

ADJOINT CONSISTENCY ANALYSIS OF DISCONTINUOUS GALERKIN DISCRETIZATIONS*

RALF HARTMANN†

Abstract. This paper is concerned with the adjoint consistency of discontinuous Galerkin (DG) discretizations. Adjoint consistency—in addition to consistency—is the key requirement for DG discretizations to be of optimal order in L^2 as well as measured in terms of target functionals. We provide a general framework for analyzing the adjoint consistency of DG discretizations which is also useful for the derivation of adjoint consistent methods. This analysis will be performed for the DG discretizations of the linear advection equation, the interior penalty DG method for elliptic problems, and the DG discretization of the compressible Euler equations. This framework is then used to derive an adjoint consistent DG discretization of the compressible Navier–Stokes equations. Numerical experiments demonstrate the link of adjoint consistency to the accuracy of numerical flow solutions and the smoothness of discrete adjoint solutions.

Key words. discontinuous Galerkin discretization, adjoint consistency, compressible Navier–Stokes equations, continuous adjoint problem, discrete adjoint problem

AMS subject classifications. 65N12, 65N15, 65N30

DOI. 10.1137/060665117

1. Introduction. The past few years have seen considerable progress in the development and analysis of discontinuous Galerkin (DG) methods. In addition to consistency in numerical analysis the so-called adjoint consistency property of discretizations has experienced increasing interest [1, 11, 12, 24]. Adjoint consistency is the key property of discretizations that ensures optimal order of convergence of the error measured in L^2 as well as in terms of specific target functionals $J(\cdot)$. A typical situation is given by the interior penalty discontinuous Galerkin methods. While both the symmetric version (SIPG) and the nonsymmetric version (NIPG) are of optimal order $\mathcal{O}(h^p)$ measured in H^1 , only the symmetric version is adjoint consistent which allows employment of a duality argument resulting in an optimal $\mathcal{O}(h^{p+1})$ order of convergence in L^2 . In contrast to that, the nonsymmetric interior penalty method is suboptimal, $\mathcal{O}(h^p)$ in L^2 . Similarly, adjoint consistency in conjunction with a duality argument leads to the so-called order doubling in the error measured in target functionals $J(\cdot)$. Whereas the adjoint consistency of the SIPG method results in $\mathcal{O}(h^{2p})$ of the error in $J(\cdot)$, the nonsymmetric version (NIPG) lacks adjoint consistency and target functionals behave like $\mathcal{O}(h^p)$. Adjoint consistency is closely linked to the smoothness of the discrete adjoint solutions. For adjoint consistent discretizations the discrete adjoint problem represents a consistent discretization of the continuous adjoint problem. Consequently, discrete adjoint solutions inherit the smoothness properties of the continuous adjoint solutions. Conversely, it has been seen that adjoint inconsistent discretizations exhibit some nonsmoothness. In particular, in [12] it has been shown that the discrete adjoint solutions arising from the SIPG method are essentially continuous. In contrast to that, the adjoint solutions arising from the

*Received by the editors July 17, 2006; accepted for publication (in revised form) July 20, 2007; published electronically December 7, 2007. This work was supported by the President's Initiative and Networking Fund of the Helmholtz Association of German Research Centres.

<http://www.siam.org/journals/sinum/45-6/66511.html>

†Institute of Aerodynamics and Flow Technology, DLR (German Aerospace Center), Lilienthalplatz 7, 38108 Braunschweig, Germany, and Institute of Scientific Computing, TU Braunschweig, 38092 Braunschweig, Germany (Ralf.Hartmann@dlr.de).

NIPG method are discontinuous between element interfaces where the jumps in the adjoint solutions persist even as the mesh is refined. The lack of regularity of the adjoint solution leads to the suboptimal rate of convergence of the NIPG method.

Adjoint consistency has been considered (cf. [1]) for linear elliptic problems with homogeneous boundary conditions which results in a characterization of element and interior face terms while ignoring the discretization of boundary terms. However, adjoint consistency is equally important for the discretization of boundary terms and for the discretization of target functionals $J(\cdot)$. In [11] it has been shown that the interior penalty method for Poisson's equation with nonhomogeneous Dirichlet in combination with a specific target functional $J(\cdot)$ results in an adjoint inconsistent discretization of boundary conditions and a nonsmooth adjoint solution even for the SIPG discretization, which is known to be adjoint consistent in the interior of the domain. Only after an appropriate modification of the target functional have adjoint consistency, smoothness of the adjoint solution, and optimal convergence rates in $J(\cdot)$ been recovered; see also [23] for elliptic problems discretized by the BR2 scheme [5]. First results for the compressible Euler equations in [23, 24] indicate that adjoint consistency is of similar importance for nonlinear problems. Whereas adjoint inconsistent discretizations of boundary terms [4, 15] lead to irregular adjoint solutions near a reflective boundary, it has been shown in [23, 24, 14] that a specific discretization of boundary conditions and target functionals is required for recovering the adjoint consistency property and a smooth discrete adjoint solution.

We note that adjoint consistency is of importance also in continuous finite element methods. In [18] it was shown that, for the streamline diffusion (SD) discretization of the linear advection equation, the discrete adjoint problem is not a consistent discretization of the continuous adjoint problem; i.e., the SD discretization is not adjoint consistent. It is, however, asymptotically adjoint consistent.

As outlined so far, adjoint consistency is a desirable property which, however, involves several issues. In addition to the adjoint consistency of element and interior face terms, it involves the discretization of boundary conditions and target functionals. The purpose of this paper is to give a general framework for analyzing the adjoint consistency property of DG discretizations for linear as well as nonlinear problems. This framework includes the derivation of the continuous adjoint problems and boundary conditions provided the primal problems and the target functionals satisfy a compatibility condition. Furthermore, it includes the derivation of the discrete adjoint problems and of primal and adjoint residuals and a discussion of under which conditions the residuals vanish for the exact primal and adjoint solutions, respectively. Additionally, a so-called consistent modification of target functionals is introduced. The analysis is performed for various model problems, recovering properties and conclusions drawn in [1, 11, 24]. In addition, this framework is used to derive an adjoint consistent DG discretization of the compressible Navier–Stokes equations. Altogether, this publication provides a general framework of an adjoint consistency analysis which can be applied to a wide range of (more complex) linear and nonlinear problems.

The paper is structured as follows: We begin by outlining the main ingredients of the framework in section 2, including the definition of adjoint consistency for linear and nonlinear problems. Then in sections 3, 4, and 5 the DG discretization of the linear advection equation, the interior penalty DG method for Poisson's equation, and the DG discretization of the compressible Euler equations are analyzed. Then in section 6 it is shown that the interior penalty DG discretization of the compressible Navier–Stokes equations in [16] is not adjoint consistent. Within the framework appropriate modifications are derived for recovering adjoint consistency. These mod-

ifications include a specific treatment of convective and diffusive fluxes at boundaries and a consistent modification of total force coefficients. Then in section 7 we show some numerical results demonstrating the effect of adjoint consistency on the accuracy of the flow solution and on the smoothness of the discrete adjoint solution.

2. General framework. We begin by defining the adjoint consistency for linear and nonlinear problems. Let Ω be a bounded open domain in \mathbb{R}^d with boundary Γ .

2.1. Linear problems. For $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ consider the following linear problem:

$$(2.1) \quad Lu = f \quad \text{in } \Omega, \quad Bu = g \quad \text{on } \Gamma,$$

where L denotes a linear differential operator on Ω and B denotes a linear differential (boundary) operator on Γ . Let J be a linear target functional given by

$$(2.2) \quad J(u) = \int_{\Omega} j_{\Omega} u \, d\mathbf{x} + \int_{\Gamma} j_{\Gamma} Cu \, ds,$$

where $j_{\Omega} \in L^2(\Omega)$, $j_{\Gamma} \in L^2(\Gamma)$, and C is a differential (boundary) operator on Γ . We say that the target functional (2.2) is *compatible* with (2.1), provided the following compatibility condition based on Green's formula holds (see, e.g., [2, 22] for elliptic problems):

$$(2.3) \quad (Lu, z)_{\Omega} + (Bu, C^*z)_{\Gamma} = (u, L^*z)_{\Omega} + (Cu, B^*z)_{\Gamma},$$

where L^* , B^* , and C^* denote the adjoint operators to L , B , and C , respectively, and $(\cdot, \cdot)_{\Omega}$ and $(\cdot, \cdot)_{\Gamma}$ denote the $L^2(\Omega)$ and $L^2(\Gamma)$ scalar products, respectively. We note that, for given operators L and B associated with the primal problem (2.1), only some target functionals (2.2) with operators C are compatible, whereas others are not. However, assuming that (2.3) holds the adjoint problem associated to (2.1), (2.2) is given by

$$(2.4) \quad L^*z = j_{\Omega} \quad \text{in } \Omega, \quad B^*z = j_{\Gamma} \quad \text{on } \Gamma.$$

In an adjoint-based optimization framework (see, e.g., [10]) this ensures that

$$(2.5) \quad \begin{aligned} J(u) &= (u, j_{\Omega})_{\Omega} + (Cu, j_{\Gamma})_{\Gamma} = (u, L^*z)_{\Omega} + (Cu, B^*z)_{\Gamma} \\ &= (Lu, z)_{\Omega} + (Bu, C^*z)_{\Gamma} = (f, z)_{\Omega} + (g, C^*z)_{\Gamma}. \end{aligned}$$

Let Ω be subdivided into shape-regular meshes $\mathcal{T}_h = \{\kappa\}$ consisting of elements κ , and let V_h be a discrete function space on \mathcal{T}_h . Furthermore, let V be a broken Sobolev space on \mathcal{T}_h appropriately chosen such that $V_h \subset V$ and $u, z \in V$, where u and z are the solutions to (2.1) and (2.4), respectively. A typical situation is $V = H^2(\mathcal{T}_h)$, with $V_h \subset V$ and $u, z \in H^2(\Omega) \subset V$; see, e.g., [1, 6]. Finally, let $\mathcal{B} : V \times V \rightarrow \mathbb{R}$ be a bilinear form such that problem (2.1) is discretized as follows: Find $u_h \in V_h$ such that

$$(2.6) \quad \mathcal{B}(u_h, v) = \mathcal{F}(v) \quad \forall v \in V_h,$$

where $\mathcal{F} : V \rightarrow \mathbb{R}$ is a linear form including the prescribed force and boundary data functions f and g . Then the discretization (2.6) is said to be consistent if the exact solution $u \in V$ to the primal problem (2.1) satisfies:

$$(2.7) \quad \mathcal{B}(u, v) = \mathcal{F}(v) \quad \forall v \in V,$$

which can then be viewed as a (broken) weak formulation of (2.1); see [6]. We note that defining \mathcal{B} via the discretization scheme (2.6) instead of the weak formulation (2.7) allows us to represent also inconsistent DG discretizations; see, e.g., [1].

Similarly, the discretization (2.6) is said to be *adjoint consistent* if the exact solution $z \in V$ to the adjoint problem (2.4) satisfies:

$$(2.8) \quad \mathcal{B}(w, z) = J(w) \quad \forall w \in V.$$

In the following we generalize this definition to nonlinear problems.

2.2. Nonlinear problems. We now consider the nonlinear problem

$$(2.9) \quad Nu = 0 \quad \text{in } \Omega, \quad Bu = 0 \quad \text{on } \Gamma,$$

where N is a nonlinear differential (and Fréchet-differentiable) operator and B is a (possibly nonlinear) boundary operator. Let $J(\cdot)$ be a nonlinear target functional

$$(2.10) \quad J(u) = \int_{\Omega} j_{\Omega}(u) \, d\mathbf{x} + \int_{\Gamma} j_{\Gamma}(Cu) \, ds,$$

with Fréchet derivative

$$(2.11) \quad J'[u](w) = \int_{\Omega} j'_{\Omega}[u] w \, d\mathbf{x} + \int_{\Gamma} j'_{\Gamma}[Cu] C'[u] w \, ds,$$

where $j_{\Omega}(\cdot)$ and $j_{\Gamma}(\cdot)$ may be nonlinear with derivatives j'_{Ω} and j'_{Γ} , respectively, and C is a differential boundary operator on Γ and may be nonlinear with derivative C' . Here $'$ denotes the (total) Fréchet derivative, and the square bracket $[\cdot]$ denotes the state about which linearization is performed. Again, we say that the target functional (2.10) is *compatible* with (2.9) provided the following compatibility condition holds:

$$(2.12) \quad (N'[u]w, z)_{\Omega} + (B'[u]w, (C'[u])^* z)_{\Gamma} = (w, (N'[u])^* z)_{\Omega} + (C'[u]w, (B'[u])^* z)_{\Gamma},$$

where $(N'[u])^*$, $(B'[u])^*$, and $(C'[u])^*$ denote the adjoint operators to $N'[u]$, $B'[u]$, and $C'[u]$, respectively. This condition is analogous to (2.3), with L , B , and C replaced by $N'[u]$, $B'[u]$, and $C'[u]$, respectively. Assuming that (2.12) holds, the continuous adjoint problem associated to (2.9) and (2.11) is given by

$$(2.13) \quad (N'[u])^* z = j'_{\Omega}[u] \quad \text{in } \Omega, \quad (B'[u])^* z = j'_{\Gamma}[Cu] \quad \text{on } \Gamma.$$

We note that, in an optimization framework [10], this ensures, analogous to (2.5), that

$$(2.14) \quad \begin{aligned} J'[u](w) &= (w, j'_{\Omega}[u])_{\Omega} + (C'[u]w, j'_{\Gamma}[Cu])_{\Gamma} = (w, (N'[u])^* z)_{\Omega} + (C'[u]w, (B'[u])^* z)_{\Gamma} \\ &= (N'[u]w, z)_{\Omega} + (B'[u]w, (C'[u])^* z)_{\Gamma}. \end{aligned}$$

Let $\mathcal{N} : V \times V \rightarrow \mathbb{R}$ be a semilinear form, nonlinear in its first and linear in its second argument, such that the nonlinear problem (2.9) is discretized as follows: Find $u_h \in V_h$ such that

$$(2.15) \quad \mathcal{N}(u_h, v) = 0 \quad \forall v \in V_h.$$

Then the discretization (2.15) is said to be consistent if the exact solution $u \in V$ to the primal problem (2.9) satisfies the following equation:

$$(2.16) \quad \mathcal{N}(u, v) = 0 \quad \forall v \in V.$$

Furthermore, the discretization (2.15) is said to be adjoint consistent if the exact solutions $u, z \in V$ to the primal and adjoint problems (2.9) and (2.13), respectively, satisfy the following equation:

$$(2.17) \quad \mathcal{N}'[u](w, z) = J'[u](w) \quad \forall w \in V,$$

where $\mathcal{N}'[u]$ denotes the Fréchet derivatives of $\mathcal{N}(u, v)$ with respect to u .

In other words, a discretization is adjoint consistent if the discrete adjoint problem is a consistent discretization of the continuous adjoint problem. Finally, we note that, in the case of a linear problem and target functional, the definition of adjoint consistency in (2.17) reduces to the definition of linear adjoint consistency given in section 2.1.

2.3. The adjoint consistency analysis. Based on the definition of adjoint consistency in the previous two sections, we outline a framework of analyzing the adjoint consistency of discontinuous Galerkin discretizations. This framework can also be used to derive adjoint consistent DG discretizations.

2.3.1. Derivation of the continuous adjoint problem. Let the primal problem be given by (2.1) or by (2.9) in the nonlinear case. Furthermore, assume that $J(\cdot)$ is a linear (2.2) or linearized (2.11) compatible target functional. Then we derive the continuous adjoint problem (2.4) or (2.13) including adjoint boundary conditions.

We note that the derivation of the adjoint operator $(N'[u])^*$ for nonlinear systems is a considerably more complicated task than deriving L^* for scalar linear problems. Still more involved is the derivation of the adjoint boundary operators $(B'[u])^*$. In the framework of optimal design, [10] gives a general approach of deriving $(B'[u])^*$ and $(C'[u])^*$ assumed to be connected to B, C, N , and $(N'[u])^*$ through (2.12). This approach is based on a matrix representation of boundary operators which for systems of equations leads to lengthy and error prone derivations. In contrast to optimization where both $(B'[u])^*$ and C^* are required, in the following analysis we require only the adjoint operator $(B'[u])^*$. Due to this we can circumvent the approach described in [10] and use a simpler way of deriving the adjoint boundary operators $(B'[u])^*$.

2.3.2. Consistency analysis of the discrete primal problem. We rewrite the discontinuous Galerkin discretization (2.15) of problem (2.9) in the following element-based primal residual form: Find $u_h \in V_h$ such that

$$(2.18) \quad \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} R(u_h) v \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u_h) v \, ds + \int_{\Gamma} r_{\Gamma}(u_h) v \, ds = 0 \quad \forall v \in V_h,$$

where $R(u_h)$, $r(u_h)$, and $r_{\Gamma}(u_h)$ denote the element, interior face, and boundary residuals, respectively. According to (2.16), the discretization (2.15) is consistent if the exact solution u to (2.9) satisfies

$$(2.19) \quad \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} R(u) v \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u) v \, ds + \int_{\Gamma} r_{\Gamma}(u) v \, ds = 0 \quad \forall v \in V,$$

which holds, provided u satisfies

$$(2.20) \quad R(u) = 0 \quad \text{in } \kappa, \kappa \in \mathcal{T}_h, \quad r(u) = 0 \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \quad r_{\Gamma}(u) = 0 \quad \text{on } \Gamma.$$

2.3.3. Derivation of the discrete adjoint problem. Given the discretization (2.15), the target functional (2.10), and its linearization (2.11), we derive the discrete

adjoint problem: Find $z_h \in V_h$ such that

$$(2.21) \quad \mathcal{N}'[u_h](w, z_h) = J'[u_h](w) \quad \forall w \in V_h.$$

$\mathcal{N}'[u_h]$ is called the Jacobian of the numerical scheme and is required also for implicit and adjoint methods, e.g., Newton iteration, a posteriori error estimation, adjoint-based adaptation (see [13]), and for optimization.

2.3.4. Adjoint consistency of element, interior face, and boundary terms.

We rewrite the discrete adjoint problem (2.21) in element-based adjoint residual form: Find $z_h \in V_h$ such that

$$(2.22) \quad \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} w R^*[u_h](z_h) \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} w r^*[u_h](z_h) \, ds + \int_{\Gamma} w r_{\Gamma}^*[u_h](z_h) \, ds = 0$$

for all $w \in V_h$, where $R^*[u_h](z_h)$, $r^*[u_h](z_h)$, and $r_{\Gamma}^*[u_h](z_h)$ denote the element, interior face, and boundary adjoint residuals, respectively. According to (2.17), the discretization (2.15) is adjoint consistent if the exact solutions u and z satisfy

$$(2.23) \quad \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} w R^*[u](z) \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} w r^*[u](z) \, ds + \int_{\Gamma} w r_{\Gamma}^*[u](z) \, ds = 0 \quad \forall w \in V,$$

\downarrow
 $j_{\Omega}(\cdot)$

\downarrow
 $j_{\Gamma}(\cdot)$

which holds, provided u and z satisfy

$$(2.24) \quad R^*[u](z) = 0 \quad \text{in } \kappa, \quad r^*[u](z) = 0 \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \quad r_{\Gamma}^*[u](z) = 0 \quad \text{on } \Gamma.$$

We note that the adjoint problem and consequently the adjoint consistency of a discretization depend on the specific target functional $J(\cdot)$ under consideration. Given a target functional of the form (2.10), we see that $R^*[u](z)$ depends on $j_{\Omega}(\cdot)$ and $r_{\Gamma}^*[u](z)$ depends on $j_{\Gamma}(\cdot)$. For obtaining an adjoint consistent discretization it is, in some cases (see following sections) necessary to modify the target functional as follows:

$$(2.25) \quad \tilde{J}(u_h) = J(i(u_h)) + \int_{\Gamma} r_J(u_h) \, ds,$$

where $i(\cdot)$ and $r_J(\cdot)$ are functions to be specified. A modification of a target functional is called consistent if $\tilde{J}(u) = J(u)$ holds for the exact solution u . Thereby, the modification in (2.25) is consistent if the exact solution u satisfies $i(u) = u$ and $r_J(u) = 0$. Although the true value of the target functional is unchanged, $J(u) = J(u)$, the computed value $J(u_h)$ of the target functional is modified, and, more importantly, $\tilde{J}'[u_h]$ differs from $J'[u_h]$. This modification can be used to recover an adjoint consistent discretization. We note that (2.25) is not a unique choice of a consistent modification of $J(\cdot)$; other examples are $\tilde{J}(u_h) = J(u_h) + \int_{\Omega} R_J(u_h) \, d\mathbf{x}$, with $R_J(u) = 0$, or $\tilde{J}(u_h) = m(J(u_h), J(i(u_h)))$, with $i(u) = u$ and $m(j, j) = j$. However, the consistent modification as given in (2.25) will be sufficient for the purposes of this work.

3. The linear advection equation. We consider the linear advection equation

$$(3.1) \quad \nabla \cdot (\mathbf{b}u) + cu = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_-,$$

where $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega)$ are real-valued and $\mathbf{b} = \{\beta_i\}_{i=1}^d$ is a vector function whose entries β_i are Lipschitz continuous real-valued functions on Ω . By $\Gamma_- = \{x \in \Gamma, \mathbf{b} \cdot \mathbf{n} < 0\}$ we denote the inflow part of the boundary $\Gamma = \partial\Omega$. Furthermore, we adopt the following hypothesis: There exists a $c_0 \in L^\infty(\Omega)$ and a number $\gamma_0 > 0$ such that $c(\mathbf{x}) + \frac{1}{2}\nabla \cdot \mathbf{b}(\mathbf{x}) = c_0^2(\mathbf{x}) \geq \gamma_0 > 0$. To demonstrate the similarities with the compressible Euler equations in section 5, we consider the linear advection equation in conservative form which is equivalent to the nonconservative form $\mathbf{b} \cdot \nabla u + \tilde{c}u = f$ (see, e.g., [19]), with $\tilde{c} - \frac{1}{2}\nabla \cdot \mathbf{b} = c_0^2$ and $\tilde{c} = c + \nabla \cdot \mathbf{b}$.

In order to derive the continuous adjoint problem, we multiply the left-hand side of (3.1) by z and integrate by parts over the domain Ω . Thereby, we obtain

$$(3.2) \quad (\nabla \cdot (\mathbf{b}u) + cu, z)_\Omega + (u, -\mathbf{b} \cdot \mathbf{n}z)_{\Gamma_-} = (u, -\mathbf{b} \cdot \nabla z + cz)_\Omega + (u, \mathbf{b} \cdot \mathbf{n}z)_{\Gamma_+}.$$

From (2.3) we see that, for $Lu = \nabla \cdot (\mathbf{b}u) + cu$ in Ω , and $Bu = u$ and $Cu = 0$ on Γ_- , and $Bu = 0$ and $Cu = u$ on Γ_+ , we have $L^*z = -\mathbf{b} \cdot \nabla z + cz$ in Ω , $B^*z = 0$ and $C^*z = -\mathbf{b} \cdot \mathbf{n}z$ on Γ_- , and $B^*z = \mathbf{b} \cdot \mathbf{n}z$ and $C^*z = 0$ on Γ_+ . In particular, the target functional $J(u) = \int_\Omega j_\Omega u \, d\mathbf{x} + \int_{\Gamma_+} j_\Gamma u \, ds$ is compatible, and the adjoint problem is given by

$$(3.3) \quad -\mathbf{b} \cdot \nabla z + cz = j_\Omega \quad \text{in } \Omega, \quad \mathbf{b} \cdot \mathbf{n}z = j_\Gamma \quad \text{on } \Gamma_+.$$

Let Ω be subdivided into shape-regular meshes $\mathcal{T}_h = \{\kappa\}$ consisting of elements κ , and let V_h^p be the discrete function space consisting of discontinuous piecewise polynomial functions of degree $p \geq 0$. Suppose that $v|_\kappa \in H^1(\kappa)$ for each $\kappa \in \mathcal{T}_h$. Let κ^+ and κ^- be two adjacent elements of \mathcal{T}_h and \mathbf{x} be an arbitrary point on the interior edge $e = \partial\kappa^+ \cap \partial\kappa^- \subset \Gamma_\mathcal{I}$, where $\Gamma_\mathcal{I}$ denotes the union of all interior edges of \mathcal{T}_h . Moreover, let v and ϕ be a scalar and a d -vector-valued function, respectively, that are smooth inside each element κ^\pm . By $u^\pm := u|_{\partial\kappa^\pm}$ and $\phi^\pm := \phi|_{\partial\kappa^\pm}$ we denote the traces of u and ϕ , respectively, on e taken from within the interior of κ^\pm . Then we define the averages at $\mathbf{x} \in e$ by $\{v\} = (v^+ + v^-)/2$ and $\{\phi\} = (\phi^+ + \phi^-)/2$. Similarly, the jump at $\mathbf{x} \in e$ is given by $[[v]] = v^+ \mathbf{n}^+ + v^- \mathbf{n}^-$ and by $[[\phi]] = \phi^+ \cdot \mathbf{n}^+ + \phi^- \cdot \mathbf{n}^-$. On a boundary edge $e \subset \Gamma$ the average and jump operators are defined by $\{v\} = v^+$, $\{\phi\} = \phi^+$, $[[v]] = v^+ \mathbf{n}^+$, and $[[\phi]] = \phi^+ \cdot \mathbf{n}^+$.

The DG discretization of (3.1) (e.g., [12, 7]) is given by: Find $u_h \in V_h^p$ such that

$$(3.4) \quad \begin{aligned} \mathcal{B}(u_h, v) \equiv & - \int_\Omega u_h \mathbf{b} \cdot \nabla_h v \, d\mathbf{x} + \int_\Omega cu_h v \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa_- \setminus \Gamma} \mathbf{b} \cdot \mathbf{n} u_h^- v^+ \, ds \\ & + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa_+} \mathbf{b} \cdot \mathbf{n} u_h^+ v^+ \, ds = \int_\Omega f v \, d\mathbf{x} - \int_{\Gamma_-} \mathbf{b} \cdot \mathbf{n} g v^+ \, ds \end{aligned}$$

for all $v \in V_h^p$. Then integration by parts on each $\kappa \in \mathcal{T}_h$ yields: Find $u_h \in V_h^p$:

$$\int_\Omega (\nabla_h \cdot (\mathbf{b}u_h) + cu_h) v \, d\mathbf{x} - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa_-} \mathbf{b} \cdot \mathbf{n} [u_h] v^+ \, ds = \int_\Omega f v \, d\mathbf{x} - \int_{\Gamma_-} \mathbf{b} \cdot \mathbf{n} g v^+ \, ds$$

for all $v \in V_h^p$. Hence we have the primal residual form (2.18) with

$$\begin{aligned} R(u_h) &= f - \nabla_h \cdot (\mathbf{b}u_h) - cu_h && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r(u_h) &= \mathbf{b} \cdot \mathbf{n} [u_h] && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_\Gamma(u_h) &= \mathbf{b} \cdot \mathbf{n} (u_h - g) && \text{on } \Gamma_-, \end{aligned}$$

and $r_\Gamma(u_h) \equiv 0$ on Γ_+ . As (2.19) holds for the exact solution u to (3.1), we conclude that (3.4) is a consistent discretization of (3.1). Substituting

$$\sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa_- \setminus \Gamma} \mathbf{b} \cdot \mathbf{n} u_h^- v^+ \, ds = - \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa_+ \setminus \Gamma} \mathbf{b} \cdot \mathbf{n} u_h^+ v^- \, ds$$

in (3.4), we find that the discrete adjoint problem to the discretization (3.4) is given by: Find $z_h \in V_h$ such that

$$\mathcal{B}(w, z_h) \equiv \int_{\Omega} w (-\mathbf{b} \cdot \nabla_h z_h + cz_h) \, d\mathbf{x} + \sum_{\kappa} \int_{\partial\kappa_+ \setminus \Gamma} w^+ \mathbf{b} \cdot \llbracket z_h \rrbracket \, ds = J(w)$$

for all $w \in V_h$. Hence we have the adjoint residual form (2.22) with

$$(3.5) \quad \begin{aligned} R^*(z_h) &= j_\Omega + \mathbf{b} \cdot \nabla_h z_h - cz_h && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ r^*(z_h) &= -\mathbf{b} \cdot \llbracket z_h \rrbracket && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \end{aligned}$$

$$(3.6) \quad r_\Gamma^*(z_h) = j_\Gamma - \mathbf{b} \cdot \mathbf{n} z_h \quad \text{on } \Gamma_+,$$

and $r_\Gamma^*(z_h) \equiv 0$ on Γ_- . As the adjoint residuals vanish for the exact solution z to (3.3), we conclude that (3.4) is an adjoint consistent discretization of (3.1).

4. Poisson's equation. We now consider the elliptic model problem

$$(4.1) \quad -\Delta u = f \quad \text{in } \Omega, \quad u = g_D \quad \text{on } \Gamma_D, \quad \mathbf{n} \cdot \nabla u = g_N \quad \text{on } \Gamma_N,$$

where $f \in L^2(\Omega)$, $g_D \in L^2(\Gamma_D)$, and $g_N \in L^2(\Gamma_N)$ are given functions. We assume that Γ_D and Γ_N are disjoint subsets with union Γ . We also assume that Γ_D is nonempty.

In order to derive the continuous adjoint problem we multiply the left-hand side of (4.1) by z and integrate twice by parts over the domain Ω . Thereby, we obtain

$$(-\Delta u, z)_\Omega + (u, -\mathbf{n} \cdot \nabla z)_{\Gamma_D} + (\mathbf{n} \cdot \nabla u, z)_{\Gamma_N} = (u, -\Delta z)_\Omega + (\mathbf{n} \cdot \nabla u, -z)_{\Gamma_D} + (u, \mathbf{n} \cdot \nabla z)_{\Gamma_N}.$$

From (2.3) we see that, for $Lu = -\Delta u$ in Ω , $Bu = u$ and $Cu = \mathbf{n} \cdot \nabla u$ on Γ_D , and $Bu = \mathbf{n} \cdot \nabla u$ and $Cu = u$ on Γ_N , we have $L^*z = -\Delta z$ in Ω , $B^*z = -z$ and $C^*z = -\mathbf{n} \cdot \nabla z$ on Γ_D , and $B^*z = \mathbf{n} \cdot \nabla z$ and $C^*z = z$ on Γ_N . Then (2.2) reduces to

$$(4.2) \quad J(u) = \int_{\Omega} j_\Omega u \, d\mathbf{x} + \int_{\Gamma_D} j_D \mathbf{n} \cdot \nabla u \, ds + \int_{\Gamma_N} j_N u \, ds.$$

This target functional is compatible, and the continuous adjoint problem is given by

$$(4.3) \quad -\Delta z = j_\Omega \quad \text{in } \Omega, \quad z = j_D \quad \text{on } \Gamma_D, \quad \mathbf{n} \cdot \nabla z = j_N \quad \text{on } \Gamma_N.$$

The method by Baumann–Oden and the symmetric and nonsymmetric interior penalty methods can be expressed as follows (see, e.g., [25]): Find $u_h \in V_h^p$ such that

$$(4.4) \quad \begin{aligned} \mathcal{B}(u_h, v) &\equiv \int_{\Omega} \nabla_h u_h \cdot \nabla_h v \, d\mathbf{x} \\ &\quad + \int_{\Gamma_T \cup \Gamma_D} (\theta \llbracket u_h \rrbracket \cdot \{\nabla_h v\} - \{\nabla_h u_h\} \cdot \llbracket v \rrbracket) \, ds + \int_{\Gamma_T \cup \Gamma_D} \delta \llbracket u_h \rrbracket \cdot \llbracket v \rrbracket \, ds \\ &= \int_{\Omega} f v \, d\mathbf{x} + \int_{\Gamma_D} \theta g_D \mathbf{n} \cdot \nabla v \, ds + \int_{\Gamma_D} \delta g_D v \, ds + \int_{\Gamma_N} g_N v \, ds \end{aligned}$$

for all $v \in V_h^p$, where the constants θ and δ are given by $\theta = 1, \delta = 0$ for Baumann–Oden, by $\theta = 1, \delta > 0$ for NIPG, and by $\theta = -1, \delta > \delta_0 > 0$ for SIPG.

The scheme in (4.4) is given in face-based form including integrals over all interior faces $\Gamma_{\mathcal{I}}$. Rewriting the interior face terms in element-based form, we obtain

$$(4.5) \quad \begin{aligned} \mathcal{B}(u_h, v) = & \int_{\Omega} \nabla_h u_h \cdot \nabla_h v \, d\mathbf{x} + \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma} \frac{1}{2} \theta \llbracket u_h \rrbracket \cdot \nabla_h v \, ds \\ & - \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma_N} \{ \nabla_h u_h \} \cdot \mathbf{n} v \, ds + \sum_{\kappa} \int_{\partial\kappa} \delta \llbracket u_h \rrbracket \cdot \mathbf{n} v \, ds + \int_{\Gamma_D} \theta u_h \mathbf{n} \cdot \nabla_h v \, ds. \end{aligned}$$

Using integration by parts and the relation

$$(4.6) \quad \nabla_h u^+ \cdot \mathbf{n}^+ v^+ = \{ \nabla_h u \} \cdot \mathbf{n}^+ v^+ + \frac{1}{2} \llbracket \nabla_h u \rrbracket v^+,$$

we can rewrite (4.4) as follows: Find $u_h \in V_h^p$ such that

$$(4.7) \quad \begin{aligned} \mathcal{B}(u_h, v) \equiv & - \int_{\Omega} \Delta_h u_h v \, d\mathbf{x} + \int_{\Gamma_N} \nabla_h u_h \cdot \mathbf{n} v \, ds + \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma} \frac{1}{2} \theta \llbracket u_h \rrbracket \cdot \nabla_h v \, ds \\ & + \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma} \left(\frac{1}{2} \llbracket \nabla_h u_h \rrbracket + \delta \llbracket u_h \rrbracket \cdot \mathbf{n} \right) v \, ds + \int_{\Gamma_D} \theta u_h \mathbf{n} \cdot \nabla_h v \, ds + \int_{\Gamma_D} \delta u_h v \, ds \\ = & \int_{\Omega} f v \, d\mathbf{x} + \int_{\Gamma_D} \theta g_D \mathbf{n} \cdot \nabla v \, ds + \int_{\Gamma_D} \delta g_D v \, ds + \int_{\Gamma_N} g_N v \, ds \end{aligned}$$

for all $v \in V_h^p$. Hence we have the element-based primal residual form

$$(4.8) \quad \begin{aligned} \int_{\Omega} R(u_h) v \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u_h) v + \boldsymbol{\rho}(u_h) \cdot \nabla v \, ds \\ + \int_{\Gamma} r_{\Gamma}(u_h) v + \boldsymbol{\rho}_{\Gamma}(u_h) \cdot \nabla v \, ds = 0 \quad \forall v \in V_h, \end{aligned}$$

where the residuals are given by $R(u_h) = f + \Delta_h u_h$ in Ω , and

$$(4.9) \quad \begin{aligned} r(u_h) &= -\frac{1}{2} \llbracket \nabla_h u_h \rrbracket - \delta \llbracket u_h \rrbracket \cdot \mathbf{n}, & \boldsymbol{\rho}(u_h) &= -\frac{1}{2} \theta \llbracket u_h \rrbracket && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ r_{\Gamma}(u_h) &= \delta(g_D - u_h), & \boldsymbol{\rho}_{\Gamma}(u_h) &= \theta(g_D - u_h) \mathbf{n} && \text{on } \Gamma_D, \\ r_{\Gamma}(u_h) &= g_N - \mathbf{n} \cdot \nabla_h u_h, & \boldsymbol{\rho}_{\Gamma}(u_h) &= 0 && \text{on } \Gamma_N. \end{aligned}$$

We note that (4.8) is a generalization to (2.18) to include ∇v terms. Furthermore, we note that the discretization is consistent as the residuals in (4.9) vanish for the exact solution u to (4.1). Given the target functional defined in (4.2), the discrete adjoint problem (2.8) to the discretization (4.4) is given by: Find $z_h \in V_h$ such that

$$\int_{\Omega} \nabla_h w \cdot \nabla_h z_h \, d\mathbf{x} + \int_{\Gamma_{\mathcal{I}} \cup \Gamma_D} (\theta \llbracket w \rrbracket \cdot \{ \nabla_h z_h \} - \{ \nabla_h w \} \cdot \llbracket z_h \rrbracket + \delta \llbracket w \rrbracket \cdot \llbracket z_h \rrbracket) \, ds = J(w)$$

for all $w \in V_h$. Then in element-based form we have: Find $z_h \in V_h$ such that

$$\begin{aligned} \int_{\Omega} \nabla_h w \cdot \nabla_h z_h \, d\mathbf{x} + \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma_N} w (\theta \mathbf{n} \cdot \{ \nabla_h z_h \} + \delta \llbracket z_h \rrbracket \cdot \mathbf{n}) \, ds \\ - \frac{1}{2} \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma} \nabla_h w \cdot \llbracket z_h \rrbracket \, ds - \int_{\Gamma_D} \nabla_h w \cdot \mathbf{n} z_h \, ds = J(w) \quad \forall w \in V_h. \end{aligned}$$

Using integration by parts and (4.6) with u_h and v replaced by z_h and w yields

$$\begin{aligned} & - \int_{\Omega} w \Delta_h z_h \, dx + \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma} w \left(\frac{1}{2} \llbracket \nabla_h z_h \rrbracket + (1 + \theta) \mathbf{n} \cdot \{ \nabla_h z_h \} + \delta \llbracket z_h \rrbracket \cdot \mathbf{n} \right) \, ds \\ & \quad - \sum_{\kappa} \int_{\partial\kappa \setminus \Gamma} \frac{1}{2} \nabla_h w \cdot \llbracket z_h \rrbracket \, ds + \int_{\Gamma_N} w \mathbf{n} \cdot \nabla_h z_h \, ds \\ & \quad + \int_{\Gamma_D} w ((1 + \theta) \mathbf{n} \cdot \nabla_h z_h + \delta z_h) \, ds - \int_{\Gamma_D} \nabla_h w \cdot \mathbf{n} z_h \, ds \\ & \quad = \int_{\Omega} w j_{\Omega} \, d\mathbf{x} + \int_{\Gamma_D} \nabla w \cdot \mathbf{n} j_D \, ds + \int_{\Gamma_N} w j_N \, ds, \end{aligned}$$

and we obtain the element-based adjoint residual form: Find $z_h \in V_h$ such that

$$(4.10) \quad \begin{aligned} & \int_{\Omega} w R^*(z_h) \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} w r^*(z_h) + \nabla w \cdot \boldsymbol{\rho}^*(z_h) \, ds \\ & \quad + \int_{\Gamma} w r_{\Gamma}^*(z_h) + \nabla w \cdot \boldsymbol{\rho}_{\Gamma}^*(z_h) \, ds = 0 \quad \forall w \in V_h, \end{aligned}$$

where the adjoint residuals are given by $R^*(z_h) = j_{\Omega} + \Delta_h z_h$ in Ω , by

$$(4.11) \quad r^*(z_h) = -\frac{1}{2} \llbracket \nabla_h z_h \rrbracket - (1 + \theta) \mathbf{n} \cdot \{ \nabla_h z_h \} - \delta \llbracket z_h \rrbracket \cdot \mathbf{n}, \quad \boldsymbol{\rho}^*(z_h) = \frac{1}{2} \llbracket z_h \rrbracket,$$

on interior faces $\partial\kappa \setminus \Gamma$, $\kappa \in \mathcal{T}_h$, and by

$$(4.12) \quad \begin{aligned} r_{\Gamma}^*(z_h) &= -(1 + \theta) \mathbf{n} \cdot \nabla_h z_h - \delta z_h, & \boldsymbol{\rho}_{\Gamma}^*(z_h) &= (j_D + z_h) \mathbf{n} && \text{on } \Gamma_D, \\ r_{\Gamma}^*(z_h) &= j_N - \mathbf{n} \cdot \nabla_h z_h, & \boldsymbol{\rho}_{\Gamma}^*(z_h) &= 0 && \text{on } \Gamma_N. \end{aligned}$$

From (4.11) we see that the exact solution z to the adjoint problem (4.3) satisfies $r^*(z) = 0$, provided $\theta = -1$. Furthermore, we have $R^*(z) = 0$. This shows the well-known result (see, e.g., [1]) that the method by Baumann–Oden and NIPG are not adjoint consistent whereas the interior face terms of SIPG are adjoint consistent. In fact, in [12] it has been demonstrated that the lack of adjoint consistency of the NIPG method leads to nonsmooth adjoint solutions and a suboptimal convergence of the method for the primal problem. In contrast to that, the adjoint consistent SIPG method shows an optimal order of convergence.

As $r_{\Gamma}^*(z) = 0$ and $\boldsymbol{\rho}_{\Gamma}^*(z) = 0$ on Γ_N the SIPG method is also adjoint consistent on Γ_N . However, on Γ_D the requirements $r_{\Gamma}^*(z) = 0$ and $\boldsymbol{\rho}_{\Gamma}^*(z) = 0$ reduce to the conditions $z = 0$ (note that $\theta = -1$) and $z = -j_D$, respectively, which are compatible for $j_D = 0$ but conflict for $j_D \neq 0$. This incompatibility can be resolved by modifying the target functional according to (2.25), with $i(u_h) = u_h$ and

$$(4.13) \quad r_J(u_h) = -\delta(u_h - g_D)j_D,$$

which in the following will be denoted by the *IP modification* of the target functional. This modification is consistent, as $i(u) = u$ and $r_J(u) = 0$ hold for the exact solution u to (4.1). As the modified functional is not linear in u_h (it is affine), the discrete adjoint problem includes its linearization as follows: Find $z_h \in V_h$ such that

$$(4.14) \quad \mathcal{B}(w, z_h) = \tilde{J}'[u](w) \quad \forall w \in V_h,$$

where

$$(4.15) \quad \tilde{J}'[u](w) = J'[u](w) + \int_{\Gamma_D} r'_J[u](w) \, ds = J(w) - \int_{\Gamma_D} w \, \delta j_D \, ds.$$

Then the adjoint residuals on Γ_D are given by

$$(4.16) \quad r_{\Gamma}^*(z_h) = -\delta j_D - (1 + \theta) \mathbf{n} \cdot \nabla_h z_h - \delta z_h, \quad \rho_{\Gamma}^*(z_h) = (j_D + z_h) \mathbf{n} \quad \text{on } \Gamma_D,$$

which vanish for $z = -j_D$. Hence the SIPG method is adjoint consistent also on Γ_D .

In contrast to the presentation in [11, 12], where the $\frac{1}{2} \llbracket \nabla_h z \rrbracket$ term in the inter-element conditions analogous to (4.11) has been omitted, we see that for $\theta = -1$ there is a clear correspondence of the adjoint residuals to the primal residuals (4.9). In fact, the discrete adjoint equations correspond to the discrete primal equations with u , f , g_D , and g_N replaced by z , j_{Ω} , $-j_D$, and j_N , respectively; i.e., the discrete adjoint equation to the SIPG discretization represents an SIPG discretization of the continuous adjoint equation.

Furthermore, we note that [11] considers the target functional $J(u) = \int_{\Gamma_0} \mathbf{n} \cdot \nabla u j_D \, ds$, $\Gamma_0 \subset \Gamma_D$, which is a special case of (4.2) with $j_{\Omega} \equiv 0$ in Ω , $j_N \equiv 0$ on Γ_N , and $j_D \equiv 0$ on $\Gamma_D \setminus \Gamma_0$. Numerical experiments in [11] have shown that the discrete adjoint solution associated with this target functional is nonsmooth near Γ_0 . Furthermore, it has been demonstrated that, either by setting $\delta = 0$ on Γ_0 or by modifying the target functional appropriately, this effect vanishes, and the adjoint solution becomes smooth. We note that the modification of the target functional proposed in [11] is connected to (4.15). However, here we derive (4.15) in the more general framework of consistent modifications of target functionals; see (2.25).

5. The compressible Euler equations. In this section we consider the two-dimensional stationary compressible Euler equations

$$(5.1) \quad \nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0 \quad \text{in } \Omega,$$

subject to various boundary conditions, e.g., slip-wall boundary conditions at solid wall boundaries $\Gamma_W \subset \Gamma$, with vanishing normal velocity $\mathbf{n} \cdot \mathbf{v} = n_1 v_1 + n_2 v_2 = 0$; i.e.,

$$(5.2) \quad B\mathbf{u} = n_1 u_2 + n_2 u_3 = 0 \quad \text{on } \Gamma_W$$

is imposed, where the vector of conservative variables $\mathbf{u} = (u_1, u_2, u_3, u_4)^{\top}$ and the convective fluxes $\mathcal{F}^c(\mathbf{u}) = (\mathbf{f}_1^c(\mathbf{u}), \mathbf{f}_2^c(\mathbf{u}))$ are defined by

$$(5.3) \quad \mathbf{u} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho E \end{bmatrix}, \quad \mathbf{f}_1^c(\mathbf{u}) = \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho H v_1 \end{bmatrix}, \quad \text{and} \quad \mathbf{f}_2^c(\mathbf{u}) = \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho H v_2 \end{bmatrix},$$

and ρ , $\mathbf{v} = (v_1, v_2)^{\top}$, p , and E denote the density, velocity vector, pressure, and specific total energy, respectively. Additionally, H is the total enthalpy given by $H = E + \frac{p}{\rho} = e + \frac{1}{2} \mathbf{v}^2 + \frac{p}{\rho}$, where e is the specific static internal energy, and the pressure is determined by the equation of state of an ideal gas $p = (\gamma - 1)\rho e$, where $\gamma = c_p/c_v$ is the ratio of specific heat capacities at constant pressure c_p and constant volume c_v ; for dry air $\gamma = 1.4$. Let us consider the target functional

$$(5.4) \quad J(\mathbf{u}) = \int_{\Gamma} j(C\mathbf{u}) \, ds = \int_{\Gamma_W} p(\mathbf{u}) \mathbf{n} \cdot \boldsymbol{\psi}_{\Gamma_W} \, ds,$$

with $C\mathbf{u} = p(\mathbf{u})$, $j(p) = p\mathbf{n} \cdot \boldsymbol{\psi}_{\Gamma_W}$ on Γ_W and $j(p) \equiv 0$ elsewhere, and $\boldsymbol{\psi}_{\Gamma_W} \in [L^2(\Gamma_W)]^2$. As we will see later, this target functional is compatible with (5.1) and (5.2). The most important target quantities of type (5.4) in inviscid compressible flows are the pressure-induced drag and lift coefficients c_{dp} and c_{lp} , where $\boldsymbol{\psi}_{\Gamma_W} = \frac{1}{c_\infty} \boldsymbol{\psi}$. Here $\boldsymbol{\psi}$ is given by $\boldsymbol{\psi}_d = (\cos(\alpha), \sin(\alpha))^\top$ and $\boldsymbol{\psi}_l = (-\sin(\alpha), \cos(\alpha))^\top$ for the drag and lift coefficients, respectively. Furthermore, $C_\infty = \frac{1}{2} \gamma p_\infty M_\infty^2 \bar{l} = \frac{1}{2} \gamma \frac{|\mathbf{v}_\infty|^2}{c_\infty^2} p_\infty \bar{l} = \frac{1}{2} \rho_\infty |\mathbf{v}_\infty|^2 \bar{l}$, where M_∞ denotes the Mach number at free-stream conditions, c_∞ is the speed of sound defined by $c_\infty^2 = \gamma p_\infty / \rho_\infty$, and \bar{l} denotes a reference length.

In order to derive the continuous adjoint problem, we multiply the left-hand side of (5.1) by \mathbf{z} , integrate by parts, and linearize about \mathbf{u} to obtain

$$(5.5) \quad (\nabla \cdot (\mathcal{F}_\mathbf{u}^c[\mathbf{u}](\mathbf{w})), \mathbf{z})_\Omega = -(\mathcal{F}_\mathbf{u}^c[\mathbf{u}](\mathbf{w}), \nabla \mathbf{z})_\Omega + (\mathbf{n} \cdot \mathcal{F}_\mathbf{u}^c[\mathbf{u}](\mathbf{w}), \mathbf{z})_\Gamma,$$

where $\mathcal{F}_\mathbf{u}^c[\mathbf{u}] := (\mathcal{F}^c)'[\mathbf{u}]$ denotes the Fréchet derivative of \mathcal{F}^c with respect to \mathbf{u} . Here we already use the subscript \mathbf{u} notation which we require in section 6 to distinguish from subscript $\nabla \mathbf{u}$ denoting the derivative with respect to $\nabla \mathbf{u}$. Thereby, the variational formulation of the continuous adjoint problem is given by: Find \mathbf{z} such that

$$(5.6) \quad -\left(\mathbf{w}, (\mathcal{F}_\mathbf{u}^c[\mathbf{u}])^\top \nabla \mathbf{z}\right)_\Omega + \left(\mathbf{w}, (\mathbf{n} \cdot \mathcal{F}_\mathbf{u}^c[\mathbf{u}])^\top \mathbf{z}\right)_\Gamma = J'[\mathbf{u}](\mathbf{w}) \quad \forall \mathbf{w} \in V,$$

$\langle \nabla f, \mathbf{g} \rangle = \langle f, \nabla^* g \rangle$

and the continuous adjoint problem is given by

$$(5.7) \quad -(\mathcal{F}_\mathbf{u}^c[\mathbf{u}])^\top \nabla \mathbf{z} = 0 \quad \text{in } \Omega, \quad (\mathbf{n} \cdot \mathcal{F}_\mathbf{u}^c[\mathbf{u}])^\top \mathbf{z} = j'[\mathbf{u}] \quad \text{on } \Gamma.$$

Using $\mathcal{F}^c(\mathbf{u}) \cdot \mathbf{n} = p(0, n_1, n_2, 0)^\top$ on Γ_W and the definition of j in (5.4), we obtain

$$p'[\mathbf{u}](0, n_1, n_2, 0) \cdot \mathbf{z} = \frac{1}{C_\infty} p'[\mathbf{u}] \mathbf{n} \cdot \boldsymbol{\psi} \quad \text{on } \Gamma_W,$$

which reduces to the boundary condition of the adjoint compressible Euler equations

$$(5.8) \quad (B'[\mathbf{u}])^* \mathbf{z} = n_1 z_2 + n_2 z_3 = \frac{1}{C_\infty} \mathbf{n} \cdot \boldsymbol{\psi} \quad \text{on } \Gamma_W.$$

Comparing (5.5) with (2.12) we see that the target functional (5.4) is compatible. Furthermore, we note that the adjoint boundary condition (5.8) has first been derived in [20]. In the framework of matrix representations of adjoint boundary operators related to (2.3), they have been derived in [10]; see also [9] for a more detailed derivation.

Let \mathbf{V}_h^p be a discrete function space consisting of discontinuous piecewise vector-valued polynomial functions of degree $p \geq 0$. Then the discontinuous Galerkin discretization in element-based form of (5.1) is given by: Find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$(5.9) \quad \mathcal{N}(\mathbf{u}_h, \mathbf{v}) \equiv - \int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v} \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial \kappa \setminus \Gamma} \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) \mathbf{v}^+ \, ds \\ + \int_\Gamma \tilde{\mathcal{H}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}^+) \mathbf{v}^+ \, ds = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^p,$$

where \mathcal{H} and $\tilde{\mathcal{H}}$ may be any Lipschitz continuous, consistent, and conservative numerical flux functions approximating the normal flux $\mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_h)$. On interior faces, \mathcal{H} takes into account the possible discontinuities of \mathbf{u}_h at element interfaces. On the boundary Γ , $\tilde{\mathcal{H}}$ may depend on the interior trace \mathbf{u}_h^+ and a consistent boundary function $\mathbf{u}_\Gamma(\mathbf{u}_h^+)$. We note that $\tilde{\mathcal{H}}$ may be different from \mathcal{H} . In fact, we will see below

that, depending on the specific choice of $\tilde{\mathcal{H}}$, the discontinuous Galerkin discretization (5.9) may be adjoint consistent or not.

Using integration by parts we obtain the residual form: Find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$(5.10) \quad \int_{\Omega} \mathbf{R}(\mathbf{u}_h) \cdot \mathbf{v} \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathbf{r}(\mathbf{u}_h) \cdot \mathbf{v}^+ \, ds + \int_{\Gamma} \mathbf{r}_{\Gamma}(\mathbf{u}_h) \cdot \mathbf{v}^+ \, ds = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^p,$$

where the primal residuals are given by

$$(5.11) \quad \begin{aligned} \mathbf{R}(\mathbf{u}_h) &= -\nabla \cdot \mathcal{F}^c(\mathbf{u}_h) && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ \mathbf{r}(\mathbf{u}_h) &= \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_h^+) - \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ \mathbf{r}_{\Gamma}(\mathbf{u}_h) &= \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_h^+) - \tilde{\mathcal{H}}(\mathbf{u}_h^+, \mathbf{u}_{\Gamma}(\mathbf{u}_h^+), \mathbf{n}^+) && \text{on } \Gamma. \end{aligned}$$

Given the consistency of the numerical flux $\mathcal{H}(\mathbf{w}, \mathbf{w}, \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{w})$ and the consistency of the boundary function, i.e., $\mathbf{u}_{\Gamma}(\mathbf{u}) = \mathbf{u}$ for the exact solution \mathbf{u} to (5.1), we find that \mathbf{u} satisfies the following equations:

$$(5.12) \quad \mathbf{R}(\mathbf{u}) = 0 \quad \text{in } \kappa, \kappa \in \mathcal{T}_h, \quad \mathbf{r}(\mathbf{u}) = 0 \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \quad \mathbf{r}_{\Gamma}(\mathbf{u}) = 0 \quad \text{on } \Gamma.$$

We conclude that (5.9) is a consistent discretization of (5.1).

For the target functional $J(\cdot)$ defined in (5.4) with Fréchet derivative $J'[\mathbf{u}](\cdot)$, the discrete adjoint problem is given by: Find $\mathbf{z}_h \in \mathbf{V}_h^p$ such that

$$(5.13) \quad \mathcal{N}'[\mathbf{u}_h](\mathbf{w}, \mathbf{z}_h) = J'[\mathbf{u}_h](\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_h^p,$$

where

$$(5.14) \quad \begin{aligned} \mathcal{N}'[\mathbf{u}_h](\mathbf{w}, \mathbf{z}_h) &\equiv - \int_{\Omega} (\mathcal{F}_{\mathbf{u}}^c[\mathbf{u}_h] \mathbf{w}) : \nabla_h \mathbf{z}_h \, d\mathbf{x} \\ &+ \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} (\mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) \mathbf{w}^+ + \mathcal{H}'_{\mathbf{u}^-}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) \mathbf{w}^-) \mathbf{z}_h^+ \, ds \\ &+ \int_{\Gamma} (\tilde{\mathcal{H}}'_{\mathbf{u}^+}(\mathbf{u}_h^+, \mathbf{u}_{\Gamma}(\mathbf{u}_h^+), \mathbf{n}^+) + \tilde{\mathcal{H}}'_{\mathbf{u}^-}(\mathbf{u}_h^+, \mathbf{u}_{\Gamma}(\mathbf{u}_h^+), \mathbf{n}^+) \mathbf{u}'_{\Gamma}[\mathbf{u}_h^+]) \mathbf{w}^+ \mathbf{z}_h^+ \, ds. \end{aligned}$$

Here $\mathbf{v} \rightarrow \mathcal{H}'_{\mathbf{u}^+}(\mathbf{v}^+, \mathbf{v}^-, \mathbf{n})$ and $\mathbf{v} \rightarrow \mathcal{H}'_{\mathbf{u}^-}(\mathbf{v}^+, \mathbf{v}^-, \mathbf{n})$ denote the derivatives of the flux function $\mathcal{H}(\cdot, \cdot, \cdot)$ with respect to its first and second arguments, respectively. As the numerical flux is conservative $\mathcal{H}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = -\mathcal{H}(\mathbf{w}, \mathbf{v}, -\mathbf{n})$, we obtain $\mathcal{H}'_{\mathbf{u}^-}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = \partial_{\mathbf{w}} \mathcal{H}(\mathbf{v}, \mathbf{w}, \mathbf{n}) = -\partial_{\mathbf{w}} \mathcal{H}(\mathbf{w}, \mathbf{v}, -\mathbf{n}) = -\mathcal{H}'_{\mathbf{u}^+}(\mathbf{w}, \mathbf{v}, -\mathbf{n})$, and

$$(5.15) \quad \begin{aligned} \int_{\Gamma_{\mathcal{T}}} \mathcal{H}'_{\mathbf{u}^-}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) \mathbf{w}^- \mathbf{z}^+ \, ds &= - \int_{\Gamma_{\mathcal{T}}} \mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}_h^-, \mathbf{u}_h^+, \mathbf{n}^-) \mathbf{w}^- \mathbf{z}^+ \, ds \\ &= - \int_{\Gamma_{\mathcal{T}}} \mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) \mathbf{w}^+ \mathbf{z}^- \, ds, \end{aligned}$$

where we exchanged notations $^+$ and $^-$ on $\Gamma_{\mathcal{T}}$. Then the discrete adjoint problem (5.13) with (5.14) is given in adjoint residual form as follows: Find $\mathbf{z}_h \in \mathbf{V}_h^p$ such that

$$(5.16) \quad \int_{\Omega} \mathbf{w} \cdot \mathbf{R}^*[\mathbf{u}_h](\mathbf{z}_h) \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathbf{w}^+ \cdot \mathbf{r}^*[\mathbf{u}_h](\mathbf{z}_h) \, ds + \int_{\Gamma} \mathbf{w}^+ \cdot \mathbf{r}_{\Gamma}^*[\mathbf{u}_h](\mathbf{z}_h) \, ds = 0$$

for all $\mathbf{w} \in \mathbf{V}_h^p$, where the adjoint residuals are given by

$$\begin{aligned} \mathbf{R}^*[\mathbf{u}_h](\mathbf{z}_h) &= (\mathcal{F}_{\mathbf{u}}^c[\mathbf{u}_h])^\top \nabla \mathbf{z}_h && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ \mathbf{r}^*[\mathbf{u}_h](\mathbf{z}_h) &= -(\mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+))^\top \llbracket \mathbf{z}_h \rrbracket \cdot \mathbf{n} && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ (5.17) \quad \mathbf{r}_\Gamma^*[\mathbf{u}_h](\mathbf{z}_h) &= j'[\mathbf{u}_h] - \left(\tilde{\mathcal{H}}'_{\mathbf{u}^+} + \tilde{\mathcal{H}}'_{\mathbf{u}^-} \mathbf{u}'_\Gamma[\mathbf{u}_h] \right)^\top \mathbf{z}_h^+ && \text{on } \Gamma, \end{aligned}$$

where $\tilde{\mathcal{H}}'_{\mathbf{u}^+} := \tilde{\mathcal{H}}'_{\mathbf{u}^+}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}^+)$ and $\tilde{\mathcal{H}}'_{\mathbf{u}^-} := \tilde{\mathcal{H}}'_{\mathbf{u}^-}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}^+)$.

Comparing the discrete adjoint boundary condition

$$(5.18) \quad \left(\tilde{\mathcal{H}}'_{\mathbf{u}^+} + \tilde{\mathcal{H}}'_{\mathbf{u}^-} \mathbf{u}'_\Gamma[\mathbf{u}_h] \right)^\top \mathbf{z}_h^+ = j'[\mathbf{u}_h] \quad \text{on } \Gamma$$

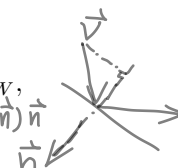
and the continuous adjoint boundary condition in (5.7), we notice that not all choices of $\tilde{\mathcal{H}}$ give rise to an adjoint consistent discretization. In fact, we require $\tilde{\mathcal{H}}$ to have the following properties: In order to incorporate boundary conditions in the primal discretization (5.9), $\tilde{\mathcal{H}}$ must depend on $\mathbf{u}_\Gamma(\mathbf{u}_h^+)$; hence, $\tilde{\mathcal{H}}'_{\mathbf{u}^-} \neq 0$. Furthermore, we require $\tilde{\mathcal{H}}'_{\mathbf{u}^+} = 0$ as otherwise the left-hand side in (5.18) involves two summands which is in contrast to the continuous adjoint boundary condition in (5.7). Finally, we recall that $\tilde{\mathcal{H}}$ is consistent $\tilde{\mathcal{H}}(\mathbf{v}, \mathbf{v}, \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{v})$ and conclude that $\tilde{\mathcal{H}}$ is given by $\tilde{\mathcal{H}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+))$. Employing a modified target functional $\tilde{J}(\mathbf{u}_h) = J(\mathbf{i}(\mathbf{u}_h))$, i.e., (2.25) with $r_j(\mathbf{u}_h) \equiv 0$, (5.18) yields

$$(5.19) \quad (\mathbf{n} \cdot (\mathcal{F}_{\mathbf{u}}^c[\mathbf{u}_\Gamma(\mathbf{u}_h^+)]) \mathbf{u}'_\Gamma[\mathbf{u}_h])^\top \mathbf{z} = j'[\mathbf{i}(\mathbf{u}_h^+)] \mathbf{i}'[\mathbf{u}_h^+].$$

We find the modification $\mathbf{i}(\mathbf{u}_h) = \mathbf{u}_\Gamma(\mathbf{u}_h)$, which is consistent as $\mathbf{i}(\mathbf{u}) = \mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{u}$ holds for the exact solution \mathbf{u} . Thereby (5.19) reduces to

$$(5.20) \quad (\mathbf{n} \cdot \mathcal{F}_{\mathbf{u}}^c[\mathbf{u}_\Gamma(\mathbf{u}_h^+)])^\top \mathbf{z} = j'[\mathbf{u}_\Gamma(\mathbf{u}_h^+)],$$

which represents a discretization of the continuous adjoint boundary condition in (5.7). In order to obtain a discretization of the adjoint boundary condition at solid wall boundaries (5.8), we require $B\mathbf{u}_\Gamma(\mathbf{u}_h^+) = 0$ on Γ_W . This condition is satisfied by

$$(5.21) \quad \mathbf{u}_\Gamma(\mathbf{u}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - n_1^2 & -n_1 n_2 & 0 \\ 0 & -n_1 n_2 & 1 - n_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{u} \quad \text{on } \Gamma_W,$$


which originates from \mathbf{u} by subtracting the normal velocity component of \mathbf{u} ; i.e., $\mathbf{v} = (v_1, v_2)$ is replaced by $\mathbf{v}_\Gamma = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ which ensures that the normal velocity component vanishes: $\mathbf{v}_\Gamma \cdot \mathbf{n} = 0$.

In summary, let \mathbf{u}_Γ be given by (5.21) and $\tilde{\mathcal{H}}$ and \tilde{J} be defined by

$$(5.22) \quad \tilde{\mathcal{H}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}_\Gamma^c(\mathbf{u}_h^+), \quad \tilde{J}(\mathbf{u}_h) = J_\Gamma(\mathbf{u}_h),$$

respectively, where $\mathcal{F}_\Gamma^c(\mathbf{u}_h^+) := \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+))$, $J_\Gamma(\mathbf{u}_h) := J(\mathbf{u}_\Gamma(\mathbf{u}_h))$, and $j_\Gamma(\mathbf{u}_h) := j(\mathbf{u}_\Gamma(\mathbf{u}_h))$; then the adjoint residuals (5.17) are given by

$$(5.23) \quad \begin{aligned} \mathbf{R}^*[\mathbf{u}_h](\mathbf{z}_h) &= (\mathcal{F}_{\mathbf{u}}^c[\mathbf{u}_h])^\top \nabla \mathbf{z}_h && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\ \mathbf{r}^*[\mathbf{u}_h](\mathbf{z}_h) &= -(\mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+))^\top \llbracket \mathbf{z}_h \rrbracket \cdot \mathbf{n} && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\ \mathbf{r}_\Gamma^*[\mathbf{u}_h](\mathbf{z}_h) &= j'_\Gamma[\mathbf{u}_h^+] - (\mathbf{n} \cdot \mathcal{F}_{\Gamma, \mathbf{u}}^c[\mathbf{u}_h^+])^\top \mathbf{z}_h^+ && \text{on } \Gamma. \end{aligned}$$

$$\mathbf{n} \cdot \mathcal{F}(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) = \begin{pmatrix} n_1 p(\mathbf{u}_\Gamma) \\ n_2 p(\mathbf{u}_\Gamma) \\ 0 \end{pmatrix}$$

In particular, the discretization (5.9) together with (5.22) is adjoint consistent as the exact solutions u and \mathbf{z} to (5.1) and (5.7), respectively, satisfy

$$\mathbf{R}^*[\mathbf{u}](\mathbf{z}) = 0 \quad \text{in } \kappa, \kappa \in \mathcal{T}_h, \quad \mathbf{r}^*[\mathbf{u}](\mathbf{z}) = 0 \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \quad \mathbf{r}_\Gamma^*[\mathbf{u}](\mathbf{z}) = 0 \quad \text{on } \Gamma.$$

Note that the adjoint residuals in (5.23) reduce to the adjoint residuals (3.5) of the linear advection equation with $b = 0$, when setting $\mathcal{F}^c(u) = \mathbf{b}u$ and $\mathcal{H}'_{u^+} = \mathbf{b} \cdot \mathbf{n}$.

Also note that the standard discontinuous Galerkin discretizations for the compressible Euler equations (see, e.g., [4, 15, 16] among several others) take the same numerical flux function on the boundary Γ as in the interior of the domain, and simply replace \mathbf{u}_h^- in $\mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n})$ by the boundary function $\mathbf{u}_\Gamma(\mathbf{u}_h^+)$, resulting in $\tilde{\mathcal{H}}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n})$. Furthermore, the definition of \mathbf{u}_Γ in [4, 15] based on $\mathbf{v}_\Gamma = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ ensures a vanishing average normal velocity $\bar{\mathbf{v}} \cdot \mathbf{n} = \frac{1}{2}(\mathbf{v} + \mathbf{v}_\Gamma) \cdot \mathbf{n} = 0$. However, $\mathbf{v}_\Gamma \cdot \mathbf{n} = 0$ and $B\mathbf{u}_\Gamma(\mathbf{u}_h^+) = 0$, as required in (5.20), are not satisfied. Thereby, the discontinuous Galerkin discretization based on the standard choice of \mathcal{H} and \mathbf{u}_Γ is not adjoint consistent. In fact, already the numerical experiments in [15] indicated large gradients, i.e., an irregular adjoint solution near solid wall boundaries. The lack of adjoint consistency of this standard approach was first analyzed by Lu [23, 24], who also proposed the adjoint consistent approach (5.22) and demonstrated that this approach gives rise to smooth adjoint solutions for an inviscid compressible flow over a Gaussian bump. The smoothness of the discrete adjoint has been confirmed in [14] for an inviscid compressible flow around a NACA0012 airfoil. Furthermore, [14] studies the effect of adjoint consistency on the accuracy of the flow solution and on error cancellation in an a posteriori error estimation approach.

6. The compressible Navier–Stokes equations. In this section we consider the two-dimensional stationary compressible Navier–Stokes equations

$$(6.1) \quad \nabla \cdot (\mathcal{F}^c(\mathbf{u}) - \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u})) = 0 \quad \text{in } \Omega,$$

subject to various boundary conditions, e.g., no-slip wall boundary conditions with vanishing velocity $\mathbf{v} = (v_1, v_2)^\top = 0$ at isothermal walls Γ_{iso} where $T = T_{wall}$ or at adiabatic walls Γ_{adia} where $\mathbf{n} \cdot \nabla T = 0$. The vector of conservative variables \mathbf{u} and the convective fluxes $\mathcal{F}^c(\mathbf{u})$ are as defined in (5.3). Furthermore, the viscous fluxes $\mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = (\mathbf{f}_1^v(\mathbf{u}, \nabla \mathbf{u}), \mathbf{f}_2^v(\mathbf{u}, \nabla \mathbf{u}))$ are defined by

$$(6.2) \quad \mathbf{f}_1^v(\mathbf{u}, \nabla \mathbf{u}) = \begin{bmatrix} 0 \\ \tau_{11} \\ \tau_{21} \\ \tau_{1j}v_j + \mathcal{K}T_{x_1} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_2^v(\mathbf{u}, \nabla \mathbf{u}) = \begin{bmatrix} 0 \\ \tau_{12} \\ \tau_{22} \\ \tau_{2j}v_j + \mathcal{K}T_{x_2} \end{bmatrix}.$$

Here T denotes the temperature given by $e = c_v T$, \mathcal{K} is the thermal conductivity coefficient, and τ is the viscous stress tensor defined by $\tau = \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top) - \frac{2}{3}(\nabla \cdot \mathbf{v})I$, where μ is the dynamic viscosity coefficient. Using the homogeneity tensor G (e.g., [16]) with $G_{ij}(\mathbf{u}) = \partial \mathbf{f}_i^v(\mathbf{u}, \nabla \mathbf{u}) / \partial u_{x_j}$, for $i, j = 1, 2$, the viscous fluxes are $\mathbf{f}_i^v(\mathbf{u}, \nabla \mathbf{u}) = G_{ij}(\mathbf{u}) \partial \mathbf{u} / \partial x_j$, $i = 1, 2$, and $\mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = G(\mathbf{u}) \nabla \mathbf{u}$. Consider the target functional

$$(6.3) \quad J(\mathbf{u}) = \int_\Gamma j(C\mathbf{u}) \, ds = \int_{\Gamma_w} (p\mathbf{n} - \tau \cdot \mathbf{n}) \cdot \boldsymbol{\psi}_{\Gamma_w} \, ds,$$

where $(C\mathbf{u})_{ij} = p(\mathbf{u})\delta_{ij} - \tau_{ij}(\mathbf{u}, \nabla\mathbf{u})$, $j(C\mathbf{u}) = (C\mathbf{u})_{ij}n_j(\psi_{\Gamma_W})_i$ on Γ_W , and $j(C\mathbf{u}) \equiv 0$ elsewhere, with $\psi_{\Gamma_W} \in [L^2(\Gamma_W)]^2$. Important target quantities of type (6.3) in viscous compressible flows are the (total) drag and lift coefficients c_d and c_l , which include both pressure-induced and viscous forces. Then $\psi_{\Gamma_W} = \frac{1}{C_\infty}\psi$, and C_∞ and ψ are as in (5.4).

In order to derive the adjoint problem, we multiply the left-hand side of (6.1) by \mathbf{z} , integrate by parts, and linearize about \mathbf{u} to obtain

$$\begin{aligned} & (\nabla \cdot (\mathcal{F}_{\mathbf{u}}^c \mathbf{w} - \mathcal{F}_{\mathbf{u}}^v \mathbf{w} - \mathcal{F}_{\nabla\mathbf{u}}^v \nabla \mathbf{w}), \mathbf{z})_\Omega \\ &= -((\mathcal{F}_{\mathbf{u}}^c - \mathcal{F}_{\mathbf{u}}^v) \mathbf{w} - \mathcal{F}_{\nabla\mathbf{u}}^v \nabla \mathbf{w}, \nabla \mathbf{z})_\Omega + (\mathbf{n} \cdot (\mathcal{F}_{\mathbf{u}}^c \mathbf{w} - \mathcal{F}_{\mathbf{u}}^v \mathbf{w} - \mathcal{F}_{\nabla\mathbf{u}}^v \nabla \mathbf{w}), \mathbf{z})_\Gamma, \end{aligned}$$

where $\mathcal{F}_{\mathbf{u}}^v := \partial_{\mathbf{u}} \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u}) = G'[\mathbf{u}] \nabla\mathbf{u}$ and $\mathcal{F}_{\nabla\mathbf{u}}^v := \partial_{\nabla\mathbf{u}} \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u}) = G(\mathbf{u})$ denote the derivatives of \mathcal{F}^v with respect to \mathbf{u} and $\nabla\mathbf{u}$, respectively. Using integration by parts once more we obtain the following variational formulation of the continuous adjoint problem: Find \mathbf{z} such that

$$\begin{aligned} & -\left(\mathbf{w}, (\mathcal{F}_{\mathbf{u}}^c - \mathcal{F}_{\mathbf{u}}^v)^\top \nabla \mathbf{z}\right)_\Omega - \left(\mathbf{w}, \nabla \cdot \left((\mathcal{F}_{\nabla\mathbf{u}}^v)^\top \nabla \mathbf{z}\right)\right)_\Omega + \left(\mathbf{w}, \mathbf{n} \cdot \left((\mathcal{F}_{\nabla\mathbf{u}}^v)^\top \nabla \mathbf{z}\right)\right)_\Gamma \\ & + \left(\mathbf{w}, (\mathbf{n} \cdot (\mathcal{F}_{\mathbf{u}}^c - \mathcal{F}_{\mathbf{u}}^v))^\top \mathbf{z}\right)_\Gamma - \left(\nabla \mathbf{w}, (\mathbf{n} \cdot \mathcal{F}_{\nabla\mathbf{u}}^v)^\top \mathbf{z}\right)_\Gamma = J'[\mathbf{u}](\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}. \end{aligned}$$

Given that

$$\begin{aligned} J'[\mathbf{u}](\mathbf{w}) &= \frac{1}{C_\infty} \int_{\Gamma_W} (p_{\mathbf{u}}[\mathbf{u}] \mathbf{n} - \mathcal{I}_{\mathbf{u}}[\mathbf{u}] \mathbf{n}) \cdot \psi \mathbf{w} - (\mathcal{I}_{\nabla\mathbf{u}}[\mathbf{u}] \mathbf{n}) \cdot \psi \nabla \mathbf{w} \, ds \\ (6.4) \quad &= \left(\mathbf{w}, \frac{1}{C_\infty} (p_{\mathbf{u}} \mathbf{n} - \mathcal{I}_{\mathbf{u}} \mathbf{n}) \cdot \psi\right)_{\Gamma_W} - \left(\nabla \mathbf{w}, \frac{1}{C_\infty} (\mathcal{I}_{\nabla\mathbf{u}} \mathbf{n}) \cdot \psi\right)_{\Gamma_W}, \end{aligned}$$

we see that the adjoint solution \mathbf{z} satisfies the following equation:

$$(6.5) \quad -(\mathcal{F}_{\mathbf{u}}^c - \mathcal{F}_{\mathbf{u}}^v)^\top \nabla \mathbf{z} - \nabla \cdot \left((\mathcal{F}_{\nabla\mathbf{u}}^v)^\top \nabla \mathbf{z}\right) = 0,$$

subject to the boundary conditions on $\Gamma_W = \Gamma_{iso} \cup \Gamma_{adia}$,

$$(6.6) \quad (\mathbf{n} \cdot (\mathcal{F}_{\mathbf{u}}^c - \mathcal{F}_{\mathbf{u}}^v))^\top \mathbf{z} + \mathbf{n} \cdot \left((\mathcal{F}_{\nabla\mathbf{u}}^v)^\top \nabla \mathbf{z}\right) = \frac{1}{C_\infty} (p_{\mathbf{u}} \mathbf{n} - \mathcal{I}_{\mathbf{u}} \mathbf{n}) \cdot \psi,$$

$$(6.7) \quad (\mathbf{n} \cdot \mathcal{F}_{\nabla\mathbf{u}}^v)^\top \mathbf{z} = \frac{1}{C_\infty} (\mathcal{I}_{\nabla\mathbf{u}} \mathbf{n}) \cdot \psi.$$

At wall boundaries Γ_W where $\mathbf{v} = (v_1, v_2)^\top = 0$, the normal viscous flux reduces to $\mathbf{n} \cdot \mathcal{F}^v(\mathbf{u}, \nabla\mathbf{u}) = (0, (\tau\mathbf{n})_1, (\tau\mathbf{n})_2, \mathbf{n} \cdot \nabla T)^\top$. Hence (6.7) is fulfilled, provided \mathbf{z} satisfies

$$(6.8) \quad \begin{pmatrix} 0 \\ (\tau_{\nabla\mathbf{u}}\mathbf{n})_1 z_2 \\ (\tau_{\nabla\mathbf{u}}\mathbf{n})_2 z_3 \\ (\mathbf{n} \cdot \nabla T_{\nabla\mathbf{u}}) z_4 \end{pmatrix} = \frac{1}{C_\infty} \begin{pmatrix} 0 \\ (\tau_{\nabla\mathbf{u}}\mathbf{n})_1 \psi_1 \\ (\tau_{\nabla\mathbf{u}}\mathbf{n})_2 \psi_2 \\ 0 \end{pmatrix},$$

which reduces to the conditions $z_2 = \frac{1}{C_\infty} \psi_1$ on Γ_W , $z_3 = \frac{1}{C_\infty} \psi_2$ on Γ_W , and $z_4 = 0$ on Γ_{iso} . At adiabatic boundaries we have $\mathbf{n} \cdot \nabla T = 0$, and the last condition in (6.8) vanishes. Substituted into (6.6) we obtain $\mathbf{n} \cdot ((\mathcal{F}_{\nabla\mathbf{u}}^v)^\top \nabla \mathbf{z}) = 0$ on Γ_W , which at adiabatic boundaries reduces to $\mathbf{n} \cdot \nabla z_4 = 0$. On isothermal boundaries no additional

boundary condition is obtained. In summary, the boundary conditions of the adjoint problem (6.5) to the compressible Navier–Stokes equations are given by

$$(6.9) \quad z_2 = \frac{1}{C_\infty} \psi_1, \quad z_3 = \frac{1}{C_\infty} \psi_2 \quad \text{on } \Gamma_W, \quad z_4 = 0 \quad \text{on } \Gamma_{iso}, \quad \mathbf{n} \cdot \nabla z_4 = 0 \quad \text{on } \Gamma_{adia}.$$

These boundary conditions have been derived by computing the adjoint equations of each of the four primal equations separately in [21] and [8].

In addition to the notation introduced in section 3, we use the standard notation $\underline{\sigma} : \underline{\tau} = \sum_{k=1}^m \sum_{l=1}^n \sigma_{kl} \tau_{kl}$ for matrices $\underline{\sigma}, \underline{\tau} \in \mathbb{R}^{m \times n}$, $m, n \geq 1$; additionally, for vectors $\mathbf{v} \in \mathbb{R}^m$, $\mathbf{w} \in \mathbb{R}^n$, the matrix $\mathbf{v} \otimes \mathbf{w} \in \mathbb{R}^{m \times n}$ is defined by $(\mathbf{v} \otimes \mathbf{w})_{kl} = v_k w_l$.

According to [16, 17] the interior penalty discontinuous Galerkin discretization of the compressible Navier–Stokes equations (6.1) is given by: Find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$(6.10) \quad \begin{aligned} \mathcal{N}(\mathbf{u}_h, \mathbf{v}) \equiv & - \int_{\Omega} \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v} \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial \kappa \setminus \Gamma} \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) \cdot \mathbf{v}^+ \, ds \\ & + \int_{\Omega} \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h) : \nabla_h \mathbf{v} \, d\mathbf{x} - \int_{\Gamma_{\mathcal{T}}} \{ \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h) \} : \llbracket \mathbf{v} \rrbracket \, ds \\ & + \int_{\Gamma_{\mathcal{T}}} \theta \{ G^\top(\mathbf{u}_h) \nabla_h \mathbf{v} \} : \llbracket \mathbf{u}_h \rrbracket \, ds + \int_{\Gamma_{\mathcal{T}}} \delta \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v} \rrbracket \, ds + \mathcal{N}_\Gamma(\mathbf{u}_h, \mathbf{v}) = 0 \end{aligned}$$

for all \mathbf{v} in \mathbf{V}_h^p . Here $\mathcal{N}_\Gamma(\mathbf{u}_h, \mathbf{v})$ includes all boundary terms which will be specified in the following. Recalling the discussion at the end of section 5, we know that the discretization of boundary terms in [16] is not adjoint consistent. In fact, [16] uses the standard discretization of convective boundary fluxes as opposed to the adjoint consistent discretization given in (5.22). Thereby, in the following we consider the boundary terms like in [16] but with an adjoint consistent treatment of convective fluxes like in (5.22). Furthermore, we treat the viscous fluxes analogous to the convective fluxes; i.e., we replace the viscous boundary flux \mathcal{F}^v by \mathcal{F}_Γ^v , where

$$(6.11) \quad \mathcal{F}_\Gamma^v(\mathbf{u}_h, \nabla \mathbf{u}_h) = \mathcal{F}^v(\mathbf{u}_\Gamma(\mathbf{u}_h), \nabla \mathbf{u}_h) = G_\Gamma(\mathbf{u}_h) \nabla \mathbf{u}_h = G(\mathbf{u}_\Gamma(\mathbf{u}_h)) \nabla \mathbf{u}_h.$$

Thereby, the discretization of boundary terms is given by

$$(6.12) \quad \begin{aligned} \mathcal{N}_\Gamma(\mathbf{u}_h, \mathbf{v}) = & \int_{\Gamma} \mathbf{n} \cdot \mathcal{F}_\Gamma^c(\mathbf{u}_h^+) \mathbf{v}^+ \, ds + \int_{\Gamma} \delta (\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{v}^+ \, ds \\ & - \int_{\Gamma} \mathbf{n} \cdot \mathcal{F}_\Gamma^v(\mathbf{u}_h^+, \nabla \mathbf{u}_h^+) \mathbf{v}^+ \, ds \\ & + \theta \int_{\Gamma} (G_\Gamma^\top(\mathbf{u}_h^+) \nabla \mathbf{v}_h^+) : (\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)) \otimes \mathbf{n} \, ds, \end{aligned}$$

where on adiabatic boundaries $\Gamma_{adia} \subset \Gamma_W$ the viscous flux \mathcal{F}_Γ^v and the corresponding homogeneity tensor G_Γ are modified such that $\mathbf{n} \cdot \nabla T = 0$. Using integration by parts in (6.10), we obtain the primal residual form as follows: Find $\mathbf{u}_h \in \mathbf{V}_h^p$ such that

$$(6.13) \quad \begin{aligned} \int_{\Omega} \mathbf{R}(\mathbf{u}_h) \cdot \mathbf{v} \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial \kappa \setminus \Gamma} \mathbf{r}(\mathbf{u}_h) \cdot \mathbf{v}^+ + \boldsymbol{\rho}(\mathbf{u}_h) : \nabla \mathbf{v}^+ \, ds \\ + \int_{\Gamma} \mathbf{r}_\Gamma(\mathbf{u}_h) \cdot \mathbf{v}^+ + \boldsymbol{\rho}_\Gamma(\mathbf{u}_h) : \nabla \mathbf{v}^+ \, ds = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^p, \end{aligned}$$

where the primal residuals are given by

$$\begin{aligned}
\mathbf{R}(\mathbf{u}_h) &= -\nabla \cdot \mathcal{F}^c(\mathbf{u}_h) + \nabla \cdot \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h) && \text{in } \kappa, \kappa \in \mathcal{T}_h, \\
\mathbf{r}(\mathbf{u}_h) &= \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_h^+) - \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) - \frac{1}{2} \llbracket \mathcal{F}^v(\mathbf{u}_h, \nabla_h \mathbf{u}_h) \rrbracket - \delta \llbracket \mathbf{u}_h \rrbracket \cdot \mathbf{n}, \\
\rho(\mathbf{u}_h) &= -\frac{\theta}{2} \left(G(\mathbf{u}_h) \llbracket \mathbf{u}_h \rrbracket \right)^\top && \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\
\mathbf{r}_\Gamma(\mathbf{u}_h) &= \mathbf{n} \cdot (\mathcal{F}^c(\mathbf{u}_h^+) - \mathcal{F}_\Gamma^c(\mathbf{u}_h^+) - \mathcal{F}^v(\mathbf{u}_h^+, \nabla_h \mathbf{u}_h^+) + \mathcal{F}_\Gamma^v(\mathbf{u}_h^+, \nabla_h \mathbf{u}_h^+)) - \delta(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)), \\
\rho_\Gamma(\mathbf{u}_h) &= -\theta \left(G_\Gamma^\top(\mathbf{u}_h^+) : (\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)) \otimes \mathbf{n} \right)^\top && \text{on } \Gamma.
\end{aligned}$$

We see that the exact solution \mathbf{u} to (6.1) satisfies

$$\mathbf{R}(\mathbf{u}) = 0, \quad \mathbf{r}(\mathbf{u}) = 0, \quad \rho(\mathbf{u}) = 0, \quad \mathbf{r}_\Gamma(\mathbf{u}) = 0, \quad \rho_\Gamma(\mathbf{u}) = 0,$$

where we used consistency of the numerical flux $\mathcal{H}(\mathbf{w}, \mathbf{w}, \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{w})$, continuity of \mathbf{u} , and the consistency of the boundary function, i.e., \mathbf{u} satisfies $\mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{u}$ on Γ . We conclude that the discretization given in (6.10) and (6.12) is consistent.

Given the target quantity $J(\cdot)$ defined in (6.3) with Fréchet derivative (6.4), we consider the following modification of $J(\cdot)$:

$$(6.14) \quad \tilde{J}(\mathbf{u}_h) = J(\mathbf{i}(\mathbf{u}_h)) + \int_\Gamma r_J(\mathbf{u}_h) \, ds = J_\Gamma(\mathbf{u}_h) + \int_\Gamma r_J(\mathbf{u}_h) \, ds.$$

As in section 5, here we set $\mathbf{i}(\mathbf{u}_h) = \mathbf{u}_\Gamma(\mathbf{u}_h)$ and $J_\Gamma(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h))$; $r_J(\mathbf{u}_h)$ will be specified later. Noting that $\mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{u}$ holds for the exact solution \mathbf{u} , $\tilde{J}(\cdot)$ in (6.14) is a consistent modification of $J(\cdot)$, provided that \mathbf{u} satisfies $r_J(\mathbf{u}) = 0$; see also (2.25).

Rewriting $\mathcal{N}(\mathbf{u}_h, \mathbf{v})$ in (6.10) in terms of the homogeneity tensor G and recalling (5.15), we see that the discrete adjoint problem is given by: Find $\mathbf{z}_h \in \mathbf{V}_h$ such that

$$(6.15) \quad \mathcal{N}'[\mathbf{u}_h](\mathbf{w}, \mathbf{z}_h) = \tilde{J}'[\mathbf{u}_h](\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V},$$

where $\mathcal{N}'[\mathbf{u}](\mathbf{w}, \mathbf{z})$ is given by

$$\begin{aligned}
\mathcal{N}'[\mathbf{u}](\mathbf{w}, \mathbf{z}) &= - \int_\Omega (\mathcal{F}_\mathbf{u}^c[\mathbf{u}]\mathbf{w}) : \nabla_h \mathbf{z} \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}^+, \mathbf{u}^-, \mathbf{n}^+) \mathbf{w}^+ \llbracket \mathbf{z} \rrbracket \cdot \mathbf{n} \, ds \\
&\quad + \int_\Omega (G'[\mathbf{u}]\mathbf{w} \nabla_h \mathbf{u}) : \nabla_h \mathbf{z} \, d\mathbf{x} + \int_\Omega (G(\mathbf{u}) \nabla_h \mathbf{w}) : \nabla_h \mathbf{z} \, d\mathbf{x} \\
&\quad - \int_{\Gamma_\mathcal{T}} \{G'[\mathbf{u}]\mathbf{w} \nabla_h \mathbf{u}\} : \llbracket \mathbf{z} \rrbracket \, ds - \int_{\Gamma_\mathcal{T}} \{G(\mathbf{u}) \nabla_h \mathbf{w}\} : \llbracket \mathbf{z} \rrbracket \, ds \\
&\quad + \int_{\Gamma_\mathcal{T}} \theta \{ (G^\top)'[\mathbf{u}]\mathbf{w} \nabla_h \mathbf{z} \} : \llbracket \mathbf{u} \rrbracket \, ds + \int_{\Gamma_\mathcal{T}} \theta \{ G^\top(\mathbf{u}) \nabla_h \mathbf{z} \} : \llbracket \mathbf{w} \rrbracket \, ds \\
(6.16) \quad &\quad + \int_{\Gamma_\mathcal{T}} \delta \llbracket \mathbf{w} \rrbracket : \llbracket \mathbf{z} \rrbracket \, ds + \mathcal{N}'_\Gamma[\mathbf{u}](\mathbf{w}, \mathbf{z}).
\end{aligned}$$

Using integration by parts, this can be rewritten as follows:

$$\begin{aligned}
 & - \int_{\Omega} \mathbf{w} (\mathcal{F}_{\mathbf{u}}^c[\mathbf{u}])^\top \nabla_h \mathbf{z} \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathbf{w}^+ (\mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}^+, \mathbf{u}^-, \mathbf{n}^+))^\top \llbracket \mathbf{z} \rrbracket \cdot \mathbf{n} \, ds \\
 & + \int_{\Omega} \mathbf{w} (G'[\mathbf{u}] \nabla_h \mathbf{u})^\top \nabla_h \mathbf{z} \, d\mathbf{x} - \int_{\Omega} \mathbf{w} \nabla_h \cdot (G^\top(\mathbf{u}) \nabla_h \mathbf{z}) \, d\mathbf{x} \\
 & - \frac{1}{2} \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} (G'[\mathbf{u}] \mathbf{w} \nabla_h \mathbf{u}) : \llbracket \mathbf{z} \rrbracket \, ds - \frac{1}{2} \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} (G(\mathbf{u}) \nabla_h \mathbf{w}) : \llbracket \mathbf{z} \rrbracket \, ds \\
 & + \frac{1}{2} \theta \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} ((G^\top)'[\mathbf{u}] \mathbf{w} \nabla_h \mathbf{z}) : \llbracket \mathbf{u} \rrbracket \, ds + \frac{1}{2} \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathbf{w} [G^\top(\mathbf{u}) \nabla_h \mathbf{z}] \, ds \\
 & + (1 + \theta) \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} (\mathbf{w} \otimes \mathbf{n}) : \{G^\top(\mathbf{u}) \nabla_h \mathbf{z}\} \, ds + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \delta \mathbf{w} \llbracket \mathbf{z} \rrbracket \cdot \mathbf{n} \, ds \\
 (6.17) \quad & + \int_{\Gamma} (\mathbf{w} \otimes \mathbf{n}) : (G^\top(\mathbf{u}) \nabla_h \mathbf{z}) \, ds + \mathcal{N}'_{\Gamma}[\mathbf{u}](\mathbf{w}, \mathbf{z}).
 \end{aligned}$$

Hence the discrete adjoint problem (6.15) in adjoint residual form is given as follows: Find $\mathbf{z}_h \in \mathbf{V}_h$ such that

$$\begin{aligned}
 & \int_{\Omega} \mathbf{w} \cdot \mathbf{R}^*[\mathbf{u}_h](\mathbf{z}_h) \, d\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} \mathbf{w} \cdot \mathbf{r}^*[\mathbf{u}_h](\mathbf{z}_h) + \nabla \mathbf{w} : \boldsymbol{\rho}^*[\mathbf{u}_h](\mathbf{z}_h) \, ds \\
 (6.18) \quad & + \int_{\Gamma} \mathbf{w} \cdot \mathbf{r}_{\Gamma}^*[\mathbf{u}_h](\mathbf{z}_h) + \nabla \mathbf{w} : \boldsymbol{\rho}_{\Gamma}^*[\mathbf{u}_h](\mathbf{z}_h) \, ds = 0 \quad \forall \mathbf{w} \in \mathbf{V}_h,
 \end{aligned}$$

where the adjoint residuals are given by

$$\begin{aligned}
 \mathbf{R}^*[\mathbf{u}_h](\mathbf{z}_h) &= (\mathcal{F}_{\mathbf{u}}^c(\mathbf{u}_h) - G'[\mathbf{u}_h] \nabla \mathbf{u}_h)^\top \nabla_h \mathbf{z}_h + \nabla_h \cdot (G^\top(\mathbf{u}_h) \nabla_h \mathbf{z}_h) \quad \text{in } \kappa, \kappa \in \mathcal{T}_h, \\
 \mathbf{r}^*[\mathbf{u}_h](\mathbf{z}_h) &= -(\mathcal{H}'_{\mathbf{u}^+}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+))^\top \llbracket \mathbf{z}_h \rrbracket \cdot \mathbf{n} \\
 (6.19) \quad & - \frac{1}{2} \llbracket G^\top(\mathbf{u}_h) \nabla_h \mathbf{z}_h \rrbracket - (1 + \theta) \mathbf{n} \cdot \{G^\top(\mathbf{u}_h) \nabla_h \mathbf{z}_h\} - \delta \llbracket \mathbf{z}_h \rrbracket \cdot \mathbf{n} \\
 & + \frac{1}{2} (G'[\mathbf{u}_h] \nabla \mathbf{u}_h)^\top \llbracket \mathbf{z}_h \rrbracket - \frac{1}{2} \theta \left(G'[\mathbf{u}_h] \llbracket \mathbf{u}_h \rrbracket \right)^\top \nabla_h \mathbf{z}_h \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \\
 \boldsymbol{\rho}^*[\mathbf{u}_h](\mathbf{z}_h) &= \frac{1}{2} G^\top[\mathbf{u}_h] \llbracket \mathbf{z}_h \rrbracket \quad \text{on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h.
 \end{aligned}$$

The adjoint boundary residuals \mathbf{r}_{Γ}^* and $\boldsymbol{\rho}_{\Gamma}^*$ will be specified below. Recalling that $\mathcal{F}_{\mathbf{u}}^v = G'[\mathbf{u}] \nabla \mathbf{u}$ and $\mathcal{F}_{\nabla \mathbf{u}}^v = G(\mathbf{u})$, we see that the exact solution \mathbf{z} to the continuous adjoint problem (6.5) satisfies $\mathbf{R}^*[\mathbf{u}](\mathbf{z}) = 0$. In the three lines in (6.19) representing the face residual term $\mathbf{r}^*[\mathbf{u}_h](\mathbf{z}_h)$, we recognize the jump $-(\mathcal{H}'_{\mathbf{u}^+})^\top \llbracket \mathbf{z}_h \rrbracket \cdot \mathbf{n}$ due to the convective part of the equations (cf. (5.23)); furthermore, the terms in the second line correspond to the adjoint face residuals of Poisson's equation (cf. (4.11)); and finally the two terms in the third line are due to the nonlinearity of the compressible Navier–Stokes equations. Whereas the last term in the third line vanishes for a smooth exact primal solution \mathbf{u} , all other terms vanish for the exact solution \mathbf{z} to the adjoint problem (6.5), provided $\theta = -1$. Thereby, the adjoint solution \mathbf{z} satisfies $\mathbf{r}^*[\mathbf{u}](\mathbf{z}) = 0$, provided that $\theta = -1$. Furthermore, \mathbf{z} satisfies $\boldsymbol{\rho}^*[\mathbf{u}](\mathbf{z}) = 0$. In summary, we see that, like for Poisson's equation, the element and interior face terms of the IP discretization of the compressible Navier–Stokes equation are adjoint consistent for the symmetric ($\theta = -1$) but not for the nonsymmetric ($\theta = 1$) version.

The boundary terms of the discrete adjoint problem are given by

$$\begin{aligned} \mathcal{N}'_{\Gamma}[\mathbf{u}_h](\mathbf{w}, \mathbf{z}_h) + \int_{\Gamma} (\mathbf{w} \otimes \mathbf{n}) : (G_{\Gamma}^{\top}(\mathbf{u}_h) \nabla_h \mathbf{z}_h) \, ds \equiv \\ + \int_{\Gamma} \mathbf{n} \cdot (\mathcal{F}_{\Gamma, \mathbf{u}}^c[\mathbf{u}_h](\mathbf{w})) \, \mathbf{z}_h \, ds + \int_{\Gamma} \delta (\mathbf{w} - \mathbf{u}'_{\Gamma}[\mathbf{u}_h] \mathbf{w}) \cdot \mathbf{z} \, ds, \\ - \int_{\Gamma} \mathbf{n} \cdot (\mathcal{F}_{\Gamma, \mathbf{u}}^v[\mathbf{u}_h, \nabla_h \mathbf{u}_h](\mathbf{w}) + \mathcal{F}_{\Gamma, \nabla \mathbf{u}}^v[\mathbf{u}_h, \nabla_h \mathbf{u}_h](\nabla_h \mathbf{w})) \, \mathbf{z}_h \, ds \\ + \theta \int_{\Gamma} \left(((G_{\Gamma}^{\top})'[\mathbf{u}_h] \mathbf{w}) \nabla_h \mathbf{z}_h \right) : (\mathbf{u}_h - \mathbf{u}_{\Gamma}(\mathbf{u}_h)) \otimes \mathbf{n} \, ds \\ + \theta \int_{\Gamma} (G_{\Gamma}^{\top}(\mathbf{u}_h) \nabla_h \mathbf{z}_h) : (\mathbf{w} - \mathbf{u}'_{\Gamma}[\mathbf{u}_h] \mathbf{w}) \otimes \mathbf{n} \, ds \\ + \int_{\Gamma} (\mathbf{w} \otimes \mathbf{n}) : (G_{\Gamma}^{\top}(\mathbf{u}_h) \nabla_h \mathbf{z}_h) \, ds = \tilde{J}'[\mathbf{u}_h](\mathbf{w}). \end{aligned}$$

Thus the adjoint boundary residuals in (6.18) on Γ_W are given by

$$\begin{aligned} \mathbf{r}_{\Gamma}^*[\mathbf{u}_h](\mathbf{z}_h) = \frac{1}{C_{\infty}} (p_{\mathbf{u}} \mathbf{n} - \mathcal{I}_{\mathbf{u}} \mathbf{n}) \cdot \psi - (\mathbf{n} \cdot (\mathcal{F}_{\Gamma, \mathbf{u}}^c - \mathcal{F}_{\Gamma, \mathbf{u}}^v))^{\top} \mathbf{z}_h - \mathbf{n} \cdot (G_{\Gamma}^{\top} \nabla \mathbf{z}_h) \\ + r'_J[\mathbf{u}_h] - \delta (I - \mathbf{u}'_{\Gamma}[\mathbf{u}_h])^{\top} \mathbf{z}_h - \theta (G'_{\Gamma}[\mathbf{u}_h] : (\mathbf{u}_h - \mathbf{u}_{\Gamma}(\mathbf{u}_h)) \otimes \mathbf{n})^{\top} \nabla_h \mathbf{z}_h \\ (6.20) \quad - \theta (G_{\Gamma}(\mathbf{u}_h) : (I - \mathbf{u}'_{\Gamma}[\mathbf{u}_h]) \otimes \mathbf{n})^{\top} \nabla_h \mathbf{z}_h, \end{aligned}$$

$$\begin{aligned} \rho_{\Gamma}^*[\mathbf{u}_h](\mathbf{z}_h) = -\frac{1}{C_{\infty}} (\mathcal{I}_{\nabla \mathbf{u}} \mathbf{n}) \cdot \psi + (\mathbf{n} \cdot \mathcal{F}_{\Gamma, \nabla \mathbf{u}}^v)^{\top} \mathbf{z}_h. \\ (6.21) \end{aligned}$$

We recall (6.7), $\mathcal{F}_{\Gamma, \nabla \mathbf{u}}^v = G_{\Gamma}(\mathbf{u})$, and see that the exact solutions \mathbf{u} and \mathbf{z} to the primal problem (6.1) and the continuous adjoint problem (6.5)–(6.9) satisfy $\rho_{\Gamma}^*[\mathbf{u}](\mathbf{z}) = 0$.

We now choose the modification $r_J(\mathbf{u}_h)$ of the target functional in (6.14) as follows:

$$(6.22) \quad r_J(\mathbf{u}_h) = \delta (\mathbf{u}_h^+ - \mathbf{u}_{\Gamma}(\mathbf{u}_h^+)) \cdot \mathbf{z}_{\Gamma} + \theta (G_{\Gamma}^{\top}(\mathbf{u}_h^+) \nabla_h \mathbf{z}_{\Gamma}) : (\mathbf{u}_h^+ - \mathbf{u}_{\Gamma}(\mathbf{u}_h^+)) \otimes \mathbf{n},$$

with Fréchet derivative

$$\begin{aligned} r'_J[\mathbf{u}_h](\mathbf{w}) = \delta (I - \mathbf{u}'_{\Gamma}[\mathbf{u}_h]) \mathbf{w} \cdot \mathbf{z}_{\Gamma} + \theta (G'_{\Gamma}[\mathbf{u}_h] : (\mathbf{u} - \mathbf{u}_{\Gamma}(\mathbf{u}_h)) \otimes \mathbf{n})^{\top} \nabla_h \mathbf{z}_{\Gamma} \\ + \theta (G_{\Gamma}(\mathbf{u}_h) : (I - \mathbf{u}'_{\Gamma}[\mathbf{u}_h]) \otimes \mathbf{n})^{\top} \nabla_h \mathbf{z}_{\Gamma}. \end{aligned}$$

As the exact solution \mathbf{u} to the primal problem satisfies $\mathbf{u}_{\Gamma}(\mathbf{u}) = \mathbf{u}$, we have $r_J(\mathbf{u}) = 0$. Hence (6.22) is a consistent modification of the target functional. Recalling (6.6) we see that the exact solutions \mathbf{u} and \mathbf{z} satisfy

$$\begin{aligned} \mathbf{r}_{\Gamma}^*[\mathbf{u}](\mathbf{z}) = \delta (I - \mathbf{u}'_{\Gamma}[\mathbf{u}])^{\top} (\mathbf{z}_{\Gamma} - \mathbf{z}) + \theta (G'_{\Gamma}[\mathbf{u}] : (\mathbf{u} - \mathbf{u}_{\Gamma}(\mathbf{u})) \otimes \mathbf{n})^{\top} (\nabla \mathbf{z}_{\Gamma} - \nabla \mathbf{z}) \\ (6.23) \quad + \theta (G_{\Gamma}(\mathbf{u}) : (I - \mathbf{u}'_{\Gamma}[\mathbf{u}]) \otimes \mathbf{n})^{\top} (\nabla \mathbf{z}_{\Gamma} - \nabla \mathbf{z}). \end{aligned}$$

Furthermore, setting $\mathbf{z}_{\Gamma} = \mathbf{z}$ on Γ_W we obtain $\mathbf{r}_{\Gamma}^*[\mathbf{u}](\mathbf{z}) = 0$ and conclude that the discretization of boundary terms is adjoint consistent.

Due to $\mathbf{n} \cdot (G_{\Gamma}^{\top}(\mathbf{u}_h^+) \nabla \mathbf{z}) = \mathbf{n} \cdot ((\mathcal{F}_{\nabla \mathbf{u}}^v)^{\top} \nabla \mathbf{z}) = 0$ on Γ_W the second term in (6.22) vanishes. Furthermore, on adiabatic boundaries Γ_{adia} we have $(\mathbf{u}_h^+ - \mathbf{u}_{\Gamma}(\mathbf{u}_h^+))_i = 0$,

$i = 1, 4$, and on isothermal boundaries Γ_{iso} we have $(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+))_1 = 0$. Together with (6.9) the consistent modification (6.22) reduces to

$$\begin{aligned} r_J(\mathbf{u}_h) &= \delta(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{z}_\Gamma \\ (6.24) \quad &= \delta(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+))_2 \frac{1}{C_\infty} \psi_1 + \delta(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+))_3 \frac{1}{C_\infty} \psi_2, \end{aligned}$$

which completes the adjoint consistency analysis of the interior penalty discontinuous Galerkin discretization of the compressible Navier–Stokes equations. Finally, we note that the consistent modification $r_J(\mathbf{u}_h)$ given in (6.24) corresponds to the IP modification of target functionals for Poisson’s equation where $r_J(u_h) = \delta(u_h - g_D)z_\Gamma$ with $z_\Gamma = -j_D$; see (4.13).

In summary, we have shown that the adjoint element and interior residuals $\mathbf{R}^*[\mathbf{u}_h](\mathbf{z}_h)$, $\mathbf{r}^*[\mathbf{u}_h](\mathbf{z}_h)$, and $\boldsymbol{\rho}^*[\mathbf{u}_h](\mathbf{z}_h)$ (see (6.19)) vanish for the exact solutions \mathbf{u} and \mathbf{z} to (6.1) and (6.5), respectively, provided $\theta = -1$. Additionally, using an adjoint consistent treatment of convective and diffusive boundary fluxes,

$$(6.25) \quad \mathbf{n} \cdot \mathcal{F}_\Gamma^c(\mathbf{u}_h^+) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)), \quad \mathbf{n} \cdot \mathcal{F}_\Gamma^v(\mathbf{u}_h^+) = \mathbf{n} \cdot \mathcal{F}^v(\mathbf{u}_\Gamma(\mathbf{u}_h^+), \nabla_h \mathbf{u}_h^+),$$

and using the following consistent modification of the target functional:

$$(6.26) \quad \tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h)) + \int_{\Gamma_W} \delta(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{z}_\Gamma \, ds,$$

with $\mathbf{z}_\Gamma = \frac{1}{C_\infty}(0, \psi_1, \psi_2, 0)^\top$, for $J(\cdot)$ representing a total force coefficient defined in (6.3), the adjoint boundary residuals $\mathbf{r}_\Gamma^*[\mathbf{u}_h](\mathbf{z}_h)$ and $\boldsymbol{\rho}_\Gamma^*[\mathbf{u}_h](\mathbf{z}_h)$ (see (6.20) and (6.21)) vanish for the exact solutions \mathbf{u} and \mathbf{z} . Thereby, using the modifications given in (6.25) and (6.26) we recover an adjoint consistent symmetric interior penalty discontinuous Galerkin discretization of the compressible Navier–Stokes equations in conjunction with total force coefficients. Finally, we note that arguments given in [23] en route to obtaining an adjoint consistent discretization based on the BR2 scheme [5] can also be covered within the presented framework and lead to analogous modifications.

7. Numerical experiments for the compressible Navier–Stokes equations. In this section we will demonstrate the effect on the smoothness of the discrete adjoint solution when employing the adjoint consistent SIPG discretization based on (6.25) and (6.26) in comparison to the original SIPG discretization of the compressible Navier–Stokes equations [16], which instead uses

$$(7.1) \quad \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}), \quad \mathbf{n} \cdot \mathcal{F}^v(\mathbf{u}_h^+, \nabla_h \mathbf{u}_h^+) \quad \text{on } \Gamma, \quad \text{and} \quad J(\mathbf{u}_h).$$

Furthermore, we compare the accuracy of the original formulation and the adjoint consistent discretization on a sequence of globally refined meshes.

To this end, we revisit the standard test case [3, 16, 17] of a $M = 0.5$ viscous flow at $\text{Re} = 5000$ and at zero angle of attack around the NACA0012 airfoil with adiabatic no-slip boundary conditions imposed on the profile. Figure 7.1 shows the primal flow solution based on the adjoint consistent discretization on a (locally) refined mesh created by repeated refinement of the coarse C-type mesh depicted in Figure 7.2. Then, in Figure 7.3 we show the components z_1 – z_4 of the corresponding discrete adjoint solution \mathbf{z}_h . In particular, we find that the second and third components $z_2 \approx 1/C_\infty = 40/7$ and $z_3 \approx 0$ are constant on the profile. Furthermore, we see that $\mathbf{n} \cdot \nabla z_4 \approx 0$ as required by the continuous adjoint boundary conditions (6.9).

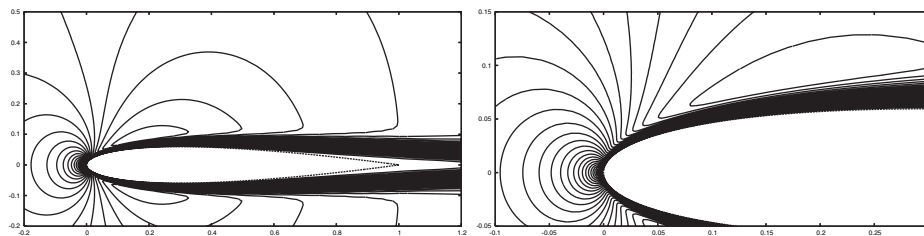


FIG. 7.1. Adjoint consistent DG discretization of the $M = 0.5, \alpha = 0^\circ, \text{Re} = 5000$ flow around the NACA0012 airfoil: Mach isolines ($0.02i, i \in \mathbb{N}$) of the (primal) flow solution.

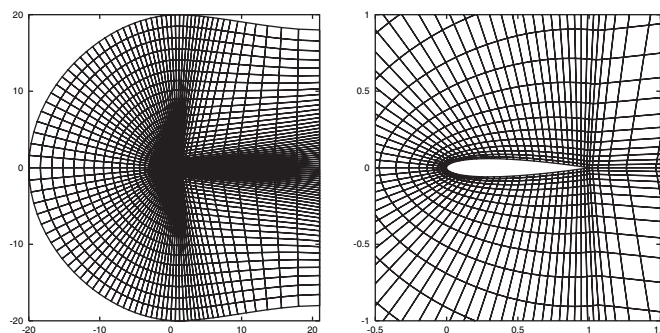


FIG. 7.2. Computational mesh with 3072 element: (left) full and (right) detailed view.

For comparison, Figure 7.4 shows the z_4 component of the discrete adjoint to the original DG discretization [16]. We clearly see an irregular adjoint solution at the profile boundary. In contrast, the discrete adjoint solution (see Figure 7.3) based on the adjoint consistent discretization is entirely smooth. The components $z_i, i = 1, 2, 3$, show a similar behavior.

For a quantitative comparison of the original and the adjoint consistent discretization, we collect the errors of the computed solutions on a sequence of globally refined meshes in Table 7.1. Here we show the number of elements, the number of degrees of freedom, and the error $J(\mathbf{u}) - J(\mathbf{u}_h)$ of the flow solution for three different discretizations. The error of $\mathbf{u}_h \in \mathbf{V}_h^1$ is measured in terms of the total drag coefficient c_d where the reference value $J(\mathbf{u}) \approx 0.05482$ is based on very fine grid computations. In columns (a) and (b) we collect the errors and rates of convergence of the original SIPG formulation (7.1) (cf. [16]) and for the adjoint consistent discretization based on (6.25) and (6.26), respectively. We see that the adjoint consistent discretization is by a factor of about 2–400 more accurate than the original discretization on the same mesh and with the same numerical complexity. Furthermore the adjoint consistent discretization shows an $\mathcal{O}(h^5)$ order of convergence which is significantly higher than that of the original discretization. In order to demonstrate the relevance of the IP modification (6.24) of the target functional, in column (c) of Table 7.1 we collect the respective errors based on the same discretization as in column (b) while omitting the IP modification; i.e., column (c) is based on (6.25) and $\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h))$. The accuracy is significantly reduced, partly even below the accuracy of the original discretization. From this, we see that the IP modification is essential for the adjoint consistency of the discretization and the accuracy of the numerical flow solution. A similar behavior is seen for $\mathbf{u}_h \in \mathbf{V}_h^2$ in Table 7.2. Here the original discretization

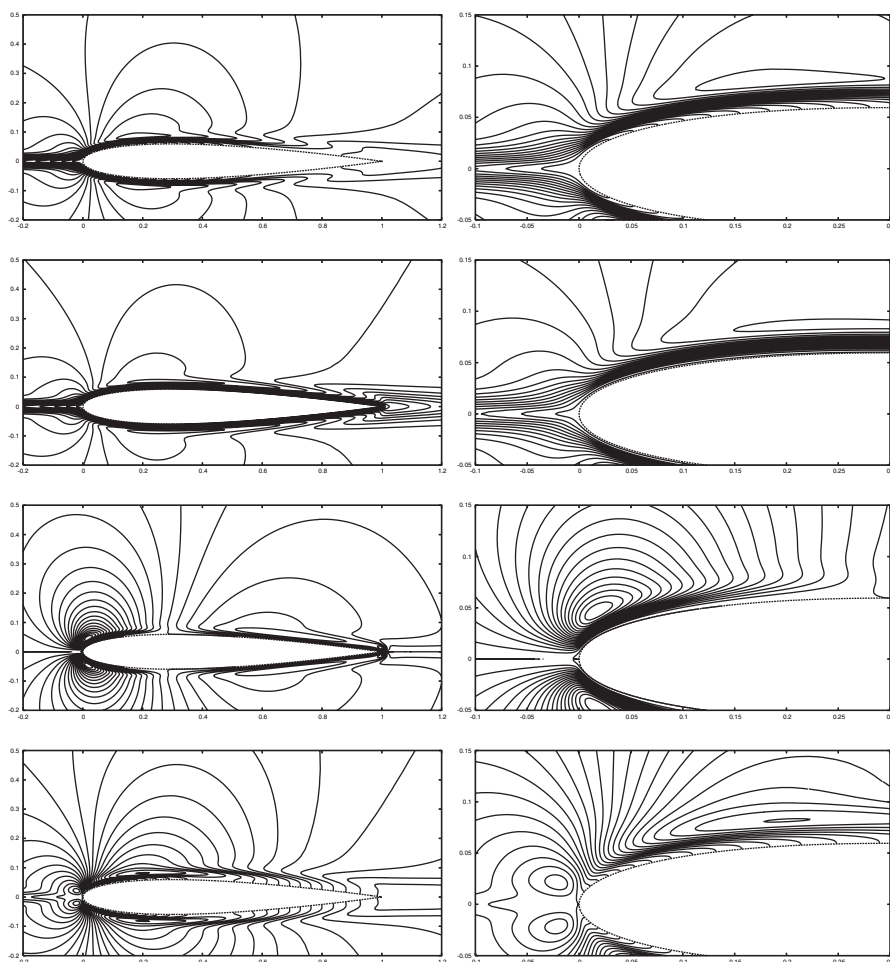


FIG. 7.3. Adjoint consistent DG discretization of the $M = 0.5, \alpha = 0^\circ, \text{Re} = 5000$ flow around the NACA0012 airfoil: i th row: Isolines of component z_i , $i = 1, \dots, 4$, of the adjoint sol. \mathbf{z} for c_d .

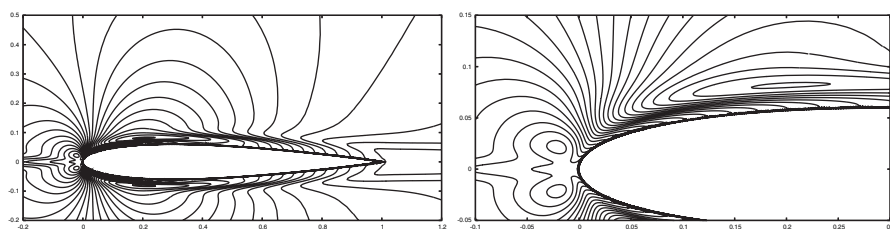


FIG. 7.4. Original DG discretization [16] of the $M = 0.5, \alpha = 0^\circ, \text{Re} = 5000$ flow around the NACA0012 airfoil: Isolines of component z_4 of the adjoint solution \mathbf{z} for c_d .

shows an $\mathcal{O}(h^3)$ convergence on average, whereas the convergence of the adjoint consistent discretization is $\mathcal{O}(h^6)$.

We note that independently Lu [23] demonstrated, for a $M = 0.5$ viscous compressible flow around the NACA0012 airfoil at $\text{Re} = 5000$ and $\alpha = 2^\circ$ angle of attack, that the discrete adjoint solution to the adjoint consistent discretization based on the BR2 scheme [5] is smooth whereas that to the standard discretization is not. Fur-

TABLE 7.1

Error $J(\mathbf{u}) - J(\mathbf{u}_h)$ and rate of convergence of $\mathbf{u}_h \in \mathbf{V}_h^1$ measured in terms of c_d (see (6.3)) for (a) the original SIPG formulation based on (7.1) (cf. [16]), for (b) the adjoint consistent discretization based on (6.25) and (6.26), and for (c) the adjoint consistent discretization without the IP modification of the target functional, i.e., based on (6.25) and $\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h))$.

# Cells	# Dofs	(a) Error	rate	(b) Error	rate	(c) Error	rate
3072	49152	-3.164e-03	-	1.502e-03	-	-1.243e-03	-
12288	196608	8.048e-04	3.9	3.682e-05	40.8	6.994e-04	1.8
49152	786432	4.519e-04	1.8	-1.139e-06	32.3	4.795e-04	1.5

TABLE 7.2

Error $J(\mathbf{u}) - J(\mathbf{u}_h)$ and rate of convergence of $\mathbf{u}_h \in \mathbf{V}_h^2$ measured in terms of c_d (see (6.3)) for (a) the original SIPG formulation based on (7.1) (cf. [16]), for (b) the adjoint consistent discretization based on (6.25) and (6.26), and for (c) the adjoint consistent discretization without the IP modification of the target functional, i.e., based on (6.25) and $\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h))$.

# Cells	# Dofs	(a) Error	rate	(b) Error	rate	(c) Error	rate
768	27648	-3.903e-02	-	5.565e-03	-	-2.054e-02	-
3072	110592	8.663e-04	45.1	6.234e-05	89.3	4.216e-04	48.7
12288	442368	4.987e-04	1.7	9.789e-07	63.7	4.139e-04	1.02

thermore, [23] showed optimal order of convergence results for the adjoint consistent discretization in comparison to reduced ones for the standard discretization which lacks adjoint consistency.

8. Conclusion. A discretization is adjoint consistent if the discrete adjoint problem is a consistent discretization of the continuous adjoint problem. In fact, adjoint consistency is the link between the so-called continuous adjoint approach (which discretizes the adjoint equations) and the discrete adjoint approach (which takes the adjoint of the discrete equations) in that the solutions to both approaches coincide. In particular, adjoint consistency is the key requirement for optimal order duality-based error estimates in L^2 as well as measured in terms of target functionals. Furthermore, adjoint consistency is closely related to the smoothness of the discrete adjoint solutions. However, in addition to the adjoint consistency of element and interior face terms, an adjoint consistent treatment of boundary terms as well as of target functionals is required for an adjoint consistent discontinuous Galerkin discretization.

In this article we have introduced a framework for analyzing consistency and adjoint consistency of discontinuous Galerkin discretizations of linear and nonlinear problems. This framework includes the derivation of the continuous adjoint problems and adjoint boundary conditions, provided the primal problem and the target functional satisfy a compatibility condition. It includes the derivation of the discrete adjoint problems and primal and adjoint residuals and a discussion of under which conditions the residuals vanish for the exact primal and adjoint solutions. In addition, we have introduced so-called consistent modifications of target functionals which allow us to modify (and possibly improve) computed target quantities without changing their exact values. We then analyzed the DG discretization of the linear advection equation, the interior penalty (IP)DG discretization of Poisson's equation, and the DG discretization of the compressible Euler equations. While recovering properties and conclusions drawn in [1, 11, 24], the outlined framework gives a unified analysis of these discretizations, including the definition of consistent modification of a target functional such as the so-called IP modification as well as a consistent modification of the force coefficients for inviscid compressible flows.

This framework has then been used to analyze the adjoint consistency property of the symmetric interior penalty discontinuous Galerkin discretization of the compressible Navier–Stokes equations. While the original formulation of the SIPG discretization [16] has been shown to be adjoint inconsistent, the analysis revealed that a special treatment of boundary terms as well as an IP modification of viscous force coefficients is required for recovering an adjoint consistent DG discretization of the compressible Navier–Stokes equations. Numerical experiments have confirmed that, in contrast to the original formulation in [16], the discrete adjoint solution to the adjoint consistent discretization is entirely smooth. Furthermore, numerical tests on globally refined meshes have shown that the adjoint consistent discretization is by a factor of 2–400 more accurate measured in terms of viscous force coefficients than the original formulation. Also, a significantly improved order of convergence has been observed.

REFERENCES

- [1] D. N. ARNOLD, F. BREZZI, B. COCKBURN, AND L. D. MARINI, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2002), pp. 1749–1779.
- [2] J. P. AUBIN, *Approximation of Elliptic Boundary-Value Problems*, Pure Appl. Math., 26, Wiley, Interscience, New York, 1972.
- [3] F. BASSI AND S. REBAY, *A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations*, J. Comput. Phys., 131 (1997), pp. 267–279.
- [4] F. BASSI AND S. REBAY, *High-order accurate discontinuous finite element solution of the 2d Euler equations*, J. Comput. Phys., 138 (1997), pp. 251–285.
- [5] F. BASSI AND S. REBAY, *GMRES discontinuous Galerkin solution of the compressible Navier-Stokes equations*, in Discontinuous Galerkin Methods, B. Cockburn, G. Karniadakis, and C.-W. Shu, eds., Lect. Notes Comput. Sci. Eng. 11, Springer-Verlag, Berlin, 1999, pp. 197–208.
- [6] F. BREZZI, B. COCKBURN, D. M. MARINI, AND E. SÜLI, *Stabilization mechanisms in discontinuous Galerkin finite element methods*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 3293–3310.
- [7] F. BREZZI, L. D. MARINI, AND E. SÜLI, *Discontinuous galerkin methods for first-order hyperbolic problems*, Math. Models Methods Appl. Sci., 14 (2004), pp. 1893–1903.
- [8] C. CASTRO, C. LOZANO, F. PALACIOS, AND E. ZUAZUA, *A systematic continuous adjoint approach to viscous aerodynamic design on unstructured grids*, AIAA, 2006-0051, 2006.
- [9] N. GAUGER, *Das Adjungiertenverfahren in der aerodynamischen Formoptimierung*, Technical report DLR-FB-2003-05 (ISSN 1434-8454), DLR, 2003.
- [10] M. GILES AND N. PIERCE, *Adjoint equations in CFD: Duality, boundary conditions and solution behaviour*, AIAA, 97-1850, 1997.
- [11] K. HARRIMAN, D. GAVAGHAN, AND E. SÜLI, *The Importance of Adjoint Consistency in the Approximation of Linear Functionals Using the Discontinuous Galerkin Finite Element Method*, Technical report, Oxford University Computing Laboratory, Oxford, 2004.
- [12] K. HARRIMAN, P. HOUSTON, B. SENIOR, AND E. SÜLI, *hp-Version discontinuous Galerkin methods with interior penalty for partial differential equations with nonnegative characteristic form*, in Recent Advances in Scientific Computing and Partial Differential Equations, Contemporary Mathematics 330, AMS, Providence, RI, 2003, pp. 89–119.
- [13] R. HARTMANN, *The role of the Jacobian in the adaptive Discontinuous Galerkin method for the compressible Euler equations*, in Analysis and Numerics for Conservation Laws, G. Warnecke, ed., Springer-Verlag, Berlin, 2005, pp. 301–316.
- [14] R. HARTMANN, *Derivation of an adjoint consistent discontinuous Galerkin discretization of the compressible Euler equations*, in Proceedings of the BAIL 2006 conference, G. Lube and G. Rapin, eds., 2006.
- [15] R. HARTMANN AND P. HOUSTON, *Adaptive discontinuous Galerkin finite element methods for the compressible Euler equations*, J. Comput. Phys., 183 (2002), pp. 508–532.
- [16] R. HARTMANN AND P. HOUSTON, *Symmetric interior penalty DG methods for the compressible Navier–Stokes equations I: Method formulation*, Int. J. Numer. Anal. Model., 3 (2006), pp. 1–20.

- [17] R. HARTMANN AND P. HOUSTON, *Symmetric interior penalty DG methods for the compressible Navier–Stokes equations II: Goal-oriented a posteriori error estimation*, Int. J. Numer. Anal. Model., 3 (2006), pp. 141–162.
- [18] P. HOUSTON, R. RANNACHER, AND E. SÜLI, *A posteriori error analysis for stabilised finite element approximations of transport problems*, Comput. Methods Appl. Mech. Engrg., 190 (2000), pp. 1483–1508.
- [19] P. HOUSTON, C. SCHWAB, AND E. SÜLI, *Discontinuous hp-finite element methods for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., 39 (2006), pp. 2133–2163.
- [20] A. JAMESON, *Aerodynamic design via control theory*, J. Sci. Comput., 3 (1988), pp. 233–260.
- [21] A. JAMESON, N. PIERCE, AND L. MARTINELLI, *Optimum aerodynamic design using the Navier–Stokes equations*, Theoretical and Computational Fluid Dynamics, 10 (1998), pp. 213–237.
- [22] J. L. LIONS AND E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin, 1972.
- [23] J. LU, *An a posteriori Error Control Framework for Adaptive Precision Optimization Using Discontinuous Galerkin Finite Element Method*, Ph.D. thesis, MIT, Cambridge, MA, 2005.
- [24] J. LU AND D. L. DARMOFAL, *Dual-consistency analysis and error estimation for discontinuous Galerkin discretization: Application to first-order conservation laws*, IMA J. Numer. Anal., submitted.
- [25] S. PRUDHOMME, F. PASCAL, J. ODEN, AND A. ROMKES, *Review of a priori error estimation for discontinuous Galerkin methods*, TICAM Report 00-27, University of Texas, 2000.