DIGITAL COMMUNICATION

Through Simulations

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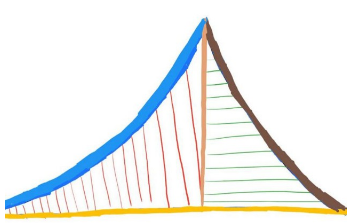

Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Introduction

Chapter 2

Axioms

2.1 Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of a football game?

Solution: Let X be a random variable which takes the values 0 and 1.

$$X = \begin{cases} 1, & \text{if coin toss results in Head} \\ 0, & \text{if coin toss results in Tail} \end{cases} \tag{2.1}$$

From law of total probability,

$$\Pr(X = 0) + \Pr(X = 1) = 1 \tag{2.2}$$

Since there is only one head,

$$\Pr(X = 1) = \frac{1}{2} \tag{2.3}$$

Similarly,

$$\Pr(X = 0) = 1 - \Pr(X = 1) = \frac{1}{2} \tag{2.4}$$

Thus,

$$\Pr(X = 0) = \Pr(X = 1) \tag{2.5}$$

which is why tossing the coin is a fair way to decide.

2.2 Which of the following cannot be the probability of an event ?

(a) $\frac{2}{3}$

(b) -1.5

(c) 15%

(d) 0.7

Solution: From the axioms of probability,

$$0 \leq \Pr(E) \leq 1 \tag{2.6}$$

(a) $\Pr(E) = \frac{2}{3}$

$$\because 0 \leq \frac{2}{3} \leq 1 \tag{2.7}$$

from (2.6), it can be probability of an event.

(b) $\Pr(E) = -1.5$

$$\because -1.5 < 0 \tag{2.8}$$

from (2.6), it cannot be a probability of any event.

(c)

$$\Pr\left(E\right) = \frac{15}{100} \tag{2.9}$$
$$\because 0 \le \frac{15}{100} \le 1, \tag{2.10}$$

from (2.6), it can be probability of an event.

(d) $\Pr\left(E\right) = 0.7$

$$\because 0 \le 0.7 \le 1 \tag{2.11}$$

from (2.6), it can be a probability of an event.

2.3 If P(E) = 0.05, what is the probability of 'not E'?

Solution: The desired probability is

$$\Pr\left(E'\right) = 1 - \Pr\left(E\right) = 0.95 \tag{2.12}$$

2.4 A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out

(a) an orange flavoured candy?

(b) a lemon flavoured candy?

Solution:

$$\Pr\left(O\right) = 0 \tag{2.13}$$

$$\Pr\left(L\right) = 1 \tag{2.14}$$

2.5 It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

Solution: Let E be the event that no 2 students in a group of 3 share a birthday. Then

$$\Pr(E) = 0.992 \implies \Pr\left(E'\right) = 1 - \Pr(E) = 0.008 \tag{2.15}$$

2.6 Check whether the following probabilities $\Pr(A)$ and $\Pr(B)$ are consistently defined

(a) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cap B) = 0.6$

(b) $\Pr(A) = 0.5$, $\Pr(B) = 0.7$, $\Pr(A \cup B) = 0.8$

Solution: To check whether the given probabilities are consistently defined, we check whether the following property holds correctly with the probability axioms

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.16}$$

(a) Given that

$$\Pr(A) = 0.5,\ \Pr(B) = 0.7,\ \Pr(AB) = 0.6 \tag{2.17}$$

From (2.16),

$$\Pr(A + B) = 0.5 + 0.7 - 0.6 = 0.6 \tag{2.18}$$

From (2.18) we have

$$0 \leq \Pr(A+B) \leq 1 \tag{2.19}$$

Hence the given probabilities are consistently defined.

(b) Given that

$$\Pr(A) = 0.5,\ \Pr(B) = 0.7,\ \Pr(A+B) = 0.8 \tag{2.20}$$

From (2.16) we get,

$$\Pr(AB) = 0.5 + 0.7 - 0.8 \tag{2.21}$$

$$= 0.4 \tag{2.22}$$

From (2.22) we have

$$0 \leq \Pr(AB) \leq 1 \tag{2.23}$$

Hence the given probabilities are consistently defined

2.7 A and B are events such that $\Pr(A) = 0.42, \Pr(B) = 0.48$ and $\Pr(\text{A and B}) = 0.16$. Determine

(a) $\Pr(\text{not A})$

(b) $\Pr(\text{not B})$

(c) $\Pr(\text{A or B})$

Solution: Solution:

(a) Pr (not A)

$$\Pr\left(A'\right) = 1 - \Pr(A) \tag{2.24}$$
$$= 1 - 0.42 \tag{2.25}$$
$$= 0.58 \tag{2.26}$$

(b) Pr (not B)

$$\Pr\left(B'\right) = 1 - \Pr(B) \tag{2.27}$$
$$= 1 - 0.48 \tag{2.28}$$
$$= 0.52 \tag{2.29}$$

(c) Pr (A or B)

$$\Pr(\text{A+B}) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.30}$$
$$= 0.42 + 0.48 - 0.16 \tag{2.31}$$
$$= 0.74 \tag{2.32}$$

2.8 In class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class,find the probability that he will be studying Mathematics or Biology

Solution: The given information is summarised in Table 2.2. Thus,

Random Variable	Subject	Probability
M	Mathematics	$\Pr(M)$=0.4
B	Biology	$\Pr(B)$=0.3
M, B	Both	$\Pr(MB)$=0.10

Table 2.2:

$$\Pr(M+B) = \Pr(M) + \Pr(B) - \Pr(M,B) \tag{2.33}$$

$$= 0.6 \tag{2.34}$$

2.9 In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(a) The student opted for NCC or NSS.

(b) The student has opted neither NCC nor NSS.

(c) The student has opted NSS but not NCC.

Solution: Define random variables X and Y as shown in Tables 2.3 and 2.4. From

$X = 0$	Student does not opt for NCC.
$X = 1$	Student opts for NCC.

Table 2.3: Definition of X.

$Y = 0$	Student does not opt for NSS.
$Y = 1$	Student opts for NSS.

Table 2.4: Definition of Y.

the given data

$$\Pr(X=1) = \frac{30}{60} = \frac{1}{2} \tag{2.35}$$
$$\Pr(Y=1) = \frac{32}{60} = \frac{8}{15} \tag{2.36}$$
$$\Pr(X=1,\ Y=1) = \frac{24}{60} = \frac{2}{5} \tag{2.37}$$

Thus, we write

$$\Pr(X=1,\ Y=0) = \Pr(X=1) - \Pr(X=1,\ Y=1) = \frac{1}{10} \tag{2.38}$$
$$\Pr(X=0,\ Y=1) = \Pr(Y=1) - \Pr(X=1,\ Y=1) = \frac{2}{15} \tag{2.39}$$
$$\Pr(X=0,\ Y=0) = \Pr(Y=0) - \Pr(X=1,\ Y=0) \tag{2.40}$$
$$= 1 - \Pr(Y=1) - \Pr(X=1,\ Y=0) \tag{2.41}$$
$$= 1 - \frac{8}{15} - \frac{1}{10} = \frac{11}{30} \tag{2.42}$$

and form the joint pmf as in Table 2.5.

	$X=0$	$X=1$
$Y=0$	$\frac{11}{30}$	$\frac{1}{10}$
$Y=1$	$\frac{2}{15}$	$\frac{2}{5}$

Table 2.5: Joint pmf of X and Y.

(a) From Table 2.5,

$$\Pr(X+Y \geq 1) = 1 - \Pr(X+Y=0) \tag{2.43}$$
$$= \frac{19}{30} \tag{2.44}$$

(b) From Table 2.5,

$$\Pr(X=0,\ Y=0)=\frac{11}{30} \tag{2.45}$$

(c) From Table 2.5,

$$\Pr(X=0,\ Y=1)=\frac{2}{15} \tag{2.46}$$

2.10 A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(a) $\Pr(2)$

(b) $\Pr(1 \text{ or } 3)$

(c) $\Pr(\text{not } 3)$

Solution: The given information is summarized in the following table 2.6

RV	**Description**	**Probability**
$X=1$	Die rolls to 1	$\frac{1}{3}$
$X=2$	Die rolls to 2	$\frac{1}{2}$
$X=3$	Die rolls to 3	$\frac{1}{6}$

Table 2.6: Random variable X

(a)

$$\Pr(X=2)=\frac{1}{2} \tag{2.47}$$

(b) Since

$$X = 1 \text{ or } X = 3 \equiv X \in \{1,3\} \tag{2.48}$$

$$X = 1 \text{ and } X = 3 \equiv X = \phi \tag{2.49}$$

$$\begin{aligned}\Pr(X \in \{1,3\}) &= \Pr(X = 1) + \Pr(X = 3) - \Pr(X = \phi) &(2.50)\\ &= \frac{1}{3} + \frac{1}{6} &(2.51)\\ &= \frac{1}{2} &(2.52)\end{aligned}$$

(c)

$$\begin{aligned}\Pr(X \neq 3) &= 1 - \Pr(X = 3) &(2.53)\\ &= 1 - \frac{1}{6} &(2.54)\\ &= \frac{5}{6} &(2.55)\end{aligned}$$

2.11 If $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$ find $\Pr(A \cap B)$ if A and B are independent events.

Solution: Since the events A, B are independent, we have

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{3}{25} \tag{2.56}$$

2.12 Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A+B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

Solution:

(a) In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \tag{2.57}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p \tag{2.58}$$

$$\therefore p = \frac{1}{10} \tag{2.59}$$

(b) Given A and B are independent events, then,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.60}$$

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{2.61}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{2.62}$$

$$\therefore p = \frac{1}{5} \tag{2.63}$$

2.13 Let E and F be events with $\Pr(E) = \frac{3}{5}, \Pr(F) = \frac{3}{10}$ and $\Pr(EF) = \frac{1}{5}$. Are E and F independent?

Solution: From the given information,

$$\Pr(E)\Pr(F) = \frac{3}{5} \times \frac{9}{50} \tag{2.64}$$

$$\Pr(EF) = \frac{1}{50} \tag{2.65}$$

$$\implies \Pr(EF) \neq P(E)P(F) \tag{2.66}$$

$\therefore$ E and F are not independent events.

2.14 If A and B are two events such that $\Pr(A) = \frac{1}{4}, \Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$, find $\Pr$ (not A and not B).

Solution: Since

$$A'B' = (A+B)', \tag{2.67}$$

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{2.68}$$

$$= 1 - \Pr(A+B) \tag{2.69}$$

Thus,

$$\Pr\left(A'B'\right) = 1 - \{\Pr(A) + \Pr(B) - \Pr(AB)\} \tag{2.70}$$

$$= \frac{3}{8} \tag{2.71}$$

2.15 A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

2.16 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even, ' and B be the event, 'the number is red'. Are A and B independent?

2.17 Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

2.18 Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

2.19 If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find $P(\text{not } A \text{ and not } B)$

2.20 Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}$. State whether A and B are independent?

2.21 Given two independent events A and B such that $P(A) = 0.3$, $P(B) = 0.6$. Find

(a) $P(A \text{ and } B)$

(b) $P(A \text{ and not } B)$

(c) $P(A \text{ or } B)$

(d) $P(\text{neither } A \text{ nor } B)$

2.22 Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved

(b) exactly one of them solves the problem

Solution: Given that $\Pr(A) = \frac{1}{2}$ and $\Pr(B) = \frac{1}{3}$ A, B are independent so

$$\Pr(AB) = \Pr(A)\Pr(B) = \frac{1}{6} \tag{2.72}$$

(a) The probability of the problem being solved is

$$\begin{aligned}
\Pr(A+B) &= \Pr(A) + \Pr(B) - \Pr(AB) && (2.73)\\
&= \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) && (2.74)\\
&= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} && (2.75)
\end{aligned}$$

(b) Probalbility that exactly one person solves problem is

$$\begin{aligned}
\Pr\left(AB'\right) + \Pr\left(A'B\right) &= \Pr(A)\Pr\left(B'\right) + \Pr\left(A'\right)\Pr(B) && (2.76)\\
&= \Pr(A) + \Pr(B) - 2\Pr(A)\Pr(B) = \frac{1}{2} && (2.77)
\end{aligned}$$

2.23 One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?

(a) E: 'the card drawn is spade'

F: 'the card drawn is an ace'

(b) E: 'the card drawn is black'

F : 'the card drawn is a king'

(c) E : 'the card drawn is a king or queen'

F : 'the card drawn is a queen or jack'

Choose the correct answer in the following exercises

2.24 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

2.25 Two events A and B will be independent, if

(a) A and B are mutually exclusive

(b) P(not $A \cap$ not B) $=$ $[1 - P(A)]$ $[1 - P(B)]$

(c) $P(A) = P(B)$

(d) $P(A) + P(B) = 1$

Solution: Two events A and B are independent if

$$\Pr(AB) = \Pr(A)\Pr(B|A) = \Pr(A)\Pr(B) \tag{2.78}$$

using Bayes' Rule. We consider the options one by one. Here, let A be the event of rolling a prime number on a fair die and B the event of rolling an odd prime number on a fair die. The joint pmf is shown in Table 2.7. Notice that A and B are independent,

	A	$\bar{A}$
B	$\frac{1}{3}$	0
$\bar{B}$	$\frac{1}{6}$	$\frac{1}{2}$

Table 2.7: Joint PMF of A and B.

as

$$\Pr(A) = \frac{1}{2},\ \Pr(B) = \frac{1}{3} \tag{2.79}$$

$$\Pr(AB) = \frac{1}{6} = \Pr(A)\Pr(B) \tag{2.80}$$

thereby satisfying (2.78)

(a) From (2.80), $\Pr(AB) > 0$, hence this option is incorrect.

(b) We have,

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{2.81}$$

$$= 1 - \Pr(A+B) \tag{2.82}$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(AB) \tag{2.83}$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(A)\Pr(B) \tag{2.84}$$

$$= (1 - \Pr(A))(1 - \Pr(B)) \tag{2.85}$$

where (2.81) follows from De-Morgan's laws and (2.84) follows from (2.78). Thus, this option is correct.

(c) Clearly from the given example, $\Pr(A) \neq \Pr(B)$. Thus, this option is incorrect.

(d) Again, from the given example, $\Pr(A) + \Pr(B) = \frac{5}{6} < 1$. Thus, this option is incorrect.

Hence, the answer is option **b)**.

2.26 If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \tag{2.86}$$

2.27 If a leap year is selected at random, what is the chance that it will contain 53 tuesdays?

Solution: The number of days in the leap year can be expressed as

$$366 = 52 \times 7 + 2 \tag{2.87}$$

The probability of one of the two remaining days being a Tuesday is $\frac{2}{7}$.

Chapter 3

Random Variables

3.1 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

Solution: The given information is summarised in Table 3.2. yielding

RV	**Values**	**Description**
X	{0,1,2,3}	Cards drawn randomly
Y	{0,1}	0:diamond ,1:spade
X,Y	{00,10,20,31}	3 diamonds and one spade out of 13 each

Table 3.2: Random variables(RV) X,Y and X,Y

$$\Pr\left(00, 10, 20, 31\right) = \frac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4} \tag{3.1}$$

$$= \frac{286}{20285} \tag{3.2}$$

3.2 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

Solution: The given information is summarised in Table 3.4 The total number of possible outcomes is ${}^{N}C_n$ and the total number of favourable outcomes is ${}^{q}C_n$ yielding

Variable	**Value**	**Description**
N	10000	Total number of tickets sold
k	10	Total number of prizes awarded
n	{0,1,2,...,N}	Number of tickets purchased
$\Pr(n)$		probability of not wining a prize
q	N-k	number of tickets with no prize

Table 3.4:

the desired probability

$$\Pr(n) = \frac{{}^{q}C_n}{{}^{N}C_n} \tag{3.3}$$

Substituting numerical values,

(a) For one ticket,

$$\Pr(1) = \frac{{}^{9990}C_1}{{}^{10000}C_1} = 0.9990 \tag{3.4}$$

(b) For two tickets,

$$\Pr(2) = \frac{{}^{9990}C_2}{{}^{10000}C_2} = 0.9980 \tag{3.5}$$

(c) For 10 tickets

$$\Pr(3) = \frac{{}^{9990}C_{10}}{{}^{10000}C_{10}} = 0.9901 \tag{3.6}$$

3.3 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

Solution: Table 3.6 summarises the given information.

RV	Values	Description
X	{0,1}	0: sectioin1, 1: section2
Y	{0,1}	0: student1, 1: student2
XY	{001,101}	Students enter same section
	{00,01,10,11}	Students enter different section

Table 3.6:

(a) When both enter the same section, the probability is

$$\Pr(001, 101) = \frac{{}^{40}C_2}{{}^{100}C_2} + \frac{{}^{60}C_2}{{}^{100}C_2} = \frac{156}{990} + \frac{354}{990} = 0.51 \tag{3.7}$$

(b) When both enter different sections, the desired probability is

$$\Pr(00, 01, 10, 11) = 1 - 0.51 = 0.49 \tag{3.8}$$

3.4 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9.The lock opens with a sequence of four digits with no repeats.What is the probability of a person getting the right sequence to open the suitcase.

Solution: The given information is represented in Tables 3.8 3.10 and 3.12.

Random variable	Value	Description
X	{1,2,3,4}	The number lock of a suitcase
Y	{0,1,2...9}	The digits labelled on each wheel

Table 3.8: Random variables X and Y

Wheel 1	Wheel 2	Wheel 3	Wheel 4
10 ways	9 ways	8 ways	7 ways

Table 3.10: Suitcase wheel

The number of possible placement of the digits are

$$10 \times 9 \times 8 \times 7 = 5040 \tag{3.9}$$

Thus, the probability of the correct sequence begin selected is

$$\Pr(A) = \frac{1}{5040} \tag{3.10}$$

3.5 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Table 3.14 summarizes the various events Given that the cards are drawn at random without replacement. Without replacement means only one card is random at a time and is excluded from the total while next card is drawn at random. Thus, the probability that both the cards are black is,

$$\Pr(00, 10) = \frac{{}^{26}C_1}{{}^{52}C_1} \times \frac{{}^{25}C_1}{{}^{51}C_1} = \frac{1}{2} \times \frac{25}{51} = 0.24 \tag{3.11}$$

3.6 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale,

Example	**Wheel**	**Outcome**
1	8 6 4 2	Not repeating
2	8 4 2 6	Not repeating
3	1 2 3 4	Not repeating
4	8 8 8 8	Repeating
5	1 1 2 2	Repeating

Table 3.12: Combinations

RV	Values	Description
X	{0,1}	number of cards drawn 2
Y	{0,1}	0: black card, 1: red card
XY	{00,10}	card drawn is black

Table 3.14:

(b) first ball is black and second is red.

(c) one of them is black and other is red.

3.8 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

Chapter 4

Conditional Probability

4.1 Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.

Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{4.1}$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{4.2}$$

4.2 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

4.3 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

4.4 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \tag{4.3}$$

4.5 If $\Pr(A) = 0.8$, $\Pr(B) = 0.5$ and $\Pr(B|A = 0.4)$, find

(a) $\Pr(AB)$

(b) $\Pr(A|B)$

(c) $\Pr(A + B)$

Solution:

(a) Since

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)}, \tag{4.4}$$

from the given information,

$$\frac{\Pr(AB)}{\Pr(A)} = 0.4 \tag{4.5}$$

$$\implies \Pr(AB) = 0.4 \times 0.8 \tag{4.6}$$

$$= 0.32 \tag{4.7}$$

(b) Similarly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{4.8}$$
$$= \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}. \tag{4.9}$$
$$= \frac{0.4 \times 0.8}{0.5} \tag{4.10}$$
$$= 0.64 \tag{4.11}$$

(c) Since,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{4.12}$$

Substituting (4.7) in (4.12),

$$\Pr(A+B) = 0.8 + 0.5 - 0.32 \tag{4.13}$$
$$= 0.98 \tag{4.14}$$

4.6 If $\Pr(A) = \frac{6}{11}$, $\Pr(B) = \frac{5}{11}$ and $\Pr(A+B) = \frac{7}{11}$, find

(a) $\Pr(AB)$

(b) $\Pr(A \mid B)$

(c) $\Pr(B \mid A)$

Solution:

(a) Since

$$\Pr(AB) = \Pr(A) + \Pr(B) - \Pr(A+B), \tag{4.15}$$

$$\Pr(AB) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11} \tag{4.16}$$

(b) Since

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}, \tag{4.17}$$

$$\Pr(A \mid B) = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5} \tag{4.18}$$

(c) Since

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)}, \tag{4.19}$$

$$\Pr(B \mid A) = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{2}{3} \tag{4.20}$$

4.7 Mother, Father and Son line up at random for a family picture. Determine $\Pr(E \mid F)$ where E: Son on one end, F: Father in middle

Solution: The total ways of arranging Father, Son, Mother in the family chart is 3! = 6. The probability that Father in middle is

$$\Pr(F) = \frac{2!}{3!} = \frac{1}{3} \tag{4.21}$$

The probability that Father in middle and Son is on one end is

$$\Pr(EF) = \frac{2!}{3!} = \frac{1}{3} \tag{4.22}$$

Thus,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} = 1 \tag{4.23}$$

4.8 A fair die is rolled. Consider events $E = 1, 3, 5 F = 2, 3$ and $G = 2, 3, 4, 5$. Find

(a) $\Pr(E \mid F)$ and $\Pr(F \mid E)$

(b) $\Pr(E \mid G)$ and $\Pr(G \mid E)$

(c) $\Pr(E \cup F \mid G)$ and $\Pr(E \cap F \mid G)$

Solution: The given information is summarised in Table 4.2.

Event	**Probability**
$\Pr(E)$	$\frac{1}{2}$
$\Pr(F)$	$\frac{1}{3}$
$\Pr(G)$	$\frac{2}{3}$
$\Pr(EF)$	$\frac{1}{6}$
$\Pr(EG)$	$\frac{1}{3}$
$\Pr(FG)$	$\frac{1}{3}$
$\Pr(EFG)$	$\frac{1}{6}$

Table 4.2: Probability of Events.

(a)

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \tag{4.24}$$
$$= \frac{\frac{1}{6}}{\frac{1}{3}} \tag{4.25}$$
$$= \frac{1}{2} \tag{4.26}$$
$$\Pr(F \mid E) = \frac{\Pr(FE)}{\Pr(E)} \tag{4.27}$$
$$= \frac{\frac{1}{6}}{\frac{1}{2}} \tag{4.28}$$
$$= \frac{1}{3} \tag{4.29}$$

(b)

$$\Pr(E \mid G) = \frac{\Pr(EG)}{\Pr(G)} \tag{4.30}$$
$$= \frac{\frac{1}{3}}{\frac{2}{3}} \tag{4.31}$$
$$= \frac{1}{2} \tag{4.32}$$
$$\Pr(G \mid E) = \frac{\Pr(GE)}{\Pr(G)} \tag{4.33}$$
$$= \frac{\frac{1}{3}}{\frac{1}{2}} \tag{4.34}$$
$$= \frac{2}{3} \tag{4.35}$$

(c)

$$\Pr(E+F \mid G) = \frac{\Pr((E+F)G)}{\Pr(G)} \tag{4.36}$$

$$= \frac{\Pr(EG+FG)}{\Pr(G)} \tag{4.37}$$

$$= \frac{\Pr(EG)+\Pr(FG)-\Pr(EFG)}{\Pr(G)} \tag{4.38}$$

$$= \frac{3}{4} \tag{4.39}$$

$$\Pr(EF \mid G) = \frac{\Pr(EFG)}{\Pr(G)} \tag{4.40}$$

$$= \frac{1}{4} \tag{4.41}$$

4.9 If $\Pr(A) = \frac{1}{2}$, $\Pr(B) = 0$, then $\Pr(A \mid B)$ is

(a) 0

(b) $\frac{1}{2}$

(c) not defined

(d) 1

Since

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}, \tag{4.42}$$

$$\Pr(A \mid B) \text{ is not defined} \tag{4.43}$$

4.10 If A and B are events such that

$$\Pr(A|B) = \Pr(B|A) \tag{4.44}$$

then

(a) $A \subset B$ but $A \neq B$

(b) $A = B$

(c) $A \cap B = \phi$

(d) $\Pr(A) = \Pr(B)$

Solution: Using Bayes' Rule,

$$\begin{aligned}\Pr(AB) &= \Pr(A)\Pr(B|A) &(4.45)\\ &= \Pr(B)\Pr(A|B) &(4.46)\end{aligned}$$

Using (4.44) in (4.45) and (4.46),

$$\Pr(A) = \Pr(B) \tag{4.47}$$

We consider the options one by one.

(a) If $A \subset B$ and $A \neq B$, then we can write $B = A + C$, where $AC = 0$ and $C \neq 0$. Thus,

$$\begin{aligned}\Pr(B) &= \Pr(A + C) &(4.48)\\ &= \Pr(A) + \Pr(C) - \Pr(AC) &(4.49)\\ &= \Pr(A) + \Pr(C) > \Pr(A) &(4.50)\end{aligned}$$

However, (4.50) contradicts (4.47).

(b) We give a counterexample to show this is wrong. Consider A as the event that an even number shows on rolling a fair die and B as the event that a prime number shows on rolling a fair die. The joint pmf is shown in Table 4.3. Clearly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{2} \tag{4.51}$$

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{2} \tag{4.52}$$

(c) The same example as before provides the required counterexample, as $\Pr(AB) = \frac{1}{6}$.

(d) This is the correct answer, as discussed above.

	A	$\bar{A}$
B	$\frac{1}{6}$	$\frac{1}{3}$
$\bar{B}$	$\frac{1}{3}$	$\frac{1}{6}$

Table 4.3: Joint pmf for events A and B.

4.11 Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Solution: Let X denote the die roll for the first trial. The pmf of X is

$$\Pr(X = k) = \begin{cases} \frac{1}{6} & 1 \le i \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{4.53}$$

Let Y be the random variable denoting the outcome of the coin toss in the second trial. The pmf of Y is

$$\Pr(Y = k) = \begin{cases} \frac{1}{2} & 0 \leq i \leq 1 \\ \\ 0 & \text{otherwise} \end{cases} \tag{4.54}$$

We are required to find $\Pr(Y = 1|X = 3)$. However, from the given data,

$$\Pr(Y = 1,\ X = k) = \begin{cases} \frac{1}{12} & k \in \{1, 2, 4, 5\} \\ \\ 0 & \text{otherwise} \end{cases} \tag{4.55}$$

Therefore, from (4.55),

$$\Pr(Y = 1|X = 3) = \frac{\Pr(X = 3,\ Y = 1)}{\Pr(X = 3)} = 0 \tag{4.56}$$

4.12 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Let E_1 denote the event that the first card drawn is Black, E_2 denote the event that the second card drawn is Black. Then

$$\Pr(E_1) = \frac{26}{52}, \Pr(E_2 \mid E_1) = \frac{25}{51} \tag{4.57}$$

$$\implies \Pr(E_1 E_2) = \Pr(E_1) \Pr(E_2 \mid E_1) = \frac{25}{102} \tag{4.58}$$

4.13 Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) $P(A|B)$

(d) $P(B|A)$

4.14 An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

Solution: The given information is summarized in Tables 4.4 and 4.5.

RV	**Values**	**Description**
X	$\{0,1\}$	1st draw - 0: Red, 1: Black
Y	$\{0,1\}$	2nd draw - 0: Red, 1: Black

Table 4.4: Random variables X,Y

Event	**Probability**
$\Pr(X=0)$	$\frac{5}{10}$
$\Pr(X=1)$	$\frac{5}{10}$
$\Pr(Y=1 \mid X=0)$	$\frac{7}{12}$
$\Pr(Y=1 \mid X=1)$	$\frac{5}{12}$

Table 4.5: Probabilities

The required probability is given by

$$\Pr(Y=0) = \Pr(X=0)\Pr(Y=0 \mid X=0) + \Pr(X=1)\Pr(Y=0 \mid X=1) \quad (4.59)$$

$$= \left(\frac{5}{10} \times \frac{7}{12}\right) + \left(\frac{5}{10} \times \frac{5}{12}\right) = \frac{1}{2} \quad (4.60)$$

4.15 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

4.16 Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

4.17 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution: Let $X \in \{0, 1\}$ where 0 denotes a guess and 1 denotes that he knows the answer. Let $Y \in \{0, 1\}$ where 0 being the case when the answer is incorrect and 1 being the case that the answer is correct. The given information is summarised in Tables 4.9 and 4.7

Random variable	**Description**
X=0	Student guesses the answer
X=1	Student knows the answer
Y=0	Answer is incorrect
Y=1	Answer is correct

Table 4.7: Random variables X and Y

Pr(Event)	**Value**
Pr(Y=1 | X=0)	0.25
Pr(Y=1|X=1)	1
Pr(X=0)	0.75
Pr(X=1)	0.25

Table 4.9: Probability of events X and Y

The probability that the student knows the answer and he answered it correctly is

$$\Pr(X=1|Y=1) = \frac{\Pr(Y=1|X=1)\Pr(X=1)}{\Pr(Y=1|X=1)\Pr(X=1)+\Pr(Y=1|X=0)\Pr(X=0)} \tag{4.61}$$

$$= \frac{0.25}{0.25+0.25\times 0.75} \tag{4.62}$$

$$= 0.571 \tag{4.63}$$

4.18 A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

4.19 There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

Solution: Define the random variable X as in Table 4.10. Clearly, the pmf of X is

$X=1$	Two-headed coin is selected.
$X=2$	75% biased coin is selected.
$X=3$	Fair coin is selected.

Table 4.10: Definition of X.

$$\Pr(X = k) = \begin{cases} \frac{1}{3} & 1 \leq k \leq 3 \\ 0 & \text{otherwise} \end{cases} \tag{4.64}$$

Let the random variables Y_1, Y_2 and Y_3 (one for each coin) be defined as

$$Y_1 \sim \text{Ber}(1) \tag{4.65}$$
$$Y_2 \sim \text{Ber}\left(\frac{3}{4}\right) \tag{4.66}$$
$$Y_3 \sim \text{Ber}\left(\frac{1}{2}\right) \tag{4.67}$$

Define Y as

$$Y \triangleq \sum_{i=1}^{3} \mathbf{1}_i(X)\, Y_i \tag{4.68}$$

where $\mathbf{1}$ denotes the indicator random variable, defined as

$$\mathbf{1}_i(X) = \begin{cases} 1 & \text{if } X = i \\ 0 & \text{otherwise} \end{cases} \tag{4.69}$$

We are required to find $\Pr(X = 1|Y = 1)$. However, from Bayes' Rule,

$$\Pr(X = 1, Y = 1) = \Pr(X = 1)\Pr(Y = 1|X = 1) \tag{4.70}$$
$$= \Pr(Y = 1)\Pr(X = 1|Y = 1) \tag{4.71}$$

Note from (4.68) that

$$X = 1 \implies Y = Y_1 \tag{4.72}$$

and also,

$$\Pr(Y=1) = \sum_{i=1}^{3} \Pr(X=i)\Pr(Y_i=1) \tag{4.73}$$

$$= \frac{1}{3}\left(1+\frac{3}{4}+\frac{1}{2}\right) = \frac{3}{4} \tag{4.74}$$

Thus, from (4.70), (4.71) and (4.74), we see that

$$\Pr(X=1|Y=1) = \frac{\Pr(X=1)\Pr(Y_1=1)}{\Pr(Y=1)} \tag{4.75}$$

$$= \frac{\frac{1}{3}}{\frac{3}{4}} = \frac{4}{9} \tag{4.76}$$

4.20 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

4.21 A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

4.22 . Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding prbability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

Solution: The given information is listed in Tables 4.12 and 4.14

RV	**Values**	**Description**
X	{1,2}	1:Group1 ,2:Group2
Y	{0,1}	0:New product not introduced ,1:New product introduced

Table 4.12: Random variables(RV) X,Y

Event	**Probability**	**Description**
$\Pr(X=1)$	0.6	First group winning
$\Pr(X=2)$	0.4	Second group winning
$\Pr(Y=1 \mid X=1)$	0.7	Introducing 1 if 1 wins
$\Pr(Y=1 \mid X=2)$	0.3	Introducing 1 if 2 wins

Table 4.14: Probabilities

$$\Pr(X=2 \mid Y=1) = \frac{\Pr(2)\Pr(1 \mid 2)}{\Pr(1)\Pr(1 \mid 1) + \Pr(2)\Pr(1 \mid 2)} = \frac{2}{9} \tag{4.77}$$

4.23 Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?]

4.24 A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for

30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

4.25 A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

4.26 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

Solution: Consider the random variables A, X as described in the table 4.15.

RV	**Values**	**Description**
A	$\{0, 1\}$	1: A speaks truth, 0: A lies
X	$\{0, 1\}$	1: Heads, 0: Tails

Table 4.15: Random variables A, X

The given information about probabilities is listed in table 4.16.

Event	**Probability**
$\Pr(A = 1)$	$\frac{4}{5}$
$\Pr(X = 1)$	$\frac{1}{2}$
$\Pr(X = 1 \mid A = 1)$	$\frac{1}{2}$

Table 4.16: Probabilities

The required probability is given by

$$\Pr(A=1 \mid X=1) = \frac{\Pr(A=1)\Pr(X=1 \mid A=1)}{\Pr(X=1)} \tag{4.78}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{1}{2}} \tag{4.79}$$

$$= \frac{4}{5} \tag{4.80}$$

4.27 If A and B are two events such that $A \subset B$ and $\Pr(B) \neq 0$, then which of the following is correct ?

(a) $\Pr(A \mid B) = \frac{\Pr(B)}{\Pr(A)}$

(b) $\Pr(A \mid B) < \Pr(A)$

(c) $\Pr(A \mid B) \geq \Pr(A)$

(d) None of these

Solution: if $A \subset B$ and $\Pr(B) \neq 0$ then

$$AB = A \tag{4.81}$$

$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \tag{4.82}$$

we know that

$$\Pr(B) \leq 1 \tag{4.83}$$

$$\implies 1 \leq \frac{1}{\Pr(B)} \tag{4.84}$$

Multiplying both sides with $\Pr(A)$,

$$\Pr(A) \leq \frac{\Pr(A)}{\Pr(B)} \tag{4.85}$$

$$= \Pr(A \mid B) \tag{4.86}$$

from (4.82).

4.28 A and B are two events such that $\Pr(A) \neq 0$. Find $\Pr(B \mid A)$, if

(a) A is a subset of B

(b) $A \cap B = \phi$

Solution: We use

$$\Pr(B \mid A) = \frac{\Pr(BA)}{\Pr(A)} \tag{4.87}$$

(a) In this case,

$$BA = A \implies \Pr(BA) = \Pr(A) \tag{4.88}$$

From (4.87),

$$\Pr(B \mid A) = 1 \tag{4.89}$$

(b) $A \cap B = \phi$. This implies

$$\Pr(BA) = 0 \tag{4.90}$$

From (4.87),

$$\Pr(B \mid A) = 0 \tag{4.91}$$

4.29 A couple has two children.

(a) Find the probability that both children are males, if it is known that at least one of the children is male.

(b) Find the probability that both children are females, if it is known that the elder child is a female.

Solution: Consider the random variables X, Y, which denotes the first child, second child gender respectively as described in table 4.17.

RV	**Values**	**Description**
X	$\{0, 1\}$	0: Male , 1: Female
Y	$\{0, 1\}$	0: Male, 1: Female

Table 4.17: Random variables X

The probabilities for the random variables X, Y is listed in table 4.18.

Event	**Probability**
$\Pr(X = 0)$	$\frac{1}{2}$
$\Pr(X = 1)$	$\frac{1}{2}$
$\Pr(Y = 0)$	$\frac{1}{2}$
$\Pr(Y = 1)$	$\frac{1}{2}$
$\Pr(X + Y = 0)$	$\frac{1}{4}$
$\Pr(X + Y = 2)$	$\frac{1}{4}$
$\Pr(XY = 0)$	$\frac{3}{4}$

Table 4.18: Probabilities

The probability $\Pr(XY = 0)$ is given by

$$= \Pr(X = 0) + \Pr(Y = 0) - \Pr(X + Y = 0) \tag{4.92}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \tag{4.93}$$

$$= \frac{3}{4} \tag{4.94}$$

(a) The event of both children being Male is when $X + Y = 0$. The event of atleast one of the children being Male is when $XY = 0$.

$$\{X + Y = 0\} \cap \{XY = 0\} \equiv \{X + Y = 0\} \tag{4.95}$$

The required probability is given by,

$$\Pr(X + Y = 0 \mid XY = 0) \tag{4.96}$$

$$= \frac{\Pr(X + Y = 0)}{\Pr(XY = 0)} \tag{4.97}$$

$$= \frac{1}{3} \tag{4.98}$$

(b) The event of both children being Female is when $X + Y = 2$. The event of elder child being Female is when $X = 1$.

$$\{X + Y = 2\} \cap \{X = 1\} \equiv \{X + Y = 2\} \tag{4.99}$$

The required probability is given by,

$$\Pr\left(X+Y=2 \mid X=1\right) \tag{4.100}$$

$$= \frac{\Pr\left(X+Y=2\right)}{\Pr\left(X=1\right)} \tag{4.101}$$

$$= \frac{1}{2} \tag{4.102}$$

Chapter 5

Distributions

5.1 A lot consists of 144 ball pens of which 20 are defective and the others are good. Nuri will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

(a) She will buy it?

(b) She will not buy it?

Solution: We can model this situation using the random variable $X \sim \text{Ber}(p)$, where p is the probability of success, *i.e.* the pen is purchased. From the given data,

$$1 - p = \frac{20}{144} \implies p = \frac{67}{72} \tag{5.1}$$

(a) Probability that the pen is purchased is

$$\Pr(X = 1) = p = \frac{67}{72} \tag{5.2}$$

(b) Probability that the pen is not purchased is

$$\Pr(X = 0) = 1 - p = \frac{5}{72} \tag{5.3}$$

5.2 A die is thrown, find the probability of following events:

(a) A prime number will appear

(b) A number greater than or equal to 3 will appear

(c) A number less than or equal to one will appear

(d) A number more than 6 will appear

(e) A number less than 6 will appear

Solution: The CDF of the random variable X representing the roll of a dice, is available in (C.3.3.1).

(a) The set of possible prime numbers in a die roll contains 2,3,5

$$\Pr\left(X \in \{2,3,5\}\right) = p_X\left(2\right) + p_X\left(3\right) + p_X\left(5\right) \tag{5.4}$$
$$= \frac{1}{2} \tag{5.5}$$

(b) The probability that a number greater than or equal to 3 will appear is given by

$$\Pr\left(X \geq 3\right) = 1 - \Pr\left(X \leq 2\right) \tag{5.6}$$
$$= 1 - F_X\left(2\right) \tag{5.7}$$
$$= \frac{2}{3} \tag{5.8}$$

(c) The probability that a number less than or equal to 1 will appear is given by

$$\Pr\left(X \leq 1\right) = F_X\left(1\right) \tag{5.9}$$
$$= \frac{1}{6} \tag{5.10}$$

(d) The probability that a number greater than 6 will appear is given by

$$\Pr(X > 6) = 1 - \Pr(X \le 6) \tag{5.11}$$
$$= 1 - F_X(6) \tag{5.12}$$
$$= 0 \tag{5.13}$$

(e) The probability that a number less than 6 will appear is given by

$$\Pr(X < 6) = \Pr(X \le 5) \tag{5.14}$$
$$= F_X(5) \tag{5.15}$$
$$= \frac{5}{6} \tag{5.16}$$

5.3 Three coins are tossed once. Find the probability of getting

(a) 3 heads

(b) 2 heads

(c) atleast 2 heads

(d) atmost 2 heads

(e) no head

(f) 3 tails

(g) exactly two tails

(h) no tail

(i) atmost two tails

Solution: Let the random variable X denote one single coin toss, where obtaining a head is considered a success. Then,

$$X \sim \mathrm{Ber}\left(p\right) \tag{5.17}$$

Suppose $X_i, 1 \le i \le n$ represent each of the n tosses. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{5.18}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \mathrm{Bin}\left(n, p\right) \tag{5.19}$$

The cdf of Y is given by

$$F_Y\left(k\right) = \Pr\left(Y \le k\right) \tag{5.20}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ i \end{pmatrix} p^i \left(1-p\right)^{n-i} & 1 \le k \le n \\ 1 & k \ge n \end{cases} \tag{5.21}$$

In this case,

$$p = \frac{1}{2}, \ n = 3 \tag{5.22}$$

(a) We require $\Pr(Y = 3)$. Thus, from (5.19),

$$\Pr(Y = 3) = \begin{pmatrix} n \\ 3 \end{pmatrix} p^3 (1-p)^{n-3} \tag{5.23}$$

$$= \frac{1}{8} \tag{5.24}$$

(b) We require $\Pr(Y = 2)$. Thus, from (5.19),

$$\Pr(Y = 2) = \begin{pmatrix} n \\ 2 \end{pmatrix} p^2 (1-p)^{n-2} \tag{5.25}$$

$$= \frac{3}{8} \tag{5.26}$$

(c) We require $\Pr(Y \geq 2)$. Since $n = 3$ in (5.21),

$$\Pr(Y \geq 2) = 1 - \Pr(Y < 2) \tag{5.27}$$

$$= F_Y(3) - F_Y(1) \tag{5.28}$$

$$= \sum_{k=2}^{3} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \tag{5.29}$$

$$= \frac{1}{2} \tag{5.30}$$

(d) We require $\Pr(Y \leq 2)$. Thus, from (5.21),

$$\Pr(Y \leq 2) = \sum_{k=0}^{2} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \tag{5.31}$$

$$= \frac{7}{8} \tag{5.32}$$

(e) We require $\Pr(Y=0)$. Thus, from (5.19),

$$\Pr(Y=0) = \begin{pmatrix} n \\ 0 \end{pmatrix} p^0 (1-p)^n \tag{5.33}$$

$$= \frac{1}{8} \tag{5.34}$$

(f) Obtaining 3 tails is the same as obtaining no heads. Hence, from (5.34), we require $\Pr(Y=0) = \frac{1}{8}$.

(g) We require $\Pr(Y=1)$ (since only one head is obtained). Thus, from (5.19),

$$\Pr(Y=1) = \begin{pmatrix} n \\ 1 \end{pmatrix} p^1 (1-p)^{n-1} \tag{5.35}$$

$$= \frac{3}{8} \tag{5.36}$$

(h) We require $\Pr(Y=3) = \frac{1}{8}$ from (5.24).

(i) We require $\Pr(Y \geq 1)$ (since at least one head is obtained). Thus, from (5.21) and (5.34),

$$\Pr(Y \geq 1) = 1 - \Pr(Y < 1) \tag{5.37}$$

$$= 1 - F_Y(0) \tag{5.38}$$

$$= 1 - \Pr(Y=0) \tag{5.39}$$

$$= \frac{7}{8} \tag{5.40}$$

5.4 Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

5.5 A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that

(a) all will be blue?

(b) atleast one will be green?

Solution: See (D.1.2). In this question,

$$N = 60,\; R = 10,\; B = 20,\; G = 30,\; n = 5 \tag{5.41}$$

(a) From (D.1.2),

$$\Pr(0, 5, 0) = \frac{{}^{20}C_5}{{}^{60}C_5} \tag{5.42}$$

(b) Since

$$\Pr(r, b, 0) = \frac{{}^{R}C_r\,{}^{B}C_b}{{}^{R+B+G}C_{r+b}} \tag{5.43}$$

The probability that at least one marble is green is given by

$$1 - \sum_{r+b=n} \Pr(r, b, 0) = 1 - \sum_{r+b=n} \frac{{}^{R}C_r\,{}^{B}C_b}{{}^{R+B+G}C_{r+b}} = 1 - \frac{{}^{R+B}C_n}{{}^{R+B+G}C_n} \tag{5.44}$$

from (D.2.1). Substituting numerical values, the desired probability is

$$1 - \frac{{}^{30}C_5}{{}^{60}C_5} \tag{5.45}$$

5.6 A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game

5.7 A coin is tossed three times, where. Determine $\Pr\left(E \mid F\right)$ where

(a) E : head on third toss, F : heads on first two tosses

(b) E : at least two heads, F : at most two heads

(c) E : at most two tails, F : at least one tail

Solution: Consider the random variables X_1, X_2, X_3, X, which denotes the first, second, third toss and number of heads in the 3 tosses respectively as described in table 5.1.

RV	**Values**	**Description**
X	$\{0,1,2,3\}$	Number of heads in 3 tosses
X_1	$\{0,1\}$	0: Heads , 1: Tails
X_2	$\{0,1\}$	0: Heads , 1: Tails
X_3	$\{0,1\}$	0: Heads , 1: Tails

Table 5.1: Random variables X_1, X_2, X_3, X

The random variable X follows binomial distribution

$$X = X_1 + X_2 + X_3 \tag{5.46}$$

The PMF of the random variable X is given by,

$$P_X\left(n\right) = {}^{N}C_n\, p^n \left(1-p\right)^{N-n} \tag{5.47}$$

Here we have

$$N = 3,\, p = \frac{1}{2} \tag{5.48}$$

The CDF of the random variable X is given by,

$$F_X(n) = \Pr(X \leq n) = \sum_{i=0}^{n} {}^{N}C_i\, p^i (1-p)^{N-i} \tag{5.49}$$

(a) The events E, F can be described by the RV as

$$E : X_3 = 0 \tag{5.50}$$

$$F : X_1 + X_2 = 0 \tag{5.51}$$

Y is another random variable which represents the number of heads in first two tosses.

$$Y = X_1 + X_2 \tag{5.52}$$

The PMF of the random variable Y is given by,

$$P_Y(n) = {}^{N}C_n\, p^n (1-p)^{N-n} \tag{5.53}$$

Here we have

$$N = 2,\, p = \frac{1}{2} \tag{5.54}$$

The event EF can be expressed as,

$$X_3 = 0 \cap X_1 + X_2 = 0 \tag{5.55}$$

$$\triangleq X_1 + X_2 + X_3 = 0 \tag{5.56}$$

$$\implies X = 0 \tag{5.57}$$

The required probability is given by,

$$\Pr\left(X_3 = 0 \mid Y = 0\right) \tag{5.58}$$

$$= \frac{\Pr\left(X = 0\right)}{\Pr\left(Y = 0\right)} \tag{5.59}$$

$$= \frac{1}{2} \tag{5.60}$$

(b) The events E, F, F' can be described by the RV as

$$E : X \leq 1 \tag{5.61}$$

$$F : X \geq 1 \tag{5.62}$$

$$F' : X = 0 \tag{5.63}$$

The required probability is given by,

$$= \frac{\Pr\left(EF\right)}{1 - \Pr\left(F'\right)} \tag{5.64}$$

The event EF can be expressed as,

$$X \leq 1 \cap X \geq 1 \tag{5.65}$$

$$\implies X = 1 \tag{5.66}$$

Hence, the required probability is given by,

$$= \frac{\Pr(X = 1)}{1 - \Pr(X = 0)} \tag{5.67}$$

$$= \frac{\frac{3}{8}}{1 - \frac{1}{8}} \tag{5.68}$$

$$= \frac{3}{7} \tag{5.69}$$

(c) For the events E, F, their complements are E' : all 3 tails, F' : zero tails. The events E', F' can be described by the RV as

$$E' : X = 3 \tag{5.70}$$

$$F' : X = 0 \tag{5.71}$$

By using property of conditional probability we have,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \tag{5.72}$$

$$= \frac{1 - \Pr(E' + F')}{\Pr(F)} \tag{5.73}$$

The required probability is given by,

$$= \frac{1 - \Pr(X = 0 + X = 3)}{1 - \Pr(X = 0)} \tag{5.74}$$

$$= \frac{1 - (\Pr(X = 0) + \Pr(X = 3) - \Pr(\phi))}{1 - \Pr(X = 0)} \tag{5.75}$$

$$= \frac{1 - \left(\frac{1}{8} + \frac{1}{8} - 0\right)}{1 - \frac{1}{8}} \tag{5.76}$$

$$= \frac{6}{7} \tag{5.77}$$

5.8 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Solution: Choosing

$$R = 12, B = 3, G = 0, n = 3, r = 3, b = 0, g = 0 \tag{5.78}$$

in (D.1.2) the desired probability is

$$\Pr(3, 0, 0) = \frac{{}^{12}C_3}{{}^{15}C_3} = \frac{44}{91} \tag{5.79}$$

5.9 A die is tossed thrice. Find the probability of getting an odd number at least once.

5.10 Find the probability distribution of

(a) number of heads in two tosses of a coin.

(b) number of tails in the simultaneous tosses of three coins.

(c) number of heads in four tosses of a coin.

Solution: Table 5.3 summarises the given information.

Variable	**Value**	**Description**
n	$\{2, 3, 4\}$	Number of trials in 2,3,4 tosses of a coin
p	$\frac{1}{2}$	Probability of getting a head
q	$1 - p$	Probability of not getting a head
X_1	$\{0, 1, 2\}$	Number of heads in 2 tosses of a coin
X_2	$\{0, 1, 2, 3\}$	Number of tails in 3 tosses of a coin
X_3	$\{0, 1, 2, 3, 4\}$	Number of heads in 4 tosses of a coin

Table 5.3: Variable Description

(a) Number of heads in two tosses of a coin.

$$p_{X_1}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 2, n = 2 \tag{5.80}$$

(b) Number of tails in the simultaneous tosses of three coins.

$$p_{X_2}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 3, n = 3 \tag{5.81}$$

(c) Number of heads in four tosses of a coin.

$$p_{X_3}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 4, n = 4 \tag{5.82}$$

5.11 The random variable X has a probability distribution $\Pr(X)$ of the following form, where k is some number

$$\Pr(X) = \begin{cases} k, & x = 0 \\ 2k, & x = 1 \\ 3k, & x = 2 \\ 0, & \text{otherwise} \end{cases} \tag{5.83}$$

(a) Determine the value of k

(b) Find $\Pr(X < 2)$,$\Pr(X \leq 2)$,$\Pr(X \geq 2)$

Solution:

(a) Using the axioms of probability,

$$k + 2k + 3k = 1 \implies k = \frac{1}{6} \tag{5.84}$$

(b) The CDF is given by

$$F_X(k) = \begin{cases} 0, & x < 0 \\ \frac{1}{6}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases} \tag{5.85}$$

Thus,

i.

$$\Pr(X < 2) = F(1) = \frac{1}{2} \tag{5.86}$$

ii.

$$\Pr(X \le 2) = F(2) = 1 \tag{5.87}$$

iii.

$$\Pr(X \ge 2) = 1 - \Pr(X < 2) = 1 - F(1) = \frac{1}{2} \tag{5.88}$$

5.12 Find the mean number of heads in three tosses of a fair coin.

Solution: Substituting $n = 3, p = \frac{1}{2}$ in (C.2.3.1), the mean is $\frac{3}{2}$.

5.13 State which of the following are not the probability distributions of a random variable. Give reasons for your answer

X	0	1	2
P(X)	0.4	0.4	0.2

i

X	0	1	2	3	4
P(X)	0.1	0.5	0.2	-0.1	0.3

ii

Y	-1	0	1
P(Y)	0.6	0.1	0.2

iii

iv

Z	3	2	1	0	-1
P(Z)	0.3	0.2	0.4	0.1	-0.05

5.14 An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that

i all will bear 'X' mark.

ii not more than 2 will bear 'Y' mark.

iii at least one ball will bear 'Y' mark.

iv the number of balls with 'X' mark and 'Y' mark will be equal.

Solution: Let the random variable X denote one single draw with p being the probability that a ball marked 'X' is drawn. Then,

$$X \sim \text{Ber}\left(p\right) \tag{5.89}$$

Suppose $X_i, 1 \leq i \leq n$ represent each of the n draws. Define Y as

$$Y = \sum_{i=1}^{n} X_i \tag{5.90}$$

Then, since the X_i are iid, the pmf of Y is given by

$$Y \sim \text{Bin}\left(n, p\right) \tag{5.91}$$

The cdf of Y is given by

$$F_Y(k) = \Pr(Y \leq k) \tag{5.92}$$

$$= \begin{cases} 0 & k < 0 \\ \sum_{i=1}^{k} \begin{pmatrix} n \\ \\ i \end{pmatrix} p^i (1-p)^{n-i} & 1 \leq k \leq n \\ 1 & k \geq n \end{cases} \tag{5.93}$$

In this case,

$$p = \frac{2}{5}, \ n = 6 \tag{5.94}$$

i We require $\Pr(Y = 6)$. Thus, from (5.91),

$$\Pr(Y = 6) = \begin{pmatrix} n \\ 6 \end{pmatrix} p^6 (1-p)^0 \tag{5.95}$$

$$= 0.004096 \tag{5.96}$$

ii We require $\Pr(Y \geq 4)$. Thus, from (5.93),

$$\Pr(Y \geq 4) = 1 - \Pr(Y < 4) \tag{5.97}$$

$$= F_Y(6) - F_Y(3) \tag{5.98}$$

$$= \sum_{k=4}^{6} \Pr(Y = k) \tag{5.99}$$

$$= \sum_{k=4}^{6} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \tag{5.100}$$

$$= 0.1792 \tag{5.101}$$

iii We require $\Pr(Y \leq 5)$. Since $n = 6$ in (5.93), using (5.96) gives

$$\Pr(Y \leq 5) = F_Y(5) \tag{5.102}$$

$$= 1 - \Pr(Y = 6) \tag{5.103}$$

$$= 0.995904 \tag{5.104}$$

iv We require $\Pr(Y = 3)$. Thus, from (5.91),

$$\Pr(Y = 3) = \begin{pmatrix} n \\ 3 \end{pmatrix} p^3 (1-p)^{n-3} \tag{5.105}$$

$$= 0.27648 \tag{5.106}$$

5.15 An urn contains 5 red and 2 black balls. Two balls are randomly drawn. Let X represent the number of black balls. What are the possible values of X? Is X a random variable ?

5.16 Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed 6 times. What are possible values of X?

5.17 Find the probability distribution of

i number of heads in two tosses of a coin.

ii number of tails in the simultaneous tosses of three coins.

iii number of heads in four tosses of a coin.

5.18 Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as

i number greater than 4

ii six appears on at least one die

5.19 From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

5.20 A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

5.21 A coin is tossed twice, what is the probability that atleast one tail occurs?

Solution: By using binomial distribution, the desired probability is given by

$$\Pr\left(Y \geq 1\right) = \sum_{k=1}^{2} \binom{n}{k} p^k (1-p)^{n-k} = \frac{3}{4} \tag{5.107}$$

upon substituting $p = \frac{1}{2}$.

5.22 A random variable X has the following probability distribution

Determine

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

i k

ii P($X < 3$)

iii P($X > 6$)

iv P($0 < X < 3$)

5.23 The random variable X has a probability distribution P(X) of the following form, where k is some number :

$$P(x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & otherwise \end{cases}$$

i Determine the value of k.

ii Find P ($X < 2$), P ($X \leq 2$), P($X \geq 2$)

5.24 Find the mean number of heads in three tosses of a fair coin.

5.25 Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

5.26 Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E(X).

5.27 Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance and standard deviation of X.

5.28 A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X? Find mean, variance and standard deviation of X.

5.29 In a meeting, 70A member is selected at random and we take X = 0 if he opposed, and X = 1 if he is in favour. Find E(X) and Var (X).

Choose the correct answer in each of the following:

5.30 The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is

i 1

ii 2

iii 5

iv $\frac{8}{3}$

5.31 Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is

i $\frac{37}{221}$

ii $\frac{5}{13}$

iii $\frac{1}{13}$

iv $\frac{2}{13}$

Chapter 6

Random Numbers

6.1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

6.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Execute the following C program.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//Uniform random numbers
uniform("uni.dat", 1000000);

//Gaussian random numbers
```

```
gaussian("gau.dat", 1000000);


//Mean of uniform
//printf("%lf",mean("uni.dat"));
return 0;
}
```

```
//Function declaration
void uniform(char *str, int len);
void gaussian(char *str, int len);
double mean(char *str);
//End function declaration




//Defining the function for generating uniform random numbers
void uniform(char *str, int len)
{
int i;
FILE *fp;


fp = fopen(str,"w");
//Generate numbers
for (i = 0; i < len; i++)
{
```

```
fprintf(fp,"%lf\n",(double)rand()/RAND_MAX);
}
fclose(fp);

}
//End function for generating uniform random numbers

//Defining the function for calculating the mean of random numbers
double mean(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
```

```
return temp;


}
//End function for calculating the mean of random numbers


//Defining the function for generating Gaussian random numbers
void gaussian(char *str, int len)
{
int i,j;
double temp;
FILE *fp;


fp = fopen(str,"w");
//Generate numbers
for (i = 0; i < len; i++)
{
temp = 0;
for (j = 0; j < 12; j++)
{
temp += (double)rand()/RAND_MAX;
}
temp-=6;
fprintf(fp,"%lf\n",temp);
}
fclose(fp);
```

```
}
//End function for generating Gaussian random numbers
```

6.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \leq x\right) \tag{6.1}$$

Solution: The following code plots Fig. 6.1

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if



x = np.linspace(-4,4,30)#points on the x axis
simlen = int(1e6) #number of samples
err = [] #declaring probability list
#randvar = np.random.normal(0,1,simlen)
```

```
randvar = np.loadtxt('uni.dat',dtype='double')
#randvar = np.loadtxt('gau.dat',dtype='double')
for i in range(0,30):
        err_ind = np.nonzero(randvar < x[i]) #checking probability condition
        err_n = np.size(err_ind) #computing the probability
        err.append(err_n/simlen) #storing the probability values in a list


plt.plot(x.T,err)#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x$')
plt.ylabel('$F_X(x)$')

#if using termux
plt.savefig('../figs/uni_cdf.pdf')
plt.savefig('../figs/uni_cdf.eps')
subprocess.run(shlex.split("termux-open ../figs/uni_cdf.pdf"))
#if using termux
#plt.savefig('../figs/gauss_cdf.pdf')
#plt.savefig('../figs/gauss_cdf.eps')
#subprocess.run(shlex.split("termux-open ../figs/gauss_cdf.pdf"))
#else
#plt.show() #opening the plot window
```

6.1.3 Find a theoretical expression for $F_U(x)$.

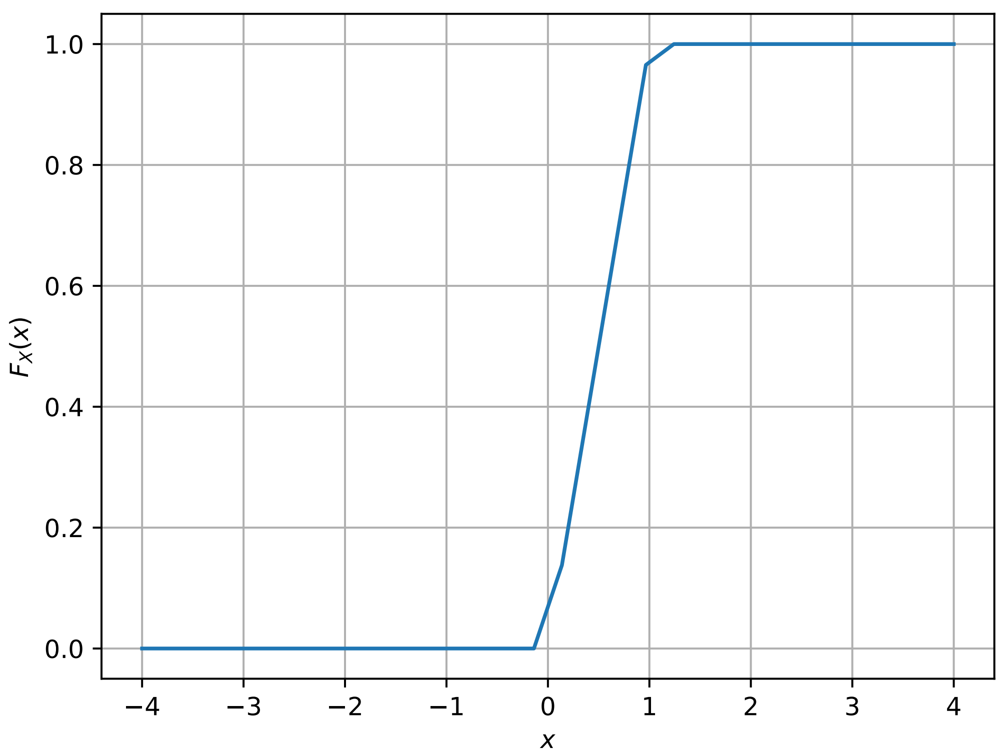

Figure 6.1: The CDF of U

6.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{6.2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{6.3}$$

Write a C program to find the mean and variance of U.

6.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{6.4}$$

6.2. Central Limit Theorem

6.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{6.5}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

6.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 6.2

6.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat.

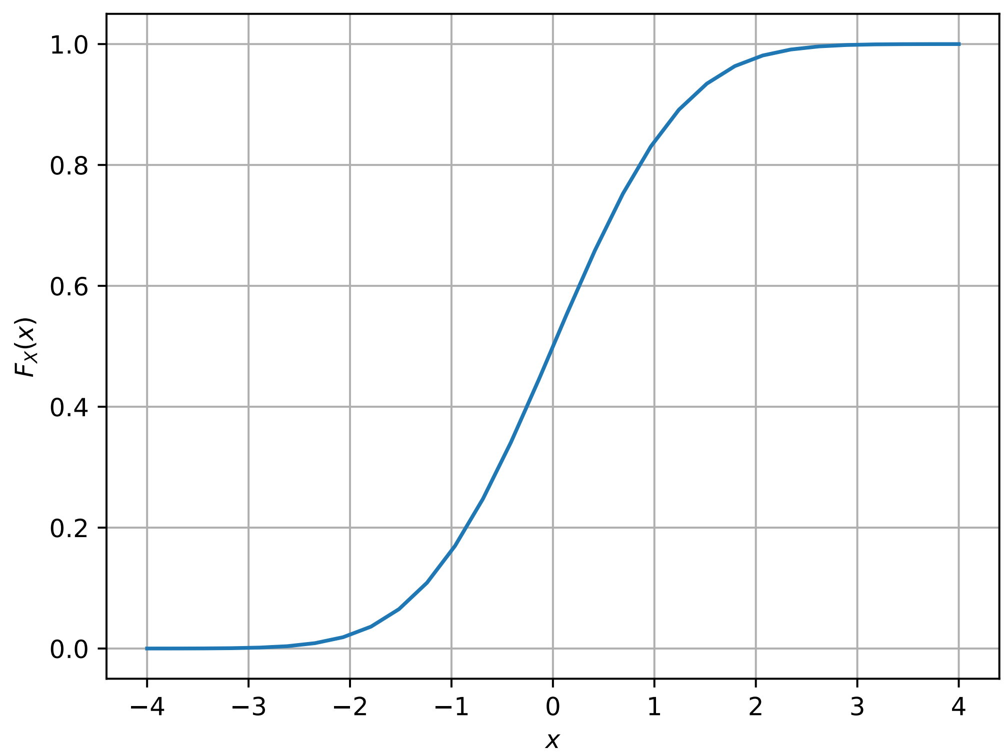

Figure 6.2: The CDF of X

The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{6.6}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 6.3 using the code below

```
#Importing numpy, scipy, mpmath and pyplot
import numpy as np
import mpmath as mp
import scipy
import matplotlib.pyplot as plt

#if using termux
import subprocess
import shlex
#end if



maxrange=50
maxlim=6.0
x = np.linspace(-maxlim,maxlim,maxrange)#points on the x axis
simlen = int(1e6) #number of samples
err = [] #declaring probability list
pdf = [] #declaring pdf list
h = 2*maxlim/(maxrange-1);
#randvar = np.random.normal(0,1,simlen)
```

```
#randvar = np.loadtxt('uni.dat',dtype='double')
randvar = np.loadtxt('gau.dat',dtype='double')

for i in range(0,maxrange):
    err_ind = np.nonzero(randvar < x[i]) #checking probability condition
    err_n = np.size(err_ind) #computing the probability
    err.append(err_n/simlen) #storing the probability values in a list


for i in range(0,maxrange-1):
    test = (err[i+1]-err[i])/(x[i+1]-x[i])
    pdf.append(test) #storing the pdf values in a list

def gauss_pdf(x):
    return 1/mp.sqrt(2*np.pi)*np.exp(-x**2/2.0)

vec_gauss_pdf = scipy.vectorize(gauss_pdf)

plt.plot(x[0:(maxrange-1)].T,pdf,'o')
plt.plot(x,vec_gauss_pdf(x))#plotting the CDF
plt.grid() #creating the grid
plt.xlabel('$x_i$')
plt.ylabel('$p_X(x_i)$')
plt.legend(["Numerical","Theory"])
```

```
#if using termux
#plt.savefig('../figs/uni_pdf.pdf')
#plt.savefig('../figs/uni_pdf.eps')
#subprocess.run(shlex.split("termux-open ../figs/uni_pdf.pdf"))
#if using termux
plt.savefig('../figs/gauss_pdf.pdf')
plt.savefig('../figs/gauss_pdf.eps')
subprocess.run(shlex.split("termux-open ../figs/gauss_pdf.pdf"))
#else
#plt.show() #opening the plot window
```

6.2.4 Find the mean and variance of X by writing a C program.

6.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{6.7}$$

repeat the above exercise theoretically.

6.3. From Uniform to Other

6.3.1 Generate samples of

$$V = -2\ln(1-U) \tag{6.8}$$

and plot its CDF.

6.3.2 Find a theoretical expression for $F_V(x)$.

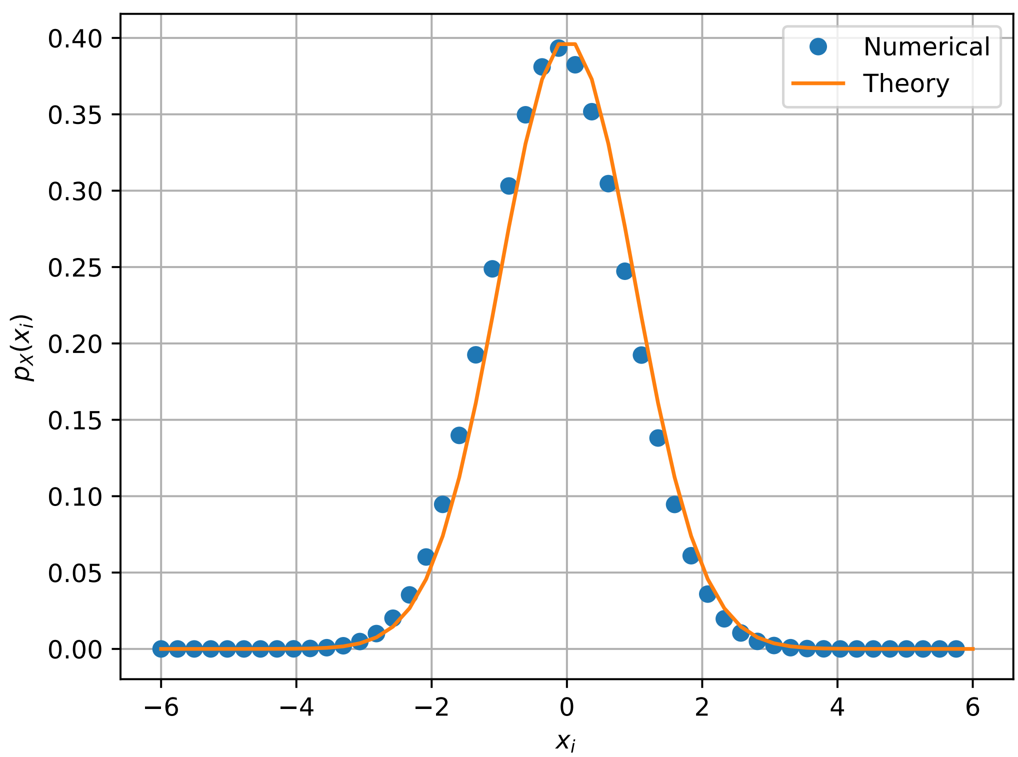

Figure 6.3: The PDF of X

6.4. Triangular Distribution

6.4.1 Generate

$$T = U_1 + U_2 \tag{6.9}$$

6.4.2 Find the CDF of T.

6.4.3 Find the PDF of T.

6.4.4 Find the theoretical expressions for the PDF and CDF of T.

6.4.5 Verify your results through a plot.

Chapter 7

Maximum Likelihood Detection: BPSK

7.1. Maximum Likelihood

7.1.1 Generate equiprobable $X \in \{1, -1\}$.

7.1.2 Generate

$$Y = AX + N, \tag{7.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

7.1.3 Plot Y using a scatter plot.

7.1.4 Guess how to estimate X from Y.

7.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{7.3}$$

7.1.6 Find P_e assuming that X has equiprobable symbols.

7.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

7.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

7.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{7.4}$$

7.1.10 Repeat the above exercise using the MAP criterion.

Chapter 8

Transformation of Random Variables

8.1. Gaussian to Other

8.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{8.1}$$

8.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{8.2}$$

find α.

8.1.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{8.3}$$

8.2. Conditional Probability

8.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{8.4}$$

for

$$Y = AX + N, \tag{8.5}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

8.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

8.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{8.6}$$

Find $P_e = E\left[P_e(N)\right]$.

8.2.4 Plot P_e in problems 8.2.1 and 8.2.3 on the same graph w.r.t γ. Comment.

Chapter 9

Bivariate Random Variables: FSK

9.1. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{9.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{9.3}$$

9.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{9.4}$$

on the same graph using a scatter plot.

9.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

9.1.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{9.5}$$

with respect to the SNR from 0 to 10 dB.

9.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Chapter 10

Exercises

10.1. BPSK

1. The signal constellation diagram for BPSK is given by Fig. 10.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

   Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{10.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{10.2}$$

   where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

2. From Fig. 10.1 obtain a decision rule for BPSK

   Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{10.3}$$

3. Repeat the previous exercise using the MAP criterion.

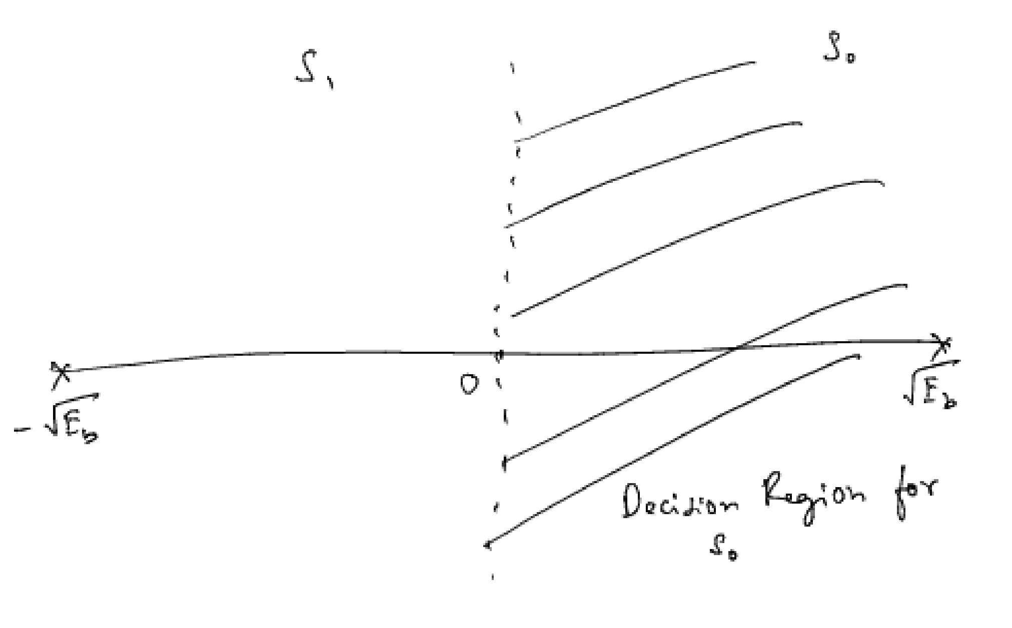

Figure 10.1:

4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0 | s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{10.4}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{10.5}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}\left(0, 1\right)$. Then $n = \sqrt{\frac{N_0}{2}} w$. Substituting this in

(10.5),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{10.6}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{10.7}$$

where $Q\left(x\right) \triangleq \Pr\left(w > x\right), x \geq 0$.

5. The PDF of $w \sim \mathcal{N}\left(0, 1\right)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{10.8}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{10.9}$$

Show that

$$Q(x) = \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{10.10}$$

6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 10.2

7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{10.11}$$

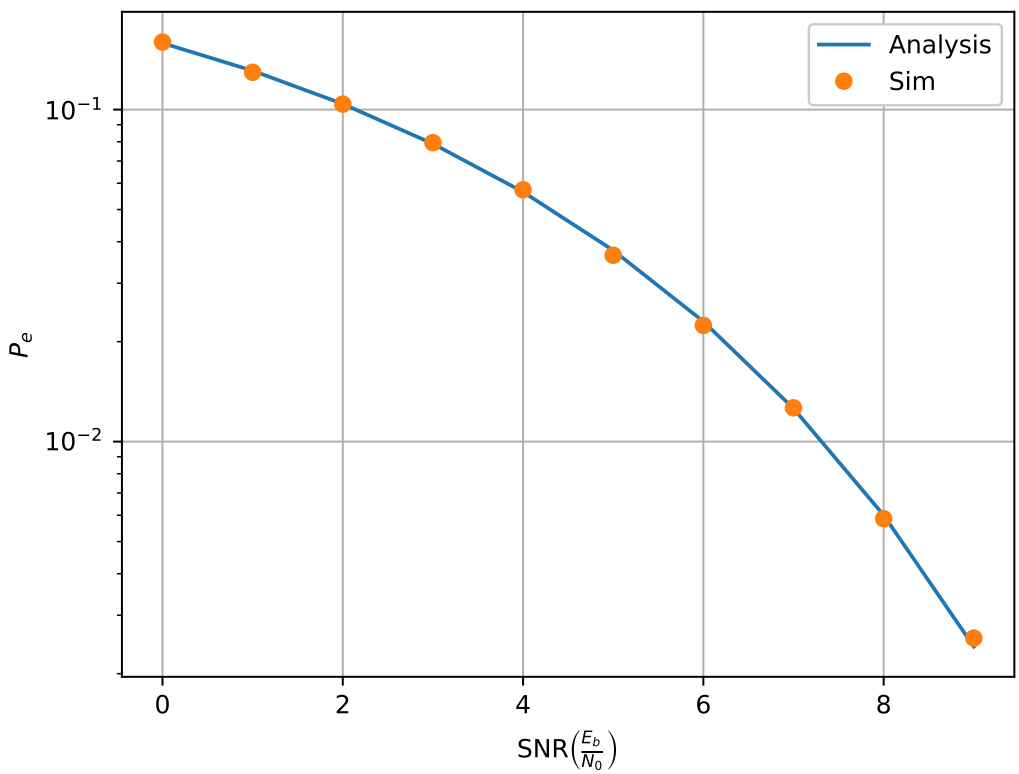

Figure 10.2:

10.2. Coherent BFSK

1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 10.3. Obtain the equations for the received symbols.

   Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{10.12}$$

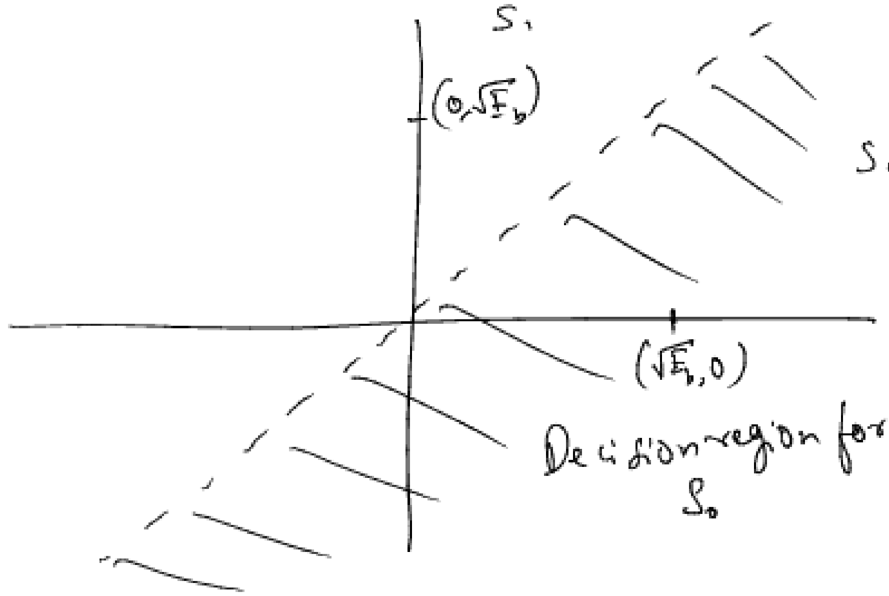

Figure 10.3:

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{10.13}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2. Obtain a decision rule for BFSK from Fig. 10.3.

   Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{10.14}$$

3. Repeat the previous exercise using the MAP criterion.

4. Derive and plot the probability of error. Verify through simulation.

10.3. QPSK

1. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{10.15}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{10.16}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{10.17}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{10.18}$$

(a) Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|r|_2 < r_1 \tag{10.19}$$

(b) Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[\left(X - \mu_x\right)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{10.20}$$

X and Y are said to be uncorrelated if $\rho = 0$

(c) Show that if X and Y are uncorrelated Verify this numerically.

(d) Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.

(e) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.

(f) Show that $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < A, Y < A\right)$.

(g) Find $\Pr\left(X < A, Y < A\right)$.

(h) Verify the above through simulation.

10.4. M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{10.21}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(a) Substituting

$$r_1 = R\cos\theta \tag{10.22}$$

$$r_2 = R\sin\theta \tag{10.23}$$

show that the joint pdf of R, θ is

$$p\left(R, \theta\right) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{10.24}$$

(b) Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\, e^{-(V-\alpha)^2}\, dV = 0 \tag{10.25}$$

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\, dV = \sqrt{\pi} \tag{10.26}$$

(c) Using the above, evaluate

$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\} dV \tag{10.27}$$

for large values of γ.

(d) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\, d\theta \tag{10.28}$$

(e) Find $P_{e|\mathbf{s}_0}$.

10.5. Noncoherent BFSK

10.5.1 Show that

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta \tag{10.29}$$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta-\phi)}\, d\theta \tag{10.30}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{m_1\cos\theta + m_2\sin\theta}\, d\theta = I_0\left(\sqrt{m_1^2+m_2^2}\right) \tag{10.31}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos n\theta\, d\theta \tag{10.32}$$

10.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b}\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\\ 0\\ 0\end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b}\begin{pmatrix}0\\ 0\\ \cos\phi_1\\ \sin\phi_1\end{pmatrix} + \mathbf{n}_1 \tag{10.33}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

(a) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p\left(\mathbf{r}|0, \phi_0\right)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.

(b) If ϕ_0 is uniformly distributed between 0 and 2π, find $p\left(\mathbf{r}|0\right)$. Note that this expression will no longer contain ϕ_0.

(c) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2+r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2+r_4^2}\right) \tag{10.34}$$

where k is a constant.

(d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.

(e) Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \tag{10.35}$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \tag{10.36}$$

(g) Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < Y|0, r_1, r_2\right). \tag{10.37}$$

(h) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(i) Now show that

$$E\left[g(r_1, r_2)\right] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \tag{10.38}$$

10.5.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R} \cos \Theta \tag{10.39}$$

$$V = \sqrt{R} \sin \Theta \tag{10.40}$$

(a) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det \begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \tag{10.41}$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v) \left. J \right|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{10.42}$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k}e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \tag{10.43}$$

assuming that Θ is uniformly distributed between 0 to 2π.

(c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k}e^{-\frac{|y|}{k}} \tag{10.44}$$

(d) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{10.45}$$

where ϕ is a constant.

(e) Find $\Pr(Y > Z)$.

(f) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{10.46}$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{10.47}$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n \left(n!\right)^2} \tag{10.48}$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \ge 0 \\ 0 & z < 0 \end{cases} \tag{10.49}$$

find $\Pr\left(R < Z\right)$.

10.6. Craig's Formula and MGF

10.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{10.50}$$

where X is a random variable and $E\left[\cdot\right]$ is the expectation.

(a) Let $Y \sim \mathcal{N}\left(0, 1\right)$. Define

$$Q(x) = \Pr\left(Y > x\right), x > 0 \tag{10.51}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \tag{10.52}$$

(b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

(c) Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{10.53}$$

Show that

$$P_e = \int_0^\infty Q\left(\sqrt{2x}\right) p_A(x)\, dx \tag{10.54}$$

where $A = \frac{E_s |h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{10.55}$$

(e) compute $M_A(s)$.

(f) Find P_e.

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.

Appendix A

Axioms

A.1. Definitions

A.1.1

$$0 \leq \Pr(A) \leq 1 \tag{A.1.1.1}$$

A.1.2 If $AB = 0$,

$$\Pr(A + B) = \Pr(A) + \Pr(B)\,. \tag{A.1.2.1}$$

A.1.3 If A, B are independent,

$$\Pr(AB) = \Pr(A)\Pr(B) \tag{A.1.3.1}$$

A.1.4

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{A.1.4.1}$$

A.2. Boolean Logic

A.2.1

$$A + A' = 1 \tag{A.2.1.1}$$

A.2.2

$$A'B' = (A + B)' \tag{A.2.2.1}$$

A.2.3

$$A + B = A\left(B + B'\right) + B \tag{A.2.3.1}$$

$$= B\left(A + 1\right) + AB' \tag{A.2.3.2}$$

$$= B + AB' \tag{A.2.3.3}$$

A.2.4

$$A = A\left(B + B'\right) = AB + AB' \tag{A.2.4.1}$$

and

$$\left(AB\right)\left(AB'\right) = 0, \because BB' = 0 \tag{A.2.4.2}$$

Hence, AB and AB' are mutually exclusive.

A.3. Properties

A.3.1

$$\Pr\left(A'\right) = 1 - \Pr\left(A\right). \tag{A.3.1.1}$$

A.3.2

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{A.3.2.1}$$

$$= 1 - \Pr\left(A+B\right) \tag{A.3.2.2}$$

A.3.3

$$\Pr\left(A+B\right) = \Pr\left(B + AB'\right) \tag{A.3.3.1}$$

$$= \Pr\left(B\right) + \Pr\left(AB'\right) \tag{A.3.3.2}$$

$$\because B\left(AB'\right) = 0 \tag{A.3.3.3}$$

A.3.4

$$\Pr\left(A\right) = \Pr\left(AB\right) + \Pr\left(AB'\right) \tag{A.3.4.1}$$

$$\implies \Pr\left(AB'\right) = \Pr\left(A\right) - \Pr\left(AB\right) \tag{A.3.4.2}$$

A.3.5 Substituting (A.3.4.2) in (A.3.3.2),

$$\Pr\left(A+B\right) = \Pr\left(A\right) + \Pr\left(B\right) - \Pr\left(AB\right) \tag{A.3.5.1}$$

Appendix B

Z-transform

B.1 The Z-transform of X is defined as

$$M_X(z) = E\left[z^{-X}\right] = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{B.1.1}$$

B.2 If X_1 and X_2 are independent, the MGF of

$$X = X_1 + X_2 \tag{B.2.1}$$

is given by

$$M_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{B.2.2}$$

The above property follows from Fourier analysis and is fundamental to signal processing.

B.3 Let X_i be independent. For

$$X = X_1 + X_2 + ... + X_n, \tag{B.3.1}$$

$$M_X(z) = \prod_{i=1}^{n} M_{X_i}(z) \tag{B.3.2}$$

B.4 The nth moment of X can be expressed as

$$E\left[X^n\right] = \frac{d^n M_X(z^{-1})}{dz^n}|_{z=1} \tag{B.4.1}$$

Appendix C

Distributions

C.1. Bernoulli

C.1.1. The pmf of a Bernoulli distribution is defined as

$$p_X(k) = \begin{cases} q = 1-p & k = 0 \\ p & k = 1 \\ 0 & \text{otherwise} \end{cases} \tag{C.1.1.1}$$

C.1.2. For a Bernoulli random variable X with success probability p,

$$M_X(z) = q + pz^{-1} \tag{C.1.2.1}$$

C.1.3. The mean of the Bernoulli distribution is

$$E(X) = p \tag{C.1.3.1}$$

C.1.4. The following code simulates 100 coin tosses

```
#Code by GVV Sharma
```

```
#November 18, 2020
#Released under GNU/GPL
#Given a Bernoulli probability and
#number of samples, the code generates the event data

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli

#100 samples
simlen=int(1e2)

#Probability of the event
prob = 0.5

#Generating sample date using Bernoulli r.v.
data_bern = bernoulli.rvs(size=simlen,p=prob)
#Calculating the number of favourable outcomes
err_ind = np.nonzero(data_bern == 1)
#calculating the probability
err_n = np.size(err_ind)/simlen

#Theory vs simulation
print(err_n,prob)
print(data_bern)
```

C.2. Binomial Distribution

C.2.1. The Binomial distribution is defined as

$$X = X_1 + X_2 + ... + X_n, \tag{C.2.1.1}$$

Where X_i are i.i.d bernoulli.

C.2.2. For a Binomial random variable X with parameters n, p,

$$M_X(z) = \left(q + pz^{-1}\right)^n \tag{C.2.2.1}$$

C.2.3. The mean for the Binomial r.v. is

$$E\left[X\right] = np \tag{C.2.3.1}$$

Solution: From (B.4.1) and (C.2.2.1),

$$E\left[X\right] = \frac{d(q + pz)^n}{dz}|_{z=1} \tag{C.2.3.2}$$

$$= np(q + pz)^{n-1}|_{z=1} \tag{C.2.3.3}$$

$$= np(q + p)^{n-1} \tag{C.2.3.4}$$

yielding (C.2.3.1)

$$\because p + q = 1 \tag{C.2.3.5}$$

C.2.4. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Solution: See the following code

```
#Code by GVV Sharma
#November 20,2020
#Released under GNU/GPL
#To find the probability of an event using the binomial distribution

import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import bernoulli
from scipy.stats import norm
from scipy.stats import binom




#Simlen
simlen=1000

#Number of hurdles
n = 10

#Probability of clearing a hurdle
p = 1-5/6

#Mean
```

```
mu = p


#Variance
sigma = np.sqrt(p*(1-p))


#Theoretical probability of knocking down fewer than 2 hurdles
k = 1
print(binom.cdf(k, n, p),3*(5/6)**10)


#Using the Gaussian approximation for the binomial pdf
print(1/(sigma*np.sqrt(n))*(norm.pdf((k-n*mu)/(sigma*np.sqrt(n)))+norm.pdf((k
    -1-n*mu)/(sigma*np.sqrt(n)))))



#Simulating the probability using the binomial random variable
data_binom = binom.rvs(n,p,size=simlen) #Simulating the event of jumping 10
    hurdles
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
#print(data_binom)



#Simulating the probability using the bernoulli random variable
data_bern_mat = bernoulli.rvs(p,size=(n,simlen))
```

```
data_binom=np.sum(data_bern_mat, axis=0)
#print(data_bern_mat)
#print(data_binom)
err_ind = np.nonzero(data_binom <=k) #checking probability condition
err_n = np.size(err_ind) #computing the probability
print(err_n/simlen)
```

C.3. Uniform Distribution

C.3.1. Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the random variables representing the outcome for a die. Assuming the die to be fair, the probability mass function (pmf) is expressed as

$$p_X(n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{C.3.1.1}$$

C.3.2. The Z-transform of X is given by

$$P_X(z) = \frac{1}{6} \sum_{n=1}^{6} z^{-n} = \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{C.3.2.1}$$

upon summing up the geometric progression.

C.3.3. From (C.3.2.1), the CDF of X is given by

$$F_X(n) = \Pr(X \le n) = \begin{cases} 0 & n < 1 \\ \frac{n}{6} & 1 \le n \le 6 \\ 1 & \text{otherwise} \end{cases} \tag{C.3.3.1}$$

and plotted in Fig. C.3.3.1.

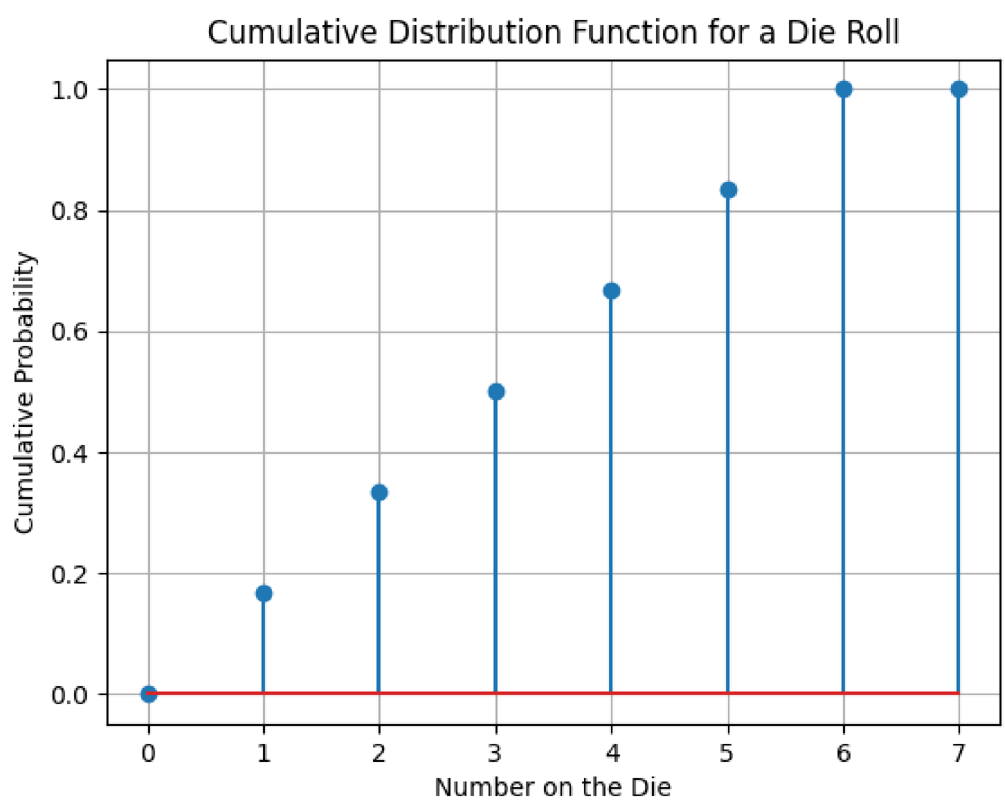

Figure C.3.3.1: CDF

C.4. Triangular Distribution

C.4.1. The desired outcome is

$$X = X_1 + X_2, \tag{C.4.1.1}$$

$$\implies X \in \{1, 2, \dots, 12\} \tag{C.4.1.2}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{C.4.1.3}$$

C.4.2. Convolution: From (C.4.1.1),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{C.4.2.1}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{C.4.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k)$$

$$= \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{C.4.2.3}$$

From (C.4.2.2) and (C.4.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{C.4.2.4}$$

where $*$ denotes the convolution operation. Substituting from (C.3.1.1) in (C.4.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{C.4.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \tag{C.4.2.6}$$

From (C.4.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \leq n-1 \leq 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \leq 6 \\ 0 & n > 12 \end{cases} \tag{C.4.2.7}$$

Substituting from (C.3.1.1) in (C.4.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \leq n \leq 7 \\ \frac{13-n}{36} & 7 < n \leq 12 \\ 0 & n > 12 \end{cases} \tag{C.4.2.8}$$

satisfying (C.4.1.3).

C.4.3. <u>The Z-transform:</u> From (C.3.2.1) and (B.2.2),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \tag{C.4.3.1}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.2}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{C.4.3.3}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.4}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{C.4.3.5}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{C.4.3.6}$$

From (B.1.1), (C.4.3.2) and (C.4.3.5)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{C.4.3.7}$$

which is the same as (C.4.2.8). Note that (C.4.2.8) can be obtained from (C.4.3.5) using contour integration as well.

C.4.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure C.4.4.1. The theoretical pmf obtained in (C.4.2.8) is plotted for comparison.

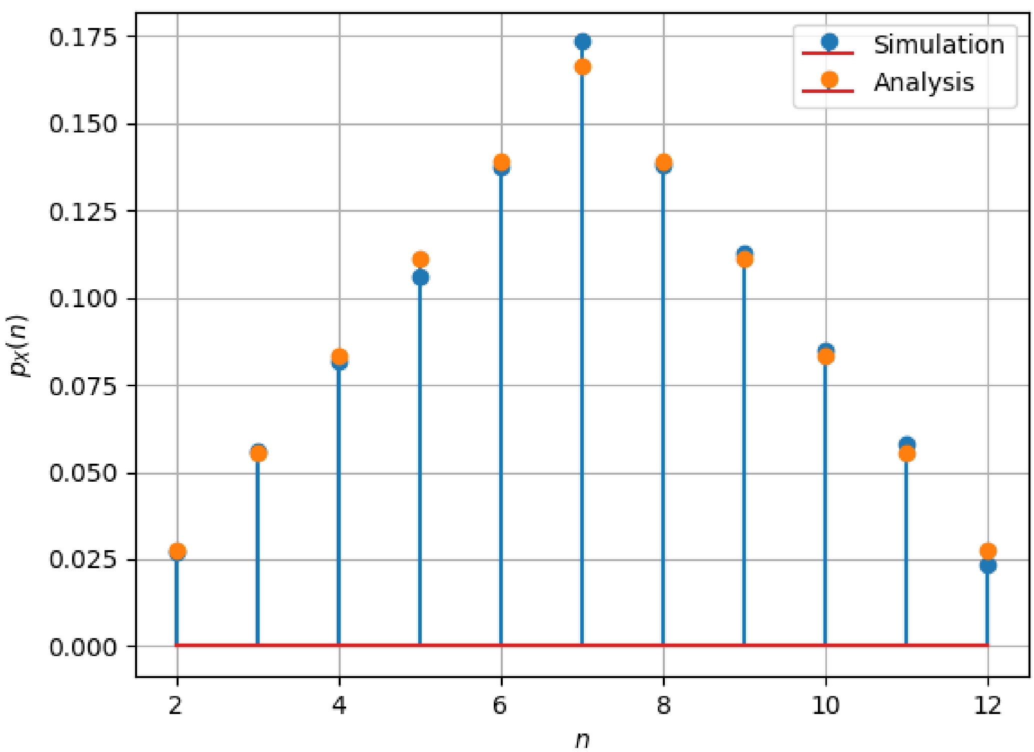

Figure C.4.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

C.4.5. The python code is available below

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

#Sample size
simlen = 10000
#Possible outcomes
n = range(2,13)
# Generate X1 and X2
y = np.random.randint(1,7, size=(2, simlen))

#Generate X
X = np.sum(y, axis = 0)
#Find the frequency of each outcome
unique, counts = np.unique(X, return_counts=True)
#Simulated probability
psim = counts/simlen
#Theoretical probability
n1 = range(2,8)
n2 = range(8,13)
panal1 = (n1 -np.ones((1,6)))
panal2 = (13*np.ones((1,5))-n2)
```

```
panal = np.concatenate((panal1,panal2),axis=None)/36


#Plotting
plt.stem(n,psim, markerfmt='o', use_line_collection=True, label='Simulation')
plt.stem(n,panal, markerfmt='o',use_line_collection=True, label='Analysis')
plt.xlabel('$n$')
plt.ylabel('$p_{X}(n)$')
plt.legend()
plt.grid()# minor


#If using termux
plt.savefig('figs/pmf.pdf')
plt.savefig('figs/pmf.png')
subprocess.run(shlex.split("termux-open figs/pmf.pdf"))
#else
#plt.show()
```

Appendix D

Identities

D.1 Let

$$N = R + B + G, n = r + b + g \tag{D.1.1}$$

where R, B, G and r, b, g represent the number of red, blue and green marbles respectively within N and n. Then

$$\Pr(r, b, g) = \frac{{}^{R}C_{r}\,{}^{B}C_{b}\,{}^{G}C_{g}}{{}^{R+B+G}C_{r+b+g}} \tag{D.1.2}$$

Solution: The number of ways of choosing n marbles from N is

$$^{N}C_{n} \tag{D.1.3}$$

The number of ways of choosing r,b,g marbles is

$$^{R}C_{r}\,{}^{B}C_{b}\,{}^{G}C_{g} \tag{D.1.4}$$

Using the definition of probability, we obtain (D.1.2).

D.2

$$ {}^{R+B}C_n = \sum_{k=0}^{R} \sum_{m=n-k}^{B} {}^{R}C_k \, {}^{B}C_m \tag{D.2.1} $$

Solution: Since

$$ (x+1)^R = \sum_{k=0}^{R} {}^{R}C_k \, x^k, \tag{D.2.2} $$

$$ (x+1)^R \, (x+1)^B = \sum_{k=0}^{R} \sum_{m=0}^{B} {}^{R}C_k \, {}^{B}C_m \, x^{k+m} \tag{D.2.3} $$

$$ \implies (x+1)^{R+B} = \sum_{k=0}^{R} \sum_{m=n-k}^{B} {}^{R}C_k \, {}^{B}C_m \, x^n + \sum_{k=0}^{R} \sum_{m \neq n-k}^{B} {}^{R}C_k \, {}^{B}C_m \, x^{k+m} \tag{D.2.4} $$

$$ \tag{D.2.5} $$

yielding (D.2.1) upon comparing the coefficients of x^n on both sides.