Generate True Trajectory

September 2024

Continuous-time to Discrete-time I

Equations of Motion (EOM)

These are the continuous-time equations that describe the motion of a projectile under gravity and drag:

$$\frac{dx}{dt} = v \cos\theta$$
$$\frac{dz}{dt} = v \sin\theta$$
$$\frac{dv}{dt} = -g \sin\theta - gkv^2$$
$$\frac{d\theta}{dt} = -\frac{g \cos\theta}{v}$$
$$\frac{dk}{dt} = 0$$

Continuous-time to Discrete-time II

State Variables

The state variables are defined as follows:

$$
\begin{aligned}
x_1 &= x \\
x_2 &= z \\
x_3 &= v \\
x_4 &= \theta \\
x_5 &= k
\end{aligned}
$$

Continuous-time to Discrete-time III

State Equations

We rearranged the EOM using the newly defined state variables as follows:

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_3 \cos(x_4) \\
\frac{dx_2}{dt} &= x_3 \sin(x_4) \\
\frac{dx_3}{dt} &= -g \sin(x_4) - g x_5 x_3^2 \\
\frac{dx_4}{dt} &= -\frac{g \cos(x_4)}{x_3} \\
\frac{dx_5}{dt} &= 0
\end{aligned}
$$

Continuous-time to Discrete-time IV

State Space Equation (Continuous-time)

The continuous-time state-space representation can be written as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_3 \cos(x_4) \\ x_3 \sin(x_4) \\ -g \sin(x_4) - g x_5 x_3^2 \\ -\frac{g \cos(x_4)}{x_3} \\ 0 \end{bmatrix}$$

where the state vector $\mathbf{x}$ is:

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T$$

Continuous-time to Discrete-time V

State Space Equation (Discrete-time)
To generate a trajectory in discrete-time, we need to discretize the continuous-time equations. This can be done using a numerical method such as the Fourth Order Runge-Kutta (RK4) method. The discrete-time state-space equation is represented as:

$$\mathbf{x_k} = \mathbf{f_d}(\mathbf{x}_{k-1}) = \mathbf{Numerical\_method}(\mathbf{f}(\mathbf{x_{k-1}}), \Delta\mathbf{t})$$

We use the RK4 method to discretize the equation as follows:

Continuous-time to Discrete-time VI

The Fourth Order-Runge Kutta Method (RK4):
The following steps show how RK4 approximates the system's state in discrete-time:

$$
\begin{aligned}
\mathbf{x_k} &= \mathbf{RK4}(\mathbf{f}(\mathbf{x_{k-1}}), \Delta\mathbf{t}) \\
\mathbf{k_1} &= \mathbf{f}(\mathbf{x_{k-1}}) \\
\mathbf{k_2} &= \mathbf{f}(\mathbf{x_{k-1}} + 0.5 \cdot \Delta\mathbf{t} \cdot \mathbf{k_1}) \\
\mathbf{k_3} &= \mathbf{f}(\mathbf{x_{k-1}} + 0.5 \cdot \Delta\mathbf{t} \cdot \mathbf{k_2}) \\
\mathbf{k_4} &= \mathbf{f}(\mathbf{x_{k-1}} + \Delta\mathbf{t} \cdot \mathbf{k_3}) \\
\mathbf{k} &= \frac{1}{6} \cdot (\mathbf{k_1} + 2 \cdot \mathbf{k_2} + 2 \cdot \mathbf{k_3} + \mathbf{k_4}) \\
\mathbf{x_k} &= \mathbf{x_{k-1}} + \mathbf{k} \cdot \Delta\mathbf{t}
\end{aligned}
$$