SIAM J. NUMER. ANAL.
Vol. 18, No. 3, June 1981

0036-1429/81/1803-0013 $01.00/0

A POSTERIORI ERROR ANALYSIS OF FINITE ELEMENT SOLUTIONS FOR ONE-DIMENSIONAL PROBLEMS*

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Abstract. The paper develops a theory of a posteriori error estimates under the L_p-energy norm for $2 \leqq p \leqq \infty$. The theory is based on a general concept of error indicators and error estimators. Several specific examples of these quantities are introduced and analyzed in detail. The results provide a variety of easily computable error estimates under all these important norms for finite element solutions of one-dimensional problems.

1. Introduction. Generally, in practical finite element computations, it is very desirable to provide for reliable and computationally inexpensive error estimates. However, in order to be broadly usable, such estimates need to be available for various physically important norms. In a sequence of papers, the authors have developed a theory of a posteriori estimates for the L_2-energy norm. More specifically, in [1] and [2] estimates for one-dimensional problems were presented while [3] concerns the multi-dimensional case. In [4], the multi-dimensional estimates are analyzed which form the basis of the experimental adaptive finite element solver FEARS developed at the University of Maryland (see also [5], [6]).

In this paper, we consider again the one-dimensional case and develop for it a more complete theory of a posteriori error estimates under any L_p-energy norm for $2 \leqq p \leqq \infty$. The theory is based on a general concept of error indicators and error estimators. Several specific examples of these quantities are introduced and analyzed in detail. The results provide a variety of easily computable error estimates under all these important norms for finite element solutions of one-dimensional problems.

Already for the above-mentioned estimates under the L_2-energy norm, there exist essential differences between the one- and the multi-dimensional cases. These differences are further accentuated for the L_p-energy norms considered here.

2. Notation. For given real α, β with $l = \beta - \alpha > 0$, let $I(\alpha, \beta)$ be the open interval $\{x \in R^1; \alpha < x < \beta\}$ and $\bar{I}(\alpha, \beta)$ its closure. Whenever there is no possibility of confusion, we shall write I or $I(l)$ instead of $I(\alpha, \beta)$.

As usual, $H_p^0(I) = L_p(I)$, $1 \leqq p \leqq \infty$, will be the space of all pth power integrable functions on I with finite norm

$$(2.1)\qquad \begin{aligned} {}_I\|u\|_{p,0} &= \left[\int_\alpha^\beta |u|^p\,dx\right]^{1/p} \quad \text{if } 1 \leqq p < \infty, \\ {}_I\|u\|_{\infty,0} &= \operatorname*{ess\,sup}_{x \in I} |u(x)| \quad \text{if } p = \infty. \end{aligned}$$

More generally, for any integer $k \geqq 0$ we denote by $H_p^k(I)$, $1 \leqq p \leqq \infty$, the spaces of all such functions with finite norm

$$(2.2)\qquad \begin{aligned} {}_I\|u\|_{p,k} &= \left[\sum_{i=0}^{k} {}_I\left\|\frac{d^iu}{dx^i}\right\|_{p,0}^p\right]^{1/p} \quad \text{if } 1 \leqq p < \infty, \\ {}_I\|u\|_{\infty,k} &= \max_{0 \leqq i \leqq k} \left\|\frac{d^iu}{dx^i}\right\|_{\infty,0} \quad \text{if } p = \infty. \end{aligned}$$

* Received by the editors December 27, 1979, and in revised form June 13, 1980. This work was supported in part by the U.S. Department of Energy under contract E(401)3443, the National Science Foundation under grant MCS-78-05299, and the U.S. Office of Naval Research under contract N0014-77-C-0623.

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Here d^iu/dx^i is the usual (weak) ith derivative of u; for $i=1,2$ we often use the standard notation u' and u''. Moreover, $\mathring{H}_p^k(I)\subset H_p^k(I)$, $1\leqq p\leqq\infty$, $k\geqq 0$, shall be the spaces of all functions of $H_p^k(I)$ which together with all their derivatives up to order $k-1$ are zero at the endpoints of I (in the sense of traces). The space $\mathring{H}_p^k(I)$ may also be supplied with the equivalent norm

$$ {}_I|u|_{p,k} = {}_I\left\|\frac{d^ku}{dx^k}\right\|_{p,0}. \tag{2.3}$$

Evidently, this is a seminorm on $H_p^k(I)$ but a norm on $\mathring{H}_p^k(I)$. The space $\mathring{H}_p^k(I)$ with the norm (2.3) is called $\mathring{L}_p^k(I)$.

Let $\mathscr{E}(I)$ be the space of all real, infinitely differentiable functions on I for which all derivatives have continuous extensions on $\bar{I}$. Moreover, let $\mathscr{D}(I)\subset\mathscr{E}(I)$ be the subspace of all functions with compact support in I. For $1\leqq p<\infty$, $\mathscr{E}(I)$ and $\mathscr{D}(I)$ are dense in $H_p^k(I)$ and $\mathring{H}_p^k(I)$, respectively. The closure of $\mathscr{E}(I)$ and $\mathscr{D}(I)$ in $H_\infty^k(I)$ are denoted by $C^k(I)$ and $\mathring{C}^k(I)$, respectively.

On the interval $I(\alpha,\beta)$ we consider partitions $\Delta(I)$ of the form

$$\Delta(I)\colon \alpha=x_0^\Delta<x_1^\Delta<\cdots<x_m^\Delta=\beta,\qquad m=m(\Delta)\geqq 1, \tag{2.4}$$

and write

$$\begin{gathered} I_j(\Delta)=I(x_{j-1}^\Delta,x_j^\Delta),\quad h_j(\Delta)=x_j^\Delta-x_{j-1}^\Delta,\quad j=1,2,\cdots,m(\Delta),\\ \bar{h}(\Delta)=\max_{j=1,\cdots,m(\Delta)} h_j(\Delta),\qquad \underline{h}(\Delta)=\min_{j=1,\cdots,m(\Delta)} h_j(\Delta).\end{gathered} \tag{2.5}$$

Again, the argument Δ will often be omitted when it is self-evident. A family $\mathscr{P}$ of partitions is said to be (λ,κ)*-regular* if for fixed $1\leqq\lambda<\infty$, $1\leqq\kappa<\infty$, we have

$$\underline{h}(\Delta)\geqq\lambda\bar{h}^\kappa(\Delta)\qquad\forall\Delta\in\mathscr{P}. \tag{2.6}$$

A family $\mathscr{P}$ of partitions is λ*-quasi-uniform* if

$$\frac{1}{\lambda}\leqq\frac{h_i(\Delta)}{h_{i-1}(\Delta)}\leqq\lambda,\qquad i=1,\cdots,m(\Delta)\quad\forall\Delta\in\mathscr{P}, \tag{2.7}$$

with a fixed constant $1<\lambda<\infty$.

For any partition (2.4) we define $\mathring{S}(I,\Delta)\subset\mathring{H}_\infty^1(I)$ as the subspace of all functions of $\mathring{H}_\infty^1(I)$ which are linear on each subinterval $I_j(\Delta)$, $j=1,\cdots,m(\Delta)$. Moreover, for any integer $k\geqq 0$ we introduce the space ${}^\Delta H_p^k(I)\subset\mathring{H}_p^k(I)$ of all functions of $\mathring{H}_p^k(I)$ which together with all their derivatives up to order $k-1$ are zero at the points x_i^Δ, $i=0,1,\cdots,m(\Delta)$. The same space with the norm (2.3) will be denoted by ${}^\Delta L_p^k(I)$.

3. The model problem and its basic properties. On $I(0,l)$, $0<l\leqq 1$, we consider the boundary value problem

$$L[u]\equiv-\frac{d}{dx}a(x)\frac{du}{dx}+b(x)u=f(x), \tag{3.1}$$

$$u(0)=u(l)=0. \tag{3.2}$$

Here a, b, f are assumed to be measurable functions on I such that

$$\left.\begin{array}{ll} \text{i)} & 0<\underline{a}\leqq a(x)\leqq\bar{a}<\infty,\\ \text{ii)} & 0\leqq b(x)\leqq\bar{b},\\ \text{iii)} & f\in H_1^0(I),\end{array}\right\}\quad\forall x\in\bar{I}. \tag{3.3}$$

Later on, we will need more restrictive conditions about those functions. In particular, we shall use

$$\text{(3.3)} \quad \text{iv)} \qquad a \in C^1(I), \qquad |a'(x)| \leqq \bar{a}', \qquad x \in I.$$

A basic result about this problem may be stated as follows:

THEOREM 3.1. *Define the bilinear form*

$$\text{(3.4)} \qquad B(u,v) = \int_0^l (a(x)u'(x)v'(x) + b(x)u(x)v(x))\,dx,$$

for all $(u,v) \in \mathring{L}_p^1(I) \times \mathring{L}_q^1(I)$, $1 \leqq p, q \leqq \infty$, $1/p + 1/q = 1$. *Then B satisfies the following inequalities*:

$$\text{(3.5)} \quad \text{i)} \qquad |B(u,v)| \leqq C_1^{[p]}\, {}_I|u|_{p,1}\, {}_I|v|_{q,1} \quad \forall (u,v) \in \mathring{L}_p^1(I) \times \mathring{L}_q^1(I),$$

where

$$\text{(3.5)} \quad \text{ii)} \qquad C_1^{[p]} = \bar{a} + \bar{b}\left(\frac{l}{2}\right)^2 \cdot p^{-1/p} \cdot q^{-1/p};$$

and

$$\text{(3.6)} \quad \text{i)} \qquad \inf_{\substack{u \in \mathring{L}_p^1(I) \\ |u|_{p,1}=1}} \sup_{\substack{v \in \mathring{L}_q^1(I) \\ |v|_{q,1}=1}} |B(u,v)| \geqq C_2^{[p]},$$

where

$$\text{(3.6)} \quad \text{ii)} \qquad C^{[p]} = \underline{a}\left\{\left[1 + \frac{\bar{b}l^2}{\underline{a}2^{1/p}}\left(1 + \frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)^2 \left(\frac{p}{2+p}\right)^{1/2}\right)\right]\left[1 + \frac{1}{\underline{a}^p}\right]^{1/p} [1 + \bar{a}^q]^{1/q}\right\}^{-1}.$$

Proof. For $u \in \mathring{L}_p^1(I)$, we have $u(0) = u(l) = 0$, and hence,

$$\text{(3.7)} \qquad \begin{aligned} |u(x)| &\leqq x^{1/q}_{I(0,l/2)}|u|_{p,1}, \qquad 0 \leqq x \leqq \frac{l}{2}, \\ |u(x)| &\leqq (l-x)^{1/q}_{I(l/2,l)}|u|_{p,1}, \qquad \frac{l}{2} \leqq x \leqq l, \end{aligned}$$

which in turn implies that on $I = I(0, l)$,

$$\text{(3.8)} \qquad {}_I\|u\|_{p,0}^p \leqq \left(\frac{l}{2}\right)^p \frac{1}{p}\, {}_I|u|_{p,1}^p.$$

Applying the Hölder inequality to (3.4) and using (3.8) we then obtain

$$\begin{aligned} |B(u,v)| &\leqq \bar{a}\|u'\|_{p,0}\|v'\|_{q,0} + \bar{b}\|u\|_{p,0}\|v\|_{q,0} \\ &\leqq \bar{a}|u|_{p,I}|v|_{q,1} + \bar{b}\left(\frac{l}{2}\right)^2 \frac{1}{p^{1/p}} \frac{1}{p^{1/q}}|u|_{p,1}|v|_{q,1} \\ &= C_1^{[p]}|u|_{p,1}|v|_{q,1}, \end{aligned}$$

which proves (3.5).

For the proof of (3.6) we define for any $u \in \mathring{L}_p^1(I)$ the function

$$\text{(3.9)} \qquad w(x) = \int_0^x (\operatorname{sgn} u'(t))|u'(t)|^{p-1}\,dt - A\int_0^x \frac{dt}{a(t)}\,dt, \qquad 0 \leqq x \leqq l,$$

with

$$A=\left[\int_0^l \frac{dt}{a(t)}\right]^{-1}\left[\int_0^l (\operatorname{sgn} u'(t))|u'(t)|^{p-1}\,dt\right]. \tag{3.10}$$

Clearly then, $w(0)=w(l)=0$ and

$$w'(x)=(\operatorname{sgn} u'(x))|u'(x)|^{p-1}-\frac{A}{a(x)}.$$

Therefore,

$$|w'(x)|\leqq[|u'(x)|^{(p-1)q}+|A|^q]^{1/q}\left[1+\frac{1}{\underline{a}^p}\right]^{1/p},$$

and by (3.10)

$$|A|\leqq \bar{a}l^{-1/q}|u|_{p,1}^{p/q}. \tag{3.11}$$

This shows that

$$|w|_{q,1}=\|w'\|_{q,0}\leqq D\|u'\|_{p,0}^{p/q}=D|u|_{p,1}^{p,q}, \tag{3.12}$$

with

$$D=\left[1+\frac{1}{\underline{a}^p}\right]^{1/p}[1+\bar{a}^q]^{1/q}, \tag{3.13}$$

and thus $w\in\overset{\circ}{L}{}_q^1$. Furthermore, as in the proof of (3.8) it follows that

$$\|w\|_{2,0}\leqq\left(\frac{l}{2}\right)^{(2+p)/2p}\left(\frac{2p}{2+p}\right)^{1/2}\|w'\|_{q,0}. \tag{3.14}$$

Now let $z\in\overset{\circ}{H}{}_2^1(I)$ be the solution of the auxiliary problem

$$L[z]=-bw,\qquad z(0)=z(l)=0.$$

The standard theory together with (3.8) and (3.14) then provides that

$$|z|_{2,1}\leqq\frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)\frac{1}{2^{1/2}}\|w\|_{2,0}\leqq\frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)^{(2+p)/2p+1}\left(\frac{p}{2+p}\right)^{1/2}\|w'\|_{q,0}, \tag{3.15}$$

and hence with (3.7) that

$$|z|_{\infty,0}\leqq\frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)^{(1/p)+2}\left(\frac{p}{2+p}\right)^{1/2}|w|_{q,1}. \tag{3.16}$$

Thus, we have

$$|z+w|_{\infty,0}\leqq\left[\frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)^{(1/p)+2}\left(\frac{p}{2+p}\right)^{1/2}+\left(\frac{l}{2}\right)^{1/p}\right]|w|_{q,1}, \tag{3.17}$$

and

$$-\frac{d}{dx}a(x)\frac{dz}{dx}=-b(x)(z(x)+w(x)),$$

implies that

$$|z'|_{\infty,0}\leqq\frac{\bar{b}}{\underline{a}}l\left[\frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)^{(1/p)+2}\left(\frac{p}{2+p}\right)^{1/2}+\left(\frac{l}{2}\right)^{1/p}\right]|w|_{q,1},$$

whence

$$\|z'\|_{q,0} \leqq \frac{\bar{b}l^2}{\underline{a}2^{1/p}}\left[\frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)^2\left(\frac{p}{2+p}\right)^{1/2}+1\right]|w|_{q,1}. \tag{3.18}$$

Now set $v = w + z$. Then

$$|v|_{q,1} \leqq K|w|_{q,1}, \tag{3.19}$$

with

$$K = 1 + \frac{\bar{b}l^2}{\underline{a}2^{1/p}}\left[\frac{\bar{b}}{\underline{a}}\left(\frac{l}{2}\right)^2\left(\frac{p}{2+p}\right)^{1/2}+1\right],$$

and by (3.12),

$$\frac{|u|_{p,1}^{p/q}}{|v|_{q,1}} \geqq \frac{1}{DK}. \tag{3.20}$$

Since

$$\begin{aligned} B(u,v) = B(u,w) + B(u,z) &= B(u,w) - \int_0^l b(x)w(x)u(x)\,dx \\ &= \int_0^l a(x)u'(x)w'(x)\,dx \\ &= \int_0^l a(x)|u'(x)|^p\,dx - A\int_0^l u'(x)\,dx \\ &\geqq \underline{a}|u|_{p,1}^p \geqq \underline{a}|u|_{p,1}|v|_{q,1}\frac{|u|_{p,1}^{p/q}}{|v|_{q,1}} \geqq \underline{a}\frac{1}{DK}, \end{aligned} \tag{3.21}$$

the inequality (3.6) now follows.

For $p = 2$, the constant $C_2^{[p]}$ can be improved.

THEOREM 3.2. *Under the conditions of Theorem* 3.1 *with* $p = 2$, *we have*

$$C_2^{[p]} = \underline{a}. \tag{3.22}$$

Proof. Let $u \in \overset{\circ}{L}{}_2^1(I(0,l))$ and set $v = u$. Then

$$\begin{aligned} B(u,v) = B(u,u) &= \int_0^l [a(x)u'(x)^2 + b(x)u(x)]\,dx \\ &\geqq \underline{a}|u|_{2,1}^2 = \underline{a}|u|_{2,1}|v|_{2,1}. \end{aligned}$$

For later reference, we mention also the following special case:

COROLLARY 3.3. *Under the conditions of Theorem* 3.1 *for* $a \equiv 1$, $b \equiv 0$ *on* $I(0,l)$ *the constants* (3.5ii), (3.6ii) *reduce to*

$$C_1^{[p]} = 1, \qquad C_2^{[p]} = \tfrac{1}{2}. \tag{3.23}$$

This is directly verifiable from the formulas of Theorem 3.1.

The following existence theorem is well known; for a proof see e.g., [7]:

THEOREM 3.4. *Let* $F \in \overset{\circ}{L}{}_q^1(I)'$, $I = I(0,l)$, $1 < q < \infty$, *be a given linear functional over* $\overset{\circ}{L}{}_q^1(I)$. *Then there exists a unique* $u_0 \in \overset{\circ}{L}{}_p^1(I)$, $1/p + 1/q = 1$, *such that*

$$B(u_0, v) = F(v) \quad \forall v \in \overset{\circ}{L}{}_q^1(I) \tag{3.24}$$

and

$$|u_0|_{p,1} \leqq \frac{1}{C_2^{[p]}} \|F\|_{\mathring{L}_q^1(I)'}, \tag{3.25}$$

where $C_2^{[p]}$ *is given by* (3.6ii).

As proved in [7], this theorem holds only for reflexive spaces and hence we had to exclude the cases $q = 1, \infty$. However, we shall use later on that when

$$F(v) = \int_0^l f(x)v(x)\,dx, \tag{3.26}$$

with some $f \in L_1^0(I)$, then Theorem 3.4 holds also for $q = 1, \infty$. In fact, in this case, we have $F \in \mathring{L}_q^1(I)'$ for all q, $1 < q < \infty$, and $\|F\|_{\mathring{L}_q^1(I)'}$ is uniformly bounded with respect to q. Hence, the standard limiting arguments for $q \to 1$ as well as $q \to \infty$ ensure that (3.25) holds for $q = 1, \infty$ whenever F is defined by (3.26).

For a given partition (2.4), the form (3.4) may also be considered on ${}^{\Delta}L_p^1(I) \times {}^{\Delta}L_p^1(I)$. For this, let

$$\left.\begin{aligned} 0 < \underline{a}_i \leqq a(x) \leqq \bar{a}_i \\ 0 \leqq b(x) \leqq \bar{b}_i \end{aligned}\right\} \quad \forall x \in I_i(\Delta), \quad i = 1, \cdots, m(\Delta), \tag{3.27}$$

and in analogy to (3.5ii), (3.6ii) define

$$\begin{aligned} {}^{[i]}C_1^{[p]} &= \bar{a}_i + \bar{b}_i\left(\frac{h_i}{2}\right)^2 p^{-1/p} q^{-1/q}, \\ {}^{[i]}C_2^{[p]} &= \underline{a}_i \left\{ \left[1 + \frac{\bar{b}_i h_i^2}{\underline{a}_i 2^{1/p}}\left(1 + \frac{\bar{b}_i}{\underline{a}_i}\left(\frac{h_i}{2}\right)^2 \left(\frac{p}{2+p}\right)^{1/2}\right)\right]\left[1 + \frac{1}{\underline{a}_i^p}\right]^{1/p} [1 + \bar{a}_i^q]^{1/q} \right\}^{-1}. \end{aligned} \tag{3.28}$$

Then it is readily seen that Theorem 3.1 holds for B defined on ${}^{\Delta}L_p^1(I) \times {}^{\Delta}L_p^1(I)$ with the constants $C_l^{[p]}$, $l = 1, 2$, replaced by

$$\begin{aligned} {}^{\Delta}C_1^{[p]} &= \max_{i=1,\cdots,m(\Delta)} {}^{[i]}C_1^{[p]}, \\ {}^{\Delta}C_2^{[p]} &= \min_{i=1,\cdots,m(\Delta)} {}^{[i]}C_2^{[p]}. \end{aligned} \tag{3.29}$$

Moreover, we have the following analogue to Theorem 3.4.

THEOREM 3.5. *Let* Δ *be an arbitrary partition* (2.4) *of* $I = I(0, l)$. *Then for given* $F \in {}^{\Delta}\mathring{L}_q^1(I)'$, $1 < q < \infty$, *there exists a unique* ${}^{\Delta}u \in {}^{\Delta}L_q^1(I)$ *such that*

$$B({}^{\Delta}u, v) = F(v) \quad \forall v \in {}^{\Delta}L_q^1(I), \tag{3.30}$$

and

$$|{}^{\Delta}u|_{p,1} \leqq \frac{1}{{}^{\Delta}C_2^{[p]}} \|F\|_{{}^{\Delta}\mathring{L}_q^1(I)'}. \tag{3.31}$$

Moreover, if F *is defined by* (3.26) *with given* $f \in L_1^0(I)$ *then the result also holds for* $q = 1, \infty$.

For any fixed subinterval $I_i(\Delta)$, $1 \leqq i \leqq m(\Delta)$, consider now the bilinear form

$$B_i(u, v) = \int_{I_i} (a(x)u'(x)v'(x) + b(x)u(x)v(x))\,dx, \tag{3.32}$$

on $\mathring{L}_p^1(I_i(\Delta)) \times \mathring{L}_q^1(I_i(\Delta))$. Let ${}^{\Delta}u_i$ be the restriction of the solution ${}^{\Delta}u$ of (3.30) to $I_i(\Delta)$.

Then it can be shown easily that ${}^{\Delta}u_i \in \mathring{L}_p^1(I_i(\Delta))$ and

$$B_i({}^{\Delta}u_i, v) = F(v) \quad \forall v \in \mathring{L}_q^1(I_i(\Delta)), \tag{3.33}$$

as well as

$$ {}_{I_i}|{}^{\Delta}u|_{p,1} \leqq \frac{1}{{}^{[i]}C_2^{[p]}} \|F\|_{\mathring{L}_q^1(I_i(\Delta))'} \tag{3.34}$$

for $1<q<\infty$. Moreover, the analogous statement as in Theorems 3.4, 3.5 for the case $q=1, \infty$ applies.

For any $u \in \mathring{L}_p^1(I(0,l))$ define now $F \in {}^{\Delta}\mathring{L}_q^1(I)'$ by

$$F(v) = B(u, v) \quad \forall v \in {}^{\Delta}\mathring{L}_q^1(I). \tag{3.35}$$

Then Theorem 3.5 guarantees the existence of a unique ${}^{\Delta}u \in {}^{\Delta}L_p^1(I)$ such that (3.30) holds. We define the operator ${}^{\Delta}P : \mathring{L}_p^1(I) \to {}^{\Delta}L_p^1(I)$ by

$${}^{\Delta}Pu = {}^{\Delta}u. \tag{3.36}$$

Clearly, ${}^{\Delta}P$ satisfies

$$\begin{aligned} |{}^{\Delta}Pu|_{p,1} = |{}^{\Delta}u|_{p,1} &\leqq \frac{1}{{}^{\Delta}C_2^{[p]}} \|F\|_{{}^{\Delta}\mathring{L}_q(I)'} \\ &= \frac{1}{{}^{\Delta}C_2^{[p]}} \sup_{\substack{v \in {}^{\Delta}\mathring{L}_q^1(I) \\ v \neq 0}} \frac{|B(u,v)|}{|v|_{q,1}} \leqq \frac{C_1^{[p]}}{{}^{\Delta}C_2^{[p]}} |u|_{p,1}. \end{aligned} \tag{3.37}$$

Since ${}^{\Delta}\mathring{L}_q^1(I) \subset \mathring{L}_p^1(I)$, the operator ${}^{\Delta}Q$ defined by

$${}^{\Delta}Qu = u - {}^{\Delta}u, \tag{3.38}$$

maps $L_p^1(I)$ into $L_p^1(I)$ and we obtain from (3.37)

$$|{}^{\Delta}Qu|_{p,1} \leqq \left(1 + \frac{C_1^{[p]}}{{}^{\Delta}C_2^{[p]}}\right) |u|_{p,1}. \tag{3.39}$$

Let now $u \in \mathring{L}_p^1(I)$ and $v \in \mathring{L}_q^1(I)$, $1<p<\infty$, $1/p+1/q=1$, be any two functions. Then we have by definition

$$B({}^{\Delta}Pu, {}^{\Delta}Pv) = B({}^{\Delta}Pu, {}^{\Delta}Pu) = B(v, {}^{\Delta}Pu) = B({}^{\Delta}Pu, v),$$

because ${}^{\Delta}Pu \in {}^{\Delta}L_p^1(I)$. Hence, we see that

$$B({}^{\Delta}Pu, {}^{\Delta}Qv) = B({}^{\Delta}Pu, v) - B({}^{\Delta}Pu, {}^{\Delta}Pv) = 0. \tag{3.40}$$

Consider for a moment the bilinear form

$$B_0(u, v) = \int_0^l u'v'\,dx;$$

then we find easily from (3.40) that ${}^{\Delta}Qu$ is linear on each subinterval $I_i(\Delta)$. Because $u = {}^{\Delta}Pu + {}^{\Delta}Qu$ and $({}^{\Delta}Pu)(x_i) = 0$, it now follows directly that ${}^{\Delta}Qu$ is the linear interpolant of u. Hence, from (3.23) and (3.37) we obtain that

$$|u - {}^{\Delta}Qu|_{p,1} = |{}^{\Delta}Pu|_{p,1} \leqq 2|u|_{p,1}, \tag{3.41}$$

whence,

$$|{}^{\Delta}Qu|_{p,1} \leqq 3|u|_{p,1}. \tag{3.42}$$

4. The finite element method and its properties. The finite element solution of (3.1), (3.2) is defined as the function $u(\Delta)\in \mathring{S}(I,\Delta)$ such that

$$B(u(\Delta),v)=\int_0^l f(x)v(x)\,dx \quad \forall v\in \mathring{S}(I,\Delta). \tag{4.1}$$

It is well known that $u(\Delta)$ is determined by the solution of a system of linear equations with positive-definite symmetric matrix. Hence, $u(\Delta)$ exists and is unique.

Let u_0 be the solution of (3.24) specified by Theorem 3.4 and set

$$e(\Delta)=u(\Delta)-u_0. \tag{4.2}$$

Then,

$$B(e(\Delta),v)=0 \quad \forall v\in \mathring{S}(I,\Delta), \tag{4.3}$$

and by (3.6i) for any function $z\in {}^{\Delta}L_p^1(I)$,

$$|e(\Delta)-z|_{p,1}\leqq \frac{1}{C_2^{[p]}}\sup_{\substack{v\in \mathring{L}^1_q(I)\\ |v|_{q,1}=1}}|B(e(\Delta)-z,v)|. \tag{4.4}$$

We prove the following result:

THEOREM 4.1. *Set*

$$\rho_j=\inf_c \sup_{x\in I_j(\Delta)}|a(x)-c|, \qquad j=1,\cdots,m(\Delta) \tag{4.5}$$

and

$$\theta_j=3\left[\rho_j+\bar{b}_j\frac{h_j}{2}\frac{l}{2}\left(\frac{1}{p}\right)^{1/p}\left(\frac{1}{q}\right)^{1/q}\right]. \tag{4.6}$$

Then, for any $z\in {}^{\Delta}\mathring{L}_p^1(I)$, *we have*

$$|e(\Delta)-z|_{p,1}\leqq \frac{1}{C_2^{[p]}}\left[\sum_{j=1}^{m(\Delta)} {}_{I_j}|z_j|_{p,1}^p\theta_j^p\right]^{1/p}+\frac{2}{C_2^{[p]}}\sup_{\substack{w\in {}^{\Delta}L_q^1(I)\\ |w|_{q,1}=1}}|B(e(\Delta)-z,w)|, \tag{4.7}$$

where z_j is the restriction of z to I_j, $j=1,\cdots,m(\Delta)$.

Proof. For $v\in \mathring{L}_q^1(I)$ let $v^{[1]}\in \mathring{S}(I,\Delta)$ be the unique linear interpolant with $v^{[1]}(x_j^\Delta)=v(x_j^\Delta)$, $j=0,1,\cdots,m(\Delta)$. Then by (3.41) and (3.42) the functions $v^{[1]}$ and $v^{[2]}=v-v^{[1]}$ satisfy

$$|v^{[1]}|_{q,1}\leqq 3|v|_{q,1}, \qquad |v^{[2]}|_{q,1}\leqq 2|v|_{q,1}. \tag{4.8}$$

Moreover, because of (4.3) we have

$$\begin{aligned} B(e-z,v)&=B(e-z,v^{[1]})+B(e-z,v^{[2]})\\ &=B(e-z,v^{[2]})-B(z,v^{[1]}). \end{aligned} \tag{4.9}$$

Let z_j, $v_j^{[i]}$, $j=1,\cdots,m(\Delta)$, $i=1,2$, be the restrictions of z and $v^{[i]}$, $i=1,2$, to $I_j(\Delta)$. Then $z_j\in \mathring{L}_p^1(I_j(\Delta))$, and since $(v_j^{[1]})'$ is constant on I_j, we have

$$\int_{I_j} a(x)z_j'(x)v_j^{[1]}(x)'\,dx=\int_{I_j}(a(x)-c)z_j'(x)v_j^{[1]}(x)'\,dx,$$

for any constant c. Hence,

$$\left|\int_{I_j} a(x)z_j'(x)v_j^{[1]}(x)'\,dx\right|\leqq \rho_j\,{}_{I_j}|z|_{p,1}\,{}_{I_j}|v^{[1]}|_{q,1}. \tag{4.10}$$

Furthermore, using (3.8) we get

$$\left|\int_{I_j} b(x)z_j(x)v_j^{[1]}(x)\,dx\right| \leqq \bar{b}_j\left(\frac{h_j}{2}\right)\left(\frac{1}{p}\right)^{1/p} {}_{I_j}|z|_{p,1}\ {}_{I_j}\|v^{[1]}\|_{q,0}$$

$$\leqq \bar{b}_j\frac{h_j}{2}\frac{l}{2}\left(\frac{1}{p}\right)^{1/p}\left(\frac{1}{q}\right)^{1/q} {}_{I_j}|z|_{p,1}\ {}_{I}|v^{[1]}|_{q,1}.$$

Together with (4.10), this gives

$$|B(z, v^{[1]})| = \left|\sum_{j=1}^{m(\Delta)}\int_{I_j}[a(x)z_j'(x)v_j^{[1]}(x)' + b(x)z_j(x)v_j^{[1]}(x)]\,dx\right|$$

$$\leqq \sum_{j=1}^{m(\Delta)} {}_{I_j}|z_j|_{p,1}\ {}_{I}|v^{[1]}|_{q,1}\left[\rho_j + \bar{b}_j\frac{h_j}{2}\frac{l}{2}\left(\frac{1}{p}\right)^{1/p}\left(\frac{1}{q}\right)^{1/q}\right],$$

or with (4.6) and (4.8)

$$|B(z, v^{[1]})| \leqq \left(\sum_{j=1}^{m(\Delta)} {}_{I_j}|z|_{p,1}^p\ \theta_j^p\right)^{1/p} {}_{I}|v|_{q,1}.$$

Now by (4.4), (4.8) and (4.9) it follows that

$$|e(\Delta)-z|_{p,1} \leqq \frac{1}{C_2^{[p]}}\sup_{v\in \mathring{L}_q^1(I)}\frac{|B(e(\Delta)-z, v)|}{|v|_{q,1}}$$

$$\leqq \frac{1}{C_2^{[p]}}\sup_{v\in \mathring{L}_q^1(I)}\left[\frac{|B(e(\Delta)-z, v^{[2]})|}{|v^{[2]}|_{q,1}}\frac{|v^{[2]}|_{q,1}}{|v|_{q,1}} + \frac{|B(z, v^{[1]})|}{|v^{[1]}|_{q,1}}\frac{|v^{[1]}|_{q,1}}{|v|_{q,1}}\right]$$

$$\leqq \frac{2}{C_2^{[p]}}\sup_{w\in {}^{\Delta}L_q^1(I)}\left(\frac{|B(e(\Delta)-z, w)|}{|w|_{q,1}}\right) + \frac{1}{C_2^{[p]}}\left[\sum_{j=1}^{m(\Delta)} {}_{I_j}|z|_{p,1}^p\theta_j^p\right]^{1/p},$$

which is (4.7).

5. Some auxiliary functions. So far the function z of Theorem 4.1 is arbitrary. Now we exhibit some special choices of z which will be used later. For these discussions, we shall assume that

$$a \in C^1(I), \quad b \in C^0(I), \quad f \in C^0(I), \tag{5.1}$$

whence, of course, $u \in C^2(I)$.

Let $u(\Delta)$ be the finite element solution of (4.1) and $u_j(\Delta)$ the restrictions of $u(\Delta)$ to $I_j(\Delta)$, $j = 1, \cdots, m(\Delta)$. Then on $I_j(\Delta)$ the residuals

$$r_j = a'u_j(\Delta)' - bu_j(\Delta) + f, \qquad j = 1, \cdots, m(\Delta) \tag{5.2}$$

are well defined. The following result shows that in Theorem 4.1, it is possible to choose z such that the second term of (4.7) disappears.

THEOREM 5.1. *Under the assumption* (5.1), *there exists a unique function* $z \in {}^{\Delta}L_p^1(I)$ *such that*

$$B(e(\Delta)-z, w) = 0 \quad \forall w \in {}^{\Delta}L_q^1(I), \tag{5.3}$$

and for $2 \leqq p \leqq \infty$,

$${}_{I_j}|z_j|_{p,1} = {}_{I_j}\|z_j'\|_{p,0} \leqq \frac{1}{\underline{a}_j}\,{}_{I_j}\|r_j\|_{p,0}h_j\left(\frac{1}{\pi}\right)^{2/p}, \qquad j = 1, \cdots, m(\Delta), \tag{5.4}$$

where z_j *is the restriction of* z *to* I_j.

Proof. The existence and uniqueness of z follows directly from Theorem 3.4. We need to prove the estimate (5.4) only for the cases $p=2$, $p=\infty$. Then for general $2<p<\infty$, it follows by application of the Riesz interpolation theorem (see, e.g., [8, p. 2]).

By definition of z and r_j, we evidently have

$$\begin{aligned}\int_{I_j}(a(z_j')^2+bz_j^2)\,dx&=\int_{I_j}r_jz_j\,dx\\&\leqq {}_{I_j}\|r_j\|_{2,0}\,{}_{I_j}\|z_j\|_{2,0},\qquad j=1,\cdots,m(\Delta).\end{aligned}\tag{5.5}$$

On the other hand, using basic eigenvalue properties, we obtain

$$\frac{1}{\|z_j\|_{2,0}^2}\int_{I_j}(a(z_j')^2+bz_j^2)\,dx\geqq \underline{a}_j\frac{\pi^2}{h_j^2},\qquad j=1,\cdots,m(\Delta),\tag{5.6}$$

that is,

$$I_j\|z_j\|_{2,0}\leqq\frac{h_j}{\pi\underline{a}_j^{1/2}}(B_j(z_j,z_j))^{1/2}.$$

Together with (5.5), this gives

$$\int_{I_j}(a(z_j')^2+bz_j^2)\,dx\leqq\frac{h_j}{\pi\underline{a}_j^{1/2}}\,{}_{I_j}\|r_j\|_{2,0},$$

and therefore

$${}_{I_j}|z_j|_{2,1}^2\leqq\frac{h_j^2}{\pi^2\underline{a}_j^2}\,{}_{I_j}\|r_j\|_{2,0},\tag{5.7}$$

which is (5.4) for $p=2$.

We now turn to the case $p=\infty$. For the following considerations let $I_j=I_j(\Delta)$ be a given subinterval, $1\leqq j\leqq m(\Delta)$, and $w_{j1}, w_{j2}\in C^2[I_j]$ two linearly independent solutions of $L[w]=0$. For $x_{j-1}^\Delta\leqq x\leqq x_j^\Delta$, $x_{j-1}^\Delta<\xi<x_j^\Delta$, we construct a function

$$G_j(x,\xi)=\begin{cases}(a_1(\xi)+b_1(\xi))w_{j1}(x)+(a_2(\xi)+b_2(\xi))w_{j2}(x), & x_{j-1}^\Delta\leqq x<\xi,\\ (a_1(\xi)-b_1(\xi))w_{j1}(x)+(a_2(\xi)-b_2(\xi))w_{j2}(x), & \xi<x\leqq x_j^\Delta,\end{cases}\tag{5.8}$$

such that

$$\begin{aligned}&\text{i)}\qquad G(x_{j-1}^\Delta,\xi)=G(x_j^\Delta,\xi)=0,\\&\text{ii)}\qquad G(\xi+0,\xi)-G(\xi-0,\xi)=-\frac{2}{a(\xi)},\\&\text{iii)}\qquad \frac{\partial}{\partial x}G(\xi+0,\xi)=\frac{\partial}{\partial x}G(\xi-0,\xi).\end{aligned}\tag{5.9}$$

As with the standard Green's functions it is easily seen that, because of the linear independence of w_{j1}, w_{j2}, the conditions (5.9) determine the four coefficients $a_i(\xi)$, $b_i(\xi)$, $i=1,2$, uniquely. Moreover, G_j as a function of x obviously satisfies $LG=0$ for $x_{j-1}^\Delta\leqq x<\xi$ and $\xi<x\leqq x_j^\Delta$. Finally, it follows from Green's formula that

$$\begin{aligned}&\int_{x_{j-1}}^{\xi}G(x,\xi)L[z_j](x)\,dx+\int_{\xi}^{x_j}G(x,\xi)L[z_j](x)\,dx\\&\quad=\int_{x_{j-1}}^{\xi}LG(x,\xi)z_j(x)\,dx+\int_{\xi}^{x_j}LG(x,\xi)z_j(x)-a(\xi)[G(\xi+0,\xi)-G(\xi-0,\xi)]z_j'(\xi),\end{aligned}$$

whence, because of $L[z_j]=r_j$ and (5.9ii)

$$z_j'(\xi)=\frac{1}{2}\int_{I_j} G(x,\xi)r_j(\xi)\,d\xi. \tag{5.10}$$

Now, suppose that $G(\xi-0,\xi)=\alpha(\xi)$. Then for $\alpha(\xi)>0$ the strong maximum principle implies that $0\leqq G(x,\xi)<\alpha(\xi)$, $x_{j-1}^{\Delta}\leqq x<\xi$ and $\partial G(\xi-0,\xi)/\partial x>0$. Similarly, for $\alpha(\xi)<0$, we obtain $0\geqq G(x,\xi)>\alpha(\xi)$, $x_{j-1}^{\Delta}\leqq x<\xi$, and $\partial G(\xi-0,\xi)/\partial x<0$. Hence, in either case, we have

$$|G(x,\xi)|<|\alpha(\xi)|,\qquad x_{j-1}^{\Delta}\leqq x<\xi, \tag{5.11a}$$

$$\frac{\partial}{\partial x}G(\xi-0,\xi)\alpha(\xi)>0. \tag{5.11b}$$

Correspondingly it follows for $\beta(\xi)=G(\xi+0,\xi)$ that

$$|G(x,\xi)|<|\beta(\xi)|,\qquad \xi<x\leqq x_{j-1}^{\Delta}, \tag{5.12a}$$

$$\frac{\partial}{\partial x}G(\xi+0,\xi)\alpha(\xi)<0. \tag{5.12b}$$

For $\alpha(\xi)=0$ we have evidently $G(x,\xi)=0$ for $x_{j-1}^{\Delta}\leqq x\leqq\xi$ and thus, $\partial G(\xi-0,\xi)/\partial x=0$. But then it follows from (5.9iii) that also $\partial G(\xi+0,\xi)/\partial x=0$, whence (5.12b) implies $\beta(\xi)=0$. However, by (5.9ii), $\alpha(\xi)$ and $\beta(\xi)$ cannot both be zero and therefore, neither can vanish. Thus, from (5.11b), (5.12b) together with (5.9iii) we conclude that always

$$\alpha(\xi)\beta(\xi)<0. \tag{5.13}$$

Now it follows from (5.9ii) that

$$|\alpha(\xi)|\leqq\frac{2}{\underline{a}_j},\qquad |\beta(\xi)|\leqq\frac{2}{\underline{a}_j}. \tag{5.14}$$

Hence (5.11a), (5.12b) imply that

$$\int_{I_j}|G(x,\xi)|dx\leqq h_j\frac{2}{\underline{a}_j},$$

and with (5.10) we find that

$$|z_j|_{\infty,1}\leqq h_j\frac{1}{\underline{a}_j}\|r_j\|_{\infty,0},$$

which is (5.4) for $p=\infty$.

The general case for $2\leqq p\leqq\infty$ now follows by an application of the Riesz interpolation theorem (see, e.g., [7, p. 2]). This completes the proof.

6. A posteriori estimates. In practical applications, the choice of the norm for the error estimates depends on the purpose of the computation. For example, the problem may require the use of the energy norm, the stress energy norm, or the stress norm. All these norms turn out to be equivalent with the norm ${}_I\|\cdot\|_{p,1}$. In other applications, norms of physical significance are equivalent with ${}_I\|\cdot\|_{p,0}$.

Here we restrict ourselves to norms which are equivalent with ${}_I\|\cdot\|_{p,1}$, and as a model problem we consider the L_p-stress energy norm

$$\|u\|_{SE,p}=\|[a(u')^2]^{1/2}\|_{p,0}. \tag{6.1}$$

The a posteriori estimates discussed here are based on a local analysis. For a given partition Δ and corresponding approximate solution $u(\Delta)$, we associate with every subinterval $I_j(\Delta)$ an *error indicator* $\eta, j=1,\cdots,m(\Delta)$. These η_j have to be computable in terms of information about a, b, f and u on $I_j(\Delta)$ or at most on I_{j-1}, $I_j(\Delta)$, $I_{j+1}(\Delta)$.

On the basis of the vector of indicators $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_{m(\Delta)})$ an error estimator $\varepsilon=\varepsilon(\boldsymbol{\eta})$ is now constructed. Of course, the η_j as well as ε depend on the chosen norm. The error estimator ε is called a *U-estimator* or *L-estimator* if there exist constants C_u, C_L such that

$$\|e(\Delta)\| \leqq C_u \varepsilon(\boldsymbol{\eta}), \tag{6.2}$$

or

$$\|e(\Delta)\| \geqq C_L \varepsilon(\boldsymbol{\eta}), \tag{6.3}$$

respectively. Here the constants C_u, C_L should depend only on a and b but not on f, $u(\Delta)$ and the partition Δ. In the case $C_u=1$ we speak of a guaranteed U-estimator or G-estimator for short.

The estimator $\varepsilon(\boldsymbol{\eta})$ is said to be *asymptotically exact* if under reasonable assumptions about a, b, f, u, Δ we have

$$\|e(\Delta)\| = \varepsilon(\boldsymbol{\eta})(1+O(\|e(\Delta)\|^{\gamma}) \quad \text{as } \bar{h}(\Delta)\to 0, \tag{6.4}$$

with some constant $\gamma>0$. In general, the bound in the O-term will depend on u, Δ, etc.

The notion of a G-estimator suggests consideration of correction vectors $\boldsymbol{\phi}=(\phi_1,\cdots,\phi_{m(\Delta)})$ and corrector $\Phi(\boldsymbol{\phi})$ such that for a given estimator $\varepsilon(\boldsymbol{\eta})$ the quantity $\varepsilon(\boldsymbol{\eta})+\Phi(\boldsymbol{\phi})$ is a G-estimator. Such a corrector $\Phi(\boldsymbol{\phi})$ is asymptotically exact if under reasonable assumptions about a, b, f, u, Δ

$$0 \leqq \Phi(\boldsymbol{\phi}) \leqq O(\varepsilon(\boldsymbol{\eta})^{\sigma}) \quad \text{as } \bar{h}(\Delta)\to 0, \tag{6.5}$$

with some $\sigma>0$. Note here that we admit the specification of "reasonable assumptions" about the problem only in the asymptotic cases and not in the other ones.

We introduce now three different error indicators and estimators.

DEFINITION 6.1. For $2\leqq p\leqq\infty$ and $j=1,\cdots,m(\Delta)$ set

$$\eta_j^{[1]} = \left(\frac{1}{p+1}\right)^{1/p} \frac{h_j}{2\tilde{a}_{j-1/2}^{1/2}}\, {}_{I_j}\|r_j\|_{p,0} \tag{6.6}$$

and

$$\varepsilon^{[1]}(\boldsymbol{\eta}) = \begin{cases} \left[\sum_{j=1}^{m(\Delta)} (\eta_j^{[1]})^p\right]^{1/p} & \text{for } 2\leqq p<\infty, \\ \max_{j=1,\cdots,m(\Delta)} \eta_j^{[1]} & \text{for } p=\infty, \end{cases} \tag{6.7}$$

where

$$\tilde{a}_{j-1/2} = a\left(\tfrac{1}{2}(x_j^{\Delta}+x_{j-1}^{\Delta})\right), \qquad j=1,\cdots,m(\Delta), \tag{6.8}$$

and r_j is given by (5.2)

DEFINITION 6.2. For $1\leqq p\leqq\infty$ and $j=1,\cdots,m(\Delta)$,

$$\eta_j^{[2]} = \left(\frac{1}{p+1}\right)^{1/p} \frac{3h_j^{-2+1/p}}{\tilde{a}_{j-1/2}^{1/2}} \left|\int_{I_j} r_j(x)(x-x_j^{\Delta})(x-x_{j-1}^{\Delta})\,dx\right|, \tag{6.9}$$

and

$$\varepsilon^{[2]}(\boldsymbol{\eta}) = \begin{cases} \left[\sum_{j=1}^{m(\Delta)} (\eta_j^{[2]})^p \right]^{1/p} & \text{for } 1 \leqq p < \infty, \\ \max_{j=1,\cdots,m(\Delta)} \eta_j^{[2]} & \text{for } p = \infty, \end{cases} \tag{6.10}$$

where $\tilde{a}_{j-1/2}$ is again given by (6.8) and r_j by (5.2).

DEFINITION 6.3. For $p = 2$ and $j = 1, \cdots, m(\Delta)$ set

$$\eta_j^{[3]} = {}_{I_j}\left\| \frac{1}{a(x)} \frac{dP_j}{dx} \right\|_{p,0}, \tag{6.11}$$

where P_j is a quadratic polynomial on I_j such that

$$P_j(x_{j-1}^{\Delta}) = 0, \quad P_j'(x_{j-1}^{\Delta}) = \alpha_{j,1}^{\Delta}, \quad P_j'(x_j^{\Delta}) = -\alpha_{j,0}^{\Delta}, \tag{6.12}$$

where

$$\alpha_{j,l}^{\Delta} = \frac{h_j}{h_{j-l+1} + h_{j-l}} d(x_{j-l}^{\Delta}) a(x_{j-l}^{\Delta}), \qquad l = 0, 1, \quad j = 1, \cdots, m(\Delta), \tag{6.13}$$

with $\alpha_{1,0}^{\Delta} = \alpha_{1,1}^{\Delta}$, $\alpha_{m,0}^{\Delta} = \alpha_{m,1}^{\Delta}$, and

$$d(x_i^{\Delta}) = \frac{1}{h_{i+1}}[u(\Delta)(x_{i+1}^{\Delta}) - u(\Delta)(x_i^{\Delta})] - \frac{1}{h_i}[u(\Delta)(x_i^{\Delta}) - u(\Delta)(x_{i-1}^{\Delta})], \quad i = 1, \cdots, m(\Delta). \tag{6.14}$$

Further, set

$$\varepsilon^{[3]}(\boldsymbol{\eta}) = \left[\sum_{j=1}^{m(\Delta)} (\eta_j^{[3]})^2 \right]^{1/2}. \tag{6.15}$$

Clearly, all these error indicators and estimators are computable. In §§ 8 and 9, we shall also introduce some correctors for these estimators.

7. Basic properties of the estimators of Definitions 6.1, 6.2.

THEOREM 7.1. *Suppose that the condition* (5.1) *holds. Then, for the L_p-stress energy norm* (6.1) *and* $2 \leqq p \leqq \infty$, *the error estimator* $\varepsilon^{[1]}(\boldsymbol{\eta})$ *is an U-estimator and the constant* C_u *of* (6.2) *depends only on* $\bar{a}$, $\underline{a}$, $\bar{b}$ *and* ${}_I\|a'\|_{\infty,0}$, *but not on* $f \in L_p(I)$ *and* u.

Proof. By (5.1), the quantities (4.5) satisfy

$$\rho_j \leqq {}_I\|a'\|_{\infty,0} h_j,$$

whence

$$\theta_j \leqq K_0 h_j, \qquad K_0 = 3\left[{}_I\|a'\|_{\infty,0} + \bar{b} \frac{l}{4} \left(\frac{1}{p}\right)^{1/p} \left(\frac{1}{q}\right)^{1/q} \right], \tag{7.1}$$

for $j = 1, \cdots, m(\Delta)$. For $2 \leqq p < \infty$ it now follows from Theorem 4.1 with the choice of the function z of Theorem 5.1 that

$$\begin{aligned} \|e(\Delta) - z\|_{SE,p} &\leqq \bar{a}^{1/2} {}_I|e(\Delta) - z|_{p,1} \\ &\leqq \frac{\bar{a}^{1/2}}{C_2^{[p]}} \left[\sum_{j=1}^{m(\Delta)} \theta_j^p \left(\frac{h_j}{\underline{a}_j}\right)^p \frac{1}{\pi^2} {}_{I_j}\|r_j\|_{p,0}^p \right]^{1/p} \\ &\leqq 2\frac{\bar{a}^{1/2}}{C_2^{[p]}} \left[\sum_{j=1}^{m(\Delta)} \theta_j^p \frac{p+1}{\pi^2} \left(\frac{\bar{a}^{1/2}}{\underline{a}_j}\right)^p (\eta_j^{[1]})^p \right]^{1/p} \leqq K_1 \bar{h} \varepsilon^{[1]}, \end{aligned} \tag{7.2}$$

where by (7.1)

$$K_1 = \frac{2K_0}{C_2^{[p]}} \frac{\bar{a}}{\underline{a}}\left(\frac{p+1}{\pi^2}\right)^{1/p}.$$

Furthermore, once again by Theorem 5.1, we have

$$\begin{aligned}\|z\|_{SE,p} &\leqq \bar{a}^{1/2}\left[\sum_{j=1}^{m(\Delta)} {}_{I_j}|z_j|_{p,1}^p\right]^{1/p}\\ &\leqq 2\bar{a}^{1/2}\left[\sum_{j=1}^{m(\Delta)} \frac{p+1}{\pi^2}\frac{\bar{a}_j^{p/2}}{\underline{a}_j^p}(\eta_j^{[1]})^p\right]^{1/p} \leqq K_2\varepsilon^{[1]},\end{aligned} \tag{7.3}$$

with

$$K_2 = 2\frac{\bar{a}}{\underline{a}}\left(\frac{p+1}{\pi^2}\right)^{1/p}.$$

From (7.2) and (7.3), we obtain

$$\|e(\Delta)\|_{SE,p} \leqq \|e(\Delta)-z\|_{SE,p} + \|z\|_{SE,p} \leqq (K_1\bar{h}+K_2)\varepsilon^{[1]},$$

which is the desired inequality (6.2). The explicit expressions given for $C_2^{[p]}$, K_0, K_1, K_2 show directly that C_U indeed depends only on $\underline{a}$, $\bar{a}$, $\bar{b}$ and ${}_I\|a'\|_{\infty,0}$. For $p=\infty$ the proof of the result follows by the standard limiting argument.

THEOREM 7.2. *Suppose that* (5.1) *holds and that the L_p-stress energy norm* (6.1) *is used. Then for $1 \leqq p \leqq \infty$ the error estimator $\varepsilon^{[2]}(\boldsymbol{\eta})$ is an L-estimator, and the constant C_L of* (6.3) *depends only on $\bar{a}$, $\underline{a}$ and ${}_I\|a'\|_{\infty,0}$ but not on $f \in L_1(I)$ and u.*

Proof. Let $1 \leqq p < \infty$. For any $v \in {}^{\Delta}L_q^1(I)$, we have

$$B(e(\Delta), v) = \sum_{j=1}^{m} \int_{I_j} r_j v_j\,dx$$

and

$$|B(e(\Delta), v)| \leqq C_1^{[p]}\,{}_I|e(\Delta)|_{p,1}\,{}_I|v|_{q,1}. \tag{7.4}$$

Set

$$s_j(x) = (x - x_{j-1}^{\Delta})(x - x_j^{\Delta}), \qquad j=1,\cdots,m(\Delta),$$

and define $v^{\Delta} \in L_q^1(I)$ by

$$v(x) = \gamma_j s_j(x) \quad \text{for } x \in I_j(\Delta), \quad j=1,\cdots,m(\Delta),$$

with

$$\gamma_j = \operatorname{sgn}\left(\int_{I_j} r_j s_j\,dx\right)(\eta_j^{[2]})^{p-1}\,h_j^{-2+1/p}, \qquad j=1,\cdots,m(\Delta). \tag{7.5}$$

Then (7.4) implies that

$$\begin{aligned}{}_I|e(\Delta)|_{p,1} &\geqq \frac{1}{C_1^{[p]}}\frac{\sum_{j=1}^{m(\Delta)}(\eta_j^{[2]})^{p-1}h_j^{-2+1/p}\left|\int_{I_j} r_j s_j\,dx\right|}{\left[\sum_{j=1}^{m(\Delta)}|\gamma_j|^q\int_{I_j}|s_j'(x)|^q\,dx\right]^{1/q}}\\ &\geqq \frac{1}{C_1^{[p]}}\frac{\sum_{j=1}^{m(\Delta)}\frac{1}{3}\tilde{a}_{j-1/2}^{1/2}(p+1)^{1/p}(\eta_j^{[2]})^p}{[\sum_{j=1}^{m(\Delta)}(\eta_j^{[2]})^{(p-1)q}h_j^{(-2+1/p)q}h_j^{q+1}(1+q)^{-1}]^{1/q}}\\ &\geqq \frac{\underline{a}^{1/2}}{3C_1^{[p]}}(p+1)^{1/p}(q+1)^{1/q}(\varepsilon^{[2]})^{p-p/q},\end{aligned}$$

whence

$$\|e(\Delta)\|_{SE,p} \geqq \frac{a^{3/2}}{3C_1^{[p]}}(p+1)^{1/p}(q+1)^{1/q}\varepsilon^{[2]},$$

as stated. For $p=\infty$, the result follows again by a limiting argument.

The next two theorems concern the asymptotic exactness of $\varepsilon^{[1]}$ and $\varepsilon^{[2]}$.

THEOREM 7.3. *Suppose that*

$$a \in C^2(I), \quad b \in C^1(I), \quad f \in C^1(I), \quad I = I(0,1), \tag{7.6}$$

and that the exact solution u_0 *of* (3.1) *satisfies*

$$|u_0''(x)| \geqq \rho > 0 \quad \forall x \in I(\alpha,\beta), \quad 0 \leqq \alpha < \beta \leqq 1. \tag{7.7}$$

Moreover, let $\mathcal{P}$ *be any* (λ,κ)*-regular family of partitions with* $1 \leqq \kappa < 2$. *Then for any* $\Delta \in \mathcal{P}$ *and the* L_p*-stress energy norm with* $2 \leqq p \leqq \infty$ *the error estimator* $\varepsilon^{[1]}$ *is asymptotically exact and*

$$\left.\begin{aligned} &\text{i)} \quad \|e(\Delta)\|_{SE,p} = \varepsilon^{[1]}(1+O(\bar h(\Delta)^{\mu}))\\ &\text{ii)} \quad \|e(\Delta)\|_{SE,p} = \varepsilon^{[1]}(1+O(\|e(\Delta)\|_{SE,p}^{\mu/\kappa})\end{aligned}\right\} \quad \text{as } \bar h(\Delta)\to 0, \quad \Delta \in \mathcal{P}, \tag{7.8}$$

where $\mu = 1-\kappa/2$. *The constants of the O-terms depend on* a, b, f, u, λ *but are independent of* $\bar h(\Delta)$ *and the error* $e(\Delta)$.

Proof. We prove the result for $2 \leqq p < \infty$; for $p = \infty$ the proof is analogous.

By definition (5.2) of r_j, we have

$$r_j = -au_0'' + a'e(\Delta)' - be(\Delta), \qquad j = 1,\cdots,m(\Delta). \tag{7.9}$$

We introduce the quantities

$$r_j^{[1]} = -au_0'', \qquad r_j^{[2]} = -\tilde a_{j-\frac12}u_0''(\tfrac12(x_{j-1}^\Delta + x_j^\Delta)), \qquad j=1,\cdots,m(\Delta). \tag{7.10}$$

Let $z_j^{[i]} \in \mathring{L}_p^1(I_j)$, $j = 1,\cdots,m(\Delta)$, be the unique solutions of

$$B_j(z_j^{[i]}, w) = \int_{I_j} r_j^{[i]} w\,dx \quad \forall w \in \mathring{L}_q^1(I_j), \quad i = 1,2, \tag{7.11}$$

and define $z^{[i]} \in {}^\Delta L_p^1(I)$, $i = 1, 2$, as the functions for which the restrictions to $I_j(\Delta)$ are the functions $z_j^{[i]}$, $i = 1, 2$. Moreover, in analogy to Definition 6.1, we set

$$\varepsilon_i^{[1]}(\boldsymbol{\eta}) = \left(\frac{1}{p+1}\right)^{1/p}\left[\sum_{j=1}^{m(\Delta)}\left(\frac{h_j}{2\tilde a_{j-\frac12}^{1/2}}{}_{I_j}\|r_j^{[i]}\|_{p,0}\right)^p\right]^{1/p}, \qquad i=1,2. \tag{7.12}$$

Then the proof of (7.8i) will be complete if we can prove the following sequence of asymptotic equalities:

$$\left.\begin{aligned} &\text{i)} && \|e\|_{SE,p} = \|z\|_{SE,p}(1+O(\bar h(\Delta))),\\ &\text{ii)} && \|z\|_{SE,p} = \|z^{[1]}\|_{SE,p}(1+O(\bar h(\Delta)^{1/2})),\\ &\text{iii)} && \|z^{[1]}\|_{SE,p} = \|z^{[2]}\|_{SE,p}(1+O(\bar h(\Delta)^{\mu})),\\ &\text{iv)} && \|z^{[2]}\|_{SE,p} = \varepsilon_2^{[1]}(1+O(\bar h(\Delta))),\\ &\text{v)} && \varepsilon_2^{[1]} = \varepsilon_1^{[1]}(1+O(\bar h(\Delta)^{\mu})),\\ &\text{vi)} && \varepsilon_1^{[1]} = \varepsilon^{[1]}(1+O(\bar h(\Delta)^{1/2})),\end{aligned}\right\} \quad \text{as } \bar h(\Delta)\to 0. \tag{7.13}$$

Here, z is the function of Theorem 5.1.

If we use this function z in the estimate (4.7) of Theorem 4.1 and observe (7.1), then we obtain

$$\begin{aligned}\|e(\Delta)\|_{SE,p} &\leqq \|z\|_{SE,p}+\|e(\Delta)-z\|_{SE,p}\\ &\leqq \|z\|_{SE,p}+C\bar{h}(\Delta)\|z\|_{SE,p}=\|z\|_{SE,p}(1+O(\bar{h}(\Delta))) \quad \text{as } \bar{h}(\Delta)\to 0,\end{aligned} \tag{7.14}$$

where C depends only on $\bar{a}$, $\underline{a}$, $\bar{b}$, and ${}_I\|a'\|_{\infty,0}$. On the other hand, we have also

$$\|z\|_{SE,p}\leqq \|e(\Delta)\|_{SE,p}+\|e(\Delta)-z\|_{SE,p}\leqq \|e\|_{SE,p}+C\bar{h}(\Delta)\|z\|_{SE,p},$$

whence for sufficiently small $\bar{h}(\Delta)$

$$\|z\|_{SE,p}\leqq \|e\|_{SE,p}(1+O(\bar{h}(\Delta))) \quad \text{as } \bar{h}(\Delta)\to 0. \tag{7.15}$$

This together with (7.15) proves (7.13i).

We turn now to the proof of (7.13vi). For this, observe that

$$\begin{aligned}\varepsilon^{[1]}(\boldsymbol{\eta}) &= \frac{1}{2}(p+1)^{-1/p}\left[\sum_{j=1}^{m(\Delta)}\frac{h_j^p}{\tilde{a}_{j-\frac12}^{p/2}}\,{}_{I_j}\|r_j^{[1]}+(r_j-r_j^{[1]})\|_{p,0}^p\right]^{1/p}\\ &\leqq \frac{1}{2}(p+1)^{-1/p}\left[\sum_{j=1}^{m(\Delta)}\frac{h_j^p}{\tilde{a}_{1-\frac12}^{p/2}}\left((1+\delta)^p\,{}_{I_j}\|r_j^{[1]}\|_{p,0}^p+\left(1+\frac{1}{\delta}\right)^p{}_{I_j}\|r_j-r_j^{[1]}\|_{p,0}^p\right)\right]^{1/p},\end{aligned} \tag{7.16}$$

with any $\delta>0$, which will be determined shortly. Since by (7.9)

$${}_{I_j}\|r_j-r_j^{[1]}\|_{p,0}={}_{I_j}\|a'e(\Delta)'-be(\Delta)\|_{p,0}, \tag{7.17}$$

it follows from (7.16) and the definition (7.12) of $\varepsilon_1^{[1]}(\boldsymbol{\eta})$ that

$$\varepsilon^{[1]}(\boldsymbol{\eta})^p\leqq(1+\delta)^p\varepsilon_1^{[1]}(\boldsymbol{\eta})^p+C\left(1+\frac{1}{\delta}\right)^p\bar{h}^p\|e(\Delta)\|_{SE,p}^p. \tag{7.18}$$

Here, C represents from now on a generic constant with different values at each instance. With the choice $\delta=\bar{h}^{1/2}$ we obtain from Theorem 7.1 that

$$\|e\|_{SE,p}^p\leqq C_U^p\varepsilon^{[1]}(\boldsymbol{\eta})^p\leqq C_U^p(1+\bar{h}^{1/2})^p\varepsilon_1^{[1]}(\boldsymbol{\eta})^p+C\bar{h}^{p/2}\|e\|_{SE,p}^p,$$

whence

$$\|e(\Delta)\|_{SE,p}\leqq C\varepsilon_1^{[1]}(\boldsymbol{\eta}) \quad \text{as } \bar{h}(\Delta)\to 0, \tag{7.19}$$

and thus, by (7.18),

$$\varepsilon^{[1]}(\boldsymbol{\eta})\leqq\varepsilon_1^{[1]}(\boldsymbol{\eta})(1+O(\bar{h}(\Delta)^{1/2})) \quad \text{as } \bar{h}(\Delta)\to 0. \tag{7.20}$$

Analogously, using (7.17) again we have

$$\begin{aligned}\varepsilon_1^{[1]}(\boldsymbol{\eta}) &\leqq \frac{1}{2}(p+1)^{-1/p}\left[\sum_{j=1}^{m(\Delta)}\frac{h_j^p}{\tilde{a}_{j-\frac12}^{p/2}}\left((1+\delta)^p\,{}_{I_j}\|r_j\|_{p,0}^p+\left(1+\frac{1}{\delta}\right)^p{}_{I_j}\|r_j^{[1]}-r_j\|_{p,0}^p\right)\right]^{1/p}\\ &\leqq\left[(1+\delta)^p\varepsilon^{[1]}(\boldsymbol{\eta})+C\left(1+\frac{1}{\delta}\right)^p\bar{h}^p\|e(\Delta)\|_{SE,p}^p\right]^{1/p}.\end{aligned}$$

With (7.19) and the choice $\delta=\bar{h}^{1/2}$, this shows that

$$\varepsilon_1^{[1]}(\boldsymbol{\eta})\leqq\varepsilon^{[1]}(\boldsymbol{\eta})(1+O(\bar{h}(\Delta)^{1/2})) \quad \text{as } \bar{h}(\Delta)\to 0, \tag{7.21}$$

which together with (7.20) proves (7.13vi). With the help of (7.13i) the proof of (7.13ii) is entirely analogous to that of (7.13vi) and we shall not repeat the details.

For the proof of (7.13v) observe that

$$|r_j^{[1]}(x)-r_j^{[2]}(x)|=|-(a(x)u_0''(x)-\tilde{a}_{j-\frac{1}{2}}u_0''(\tfrac{1}{2}(x_{j-1}^\Delta+x_j^\Delta)))|<Ch_j \quad \forall x\in I_j. \tag{7.22}$$

Hence,

$$\begin{aligned}\varepsilon_1^{[1]}(\boldsymbol{\eta})^p&=\frac{1}{2^p(1+p)}\sum_{j=1}^{m(\Delta)}\frac{h_j^p}{\tilde{a}_{j-\frac{1}{2}}^{p/2}}\Big[(1+\delta)^p\,{}_{I_j}\|r_j^{[2]}\|_{p,0}^p+\Big(1+\frac{1}{\delta}\Big)^p\,{}_{I_j}\|r_j^{[1]}-r^{[2]}\|_{p,0}^p\Big]\\&\leqq(1+\delta)^p\varepsilon_2^{[1]}(\boldsymbol{\eta})^p+\frac{1}{2^p(1+p)}\sum_{j=1}^{m(\Delta)}\frac{h_j^p}{\tilde{a}_{j-\frac{1}{2}}^{p/2}}C\Big(1+\frac{1}{\delta}\Big)^p h_j^{p+1}.\end{aligned} \tag{7.23}$$

Now (7.7) and the (λ,κ)-regularity of Δ imply for sufficiently small $\bar{h}$ that

$$\sum_{j=1}^{m(\Delta)}h_j^p\,{}_{I_j}\|r_j^{[2]}\|_{p,0}^p\geqq \underline{h}^p\sum_{j=1}^{m(\Delta)}\|r_j^{[2]}\|_{p,0}^p\geqq C\underline{h}^p\rho^p\geqq C\bar{h}^{\kappa p}\rho^p\lambda^p. \tag{7.24}$$

When applied to the second term on the right of (7.23), this gives

$$\varepsilon_1^{[1]}(\boldsymbol{\eta})^p\leqq(1+\delta)^p\varepsilon_2^{[1]}(\boldsymbol{\eta})^p+C\Big(1+\frac{1}{\delta}\Big)^p\bar{h}^{2p-\kappa p}\varepsilon_2^{[1]}(\boldsymbol{\eta})^p,$$

and with the choice of $\delta=\bar{h}^\mu$ we obtain

$$\varepsilon_1^{[1]}(\boldsymbol{\eta})^p\leqq\varepsilon_2^{[1]}(\boldsymbol{\eta})^p(1+\bar{h}^\mu)^p(1+C\bar{h}^\mu)^p,$$

or

$$\varepsilon_1^{[1]}(\boldsymbol{\eta})\leqq\varepsilon_2^{[1]}(\boldsymbol{\eta})(1+O(\bar{h}(\Delta)^\mu))\quad\text{as }\bar{h}(\Delta)\to 0. \tag{7.25}$$

As in the proof of (7.21), we may proceed to show that

$$\varepsilon_2^{[1]}(\boldsymbol{\eta})\leqq\varepsilon_1^{[1]}(\boldsymbol{\eta})(1+O(\bar{h}(\Delta)^\mu))\quad\text{as }\bar{h}(\Delta)\to 0,$$

which together with (7.25) proves (7.13v). Once again the proof of the corresponding relation (7.13iii) is entirely analogous and will be omitted.

Since $r_j^{[2]}$ is constant on each subinterval $I_j(\Delta)$, it follows easily that

$$\|z_j^{[2]}\|_{SE,p}=\frac{1}{2(p+1)^{1/p}}\frac{h_j}{\tilde{a}_{j-\frac{1}{2}}^{1/2}}\|r_j^{[2]}\|_{SE,p}(1+O(\bar{h}(\Delta)))\quad\text{as }\bar{h}(\Delta)\to 0,$$

which directly gives (7.13iv) and thereby concludes the proof of (7.8i) for $z\leqq p<\infty$.

By the definition (7.12) of $\varepsilon_2^{[1]}$ and (7.7), we have

$$\varepsilon_2^{[1]}(\boldsymbol{\eta})\geqq C\underline{h}\geqq C\bar{h}^\kappa.$$

Hence, (7.13i–iv) imply that

$$\|e(\Delta)\|_{SE,p}\geqq C\varepsilon_2^{[1]}(\boldsymbol{\eta})\geqq C\bar{h}^\kappa\quad\text{as }\bar{h}(\Delta)\to 0, \tag{7.26}$$

or

$$\bar{h}^\mu\leqq C\|e(\Delta)\|_{SE,p}^{\mu/\kappa}.$$

Thus, (7.8ii) follows directly from (7.8i) and the proof for $2\leqq p<\infty$ is complete. As mentioned earlier, the proof for $p=\infty$ proceeds exactly in the same manner.

THEOREM 7.4. *Under the assumptions of Theorem* 7.3, $\varepsilon^{[2]}$ *is an asymptotically exact estimator and*

$$\text{(7.27)}\qquad \left.\begin{aligned} &\text{i)} && \|e(\Delta)\|_{SE,p} = \varepsilon^{[2]}(1+O(\bar{h}(\Delta)^{\mu})) \\ &\text{ii)} && \|e(\Delta)\|_{SE,p} = \varepsilon^{[2]}(1+O(\|e(\Delta)\|_{SE,p}^{\mu/\kappa})) \end{aligned}\right\} \quad \text{as } \bar{h}(\Delta)\to 0,$$

where again $\mu = 1-\kappa/2$, *and the constants of the O-terms depend on* a, b, f, u, λ *but not on* $\bar{h}(\Delta)$ *or* $\|e(\Delta)\|_{SE,p}$.

Proof. We use the same notation as in the proof of Theorem 7.3. If we can show that

$$\|z^{[2]}\|_{SE,p} = \varepsilon^{[2]}(1+O(\bar{h}(\Delta)^{\mu})) \quad \text{as } \bar{h}(\Delta)\to 0,$$

then (7.27i) follows directly from (7.13i–iii). Evidently, we have

$$\begin{aligned} \varepsilon^{[2]}(\boldsymbol{\eta}) &= \frac{3}{(p+1)^{1/p}}\left[\sum_{j=1}^{m(\Delta)} \frac{h_j^{(-2+1/p)p}}{a_{j-\frac12}^{p/2}}\left|\int_{I_j} r_j(x)(x-x_j^{\Delta})(x-x_{j-1}^{\Delta})\,dx\right|^p\right]^{1/p} \\ \text{(7.28)}\qquad &\le \frac{3}{(p+1)^{1/p}}\left\{\sum_{j=1}^{m(\Delta)} \frac{h_j^{-2p+1}}{a_{j-\frac12}^{p/2}}\left[(1+\delta)^p\left|\int_{I_j} r_j^{[2]}(x-x_j^{\Delta})(x-x_{j-1}^{\Delta})\,dx\right|^p \right.\right. \\ &\qquad\qquad \left.\left. +\left(1+\frac{1}{\delta}\right)^p h_j^{3p-1}\,{}_{I_j}\|r_j-r_j^{[2]}\|_{p,0}^p\right]\right\}^{1/p}. \end{aligned}$$

By (7.17) and (7.22), we have

$$\begin{aligned} \sum_{j=1}^{m(\Delta)} {}_{I_j}\|r_j-r_j^{[2]}\|_{p,0}^p &\le C\sum_{j=1}^{m(\Delta)} [{}_{I_j}\|r_j-r_j^{[1]}\|_{p,0}^p + {}_{I_j}\|r_j^{[1]}-r_j^{[2]}\|_{p,0}^p] \\ &\le C[{}_{I_j}\|e(\Delta)\|_{SE,p}^p + \bar{h}^p], \end{aligned}$$

and, since $r_j^{[2]}$ is constant, a simple calculation shows that

$${}_{I_j}\|z_j^{[2]}\|_{SE,p}^p = \frac{3^p}{p+1}\,\frac{h_j^{(-2+1/p)p}}{\tilde{a}_{j-\frac12}^{p/2}}\left|\int_{I_j} r_j^{[2]}(x-x_j^{\Delta})(x-x_{j-1}^{\Delta})\,dx\right|^p (1+O(h_j)).$$

Using these two estimates in (7.28) together with (7.26) and (7.13i–iii) we find that

$$\begin{aligned} \varepsilon^{[2]}(\boldsymbol{\eta}) &\le \left\{(1+\delta)^p\|z^{[2]}\|_{SE,p}^p + C\left(1+\frac{1}{\delta}\right)^p[\bar{h}^p\|e(\Delta)\|_{SE,p}^p+\bar{h}^{2p}]\right\}^{1/p} \\ &\le \|z^{[2]}\|_{SE,p}\left[(1+\delta)^p + C\left(1+\frac{1}{\delta}\right)^p(\bar{h}^p+\bar{h}^{(2-\kappa)p})\right]^{1/p}. \end{aligned}$$

With $\delta = \bar{h}^{\mu}$, $\mu = 1-\kappa/2$, this gives

$$\text{(7.29)}\qquad \varepsilon^{[2]}(\boldsymbol{\eta}) \le \|z^{[2]}\|_{SE,p}(1+O(\bar{h}^{\mu})).$$

The proof of the opposite inequality,

$$\|z^{[2]}\|_{SE,p} \le \varepsilon^{[2]}(\boldsymbol{\eta})(1+O(\bar{h}^{\mu})),$$

proceeds entirely along the same lines and there is no need to enter into the details. The asymptotic equality (7.27i) now follows directly from (7.13i–iii). This proves the theorem for $2\le p<\infty$. For $p=\infty$, the proof is analogous. The proof of (7.27ii) is obvious.

The results of Theorems 7.3 and 7.4 raise the question about the optimality of the exponent in the estimates (7.8) and (7.27). This is as yet unresolved. We illustrate the problem with a numerical example.

Consider the problem

$$(7.30)\qquad \begin{aligned} -((x+\delta)^{1/10}u')' + u &= f, \qquad 0<x<1, \quad \delta = \tfrac{1}{10},\\ u(0)=\delta^{1/2}, &\qquad u(1)=(1+\delta)^{1/2}, \end{aligned}$$

where f is chosen such that the exact solution is

$$(7.31)\qquad u(x)=(x+\delta)^{1/2}.$$

Some numerical results for the finite element solution on uniform meshes are given in Tables 7.1 and 7.2. More specifically, for different $m(\Delta)$ we show the behavior of the estimator $\varepsilon^{[1]}$ in relation to the exact error $\|e(\Delta)\|_{SE,p}$ for $p=2$ and $p=8$.

TABLE 7.1

$m(\Delta)$	$\|e(\Delta)\|_{SE,2}$	$\varepsilon^{[1]}$	$\dfrac{\varepsilon^{[1]}}{\|u\|_{SE,2}}\cdot 100$	$\varepsilon^{[1]}/\|e(\Delta)\|_{SE,2}$
20	0.2287 (−1)	0.2313 (−1)	3.22%	1.01168
40	0.1156 (−1)	0.1159 (−1)	1.62%	1.00309
80	0.5796 (−2)	0.5801 (−1)	0.809%	1.00076

TABLE 7.2

$m(\Delta)$	$\|e(\Delta)\|_{SE,8}$	$\varepsilon^{[1]}$	$\dfrac{\varepsilon^{[1]}}{\|u\|_{SE,8}}\cdot 100$	$\varepsilon^{[1]}/\|e(\Delta)\|_{SE,8}$
20	0.6933 (−1)	0.7569 (−1)	4.79%	1.09174
40	0.3647 (−1)	0.3198 (−1)	2.42%	1.04975
80	0.1855 (−1)	0.9122 (−1)	1.21%	1.03632

Theorem 7.3 guarantees that the estimator $\varepsilon^{[1]}$ is asymptotically exact under both norms and we see clearly the expected convergence. However, under the $\|\cdot\|_{SE,2}$ norm the data indicate that

$$(7.32)\qquad \begin{aligned} \|e(\Delta)\|_{SE,2} &= \varepsilon^{[1]}(1+O(\bar{h}(\Delta)^2)),\\ \|e(\Delta)\|_{SE,2} &= \varepsilon^{[1]}(1+O(\|e(\Delta)\|^2)). \end{aligned}$$

On the other hand, since here $\kappa=1$, the estimates (7.8) give only the factors $(1+O(\bar{h}(\Delta)^{1/2}))$ and $(1+O(\|e(\Delta)\|^{1/2}))$, respectively. Of course, Theorem 7.3 concerns any (λ,κ)-regular family of partitions, while we used only uniform meshes. It appears that for $p=2$ such a faster rate of convergence can be proved more generally even for nonuniform meshes. But the question about lower bounds of the exponents in the estimates of Theorem 7.3 remains open. In the case $p=8$ we do have approximately the convergence provided for in Theorem 7.3.

It may be noted that in our example, $\varepsilon^{[1]}$ consistently provides upper bounds for the error even though $\varepsilon^{[1]}$ is only an U-estimator and not a guaranteed one. The tables also show that for all practical purposes, the reliability of the estimates is already sufficient when the accuracy is in the range of up to 5%. This has been observed for numerous practical examples.

8. Properties of the estimator of Definition 6.3. In this section, we restrict ourselves to the case $p=2$. The error $e=e(\Delta)$ of (4.2) satisfies the boundary value problem

$$Le = \tilde{r}, \qquad e(0) = e(1) = 0, \tag{8.1}$$

where

$$\tilde{r}(x) = r(x) + \sum_{j=1}^{m-1} \delta(x - x_j) a(x_j)\, d(x_j^{\Delta}). \tag{8.2}$$

Here, $d(x_j^{\Delta})$ is given by (6.14), δ is the Dirac function, and $r(x)$ the residual for which the restriction $I_j(\Delta)$ is defined by (5.2). Because of (4.3), we have

$$\int \tilde{r}(x) w(x)\, dx = 0 \quad \forall w \in \mathring{S}(I, \Delta). \tag{8.3}$$

We introduce the auxiliary form

$$\hat{B}(v, w) = \int_I (a(x) v'(x) w'(x) + K \bar{h}^{-\alpha} v(x) w(x))\, dx, \qquad v, w \in \mathring{H}_2^1, \tag{8.4}$$

with given $K > 0$ and $0 < \alpha < 1$ to be specified later. Then there exists a unique solution $y \in \mathring{H}_2^1$ of

$$\hat{B}(y, v) = \int_I \tilde{r}(x) v(x)\, dx \quad \forall v \in \mathring{H}_2^1. \tag{8.5}$$

Let P_j again be the quadratic polynomial on I_j specified in Definition 6.3. By applying the Castigliano complementary variational principle (see, e.g., [9]) it then follows that

$$\begin{aligned} \hat{B}(y, y) &\leqq \sum_{j=1}^{m(\Delta)} \int_{I_j} \left[\frac{1}{a(x)} P_j'(x)^2 + \frac{1}{K} \bar{h}^{\alpha} (r_j(x) + P_j''(x))^2 \right] dx \\ &= \varepsilon^{[3]}(\boldsymbol{\eta})^2 + \Lambda^2, \end{aligned} \tag{8.6}$$

where $\varepsilon^{[3]}$ is the estimator of Definition 6.3 and

$$\Lambda^2 = \frac{1}{K} \bar{h}^{\alpha} \sum_{j=1}^{m(\Delta)} \int_{I_j} (r_j(x) + P_j''(x))^2\, dx. \tag{8.7}$$

Now the following result holds:

THEOREM 8.1. *For $K > 0$, $0 < \alpha < 1$, and Λ defined by* (8.7) *we have*

$$\|e\|_{SE,2} \leqq (\varepsilon^{[3]}(\boldsymbol{\eta})^2 + \Lambda^2)^{1/2} (1 + O(\bar{h}^{1-\alpha})), \quad \text{as } \bar{h} \to 0. \tag{8.8}$$

The O-term is computable and depends only on a, b, K, α, but not on f and the solution u_0.

Proof. With the solution $y \in \mathring{H}_2^1$ of (8.5) there exists a unique $z \in \mathring{H}_2^1$ such that

$$\hat{B}(z, v) = \int_I y(x) v(x)\, dx \quad \forall v \in \mathring{H}_2^1. \tag{8.9}$$

From

$$\hat{B}(z, z) \leqq \|y\|_{2,0} \|z\|_{2,0} \leqq K^{-1/2} \bar{h}^{\alpha/2} \hat{B}(z, z)^{1/2} \|y\|_{2,0},$$

it then follows that

$$\|z\|_{2,0} \leqq K^{-1} \bar{h}^{\alpha} \|y\|_{2,0}. \tag{8.10}$$

By (8.9), we have

$$-(a\,z')' = y - K\bar{h}^{-\alpha} z,$$

and therefore, altogether,

$$\|z\|_{2,2} \leqq C\|y\|_{2,0}, \tag{8.11}$$

with some generic constant C. Hence, there exists a $w \in \mathring{S}(I, \Delta)$ such that

$$\begin{aligned} &\|w - z\|_{2,1} \leqq C\bar{h}\|z\|_{2,2} \leqq C\bar{h}\|y\|_{2,0}, \\ &\|w - z\|_{2,0} \leqq C\bar{h}^2\|z\|_{2,2} \leqq C\bar{h}^2\|y\|_{2,0}. \end{aligned} \tag{8.12}$$

Now using $v = y$ in (8.9) and together with (8.3) it follows that

$$\|y\|_{2,0}^2 = \hat{B}(z - w, y) + \hat{B}(w, y) = \hat{B}(z - w, y).$$

Thus,

$$\|y\|_{2,0}^2 \leqq \hat{B}(y, y)^{1/2} C_1[\bar{h}^2\|y\|_{2,0}^2 + C_2\bar{h}^{4-\alpha}\|y\|_{2,0}^2]^{1/2},$$

and with (8.6),

$$\|y\|_{2,0} \leqq C_1(\varepsilon^{[3]}(\boldsymbol{\eta})^2 + \Lambda^2)^{1/2}\bar{h}(1 + C_2\bar{h}^{2-\alpha})^{1/2}. \tag{8.13}$$

By definition of y and (8.1), it follows that

$$-\frac{d}{dx}a\frac{d}{dx}(y - e) = -K\bar{h}^{-\alpha}y + be,$$

and hence,

$$\|y - e\|_{SE,2} \leqq (8\underline{a})^{-1/2}[K\bar{h}^{-\alpha}\|y\|_{2,0} + \bar{b}\|e\|_{2,0}].$$

By the duality principle, we obtain

$$\|e\|_{2,0} \leqq C_3\bar{h}\|e\|_{SE,2},$$

where C_3 depends on a and b. This, together with (8.13), gives

$$\begin{aligned} \|e\|_{SE,2} &\leqq \|y\|_{SE,2} + \|y - e\|_{SE,2} \\ &\leqq \hat{B}(y, y)^{1/2} + (8\underline{a})^{-1/2}\bar{b}C_3\bar{h}\|e\|_{SE,2} \\ &\quad + (8\underline{a})^{-1/2}KC_1\bar{h}^{1-\alpha}(1 + C_2\bar{h}^{2-\alpha})^{1/2}(\varepsilon^{[3]}(\boldsymbol{\eta})^2 + \Lambda^2)^{1/2}, \end{aligned}$$

which by (8.6) implies (8.8). This proves the theorem.

Evidently, for given K and α the quantity Λ of (8.7) is computable. Hence, Theorem 8.1 states that $(\varepsilon^{[3]}(\boldsymbol{\eta})^2 + \Lambda^2)^{1/2}$ is an asymptotically guaranteed U-estimator. Moreover, the quantity

$$\phi = (\varepsilon^{[3]}(\boldsymbol{\eta})^2 + \Lambda^2)^{1/2}(1 + O(\bar{h}^{1-\alpha})) - \varepsilon^{[3]}(\boldsymbol{\eta}) \geqq 0, \tag{8.14}$$

represents a corrector for $\varepsilon^{[3]}$.

The next theorem shows that under certain additional conditions $\varepsilon^{[3]}$ is asymptotically exact.

THEOREM 8.2. *Suppose that the conditions of Theorem 7.3 hold and that the family $\mathcal{P}$ of partitions is λ-quasi-uniform. Then $\varepsilon^{[3]}(\boldsymbol{\eta})$ is asymptotically exact and*

$$\left.\begin{aligned} &\text{i)} \quad \|e\|_{SE,2} = \varepsilon^{[3]}(\boldsymbol{\eta})(1 + O(\bar{h}(\Delta)^{\mu})) \\ &\text{ii)} \quad \|e\|_{SE,2} = \varepsilon^{[3]}(\boldsymbol{\eta})(1 + O(\|e\|_{SE,2}^{\mu/\varkappa})) \end{aligned}\right\} \quad \text{as } \bar{h}(\Delta) \to 0, \quad \Delta \in \mathcal{P}. \tag{8.15}$$

Proof. By (8.2) and (8.3), we have

$$-a(x_j)\, d(x_j) = \frac{1}{h_j}\int_{I_j} r_j(x)(x-x_{j-1})\, dx + \frac{1}{h_{j+1}}\int_{I_{j+1}} r_{j+1}(x)(x_{j+1}-x)\, dx. \tag{8.16}$$

As in Theorem 7.3, we decompose r_j as follows:

$$\begin{aligned} r_j^{[2]} &= -\tilde{a}_{j-\frac12} u_0''(\tilde{x}_{j-\frac12}),\\ r_j^{[3]} &= -a'(x)e'(x) - b(x)e(x),\\ r_j^{[4]} &= -a(x)u_0''(x) + \tilde{a}_{j-\frac12}u_0''(\tilde{x}_{j-\frac12}) = r_j - r_j^{[2]} - r_j^{[3]}, \end{aligned} \tag{8.17}$$

where $\tilde{x}_{j-\frac12} = \frac12(x_j + x_{j-1})$ and, as before, $\tilde{a}_{j-\frac12} = a(\tilde{x}_{j-\frac12})$. Moreover, we set

$$\begin{aligned} d^{[l]}(x_j) = -\frac{1}{a(x_j)}\bigg[&\frac{1}{h_j}\int_{I_j} r_j^{[l]}(x_j)(x-x_{j-1})dx\\ &+\frac{1}{h_{j+1}}\int_{I_{j+1}} r_{j+1}^{[l]}(x)(x_{j+1}-x)\, dx\bigg], \qquad l = 2, 3, 4, \end{aligned} \tag{8.18}$$

whence, by (8.17),

$$d(x_j) = d^{[2]}(x_j) + d^{[3]}(x_j) + d^{[4]}(x_j). \tag{8.19}$$

Let $\alpha_{j,1,l}$, $\alpha_{j,0,l}$, $l = 2, 3, 4$, be the quantities (6.13) formed with $d^{[l]}(x_j)$, $d^{[l]}(x_{j-1})$, $l = 2, 3, 4$, respectively, and set

$$h_j P_{jl}'(x) = (x_j - x)\alpha_{j,1,l} - (x - x_{j-1})\alpha_{j,0,l}, \qquad l = 2, 3, 4.$$

Then we have, with any $\varepsilon > 0$,

$$\begin{aligned} \int_{I_j} \frac{1}{a(x)} P_j'(x)^2\, dx &= \frac{1+O(\bar{h})}{\tilde{a}_{j-\frac12}}\int_{I_j} P_j'(x)^2\, dx\\ &\leqq \frac{(1+O(\bar{h})}{\tilde{a}_{j-\frac12}}\bigg\{(1+C_1\varepsilon)^2\int_{I_j} P_{j2}'(x)^2\, dx\\ &\qquad + \left(1+\frac{1}{\varepsilon}\right)^2\bigg[C_2\int_{I_j} P_{j3}'(x)^2\, dx + \int_{I_j} P_{j,4}'(x)^2\, dx\bigg]\bigg\}, \end{aligned} \tag{8.20}$$

and

$$\int_{I_j} P_{jl}'(x)^2\, dx = \frac{h_j}{3}[\alpha_{j,1,l}^2 - \alpha_{j,1,l}\alpha_{j,0,l} + \alpha_{j,0,l}^2], \qquad l = 2, 3, 4.$$

From (8.17), (8.18) and the definition (6.13) it then follows that

$$\int_{I_j} P_{j2}'(x)^2\, dx = \frac{1}{12}\tilde{a}_{j-\frac12}^2 u_0''(\tilde{x}_{j-\frac12})^2 h_j^3 + h_j O(\bar{h}^4),$$

$$\int_{I_j} P_{j3}'(x)^2\, dx \leqq C\bar{h}^2[{}_{I_{j-1}}\|e\|_{SE,2}^2 + {}_{I_j}\|e\|_{SE,2}^2 + {}_{I_{j+1}}\|e\|_{SE,2}^2 + {}_{I_{j-1}}\|e\|_{0,2}^2 + {}_{I_j}\|e\|_{0,2}^2 + {}_{I_{j+1}}\|e\|_{0,2}^2],$$

$$\int_{I_j} P_{j4}'(x)^2\, dx \leqq C h_j \bar{h}^4.$$

Thus, with $\varepsilon = \bar{h}^{\mu}$ in (8.20) and after summation over j we obtain

$$\varepsilon^{[3]}(\boldsymbol{\eta})^2 \leqq \left(\frac{1}{12}\sum_{j=1}^{m(\Delta)} \tilde{a}_{j-\frac12}u_0''(\tilde{x}_{j-\frac12})^2 h_j^3\right)(1+O(\bar{h}^{\mu})) + C_1\bar{h}_I^{\varkappa}\|e\|_{SE,2}^2 + C_2\bar{h}^{2+\varkappa}. \tag{8.21}$$

Now (7.7), (7.12) and (7.13) show that

$$\begin{aligned}\varepsilon_2^{[1]}(\boldsymbol{\eta}) &= \left[\frac{1}{12}\sum_{j=1}^{m} \tilde{a}_{j-\frac{1}{2}} u_0''(\tilde{x}_{j-\frac{1}{2}})^2 h_j^3\right]^{1/2} \geqq C\underline{h}, \\ {}_I\|e\|_{SE,2} &= \varepsilon_2^{[1]}(\boldsymbol{\eta})(1+O(\bar{h})^{\mu}),\end{aligned} \tag{8.22}$$

and hence it follows that

$$\begin{aligned}\varepsilon^{[3]}(\boldsymbol{\eta})^2 &\leqq \varepsilon_2^{[1]}(\boldsymbol{\eta})^2[(1+O(\bar{h}^{\mu})+C_1\bar{h}^{\varkappa}(1+O(\bar{h}^{\mu}))+C_2\bar{h}^{2\mu}] \\ &= \varepsilon_2^{[1]}(\boldsymbol{\eta})^2(1+O(h^{\mu}))\leqq {}_I\|e\|_{SE,2}^2(1+O(h^{\mu})),\end{aligned}$$

that is,

$$\varepsilon^{[3]}(\boldsymbol{\eta})^2 \leqq {}_I\|e\|_{SE,2}^2(1+O(\bar{h}^{\mu})).$$

The reverse inequality is obtained in essentially the same way and this proves the theorem.

THEOREM 8.3. *Suppose that the conditions of Theorem* 8.2 *hold. Then the quantity* ϕ *of* (8.14) *is an asymptotically exact corrector of* $\varepsilon^{[3]}(\boldsymbol{\eta})$.

Proof. Theorem 8.1 guarantees that ϕ is a corrector of $\varepsilon^{[3]}$. Since

$$\phi = \varepsilon^{[3]}(\boldsymbol{\eta})\left[\left(1+\left(\frac{\Lambda}{\varepsilon^{[3]}(\boldsymbol{\eta})}\right)^2\right)^{1/2}(1+O(\bar{h}^{1-\alpha})-1\right],$$

we need to show only that

$$\Lambda = o(\varepsilon^{[3]}) \quad \text{as } \bar{h}\to 0. \tag{8.23}$$

From (8.17), (8.18) it follows that

$$\int_{I_j}(r_j+P_j'')^2\,dx \leqq C\left(\sum_{l=-1}^{1}({}_{I_{j-l}}\|e\|_{SE,2}^2+{}_{I_{j-l}}\|e\|_{0,2}^2)+h_j^5\right),$$

whence

$$\Lambda^2 = \frac{\bar{h}^{\alpha}}{K}\sum_{j=1}^{m}\int_{I_j}(r_j+P_j'')^2\,dx \leqq C_1\bar{h}^{\alpha}(\|e\|_{SE,2}^2+C_2\bar{h}^4).$$

On the other hand, we have for λ-quasi-uniform partitions

$${}_I\|e\|_{SE,2}^2 \geqq C\bar{h}^{2\varkappa},$$

and therefore,

$$\Lambda^2 \leqq C\bar{h}^{\alpha}{}_I\|e\|_{SE,2}^2 = C\bar{h}^{\alpha}\varepsilon^{[3]}(\boldsymbol{\eta})^2(1+O(\bar{h}^{\mu})),$$

which proves (8.23).

9. Correctors for the estimators of Definitions 6.1, 6.2. In § 8, we considered the estimator $\varepsilon^{[3]}$ of Definition 6.3 and discussed, in particular, the construction of an asymptotically exact corrector. In this section, we outline the construction of such correctors for the estimators $\varepsilon^{[2]}$ and $\varepsilon^{[1]}$.

THEOREM 9.1. *Suppose that the conditions of Theorem* 5.1 *hold. Let* r_j *be given by* (5.2) *and define* $\tilde{z}$ *as the function on I with the restrictions*

$$\tilde{z}_j(x) = \tfrac{1}{2}C_j(x_j-x)(x-x_{j-1}) \quad \forall x\in I_j(\Delta), \quad j=1,\cdots,m, \tag{9.1}$$

where

$$C_j = -\frac{3h_j^{-3}}{\tilde{a}_{j-\frac{1}{2}}}\int_{I_j} r_j(x)(x-x_j)(x-x_{j-1})\,dx, \qquad j=1,\cdots,m. \tag{9.2}$$

and set

$$\sigma_j(x) = r_j(x) - L[\tilde{z}_j](x) \quad \forall x \in I_j, \quad j = 1, \cdots, m. \tag{9.3}$$

Then with the corrector vector $\phi = (\phi_1, \cdots, \phi_m)$ *defined by*

$$\phi_j = \frac{1}{\underline{a}_j}\left(\frac{1}{\pi}\right)^{2/p} h_j \, {}_{I_j}\|\sigma_j\|_{p,0}, \qquad j = 1, \cdots, m, \tag{9.4}$$

and with the constants θ_j *of* (4.6) *and* $\eta_j^{[2]}$ *of* (6.9), *the quantity*

$$\begin{aligned}\Phi(\boldsymbol{\eta}) = \frac{\bar{h}}{2}\frac{\|a'\|_{\infty,0}}{\underline{a}}\varepsilon^{[2]} + \frac{2\bar{a}^{1/2}}{C_2^{[p]}} C_1^{[p]}\left[\sum_{j=1}^{m(\Delta)} |\phi_j|^p\right]^{1/p} \\ + \frac{1}{C_2^{[p]}}\left[\sum_{j=1}^{m(\Delta)} \left(\frac{\bar{a}}{\tilde{a}_{j-\frac{1}{2}}}\right)^{p/2} |\eta_j^{[2]}|^p \theta_j^p\right]^{1/p}\end{aligned} \tag{9.5}$$

is a corrector of $\varepsilon^{[2]}$. *Moreover, under the conditions of Theorem* 7.4 *the corrector is asymptotically exact.*

Proof. A simple computation shows that for $1 \leqq p < \infty$

$$a_{j-\frac{1}{2}}^{1/2}\left(\int_{I_j} |\tilde{z}_j'(x)|^p \, dx\right)^{1/p} = \eta_j^{[2]}. \tag{9.6}$$

This implies that

$${}_{I_j}\|\tilde{z}\|_{SE,p} \leqq |\eta_j^{[2]}|(1 + Ah_j)^{1/2}, \qquad A = \frac{1}{2}\frac{1}{\underline{a}}\|a'\|_{\infty,0}, \tag{9.7}$$

and therefore

$${}_I\|\tilde{z}\|_{SE,p} \leqq (1 + A\bar{h})^{1/2}\varepsilon^{[2]}(\boldsymbol{\eta}). \tag{9.8}$$

Moreover, using Theorems 5.1 and 3.1 we have

$$\sup_{\substack{w \in \mathring{L}_q^1(I_j) \\ |w|_{q,1} = 1}} |B(e - \tilde{z}_j, w)| \leqq C_1^{[p]}\phi_j, \qquad j = 1, \cdots, m. \tag{9.9}$$

Hence, it follows from Theorem 4.1 that

$$|e(\Delta) - \tilde{z}|_{p,1} \leqq \frac{1}{C_2^{[p]}}\left[\sum_{j=1}^{m(\Delta)} \frac{1}{\tilde{a}_{j-\frac{1}{2}}^{p/2}}|\eta_j^{[2]}|^p\theta_j^p\right]^{1/p} + \frac{2C_1^{[p]}}{C_2^{[p]}}\left(\sum_{j=1}^{m(\Delta)} (\phi_j^p)^p\right)^{1/p},$$

and

$$\begin{aligned}{}_I\|e(\Delta) - \tilde{z}\|_{SE,p} \leqq \frac{1}{C_2^{[p]}}\left[\sum_{j=1}^{m(\Delta)}\left(\frac{\bar{a}}{\tilde{a}_{j-\frac{1}{2}}}\right)^{p/2} |\eta_j^{[2]}|^p\theta_j^p\right]^{1/p} \\ + \frac{2\bar{a}^{1/2}}{C_2^{[p]}} C_1^{[p]}\left[\sum_{j=1}^{m(\Delta)} \phi_j^p\right]^{1/p}.\end{aligned} \tag{9.10}$$

Now (9.5) follows directly from

$${}_I\|e\|_{SE,p} \leqq {}_I\|\tilde{z}\|_{SE,p} + {}_I\|e(\Delta) - \tilde{z}\|_{SE,p},$$

together with (9.7) and (9.10). The proof of the asymptotic exactness is virtually a repetition of the proof of Theorem 7.4 concerning the asymptotic exactness of $\varepsilon^{[2]}$ itself.

THEOREM 9.2. *Suppose that the conditions of Theorem* 5.1 *hold and set*

$$\tilde{z}_j = -C_j(x-x_j)(x-x_{j-1}) \operatorname{sgn} \int_{I_j} r_j(x)(x-x_j)(x-x_{j-1})\,dx, \qquad j=1,\cdots,m, \tag{9.11}$$

where C_j is determined such that

$$_{I_j}\|\tilde{z}_j\|_{SE,p} = \eta_j^{[1]}. \tag{9.12}$$

Further, let the σ_j and $\phi_j^{[2]}$, $j=1,\cdots,m$ be defined by (9.3), (9.4). *Then*

$$\begin{aligned}\Phi(\boldsymbol{\phi}) &= \frac{h}{2}\frac{\|a'\|_{\infty,0}}{\underline{a}}\varepsilon^{[1]} + \frac{2\bar{a}^{1/2}}{C_2^{[p]}}C_1^{[p]}\left[\sum_{j=1}^{m(\Delta)} |\phi_j|^p\right]^{1/p} \\ &\quad + \frac{1}{C_2^{[p]}}\left[\sum_{j=1}^{m(\Delta)} \left(\frac{\underline{a}}{\tilde{a}_{j-\frac{1}{2}}}\right)^{p/2} |\eta_j^{[1]}|^p \theta_j^p\right]^{1/p},\end{aligned} \tag{9.13}$$

is a corrector of $\varepsilon^{[1]}$. Moreover, under the conditions of Theorem 7.3 *the corrector is asymptotically exact.*

The proof is entirely analogous to that of Theorem 9.1 and is therefore omitted.

It should be noted that from a practical viewpoint, these correctors are not very important because of the obvious cost in computing them. Moreover, it may be possible to use more refined arguments to obtain better correctors which do not involve constants such as $C_2^{[p]}$, etc.

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