Equidistributed

The content is from [1, Chapter 4, Section 2].

Definition 1. *A sequence of number* $\{\xi_n\}_{n\in\mathbb{N}} \subset [0,1)$ *is said to be equidistributed if for any interval* $(a,b) \subset [0,1)$*,*

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \xi_n \in (a,b)\}}{N} = b - a.$$

where the $\#A$ *denotes the cardinality of the finite set* A*.*

Definition 2. *Let* $[x]$ *denote the greatest integer less than or equal to x and call* $[x]$ *the integer part of* x*. Let* $\langle x\rangle := x - [x]$ *and call* $\langle x\rangle$ *the fractional part of* x*.*

Theorem 1 (Weyl's criterion)**.** *A sequence of number* $\{\xi_n\}_{n\in\mathbb{N}} \subset [0,1)$ *is equidistributed if and only if for any* $k \in \mathbb{Z} \setminus \{0\}$*,*

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0, \quad as\ N \to \infty.$$

From Weyl's criterion, we have following corollaries.

Corollary 1. *Let* $P(n) := c_n x^n + \cdots + c_0$*, for any* $k \in \{0,\ldots,n\}$*,* $c_n \in \mathbb{R}$ *and one of them is irrational number. Then* $\{\langle P(n)\rangle\}_{n\in\mathbb{N}} \subset [0,1)$ *is equidistributed.*

Corollary 2. $\{\sin n\}_{n\in\mathbb{N}}$ *is dense in* $[0,1]$*.*

References

[1] E. M. Stein, and R. Shakarchi, Fourier Analysis: An Introduction, Princeton University Press, 2011.