Name: Manas Sachin Deshmukh
Roll No: 2022102040

Date: 27/12/2022
Day: Tuesday

Real Analysis Project

Acknowledgements and references :—

I would like to thank our group & T.A. Shreya Patil sir for always clearing my doubts regarding the project and always helping me in understanding crucial concepts of R.A. (Real Analysis)

Following are the references which I used while making this project:-

1. Real Analysis notes:-
2. biology reader. com / growth-curve-of-bacteria.html
3. For Conceptual doubts I used:-
   Mathematical methods for physicists - Weber & Harris

Name: Manas Deshmukh

Roll No: 2022102040

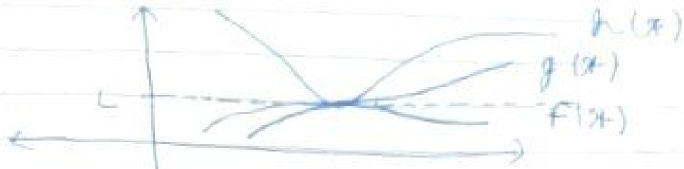
R.A. Project:- Date: 27/12/2022 1

1. Squeeze Theorem:- + Root theorem (same concept) of modulus of value of limit of summation not exactly root's theorem

If $f(x) \le g(x) \le h(x)$

and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$

then $\lim_{x \to c} g(x) = L$



this is called squeeze / sandwich theorem.

We in our solution to a real analysis problem will use this theorem.

Question 1) Concept of Convergence / divergence of series in real analysis

Check whether the sequence $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{k+n^2}$ is converging or a diverging summation.

(Evaluating limit for root test is difficult

Sol^n The conventional way of attempting this problem may be ratio/root test but another interesting and unique approach would be Sandwich Theorem:

$$S = \frac{1}{1+n^2} + \frac{2}{2+n^2} \text{------} \frac{n}{n+n^2}$$

By simple Mathematical inequalities we can write:-

$$\lim_{n\to\infty} \frac{1}{n+n^2} + \frac{2}{n+n^2} \text{---} \frac{n}{n+n^2} < S < \lim_{n\to\infty} \frac{1}{1+n^2} + \frac{2}{1+n^2} \text{------} \frac{n}{1+n^2}$$

↓ ↓ divide by n^2

$$\lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n+n^2} = \lim_{n\to\infty} \frac{(n+\cancel{1})}{2(\cancel{1}+n)} = \boxed{\frac{1}{2}}$$

$$\lim_{n\to\infty} \frac{n(n+1)}{2(1+n^2)} = \lim_{n\to\infty} \frac{(n+1)/n}{2(\frac{1}{n^2}+ \ldots} = \frac{1}{2}$$

Since both the limits are equal to 1/2

$\therefore \lim_{n\to\infty} S = \frac{1}{2}$ by squeeze theorem

Therefore by basic concept of root test
we can conclude that S (summation) is converging :-
$|\lim_{n\to\infty} S| < 1$ it is converging summation.

$\therefore$ We can similarly utilize this theorem for other series where finding limit is not easy

other example for reference:

To check convergence/divergence of

Q2 ⇒ $$\lim_{n\to\infty} x_n = \frac{\frac{n(n+1)}{2}\alpha - \{\alpha\} - \{2\alpha\} \cdots \{n\alpha\}}{n^2}$$

Sol^n:- x_n^* can be cleverly written

as $$x_n = \frac{(\alpha + 2\alpha \cdots n\alpha) - (\{\alpha\} + \{2\alpha\} \cdots \{n\alpha\})}{n^2}$$

$$x_n = \frac{(\alpha - \{\alpha\}) + (2\alpha - \{2\alpha\}) \cdots (n\alpha - \{n\alpha\})}{n^2}$$

$$x_n = \frac{[\alpha] + [2\alpha] \cdots [n\alpha]}{n^2} = x_n$$

Now,

We know that
for any x
$[x] \leq x \leq [x] + 1$

$\therefore$ We can clearly say $x-1 < [x] < x$

$\therefore$ we can say after simplification

$$\frac{(\alpha-1)+(2\alpha-1)+\cdots(n\alpha-1)}{n^2} < \frac{[\alpha]+[2\alpha]\cdots\cdots+[n\alpha]}{n^2} < \frac{\alpha+2\alpha\cdots\cdots n\alpha}{n^2}$$

Trying to apply squeeze theorem

$$\frac{(1+2+3\cdots+n)\alpha-n}{n^2} < x_n < \frac{(1+2+3\cdots n)\alpha}{n^2}$$

$$\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}\alpha-n}{n^2} < \lim_{n\to\infty} x_n < \frac{n(n+1)\alpha}{2n^2}\lim_{n\to\infty}$$

$$\lim_{n\to\infty}\alpha\frac{(n+1)}{2n}-\frac{1}{n} < \lim_{n\to\infty} x_n < \lim_{n\to\infty}\frac{(n+1)\alpha}{2n}$$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

value $= \frac{\alpha}{2}$ $\qquad\qquad\qquad$ value $= \frac{\alpha}{2}$

$\therefore$ So by sandwich theorem we can conclude that

$$\lim_{n\to\infty} x_n = \frac{\alpha}{2}$$

If $\frac{\alpha}{2} < 1 \longrightarrow$ ~~converging~~

$\alpha < 2 \longrightarrow$ converging

&

If $\frac{\alpha}{2} > 1 \longrightarrow$ Diverging

$\alpha > 2 \rightarrow$ ~~Diverging~~

Another ex:-

Let a_n be a sequence such that $(n \geq 1)$

$a_1 = 1 \quad a_{n+1} = a_n + 2$

Find $\lim_{n \to \infty} \left(\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots \frac{1}{a_{n-1} a_n} \right)$

Sol^n: We know that for $n \geq 1$

$$\boxed{a_{n+1} - a_n = 2}$$

$\therefore$ We can conclude $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots a_n - a_{n-1} = 2$

So $\lim_{n \to \infty} \frac{1}{2} \left(\frac{2}{a_1 a_2} + \frac{2}{a_2 a_3} + \cdots \frac{2}{a_{n-1} a_n} \right)$

$$\lim_{n \to \infty} \frac{1}{2} \left(\frac{a_2 - a_1}{a_1 a_2} + \frac{a_3 - a_2}{a_2 a_3} + \cdots \frac{a_n - a_{n-1}}{a_{n-1} a_n} \right)$$

$$\lim_{n \to \infty} \frac{1}{2} \left(\frac{1}{a_1} \cancel{- \frac{1}{a_2}} \cancel{+ \frac{1}{a_2}} \cancel{- \frac{1}{a_3}} \cancel{+ \frac{1}{a_3}} \cancel{- \frac{1}{a_4}} + \cdots \cancel{\frac{1}{a_{n-1}}} - \frac{1}{a_n} \right)$$

$$\boxed{a_1 = 1}$$

as $n \to \infty$

a_n will be very large

$\frac{1}{a_n} \to 0$

So values of limit is $\lim_{n \to \infty} \frac{1}{2} \left(1 - \frac{1}{a_n} \right)$

$$\lim_{n \to \infty} \frac{1}{2} \left(1 - \cancel{\frac{1}{a_n}}^{0} \right) = \frac{1}{2}$$

Ans

Specific Applications of Real Analysis:— Conceptual Overview

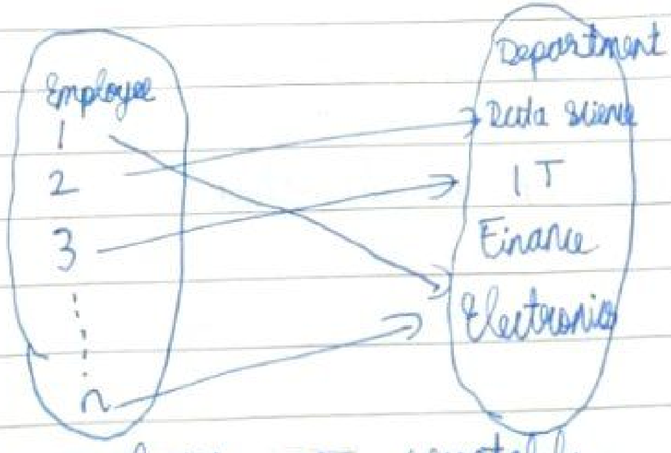
① The use of mapping of functions (usually one one and onto) in data science and machine learning and data keeping; finance and economics.

Mapping is applied on input data that transforms qualitative / excessive / repetitive data into machine understandable data.

For Example:-

Mathematics analysis:—

Ⓐ For doing data keeping, for example in a big company there are huge number of employees all details like contact number, department, salary should be mapped to the particular employee.



These employees are countable and can easily be mapped to their sectors etc.
This helps in efficient storage of data.

Such kind of mappings are also used in banking, finance and economics.

(3) In Machine Learning (in machine learning)

Consider we are predicting whether customer will buy certain product provide we have provided qualitative data from previous buyers. This is basically a learning model and a categorial formulation to derive a logical inference pattern.

***

	Table	Meaning
$M(x) =$	1	Bad
	2	Decent
	3	Satisfactory

Such mappings have great use in machine learning.

Excessive use of various fields of real analysis in machine learning for "algorithm optimization"

In machine learning we often encounter functions which are not linear with complex and lengthy definition and landscape. It is possible for function to have lowest value within small/local region. This point is infimum/ local minima of that function in that particular zone. For Example:-

Also there are many problems in machine learning, where we need optimization in specific space/whole zone. Such problems are unconstrained optimization

For example :-

If we need to minimize $x^2 + y^2$ subject to $x + y \leq 2$ — This is kind of constrained optimization

(A) We need to make sure
if $x \in S_1$ and $y \in S_2$
where S_1 is domain of x
and S_2 is domain of y
We know that we need minimum of function when sum of variables in domain must sum up maximum to two.

(B) Sometimes we need to minize such that if we consider some vectors in direction of some function they are orthogonal to each other

(C) For optimization we need graph of constraint and calculus for reaching the answer.

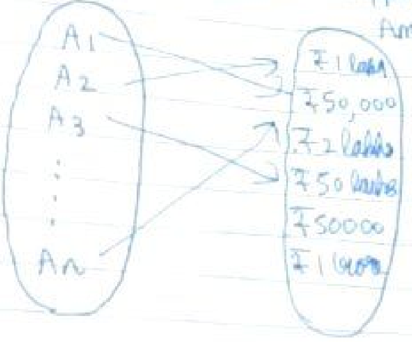
Details of specific parts of machine learning :-

1. Gradient descent in neutral networks (unconstrained optimization)
2. Lagrange multipliers in vector machines (constrained optimization)
3. Clustering via expectation maximization algorithm (constrained optimization)
4. Logistic regression (unconstrained optimization)

Also in finance, banking and economics :-

For example: There is a bank having "n" people as account holders who have their money deposited in the bank.
Let A_i be the account of that person who registered i^{th} time
Again here functional mapping can be used

Soln:



Separately A_i $(i \in N)$ can be mapped to fluctuating and different rate of interests before calculating the actual amount left in the account of the i^{th} person.

#(2) Use of real analysis in mathematical modelling of bacterial population :- (Growth / Decay) / Radioactivity

In general the growth pattern of bacteria is in geometric series that is diverging i.e a bacterial cell first divides into two then four, then eight and so on

To mathematically express growth of bacteria there is relationship between initial no. of cells and final no. of cells

$\boxed{N = N_0 \times 2^n}$

This is a diverging sequence

Taking log on both sides

$$\log_{10} N = \log_{10} N_0 + n \log_{10} 2$$

$$n = \frac{\log_{10} N - \log_{10} N_0}{0.301}$$

** $\boxed{n = 3.3 \left(\log_{10} N - \log_{10} N_0\right)}$

There are four distinct ~~four~~ phases in development of bacteria :-

1. Lag phase
2. Log phase
3. Stationary phase
4. Death phase

1 → Lag phase
2 → Log phase
3 → Stationary phase
4 → Death phase

** This exponential growth of bacteria is diverging and has four phases

Also, the decay of bacteria :- / Decay in radioactivity :-

In general decay of bacteria is very fast, it decreases quickly in very less time. Let N_0 be initial number of bacteri

$$\boxed{N = N_0 \left(\frac{1}{2}\right)^n}$$

So the population of bacteria halfs when $n=1$, one fourth when $n=2$ and so on. This is therefore a converging sum / sequence, is an infinite geometric progression.

Sum

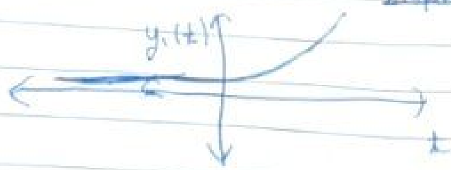
$$N_s = N_0 + \frac{N_0}{2} + \frac{N_0}{4} - - - \cdot \infty = \frac{N_0}{1-\frac{1}{2}} = 2N_0$$

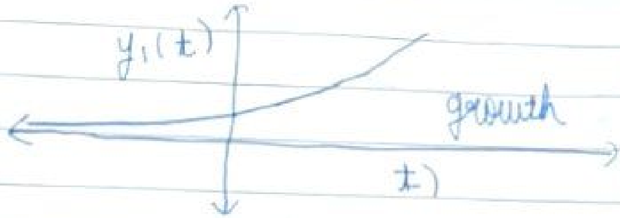
converges to a finite number

Generally the half factor is in radioactivity
for bacterial growth and decay we generally use

$$y_1(t) = y_0\, e^{kt} \quad \text{and} \quad y_2(t) = y_0\, e^{-kt} \;\forall\; k > 0$$

exponential functions:-





* Some special sequences/series like fibonacci sequence with a_n

where $\boxed{a_n = a_{n-1} + a_{n-2}}$

The fibonacci series represents in graphic design the "Golden Ratio" which helps tremendously to create art, architecture with perfect harmony and symmetry

0, 1, 1, 2, 3, 5, 8, 13, 21, 34 - - - - - -

Completed