

Computational Thinking

# Discrete Mathematics

## Topic 04 — Relations and Functions

Number Theory

Logic

### Lecture 01 — Relation Concepts and Definitions

Dr Kieran Murphy 

Department of Computing and Mathematics,  
SETU (Waterford).  
(kieran.murphy@setu.ie)

Graphs and  
Networks

Collections

Autumn Semester, 2023

#### Outline

- Defining a relation via Cartesian product
- Relation Terminology

Enumeration

Relations & Functions

# Outline

---

1. Relation Definition	2
1.1. Cartesian product and Relations	3
1.2. Graphical Representation of Relations using Venn Diagrams	9

# Cartesian product

Recall that the Cartesian product of two sets,  $A$  and  $B$ , is the set of all ordered pairs of all elements where the first element is from set  $A$  and the second element is from  $B$ .

## Definition 1 (Cartesian product)

The **Cartesian product** of two sets  $A$  and  $B$ , denoted by  $A \times B$  is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

- The order within the pair matters, so  $(a, b) \neq (b, a)$ .
- But, since  $A \times B$  is a set, the order between the pairs is not important.

$$\{(a, b), (c, d)\} = \{(c, d), (a, b)\}$$

- The set  $A \times B$  has  $|A||B|$  elements.

# Cartesian product

Recall that the Cartesian product of two sets,  $A$  and  $B$ , is the set of all ordered pairs of all elements where the first element is from set  $A$  and the second element is from  $B$ .

## Definition 1 (Cartesian product)

The **Cartesian product** of two sets  $A$  and  $B$ , denoted by  $A \times B$  is

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

- The order within the pair matters, so  $(a, b) \neq (b, a)$ .
- But, since  $A \times B$  is a set, the order between the pairs is not important.

$$\{(a, b), (c, d)\} = \{(c, d), (a, b)\}$$

- The set  $A \times B$  has  $|A||B|$  elements.

# Example

## Example 2

The Cartesian product of  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

Or in Python\*...

cartesian\_product .py

```
1 A = {0,1,2,3}
2 B = {0,1,4}
3
4 C = {(a,b) for a in A for b in B}
5
6 print (C)
```

```
1 {(0, 1), (0, 0), (3, 0), (3, 1), (1, 4), (2, 1), (2, 0), (2, 4), (0, 4),
```

---

\*From now on I will break with the upper/lower case programming naming convention that you are probably using in Processing/Java and follow conventions in Mathematics.

# Example

## Example 2

The Cartesian product of  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$

Or in Python\*...

cartesian\_product .py

```
1 A = {0,1,2,3}
2 B = {0,1,4}
3
4 C = {(a,b) for a in A for b in B}
5
6 print (C)
```

```
1 {(0, 1), (0, 0), (3, 0), (3, 1), (1, 4), (2, 1), (2, 0), (2, 4), (0, 4),
```

---

\*From now on I will break with the upper/lower case programming naming convention that you are probably using in Processing/Java and follow conventions in Mathematics.

# Example

## Example 2

The Cartesian product of  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$

Or in Python\*...

cartesian\_product .py

```
1 A = {0,1,2,3}
2 B = {0,1,4}
3
4 C = {(a,b) for a in A for b in B}
5
6 print (C)
```

```
1 {(0, 1), (0, 0), (3, 0), (3, 1), (1, 4), (2, 1), (2, 0), (2, 4), (0, 4),
```

---

\*From now on I will break with the upper/lower case programming naming convention that you are probably using in Processing/Java and follow conventions in Mathematics.

# Relation

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$

So possible relations between  $A$  and  $B$  include

- $R = \{(0, 0), (1, 1), (2, 4)\}$  (relation is based on  $x \mapsto x^2, x < 3$ )
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$  (relation is based on  $x \mapsto x \bmod 2$ )
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$  (relation is based on  $x \mapsto \text{isPrime}(x)$ )
- $R = \{(0, 0), (1, 4)\}$  (relation is ... unknown?)
- $R = \{\}$  (remember, empty set is a ...)



# Relation

## I

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$$

So possible relations between  $A$  and  $B$  include

- $R = \{(0, 0), (1, 1), (2, 4)\}$  (relation is based on  $x \mapsto x^2, x < 3$ )
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$  (relation is based on  $x \mapsto x \bmod 2$ )
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$  (relation is based on  $x \mapsto \text{isPrime}(x)$ )
- $R = \{(0, 0), (1, 4)\}$  (relation is ... unknown?)
- $R = \{\}$  (remember, empty set is a ...)

# Relation

## I

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$$

So possible relations between  $A$  and  $B$  include

- $R = \{(0, 0), (1, 1), (2, 4)\}$  (relation is based on  $x \mapsto x^2, x < 3$ )
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$  (relation is based on  $x \mapsto x \bmod 2$ )
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$  (relation is based on  $x \mapsto \text{isPrime}(x)$ )
- $R = \{(0, 0), (1, 4)\}$  (relation is ... unknown?)
- $R = \{\}$  (remember, empty set is a ...)

# Relation

## I

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$$

So possible relations between  $A$  and  $B$  include

- $R = \{(0, 0), (1, 1), (2, 4)\}$  (relation is based on  $x \mapsto x^2, x < 3$ )
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$  (relation is based on  $x \mapsto x \bmod 2$ )
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$  (relation is based on  $x \mapsto \text{isPrime}(x)$ )
- $R = \{(0, 0), (1, 4)\}$  (relation is ... unknown?)
- $R = \{\}$  (remember, empty set is a ...)

# Relation

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$$

So possible relations between  $A$  and  $B$  include

- $R = \{(0, 0), (1, 1), (2, 4)\}$  (relation is based on  $x \mapsto x^2, x < 3$ )
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$  (relation is based on  $x \mapsto x \bmod 2$ )
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$  (relation is based on  $x \mapsto \text{isPrime}(x)$ )
- $R = \{(0, 0), (1, 4)\}$  (relation is ... unknown?)
- $R = \{\}$  (remember, empty set is a ...)

# Relation

## I

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$$

So possible relations between  $A$  and  $B$  include

- $R = \{(0, 0), (1, 1), (2, 4)\}$  (relation is based on  $x \mapsto x^2, x < 3$ )
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$  (relation is based on  $x \mapsto x \bmod 2$ )
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$  (relation is based on  $x \mapsto \text{isPrime}(x)$ )
- $R = \{(0, 0), (1, 4)\}$  (relation is ... unknown?)
- $R = \{\}$  (remember, empty set is a ...)

# Relation

## I

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$$

So possible relations between  $A$  and  $B$  include

- $R = \{(0, 0), (1, 1), (2, 4)\}$  (relation is based on  $x \mapsto x^2, x < 3$ )
- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$  (relation is based on  $x \mapsto x \bmod 2$ )
- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$  (relation is based on  $x \mapsto \text{isPrime}(x)$ )
- $R = \{(0, 0), (1, 4)\}$  (relation is ... unknown?)
- $R = \{\}$  (remember, empty set is a ...)

# Relation

## I

The formal definition of a relation is based on the Cartesian product between two sets, later we will see more initiative but less general definitions.

## Definition 3 (Relation)

Given two sets  $A$  and  $B$ . **Any** subset of the Cartesian product between  $A$  and  $B$  is called a **relation**.

## Example 4

The Cartesian product of the sets  $A = \{0, 1, 2, 3\}$  and  $B = \{0, 1, 4\}$  is

$\{(0, 0), (0, 1), (0, 4), (1, 0), (1, 1), (1, 4), (2, 1), (2, 1), (2, 4), (3, 0), (3, 1), (3, 4)\}$

So possible relations between  $A$  and  $B$  include

4096 possible relations!

- $R = \{(0, 0), (1, 1), (2, 4)\}$

(relation is based on  $x \mapsto x^2, x < 3$ )

- $R = \{(0, 0), (1, 1), (2, 0), (3, 1)\}$

(relation is based on  $x \mapsto x \bmod 2$ )

- $R = \{(0, 0), (1, 0), (2, 1), (3, 1)\}$

(relation is based on  $x \mapsto \text{isPrime}(x)$ )

- $R = \{(0, 0), (1, 4)\}$

(relation is ... unknown?)

- $R = \{\}$

(remember, empty set is a ...)

# Relation

## II

You should spend some time thinking about the consequences of the definition that we have just covered ...

- *Given two sets,  $A$  and  $B$ , how many distinct relations can we construct?*
- *Relation vs. Cartesian product vs. power set of the Cartesian product*



# Relation

## II

You should spend some time thinking about the consequences of the definition that we have just covered ...

- *Given two sets,  $A$  and  $B$ , how many distinct relations can we construct?*
  - Set  $A$  and  $B$  have  $|A|$  and  $|B|$  elements respectively.
  - The Cartesian product,  $A \times B$ , has  $|A||B|$  elements.
  - Relation between  $A$  and  $B$  is **any** subset of  $A \times B$ .
  - Sets of size  $n$  have  $2^n$  subsets.
- *Relation vs. Cartesian product vs. power set of the Cartesian product*

# Relation

## II

You should spend some time thinking about the consequences of the definition that we have just covered ...

- *Given two sets,  $A$  and  $B$ , how many distinct relations can we construct?*
  - Set  $A$  and  $B$  have  $|A|$  and  $|B|$  elements respectively.
  - The Cartesian product,  $A \times B$ , has  $|A||B|$  elements.
  - Relation between  $A$  and  $B$  is **any** subset of  $A \times B$ .
  - Sets of size  $n$  have  $2^n$  subsets.

Number of distinct relations between sets  $A$  and  $B$  is  $2^{|A||B|}$

- *Relation vs. Cartesian product vs. power set of the Cartesian product*

# Relation

## II

You should spend some time thinking about the consequences of the definition that we have just covered ...

- *Given two sets,  $A$  and  $B$ , how many distinct relations can we construct?*
  - Set  $A$  and  $B$  have  $|A|$  and  $|B|$  elements respectively.
  - The Cartesian product,  $A \times B$ , has  $|A||B|$  elements.
  - Relation between  $A$  and  $B$  is **any** subset of  $A \times B$ .
  - Sets of size  $n$  have  $2^n$  subsets.

Number of distinct relations between sets  $A$  and  $B$  is  $2^{|A||B|}$

- *Relation vs. Cartesian product vs. power set of the Cartesian product*

Let  $R$  be a relation between sets  $A$  and  $B$ . Then

- $R$  is a subset of the Cartesian product of  $A$  and  $B$ .

$$R \subseteq A \times B$$

# Relation

## II

You should spend some time thinking about the consequences of the definition that we have just covered ...

- Given two sets,  $A$  and  $B$ , how many distinct relations can we construct?
  - Set  $A$  and  $B$  have  $|A|$  and  $|B|$  elements respectively.
  - The Cartesian product,  $A \times B$ , has  $|A||B|$  elements.
  - Relation between  $A$  and  $B$  is **any** subset of  $A \times B$ .
  - Sets of size  $n$  have  $2^n$  subsets.

Number of distinct relations between sets  $A$  and  $B$  is  $2^{|A||B|}$

- *Relation vs. Cartesian product vs. power set of the Cartesian product*

Let  $R$  be a relation between sets  $A$  and  $B$ . Then

- $R$  is a subset of the Cartesian product of  $A$  and  $B$ .

$$R \subseteq A \times B$$

- $R$  is an element of the power set of the Cartesian product of  $A$  and  $B$ .

$$R \in \mathcal{P}(A \times B)$$

## Example

### Example 5

Let  $A = \{2, 3, 5, 6\}$  and define a relation  $R$  from  $A$  to  $A$  by  $(a, b) \in R$  if and only if  $a$  divides evenly into  $b$ .

The relation  $R$  is defined by

$$R = \{(a, b) \mid a \in A, b \in A, a \text{ divided evenly into } b\}$$

The set of pairs that qualify for membership of  $R$  is

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

### Definition 6 (Relation on a Set)

A relation from set  $A$  to  $A$  is called a **relation on  $A$** .

## Notation Warning — Divisibility

When explaining relations we will often use (as in the previous example) the idea of “divides”. Lets make sure we all agree on what this means ...

### Definition 7 (Divides)

Let  $a, b \in \mathbb{Z}$ . We say that  $a$  **divides**  $b$ , denoted  $a \mid b$ , if and only if there exists an integer  $k$  such that  $ak = b$ .

- Be careful in writing about the relation “divides.” The vertical line symbol use for this relation, if written carelessly, can look like division. While  $a \mid b$  is either **True** or **False**,  $a/b$  is a number<sup>†</sup>.
- Even worse. We, mathematicians, use the same symbol “|” for “such that” in set builder notation and for “divides”.
  - Usually this is not a problem as the intended meaning for “|” will be clear from the context.
  - Use alternative symbols: “|” is replaced by “:” in set builder notation.

---

<sup>†</sup>Also the direction is different. “ $a \mid b$ ” means “ $a$  divides (evenly) into  $b$ ”, while “ $a/b$ ” means “the value of  $a$  divided by  $b$ ”.

## Notation Warning — Divisibility

When explaining relations we will often use (as in the previous example) the idea of “divides”. Lets make sure we all agree on what this means ...

### Definition 7 (Divides)

Let  $a, b \in \mathbb{Z}$ . We say that  $a$  **divides**  $b$ , denoted  $a \mid b$ , if and only if there exists an integer  $k$  such that  $ak = b$ .

- Be careful in writing about the relation “divides.” The vertical line symbol use for this relation, if written carelessly, can look like division. While  $a \mid b$  is either **True** or **False**,  $a/b$  is a number<sup>†</sup>.
- Even worse. We, mathematicians, use the same symbol “|” for “such that” in set builder notation and for “divides”.
  - Usually this is not a problem as the intended meaning for “|” will be clear from the context.
  - Use alternative symbols: “|” is replaced by “:” in set builder notation.

---

<sup>†</sup> Also the direction is different. “ $a \mid b$ ” means “ $a$  divides (evenly) into  $b$ ”, while “ $a/b$ ” means “the value of  $a$  divided by  $b$ ”.

## Notation Warning — Divisibility

When explaining relations we will often use (as in the previous example) the idea of “divides”. Lets make sure we all agree on what this means ...

### Definition 7 (Divides)

Let  $a, b \in \mathbb{Z}$ . We say that  $a$  **divides**  $b$ , denoted  $a \mid b$ , if and only if there exists an integer  $k$  such that  $ak = b$ .

- Be careful in writing about the relation “divides.” The vertical line symbol use for this relation, if written carelessly, can look like division. While  $a \mid b$  is either **True** or **False**,  $a/b$  is a number<sup>†</sup>.
- Even worse. We, mathematicians, use the same symbol “ $\mid$ ” for “such that” in set builder notation and for “divides”.
  - Usually this is not a problem as the intended meaning for “ $\mid$ ” will be clear from the context.
  - Use alternative symbols: “ $\mid$ ” is replaced by “ $:$ ” in set builder notation.

---

<sup>†</sup> Also the direction is different. “ $a \mid b$ ” means “ $a$  divides (evenly) into  $b$ ”, while “ $a/b$ ” means “the value of  $a$  divided by  $b$ ”.



# Graphical Representation of Relations I — Venn Diagrams

Representing relations graphically can help in identifying its properties ...

Consider the relation  $R$  from  $A$  into  $A$ , where  $A = \{2, 3, 5, 6\}$  and  $(a, b) \in R$  if and only if  $a$  divides evenly into  $b$ .

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

- Draw set  $A$ .
- This relation is from  $A$  to  $A$ , so we make a copy of set  $A$  and called it  $B$ .
- Indicate each of the ordered pairs in  $R$  using an arrow.

# Graphical Representation of Relations I — Venn Diagrams

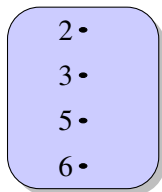
Representing relations graphically can help in identifying its properties ...

Consider the relation  $R$  from  $A$  into  $A$ , where  $A = \{2, 3, 5, 6\}$  and  $(a, b) \in R$  if and only if  $a$  divides evenly into  $b$ .

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

- Draw set  $A$ .
- This relation is from  $A$  to  $A$ , so we make a copy of set  $A$  and called it  $B$ .
- Indicate each of the ordered pairs in  $R$  using an arrow.

$A$



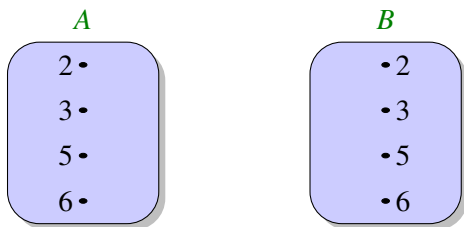
# Graphical Representation of Relations I — Venn Diagrams

Representing relations graphically can help in identifying its properties ...

Consider the relation  $R$  from  $A$  into  $A$ , where  $A = \{2, 3, 5, 6\}$  and  $(a, b) \in R$  if and only if  $a$  divides evenly into  $b$ .

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

- Draw set  $A$ .
- This relation is from  $A$  to  $A$ , so we make a copy of set  $A$  and called it  $B$ .
- Indicate each of the ordered pairs in  $R$  using an arrow.



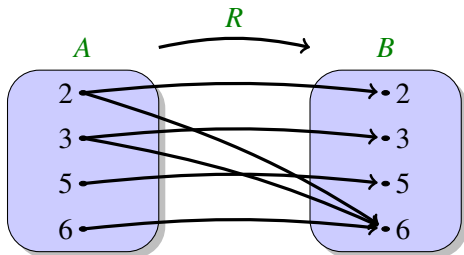
# Graphical Representation of Relations I — Venn Diagrams

Representing relations graphically can help in identifying its properties ...

Consider the relation  $R$  from  $A$  into  $A$ , where  $A = \{2, 3, 5, 6\}$  and  $(a, b) \in R$  if and only if  $a$  divides evenly into  $b$ .

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

- Draw set  $A$ .
- This relation is from  $A$  to  $A$ , so we make a copy of set  $A$  and called it  $B$ .
- Indicate each of the ordered pairs in  $R$  using an arrow.



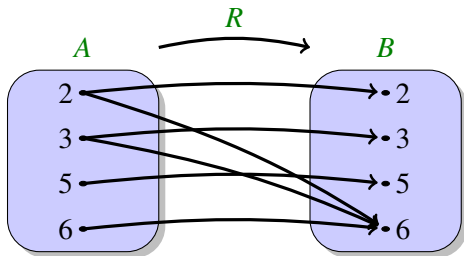
# Graphical Representation of Relations I — Venn Diagrams

Representing relations graphically can help in identifying its properties ...

Consider the relation  $R$  from  $A$  into  $A$ , where  $A = \{2, 3, 5, 6\}$  and  $(a, b) \in R$  if and only if  $a$  divides evenly into  $b$ .

$$R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$$

- Draw set  $A$ .
- This relation is from  $A$  to  $A$ , so we make a copy of set  $A$  and called it  $B$ .
- Indicate each of the ordered pairs in  $R$  using an arrow.



Things we are interested in seeing ...

- Is there an arrow from every element in the first set?
- Is there an arrow to every element in the second set?
- Are there multiple arrows from some elements?
- Are there multiple arrows into some elements?

# Relation Terminology

## I

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

# Relation Terminology

## I

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\}$$

# Relation Terminology

## I

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

In python  $R = \{ (a,b) \text{ for } a \text{ in } A \text{ for } b \text{ in } B \text{ if } a < b \}$

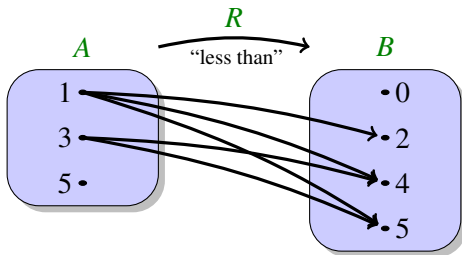


# Relation Terminology

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

In python  $R = \{ (a,b) \text{ for } a \text{ in } A \text{ for } b \text{ in } B \text{ if } a < b \}$

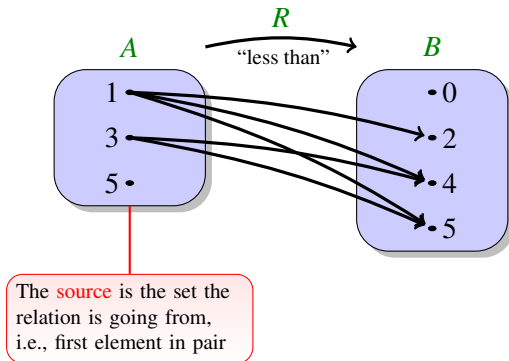


# Relation Terminology

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

In python  $R = \{ (a,b) \text{ for } a \text{ in } A \text{ for } b \text{ in } B \text{ if } a < b \}$

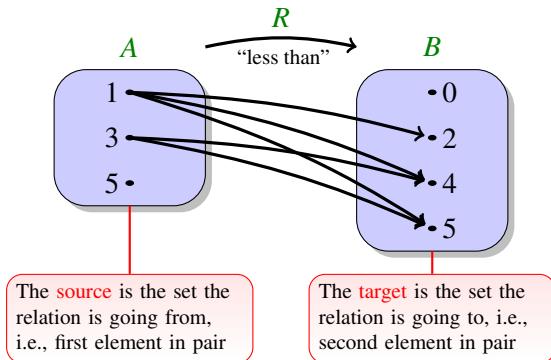


# Relation Terminology

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

In python  $R = \{ (a,b) \text{ for } a \text{ in } A \text{ for } b \text{ in } B \text{ if } a < b \}$

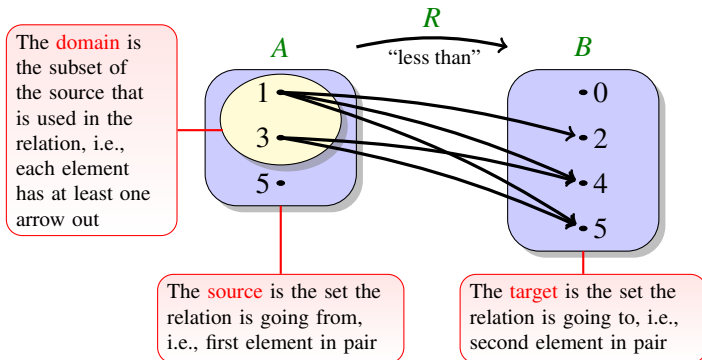


# Relation Terminology

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

In python  $R = \{ (a,b) \text{ for } a \text{ in } A \text{ for } b \text{ in } B \text{ if } a < b \}$

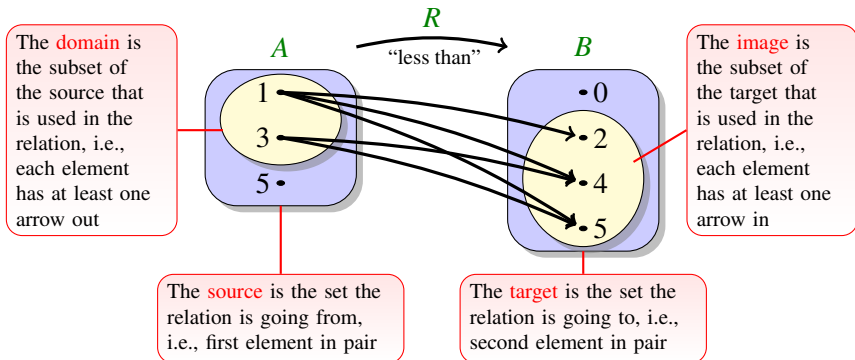


# Relation Terminology

Consider the relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{0, 2, 4, 5\}$ . We have

$$R = \{(a, b) \mid a \in A, b \in B, a < b\} = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

In python  $R = \{ (a,b) \text{ for } a \text{ in } A \text{ for } b \text{ in } B \text{ if } a < b \}$



# Relation Terminology

## II

Given relation  $R$  from set  $S$  to set  $T$  we have:

- The **source**,  $S$ , is the set that the relation is going from.
- The **target**,  $T$ , is the set that the relation is going to.
- The **domain** of  $R$ , denoted by  $\text{Dom}(R)$ , is the subset of the source for which there is at least one arrow leaving each element.

$$\text{Dom}(R) = \{s \mid s \in S, \underbrace{\exists t \in T((s, t) \in R)}_{\text{exists at least one arrow leaving each element}}\} \subseteq S$$

- The **image** of  $R$ , denoted by  $\text{Im}(R)$ , is the subset of the target for which there is at least one arrow entering each element.

$$\text{Im}(R) = \{t \mid t \in T, \underbrace{\exists s \in S((s, t) \in R)}_{\text{exists at least one arrow entering each element}}\} \subseteq T$$

# Relation Terminology — Into vs. Onto

## III

From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\text{Im}(R) \subseteq T$$

This gives us two possibilities ...

|  
or  
|

# Relation Terminology — Into vs. Onto

## III

From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\text{Im}(R) \subseteq T$$

This gives us two possibilities ...

$$\text{Im}(R) \subset T$$



$$\text{Im}(R) = T$$

or





# Relation Terminology — Into vs. Onto

## III

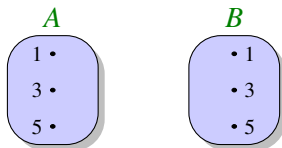
From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\text{Im}(R) \subseteq T$$

This gives us two possibilities ...

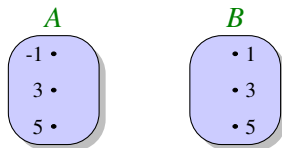
$$\text{Im}(R) \subset T$$

Example, consider relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{1, 3, 5\}$  is



$$\text{Im}(R) = T$$

Example, consider relation “is less than” from set  $A = \{-1, 3, 5\}$  to set  $B = \{1, 3, 5\}$  is



or

# Relation Terminology — Into vs. Onto

## III

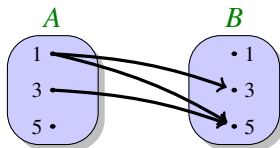
From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\text{Im}(R) \subseteq T$$

This gives us two possibilities ...

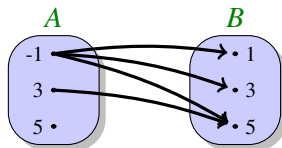
$$\text{Im}(R) \subset T$$

Example, consider relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{1, 3, 5\}$  is



$$\text{Im}(R) = T$$

Example, consider relation “is less than” from set  $A = \{-1, 3, 5\}$  to set  $B = \{1, 3, 5\}$  is



or

# Relation Terminology — Into vs. Onto

## III

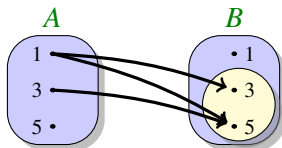
From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\text{Im}(R) \subseteq T$$

This gives us two possibilities ...

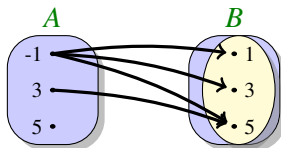
$$\text{Im}(R) \subset T$$

Example, consider relation “is less than” from set  $A = \{1, 3, 5\}$  to set  $B = \{1, 3, 5\}$  is



$$\text{Im}(R) = T$$

Example, consider relation “is less than” from set  $A = \{-1, 3, 5\}$  to set  $B = \{1, 3, 5\}$  is



or

# Relation Terminology — Into vs. Onto

## III

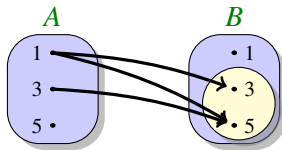
From our definitions, we have that the image of a relation is a subset of its target, i.e.,

$$\text{Im}(R) \subseteq T$$

This gives us two possibilities ...

$$\text{Im}(R) \subset T$$

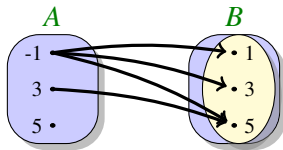
Example, consider relation “is less than” from set  $A = \{1, 3, 5\}$  ~~to~~ **into** set  $B = \{1, 3, 5\}$  is



A relation,  $R$ , in which the image is a proper subset of the target is said to be an **into** relation.

$$\text{Im}(R) = T$$

Example, consider relation “is less than” from set  $A = \{-1, 3, 5\}$  ~~to~~ **onto** set  $B = \{1, 3, 5\}$  is



A relation,  $R$ , in which the image is equal to the target is said to be an **onto** relation.

or

# Relation Terminology — Injective (one-to-one)

## III

### Definition 8 (Injective)

A relation is said to be **injective** (or **one-to-one**) if there is at most one arrow into every element in the target set.

|  
or  
|

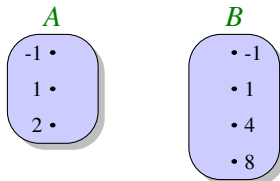
# Relation Terminology — Injective (one-to-one)

## III

### Definition 8 (Injective)

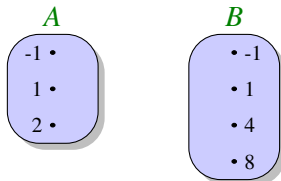
A relation is said to be **injective** (or **one-to-one**) if there is at most one arrow into every element in the target set.

Consider the relation “is square root of” from set  $A = \{-1, 1, 2\}$  to set  $B = \{-1, 1, 4, 8\}$ .



or

Consider the relation “is cube root of” from set  $A = \{-1, 1, 2\}$  to set  $B = \{-1, 1, 4, 8\}$ .



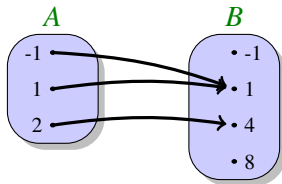
# Relation Terminology — Injective (one-to-one)

## III

### Definition 8 (Injective)

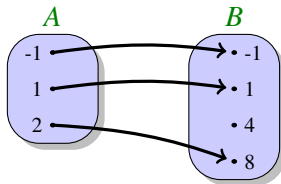
A relation is said to be **injective** (or **one-to-one**) if there is at most one arrow into every element in the target set.

Consider the relation “is square root of” from set  $A = \{-1, 1, 2\}$  to set  $B = \{-1, 1, 4, 8\}$ .



or

Consider the relation “is cube root of” from set  $A = \{-1, 1, 2\}$  to set  $B = \{-1, 1, 4, 8\}$ .



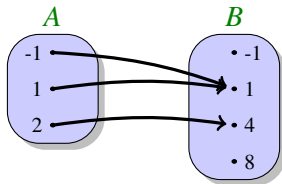
# Relation Terminology — Injective (one-to-one)

## III

### Definition 8 (Injective)

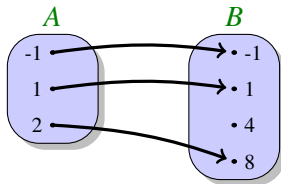
A relation is said to be **injective** (or **one-to-one**) if there is at most one arrow into every element in the target set.

Consider the relation “is square root of” from set  $A = \{-1, 1, 2\}$  to set  $B = \{-1, 1, 4, 8\}$ .



Not injective, since there exists at least one element in the target, ( $1$ ), which has more than one incoming arrows.

Consider the relation “is cube root of” from set  $A = \{-1, 1, 2\}$  to set  $B = \{-1, 1, 4, 8\}$ .



Is injective, since there is at most one arrow into each element in the target.



# Review Exercises 1 (Relation Definition)

## Question 1:

Consider the sets  $A = \{0, 1, \dots, 6\}$  and  $B = \{0, 1, \dots, 12\}$ . Draw each of the following relations, and specify the domain and image of  $R$  from  $A$  to  $B$  and whether it is into or onto, and injective or not.

- a)  $(a, b) \in R$  iff  $a \mid b$
- b)  $(a, b) \in R$  iff  $a > b$
- c)  $(a, b) \in R$  iff number of primes less than  $a$  is equal to number of primes less than  $b$
- d)  $(a, b) \in R$  iff number of factors of  $a$  is equal to number of factors of  $b$ .
- e)  $(a, b) \in R$  iff number of letters in writing  $a$  in English is equal number of letters in writing  $b$  in English.

## Question 2:

Let  $R$  be the relation from  $\mathbb{N}$  to  $\mathbb{N}$  where  $(a, b) \in R$  iff  $b = a + 2$ . Is  $R$  onto?

## Question 3:

Let  $R$  be the relation from  $\mathbb{Z}$  to  $\mathbb{Z}$  where  $(a, b) \in R$  iff  $b = a + 2$ . Is  $R$  onto?

## Question 4:

Let  $R$  be the relation from  $\mathbb{N}$  to  $\mathbb{N}$  where  $(a, b) \in R$  iff  $b = a^2$ . Is  $R$  one-to-one?

## Question 5:

Let  $R$  be the relation from  $\mathbb{Z}$  to  $\mathbb{Z}$  where  $(a, b) \in R$  iff  $b = a^2$ . Is  $R$  one-to-one?