Lebesgue's number lemma

In [1, Lemma 7.2], Munkres proof the following Lebesgue's number lemma.

Theorem 1. *If the metric space (X, d) is compact and an open cover of X is given, then there exists $\delta \in (0, \infty)$ such that every subset of X having diameter less than δ is contained in some member of the cover. Such a number δ is called a Lebesgue number of this cover.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is compact we can extract a finite subcover $\{A_1, \ldots, A_n\} \subset \mathcal{U}$. If there exists $k_0 \in \{1, \ldots, n\}$ such that $A_{k_0} = X$, then any $\delta \in (0, \infty)$ will serve as a Lebesgue number. Otherwise for any $k \in \{1, \ldots, n\}$, let $C_k := X \setminus A_k$, note that C_k is not empty closed set, and define a function

$$f : X \to \mathbb{R}, x \mapsto \frac{1}{n} \sum_{k=1}^{n} d(x, C_k).$$

Since f is continuous on a compact set, let f attains a minimum δ at $x_0 \in X$. From this and $\{A_1, \ldots, A_n\}$ is an open cover of X, there exists $k \in \{1, \ldots, n\}$ such that $x_0 \in A_k$, then $x_0 \notin C_k$ so that

$$\delta := f(x_0) \geq \frac{1}{n} d(x_0, C_k) > 0.$$

Now we can verify that this δ is the desired Lebesgue number. In fact, for any $Y \subset X$ of diameter less than δ, let $y \in Y$, we have

$$Y \subset B(y, \delta). \tag{1}$$

Since $f(y) \geq \delta$, there exist $k \in \{1, \ldots, n\}$ such that $d(y, C_k) \geq \delta$. It means that $B(y, \delta) \subset X \setminus C_k$, then $B(y, \delta) \subset A_k$. From this and (1), $Y \subset A_k$. So δ is a Lebesgue number of cover $\mathcal{U}$. □

References

[1] J. R. Munkres, Topology: a first course, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.