OSCILLATIONS & WAVES

① Hooke's Law

$$F = -kx$$

$$ma = -kx$$

$$a = -\frac{k}{m}x$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

② $x(t) = A\cos(\omega t + \phi)$

$v(t) = -\omega A \sin(\omega t + \phi)$

$v_{max} = \omega A$

$a(t) = -\omega^2 A\cos(\omega t + \phi)$

$a_{max} = \omega^2 A$

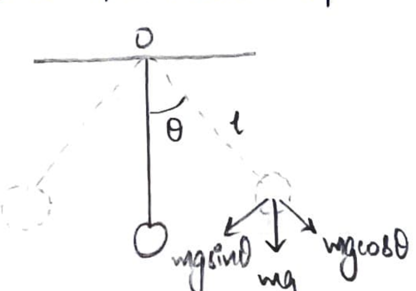
$T = 2\pi\sqrt{\frac{m}{k}}$, $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$KE = \frac{1}{2}k(A^2 - x^2)$

$PE = \frac{1}{2}kx^2$

$TE = \frac{1}{2}kA^2$

③ Simple Pendulum



$v = r\omega = l\omega$

$v = \frac{l\,d\theta}{dt}$

$a = \frac{dv}{dt} = \frac{l\,d^2\theta}{dt^2}$

$F = ma = ml\frac{d^2\theta}{dt^2}$

also $F = -mg\sin\theta$

$$-\cancel{m}g\sin\theta = \cancel{m}l\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

$$\sin\theta \simeq \theta$$

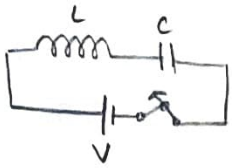
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\theta = \theta_0\cos(\omega t + \phi)$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

4 LC oscillator



$$V_L + V_C = 0$$

$$L\frac{dI}{dt} + \frac{Q}{C} = 0$$

$$L\frac{d}{dt}\left(\frac{dQ}{dt}\right) + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$Q = Q_0\cos(\omega t + \phi)$$

$$U_L = \frac{1}{2}LI_{max}^2$$

$$= \frac{1}{2}L(Q_0\omega)^2$$

$$= \frac{1}{2}\cancel{L}Q_0^2\frac{1}{\cancel{L}C}$$

$$= \frac{Q_0^2}{2C}$$

$$U_L = U_C$$

- $t=0$, $I=0$, $Q=Q_{max}$ → expanded string-mass (Capacitor fully charged)
- $t=T/4$, $I=I_{max}$, $Q=0$ → equilibrium (Conductor fully charged)
- $t=T/2$, $I=0$, Q_{max} (opp. charges) → compressed string-mass

Damped Mechanical Oscillator

$F_x = -kx$

$F_R = -R\dfrac{dx}{dt}$ damping coefficient

$F = F_x + F_R$

$$m\frac{d^2x}{dt^2} = -kx - R\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + \frac{R}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \omega^2 x = 0$$

where $K = \dfrac{R}{2m}$, $\omega^2 = \dfrac{k}{m}$

damping factor

Case i Overdamped

$K^2 > \omega^2$

or $\dfrac{R^2}{4m^2} > \dfrac{k}{m}$

Case ii Critically damped

$K^2 = \omega^2$

or $\dfrac{R^2}{4m^2} = \dfrac{k}{m}$

Case iii Underdamped

$K^2 < \omega^2$

$\dfrac{R^2}{4m^2} < \dfrac{k}{m}$

Solⁿ → $\cancel{x} = Ce^{-Kt}\cos(\beta t + \phi)$

$\beta = \sqrt{\omega^2 - K^2}$

$= \sqrt{\dfrac{k}{m} - \dfrac{R^2}{4m^2}}$

ogarithmic Decrement

$$\delta = \ln \frac{x_n}{x_{n+1}}$$

$$x_n = c e^{-\frac{R}{2m}t} . \cos(\omega t + \phi)$$

$$x_{n+1} = c e^{-\frac{R}{2m}(t+T)} \cos(\omega t + \phi)$$

$$\delta = \ln \left(e^{\frac{R}{2m}T} \right) = \frac{R}{2m} T$$

⑦ Relaxation Time

The time (τ_a) during which the energy of an oscillatory motion decays to 1/e of its initial value.

$$\cancel{c} e^{-1} = \cancel{c} e^{-\frac{R}{2m}\tau}$$

$$e^{\frac{R}{2m}\tau} = e$$

$$\tau_a = \frac{2m}{R}$$

$$\delta = \frac{R}{2m} T = \frac{T}{\tau_a}$$

⑧ Quality Factor

Less the losses (due to damping), more the quality

$$Q = \frac{\text{Energy Stored}}{\text{Energy loss per cycle}} \times 2\pi$$

$$Q = 2\pi \left(\frac{E}{-dE} \right)$$

$$E = E_0 e^{-\frac{R}{2m}t}$$

$$dE = \frac{-R}{2m} E_0 e^{-\frac{R}{2m}t} dt = \frac{-R}{2m} E_0 e^{-\frac{R}{2m}t} T$$

$$Q = 2\pi . \frac{2m}{RT} = \frac{2m}{R} \omega$$

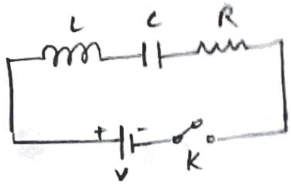
-recal System

$-V_c + V_R = 0$

$-\frac{dI}{dt} + \frac{q}{C} + IR = 0$

$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{...}$

(9) Electrical System



$$V_L + V_C + V_R = 0$$

$$L\frac{dI}{dt} + \frac{Q}{C} + IR = 0$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$

$$\frac{d^2Q}{dt^2} + 2K\frac{dQ}{dt} + \omega^2 Q = 0$$

where $K = \frac{R}{2L}$, $\omega^2 = \frac{1}{LC}$

$$Q = Q_0 e^{-\frac{R}{2L}t}\cos(\beta t + \phi)$$

$$\beta = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$T = \frac{2\pi}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}$$

There exists a critical resistance value above which no oscillations occur $\Rightarrow \beta = 0$

$$\therefore R_c = \sqrt{\frac{4L}{C}}$$

$R = R_c \Rightarrow$ critically damped

$R > R_c \Rightarrow$ overdamped

$$\delta = \frac{R}{2L}T$$

$$\tau_a = \frac{2L}{R}$$

ACOUSTICS

① Infrasonic – below 20 Hz

Audible – 20 to 20,000 Hz

Ultrasonic – above 20,000 Hz

② Loudness (in decibels) $= 10 \log_{10} \frac{\text{power at o/p}}{\text{power at i/p}}$

③ Sabine's Formula

$$T_s = \frac{KV}{\Sigma S\bar{\alpha}} \quad ; \quad K = 0.05 \text{ (in ft)}, \quad K = 0.162 \text{ (in m)}$$

$\bar{\alpha} \rightarrow$ avg. absorption coefficient

$V \rightarrow$ volume of hall

$S \rightarrow$ surface area

$\Sigma S\bar{\alpha} \rightarrow$ absorbing power

④ Eyring's Formula

$$T_e = \frac{-KV}{\Sigma S \ln(1-\bar{\alpha})} \quad ; \quad K = 0.05 \text{ (in ft)}, \quad K = 0.162 \text{ (in m)}$$

⑤ Measurement of Absorption Coefficient

$$\alpha = \frac{KV}{S}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

or $\alpha = \frac{4V(\ln I_m - \ln I_m')}{Sc(T_1 - T_2)}$

$$I = I_0 e^{-Mt}$$

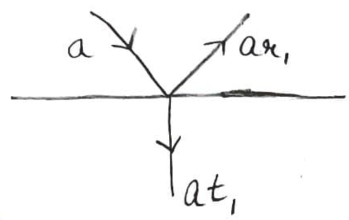
$$M = \frac{\alpha S v}{4V}$$

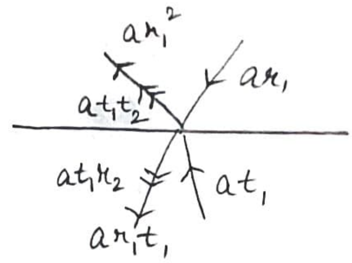
ULTRASONIC WAVES

① $\nu = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$

INTERFERENCE

① Stoke's Law





$$ar_1^2 + at_1t_2 = a \quad — ①$$

$$r_1^2 + t_1t_2 = 1$$

$$t_1t_2 = 1 - r^2$$

$$t_1t_2 = t_1^2$$

$$t_1 = t_2$$

$\because r_1^2 + t_1^2 = 1$

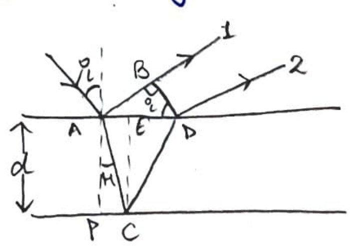
$r_2^2 + t_2^2 = 1$

$$\cancel{a}\cancel{t_1}r_2 + \cancel{a}r_1\cancel{t_1} = 0 \quad - ②$$

$$r_1 = -r_2$$

② Optical path = $\mu \times$ Geometrical path

③ Interference in II film due to reflected light



Optical path difference $= \mu(AC + CD) - AB$

$$\Rightarrow \frac{2\mu d}{\cos r} - 2\mu d \sin r \tan r$$

$$\Rightarrow 2\mu d\left(\frac{1}{\cos r} - \frac{\sin^2 r}{\cos r}\right)$$

$$\Rightarrow 2\mu d \frac{\cos^2 r}{\cos r}$$

$$= 2\mu d \cos r$$

$\because AP = AC\cos r$

$AC = \dfrac{d}{\cos r}$

$AB = AD \sin i$

$= 2AE \sin i$

$= 2d \tan r \sin i \mu$

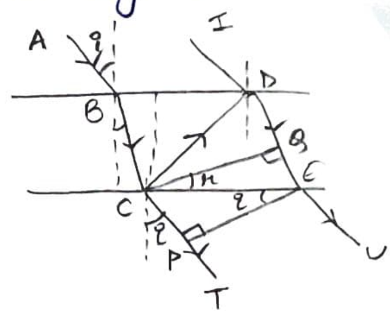
Condⁿ for maxima $\rightarrow 2\mu d\cos r + \lambda/2 = n\lambda$

$$2\mu d\cos r = (2n-1)\frac{\lambda}{2}$$

Condⁿ for minima $\rightarrow 2\mu d\cos r + \lambda/2 = (2n+1)\lambda/2$

$$2\mu d\cos r = n\lambda$$

④ Interference in ‖ film due to transmitted light



Optical path difference $= \mu(CD+DE) - CP$

$= \mu(CD+DE-QE)$

$= \mu(CD+DQ+\cancel{QE}-\cancel{QE})$

$= \mu(ID+DQ)$

$= \mu(IQ)$

$= 2\mu d\cos r$

$\because \mu = \dfrac{\sin i}{\sin r}$

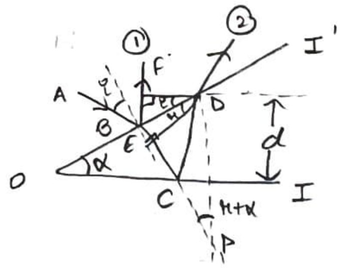
$= \dfrac{CP/\cancel{CE}}{QE/\cancel{CE}}$

$CP = \mu QE$

Cond^n for maxima $\rightarrow 2\mu d\cos r = n\lambda$

Cond^n for minima $\rightarrow 2\mu d\cos r = (2n+1)\dfrac{\lambda}{2}$

⑤ Wedge Film Interference



Optical path difference $= \mu(BC+CD) - BF$

$= \mu(BE+EC+CD) - BF$

$= \mu(\cancel{BE}+EC+CD) - \mu\cancel{BE}$

$= \mu(EC+CP)$

$= \mu EP$

$= \mu DP\cos(r+\alpha)$

$= 2\mu d\cos(r+\alpha)$

$\because \mu = \dfrac{\sin i}{\sin r}$

$= \dfrac{BF/\cancel{BD}}{BE/\cancel{BD}}$

$BF = \mu BE$

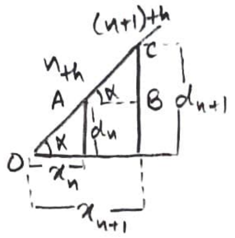
Cond^n for maxima $\rightarrow 2\mu d\cos(r+\alpha) + \lambda/2 = n\lambda$

$2\mu d\cos(r+\alpha) = (2n-1)\dfrac{\lambda}{2}$

Cond^n for minima $\rightarrow 2\mu d\cos(r+\alpha) + \lambda/2 = (2n+1)\dfrac{\lambda}{2}$

$2\mu d\cos(r+\alpha) = n\lambda$

⑥ Fringe Width



$\beta \Rightarrow x_{n+1} - x_n$

$\Rightarrow \tan\alpha = \dfrac{BC}{AB} = \dfrac{d_{n+1} - d_n}{x_{n+1} - x_n}$

$\beta = x_{n+1} - x_n = \dfrac{d_{n+1} - d_n}{\tan\alpha}$

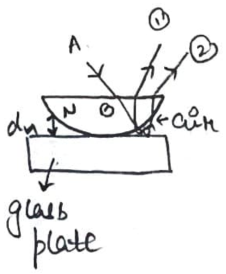
at A, $2\mu\cos(r+\alpha)\, d_n = n\lambda$

at C, $2\mu\cos(r+\alpha)\, d_{n+1} = (n+1)\lambda$

$\therefore\ d_{n+1} - d_n = \dfrac{\lambda}{2\mu\cos(r+\alpha)}$

$\therefore\ \beta = \dfrac{\lambda}{2\mu\cos(r+\alpha)\tan\alpha}$

⑦ Newton's Ring



Maxima (constructive)

$\rightarrow 2\mu d\cos(r+\theta) + \dfrac{\lambda}{2} = n\lambda$

Minima (Destructive)

$\rightarrow 2\mu d\cos(r+\theta) + \dfrac{\lambda}{2} = \dfrac{(2n+1)\lambda}{2}$

for N to be dark →

$2\mu d_n = n\lambda$ $\qquad \because r^2 = 2dR$

$\cancel{2}\mu \dfrac{r^2}{\cancel{2}R} = n\lambda$

$r^2 = \dfrac{n\lambda R}{\mu}$

for N to be bright →

$2\mu d_n = (2n-1)\lambda/2$

$r^2 = \dfrac{(2n-1)R\lambda}{2\mu}$

$D_n = 2r_n$

Diameter of dark ring →

$$D_n^2 = \frac{4n\lambda R}{\mu}$$

Diameter of bright ring →

$$D_n^2 = \frac{2(2n-1)R\lambda}{\mu}$$

⑧ Measurement of λ of light

d of n_{th} ring → $D_n^2 = \dfrac{4n\lambda R}{\mu}$

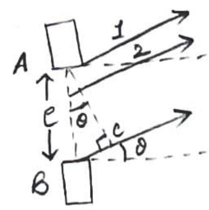
d of $(n+p)_{th}$ ring → $D_{n+p}^2 = \dfrac{4(n+p)\lambda R}{\mu}$

$$D_{n+p}^2 - D_n^2 = \frac{4p\lambda R}{\mu}$$

$$\therefore \lambda = \frac{(D_{n+p}^2 - D_n^2)\mu}{4pR}$$

DIFFRACTION

① Fraunhofer diffraction at single slit

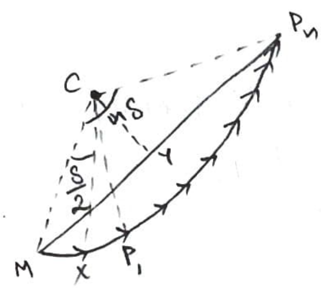


path difference $= BC = e\sin\theta$ → slit width

phase difference, $\phi = \frac{2\pi}{\lambda} e\sin\theta$

phase difference b/w 2 consecutive waves, $\delta = \frac{1}{n} \cdot \frac{2\pi}{\lambda} e\sin\theta$ → slit divided into 'n' equal parts

② Resultant amplitude on the screen



$\frac{MX}{MC} = \sin \delta/2$ → phase difference

$MX = \frac{1}{2} MP_1 = \frac{a}{2}$ → amplitude of secondary light ray

$\frac{a}{2} = MC \sin\frac{\delta}{2}$

$\frac{MY}{MC} = \sin n\delta/2$

$MY = \frac{1}{2} MP_n = \frac{R_\theta}{2}$ → resultant of 'n' waves

$\frac{R_\theta}{2} = MC \sin\frac{n\delta}{2}$

$$\frac{\frac{R_\theta}{\cancel{2}}}{\frac{a}{\cancel{2}}} = \frac{\sin\frac{n\delta}{2}}{\sin\frac{\delta}{2}}$$

$$R_\theta = \frac{a\sin(n\delta/2)}{\sin(\delta/2)}$$

$$R_\theta = \frac{a\sin\alpha}{\sin(\alpha/n)} \rightarrow \alpha = \frac{n\delta}{2} = \frac{\pi e\sin\theta}{\lambda}$$

n is very large

$$\therefore R_\theta = \frac{a\sin\alpha}{\alpha/n} = \frac{an\sin\alpha}{\alpha} = \frac{A_0\sin\alpha}{\alpha}$$

$A_0 = na$ → amplitude of incident light beam

$I \propto R^2$

$$I = I_0 \left[\frac{\sin\alpha}{\alpha}\right]^2$$

(3) Intensity Distribution

$$\frac{dI}{d\alpha} = 0$$

$$2\alpha\sin\alpha(\alpha\cos\alpha - \sin\alpha) = 0$$

Case i

$$\sin\alpha = 0$$

$$\alpha = \pm m\pi$$

at $m = 0 \Rightarrow \alpha = 0$

$$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1$$

$\therefore$ Central principle maxima at $\alpha = 0°$ (or $\theta = 0°$)

Minimum Intensity,

$$e\sin\theta = \pm m\lambda$$

Case ii

$$\alpha\cos\alpha - \sin\alpha = 0$$

$$\alpha = \tan\alpha$$

$$\alpha \simeq \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ---$$

Exactly, $\alpha = \pm 1.43\pi, \pm 2.46\pi, \pm 3.47\pi ----$

$$e\sin\theta_m = \pm(m + 1/2)\lambda$$

Secondary Maxima

(4) Fringe Width (z)

$$\tan\theta = \theta = \frac{z}{f}$$

m^{th} minima $\Rightarrow z_{min} = \pm\dfrac{mf\lambda}{e}$

m^{th} maxima $\Rightarrow z_{max} = \pm\dfrac{(m+1/2)f\lambda}{e}$

⑤ Fraunhofer Diffraction at Double Slit

$$\delta = \frac{2\pi}{\lambda}(e+d)\sin\theta$$

slit width

separation of slits

$$I = 4I_0 \underbrace{\left(\frac{\sin\alpha}{\alpha}\right)^2}_{\text{due to diffraction}} \cos^2\beta \rightarrow \text{contribution from interference}$$

$$\alpha = \frac{\pi}{\lambda} e\sin\theta$$

$$\beta = \frac{\pi(e+d)\sin\theta}{\lambda} = \frac{\delta}{2}$$

Maxima,

$$\cos^2\beta = 1$$

$$\beta = \pm n\pi$$

$$(e+d)\sin\theta_n = \pm n\lambda$$

Minima,

$$\cos^2\beta = 0$$

$$\beta = \pm(2n+1)\pi/2$$

$$(e+d)\sin\theta_n = \pm(2n+1)\lambda/2$$

Missing Fringes,

$m\lambda = e\sin\theta \rightarrow$ due to minima of diffraction

$n\lambda = (e+d)\sin\theta \rightarrow$ due to maxima of interference

$$\frac{n}{m} = \frac{e+d}{e}$$

⑥ Diffraction by n-slits

$$R^* = \frac{R\sin(N\delta/2)}{\sin(\delta/2)} = \frac{R\sin\left(\frac{N.2\beta}{2}\right)}{\sin\left(\frac{2\beta}{2}\right)} = \frac{R\sin(N\beta)}{\sin\beta}$$

$$I = I_0\left[\frac{\sin\alpha}{\alpha}\right]^2\left[\frac{\sin(N\beta)}{\sin\beta}\right]^2$$

Principle Maxima,

$$\sin\beta = 0$$

$$\beta = \pm n\pi$$

$$(e+d)\sin\theta = \pm n\lambda$$

$$I = I_0\left[\frac{\sin\alpha}{\alpha}\right]^2 N^2$$

Minima,

$$\sin N\beta = 0$$

$$N\beta = \pm m\pi$$

$$N(e+d)\sin\theta = \pm m\lambda$$

⑦ Dispersive Power

$$(e+d)\sin\theta = n\lambda$$

$$(e+d)\cos\theta \, d\theta = n \, d\lambda$$

$$\frac{d\theta}{d\lambda} = \frac{n}{(e+d)\cos\theta}$$

⑧ Resolving Power

$$\frac{\lambda}{d\lambda} = \underbrace{n}_{\text{no. of principle maxima}}\overbrace{N}^{\text{no. of slits}} = \frac{N(e+d)\sin\theta_n}{\lambda}$$

POLARIZATION

① Linear

Two orthogonal plane waves with same phase, but possibly different amplitudes.

- Circular

Two orthogonal plane waves with 90° phase difference and same amplitudes.

- Elliptical

Two orthogonal plane waves with 90° phase different and different amplitudes.

② Polarization by Reflection

$$\tan \alpha = \mu$$

$$\mu = \frac{\sin \alpha}{\sin \beta}$$

α → angle of polarization

β → angle of reflection

$$\cos \alpha = \sin \beta = \cos(90 - \beta)$$

$$\alpha + \beta = 90^\circ$$

Reflected & Refracted waves are ⊥ to each other.

Reflection Coefficient, $r = \frac{\mu - 1}{\mu + 1} \simeq 0.15 - 0.25$

③ Law of Malus

$$E = E_0 \cos \theta$$

(Amplitude)

$$I = I_0 \cos^2 \theta$$

(Intensity)

④ Phase Retardation Plates

$$t_{quarter}(\mu_o - \mu_e) = \lambda/4$$

$$t_{half}(\mu_o - \mu_e) = \lambda/2$$

$$t_{full}(\mu_o - \mu_e) = \lambda$$

$$\phi = \frac{2\pi}{\lambda}(\mu_o - \mu_e)t$$

⑤ Optical Activity

$$S = \frac{\theta}{LC}$$

S → specific rotation

L → length of substance

C → concentration of substance

LASER

① $\phi = \frac{1.27\lambda}{D}$

$\phi \rightarrow$ divergence angle

$D \rightarrow$ diameter

Size of spot = ϕ x Distance

② $L_\phi = c\,\tau_\phi$

$L_\phi \rightarrow$ coherence length

$\tau_\phi \rightarrow$ coherence time

$(\tau_c \text{ or}) \tau_\phi = \frac{1}{\Delta\nu}$

$L_\phi (\text{or } L_c) = \frac{c}{\Delta\nu}$

$L_c = \frac{\lambda^2}{\Delta\lambda}$

$t_c = \frac{L_c}{c}$

③ Probability of Emission

$P_{21} = A_{21} + B_{21} \cdot u(\nu)$

$\frac{B_{21}}{B_{12}} = 1$

$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}$

④ Boltzmann Distribution Law

$N_2 = N_1 e^{-\left(\frac{\Delta E}{kT}\right)}$

if $N_2 > N_1 \Rightarrow$ population inversion

(-ve state temp. or non. equilibrium state)

...UM MECHANICS

Wave - Particle Duality

- $E = h\nu \rightarrow$ Planck's hypothesis (wave nature)

  $E = mc^2 \rightarrow$ particle nature

  $h\nu = mc^2$

  $\frac{hc}{\lambda} = mc^2$

  $\lambda = \frac{h}{mc} = \frac{h}{p}$

- de-broglie wavelength of an accelerating electron

  $E = eV$

  $E = \frac{1}{2}mv^2 = \frac{p^2}{2m} = eV$

  $p = \sqrt{2meV}$

  $p = \sqrt{2mE}$

  $\therefore \lambda = \frac{h}{\sqrt{2meV}} = \frac{h}{\sqrt{2mE}}$

  $\lambda = \frac{1.228}{\sqrt{V}}$ nm   valid for slow-moving e^- only

- Relativistic Correction in λ

  $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ $m_0 \rightarrow$ rest mass of e^- $m > m_0$

  $mc^2 = K + U$

  $m_0c^2 = U$

  $\therefore K = mc^2 - m_0c^2 = (m - m_0)c^2 = eV$

  $m = m_0\left(1 + \frac{eV}{m_0c^2}\right)$

  $\lambda = \frac{h}{\sqrt{2m_0eV}}\left(1 + \frac{eV}{m_0c^2}\right)^{-1/2}$

  or $\lambda = \frac{1.228}{\sqrt{V}}\left(1 + \frac{eV}{m_0c^2}\right)^{-1/2}$ nm

Few Estimates:

1. de-Broglie λ of $\bar{e}$ moving with 1% velocity of light is 2.4 Å.
2. de-Broglie λ of an $\bar{e}$ moving in influence of 100V is 1.22 Å

② Heisenberg Uncertainty Principle

$$\Delta p_x \cdot \Delta x \geq \hbar \ (\text{or } \hbar/2)$$ true only for conjugate pairs

$$\Delta E \cdot \Delta t \geq \hbar$$

③ Eqⁿ of a wave ⇒ $y = A\sin\{\omega(t - x/v)\}$; $y = A\sin(\omega t - kx)$

Phase angle ⇒ $\phi = \omega(t - x/v)$; $\phi = \omega t - kx$

$$\frac{d\phi}{dt} = \omega\left(1 - \frac{1}{v}\frac{dx}{dt}\right) = 0$$

$$\frac{dx}{dt} = v = \frac{\omega}{k}$$

phase velocity

$$\therefore v_p = \frac{\omega}{k}$$

$$y = y_1 + y_2 = A\sin(\omega t - kx) + A\sin\{(\omega + d\omega)t - (k + dk)x\}$$

$$= 2A\cos\left(\frac{d\omega}{2}t - \frac{dk}{2}x\right)\sin\left\{\frac{2\omega + d\omega}{2}t - \frac{2k + dk}{2}x\right\}$$

$$y = 2A\cos\left(\frac{d\omega}{2}t - \frac{dk}{2}x\right)\sin(\omega t - kx)$$

group velocity

$$\therefore v_g = \frac{d\omega}{dk}$$

$$\omega = 2\pi\nu = \frac{2\pi E}{h} \quad ; \quad k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}$$

$$v_g = \frac{dE}{dp} = \frac{p}{m} = v_{particle} \quad \because E = \frac{p^2}{2m} + U$$

④ Postulate 1 : Wave Function

$$\text{Probability}, P = \int_{x_1}^{x_2} |\Psi(x,t)|^2 dx$$

- wave fⁿ must be single valued
- must vanish at ∞ (convergent fⁿ)
- derivative of the fⁿ must be continuous
- $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$ (normalization)

(5) Postulate 2 : Operator

$$\psi(x,t) = a e^{i(kx - \omega t)} \qquad \because k = \frac{2\pi}{\lambda} \; ; \; \lambda = \frac{h}{p}$$

$$= a e^{i(px - Et)/\hbar} \qquad \omega = 2\pi\nu \; ; \; \nu = \frac{E}{h}$$

$$\frac{\partial}{\partial x}\psi(x,t) = \frac{ip}{\hbar}\psi(x,t)$$

$$p\,\psi(x,t) = \left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x,t) \qquad \hat{p}_x \equiv \left(-i\hbar\frac{\partial}{\partial x}\right)$$

$\hookrightarrow$ momentum operator

$$\frac{\partial}{\partial t}\psi(x,t) = \frac{-iE}{\hbar}\psi(x,t)$$

$$E\,\psi(x,t) = \left(i\hbar\frac{\partial}{\partial t}\right)\psi(x,t) \qquad \hat{E} \equiv \left(i\hbar\frac{\partial}{\partial t}\right)$$

$\hookrightarrow$ energy operator

- Hamiltonian (total energy) operator

$$K = \frac{p_x^2}{2m}$$

$$\hat{K} \equiv \left(-\frac{\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}$$

$$H = K + U$$

$$\hat{H} \equiv \hat{K} + \hat{U} \equiv \left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2} + \hat{U}$$

(6) Postulate 3 : Measurement

$$\hat{O}\,\psi(x,t) = O.\,\psi(x,t)$$

operator

eigen f^n of operator

eigen value of operator

(7) Postulate 4 : Expectation value

$$\langle \hat{O} \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{O}\,\psi(x,t)\,dx = \langle \psi(x,t)|\hat{O}|\psi(x,t)\rangle$$ for normalized ψ

$$\langle \hat{O} \rangle = \frac{\int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{O}\,\psi(x,t)\,dx}{\int_{-\infty}^{\infty} \psi^*(x,t)\,\psi(x,t)\,dx} = \frac{\langle \psi(x,t)|\hat{O}|\psi(x,t)\rangle}{\langle \psi(x,t)|\psi(x,t)\rangle}$$ for non-normalized ψ

⑧ Postulate 5: Schrodinger Eqn

- Time Dependent

$$\hat{H}\,\psi(x,t) = \hat{E}\,\psi(x,t)$$

$$\boxed{\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + \hat{U}\,\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{dt}}$$

- Time Independent

$$\psi(x,t) = \phi(t)\,\psi(x)$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + \hat{U}\,\psi(x,t) = \frac{i\hbar\partial\psi(x,t)}{\partial t}$$

$$\frac{-\hbar^2}{2m}\phi(t)\frac{\partial^2\psi(x)}{\partial x^2} + \hat{U}\,\phi(t)\psi(x) = \frac{i\hbar\partial\phi(t)}{dt}\psi(x)$$

$$\frac{-\hbar^2}{2m}\cdot\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} + \hat{U} = \frac{i\hbar}{\phi(t)}\frac{\partial\phi(t)}{\partial t} = E \text{ (constant)}$$

- $$\frac{\partial^2\psi(x)}{\partial x^2} - \frac{2m}{\hbar^2}\hat{U}\,\psi(x) = \frac{-2m}{\hbar^2}E\,\psi(x)$$

$$\hat{U}\,\psi(x) = V(x)\,\psi(x)$$

$$\boxed{\frac{\partial^2\psi(x)}{\partial x^2} + \frac{2m\,(E-V)}{\hbar^2}\psi(x) = 0}$$

Time independent Schrodinger eqn

- $$\frac{1}{\phi(t)}d\phi(t) = -\frac{iE}{\hbar}dt$$

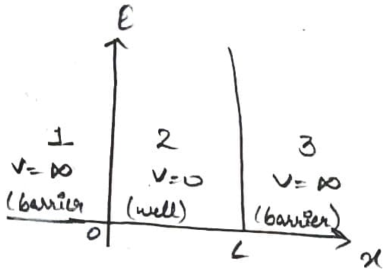
$$\ln(\phi(t)) = \frac{-iEt}{\hbar}$$

$$\boxed{\phi(t) = e^{-iEt/\hbar}}$$

Time evolution of wave fn

(4) Particle in a box (infinite square well potential)

$$V = \begin{cases} \infty , & x \leq 0 \\ 0 ; & 0 < x < L \\ \infty ; & x \geq L \end{cases}$$



$$\psi_1(x) = 0 = \psi_3(x)$$

$$\psi_2(x) \neq 0$$

for $\psi_2(x)$, $V = 0$

$$\frac{\partial^2 \psi_2(x)}{\partial x^2} + \frac{2mE}{\hbar^2}\psi_2(x) = 0$$

Let, $\dfrac{2mE}{\hbar^2} = k^2$

$$\frac{\partial^2 \psi_2(x)}{\partial x^2} + k^2 \psi_2(x) = 0$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 = \underbrace{\int_{-\infty}^{0} |\psi_1(x)|^2 dx}_{0} + \underbrace{\int_{0}^{L} |\psi_2(x)|^2 dx}_{1} + \underbrace{\int_{L}^{\infty} |\psi_3(x)|^2 dx}_{0} = 1$$

(normalization)

$$\psi_2(x) = A\sin kx + B\cos kx$$

$$\psi_2(0) = 0 = A \times 0 + B \times 1$$

$$\therefore B = 0$$

$$\psi_2(L) = 0 = A\sin kL = \sin n\pi$$

$$\therefore k = \frac{n\pi}{L}$$

$$\boxed{\psi_2(x) = A\sin\left(\frac{n\pi x}{L}\right)}$$

Normalization,

$$\int_0^L |\psi_2(x)|^2 dx = 1$$

$$\Rightarrow A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$\Rightarrow \frac{A^2}{2} \int_0^L 1 - \cos\left(\frac{2\pi n x}{L}\right) dx = 1$$

$$\Rightarrow \frac{A^2}{2} x \Big|_0^L \Rightarrow \frac{A^2 L}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{L}}$$

$$\therefore \boxed{\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)}$$

$$E = \frac{k^2\hbar^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

$$\boxed{E_n = \frac{\hbar^2\pi^2}{2mL^2} n^2}$$

- Expectation value

$$\hat{x} = x$$

$$\psi_n(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad \langle \hat{O} \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, \hat{O}\, \psi(x,t)\, dz$$

$$\langle \hat{x} \rangle = \int_0^L \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right]^* x \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right] dx$$

$$\langle \hat{x} \rangle = \frac{2}{L} \int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}$$

- Momentum of particle

① $$\hat{p}\, \psi_n(x,t) = -i\hbar \frac{d}{dx} \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right] \neq p_n \psi_n(x,t)$$

② $$\hat{p} \equiv -i\hbar \frac{\partial}{\partial x}$$

$$\langle \hat{p} \rangle = -\int_0^L \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right]^* \left(i\hbar \frac{\partial}{\partial x}\right) \left[\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right] dx$$

$$= -\frac{2}{L} i\hbar \frac{n\pi}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{-1}{L} i\hbar \frac{n\pi}{L} \int_0^L \sin\left(\frac{2n\pi x}{L}\right) dx$$

$$= 0$$

③ $$E_n = \left(\frac{\hbar^2 \pi^2}{2mL^2}\right) n^2$$

$$E = \frac{p^2}{2m}$$

$$p_n = \pm \sqrt{2mE_n} = \pm \frac{n\pi\hbar}{L}$$

$$\therefore \langle \hat{p} \rangle = 0$$

① can be modified

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) = \sqrt{\frac{2}{L}} \times \frac{e^{\left(\frac{i n\pi x}{L}\right)} - e^{-\left(\frac{i n \pi x}{L}\right)}}{2i}$$

$$\psi_n(x) = \psi_n^+(x) - \psi_n^-(x)$$

$$\hat{p}\psi_n^+(x) = -i\hbar \frac{\partial \psi_n^+(x)}{\partial x} = -i\hbar \frac{i n\pi}{L} \psi_n^+(x)$$

$$p^+ = \frac{n\hbar\pi}{L}$$

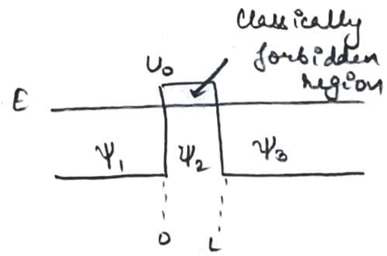
$$\hat{p}\psi_n^-(x) = -i\hbar \frac{\partial \psi_n^-(x)}{\partial x} = -i\hbar \frac{-i n\pi}{L} \psi_n^-(x)$$

$$p^- = \frac{-n\pi\hbar}{L}$$

$$\therefore \langle \hat{p} \rangle = 0$$

⑩ Potential Barrier

$$V = \begin{cases} 0 ; & x \le 0 \\ U_0 ; & 0 < x < L \\ 0 ; & x \ge L \end{cases}$$



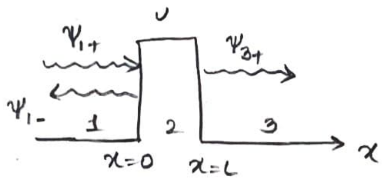
$$\frac{\partial^2 \psi_1(x)}{\partial x^2} + \frac{2mE}{\hbar^2}\psi_1(x) = 0$$

$$k_1^2 = \frac{2mE}{\hbar^2} \quad (\text{for } 1 \ \& \ 3)$$

$$\frac{\partial^2 \psi_2(x)}{\partial x^2} + \frac{2m(E-U_0)}{\hbar^2}\psi_2(x) = 0 \Rightarrow \frac{\partial^2 \psi_2(x)}{\partial x^2} = k_2^2 \psi_2(x)$$

$$k_2^2 = \frac{2m(U_0 - E)}{\hbar^2} \quad (\text{for } 2)$$

$$\frac{\partial^2 \psi_3(x)}{\partial x^2} + \frac{2mE}{\hbar^2}\psi_3(x) = 0$$



$$\psi_1(x) = Ae^{ik_1 x} + Be^{-ik_1 x} \quad (\psi_{1+} + \psi_{1-})$$

$$\psi_2(x) = \cancel{Ce^{k_2 x}} + De^{-k_2 x}$$

$$\psi_3(x) = Fe^{ik_1 x} + \cancel{Ge^{-ik_1 x}} \quad (\psi_{3+} + \cancel{\psi_{3-}})$$

Transmission Probability $\boxed{T \propto e^{-2k_2 L}}$