Schwartz Space and Tempered Distributions

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在本文中,
$$\mathbb{Z}_+ := \{0, 1, 2, \ldots\}.$$
对$\forall\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$,
$$|\alpha| := \sum_{k=1}^{n} \alpha_k.$$
对$\forall\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ 和$\forall f \in \mathcal{S}$,
$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} f.$$
对$\forall\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ 和$\forall x := (x_1, \ldots, x_n) \in \mathbb{R}^n$,
$$x^\alpha := \prod_{k=1}^{n} x_k^{\alpha_k}.$$
$\mathcal{S}$ 代表Schwartz 空间.

0.1 Schwartz Space

Proposition 0.1. 设$f, g \in \mathcal{S}(\mathbb{R}^n)$, 则$fg, f * g \in \mathcal{S}(\mathbb{R}^n)$ 且对$\forall\alpha \in \mathbb{Z}_+^n$,
$$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g).$$

0.2 The Space of Tempered Distributions

Definition 0.2. 设$T \in \mathcal{S}'(\mathbb{R}^n)$, 对$\forall f \in \mathcal{S}(\mathbb{R}^n)$, 定义
$$\widehat{T}(f) := T(\hat{f}).$$

Remark 0.3. $\widehat{T} \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 0.4. 设$T \in \mathcal{S}'(\mathbb{R}^n)$. 称$T$ 在分布意义下与$\mathbb{R}^n$ 上的可测函数h 是一致的, 若对$\forall f \in \mathcal{S}(\mathbb{R}^n)$, 有
$$T(f) = \int_{\mathbb{R}^n} h(x) f(x) dx.$$

0.3 Space of Tempered Distributions Modulo Polynomials

本section 的内容来自[1, 1.1.1]

首先介绍多项式空间$\mathcal{P}(\mathbb{R}^n)$.

Definition 0.5.

$$\mathcal{P}(\mathbb{R}^n) := \left\{ \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \le m} c_\alpha x^\alpha : m \in \mathbb{Z}_+, \text{ for any } \alpha \in \mathbb{Z}_+^n \text{ and } |\alpha| \le m, c_\alpha \in \mathbb{C} \right\}.$$

设$T_1, T_2 \in \mathcal{S}'(\mathbb{R}^n)$. 称$T_1 \equiv T_2$, 若存在$p \in \mathcal{P}(\mathbb{R}^n)$, 使得$T_1 - T_2$ 在分布意义下与p 一致, 则$\equiv$ 是个等价关系, 记$\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ 为该等价关系诱导的商空间.

Definition 0.6.

$$\mathcal{S}_0(\mathbb{R}^n) := \left\{ f \in \mathcal{S} : \text{ for any } \alpha \in \mathbb{Z}_+^n, \int_{\mathbb{R}^n} x^\alpha f(x) dx = 0 \right\}.$$

$\mathcal{S}_0(\mathbb{R}^n)$ 是$\mathcal{S}(\mathbb{R}^n)$ 的子空间.

对$\forall \alpha \in \mathbb{Z}_+^n$ 和$f \in \mathcal{S}$, 由

$$\partial^\alpha(\hat{f}) = ((-2\pi i x)^\alpha f)\hat{}$$

知$\partial^\alpha(\hat{f})(0) = 0$ if and only if

$$\int_{\mathbb{R}^n} x^\alpha f(x) dx = 0.$$

故

$$\mathcal{S}_0(\mathbb{R}^n) = \left\{ f \in \mathcal{S} : \text{ for any } \alpha \in \mathbb{Z}_+^n, \partial^\alpha(\hat{f})(0) = 0 \right\}.$$

References

[1] L. Grafakos, Modern Fourier Analysis, Third edition, Graduate Texts in Mathematics, 249, Springer, New York, 2014.