Cauchy-Schwarz Inequality

November 16, 2020

[H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations] 中有提到, Cauchy–Schwarz inequality 不需要

$$(x, x) = 0 \Longleftrightarrow x = \theta.$$

本文中的X 均为线性空间.

1 复数的Cauchy–Schwarz inequality

Definition 1.1. 若二元函数$a : X \times X \to \mathcal{K}$ 满足: 对任意$\forall \alpha_1,\ \alpha_2 \in \mathcal{K}$ 和$\forall x,\ x_1,\ x_2, y,\ y_1,\ y_2 \in X$,

(i) $a(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 a(x_1, y) + \alpha_2 a(x_2, y)$;

(ii) $a(x, \alpha_1 y_1 + \alpha_2 y_2) = \overline{\alpha_1} a(x, y_1) + \overline{\alpha_2} a(x, y_2)$.

我们称$a(\cdot, \cdot)$ 为X 上的bilinear form.

Remark 1.2. 简单来说, bilinear form 就是两个变量都是线性的双变量映射.

Theorem 1.3 (Cauchy–Schwarz inequality)**.** 设$a(\cdot, \cdot)$ 为X 上的共轭双线性函数且*for any* $x \in X$,

$$a(x, x) \geq 0,$$

then for any $x, y \in X$,

$$|a(x, y)|^2 \leq a(x, x) a(y, y). \tag{1}$$

Remark 1.4. 注意, $a(x, x) \geq 0$ 能推出$a(x, y) = \overline{a(y, x)}$, 这个结论在证明中会用到. 此外, 由于该Cauchy–Schwarz inequality 减弱了条件, 取等条件变得复杂了许多. 令$\mathcal{X}_0 := \{x \in \mathcal{X} :\ a(x, x) = 0\}$, 则$\mathcal{X}_0$ 是闭线性子空间, 故可定义商空间. 对$\forall x \in \mathcal{X}$,

$$[x] := \{y \in \mathcal{X} :\ x - y \in \mathcal{X}_0\}$$

且

$$\mathcal{X}/\mathcal{X}_0 := \{[x]:\ \forall x \in \mathcal{X}\}.$$

此时我们有, 对$\forall x \in \mathcal{X}$,

$$a(x,x) = 0 \Longleftrightarrow x \in [\theta].$$

且对$\forall x,\ y \in \mathcal{X}$,

$$|a(x,y)|^2 = a(x,x)a(y,y) \Longleftrightarrow [x] \text{ and } [y] \text{ linearly dependent}.$$

Proof of Theorem 1.3. For any $x, y \in X$, if $a(x,x) = a(y,y) = 0$, then

$$\begin{aligned}
0 &\le a(x+y, x+y) = a(x,x) + a(y,y) + 2\mathrm{Re}a(x,y) = 2\mathrm{Re}a(x,y),\\
0 &\le a(x-y, x-y) = a(x,x) + a(y,y) - 2\mathrm{Re}a(x,y) = -2\mathrm{Re}a(x,y),\\
0 &\le a(x+iy, x+iy) = a(x,x) + a(y,y) + 2\mathrm{Im}a(x,y) = 2\mathrm{Im}a(x,y),\\
0 &\le a(x-iy, x+iy) = a(x,x) + a(y,y) - 2\mathrm{Im}a(x,y) = -2\mathrm{Im}a(x,y),
\end{aligned}$$

So $\mathrm{Re}a(x,y) = 0 = \mathrm{Im}a(x,y)$, i.e $a(x,y) = 0$.

$$|a(x,y)|^2 = 0 = a(x,x)a(y,y).$$

Otherwise, we assume $a(y,y) \ne 0$, let $\lambda := a(x,y)/a(y,y)$, then

$$\begin{aligned}
0 &\le a(x-\lambda y, x-\lambda y)\\
&= a(x,x) - \overline{\lambda}a(x,y) - \lambda a(y,x) + |\lambda|^2 a(y,y)\\
&= a(x,x) - \frac{|a(x,y)|^2}{a(y,y)}.
\end{aligned}$$

From this we have

$$|a(x,y)|^2 \le a(x,x)a(y,y).$$

The proof of Cauchy-Schwarz inequality is finished. □

2 实数的Cauchy–Schwarz inequality

Theorem 2.1 (Cauchy–Schwarz inequality)**.** *设$a(\cdot,\cdot)$ 为X 上的双线性函数, 对$\forall x$, $y \in X$, $a(x,y) = a(y,x)$, 且对$\forall x \in X$,*

$$a(x,x) \ge 0.$$

则对$\forall x, y \in X$,

$$|a(x,y)|^2 \le a(x,x)a(y,y).$$

证明类似.