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THE P AND H-P VERSIONS OF THE FINITE ELEMENT METHOD, BASIC PRINCIPLES AND PROPERTIES*

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Abstract. In the classical form of the finite element method called the *h version*, piecewise polynomials of fixed degree p are used and the mesh size h is decreased for accuracy. In this paper, we discuss the fundamental theoretical ideas behind the relatively recent p version and h-p version. In the p version, a fixed mesh is used and p is allowed to increase. The h-p version combines both approaches. The authors describe and explain the basic properties and characteristics of these newer versions, especially in areas where their behavior is significantly different from that of the h version. Simplified proofs of key concepts are included and computational illustrations of several results are provided. A benchmark comparison between the various versions in included.

Key words. h-p version, h-version, p-version

AMS subject classifications. 65N30, 65-02

1. Introduction. A major tool in computational mechanics today is the finite element method (FEM). Although this method has a long history (see, e.g., [Ba], [Cl2], [Od], and [Wl]) the beginning of its computational success can be related to [TC] and [Cl1]. Originally, the main application of the FEM was in the area of structural mechanics but today, its use has expanded to various other fields such as fluid mechanics, thermal analysis, electrical engineering, etc., with applications to both linear/nonlinear and stationary/transient problems. Consequently, there is a vast body of published research in the field of finite element theory and applications. In Fig. 1.1, we have presented data showing the number of papers related to the FEM. (See also [Ma].)

Most of the papers represented in Figure 1.1 are related to the classical form of the FEM, called the *h version*, in which polynomials of fixed degree p are used and the mesh is refined to increase accuracy. There are several commercial finite element programs based on the h version of the FEM which are currently available (e.g., MSC/NASTRAN, Cosmos/M, Abaqus, Aska, Adina, Ansys, etc.).

In contrast, only a few (less than 100) of the above papers are related to the relatively recent *p* and *h-p versions*. The p version of the FEM uses a fixed mesh but increases the polynomial degree p to increase accuracy. In the h-p version, the two approaches of mesh refinement and degree enhancement are combined. (The h and p versions can be considered as special cases of the h-p version.) The first theoretical papers on these new versions [BSK], [BD] were published in 1981. In contrast to the many books devoted to the h version of the FEM, the only book addressing the p and h-p versions is [SzB].

In the field of industrial finite element computation, the p and h-p versions have already spawned several new codes. These include such p version commercial codes as Applied Structure (Rasna Corp., CA, USA), MSC/PROBE (MacNeal Schwendler, CA, USA) and Poly FEM (IBM, MA, USA), the adaptive h-p version commercial code PHLEX (Computational Mechanics, TX, USA) and the 3-d adaptive p version research code STRIPE (Aeronautial Research Institute of Sweden), with other codes such as PEGASYS (ESRD, MO, USA) under development. The availability of the above codes and their success shows the practical importance of these methods.

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Chronological distribution of FE papers.

FIG. 1.1. *Number of* FEM *papers in* MAKEBASE *database* (1993 *is not completely correct—delayed literature acquisition. Computational fluid mechanics is not included*). (*Courtesy: Jaroslav Mackerle, Linköping Inst. of Technology, Sweden.*)

An obvious question that arises is how well these methods compare with the traditional h version. The answer is very dependent upon the goals of the computation and the problem under consideration, since both h-type and p-type methods may be particularly suitable in different situations. However, *for certain problems with piecewise analytic input data, the capability of increasing* p *coupled with mesh refinement* (*the* h-p *version*) *leads to exponential rates of convergence, which are not possible with just the* h *version alone*. Moreover, in many situations, this high rate of convergence can be practically observed by just using the pure p version without mesh refinement (see Remark 4.1). A benchmark comparison between the p and h versions from [CM] is included in §8. Of course, perhaps the most important test for any method is its use in successful codes in today's competitive finite element software world.

Our main concern in this paper is not so much to evaluate the relative advantages and disadvantages of the p and h-p versions in comparison with the h version. Rather, it is to describe the particular characteristics of these methods that differentiate them from the h version (such as, for example, the exponential convergence mentioned above) and to discuss the basic theoretical reasons behind such behavior. Rather than an exhaustive summary of published work, we therefore only concentrate on the most salient (and practically significant) of these characteristics. We give numerous elaborations and examples of key theoretical results to help explain the underlying reasons for the behavior of these versions. We also include informal proofs of several important theorems to make them more accessible and to help introduce the reader to p version theoretical techniques of approximation (which can be quite different from standard h version techniques). Throughout this survey, to keep the exposition focussed, we lay special emphasis on problems related to structural mechanics, as opposed to other applications, such as fluid dynamics, for which p-type methods are also popular (see, e.g., the survey [BMd] on the related *spectral element methods*).

The plan of our paper is as follows. In §2, we consider the one-dimensional case, which allows us to present various ideas and results (which hold for higher dimensions as well) in a simplified setting. Perhaps the most important characteristic of the p and h-p versions, the enhanced asymptotic rates of convergence that are possible, is discussed at length in §§2.3–2.4. We show, by elementary arguments, exactly what leads to the well-documented phenomenon

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of the "doubling" in the rate of convergence of the p version (over the quasiuniform h version) in the presence of singularities. We also demonstrate (by means of some simple proofs) the following basic difference: the rate for the h version can never be better than *algebraic* (no matter which mesh is used), while the h-p version can yield *exponential* rates. In §§2.5 and 2.6, we discuss, respectively, the error in the L_2 norm and the problem of "A-pollution," two areas in which the p version behaves differently from the h version. Our results in the one-dimensional case are quite detailed, and most of our proofs are presented in this simplified setting, since the higher-dimensional proofs are quite involved.

The two- and three-dimensional analogs of results in §2 are discussed in §4. The refined approximation theorems for the p and h-p versions presented in §4 are extremely dependent on precise regularity results for the solution (which are more detailed than those required for the h version). Therefore, §3 is devoted to a survey of various types of regularity theorems for elliptic problems that are used in the development of such convergence results. In particular, regularity results in terms of countably normed spaces, which are crucial in deriving exponential rates of convergence for the h-p version, are presented in §3.3.

An inherent property of the p version is its *robustness* in most situations, a characteristic that can be put to good use in applications to problems that are susceptible to *numerical locking*. We explain this phenomenon in §5 and survey some recent related results, using the equations of linear elasticity as a model.

The convergence results in §§2 and 4 are mostly in the energy norm. Often, instead of the energy, some other functional of the solution is of interest. In §6, we discuss *extraction procedures* by which these functionals can be recovered from the solution. As a result, the high rates of convergence for the h-p version in the energy are preserved for these quantities of interest as well.

In §7, we consider the use of the p version in plate modeling. We show how a *hierarchy* of plate models may be set up, which approximates the three-dimensional plate with increasing accuracy. The implementation of such a hierarchy of plate models can then be effected in a natural way by using the p version. This may be used to ensure convergence to the solution of the three-dimensional plate. This is in contrast to the usual method of plate modeling in which a fixed model (e.g., the Reissner–Mindlin plate) is chosen and discretized. The approximations then only converge to the solution of the two-dimensional plate model.

Implementational aspects of the p and h-p version are briefly discussed in §8, where a benchmark comparison is given.

We have not tried to compose an exhaustive bibliography of publications on the p and h-p version (as could, for instance, be obtained using MAKEBASE). Rather, we have provided only selected representative references, for the interested reader to obtain more information on the topics discussed. In this connection, some representative references on the p and h-p versions for areas not discussed in this paper are as follows: *boundary element methods* [PS], [SS], *mixed methods* [JV], [S1], [StS], *parabolic problems* [BJ], *shape optimization* [Sh], *nonlinear problems* [Ng], *solution of large systems* [Mn]. In addition, see [Sz] for a survey on computational questions related to the p and h-p version.

Throughout our discussion, Ω will be a connected bounded domain in $\mathbb{R}^n$. For $k \geq 0$ integer, $1 \leq t \leq \infty$, $W^{k,t}(\Omega)$ will denote the usual Sobolev space of functions with k derivatives lying in $L_t(\Omega)$. We denote $W^{k,2}(\Omega) = H^k(\Omega)$ and let $H_0^k(\Omega)$ be the subset of functions with vanishing trace for $k-1$ derivatives at the boundary of Ω. The spaces $W^{k,t}(\Omega)$ for $k < 0$ will be defined by duality and for k fractional by interpolation ([LM]). For Ω an interval in $\mathbb{R}$, we will also use the space $H_{00}^{1/2}(\Omega)$, which is defined by interpolating between $L_2(\Omega)$ and $H_0^1(\Omega)$ (see [LM]).

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2. The h, p, and h-p versions in one dimension. The one-dimensional setting allows a simple and convenient vehicle for the exposition of our basic ideas and results. The fundamental properties of and differences between the h, p and h-p versions are most clearly explained and understood through the one-dimensional results and numerical illustrations in this section. These also indicate what can be expected in two and three dimensions (discussed in succeeding sections), for which strongly analogous results hold. A detailed analysis of the 1-d case has been made in [GiB], from which most of the results in this section have been cited.

2.1. The model problem and its discretization. Let us consider the model problem

$$\begin{aligned} -u'' &= f \quad \text{on } \Omega = (0,1) = I, \\ u(0) &= u(1) = 0. \end{aligned} \tag{2.1}$$

We will consider the general case when it is only assumed that $f \in H^{k-2}(\Omega)$, $k \geq 1$, as well as the particular case when f is chosen such that the exact solution of (2.1) is

$$u(x) = x^\alpha - x, \qquad \alpha > \frac{1}{2}. \tag{2.2}$$

It will be seen shortly that much more refined convergence results exist for the particular case (2.2) than the general case. The form (2.2) is obviously singular at the endpoint $x = 0$. The reason we choose it is that it serves as a model for two (and higher) dimensions where analogous singularities exist at the corners of domains which are curvilinear polygons (see §3). The restriction $\alpha > \frac{1}{2}$ is made to ensure that $u \in H^1(\Omega)$.

In general, the regularity of the solution u is determined by the regularity of f. Obviously, if $f \in H^{k-2}(\Omega)$, then $u \in H^k(\Omega)$. An analogous but more complicated relation between the regularity of the input data and of the solution occurs in two and three dimensions and will be discussed in §3.

As a prelude to using the finite element method, we cast the problem (2.1) into the following variational form. Find $u \in H_0^1(\Omega)$ such that for any $v \in H_0^1(\Omega)$,

$$B(u,v) = \int_\Omega u'(x)v'(x)dx = f(v) \tag{2.3}$$

with

$$f(v) = \int_\Omega f(x)v(x)dx.$$

The above may now be discretized by constructing a sequence of finite-dimensional (finite element) subspaces $V^n \subset H_0^1(\Omega)$ of dimension $N_n = N(V^n), n = 1, 2, \ldots,$ called the number of degrees of freedom. These spaces consist of piecewise polynomials on a sequence of meshes $\{\mathcal{T}^n\}, n = 1, 2, \ldots,$ on Ω where

$$\mathcal{T}^n = \left\{0 = x_0^n < x_1^n < \cdots < x_{m_n}^n = 1\right\}.$$

We denote $I_j^n = (x_{j-1}^n, x_j^n)$, $h_{n,j} = x_j^n - x_{j-1}^n$, $j = 1, \ldots, m_n$, $h_n = \max_{1 \leq j \leq m_n} h_{n,j}$, and call I_j^n elements.[1]

Let us now consider three basic types of meshes, which are analogous to those used in the two- and three-dimensional settings.

[1] We consider here only a denumerable family of meshes. Of course, we could consider a one (or more) parametric family of meshes as well.

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1. *The quasiuniform mesh.* Here $\{T^n\}$ is such that we have for all n

$$\frac{h_n}{h_{n,j}} \leq \tau,$$

where $\tau > 0$ will be called the quasiuniformity constant. A special case is the uniform mesh, where $\tau = 1$ and

$$x_i^n = \frac{i}{m_n}, i = 0, \ldots, m_n. \tag{2.5}$$

2. *The radical mesh* with power $\beta > 0$ (with respect to $x = 0$). This is defined by choosing

$$x_i^n = \left(\frac{i}{m_n}\right)^\beta, \qquad i = 1, \ldots, m_n \tag{2.6}$$

(the uniform mesh is obviously a radical mesh with $\beta = 1$).

The radical mesh is a special case of a "graded mesh" with "grading function" $g(y) = y^\beta$, defined by

$$x_i^n = g\left(\frac{i}{m_n}\right), \qquad i = 1, \ldots, m_n.$$

Remark 2.1. A generalization of graded meshes uses the notion of density $\psi(\psi = g'$ in the one-dimensional case). It is naturally applicable to two dimensions. See [Hu].

3. *The geometric mesh* with ratio $q, 0 < q < 1$ (with respect to $x = 0$). Here

$$x_i^n = q^{m_n - i}, \qquad i = 1, \ldots, m_n, \tag{2.7a}$$

$$x_0^n = 0. \tag{2.7b}$$

A more general definition will be presented in §4 for the 2-d case.

Let $\boldsymbol{p_n} = (p_{n,1}, p_{n,2}, \ldots, p_{n,n})$ with $p_{n,j} \geq 1$ integer, be called the *degree vector.* By $V^n(\mathcal{T}^n, \boldsymbol{p_n}) \subset H_0^1(\Omega)$, we denote the set of all functions v such that the restriction $v|_{I_j^n}$ is a polynomial of degree $p_{n,j}$, i.e., it lies in $\mathcal{P}_{p_{n,j}}$. If $p_{n,j} = p_n$, then we speak about the *uniform degree vector* $\boldsymbol{p_n}$, for which we write p_n. $V^n = V^n(\mathcal{T}^n, \boldsymbol{p_n})$ will be called the space of elements of degree $\boldsymbol{p_n}$.

Having constructed a finite-dimensional subspace as above, we now define the finite element solution as $u_n \in V^n$, satisfying (2.3) for all $v \in V^n$. Then we immediately obtain, with $e_n = u - u_n$,

$$\|e_n\|_E \leq \inf_{v \in V^n} \|u - v\|_E, \tag{2.8}$$

where $\|u\|_E = (B(u, u))^{1/2}$ is the energy norm, which is equivalent to the $H^1(\Omega)$ norm.

In this paper, we will be mostly interested in the energy norm measure of the error in the finite element solution. Any change of the norm can (and usually does) change the results and conclusions following from them. In practice, many different accuracy assessments are used.

2.2. Extension processes. In order to increase the accuracy of the finite element solution, we define an *extension process* which entails the construction of a sequence of subspaces V^n

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(of particular character). The classical extension process uses $V^n(\mathcal{T}^n, \boldsymbol{p}_n)$, where $\mathcal{T}^n, n = 1, 2, \ldots,$ is a sequence of meshes with h_n decreasing ($h_n \to 0$ as $n \to \infty$) and $\boldsymbol{p}_n = \bar{p}$, usually $\bar{p} = 1, 2$. Here the meshes $\mathcal{T}^n$ could be a sequence of uniform, quasiuniform or non-quasiuniform meshes. This choice of V^n is called the *h version* of the FEM.

The *p version* of the FEM on the other hand uses $V^n(\mathcal{T}^n, \boldsymbol{p}_n)$, where $\mathcal{T}^n = \mathcal{T}$ is a fixed mesh and the degrees $p_{n,j} \to \infty$ as $n \to \infty$. The increase of $\boldsymbol{p}_n$ can be uniform ($\boldsymbol{p}_n = p_n$), or selective (i.e., $p_{n,j}$ are different for different j). The mesh $\mathcal{T}$ can be essentially uniform or it may be strongly refined in certain areas.

Finally, the *h-p version* of the FEM consists of $V^n(\mathcal{T}^n, \boldsymbol{p}_n)$ with $\mathcal{T}^n$ in general being refined and $\boldsymbol{p}_n$ increasing with n. Obviously, the h-p version is a generalization of the h and p versions.

It is useful to distinguish between *hierarchic* and *nonhierarchic* extensions. Hierarchic ones are such that $V^{n+1} \supset V^n$. This assures that the error decreases monotonically, i.e., $\|e_{n+1}\|_E \leq \|e_n\|_E$. If the spaces V^n are not hierarchic, then, in general, this monotone behavior does not hold. The basis functions for the p version are in general hierarchic.

In practice, of course, we always use finite meshes with elements of finite degree so that for concrete computations, the distinction between different extensions is blurred. Nevertheless, this distinction is important not only theoretically, but also for practical purposes, since while performing an actual computation, it is usually assumed that the results are in the asymptotic range of the extension process. Since, in practice, one or more steps of the extension process are usually carried out, it is here that the behavior and asymptotic properties of various extensions play an important role.

2.3. Approximation theorems for quasiuniform meshes. The rate at which the finite element approximations converge to the true solution will depend upon two factors: the definition of each of the subspaces V^n (i.e., quasiuniform or non-quasiuniform meshes, choice of $\boldsymbol{p}_n$, etc.) as well as the type of extension procedure (h, p or h-p). We now present various approximation theorems which bound the right-hand side of (2.8), in terms of the number of degrees of freedom $N_n = \dim V^n$. If various methods are to be compared, they should be assessed in terms of the total work required to obtain a given accuracy. Although N_n is one measure of the work required, it does not give the complete picture, since various other factors like amount of user time, special solution techniques, etc., are involved (see [BE], [BEM], and [CM]). With these qualifications in mind, we first discuss the case of the h-p version using quasiuniform meshes. We will omit the subscript n where no confusion would occur.

THEOREM 2.1. *Let $\boldsymbol{p}_n = p_n = p$, and $\mathcal{T}^n$ be quasiuniform. Then for $k \geq 1$,*

$$\|e\|_E \leq Ch^{(\mu-1)} p^{-(k-1)} \|u\|_{H^k(\Omega)}, \tag{2.9}$$

where $\mu = \min(k, p+1)$ and C is independent of n (i.e., of h, p) and u, but depends on k and the quasiuniformity constant τ.

From this, noting that $N = N(V^n) \approx Ch^{-1}p$, we obtain the following corollaries.

COROLLARY 2.1. *For the h version (i.e., $\boldsymbol{p}_n = \bar{p}$) we get*

$$\|e\|_E \leq Ch^{(\mu-1)} \|u\|_{H^k(\Omega)} \leq CN^{-(\mu-1)}. \tag{2.10}$$

COROLLARY 2.2. *For the p version (i.e., $\mathcal{T}^n = \mathcal{T}$) we get*

$$\|e\|_E \leq Cp^{-(k-1)} \|u\|_{H^k(\Omega)} \leq CN^{-(k-1)}. \tag{2.11}$$

The estimate (2.9) is a refined version of the estimate (2.10) which is the one found in many books on the FEM. It shows explicitly how the constant C in (2.10) behaves with respect

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to p. The two-dimensional version of this estimate (see also Theorem 4.1) was proven in [BS1], while that of the estimate (2.11) was proven in [BS2].

Notice that (2.9) shows that if the h version is used in a case that $k < p+1$, then increasing p will decrease the error by decreasing the "constant" $Cp^{-(k-1)}$; it will not affect the exponent of h. This exponent of N (or h) is often called the asymptotic order of convergence, and, for example, for (2.10) will often be written as

$$\|e\|_E = O(N^{-(\mu-1)}).$$

The above estimates are optimal in the following sense. For any N, there is a function $u \in H^k(\Omega)$, possibly depending on N, such that

$$\|e\|_E \geq CN^{-(k-1)}\|u\|_{H^k(\Omega)} \tag{2.12}$$

with $C > 0$ independent of u, N, the mesh $\mathcal{T}$, and element degrees p. The inequality (2.12) follows from the theory of n-widths (see [Pi]).

If $k \geq p+1$, then for the h version there is a $u \in H^k(\Omega)$ such that

$$\|e\|_E \geq C(p)h^p\|u\|_{H^k(\Omega)} \geq CN^{-p} \tag{2.13}$$

with C independent of $\mathcal{T}^n$.

Let us remark that the error bounds (2.12) and (2.13) must be distinguished from asymptotical expansions of the type

$$\|e\|_E = C(u)N^{-(k-1)} + o(N^{-(k-1)}) \tag{2.14}$$

which are stronger estimates.

In practice, we are usually interested in the convergence of the FEM for a specific u, namely, the solution of the problem of interest. For this specific u, the rate of convergence could be higher than the rate for the "worst" case (which is the one addressed in (2.9) and (2.11)). For example, the function u defined in (2.2) belongs maximally to $H^{\alpha-1/2-\varepsilon}(I)$, but the convergence rate in (2.9) can be too pessimistic since for particular relations between h and p, better asymptotic rates of the form (2.14) hold (see, e.g., Theorem 2.2). In general, we must therefore distinguish between the convergence rate for the worst case (which may sometimes only be achievable for a sequence of exact solutions dependent on N) and the convergence rate for a particular case, especially when the solution under consideration is "reasonable." Consequently, when the accuracy of the FEM is experimentally studied by numerical determination of the exponent $(k-1)$ of N in (2.14), it should be kept in mind that this only gives an *upper bound* for the true exponent in (2.12), (2.13) and cannot always serve as an indicator of the performance of the method for all cases.

The estimate (2.13) shows that the maximum possible rate of convergence using the h version is always bounded by the fixed degree p of the polynomials used. In contrast, (2.11) shows that the p version gives arbitrarily high rates of convergence, depending only on the smoothness of the function u.

The above estimates are based *only* on the information that $u \in H^k(\Omega)$. If more information is available, then better estimates can be obtained. For example, if u is analytic on [0,1], then the error for the p version decays exponentially.

Let us now assume that the solution has a singularity at a point, or specifically, that u is given by (2.2). Then we have the following.

THEOREM 2.2. *Let the solution u be given by* (2.2). *Then*

$$\|e\|_E \leq C\min[h^{\alpha-1/2}, h^{\min(\alpha-1/2,\, p-\alpha+1/2)}p^{-2(\alpha-1/2)}].$$

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This immediately gives the following.

COROLLARY 2.3. *For the h version*

$$\|e\|_E \leq Ch^{\alpha-1/2} = O(N^{-(\alpha-1/2)}). \tag{2.15}$$

COROLLARY 2.4. *For the p version*

$$\|e\|_E \leq Cp^{-2(\alpha-1/2)} = O(N^{-2(\alpha-1/2)}). \tag{2.16}$$

The estimate (2.15) follows from (2.10) when Besov spaces $B^{\alpha}_{2,\infty}$ are used instead of fractional Sobolev spaces $H^k(\Omega)$. It is sharp when quasiuniform meshes are used, in the sense that there exists $\bar{C} > 0$ such that

$$\|e\|_E \geq \bar{C}h^{\alpha-1/2}. \tag{2.17}$$

Also, (2.16) is sharp as well. Theorem 2.2 is once more a special version of the theorem for the two-dimensional case proven in [BS1], [BD] (where $\alpha - \frac{1}{2}$ has to be replaced by α).

We see that the p version gives twice the rate of convergence as the h version (on quasiuniform meshes) when the solution has an x^{α} type of singularity at an endpoint of the element. If the singularity is located inside the element, then this doubling in the rate does not occur. The above result about doubling of the rate was first proven for the two-dimensional case in [BSK], where a slightly weaker result, showing the doubled rate of convergence reduced by an amount $O(N^{\varepsilon})(\varepsilon > 0$ arbitrary), was obtained. The dependence on ε was removed in [BS2]. (Note that for the h version with nonquasiuniform (radical) meshes, the rate obtained will be $O(N^{-p})$, which is generally higher than (2.15)—see §2.4).

Let us outline the reason for obtaining the rate of convergence $O(N^{-2(\alpha-1/2)})$ in Corollary 2.4. Suppose, for simplicity, we have just one element and the domain is $\Omega = (-1, 1)$ instead of $\Omega = (0, 1)$, with the solution (2.2) correspondingly scaled. Then if we were to use the standard estimate (2.11) with u being imbedded in the Besov space like we did for the h version, we would get an identical rate $O(N^{-(\alpha-1/2)})$ as in (2.15). It turns out, however, that using various orthogonality properties of Jacobi polynomials, one can show that [GiB]

$$\|u - u_p\|_E \leq Cp^{-k}\|u'\|_{V^k_k(\Omega)}, \tag{2.18}$$

where

$$\|v\|^2_{V^k_k(\Omega)} \equiv \int_{-1}^{1} (1-t^2)^k (v^{(k)})^2 dt \tag{2.19}$$

so that the usual Sobolev norm in (2.11) is replaced by the weighted Sobolev norm in (2.19). Now it may be readily verified that although the solution (2.2) satisfies $u' \in H^k(\Omega)$ for $k < \alpha - \frac{1}{2}$ only, due to the weight $(1-t^2)^k$ in (2.19), we have instead, $u' \in V^k_k(\Omega)$ for $k < 2(\alpha - \frac{1}{2})$, leading to the doubled rate up to an arbitrary $\varepsilon > 0$.

In order to obtain a rate of $O(p^{-k})$ with k actually equal to $2(\alpha - \frac{1}{2})$ without the ε, a more detailed analysis is required. Specifically, we expand u' in terms of the Legendre polynomials $\{P_n\}$

$$u'(x) = \sum_{n=1}^{\infty} a_n P_n(x) \tag{2.20}$$

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and choose u_p such that $u_p(\pm 1) = 0$ and u'_p is the sum of the first $(p-1)$ terms of (2.20). Then we have, by the orthogonality property of $\{P_n\}$,

$$\|u - u_p\|_E^2 = \sum_p^\infty a_n^2 \frac{2}{2n+1}. \tag{2.21}$$

Here, a_n may be explicitly computed by the formula

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} u' P_n(t)dt = \frac{2n+1}{2}\int_{-1}^{1} (\alpha(1+t)^{\alpha-1} - 2^\alpha)\frac{1}{2^n n!}[(1-t^2)^n]^{(n)}dt,$$

using the Rodriguez form of Legendre polynomials. Once again, the weight $(1-t^2)^n$ leads to an "absorption" of the singularity at the end of Ω. Integrating by parts and using some properties of the gamma function leads to the following estimate, proved in [GiB]:

$$a_n = (-1)^{n-1}\frac{C(\alpha)}{n^{2\alpha-1}}\left(1 + \frac{C_1(\alpha)}{n} + \frac{C_2(\alpha)}{n^2} + \cdots\right) \tag{2.22}$$

with

$$C(\alpha) = \frac{2^\alpha |\Gamma(\alpha)|^2 \sin \pi(\alpha - 1)}{\pi}.$$

Equations (2.21), (2.22) lead to the following theorem, which has been stated in terms of the original domain $\Omega = (0, 1)$ [GiB].

THEOREM 2.3. *For the p version in the case of a single element, we have*

$$\|e\|_E = C_0(\alpha)\frac{1}{p^{2\alpha-1}}\left(1 + O\left(\frac{1}{p}\right)\right) \tag{2.23}$$

as $p \to \infty$, *where*

$$C_0(\alpha) = \frac{\alpha(\Gamma(\alpha))^2 |\sin \pi\alpha|}{\pi\sqrt{2\alpha - 1}}.$$

Let us give a numerical example. Consider the cases $\alpha = 0.7$ and $\alpha = 3.5$. In Table 1 we show computed values of $\|e_n\|_E = E_p$ and $R_p^A = E_p/E_p^A$, where E_p^A is the leading term in the right-hand side of (2.23). The numerical results are seen to be in close agreement with (2.23).

2.4. Approximation theorems for nonquasiuniform meshes. Let us now consider the case when nonquasiuniform meshes are used and the solution is given by (2.2). We will show that it is possible to get an exponential rate of convergence (in N) using the h-p version of the FEM. First, however, let us discuss the h version. Let us show that the best possible rate is $O(N^{-p})$ (no matter what mesh is used), i.e., the convergence is always *algebraic*. As before, N is the number of degrees of freedom and p is the degree of polynomials used in the h version.

For simplicity, we take $p = 1$. Let $w(x) = u'(x) = \alpha x^{\alpha-1} - 1$ on $\Omega = (0, 1)$ (α not an integer). Then $w(x)$ is monotonic and $|w'(x)| \geq \alpha|\alpha - 1| = \kappa$. Now over $I_j = (x_{j-1}, x_j)$ we have

$$\int_{I_j} |e'|^2 dx \geq \inf_{c\in\mathbb{R}} \int_{I_j} |w(x) - c|^2 dx \geq \frac{2}{3}\left(\frac{h_j}{2}\right)^3 \kappa^2.$$

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TABLE 1
The error of the finite element solution.

p	$\alpha = 0.7$		$\alpha = 3.5$	
	E_p	R_p^A	E_p	R_p^A
1	4.743E-1	0.9877	1.021	0.203
2	3.637E-1	1.0067	3.402E-1	4.335
3	3.090E-1	0.9985	3.093E-2	4.488
4	2.766E-1	0.9992	2.379E-3	1.949
5	2.522E-1	0.9995	4.760E-4	1.480
6	2.344E-1	0.9996	1.400E-4	1.300
7	2.204E-1	0.9997	5.154E-5	1.208
8	2.090E-1	0.9998	2.210E-5	1.153
9	1.994E-1	0.9998	1.057E-5	1.118
10	1.912E-1	0.9999	5.495E-6	1.094
11	1.840E-1	0.9999	3.053E-6	1.077
12	1.777E-1	0.9999	1.790E-6	1.064
13	1.722E-1	1.000	8.999E-7	1.054
14	1.671E-1	1.000	6.978E-7	1.046
15	1.626E-1	1.000	4.585E-7	1.040

Hence

$$\|e\|_E^2 \geq C \sum_{j=1}^{m_n} h_j^3.$$

(Here m_n is the number of elements.) Since the h_j's in the above are arbitrary, we estimate the above sum as follows. Let

$$J = \left\{ j | h_j \geq \frac{1}{2m_n} \right\}, \qquad J^* = \left\{ j | h_j < \frac{1}{2m_n} \right\}.$$

Then obviously

$$\sum_{j \in J^*} h_j < m_n \frac{1}{2m_n} \leq \frac{1}{2},$$

so that since $I = (0, 1)$ we have

$$\sum_{j \in J} h_j \geq \frac{1}{2}.$$

Therefore

$$\sum_{j=1}^{m_n} h_j^3 \geq \sum_{j \in J} h_j^3 \geq \left(\frac{1}{2m_n} \right)^2 \sum_{j \in J} h_j \geq \frac{C}{(m_n)^2}.$$

Because $N = m_n - 1$ we get

$$\|e\|_E \geq CN^{-1}, \tag{2.24}$$

no matter what kind of mesh is used. A similar result holds for $p > 1$.

The following theorem shows that we also have $\|e\|_E \leq CN^{-p}$, provided the mesh is properly chosen (i.e., the above bound is attained). It also shows that the radical mesh is optimal in the following sense. Consider the sequences of meshes defined by grading functions g that satisfy

(i) $g \in C^0[0,1] \cap C^1(0,1)$

(ii) $g(0) = 0, \quad g(1) = 1$ and g is strictly increasing.

Then the error of the FEM with the radical mesh ($g = x^{\beta_0}$) attains the minimum among all grading meshes satisfying (i) and (ii).

THEOREM 2.4. *For the h version, using polynomials of fixed degree (i.e., $p_n = p$), the radical mesh* (2.6) *using m elements with*

$$\beta_0 = \frac{p + 1/2}{\alpha - 1/2} \tag{2.25}$$

is the optimal choice and the following estimate holds with $h = 1/m$:

$$\lim_{h \to \infty} \frac{\|e\|_E}{h^p} = C(\alpha, p)\beta_0^{p+1/2}, \tag{2.26}$$

where

$$C(\alpha, p) = \frac{\alpha\Gamma(\alpha)|\sin \pi\alpha|\Gamma(p - \alpha + 1)}{\sqrt{\pi}4^p\sqrt{2p+1}\Gamma(p + 1/2)}. \tag{2.27}$$

It is better to (mildly) overrefine the mesh than underrefine it. This can be seen from the detailed results proven in [GiB]. Mild overrefinement is advisable in two and three dimensions as well.

We see that proper refinement leads to the highest possible rate in the h version, algebraic in N, which is the same as would be achieved if the solution were smooth. We also see that for fixed p (uniform) the best mesh to use (for the h version) is a radical one, with the refinement strength (i.e., β) being increased linearly with p.

Let us now show that an *exponential* rate of convergence is possible using the h-p version. For simplicity, we will take a uniform degree vector. Let us take the mesh to be the geometric one defined by (2.7) and let

$$x_{i-1/2} = \tfrac{1}{2}(x_{i-1} + x_i).$$

As before, let $w(x) = u'(x)$, with $u(x)$ given by (2.2), and let us expand $w(x)$ in terms of its Taylor expansion about the point $x_{i-1/2}$ so that

$$\begin{aligned} w(x) = w_{i,p}(x) + r_{i,p}(x) &= \sum_{k=0}^{p-1} a_k^{(i)}(x - x_{i-1/2})^k \\ &+ \sum_{k=p}^{\infty} a_k^{(i)}(x - x_{i-1/2})^k, \end{aligned} \tag{2.28}$$

where $w_{i,p}(x)$ is the Taylor polynomial of degree $p - 1$ and $r_{i,p}(x)$ is the remainder. Using (2.7a), (2.7b) we see that

$$x_{i-1/2} = \frac{x_i + x_{i-1}}{2} = q^{m-i}\left(\frac{1+q}{2}\right)$$

and

$$\frac{x_i - x_{i-1}}{2} = q^{m-1}\left(\frac{1-q}{2}\right).$$

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Now for $i = 2, 3, \ldots, m$, the function $x^{\alpha-1}$ is analytic in the closed disk with center $x_{i-1/2}$ and radius $q^{(m-i)}(((1+q)/2) - q^2)(= x_{i-1/2} - x_{i-2}$ for $i \geq 3)$. Also $x^{\alpha-1}$ is bounded by $C_q^{(m-i)(\alpha-1)}$ there. Hence, applying the Cauchy theorem,

$$|a_k^{(i)}||\frac{x_i - x_{i-1}}{2}|^k \leq Cq^{(m-i)(\alpha-1)}\left(\frac{1-q}{2}\right)^k\left(\frac{1+q}{2} - q^2\right)^{-k} \leq Cq^{(m-i)(\alpha-1)}\rho^k$$

where $|\rho| = |2q+1|^{-1} < 1$, and C is independent of m, i, and k. Hence

$$|w(x) - w_{i,p}(x)| \leq Cq^{(m-i)(\alpha-1)}\rho^p. \tag{2.29}$$

Using this,

$$\begin{aligned} E_i &= \inf_{r\in\mathcal{P}_{p-1}} \int_{I_i} |r-w|^2 dx \leq \int_{I_i} |w_{i,p} - w|^2 dx \\ &\leq C\rho^{2p} q^{(m-i)(2(\alpha-1)+1)}. \end{aligned} \tag{2.30}$$

Therefore

$$\sum_{i=2}^{m} E_i \leq C\rho^{2p}. \tag{2.31}$$

Next, we note that

$$\inf_{r\in\mathcal{P}_{p-1}} \int_{I_1} |r-w|^2 dx \leq \int_{I_1} |w(x)|^2 dx = Cq^{(m-1)(2\alpha-1)} \leq C\rho^{m_1} \tag{2.32}$$

for some $0 < \rho_1 < 1$. Choosing $p = \kappa m$ and noting that $N = m(p+1)$ we see, using (2.31) and (2.32), that

$$\sum_{i=1}^{m} E_i \leq Ce^{-\gamma N^{1/2}}$$

for some $\gamma > 0$ and C depending on q and α.

Now let r_i denote the minimizer of E_i in (2.30). Then obviously

$$\int_{I_i} (r_i - w)dx = 0.$$

Hence defining $R(x)$, $0 < x < 1$, $R(x)|_{I_i} = r_i(x)$ we can define

$$\omega(x) = \int_0^x R(t)dt$$

and get $\omega(1) = 0$. Hence $\omega \in V^n(\mathcal{T}, p)$ such that

$$\|u - \omega\|_{H^1(I)} \leq Ce^{-\gamma N^{1/2}}$$

and the exponential rate is proven. See [GiB] for a more precise analysis.

We have used **p** uniform. We could also analyze the case for **p** nonuniform. In fact, the optimal rate is obtained by choosing a nonuniform degree vector as described in the following theorems.

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THEOREM 2.5. *Let* [a] *denote the integral part of a. Then the optimal mesh for the h-p version is the geometric one with ratio* $q_0 = (\sqrt{2}-1)^2 = e^{-1.7626} \approx 0.17$ *and linear degree vector* $\mathbf{p} = \{p_j\}$, $p_j = [s_0 j] + 1$ *with* $s_0 = 2\alpha - 1$. *With this choice,*

$$\|e\|_E \le C(\alpha) q_0^{\sqrt{(\alpha-1/2)N}} = C(\alpha) e^{-1.7626\sqrt{(\alpha-1/2)N}}. \tag{2.34}$$

Let us be more precise about the notion of optimality. We have the following theorem.

THEOREM 2.6. *For any mesh* $\mathcal{T}$, *and any degree vector p, we have*

$$\|e\|_E \ge C(\alpha) \frac{q_0^{\sqrt{(\alpha-1/2)N}}}{\sqrt{N^{(\alpha-1/2)}}}.$$

Comparing the lower bound given in Theorem 2.6 and the upper one in Theorem 2.5, we see that the geometrical mesh with linear degree vector gives the maximal possible exponential rate. This is the justification for calling the mesh optimal.

For any given geometrical mesh with ratio q there is an optimal p, which is linear, with slope

$$s_0(q) = \left(\alpha - \frac{1}{2}\right) \frac{\log q}{\log r}, \qquad r = \frac{1-\sqrt{q}}{1+\sqrt{q}}. \tag{2.35}$$

For this choice, the error is

$$\|e\|_E \le C(\alpha, q) e^{-\sqrt{(\alpha-1/2)N}\sqrt{2\log q \log r}}. \tag{2.36}$$

This shows that for small q (i.e., strong refinement) the optimal degree vector is less uniform.

In Theorems 2.5 and 2.6, we have addressed the case when the degree vector was nonuniform. For uniform degree vector, the situation is slightly different.

THEOREM 2.7. *Let q be the ratio of the geometric mesh with m elements and* $p = s_0 m$, *where* s_0 *is as in* (2.35). *Then*

$$\|e\|_E \le C(\alpha, q) e^{-\sqrt{(\alpha-1/2)N}\sqrt{\log q \log r}} N^{-\sigma/2}$$

where

$$\sigma = \min(2\alpha - 1, \alpha).$$

Substituting the optimal $q = q_0 = (\sqrt{2}-1)^2$ in the above, we get

$$\|e\|_E \le C(\alpha) e^{-1.2464\sqrt{(\alpha-1/2)N}} N^{-\sigma/2}. \tag{2.37}$$

Hence we see that the exponential rate is smaller in this case.

In practice (in higher dimensions) one often uses a geometric mesh with a fixed number of elements m and determines p more or less adaptively. For every m, the convergence will have two parts. In the pre-asymptotic portion, the rate is exponential, while in the asymptotic region, one obtains the algebraic rate of the p version. If the result of a computation lies in the algebraic part, then this indicates the mesh was not sufficiently refined.

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If a uniform degree vector p is to be used, then it turns out that the geometric mesh does not give the best rate of convergence, i.e., (2.37) is not optimal. We have seen that for fixed uniform degree vector (i.e., the h version), the radical (and not the geometric) mesh is optimal. If we consider a sequence of uniform degree vectors and perform the h version for each of them with the corresponding optimal radical mesh, then the envelope of the resulting error curves leads to an h-p version which gives better results than the one in Theorem 2.7. We have the following theorem.

THEOREM 2.8. *There is an h-p version with uniform degree vector p which has the following estimate*:

$$\|e\|_E \leq \frac{C(\alpha)}{N^{(\alpha-1/2)/2}} e^{-(4/e)\sqrt{(\alpha-1/2)N}} = \frac{C(\alpha)}{N^{(\alpha-1/2)/2}} e^{-1.4715\sqrt{(\alpha-1/2)N}}. \tag{2.38}$$

As $N \to \infty$, the meshes (inside (0,1)) tend to the geometric mesh with ratio $q = e^{-4/e^2} \approx 0.5820$ and the relation between the degree p and the number of elements m tends to a linear one,

$$p \approx \frac{4}{e^2}\left(\alpha - \frac{1}{2}\right) m.$$

Various numerical examples illustrating these results may be found in [GiB].

2.5. The rate of convergence in the L_2 norm. So far, we have discussed the h, p, and h-p version in terms of their convergence in the energy norm. An important difference between the h version on the one hand and the p (and h-p) version on the other, which is apparent from the above, is that the p (and h-p) version leads to different asymptotic rates of convergence, depending upon the nature of the solution (for example, the "doubling" effect for x^α singularities). In contrast, the only way that the h version could lead to a rate better than $O(h^p)$ is if the exact solution were a polynomial of degree $\leq p$. This difference leads to an interesting effect in the error in the L_2 and other lower-order norms.

Let us discuss the error in the L_2 norm for the p version. Suppose we first consider the case where $u \in H^k(\Omega)$ and u satisfies (2.1). Then, by the usual duality argument (which is proved for the h version in [Ci]), we may show

$$\|e\|_{L_2(\Omega)} \leq Cp^{-1}\|e\|_E \tag{2.39}$$

so that using Corollary 2.2,

$$\|e\|_{L_2(\Omega)} \leq Cp^{-k}\|u\|_{H^k(\Omega)}.$$

In fact, the usual duality agrument gives the following theorem.

THEOREM 2.9. *Let $\boldsymbol{p}_n = p_n = p$ and $\{\mathcal{T}^n\}$ be quasiuniform. Then for $k \geq 1$,*

$$\|e\|_{L_2(\Omega)} \leq Ch^\mu p^{-k}\|u\|_{H^k(\Omega)} \tag{2.40}$$

where $\mu = \min(k, p+1)$.

This result is optimal in the sense that

$$\sup_{\substack{u \in H^k(\Omega) \\ \|u\|_{H^k(\Omega)} \leq 1}} \inf_{v \in V^n} \|u - v\|_{L_2(\Omega)} \geq Ch^\mu p^{-k}.$$

If we now consider the special singular solution of the form (2.2) (scaled on to $\Omega = (-1, 1)$), then Theorem 2.9 predicts a rate of $O(h^{\min(\alpha+1/2, p+1)})$ for the h version, which is

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the best L_2 error possible with a quasiuniform mesh, so that the h version is again optimal. For the p version (using a single element), we may now use (2.39) with (2.23) to obtain an $O(p^{-2\alpha})$ rate. The *optimal* L_2 approximation estimate on the other hand, is given by [GiB]:

$$\inf_{v\in V^n} \|u - v\|_{L_2(\Omega)} = Cp^{-2(\alpha+\frac{1}{2})}(1 + O(p^{-1})). \tag{2.41}$$

Hence the estimate through duality does not give the optimal rate (2.41).

An improved estimate of $O(p^{-2\alpha-\frac{1}{2}})$ for $\|e\|_{L_2(\Omega)}$ may be obtained using the fact that

$$\int_{-1}^{x} P_p(t)dt = \frac{1}{2p+1}(P_{p+1}(x) - P_{p-1}(x)). \tag{2.42}$$

Then by (2.20), the definition of u_p, and (2.42),

$$\begin{aligned} u - u_p &= \sum_{r=p}^{\infty} \frac{a_r}{2r+1} \int_{-1}^{x} P_r(t)dt \\ &= -\frac{a_p}{2p+1} P_{p-1} - \frac{a_{p+1}}{2p+3} P_p + \sum_{r=p}^{\infty} \left(\frac{a_r}{2r+1} - \frac{a_{r+2}}{2r+5} \right) P_{r+1}. \end{aligned} \tag{2.43}$$

Define

$$b_r = \frac{a_r}{2r+1} - \frac{a_{r+2}}{2r+5}.$$

Then, using (2.22), we obtain

$$b_r = (-1)^{r-1} \frac{\tilde{C}(\alpha)}{r^{2\alpha+1}} \left(1 + \frac{\tilde{C}_1(\alpha)}{r} + \frac{\tilde{C}_2(\alpha)}{r^2} + \cdots \right). \tag{2.44}$$

Now by (2.43),

$$\|u - u_p\|_{L_2(\Omega)} = \frac{a_p^2}{(2p+1)^2} \frac{2}{2p-1} + \frac{a_{p+1}^2}{(2p+3)^2} \frac{2}{2p+1} + \sum_{r=p}^{\infty} b_r^2 \frac{2}{2r+3}. \tag{2.45}$$

Using (2.44), we see that

$$\sum_{r=p}^{\infty} b_r^2 \frac{2}{2r+3} = \frac{C(\alpha)}{p^{4\alpha+2}}(1 + O(p^{-1})).$$

Also, (2.22) may be used to estimate the first two terms in (2.45), yielding

$$\|u - u_p\|_{L_2(\Omega)} = \frac{C(\alpha)}{p^{2\alpha+1/2}}(1 + O(p^{-1})). \tag{2.46}$$

This estimate, which is sharp, improves upon the rate through duality, but it still does not give the optimal rate of (2.41).

Hence we observe the following interesting difference when the projection of the function (2.2) is taken in the energy norm (equivalent to the $H^1(\Omega)$ norm). For the h version with quasiuniform meshes, the convergence in the $L_2(\Omega)$ norm is $O(h^{\min(\alpha+\frac{1}{2}, p+1)})$ which is *optimal*. For the p version, this L_2 convergence rate is higher, $O(p^{-2\alpha-\frac{1}{2}})$. However, the best L_2 approximation is $O(p^{-2\alpha-1})$, which is even higher. As a result, the projection in the energy norm does not give the optimal rate in the $L_2(\Omega)$ norm for (2.2) (see [Er]). Let us mention that in two dimensions, for nonconvex domains, the h version *also* fails to give the optimal L_2 rate (see §4.2).

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2.6. The A-pollution problem. Let us now discuss another difference between the h and p versions, the phenomenon of *A-pollution* (for "Approximability pollution"). This effect is related to the unsmoothness of u caused by unsmoothness of the input data (such as the right hand side). If such unsmoothness in one area influences the accuracy of u_n in a different area, we say A-pollution is present. Obviously, here the accuracy of u_n is not governed by the local behavior of u. Let us elaborate by means of an example.

Consider once again the approximation of u given by (2.2) (scaled to $(-1, 1)$). Obviously, if (2.2) is a solution to (2.1), then the data f is unsmooth at $x = -1$. Then u is singular at $x = -1$ but analytic everywhere else. Suppose u_n is the solution using the h version (say with $p = 1$). Then it is easily proven that u_n is simply the linear interpolant of u (an analogous result holds for higher p). This obviously implies that the error $|(u - u_n)(0)|$, for instance, will only depend upon the behavior of u over some $I_\delta = (\delta, \delta)$, $\delta < 1$. In fact, this error will converge asymptotically at the optimal rate of $O(h^2)$ for *any* $\alpha > 1/2$. Also, for fixed δ, $\|u - u_n\|_{H^r(I_\delta)}$ for $r = 0, 1$ will similarly converge at the optimal rate $O(h^{2-r})$, even if $1/2 < \alpha < 3/2$. We say, therefore, that the h version is *free from A-pollution*. The h-p version also turns out to be free from A-pollution, so that $|(u - u_n)(0)|$, $\|u - u_n\|_{L_2(I_\delta)}$, $\|u - u_n\|_{H^1(I_\delta)}$ all converge at the optimal exponential asymptotic rate determined only by the analytic behavior of u over $I_{\delta'}$, δ' close to δ, and not by the singular behavior of f and u at $x = -1$.

The p version, however, will exhibit A-pollution. Suppose we use a single element, then with $u_n \equiv u_p$, $\|u - u_p\|_{H^r(I_\delta)}$, $r = 0, 1$, will be no better than the rates given in §§2.3, 2.5 for $\|u - u_p\|_{H^r(I)}$, even though u is analytic over I_δ. More precisely, let us estimate $|(u - u_p)(0)|$. Noting that

$$P_{2m}(0) = (-1)^m \frac{(2m-1)!!}{(2m)!!} = \frac{(-1)^m}{\sqrt{\pi}} \frac{1}{\sqrt{m}} \left(1 + \frac{C_1}{m} + \cdots\right)$$

$$P_{2m+1}(0) = 0,$$

we get for $p = 2k + 1$ (using (2.22), (2.43)),

$$(u - u_p)(0) = \frac{C}{p^{2\alpha+1/2}}(1 + O(p^{-1})) + \sum_{m=k}^{\infty} \frac{(-1)^{m+1} b_{2m+1}}{\sqrt{\pi}(m+1)^{1/2}}(1 + O(m^{-1})).$$

Using (2.44) and the fact that the terms in the sum are of alternating sign, this gives

$$|(u - u_p)(0)| = \frac{C}{p^{2\alpha+1/2}}(1 + O(p^{-1})).$$

The above estimate clearly shows the A-pollution effect, since in the absence of pollution, the rate should be exponential because u is analytic over I_δ. The reason this A-pollution occurs should be clear; unlike the h and h-p versions, there is no "buffer" of elements now to isolate the singularity from the rest of the domain.

3. The model problem and its regularity. In §2 we have analyzed a simple one-dimensional problem. In this section we will address the basic properties of the analogous problem in two and three dimensions. We will concentrate on model problems which are typical in the field of computational (structural) mechanics. Mathematically, these problems are described by elliptic partial differential equations with piecewise analytic input data. Hence this class of practical problems is relatively narrow and well-defined, leading to several important common properties which can be exploited in their numerical treatment.

Let us immediately note one important difference between the one and higher dimensional cases. In the previous section, the regularity of the exact solution was completely determined

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by the regularity of the input data. In higher dimensions, this is no longer the case, since in addition to the right hand side, unsmoothness can also be present in the boundary of the domain, leading to singular components arising in the solution. For instance, consider the L-shaped domain of Fig. 3.1, which we will encounter again later in §§4.2 and 4.4. Then the solution of various model elliptic problems will have singular components in neighborhoods of each of the vertices A, B, C, D, E, O, with the worst singularity occurring at O, the vertix with the largest interior angle. Moreover, these singular components behave like r^α, which is of the same nature as the x^α singularities discussed in the previous section. (These x^α singularities were artificial in the one-dimensional context and were analyzed primarily because of their similarity to the r^α singularities.) The greater part of this section is devoted to characterizing the above singular behavior in precise mathematical terms needed for the subsequent convergence analysis, both for the two-dimensional and more complicated three-dimensional case.

FIG. 3.1. *The* L-*shaped domain.*

3.1. The model problem. We consider a curvilinear polygon $\Omega \subset \mathbb{R}^2$ with piecewise analytic boundary $\partial\Omega = \bigcup_{i=1}^{M} \bar{\Gamma}_i$, where Γ_i is an open arc and $\bar{\Gamma}_i$ is analytic with endpoints A_i, A_{i+1}. $A_1, A_2, \ldots, A_M (A_{M+1} = A_1)$ will denote the vertices with internal angles $\omega_1, \ldots \omega_M (0 < \omega_i \leq 2\pi)$. (If $\omega_i = 2\pi$ then we have a slit domain, where the boundary is taken to be two-sided.) $\partial\Omega$ will consist of two parts, $\Gamma_D = \bigcup_{i \in \mathcal{D}} \bar{\Gamma}_i$ being the Dirichlet boundary and $\Gamma_N = \partial\Omega - \Gamma_D$ the Neumann boundary. It should be noted that the boundary could be smooth everywhere, with $\omega_i = \pi$ and the vertices A_i corresponding to points in $\bar{\Gamma}_D \cap \bar{\Gamma}_N$. We restrict ourselves to simply connected domains in order to simplify the exposition. An example of such a domain and the notation used is shown in Fig. 3.2.

In three dimensions, $\Omega \subset \mathbb{R}^3$ will be a bounded domain with piecewise analytic boundary $\partial\Omega$ consisting of faces $\Gamma_i, i = 1, 2, \ldots, M$ which are curved polygons in $\mathbb{R}^3$, joined by edges $\gamma_i, i = 1, \ldots, n_e$ (curves in $\mathbb{R}^3$) and vertices $A_i, i = 1, \ldots, n_w$. Γ_D and Γ_N will now be the union of some faces.

Analogously to (2.1) we will now consider a second order elliptic system of m differential equations for the unknown vector $\vec{u} = (u_1, \ldots, u_m)$ which, when cast into the variational form $B(\vec{u}\,\vec{v})$ (analogous to (2.3)), is given by

$$B(\vec{u}, \vec{v}) = \int_\Omega \left(\sum_{i,j=1}^{m} \sum_{k,\ell=1}^{t} a_{ijk\ell} \frac{\partial u_i}{\partial x_k} \frac{\partial v_j}{\partial x_\ell} \right) dx, \tag{3.1}$$

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FIG. 3.2. *The domain* Ω.

where $t = 2$ or 3 and $a_{ijk\ell} = a_{ji\ell k}$ are analytical functions on Ω satisfying an ellipticity condition that ensures

$$B(\vec{u}, \vec{v}) \geq \gamma | \vec{u} |^2_{H^1(\Omega)}, \qquad \gamma > 0. \tag{3.2}$$

Let

$$H^1_D(\Omega) = \{\vec{v} \in H^1(\Omega), \vec{v} = 0 \text{ on } \Gamma_D\}.$$

Furthermore, let F be a continuous functional, defined on $H^1_D(\Omega)$, of the form

$$F(\vec{v}) = \int_\Omega \vec{f} \cdot \vec{v}\, dx + \int_{\Gamma_N} \vec{h} \cdot \vec{v}\, ds, \tag{3.3}$$

where $\vec{f}$ is analytic on $\bar{\Omega}$ and $\vec{h}$ is analytic on every $\bar{\Gamma}_i$, $\Gamma_i \subset \Gamma_N$. Moreover, let $\vec{g} \in C^0(\Gamma_D)$ and be analytic on each $\bar{\Gamma}_i \subset \Gamma_D$. Then our model problem consists of finding $\vec{u} \in H^1(\Omega)$ satisfying

$$B(\vec{u}, \vec{v}) = F(\vec{v}) \quad \forall\, \vec{v} \in H^1_D(\Omega), \tag{3.4}$$

$$\vec{u} = \vec{g} \quad \text{on } \Gamma_D. \tag{3.5}$$

For $\Gamma_D \neq \phi$ condition (3.2) ensures a unique solution for (3.4), (3.5). For $\Gamma_D = \phi$, we need an additional solvability condition on Γ and Ω (such as the absence of rigid body motions for the elasticity problem).

We define the energy norm $\|\vec{u}\|_E = (B(\vec{u}, \vec{u}))^{1/2}$, equivalent to the $H^1(\Omega)$ norm. By restricting our formulation to the above class of problems with piecewise analytic input data of the type assumed, we ensure that the solution will possess certain special features.

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Let us now describe two concrete examples of (3.4), (3.5) in which we shall be interested. The first problem is a scalar one ($m = 1$) on $\Omega \subset \mathbb{R}^t$ with

$$B(u, v) = \int_\Omega \left(\sum_{i=1}^{t} k_i \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx, \qquad k_i = k_i(x) > 0. \tag{3.6}$$

For $k_i \equiv 1$, we get the Poisson equation. Most of our cited results are in two dimensions but the three-dimensional problem will be mentioned as well.

In Section 5 we will be interested in the 2-d case that k_i are very different in magnitude. Obviously, in this case the bilinear form becomes degenerate as $k_1/k_2 \to 0$. Such degeneracies occur in various practical problems and their numerical treatment will be discussed in the context of the robustness of the method (the locking problem).

Our second example is the typical model problem of two-dimensional elasticity. We will restrict ourselves here to the case of isotropic homogeneous materials (plane strain) for which $m = 2, t = 2$, and

$$B(\vec{u}, \vec{v}) = \frac{E}{1+\nu} \int_\Omega \left[\sum_{i,j=1}^{t} \varepsilon_{ij}(\vec{u})\varepsilon_{ij}(\vec{v}) + \frac{\nu}{1-2\nu}(\operatorname{div} \vec{u}, \operatorname{div} \vec{v}) \right] dx, \tag{3.7}$$

where $E > 0$ is the Young's modulus of elasticity, $0 \le \nu < \frac{1}{2}$ is the Poisson ratio and

$$\varepsilon_{ij}(\vec{u}) = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]. \tag{3.8}$$

For $\nu \to \frac{1}{2}$, the bilinear form becomes degenerate as in the first example. Nevertheless, there is a large difference in these two cases. In the first one, the solution becomes less smooth as the form degenerates and in the limiting case the problem is not elliptic. In the second example, the degeneracy does not influence the smoothness of the solution and the limiting case is still an elliptic problem. See, e.g., [AF1] for a general analysis of the degeneracy with the smoothness being preserved.

In three-dimensional elasticity the formulation is analogous to (3.7) and (3.8).

3.2. Regularity of the two-dimensional problem. Let us now analyze the regularity of the solution of our model problem for $\Omega \subset \mathbb{R}^2$. First assume that $\partial\Omega$ is smooth and that either $\Gamma_N = \partial\Omega$ or $\Gamma_D = \partial\Omega$. Then (see [LM])

$$\|\vec{u}\|_{H^k(\Omega)} \le C \left[\|\vec{f}\|_{H^{k-2}(\Omega)} + \|\vec{g}\|_{H^{k-1/2}(\Gamma_D)} + \|\vec{h}\|_{H^{k-3/2}(\Gamma_N)} \right]. \tag{3.9}$$

If the data (i.e., $\partial\Omega, \vec{g}, \vec{h}$) are not sufficiently smooth (at vertices) or both parts Γ_N and Γ_D are present, then (3.9) does not hold. (Note that the analyticity of $\vec{g}, \vec{h}$ is only on $\bar{\Gamma}_i$.) However, under our assumptions, $\vec{u}$ will be analytic on $\bar{\Omega} - \cup_{i=1}^{M} A_i$ and in the neighborhood of a vertex A_i. For the scalar equation ($m = 1$), u will have the form (see [CS], [Gr2], [Gr4], and [Da])

$$u = \sum_{j=1}^{J} \sum_{s=0}^{S} \sum_{t=0}^{T} c_{jst} \psi_{jst}(\varphi) r^{\alpha_j + t} \ln^s r + u_0 = \sum_{j=1}^{J} u_j + u_0, \tag{3.10}$$

in the neighborhood of A_i. Here, (r, φ) are local polar coordinates at the vertex A_i and ψ_{jst} are analytic functions in φ. The decomposition (3.10) for a proper choice of J, S, T is such

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that the remainder u_0 is smoother than the terms in the sum and belongs to a weighted space with the weight $w(r) \to \infty$ as $r \to 0$.

In the simpler case of the Laplace operator on a straight-sided polygonal domain and homogeneous boundary conditions, one has $S = T = 0$, i.e.,

$$u = \sum_{j=1}^{J} c_j \psi_j(\varphi) r^{\alpha_j} + u_0, \tag{3.10$'$}$$

except for the case of α_j being an integer, when $S = 1$. The ψ_j and α_j are explicitly known:

$$\alpha_j = \begin{cases} j\pi/\omega & \text{for Dirichlet or Neumann boundary conditions,} \\ (2j-1)\pi/2\omega & \text{for mixed boundary conditions,} \end{cases}$$

$$\psi_j(\varphi) = \begin{cases} \sin \alpha_j \varphi & \text{for the Dirichlet or mixed problem,} \\ \cos \alpha_j \varphi & \text{for the Neumann problem.} \end{cases}$$

The mixed problem here means Dirichlet conditions for $\varphi = 0$ and Neumann conditions for $\varphi = \omega$.

The number T in (3.10) could be different from 0 in the following cases:

(i) The coefficients of the differential operator are not constant;

(ii) the operator contains lower order terms;

(iii) the arcs Γ_i are curvilinear.

In the case of the Laplacian, the singular term due to the curvature of Γ_i (with $j = 1, t = 1$) may be less regular than the second term ($j = 2, t = 0$). This happens if

$$\alpha_1 + 1 < \alpha_2, \quad \text{i.e.,} \quad \omega < \pi$$

for all three boundary conditions.

The coefficients c_{jst} have physical meaning and often are called *stress intensity factors*. For details, see [CS]. We see here that the solution is more complicated, but the singularity of the solution is analogous to the singularity in the one-dimensional setting, (2.2).

Taking the correct number of terms in (3.10), we can write

$$u = \sum_{i=1}^{M} \sum_{j} u_j^{(i)} \chi_i + u_0, \tag{3.11}$$

where χ_i is a smooth cut-off function in the neighborhood of A_i. Then we have

$$\|u_0\|_{H^k(\Omega)} \le C \left(\|f\|_{H^{k-2}(\Omega)} + \sum_{\Gamma_i \subset \Gamma_D} \|g\|_{H^{k-1/2}(\Gamma_i)} + \sum_{\Gamma_i \subset \Gamma_N} \|h\|_{H^{k-3/2}(\Gamma_i)} \right), \tag{3.12}$$

i.e., the smoothness of u_0 is analogous to that when the domain and data are smooth as in (3.9). Note that although the exponents α_j in (3.10) depend continuously on the angles ω_i, the value of S does not. As a result, the constant C in (3.12) can be very large for angles ω_i close to the exceptional values for which $S = 1$ instead of 0. This problem can be avoided, however, by adjusting the expansion of u.

So far, we have only discussed the scalar equation, which was represented by the example of Laplace's equation. The case of a system (like the elasticity problem) is more complicated but very analogous. See, for example, [Sa]. The decomposition (3.10) still holds with $\psi_{jst}(\varphi)$ being a vector. In contrast to the previous case, α_j may now be complex, and there are more exceptional cases than before.

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3.3. Countably normed spaces. The regularity of the solution in our setting of piecewise analytic input data can be very advantageously described in the form of countably normed spaces. (For details, see [BG2] and [BG3].) For $\delta < 1$, let $S_\delta^z = \{x \in \Omega, r < \delta\}$ with r measured from the origin, situated at the point z, and let $\Phi_\beta = r^\beta$. By $H_\beta^{m,\ell}(S_\delta^z)$, $m \geq \ell > 0$ being integers, $0 < \beta < 1$, we denote the completion of the set of all infinitely differentiable functions under the norm

$$\|u\|^2_{H_\beta^{m,\ell}(S_\delta^z)} = \|u\|^2_{H^{\ell-1}(S_\delta^z)} + \sum_{k=\ell}^{m} \|\Phi_{\beta+k-\ell}|D^k u|\|^2_{L_2(S_\delta^z)},$$

where

$$|D^m u|^2 = \sum_{|\alpha|=m} |D^\alpha u|^2$$

and

$$\|u\|^2_{H_\beta^{m,0}(S_\delta^z)} = \sum_{k=0}^{m} \|\Phi_{\beta+k}|D^k u|\|^2_{L_2(S_\delta^z)}.$$

Furthermore, let us introduce $\mathcal{B}_\beta^\ell(S_\delta^z)$, $\ell \geq 0$ consisting of those $u \in H_\beta^{\ell,\ell}(S_\delta^z)$ for which

$$\|\Phi_{\beta+k-\ell}|D^k u|\|_{L_2(S_\delta^z)} \leq C d^{k-1}(k-\ell)!,$$

$k = \ell, \ell+1, \ldots, C \geq 1, d \geq 1$ independent of k.

We introduce the space $\mathcal{C}_\beta^2(S_\delta^z)$,

$$\mathcal{C}_\beta^2(S_\delta^z) = \{u \in H_\beta^{2,2}(S_\delta^z) | \quad |D^k u|(x) \leq C d^k k!(\Phi_{\beta+k-1})^{-1}\}.$$

We then have

$$\mathcal{B}_\beta^2(S_\delta^z) \subset \mathcal{C}_\beta^2(S_\delta^z) \subset \mathcal{B}_{\beta+\varepsilon}^2(S_\delta^z), \qquad \varepsilon > 0 \text{ arbitrary}.$$

Furthermore, let Ω be the curvilinear polygon with vertices A_i, $i = 1, \ldots, M$, as described at the beginning of §3.1, and let $\beta = (\beta_1, \ldots, \beta_M)$, $0 < \beta_i < 1$. Let $\nu > 0$ be such that

$$S_\nu^{A_i} \cap S_\nu^{A_j} = \phi \quad \text{for } i \neq j, \quad i, j = 1, \ldots, M.$$

Then we define

$$\mathcal{B}_\beta^\ell(\Omega) = \left\{ u | u \text{ is analytic in } \bar{\Omega} - \bigcup_{i=1}^{M} A_i \text{ and } u \in \mathcal{B}_{\beta_i}^\ell(S_\nu^{A_i}), i = 1, \ldots, M \right\}$$

$$\mathcal{C}_\beta^2(\Omega) = \left\{ u | u \text{ is analytic in } \bar{\Omega} - \bigcup_{i=1}^{M} A_i \text{ and } u \in \mathcal{C}_{\beta_i}^2(S_\nu^{A_i}), i = 1, \ldots, M \right\}.$$

Under the assumptions made above, we have the following theorem.

THEOREM 3.1 [BG2] [BG3]. *Let u be the solution to* (3.4) *corresponding to a second-order* (*scalar*) *operator.*

(i) *Let Ω be a polygon* (*i.e., Γ_i are straight lines*). *Then for $\beta = (\beta_1, \ldots, \beta_M)$, $0 < \beta_i < 1$ dependent on ω_i we have $u \in \mathcal{B}_\beta^2(\Omega)$.*

(ii) *Let Ω be a curvilinear polygon. Then for $\beta = (\beta_1, \ldots, \beta_M)$, $0 < \beta_i < 1$ dependent on ω_i, we have $u \in \mathcal{C}_\beta^2(\Omega)$.*

The characterization given in Theorem 3.1 is very useful in practice because it allows us to construct the h-p version so that its convergence rate is exponential, without knowing the decomposition form (3.10) exactly. For analogous results for problem (3.7) we refer to [GoB].

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3.4. Regularity of the three-dimensional problem. In three dimensions, the situation is more complicated (see [Da]). The regularity of the solution has a different character in various regions of the domain. These are as follows and are depicted in Fig. 3.3.

FIG. 3.3. *Neighborhoods of singularities in three-dimensional domains.*

(1) *The neighborhood of an edge not close to a vertex.* Here the solution is smooth along the edge and singular in the direction perpendicular to the edge, giving rise to *edge singularities.*

(2) *The neighborhood of a vertex.* For such neighborhoods, we can distinguish between two kinds of subregions: (A) Cone neighborhoods of the edges and (B) The complement of these cones in the vertex neighborhood. In (B), the solution has an essentially radial singularity, once again of the form r^α, called a *vertex singularity.* The cone neighborhoods (A) contain transitional *edge-vertex singularities.*

(3) *The remainder of the domain* (*besides regions of the type* (1) *and* (2)). Here the solution is smooth (analytic).

The stress intensity factors, i.e., analogues to c_{jst}, are now functions along the edges which are smooth but have singular behavior in the neighborhood of the endpoints (the vertices). These stress intensity factors may be bounded or unbounded in dependence of the geometry. In addition, there are stress intensity factors for the vertex singularity.

As an illustration, let us present some regularity results from [Gr1] and [Gr2], (which were used in [D2]) for the following model problem: Given $f \in C^\infty(\bar{\Omega})$, find u such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.13}$$

where Ω is a polyhedral domain in $\mathbb{R}^3$.

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First we describe the singularity close to γ, an open segment of an edge γ_0 of Ω assumed to be along the x_3-axis. Let the dihedral angle at γ_0 be ω, then this singularity is characterized only by the values $\alpha_j = j\pi/\omega$. Then, in a neighborhood G of γ (such that $\bar{G}$ does not intersect any corners or edges other than γ_0) we have, analogously to (3.10) (for $f \in C^\infty(\bar{\Omega})$),

$$u = \sum_{j=1}^{J}\sum_{s=0}^{S}\sum_{t=0}^{T} c_{jst}(x_3)\psi_{jst}(\phi)r^{\alpha_j+t}\ln^s(r) + u_0, \tag{3.14}$$

where (r, ϕ, x_3) are local cylindrical coordinates, $c_{jst} \in C^\infty(\gamma)$ and ψ_{jst} are analytic in ϕ. As before, $S = 0$ for the problem (3.13) except for integral values of α_j, for which $S = 1$. It is seen from (3.14) that the solution is smooth in the direction of the edges and unsmooth in the perpendicular direction.

Next, we consider the behavior of u in the neighborhood K_0 of a vertex A_0. We first describe K_0. Let A_0 be situated at the origin and be an endpoint of the edge γ_0. Let K be the intersection of Ω and a solid circular cone with vertex A_0 and axis along γ_0, which does not intersect any edge having an end point at A_0 other than γ_0. Then $K_0 = K \cap B_0$ where B_0 is a small ball of radius ρ_0 about the origin.

Next, let C be the infinitely extended cone coinciding with Ω in a neighborhood of the origin and let $R = C \cap S$, where S is the unit sphere. Let $0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of the Laplace–Beltrami operator on R with homogeneous Dirichlet boundary data on ∂R, let $\beta_i = \sqrt{\lambda_i + \frac{1}{4}}$, and let $\tilde{\alpha}_j$ be the coefficients (corresponding to the singular coefficients α_j above) for this operator for the singularity in a neighborhood of a vertex with angle ω. Then, with $f \in C^\infty(\Omega)$, we have on K_0

$$\begin{aligned} u = &\sum_{j=1}^{J}\sum_{s=0}^{S}\sum_{t=0}^{T}\left(\sum_{i=0}^{I}\tilde{c}_{jsti}\rho^{-1/2+\beta_i} + f_{jst}(\rho)\right)\tilde{\psi}_{jst}(\phi)\theta^{\tilde{\alpha}_j+t}\ln^s(\theta) \\ &+ \sum_{i=0}^{I} d_i\rho^{-1/2+\beta_i}\tilde{g}_i(\theta,\phi) + \tilde{u}_0, \end{aligned} \tag{3.15}$$

where (ρ, θ, ϕ) are the local spherical coordinates, with origin at A_0, and with $\theta = 0$ corresponding to edge γ_0. The functions f_{jst}, $\tilde{\psi}_{jst}$, $\tilde{g}_i$ and $\tilde{u}_0$ are all smooth and their exact regularity is stated in Lemma 4.3 of [D2]. Here, the first term is a transitional edge-vertex singularity corresponding to edge γ_0 and vertex A_0, while the second term is the vertex singularity of radial type which will be present throughout R.

Equations (3.14) and (3.15) then characterize the two types of singular behavior (near edges and vertices) for three-dimensional problems. For the singular behavior of the elasticity problem we refer to [Gr3] and [Pe].

We see that in two dimensions (for the Laplace equation), the singularity depends only on ω_j, the interior angles of the domain and upon the boundary conditions. In three dimensions, the situation is similar for edge singularities, but not for the singularities near vertices, which depend upon the Laplace-Beltrami eigenvalues. This makes the a priori determination of the exact form of the singularities more difficult. In general, for any vertex A_ℓ, $\ell = 1, 2, \ldots, n_w$, we now define S_ℓ to be the portion of the unit sphere subtended by the infinite cone which coincides with Ω in a neighborhood of A_ℓ. Then, in addition to the dihedral angles w_j, $j = 1, 2, \ldots, n_e$, the regularity also depends upon the coefficients $\beta_{\ell,i} = \sqrt{\lambda_{\ell,i} + \frac{1}{4}}$, where $0 < \lambda_{\ell,1} \le \lambda_{\ell,2} \le \cdots$ are the eigenvalues of the Laplace-Beltrami operator on S_ℓ with homogeneous boundary conditions on ∂S_ℓ. These eigenvalues are directly related to the Steklov eigenvalue problem on K_0. See [BPA].

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Let us mention that we can also describe the regularity of the solution using weighted spaces, as was done in Theorem 3.1 for the two-dimensional case. Unlike the *isotropic* weighted spaces used in the two-dimensional case, however, the natural spaces in three dimensions to describe the different types of singular behavior in the neighborhood of edges and vertices are *anisotropic*. We will not elaborate on this here. For a survey of regularity and error estimates in terms of countably normed spaces in 3-d, we refer to [BG4] where numerical results are also presented.

4. Finite element spaces and basic approximation results. We will proceed here analogously to the one-dimensional case. Assume for now that $\vec{g} = 0$. Then as in §§2 and 3 we will consider $V^n \subset H_D^1(\Omega)$ to be the finite element space of dimension N_n (number of degrees of freedom). The finite element solution $\vec{u}_n \in V^n$ is then such that

$$B(\vec{u}_n, \vec{v}) = F(\vec{v}) \quad \forall\, \vec{v} \in V^n.$$

4.1. The finite element spaces. We will now restrict ourselves mainly to the two-dimensional problem and define the meshes and elements analogously as before.

Let $S = (-1, 1)^2$ and $T = \{(\xi, \eta) | 0 < \eta < (\xi + 1)\sqrt{3}, -1 < \xi \leq 0, 0 < \eta < (1 - \xi)\sqrt{3}, 0 \leq \xi < 1\}$ be the standard square and standard triangle respectively. The mesh $\mathcal{T}^n$ is now a partition of Ω into the open curvilinear triangles or quadrilaterals denoted by $\tau_j^{(n)}$, $j = 1, 2, \ldots, m_n$ such that to every $\tau_j^{(n)}$, we may associate an invertible mapping $F_j^{(n)}$ given by

$$x = X_j^{(n)}(\xi, \eta), \qquad y = Y_j^{(n)}(\xi, \eta),$$

which is a one-to-one mapping of S onto $\tau_j^{(n)}$ (if $\tau_j^{(n)}$ is a quadrilateral), or T onto $\tau_j^{(n)}$ (if $\tau_j^{(n)}$ is a triangle). The functions $X_j^{(n)}$ and $Y_j^{(n)}$ are analytic functions in ξ, η and have certain uniform bounds of the form

$$|D^\alpha X_j^{(n)}|, |D^\alpha Y_j^{(n)}| \leq C_1 h_{n,j}, \qquad |\alpha| = 1, 2, \tag{4.1a}$$

$$C_2 h_{n,j}^2 < |J_j| \leq C_3 h_{n,j}^2 \tag{4.1b}$$

where J_j is the Jacobian and the constants C_i, $i = 1, 2, 3$, are independent of the partition. Here, $h_{n,j}$ is the maximal side length of $\tau_j^{(n)}$. There are some additional assumptions about uniform analytic extendibility of functions $X_j^{(n)}(\xi, \eta)$ and $Y_j^{(n)}(\xi, \eta)$ outside S or T, which we will not discuss here (see [BG1]). In the case when $\tau_j^{(n)}$ are straight sided triangles, the mapping $F_j^{(n)}$ is linear and the uniformity condition is satisfied if the minimal (and maximal) angle is bounded from below (and above) by constants independent of the partition under consideration. Further, we assume that $\bar{\tau}_j^{(n)} \cap \bar{\tau}_k^{(n)}$ is either empty, a single common vertex or a single common side and if $s_{j,k} = \bar{\tau}_j^{(n)} \cap \bar{\tau}_k^{(n)}$ is the common side, then $x \in s_{j,k}$ is mapped by the mappings $(F_j^{(n)})^{-1}$ and $(F_k^{(n)})^{-1}$ into the same relative position on the side of the standard square or triangle. The above assumptions about the partitioning are analogous to the ones for the partitioning of (0,1) into I_j^n in §2.

Now we will define the quasiuniform, radical and geometric meshes.

(1) *The quasiuniform mesh.* Here $\{\mathcal{T}^n\}$ is such that

$$\frac{h_n}{h_{n,j}} \leq \tau,$$

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where $h_n = \max h_{n,j}$ and $\tau > 0$ is uniform for all meshes $\mathcal{T}^n$ under consideration.

(2) *The radical mesh* with power $\beta \geq 1$ (with respect to A_i). If $A_i \notin \bar{\tau}_j^{(n)}$, then with $\gamma = 1 - (1/\beta)$,

$$C_1 h_n \left(\min_{x \in \bar{\tau}_j^{(n)}} r^\gamma(x) \right) \leq h_{n,j} \leq C_2 h_n \left(\max_{x \in \bar{\tau}_j^{(n)}} r^\gamma(x) \right),$$

while if $A_i \in \bar{\tau}_j^{(n)}$, then

$$C_1 h_n \left(\max_{x \in \bar{\tau}_j^{(n)}} r^\gamma(x) \right) \leq h_{n,j} \leq C_2 h_n \left(\max_{x \in \bar{\tau}_j^{(n)}} r^\gamma(x) \right).$$

Here $r(x)$ is the distance of x from A_i. This definition, which holds in a neighborhood of A_i, is the two-dimensional version of the definition of the radical mesh introduced in §2. It was employed in [BKP].

(3) *The geometric mesh* with ratio $q, 0 < q < 1$ (in the neighborhood of A_i). Let us number the elements of $\mathcal{T}^n$ by a double index $\tau_{j,k}^{(n)}$ with $j = 1, \ldots, \rho(k)$, $\rho(k) \leq \rho_0$ and $k = 1, 2, \ldots, n+1$. Then if $d_{n,j,k}$ denotes the distance between $\tau_{j,k}^{(n)}$ and A_i and $A_i \notin \bar{\tau}_{n,j,k}^{(n)}$,

$$C_1 q^{n+2-k} \leq d_{n,j,k} \leq C_2 q^{n+1-k},$$

$$\kappa_1 d_{n,j,k} \leq h_{n,j,k} \leq \kappa_2 d_{n,j,k},$$

$j = 1, \ldots, \rho(k), k = 2, \ldots, n+1$.

If $A_i \in \bar{\tau}_{j,k}^{(n)}$, then $k = 1$ and

$$\kappa_3 q^n \leq h_{n,j,k} \leq \kappa_4 q^n, \qquad j = 1, \ldots, \rho(1),$$

where constants C_i and κ_i are uniform for all meshes under consideration. The geometric mesh will be denoted by $\{\mathcal{T}_q^n\}$. For more about these meshes we refer to [BG1].

As in one dimension, the radical and geometric meshes are refined in neighborhoods of the vertices A_i where the singularities of the solution are located.

Now let us define

$$\tilde{V}^n(\Omega) = \{u \in H^1(\Omega) | \text{ For } x \in \tau_j^{(n)}, u \circ F_j^{(n)} \in \mathcal{U}_{p_j}(K)\} \tag{4.2}$$

where, if $\tau_j^{(n)}$ is a curvilinear triangle, then we take $K \equiv T$ and $\mathcal{U}_{p_j} \equiv \mathcal{P}_{p_j}$, while if $\tau_j^{(n)}$ is a curvilinear quadrilateral, then we take $K \equiv S$ and $\mathcal{U}_{p_j} \equiv Q_{p_j}$ or Q'_{p_j}. Here, $\mathcal{P}_{p_j}(K)$ denotes the set of all polynomials of total degree $\leq p_j$ on $K = T$ or $K = S$. Further, $Q_{p_j}(S)$ is the set of all polynomials of degree $\leq p_j$ in each variable and $Q'_{p_j}(S) = \mathcal{P}_{p_j}(S) \oplus \{\xi^{p_j}\eta, \xi\eta^{p_j}\}$. The elements Q_p are sometimes called product elements, while Q'_{p_j} are called serendipity or trunk elements, see, e.g., [Ci], [SzB]. They are used for example, in the programs MSC/PROBE, FIESTA, PEGASYS, and STRIPE.

The theoretical question of under what conditions the space $Q_p(S)$ or $Q'_p(S)$ is preferable has not yet been resolved sufficiently. For the analysis of some aspects of this problem, we refer to [BE].

We now set $V^n = \tilde{V}^n \cap H_D^1$. In general, the condition $u \in H_D^1(\Omega)$ in the definition of $V^n(\Omega)$ restricts the functions $u(F_j^{(n)}(x))$ to belong to a proper subset of $\mathcal{P}_{p_j}(T)$ (or

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$Q_{p_j}(S)$, $Q'_{p_j}(S)$). This is necessary to enforce continuity of $u \in H^1_D(\Omega)$, for example, when p_j is different over adjacent elements.

As in §2, we now deal with the space $V^n(\mathcal{T}^n, \boldsymbol{p_n}) \subset H^1_D(\Omega)$ where $\boldsymbol{p_n}$ is the degree vector. We note that for quasiuniform meshes and uniform $\boldsymbol{p_n} = p_n$, we have

$$N_n \cong h_n^{-2} p_n^2,$$

which also holds for the case of radical meshes and uniform p_n. For the geometric mesh and uniform p_n we have

$$N_n \cong n\rho_0 p_n^2$$

Analogously as in §2, we have in our model problem (3.1), $B(\vec{u}, \vec{u}) := \|\vec{u}\|_E^2 \approx \|\vec{u}\|^2_{H^1(\Omega)}$, and hence, with $\vec{e}_n = \vec{u} - \vec{u}_n$,

$$\|\vec{e}_n\|_E \leq \inf_{\vec{v} \in V^n(\mathcal{T}^n, \boldsymbol{p_n})} \|\vec{u} - \vec{v}\|_E.$$

4.2. Approximation theorems. Let us now formulate theorems analogous to the ones in §2 from which the error bounds for the finite element method immediately follow.

As in §2.3, we again remark that the approximation theorems below (which are given in terms of the number of degrees of freedom) do not completely address the question of the *performance* of various elements (in terms of say CPU time). A further qualification is that even if the asymptotic *rate* at which the error decreases may be the same for two elements (e.g. Q_p and Q'_p), the actual *magnitude* of the error (which is governed by the constant C in estimates like (2.9)) may be different (see [BE]).

Our first theorem is for the h-p version with quasiuniform meshes ($\vec{u}$ is the solution of (3.1)–(3.4)), which gives the same estimate as the one-dimensional case (Theorem 2.1).

THEOREM 4.1 [BS1]. *Let us consider* $\boldsymbol{p_n} = p$ *and a family of quasiuniform meshes* $\mathcal{T}^n$. *Then*

$$\inf_{\vec{v} \in V^n} \|\vec{u} - \vec{v}\|_{H^1(\Omega)} \leq C h^{\mu-1} p^{-(k-1)} \|\vec{u}\|_{H^k(\Omega)}, \tag{4.3}$$

where $\mu = \min(p+1, k)$ *and* C *is a constant independent of* h, p *and* $\vec{u}$.

From the above theorem, it is clear that for the p version, we have a rate of $O(p^{-(k-1)})$. Let us outline how this rate is established. This will allow us to explain some key techniques used in p version approximation theory.

Suppose we consider a mesh of triangles, then for the p version, there will only be a finite number of elements, this number being independent of n. Consider the triangular element $\tau_j^{(n)}$ (with $h_{n,j} < 1$). The first step is to extend the function $\vec{u}$ defined on $\tau_j^{(n)}$ to an open unit square R satisfying $R \supset \bar{\tau}_j^{(n)}$, such that the norm-preserving property $\|\vec{u}\|_{H^k(R)} \leq C\|\vec{u}\|_{H^k(\tau_j^{(n)})}$ holds.

Next, we use the Chebyshev transformation, which transforms algebraic polynomials into periodic trigonometric polynomials. $\vec{u}$ gets transformed into $\vec{\tilde{u}}$ on the mapped square $\tilde{R}$ such that the norm is once again preserved, i.e., $\|\vec{\tilde{u}}\|_{H^k(\tilde{R})} \leq C\|\vec{u}\|_{H^k(R)}$. The function $\vec{\tilde{u}}$, of course, is now periodic, and therefore it can be approximated by Fourier expansions. For each $p = 1, 2, \ldots$ we therefore obtain a trigonometric polynomial $\vec{\tilde{\omega}}_p$ satisfying

$$\|\vec{\tilde{u}} - \vec{\tilde{\omega}}_p\|_{H^s(\tilde{R})} \leq C p^{-(k-s)} \|\vec{u}\|_{H^k(\tilde{R})}, \; 0 \leq s \leq k. \tag{4.4}$$

Transforming the functions $\vec{\tilde{u}}$, $\vec{\tilde{\omega}}_p$ back, we get a $\vec{\omega}_p^{(j)} \in \mathcal{P}_p(\tau_j^{(n)})$ for each p, j, such that

$$\|\vec{u} - \vec{\omega}_p^{(j)}\|_{H^s(\tau_j^{(n)})} \leq Cp^{-(k-s)}\|\vec{u}\|_{H^k(\tau_j^{(n)})}. \tag{4.5}$$

Note that in the above, we used the fact that $\bar{\tau}_j^{(n)}$ is a proper subset of R, i.e., the distance of $\bar{\tau}_j^{(n)}$ from the boundary of R is strictly positive.

The above construction yields a separate polynomial for every triangle. The next step is to adjust these polynomials so that they will be continuous on Ω. As shown in [BS2], for the case $k > 3/2$, we may first adjust $\vec{\omega}_p^{(j)}$ so that it coincides at the vertices with the function $\vec{u}$ and estimate (4.5) is preserved. Then the jump (discontinuity) $\vec{\psi}$ on a side γ of $\tau_j^{(n)}$ is a polynomial which vanishes at the ends of γ and satisfies by (4.4) and the trace theorem,

$$\|\vec{\psi}\|_{H_{00}^{1/2}(\gamma)} \leq Cp^{-(k-1)}\|\vec{u}\|_{H^k(\tau_j^{(n)})}, \tag{4.6}$$

where $H_{00}^{1/2}(\gamma)$ is as defined at the end of §1.

The final step is to extend $\vec{\psi}$ to a polynomial $\vec{z}_p^{(j)}$ over $\tau_j^{(n)}$ which coincides with $\vec{\psi}$ on γ and vanishes on the other two sides of γ and which has a suitably bounded $H^1(\tau_j^{(n)})$ norm.

In the original proof from [BSK], the above extension was accomplished in a nonoptimal way, giving an estimate

$$\|\vec{z}_p^{(j)}\|_{H^1(\tau_j^{(n)})} \leq C\|\vec{\psi}\|_{H^1(\gamma)}. \tag{4.7}$$

The right side of (4.7) was bounded above by $Cp^{-(k-2)}\|\vec{u}\|_{H^k(\tau_j^{(n)})}$, so that subtracting $\vec{z}_p^{(j)}$ from $\vec{\omega}_p^{(j)}$ for each j gave a polynomial $\vec{w}_p \in V^n$ satisfying

$$\|\vec{u} - \vec{w}_p\|_{H^1(\Omega)} \leq Cp^{-(k-2)}\|\vec{u}\|_{H^k(\Omega)}. \tag{4.8}$$

Hence, we obtained the desired estimate, except that a factor p^{-1} was lost. This loss can be reduced to $p^{-\varepsilon}$, for $\varepsilon > 0$ arbitrary, by a useful interpolation argument, which we illustrate below.

We obviously have

$$\inf_{\vec{w} \in V^n} \|\vec{u} - \vec{w}\|_{H^1(\Omega)} = \|\vec{u} - P_p\,\vec{u}\|_{H^1(\Omega)} \leq C\|\vec{u}\|_{H^1(\Omega)}$$

where $P_p\,\vec{u}$ is the H^1 projection of $\vec{u}$ onto V^n. From this,

$$\|I - P_p\|_{\mathcal{L}(H^1(\Omega), H^1(\Omega))} \leq C. \tag{4.9}$$

Also, from (4.8), we get

$$\|\vec{u} - P_p\,\vec{u}\|_{H^1(\Omega)} \leq Cp^{-(K-2)}\|u\|_{H^K(\Omega)},$$

from which, for any $K \geq 1$,

$$\|I - P_p\|_{\mathcal{L}(H^K(\Omega), H^1(\Omega))} \leq Cp^{-(K-2)}. \tag{4.10}$$

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Now given k, we choose $K \geq 4$ and using (4.9), (4.10), and the interpolation results for Banach spaces [LM], obtain

$$\|I - P_p\|_{\mathcal{L}(H^1(\Omega),(H^1(\Omega),H^K(\Omega))_\mu)} \leq C\left(p^{-(K-2)}\right)^\mu$$

where $\mu = \frac{k-1}{K-1}$. Noting that $(H^1(\Omega), H^K(\Omega))_\mu = H^K(\Omega)$, this gives

$$\|\vec{u} - P_p\,\vec{u}\|_{H^1(\Omega)} \leq Cp^{-(k-1)(\frac{K-2}{K-1})}\|\vec{u}\|_{H^k(\Omega)}.$$

Selecting K large enough, we get $(K-2)/(K-1) = 1 - (\varepsilon/(k-1))$, which leads to the desired result, with the constant C in (4.3) depending on k and ε.

Let us state the above interpolation argument as a theorem.

THEOREM 4.2. *Suppose $P_p : H^k(\Omega) \longrightarrow V^n$ is a projection that satisfies*

$$\|u - P_p u\|_{H^k(\Omega)} \leq C\|u\|_{H^k(\Omega)}$$
$$\|u - P_p u\|_{H^k(\Omega)} \leq C(s)p^{-(s-\ell)}\|u\|_{H^s(\Omega)} \quad \textit{for arbitrary } s > k, s \geq \ell.$$

Then given $\varepsilon > 0$, there exists $C = C(\varepsilon, s)$ such that

$$\|u - P_p u\|_{H^k(\Omega)} \leq Cp^{-(s-k)+\varepsilon}\|u\|_{H^s(\Omega)}.$$

The above interpolation argument has been used frequently in the p version literature, e.g., in [BSK], [D1], [JV], [S1], and [StS]. Note that such an argument does not work for the h version, since the rate of convergence is bounded by the polynomial degree.

To obtain the *optimal* order (4.3), without the ε, the extensions $\vec{z}_p^{(j)}$ in (4.7) must be constructed in an *optimal* way. This was accomplished in [BS1], using Theorems 4.3 and 4.4 below, which have found applications in other contexts as well (see [BS3], for example). Using these theorems together with (4.6) gives the desired optimal estimate for $\vec{z}_p^{(j)}$, leading to (4.3).

THEOREM 4.3 [BS1], [BCM]. *Let T be the standard triangle with sides γ_i, $i = 1, 2, 3$, and let f be continuous on ∂T with $f|_{\gamma_i} \in \mathcal{P}_p(\gamma_i)$, $i = 1, 2, 3$. Then there exists $U \in \mathcal{P}_p(T)$ such that $U = f$ on ∂T and*

$$\|U\|_{H^1(T)} \leq C\|f\|_{H^{1/2}(\partial T)},$$

where the constant C is independent of p and f.

THEOREM 4.4 [BS1]. *Let S be the standard square with sides γ_i, $i = 1, 2, 3, 4$. Let f be a continuous function on ∂S with $f|_{\gamma_i} \in \mathcal{P}_p(\gamma_i)$, $i = 1, 2, 3, 4$. Then there exists $U \in Q_p(S)$ such that $U = f$ on ∂S and*

$$\|U\|_{H^1(S)} \leq C\|f\|_{H^{1/2}(\partial S)},$$

where the constant C is independent of p and f. (*See also* [BCM].)

Theorems 4.3 and 4.4 are the analogs of the classical extension theorems. Here the extension is constrained to polynomials of the same degree as the traces. We mention that when using the space $Q'_p(S)$ instead of $Q_p(S)$ the constant C in Theorem 4.4 is not independent of p [Md].

Let us now consider singular functions.

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THEOREM 4.5 [BS1]. *Let* $\vec{u} = r^\alpha \log^s r\ \vec{\psi}\ (\theta)$ *(where by* r^α *for* α *complex, we denote* Re r^α*), and let the meshes be quasiuniform with uniform p. Then*

$$\inf_{\vec{v} \in V^n(\mathcal{T}^n, p)} \|\vec{u} - \vec{v}\|_{H^1(\Omega)} \leq Cg(h, p, s) \min\left[h^{\hat{\alpha}}, \frac{h^{\min(\hat{\alpha}, p-\hat{\alpha})}}{p^{2\hat{\alpha}}}\right],$$

where $g(h, p, s) = \max(|\log h|^s, |\log p|^s)$, $\hat{\alpha} = \text{Re}\ \alpha$ *and C is a constant independent of* h, p.

See also [BD]. The next theorem is analogous to Theorem 2.5 and gives the rate of convergence of the h version with radical meshes.

THEOREM 4.6. *Let* $\vec{u} = r^\alpha \log^s r\ \vec{\psi}\ (\theta)$. *Then for the radical mesh with power* β *and uniform p,*

$$\inf_{\vec{v} \in V^n} \|\vec{u} - \vec{v}\|_{H^1(\Omega)} \leq C(\hat{\alpha}, \beta, s) h^p,$$

provided that $1 > \gamma = 1 - (1/\beta) > 1 - (\hat{\alpha}/p) > 0$.

The proof is a simple generalization of the results in [BKP].

Finally, let us state the following theorem which shows that the h-p version with geometric meshes leads to exponential convergence.

THEOREM 4.7 [BG1]. *Let* $u \in \mathcal{B}^2_\beta(\Omega), 0 < \beta < 1$. *Let* $\{\mathcal{T}^n\}$ *be geometric meshes and* $\mu n \leq p \leq \nu n$, $p \geq 1, 0 < \mu, \nu < \infty$. *Then we have*

$$\inf_{\vec{v} \in V^n} \|\vec{u} - \vec{v}\|_{H^1(\Omega)} \leq Ce^{-\gamma \sqrt[3]{N_n}} \tag{4.11}$$

with $\gamma > 0$.

See also [BD]. The proof of Theorem 4.7 uses a combination of the approaches used in the outline of the proof of the convergence of the p version in this section and the proof in the one dimensional setting presented in §2 for the h-p version exponential rate.

It is more advantageous to use a linear degree vector than a constant one. The use of the geometric mesh is advantageous when the effort of mesh generation is considered for complex geometries, since the refinement is restricted to a small subset. It is also possible to use a sequence of radical meshes as in one dimension and by means of the envelope obtain an exponential rate, but the mesh generation effort is now much more complex. In practice the p version on a properly refined (geometric) mesh is often used instead (see Remark 4.1 ahead). Then in the range of practically desirable accuracy, the p version performs as the h-p version.

So far we have only mentioned the problems where the approximation in the spaces $H^1(\Omega)$ was relevant. There are problems where the norms in $H^m(\Omega), m > 1$ have to be used. Various analogous results can be proven here. We refer to [Go] and [S2] for details. An analysis of the error in the $L_2(\Omega)$ and negative norms has been presented in [JS]. Let us briefly describe a result of interest from this reference, related to the duality results presented in §2.

Consider the L-shaped domain in Fig. 3.1. The solution of Laplace's equation on Ω is known to have a singularity behaving like $r^{2/3}$ by the results of the previous section. If the h version is used to approximate the solution of Laplace's equation (using a uniform mesh), the error will satisfy

$$\|e\|_{H^1(\Omega)} \approx Ch^{2/3}, \|e\|_{L_2(\Omega)} \approx Ch^{4/3}. \tag{4.12}$$

The usual duality argument does not lead to a full extra power in h for the $L_2(\Omega)$ norm over the $H^1(\Omega)$ norm. This is because of the nonconvexity of the domain: the solution of the dual

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problem

$$-\Delta w = e \quad \text{on } \Omega \tag{4.13}$$

will satisfy

$$\|w\|_{H^k(\Omega)} \leq C\|e\|_{L_2(\Omega)} \tag{4.14}$$

with $k = 4/3$ only and not $k = 2$, as required. We note that the estimates in (4.12) cannot be improved; see [W1].

For the p version, however, we obtain the *full* extra power of convergence, i.e.,

$$\|e\|_{H^1(\Omega)} \approx Cp^{-4/3}, \qquad \|e\|_{L_2(\Omega)} \leq Cp^{-1}\|e\|_{H^1(\Omega)} \leq Cp^{-7/3}. \tag{4.15}$$

This is because the solution of (4.13) can now be decomposed into a singular and smooth part as in equation (3.10)′. The smooth part will satisfy (4.14) with $k = 2$. The singular part will not; however, since it behaves like $r^{2/3}$, it is approximated at double the usual rate by the p version, and the duality argument easily follows. The same argument works for other nonconvex domains (see [JS]).

So far, we have assumed that $\vec{g} = 0$ on Γ_D. In the inhomogeneous case, u is approximated by first finding a suitable approximation $\vec{g}_n$ to $\vec{g}$ in the trace space $W^n = W^n(\Gamma_D) = \{w|_{\Gamma_D}, w \in \tilde{V}^n\}$ where $\tilde{V}^n$ is the finite element space when $\Gamma_D = \phi$. This introduces a new component, depending upon the difference $\vec{g} - \vec{g}_n$, into the error. There are various interesting results on the effect of this component on the total error, depending upon how $\vec{g}_n$ is defined. We refer to [BS3] for example.

Let us mention here that the spectral element method, which has many features in common with the p version, also leads to various approximation results. We will not survey them here but refer instead to [BMd] and the references therein.

In the three-dimensional case the solution is essentially similar but more complex. Let us mention the following theorem for the p version.

THEOREM 4.8 [D2]. *Consider a fixed mesh $\mathcal{T}$ and a sequence of degrees $p_n \geq 0$. Let $f \in C^\infty(\bar{\Omega})$. If u is the solution of* (3.13) *and $\omega_j, \beta_{\ell,i}, j = 1, \ldots, n_e, \ell = 1, \ldots, n_w, i = 1, 2, \ldots,$, are as described in* §3, *then for arbitrary $\varepsilon > 0$*

$$\inf_{v \in V^n} \|u - v\|_{H^1(\Omega)} \leq Cp^{-2\mu+\varepsilon}$$

where

$$\mu = \min_{j,\ell}\left\{\frac{\pi}{\omega_j}, \beta_{\ell,1}\right\}$$

and C depends on u and ε but is independent of p.

The above theorem is based on the regularity results (3.14) and (3.15) for the solution u. Let us remark that in most practical cases, $\mu = \min_j\{\pi/w_j\}$ and hence the edge singularity (and not the vertex one) governs the rate of convergence for the p version.

In three dimensions the proper mesh selection is complex because of the (anisotropic) singular behavior of the solution. Here, in the neighborhood of edges, long (needle) elements have to be used because while the solution is smooth along the edge, it is unsmooth in the direction perpendicular to the edge. In the neighborhood of the vertices, the singularity is isotropic and elements of the usual type are used. Hence, the class of meshes (radical and

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geometric) introduced in two dimensions has to be enlarged. In addition, the mesh generation has to be more sophisticated, to take into account the transition between various meshes. For the approximation properties of "needle" elements which violate the usual minimal angle condition, we refer to [BA] and [Kr].

As we saw in Theorem 4.7, for a proper sequence of meshes and degree distributions, we have an exponential rate of convergence (with the h-p version). In the three dimensional case, we get a result which is quite analogous to Theorem 4.7, with the exponent $\sqrt[3]{N_n}$ in (4.11) now being replaced by $\sqrt[5]{N_n}$. For more see [Go2], [BG4].

4.3. The effect of numerical quadrature. So far, we have assumed throughout that exact integration is used in the calculations of stiffness matrices and load vectors. In practice, of course, these are generally computed using numerical integration. Unlike the h version, where a rule of fixed precision may be used as $h \to 0$, in the p version, the precision of the rule must also increase as $p \to \infty$. In [BnS], the optimal convergence of the p version using numerical integration in $\mathbb{R}^t$, $t = 1, 2, 3$ was established under a set of sufficient conditions on the quadrature rule. Let us describe these results for the model problem (3.6), where $\partial\Omega \equiv \Gamma_D$, $g = 0$, a fixed mesh $\mathcal{T}^n \equiv \mathcal{T}$ is used and $p_j \equiv p$ in (4.2).

Suppose K is a reference element, $K = S$ or T. Then a quadrature rule on K induces a corresponding rule on each element τ_j. We then use a family of rules $\{R_p\}$ satisfying the following two assumptions.

(A) The weights are positive and the quadrature points lie within K.

(B) R_p is exact for all $v \in \mathcal{U}_m(K)$ with $m \geq 2p$.

Condition (B) can be weakened; see [BnS].

Newton–Cotes rules are unsuitable for the p version, since they have low degree of precision and violate the positivity of weights when p is large. The most advantageous rules are Gaussian rules, which obviously satisfy (A). In [BnS], it is shown that in one dimension, the p point Gauss–Legendre and $p-1$ point Gauss–Lobatto rules are the minimal required when polynomials of degree p are used. For Q_p type elements in higher dimensions, the corresponding tensor product rules using $p+1$ points in each direction are minimally required. For other rules, see [BnS].

Let $A = \operatorname{diag}(k_i)$, where $k_i = k_i(x)$ are the coefficients in (3.6). Also, let $DF_j^{-1}(x)$ denote the Jacobian matrix of the inverse of F_j and define the matrix $B^j = B^j(x)$ by

$$B^j = [b^j_{k\ell}] = J_j(DF_j^{-1})A(DF_j^{-1})^T.$$

Moreover, let $\beta_{s,q} = \max_{j,k,\ell} \|b^j_{k\ell}\|_{W^{s,q}(K)}$. Then the following theorem is proved in [BnS].

THEOREM 4.9. *Let $\Omega \subset \mathbb{R}^t$. Let f in* (3.3) *be in $H^s(\Omega)$, $s > t/2$ and let u, the solution of the model problem* (3.6), *lie in $H^k(\Omega)$, $k > 1$. Let $b^j_{k\ell} \in H^d(K)$ for each j, k, ℓ, with $d > t/2$. Let $\tilde{e} = \tilde{e}_p$ be the error when the finite element solution $\tilde{u}_p$ is computed using rules $\{R_p\}$ which satisfy* (A) *and* (B). *Then for any $0 \leq q < m - p$, $r = \min(p, m-p-q)$,*

$$\|\tilde{e}\|_{H^1(\Omega)} \leq C \left\{ (m-p)^{-(s-\frac{t}{2})} \|f\|_{H^s(\Omega)} + q^{-(d-\frac{t}{2})} \beta_{d,2} \|u\|_{H^1(\Omega)} \right.$$
$$\left. + r^{-(k-1)} \left(\beta_{0,\infty} + q^{-(d-\frac{t}{2})} \beta_{d,2} \right) \|u\|_{H^k(\Omega)} \right\},$$

where the constant C is independent of u, m, p, and q.

The above estimate contains three terms. The first term, which is $O((m-p)^{-(s-(t/2))})$, arises from the approximation of f, the second term, which is $O(q^{-(d-(t/2))})$, arises from approximating the mappings F_j and coefficients k_i, and the final term is $O(r^{-(k-1)})$ and arises from the solution u.

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For the case that f, F_j, and $k_i(x)$ are sufficiently smooth, the above theorem shows that if $m \approx 2p$, then we once again recover the $O(p^{-(k-1)})$ convergence predicted by Theorem 4.1. A similar result holds for the case that the solution is singular, for which it may be shown that the optimal rate of $O(p^{-2\alpha})$ may be recovered.

If, however, d is small, then "overintegration" (i.e., taking $m > 2p$) may be needed to preserve the error bound. See [BnS] and [KS] for additional theorems and various computational results.

4.4. Numerical examples. The theorems in §4.2 are asymptotic in nature. To verify that they hold for a practical range of parameters as well, we consider a computational example involving the elasticity problem (3.7) over the L-shaped domain shown in Fig. 3.1.

We specify tractions on $\partial\Omega$ (with $\Gamma_D = \phi$, $\vec{f} = 0$ and sides OA, OE being traction free) such that the true solution $\vec{u} = (u_1, u_2)$ is given by

$$
\begin{aligned}
(4.16)\qquad u_1^I &= \frac{1}{2G} r^{\alpha_1}[(\kappa - Q_1(\alpha_1+1))\cos\alpha_1\theta - \alpha_1\cos(\alpha_1-2)\theta] \quad \text{(Mode I)}\\
u_2^I &= \frac{1}{2G} r^{\alpha_1}[(\kappa + Q_1(\alpha_1+1))\sin\alpha_1\theta + \alpha_1\sin(\alpha_1-2)\theta],\\
u_1^{II} &= \frac{1}{2G} r^{\alpha_2}[(\kappa - Q_2(\alpha_2+1))\sin\alpha_2\theta - \alpha_2\sin(\alpha_2-2)\theta] \quad \text{(Mode II)}\\
u_2^{II} &= \frac{1}{2G} r^{\alpha_2}[(\kappa + Q_2(\alpha_2+1))\cos\alpha_2\theta + \alpha_2\cos(\alpha_2-2)\theta],
\end{aligned}
$$

with Q_i being functions of α_i (see, e.g., [SzB, p. 175]). In our case, we use mode I where $\alpha_1 = 0.5444837$ and $Q_1 = 0.5430756$. Here, G is the modulus of rigidity and $\kappa = 3 - 4\nu$, with $\nu = 0.3$. Equation (4.16) shows that the solution has an r^α type singularity at the origin 0, of the same type as in the decomposition (3.10)–(3.11). It may also be mentioned that the solution lies in $H^{1+\alpha-\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$ and in the Besov space $B^{1+\alpha}_{2,\infty}(\Omega)$.

We first consider the FEM approximation of this problem using a uniform mesh with square elements, as shown in Fig. 4.1.

Since we have a quasiuniform mesh, by Theorem 4.5, for $p \geq 1$ we obtain the estimate

$$\|e\|_E \leq C \min\left[h^\alpha, \frac{h^{\min(\alpha, p-\alpha)}}{p^{2\alpha}}\right].$$

This leads to the predicted asymptotic rates h^α and $p^{-2\alpha}$ for the h and p versions, respectively.

Figures 4.2 and 4.3 in which the relative error in the energy norm $\|e\|_{E,R}$ is plotted respectively as a function of h and p, show that the error is in the asymptotic range even for moderate h and p. We reach the same conclusions as in the one-dimensional case.

Let us now consider the case of strongly refined (geometric) meshes. Such a mesh is shown in Fig. 4.4 for $n = 2$ layers. This is a geometric mesh with ratio 0.15 (which leads to nearly optimal convergence).

Figure 4.5 shows the relative error vs. the number of degrees of freedom for various n (with p being increased in each case to increase N) in $\log \|e\|_{E,R}$ vs $N^{1/3}$ scale. In Fig. 4.6, $\log \|e\|_{E,R}$ for various combinations of n and p has been plotted. It is observed that $\|e\|_{E,R} \approx Ce^{-\gamma N^{1/3}}$, which is consistent with Theorem 4.7.

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FIG. 4.1. *The uniform mesh.*

FIG. 4.2. *The error for the* h *version.*

Remark 4.1. The meshing required by the h-p version can be expensive to implement. Often (especially in the context of commercial p version codes), it is enough to select a fixed "good" mesh, i.e., one that is sufficiently refined near potential singularities, so that the error decreases largely in the pre-asymptotic exponential region. This region is clearly seen in Fig. 4.5 for any fixed n—when p is increased further, the error goes into the asymptotic region, which is algebraic. In general, for small required accuracies, the mesh should be over-refined (low p), while for high accuracies, it is underrefined (high p). The optimal design of the mesh would require either a fully adaptive approach (see, e.g., [DO]) or an approach based on an expert system (see, e.g., [BR]).

FIG. 4.3. *The error for the* p *version.*

FIG. 4.4. *The geometric mesh.*

We presented here only examples of two-dimensional problems. For some computational results of three-dimensional problems, we refer, for example, to [AB], [BG4] and [CM]. The basic features are completely similar to the two-dimensional case.

4.5. The pollution problem. In §2.6, we discussed the effect of "A-pollution." This occurs when the solution u is locally singular due to unsmooth input data, for example, an unsmooth right hand side f (as in §2.6). In higher dimensions, this could also happen due to unsmooth boundary conditions. In either case, one feature of A-pollution is that it can occur even when the Green's function for the problem is smooth. We saw from the one-dimensional case that A-pollution does not occur in the h version (or h-p version). The same is true in higher dimensions—for the case of unsmooth right hand sides with *smooth* domains and coefficients, there is no A-pollution asymptotically (i.e., for sufficiently small h), so that the quality of approximations is related only to the local behavior of the solution (see §11 of [W3]). This is *not* the case with the p version, as the first example below (where the singularity occurs due to a concentrated load) demonstrates. In fact, with large elements (for example, if the square domain consists of one element only), the effect of a singularity is propagated to all points in the element containing the singularity. This effect is even more pronounced when it is the derivatives of the solution (like stresses) that are of interest.

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FIG. 4.5. *The error $\|e\|_{E,R}$ as a function of n and p.*

FIG. 4.6. *The error $\|e\|_E$ for selected combinations of (n, p).*

In two and three dimension, there is another type of pollution which is quite common, which occurs due to the unsmoothness of the domain. This is the phenomenon classically referred to as "pollution" (see, e.g., [W1], [W2], and [W3])—we label it "B-pollution" to differentiate it from "A-pollution." B-pollution can also occur if the boundary condition changes or if the coefficients of the problem degenerate. It is only present when the Green's

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function is *unsmooth*, and once again manifests itself by a singularity in the solution at one point influencing the behavior of the approximation in another region.

B-pollution is closely related to the theory of local interior estimates on a subdomain $\tilde{\Omega} \subset \Omega$ (see [W3, §9]). For the h version, the error on $\tilde{\Omega}$ is governed by the local optimal error in the $H^1(\tilde{\Omega})$ norm and the global error in a lower norm on Ω. This result holds for small h when the domain $\tilde{\Omega}$ is separated by many elements from the region where the solution (or Green's function) is unsmooth [W3]. In the p version, we have a better lower norm global error than in the h version, as discussed in §4.2. On the other hand, we do not have enough elements separating the region $\tilde{\Omega}$ from the one where the solution (or Green's function) is unsmooth. Hence, the results from [W3] cannot be applied to the p version.

Let us first show the effect of A-pollution by an example from [BO]. Consider the plane strain problem ($E = 10^6$, $\nu = 0.3$) of a plane loaded by a concentrated load as shown in Fig. 4.7. The boundary conditions are traction conditions such that the solution is the well known Boussinesq solution on a half plane. The solution then has *infinite* energy but it has finite energy on any subdomain, particularly in the shaded area shown in Fig. 4.7.

FIG. 4.7. *A panel loaded by a concentrated force.*

In the case of the h version, the energy error in the shaded area will be essentially the same as the error of the best approximation on the domain (let us call it A-pollution free). This property of the h version is caused by separation of the area of interest from the singular area by a sufficiently large number of elements, as mentioned above [W3]. The same effect of filtering occurs in the h-p version, as long as, again, there are a sufficient number of layers of elements between the singularity and the shaded domain. In Fig. 4.8, we show a sequence of four meshes (on half of the panel because of symmetry) used for the computation of the error in the shaded area. Figure 4.9 shows the relative error. We have also depicted the A-pollution-free error which is the error of the best approximation of the solution on the shaded area. We see clearly the decrease in A-pollution in the p version when the number of separating layers between the shaded area and the point P is increased.

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FIG. 4.8. *The sequence of refined meshes.*

FIG. 4.9. *The error in the shaded area for different numbers of layers.*

In our second example, we show pollution effects (both types) for the p version in the neighborhood of a crack tip for the elasticity problem. We consider the cracked square shown in Fig. 4.10, with the crack along the line joining $(-1, 0)$ and $(0, 0)$. We use $E = 10^6$, $\nu = 0.3$ as in the previous example, with traction boundary conditions. The exact solution is a combination of the mode I and mode II crack solutions. We use curvilinear Q'_p (i.e., trunk) and triangular elements of degree $p = 6, 7, 8$ on the mesh shown.

In Fig. 4.11, we have shown the error in the stress σ_x on the line indicated in Fig. 4.10. There is a marked difference in the behavior of the error in the first element (where we see mainly A-pollution) and in the second (mainly B-pollution). This shows that to control the oscillations, a practical solution is to separate the area of interest from the singular area by one or two layers of elements. Merely taking a single layer of elements around the crack tip and increasing p would continue to cause oscillations (as observed in the first element in Fig. 4.11). Let us mention that the stresses in the element containing the crack tip cannot be reliably computed by the p version. They should be computed by first computing the stress intensity factors c_{jst} and then using the expansion (3.10) (for the elasticity problem).

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FIG. 4.10. *The cracked square panel.*

FIG. 4.11. *Error in σ_x.*

5. Locking and robustness. Suppose we consider (3.7) when the Poisson ratio ν is very close to 0.5 and use the h version with piecewise linear elements to approximate this problem. Then it is well-known (see, e.g., [SV] and [Vo]) that the finite element solution for practical choices of the parameter h is generally not very accurate. This is due to a phenomenon called *locking* (*Poisson ratio locking* in the particular case mentioned), which is said to occur when the accuracy of numerical schemes for the approximation of certain parameter-dependent problems deteriorates as the parameter converges to a limiting value. Another example of locking occurs when the h version is used on (3.6) when the parameter $d = k_1/k_2$ is close to

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0 (highly anisotropic materials) [BS4]. Other examples of locking include various plate and shell models when the thickness t is close to 0 (see, for example, [AF2], [Pt], and [BrF]).

For such problems, one needs methods that are *robust*, i.e., which work equally well and are essentially independent of the parameter value. In this connection, it has been demonstrated for various problems that the p version is often robust in situations where low-order standard h version schemes are not. For example, in [Vo], it has been shown that in the case of Poisson ratio locking, the p version is essentially free from locking for the displacements, while in [Li], the Timoshenko beam model (for which the standard h version shows locking; see [Ar]) was shown to be free from locking when the p and h-p versions are used. Analogous results have been shown for shell models; see [Pt].

Let us point out that other robust, locking-free methods exist. For example, in [SV], it was established that for the case of Poisson ratio locking, using the h version with high order elements ($p \geq 4$) also avoids locking. Another strategy to overcome locking is to use mixed methods, as done for the Timoshenko beam problem in [Ar] and for various other problems in, e.g., [AF2], [Pt], and [BrF]. However, the reformulations of the variational form necessary for mixed methods may not be feasible in the context of available commercial codes, so that it may be more desirable to have a method (like the p version) which is robust in terms of the *standard* formulation of the problem.

In [BS4], we have presented a general theory of locking and robustness. Let us present some definitions and results from there, adapted to the context of the model problem (3.7), when the material is nearly incompressible.

Let us denote the bilinear form (3.7) by $B_\nu(\cdot,\cdot)$ and let the exact solution be $\vec{u}^\nu$ when the Poisson ratio is $\nu (0 \leq \nu < 0.5)$. Then the corresponding finite element solution $\vec{u}^\nu_n$ will satisfy

$$B_\nu(\vec{u}^\nu_n, \vec{v}) = B_\nu(\vec{u}^\nu, \vec{v}) \quad \forall\, \vec{v} \in V^n. \tag{5.1}$$

Let us denote the energy norm $(B_\nu(\vec{w}, \vec{w}))^{1/2}$ by $\|\vec{w}\|_{E,\nu}$. We will restrict our attention to the case when the exact solutions $\vec{u}^\nu$ belong to the closed unit balls $H^B_{k,\nu} \subset H^k(\Omega)$ $(k \geq 2)$ defined in terms of the norms

$$\|\vec{w}\|^2_{k,\nu} = \|\vec{w}\|^2_{H^k(\Omega)} + (1-2\nu)^{-2}\|\mathrm{div}\ \vec{w}\|^2_{H^{k-1}(\Omega)}.$$

As shown in [BS5], $H^B_{k,\nu}$ correspond to precisely the correct weighted spaces that characterize the regularity of the solution when the data is appropriately bounded.

We assume that the sequence $\{V^n\}$ is such that

$$C_1 F_0(N_n) \leq \sup_{\vec{w} \in \vec{H}^B_k} \inf_{\vec{v} \in V^n} \|\vec{w} - \vec{v}\|_{1,\Omega} \leq C_2 F_0(N_n) \tag{5.2}$$

where H^B_k is the unit ball in the $H^k(\Omega)$ norm. Here, C_1, C_2 are independent of N and $F_0(N)$ represents the asymptotic rate of approximation, $F_0(N) \to 0$ as $N \to \infty$. Equations (5.1), (5.2) imply that for any $0 < \nu_0 < 0.5$, we can find constants $C_1(\nu_0)$, $C_2(\nu_0)$, such that the following holds uniformly for all $0 \leq \nu < \nu_0$ $(N = N_n)$:

$$C_1(\nu_0) F_0(N) \leq \sup_{\vec{u}^\nu \in \vec{H}^B_{k,\nu}} \|\vec{u}^\nu - \vec{u}^\nu_n\|_E \leq C_2(\nu_0) F_0(N). \tag{5.3}$$

If (5.3) can be made to hold uniformly for all $0 \leq \nu < 0.5$, we say that $\{V^n\}$ is *free from locking*. More precisely, we have the following definitions.

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DEFINITION 5.1. *Let f be an increasing function with* $\lim_{N\to\infty} f(N) = \infty$. *The extension process* $\{V^n\}$ *shows* locking of order $f(N)$ *for the family of problems* (5.1), $\nu \in [0,0.5)$, *with respect to the solution sets* $\{H^B_{k,\nu}\}$ *in the energy norm if and only if*

$$(5.4)\quad 0 < \lim_{N\to\infty} \sup \left(\sup_{\nu\in[0,0.5)} \left(\sup_{\vec{u}^\nu \in H^B_{k,\nu}} \|\vec{u}^\nu - \vec{u}^\nu_n\|_E \right) \right) (F_0(N) f(N))^{-1} = M < \infty.$$

For the case that M (depending on f) is bounded (respectively, infinite), we say that the order of locking is at most (*respectively,* at least) $f(N)$. *If* (5.4) *holds with* $f(N) \equiv 1$, *we say that* $\{V^n\}$ *is* free from locking.

DEFINITION 5.2. *Let g be a decreasing function, $g(N) \to 0$ as $N \to \infty$. The extension process* $\{V^n\}$ *is* robust *with uniform order $g(N)$ for the family of problems* (5.1), $\nu \in [0, 0.5)$, *with respect to the solution sets* $\{H^B_{k,\nu}\}$ *in the energy norm if and only if*

$$(5.5)\qquad \lim_{N\to\infty} \sup \left(\sup_{\nu\in[0,0.5)} \left(\sup_{\vec{u}^\nu \in H^B_{k,\nu}} \|\vec{u}^\nu - \vec{u}^\nu_n\|_E \right) \right) (g(N))^{-1} = K < \infty.$$

We see from the above definitions that if $f(N)$ is such that $f(N)F_0(N) = g(N) \to 0$ as $N \to \infty$, then $\{V^n\}$ shows locking or order $f(N)$ if and only if it is robust with maximum uniform order $g(N)$. Note that in the above definitions, we could choose other error measures instead of the energy norm (e.g., the $H^1(\Omega)$ norm); see [BS4].

As $\nu \to 0.5$, we see that the *incompressibility constraint*,

$$(5.6)\qquad C\,\vec{w} = \operatorname{div}\,\vec{w} = 0$$

gets imposed on the exact solution. The reason locking occurs is that (5.6) gets imposed on the finite element solution as well. For there to be no locking in the energy or $H^1(\Omega)$ norm, the following necessary condition must be satisfied

$$(5.7)\qquad \tilde{C}_1 F_0(N) \le \sup_{\vec{u}\in H^B_{k,0}} \inf_{\substack{\vec{v}\in V^n \\ C\vec{v}=0}} \|\vec{u} - \vec{v}\|_1 \le \tilde{C}_2 F_0(N)$$

where $H^B_{k,0} = \cap_{\nu\in[0,0.5)} H^B_{k,\nu}$. Condition (5.7) is sufficient as well, under certain general conditions (see [BS4]) which are shown in [BS5] to be satisfied for our model problem. This reduces the whole question of locking to one of approximability alone. Let us give some sample results from [BS5].

Let us take Ω to be a square domain. Let $\mathcal{T}^h_1$ and $\mathcal{T}^h_2$ be the uniform triangular and uniform square meshes, respectively (Fig. 5.1). Define, for $i = 1, 2, 3$,

$$V^i_{p,h} = \left\{u \in C^0(\Omega), u|_S \in R^i_p(S), \forall S \subset \mathcal{T}^h_i\right\}$$

where $R^1_p(S) = \mathcal{P}_p(S)$, $R^2_p(S) = Q_p(S)$, $R^3_p(S) = Q'_p(S)$ and $\mathcal{T}^h_3 = \mathcal{T}^h_2$.

THEOREM 5.1. *Let the extension process be the h version for the problem* (3.7), (5.1) *using subspaces* $V^n = (V^i_{p,h})^2$ *with p fixed. Let the solution sets be* $H^B_{k,\nu}$ *and let* $k \ge p + 1$. *Then the locking in the energy norm and also in the* H^1 *norm is as shown in Table* 2.

Table 2 shows that locking cannot be avoided for $p \le 3$ for the h version. Also, using rectangular elements leads to locking for all p. The result for $p \ge 4$ for $(V^1_{p,h})^2$ was first proved in [SV] and holds for more general meshes.

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FIG. 5.1. *Uniform triangular and square mesh.*

TABLE 2

Space V^n	Degree p	Order of locking ℓ $f(N) \approx CN^{\ell}$	Robustness order r $g(N) \approx CN^{-r}$
$(V^1_{p,h})^2$	$1 \le p \le 3$	$\ell = 1/2$	$r = (p-1)/2$
	$p \ge 4$	$\ell = 0$	$r = p/2$
$(V^2_{p,h})^2$	$p \ge 1$	$\ell = 1/2$	$r = (p-1)/2$
$(V^3_{p,h})^2$	$1 \le p \le 2$	$\ell = 1/2$	$r = (p-1)/2$
	$p \ge 3$	$\ell = 1$	$r = (p-2)/2$

In contrast, as mentioned earlier, it was shown in [Vo] that the p version (using straight sided triangles) leads to uniform robustness of optimal order for the displacements, so that locking is eliminated. However, the use of curvilinear elements is indispensable for the p version in most cases. In this connection, we have the following computational results.

Let us consider the p version for a single curvilinear element of type Q'_p for problem (3.7). We assume that this element is the image of the standard square under a curvilinear mapping F. Below, we show the relative errors in the energy norm for the cases that the mapped element is a square, a trapezoid, a quadrilateral with one side a parabola, and a quadrilateral with one side a circle. We do this for $\nu = 0.3$ (Fig. 5.2) and $\nu = 0.5 - 10^{-10}$ (Fig. 5.3). It is observed that the rate of convergence now behaves like $(p-s)^{-(k-1)}$, with $s \ge 0$ depending on F ($s = 0$ for affine F). There is no change in the apparent asymptotic rate so that in this sense, there is no observed locking.

We have treated various other examples of locking in [BS4] using the above definitions and principles. Like the above case, the reason locking occurs in general is due to the imposition of a constraint similar to (5.6). Such constraints usually can be interpreted as inter-element continuity requirements, with (5.7) breaking down when the constrained finite element space is too small. (For example, (5.6) can be interpreted as imposing $C^{(1)}$ continuity on a space related to V^n; see [BS5].) Unlike low-order h version schemes, the p version remains robust under such continuity constraints, which is why it is more resistent to locking.

As another example, consider problem (3.6) with k_1/k_2 close to zero. In this case, the constraint is $\partial w/\partial x_2 \cong 0$. Suppose the h version is used, with piecewise polynomials of degree $\le p$ on either of the meshes in Fig. 5.1. Then, since the meshes are aligned with the x_1 and x_2 axes, there is no locking (see [BS4]) since there are a number of functions satisfying $\partial v/\partial x_2 = 0$. However, if these meshes are not aligned with the x_1 and x_2 axes (for example, as in Fig. 5.4), then the only functions satisfying $\partial v/\partial x_2 = 0$ will be *actual* (rather than

FIG. 5.2. *Error behavior for curved elements for* $\nu = 0.3$.

FIG. 5.3. *Error behavior for curved elements for* $\nu = 0.5 - 10^{10}$.

piecewise) polynomials and (5.7) will not hold. There will be complete locking in this case, with the method being nonrobust. On the other hand, if we use the p version with just one element, then the constraint $\partial v/\partial x_2 = 0$ has no effect and there is no locking, i.e. the method is robust. See [BS4] for a detailed analysis. Also, see [SBS] for an analysis of locking in the Reissner–Mindlin plate.

6. Extraction and post-processing techniques. Usually in computational practice, the solution u of the differential equation considered is only a tool to get the primary quantity of interest. For example, the goal of the computation may be to find the stresses at a point, the stress intensity factors, or the resultant (e.g., the moments) in shell and plate theory.

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FIG. 5.4. *Rotated triangular and square meshes.*

The results in §§2 and 4 indicate the high asymptotic rates of convergence, including exponential convergence, that can be obtained by the h-p version, when the error is measured in the *energy* norm. The question that we ask here is how these rates can be obtained for quantities of interest other than the energy. For instance, if high-order derivatives of the h-p solution were computed directly, then severe oscillations could result and these rates would be lost.

Mathematically, we are interested in evaluating the value of a certain function of the solution. There are various forms in which this function may be expressed, using properties of the equation under consideration. Choosing one such form, we then substitute the finite element solution instead of the exact one to get an approximation of the desired value. If the exact solution were substituted, then the computed value would be the same, no matter which form of the function was used. However, when the finite element method is used, the accuracy of the computed value can be very different. In this section, we indicate how the convergence rate in the energy (i.e., the square of the energy norm) can be achieved for other quantities as well (see, e.g., [BM], [Wh]).

We illustrate the main idea by considering the simple one-dimensional problem,

$$\begin{aligned} -u'' + u &= f \quad \text{on } \Omega = (0, 1) = I, \\ u(0) &= u(1) = 0, \end{aligned} \tag{6.1}$$

where f is assumed to be in $L_2(\Omega)$ so that $u \in H^2(\Omega)$. Suppose the quantity we are interested in is the value of $u'(0)$.

Problem (6.1) can be cast in the variational form

$$B(u, v) = \int_0^1 (u'v' + uv)dx = \int_0^1 fvdx \tag{6.2}$$

with $u \in H_D^1(\Omega) = H_0^1(\Omega)$, $v \in H_D^1(\Omega)$.

Let $\psi \in H^1(\Omega)$ satisfy $\psi(0) = 1$, $\psi(1) = 0$ (for example, ψ could be a linear function in our case). Then

$$B(u, \psi) = \int_0^1 (-u'' + u)\psi dx - u'(0) = \int_0^1 f\psi dx - u'(0)$$

so that

$$u'(0) = \int_0^1 f\psi dx - B(u, \psi). \tag{6.3}$$

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Hence, we have two possible ways of obtaining the desired value $u'(0)$: we could either use differentiation, or equation (6.3). The function ψ is called an *extraction function*. Let us compare the accuracy of these two approaches.

Suppose that $\{V^n\}$ is the sequence of subspaces defining the extension process ($V^n \subset H_D^1$) and $\{u_n\}$ are the corresponding finite element solutions. Define

$$F = F(u) = \int_0^1 f\psi dx - B(u, \psi)(= u'(0)), \tag{6.4}$$

$$F_n = F(u_n) = \int_0^1 f\psi dx - B(u_n, \psi). \tag{6.5}$$

Then

$$|F - F_n| = |B(u_n - u, \psi)|. \tag{6.6}$$

Now let $z \in H_D^1(\Omega)$ be such that

$$B(v, z) = B(v, \psi) \quad \forall v \in H_D^1(\Omega). \tag{6.7}$$

Then for any $\omega^{(n)} \in V^n$ we have

$$|F - F_n| = |B(u_n - u, z)| = |B(u_n - u, z - \omega^{(n)})| \le C\|u_n - u\|_{H^1(\Omega)}\|z - \omega^{(n)}\|_{H^1(\Omega)}.$$

Hence,

$$|F - F_n| \le C\|u_n - u\|_{H^1(\Omega)} \inf_{\omega \in V^n} \|z - \omega\|_{H^1(\Omega)},$$

and the theory of §2 is applicable.

The smoothness of the function z can be easily established. The function z is analytic in our case, and if f is smooth, we get for the h version $\|z - w\|_{H^1(\Omega)} \le Ch^p$ as well as $\|u_n - u\|_{H^1(\Omega)} \le Ch^p$, so that $\|z - w\|_{H^1(\Omega)} \approx \|u_n - u\|_{H^1(\Omega)}$. Usually, either this or $\|z - w\|_{H^1(\Omega)} \le C\|u_n - u\|_{H^1(\Omega)}$ occurs in practical cases of the extraction method. We see therefore that the error $|F - F_n|$ is, in general, of the order $\|u_n - u\|^2_{H^1(\Omega)}$, i.e., the order of the square of the energy norm error. In contrast, if we defined F_n by direct differentiation, i.e., $F_n - u_n'(0)$, then we would only get $\|u_n - u\|_{H^1(\Omega)}$ (modulo a logarithmic term) as the rate of convergence of $|F - F_n|$.

We therefore see that by methods of the type above, called *extraction methods*, more accurate values can be obtained when ψ is properly selected (ψ smooth). (In the case above, we could, for instance, take $\psi = 1 - x$.) Let us note that the extracted value is not changed when we replace ψ by $\psi - \omega, \omega \in V^N$, so that, for example, in the h version, ψ can be restricted only to the element containing $x = 0$. (See [BM] for more.)

Similar ideas can be used for other values of interest. One important case is the extraction of stress intensity factors (see §3). Let us elaborate briefly on this case. Assume that we are solving the problem

$$-\Delta u = 0 \quad \text{on } \Omega$$

where Ω is the L-shaped domain shown in Fig. 3.1. Suppose on $\partial\Omega$ we have the boundary condition

$$\frac{\partial u}{\partial n} = g \quad \text{on } ABCDE,$$
$$u = 0 \quad \text{on } OE, OA.$$

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Then using (3.10)′ we have

$$u = Fr^{2/3}\cos\left(\frac{2\theta}{3}\right) + v$$

with

$$|v||r^{-2/3}|, \qquad |\text{grad } v||r^{1/3}| = o(1).$$

Here, F is the stress intensity factor. Let us define

$$\psi = \left(r^{-2/3}\cos\left(\frac{2\theta}{3}\right)\right)\chi(r)$$

where $\chi = 1$ for $0 \le r < a, \chi = 0$ for $r > b, a < b < 1, \chi \in C^\infty(\Omega)$. Obviously, $w = \Delta\psi \in C^\infty(\Omega)$ and

$$w = 0 \quad \text{for } r < a \quad \text{and} \quad r > b.$$

Now denoting $\Omega_\varepsilon = \Omega - \bar{R}_\varepsilon$ where $\bar{R}_\varepsilon = \{(r,\theta)|r \le \varepsilon\}$ we have

$$\begin{aligned}\int_{\Omega_\varepsilon} u\Delta\psi dx = \int_\Omega uwdx &= \int_{\partial R_\varepsilon}\frac{\partial\psi}{\partial n}uds - \int_{\partial R_\varepsilon}\frac{\partial u}{\partial n}\psi ds \\ &= -\frac{4}{3}F\int_0^{3\pi/2}\cos^2\left(\frac{2\theta}{3}\right)d\theta + o(1).\end{aligned}$$

Hence, taking the limit as $\varepsilon \to 0$,

$$F = -\left(\int_\Omega uw\,dx\right)\frac{1}{\pi}.$$

Proceeding as before, we define

$$F_n = -\left(\int_\Omega u_n w\,dx\right)\frac{1}{\pi}$$

and get

$$|F - F_n| \le C\|u - u_n\|_{H^1(\Omega)}\|z - z_n\|_{H^1(\Omega)},$$

where z is the solution of the problem

$$\begin{aligned}-\Delta z &= w,\\ \frac{\partial z}{\partial n} &= 0 \quad \text{on } ABCDE,\\ z &= 0 \quad \text{on } OE, OA.\end{aligned}$$

Because $w \in C^{(\infty)}(\Omega)$, and all internal angles except at 0 are $\frac{\pi}{2}$, z is C^∞ except in a neighborhood of the origin. In this case (when $F \ne 0$) we have

$$\|z - z_n\|_{H^1(\Omega)} \le C\|u - u_n\|_{H^1(\Omega)},$$

and hence

$$|F - F_n| \approx \|u - u_n\|^2_{H^1(\Omega)}$$

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(or better), i.e., $|F - F_n|$ is once more of the order of the *square* of the energy norm error. This is typical for this kind of extraction.

The stress intensity factor extraction method obviously requires a priori knowledge of the form of the singularities. In two dimensions, a numerical algorithm for obtaining the singular form for the general elasticity problem may be found in [Pa] (see also [Sa]).

As a numerical example, we consider the L-shaped domain shown in Fig. 3.1. We consider the elasticity problem described in Section 4 with prescribed traction conditions on $\partial\Omega$ such that the exact solution is given by

$$(u_1, u_2) = \left(u_1^I + 2u_1^{II}, u_2^I + 2u_2^{II}\right)$$

where u_i^I, u_i^{II} are given by (4.16). The mesh shown in Fig. 4.5 with two layers is used. In Table 3, we report results using the p-extension for the given mesh, showing the number of degrees of freedom N, the strain energy, as well as the values of the first two stress intensity factors. In addition, we present the exact values ($p = \infty$) of these stress intensity factors. (Note that the strain energy is $\frac{1}{2}\|\vec{u}\|_E^2 = \frac{1}{2}B(\vec{u}, \vec{u})$, where $B(\vec{u}, \vec{u})$ is as defined in (3.7).) In Fig. 6.1, we show the relative error in the stress intensity factors as well as the relative error in the strain energy of the solution (i.e., square of the error in the energy norm). We see that the errors in the stress intensity factors are in fact of the order of those in the strain energy. For more, we refer to [SzB]. For the extraction of stress intensity factors in 3-d, we refer to [AB] and [BPA].

TABLE 3
Strain energy and stress intensity factors.

P	N	Strain Energy	Stress Intensity Factor-Mode 1	Stress Intensity Factor-Mode 2
1	41	6.42072796	0.95268	2.29075
2	119	6.74137580	1.02177	2.08422
3	209	6.77029847	1.00250	2.02239
4	335	6.77575144	1.00073	2.00437
5	497	6.77683967	0.99991	2.00097
6	695	6.77719530	0.99985	2.00022
7	929	6.77736281	0.99987	2.00005
8	1199	6.77749228	0.99990	2.00001
∞	∞	6.77776914	1.00000	2.00000

7. Plate modelling. A very common problem in structural mechanics is the problem of solving partial differential equations on a *thin* domain. A typical problem of this type is the *plate problem*, which we elaborate on below.

Let

$$\Omega = \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 | (x_1, x_2) \in \omega \subset \mathbb{R}^2, |x_3| < \frac{d}{2}\right\},$$

where ω is a bounded domain with piecewise smooth boundary. Denote

$$S = \left\{x \in \mathbb{R}^3 | (x_1, x_2) \in \partial\omega, |x_3| < \frac{d}{2}\right\} \quad \text{and} \quad R_\pm = \left\{x \in \mathbb{R}^3 | (x_1, x_2) \in \omega, x_3 = \pm\frac{d}{2}\right\}.$$

We assume that $d << \text{diam}\,(\omega)$.

We are interested in the three-dimensional elasticity problem which we mentioned in §3. Denoting

$$\vec{u} = (u_1, u_2, u_3),\ H_D^1(\Omega) = \{\vec{u} \in (H^1(\Omega))^3 | u_3 = 0 \text{ on } S\}$$

FIG. 6.1. *Error of the stress intensity factors and strain energy.*

and

$$F(\vec{v}) = \frac{1}{2}\int_{\omega} g(x_1, x_2)\left(v_3\left(x_1, x_2, \frac{d}{2}\right) + v_3\left(x_1, x_2, -\frac{d}{2}\right)\right) dx_1 dx_2, \tag{7.1}$$

the solution $\vec{u} \in H_D^1(\Omega)$ of the plate problem satisfies (3.4) for every $\vec{v} \in H_D^1(\Omega)$, with $B(\vec{u}, \vec{v})$ given by (3.7) (with $t = 3$). This problem is the so-called plate problem with soft simple support. Using other constrained spaces, we can describe other physical conditions (for example, using the condition $\vec{u} = 0$ instead of $u_3 = 0$ on S gives the clamped boundary condition).

In order to numerically approximate this problem, a common technique is to first replace the *three-dimensional* model by a *two-dimensional* one, which leads to a considerable reduction in the number of degrees of freedom and makes the problem more numerically tractable. The simplest model that could be considered is the Kirchhoff plate. However, this has the disadvantage of requiring $C^{(1)}$ elements.

Two-dimensional models that only require $C^{(0)}$ continuous elements can often be viewed as one of a *hierarchy* of models in the sense described below. To obtain such models, the solution is essentially "expanded" in terms of polynomials (or other functions) in the x_3 direction. More precisely, let $n = (n_1, n_2, n_3)$, $n_i \geq 0$ are integers, and define

$$\begin{aligned} {}^{n}H_D^1(\Omega) = \Bigg\{ \vec{u} \in H_D^1(\Omega) | u_i = \sum_{j=0}^{n_i} \varphi_{ij}(x_1, x_2) L_j\left(\frac{2x_3}{d}\right), \qquad i = 1, 2, \\ u_3 = \sum_{j=0}^{n_3} \psi_j(x_1, x_2) L_j\left(\frac{2x_3}{d}\right) \Bigg\}, \end{aligned} \tag{7.2}$$

where $L_j(x)$ are Legendre polynomials. Then by the *hierarchy* of plate models we mean the problem of finding $\vec{u}^{(n)}$ such that

$$B(\vec{u}^{(n)}, v) = F(\vec{v}) \quad \forall\ \vec{v} \in {}^{n}H_D^1(\Omega). \tag{7.3}$$

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Let us note that because of symmetry, the models $(2n_1 + 1, 2n_2 + 1, 2n_3)$ are the same as models $(2n_1+2, 2n_2+2, 2n_3+1)$. Typical models used are $(n, n, n+1)$ models. As we note below, such hierarchies of models are easily implemented in the framework of the p version. The exact plate solution and the solution of the particular model depends on the thickness d. It is possible to show that the models $n = (n_1, n_2, n_3)$, $n_1 \geq 1$, $n_2 \geq 1$, and $n_3 \geq 2$ converge to a limiting solution as $d \to 0$, which is the same as the limiting solution as $d \to 0$ of the exact (three-dimensional) solution introduced above. The model (1,1,0) which is very often used in engineering (the so-called Reissner–Mindlin model) does not have the above property, namely, the same limiting solutions as the exact one, unless modified elastic constants are used (see, e.g., [BL1]). We note that for $\nu = 0$, no modification is needed. Further, it is possible to show that the polynomial choice in x_3 used in (7.2) is asymptotically optimal.

For fixed d, the solutions of the models converge to the exact 3-d solution as $\min(n_1, n_2, n_3) \to \infty$. If $g(x)$ in (7.1) satisfies some smoothness conditions, then, for larger n (n fixed), the rate with respect to d (as $d \to 0$) is higher. The various models have different properties with respect to boundary layers, corner singularities, etc.; see [BL2].

Assume now that the domain ω is partitioned into the elements τ_j based on triangles T or squares S (see §4). Then $T_j = \tau_j \times (-d/2, d/2)$ is a three-dimensional element. Now denote $Q_j(p, q)$ to be the set of polynomials of degree q in x_3 and of degree p in x_1, x_2 (this could include the case of $\mathcal{P}_p$, Q_p or Q'_p type elements in x_1, x_2—see §4). Then the finite element solution with elements $Q_j(p, q)$ of the three-dimensional plate problem with degree q fixed can be interpreted as the finite element solution (using elements of degree p) of the plate model with $n_1 = n_2 = n_3 = q$ in (7.1). Hence, the hierarchic modelling can be easily implemented in the frame of the p (or h-p) version of three-dimensional elasticity. (In fact, this implementation is available in the commercial program MSC/PROBE.) Let us comment on three main aspects of the solution of the plate problem in this hierarchical framework.

(a) *The locking problem.* Here, the theory addressed in Section 5 fully applies. The analog to constraint (5.6) for the (1,1,0) model is

$$\varphi_{i,1} = -\frac{\partial \psi_0}{\partial x_i}, \qquad i = 1, 2.$$

For more, see [SBS].

(b) *The problem of optimal meshes.* In the neighborhood of corners, refined meshes similar to the ones discussed in the previous sections (radical or geometric) should be used. In the neighborhood of edges, the use of "needle" elements analogous to those used in the neighborhood of edges in three dimensions is of importance to resolve boundary layers. Let us mention, however, that the singularity here is of different character than the edge singularity in three-dimensional problems. The problem of the optimal mesh has not been satisfactory resolved as yet.

(c) *The pollution problem.* The error in the neighborhood of the boundary can now influence the accuracy of the solution far from the boundary. Essentially, there is pollution present.

8. Implementational aspects. In this paper, our emphasis has been mainly on theoretical aspects of the p and h-p versions. In this section, we briefly focus on implementational aspects which, of course, are of essential importance in terms of practical use.

8.1. Differences between h-p and h version codes. As mentioned in the introduction, there are already various research and commercial codes based on the p and h-p versions. The implementation of p and h-p version codes such as these is significantly different from that of h version codes. Let us summarize some of the basic differences.

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(a) *Mesh construction.* This is much simpler for the p version, which uses a relatively small number of elements of large size. With p version codes, it is important that the user select a "good" mesh (in the sense of Remark 4.1) to obtain a high rate of convergence (i.e. one refined in the areas of singularities).

(b) *Sparsity.* The local stiffness matrices are much less sparse for the p version than the h version. Hence, the computation of stiffness matrices has to be given special care, for instance by taking into account the computer architecture used. For example, the program STRIPE successfully exploits the architecture of the CRAY computer to deal with this problem. Reduced sparsity, especially in the case of three-dimensional problems with more than 100,000–500,000 unknowns, has serious implications for the I/O time of communication between disk and fast memory.

(c) *Adaptivity.* In the case of the p version, the structure of an adaptive program is much simpler than the h version, since only new shape functions are added, without changing the mesh (see [AB]). The architecture for the adaptive h-p version (e.g., PHLEX) is much more complex (see [DO]). The adaptive selection of shape functions is usually determined by considerations based on the error in the energy. Essentially, those shape functions are added which maximally change the computed energy. The reason that adaptive principles for the h-p version are more complex is that simultaneous decisions about mesh changes and shape function selection have to be made.

(d) *Accuracy assessment.* The p version allows the user to control the desired accuracy of the output quantities of interest. This control is essentially based on comparison of the computed quantities by increasing the degree p, so that the user can a priori specify the required accuracy. This feature has been incorporated, e.g., in Applied Structure—see also [AB]. It is also possible to use error estimators analogous to the h version. For more, see [BSt1] and [BSt2].

(e) *Linear solver.* This can either be *direct*, when many right hand sides (loads) are present and the number of unknowns is not too large (say $\ll 10-20k$), or *iterative*, when the number of unknowns is large ($> 100k$, say) (see, e.g, [Mn1], [Mn2]). An effective preconditioner can be based on the low p ($p = 1, 2$) discretization. For various aspects of the computation on parallel computers (Alliant FX/8) we refer to [BEM].

(f) *Hierarchic bases.* As mentioned in Section 2.2, the shape functions for the p version have a hierarchic character. This allows the use of different degrees p in different elements simply by adding or suppressing shape functions. Such a technique is used in the p-adaptive approach as, for example, in the code STRIPE. The hierarchic bases can also be used in the construction of preconditioners through elements of low degree.

(g) *Overall effectivity.* The effectivity of any method, h, p or h-p, depends upon many factors. In this paper, we have focused primarily on one factor, the asymptotic rate of convergence (with respect to the number of degrees of freedom). Some other important factors are computational cost, man-power cost for data preparation, and proper usage. Further, estimations of efficiency must also take into account various features that may be present, such as adaptive features, a posteriori error estimation, etc. (The capability of a posteriori error estimations is essential if a program is to give reliable results.)

8.2. A benchmark comparison. The complex question of comparison of different methods and codes is usually accomplished through various benchmark problems. In this section, we present some results from an engineering comparison study made in [CM]. Here, a benchmark (Pillow Block Test Case) was computed by the p version code POLYFEM/IBM and the h version code MAC/NASTRAN/65E (both on the IBM 3090-600J/VF). The p version and h version meshes for the pillow block are shown respectively in Figs. 8.1 and 8.2. The error as a function of the degrees of freedom is shown in Fig. 8.3, while Fig. 8.4 shows the error as a

function of the CPU time. (Note that here, the error measure used is the strain energy, which is the *square* of the energy norm, so that a 1% error in the strain energy corresponds to a 10% error in the energy norm.)

FIG. 8.1. p *version mesh for pillow block test case.* (*From* New basis functions and computational procedures for p-type finite element analysis, *P. Carneval, R. B. Morris, Y. Tsuji, and G. Taylor,* International Journal for Numerical Methods in Engineering, 36 (1993), pp. 3759–3780. *Reprinted by permission of John Wiley & Sons, Ltd.*)

FIG. 8.2. h *version meshes for pillow block test case.* (*From* New basis functions and computational procedures for p-type finite element analysis, *P. Carneval, R. B. Morris, Y. Tsuji, and G. Taylor,* International Journal for Numerical Methods in Engineering, 36 (1993), pp. 3759–3780. *Reprinted by permission of John Wiley & Sons, Ltd.*)

Obviously, the results demonstrate the viability of the p version not only in terms of the number of degrees of freedom but also in terms of CPU time. Comparative studies using other

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FIG. 8.3. *Strain energy error vs. degrees of freedom. Dotted Line*: h *version. Solid line*: p *version.* (*From* New basis functions and computational procedures for p-type finite element analysis, *P. Carneval, R. B. Morris, Y. Tsuji, and G. Taylor,* International Journal for Numerical Methods in Engineering, 36 (1993), pp. 3759–3780. *Reprinted by permission of John Wiley & Sons, Ltd.*)

FIG. 8.4. *Strain energy error vs. CPU time. Dotted line*: h *version. Solid line*: p *version.* (*From* New basis functions and computational procedures for p-type finite element analysis, *P. Carneval, R. B. Morris, Y. Tsuji, and G. Taylor,* International Journal for Numerical Methods in Engineering, 36 (1993), pp. 3759–3780. *Reprinted by permission of John Wiley & Sons, Ltd.*)

h-p programs show similar features. For example, in [BEM], a simple example is analyzed and the optimal degrees which lead to minimum CPU times for various accuracies are given, in the context of parallel computation on Alliant FX/8.

Finally, let us mention that the h-p version, by definition, is a generalization of both the h and p versions. Hence, it can be expected that a proper implementation of the adaptive h-p version will necessarily overperform both the h and p versions.

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