Muckenhoupt Weights

February 16, 2020

如下内容来自[1].

Definition 0.1 (Definition 7.1.1)**.** 设函数$w : \mathbb{R}^n \to [0, \infty)$ 满足对a.e. $x \in \mathbb{R}^n$, $w(x) > 0$. 称$w \in A_1(\mathbb{R}^n)$, 若对a.e. $x \in \mathbb{R}^n$,

$$M(w)(x) \leq Cw(x).$$

对于$w \in A_1(\mathbb{R}^n)$, 称

$$[w]_{A_1} := \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w(t)dt\right] \|w^{-1}\|_{L^\infty(Q)}$$

为w 的A_1 Muckenhoupt 特征常数.

Definition 0.2 (Definition 7.1.3)**.** 设$p \in (1, \infty)$, $w : \mathbb{R}^n \to [0, \infty)$ 是局部可积函数且对a.e. $x \in \mathbb{R}^n$, $w(x) > 0$. 称$w \in A_p(\mathbb{R}^n)$, 若存在正常数C 使得对任意方体$Q \in \mathbb{R}^n$,

$$\left[\frac{1}{|Q|} \int_Q w(t)dt\right] \left\{\frac{1}{|Q|} \int_Q [w(t)]^{-\frac{1}{p-1}} dt\right\}^{p-1} \leq C.$$

对于$w \in A_p(\mathbb{R}^n)$, 称

$$[w]_{A_p} := \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w(t)dt\right] \left\{\frac{1}{|Q|} \int_Q [w(t)]^{-\frac{1}{p-1}} dt\right\}^{p-1}$$

为w 的A_p Muckenhoupt 特征常数.

Lemma 0.3. 设$f \in L^1_{loc}(\mathbb{R}^n)$ 且$\alpha \in [1, \infty)$. 对任意方体$Q \subset \mathbb{R}^n$,

$$\frac{1}{|Q|} \int_Q |f(t)|dt \leq \left\{\frac{1}{|Q|} \int_Q |f(t)|^\alpha dt\right\}^{\frac{1}{\alpha}}.$$

Proof. 对任意方体$Q \subset \mathbb{R}^n$, 由Hölder 不等式知

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f(t)|dt &\leq \frac{1}{|Q|} \left[\int_Q |f(t)|^\alpha dt\right]^{\frac{1}{\alpha}} \left(\int_Q 1^{\alpha'} dt\right)^{\frac{1}{\alpha'}} \\ &= \left[\int_Q |f(t)|^\alpha dt\right]^{\frac{1}{\alpha}} |Q|^{\frac{1}{\alpha'}-1} = \left\{\frac{1}{|Q|} \int_Q |f(t)|^\alpha dt\right\}^{\frac{1}{\alpha}}. \end{aligned}$$

□

Theorem 0.4 (习题7.1.10)**.** 设$p_1, p_2 \in [1, \infty)$ 且$w_1 \in A_{p_1}(\mathbb{R}^n), w_2 \in A_{p_2}(\mathbb{R}^n)$, 则

$$[w_1 + w_2]_{A_p} \le [w_1]_{A_{p_1}} + [w_2]_{A_{p_2}}.$$

其中$p = \max(p_1, p_2)$.

Proof. Case 1), $p_1 = p_2 = 1$. 此时$p = \max(p_1, p_2) = 1$,

$$\begin{aligned}
[w_1 + w_2]_{A_1} = & \sup_{\text{cube } Q \subset \mathbb{R}^n} \left\{ \frac{1}{|Q|} \int_Q [w_1(t) + w_2(t)] dt \right\} \left\| [w_1(t) + w_2(t)]^{-1} \right\|_{L^\infty(Q)} \\
& \le \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_1(t) dt \right] \left\| [w_1(t) + w_2(t)]^{-1} \right\|_{L^\infty(Q)} \\
& \quad + \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_2(t) dt \right] \left\| [w_1(t) + w_2(t)]^{-1} \right\|_{L^\infty(Q)} \\
& \le \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_1(t) dt \right] \left\| [w_1(t)]^{-1} \right\|_{L^\infty(Q)} \\
& \quad + \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_2(t) dt \right] \left\| [w_2(t)]^{-1} \right\|_{L^\infty(Q)} \\
& = [w_1]_{A_1} + [w_2]_{A_1}.
\end{aligned}$$

Case 2), $p_1 = 1$, $p_2 \in (1, \infty)$. 此时$p = \max(p_1, p_2) = p_2$,

$$\begin{aligned}
[w_1 + w_2]_{A_{p_2}} = & \sup_{\text{cube } Q \subset \mathbb{R}^n} \left\{ \frac{1}{|Q|} \int_Q [w_1(t) + w_2(t)] dt \right\} \left\{ \frac{1}{|Q|} \int_Q [w_1(t) + w_2(t)]^{-\frac{1}{p_2 - 1}} dt \right\}^{p_2 - 1} \\
& \le \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_1(t) dt \right] \left\{ \frac{1}{|Q|} \int_Q [w_1(t) + w_2(t)]^{-\frac{1}{p_2 - 1}} dt \right\}^{p_2 - 1} \\
& \quad + \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_2(t) dt \right] \left\{ \frac{1}{|Q|} \int_Q [w_1(t) + w_2(t)]^{-\frac{1}{p_2 - 1}} dt \right\}^{p_2 - 1} \\
& \le \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_1(t) dt \right] \left\| [w_1(t)]^{-1} \right\|_{L^\infty(Q)} \\
& \quad + \sup_{\text{cube } Q \subset \mathbb{R}^n} \left[\frac{1}{|Q|} \int_Q w_2(t) dt \right] \left\{ \frac{1}{|Q|} \int_Q [w_2(t)]^{-\frac{1}{p_2 - 1}} dt \right\}^{p_2 - 1} \\
& = [w_1]_{A_1} + [w_2]_{A_{p_2}}.
\end{aligned}$$

Case 3), $p_1 \in (1, \infty)$, $p_2 = 1$. 类似Case 2) 可证.

Case 4), $p_1, p_2 \in (1, \infty)$. 由Lemma 0.3 知,

$$\begin{aligned}
[w_1+w_2]_{A_p} = & \sup_{\text{cube } Q\subset\mathbb{R}^n} \left\{\frac{1}{|Q|}\int_Q [w_1(t)+w_2(t)]dt\right\}\left\{\frac{1}{|Q|}\int_Q [w_1(t)+w_2(t)]^{-\frac{1}{p-1}}dt\right\}^{p-1} \\
\leq & \sup_{\text{cube } Q\subset\mathbb{R}^n} \left[\frac{1}{|Q|}\int_Q w_1(t)dt\right]\left\{\frac{1}{|Q|}\int_Q [w_1(t)+w_2(t)]^{-\frac{1}{p-1}}dt\right\}^{p-1} \\
& + \sup_{\text{cube } Q\subset\mathbb{R}^n} \left[\frac{1}{|Q|}\int_Q w_2(t)dt\right]\left\{\frac{1}{|Q|}\int_Q [w_1(t)+w_2(t)]^{-\frac{1}{p-1}}dt\right\}^{p-1} \\
\leq & \sup_{\text{cube } Q\subset\mathbb{R}^n} \left[\frac{1}{|Q|}\int_Q w_1(t)dt\right]\left\{\frac{1}{|Q|}\int_Q [w_1(t)]^{-\frac{1}{p-1}}dt\right\}^{p-1} \\
& + \sup_{\text{cube } Q\subset\mathbb{R}^n} \left[\frac{1}{|Q|}\int_Q w_2(t)dt\right]\left\{\frac{1}{|Q|}\int_Q [w_2(t)]^{-\frac{1}{p-1}}dt\right\}^{p-1} \\
\leq & \sup_{\text{cube } Q\subset\mathbb{R}^n} \left[\frac{1}{|Q|}\int_Q w_1(t)dt\right]\left\{\frac{1}{|Q|}\int_Q [w_1(t)]^{-\frac{1}{p_1-1}}dt\right\}^{p_1-1} \\
& + \sup_{\text{cube } Q\subset\mathbb{R}^n} \left[\frac{1}{|Q|}\int_Q w_2(t)dt\right]\left\{\frac{1}{|Q|}\int_Q [w_2(t)]^{-\frac{1}{p_2-1}}dt\right\}^{p_2-1} \\
= & [w_1]_{A_{p_1}} + [w_2]_{A_{p_2}}.
\end{aligned}$$

$\square$

References

[1] L. Grafakos, Classical Fourier Analysis, Third edition, Graduate Texts in Mathematics, 249, Springer, New York, 2014.