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A review of some a posteriori error estimates for adaptive finite element methods

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Abstract

Recently, the adaptive finite element methods have gained a very important position among numerical procedures for solving ordinary as well as partial differential equations arising from various technical applications. While the classical a posteriori error estimates are oriented to the use in *h*-methods the contemporary higher order *hp*-methods usually require new approaches in a posteriori error estimation.

We present a brief review of some error estimation procedures for some particular both linear and nonlinear differential problems with special regards to the needs of the *hp*-method.

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1. Introduction

Numerical computation has always been connected with some control procedures. It means that not only the approximate result is of importance, but also the error of this computed result, i.e. some norm of the difference between the exact and approximate solution. The exact solution usually is not known. This means that we can get only some estimates of the error.

The development of numerical procedures has been accompanied with the development of *a priori error estimates* that are very useful in theory but usually include constants that are completely unknown, in better cases can be estimated. In particular, the development of the finite element method, and its *h*-version and *hp*-version required reliable and computable estimates of the error that depend only on the approximate solution just computed, if possible. This is the means for the local mesh refinement in the *h*-version and, moreover, also for the increase of the polynomial degree in the *p*-version.

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We introduce several quantities called *a posteriori error estimators* in the paper to assess the error of the approximate solution in a triangulation employed. The quality of a posteriori error estimators is often measured by its *effectivity index*, i.e. the ratio of some norm of the error estimate and the true error. An error estimator is called *effective* if both its effectivity index and the inverse of the index remain bounded for all meshsizes of triangulation. It is called *asymptotically exact* if its effectivity index converges to 1 as the meshsize tends to 0.

Undoubtedly, obtaining efficient and computable a posteriori error estimates is not easy. Analytical ones may not be the most reliable. Moreover, the local nature of the estimates is very advantageous. It provides for the local mesh refinement or the local increase of the polynomial degree and usually means lower time requirements of the computation. The papers [2,3] by Babuška and Rheinboldt represent the pioneering work in this field. The book [1] appeared in 2000 and presents basic error estimation techniques.

We survey estimators only for some classes of rather simple differential problems in this contribution. We pay much attention to linear second order elliptic equations with some generalization to quasilinear ones. As further examples, we mention stationary incompressible Navier–Stokes equations, equations of linearized elasticity, the biharmonic equation, and the convection–diffusion–reaction equation. We also treat briefly a nonlinear parabolic equation both in the semidiscrete and fully discrete case.

Nevertheless, the scope of a posteriori error estimates is much wider. It would be nice to add later also a review of estimates for nonlinear hyperbolic conservation laws, further fluid dynamics models, Maxwell equations, a wide variety of civil engineering problems (not only elasticity ones), discontinuous coefficient problems, and, in particular, coupled problems. Time-dependent models and discontinuous finite elements are of special importance, too.

There are several classes of local a posteriori error estimators based on different approaches and their names slightly vary in the literature. Let us name residual estimators: explicit, implicit (based on the solution of local problems), and hierarchic (multilevel) estimators, further also recovery-based estimators (based on the averaging of gradient), and some other. A weighted (complementary) energy norm global a posteriori estimator is presented in Section 4.

Goal-oriented error estimators have been intensively treated during the last two decades. They are not concerned with the norm of the error but with the value of some functional defined for the solution of the problem treated and are very efficient in goal-oriented adaptive computation.

Let $\Omega \subset R^n$, $n \geq 2$, be a connected bounded domain with polyhedral boundary Γ. We use the notation $\|\cdot\|_0$ for the $L_2(\Omega)$ norm and $\|\cdot\|_1$ for the $H^1(\Omega)$ norm for scalar as well as vector-valued functions. The norm may be restricted to any open set $\omega \subset \Omega$ with Lipschitz boundary γ. We write $\|\cdot\|_{0;\omega}$ for the $L_2(\omega)$ norm, $\|\cdot\|_{1;\omega}$ for the $H^1(\omega)$ norm, and $\|\cdot\|_{0;\gamma}$ for the $L_2(\gamma)$ norm. On Ω, ω, and γ, we further consider the spaces $W^{k,s}(\Omega)$, $W^{k,s}(\omega)$, and $L_s(\gamma)$ with an integer k and $1 \leq s \leq \infty$. We also employ the space $H_0^1(\Omega)$ etc.

We use the notation $\nabla \cdot a = \operatorname{div} a$ if a is a vector-valued function and $\nabla b = \operatorname{grad} b$ if b is a (scalar) function. Further, $f \cdot g$ is the inner product of two vector-valued functions f and g. We consider also the space $H(\mathrm{div},\Omega) = \{u \in [L_2(\Omega)]^n | \operatorname{div} u \in L_2(\Omega)\}$ of vector-valued functions defined for $n = 2, 3$.

Symbols $c, c_1, \ldots$ are generic. They may represent different quantities (depending possibly on other different quantities) at different occurences.

2. Linear second order elliptic equation

Problem

Poisson equation with Dirichlet and Neumann boundary conditions

$$-\Delta u = f \quad \text{in} \quad \Omega, \tag{2.1}$$

where $\Omega \subset R^2$ is a connected bounded polygonal domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, with the boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_D, \tag{2.2}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma_N. \tag{2.3}$$

We assume that Γ_D is closed relatively to Γ and has a positive length, and that f and g are square integrable functions on Ω and Γ_N, respectively.

We have chosen the simplest Eq. (2.1) for the model problem to show the results of a posteriori error estimation in the easiest way. A more general linear second order elliptic equation is, e.g.,

$$-\nabla \cdot (a\nabla u) + bu = f \quad \text{in} \quad \Omega,$$

where a is a positive scalar function or a positive definite matrix and b a nonnegative function.

Note that this problem is treated in [23] by the combination of the equilibrated residual method and the method of hypercircle.

Further, the problem is treated in [13] with the help of the technique presented in Section 4 of this paper, i.e., the approximate solution can be obtained in any way (no particular numerical procedure is assumed) and the number of unknown constants in the global estimate is minimized to the constants in the definition of positive defineteness of a, Friedrichs inequality, and Trace Theorem inequality.

Several classes of nonlinear elliptic problems (second-order and fourth-order problems in arbitrary dimension, elasticity problem) are investigated in [12] using the same approach.

Weak solution

We set

$$X = \{\varphi | \varphi \in H^1(\Omega), \quad \varphi = 0 \quad \text{on} \quad \Gamma_D\}.$$

The weak solution $u \in X$ of the problem is defined by the identity

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\Gamma_N} gv \tag{2.4}$$

to be satisfied by all test functions $v \in X$. It is well-known that the problem (2.4) has a unique solution (see, e.g., [9]).

Approximate solution

Let $\mathcal{T}_h$, $h > 0$, be a family of *regular triangulations* (see, e.g., [1,4]). For any triangle $T \in \mathcal{T}_h$ we denote by h_T its diameter, by ρ_T the diameter of the largest ball inscribed into T, and by $\mathcal{E}(T)$ the set of all its edges without endpoints. We set

$$\mathcal{E}_h = \bigcup_{T \in \mathcal{T}_h} \mathcal{E}(T).$$

We split $\mathcal{E}_h$ in the form

$$\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,D} \cup \mathcal{E}_{h,N}$$

with

$$\mathcal{E}_{h,\Omega} = \{E \in \mathcal{E}_h | E \subset \Omega\}, \quad \mathcal{E}_{h,D} = \{E \in \mathcal{E}_h | E \subset \Gamma_D\}, \quad \mathcal{E}_{h,N} = \{E \in \mathcal{E}_h | E \subset \Gamma_N\}.$$

For $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$ we define

$$\omega_T = \bigcup_{\mathcal{E}(T) \cap \mathcal{E}(T') \neq \emptyset} T', \quad \omega_E = \bigcup_{E \in \mathcal{E}(T')} T'.$$

The length of $E \in \mathcal{E}_h$ is denoted by h_E. Further put

$$f_T = |T|^{-1} \int_T f, \quad g_E = h_E^{-1} \int_E g,$$

where $|T|$ is the area of T.

With every edge $E \in \mathcal{E}_h$ we associate a unit vector n_E such that n_E is orthogonal to E and equals the unit outer normal to Γ if $E \subset \Gamma$. Given any $E \in \mathcal{E}_{h,\Omega}$ and any $\varphi \in L_2(\omega_E)$ with $\varphi|_{T'} \in C(T')$ for all $T' \subset \omega_E$, we denote by $[\varphi]_E$

the jump of φ across E in the direction n_E,

$$[\varphi]_E(x) = \lim_{t\to 0+} \varphi(x+tn_E) - \lim_{t\to 0+} \varphi(x-tn_E) \quad \text{for all} \quad x \in E.$$

Now, denote by X_h the space of all continuous piecewise linear finite element functions corresponding to $\mathcal{T}_h$ and vanishing on Γ_D. This is the simplest case, the error estimators usually work well, and many of them are constructed only for this case. We look for the approximate solution $u_h \in X_h$ such that it satisfies, for all $v_h \in X_h$, the identity

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h + \int_{\Gamma_N} g v_h \tag{2.5}$$

corresponding to the identity (2.4). It is well-known that the problem (2.5) has a unique solution (see, e.g., [9]).

For the sake of simplicity, we restrict ourselves in this section to linear finite elements. The results of Sections 2.1 and 2.2, however, can be immediately generalized to higher order finite elements [24]. Moreover, we assume that all the integrals in the Eq. (2.4) are evaluated exactly.

2.1. Explicit residual error estimator

Let u and u_h be the solutions of the problems (2.4) and (2.5). They satisfy the identity

$$\int_\Omega \nabla(u-u_h)\cdot\nabla v = \int_\Omega f v + \int_{\Gamma_N} g v - \int_\Omega \nabla u_h \cdot \nabla v \tag{2.6}$$

for all $v \in X$ [1]. The right-hand part of the Eq. (2.6) implicitly defines the *residual* of u_h as an element of the space dual to X.

We introduce the *explicit residual a posteriori error estimator* $\eta_{R,T}$ on the triangle T by

$$\eta_{R,T} = \left(h_T^2 \|f_T\|_{0;T}^2 + \frac{1}{2} \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} h_E \|[n_E\cdot\nabla u_h]_E\|_{0;E}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,N}} h_E \|g_E - n_E\cdot\nabla u_h\|_{0;E}^2 \right)^{1/2}$$

and present its properties proven in [25].

Notice that the error estimator $\eta_{R,T}$ consists of three parts. The first one is connected with the residual of the "strong" solution and the rest is formed by "boundary terms". The second part of the estimator expresses the fact that $u_h \notin H^2(\Omega)$, i.e., that ∇u_h may have jumps across triangle edges. The last part expresses that u_h may not satisfy the Neumann boundary condition exactly.

Further error estimators we mention in what follows are often similar in their nature.

Theorem 2.1. *Let u and u_h be solutions of the problems (2.4) and (2.5). Then there are positive constants c_1, c_2 that depend only on the smallest angle in the triangulation such that the estimates*

$$\|u-u_h\|_1 \le c_1 \left(\sum_{T\in\mathcal{T}_h} \eta_{R,T}^2 + \sum_{T\in\mathcal{T}_h} h_T^2 \|f - f_T\|_{0;T}^2 + \sum_{E\in\mathcal{E}_{h,N}} h_E \|g - g_E\|_{0;E}^2 \right)^{1/2}$$

and

$$\eta_{R,T} \le c_2 \left(\|u-u_h\|_{1;\omega_T}^2 + \sum_{T'\subset\omega_T} h_{T'}^2 \|f - f_{T'}\|_{0;T'}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_{h,N}} h_E \|g - g_E\|_{0;E}^2 \right)^{1/2}$$

hold for all $T \in \mathcal{T}_h$.

Notice that the error estimator $\eta_{R,T}$ really provides, like many other estimators, a two-sided estimate of the error and that its nature is local. The constants c_1, c_2 cannot be calculated, in general. Their values depend on the smallest angle in the triangulation (see, e.g., [8] for the estimates of these constants).

The error estimator $\eta_{R,T}$ was first proposed and analyzed for the problem (2.1)–(2.3) in one dimension in [2,3]. Some further, more refined error estimates can be proven if Ω is convex [25].

2.2. Some other error estimators

The residual treated in Section 2.1 can be considered in suitable subspaces of X where we solve auxiliary problem similar to, but simpler than the original discrete problem (2.5)[25]. We can thus consider i mplicit residual error estimators based on the solution of local problems. The most often used two ways of the construction of the local problems yield *implicit Dirichlet local problem a posteriori error estimator* $\eta_{\mathrm{D},T}$ and *implicit Neumann local problem a posteriori error estimator* $\eta_{\mathrm{N},T}$. For their exact definition and properties we refer to [6,7,25]. It can be easily seen there that the computation of $\eta_{\mathrm{D},T}$ and $\eta_{\mathrm{N},T}$ is more expensive than that of $\eta_{\mathrm{R},T}$.

Using superconvergence results, one can prove that on special meshes the error estimators of Sections 2.1 and 2.2 are asymptotically exact in the sense of Section 1, see [28].

The *hierarchic a posteriori error estimators* $\eta_{\mathrm{H},E}$ and $\eta_{\mathrm{H},T}$ bound the error $u-u_h$ by evaluating the residual of u_h with respect to certain basis functions of another finite element space W_h that satisfies $X_h \subset W_h \subset X$ and that either consists of higher order finite elements or corresponds to a refinement of $\mathcal{T}_h$, or both [25]. If we scale the corresponding basis functions suitably, the identity (2.6) and the Schwarz inequality imply that we have a lower bound on the error. A proper choice of W_h can lead to upper bounds, too. The estimates were first presented and proven in [25]. Another error estimate (based on the saturation assumption) is analyzed in [5].

Consider the problem (2.1) and (2.2) with $\Gamma_N = \emptyset$ and denote by u and u_h the unique solutions of problems (2.4) and (2.5). Suppose that we can easily compute, by some postprocessing, an approximation Gu_h of ∇u_h. We then introduce the *a posteriori error estimator based on the averaging of gradient* $\eta_{\mathrm{Z},T}$ that was first proposed in [29] where its properties were analyzed.

3. Stationary incompressible Navier–Stokes equations

Problem

Consider the elliptic system of stationary incompressible Navier–Stokes equations

$$\begin{aligned} &-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in} \quad \Omega \subset R^n,\\ &\nabla\cdot u = 0 \quad \text{in} \quad \Omega,\\ &u = 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{3.1}$$

where u and f are vector-valued functions, u is the velocity, p the pressure, and $\nu > 0$ the constant viscosity of the fluid.

Weak solution

We put

$$M = \{u \in [W^{1,2}(\Omega)]^n \,|\, u = 0 \quad \text{on} \quad \Gamma\}, \qquad Q = \left\{ p \in L_2(\Omega) \,\middle|\, \int_\Omega p = 0 \right\}$$

and define

$$\begin{aligned} &X = Y = M \times Q,\\ &\|\cdot\|_X = \|\cdot\|_Y = (\|\cdot\|_1^2 + \|\cdot\|_0^2)^{1/2},\\ &\langle F([u,p]),[v,q]\rangle = \nu\int_\Omega (\nabla\cdot u)(\nabla\cdot v) + \int_\Omega (u\cdot\nabla)u\cdot v - \int_\Omega p\nabla\cdot v + \int_\Omega q\nabla\cdot u - \int_\Omega f\cdot v, \end{aligned}$$

where $[u, p]$ and $[v, q]$ are $(n+1)$-component vector-valued functions. We then say that $[u, p] \in X$ is the weak solution of the problem (3.1) if

$$\langle F([u,p]),[v,q]\rangle = 0$$

for all $[v, q] \in Y$.

Approximate solution

Let $M_h \subset M$ and $Q_h \subset Q$ be two finite element spaces corresponding to $\mathcal{T}_h$ consisting of affine equivalent elements in the sense of [9].

Let us put

$$X_h = Y_h = M_h \times Q_h, \langle F_h([u_h, p_h]), [v_h, q_h]\rangle = \langle F([u_h, p_h]), [v_h, q_h]\rangle + \delta \sum_{T \in \mathcal{T}_h} h_T^2 \int_T (-\nu \Delta u_h + (u_h \cdot \nabla) u_h$$

$$+ \nabla p_h - f) \cdot ((u_h \cdot \nabla) v_h + \nabla q_h) + \delta \sum_{E \in \mathcal{E}_{h,\Omega}} h_E \int_E [p_h]_E [q_h]_E + \alpha\delta \int_\Omega (\nabla \cdot u_h)(\nabla \cdot v_h) \tag{3.2}$$

Here, $\alpha \geq 0$ and $\delta \geq 0$ are stability parameters. If $\alpha > 0$ and $\delta > 0$ the above discretization is capable of stabilizing both the influence of the convection term and the divergence constraint without any conditions on the spaces M_h and Q_h, or the Péclet number h_T/ν [21]. The case $\alpha = \delta = 0$ corresponds to the standard mixed finite element discretization of the problem (3.1).

We say that $[u_h, p_h] \in X_h$ is the approximate solution of the problem (3.1) if

$$\langle F_h([u_h, p_h]), [v_h, q_h]\rangle = 0$$

for all $[v_h, q_h] \in Y_h$, where F_h is given by Eq. (3.2).

Put

$$\eta_{\mathrm{NS},T} = \left(h_T^2 \| -\nu\Delta u_h + (u_h \cdot \nabla) u_h + \nabla p_h - \pi_{0,T} f \|_{0;T}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} h_E \| [(\nabla \cdot u_h)\nu n_E - p_h n_E]_E \|_{0;E}^2 + \| \nabla \cdot u_h \|_{0;T}^2 \right)^{1/2}, \tag{3.3}$$

where Π_k, $k \geq 0$, is the space of polynomials of degree at most k and $\pi_{k,S}$, $S \in \mathcal{T}_h \cup \mathcal{E}_h$, is the L_2 projection of $L_2(S)$ onto $\Pi_{k|S}$.

Theorem 3.1. *Let $[u, p]$ be a weak solution of the problem (3.1) that is regular and let $[u_h, p_h] \in X_h$ be its approximate solution. Then the following a posteriori estimates hold.*

$$(\|u - u_h\|_1^2 + \|p - p_h\|_0^2)^{1/2} \leq c_1(1 + (1+\alpha)\delta(1 + \|u_h\|_1)) \left(\sum_{T \in \mathcal{T}_h} \eta_{\mathrm{NS},T}^2 \right)^{1/2}$$

$$+ c_2 \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|f - \pi_{0,T} f\|_{0;T}^2 \right)^{1/2},$$

$$\eta_{\mathrm{NS},T} \leq c_3 (\|u - u_h\|_{1;\omega_T}^2 + \|p - p_h\|_{0;\omega_T}^2)^{1/2} + c_4 \left(\sum_{T' \subset \omega_T} h_{T'}^2 \|f - \pi_{0,T'} f\|_{0;T'}^2 \right)^{1/2},$$

where $\eta_{NS,T}$ is the residual error estimator given by (3.3) and the positive constants $c_1, \ldots, c_4$ depend only on the polynomial degrees of the spaces M_h and Q_h, and on the ratio h_T/ρ_T.

Theorem 3.1 is proven in [25] and can be extended to the case of the slip boundary condition as well as to non-Newtonian fluids. We see, in general, that the nature of the estimate of Theorem 3.1 is similar to that of Theorem 2.1.

4. Stationary convection–diffusion–reaction equation

We consider a convection–diffusion–reaction problem in this section to present an a posteriori error estimate that is as free of unknown constants as possible.

Problem

We assume the convection–diffusion–reaction problem in the form [14]

$$\begin{aligned} &-\varepsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{in} \quad \Omega,\\ &u = 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{4.1}$$

where u is a scalar function to be found, ε a positive constant diffusion coefficient, $b \in [W^{1,\infty}(\Omega)]^n$ a velocity vector field, $c \in L_\infty(\Omega)$ a reaction rate, and f a source term. Further, $\Omega \subset R^n$, $n \geq 1$, is a bounded domain with Lipschitz continuous boundary Γ.

Weak solution

The weak formulation of the problem (4.1) reads: Find the function $u \in H_0^1(\Omega)$ such that

$$a(u, w) = F(w) \tag{4.2}$$

for all test functions $w \in H_0^1(\Omega)$, where

$$a(v, w) = \int_\Omega \varepsilon\nabla v\cdot\nabla w + \int_\Omega b\cdot\nabla v w + \int_\Omega cvw \quad \text{and} \quad F(w) = \int_\Omega fw$$

are a bilinear form and linear functional defined for all $v, w \in H_0^1(\Omega)$. It is well-known that the weak solution $u \in H_0^1(\Omega)$ of the variational problem (4.2) exists and is unique [9] provided that the condition

$$\tilde{c}(x) = c(x) - \frac{1}{2}\nabla\cdot b(x) \geq 0$$

holds almost everywhere in Ω.

Error estimator

Let $\bar{u}$ be a function from $H_0^1(\Omega)$ considered as an approximation of the weak solution u. In [14], no specification of the way $\bar{u}$ has been computed is required, it is just an arbitrary function of the admissible class. We put $e = u - \bar{u}$ for the error.

For $w \in H_0^1(\Omega)$, we introduce the global weighted energy norm by the identity

$$|||w|||_{\lambda,\mu}^2 = \lambda\int_\Omega |\nabla w|^2 + \mu\int_\Omega \tilde{c}w^2, \tag{4.3}$$

where λ and μ are nonnegative real numbers to be chosen later. In particular, we have $a(e, e) = |||e|||_{\varepsilon,1}^2$.

Let us put

$$\begin{aligned} \eta_{\alpha,\beta}(\gamma, y, v, \bar{u}) = {} & \frac{1}{2\alpha}\left((1+\gamma)\|y - \varepsilon\nabla\bar{u} + \varepsilon\nabla v\|_0^2 + \left(1+\frac{1}{\gamma}\right)C_\Omega^2\|f - b\cdot\nabla\bar{u} - c\bar{u} + \nabla\cdot y - cv - b\cdot\nabla v\|_0^2\right)\\ & + \int_\Omega(\varepsilon\nabla v\cdot\nabla\bar{u} + b\cdot\nabla\bar{u}v + cv\bar{u} - fv) + \beta\|v\sqrt{\tilde{c}}\|_0^2, \end{aligned}$$

where α, β, and γ are arbitrary positive numbers, $y \in H(\text{div},\Omega)$ is an arbitrary vector-valued function, and $v, \bar{u} \in H_0^1(\Omega)$ are arbitrary functions. Moreover, C_Ω is the constant from the Friedrichs–Poincaré inequality [14]

$$\|w\|_0 \leq C_\Omega\|\nabla w\|_0$$

valid for all functions $w \in H_0^1(\Omega)$. The following theorem is proven in [14](cf. [17]).

Theorem 4.1. *Let α and β be fixed real numbers such that $2\varepsilon \geq \alpha > 0$, $\beta \geq 1$. Then*

$$|||e|||_{\lambda,\mu} \leq \eta_{\alpha,\beta}(\gamma, y, v, \bar{u}), \tag{4.4}$$

where $\lambda = \varepsilon - \frac{1}{2}\alpha$, $\mu = 1 - 1/\beta$, and γ, y and v are arbitrary quantities whose nature is described above.

Notice that the above introduced error estimator $\eta_{\alpha,\beta}(\gamma, y, v, \bar{u})$ is of global nature and the estimate (4.4) of Theorem 4.1 as well.

Taking different values of numbers α and β that satisfy the assumptions of Theorem 4.1, we get a family of a posteriori error estimates in various weighted energy norms (4.3). The choice of all the parameters, including γ, y, and v, can be optimized. It is shown in [14] that putting, e.g.,

$$y = \varepsilon \nabla u, \quad v = u - \bar{u}$$

yields

$$|||e|||^2_{\nu,\rho} \leq \eta_{\alpha,\beta}(\gamma, \varepsilon\nabla u, u - \bar{u}, \bar{u}) \leq |||e|||^2_{\sigma,\tau},$$

where $\nu = \varepsilon - \frac{1}{2}\alpha$, $\rho = 1 - 1/\beta$, $\sigma = 2(1+\gamma)\varepsilon^2/\alpha - \varepsilon$, and $\tau = \beta - 1$.

In the definition of the estimator $\eta_{\alpha,\beta}(\gamma, y, v, \bar{u})$, there is a global constant C_Ω. Its realistic estimate

$$C_\Omega \leq \left(\pi \sqrt{\frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2}}\right)^{-1}$$

can be readily obtained by enclosing the domain Ω into a rectangular box of dimensions $a_1, \ldots, a_n$ [15].

Another approach to this differential problem can be found, e.g., in [27].

5. Some other problems

Let us at least list some of other differential problems that can be treated by the technology of a posteriori error estimates.

5.1. Quasilinear second order equation

Consider the boundary value problem

$$\begin{aligned} -\nabla \cdot a(x, u, \nabla u) &= b(x, u, \nabla u) \quad \text{in} \quad \Omega \subset R^n, \\ u &= 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

where $a(x, y, z)$, $a \in C_1(\Omega \times R \times R^n, R^n)$, is a vector-valued function and $b(x, y, z)$, $b \in C(\Omega \times R \times R^n, R)$, is a function. Let the matrix

$$A(x, y, z) = \left(\frac{1}{2}(\partial_{z_j} a_i(x, y, z) + \partial_{z_i} a_j(x, y, z))\right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

be positive definite for all $x \in \Omega$, $y \in R$, and $z \in R^n$.

Examples of problems of this class are subsonic flow of irrotational ideal compressible gas, α-Laplacian or stationary heat equation with convection and nonlinear diffusion coefficients. Further examples falling into this class are, e.g., the equations of prescribed mean curvature, Bratu's equation, or a nonlinear eigenvalue problem (cf. [25]). The subject (including linear equations) is treated also in [17].

5.2. Equations of linearized elasticity

Let us consider the elliptic system

$$\begin{aligned} -\nabla \cdot \sigma(u) &= f \quad \text{in} \quad \Omega \subset R^n, \\ u &= 0 \quad \text{on} \quad \Gamma_D, \\ n \cdot \sigma(u) &= g \quad \text{on} \quad \Gamma_N, \end{aligned} \tag{5.1}$$

where $u : \Omega \to R^n$ is the sought displacement of the elastic structure, f denotes the external load, and g represents prescribed boundary tractions. The stress tensor $\sigma(u)$ is related to the strain tensor

$$\varepsilon(u) = \left(\frac{1}{2}(\partial_i u_j + \partial_j u_i)\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

by the constitutive law

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda \mathrm{tr}\varepsilon(u) I,$$

where

$$I = (\delta_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \quad \text{and} \quad \mathrm{tr}\,\varepsilon(u) = \sum_{i=1}^{n} \varepsilon_{ii}(u)$$

are the *unit tensor* and *trace of* $\varepsilon(u)$, and $\lambda > 0$ and $\mu > 0$ are the Lamé constants. As in the previous sections we assume for simplicity that Ω is a polyhedron. In order to guarantee the unique solvability of the problem (5.1), the Dirichlet part Γ_D of the boundary has to have a positive $(n-1)$-dimensional Lebesgue measure. For simplicity we assume that the displacement u vanishes on Γ_D. Other cases can be treated in a similar way.

A posteriori error estimates are presented in, e.g., [12,25]. Moreover, it is possible to extend them to problems of nonlinear elasticity where the load and boundary tractions may depend on the displacement and the stress tensor and where the Lamé coefficients λ, μ may be functions of the displacement. In order to avoid locking phenomena it may be advantageous to formulate the problem (5.1) in a mixed way.

5.3. Biharmonic equation

Let us consider the two-dimensional biharmonic equation

$$\begin{aligned} \Delta^2 u &= f \quad \text{in} \quad \Omega \subset R^2, \\ u = \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \Gamma \end{aligned} \tag{5.2}$$

that models the vertical displacement of the mid-surface of a clamped plate.

Let us put

$$X = Y = \{u \in W^{2,2}(\Omega)\,|\,u = \partial u/\partial n = 0 \quad \text{on} \quad \Gamma\}, \quad \|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|_2, \quad \langle F(u), \varphi\rangle = \int_\Omega \Delta u \Delta\varphi - \int_\Omega f\varphi. \tag{5.3}$$

We then say that $u \in X$ is the weak solution of the problem (5.2) if

$$\langle F(u), \varphi\rangle = 0$$

for all $\varphi \in Y$.

For the discretization of the problem (5.2) we assume that $X_h \subset X$ and $Y_h \subset Y$ are finite element spaces corresponding to $\mathcal{T}_h$ and consisting of piecewise polynomials. These conditions imply in particular that the functions in X_h and Y_h are of class C_1. Denote by k, $k \ge 1$, the maximum polynomial degree of the functions in X_h. Further, put

$$f_h = \sum_{T \in \mathcal{T}_h} \pi_{l,T} f,$$

where l is an arbitrary integer to be kept fixed in this section and the projection $\pi_{l,T}$ is introduced in (3.3).

Replacing f in the definition (5.3) by f_h to get the functional F_h, we say that $u_h \in X_h$ is the approximate solution of the problem (5.2) if

$$\langle F_h(u_h), \varphi \rangle = 0$$

for all $\varphi \in Y_h$.

Denote, in this case, the error of the approximation of f by f_h by

$$\varepsilon_T = h_T^2 \| f - f_h \|_{0;T}$$

and define the *residual a posteriori error estimator*

$$\eta_{\mathrm{B},T} = \left(h_T^4 \| \Delta^2 u_h - f_h \|_{0;T} + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} (h_E \| [\Delta u_h]_E \|_{0;E}^2 + h_E^3 \| [n_E \cdot \nabla \Delta u_h]_E \|_{0;E}^2) \right)^{1/2}$$

for all $T \in \mathcal{T}_h$.

A two-sided a posteriori error estimate for $\| u - u_h \|_2$ based on the quantities ε_T and $\eta_{\mathrm{B},T}$ is proven in [25]. Another estimate is presented, for example, in [12].

6. Nonlinear parabolic equation

Problem

We are concerned with a single nonlinear parabolic equation with a scalar solution in 1D space variable

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\left(a(u) \frac{\partial u}{\partial x}(x,t) \right) + f(u) = 0 \quad \text{for} \quad 0 < x < 1, \quad 0 < t \le T \tag{6.1}$$

with the Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T, \tag{6.2}$$

and with the initial condition

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{6.3}$$

where u_0 is a given function.

The weak solution of the problem (6.1)–(6.3) is defined in a standard way [19].

Semidiscrete approximate solution

To define the finite element solution of the problems (6.1)–(6.3), we start with the space discretization (semidiscretization) (see, e.g., [16,19]). Let us denote the error of the semidiscrete solution $\bar{U}(x,t)$by

$$e(x,t) = u(x,t) - \bar{U}(x,t)$$

and use the symbol $\| \cdot \|_1$ for the $H^1(0,1)$ norm.

We can obtain several hierarchic semidiscrete error indicators with the following property.

Theorem 6.1. *Let $u(x,t)$ be the smooth weak solution of the problem* (6.1) to (6.3), *l* et $\bar{U}(x,t)$*be the corresponding semidiscrete approximate solution. Assume that $\bar{E}$ is the nonlinear parabolic, or linear parabolic, or linear elliptic error indicator introduced in* [19]. *Let*

$$\| e \|_1 \ge C h^p.$$

Then

$$\lim_{h \to 0} \frac{\| \bar{E} \|_1}{\| e \|_1} = 1. \tag{6.4}$$

The quantity $\|\bar{E}\|_1/\|e\|_1$ was introduced as the effectivity index in Section 1 and the limit (6.4) says that the estimator $\|\bar{E}\|_1$ is asymptotically exact. For the parabolic as well as linear elliptic error indicator (but not for the nonlinear elliptic one), this theorem is proven in [19], too.

Analysis of the semidiscrete error does not include analysis of the error of solution of the corresponding system of ordinary differential equations in time. In practice, this system is solved by standard software that admits the required accuracy to be given by the user. This required accuracy is then prescribed several orders less than the total prescribed accuracy of the fully discrete solution. There are several papers concerned with the analysis of fully discrete error, see, e.g., [16,22,26].

7. Conclusion

The general experience shows that, among various adaptive strategies for finite elements, the best results can be achieved by the goal-oriented *hp*-adaptivity. *Goal-oriented adaptivity* is based on adaptation of the finite element mesh with aim of improving the resolution of a specific quantity of interest (instead of minimizing the error of the approximation in some global norm), and *hp*-adaptivity is based on the combination of spatial refinements (*h*-adaptivity) with simultaneous variation of the polynomial order of the approximation (*p*-adaptivity). Automatic *hp*-adaptivity belongs to the most advanced topics in the higher order finite element technology and it is subject to active research. We refer to, e.g., [10,11,18,20] and references therein.

We have met, in the individual sections of the paper, a large number of a posteriori error estimators whose quality is measured by inequalities, usually with some unknown constants on the right-hand part. The estimators are easily computable from the approximate solution only, but their quantitative properties cannot be easily assessed. We then are in a position similar to that with a priori estimators. The use of a posteriori estimators mentioned in the paper can fit the *h*-adaptivity process.

There are, however, global error estimates for some classes of problems (see, e.g., [12–14,17]) that require as few unknown constants as possible. Moreover, some papers provide for the estimation of these constants.

Unfortunately, some error estimators are constructed only for the lowest order polynomial approximation.

The best situation (cf. Section 6) occurs if the estimator is asymptotically exact. However, the asymptotic exactness of estimator may be of little practical advantage. Fortunately enough, many asymptotically exact estimators behave on many classes of problems very properly: they give sharp estimates not only as $h \to 0$, but for particular finite h, too.

The automatic *hp*-adaptivity gives many h as well p possibilities for the next step of the solution process. A single number provided by the a posteriori error estimator for each mesh element may not be enough information for the decision. This is the reason for using the *reference solution* (*computational error estimate*), which is employed as the standard approach in solving ordinary differential equations.

Reference solutions (see, e.g., [20]) are approximations of the exact solution that are substantially more accurate than the current finite element approximation itself. They include the global refinement of the mesh geometry as well as the general increase of the polynomial degree. They serve for the *hp*-adaptive construction of the next approximate solution.

The computation of the reference solution is time-consuming but need not be carried out at each step of the adaptive solution process. The reference solution is obtained by the same software that is used to compute the approximate solution. We use reference solutions as robust error estimators to guide the adaptive (in particular, h p-adaptive) strategies. They are also a good means in constructing multilevel solution procedures.

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References

[1] M. Ainsworth, J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, New York, 2000.
[2] I. Babuška, W.C. Rheinboldt, A posteriori error estimates for the finite element method, Internat. J. Numer. Methods Eng. 12 (1978) 1597–1615.

[3] I. Babuška, W.C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal. 15 (1978) 736–754.
[4] I. Babuška, T. Strouboulis, The Finite Element Method and Its Reliability, Clarendon Press, Oxford, 2001.
[5] R.E. Bank, K. Smith, A posteriori error estimates based on hierarchical bases, SIAM J. Numer. Anal. 30 (1993) 921–935.
[6] R.E. Bank, A. Weiser, Some a posteriori error estimators for elliptic partial differential equations, Math. Comp. 44 (1985) 283–301.
[7] C. Bernardi, B. Métivet, R. Verfürth, Analyse numérique d'indicateurs d'erreur, Report 93025, Université Pierre et Marie Curie, Paris, 1993.
[8] C. Carstensen, S.A. Funken, Constants in Clément-interpolation error and residual based a posteriori error estimates in finite element methods, East-West J. Numer. Math. 8 (2000) 153–175.
[9] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.
[10] L. Demkowicz, Computing with *hp*-Adaptive Finite Elements, vols. 1 and 2, Chapman & Hall/CRC, Boca Raton, FL, 2007/2008.
[11] L. Demkowicz, W. Rachowitz, P. Devloo, A fully automatic *hp*-adaptivity, J. Sci. Comput. 17 (2002) 117–142.
[12] J. Karátson, S. Korotov, Sharp Upper Global A Posteriori Error Estimates for Nonlinear Elliptic Variational Problems, Research Report A527, Institute of Mathematics, Helsinki University of Technology, Espoo, 2007.
[13] S. Korotov, Two-sided a posteriori error estimates for linear elliptic problems with mixed boundary conditions, Appl. Math. 52 (2007) 235–249.
[14] S. Korotov, Global a posteriori error estimates for convection–reaction–diffusion problems, Appl. Math. Model. 32 (2008) 1579–1586.
[15] S.G. Mikhlin, Constants in Some Inequalities of Analysis, John Wiley & Sons, Chichester, 1986.
[16] P.K. Moore, A posteriori error estimation with finite element semi- and fully discrete methods for nonlinear parabolic equations in one space dimension, SIAM J. Numer. Anal. 31 (1994) 149–169.
[17] P. Neittaanmäki, S. Repin, Reliable Methods for Computer Simulation: Error Control and A Posteriori Estimates, Elsevier, Amsterdam, 2004.
[18] C. Schwab, *p*- and *hp*-Finite Element Methods, Clarendon Press, Oxford, 1998.
[19] K. Segeth, A posteriori error estimation with the finite element method of lines for a nonlinear parabolic equation in one space dimension, Numer. Math. 83 (1999) 455–475.
[20] P. Šolín, K. Segeth, I. Doležel, Higher-Order Finite Element Methods, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[21] L. Tobiska, R. Verfürth, Analysis of a streamline diffusion finite element method for the Stokes and Navier–Stokes equations, SIAM J. Numer. Anal. 33 (1996) 107–127.
[22] T. Vejchodský, Fully discrete error estimation by the method of lines for a nonlinear parabolic problem, Appl. Math. 48 (2003) 129–151.
[23] T. Vejchodský, Guaranteed and locally computable a posteriori error estimate, IMA J. Numer. Anal. 26 (2006) 525–540.
[24] R. Verfürth, A posteriori error estimation and adaptive mesh-refinement techniques, J. Comput. Appl. Math. 50 (1994) 67–83.
[25] R. Verfürth, A Review of A Posteriori Error Estimation and Adaptive Mesh Refinement Techniques, John Wiley & Sons, Chichester, and B.G. Teubner, Stuttgart, 1996.
[26] R. Verfürth, A posteriori error estimates for nonlinear problems: $L^r(0, T; W^{1,\rho}(\Omega))$-error estimates for finite element discretizations of parabolic equations, Numer. Methods Part. Diff. Eq. 14 (1998) 487–518.
[27] M. Vohralík, A posteriori error estimates for lowest-order mixed finite element discretizations of convection–diffusion–reaction equations, SIAM J. Numer. Anal. 45 (2007) 1570–1599.
[28] M.F. Wheeler, J.R. Whiteman, Superconvergent recovery of gradients on subdomains from piecewise linear finite element approximations, Numer. Methods Part. Diff. Eq. 3 (1987) 357–374.
[29] O.C. Zienkiewicz, J.Z. Zhu, A simple error estimator and adaptive procedure for practical engineering analysis, Internat. J. Numer. Methods Eng. 24 (1987) 337–357.