1 Boundedness of S_N on $L^p(\mathbb{T})$ with $p \in (1, \infty)$

Recall that, for any given $N \in \mathbb{N} := \{1, 2, 3, \dots\}$, the N-th symmetric partial sum of $f \in L^1(\mathbb{T})$ is defined by setting, for any $x \in \mathbb{R}$,

$$S_N(f)(x) := \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k x} := \sum_{k=-N}^{N} \int_{\mathbb{T}} f(y) e^{-2\pi i k y}\, dy\, e^{2\pi i k x}.$$

This section is devoted to the uniform (in N) $L^p(\mathbb{T})$ boundedness of S_N with $p \in (1, \infty)$; see also [2, p. 241, Section 4.1] and [1, p. 62 and p. 68].

We first sketch the $L^2(\mathbb{T})$ boundedness.

Proposition 1.1. *Let $f \in L^2(\mathbb{T})$. Then, for any $N \in \mathbb{N}$,*

$$\|S_N(f)\|_{L^2(\mathbb{T})} \le \|f\|_{L^2(\mathbb{T})}. \tag{1.1}$$

Proof. We first claim that the Parseval identity

$$\|f\|_{L^2(\mathbb{T})}^2 = \sum_{k \in \mathbb{Z}} \left| \widehat{f}(k) \right|^2 \tag{1.2}$$

holds true for any given $f \in L^2(\mathbb{T})$. Indeed, to show this identity, by [K. C. Chang, Functional Analysis, p. 61, Theorem 1.6.25], it suffices to show that, if

$$\langle f, e^{2\pi i k(\cdot)} \rangle = 0 \tag{1.3}$$

for any $k \in \mathbb{Z}$, then $f = 0$. Recall that the trigonometric polynomials are dense in $L^2(\mathbb{T})$; see, for instance, [1, p. 10, Corollary 1.11(2)]. Then there exist trigonometric polynomials $\{g_\ell\}_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \to \infty} \|f - g_\ell\|_{L^2(\mathbb{T})} = 0$. From this, the continuity of inner product, and (1.3), it follows that

$$\|f\|_{L^2(\mathbb{T})}^2 = \lim_{\ell \to \infty} \langle f, g_\ell \rangle = 0.$$

This shows that the above claim holds true.

Now, from (1.2), we deduce that

$$\|S_N(f)\|_{L^2(\mathbb{T})}^2 = \sum_{\{k \in \mathbb{N}:\ |k| \le N\}} \left| \widehat{f}(k) \right|^2 \le \sum_{k \in \mathbb{Z}} \left| \widehat{f}(k) \right|^2 = \|f\|_{L^2(\mathbb{T})}^2,$$

which shows that (1.1) holds true. This finishes the proof of Proposition 1.1. □

Next, we consider general $p \in [1, \infty)$ by means of the following conjugate function and the Riesz projections.

Definition 1.2. Let $f \in C^\infty(\mathbb{T})$. The *conjugate function* $\mathscr{C}(f)$ of f is defined by setting, for any $x \in \mathbb{T}$,

$$\mathscr{C}(f)(x) := -i \sum_{k \in \mathbb{Z}} \operatorname{sgn}(k) \widehat{f}(k) e^{2\pi i k x},$$

where

$$\operatorname{sgn}(k) := \begin{cases} 1 & \text{if} \quad k \in \mathbb{N}, \\ 0 & \text{if} \quad k = 0, \\ -1 & \text{if} \quad k \in \{-1, -2, \dots\}. \end{cases}$$

Moreover, the *Riesz projections* $P_+(f)$ and $P_-(f)$ of f are defined by setting, for any $x \in \mathbb{T}$,

$$P_+(f)(x) := \sum_{k=1}^{\infty} \widehat{f}(k) e^{2\pi i k x} \quad \text{and} \quad P_-(f)(x) := \sum_{k=-\infty}^{-1} \widehat{f}(k) e^{2\pi i k x}.$$

Remark 1.3. (i) $\mathscr{C}$, P_+, and P_- are well defined on $C^\infty(\mathbb{T})$ because the smoothness of f implies the decay of its Fourier coefficients $\{\widehat{f}(k)\}_{k\in\mathbb{Z}}$ as $|k| \to \infty$. More precisely, for any given $f \in C^\infty(\mathbb{T})$ and $s \in \mathbb{N}$, and any $k \in \mathbb{Z} \setminus \{0\}$, integrating by parts s times, we obtain

$$\widehat{f}(k) = \int_{\mathbb{T}} f(y) e^{-2\pi i k y} \, \mathrm{d}y = (-1)^s \int_{\mathbb{T}} f^{(s)}(y) \frac{e^{-2\pi i k y}}{(-2\pi i k)^s} \, \mathrm{d}y,$$

where the boundary terms all vanish because of the periodicity of the integrand. Taking absolute values, we then have

$$\left|\widehat{f}(k)\right| \le (2\pi)^{-s} \frac{\|f^{(s)}\|_{L^\infty(\mathbb{T})}}{|k|^s}, \tag{1.4}$$

which, combined with the convergence of $\{k^{-s}\}_{k\in\mathbb{N}}$ for any $s > 1$, further implies that $\mathscr{C}$, P_+, and P_- are well-defined.

(ii) Let $f \in C^\infty(\mathbb{T})$. Then f satisfies the Dini condition and hence

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x}, \quad \forall\, x \in \mathbb{T}.$$

This implies that

$$f = P_+(f) + \widehat{f}(0) + P_-(f).$$

From this and the observation

$$\mathscr{C}(f) = -iP_+(f) + iP_-(f),$$

we deduce that

$$P_+(f) = \frac{1}{2}\left[f + i\mathscr{C}(f)\right] - \frac{1}{2}\widehat{f}(0) \quad \text{and} \quad P_-(f) = \frac{1}{2}\left[f - i\mathscr{C}(f)\right] - \frac{1}{2}\widehat{f}(0). \tag{1.5}$$

Lemma 1.4. *Let $p \in [1, \infty)$. Then $C^\infty(\mathbb{T})$ is dense in $L^p(\mathbb{T})$.*

Proof. Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{T})$. Then $|f|^p$ is integrable on $[-\frac{1}{2}, \frac{1}{2}]$. From this and the dominated convergence theorem, it follows that, for any given $\epsilon \in (0, \infty)$, there exists an $\ell_{(\epsilon)} \in \mathbb{N}$ such that

$$\left\|f - f_{(\epsilon)}\right\|_{L^p([-\frac{1}{2}, \frac{1}{2}])} < \epsilon/2, \tag{1.6}$$

where $f_{(\epsilon)} := f\mathbf{1}_{[-\frac{1}{2}+\frac{1}{\ell_{(\epsilon)}},\frac{1}{2}-\frac{1}{\ell_{(\epsilon)}}]}$.

Now, let φ be a infinitely differentiable function supported in $[-\frac{1}{2},\frac{1}{2}]$ such that $\int_{-\frac{1}{2}}^{\frac{1}{2}}\varphi(y)\,\mathrm{d}y=1$, and, for any $t\in(0,1)$ and $x\in[-\frac{1}{2},\frac{1}{2}]$, define $\varphi_t(x) := t^{-n}\varphi(t^{-1}x)$. Then, by [1, p. 25, Theorem 2.1], we conclude that there exists a $T_{(\epsilon)}\in(0,1)$ such that, for any $t\in(0,T_{(\epsilon)}]$,

$$\left\|f_{(\epsilon)}-f_{(\epsilon)}*\varphi_t\right\|_{L^p([-\frac{1}{2},\frac{1}{2}])}=\left\|\widetilde{f_{(\epsilon)}}-\widetilde{f_{(\epsilon)}}*\widetilde{\varphi_t}\right\|_{L^p([-\frac{1}{2},\frac{1}{2}])}\le\left\|\widetilde{f_{(\epsilon)}}-\widetilde{f_{(\epsilon)}}*\widetilde{\varphi_t}\right\|_{L^p(\mathbb{R})}<\epsilon/2, \tag{1.7}$$

where $\widetilde{f_{(\epsilon)}} := f_{(\epsilon)}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$ and $\widetilde{\varphi} := \varphi\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$ are function on $\mathbb{R}$, and the first identity holds true because, for any $x\in[-\frac{1}{2},\frac{1}{2}]$,

$$\widetilde{f_{(\epsilon)}}(x)=f_{(\epsilon)}(x)\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)=f_{(\epsilon)}(x)$$

and, by $\operatorname{supp}(f_{(\epsilon)})\subset[-\frac{1}{2},\frac{1}{2}]$ as well as $\operatorname{supp}(\varphi_t)\subset[-\frac{1}{2},\frac{1}{2}]$,

$$\begin{aligned}\widetilde{f_{(\epsilon)}}*\widetilde{\varphi_t}(x)&=\int_{\mathbb{R}}\widetilde{f_{(\epsilon)}}(x-y)\widetilde{\varphi_t}(y)\,\mathrm{d}y=\int_{\mathbb{R}}f_{(\epsilon)}(x-y)\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x-y)\varphi_t(y)\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(y)\,\mathrm{d}y\\&=\int_{-\frac{1}{2}}^{\frac{1}{2}}f_{(\epsilon)}(x-y)\varphi_t(y)\,\mathrm{d}y=f_{(\epsilon)}*\varphi_t(x).\end{aligned}$$

Notice that

$$\operatorname{supp}(f_{(\epsilon)}*\varphi_t)\subset\operatorname{supp}(f_{(\epsilon)})+\operatorname{supp}(\varphi_t)\subset\left[-\frac{1}{2}+\frac{1}{\ell_{(\epsilon)}}-\frac{t}{2},\frac{1}{2}-\frac{1}{\ell_{(\epsilon)}}+\frac{t}{2}\right],$$

and let $t_{(\epsilon)}\in(0,T_{(\epsilon)}]$ satisfy $\frac{1}{\ell_{(\epsilon)}}-\frac{t_{(\epsilon)}}{2}>0$. Then $h_{(\epsilon)} := f_{(\epsilon)}*\varphi_{t_{(\epsilon)}}\in C^\infty([-\frac{1}{2},\frac{1}{2}])$ and equals to zero in some neighborhoods of $-\frac{1}{2}$ and $\frac{1}{2}$, which further implies that $h_{(\epsilon)}\in C^\infty(\mathbb{T})$. Moreover, from this, (1.6), and (1.7), we deduce that

$$\left\|f-h_{(\epsilon)}\right\|_{L^p(\mathbb{T})}=\left\|f-h_{(\epsilon)}\right\|_{L^p([-\frac{1}{2},\frac{1}{2}])}\le\left\|f-f_{(\epsilon)}\right\|_{L^p([-\frac{1}{2},\frac{1}{2}])}+\left\|f_{(\epsilon)}-f_{(\epsilon)}*\varphi_t\right\|_{L^p([-\frac{1}{2},\frac{1}{2}])}<\epsilon,$$

which, together with the arbitrariness of $\epsilon\in(0,\infty)$, completes the proof of Lemma 1.4. □

Observe that, for any $N\in\mathbb{N}$, $f\in L^1(\mathbb{T})$, and $x\in\mathbb{T}$,

$$\begin{aligned}\sum_{k=-N}^{N}\widehat{f}(k)e^{2\pi ikx}&=\sum_{k=-N}^{N}\int_{\mathbb{T}}f(y)e^{-2\pi iky}\,\mathrm{d}y\,e^{2\pi ikx}\\&=\sum_{k=-N}^{N}\int_{\mathbb{T}}f(y)e^{2\pi iNy}e^{-2\pi i(k+N)y}\,\mathrm{d}y\,e^{-2\pi iNx}e^{2\pi i(k+N)x}\\&=e^{-2\pi iNx}\sum_{\ell=0}^{2N}\int_{\mathbb{T}}f(y)e^{2\pi iNy}e^{-2\pi i\ell y}\,\mathrm{d}y\,e^{2\pi i\ell x}\\&=e^{-2\pi iNx}\sum_{\ell=0}^{2N}\left(f(\cdot)e^{2\pi iN\cdot}\right)^{\widehat{}}(\ell)e^{2\pi i\ell x}.\end{aligned}$$

Since multiplication by exponential does not affect $L^p(\mathbb{T})$ norms, this identity implies that the norm of the operator S_N from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ is equal to that of the operator

$$\widetilde{S}_N(f)(x) := \sum_{k=0}^{2N} \widehat{f}(k) e^{2\pi i k x}$$

from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$. Indeed, let $g(\cdot) := f(\cdot)e^{2\pi i N\cdot}$. Then

$$\begin{aligned}
\|S_N\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} &= \sup_{\{f\in L^p(\mathbb{T}):\ \|f\|_{L^p(\mathbb{T})}=1\}} \frac{\|S_N(f)\|_{L^p(\mathbb{T})}}{\|f\|_{L^p(\mathbb{T})}} \\
&= \sup_{\{f\in L^p(\mathbb{T}):\ \|f\|_{L^p(\mathbb{T})}=1\}} \frac{\|e^{-2\pi i N\cdot} S_N(g)(\cdot)\|_{L^p(\mathbb{T})}}{\|g(\cdot)e^{-2\pi i N\cdot}\|_{L^p(\mathbb{T})}} \\
&= \sup_{\{g\in L^p(\mathbb{T}):\ \|g\|_{L^p(\mathbb{T})}=1\}} \frac{\|S_N(g)(\cdot)\|_{L^p(\mathbb{T})}}{\|g\|_{L^p(\mathbb{T})}} = \left\|\widetilde{S}_N\right\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})}.
\end{aligned} \tag{1.8}$$

Moreover, also notice that, for any $f \in L^p(\mathbb{T})$ with $p \in (1, \infty)$, by the Hölder inequality, we have

$$\left\|\widehat{f}(0)\right\|_{L^p(\mathbb{T})} = \left|\widehat{f}(0)\right| = \left|\int_{\mathbb{T}} f(y)\,\mathrm{d}y\right| \le \|f\|_{L^1(\mathbb{T})} \le \|f\|_{L^p(\mathbb{T})}. \tag{1.9}$$

We then give some equivalent formulations for the uniform (in N) $L^p(\mathbb{T})$ boundedness of S_N with $p \in [1, \infty)$.

Proposition 1.5. *Let $p \in [1, \infty)$. Then the following three statements are mutually equivalent.*

(i) *There exists a positive constant $C_{(p)}$ such that, for any $N \in \mathbb{N}$ and $f \in L^p(\mathbb{T})$,*

$$\|S_N(f)\|_{L^p(\mathbb{T})} \le C_{(p)}\|f\|_{L^p(\mathbb{T})}.$$

(ii) *There exists a positive constant $C_{(p)}$ such that, for any $f \in C^\infty(\mathbb{T})$,*

$$\|P_+(f)\|_{L^p(\mathbb{T})} \le C_{(p)}\|f\|_{L^p(\mathbb{T})}.$$

(iii) *There exists a positive constant $C_{(p)}$ such that, for any $f \in C^\infty(\mathbb{T})$,*

$$\|P_-(f)\|_{L^p(\mathbb{T})} \le C_{(p)}\|f\|_{L^p(\mathbb{T})}.$$

(iv) *There exists a positive constant $C_{(p)}$ such that, for any $f \in C^\infty(\mathbb{T})$,*

$$\|\mathscr{C}(f)\|_{L^p(\mathbb{T})} \le C_{(p)}\|f\|_{L^p(\mathbb{T})}.$$

Proof. We first assume that (i) holds true, and show (i) $\Longrightarrow$ (ii). From (1.8), it follows that $\widetilde{S}_N$ is also uniformly (in N) bounded on $L^p(\mathbb{T})$. By this and the Fatou lemma, we conclude that, for any $f \in C^\infty(\mathbb{T})$,

$$\left\|P_+(f) + \widehat{f}(0)\right\|_{L^p(\mathbb{T})} = \left\|\lim_{N\to\infty} \widetilde{S}_N(f)\right\|_{L^p(\mathbb{T})} \le \liminf_{N\to\infty} \left\|\widetilde{S}_N(f)\right\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})},$$

where the implicit positive constant depends only on p. From this and (1.9), we deduce that (ii) holds true.

Next, we assume that (ii) holds true, and show (ii) $\Longrightarrow$ (i). Let $f \in C^\infty(\mathbb{T})$. For any $N \in \mathbb{N}$ and $x \in \mathbb{T}$, we have

$$\begin{aligned}\widetilde{S}_N(f)(x) &= \sum_{k=0}^{\infty} \widehat{f}(k)e^{2\pi ikx} - \sum_{k=2N+1}^{\infty} \widehat{f}(k)e^{2\pi ikx} \\ &= \sum_{k=1}^{\infty} \widehat{f}(k)e^{2\pi ikx} + \widehat{f}(0) - e^{2\pi i(2N)x} \sum_{k=1}^{\infty} \widehat{f}(k+2N)e^{2\pi ikx} \\ &= P_+(f)(x) - e^{2\pi i(2N)x} P_+(e^{-2\pi i(2N)(\cdot)} f) + \widehat{f}(0),\end{aligned}$$

where we used the fact

$$\widehat{f}(k+2N) = \int_{\mathbb{T}} f(y)e^{-2\pi i(k+2N)y}\, dy = \int_{\mathbb{T}} \left[f(y)e^{-2\pi i(2N)y}\right] e^{-2\pi iky}\, dy = (f(\cdot)e^{-2\pi i(2N)\cdot})^\frown(k)$$

in the last equality. Combining this with (1.9), we obtain

$$\sup_{N\in\mathbb{N}} \left\|\widetilde{S}_N(f)\right\|_{L^p(\mathbb{T})} \le [2\|P_+\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} + 1]\, \|f\|_{L^p(\mathbb{T})}. \tag{1.10}$$

Now, we claim that (1.10) also holds true for any given $f \in L^p(\mathbb{T})$. Indeed, by Lemma 1.4, there exist functions $\{f_\ell\}_{\ell\in\mathbb{N}} \subset C^\infty(\mathbb{T})$ such that $\lim_{\ell\to\infty} \|f - f_\ell\|_{L^p(\mathbb{T})} = 0$. For any given $N \in \mathbb{N}$ and any $\ell \in \mathbb{N}$, from the Hölder inequality, we deduce that

$$\begin{aligned}\left|\widetilde{S}_N(f - f_\ell)(x)\right| &= \left|\sum_{k=0}^{2N} (f - f_\ell)^\frown(k)e^{2\pi ikx}\right| \le \sum_{k=0}^{2N} \left|\int_{\mathbb{T}} [f(y) - f_\ell(y)]e^{-2\pi iky}\, dy\right| \\ &\le (2N+1)\|f - f_\ell\|_{L^1(\mathbb{T})} \le (2N+1)\|f - f_\ell\|_{L^p(\mathbb{T})}\end{aligned}$$

and hence

$$\begin{aligned}\left\|\widetilde{S}_N(f)\right\|_{L^p(\mathbb{T})} &\le \left\|\widetilde{S}_N(f - f_\ell)\right\|_{L^p(\mathbb{T})} + \left\|\widetilde{S}_N(f_\ell)\right\|_{L^p(\mathbb{T})} \\ &\le (2N+1)\|f - f_\ell\|_{L^p(\mathbb{T})} + [2\|P_+\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} + 1]\, \|f_\ell\|_{L^p(\mathbb{T})}.\end{aligned}$$

Letting $\ell \to \infty$, it follows that the above claim holds true, which implies that $\widetilde{S}_N$ is uniformly (in N) bounded on $L^p(\mathbb{T})$, and so is S_N. This shows that (i) holds true and hence (i) is equivalent to (ii).

Finally, by (1.5) and (1.9), we easily conclude that both (ii) and (iii) are equivalent to (iv). This finishes the proof of Proposition 1.5. □

Now, we show the $L^p(\mathbb{T})$ boundedness of the conjugate function for any $p \in (1, \infty)$.

Theorem 1.6. *Let $p \in (1, \infty)$. Then there exists a positive constant $C_{(p)}$ such that, for any $f \in C^\infty(\mathbb{T})$,*

$$\|\mathscr{C}(f)\|_{L^p(\mathbb{T})} \le C_{(p)}\|f\|_{L^p(\mathbb{T})}. \tag{1.11}$$

Proof. To prove this theorem, we first make the following reductions:

(a) Assume that f is a trigonometric polynomial with its degree $\deg(f) < \infty$.

(b) Assume that $\widehat{f}(0) = 0$.

(c) Assume that f is real valued.

Then, from (c), it follows that, for any $k \in \mathbb{Z}$,

$$\widehat{f}(-k) = \int_{\mathbb{T}} f(y)e^{2\pi iky}\,\mathrm{d}y = \overline{\int_{\mathbb{T}} f(y)e^{-2\pi iky}\,\mathrm{d}y} = \overline{\widehat{f}(k)}$$

and hence, for any $x \in \mathbb{T}$,

$$\begin{aligned}\mathscr{C}(f)(x) &= -i\sum_{k=1}^{\infty}\widehat{f}(k)e^{2\pi ikx} + i\sum_{k=1}^{\infty}\widehat{f}(-k)e^{-2\pi ikx}\\ &= -i\sum_{k=1}^{\infty}\widehat{f}(k)e^{2\pi ikx} + \overline{-i\sum_{k=1}^{\infty}\widehat{f}(k)e^{2\pi ikx}}\\ &= 2\mathrm{Re}\left[-i\sum_{k=1}^{\infty}\widehat{f}(k)e^{2\pi ikx}\right],\end{aligned}$$

which implies that $\mathscr{C}(f)$ is also real valued. Moreover, from (a), we can write

$$f(x) = \sum_{k=-\deg(f)}^{\deg(f)}\widehat{f}(k)e^{2\pi ikx}, \quad \forall\, x \in \mathbb{T},$$

and hence, by (b),

$$f(x) + i\mathscr{C}(f)(x) = \widehat{f}(0) + 2\sum_{k=1}^{\deg(f)}\widehat{f}(k)e^{2\pi ikx} = 2\sum_{k=1}^{\deg(f)}\widehat{f}(k)e^{2\pi ikx}, \quad \forall\, x \in \mathbb{T}.$$

Combining this with the observation

$$\int_{\mathbb{T}} e^{2\pi i\ell x}\,\mathrm{d}x = 0, \quad \forall\, \ell \in \mathbb{N},$$

we deduce that, for any given $\ell \in \mathbb{N}$,

$$\int_{\mathbb{T}}[f(x) + i\mathscr{C}(f)(x)]^{2\ell}\,\mathrm{d}x = 4^{\ell}\int_{\mathbb{T}}\left[\sum_{k=1}^{\deg(f)}\widehat{f}(k)e^{2\pi ikx}\right]^{2\ell}\mathrm{d}x = 0.$$

Expending the 2ℓ power and taking real parts, we obtain

$$\sum_{j=0}^{\ell}(-1)^{\ell-j}\binom{2\ell}{2j}\int_{\mathbb{T}}[\mathscr{C}(f)(x)]^{2\ell-2j}[f(x)]^{2j}\,\mathrm{d}x = 0,$$

where we used the fact that both f and $\mathscr{C}(f)$ are real valued. Therefore, by the Hölder inequality, we conclude that

$$\begin{aligned}
\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})}^{2\ell} &= \int_{\mathbb{T}} |\mathscr{C}(f)(x)|^{2\ell}\,\mathrm{d}x = -\sum_{j=1}^{\ell}(-1)^{\ell-j}\binom{2\ell}{2j}\int_{\mathbb{T}}|\mathscr{C}(f)(x)|^{2\ell-2j}\,|f(x)|^{2j}\,\mathrm{d}x \\
&\le \sum_{j=1}^{\ell}\binom{2\ell}{2j}\int_{\mathbb{T}}|\mathscr{C}(f)(x)|^{2\ell\frac{2\ell-2j}{2\ell}}\,|f(x)|^{2\ell\frac{2j}{2\ell}}\,\mathrm{d}x \\
&\le \sum_{j=1}^{\ell}\binom{2\ell}{2j}\left[\int_{\mathbb{T}}|\mathscr{C}(f)(x)|^{2\ell}\,\mathrm{d}x\right]^{\frac{2\ell-2j}{2\ell}}\left[\int_{\mathbb{T}}|f(x)|^{2\ell}\,\mathrm{d}x\right]^{\frac{2j}{2\ell}} \\
&= \sum_{j=1}^{\ell}\binom{2\ell}{2j}\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})}^{2\ell-2j}\,\|f\|_{L^{2\ell}(\mathbb{T})}^{2j},
\end{aligned}$$

which further implies that

$$\left[\frac{\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})}}{\|f\|_{L^{2\ell}(\mathbb{T})}}\right]^{2\ell} \le \sum_{j=1}^{\ell}\binom{2\ell}{2j}\left[\frac{\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})}}{\|f\|_{L^{2\ell}(\mathbb{T})}}\right]^{2\ell-2j}. \tag{1.12}$$

Let $g(t) := t^{2\ell} - \sum_{j=1}^{\ell}\binom{2\ell}{2j}t^{2\ell-2j}$ for any $t \in \mathbb{R}$. Since $g(0) = -1 < 0$ and $\lim_{t\to\infty} g(t) = \infty$, g has at least one positive root. Choosing $C_{(2\ell)}$ to be the largest positive root of g, by (1.12), we have

$$\frac{\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})}}{\|f\|_{L^{2\ell}(\mathbb{T})}} \le C_{(2\ell)}.$$

To sum up, if f satisfies (a), (b), and (c), then, for any $\ell \in \mathbb{N}$,

$$\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})} \le C_{(2\ell)}\|f\|_{L^{2\ell}(\mathbb{T})}. \tag{1.13}$$

We now remove assumptions (a), (b), and (c). We first remove assumption (c). For any given complex-valued trigonometric polynomial f with $\widehat{f}(0) = 0$, we write

$$f(x) = \sum_{k=-N}^{N} c_k e^{2\pi ikx} = \left[\sum_{k=-N}^{N}\frac{c_k + \overline{c_{-k}}}{2}e^{2\pi ikx}\right] + i\left[\sum_{k=-N}^{N}\frac{c_k - \overline{c_{-k}}}{2i}e^{2\pi ikx}\right], \quad \forall\, x \in \mathbb{T},$$

(with $c_0 := 0$), and observe that the expressions inside the square brackets are real-valued trigonometric polynomials. Thus, we can express f as $P+iQ$ where P and Q are real-valued trigonometric polynomials. Applying (1.13) to P and Q, we obtain the inequality

$$\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})} \le 2C_{(2\ell)}\|f\|_{L^{2\ell}(\mathbb{T})} \tag{1.14}$$

for any complex-valued trigonometric polynomial f with $\widehat{f}(0) = 0$, and any $\ell \in \mathbb{N}$.

Next, we remove assumption (b). To achieve this, write $f = [f - \widehat{f}(0)] + \widehat{f}(0)$, and notice that the conjugate function of a constant is zero. Then, by (1.14) and (1.9), we have

$$\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})} \lesssim \left\|f - \widehat{f}(0)\right\|_{L^{2\ell}(\mathbb{T})} \lesssim \|f\|_{L^{2\ell}(\mathbb{T})} \tag{1.15}$$

for any complex-valued trigonometric polynomial f, and any $\ell \in \mathbb{N}$.

It remains to remove assumption (a). Recall that, for any given $g \in L^1(\mathbb{T})$ and $j \in \mathbb{N}$, and any $x \in \mathbb{T}$, the *Fejér sum*

$$\sigma_j(g)(x) := \frac{1}{j+1}\sum_{\ell=0}^{j}\sum_{k=-\ell}^{\ell}\widehat{g}(k)e^{2\pi ikx} = \widehat{g}(0) + \frac{1}{j+1}\sum_{\ell=1}^{j}\sum_{0<|k|\le\ell}\widehat{g}(k)e^{2\pi ikx}.$$

Moreover, notice that, for any given $g \in L^1(\mathbb{T})$,

$$\widehat{g''}(k) = (2\pi ik)^2\widehat{g}(k), \quad \forall\, k \in \mathbb{N},$$

and hence, for any given $j \in \mathbb{N}$,

$$\sigma_j(g)(x) = \widehat{g}(0) + \frac{1}{j+1}\sum_{\ell=1}^{j}\sum_{0<|k|\le\ell}\widehat{g}(k)e^{2\pi ikx} = \widehat{g}(0) + \frac{1}{j+1}\sum_{\ell=1}^{j}\sum_{0<|k|\le\ell}\widehat{g''}(k)\frac{e^{2\pi ikx}}{(2\pi ik)^2},$$

which, combined with the observation $\widehat{g''}(0) = 0$, further implies that, for any $x \in \mathbb{T}$,

$$\left[\sigma_j(g)\right]''(x) = \frac{1}{j+1}\sum_{\ell=1}^{j}\sum_{0<|k|\le\ell}\widehat{g''}(k)e^{2\pi ikx} = \sigma_j(g'')(x).$$

Thus, for any given $\ell \in \mathbb{N}$ and $f \in C^\infty(\mathbb{T})$, by [1, p. 10, Theorem 1.10], we have

$$\lim_{j\to\infty}\left\|f - \sigma_j(f)\right\|_{L^{2\ell}(\mathbb{T})} = 0 \tag{1.16}$$

and

$$\lim_{j\to\infty}\left\|\left[f - \sigma_j(f)\right]''\right\|_{L^\infty(\mathbb{T})} = \lim_{j\to\infty}\left\|f'' - \sigma_j(f'')\right\|_{L^\infty(\mathbb{T})} = 0. \tag{1.17}$$

From (1.15) and (1.4), it follows that

$$\begin{aligned}
\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})} &\le \left\|\mathscr{C}(f - \sigma_j(f))\right\|_{L^{2\ell}(\mathbb{T})} + \left\|\mathscr{C}(\sigma_j(f))\right\|_{L^{2\ell}(\mathbb{T})} \\
&\lesssim \left\|\mathscr{C}(f - \sigma_j(f))\right\|_{L^\infty(\mathbb{T})} + \left\|\sigma_j(f)\right\|_{L^{2\ell}(\mathbb{T})} \\
&\lesssim \sum_{k\in\mathbb{N}}\left|\left[f - \sigma_j(f)\right]^\frown(k)\right| + \left\|\sigma_j(f)\right\|_{L^{2\ell}(\mathbb{T})} \\
&\lesssim \sum_{k\in\mathbb{N}}\frac{2(2\pi)^{-2}}{|k|^2}\left\|\left[f - \sigma_j(f)\right]''\right\|_{L^\infty(\mathbb{T})} + \left\|\sigma_j(f)\right\|_{L^{2\ell}(\mathbb{T})} \\
&\sim \left\|\left[f - \sigma_j(f)\right]''\right\|_{L^\infty(\mathbb{T})} + \left\|\sigma_j(f)\right\|_{L^{2\ell}(\mathbb{T})}.
\end{aligned}$$

Combining this with (1.16) and (1.17), and letting $j \to \infty$, we then have

$$\|\mathscr{C}(f)\|_{L^{2\ell}(\mathbb{T})} \lesssim \|f\|_{L^{2\ell}(\mathbb{T})} \tag{1.18}$$

for any $\ell \in \mathbb{N}$ and $f \in C^\infty(\mathbb{T})$. Therefore assumptions (a), (b), (c) can be removed, and hence (1.11) holds true for any $p \in \{2\ell\}_{\ell\in\mathbb{N}}$.

Finally, we extend this $L^p(\mathbb{T})$ boundedness from $p \in \{2\ell\}_{\ell\in\mathbb{N}}$ to $p \in (1, \infty)$. To do this, for any given $\ell \in \mathbb{N}$, using (1.18) and Lemma 1.4, we can extend $\mathscr{C}$ from $C^\infty(\mathbb{T})$ to $L^{2\ell}(\mathbb{T})$, and denote it by $\widetilde{\mathscr{C}}$. Then $\widetilde{\mathscr{C}}$ is well defined on $L^{2\ell}(\mathbb{T}) + L^{2\ell+2}(\mathbb{T}) = L^{2\ell}(\mathbb{T})$, and coincides with $\mathscr{C}$ on $C^\infty(\mathbb{T})$. By this and the Marcinkiewicz interpolation theorem (see, for instance, [1, p. 29, Theorem 2.4]), we conclude that $\widetilde{\mathscr{C}}$ is bounded on $L^p(\mathbb{T})$ for any $p \in [2\ell, 2\ell + 2]$. Thus, for any $f \in C^\infty(\mathbb{T})$ and any $p \in [2\ell, 2\ell + 2]$, we have $\mathscr{C}(f) = \widetilde{\mathscr{C}}(f)$ and hence

$$\|\mathscr{C}(f)\|_{L^p(\mathbb{T})} = \left\|\widetilde{\mathscr{C}}(f)\right\|_{L^p(\mathbb{T})} \lesssim \|f\|_{L^p(\mathbb{T})}. \tag{1.19}$$

Notice that every real number $p \geq 2$ lies in an interval of the form $[2\ell, 2\ell + 2]$ for some $\ell \in \mathbb{N}$. From this and (1.19), we deduce that, for any given $p \in [2, \infty)$, there exists a positive constant $C_{(p)}$ such that (1.11) holds true. Moreover, to obtain the case $p \in (1, 2]$, we use the duality. Observe that, for any $f, g \in C^\infty(\mathbb{T})$, by Remark 1.3(i) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\langle \mathscr{C}(f), g\rangle &:= \int_{\mathbb{T}} \mathscr{C}(f)(x)g(x)\,\mathrm{d}x \\
&= \int_{\mathbb{T}} \left[-i \sum_{k\in\mathbb{Z}} \operatorname{sgn}(k)\widehat{f}(k)e^{2\pi ikx}\right] g(x)\,\mathrm{d}x \\
&= -\, i \sum_{k\in\mathbb{Z}} \operatorname{sgn}(k)\widehat{f}(k) \int_{\mathbb{T}} e^{2\pi ikx} g(x)\,\mathrm{d}x \\
&= -\, i \sum_{k\in\mathbb{Z}} \operatorname{sgn}(k)\widehat{f}(k)\widehat{g}(-k) = -i \sum_{k\in\mathbb{Z}} \operatorname{sgn}(-k)\widehat{f}(-k)\widehat{g}(k) \\
&= -\, i \sum_{k\in\mathbb{Z}} -\operatorname{sgn}(k) \int_{\mathbb{T}} f(x)e^{2\pi ikx}\,\mathrm{d}x\, \widehat{g}(k) \\
&= -\int_{\mathbb{T}} f(x) \left[-i \sum_{k\in\mathbb{Z}} \operatorname{sgn}(k)\widehat{g}(k)e^{2\pi ikx}\right] \mathrm{d}x = -\langle f, \mathscr{C}(g)\rangle.
\end{aligned}$$

Thus, for any $f \in C^\infty(\mathbb{T})$ and $p \in (1, 2]$, by the density (see Lemma 1.4) and the Hölder inequality, we conclude that

$$\begin{aligned}
\|\mathscr{C}(f)\|_{L^p(\mathbb{T})} &= \sup_{\{g\in L^{p'}(\mathbb{T}):\ \|g\|_{L^{p'}(\mathbb{T})}\leq 1\}} |\langle \mathscr{C}(f), g\rangle| = \sup_{\{g\in C^\infty(\mathbb{T}):\ \|g\|_{L^{p'}(\mathbb{T})}\leq 1\}} |\langle \mathscr{C}(f), g\rangle| \\
&= \sup_{\{g\in C^\infty(\mathbb{T}):\ \|g\|_{L^{p'}(\mathbb{T})}\leq 1\}} |\langle f, \mathscr{C}(g)\rangle| \leq \sup_{\{g\in C^\infty(\mathbb{T}):\ \|g\|_{L^{p'}(\mathbb{T})}\leq 1\}} \|f\|_{L^p(\mathbb{T})}\|\mathscr{C}(g)\|_{L^{p'}(\mathbb{T})} \\
&\leq \sup_{\{g\in C^\infty(\mathbb{T}):\ \|g\|_{L^{p'}(\mathbb{T})}\leq 1\}} \|f\|_{L^p(\mathbb{T})} C_{(p')}\|g\|_{L^{p'}(\mathbb{T})} \leq C_{(p')}\|f\|_{L^p(\mathbb{T})},
\end{aligned}$$

where the second identity holds true because, if $\lim_{j\to\infty} \|g - g_j\|_{L^{p'}(\mathbb{T})} \to 0$ with $\{g_j\}_{j\in\mathbb{N}} \subset C^\infty(\mathbb{T})$, then, by Remark 1.3(i) and the Hölder inequality,

$$\left|\langle \mathscr{C}(f), g - g_j\rangle\right| = \left|\int_{\mathbb{T}} \mathscr{C}(f)(x) \left[g(x) - g_j(x)\right] \mathrm{d}x\right|$$

$$\leq 2(2\pi)^{-2}\sum_{k=1}^{\infty}\frac{\|f''\|_{L^\infty(\mathbb{T})}}{k^2}\int_{\mathbb{T}}\left|g(x)-g_j(x)\right|\,\mathrm{d}x$$
$$\lesssim \|g-g_j\|_{L^{p'}(\mathbb{T})} \to 0.$$

Therefore (1.11) also holds true for any $p \in (1,2]$ with constant $C_{(p)} := C_{(p')}$. This finishes the proof of Theorem 1.6. □

As a direct corollary of Proposition 1.5 and Theorem 1.6, we obtain the following boundedness of S_N.

Corollary 1.7. *Let $p \in (1,\infty)$. There exists a positive constant $C_{(p)}$ such that, for any $N \in \mathbb{N}$ and $f \in L^p(\mathbb{T})$,*

$$\|S_N(f)\|_{L^p(\mathbb{T})} \leq C_{(p)}\|f\|_{L^p(\mathbb{T})}.$$

Combining Corollary 1.7 with [1, Lemma 1.8], one further has the following convergence.

Corollary 1.8. *Let $p \in (1,\infty)$. Then, for any $f \in L^p(\mathbb{T})$,*

$$\lim_{N\to\infty}\|S_N(f)-f\|_{L^p(\mathbb{T})} = 0.$$

Acknowledgements. Jin TAO would like to thank Professor Dachun YANG for lots of useful suggestions as well as improvements on this subject, and also Fan WANG, Duo ZHANG, Yangyang ZHANG for some useful discussions on the proof of Theorem 1.6.

References

[1] J. Duoandikoetxea, Fourier Analysis, Graduate Studies in Mathematics, 29, American Mathematical Society, Providence, RI, 2001.

[2] L. Grafakos, Classical Fourier Analysis, third edition, Graduate Texts in Mathematics, 249, Springer, New York, 2014.