Adaptive FEM, Approach with hp- and Goal-Oriented A Posteriori Error Estimator

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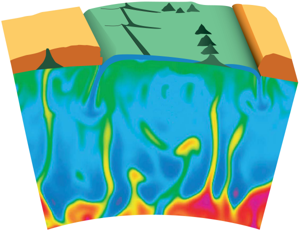
Outline

Motivation

Motivation

The Boussinesq Equations:

$$-\nu\Delta u + \nabla\varrho = f(T, g, C)$$
$$\nabla \cdot u = 0$$
$$\frac{\partial u}{\partial t} + u \cdot \nabla T - \nabla \cdot \kappa\nabla T = \gamma$$



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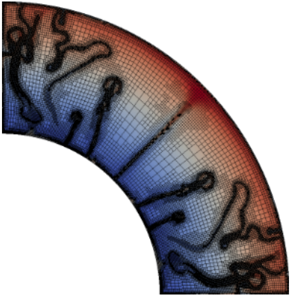

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Stokes or Creeping Flow

Given $f \in L^2(\Omega)^d$, $\{d = 2, 3\}$, $\nu \geq 1$, consider the Stokes equations as our model problem: Find $u : \overline{\Omega} \to \mathbb{R}^d$ and $\varrho : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nu \Delta u + \nabla \varrho &= f \qquad \text{in} \quad \Omega \\ \nabla \cdot u &= 0 \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \Gamma. \end{aligned}$$

Weak Formulation

We denote The standard weak formulation of problem; Seek $[u, \varrho] \in \mathcal{H}$ such that

$$\mathcal{L}([u, \varrho]; [v, q]) = (f, v)_\Omega \qquad \forall [v, q] \in \mathcal{H}.$$

Where

$$\mathcal{H} := H_0^1(\Omega)^d \times L_0^2(\Omega).$$

the bilinear form $\mathcal{L} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is defined as:

$$\mathcal{L}([u, \varrho]; [v, q]) := (\nu \nabla u, \nabla v)_\Omega - (\varrho, \nabla \cdot v)_\Omega - (\nabla \cdot u, q)_\Omega \qquad \forall [u, \varrho], [v, q] \in \mathcal{H}.$$

Discretization

Due to the continuous inf-sup condition

$$\inf_{[u,\varrho]\in\mathcal{H}} \sup_{[v,q]\in\mathcal{H}} \frac{\mathcal{L}([u,\varrho];[v,q])}{\left(\|\nabla u\|_\Omega + \|\varrho\|_\Omega\right)\left(\|\nabla v\|_\Omega + \|q\|_\Omega\right)} \geq \kappa > 0,$$

we define the finite element spaces $V_u^p(\mathcal{T})$ and $V_\varrho^p(\mathcal{T})$ by

$$V_u^p(\mathcal{T}) := \left\{u \in H_0^1(\Omega) : \;\; u|_K \circ T_K \in \mathcal{Q}_{p_K}\left(\hat{K}\right) \text{ for all } K \in \mathcal{T}\right\}$$

and

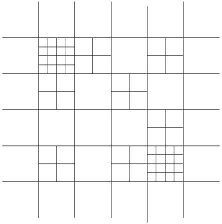
$$V_\varrho^p(\mathcal{T}) := \left\{\varrho \in L_0^2(\Omega) : \;\; \varrho|_K \circ T_K \in \mathcal{Q}_{p_K-1}\left(\hat{K}\right) \text{ for all } K \in \mathcal{T}\right\},$$

the discrete approximation is obtained by finding $[u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}] \in \mathcal{V}^p(\mathcal{T})$ such that : $\mathcal{V}^p(\mathcal{T}) := V_u^p(\mathcal{T})^d \times V_\varrho^p(\mathcal{T})$

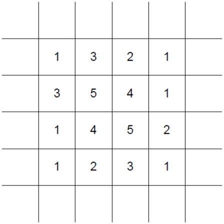
$$\mathcal{L}\left([u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}]\,;[v_{\mathrm{FE}}, q_{\mathrm{FE}}]\right) = \left(f, v_{\mathrm{FE}}\right)_\Omega \qquad \forall\,[v_{\mathrm{FE}}, q_{\mathrm{FE}}] \in \mathcal{V}^p(\mathcal{T}).$$

hp-Adaptive Finite Element Method

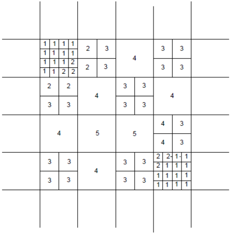
Adaptivity



h- Adaptive FEM



p- Adaptive FEM



hp- Adaptive FEM

A Posteriori Error Estimator

The idea behind a posteriori error estimation is to access the error between the exact solution and its finite element approximation, in terms of known quantities only!

- Reliability:

$$\left\|\nabla\left(u-u_{\mathrm{FE}}\right)\right\|_{\Omega}+\left\|\varrho-\varrho_{\mathrm{FE}}\right\|_{\Omega} \leq C_{rel}\eta\left(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f\right).$$

- Local error estimators:

$$\eta^{2}(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f)=\sum_{K \in \mathcal{T}} \eta_{K}^{2}\left(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f\right),$$

- Computational Efficiency

$$\eta_{K}\left(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f\right) \leq C_{eff}\left(\left\|\nabla\left(u-u_{\mathrm{FE}}\right)\right\|_{K}+\left\|\varrho-\varrho_{\mathrm{FE}}\right\|_{K}\right) \qquad \forall K \in \mathcal{T}$$

Residual Based A Posteriori Error Estimator

A posteriori error estimator η shall be the sum of local error indicators η_K:

$$\eta^2 := \sum_{K \in \mathcal{T}} \eta_K^2$$

- Local Error Estimator: The local a posteriori error estimator η_K can be decomposed into cell contribution and interface contribution:

$$\eta_K^2 := \eta_{K;R}^2 + \eta_{K;B}^2,$$

where $\eta_{K;R}$ denotes the residual-based term and $\eta_{K;B}$ indicates the jump-based term. These are defined by

$$\eta_{K;R}^2 := \frac{h_K^2}{p_K^2} \left\| \left(I_{p_K}^K f + \nu \Delta u_{\mathrm{FE}} - \nabla \varrho_{\mathrm{FE}} \right) \right\|_K^2 + \left\| (\nabla \cdot u_{\mathrm{FE}}) \right\|_K^2$$

and

$$\eta_{K;B}^2 := \sum_{e \in \mathcal{E}(K)} \frac{h_e}{2 p_e} \left\| \left[\nu \frac{\partial u_{\mathrm{FE}}}{\partial n_K} \right] \right\|_e^2 .$$

Reliability and Efficiency of Estimator

- Reliability: Let $[u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}] \in \mathcal{V}^p(\mathcal{T})$ be the solution of discrete problem and $[u, \varrho] \in \mathcal{H}$ be solution of weak problem. Further, assume that triangulation $\mathcal{T}$ is (γ_h, γ_p)-regular. Then, there exists some constant $C_{rel} > 0$ independent of mesh size vector h and polynomial degree vector p such that

$$\|\nabla (u - u_{\mathrm{FE}})\|_{\Omega}^2 + \|\varrho - \varrho_{\mathrm{FE}}\|_{\Omega}^2 \leq C_{rel} \sum_{K \in \mathcal{T}} \left(p_K^2 \eta_K^2 + \frac{h_K^2}{p_K^2} \left\| I_{p_K}^K f - f \right\|_K^2 \right).$$

- Efficiency: Let $[u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}] \in \mathcal{V}^p(\mathcal{T})$ be the solution of discrete problem, and $[u, \varrho] \in \mathcal{H}$ be solution of weak problem. Further, we assume that triangulation $\mathcal{T}$ is (γ_h, γ_p)-regular. Then, there exists some constant $C_{eff} > 0$ independent of mesh size vector h and polynomial degree vector p such that

$$\eta_K^2 \leq C_{eff} \left(p_K \left(\nu^2 \|\nabla (u - u_{\mathrm{FE}})\|_{\omega_K}^2 + \|\varrho - \varrho_{\mathrm{FE}}\|_{\omega_K}^2 \right) + \frac{h_K^2}{p_K^2} \left\| I_{p_K}^K f - f \right\|_{\omega_K}^2 \right)$$

for all $K \in \mathcal{T}$.

hp-Adaptive Refinement Loop

The fully automatic hp-adaptive refinement strategy is based on the standard adaptive loop

$$\textbf{SOLVE} \longrightarrow \textbf{ESTIMATE} \longrightarrow \textbf{MARK} \longrightarrow \textbf{REFINE}.$$

- **SOLVE** and **REFINE** are almost the same in all adaptive refinement patterns.
- **ESTIMATE** and **MARK** are the two most crucial modules in hp-adaptive method.

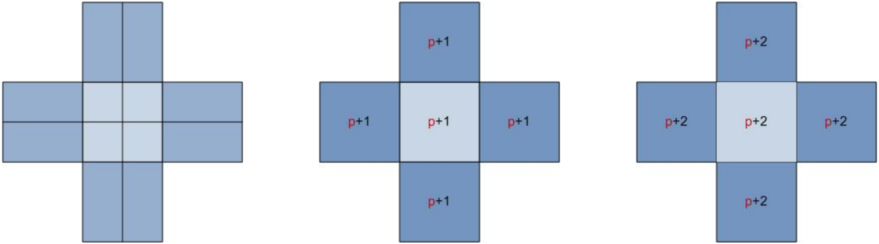
hp- Adaptive Refinement Algorithm

- **Initialization**: Set $N = 0$, a coarse mesh $\mathcal{T}_0$, $\theta \in [0, 1]$ and also tolerance TOL.
- **SOLVE**: Find the solution $(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}})$ of discrete problem.
- **ESTIMATE**: Compute a posteriori error estimation. If $\eta < TOL$ then **STOP** the algorithm, **else**,
- **MARK**: select set of cells $\mathcal{A}$ to be marked either with h- or p-refinement
- **REFINE**: Given $(\mathcal{A}, (j_K)_{K\in\mathcal{A}})$, we refine the cells contained in $\mathcal{A}$ with refinement patterns j_K corresponding to each cell. Then set $N = N + 1$ and go to step SOLVE.

Module MARK

More information needed in module **MARK** to choose the best adaptive strategy:

- Convergence Estimator: $k_{K,j}, \;\; j \in \{1,2,3\}$



j=1 j=2 j=3

- Efficiency ≈ Workload number: $\varpi_{K,j} = \text{n } \mathbf{DoFs} \quad j \in \{1,2,3\}$
- Choose the best refinement pattern ⇒ find integer $j_K \in \{1,2,3\}$ such that:

$$\frac{k_{K,j_K}}{\varpi_{K,j_K}} = \max_{j\in\{1,2,3\}} \frac{k_{K,j}}{\varpi_{K,j}}, \qquad \sum_{K\in\mathcal{A}} k^2_{K,j_k}\eta^2_K \geq \theta^2\eta^2$$

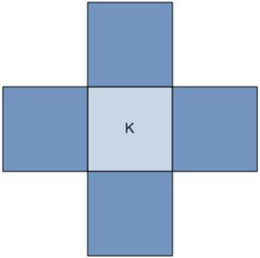
Challenges to calculate the Convergence Estimator: $k_{K,j}$

Considering the residual of Stokes problem on the local patch domain ω_K, we have:

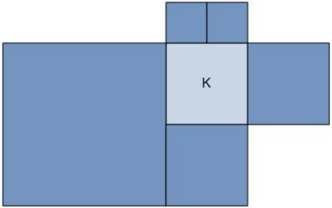
$$(\nabla v, \nabla(w_u^{N,j}))_{\omega_K} + (q, w_\varrho^{N,j})_{\omega_K} = \mathcal{L}([v,q];[e,E])_{\omega_K}, \qquad \forall (v,q) \in \mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K}).$$

The pair $(w_u^{N,j}, w_\varrho^{N,j}) \in \mathcal{V}_{K,j}^p(\mathcal{T}_N|_{\omega_K})$ is defined to be the Ritz representation of the residual.

$$k_{K,j} = \frac{1}{\eta_K(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}})} \left(\left\| \nabla w_u^{N,j} \right\|_{\omega_K}^2 + \left\| w_\varrho^{N,j} \right\|_{\omega_K}^2 \right)^{\frac{1}{2}}$$

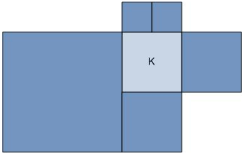


regular patch

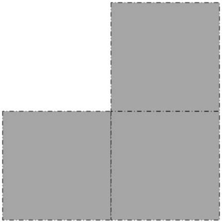


non-uniform patch

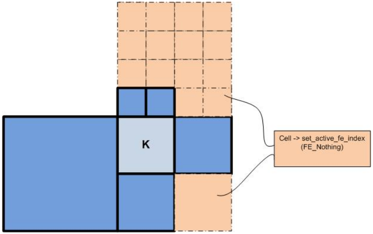
Build the local triangulation to compute $\Rightarrow k_{K_j}$



non-uniform patch



get-cells-at-coarsest-common-level



set-FE-Nothing

Contraction Convergence Results

- Contraction for Error in Energy Norm:

$$\|\nabla(u-u_{\mathrm{FE}}^{N+1})\|_{\Omega^d}^2+\|\varrho-\varrho_{\mathrm{FE}}^{N+1}\|_{\Omega}^2 \leq \mu\left(\|\nabla(u-u_{\mathrm{FE}}^{N})\|_{\Omega^d}^2+\|\varrho-\varrho^{N}{}_{\mathrm{FE}}\|_{\Omega}^2\right)$$

- Quasi- Convergence :

$$\|\nabla(u-u_{\mathrm{FE}}^{N+1})\|_{\Omega^d}^2+\|\varrho-\varrho_{\mathrm{FE}}^{N+1}\|_{\Omega}^2+\vartheta\eta^2(\mathcal{T}_{N+1}) \leq \mu\Bigg(\|\nabla(u-u_{\mathrm{FE}}^{N})\|_{\Omega^d}^2+$$

$$\|\varrho-\varrho_{\mathrm{FE}}^{N}\|_{\Omega}^2+\vartheta\eta^2(\mathcal{T}_{\mathcal{N}})\Bigg)$$

for constants $\vartheta > 0$ and $\mu \in (0, 1)$ independent of mesh size h and polynomial degree vector p.

Important Equivalence Result

- Total Error:

$$\|\nabla(u - u_{\mathrm{FE}}^N)\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^N\|_{\Omega}^2 + osc_N^2$$

- Quasi Error:

$$\|\nabla(u - u_{\mathrm{FE}}^N)\|_{\Omega^d}^2 + \|\varrho - \varrho_{\mathrm{FE}}^N\|_{\Omega}^2 + \vartheta\eta^2(\mathcal{T}_{\mathcal{N}})$$

- Quasi Error $\approx \eta^2 \approx$ Total Error

Numerical Results

Example-1

- Manufactured solution on L-shaped domain:

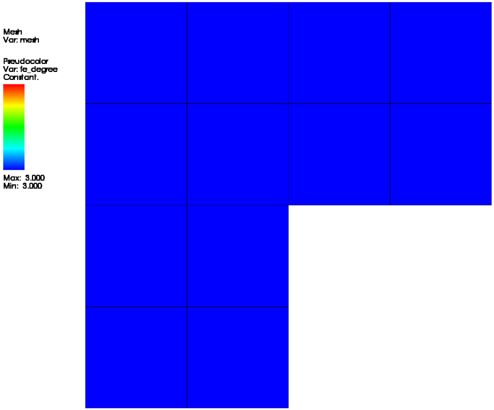
Let $\Omega \in \mathbb{R}^2$ be L-shaped domain,

$$(-1,1)^2 \backslash ([0,1] \times [-1,0])$$

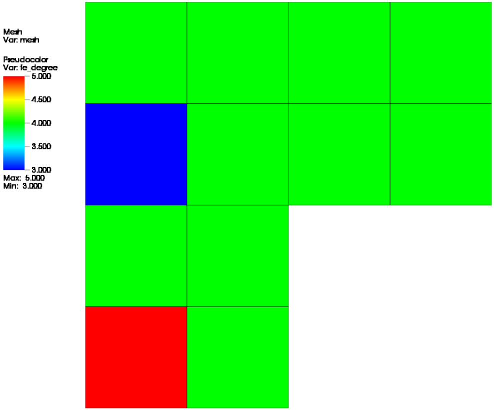
we enforce appropriate inhomogeneous boundary conditions for velocity u on Γ such that the analytical solution $u : \overline{\Omega} \to \mathbb{R}^2$ and $\varrho : \Omega \to \mathbb{R}$ are given by:

$$u = \begin{bmatrix} -e^x(y\cos(y) + \sin(y)) \\ e^x y \sin(y) \end{bmatrix}, \qquad \varrho = 2e^x \sin(y) - (2(1-e)(\cos(1)-1))/3.$$

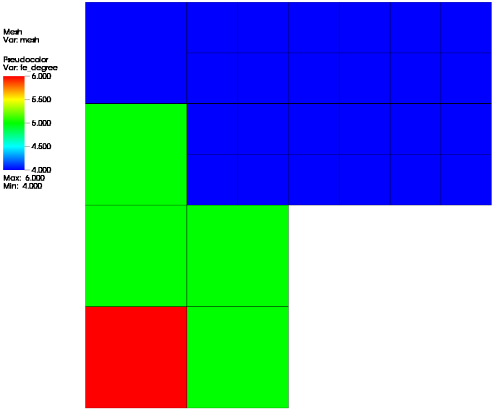
hp- adaptive refinement, cycle = 0



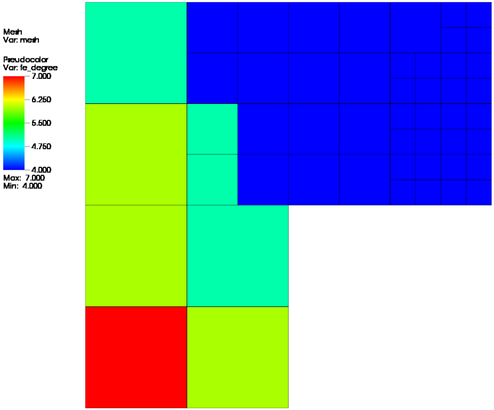
hp- adaptive refinement, cycle = 2



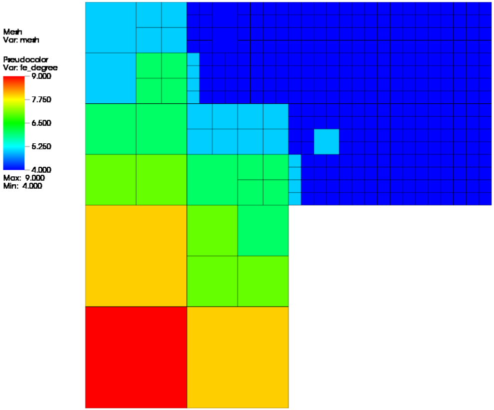
hp- adaptive refinement, cycle = 4



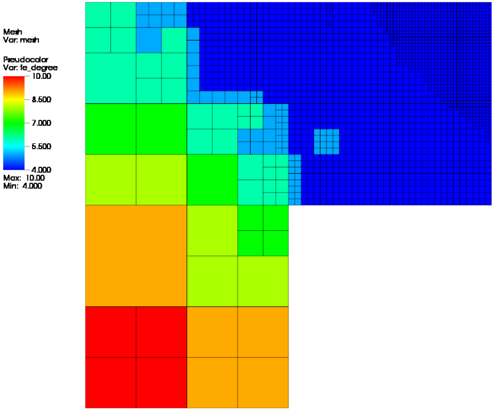
hp- adaptive refinement, cycle = 6



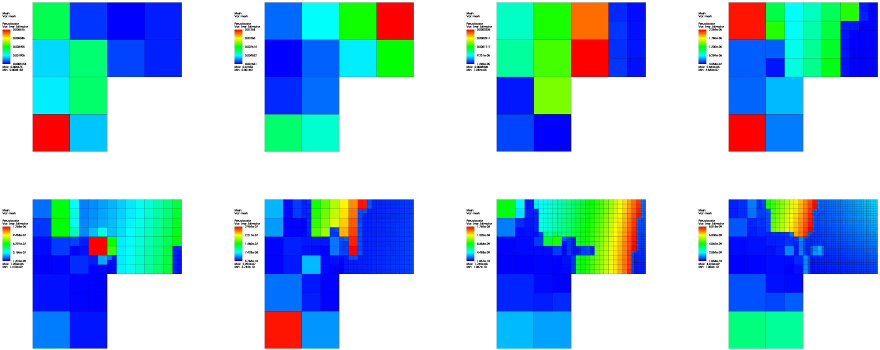
hp- adaptive refinement, cycle = 9



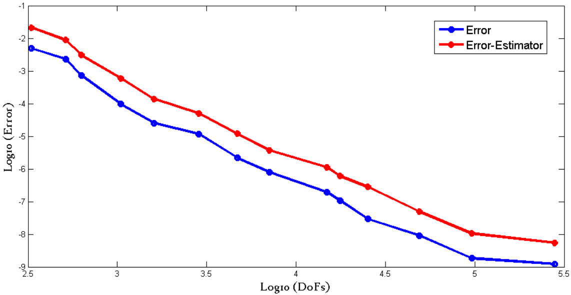
hp- adaptive refinement, cycle = 12



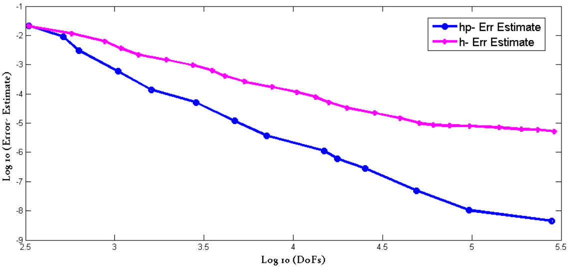
Distribution of a posteriori error estimator in hp-adaptive refinement



Error- Error Estimator in hp-AFEM



Comparison of actual and estimated energy error vs DoFs.



Comparison of the estimated error with hp- and h- adaptive mesh refinement.

Example-2

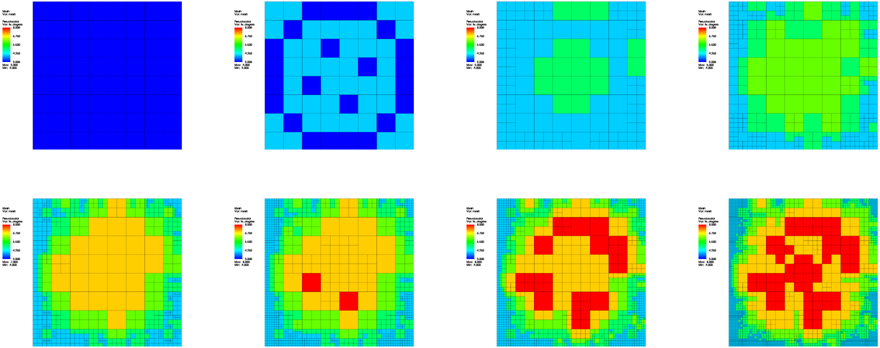
- Smooth solution in two dimensions:

Let $\Omega : (-1,1) \times (-1,1)$ and let velocity $u : \overline{\Omega} \to \mathbb{R}^2$ and $\varrho : \Omega \to \mathbb{R}$ be give by

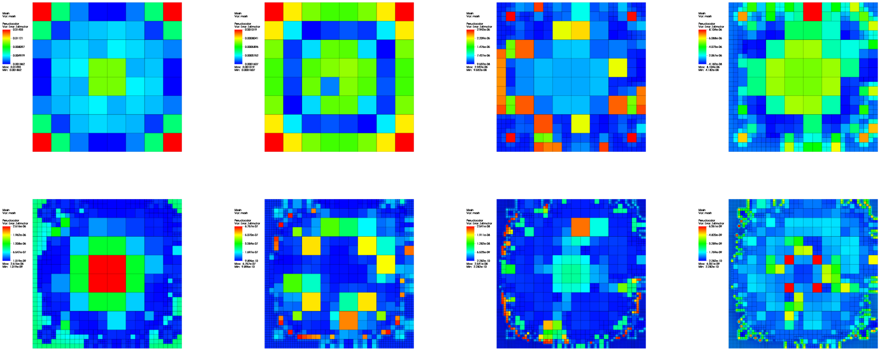
$$u = \begin{bmatrix} 2y\cos(x^2+y^2) \\ -2x\cos(x^2+y^2) \end{bmatrix}, \qquad \varrho = e^{-10(x^2+y^2)} - \varrho_m$$

where p_m is such that $\int_\Omega \varrho = 0$.

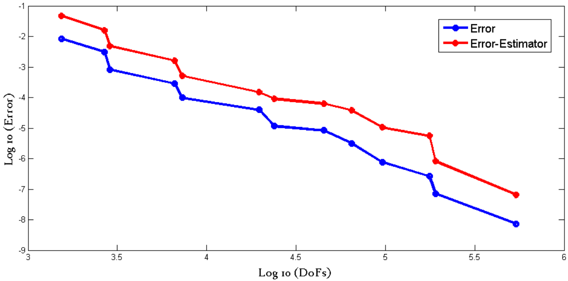
hp- adaptive refinement, cycles: 0, 2, 4, 5, 7, 8, 9, 11
Min poly. degree=3, Max poly. degree=8



Distribution of posteriori error estimator in hp- AFEM

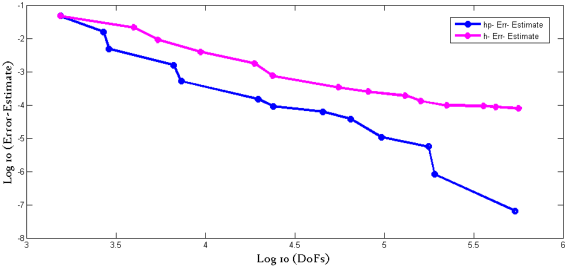


Error- Error Estimator in hp-AFEM



Comparison of actual and estimated energy error vs DoFs.

hp- and h- Error-Estimator



Comparison of the estimated error with hp- and h- adaptive mesh refinement.

- Singular solution in two dimensions

We consider the L-shaped domain

$$\Omega := (-1,1)^2 \;\; ([0,1] \times [-1,0])$$

with reentrant angle $\omega = 3\pi/2$ at the origin. Let $\alpha \approx 0.544$ be an approximation of the smallest root of a nonlinear equation: The exact velocity u and pressure ϱ are given in polar coordinates by

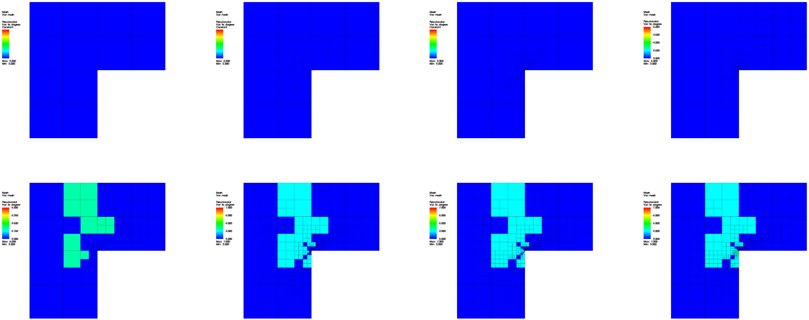
$$u(r,\varphi) = r^{\alpha} \begin{bmatrix} \cos(\varphi)\psi^{'}(\varphi) + (1+\alpha)\sin(\varphi)\psi(\varphi) \\ \sin(\varphi)\psi^{'}(\varphi) - (1-\alpha)\cos(\varphi)\psi(\varphi) \end{bmatrix}$$

and

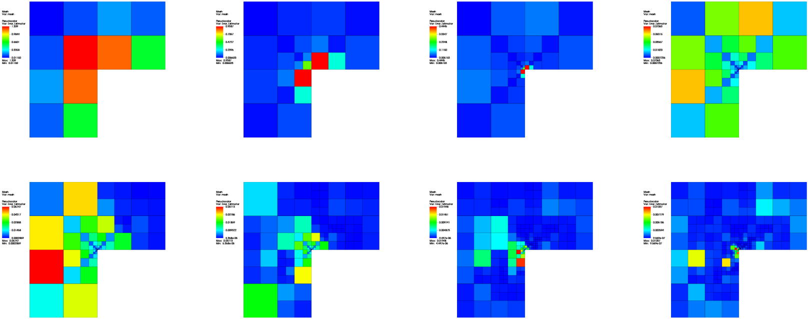
$$\varrho(r,\varphi) = -r^{\alpha-1} \frac{(1+\alpha)^2\psi^{'}(\varphi) + \psi^{'''}(\phi)}{1-\alpha}$$

where $\psi(\varphi)$ is a smooth function

Example-3, hp- adaptive refinement, cycles: 0, 2, 5, 10, 11, 16, 19, 20



Distribution of a posteriori error estimator in hp-adaptive refinement



Example-3

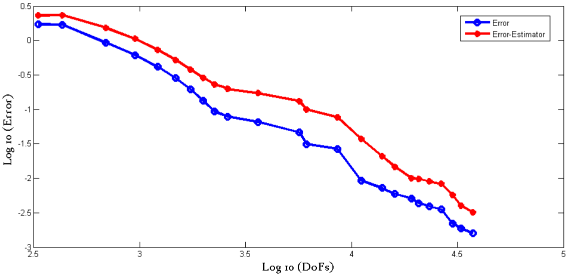


Figure: Comparison of actual and estimated energy error vs DoFs.

Goal Oriented A Posteriori Error Estimator in h- and hp-AFEM

Primal-Dual problem

- Primal Problem:

$$\left.\begin{aligned} -\nu\Delta u + \nabla\varrho &= f \qquad \text{in} \quad \Omega \\ \nabla\cdot u &= 0 \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \Gamma. \end{aligned}\right. \Longrightarrow \mathcal{L}([u,\varrho];[v,q]) = (f,v)_\Omega$$

- Dual Problem:

$$\left.\begin{aligned} -\nu\Delta z_u + \nabla z_\varrho &= j(u,\varrho) \qquad \text{in} \quad \Omega \\ \nabla\cdot z_u &= 0 \qquad \text{in} \quad \Omega \\ z_u &= 0 \qquad \text{on} \quad \Gamma. \end{aligned}\right. \Longrightarrow \mathcal{L}([v,q];[z_u,z_\varrho]) = (v,j)_\Omega$$

Goal-Oriented A Posteriori Error Estimator

Goal-Oriented estimator:

$$\zeta := \sum_{K \in \mathcal{T}} \zeta_K$$

Goal-Oriented local error estimator ζ_K are defined as

$$\zeta_K := \rho_K \cdot \eta_K(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f)$$

the local weight ρ_K is given by

$$\rho_K := \tilde{\eta}_K + \|\nabla v_{u,\mathrm{FE}}\|_{(\omega_{K,2})} + \|v_{\varrho,\mathrm{FE}}\|_{(\omega_{K,2})}$$

Reliability and Efficiency:

- Theorem -1: Reliable Goal-Orineted A Posteriori Error Estimator

$$|J(u,\varrho) - J(u_{\mathrm{FE}},\varrho_{\mathrm{FE}})| \leq C_{rel} \sum_{K\in\mathcal{T}} \left(\eta_K + \frac{h_K}{p_K} \|f - I_{p_K}^K f\|_K \right) \cdot \left(\rho_K + \frac{h_K}{p_K} \|j - I_{p_K}^K\|_{\omega_{K,2}} \right)$$

- Theorem -2: Efficient Goal-Orineted A Posteriori Error Estimator

$$\zeta_K \leq C_{eff} \left(p_K (\|\nabla(u - u_{\mathrm{FE}})\|_{\omega_{K,2}} + \|\varrho - \varrho_{\mathrm{FE}}\|_{\omega_{K,2}}) + \frac{h_K}{p_K^{\frac{1}{2}}} \|f - I_{p_K} f\|_{\omega_{K,2}} \right) \left(p_K^{\frac{3}{4}} (\|\nabla(z_u - z_{u,\mathrm{FE}})\|_{\omega K,2} + \|z_\varrho - z_{\varrho,\mathrm{FE}}\|_{\omega_{K,2}}) + \frac{h_K}{p_K^{\frac{1}{4}}} \|j - I_{p_K} j\|_{\omega_{K,2}} \right)$$

Go-AFEM Algorithm for hp-Refinement

1. Set $N = 0$, $Tol > 0$, $\theta \in (0, 1]$, and initialize coarse grid $\mathcal{T}_0$.
2. Solve the Primal problem
3. Solve the Dual problem
4. For every cell $K \in \mathcal{T}_N$, solve the local variational problem over patches
5. Compute A Posteriori Error Estimator given as

$$\zeta_K := \rho_K \cdot \eta_K(u_{\mathrm{FE}}, \varrho_{\mathrm{FE}}, f)$$

$$\rho_K := \tilde{\eta}_K + \|\nabla v_{u,\mathrm{FE}}\|_{(\omega_{K,2})} + \|v_{\varrho,\mathrm{FE}}\|_{(\omega_{K,2})}$$

6. If $\zeta \leq TOL$, STOP!
7. For every cell $K \in \mathcal{T}_N$, and $j \in \{1, 2, \cdots, n\}$, Compute the Convergence Estimator $k_{K,j}$.
8. Refine cells according to the modified fraction scheme, "Dorfler property"

$$\sum_{K \in \mathcal{A}} k^2_{K,j_k} \zeta^2_K \geq \theta^2 \zeta^2$$

Conclusion

- Residual based a posteriori error estimate for conforming hp-AFEM.
- Validate theoretically and also numerically the reliability and the efficiency of estimator
- Showing that our hp-adaptive algorithm is a contraction.
- Introducing a new locally defined reliable and efficient Goal-Oriented a posteriori error estimator for h- and hp- AFEM.

Future Work

- Providing an optimal rate for our Goal-Oriented h-adaptive FEM.
- Parallelize the code for 3D Stokes problem in Goal-Oriented h-adaptive FEM.
- Using the above error estimators in both hp- and Goal-Oriented AFEM in the Advanced Solver for Problem in Earth Convection.
- Parallelizing hp-AFEM?!

Thank You