MTH 659, Notes on Aubin-Nitsche method

Recall the following assumptions on the data of the variational problem, set in a Hilbert space V with a norm $\| \cdot \|_V$: find

$$(\mathbf{V}) u \in V : a(u,v) = F(v), \quad \forall v \in V \tag{1}$$

where a is a bilinear, (symmetric), continuous and V-elliptic form on $V \times V$, and F is a linear, continuous functional on V.

Specifically, we assume that there are constants which depend on a and Ω such that

$$a(v,v) \geq \alpha \parallel v \parallel_V^2, \ \forall v \in V \tag{2}$$
$$|a(v,w)| \leq C_a \parallel v \parallel_V \parallel w \parallel_V, \ \forall v,w \in V \tag{3}$$
$$a(v,w) = a(w,v), \ \forall v,w \in V \tag{4}$$
$$|F(v)| \leq C_F \parallel v \parallel_V, \ \forall v \in V \tag{5}$$

Also, we denote the energy norm by $\parallel w \parallel_a := \sqrt{a(w,w)}$.

Context: in order to relate the form $a(\cdot,\cdot)$ to the weak form of a second order elliptic problem over some region Ω, we need somehow to incorporate the ∇ into definition of both the form and of the norm in V. In other words, both the energy norm $\| \cdot \|_a$ and the V norm need to be equivalent to the "full" $\| \cdot \|_1$ norm. From another point of view, the $\| \cdot \|_0$ cannot be used as the norm on the set of elements of V, since it does not make it V complete (i.e., a Hilbert space). (In fact, also, the bilinear form $a(u,v) = \int_\Omega cuvdx$, cannot be associated with a second order PDE.)

Example: Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with boundary $\partial\Omega$. Let $V = H_0^1(\Omega)$ with norm $\parallel w \parallel_V := |w|_1 := \sqrt{\int_\Omega (\nabla v)^2 dx}$. (The fact that this is a norm on V follows from Poincaré-Friedrichs inequality

$$\parallel w \parallel_0 \leq C_{PF} |w|_1$$

where the constant C_{PF} depends on the domain Ω. Let

$$a(v,w) = \int_\Omega \nabla v \nabla w dx$$

and

$$F(v) = (f,v)$$

with some given $f \in L^2(\Omega)$.

We find that the appropriate constants in (2), (3), (5) are $\alpha = 1$ (from the def. of the norm, $C_\alpha = 1$ (from Cauchy-Schwarz inequality) and $C_F = C_{PF} \parallel f \parallel_0$.

Exercise: Find appropriate constants in Example above when the norm chosen on V is the full $\| \cdot \|_1$ norm.

Existence/uniqueness/stability result for problem (V) state that under the above assumptions, there exists a unique solution to (V) which satisfies the stability estimate $\| u \|_V \leq \frac{C_F}{\alpha}$.

A finite element problem is: given a finite dimensional subspace V_h of V or of a larger set, find

$$(\mathbf{FE}) u_h \in V_h : a(u_h, v_h) = F(v_h), \ \ \forall v_h \in V_h. \tag{6}$$

Céa's Lemma states that for u solving (V) (1) and u_h solving (FE) (6), we have

$$\| u - u_h \|_a \leq \| u - w_h \|_a, \ \forall w_h \in V_h.$$

It is proven by noticing the orthogonality of the error $e = u - u_h$ to the space V_h in the scalar product induced by the norm $a(\cdot, \cdot)$. Thus for any $w_h \in V_h$ we have $\| u - u_h \|_a^2 = a(u - u_h, u - u_h) = a(u - u_h + u_h - w_h) = a(u - u_h, u - w_h) \leq \| u - u_h \|_a \| u - w_h \|_a$ where the lasi inequality follows form Cauchy-Schwartz inequality applied to the $a(\cdot, \cdot)$ product.

Exercise: Redo the proof (and reconcile the constants) of Céa's Lemma using instead of $\| \cdot \|_a$ the norm $\| \cdot \|_V$ for an abstract case and for the two specific examples of Example .. and Exercise ..

Interpolation results and constants. Assume that Ω is a convex polygonal domain. For linear finite elements (akin to linear spline interpolation) we have the following abstract result, with $\tilde{w}_h := I_h w$ denoting the interpolant of w in the space V_h, assuming $w \in H^2(\Omega)$:

$$\| w - I_h w \|_1 \leq C_{interp} h |w|_2$$

In fact, a better known result is that

$$\| w - I_h w \|_0 \leq C^0_{interp} h^2 |w|_2$$

but this is less frequently used in the finite element analysis even though it gives us "hope" to expect the same order $(O(h^2)$ of approximation of u from finite element solution u_h as the one given by the interpolant $I_h u$.

Basic error estimate. Assume for now that we know constants which relate all the norms on V (the V norm, the energy norm, and the H^1 norm). For example, assume that for any $v \in V$, we have

$$\| v \|_a \leq C_{a \to 1} \| v \|_1$$

and

$$\| v \|_V \leq C_{V \to 1} \| v \|_1$$

Then by combining Céa's Lemma with the interpolation result, we obtain

$$\| e \|_a := \| u - u_h \|_a \leq C_{a \to 1} \| u - u_h \|_1 \leq C_{a \to 1} C_{interp} h |u|_2 \tag{7}$$

Exercise Find the error estimate for $\| u - u_h \|_V$ and $\| u - u_h \|_1$.

Elliptic regularity. Here we assume that Ω is convex and has a smooth boundary. Also, we assume that the coefficients of the bilinear form $a(u, v)$ are smooth. In addition, we only consider problems with homogeneous Dirichlet or homogeneous Neumann boundary conditions. Also, set $F(v) = (f, v)$ with $f \in L^2(\Omega)$, then the solution to (V) (1) satisfies

$$|u|_2 \leq C_{reg} \| f \|_0 \ . \tag{8}$$

For example, if a is such as in Example 1, then the problem is: find u so that $-\nabla^2 u = f, \quad \text{in } \Omega, u|_{\partial\Omega} = 0$, and the regularity estimate tells us how to relate second order (distributional) derivaties of u *other* than the ∇^2 to f.

(Note: the stability results we derived above is weaker as it only provides the bound for $\| u \|_V$.)

Aubin-Nitsche-duality method for deriving *a-priori* estimates of $\| e \|_0$.

Here we need Céa's Lemma, the interpolation results, and the regularity results. Their assumptions have to be satisfied. In other words, for optimal results, we need a "good" domain Ω, "good" coefficients of $a(\cdot, \cdot)$, and a "good" source term f. In the proof below we assume at every step that it is justified (as an exercise, you should retrace and label the steps yourself).

We discuss the *dual* problem V' to V that is, one in which we find $\phi \in V$ such that $a(\phi, w) = (e, w), \ \forall w \in V$.

We know $|\phi|_2 \leq C_{reg} \| e \|_0$.

Also, we know $a(e, w_h) = 0, \ \forall w_h \in V_h$.

We calculate $\| e \|_0^2 = (e, e) = a(\phi, e) = a(e, \phi) = a(e, \phi - I_h\phi)$.

We estimate $a(e, \phi - I_h\phi) \leq \| e \|_a \| \phi - I_h\phi \|_a$.

Next we estimate

$$\| \phi - I_h\phi \|_a \leq C_{a\to 1} C_{interp} h |\phi|_2 \leq C_{a\to 1} C_{interp} C_{reg} h \| e \|_0 \ .$$

Combining these together we get

$$\| e \|_0^2 \leq \| e \|_a \ C_{a\to 1} C_{interp} C_{reg} h \| e \|_0 \ .$$

Finally, the basic error estimate (7) gives another order of h in

$$\| e \|_0 \leq C_{a\to 1} C_{interp} C_{reg} C_{a\to 1} C_{interp} h^2 |u|_2.$$

Note: to make this result **really** *a-priori*, we can replace $|u|_2$ by the bound from regularity result for the original problem (not dual) $C_{reg} \| f \|_0$.

Exercise: Redo the above proof and assume that you know (i) $u \in H^3(\Omega) \cap H_0^1(\Omega)$, V_h contains (complete) piecewise polynomials of degree (at most) k, with a fixed $k \geq 2$. (ii) same but with $k = 1$, (iii) $k \geq 2$, $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Derive the best error estimate you can for $\| e \|_0$. Clearly label all the steps of the proof.

Last word. The interesting fact is that, in general, for finite element spaces (of lower order) to have the optimal interpolation properties, the domain should be polygonal, therefore not smooth. For such a domain, there can be rarely high degree of elliptic regularity and therefore solutions are rarely more than H^2 regular.