1 Vector Spherical Harmonics

In a first step the VSH coefficients are calculated from raw data describing the full sampled complex radiation pattern over the whole sphere and for the two orthogonal linear polarizations.

This step can be done once off-line to build an antenna database which exploits the sparse VSH decomposition.

$$S_1 : \mathbb{C}^{2\times N_\theta \times N_\phi \times N_f} \to \mathbb{C}^{4\times N_c}$$

In a second step field quantities are generated from the VSH coefficients for a given set of directions corresponding to the DOD/DOA of the rays coming out the RT engine.

$$S_2 : \mathbb{C}^{4\times N_c} \to \mathbb{C}^{2\times N_r \times N_f}$$

One of the interest in doing this decomposition comes from the fact that the number of directions of interest N_r is by far lower than the number required for the exhaustive sampling of the sphere needed at the starting point of the synthesis step. In practice, for a given channel, we do not need to calculate the field in all directions but only for a limited number of them. The alternative approach would consist in going through an interpolation procedure into the initial data structure, what could be either slower if the grid is tight or poorly accurate.

We are dealing with a set of complex quantities that belongs to
where N_θ,N_ϕ,N_f states respectively for the number of angle θ ϕ and frequency point. In practice in a ray tracing tool a direction of

$$\mathbb{C}^{2\times N_r \times N_f}$$

N_r is the number of rays.

$$\mathbf{F}(f,\theta,\phi) = \begin{bmatrix} F_\theta(f,\theta,\phi) \\ F_\phi(f,\theta,\phi) \end{bmatrix}$$

1.1 Vector spherical Harmonics function

The vector spherical harmonics are based on the function $\bar{V}_l^{(m)}$ and $\bar{W}_l^{(m)}$

$$\bar{V}_l^{(m)}(x) = (-1)^n \frac{\sqrt{1-x^2}}{\sqrt{l(l+1)}} \bar{P}_l^{(m)'}(x)$$

$$\bar{V}_l^{(m)}(x) = \frac{(-1)^n}{2\sqrt{l(l+1)}} \left(\sqrt{(l+m)(l-m+1)} \bar{P}_l^{(m-1)}(x) - \sqrt{(l-m)(l+m+1)} \bar{P}_l^{(m+1)}(x) \right)$$

$$\bar{W}_l^{(m)}(x) = \frac{(-1)^n m}{\sqrt{1-x^2}\sqrt{l(l+1)}} \bar{P}_l^{(m)}(x)$$

$$\bar{W}_l^{(m)}(x) = \frac{(-1)^n}{2x\sqrt{l(l+1)}} \left[\sqrt{(l+m)(l-m+1)} \bar{P}_l^{(m-1)}(x) + \sqrt{(l-m)(l+m+1)} \bar{P}_l^{(m+1)}(x) \right]$$

1.2 Vector Spherical Harmonics transform step

This transform is implemented in the spherepack library in Fortran. The input is the measured complex pattern and the outpur are

4 complex coefficients arrays which are obtained through the following projection relations.

$$br_l^{(m)} = \alpha_l^{(m)} \int_0^{2\pi} \int_{-\pi/2}^{-\pi/2} (F_\theta(\theta,\phi) V_l^{(m)} \cos m\phi - F_\phi(\theta,\phi) W_l^{(m)} \sin m\phi) \cos\theta d\theta d\phi$$

$$bi_l^{(m)} = \alpha_l^{(m)} \int_0^{2\pi} \int_{-\pi/2}^{-\pi/2} (F_\phi(\theta,\phi) W_l^{(m)} \cos m\phi + F_\theta(\theta,\phi) V_l^{(m)} \sin m\phi) \cos\theta d\theta d\phi$$

$$cr_l^{(m)} = \alpha_l^{(m)} \int_0^{2\pi} \int_{-\pi/2}^{-\pi/2} (F_\phi(\theta,\phi) V_l^{(m)} \cos m\phi + F_\theta(\theta,\phi) W_l^{(m)} \sin m\phi) \cos\theta d\theta d\phi$$

$$ci_l^{(m)} = \alpha_l^{(m)} \int_0^{2\pi} \int_{-\pi/2}^{-\pi/2} (-F_\theta(\theta,\phi) W_l^{(m)} \cos m\phi + F_\phi(\theta,\phi) V_l^{(m)} \sin m\phi) \cos\theta d\theta d\phi$$

The normalized Legendre polynomial respectively of order n and $l-1$ are satisfying the following relations

The first step consists in evaluating the vector spherical coefficient for a set of frequencies over the antenna bandwidth.

The two components of the radiated far field are given by the following expansion which makes use of 4 coefficients.

The problem here addressed is the one of reconstructing as fast as possible the radiated field from this set of coefficients.

$$F_\phi(\theta,\phi) = \sum_{l=0}^{N} \sum_{m=0}^{n} W_l^{(m)} (bi_l^{(m)} \cos m\phi - br_l^{(m)} \sin m\phi) + V_l^{(m)} (cr_l^{(m)} \cos m\phi + ci_l^{(m)} \sin m\phi)$$

$$F_\theta(\theta,\phi) = \sum_{l=0}^{N} \sum_{m=0}^{n} V_l^{(m)} (br_l^{(m)} \cos m\phi + bi_l^{(m)} \sin m\phi) + W_l^{(m)} (-ci_l^{(m)} \cos m\phi + cr_l^{(m)} \sin m\phi)$$

In those expressions the expansion functions along the angular parameter θ $V_l^{(m)}$ and $W_l^{(m)}$

$$\sqrt{l(l+1)} V_l^{(m)}(\theta) = \frac{dP_l^{(m)}}{d\theta} = \frac{1}{2} P_l^{(m+1)} - (l+m)(l-m+1) P_l^{(m-1)}$$

$$\sqrt{l(l+1)} W_l^{(m)}(\theta) = \frac{m}{\cos\theta} P_l^{(m)} = \frac{1}{2} P_{l-1}^{(m+1)} + (l+m)(l+m-1) P_{l-1}^{(m-1)}$$

Why x and $\cos(\theta)$?

$$P_l^{(m)}(x) = \frac{1}{2^n l!} (-\cos\theta)^m \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^n$$

$$P_{l-1}^{(m)}(x) = \frac{1}{2^{l-1} l!} (-\cos\theta)^m \frac{d^{l+m-1}}{dx^{l+m-1}} (x^2-1)^{n-1}$$

1.3 Associated Legendre Function

The SciPy function *lpmn(m,n,z)* is an implementation of the Associated Legendre functions of the first kind, $P_l^{(m)}(z)$ and its derivative, $P_l^{'(m)}(z)$ of order m and degree n.

It returns two arrays of size $(m+1, l+1)$ containing $P_l^{(m)}(z)$ and $P_l^{'(m)}(z)$ for all orders from $0 \dots m$ and degrees from $0 \dots n$. The z argument can be complex.

```
from scipy.special import *
```

```
m = 3
n = 3
z = 0.5
lpmn(m,n,z)
```

```
(array([[ 1.        ,  0.5       , -0.125     , -0.4375    ],
       [ 0.        , -0.8660254 , -1.29903811, -0.32475953],
       [ 0.        ,  0.        ,  2.25      ,  5.625     ],
       [ 0.        ,  0.        ,  0.        , -9.74278579]]),
 array([[  0.        ,   1.        ,   1.5       ,   0.375     ],
       [  0.        ,   0.57735027,  -1.73205081,  -6.27868418],
       [  0.        ,   0.        ,  -3.        ,   3.75      ],
       [  0.        ,   0.        ,   0.        ,  19.48557159]]))
```

1.3.1 Definition

In this scipy implementation the Condol-Shortley phase $(-1)^m$ is already included, thus the expression of the implemented function is given by :

$$P_l^{(m)}(x) = \frac{(-1)^m}{2^n l!}(1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^n$$

1.3.2 Properties

For associated the two orthogonality relations respectively for associated Legendre function of different

1.4 Derivation of Recurrence Formulas on $\bar{P}_l^{(m)}(x)$

For the Legendre function of the first kind, we have the following recurrence property

$$2mxP_l^{(m)}(x) = -\sqrt{1-x^2}\left[P_l^{(m+1)}(x) + (l+m)(l-m+1)P_l^{(m-1)}(x)\right]$$

Introducing the relation which relates $P_l^{(m)}(x)$ to $\bar{P}_l^{(m)(x)}$

$$P_l^{(m)}(x) = \sqrt{\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}}\bar{P}_l^{(m)}(x)$$

$$\frac{2mx}{\sqrt{1-x^2}}\sqrt{\frac{(l+m)!}{(l-m)!}}\bar{P}_l^{(m)}(x) = -\left[\sqrt{\frac{(l+m+1)!}{(l-m-1)!}}\bar{P}_l^{(m+1)}(x) + (l+m)(l-m+1)\sqrt{\frac{(l+m-1)!}{(l-m+1)!}}\bar{P}_l^{(m-1)}(x)\right]$$

$$\frac{2mx}{\sqrt{1-x^2}}\bar{P}_l^{(m)}(x) = -\sqrt{\frac{(l-m)!}{(l+m)!}}\left[\sqrt{\frac{(l+m+1)!}{(l-m-1)!}}\bar{P}_l^{(m+1)}(x) + (l+m)(l-m+1)\sqrt{\frac{(l+m-1)!}{(l-m+1)!}}\bar{P}_l^{(m-1)}(x)\right]$$

$$\bar{P}_l^{(m)}(x) = -\frac{\sqrt{1-x^2}}{2mx}\left[\sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(x) + \sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(x)\right]$$

1.5 Derivation of Recurrence formulas on $\frac{d\bar{P}_l^{(m)}}{dx}(x)$

$$P_l^{(m)'}(x) = \frac{1}{x^2-1}\left[-(l+m)(l-m+1)\sqrt{1-x^2}P_l^{(m-1)}(x) - mxP_l^{(m)}(x)\right]$$

$$P_l^{(m)'}(x) = \frac{1}{\sqrt{1-x^2}}\left[(l+m)(l-m+1)P_l^{(m-1)}(x) + \frac{mx}{\sqrt{1-x^2}}P_l^{(m)}(x)\right]$$

$$\sqrt{\frac{(l+m)!}{(l-m)!}}\bar{P}_l^{(m)'}(x) = \frac{1}{\sqrt{1-x^2}}\left[(l+m)(l-m+1)\sqrt{\frac{(l+m-1)!}{(l-m+1)!}}\bar{P}_l^{(m-1)}(x) + \frac{mx}{\sqrt{1-x^2}}\sqrt{\frac{(l+m)!}{(l-m)!}}\bar{P}_l^{(m)}(x)\right]$$

$$\bar{P}_l^{(m)'}(x) = \frac{1}{\sqrt{1-x^2}}\sqrt{\frac{(l-m)!}{(l+m)!}}\left[(l+m)(l-m+1)\sqrt{\frac{(l+m-1)!}{(l-m+1)!}}\bar{P}_l^{(m-1)}(x) + \frac{mx}{\sqrt{1-x^2}}\sqrt{\frac{(l+m)!}{(l-m)!}}\bar{P}_l^{(m)}(x)\right]$$

$$\bar{P}_l^{(m)'}(x) = \frac{1}{\sqrt{1-x^2}}\left[\sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(x)P_l^{(m-1)}(x) + \frac{mx}{\sqrt{1-x^2}}P_l^{(m)}(x)\bar{P}_l^{(m)}(x)\right]$$

$$\bar{P}_l^{(m)'}(x) = \frac{1}{2\sqrt{1-x^2}}\left[\sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(x) - \sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(x)\right]$$

1.5.1 Evaluation of negative order

$$P_l^{(-m)}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^{(m)}(x)$$

$$\sqrt{\frac{(l-m)!}{(l+m)!}}\bar{P}_l^{(-m)}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}\sqrt{\frac{(l+m)!}{(l-m)!}}\bar{P}_l^{(m)}(x)$$

$$\bar{P}_l^{(-m)}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}\sqrt{\frac{(l+m)!}{(l-m)!}\frac{(l+m)!}{(l-m)!}}\bar{P}_l^{(m)}(x)$$

We get the nice and simple relation

$$\bar{P}_l^{(-m)}(x) = (-1)^m\bar{P}_l^{(m)}(x)$$

1.6 Practical Implementation

$$\Re\{F\_\theta(\theta,\phi)\} = \Re\{\frac{1}{2}\sum\_l = 0^L(br_l^{(0)}\bar{V}_l^{(0)} - jcr_l^{(0)}\bar{W}_l^{(0)}) + \sum\_m = 1^M\sum\_l = m^N(br_l^{(m)}\bar{V}_l^m - jcr_l^{(m)}\bar{W}_l^m)e^{jm\phi}\}$$

$$\Im\{F\_\theta(\theta,\phi)\} = \Re\{\frac{1}{2}\sum\_l = 0^L(bi_{0,n}\bar{V}_l^0 - jci_l^{(0)}\bar{W}_l^{(0)}) + \sum\_m = 1^M\sum\_l = m^N(bi_l^{(m)}\bar{V}_l^m - jci_l^{(m)}\bar{W}_l^m)e^{jm\phi}\}$$

$$\Re\{F\_\phi(\theta,\phi)\} = \Re\{\frac{1}{2}\sum\_l = 0^N(jbr_l^{(0)}\bar{W}_l^{(0)} + cr_l^{(0)}\bar{V}_l^{(0)}) + \sum\_m = 1^M\sum\_l = 0^N(jbr_l^{(m)}\bar{W}_l^m + cr_l^{(m)}\bar{V}_l^m)e^{jm\phi})\}$$

$$\Im\{F\_\phi(\theta,\phi)\} = \Re\{\frac{1}{2}\sum\_l = 0^N(jbi_l^{(0)}\bar{W}_l^{(0)} + ci_l^{(0)}\bar{V}_l^{(0)}) + \sum\_m = 1^M\sum\_l = 0^N(jbi_l^{(m)}\bar{W}_l^m + ci_l^{(m)}\bar{V}_l^m)e^{jm\phi}\}$$

1.7 The Vector Spherical Harmonics Functions $\bar{V}_l^{(m)}$

$$\bar{V}_l^{(m)}(x) = \frac{(-1)^n}{2\sqrt{l(l+1)}} \left(\sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(x) - \sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(x) \right)$$

1.8 Calculation of $\bar{W}_l^{(m)}$

$$\bar{W}_l^{(m)}(x) = \frac{(-1)^n m}{\sqrt{1-x^2}\sqrt{l(l+1)}}\bar{P}_l^{(m)}(x)$$

$$\bar{W}_l^{(m)}(x) = (-1)^n \frac{m}{\sqrt{1-x^2}\sqrt{l(l+1)}} \frac{\sqrt{1-x^2}}{2mx} \left[\sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(x) + \sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(x) \right]$$

$$\bar{W}_l^{(m)}(x) = \frac{(-1)^n}{2x\sqrt{l(l+1)}} \left[\sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(x) + \sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(x) \right]$$

$$\bar{W}_l^{(m)}(\theta) = \frac{(-1)^n}{2\cos\theta\sqrt{l(l+1)}} \left[\sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(\cos\theta) + \sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(\cos\theta) \right]$$

There is a singularity of $\bar{W}_l^{(m)}(\theta)$ in $\theta = \pi/2$

$$\bar{W}_l^{(m)}(\theta) = \frac{(-1)^n}{2\cos\theta\sqrt{l(l+1)}} \left[\sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(\cos\theta) + \sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(\cos\theta) \right]$$

1.9 Norm of $V_l^{(m)}(x)$

Those basis functions are not unitary

$$\int_{-1}^{1} \bar{V}_l^{2(m)}(x)dx = \frac{1}{4l(l+1)} \int_{-1}^{1} \left(\sqrt{(l+m)(l-m+1)}\bar{P}_l^{(m-1)}(x) - \sqrt{(l-m)(l+m+1)}\bar{P}_l^{(m+1)}(x) \right)^2 dx$$

$$\int_{-1}^{1} \bar{V}_l^{2(m)}(x)dx = \frac{1}{4l(l+1)} \int_{-1}^{1} ((l{+}m)(l{-}m{+}1)\bar{P}_l^{2(m-1)}(x){+}(l{-}m)(l{+}m{+}1)\bar{P}_l^{2(m+1)}(x){-}2\sqrt{(l+m)(l-m+1)(l}$$

$$\int_{-1}^{1} \bar{V}\_l^{2(m)}(x)dx = \frac{1}{4l(l+1)} \left((l+m)(l-m+1) + (l-m)(l+m+1) - 2\sqrt{(l+m)(l-m+1)(l-m)(l+m+1}\right.$$

$$\int_{-1}^{1} \bar{V}\_l^{2(m)}(x)dx = \frac{1}{4l(l+1)} \left((l^2-m^2+l+m) + (l^2-m^2+l-m) - 2\sqrt{(l^2-m^2+l+m)(l^2-m^2+l-m)} \int \right.$$

$$\int_{-1}^{1} \bar{V}_l^{2(m)}(x)dx = \frac{l^2-m^2+l}{2l(l+1)} \left(1 - \sqrt{1 - \frac{m^2}{(l^2-m^2+l)^2}} \int_{-1}^{1} \bar{P}\_l^{(m-1)}(x)\bar{P}\_l^{(m+1)}(x)dx \right)$$

1.10 Norm of $\bar{W}_l^{(m)}(x)$

$$\int_{-1}^{1} \bar{W}_l^{2(m)}(x)dx = \frac{l^2-m^2+l}{l(l+1)} \left(1 - \sqrt{1 - \frac{m^2}{(l^2-m^2+l)^2}} \int_{-1}^{1} \bar{P}_l^{(m-1)}(x)\bar{P}_l^{(m+1)}(x)dx \right)$$