
DIGITAL COMMUNICATION

Through Simulations

G. V. V. Sharma



Copyright ©2022 by G. V. V. Sharma.

<https://creativecommons.org/licenses/by-sa/3.0/>

and

<https://www.gnu.org/licenses/fdl-1.3.en.html>

Contents

Introduction	iii
1 Introduction	1
2 Axioms	3
3 Definitions	11
4 Conditional Probability	17
5 Two Dice	19
5.1 Sum of Independent Random Variables	19
6 Random Numbers	25
6.1 Uniform Random Numbers	25
6.2 Central Limit Theorem	27
6.3 From Uniform to Other	30
6.4 Triangular Distribution	30
7 Maximum Likelihood Detection: BPSK	31
7.1 Maximum Likelihood	31
8 Transformation of Random Variables	33

8.1	Gaussian to Other	33
8.2	Conditional Probability	34
9	Bivariate Random Variables: FSK	35
9.1	Two Dimensions	35
10	Exercises	37
10.1	BPSK	37
10.2	Coherent BFSK	40
10.3	QPSK	42
10.4	M -PSK	43
10.5	Noncoherent BFSK	44
10.6	Craig's Formula and MGF	48

Introduction

This book introduces digital communication through probability.

Chapter 1

Introduction

Chapter 2

Axioms

2.1 A and B are events such that $\Pr(A) = 0.42$, $\Pr(B) = 0.48$ and $\Pr(A \text{ and } B) = 0.16$.

Determine

(a) $\Pr(\text{not } A)$

(b) $\Pr(\text{not } B)$

(c) $\Pr(A \text{ or } B)$

Solution:

(a) $\Pr(\text{not } A)$

$$\Pr(A') = 1 - \Pr(A) \tag{2.1}$$

$$= 1 - 0.42 \tag{2.2}$$

$$= 0.58 \tag{2.3}$$

(b) $\Pr(\text{not } B)$

$$\Pr(B') = 1 - \Pr(B) \quad (2.4)$$

$$= 1 - 0.48 \quad (2.5)$$

$$= 0.52 \quad (2.6)$$

(c) $\Pr(A \text{ or } B)$

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (2.7)$$

$$= 0.42 + 0.48 - 0.16 \quad (2.8)$$

$$= 0.74 \quad (2.9)$$

2.2 A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(a) $\Pr(2)$

(b) $\Pr(1 \text{ or } 3)$

(c) $\Pr(\text{not } 3)$

Solution: Table 2.2 summarises the given information.

Variable	Value	Description	Probability	Pr Value
X_1	1	Face of die '1'	$\Pr(X_1)$	$\frac{1}{3}$
X_2	2	Face of die '2'	$\Pr(X_2)$	$\frac{1}{2}$
X_3	3	Face of die '3'	$\Pr(X_3)$	$\frac{1}{6}$

Table 2.2:

(a)

$$\Pr(X_2) = \frac{1}{2} \quad (2.10)$$

(b)

$$\Pr(X_1 + X_3) = \Pr(X_1) + \Pr(X_3) - \Pr(X_1 X_3) \quad (2.11)$$

$$= \frac{1}{3} + \frac{1}{6} \quad (\because \Pr(X_1 X_3) = 0) \quad (2.12)$$

$$= \frac{1}{2} \quad (2.13)$$

(c)

$$\Pr(X'_3) = 1 - \Pr(X_3) \quad (2.14)$$

$$= 1 - \frac{1}{6} \quad (2.15)$$

$$= \frac{5}{6} \quad (2.16)$$

2.3 If $\Pr(A) = 0.8$, $\Pr(B) = 0.5$ and $\Pr(B|A) = 0.4$, find

(a) $\Pr(AB)$

(b) $\Pr(A|B)$

(c) $\Pr(A + B)$

Solution:

(a) Since

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)}, \quad (2.17)$$

from the given information,

$$\frac{\Pr (AB)}{\Pr (A)} = 0.4 \quad (2.18)$$

$$\implies \Pr (AB) = 0.4 \times 0.8 \quad (2.19)$$

$$= 0.32 \quad (2.20)$$

(b) Similarly,

$$\Pr (A|B) = \frac{\Pr (AB)}{\Pr (B)} \quad (2.21)$$

$$= \frac{\Pr (B|A) \Pr (A)}{\Pr (B)}. \quad (2.22)$$

$$= \frac{0.4 \times 0.8}{0.5} \quad (2.23)$$

$$= 0.64 \quad (2.24)$$

(c) Since,

$$\Pr (A + B) = \Pr (A) + \Pr (B) - \Pr (AB) \quad (2.25)$$

Substituting (2.20) in (2.25),

$$\Pr (A + B) = 0.8 + 0.5 - 0.32 \quad (2.26)$$

$$= 0.98 \quad (2.27)$$

2.4 If $\Pr (A) = \frac{3}{5}$ and $\Pr (B) = \frac{1}{5}$ find $\Pr (A \cap B)$ if A and B are independent events.

Solution: From the given information,

$$\Pr(A) = \frac{3}{5}, \Pr(B) = \frac{1}{5}. \quad (2.28)$$

Since A and B are independent,

$$\Pr(AB) = \Pr(A) \times \Pr(B) \quad (2.29)$$

$$= \frac{3}{5} \times \frac{1}{5} \quad (2.30)$$

$$= \frac{3}{25} \quad (2.31)$$

2.5 Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A+B) = \frac{3}{5}$ and $P(B) = p$.

Find p if they are

(a) mutually exclusive

(b) independent

Solution:

(a) In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \quad (2.32)$$

$$\implies \frac{3}{5} = \frac{1}{2} + p \quad (2.33)$$

$$\therefore p = \frac{1}{10} \quad (2.34)$$

(b) Given A and B are independent events, then,

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (2.35)$$

$$\implies \Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(A) \Pr(B) \quad (2.36)$$

$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \quad (2.37)$$

$$\therefore p = \frac{1}{5} \quad (2.38)$$

2.6 Let E and F be events with $\Pr(E) = \frac{3}{5}$, $\Pr(F) = \frac{3}{10}$ and $\Pr(EF) = \frac{1}{5}$. Are E and F independent?

Solution: From the given information,

$$\Pr(E) \Pr(F) = \frac{3}{5} \times \frac{9}{50} \quad (2.39)$$

$$\Pr(EF) = \frac{1}{50} \quad (2.40)$$

$$\implies \Pr(EF) \neq \Pr(E) \Pr(F) \quad (2.41)$$

$\therefore E$ and F are not independent events.

2.7 If A and B are two events such that $A \subset B$ and $\Pr(B) \neq 0$, then which of the following is correct ?

(a) $\Pr(A | B) = \frac{\Pr(B)}{\Pr(A)}$

(b) $\Pr(A | B) < \Pr(A)$

(c) $\Pr(A | B) \geq \Pr(A)$

(d) None of these

Solution: if $A \subset B$ and $\Pr(B) \neq 0$ then

$$AB = A \tag{2.42}$$

$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \tag{2.43}$$

we know that

$$\Pr(B) \leq 1 \tag{2.44}$$

$$\implies 1 \leq \frac{1}{\Pr(B)} \tag{2.45}$$

Multiplying both sides with $\Pr(A)$,

$$\Pr(A) \leq \frac{\Pr(A)}{\Pr(B)} \tag{2.46}$$

$$= \Pr(A | B) \tag{2.47}$$

from (2.43).

Chapter 3

Definitions

3.1 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

Solution: The given information is summarised in Table 3.2. yielding

RV	Values	Description
X	{0,1,2,3}	Cards drawn randomly
Y	{0,1}	0:diamond ,1:spade
X,Y	{00,10,20,31}	3 diamonds and one spade out of 13 each

Table 3.2: Random variables(RV) X,Y and X,Y

$$\Pr (00, 10, 20, 31) = \frac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4} \quad (3.1)$$

$$= \frac{286}{20285} \quad (3.2)$$

3.2 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

Solution: The given information is summarised in Table 3.4 The total number of possible outcomes is ${}^N C_n$ and the total number of favourable outcomes is ${}^q C_n$ yielding

Variable	Value	Description
N	10000	Total number of tickets sold
k	10	Total number of prizes awarded
n	$\{0,1,2,\dots,N\}$	Number of tickets purchased
$\Pr(n)$		probability of not wining a prize
q	N-k	number of tickets with no prize

Table 3.4:

the desired probability

$$\Pr(n) = \frac{{}^qC_n}{{}^NC_n} \quad (3.3)$$

Substituting numerical values,

(a) For one ticket,

$$\Pr(1) = \frac{{}^{9990}C_1}{{}^{10000}C_1} = 0.9990 \quad (3.4)$$

(b) For two tickets,

$$\Pr(2) = \frac{{}^{9990}C_2}{{}^{10000}C_2} = 0.9980 \quad (3.5)$$

(c) For 10 tickets

$$\Pr(3) = \frac{{}^{9990}C_{10}}{{}^{10000}C_{10}} = 0.9901 \quad (3.6)$$

3.3 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

Solution: Table 3.6 summarises the given information.

RV	Values	Description
X	{0,1}	0: section1, 1: section2
Y	{0,1}	0: student1, 1: student2
XY	{001,101}	Students enter same section
	{00,01,10,11}	Students enter different section

Table 3.6:

(a) When both enter the same section, the probability is

$$\Pr(001, 101) = \frac{{}^{40}C_2}{{}^{100}C_2} + \frac{{}^{60}C_2}{{}^{100}C_2} = \frac{156}{990} + \frac{354}{990} = 0.51 \quad (3.7)$$

(b) When both enter different sections, the desired probability is

$$\Pr(00, 01, 10, 11) = 1 - 0.51 = 0.49 \quad (3.8)$$

3.4 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase.

Solution: The given information is represented in Tables 3.8 3.10 and 3.12.

Random variable	Value	Description
X	$\{1,2,3,4\}$	The number lock of a suitcase
Y	$\{0,1,2...9\}$	The digits labelled on each wheel

Table 3.8: Random variables X and Y

Wheel 1	Wheel 2	Wheel 3	Wheel 4
10 ways	9 ways	8 ways	7 ways

Table 3.10: Suitcase wheel

The number of possible placement of the digits are

$$10 \times 9 \times 8 \times 7 = 5040 \quad (3.9)$$

Thus, the probability of the correct sequence begin selected is

$$\Pr(A) = \frac{1}{5040} \quad (3.10)$$

3.5 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Table 3.14 summarizes the various events Given that the cards are drawn at random without replacement. Without replacement means only one card is random at a time and is excluded from the total while next card is drawn at random. Thus, the probability that both the cards are black is,

$$\Pr(00, 10) = \frac{{}^{26}C_1}{{}^{52}C_1} \times \frac{{}^{25}C_1}{{}^{51}C_1} = \frac{1}{2} \times \frac{25}{51} = 0.24 \quad (3.11)$$

Example	Wheel	Outcome
1	8 6 4 2	Not repeating
2	8 4 2 6	Not repeating
3	1 2 3 4	Not repeating
4	8 8 8 8	Repeating
5	1 1 2 2	Repeating

Table 3.12: Combinations

RV	Values	Description
X	$\{0,1\}$	number of cards drawn 2
Y	$\{0,1\}$	0: black card, 1: red card
XY	$\{00,10\}$	card drawn is black

Table 3.14:

Chapter 4

Conditional Probability

4.1 Given that E and F are events such that $P(E) = 0.6$, $P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E | F)$ and $P(F | E)$.

Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \quad (4.1)$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \quad (4.2)$$

4.2 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \quad (4.3)$$

Chapter 5

Two Dice

5.1. Sum of Independent Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

5.1.1. The Uniform Distribution: Let $X_i \in \{1, 2, 3, 4, 5, 6\}$, $i = 1, 2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases} \quad (5.1.1.1)$$

The desired outcome is

$$X = X_1 + X_2, \quad (5.1.1.2)$$

$$\implies X \in \{1, 2, \dots, 12\} \quad (5.1.1.3)$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \quad (5.1.1.4)$$

5.1.2. Convolution: From (5.1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \quad (5.1.2.1)$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k) p_{X_2}(k) \quad (5.1.2.2)$$

after unconditioning, $\because X_1$ and X_2 are independent,

$$\begin{aligned} \Pr(X_1 = n - k | X_2 = k) \\ = \Pr(X_1 = n - k) = p_{X_1}(n - k) \end{aligned} \quad (5.1.2.3)$$

From (5.1.2.2) and (5.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \quad (5.1.2.4)$$

where $*$ denotes the convolution operation. Substituting from (5.1.1.1) in (5.1.2.4),

$$p_X(n) = \frac{1}{6} \sum_{k=1}^6 p_{X_1}(n - k) = \frac{1}{6} \sum_{k=n-6}^{n-1} p_{X_1}(k) \quad (5.1.2.5)$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \quad (5.1.2.6)$$

From (5.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6} \sum_{k=1}^{n-1} p_{X_1}(k) & 1 \leq n-1 \leq 6 \\ \frac{1}{6} \sum_{k=n-6}^6 p_{X_1}(k) & 1 < n-6 \leq 6 \\ 0 & n > 12 \end{cases} \quad (5.1.2.7)$$

Substituting from (5.1.1.1) in (5.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \leq n \leq 7 \\ \frac{13-n}{36} & 7 < n \leq 12 \\ 0 & n > 12 \end{cases} \quad (5.1.2.8)$$

satisfying (5.1.1.4).

5.1.3. The Z-transform: The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \quad (5.1.3.1)$$

From (5.1.1.1) and (5.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6} \sum_{n=1}^6 z^{-n} \quad (5.1.3.2)$$

$$= \frac{z^{-1} (1 - z^{-6})}{6 (1 - z^{-1})}, \quad |z| > 1 \quad (5.1.3.3)$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \quad (5.1.3.4)$$

$$P_X(z) = P_{X_1}(z)P_{X_2}(z) \quad (5.1.3.5)$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (5.1.3.3) and (5.1.3.5),

$$P_X(z) = \left\{ \frac{z^{-1}(1 - z^{-6})}{6(1 - z^{-1})} \right\}^2 \quad (5.1.3.6)$$

$$= \frac{1}{36} \frac{z^{-2}(1 - 2z^{-6} + z^{-12})}{(1 - z^{-1})^2} \quad (5.1.3.7)$$

Using the fact that

$$p_X(n - k) \xleftrightarrow{\mathcal{H}} Z P_X(z) z^{-k}, \quad (5.1.3.8)$$

$$nu(n) \xleftrightarrow{\mathcal{H}} Z \frac{z^{-1}}{(1 - z^{-1})^2} \quad (5.1.3.9)$$

after some algebra, it can be shown that

$$\begin{aligned} & \frac{1}{36} [(n - 1)u(n - 1) - 2(n - 7)u(n - 7) \\ & \quad + (n - 13)u(n - 13)] \\ & \quad \xleftrightarrow{\mathcal{H}} Z \frac{1}{36} \frac{z^{-2}(1 - 2z^{-6} + z^{-12})}{(1 - z^{-1})^2} \end{aligned} \quad (5.1.3.10)$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (5.1.3.11)$$

From (5.1.3.1), (5.1.3.7) and (5.1.3.10)

$$p_X(n) = \frac{1}{36} [(n-1)u(n-1) - 2(n-7)u(n-7) + (n-13)u(n-13)] \quad (5.1.3.12)$$

which is the same as (5.1.2.8). Note that (5.1.2.8) can be obtained from (5.1.3.10) using contour integration as well.

5.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 5.1.4.1. The theoretical pmf obtained in (5.1.2.8) is plotted for comparison.

5.1.5. The python code is available in

`/codes/sum/dice.py`

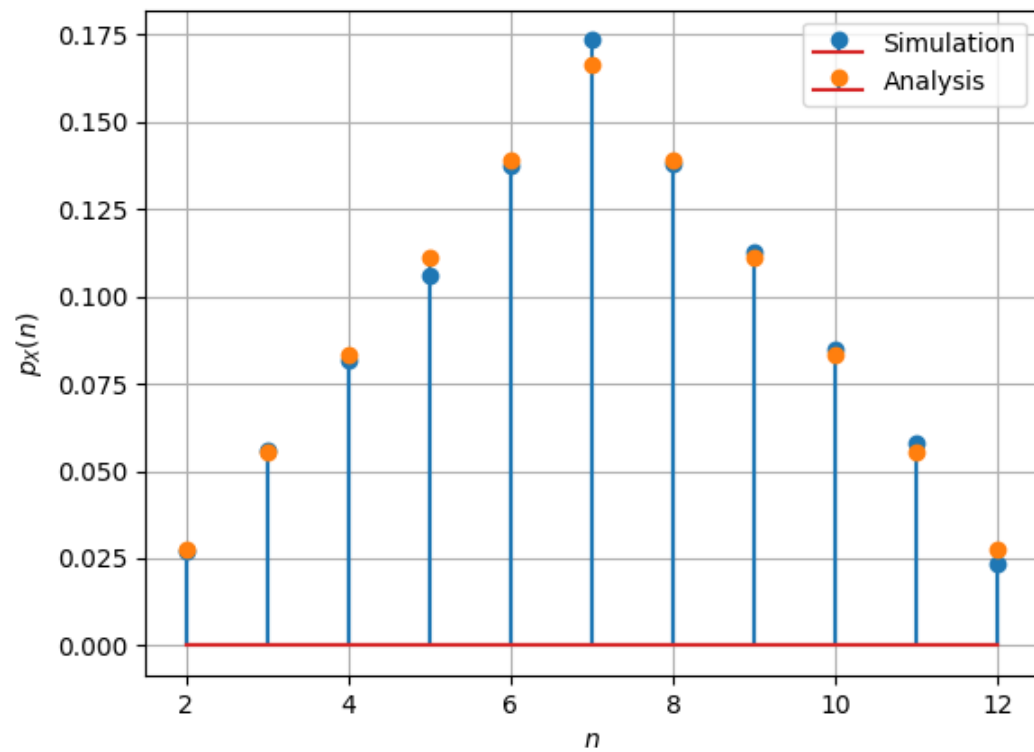


Figure 5.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

Chapter 6

Random Numbers

6.1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

6.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

`codes/exrand.c`

`codes/coeffs.h`

6.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{6.1.2.1}$$

Solution: The following code plots Fig. 6.1.2.1

`codes/cdf_plot.py`

6.1.3 Find a theoretical expression for $F_U(x)$.

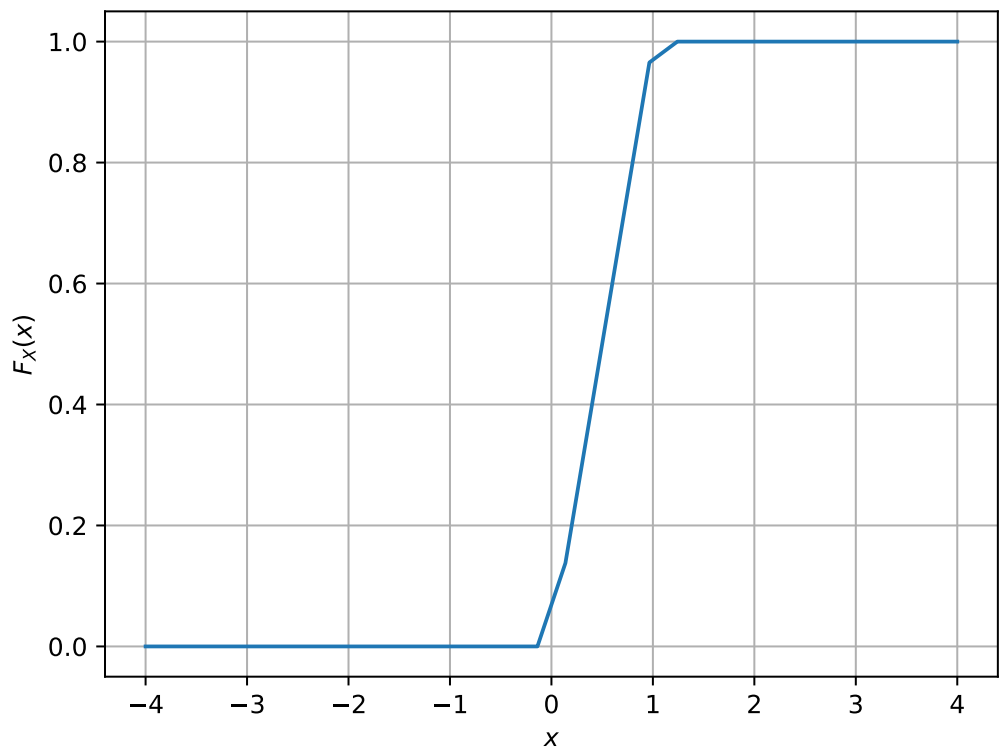


Figure 6.1.2.1: The CDF of U

6.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (6.1.4.1)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (6.1.4.2)$$

Write a C program to find the mean and variance of U .

6.1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (6.1.5.1)$$

6.2. Central Limit Theorem

6.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (6.2.1.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

6.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat.

What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 6.2.2.1

6.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat.

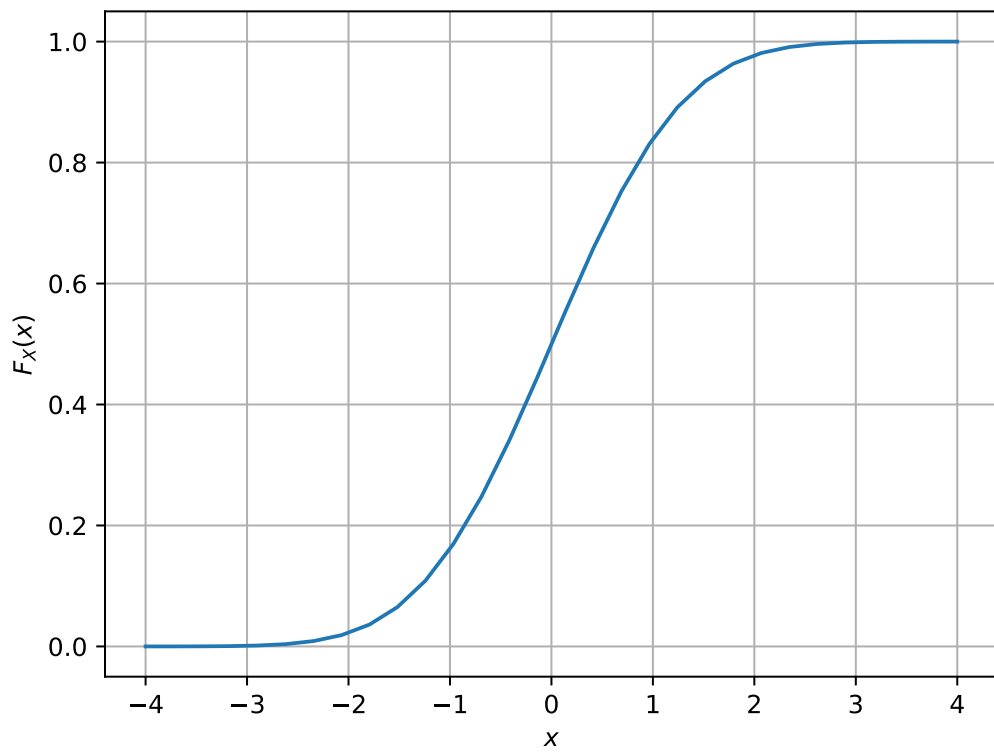


Figure 6.2.2.1: The CDF of X

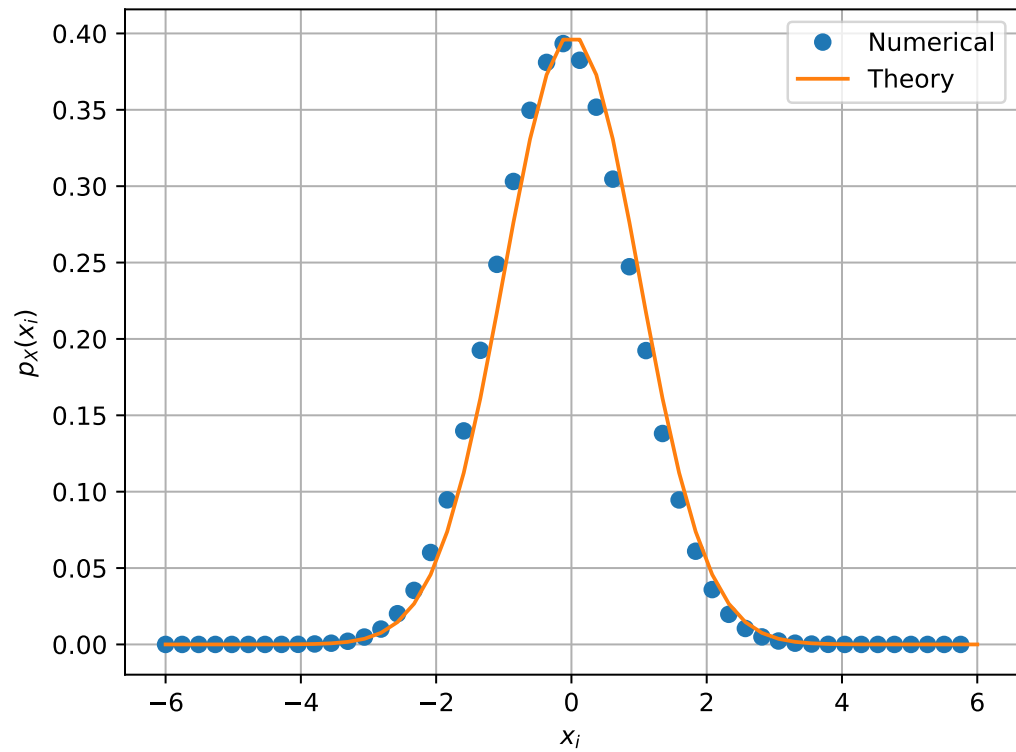


Figure 6.2.3.1: The PDF of X

The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (6.2.3.1)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 6.2.3.1 using the code below

`codes/pdf_plot.py`

6.2.4 Find the mean and variance of X by writing a C program.

6.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (6.2.5.1)$$

repeat the above exercise theoretically.

6.3. From Uniform to Other

6.3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (6.3.1.1)$$

and plot its CDF.

6.3.2 Find a theoretical expression for $F_V(x)$.

6.4. Triangular Distribution

6.4.1 Generate

$$T = U_1 + U_2 \quad (6.4.1.1)$$

6.4.2 Find the CDF of T .

6.4.3 Find the PDF of T .

6.4.4 Find the theoretical expressions for the PDF and CDF of T .

6.4.5 Verify your results through a plot.

Chapter 7

Maximum Likelihood Detection: BPSK

7.1. Maximum Likelihood

7.1.1 Generate equiprobable $X \in \{1, -1\}$.

7.1.2 Generate

$$Y = AX + N, \tag{7.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

7.1.3 Plot Y using a scatter plot.

7.1.4 Guess how to estimate X from Y .

7.1.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \tag{7.1.5.1}$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \tag{7.1.5.2}$$

7.1.6 Find P_e assuming that X has equiprobable symbols.

7.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

7.1.8 Now, consider a threshold δ while estimating X from Y . Find the value of δ that maximizes the theoretical P_e .

7.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{7.1.9.1}$$

7.1.10 Repeat the above exercise using the MAP criterion.

Chapter 8

Transformation of Random Variables

8.1. Gaussian to Other

8.1.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{8.1.1.1}$$

8.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{8.1.2.1}$$

find α .

8.1.3 Plot the CDF and PDF of

$$A = \sqrt{V} \tag{8.1.3.1}$$

8.2. Conditional Probability

8.2.1 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (8.2.1.1)$$

for

$$Y = AX + N, \quad (8.2.1.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

8.2.2 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

8.2.3 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x) dx \quad (8.2.3.1)$$

Find $P_e = E[P_e(N)]$.

8.2.4 Plot P_e in problems 8.2.1 and 8.2.3 on the same graph w.r.t γ . Comment.

Chapter 9

Bivariate Random Variables: FSK

9.1. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (4.1)$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (4.3)$$

9.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (9.1.1.1)$$

on the same graph using a scatter plot.

9.1.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

9.1.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (9.1.3.1)$$

with respect to the SNR from 0 to 10 dB.

- 9.1.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.

Chapter 10

Exercises

10.1. BPSK

1. The signal constellation diagram for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \quad (1.1)$$

$$y|s_1 = -\sqrt{E_b} + n \quad (1.2)$$

where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

2. From Fig. 1.1 obtain a decision rule for BPSK

Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \quad (2.1)$$

3. Repeat the previous exercise using the MAP criterion.

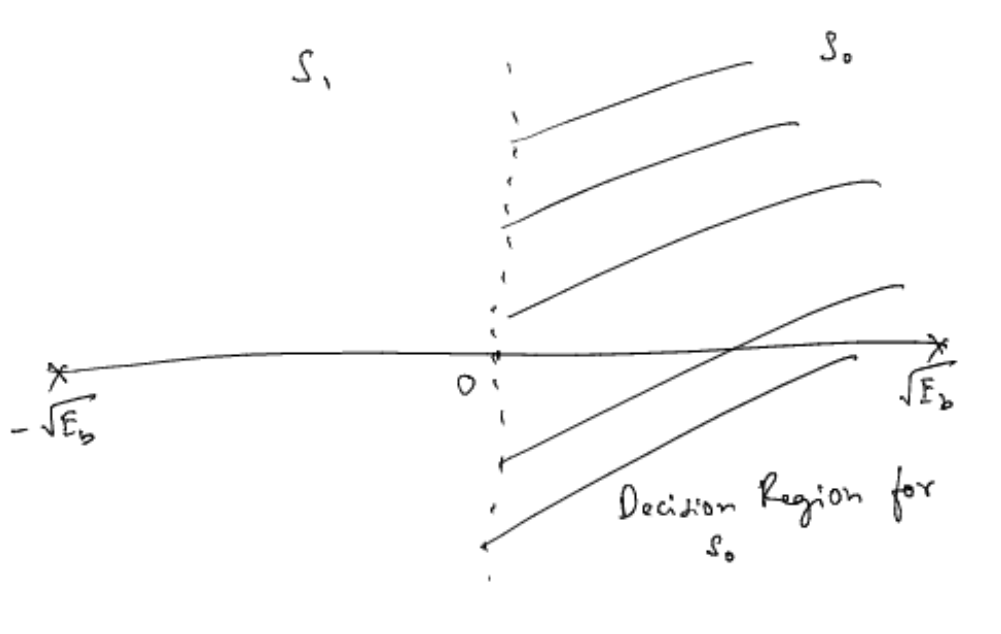


Figure 1.1:

4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr(y < 0 | s_0) = \Pr(\sqrt{E_b} + n < 0) \quad (4.1)$$

$$= \Pr(-n > \sqrt{E_b}) = \Pr(n > \sqrt{E_b}) \quad (4.2)$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0, 1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in

(4.2),

$$P_e = \Pr \left(\sqrt{\frac{N_0}{2}} w > \sqrt{E_b} \right) = \Pr \left(w > \sqrt{\frac{2E_b}{N_0}} \right) \quad (4.3)$$

$$= Q \left(\sqrt{\frac{2E_b}{N_0}} \right) \quad (4.4)$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

5. The PDF of $w \sim \mathcal{N}(0, 1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x^2}{2} \right), -\infty < x < \infty \quad (5.1)$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (5.2)$$

Show that

$$Q(x) = \frac{1}{2} \text{erfc} \left(\frac{x}{\sqrt{2}} \right) \quad (5.3)$$

6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

`codes/bpsk_ber.py`

yields Fig. 6.1

7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} d\theta \quad (7.1)$$

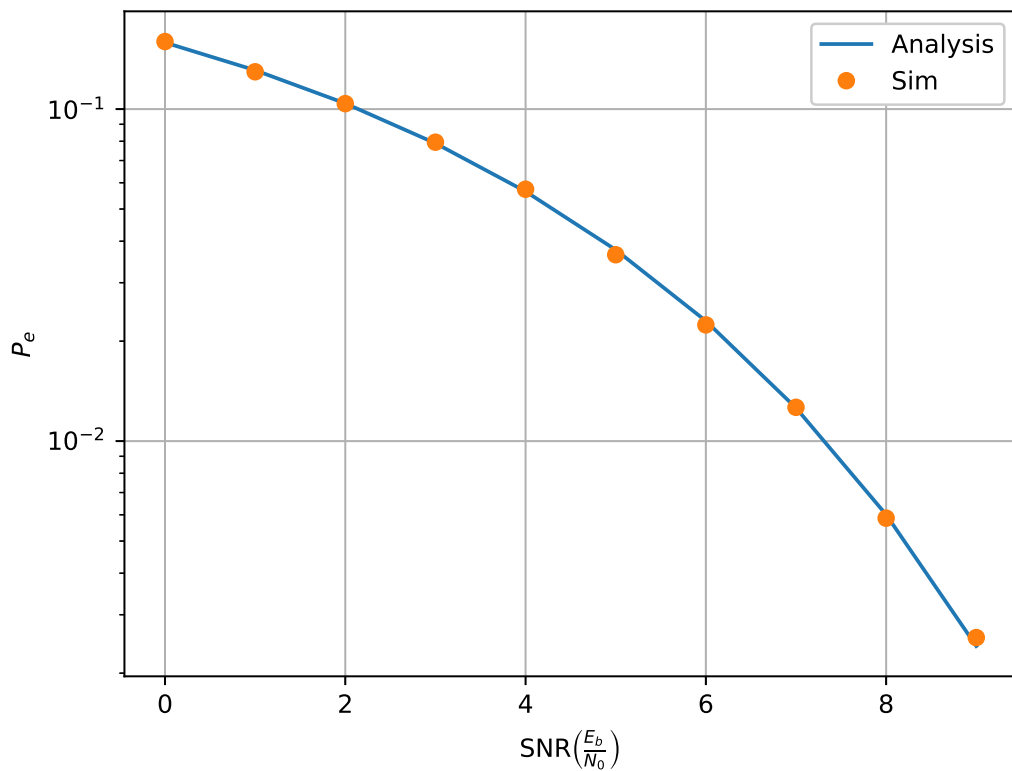


Figure 6.1:

10.2. Coherent BFSK

1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 1.1.

Obtain the equations for the received symbols.

Solution: The received symbols are given by

$$\mathbf{y}|_{s_0} = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (1.1)$$

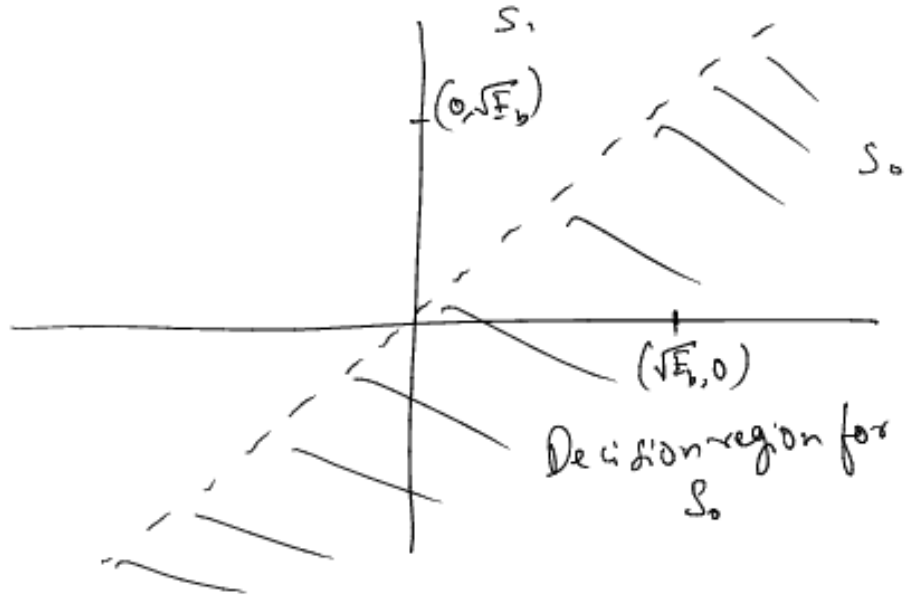


Figure 1.1:

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad (1.2)$$

where $n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2. Obtain a decision rule for BFSK from Fig. 1.1.

Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \quad (2.1)$$

3. Repeat the previous exercise using the MAP criterion.

4. Derive and plot the probability of error. Verify through simulation.

10.3. QPSK

1. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \quad (1.1)$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \quad (1.2)$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \quad (1.3)$$

$$E[\mathbf{n}] = \mathbf{0}, E[\mathbf{n}\mathbf{n}^T] = \sigma^2 \mathbf{I} \quad (1.4)$$

- (a) Show that the MAP decision for detecting \mathbf{s}_0 results in

$$|r|_2 < r_1 \quad (1.5)$$

- (b) Express $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0)$ in terms of r_1, r_2 . Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \sigma_y} \quad (1.6)$$

X and Y are said to be uncorrelated if $\rho = 0$

- (c) Show that if X and Y are uncorrelated Verify this numerically.

- (d) Show that X and Y are independent, i.e. $p_{XY}(x, y) = p_X(x)p_Y(y)$.
- (e) Show that $X, Y \sim \mathcal{N}(0, 2\sigma^2)$.
- (f) Show that $\Pr(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0) = \Pr(X < A, Y < A)$.
- (g) Find $\Pr(X < A, Y < A)$.
- (h) Verify the above through simulation.

10.4. M -PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos(\frac{2\pi i}{M}) \\ \cos(\frac{2\pi i}{M}) \end{pmatrix}, i = 0, 1, \dots, M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \quad (1.1)$$

where $n_1, n_2 \sim \mathcal{N}(0, \frac{N_0}{2})$.

- (a) Substituting

$$r_1 = R \cos \theta \quad (1.2)$$

$$r_2 = R \sin \theta \quad (1.3)$$

show that the joint pdf of R, θ is

$$p(R, \theta) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s} \cos \theta + E_s}{N_0}\right) \quad (1.4)$$

(b) Show that

$$\lim_{\alpha \rightarrow \infty} \int_0^{\infty} (V - \alpha) e^{-(V-\alpha)^2} dV = 0 \quad (1.5)$$

$$\lim_{\alpha \rightarrow \infty} \int_0^{\infty} e^{-(V-\alpha)^2} dV = \sqrt{\pi} \quad (1.6)$$

(c) Using the above, evaluate

$$\int_0^{\infty} V \exp \{ - (V^2 - 2V\sqrt{\gamma} \cos \theta + \gamma) \} dV \quad (1.7)$$

for large values of γ .

(d) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}} \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma \sin^2 \theta} \cos \theta d\theta \quad (1.8)$$

(e) Find $P_{e|\mathbf{s}_0}$.

10.5. Noncoherent BFSK

10.5.1 Show that

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos \theta} d\theta \quad (10.5.1.1)$$

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos(\theta-\phi)} d\theta \quad (10.5.1.2)$$

$$\frac{1}{2\pi} \int_0^{2\pi} e^{m_1 \cos \theta + m_2 \sin \theta} d\theta = I_0 \left(\sqrt{m_1^2 + m_2^2} \right) \quad (10.5.1.3)$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \theta} \cos n\theta d\theta \quad (10.5.1.4)$$

10.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b} \begin{pmatrix} \cos \phi_0 \\ \sin \phi_0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b} \begin{pmatrix} 0 \\ 0 \\ \cos \phi_1 \\ \sin \phi_1 \end{pmatrix} + \mathbf{n}_1 \quad (10.5.2.1)$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}(\mathbf{0}, \frac{N_0}{2} \mathbf{I})$.

- (a) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$, find the pdf $p(\mathbf{r}|0, \phi_0)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0 . Assume that all noise variables are independent.
- (b) If ϕ_0 is uniformly distributed between 0 and 2π , find $p(\mathbf{r}|0)$. Note that this expression will no longer contain ϕ_0 .
- (c) Show that the ML detection criterion for this scheme is

$$I_0 \left(k \sqrt{r_1^2 + r_2^2} \right) \stackrel{0}{\underset{1}{\gtrless}} I_0 \left(k \sqrt{r_3^2 + r_4^2} \right) \quad (10.5.2.2)$$

where k is a constant.

- (d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.
- (e) Show that

$$P_{e|0} = \Pr(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0) \quad (10.5.2.3)$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ is

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \quad (10.5.2.4)$$

(g) Find

$$g(r_1, r_2) = \Pr(r_1^2 + r_2^2 < Y | 0, r_1, r_2). \quad (10.5.2.5)$$

(h) Show that $E \left[e^{-\frac{X^2}{2\sigma^2}} \right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}(\mu, \sigma^2)$.

(i) Now show that

$$E[g(r_1, r_2)] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \quad (10.5.2.6)$$

10.5.3 Let $U, V \sim \mathcal{N}(0, \frac{k}{2})$ be i.i.d. Assuming that

$$U = \sqrt{R} \cos \Theta \quad (10.5.3.1)$$

$$V = \sqrt{R} \sin \Theta \quad (10.5.3.2)$$

(a) Compute the jacobian for U, V with respect to R and Θ defined by

$$J = \det \begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \quad (10.5.3.3)$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v) J|_{u=\sqrt{r} \cos \theta, v=\sqrt{r} \sin \theta} \quad (10.5.3.4)$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k} e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \quad (10.5.3.5)$$

assuming that Θ is uniformly distributed between 0 to 2π .

- (c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k} e^{-\frac{|y|}{k}} \quad (10.5.3.6)$$

- (d) Find the pdf of

$$Z = p + \sqrt{p} [U \cos \phi + V \sin \phi] \quad (10.5.3.7)$$

where ϕ is a constant.

- (e) Find $\Pr(Y > Z)$.

- (f) If $U \sim \mathcal{N}(m_1, \frac{k}{2})$, $V \sim \mathcal{N}(m_2, \frac{k}{2})$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \quad (10.5.3.8)$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \quad (10.5.3.9)$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \quad (10.5.3.10)$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \geq 0 \\ 0 & z < 0 \end{cases} \quad (10.5.3.11)$$

find $\Pr(R < Z)$.

10.6. Craig's Formula and MGF

10.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E[e^{sX}] \quad (10.6.1.1)$$

where X is a random variable and $E[\cdot]$ is the expectation.

(a) Let $Y \sim \mathcal{N}(0, 1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \quad (10.6.1.2)$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2 \sin^2 \theta}} d\theta \quad (10.6.1.3)$$

(b) Let $h \sim \mathcal{CN}(0, \frac{\Omega}{2})$, $n \sim \mathcal{CN}(0, \frac{N_0}{2})$. Find the distribution of $|h|^2$.

(c) Let

$$P_e = \Pr(\Re\{h^*y\} < 0), \text{ where } y = \left(\sqrt{E_s}h + n\right), \quad (10.6.1.4)$$

Show that

$$P_e = \int_0^\infty Q\left(\sqrt{2x}\right) p_A(x) dx \quad (10.6.1.5)$$

where $A = \frac{E_s|h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \quad (10.6.1.6)$$

(e) compute $M_A(s)$.

(f) Find P_e .

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.

