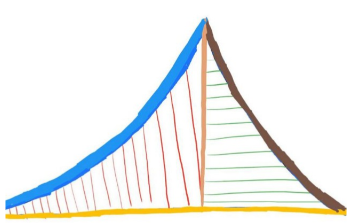
DIGITAL COMMUNICATION

Through Simulations

G. V. V. Sharma



Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Introduction

Chapter 2

Axioms

2.1 A and B are events such that $\Pr(A) = 0.42, \Pr(B) = 0.48$ and $\Pr(\text{A and B}) = 0.16$. Determine

(a) $\Pr(\text{not A})$

(b) $\Pr(\text{not B})$

(c) $\Pr(\text{A or B})$

Solution:

(a) $\Pr(\text{not A})$

$$\Pr\left(A'\right) = 1 - \Pr(A) \tag{2.1}$$

$$= 1 - 0.42 \tag{2.2}$$

$$= 0.58 \tag{2.3}$$

(b) Pr (not B)

$$\Pr\left(B'\right) = 1 - \Pr(B) \tag{2.4}$$
$$= 1 - 0.48 \tag{2.5}$$
$$= 0.52 \tag{2.6}$$

(c) Pr (A or B)

$$\Pr(\text{A+B}) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.7}$$
$$= 0.42 + 0.48 - 0.16 \tag{2.8}$$
$$= 0.74 \tag{2.9}$$

2.2 In class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class,find the probability that he will be studying Mathematics or Biology

Solution: The given information is summarised in Table 2.2. Thus,

Random Variable	**Subject**	**Probability**
M	Mathematics	$\Pr(M)$=0.4
B	Biology	$\Pr(B)$=0.3
M, B	Both	$\Pr(MB)$=0.10

Table 2.2:

$$\Pr(M+B) = \Pr(M) + \Pr(B) - \Pr(M,B) \tag{2.10}$$
$$= 0.6 \tag{2.11}$$

2.3 A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(a) $\Pr(2)$

(b) $\Pr(1 \text{ or } 3)$

(c) $\Pr(\text{not } 3)$

Solution: Table 2.4 summarises the given information.

Variable	**Value**	**Description**	**Probability**	**Pr Value**
X_1	1	Face of die '1'	$\Pr(X_1)$	$\frac{1}{3}$
X_2	2	Face of die '2'	$\Pr(X_2)$	$\frac{1}{2}$
X_3	3	Face of dir '3'	$\Pr(X_3)$	$\frac{1}{6}$

Table 2.4:

(a)

$$\Pr(X_2) = \frac{1}{2} \tag{2.12}$$

(b)

$$\Pr(X_1 + X_3) = \Pr(X_1) + \Pr(X_3) - \Pr(X_1 X_3) \tag{2.13}$$

$$= \frac{1}{3} + \frac{1}{6} \quad (\because \Pr(X_1 X_3) = 0) \tag{2.14}$$

$$= \frac{1}{2} \tag{2.15}$$

(c)

$$\Pr\left(X_3'\right) = 1 - \Pr\left(X_3\right) \tag{2.16}$$

$$= 1 - \frac{1}{6} \tag{2.17}$$

$$= \frac{5}{6} \tag{2.18}$$

2.4 If $\Pr(A) = 0.8$, $\Pr(B) = 0.5$ and $\Pr(B|A = 0.4)$, find

(a) $\Pr(AB)$

(b) $\Pr(A|B)$

(c) $\Pr(A + B)$

Solution:

(a) Since

$$\Pr(B|A) = \frac{\Pr(AB)}{\Pr(A)}, \tag{2.19}$$

from the given information,

$$\frac{\Pr(AB)}{\Pr(A)} = 0.4 \tag{2.20}$$

$$\implies \Pr(AB) = 0.4 \times 0.8 \tag{2.21}$$

$$= 0.32 \tag{2.22}$$

(b) Similarly,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} \tag{2.23}$$
$$= \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}. \tag{2.24}$$
$$= \frac{0.4 \times 0.8}{0.5} \tag{2.25}$$
$$= 0.64 \tag{2.26}$$

(c) Since,

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.27}$$

Substituting (2.22) in (2.27),

$$\Pr(A + B) = 0.8 + 0.5 - 0.32 \tag{2.28}$$
$$= 0.98 \tag{2.29}$$

2.5 A fair die is rolled. Consider events $E = 1, 3, 5 F = 2, 3$ and $G = 2, 3, 4, 5$. Find

(a) $\Pr(E \mid F)$ and $\Pr(F \mid E)$

(b) $\Pr(E \mid G)$ and $\Pr(G \mid E)$

(c) $\Pr(E \cup F \mid G)$ and $\Pr(E \cap F \mid G)$

Solution: The given information is summarised in Table 2.6.

Event	**Probability**
$\Pr(E)$	$\frac{1}{2}$
$\Pr(F)$	$\frac{1}{3}$
$\Pr(G)$	$\frac{2}{3}$
$\Pr(EF)$	$\frac{1}{6}$
$\Pr(EG)$	$\frac{1}{3}$
$\Pr(FG)$	$\frac{1}{3}$
$\Pr(EFG)$	$\frac{1}{6}$

Table 2.6: Probability of Events.

(a)

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \tag{2.30}$$

$$= \frac{\frac{1}{6}}{\frac{1}{3}} \tag{2.31}$$

$$= \frac{1}{2} \tag{2.32}$$

$$\Pr(F \mid E) = \frac{\Pr(FE)}{\Pr(E)} \tag{2.33}$$

$$= \frac{\frac{1}{6}}{\frac{1}{2}} \tag{2.34}$$

$$= \frac{1}{3} \tag{2.35}$$

(b)

$$\Pr(E \mid G) = \frac{\Pr(EG)}{\Pr(G)} \tag{2.36}$$
$$= \frac{\frac{1}{3}}{\frac{2}{3}} \tag{2.37}$$
$$= \frac{1}{2} \tag{2.38}$$
$$\Pr(G \mid E) = \frac{\Pr(GE)}{\Pr(G)} \tag{2.39}$$
$$= \frac{\frac{1}{3}}{\frac{1}{2}} \tag{2.40}$$
$$= \frac{2}{3} \tag{2.41}$$

(c)

$$\Pr(E + F \mid G) = \frac{\Pr((E+F)G)}{\Pr(G)} \tag{2.42}$$
$$= \frac{\Pr(EG + FG)}{\Pr(G)} \tag{2.43}$$
$$= \frac{\Pr(EG) + \Pr(FG) - \Pr(EFG)}{\Pr(G)} \tag{2.44}$$
$$= \frac{3}{4} \tag{2.45}$$
$$\Pr(EF \mid G) = \frac{\Pr(EFG)}{\Pr(G)} \tag{2.46}$$
$$= \frac{1}{4} \tag{2.47}$$

2.6 If $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{5}$ find $\Pr(A \cap B)$ if A and B are independent events.

Solution: From the given information,

$$\Pr(A) = \frac{3}{5}, \Pr(B) = \frac{1}{5}. \tag{2.48}$$

Since A and B are independent,

$$\Pr(AB) = \Pr(A) \times \Pr(B) \tag{2.49}$$
$$= \frac{3}{5} \times \frac{1}{5} \tag{2.50}$$
$$= \frac{3}{25} \tag{2.51}$$

2.7 Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A+B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

Solution:

(a) In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \tag{2.52}$$
$$\implies \frac{3}{5} = \frac{1}{2} + p \tag{2.53}$$
$$\therefore p = \frac{1}{10} \tag{2.54}$$

(b) Given A and B are independent events, then,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.55}$$

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{2.56}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{2.57}$$

$$\therefore p = \frac{1}{5} \tag{2.58}$$

2.8 Let E and F be events with $\Pr(E) = \frac{3}{5}, \Pr(F) = \frac{3}{10}$ and $\Pr(EF) = \frac{1}{5}$. Are E and F independent?

Solution: From the given information,

$$\Pr(E)\Pr(F) = \frac{3}{5} \times \frac{9}{50} \tag{2.59}$$

$$\Pr(EF) = \frac{1}{50} \tag{2.60}$$

$$\implies \Pr(EF) \neq P(E)P(F) \tag{2.61}$$

$\therefore E$ and F are not independent events.

2.9 If A and B are two events such that $\Pr(A) = \frac{1}{4}, \Pr(B) = \frac{1}{2}$ and $\Pr(AB) = \frac{1}{8}$, find $\Pr$ (not A and not B).

Solution: Since

$$A'B' = (A+B)', \tag{2.62}$$

$$\Pr\left(A'B'\right) = \Pr\left((A+B)'\right) \tag{2.63}$$

$$= 1 - \Pr(A+B) \tag{2.64}$$

Thus,

$$\Pr\left(A'B'\right) = 1 - \{\Pr(A) + \Pr(B) - \Pr(AB)\} \tag{2.65}$$
$$= \frac{3}{8} \tag{2.66}$$

2.10 If A and B are two events such that $A \subset B$ and $\Pr(B) \neq 0$, then which of the following is correct ?

(a) $\Pr(A \mid B) = \frac{\Pr(B)}{\Pr(A)}$

(b) $\Pr(A \mid B) < \Pr(A)$

(c) $\Pr(A \mid B) \geq \Pr(A)$

(d) None of these

Solution: if $A \subset B$ and $\Pr(B) \neq 0$ then

$$AB = A \tag{2.67}$$
$$\text{or, } P(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} \tag{2.68}$$

we know that

$$\Pr(B) \leq 1 \tag{2.69}$$
$$\implies 1 \leq \frac{1}{\Pr(B)} \tag{2.70}$$

Multiplying both sides with $\Pr(A)$,

$$\Pr(A) \le \frac{\Pr(A)}{\Pr(B)} \tag{2.71}$$

$$= \Pr(A \mid B) \tag{2.72}$$

from (2.68).

2.11 A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

2.12 A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even, ' and B be the event, 'the number is red'. Are A and B independent?

2.13 Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

2.14 Given that the events A and B are such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

2.15 Let A and B be independent events with $P(A) = 0.3$ and $P(B) = 0.4$. Find

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) $P(A|B)$

(d) $P(B|A)$

2.16 If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$, find $P(\text{not } A \text{ and not } B)$

2.17 Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\text{not } A \text{ or not } B) = \frac{1}{4}$. State whether A and B are independent?

2.18 Given two independent events A and B such that $P(A) = 0.3$, $P(B) = 0.6$. Find

(a) $P(A \text{ and } B)$

(b) $P(A \text{ and not } B)$

(c) $P(A \text{ or } B)$

(d) $P(\text{neither } A \text{ nor } B)$

2.19 Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved

(b) exactly one of them solves the problem

2.20 One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent ?

(a) E: 'the card drawn is spade'

F: 'the card drawn is an ace'

(b) E: 'the card drawn is black'

F : 'the card drawn is a king'

(c) E : 'the card drawn is a king or queen'

F : 'the card drawn is a queen or jack'

Choose the correct answer in the following exercises

2.21 The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{12}$

(d) $\frac{1}{36}$

2.22 Two events A and B will be independent, if

(a) A and B are mutually exclusive

(b) P(not $A \cap$ not B) $=$ $[1 - P(A)]\ [1 - P(B)]$

(c) $P(A) = P(B)$

(d) $P(A) + P(B) = 1$

Chapter 3

Definitions

3.1 Four cards are drawn from a well-shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade.

Solution: The given information is summarised in Table 3.2. yielding

RV	Values	Description
X	{0,1,2,3}	Cards drawn randomly
Y	{0,1}	0:diamond ,1:spade
X,Y	{00,10,20,31}	3 diamonds and one spade out of 13 each

Table 3.2: Random variables(RV) X,Y and X,Y

$$\Pr(00,10,20,31) = \frac{{}^{13}C_3 \times {}^{13}C_1}{{}^{52}C_4} \tag{3.1}$$

$$= \frac{286}{20285} \tag{3.2}$$

3.2 In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets ?

Solution: The given information is summarised in Table 3.4 The total number of possible outcomes is ${}^{N}C_n$ and the total number of favourable outcomes is ${}^{q}C_n$ yielding

Variable	**Value**	**Description**
N	10000	Total number of tickets sold
k	10	Total number of prizes awarded
n	{0,1,2,...,N}	Number of tickets purchased
$\Pr(n)$		probability of not wining a prize
q	N-k	number of tickets with no prize

Table 3.4:

the desired probability

$$\Pr(n) = \frac{{}^{q}C_n}{{}^{N}C_n} \tag{3.3}$$

Substituting numerical values,

(a) For one ticket,

$$\Pr(1) = \frac{{}^{9990}C_1}{{}^{10000}C_1} = 0.9990 \tag{3.4}$$

(b) For two tickets,

$$\Pr(2) = \frac{{}^{9990}C_2}{{}^{10000}C_2} = 0.9980 \tag{3.5}$$

(c) For 10 tickets

$$\Pr(3) = \frac{{}^{9990}C_{10}}{{}^{10000}C_{10}} = 0.9901 \tag{3.6}$$

3.3 Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

Solution: Table 3.6 summarises the given information.

RV	**Values**	**Description**
X	{0,1}	0: sectioin1, 1: section2
Y	{0,1}	0: student1, 1: student2
XY	{001,101}	Students enter same section
	{00,01,10,11}	Students enter different section

Table 3.6:

(a) When both enter the same section, the probability is

$$\Pr(001, 101) = \frac{^{40}C_2}{^{100}C_2} + \frac{^{60}C_2}{^{100}C_2} = \frac{156}{990} + \frac{354}{990} = 0.51 \tag{3.7}$$

(b) When both enter different sections, the desired probability is

$$\Pr(00, 01, 10, 11) = 1 - 0.51 = 0.49 \tag{3.8}$$

3.4 The number lock of a suitcase has 4 wheels each labelled with ten digits i.e. from 0 to 9.The lock opens with a sequence of four digits with no repeats.What is the probability of a person getting the right sequence to open the suitcase.

Solution: The given information is represented in Tables 3.8 3.10 and 3.12.

Random variable	**Value**	**Description**
X	{1,2,3,4}	The number lock of a suitcase
Y	{0,1,2...9}	The digits labelled on each wheel

Table 3.8: Random variables X and Y

Wheel 1	Wheel 2	Wheel 3	Wheel 4
10 ways	9 ways	8 ways	7 ways

Table 3.10: Suitcase wheel

The number of possible placement of the digits are

$$10 \times 9 \times 8 \times 7 = 5040 \tag{3.9}$$

Thus, the probability of the correct sequence begin selected is

$$\Pr(A) = \frac{1}{5040} \tag{3.10}$$

3.5 Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.

Solution: Table 3.14 summarizes the various events Given that the cards are drawn at random without replacement. Without replacement means only one card is random at a time and is excluded from the total while next card is drawn at random. Thus, the probability that both the cards are black is,

$$\Pr(00, 10) = \frac{{}^{26}C_1}{{}^{52}C_1} \times \frac{{}^{25}C_1}{{}^{51}C_1} = \frac{1}{2} \times \frac{25}{51} = 0.24 \tag{3.11}$$

3.6 A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale,

Example	**Wheel**	**Outcome**
1	8 6 4 2	Not repeating
2	8 4 2 6	Not repeating
3	1 2 3 4	Not repeating
4	8 8 8 8	Repeating
5	1 1 2 2	Repeating

Table 3.12: Combinations

RV	Values	Description
X	{0,1}	number of cards drawn 2
Y	{0,1}	0: black card, 1: red card
XY	{00,10}	card drawn is black

Table 3.14:

(b) first ball is black and second is red.

(c) one of them is black and other is red.

3.8 In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

Chapter 4

Conditional Probability

4.1 Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.

Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{4.1}$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{4.2}$$

4.2 Compute $\Pr(A|B)$, if $\Pr(B) = 0.5$ and $\Pr(AB) = 0.32$.

Solution: By using property of conditional probability we have,

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr B} = \frac{0.32}{0.5} = 0.64 \tag{4.3}$$

4.3 An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. What is the probability that the second ball is red?

4.4 A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

4.5 Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

4.6 In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Solution: Let $X \in \{0,1\}$ where 0 denotes a guess and 1 denotes that he knows the answer. Let $Y \in \{0,1\}$ where 0 being the case when the answer is incorrect and 1 being the case that the answer is correct. The given information is summarised in Tables 4.4 and 4.2

Random variable	**Description**
X=0	Student guesses the answer
X=1	Student knows the answer
Y=0	Answer is incorrect
Y=1	Answer is correct

Table 4.2: Random variables X and Y

The probability that the student knows the answer and he answered it correctly is

Pr(Event)	**Value**
Pr(Y=1 | X=0)	0.25
Pr(Y=1|X=1)	1
Pr(X=0)	0.75
Pr(X=1)	0.25

Table 4.4: Probability of events X and Y

$$\Pr(X=1|Y=1) = \frac{\Pr(Y=1|X=1)\Pr(X=1)}{\Pr(Y=1|X=1)\Pr(X=1)+\Pr(Y=1|X=0)\Pr(X=0)} \tag{4.4}$$

$$= \frac{0.25}{0.25+0.25\times 0.75} \tag{4.5}$$

$$= 0.571 \tag{4.6}$$

4.7 A laboratory blood test is 99% effective in detecting a certain disease when it is in fact, present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1 percent of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive ?

4.8 There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

4.9 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

4.10 A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

4.11 . Two groups are competing for the position on the Board of directors of a corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding prbability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

4.12 Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?]

4.13 A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

4.14 A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

4.15 Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$

(b) $\frac{1}{2}$

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

4.16 If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.

Chapter 5

Distributions

5.1 A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game

5.2 A die is tossed thrice. Find the probability of getting an odd number at least once.

5.3 Find the probability distribution of

(a) number of heads in two tosses of a coin.

(b) number of tails in the simultaneous tosses of three coins.

(c) number of heads in four tosses of a coin.

Solution: Table 5.2 summarises the given information.

(a) Number of heads in two tosses of a coin.

$$p_{X_1}(k) = {}^nC_k p^k q^{n-k}, 0 \le k \le 2, n = 2 \tag{5.1}$$

Variable	Value	Description
n	$\{2,3,4\}$	Number of trials in 2,3,4 tosses of a coin
p	$\frac{1}{2}$	Probability of getting a head
q	$1-p$	Probability of not getting a head
X_1	$\{0,1,2\}$	Number of heads in 2 tosses of a coin
X_2	$\{0,1,2,3\}$	Number of tails in 3 tosses of a coin
X_3	$\{0,1,2,3,4\}$	Number of heads in 4 tosses of a coin

Table 5.2: Variable Description

(b) Number of tails in the simultaneous tosses of three coins.

$$p_{X_2}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 3, n = 3 \tag{5.2}$$

(c) Number of heads in four tosses of a coin.

$$p_{X_3}(k) = {}^nC_k p^k q^{n-k}, 0 \leq k \leq 4, n = 4 \tag{5.3}$$

5.4 Find the mean number of heads in three tosses of a fair coin.

Solution: Substituting $n = 3, p = \frac{1}{2}$ in (A.6.1), the mean is $\frac{3}{2}$.

Chapter 6

Two Dice

6.1. Sum of Independent Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

6.1.1. The Uniform Distribution: Let $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{6.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{6.1.1.2}$$

$$\implies X \in \{1, 2, \dots, 12\} \tag{6.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{6.1.1.4}$$

6.1.2. Convolution: From (6.1.1.2),

$$p_X(n) = \Pr\left(X_1 + X_2 = n\right) = \Pr\left(X_1 = n - X_2\right) \tag{6.1.2.1}$$

$$= \sum_k \Pr\left(X_1 = n - k | X_2 = k\right) p_{X_2}(k) \tag{6.1.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr\left(X_1 = n - k | X_2 = k\right)$$
$$= \Pr\left(X_1 = n - k\right) = p_{X_1}(n-k) \tag{6.1.2.3}$$

From (6.1.2.2) and (6.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{6.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (6.1.1.1) in (6.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{6.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \tag{6.1.2.6}$$

From (6.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{6.1.2.7}$$

Substituting from (6.1.1.1) in (6.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{6.1.2.8}$$

satisfying (6.1.1.4).

6.1.3. The Z-transform: The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{6.1.3.1}$$

From (6.1.1.1) and (6.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{6.1.3.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{6.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{6.1.3.4}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{6.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (6.1.3.3) and (6.1.3.5),

$$P_X(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2 \tag{6.1.3.6}$$

$$= \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{6.1.3.7}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{6.1.3.8}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{6.1.3.9}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36} \frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{6.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{6.1.3.11}$$

From (6.1.3.1), (6.1.3.7) and (6.1.3.10)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) \right.$$
$$\left. -2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{6.1.3.12}$$

which is the same as (6.1.2.8). Note that (6.1.2.8) can be obtained from (6.1.3.10) using contour integration as well.

6.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 6.1.4.1. The theoretical pmf obtained in (6.1.2.8) is plotted for comparison.

6.1.5. The python code is available in

```
/codes/sum/dice.py
```

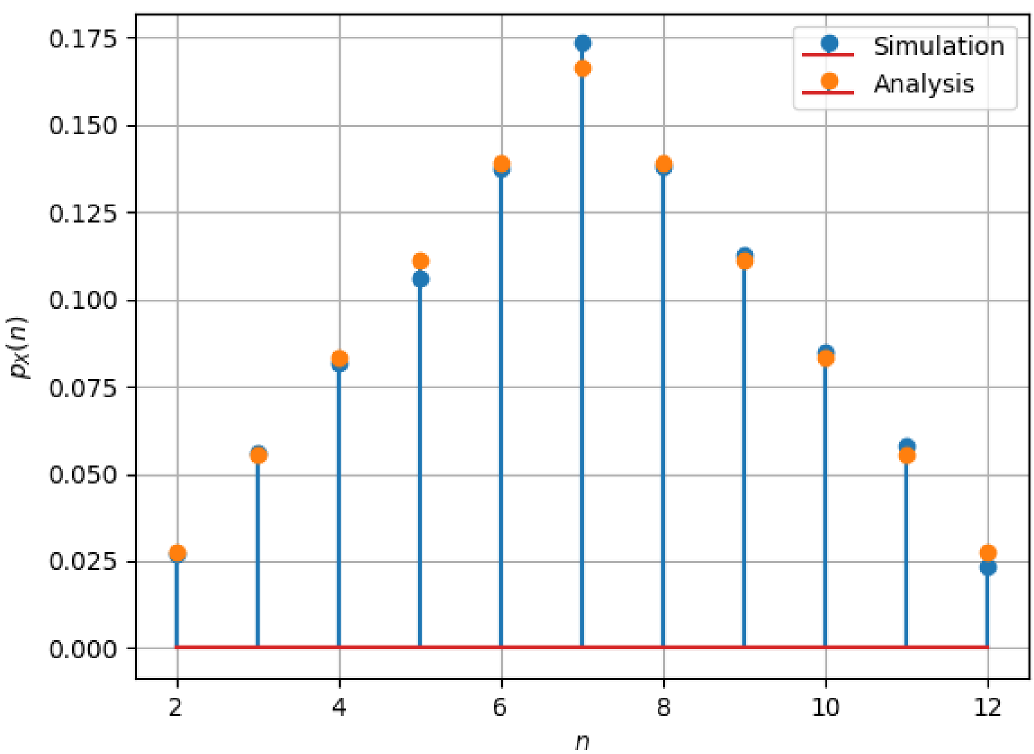


Figure 6.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

Chapter 7

Random Numbers

7.1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

7.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
codes/exrand.c
codes/coeffs.h
```

7.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{7.1.2.1}$$

Solution: The following code plots Fig. 7.1.2.1

```
codes/cdf_plot.py
```

7.1.3 Find a theoretical expression for $F_U(x)$.

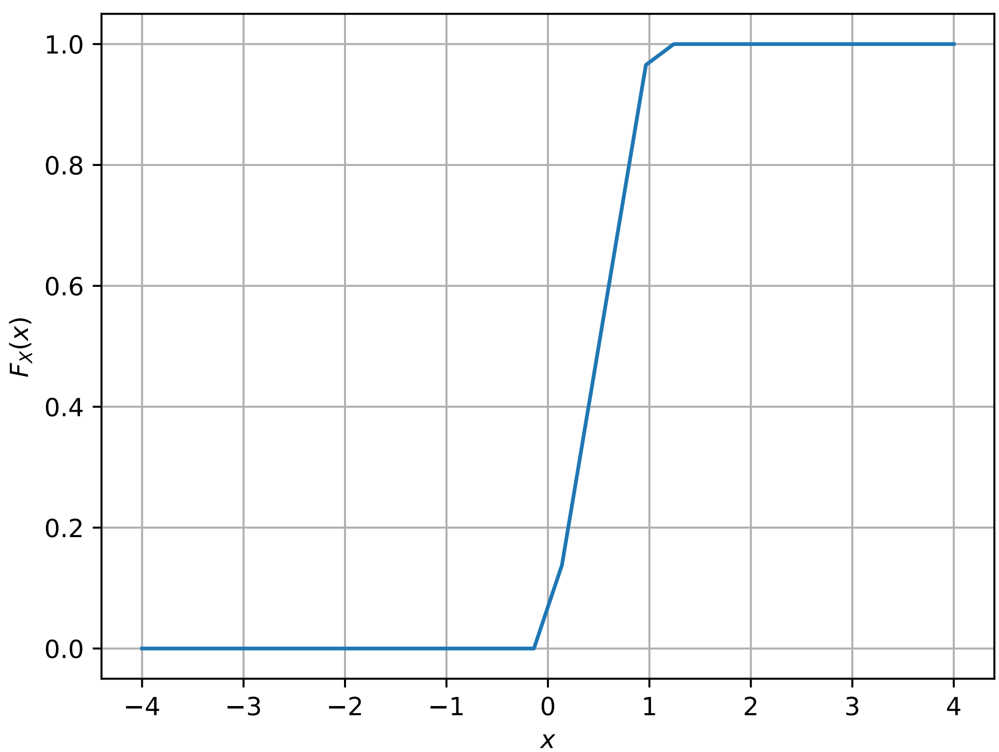


Figure 7.1.2.1: The CDF of U

7.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{7.1.4.1}$$

and its variance as

$$\operatorname{var}[U] = E[U - E[U]]^2 \tag{7.1.4.2}$$

Write a C program to find the mean and variance of U.

7.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{7.1.5.1}$$

7.2. Central Limit Theorem

7.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{7.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

7.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 7.2.2.1

7.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat.

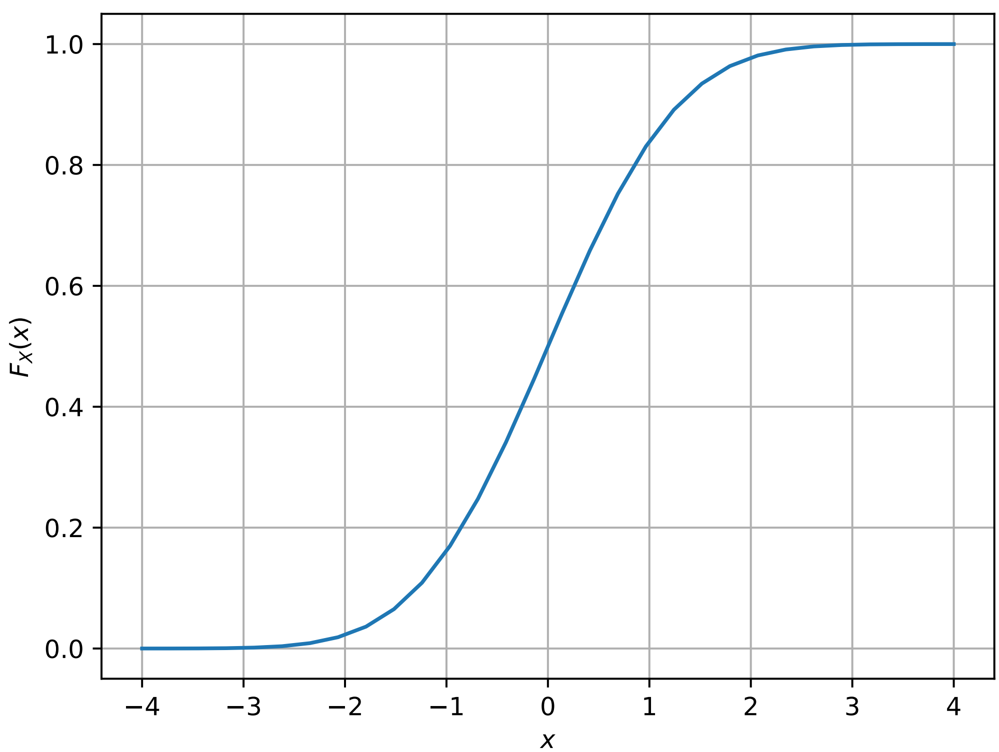


Figure 7.2.2.1: The CDF of X

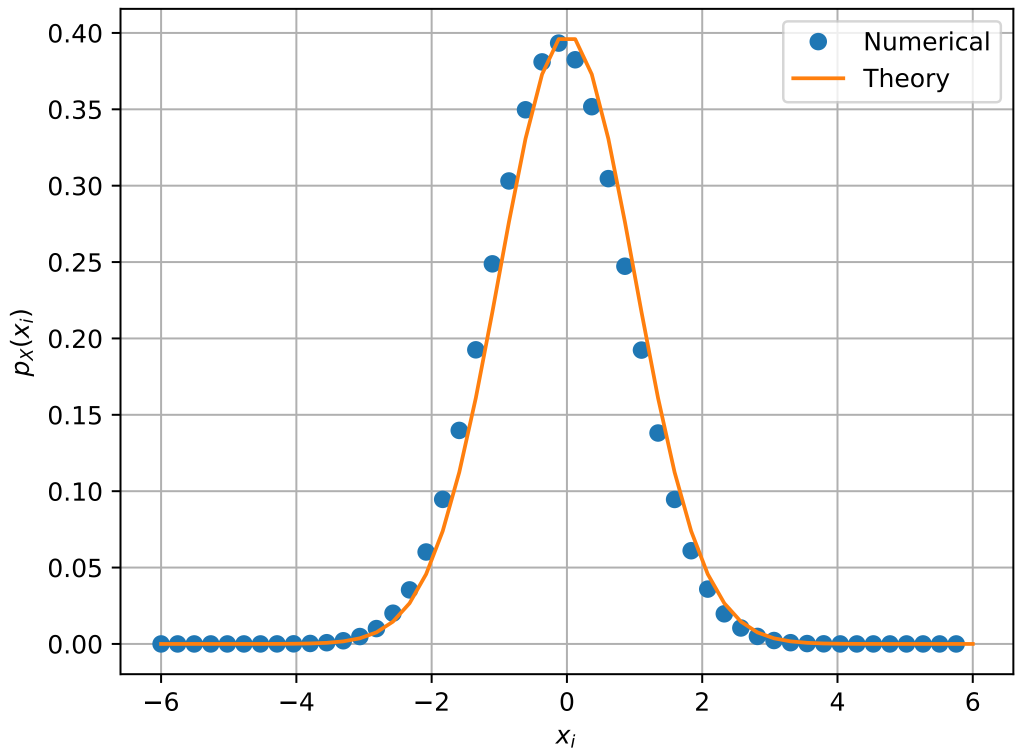


Figure 7.2.3.1: The PDF of X

The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{7.2.3.1}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 7.2.3.1 using the code below

```
codes/pdf_plot.py
```

7.2.4 Find the mean and variance of X by writing a C program.

7.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{7.2.5.1}$$

repeat the above exercise theoretically.

7.3. From Uniform to Other

7.3.1 Generate samples of

$$V = -2\ln(1-U) \tag{7.3.1.1}$$

and plot its CDF.

7.3.2 Find a theoretical expression for $F_V(x)$.

7.4. Triangular Distribution

7.4.1 Generate

$$T = U_1 + U_2 \tag{7.4.1.1}$$

7.4.2 Find the CDF of T.

7.4.3 Find the PDF of T.

7.4.4 Find the theoretical expressions for the PDF and CDF of T.

7.4.5 Verify your results through a plot.

Chapter 8

Maximum Likelihood Detection: BPSK

8.1. Maximum Likelihood

8.1.1 Generate equiprobable $X \in \{1, -1\}$.

8.1.2 Generate

$$Y = AX + N, \tag{8.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

8.1.3 Plot Y using a scatter plot.

8.1.4 Guess how to estimate X from Y.

8.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{8.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{8.1.5.2}$$

8.1.6 Find P_e assuming that X has equiprobable symbols.

8.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

8.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

8.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{8.1.9.1}$$

8.1.10 Repeat the above exercise using the MAP criterion.

Chapter 9

Transformation of Random Variables

9.1. Gaussian to Other

9.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{9.1.1.1}$$

9.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{9.1.2.1}$$

find α.

9.1.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{9.1.3.1}$$

9.2. Conditional Probability

9.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{9.2.1.1}$$

for

$$Y = AX + N, \tag{9.2.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

9.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

9.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{9.2.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

9.2.4 Plot P_e in problems 9.2.1 and 9.2.3 on the same graph w.r.t γ. Comment.

Chapter 10

Bivariate Random Variables: FSK

10.1. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{4.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{4.3}$$

10.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{10.1.1.1}$$

on the same graph using a scatter plot.

10.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

10.1.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{10.1.3.1}$$

with respect to the SNR from 0 to 10 dB.

10.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Chapter 11

Exercises

11.1. BPSK

1. The signal constellation diagram for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

   Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{1.2}$$

   where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

2. From Fig. 1.1 obtain a decision rule for BPSK

   Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{2.1}$$

3. Repeat the previous exercise using the MAP criterion.

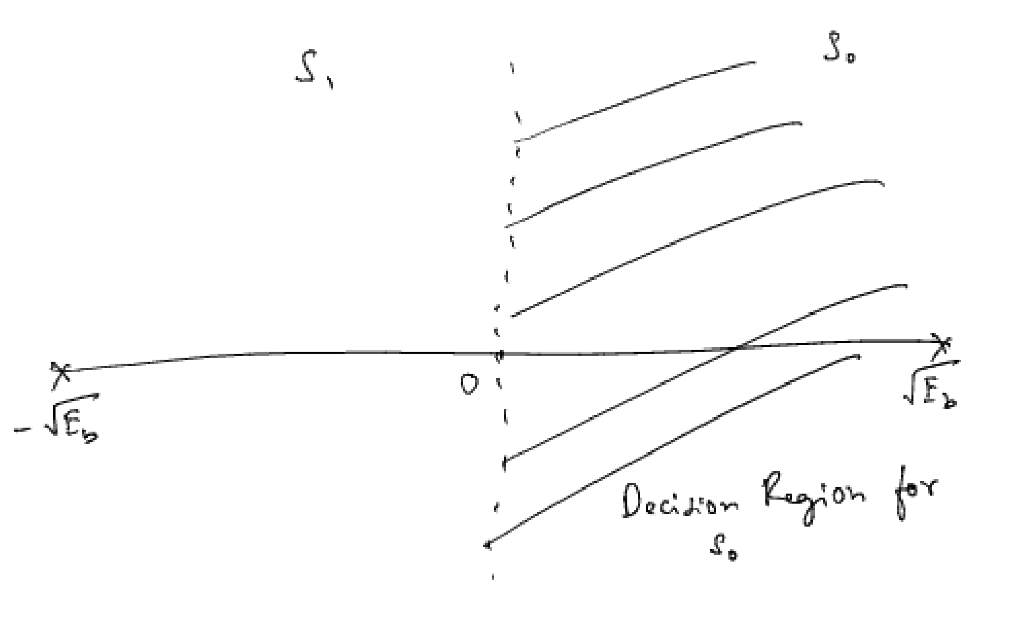


Figure 1.1:

4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0 | s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{4.1}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in

(4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{4.3}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{4.4}$$

where $Q(x) \triangleq \Pr(w > x), x \geq 0$.

5. The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{5.1}$$

and the complementary error function is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt. \tag{5.2}$$

Show that

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{5.3}$$

6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 6.1

7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{7.1}$$

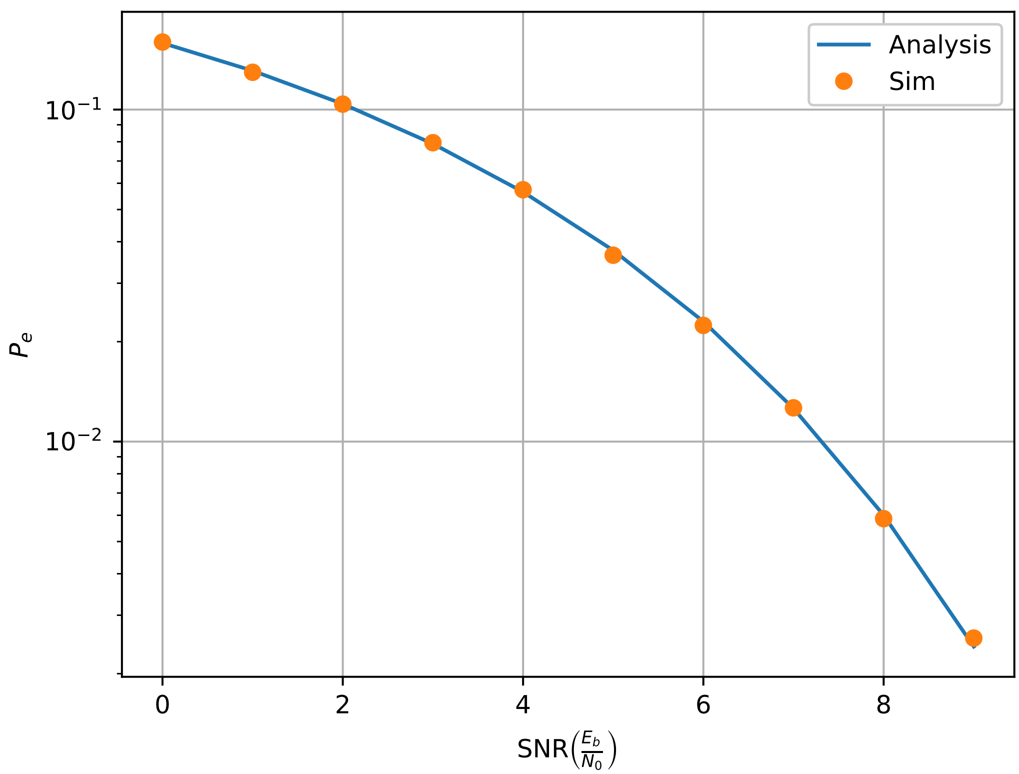


Figure 6.1:

11.2. Coherent BFSK

1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 1.1. Obtain the equations for the received symbols.

   Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{1.1}$$

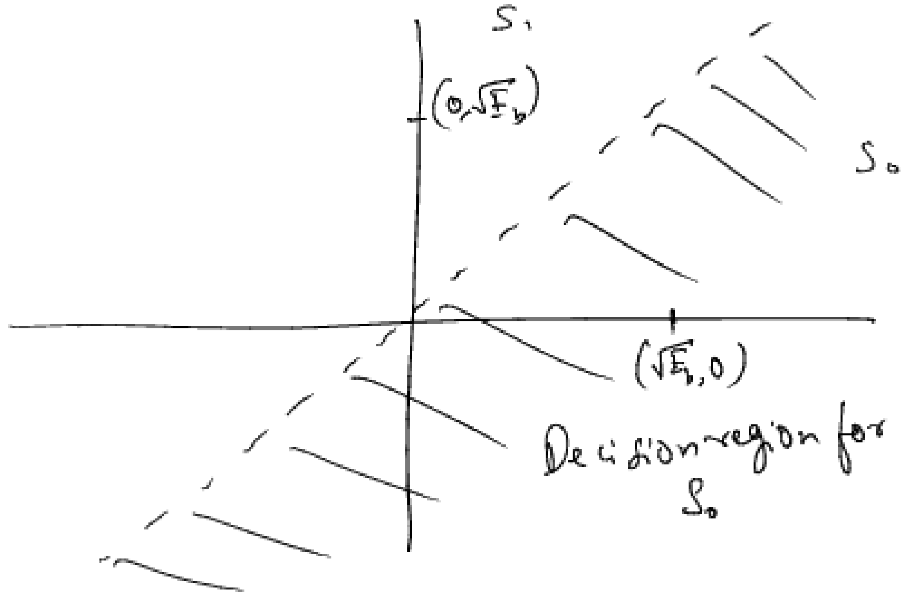


Figure 1.1:

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2. Obtain a decision rule for BFSK from Fig. 1.1.

   Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{2.1}$$

3. Repeat the previous exercise using the MAP criterion.

4. Derive and plot the probability of error. Verify through simulation.

11.3. QPSK

1. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{1.1}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{1.2}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{1.3}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{1.4}$$

(a) Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|r|_2 < r_1 \tag{1.5}$$

(b) Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[\left(X - \mu_x\right)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{1.6}$$

X and Y are said to be uncorrelated if $\rho = 0$

(c) Show that if X and Y are uncorrelated Verify this numerically.

(d) Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.

(e) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.

(f) Show that $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < A, Y < A\right)$.

(g) Find $\Pr\left(X < A, Y < A\right)$.

(h) Verify the above through simulation.

11.4. M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{1.1}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(a) Substituting

$$r_1 = R\cos\theta \tag{1.2}$$

$$r_2 = R\sin\theta \tag{1.3}$$

show that the joint pdf of R, θ is

$$p\left(R,\theta\right) = \frac{R}{\pi N_0}\exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{1.4}$$

(b) Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\, e^{-(V-\alpha)^2}\, dV = 0 \tag{1.5}$$

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\, dV = \sqrt{\pi} \tag{1.6}$$

(c) Using the above, evaluate

$$\int_0^\infty V \exp\left\{-\left(V^2 - 2V\sqrt{\gamma}\cos\theta + \gamma\right)\right\}\, dV \tag{1.7}$$

for large values of γ.

(d) Find a compact expression for

$$I = 1 - \sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\, d\theta \tag{1.8}$$

(e) Find $P_{e|\mathbf{s}_0}$.

11.5. Noncoherent BFSK

11.5.1 Show that

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\, d\theta \tag{11.5.1.1}$$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta-\phi)}\, d\theta \tag{11.5.1.2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{m_1\cos\theta + m_2\sin\theta}\, d\theta = I_0\left(\sqrt{m_1^2+m_2^2}\right) \tag{11.5.1.3}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos n\theta\, d\theta \tag{11.5.1.4}$$

11.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b}\begin{pmatrix}\cos\phi_0 \\ \sin\phi_0 \\ 0 \\ 0\end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b}\begin{pmatrix}0 \\ 0 \\ \cos\phi_1 \\ \sin\phi_1\end{pmatrix} + \mathbf{n}_1 \tag{11.5.2.1}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

(a) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p\left(\mathbf{r}|0, \phi_0\right)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.

(b) If ϕ_0 is uniformly distributed between 0 and 2π, find $p\left(\mathbf{r}|0\right)$. Note that this expression will no longer contain ϕ_0.

(c) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \tag{11.5.2.2}$$

where k is a constant.

(d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.

(e) Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \tag{11.5.2.3}$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \tag{11.5.2.4}$$

(g) Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < Y | 0, r_1, r_2\right). \tag{11.5.2.5}$$

(h) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(i) Now show that

$$E\left[g(r_1, r_2)\right] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \tag{11.5.2.6}$$

11.5.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R}\cos\Theta \tag{11.5.3.1}$$

$$V = \sqrt{R}\sin\Theta \tag{11.5.3.2}$$

(a) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det\begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \tag{11.5.3.3}$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v)\, J|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{11.5.3.4}$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k}e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \tag{11.5.3.5}$$

assuming that Θ is uniformly distributed between 0 to 2π.

(c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k}e^{-\frac{|y|}{k}} \tag{11.5.3.6}$$

(d) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{11.5.3.7}$$

where ϕ is a constant.

(e) Find $\Pr(Y > Z)$.

(f) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{11.5.3.8}$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k}I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{11.5.3.9}$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \tag{11.5.3.10}$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \ge 0 \\ 0 & z < 0 \end{cases} \tag{11.5.3.11}$$

find $\Pr(R < Z)$.

11.6. Craig's Formula and MGF

11.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{11.6.1.1}$$

where X is a random variable and $E[\cdot]$ is the expectation.

(a) Let $Y \sim \mathcal{N}(0, 1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \tag{11.6.1.2}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{11.6.1.3}$$

(b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

(c) Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{11.6.1.4}$$

Show that

$$P_e = \int_0^\infty Q\left(\sqrt{2x}\right) p_A(x)\, dx \tag{11.6.1.5}$$

where $A = \frac{E_s |h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{11.6.1.6}$$

(e) compute $M_A(s)$.

(f) Find P_e.

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.

Appendix A

Z-transform

A.1 The Z-transform of X is defined as

$$M_X(z) = E\left[z^{-X}\right] = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{A.1.1}$$

A.2 The nth moment of X can be expressed as

$$E\left[X^n\right] = \frac{d^n M_X(z^{-1})}{dz^n}|_{z=1} \tag{A.2.1}$$

A.3 For a Bernoulli random variable X with success probability p,

$$M_X(z) = q + pz^{-1} \tag{A.3.1}$$

A.4 Let X_i be i.i.d. For

$$X = X_1 + X_2 + ... + X_n, \tag{A.4.1}$$

$$M_X(z) = \prod_{=1}^{n} M_{X_i}(z) \tag{A.4.2}$$

A.5 For a Binomial random variable X with parameters n, p,

$$M_X(z) = \left(q + pz^{-1}\right)^n \tag{A.5.1}$$

A.6 The mean for the Binomial r.v. is

$$E\left[X\right] = np \tag{A.6.1}$$

Solution: From (A.2.1) and (A.5.1),

$$E\left[X\right] = \frac{d(q+pz)^n}{dz}|_{z=1} \tag{A.6.2}$$

$$= np(q+pz)^{n-1}|_{z=1} \tag{A.6.3}$$

$$= np(q+p)^{n-1} \tag{A.6.4}$$

yielding (A.6.1)

$$\because p + q = 1 \tag{A.6.5}$$