Rightarrow LD$

11 Basis

Subset B of vector space V is a basis of V if -

- B is LI
- B spans V i.e. $V = [B]$

12 Dimension

No. of elements in the basis of a vector space is dimension of VS.

* If B contains n elements in VS of dimension n, then if B is LI, it is a Basis.
(no need to check spanning)

If VS has a basis of n elements, then every set in V with more than n elements is LD.

If U & W are two subspaces of VS V, then -

- $U \cap W$ is also a subspace of V
- $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

(13) Linear Transformation

Let U & V be vector spaces.

$T: U \to V$ is called a linear mapping if -

- $T(u_1 + u_2) = T(u_1) + T(u_2)$, $u_1, u_2 \in U$
- $T(\alpha u) = \alpha T(u)$, $\alpha \in \mathbb{R}, u \in U$