Weighted A Posteriori Error Control in FE Methods

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Summary. The conventional strategy for controlling the error in finite element (FE) methods is based on a posteriori estimates for the error in the global energy or L^2 norm involving local residuals of the computed solution (error indicators). Such estimates are derived via duality arguments where the approximation properties of the finite element space enter through *local* interpolation constants while the stability of the (linearized) dual problem is usually expressed in terms of a *global* stability constant. The mesh refinement process is then oriented at the equilibration of the local error indicators. However, meshes generated in this way via controlling the error in a global norm may not be appropriate for local error quantities like point values or contour integrals. In this note a refined approach to residual–based error estimation is proposed where certain quantities involving the dual solution are used as weight-factors in order to capture information about the local error propagation. In this way "optimal" meshes may be generated for various types of error measures. This is illustrated at simple model cases. One practical application of this approach is, for example, the computation of drag and lift coefficients of blunt bodies in viscous flows governed by the Navier-Stokes equations. [1]

1 Introduction

We illustrate our objective at a simple model situation. Consider the Dirichlet problem of the Laplace operator posed on a bounded domain $\Omega \subset \mathbb{R}^2$,

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

We discretize (1) by a standard finite element Galerkin method using piecewise linear or bilinear shape functions on meshes $\mathbb{T}_h = \{K\}$ which are supposed to satisfy the usual regularity conditions (see, e.g., [7]). For each $\mathbb{T}_h$, let $h = \max_{K\in\mathbb{T}_h} h_K$, where $h_K = \mathrm{diam}(K)$. Using the notation $V_h \subset V = H_0^1(\Omega)$ for the corresponding finite element subspaces and $(\cdot,\cdot)$ for the L^2 inner product on Ω, the approximate problems to (1) read as follows:

$$U \in V_h: \quad (\nabla U, \nabla\varphi) = (f,\varphi) \quad \forall\varphi \in V_h. \tag{2}$$

Subtracting (2) from the variational formulation of (1) results in the orthogonality relation

$$(\nabla(u-U), \nabla\varphi) = 0 \quad \forall\varphi \in V_h. \tag{3}$$

[1] This work has been supported by the German Research Association (DFG) through the SFB 359 "Reactive Flow, Diffusion and Transport" at the Universität Heidelberg.

This discretization allows for optimal-order a priori estimates in the energy and L^2 norm,

$$\|u - U\| + h\|\nabla(u - U)\| \le c\, h^2 \|\nabla^2 u\|, \tag{4}$$

and also with respect to the maximum norm (for references see, e.g., [7])

$$|\log(h)|^{-1}\|u - U\|_\infty + h\|\nabla(u - U)\|_\infty \le c\, h^2 \|\nabla^2 u\|_\infty. \tag{5}$$

For error control in the energy or L^2 norm, one proceeds as follows. Using the Galerkin orthogonality (3) and integration by parts on each element K yields the identity

$$(\nabla e, \nabla z) = \sum_{K \in \mathbb{T}_h} \left\{ (f + \Delta U, z - \bar{z})_K - \frac{1}{2}([\partial_n U], z - \bar{z})_{\partial K} \right\}, \tag{6}$$

for the error $e = u - U$ and any $z \in V$, where $[\partial_n U]$ denotes the jump of $\partial_n U$ across the element boundary, and $\bar{z} \in V_h$ is a suitable approximation of z. By the local approximation properties of finite elements it follows that

$$|(\nabla e, \nabla z)| \le C_I \left(\sum_{K \in \mathbb{T}_h} \left\{ h_K^{r+1} \|f + \Delta U\|_K + \frac{1}{2} h_K^{r+1/2} \|[\partial_n U]\|_{\partial K} \right\}^2 \right)^{1/2} \|\nabla^{r+1} z\|, \tag{7}$$

for $z \in H^{r+1}(\Omega)$, $r = 0$ or $r = 1$. The constant C_I ("interpolation constant") is of size $C_I \approx 1$. Taking the supremum over $z \in V$, or over $z \in V \cap H^2(\Omega)$, one obtains the following a posteriori error bound in the energy norm, for $r = 0$, and in the L^2 norm, for $r = 1$,

$$\|\nabla^{1-r} e\| \le \eta(U) := C_S C_I \Big(\sum_{K \in \mathbb{T}_h} \eta_K^2 \Big)^{1/2}, \tag{8}$$

with the local *error indicators*

$$\eta_K := h_K^{r+1} \|f + \Delta U\|_K + \frac{1}{2} h_K^{r+1/2} \|[\partial_n U]\|_{\partial K}. \tag{9}$$

The constant C_S ("stability constant") measures the stability properties of the "dual" problem

$$z \in V \cap H^{r+1}(\Omega): \quad (\nabla \varphi, \nabla z) = (\nabla^{1-r} e, \nabla^{1-r} \varphi) \quad \forall \varphi \in V, \tag{10}$$

in terms of the global a priori estimate (trivial for $r = 0$)

$$\|\nabla^{r+1} z\| \le C_S \|\nabla^{1-r} e\|. \tag{11}$$

For earlier work on a posteriori error estimation of this type, we refer to the pioneering paper of Babuška and Rheinboldt [1], and also to the survey paper of Verfürth [18].

On the basis of the a posteriori error estimate (8), the mesh generation aims to equilibrate the local error indicators η_K, i.e., for a given tolerance eps, the elements $K \in \mathbb{T}_h$ are refined (or coarsened) according to the criterion

$$\eta_K \approx \frac{eps}{C_S C_I N^{1/2}}, \qquad N = \#\{K \in \mathbb{T}_h\}. \tag{12}$$

Since N depends on the result of the refinement, this strategy is implicit and would need iteration. However, it is common practice to use simply the value of N from the preceding

refinement level which appears appropriate if the refinement process does not proceed too fast. This strategy is based on the pre-assumption that the error indicators η_K properly describe the dependence of the global error term considered on the mesh size h_K. However, this may not be the case, particularly not for other more local error quantities, since the a posteriori error estimate (8) contains information about the mechanism of error propagation only through the *global* stability constant C_S.

To overcome this deficiency, we propose to carry along the terms involving the dual solution in (6) on each element K as weight–factors,

$$\omega_K \approx h_K^{-r-1}\|z-\bar{z}\|_K, \quad r=0,1. \tag{13}$$

The weights ω_K contain all information about the local approximation properties of the spaces V_h, as well as the local stability properties of the dual solution itself. This will allow us to construct almost optimal meshes for various kinds of error quantities, where "optimal" can mean "*most economical for achieving a prescribed accuracy eps*" or "*most accurate for a given maximal number N of mesh points*".

To illustrate this idea, let Ω be the unit circle and the quantity to be computed with best accuracy be the point-value $u(0)$. It is not likely that the meshes obtained through energy or L^2 error control are optimal for this purpose. To do better, we have first to derive an a posteriori error estimate for the pointwise error. This can again be done by a duality argument. Let $g \in W_0^{1,2-\epsilon}(\Omega)$ be the Green function corresponding to the origin, i.e., the solution of the "dual" problem

$$(\nabla\varphi, \nabla g) = \varphi(0) \quad \forall \varphi \in V \cap C(\bar{\Omega}). \tag{14}$$

Then, we have

$$e(0) = (\nabla e, \nabla(g-\bar{g})), \tag{15}$$

with the nodal interpolant $\bar{g} \in V_h$, and

$$|e(0)| = \left| \sum_{K\in\mathbb{T}_h} \left\{ (f+\Delta U, g-\bar{g})_K - \frac{1}{2}([\partial_n U], g-\bar{g})_{\partial K} \right\} \right| \leq \sum_{K\in\mathbb{T}_h} \omega_K \eta_K, \tag{16}$$

with the error indicators η_K and the weights ω_K defined by

$$\eta_K := h_K^2\|f+\Delta U\|_K + \frac{1}{2}h_K^{3/2}\|[\partial_n U]\|_{\partial K},$$
$$\omega_K := \max\{h_K^{-2}\|g-\bar{g}\|_K, h_K^{-3/2}\|g-\bar{g}\|_{\partial K}\}.$$

Consequently, observing that in this case $g(x) = \frac{1}{2\pi}\log(|x|)$, the estimator takes the form

$$|e(0)| \approx C_I \sum_{K\in\mathbb{T}_h} \frac{h_K^{1/2}}{\sigma_K^2}\eta_K, \tag{17}$$

where $\sigma_K := \max(\mathrm{dist}(K,0), h_K)$.

The meshes generated for the particular solution $u = \sin(\pi(2x_1+x_2+2))$ by the equilibration strategy based on the the weighted estimator (17) compared to the global H^1 or L^2 estimators

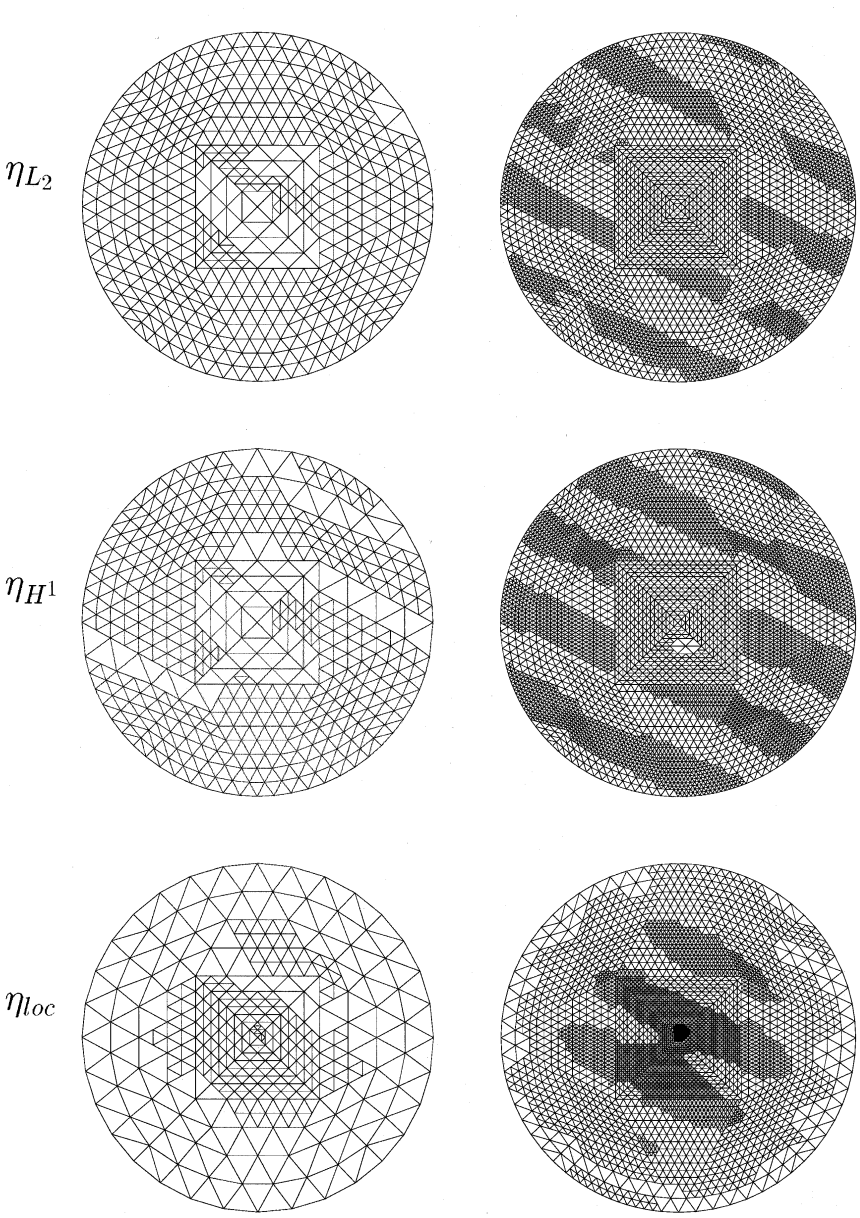

Figure 1: Refined meshes for computing $u(0)$ for $u = \sin(\pi(2x_1 + x_2 + 2))$, via the energy and L^2 error estimators η_{L^2} and η_{H^1} compared to the point error estimator η_{loc}

(8) are shown in Figure 1. The corresponding errors for a sequence of refinement steps are listed in Table 1. In this case the weighted a posteriori estimate (17) for the point error is asymptotically optimal. In comparison, computing the weights ω_K via second-order difference quotients from an approximation to the Green function calculated on the same mesh does not essentially effect the quality of the estimator, as shown in Table 2. From the structure of the

η_{H^1}			η_{L_2}			η_{loc}		
i	# n	$e(0)$	i	# n	$e(0)$	i	# n	$e(0)$
1	556	$3.857-2$	1	493	$3.811-2$	1	262	$3.764-2$
2	781	$1.749-2$	2	682	$2.568-2$	2	331	$3.101-2$
3	982	$1.386-2$	3	895	$2.411-2$	3	598	$7.269-3$
4	1453	$1.449-2$	4	982	$1.386-2$	4	1189	$9.956-4$
5	2272	$1.469-2$	5	1024	$1.477-2$	5	2014	$2.880-3$
6	3022	$9.625-3$	6	1810	$1.465-2$	6	3232	$4.965-3$
7	3637	$5.744-3$	7	2476	$1.406-2$	7	4258	$2.058-3$
8	4732	$5.174-3$	8	3028	$1.052-2$	8	6628	$7.357-4$
9	8536	$4.839-3$	9	3712	$7.944-3$	9	7537	$3.755-4$

Table 1: Results obtained for computing $u(0)$ for $u = \sin(\pi(2x_1 + x_2 + 2))$, via the energy and L^2 error estimators η_{H^1} and η_{L^2} compared to the point error estimator η_{loc}

mesh obtained for our sample solution we see that one would hardly have come up with such a mesh by mere heuristic reasoning. The above concept for deriving a posteriori error estimates

L	# n	$e(0)$	η_{loc}	$\eta_{loc}/e(0)$	$\tilde{\eta}_{loc}$	$\tilde{\eta}_{loc}/e(0)$
1	64	$1.195-1$	$1.380-1$	1.155	$9.705-1$	8.122
2	256	$5.429-2$	$5.950-2$	1.096	$3.429-1$	6.317
3	1024	$1.995-2$	$2.130-2$	1.067	$8.789-2$	4.405
4	4096	$6.517-3$	$6.855-3$	1.052	$2.084-2$	3.197
5	16384	$1.997-3$	$2.082-3$	1.043	$4.822-3$	2.415
6	65536	$5.899-4$	$6.111-4$	1.036	$1.102-3$	1.869

Table 2: Comparison between exact (η_{loc}) and numerical evaluation ($\tilde{\eta}_{loc}$) of the weights ω_K for a sequence of uniformly refined meshes

via a duality argument directly generalizes to the case that the quantity to be computed is given in terms of an arbitrary linear functional $J(\cdot)$ defined on the space V (or on a suitable subspace containing the finite element space V_h and the exact solution u). Other relevant cases are, for instance, the mean value of u over Ω (torsion moment), the value of its gradient $\nabla u(a)$ at some point $a \in \bar{\Omega}$ (stress values), or the line integral of some directional derivative $\beta \cdot \nabla u$ over the boundary $\partial\Omega$ (total surface tension),

$$J(\varphi) = \int_\Omega \varphi(x)\,dx, \qquad J(\varphi) = \nabla\varphi(a), \qquad J(\varphi) = \int_{\partial\Omega} \beta \cdot \nabla\varphi\,ds. \tag{18}$$

Weighted a posteriori error estimate for the energy or L^2 norm can be obtained within this framework by taking the error functional

$$J(\varphi) = \|\nabla^r e\|^{-1} \int_\Omega \nabla^r\varphi\nabla^r e\,dx, \tag{19}$$

for $r = 0$ or $r = 1$, respectively. In this case, the evaluation of the weight factors ω_K, i.e., the computation of the dual solution z, requires at first an approximation of the functional $J(\cdot)$. This may be achieved by replacing the unknown error e by some approximation $\tilde{e} = \tilde{U} - U$ obtained by extrapolation from two consecutively refined meshes.

2 A general approach towards residual-based error control

We describe a general approach to residual-based error control for abstract variational problems which follows the paradigm proposed in [8], [9], and also in [13]. Let V be a Hilbert space with inner product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$, and $a(\cdot;\cdot)$ a semi-linear form. We seek a solution to the abstract variational problem

$$u \in V: \quad a(u;\varphi) = 0 \quad \forall \varphi \in V. \tag{20}$$

This problem is approximated by a Galerkin method using a sequence of finite dimensional subspaces $V_h \subset V$ parameterized by a discretization parameter h, such that

$$\inf_{\varphi \in V_h} \|u - \varphi\| \to 0 \quad (h \to 0). \tag{21}$$

The discrete problems read

$$U \in V_h: \quad a(U;\varphi) = 0 \quad \forall \varphi \in V_h. \tag{22}$$

With the Fréchet derivative a' of a, we have the following orthogonality relation for the error $e = u - U$:

$$\int_0^1 a'(U + te; e, \varphi)\, dt = a(u;\varphi) - a(U;\varphi) = 0 \quad \forall \varphi \in V_h. \tag{23}$$

This suggests to use the bilinear form

$$L(u, U; \varphi, z) = \int_0^1 a'(U + te; \varphi, z)\, dt \tag{24}$$

in the duality arguments. Suppose that the quantity $J(u)$ has to be computed, where J is a linear functional defined on V, or on a suitable subspace containing the ansatz spaces V_h and the solution u. For representing the error $J(e)$, we use the duality argument

$$z \in V: \quad L(u, U; \varphi, z) = J(\varphi) \quad \forall \varphi \in V. \tag{25}$$

Assuming that this problem has a unique solution, and using the Galerkin orthogonality (23), we have the error representation

$$J(e) = L(u, U; e, z - \bar{z}), \tag{26}$$

with some approximation $\bar{z} \in V_h$. The goal is to evaluate the right hand side of (26) numerically, in order to get a criterion for the local adjustment of the discretization. For the further discussion, we need to become more specific about the setting of the above problem. Let the variational problem (20) originate from a semi-linear second-order partial differential equation of the from

$$A(u)u = -\sum_{i,j=1}^{d} \partial_i \{a_{ij}(\cdot, u)\partial_j u\} + \sum_{j=1}^{d} a_{0j}(\cdot, u)\partial_j u + a_{00}(\cdot, u)u = f, \tag{27}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$, with, for simplicity, homogeneous Dirichlet boundary conditions, $u|_{\partial\Omega} = 0$. Hence, the natural solution space is $V = H_0^1(\Omega)$. The following discussion assumes (27) to be a scalar equation, but everything directly carries over to systems. Accordingly, the semi-linear form a and its linearization a' have their natural meaning. In this setting, the error representation (26) has the following concrete form:

$$J(e) = \sum_{K \in \mathbb{T}_h} \left\{ (f - A(U)U, z - \bar{z})_K - \frac{1}{2}([\partial_n^A U], z - \bar{z})_{\partial K} \right\}, \tag{28}$$

where $\partial_n^A = \sum_{i,j=1}^d n_i a_{ij}(\cdot, U)\, \partial_j$. To evaluate this formula, one may replace the unknown solution u in the bilinear form $L(u, U; \cdot, \cdot)$ by the computed approximation U, and solve the corresponding perturbed dual problem by the same method as used in computing U, yielding an approximation $Z \in V_h$ to the exact dual solution z,

$$Z \in V_h : \quad L(U, U; \varphi, Z) = J(\varphi) \quad \forall \varphi \in V_h. \tag{29}$$

Then, one may try to evaluate (28) directly by replacing the local errors $z - \bar{z}$ by $\tilde{Z} - \bar{Z}$, where $\tilde{Z}$ is some local higher-order interpolant of Z and $\bar{Z}$ its interpolation in V_h. We prefer instead to convert (28) into an estimate,

$$|J(e)| \leq \sum_{K \in \mathbb{T}_h} \omega_K \left\{ h_K^2 \|f - A(U)U\|_K + \frac{1}{2} h_K^{3/2} \|[\partial_n^A U]\|_{\partial K} \right\}, \tag{30}$$

with the weights

$$\omega_K := \max \left\{ h_K^{-2} \|z - \bar{z}\|_K, h_K^{-3/2} \|z - \bar{z}\|_{\partial K} \right\}. \tag{31}$$

By the usual approximation properties of finite elements there holds

$$\omega_K \leq C_I \|\nabla^2 z\|_K, \tag{32}$$

where the interpolation constant C_I may be assumed to be of size $C_I \approx 1$. Estimates for the weights ω_K may again be obtained from the approximate dual solution Z simply by taking appropriate second-order difference quotients,

$$\omega_K \approx |K|^{1/2} |D_h^2 Z(x_K)|, \quad x_K \text{ midpoint of } K. \tag{33}$$

We can give a heuristic interpretation of the meaning of the resulting a posteriori error estimate. We expect that the weights ω_K converge to certain limits, as $h \to 0$, in the sense

$$|K|^{-1/2} \omega_K \to |D^2 z(x_K)|, \tag{34}$$

where x_K denotes a point contained in an infinite sequence of nested elements K. Further, the quantity $h_K^{-1}[\partial_n^A U]$ can be viewed as a second-order difference quotient of U. Hence it may be expected that, for a sequence of properly refined meshes, the residual terms in (30) also converge to certain limits as $h \to 0$,

$$|K|^{-1/2} \|f - A(U)U\|_K \to |(f - A(u)u)(x_K)| = 0, \tag{35}$$

$$|\partial K|^{-1/2} \|h_K^{-1} [\partial_n^A U]_{\partial K}\| \to |D_A^2 u(x_K)|. \tag{36}$$

In this sense, the a posteriori error estimate (30) asymptotically, for $h \to 0$, takes the form

$$|J(e)| \approx \sum_{K \in \mathbb{T}_h} h_K^4 |D^2 z(x_K)| |D_A^2 u(x_K)|. \tag{37}$$

This tells us that the variation of the error quantity $J(e)$ with respect to changes of the local mesh width h_K is essentially determined by the dual solution.

3 Application to the Navier-Stokes equations

Weighted a posteriori error estimates may be useful in solving certain problems in fluid mechanics. As an example we consider the computation of the drag force or a pressure difference for the viscous incompressible (stationary) flow around a cylinder, as shown in Figure 2. This is one of a set of benchmark problems discussed in [15]. The governing model are the two-dimensional stationary Navier-Stokes equations

$$-\nu\Delta u + u\cdot\nabla u + \nabla p = f, \quad \nabla\cdot u = 0, \quad \text{in } \Omega, \tag{38}$$

for the velocity u and the pressure p (density $\rho = 1$) on a bounded region Ω. Here, vector functions are also denoted by normal type, and no distinction is made in the notation of the corresponding inner products and norms. At the boundary $\partial\Omega$, the usual no-slip condition is posed along rigid parts together with suitable inflow and free-stream outflow conditions at inlets and outlets, respectively,

$$u|_{\Gamma_{\text{rigid}}} = 0, \quad u|_{\Gamma_{in}} = u_{\text{in}}, \quad (\nu\partial_n u - pn)|_{\Gamma_{out}} = 0. \tag{39}$$

The quantities to be computed are the drag force,

$$J_{\text{drag}}(u,p) = \int_S \{\nu\partial_n u_t n_y - pn_x\}\, ds \tag{40}$$

(u_t denoting the tangential component of u), and the pressure difference between the front point a_{front} and back point a_{back} of the cylinder,

$$J_{\Delta p}(u,p) = p(a_{\text{front}}) - p(a_{\text{back}}). \tag{41}$$

For discretizing this problem, we use a finite element method based on the triangular P1/P1-

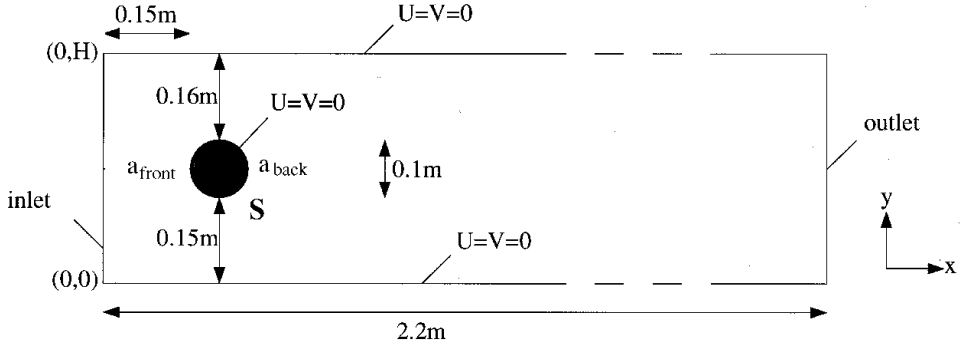

Figure 2: Geometry of the cylinder flow problem in 2D

or the (isoparametric) quadrilateral Q1/Q1-Stokes element. In this case the shape functions for velocity and pressure are both piecewise linear or (isoparametric) bilinear and continuous (see, e.g., [10]). For treating the velocity–pressure coupling as well as the convective term consistent

least–squares–stabilization á la Hughes, Franca and Balestra [11] is employed. In order to allow for easy local refinement and coarsening, one "hanging" node (denoted by circle) is allowed on each edge, as shown in Figure 3. The unknowns corresponding to these fictitious nodes are eliminated. The ansatz spaces for the velocity are denoted by $V_h \subset V$ and those for the pressure

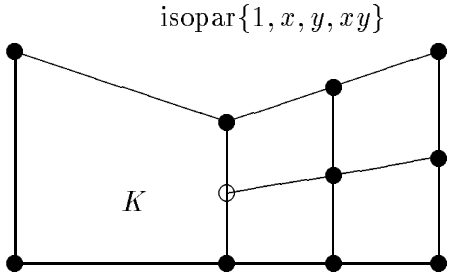

Figure 3: Structure of the Q1/Q1-Stokes element with a hanging node

by $L_h \subset L^2(\Omega)$. Then, with a suitable finite element approximation $U_{in} \approx u_{in}$, the discrete problems seek to determine $U \in V_h + U_{in}$, $P \in L_h$, such that

$$\begin{aligned}(U \cdot \nabla U, \varphi) + \nu(\nabla U, \nabla\varphi) - (P, \nabla\cdot\varphi) + \sum_K \delta_K (U\cdot\nabla U - \nu\Delta U + \nabla P, U\cdot\nabla\varphi)_K \\ = (f,\varphi) + \sum_K \delta_K (f, U\cdot\nabla\varphi)_K \qquad (42)\\ (\nabla\cdot U,\xi) + \sum_K \delta_K (U\cdot\nabla U - \nu\Delta U + \nabla P, \nabla\xi)_K = \sum_K \delta_K (f,\nabla\xi)_K,\end{aligned}$$

for all $\{\varphi,\xi\} \in V_h \times L_h$, with appropriate parameters $\delta_K = \delta_K(U) \sim h_K \min(h_K/\nu, 1/|U|)$.

This nonlinear problem is solved by an outer fixed point iteration. In each step the resulting linear quasi-Stokes problems

$$\begin{aligned}(\bar U \cdot \nabla U, \varphi) + \nu(\nabla U, \nabla\varphi) - (P, \nabla\cdot\varphi) + \sum_K \bar\delta_K (\bar U\cdot\nabla U - \nu\Delta U + \nabla P, \bar U\cdot\nabla\varphi)_K \\ = (f,\varphi) + \sum_K \bar\delta_K (f, \bar U\cdot\nabla\varphi)_K \qquad (43)\\ (\nabla\cdot U,\xi) + \sum_K \bar\delta_K (\bar U\cdot\nabla U - \nu\Delta U + \nabla P, \nabla\xi)_K = \sum_K \bar\delta_K (f,\nabla\xi)_K,\end{aligned}$$

for all $\{\varphi,\xi\} \in V_h \times L_h$, with $\bar U$ being the approximation from the preceding iteration and $\bar\delta_K = \delta_K(\bar U)$, are solved by a multigrid method employing a finite element analogue of the Vanka smoother (see [16] and [2]). Since local defect computation and smoothing is used within the mesh refinement process, this algorithm has optimal complexity. For the details of the implementation see [3] and [4]. In this discretization the analogue of the global energy a posteriori error bound (8) reads

$$\begin{aligned}\|\nabla(u-U)\| + \|p-P\| \le C_I C_S \Big[\sum_{K\in\mathbb{T}_h} \Big\{ h_K^2 \|f + \nu\Delta U - U\cdot\nabla U - \nabla P\|_K^2 + \|\nabla\cdot U\|_K^2 \\ + \nu h_K \|[\partial_n U]\|_{\partial K}^2 + \delta_K \|f + \nu\Delta U - U\cdot\nabla U - \nabla P\|_K^2 \Big\}\Big]^{1/2}. \qquad (44)\end{aligned}$$

Estimates of this type have been proven by Verfürth [17] for the Stokes problem, and by Johnson and Rannacher [12] and Bernardi et al. [6] for the Navier-Stokes problem.

Corresponding *weighted* a posteriori error estimates can be obtained following the argument described in the preceding section. In computing the drag force, the (approximate) dual problem reads

$$\nu(\nabla\varphi,\nabla z)+(U\cdot\nabla\varphi,z)-(q,\nabla\cdot\varphi)-(\nabla\cdot z,\xi)=J_{\rm drag}(\varphi,\xi), \tag{45}$$

for all $\{\varphi,\xi\}\in V_h\times L_h$, where

$$J_{\rm drag}(\varphi,\xi)=\int_S\{\nu\partial_n\varphi_t-\xi n_x\}\,ds. \tag{46}$$

Denoting its solution by $\{z,q\}$, the a posteriori estimate for the drag error becomes

$$|J_{\rm drag}(u,p)-J_{\rm drag}(U,P)|\le C_I\sum_{K\in\mathbb{T}_h}\left\{\omega_K^{(1)}\left(\eta_K^{(1)}+\eta_K^{(2)}\right)+\omega_K^{(2)}\eta_K^{(3)}+\omega_K^{(3)}\eta_K^{(4)}\right\} \tag{47}$$

with the local error indicators

$$\begin{aligned}
\eta_K^{(1)}&=h_K^2\|f+\nu\Delta U-U\cdot\nabla U-\nabla P\|_K,\\
\eta_K^{(2)}&=h_K^{3/2}\nu\|[\partial_n U]\|_{\partial K},\\
\eta_K^{(3)}&=h_K\|\nabla\cdot U\|_K,\\
\eta_K^{(4)}&=\delta_K\|f+\nu\Delta U-U\cdot\nabla U-\nabla P\|_K,
\end{aligned}$$

and the weights

$$\begin{aligned}
\omega_K^{(1)}&=\|\nabla^2 z\|_K,\\
\omega_K^{(2)}&=\|\nabla q\|_K,\\
\omega_K^{(3)}&=\|U\cdot\nabla z+\nabla q\|_K.
\end{aligned} \tag{48}$$

The bounds for the dual solution $\{z,p\}$ are obtained computationally by replacing the unknown solution u in the convection term by its approximation U and solving the resulting linearized problem on the same mesh. From the obtained approximate dual solution $\{Z,Q\}$ suitable difference quotients are taken to approximate the weights $\omega_K^{(i)}$. The interpolation constant may again be determined analytically or simply taken as $C_I=1$. The implementation of the above strategy for error control via weighted a posteriori error estimates for the benchmark problem "flow around a cylinder" is under work. In a preliminary version of the code a more heuristic strategy for the mesh refinement based on the global error bound (44) has been used. Since the weights corresponding to the error in the drag force as well as those for the pressure difference are expected to become large in the neighborhood of the cylinder contour S, in each refinement step additionally all elements adjacent to S are refined. Table 3 shows the corresponding results for the pressure difference computed on 1) hierarchically refined meshes, starting from a coarse mesh with almost uniform width, 2) hierarchically refined meshes, starting from a coarse mesh which is refined towards the cylinder contour, and 3) adaptively generated meshes starting from the coarse mesh in 1). These results demonstrate clearly the superiority of the adaptive algorithm,

as it produces an error of less than 1% already after 6 refinement steps on a mesh with about 20000 unknowns in about 50 seconds while the other algorithms need at least 75000 unknowns and 200 seconds to achieve the same accuracy. Further, it is obvious that during the refinement process the complexity of the adaptive algorithm scales linearly with the number of unknowns.

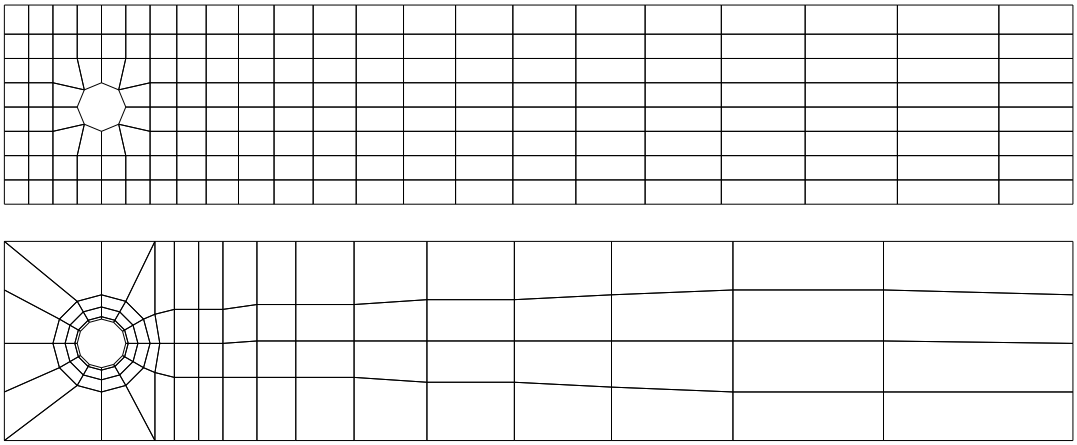

Figure 4: Almost uniform and pre-adapted mesh

	Uniform Refinement, Grid1		
i	$\#n$	Δp	*sec.*
1	2268	0.109389	4.0
2	8664	0.110513	15.2
3	33840	0.113617	61.7
4	133728	0.115488	254.2
5	531648	0.116486	1001.1
	Uniform Refinement, Grid2		
1	1296	0.106318	3.3
2	4896	0.112428	11.7
3	19008	0.115484	46.9
4	74880	0.116651	195.1
5	297216	0.117098	796.2

	Adaptive Refinement, Grid1		
i	$\#n$	Δp	*sec.*
1	825	0.105085	2.2
2	1362	0.105990	2.9
3	2616	0.111482	5.4
4	5334	0.113978	11.3
5	10281	0.115794	23.4
6	21546	0.116915	51.6
7	40584	0.117253	108.2
8	86259	0.117379	233.0
9	164046	0.117499	406.9
10	330930	0.117530	830.4

Table 3: Results of the cylinder flow computations for various types of mesh refinements, (reference value $\Delta p = 0.1175\ldots$)

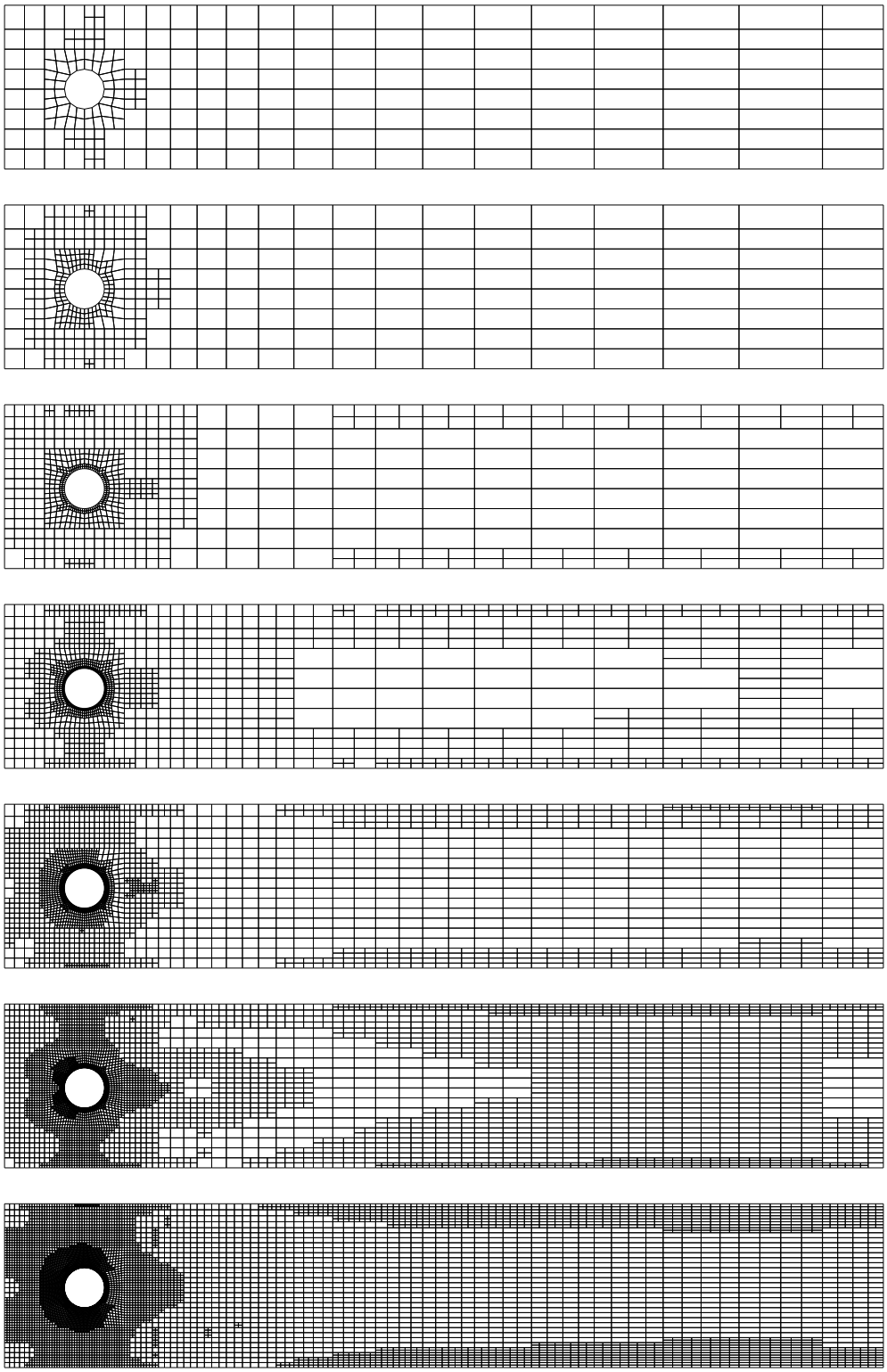

Figure 5: A sequence of successively refined meshes for the cylinder flow problem

4 Further examples

In the following, we present a partial theoretical justification of the heuristic refinement strategy proposed for computing the drag force for the cylinder flow considered in the preceding section. This is done for the model situation of Section 1, Poisson equation on the unit circle $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$. Let us aim to compute the line integral

$$J(u) = \int_{\partial\Omega} \partial_n u \, ds. \tag{49}$$

In virtue of the pointwise a priori error estimate (5), we have $J(e) = O(h)$, so that convergence is no problem. The question is how the mesh refinement process should be organized in order to obtain the quantity $J(u)$ to best accuracy. We will see that the answer is somewhat surprising and seems, at the first glance, to conflict with what one is used to from a finite element method.

Following the above approach for deriving a posteriori error estimates, we would like to use a duality argument where the functional $J(\cdot)$ is taken as right hand side. This causes a complication since $J(\cdot)$ is not contained in the dual space of $H_0^1(\Omega) \cap C(\bar{\Omega})$, so that the corresponding dual solution would be only a measure. In fact, this measure is related to a density function z, satisfying $z = -1$ in Ω, and $z = 0$ on $\partial\Omega$. In order to avoid dealing with measures, we regularize the functional $J(\cdot)$ as follows:

$$J_\epsilon(\varphi) = \frac{1}{\epsilon} \int_{1-\epsilon \le |x| \le 1} \partial_n \varphi \, dx, \quad 0 < \epsilon << 1. \tag{50}$$

The corresponding dual solution z_ϵ is given by

$$z_\epsilon = \begin{cases} -1 & , \quad 0 \le r < 1 - \epsilon, \\ -(1-r)/\epsilon & , \quad 1 - \epsilon \le r < 1. \end{cases} \tag{51}$$

This implies that the weights $\omega_K \approx \|\nabla^2 z_\epsilon\|_K$ in the a posteriori error estimate (30) are all zero on elements K outside a boundary strip of width ϵ, i.e., there is no contribution to the error $J(e)$ from those interior elements. This observation remains true if we let $\epsilon \to 0$. Therefore, starting from some coarse mesh, the "optimal" refinement strategy would be in each step only to refine those elements which are adjacent to the boundary and to leave the others unchanged. The resulting meshes are maximally refined along the boundary down to a minimal mesh size h_{min}, while in the interior of Ω the elements are kept of width $h_{max} \approx 1$. If only local mesh bisection is allowed in each refinement step, the mesh obtained after k steps has minimal width $h_{min} \approx 2^{-k}$ and consists of $N_k \approx 2^k$ elements. To reach an error of size $|J(e)| \le eps$, about $k \approx \log_2(1/eps)$ refinement cycles are needed leading to an "optimal" mesh with $N \approx 1/eps$ elements. The apparent contradiction in this result to what one is used to from finite element methods can be resolved as follows: The quantity to be calculated in this example can be expressed in terms of an integral evaluation of the data, $J(u) = \int_\Omega f \, dx$, which explains why there is no global error propagation. In turn, to realize the full accuracy on the constructed mesh, the right hand side consisting of integrals over the force f multiplied by the nodal basis functions has to be calculated almost exactly also on the coarse elements.

Figure 6 shows the approximation to the dual solution $z_\epsilon, \epsilon = h_{min}$, computed on the finest mesh obtained after 10 refinement steps. The resulting locally refined meshes obtained by using the exact and the approximate dual solution are shown in Figure 7. Some computational results are listed in Table 4.

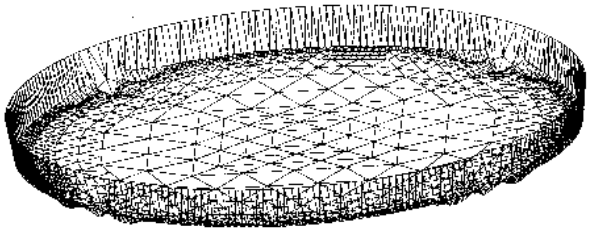

Figure 6: Discrete dual solution for the normal slope error

η_{H^1}			η		
i	# n	$J(e)$	i	# n	$J(e)$
0	256	$5.09+0$	0	256	$5.09+0$
1	1024	$2.91+0$	1	559	$2.90+0$
2	4096	$1.50+0$	2	1294	$1.50+0$
3	16384	$7.49-1$	3	3079	$7.48-1$
4	64768	$3.77-1$	4	7174	$3.62-1$
5	253858	$1.73-1$	5	16738	$1.67-1$
6	*memory*	*exhausted*	6	38146	$6.93-2$
7	*memory*	*exhausted*	7	86128	$2.03-2$

Table 4: Results obtained with the energy norm estimator η_{H^1} and by the "optimal" refinement strategy

Finally, we like to pose an exercise which may be solved by applying the approach described above. Consider the Laplace equation on the slit domain

$$\Omega = \left\{x \in \mathbb{R}^2 : |x| < 1\right\} - \left\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\right\}, \tag{52}$$

with homogeneous Dirichlet boundary conditions. Taking polar coordinates (r, θ) at the origin, the solution to this problem can be written as

$$u(r,\theta) = k_1 r^{1/2} \sin(\theta/2) + \bar{u}(r,\theta), \tag{53}$$

where $\bar{u} \in H^2(\Omega)$. The coefficient k_1 is the so-called "stress intensity factor" which is of particular interest in structural mechanics. It can be expressed in the form (see [14]).

$$k_1(u) = \frac{1}{\pi} \int_\Omega \left\{ f(x) r^{-1/2} \sin(\theta/2) + u(x)\, \Delta(r^{-1/2} \sin(\theta/2)\tau(x)) \right\} dx, \tag{54}$$

with the smooth cut-off function $\tau(x) = \exp(-1/(1-x^2))e$. We pose the question of how a most economical mesh should look like for computing k_1 with best accuracy. Setting

$$J(u) = \int_\Omega u(x)\, \Delta(r^{-1/2} \sin(\theta/2)\tau(x))\, dx, \tag{55}$$

the optimal degree of mesh refinement is determined through the solution of the corresponding dual problem according to the a posteriori error estimate (30). The further discussion is left to the reader.

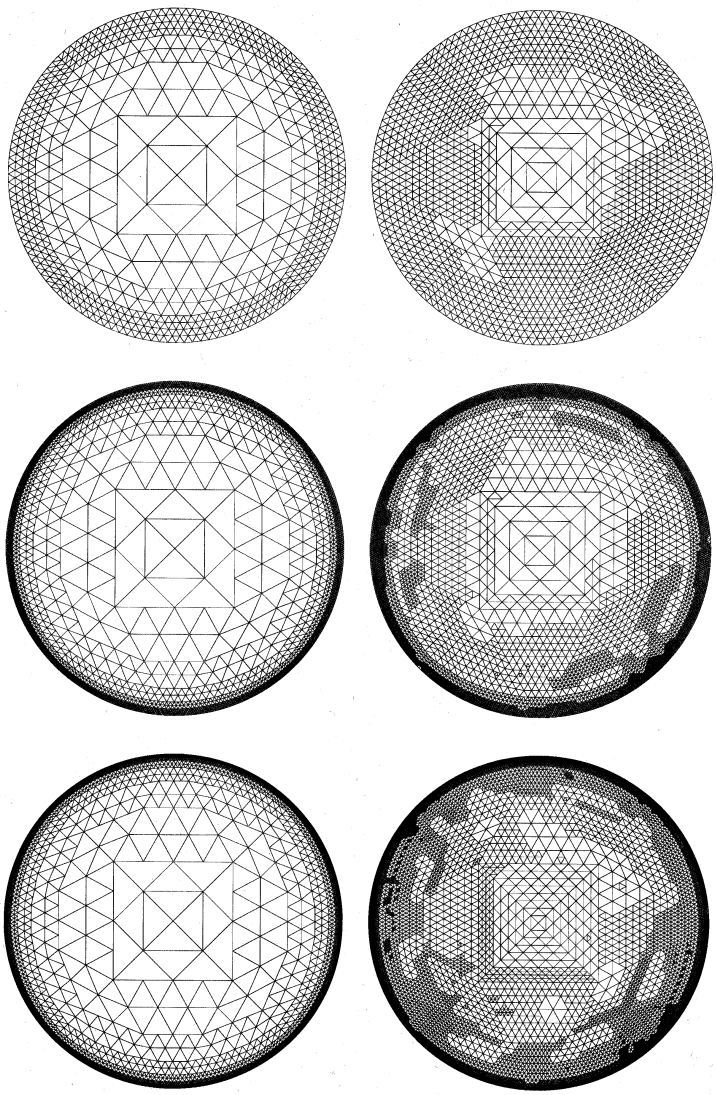

Figure 7: Grids obtained using the exact dual solution (left) and its discrete approximation (right)

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