Derivation of $t(\theta)$ for Projectile Motion with Quadratic Drag

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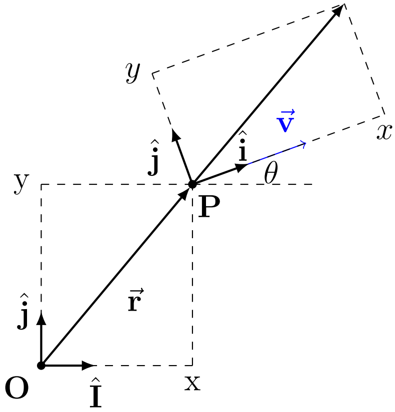
Coordinate System I



- $\mathbf{O}$ is the origin of the reference frame.
- $\hat{\mathbf{I}}$, $\hat{\mathbf{J}}$ are the unit vectors of the reference frame, aligned along the x-axis and y-axis, respectively.
- $\vec{\mathbf{r}}$ is the position vector of the projectile.
- $\mathbf{P}$ is the origin of the body-fixed frame.
- $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ are the unit vectors of the body-fixed frame, aligned along the x-axis and y-axis, respectively.
- The unit vector $\hat{\mathbf{i}}$ in the body-fixed frame is aligned with $\vec{\mathbf{v}}$.
- θ is the angle measured from $\hat{\mathbf{I}}$ to the velocity vector $\vec{\mathbf{v}}$.

Step-by-Step Derivation (Inertial Fixed Frame) I

Step 1: Define the Position Vector
The position vector of the projectile in the reference frame is given by:

$$\vec{\mathbf{r}} = \mathrm{x}\hat{\mathbf{I}} + \mathrm{y}\hat{\mathbf{J}}.$$

Step 2: Take the Time Derivative of the Position Vector
The time derivative of the position vector represents the velocity vector:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d\mathrm{x}}{dt}\hat{\mathbf{I}} + \frac{d\mathrm{y}}{dt}\hat{\mathbf{J}},$$

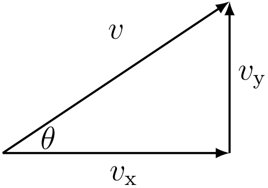
Step-by-Step Derivation (Inertial Fixed Frame) II

The velocity components are defined as:

$$v_{\mathrm{x}} \triangleq \frac{d\mathrm{x}}{dt}, \tag{1}$$

$$v_{\mathrm{y}} \triangleq \frac{d\mathrm{y}}{dt}, \tag{2}$$

$$v \triangleq \sqrt{(v_{\mathrm{x}})^2 + (v_{\mathrm{y}})^2}. \tag{3}$$



The components of the velocity vector are:

$$v_{\mathrm{x}} = v\cos\theta, \tag{4}$$

$$v_{\mathrm{y}} = v\sin\theta. \tag{5}$$

Step-by-Step Derivation (Inertial Fixed Frame) III

Step 3: Take the Time Derivative of the Velocity Vector
The time derivative of the velocity vector represents the acceleration vector:

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{dv_{\mathrm{x}}}{dt}\hat{\mathbf{I}} + \frac{dv_{\mathrm{y}}}{dt}\hat{\mathbf{J}}. \tag{6}$$

Determine unit vector of Body fixed frame I

Step 1: Determine $\hat{\mathrm{i}}$ by definition of body-fixed frame

Since the body-fixed frame defines $\hat{\mathbf{i}}$ as a unit vector aligned with $\vec{\mathbf{v}}$, we have:

$$\hat{\mathbf{i}} \triangleq \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|},$$

where

$$v \triangleq \|\vec{\mathbf{v}}\|.$$

Thus, we express $\hat{\mathbf{i}}$ as:

$$\hat{\mathbf{i}} \triangleq \frac{v_{\mathrm{x}}\hat{\mathbf{I}} + v_{\mathrm{y}}\hat{\mathbf{J}}}{v}. \tag{7}$$

Determine unit vector of Body fixed frame II

Step 2: Determine $\hat{\mathrm{j}}$ Using Orthogonality

Since $\hat{\mathbf{j}}$ is perpendicular to $\hat{\mathbf{i}}$, their dot product is zero:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0 \quad \text{(orthogonality condition)}, \tag{8}$$

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{J}} = 1 \quad \text{when } \theta = 0 \quad \text{(alignment at zero rotation)}. \tag{9}$$

Express $\hat{\mathbf{j}}$ in terms of $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$:

$$\hat{\mathbf{j}} = j_{\mathrm{x}}\hat{\mathbf{I}} + j_{\mathrm{y}}\hat{\mathbf{J}}.$$

Determine unit vector of Body fixed frame III

Using (8):

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \left(\frac{v_\mathrm{x}\hat{\mathbf{I}} + v_\mathrm{y}\hat{\mathbf{J}}}{v} \right) \cdot \left(j_\mathrm{x}\hat{\mathbf{I}} + j_\mathrm{y}\hat{\mathbf{J}} \right) = 0,$$
$$\Rightarrow \frac{j_\mathrm{x} v_\mathrm{x} + j_\mathrm{y} v_\mathrm{y}}{v} = 0.$$

Step 3: Solve for j_x and j_y
From the above equation:

$$v_\mathrm{x} j_\mathrm{x} + v_\mathrm{y} j_\mathrm{y} = 0,$$

Let:

$$j_\mathrm{x} = -v_\mathrm{y}, \quad j_\mathrm{y} = v_\mathrm{x}.$$

Determine unit vector of Body fixed frame IV

Therefore, $\hat{\mathbf{j}}$ becomes:

$$\hat{\mathbf{j}} = \left(\frac{-v_{\mathrm{y}}\hat{\mathbf{I}} + v_{\mathrm{x}}\hat{\mathbf{J}}}{v} \right).$$

Finally, using (9) to check if the derivation is correct:

$$\begin{aligned} \hat{\mathbf{j}} \cdot \hat{\mathbf{J}} &= 1 \\ \Rightarrow \left(\frac{-v_{\mathrm{y}}\hat{\mathbf{I}} + v_{\mathrm{x}}\hat{\mathbf{J}}}{v} \right) \cdot \hat{\mathbf{J}} &= 1 \\ \frac{v_{\mathrm{x}}}{v} &= 1, \end{aligned}$$

Determine unit vector of Body fixed frame V

and $v_{\mathrm{x}} = v$ when $\theta = 0$, so:

$$\frac{v}{v} = 1,$$

confirming that the derivation is correct.

Now we have defined $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in the inertial fixed frame.

Step-by-Step Derivation (Body Fixed Frame) I

Step 1: Deriving the Velocity in Body-Fixed Coordinates from the Position Vector

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}},$$

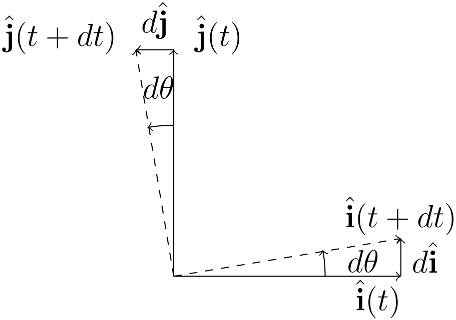
Take the time derivative:

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + x\frac{d\hat{\mathbf{i}}}{dt} + y\frac{d\hat{\mathbf{j}}}{dt}.$$

Step-by-Step Derivation (Body Fixed Frame) II

Time Change of Axes

Since the body-fixed frame is rotating, its axes change over time. The time derivatives of $\frac{d\hat{\mathbf{i}}}{dt}$ and $\frac{d\hat{\mathbf{j}}}{dt}$ are derived below:



$$d\hat{\mathbf{i}} = d\theta\hat{\mathbf{j}},$$

$$\Rightarrow \frac{d\hat{\mathbf{i}}}{dt} = \frac{d\theta}{dt}\hat{\mathbf{j}}, \tag{10}$$

$$d\hat{\mathbf{j}} = -d\theta\hat{\mathbf{i}},$$

$$\Rightarrow \frac{d\hat{\mathbf{j}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{i}}. \tag{11}$$

Step-by-Step Derivation (Body Fixed Frame) III

Express (12) and (13) in vector product form as:

$$\frac{d\hat{\mathbf{i}}}{dt} = \vec{\Omega} \times \hat{\mathbf{i}}, \tag{12}$$

$$\frac{d\hat{\mathbf{j}}}{dt} = \vec{\Omega} \times \hat{\mathbf{j}}, \tag{13}$$

where

$$\vec{\Omega} = \frac{d\theta}{dt}\hat{\mathbf{k}}.$$

The vector $\vec{\Omega}$ represents the angular velocity along the $\hat{\mathbf{k}}$-axis, and the cross products describe the change in $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ due to the rotation.

Step-by-Step Derivation (Body Fixed Frame) IV

Step 2: Substitute the Derivatives into the Velocity Expression Substitute equations (12) and (13) into the expression for $\frac{d\vec{\mathbf{r}}}{dt}$:

$$\vec{\mathbf{v}} = \left(\frac{dx}{dt} - y\frac{d\theta}{dt}\right)\hat{\mathbf{i}} + \left(\frac{dy}{dt} + x\frac{d\theta}{dt}\right)\hat{\mathbf{j}}.$$

Step 3: Body-Fixed Frame Assumptions Assuming that the projectile moves along $\hat{\mathbf{i}}$ in the body-fixed frame:

$$\frac{dx}{dt} - y\frac{d\theta}{dt} = v,$$
$$\frac{dy}{dt} + x\frac{d\theta}{dt} = 0.$$

Then we can get:

$$\vec{\mathbf{v}} = v\hat{\mathbf{i}}.$$

Step-by-Step Derivation (Body Fixed Frame) V

Step 4: Time Derivative of the Velocity Vector

$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{dv}{dt}\hat{\mathbf{i}} + v\frac{d\hat{\mathbf{i}}}{dt}.$$

Step 5: Express the Acceleration by substitute (12),

$$\vec{\mathbf{a}} = \frac{dv}{dt}\hat{\mathbf{i}} + v\frac{d\theta}{dt}\hat{\mathbf{j}}. \tag{14}$$

What We Need to Get the Equation of Motion

- To derive the equation of motion (EOM), we need to analyze the forces acting on the particle.
- In this condition, the particle is subject to gravity and air resistance.
- Since Newton's laws are only valid in an inertial frame of reference, we will use (6) and (14) for our analysis.

Equation of Motion in the Inertial Frame I

The general equation of motion can be written as:

$$m\vec{\mathbf{a}} = m\vec{\mathbf{g}} - R\hat{\mathbf{i}}. \tag{15}$$

where

- $m\vec{\mathbf{g}}$: Gravity force, where:
  - m: Mass of the projectile,
  - $\vec{\mathbf{g}}$: Gravity vector.

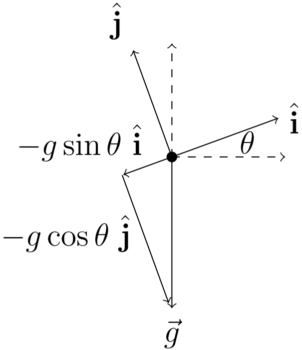
  Gravity Vector Components in the Inertial Fixed Frame:

  $$\vec{\mathbf{g}} = -g\hat{\mathbf{K}}$$

  where $\hat{\mathbf{K}}$ is the unit vector along the z-axis in the inertial frame.

Equation of Motion in the Inertial Frame II

Gravity Vector Components in the Body-Fixed Frame:
To analyze the motion in the body-fixed frame, we decompose the gravity vector into components along the body-fixed axes:



The gravity vector in the body-fixed frame is:

$$\vec{\mathbf{g}} = -g \sin\theta \hat{\mathbf{i}} - g \cos\theta \hat{\mathbf{j}}. \quad (16)$$

Equation of Motion in the Inertial Frame III

- R: Air resistance, $R = mgkv^2$.
- $k = \frac{\rho_a C_D S}{2mg}$: Proportionality factor, where:
  - ρ_a: Air density,
  - C_D: Drag coefficient,
  - S: Cross-sectional area.

Substituting into the Equation in the Inertial Fixed Frame

Substituting (6) and (7) into (15), we get:

$$m\left(\frac{dv_{\mathrm{x}}}{dt}\hat{\mathbf{I}}+\frac{dv_{\mathrm{y}}}{dt}\hat{\mathbf{j}}\right)=-mg\hat{\mathbf{j}}-mgkv^2\frac{v_{\mathrm{x}}\hat{\mathbf{I}}+v_{\mathrm{y}}\hat{\mathbf{j}}}{v},$$

By separating the $\hat{\mathbf{I}}$ and $\hat{\mathbf{j}}$ components, the equations become:

$$m\frac{dv_{\mathrm{x}}}{dt}=-mgkv^2\frac{v_{\mathrm{x}}}{v},$$
$$m\frac{dv_{\mathrm{y}}}{dt}=-mg-mgkv^2\frac{v_{\mathrm{y}}}{v}.$$

Rearranging the Equations in the Inertial Frame

Rearranging (4),(5) and substituting into the equations:

$$m\frac{d(v\cos\theta)}{dt} = -mgkv^2\cos\theta,$$
$$m\frac{d(v\sin\theta)}{dt} = -mg - mgkv^2\sin\theta.$$

Rearranging (3), (2), (4) and (5), we can express the velocity components as:

$$\frac{d\mathrm{x}}{dt} = v\cos\theta,$$
$$\frac{d\mathrm{z}}{dt} = v\sin\theta.$$

Differential Equations of Motion in the Inertial Frame

The complete differential equations of motion in the inertial frame are:

$$\frac{d(v\cos\theta)}{dt} = -gkv^2\cos\theta, \tag{17}$$

$$\frac{d(v\sin\theta)}{dt} = -g - gkv^2\sin\theta, \tag{18}$$

$$\frac{d\mathrm{x}}{dt} = v\cos\theta, \tag{19}$$

$$\frac{d\mathrm{z}}{dt} = v\sin\theta. \tag{20}$$

Substituting into the Equation in the Body-Fixed Frame

Substituting (14) and (16) into (15), we get:

$$m\left(\frac{dv}{dt}\hat{\mathbf{i}} + v\frac{d\theta}{dt}\hat{\mathbf{j}}\right) = -mg\sin\theta\hat{\mathbf{i}} - mg\cos\theta\hat{\mathbf{j}} - mgkv^2\hat{\mathbf{i}},$$

Separating into $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components, we get the following two differential equations:

$$\begin{aligned} m\frac{dv}{dt} &= -mg\sin\theta - mgkv^2, \\ mv\frac{d\theta}{dt} &= -mg\cos\theta. \end{aligned}$$

Differential Equations of Motion in the Body-Fixed Frame

The differential equations of motion in the body-fixed frame are:

$$\frac{dv}{dt} = -g\sin\theta - gkv^2, \tag{21}$$

$$\frac{d\theta}{dt} = -\frac{g\cos\theta}{v}. \tag{22}$$

Derivation Process

Deriving $t(\theta)$ from the Equation of Motion

To obtain an approximation analytical solution, we first derive $t(\theta)$.

Rearranging the equation (22), we get:

$$dt = -\frac{v(\theta)}{g\cos\theta}d\theta, \tag{23}$$

Deriving dv_{x} from the Equation of Motion

Rearranging (17), we get:

$$d(v(\theta)\cos\theta) = -gkv^2(\theta)\cos\theta \; dt.$$

Substitute dt from (23) and let $v_{\mathrm{x}} = v(\theta)\cos\theta$:

$$d(v(\theta)\cos\theta) = -gkv^2(\theta)\cos\theta \left(-\frac{v(\theta)}{g\cos\theta}d\theta\right),$$

$$\Rightarrow dv_{\mathrm{x}} = kv^3(\theta)d\theta. \tag{24}$$

Integration with v_x

Then we continue with (23):

$$t_{k+1} = t_k - \frac{1}{g} \int_{\theta_k}^{\theta_{k+1}} \frac{v(\theta)}{\cos^2 \theta} d\theta,$$

Substituting $v_\mathrm{x} = v(\theta) \cos \theta$ into the second term:

$$\frac{1}{g} \int_{\theta_k}^{\theta_{k+1}} \frac{v(\theta)}{\cos \theta} d\theta = \frac{1}{g} \int_{\theta_k}^{\theta_{k+1}} \frac{v_\mathrm{x}}{\cos^2 \theta} d\theta.$$

Here, $v_\mathrm{x} = v(\theta) \cos \theta$ is introduced to simplify the integration process.

Integration by Parts

Then, we perform integration by parts:

$$\frac{1}{g}\int_{\theta_k}^{\theta_{k+1}} \frac{v_{\mathrm{x}}}{\cos^2\theta} d\theta = \frac{1}{g}\left(v_{\mathrm{x}} \tan\theta \Big|_{\theta_k}^{\theta_{k+1}} - \int_{v_{\mathrm{x}\,k}}^{v_{\mathrm{x}\,k+1}} \tan\theta \, dv_{\mathrm{x}} \right).$$

- $v_{\mathrm{x}k} = v(\theta_k)\cos\theta_k$
- $v_{\mathrm{x}k+1} = v(\theta_{k+1})\cos\theta_{k+1}$

Taking (24) and rearranging, we get:

$$\begin{aligned}\frac{1}{g}\int_{\theta_k}^{\theta_{k+1}} \frac{v_{\mathrm{x}}}{\cos^2\theta} d\theta &= \frac{v(\theta_{k+1})\sin\theta_{k+1} - v(\theta_k)\sin\theta_k}{g} \\ &\quad - \frac{1}{g}\int_{\theta_k}^{\theta_{k+1}} \tan\theta \; kv^3(\theta) \; d\theta.\end{aligned}$$

Further Steps

Taking $d\theta = -\frac{g\cos\theta}{v(\theta)}\,dt$:

$$\frac{1}{g}\int_{\theta_k}^{\theta_{k+1}} \frac{v_{\mathrm{x}}}{\cos^2\theta} d\theta = \frac{v(\theta_{k+1})\sin\theta_{k+1} - v(\theta_k)\sin\theta_k}{g}$$

$$-\frac{1}{g}\int_{t_k}^{t_{k+1}} \tan\theta\, k v^3(\theta)\left(-\frac{g\cos\theta}{v(\theta)}\right) dt,$$

Finishing the Last Term

Rearranging:

$$\Rightarrow \frac{v(\theta_{k+1})\sin\theta_{k+1} - v(\theta_k)\sin\theta_k}{g} + \int_{t_k}^{t_{k+1}} kv^2(\theta)\sin\theta \; dt,$$

Then, we perform integration by parts again:

$$\Rightarrow \underbrace{\frac{v(\theta_{k+1})\sin\theta_{k+1} - v(\theta_k)\sin\theta_k}{g}}_{1} + \underbrace{tkv^2(\theta)\sin\theta\Big|_{\theta_k}^{\theta_{k+1}}}_{2}$$

$$-k \underbrace{\int_{v^2(\theta_k)\sin\theta_k}^{v^2(\theta_{k+1})\sin\theta_{k+1}} t\, d(v^2(\theta)\sin\theta)}_{3}. \qquad (25)$$

Trapezoidal Rule

The Trapezoidal Rule is a numerical integration method used to approximate a definite integral. In this case, the integral is:

$$\int_{v^2(\theta_k)\sin\theta_k}^{v^2(\theta_{k+1})\sin\theta_{k+1}} t\, d(v^2(\theta)\sin\theta),$$

and it can be approximated using the Trapezoidal Rule as:

$$\int_{v^2(\theta_k)\sin\theta_k}^{v^2(\theta_{k+1})\sin\theta_{k+1}} t\, d(v^2(\theta)\sin\theta) \approx \frac{v^2(\theta_{k+1})\sin\theta_{k+1} - v^2(\theta_k)\sin\theta_k}{2}\left[t_k + t_{k+1}\right].$$

where

$$t_k \triangleq t(v^2(\theta_k)\sin\theta_k)$$
$$t_{k+1} \triangleq t(v^2(\theta_{k+1})\sin\theta_{k+1})$$

Combining the Terms

Combining them together:

$$t_{k+1} = t_k + \frac{v(\theta_k)\sin\theta_k - v(\theta_{k+1})\sin\theta_{k+1}}{g} - t_{k+1}kv^2(\theta_{k+1})\sin\theta_{k+1} + t_k kv^2(\theta_k)\sin\theta_k$$
$$+ k\left(v^2(\theta_{k+1})\sin\theta_{k+1} - v^2(\theta_k)\sin\theta_k\right)\frac{t_k + t_{k+1}}{2}.$$

Simplified Equation

Simplified equation:

$$
\begin{aligned}
t_{k+1} &= t_k + \frac{v(\theta_k)\sin\theta_k - v(\theta_{k+1})\sin\theta_{k+1}}{g} \\
&\quad + t_{k+1}k\left[-v^2(\theta_{k+1})\sin\theta_{k+1} + \frac{v^2(\theta_{k+1})\sin\theta_{k+1} - v^2(\theta_k)\sin\theta_k}{2}\right] \\
&\quad + t_k k\left[v^2(\theta_k)\sin\theta_k + \frac{v^2(\theta_{k+1})\sin\theta_{k+1} - v^2(\theta_k)\sin\theta_k}{2}\right], \\
\Rightarrow t_{k+1} &= t_k + \frac{v(\theta_k)\sin\theta_k - v(\theta_{k+1})\sin\theta_{k+1}}{g} - t_{k+1}k\left[\frac{v^2(\theta_k)\sin\theta_k + v^2(\theta_{k+1})\sin\theta_{k+1}}{2}\right] \\
&\quad + t_k k\left[\frac{v^2(\theta_k)\sin\theta_k + v^2(\theta_{k+1})\sin\theta_{k+1}}{2}\right].
\end{aligned}
$$

Final Equation

Final equation: Let $\beta = k(v^2(\theta_k)\sin\theta_k + v^2(\theta_{k+1})\sin\theta_{k+1})$:

$$t_{k+1}\left[1+\frac{\beta}{2}\right] = t_k\left[1+\frac{\beta}{2}\right] + \frac{v(\theta_k)\sin\theta_k - v(\theta_{k+1})\sin\theta_{k+1}}{g},$$

and dividing by $1+\frac{\beta}{2}$:

$$t_{k+1} = t_k + \frac{2(v(\theta_k)\sin\theta_k - v(\theta_{k+1})\sin\theta_{k+1})}{g(2+\beta)}.$$

Approximate Solution

Finally, we can use the approximate solution to simulate t as follows:

$$t_{k+1} = t_k + \frac{2(v(\theta_k)\sin\theta_k - v(\theta_{k+1})\sin\theta_{k+1})}{g(2+\beta)}, \tag{26}$$

where

$$v(\theta) = \frac{v(\theta_0)\cos\theta_0}{\cos\theta\sqrt{1 + kv^2(\theta_0)\cos^2\theta_0(f(\theta_0) - f(\theta))}}, \tag{27}$$

$$f(\theta) = \frac{\sin\theta}{\cos^2\theta} + \ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right).$$

Local Nature of the Approximation

Hence, in a small interval $[\theta_k, \theta_{k+1}]$, the trajectory of the point mass can be approximated by Eq. (26). These formulas have a local nature. We can calculate the entire trajectory very accurately in steps by calculating $v(\theta)$ and t using Eqs. (27) and (26) at the right-hand end of the interval $[\theta_k, \theta_{k+1}]$ and taking them as the initial values for the following step.

Updating Initial Values

$$v(\theta_k) = v(\theta_k), \quad t_k = t(\theta_k).$$

where θ_k is the current angle at the k-th step.

$$v(\theta_{k+1}) = \frac{v(\theta_k)\cos\theta_k}{\cos\theta_{k+1}\sqrt{1 + kv(\theta_k)^2\cos^2\theta_k(f(\theta_k) - f(\theta_{k+1}))}},$$

$$f(\theta_k) = \frac{\sin\theta_k}{\cos^2\theta_k} + \ln\tan\left(\frac{\theta_k}{2} + \frac{\pi}{4}\right),$$

$$\beta = k(v^2(\theta_{k+1})\sin\theta_{k+1} + v^2(\theta_{k+1})\sin\theta_{k+1}), \quad v^2(\theta_{k+1})\sin\theta_{k+1} = v^2$$

$$t(\theta_{k+1}) = t(\theta_k) + \frac{2(v(\theta_k)\sin\theta_k - v(\theta_{k+1})\sin\theta_{k+1})}{g(2+\beta)}.$$

Updating Values for x and z

We can follow the local nature of the approximation of t to obtain x and z:

$$\mathrm{x}(\theta_{k+1}) = \mathrm{x}(\theta_k) + \frac{v^2(\theta_k)\sin(2\theta_k) - v^2(\theta_{k+1})\sin(2\theta_{k+1})}{2g(1+\beta)},$$

$$\mathrm{y}(\theta_{k+1}) = \mathrm{y}(\theta_k) + \frac{v^2(\theta_k)\sin^2(\theta_k) - v^2(\theta_{k+1})\sin^2(\theta_{k+1})}{g(2+\beta)}.$$

Updating Values in $\frac{d}{d\theta}$ Form to use in RK4

Then we create numerical solutions to verify if our approximate solution is correct.

$$\begin{aligned}
\frac{dt}{d\theta} &= -\frac{v(\theta)}{g\cos(\theta)}, \\
\frac{dv}{d\theta} &= v\tan\theta + \frac{kv^3(\theta)}{\cos\theta}, \\
\frac{d\mathrm{x}}{d\theta} &= -\frac{v^2(\theta)}{g}, \\
\frac{d\mathrm{y}}{d\theta} &= -\frac{v^2(\theta)\tan(\theta)}{g}.
\end{aligned}$$

Numerical Simulation

MATLAB Code: main_code I

```
clc
clear
close all

% Define parameters
global g k
v0 = 45; % Initial velocity
x0 = 0;
z0 = 0;
t0 = 0; % Initial time
theta0 = (80)*(pi/180); % Initial angle (rad)
initial_conditions = [t0;v0;x0;z0];
g = 9.81; % Acceleration due to gravity (m/s^2)
k = 0; % Damping coefficient
step = (-0.01)*(pi/180); % (rad)

% Define the angle range, slightly above -90 degrees
theta_span = theta0 : step : (-90+1e-10)*(pi/180);

% Preallocate arrays for better performance
t = zeros(1, length(theta_span));
```

MATLAB Code: main_code II

```
x = zeros(1, length(t));
z = zeros(1, length(t));
v = zeros(1, length(t));
y = zeros(4, length(t));



% Initial conditions
t(1) = t0;
v(1) = v0;
x(1) = x0;
z(1) = z0;
y(:,1) = initial_conditions;


% loop
for i = 2:length(theta_span)

    % v(theta)
    v(i) = v_theta( theta_span(i), theta_span(i-1), v(i-1)
   , k);
```

MATLAB Code: main_code III

```
    % parameters
    a = v(i-1)^2*sin(theta_span(i-1));
    b = v(i).^2*sin(theta_span(i));
    beta = k*(a+b);
    % t(theta)
    t(i) = t_theta(t(i-1), v(i), v(i-1), theta_span(i),
   theta_span(i-1), g, beta);
    % x(theta)
    x(i) = x_theta(x(i-1), v(i), v(i-1), theta_span(i),
   theta_span(i-1), g, beta);
    % z(theta)
    z(i) = z_theta(z(i-1), v(i), v(i-1), theta_span(i),
   theta_span(i-1), g, beta);
    % numerical
    y(:,i) = RK4(@f_theta,theta_span(i-1),y(:,i-1),step);

    % break if hit the ground
    if y(4,i) < 0
        t = t(1:i);
        v = v(1:i);
        x = x(1:i);
```

MATLAB Code: main_code IV

```
        z = z(1:i);
        theta_span = theta_span(1:i);
        y = y(:, 1:i);
        break;
    end
end

% Extract the results
ttheta = y(1,:);
vtheta = y(2,:);
xtheta = y(3,:);
ztheta = y(4,:);

% Calculate RMSE
RMSE_t = sqrt(mean((t - ttheta).^2));
RMSE_v = sqrt(mean((v - vtheta).^2));
RMSE_x = sqrt(mean((x - xtheta).^2));
RMSE_z = sqrt(mean((z - ztheta).^2));
RMSE_traj = sqrt(RMSE_x^2 + RMSE_z^2);

% Plotting results
```

MATLAB Code: main_code V

```
figure;

subplot(3, 1, 1);
plot(theta_span, t, 'b-', 'LineWidth', 2);
hold on;
plot(theta_span, ttheta, 'r--', 'LineWidth', 2);
xlabel('Theta (rad)');
ylabel('Time (s)');
title(['Time over Angle (RMSE: ', num2str(RMSE_t), ')']);
legend('Analytical', 'Numerical', 'Location', 'best');
grid on;
xlim([min(theta_span) max(theta_span)]);
ylim([min(t) max(t)]);

subplot(3, 1, 2);
plot(theta_span, v, 'b-', 'LineWidth', 2);
hold on;
plot(theta_span, vtheta, 'r--', 'LineWidth', 2);
xlabel('Theta (rad)');
ylabel('Velocity (m/s)');
```

MATLAB Code: main_code VI

```
title(['Velocity over Angle (RMSE: ', num2str(RMSE_v), ')'
    ]);
legend('Analytical', 'Numerical', 'Location', 'best');
grid on;
xlim([min(theta_span) max(theta_span)]);
ylim([min(v) max(v)]);

subplot(3, 1, 3);
plot(x, z, 'b-', 'LineWidth', 2);
hold on;
plot(xtheta, ztheta, 'r--', 'LineWidth', 2);
xlabel('X Position (m)');
ylabel('Z Position (m)');
title(['Trajectory (RMSE: ', num2str(RMSE_traj), ')']);
legend('Analytical', 'Numerical', 'Location', 'best');
grid on;
xlim([min([x xtheta])*1.1 max([x xtheta])*1.1]);
ylim([min([z ztheta])*1.1 max([z ztheta])*1.1]);
```

MATLAB Code: v(θ), $t(\theta)$

```
function v = v_theta( theta, theta0, v0, k)
    f = (sin(theta) ./ cos(theta).^2) + log(tan(theta / 2 + pi / 4));
    f0 = (sin(theta0) ./ cos(theta0).^2) + log(tan(theta0 / 2 + pi / 4));
    v = (v0 * cos(theta0)) ./ (cos(theta) .* sqrt(1 + k * v0^2 * cos(theta0)^2 .* (f0 - f)));
end
```

```
function t = t_theta(t0, v, v0, theta, theta0, g, beta)
    t = t0 + 2 * (v0 * sin(theta0) - v * sin(theta)) / (g * (2 + beta));
end
```

MATLAB Code: x(θ), $z(\theta)$

```
function x = x_theta(x0, v, v0, theta, theta0, g, beta)
    x = x0 + (v0^2 * sin(2 * theta0) - v^2 * sin(2 * theta
    )) / (2 * g * (1 + beta));
end
```

```
function z = z_theta(z0, v, v0, theta, theta0, g, beta)
    z = z0 + (v0^2 * sin(theta0)^2 - v^2 * sin(theta)^2) /
    (g * (2 + beta));
end
```

MATLAB Code: f$_{(}\theta)$

```matlab
function dydtheta = f_theta(theta, y)
    % Define global variables
    global g k

    % Extract state variables
    t = y(1);
    v = y(2);
    x = y(3);
    z = y(4);

    % Define differential equations
    dtdtheta = -v / (g * cos(theta));
    dvdtheta = (v * tan(theta) + k * v^3 / cos(theta));
    dxdt = -v^2 / g;
    dzdt = -v^2 * tan(theta) / g;

    % Return derivatives
    dydtheta = [dtdtheta; dvdtheta; dxdt; dzdt];
end
```

MATLAB Code: RK4

```
function y_next = RK4(f, t, y, h)
    k1 = f(t, y);
    k2 = f(t + 0.5 * h, y + 0.5 * h * k1);
    k3 = f(t + 0.5 * h, y + 0.5 * h * k2);
    k4 = f(t + h, y + h * k3);
    k = (k1 + 2 * k2 + 2 * k3 + k4) /6;
    y_next = y + k*h;
end
```

k Selection for a Small Sounding Rocket I

The drag coefficient k is calculated using the formula:

$$k = \frac{\rho_a C_D S}{2mg}$$

Here are the parameters for a small sounding rocket:

Parameter	**Value**
ρ_a	$1.225\ \mathrm{kg/m^3}$ (Air density)
C_D	0.24 (Drag coefficient)
S	$(0.699223)^2 \pi m^2$ (Cross-sectional area)
m	42 kg (Mass of the rocket)
g	$9.81\ \mathrm{m/s^2}$ (Acceleration due to gravity)

k Selection for a Small Sounding Rocket II

Substituting these values into the formula, we get:

$$k = \frac{1.225 \times 0.24 \times (0.699223)^2 \pi}{2 \times 42 \times 9.81} = 0.000548$$

Numerical Simulation k=0.000548 ($C_D = 0.24$)

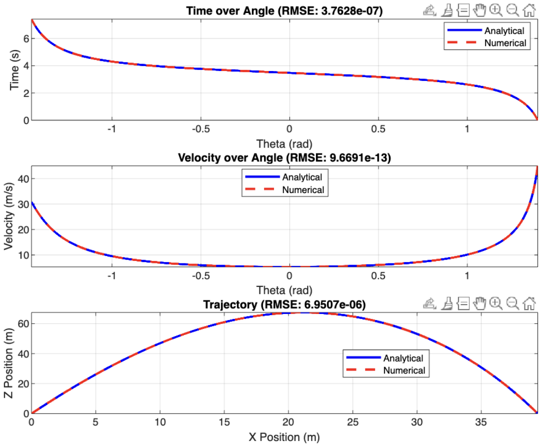


Figure 2: Numerical Simulation k=0.000548

Numerical Simulation k=0.00548 ($C_D = 2.4$)

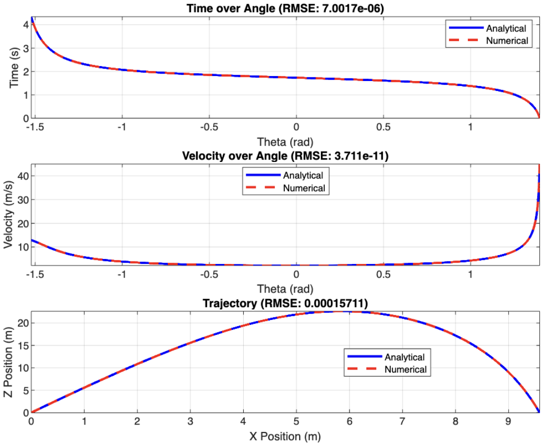


Figure 3: Numerical Simulation k=0.00548

Numerical Simulation k=0.0548 ($C_D = 24$)

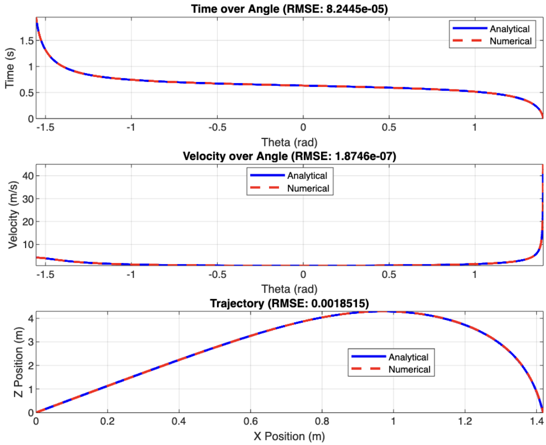


Figure 4: Numerical Simulation k=0.0548

Numerical Simulation k=0.548 ($C_D = 240$)

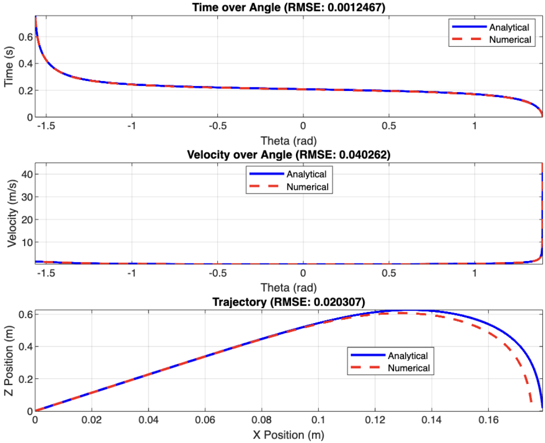


Figure 5: Numerical Simulation k=0.548

Numerical Simulation k=0.548 ($C_D = 240$) fixed

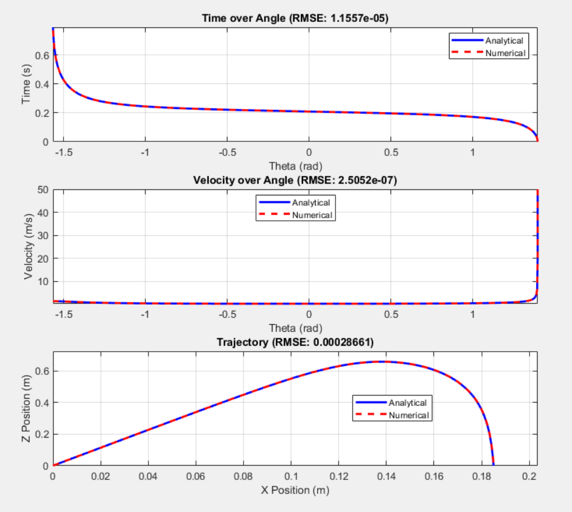


Figure 6: Numerical Simulation k=0.548, step = (-0.001)*(pi/180);

Numerical Simulation k=0 ($C_D = 0$)

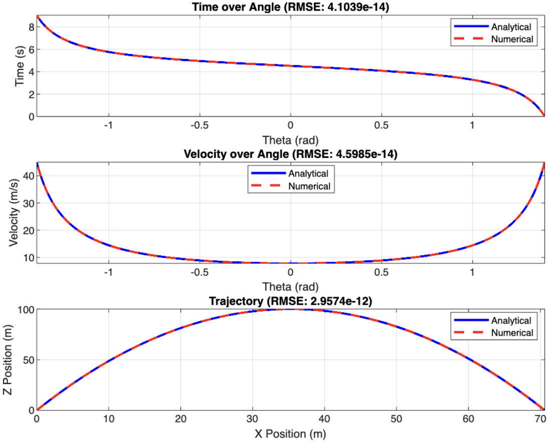


Figure 7: Numerical Simulation k=0

References

References I

[1] P.S. Chudinov, "The motion of a point mass in a medium with a square law of drag," J. Appl. Maths Mechs, Vol. 65, No. 3, pp. 421-426, 2001.