WEIGHTED INEQUALITIES FOR FRACTIONAL INTEGRALS ON EUCLIDEAN AND HOMOGENEOUS SPACES

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1. Introduction. Weighted Poincaré and Sobolev inequalities of the form

$$(1.1)\qquad \begin{aligned} &\text{(i)}\ \left(\int_Q |f(x)|^q w(x)dx\right)^{1/q} \le C\left(\int_Q |\nabla f(x)|^p v(x)dx\right)^{1/p}, \\ &\text{(ii)}\ \left(\int_Q |f(x)|^q w(x)dx\right)^{1/q} \le C\left(\int_Q |\Delta f(x)|^p v(x)dx\right)^{1/p}, \end{aligned}$$

where $1 < p \le q < \infty$, Q is a cube in $\mathbb{R}^n$ and f is once or twice continuously differentiable with either mean value zero on Q in (i) (the Poincaré inequality) or support in Q in (i), (ii) (the Sobolev inequality) arise in a number of applications to partial differential equations. For example, the case $p = 2 < q$ of (i) arises in Harnack's inequality and regularity estimates for degenerate second order differential operators $P = \nabla \cdot A\nabla$ where $A = A(x)$ is a nonnegative matrix with least and greatest eigenvalues $v(x)$ and $w(x)$ respectively (see [FKS] and [CW3]); the case $p = q = 2$, $v(x) \equiv 1$ of (i) arises in estimating the negative eigenvalues of the Schrödinger operator $H = -\Delta - w$ (see [F2], [KS], and [CW2]), while the case of general v applies to $-\nabla \cdot v\nabla - w$ (see [G]); the case $p = q = 2$, $v(x) = w(x)^{-1}$ of (ii) arises in unique continuation for the differential inequality $|\Delta u| \le w|u|$ and the absence of positive eigenvalues for the Schrödinger operator $H = -\Delta + v$ (see [JK] and [CS]).

A common approach to dealing with (1.1)(i) is to use the inequality (see [FKS])

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$$(1.2)\quad |f(x)| \leq CI_1(\chi_Q|\nabla f|)(x) = C\int_Q |x-y|^{1-n}|\nabla f(y)|dy, \quad x \in Q,$$

valid whenever f has mean value zero on Q or support in Q. The identity $f = I_2\Delta f$ is used for (1.1)(ii). Here I_α, $0 < \alpha < n$, denotes fractional integration of order α on $\mathbb{R}^n$,

$$I_\alpha g(x) = \int_{\mathbb{R}^n} |x - y|^{\alpha-n} g(y)dy.$$

One is thus led to consider the weighted inequality for fractional integrals,

$$(1.3)\qquad \left(\int_{\mathbb{R}^n} [I_\alpha f(x)]^q w(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} f(x)^p v(x)dx\right)^{1/p},$$

for all $f \geq 0$ on $\mathbb{R}^n$. Inequality (1.3) has been studied extensively in [CW1], [S4] and the many references given there. For example, in [S4] it is shown that (1.3) holds if and only if both of the following conditions hold (which essentially amount to testing (1.3) with $f = \chi_Q v^{1-p'}$, $p' = p/(p-1)$, and the inequality dual to (1.3) with $\chi_Q w$):

$$(1.4)\quad \int_Q [I_\alpha(\chi_Q v^{1-p'})]^q w \leq C\left(\int_Q v^{1-p'}\right)^{q/p} < \infty,$$

for all cubes $Q \subset \mathbb{R}^n$.

$$(1.5)\quad \int_Q [I_\alpha(\chi_Q w)]^{p'} v^{1-p'} \leq C\left(\int_Q w\right)^{p'/q'} < \infty,$$

for all cubes $Q \subset \mathbb{R}^n$.

Actually, in [S4], the integrations on the left of (1.4) and (1.5) are extended over all of $\mathbb{R}^n$. However, these stronger conditions are already implied by (1.4) and (1.5) by the result in [S3]. In a recent paper ([LN]), R. Long and F. Nie have shown that (1.1)(i) with $Q = \mathbb{R}^n$ and compactly supported f, is implied by the weak type inequality

$$\sup_{\lambda>0} \lambda \left|\{x : |f(x)| > \lambda\}\right|_w^{1/q} \leq C\left(\int |\nabla f(x)|^p v(x)dx\right)^{1/p},$$

for all continuously differentiable f with compact support, and thus by the case $\alpha = 1$ of condition (1.5) alone (which by [S3] is equivalent to the weak type analogue of (1.3)). Here $|E|_w = \int_E w(x)dx$ for any measurable set E. The corresponding result for (1.1)(ii) fails since the case $Q = \mathbb{R}^n$ of this inequality is equivalent to the case $\alpha = 2$ of (1.3) and the weak and strong type inequalities for fractional integrals are known to be inequivalent (see e.g. [S3]).

In the case $1 < p < q < \infty$, M. Gabidzashvili and V. Kokilashvili show in another recent paper ([GK]; Proposition on p. 8) that (1.5) and (1.4) are equivalent respectively to the more easily verifiable conditions

$$(1.6) \quad \left(\int_Q w\right)^{1/q}\left(\int_{\mathbb{R}^n} (|Q|^{1/n} + |x_Q - y|)^{(\alpha-n)p'} v(y)^{1-p'}dy\right)^{1/p'} \leq C$$

for all cubes Q,

$$(1.7) \quad \left(\int_Q v^{1-p'}\right)^{1/p'}\left(\int_{\mathbb{R}^n} (|Q|^{1/n} + |x_Q - y|)^{(\alpha-n)q} w(y)dy\right)^{1/q} \leq C$$

for all cubes Q,

where x_Q denotes the center of Q, and $|Q|$ its Lebesgue measure. Combining this with the results in [S4] thus yields a simple characterization of the strong type inequality (1.3) when $p < q$. Also, for $1 < p < q < \infty$ and $\alpha = 1$, condition (1.6) implies the case $Q = \mathbb{R}^n$ of (1.1)(i). Finally, as we shall see, each of (1.6) and (1.7) is equivalent to the simple condition

$$(A^{\alpha}_{p,q}) \quad |Q|^{\alpha/n-1}\left(\int_Q w\right)^{1/q}\left(\int_Q v^{1-p'}\right)^{1/p'} \leq C, \text{ for all cubes } Q \subset \mathbb{R}^n,$$

provided w and $v^{1-p'}$ respectively satisfy the reverse doubling condition

(RD) There exist δ, ϵ in (0,1) such that

$$\int_{\delta Q} w(x)dx \leq \epsilon \int_Q w(x)dx \text{ for all cubes } Q \subset \mathbb{R}^n.$$

Here δQ denotes the cube concentric with Q and having side length δ times that of Q. Note that (RD) is weaker than the doubling condition

$$\text{(D)} \qquad \int_{2Q} w(x)dx \leq C \int_Q w(x)dx, \qquad \text{for all cubes } Q \subset \mathbb{R}^n,$$

(for example, $w(x) = e^{|x|}$ satisfies (RD) but not (D)) which is itself weaker than any A_p condition ([FM]).

Unfortunately, the simple conditions $(A^{\alpha}_{p,q})$, (1.6) and (1.7) are not sufficient for the weighted inequality (1.3), nor for the corresponding weak type inequality (see (2.12) below), in the case $p = q$ (see [A] and [KS]; p. 208). As a substitute, our main result on fractional integrals in $\mathbb{R}^n$ states that for $1 < p = q < \infty$, the weighted inequality (1.3) is implied by the following two-weight analogue of a condition introduced by C. Fefferman and D. H. Phong [F2]:

$$\text{(1.8)} \quad |Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q w^r\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q v^{(1-p')r}\right)^{1/p'r} \leq C_r, \qquad \text{for all cubes } Q \subset \mathbb{R}^n,$$

for some $r > 1$. This improves the result in [CW1] for $q = p$ by not requiring $v \in A_p$. We remark that we are unable to find a condition weaker than (1.8) that is sufficient for the weak type analogue of (1.3) (see inequality (2.12) below). The following theorem is proved in Section 2.

THEOREM 1. *Suppose* $0 < \alpha < n$, $1 < p \leq q < \infty$ *and that* $w(x)$ *and* $v(x)$ *are nonnegative measurable functions on* $\mathbb{R}^n$, $n \geq 1$.

(A) *If the weighted inequality* (1.3) *holds, then the weights* w, v *satisfy*

$$\text{(1.9)} \quad |Q|^{1-\alpha/n}\left(\int s_Q{}^q w\right)^{1/q}\left(\int s_Q{}^{p'} v^{1-p'}\right)^{1/p'} \leq C, \text{ for all cubes } Q,$$

where $s_Q(x) = (|Q|^{1/n} + |x - x_Q|)^{\alpha - n}$ *and* x_Q *is the centre of* Q. *Hence the* $A_{p,q}^{\alpha}$ *condition also holds.*

Conversely, if for some $r > 1$,

$$(1.10)\quad |Q|^{\alpha/n + 1/q - 1/p}\left(\frac{1}{|Q|}\int_Q w^r\right)^{1/qr}\left(\frac{1}{|Q|}\int_Q v^{(1-p')r}\right)^{1/p'r} \leq C_r,$$

for all cubes $Q \subset \mathbb{R}^n$,

then the weighted inequality (1.3) *holds.*

(B) *If* $p < q$, *then the weighted inequality* (1.3) *holds if and only if both* (1.6) *and* (1.7) *hold, or equivalently if and only if* (1.9) *holds. If in addition* w *and* $v^{1-p'}$ *satisfy the reverse doubling condition* (RD), *then* (1.3) *holds if and only if the weights* w, v *satisfy the* $A_{p,q}^{\alpha}$ *condition.*

It follows that for $p \leq q$, (1.10) implies (1.9) (the purpose of including the case $p < q$ in part (A) is to provide a model proof for an extension to homogeneous spaces). Note also that when $r = 1$, (1.10) becomes the $A_{p,q}^{\alpha}$ condition, but for $r > 1$, (1.10) is not in general necessary for (1.3). There is a useful (in the case $p = q$) exception: if both w and $v^{1-p'}$ satisfy the A_∞ condition (see [CF] and [M]), then for r sufficiently close to 1, $A_{p,p}^{\alpha}$ and (1.8) are equivalent, and so the case $p = q$ of (1.3) holds if and only if $A_{p,p}^{\alpha}$ holds. This last result has been obtained independently by C. Perez [P] and can also be derived from (1.4) and (1.5) as well as from earlier work of the authors in [MW] and [S1] (see the end of Section 2). However, $A_{p,p}^{\alpha}$ implies (1.3) with $p = q$ for weights satisfying a weaker β-dimensional A_∞, A_∞^{β}, for $\beta > n - \alpha$ (see below). In fact, (1.3) holds for $1 < p = q < \infty$ if condition (1.8) is replaced by

$$(1.11)\qquad |Q|^{(\alpha/n) - 1}\mathcal{B}(Q)^{1/p}\mathcal{A}(Q)^{1/p'} \leq C$$

where

$$\mathcal{A}(Q) = \left(\int_Q \left(\frac{\sigma}{\nu}\right)^r \nu\right)^{1/r}\left(\int_Q \nu\right)^{1/r'}, \qquad \sigma = v^{1-p'},$$

and

$$\mathcal{B}(Q) = \left(\int_Q \left(\frac{w}{\mu}\right)^r \mu\right)^{1/r}\left(\int_Q \mu\right)^{1/r'}$$

for some $r > 1$ and any weights ν, μ in A_∞, or more generally, in A_∞^β for $\beta > n - \alpha$. Note by Holder's inequality that $\int_Q \sigma \leq \mathcal{A}(Q)$ and $\int_Q w \leq \mathcal{B}(Q)$ for any μ, ν. A weight u satisfies A_∞^β if there are positive constants C, δ such that

$$(A^\beta{}_\infty) \qquad \frac{|E|_u}{|Q|_u} \leq C\left[\frac{\|E\|_{\beta,Q}}{|Q|^{\beta/n}}\right]^\delta$$

whenever E is a measurable subset of a cube Q. Here $\|E\|_{\beta,Q} = \inf\{\Sigma_i\ |Q_i|^{\beta/n} : E \subset \cup_i Q_i \subset Q\}$. If μ is Lebesgue measure on a k-dimensional plane in $\mathbb{R}^n$, then μ satisfies (the analogue for measures of) the condition A_∞^k with $\delta = 1$. Thus the A_∞^β condition is a kind of β-dimensional A_∞ condition, and in fact, when $\beta = n$, A_∞^n is the A_∞ condition. In case $\nu \equiv \mu \equiv 1$, (1.11) is the same as (1.8) and in the case $\nu = \sigma = v^{1-p'}$, $\mu = w$, (1.11) is $A_{p,p}^\alpha$. Thus if w and σ satisfy A_∞^β, $\beta > n - \alpha$, then $A_{p,p}^\alpha$ is sufficient for the weighted inequality (1.3) with $p = q$. A simple condition sufficient for a weight u to satisfy A_∞^β is the following reverse doubling condition of order β:

$$(\mathrm{RD}_\beta) \quad \frac{|Q'|_u}{|Q|_u} \leq C\left[\frac{|Q'|}{|Q|}\right]^{\beta/n}, \qquad \text{for all pairs of cubes } Q' \subset Q.$$

Indeed, if $E \subset \cup_i Q_i \subset Q$ then

$$\frac{|E|_u}{|Q|_u} \leq \sum_i \frac{|Q_i|_u}{|Q|_u} \leq C \sum_i \frac{|Q_i|^{\beta/n}}{|Q|^{\beta/n}}, \qquad \text{by } (\mathrm{RD}_\beta),$$

and this shows that u satisfies A_∞^β with $\delta = 1$. C. Perez has shown that $A_{p,p}^\alpha$ is sufficient for the case $p = q$ of (1.3) provided w and $\sigma = v^{1-p'}$ satisfy (RD_β), $\beta > n - \alpha$ ([P]). The following theorem will also be proved in Section 2.

THEOREM 2. *Suppose* $1 < p = q < \infty$, $0 < \alpha < n$, $\beta > n - \alpha$ *and that* $w(x)$ *and* $v(x)$ *are nonnegative measurable functions on* $\mathbb{R}^n$ *satisfying* (1.11) *for some* $r > 1$ *and weights* ν, μ *in* A_∞^β. *Then the weighted inequality* (1.3) *holds. In particular, if* w *and* $v^{1-p'}$ *are in* A_∞^β *and satisfy* $A_{p,p}^\alpha$, *then* (1.3) *holds.*

Remark. If instead of (1.3) we consider the more general inequality

$$\left(\int_{\mathbb{R}^n} [I_\alpha(f d\sigma)(x)]^q d\omega(x)\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} f(x)^p d\sigma(x)\right)^{1/p}, \qquad f \geq 0,$$

where ω and σ are positive Borel measures (the case $d\omega(x) = w(x)dx$ and $d\sigma(x) = v(x)^{1-p'}dx$ recovers (1.3)), then the analogues of Theorems 1 and 2 remain true. In particular, the analogue of Theorem 1(B) holds for locally finite Borel measures ω and σ without the continuity assumptions made on the measures in [GK]—see Section 2.

In a number of applications to more singular and subelliptic equations (see e.g., [J], [N], [NSW], [FL] and [FrS]), variations of (1.1) and (1.3) arise involving operators of the form

$$(1.12) \qquad Tf(x) = \int_X \mathcal{K}(x, y) f(y) d\mu(y), \qquad x \in X,$$

where (X, d) is a quasi-metric space, μ is a doubling measure on X, and the kernel $\mathcal{K}(x, y)$ is nonnegative. At times we will impose on $\mathcal{K}$ the monotonicity conditions

$$(1.13) \qquad \begin{aligned} &\text{(i) } \mathcal{K}(x, y) \leq C_1 \mathcal{K}(x', y) \text{ whenever } d(x', y) \leq C_2 d(x, y), \\ &\text{(ii) } \mathcal{K}(x, y) \leq C_1 \mathcal{K}(x, y') \text{ whenever } d(x, y') \leq C_2 d(x, y), \end{aligned}$$

where C_1 and C_2 are constants greater than one. In particular, these conditions hold when $\mathcal{K}(x, y) = d(x, y)^{-1}$. A function $d : X \times X \to [0, \infty)$ is a quasi-metric on X provided (i) $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$ for all x, y in X, and (iii) for some positive constant K,

$$(1.14) \qquad d(x, y) \leq K[d(x, z) + d(z, y)], \qquad \text{for all } x, y, z \text{ in } X.$$

A positive measure μ is a doubling measure on X if for some positive constant C,

$$(1.15) \qquad \mu(B(x, 2r)) \leq C\mu(B(x, r)), \qquad \text{for all } x \in X,\ r > 0,$$

where $B(x, r) = \{y \in X : d(x, y) < r\}$ is the ball of radius r about x. In particular, such operators include the nonisotropic fractional integrals

$$Tf(x) = \int_{\mathbb{R}^n} \frac{f(y)}{(|x_1 - y_1|^{a_1} + \cdots + |x_n - y_n|^{a_n})} dy, \qquad x \in \mathbb{R}^n,$$

as well as the operators in [N], [NSW] and [FrS]. Our results in this setting are similar to those for fractional integrals in Theorem 1, except that certain doubling conditions are needed on the weights, and part A requires a growth condition on the kernel $\mathcal{K}(x, y)$ (see (1.20) below). However, if X has a group structure on which μ and d are either both right or both left translation invariant, then the doubling condition on the weights can be dropped (but (1.20) is then needed for part (B) also). One point here is that translation invariance avoids the use of the Besicovitch covering lemma (used in [GK]), not generally available in a homogeneous space. For example, the Besicovitch covering lemma fails on the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$. See Section 4 for details. Finally, the equivalence of (1.3) and (1.4)–(1.5) is extended to general homogeneous spaces.

Our results involve the quantity

$$(1.16) \qquad \varphi(B) = \sup\{\mathcal{K}(x, y) : x, y \in B,\ d(x, y) \geq c(K)r(B)\}$$

where B is a ball in X of radius $r(B)$ and $c(K)$ is a sufficiently small positive constant depending only on the quasi-metric constant K in (1.14). To ensure that $\varphi(B)$ is defined and to avoid annoying technicalities, we assume that the annuli $B(x, R)\backslash B(x, r)$ are nonempty for $0 < r < R$ and x in X. For example, with this assumption, condition (1.13) implies that for any $C_2' > 1$, there exists C_1' such that (1.13) holds with C_1 and C_2 replaced by C_1' and C_2' respectively.

THEOREM 3. *Suppose* $1 < p \leq q < \infty$, (X, d) *is a quasi-metric space*, μ *is a doubling measure on* X, *i.e.*, (1.15) *holds*, *and* $w(x)$ *and* $v(x)$ *are nonnegative* μ-*measurable functions on* X. *Let* T *be given by* (1.12) *and* φ *by* (1.16) *with* $c(K) = K^{-4}/9$.

(A) *If the following weighted inequality for* T *holds*,

$$(1.17) \qquad \left(\int_X [Tf(x)]^q w(x) d\mu(x)\right)^{1/q} \leq C\left(\int_X f(x)^p v(x) d\mu(x)\right)^{1/p},$$

for all $f \geq 0$,

and if the kernel $\mathcal{K}(x, y)$ *in* (1.12) *satisfies* (1.13), *then the weights* w, v *satisfy*

$$(1.18) \quad \varphi(B)\left(\int_B w\,d\mu\right)^{1/q}\left(\int_B v^{1-p'}d\mu\right)^{1/p'} \leq C, \qquad \text{for all balls } B \subset X.$$

Conversely, if for some $r > 1$, $w^r d\mu$ *and* $v^{(1-p')r}d\mu$ *are doubling measures that satisfy*

(1.19)

$$\varphi(B)\mu(B)^{1/q+1/p'}\left(\frac{1}{\mu(B)}\int_B w^r d\mu\right)^{1/qr}\left(\frac{1}{\mu(B)}\int_B v^{(1-p')r}d\mu\right)^{1/p'r} \leq C_r,$$

for all balls $B \subset X$,

and if in addition there is $\epsilon > 0$ *such that*

$$(1.20) \qquad \frac{\mu(B')}{\mu(B)} \leq C_\epsilon\left(\frac{\varphi(B)}{\varphi(B')}\right)\left(\frac{r(B')}{r(B)}\right)^\epsilon,$$

for all pairs of balls $B' \subset B$, *then the weighted inequality* (1.17) *holds* ((1.13) *is not assumed here*).

(B) *If* $p < q$ *and* $wd\mu$ *and* $v^{1-p'}d\mu$ *are doubling measures, then the weighted inequality* (1.17) *holds if condition* (1.18) *holds.*

(C) *If* $\mathcal{K}(x, y)$ *satisfies* (1.13), *then the weighted inequality* (1.17) *holds if and only if both*

$$(1.21) \qquad \int_X [T(\chi_Q v^{1-p'})]^q w\,d\mu \leq C\left(\int_Q v^{1-p'}d\mu\right)^{q/p} < \infty,$$

$$(1.22) \qquad \int_X [T^*(\chi_Q w)]^{p'} v^{1-p'}d\mu \leq C\left(\int_Q w\,d\mu\right)^{p'/q'} < \infty,$$

hold for all sets Q that are "almost" balls, i.e., with the property that there is a ball $B \subset Q$ with $Q \subset \lambda B$ where $\lambda = 8K^5$. Here T^* has kernel $\mathcal{K}^*(x, y) = \mathcal{K}(y, x)$.

In order to extend the simple characterization of (1.3) for the case $p < q$ in Theorem 1(B), we need, in addition to Theorem 3(C), an extension to homogeneous spaces of the weak type characterization in [GK]. The argument in [GK], however, relies on the Besicovitch covering lemma and this is not generally available. For example, with a view toward proving Harnack's inequality and regularity estimates for degenerate subelliptic operators modelled on sums of squares of vector fields satisfying Hormander's condition (see e.g. [CW3], [FKS] and [J]), the first step would be to study weighted Poincaré and Sobolev inequalities on nilpotent Lie groups, such as the Heisenberg groups $\mathbb{C}^n \times \mathbb{R}$, $n \geq 1$, where the Besicovitch covering lemma fails. However, the presence of such a group structure on a homogeneous space X permits the extension of Theorem 1(B) to X and also eliminates the need for doubling assumptions on the weights in Theorem 3(A).

THEOREM 4. *Suppose* $1 < p \leq q < \infty$, (X, d) *is a quasi-metric space,* μ *is a doubling measure on* X, *i.e.,* (1.15) *holds, and* $w(x)$ *and* $v(x)$ *are nonnegative* μ*-measurable functions on* X. *Let* T *be given by* (1.12) *and* φ *by* (1.16) *with* $c(K) = K^{-15}/512$.

Suppose, in addition, that X *admits a (not necessarily commutative) group structure such that for all* $x, y, z \in X$ *and all balls* $B \subset X$,

$$(1.23)\qquad \begin{aligned} &\text{(i) } d(x + z, y + z) = d(x, y),\\ &\text{(ii) } d(0, x) = d(0, -x),\\ &\text{(iii) } \mu(-B + z) = \mu(-B), \qquad \textit{where } -B = \{x : -x \in B\},\\ &\text{(iv) } \mu(B) = \mu(-B). \end{aligned}$$

Alternatively, one can replace the right invariance in (1.23)(i) *and* (iii) *by left invariance. Of course,* (ii) *follows from* (i) *and the symmetry of* d, *and* (iii) *and* (iv) *can be combined into the single condition* $\mu(-B + z) = \mu(B)$.

(A) *If, for some* $r > 1$, (1.19) *holds (but we do not assume that either* $w^r d\mu$ *or* $v^{(1-p')r} d\mu$ *is a doubling measure), and if, for some* $\epsilon > 0$, (1.20) *holds for all pairs of balls with* $B' \subset 2KB$, *then the weighted inequality* (1.17) *holds*.

(B) *If* $1 < p < q < \infty$, $\mathcal{K}$ *satisfies* (1.13) *and, for some* $\epsilon > 0$,

$$\frac{\varphi(B)}{\varphi(B')} \leq C\left(\frac{r(B')}{r(B)}\right)^{\epsilon}, \tag{1.24}$$

for all pairs of balls with $B' \subset 2KB$, *then the weighted inequality* (1.17) *holds provided*

$$\varphi(B)^{-1}\left(\int s_B{}^q w d\mu\right)^{1/q}\left(\int s_B{}^{*p'} v^{1-p'} d\mu\right)^{1/p'} \leq C, \tag{1.25}$$

for all balls $B \subset X$,

where $s_B(x) = \min\{\varphi(B), \mathcal{K}(x_B, x)\}$, $s_B{}^*(x) = \min\{\varphi(B), \mathcal{K}^*(x_B, x)\}$ *and* x_B *is the centre of* B. *In the case* $\mathcal{K}(x, y) = d(x, y)^{-1}$, (1.25) *is necessary for* (1.17).

Again, when $r = 1$, condition (1.19) reduces to (1.18). Theorem 4 includes Theorem 1 if we take $X = \mathbb{R}^n$, $d(x, y) = |x - y|^{n-\alpha}$, $\mathcal{K}(x, y) = d(x, y)^{-1}$, and $d\mu = dx$; then (1.20) holds with $\epsilon = \alpha/(n - \alpha)$ and (1.24) holds with $\epsilon = 1$. We emphasize that part B of Theorem 3 doesn't require the growth condition (1.20) and that Theorem 4 doesn't require doubling conditions on the weights. In particular, Theorem 4 applies to the operators in Section 4 of [N] on the Heisenberg group. Theorem 3 is proved in Section 3 and Theorem 4 in Section 4.

Many of the above results generalize to Poisson integrals and this topic will be pursued in a subsequent paper ([SW]).

We close the introduction by giving a version of local inequalities of Poincaré and Sobolev type that arise as corollaries of (the proofs of) Theorems 1 and 2 together with the results of [GK] and [LN].

THEOREM 5. *Suppose* Q_o *is a cube in* $\mathbb{R}^n$ *and that* f *is Lipschitz continuous on* Q_o *with either support in* Q_o, $\int_{Q_o} f(x)dx = 0$ *or* $\int_{Q_o} f(x)w(x)dx = 0$. *Then*

$$\left[\int_{Q_o} |f(x)|^q w(x)dx\right]^{1/q} \leq \mathcal{C}(v, w, Q_o)\left[\int_{Q_o} |\nabla f(x)|^p v(x)dx\right]^{1/p} \tag{1.26}$$

where

$$\mathscr{C}(v, w, Q_o) = C_{p,q} \sup_{Q\subset 8Q_o} \left(\int_Q w\right)^{1/q} \left(\int_{8Q_o} s_Q{}^{p'} v^{1-p'}\right)^{1/p'}$$

with $S_Q(x) = (|Q|^{1/n} + |x - x_Q|)^{1-n}$ *if* $p < q$, *while*

$$\mathscr{C}(v, w, Q_o) = C_{p,r} \sup_{Q\subset 8Q_o} |Q|^{1/n}\left(\frac{1}{|Q|}\int_Q w^r\right)^{1/pr} \left(\frac{1}{|Q|}\int_Q v^{(1-p')r}\right)^{1/p'r}$$

for any $r > 1$ *if* $p = q$. *Finally, we can take*

$$\mathscr{C}(v, w, Q_o) = C_{p,q} \sup_{Q\subset 8Q_o} |Q|^{1/n-1}\left(\int_Q w\right)^{1/q} \left(\int_Q v^{1-p'}\right)^{1/p'}$$

if either $p < q$ *and* w *satisfies* (RD), *or* $p = q$ *and both* w *and* $v^{1-p'}$ *satisfy* (RD_β) *or* (A_∞^β) *for some* $\beta > n - 1$.

We remark that Theorem 5 remains valid for balls in place of cubes. Theorem 5 will be proved at the end of Section 2. See also the remark in Section 2 concerning (D_β).

2. Fractional integrals on Euclidean space. We begin with the

Proof of Theorem 1(A). We first show that (1.3) implies (1.9). Fix a cube Q and let $f_R(y) = \chi_{B(0,R)}(y)s_Q(y)^{p'-1}v(y)^{1-p'}$ where $s_Q(y) = (|Q|^{1/n} + |x_Q - y|)^{\alpha-n}$, x_Q is the centre of Q and $B(0, R)$ denotes the ball of radius R about the origin. Since

$$\begin{aligned}|Q|^{1/n}|x - y| &\le |Q|^{1/n}|x_Q - x| + |Q|^{1/n}|x_Q - y| \\ &\le (|Q|^{1/n} + |x_Q - x|)(|Q|^{1/n} + |x_Q - y|),\end{aligned}$$

we have

$$(2.1) \qquad |x - y|^{\alpha-n} \ge |Q|^{1-\alpha/n}s_Q(x)s_Q(y), \qquad \text{for all } x \text{ and } y.$$

Thus

$$I_\alpha f_R(x) = \int_{B(0,R)} |x - y|^{\alpha-n} s_Q(y)^{p'-1} v(y)^{1-p'} dy$$

$$\geq \int_{B(0,R)} |Q|^{1-\alpha/n} s_Q(x) s_Q(y)^{p'} v(y)^{1-p'} dy$$

by (2.1), and substituting this into (1.3) gives

$$|Q|^{1-\alpha/n}\left(\int s_Q{}^q w\right)^{1/q}\left(\int_{B(0,R)} s_Q{}^{p'} v^{1-p'}\right) \leq \left(\int (I_\alpha f_R)^q w\right)^{1/q}$$

$$\leq C\left(\int f_R{}^p v\right)^{1/p} = C\left(\int_{B(0,R)} s_Q{}^{p'} v^{1-p'}\right)^{1/p}$$

which yields (1.9) upon dividing through by the right hand side and then letting R tend to infinity.

Conversely we show that (1.10) for $r > 1$ implies the weighted inequality (1.3). We first reduce matters in the spirit of ([FS]; p. 112) to consideration of the more tractable dyadic fractional integral,

$$I_\alpha^{dy} f(x) = \sum_{\substack{\text{dyadic cubes} \\ Q \text{ containing } x}} |Q|^{\alpha/n-1} \int_Q f(y) dy.$$

Here "dyadic cube" means a cube in $\mathbb{R}^n$ of the form $[j_1 2^k, (j_1 + 1)2^k) \times \cdots \times [j_n 2^k, (j_n + 1)2^k)$; $j_1, j_2, \ldots, j_n, k \in \mathbb{Z}$. We also consider translates by $z \in \mathbb{R}^n$ of this operator:

$$I_{\alpha,z}^{dy} f(x) = \sum_{\substack{\text{dyadic cubes } Q \\ \text{such that } x \in Q+z}} |Q|^{\alpha/n-1} \int_{Q+z} f(y) dy.$$

LEMMA 2.2. *For* $0 < \alpha < n$, $1 \leq q \leq \infty$ *and* $w(x) \geq 0$ *on* $\mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n} [I_\alpha f]^q w\right)^{1/q} \leq C \sup_{z\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n} [I_{\alpha,z}^{dy} f]^q w\right)^{1/q}, \qquad \textit{for all } f \geq 0.$$

Proof of Lemma 2.2. For $k \in \mathbb{Z}$, define the truncated fractional integral I_α^k by

$$I_\alpha^k f(x) = \int_{|x-y|\leq 2^k} |x - y|^{\alpha-n} f(y)dy,$$

and let $B_k = [-2^{k+2}, 2^{k+2}]^n$ be the cube of side length 2^{k+3} centered at the origin. Following ([S2]: p. 8), we claim that

$$(2.3) \quad I_\alpha^k f(x) \leq C_n \frac{1}{|B_k|}\int_{B_k} I_{\alpha,z}^{dy} f(x)dz, \qquad \text{for all } f \geq 0,\ k \in \mathbb{Z}.$$

To see this, fix x in $\mathbb{R}^n$ and k in $\mathbb{Z}$. Then

$$\frac{1}{|B_k|}\int_{B_k} I_{\alpha,z}^{dy} f(x)dz = \frac{1}{|B_k|}\int_{B_k}\left[\sum_{\substack{Q \text{ dyadic} \\ x\in Q+z}} |Q|^{\alpha/n-1}\int_{Q+z} f(y)dy\right]dz$$

$$= \int_{\mathbb{R}^n}\left[\frac{1}{|B_k|}\int_{B_k}\left[\sum_{\substack{Q \text{ dyadic} \\ x,y\in Q+z}} |Q|^{\alpha/n-1}\right]dz\right]f(y)dy.$$

Momentarily fix y with $|x - y| \leq 2^k$ and choose $\ell \in \mathbb{Z}$ so that $2^{\ell-1} < |x - y| \leq 2^\ell$, $\ell \leq k$. Let Ω consist of those z in B_k such that there is a dyadic cube Q of side length $2^{\ell+1}$ with both x and y in $Q + z$. It is geometrically evident that $|\Omega| \geq 2^{-n}|B_k|$, and it follows that

$$\frac{1}{|B_k|}\int_{B_k}\left[\sum_{\substack{Q \text{ dyadic} \\ x,y\in Q+z}} |Q|^{\alpha/n-1}\right]dz \geq 2^{(\ell+1)(\alpha-n)}\frac{|\Omega|}{|B_k|}$$

$$\geq C_n|x - y|^{\alpha-n}, \qquad \text{for } |x - y| \leq 2^k.$$

Using this estimate in the previous identity for $f \geq 0$ yields (2.3). For each $k \in \mathbb{Z}$, Minkowski's inequality now shows that

$$\left[\int_{\mathbb{R}^n} [I_\alpha^k f]^q w\right]^{1/q} \leq C_n \frac{1}{|B_k|} \int_{B_k} \left[\int_{\mathbb{R}^n} [I_{\alpha,z}^{dy} f]^q w\right]^{1/q} dz$$

$$\leq C_n \sup_{z \in B_k} \left[\int_{\mathbb{R}^n} [I_{\alpha,z}^{dy} f]^q w\right]^{1/q}.$$

Now let $k \to \infty$ to obtain Lemma 2.2.

Lemma 2.2 reduces matters to proving (1.3) with I_α replaced by $I_{\alpha,z}^{dy}$ and with a constant C which is independent of z. Since it is a simple matter to observe at each step that the constants obtained are independent of z, we shall consider (1.3) only for I_α^{dy}. By replacing f by $fv^{1-p'}$, this is in turn equivalent by duality to

$$\text{(2.4)} \quad \int_{\mathbb{R}^n} I_\alpha^{dy}(f\sigma) g w \leq C\left(\int_{\mathbb{R}^n} f^p \sigma\right)^{1/p} \left(\int_{\mathbb{R}^n} g^{q'} w\right)^{1/q'}, \qquad \sigma = v^{1-p'},$$

for all $f, g \geq 0$ bounded with compact support in $\mathbb{R}^n$. For the remainder of this proof, Q will be used to denote only dyadic cubes. Now

$$\text{(2.5)} \quad \begin{aligned} \int_{\mathbb{R}^n} I_\alpha^{dy}(f\sigma) g w &= \int_{\mathbb{R}^n} \left[\sum_Q |Q|^{\alpha/n-1}\left(\int_Q f\sigma\right)\chi_Q(x)\right] g(x) w(x)\,dx \\ &= \sum_Q |Q|^{\alpha/n-1}\left(\int_Q f\sigma\right)\left(\int_Q gw\right). \end{aligned}$$

Since g is bounded with compact support,

$$\frac{1}{|Q|}\int_Q gw \to 0 \text{ as } Q \uparrow \mathbb{R}^n,$$

and it follows that if there are any dyadic cubes Q with $|Q|^{-1} \int_Q gw > 2^{kn}$ for a given $k \in \mathbb{Z}$, then they are contained in cubes of this type which are maximal with respect to inclusion. For each $k \in \mathbb{Z}$, let $\{Q_j^k\}_j$

be the maximal dyadic cubes over which the average of gw exceeds 2^{kn}. Note that the Q_j^k are nonoverlapping in j for fixed k. Also,

$$2^{kn} < \frac{1}{|Q_j^k|}\int_{Q_j^k} gw \leq 2^{(k+1)n},$$

where the second inequality can be seen as follows. If $\tilde{Q}_j^k$ is the dyadic cube containing Q_j^k whose side length is twice that of Q_j^k, then

$$\frac{1}{|Q_j^k|}\int_{Q_j^k} gw \leq \frac{|\tilde{Q}_j^k|}{|Q_j^k|}\left(\frac{1}{|\tilde{Q}_j^k|}\int_{\tilde{Q}_j^k} gw\right) \leq 2^n \cdot 2^{nk} = 2^{n(k+1)}$$

by the maximality of Q_j^k.

Now, for each $k \in \mathbb{Z}$, let

$$\mathscr{C}^k = \left\{Q \text{ dyadic} : 2^{kn} < \frac{1}{|Q|}\int_Q gw \leq 2^{(k+1)n}\right\}.$$

Every dyadic Q for which gw is not identically zero on Q belongs to exactly one $\mathscr{C}^k$. Also, $Q_j^k \in \mathscr{C}^k$. In particular,

$$\frac{1}{|Q|}\int_Q gw \leq 2^n \frac{1}{|Q_j^k|}\int_{Q_j^k} gw \quad \text{if} \quad Q \in \mathscr{C}^k, \text{ for all } j.$$

If $Q \in \mathscr{C}^k$ then since $|Q|^{-1}\int_Q gw > 2^{kn}$, it follows that $Q \subset Q_j^k$ for some j. The right side of (2.5) is

$$(2.6)\quad \sum_Q |Q|^{\alpha/n-1}\left[\int_Q f\sigma\right]\left[\int_Q gw\right]$$

$$= \sum_k \sum_{Q\in\mathscr{C}^k}\left[|Q|^{\alpha/n}\int_Q f\sigma\right]\left[\frac{1}{|Q|}\int_Q gw\right]$$

$$\leq 2^n \sum_k \sum_j \left[\sum_{Q\subset Q_j^k} |Q|^{\alpha/n}\int_Q f\sigma\right]\frac{1}{|Q_j^k|}\int_{Q_j^k} gw$$

$$\leq C_{n,\alpha}\sum_{k,j}|Q_j^k|^{\alpha/n-1}\left(\int_{Q_j^k} f\sigma\right)\left(\int_{Q_j^k} gw\right),$$

where the last inequality will follow if we show that there is a constant C_α such that for any dyadic cube Q_o,

$$\sum_{Q\subset Q_o} |Q|^{\alpha/n} \int_Q f\sigma \le C_\alpha |Q_o|^{\alpha/n} \int_{Q_o} f\sigma.$$

However,

$$\begin{aligned}\sum_{Q\subset Q_o} |Q|^{\alpha/n} \int_Q f\sigma &= \sum_{\ell=0}^{\infty} \sum_{\substack{Q:Q\subset Q_o\\ |Q|=2^{-n\ell}|Q_o|}} |Q|^{\alpha/n} \int_Q f\sigma \\ &= \sum_{\ell=0}^{\infty} (2^{-n\ell})^{\alpha/n} |Q_o|^{\alpha/n} \Bigg(\sum_{\substack{Q:Q\subset Q_o\\ |Q|=2^{-n\ell}|Q_o|}} \int_Q f\sigma \Bigg) \\ &= \left(\sum_{\ell=0}^{\infty} 2^{-\alpha\ell} \right) |Q_o|^{\alpha/n} \int_{Q_o} f\sigma = C_\alpha |Q_o|^{\alpha/n} \int_{Q_o} f\sigma.\end{aligned}$$

Combining (2.5) and (2.6) yields

$$\int_{\mathbb{R}^n} I_\alpha^{dy}(f\sigma) gw \le C \sum_{k,j} |Q_j^k|^{\alpha/n-1} \left(\int_{Q_j^k} f\sigma \right) \left(\int_{Q_j^k} gw \right). \tag{2.7}$$

The reason for replacing the sum over all dyadic cubes in (2.5) by a sum over the Calderón-Zygmund collection $\{Q_j^k\}$ in (2.7) is to gain the following property:

$$\sum_{i:Q_i^\ell \subset Q_j^k} |Q_i^\ell| \le 2^n 2^{n(k-\ell)} |Q_j^k|, \text{ for all } k, j, \ell, \tag{2.8}$$

$$\text{and if } Q_i^\ell \subsetneqq Q_j^k \text{ then } \ell > k.$$

To see this, note that $|Q_i^\ell|^{-1} \int_{Q_i^\ell} gw > 2^{n\ell}$ while $|Q_j^k|^{-1} \int_{Q_j^k} gw \le 2^{n(k+1)}$, so that the sum on the left side of (2.8) is at most

$$\sum_{i:Q_i^\ell \subset Q_j^k} 2^{-n\ell} \int_{Q_i^l} gw \le 2^{-n\ell} \int_{Q_j^k} gw \le 2^{-n\ell} 2^{n(k+1)} |Q_j^k|,$$

as desired. Also note that if $Q_i^\ell \subsetneqq Q_j^k$, then $\ell > k$ since $2^{nk} < |Q_j^k|^{-1} \int_{Q_j^k} gw \le 2^{n\ell}$, where the last inequality follows from the maximality of Q_i^ℓ together with $Q_i^\ell \subsetneqq Q_j^k$. We emphasize that (2.8) is the only property of the cubes Q_j^k in (2.7) that we need. In particular, we could just as well have taken for Q_j^k the Calderon-Zygmund cubes for $f\sigma$ in place of gw.

Set $\mathcal{A}_j^k = \mathcal{A}(Q_j^k) = |Q_j^k|^{1/r'}(\int_{Q_j^k} \sigma^r)^{1/r}$ and $\mathcal{B}_j^k = \mathcal{B}(Q_j^k) = |Q_j^k|^{1/r'}(\int_{Q_j^k} w^r)^{1/r}$ where r is as in condition (1.10). Then, noting that (1.10) amounts to $|Q|^{\alpha/n-1}\mathcal{A}(Q)^{1/p'}\mathcal{B}(Q)^{1/q} \le C$, we have

$$(2.9)\quad \sum_{k,j} |Q_j^k|^{\alpha/n-1}\left(\int_{Q_j^k} f\sigma\right)\left(\int_{Q_j^k} gw\right)$$

$$= \sum_{k,j} |Q_j^k|^{\alpha/n-1}\mathcal{A}_j^k\mathcal{B}_j^k\left(\frac{1}{\mathcal{A}_j^k}\int_{Q_j^k} f\sigma\right)\left(\frac{1}{\mathcal{B}_j^k}\int_{Q_j^k} gw\right)$$

$$\le C\sum_{k,j} (\mathcal{A}_j^k)^{1/p}\left(\frac{1}{\mathcal{A}_j^k}\int_{Q_j^k} f\sigma\right)(\mathcal{B}_j^k)^{1/q'}\left(\frac{1}{\mathcal{B}_j^k}\int_{Q_j^k} gw\right), \qquad \text{by (1.8)},$$

$$\le C\left(\sum_{k,j} \mathcal{A}_j^k\left(\frac{1}{\mathcal{A}_j^k}\int_{Q_j^k} f\sigma\right)^p\right)^{1/p}\left(\sum_{k,j} (\mathcal{B}_j^k)^{p'/q'}\left(\frac{1}{\mathcal{B}_j^k}\int_{Q_j^k} gw\right)^{p'}\right)^{1/p'},$$

by Hölder's inequality

$$\le C\left(\sum_{k,j} \mathcal{A}_j^k\left(\frac{1}{\mathcal{A}_j^k}\int_{Q_j^k} f\sigma\right)^p\right)^{1/p}\left(\sum_{k,j} \mathcal{B}_j^k\left(\frac{1}{\mathcal{B}_j^k}\int_{Q_j^k} gw\right)^{q'}\right)^{1/q'},$$

since $p \le q$ implies $q' \le p'$. We now need

LEMMA 2.10. *Suppose $u(x) \ge 0$ on $\mathbb{R}^n$, $\beta \ge 1$, $\{Q_i\}_{i\in I}$ is a countable collection of dyadic cubes in $\mathbb{R}^n$ and $\{a_i\}_{i\in I}$ are positive numbers satisfying*

$$\text{(i)}\qquad \int_{Q_i} u \le Ca_i, \qquad \textit{for all } i \in I,$$

$$\text{(ii)}\qquad \sum_{j:Q_j\subset Q_i} a_j^\beta \le Ca_i^\beta, \qquad \textit{for all } i \in I.$$

Then

$$\left(\sum_{i\in I} a_i^\beta \left(\frac{1}{a_i}\int_{Q_i} fu\right)^t\right)^{1/t} \le C_{s,t}\left(\int_{\mathbb{R}^n} f^s u\right)^{1/s},$$

for all $f \ge 0$ *on* $\mathbb{R}^n$ *and* $t = s\beta$, $1 < s < \infty$.

Deferring the proof of the lemma for the moment, we claim that

$$\sum_{\ell,i:Q_i^\ell\subset Q_j^k} \mathcal{A}_i^\ell \le C_{n,r}\mathcal{A}_j^k, \qquad \text{for all } k, j. \tag{2.11}$$

To see (2.11), first recall that if $Q_i^\ell \subsetneqq Q_j^k$, then $\ell > k$. Thus, the sum on the left in (2.11) equals

$$\begin{aligned}\sum_{\ell=k}^\infty \sum_{i:Q_i^\ell\subset Q_j^k} \mathcal{A}_i^\ell &\le \sum_{\ell=k}^\infty \left(\sum_{i:Q_i^\ell\subset Q_j^k} |Q_i^\ell|\right)^{1/r'}\left(\sum_{i:Q_i^\ell\subset Q_j^k}\int_{Q_i^\ell}\sigma^r\right)^{1/r}\\ &\le \sum_{\ell=k}^\infty (2^n 2^{n(k-\ell)}|Q_j^k|)^{1/r'}\left(\int_{Q_j^k}\sigma^r\right)^{1/r} \qquad \text{by (2.8)}\\ &= C_{n,r}|Q_j^k|^{1/r'}\left(\int_{Q_j^k}\sigma^r\right)^{1/r} = C_{n,r}\mathcal{A}_j^k.\end{aligned}$$

A similar inequality holds for the $\mathcal{B}_j^k$ and since $\int_{Q_j^k}\sigma \le \mathcal{A}_j^k$, and $\int_{Q_j^k} w \le \mathcal{B}_j^k$ by Hölder's inequality, the case $\beta = 1$ of Lemma 2.10 applies to show that the final line in (2.9) is dominated by

$$C\left(\int f^p\sigma\right)^{1/p}\left(\int g^{q'}w\right)^{1/q'}.$$

Together with (2.7), this proves (1.3) for the dyadic fractional integral I_α^{dy} in place of I_α, as required. This completes the proof of part (A) of Theorem 1 save for

Proof of Lemma 2.10. The proof is a simple adaptation of arguments in [D], [H] and [St]. The map $f \to (1/a_i \int_{Q_i} fu)_{i\in I}$ takes $L^\infty(\mathbb{R}^n, u)$ to $\ell^\infty(I, a_i^\beta)$ by condition (i) and $L^1(\mathbb{R}^n, u)$ to weak $\ell^\beta(I, a_i^\beta)$ by condition (ii), as we now show. Indeed, for f bounded with compact support in $\mathbb{R}^n$ and $\lambda > 0$, let $\{Q_j\}_{j\in J}$ be the maximal dyadic cubes from the collection $\{Q_i\}_{i\in I}$ such that $1/a_i \int_{Q_i} |f|u$ exceeds λ (we may assume the collection $\{Q_i\}_{i\in I}$ is finite). Then

$$\sum_{i:|1/a_i \int_{Q_i} fu|>\lambda} a_i^\beta \le \sum_{j\in J} \sum_{i:Q_i\subset Q_j} a_i^\beta$$

$$\le \sum_{j\in J} Ca_j^\beta, \qquad \text{by (ii)}$$

$$\le \frac{C}{\lambda^\beta} \sum_{j\in J} \left(\int_{Q_j} |f|u\right)^\beta \le \frac{C}{\lambda^\beta} \left(\int_{\mathbb{R}^n} |f|u\right)^\beta,$$

since $\beta \ge 1$ and the maximal cubes Q_j, $j \in J$, are pairwise disjoint. The Marcinkiewicz interpolation theorem now completes the proof of Lemma 2.10.

Proof of Theorem 1(B). We have already shown that (1.3) implies (1.9) in the proof of part (A).

Conversely, (1.9) immediately implies both (1.6) and (1.7) and, as has already been observed in the introduction, these conditions are equivalent to (1.4) and (1.5) by the proposition on page 8 of [GK]. In fact, (1.6) is equivalent to the weak type inequality

(2.12)

$$\sup_{\lambda>0} \lambda |\{x : I_\alpha f(x) > \lambda\}|_w{}^{1/q} \le C\left(\int f(x)^p v(x)dx\right)^{1/p}, \qquad \text{for all } f \ge 0,$$

by Theorem 1.1 of [GK], and this is in turn equivalent to (1.5) by the theorem in [S3]. Conditions (1.4) and (1.5) imply (1.3) by Theorem 1 in [S4].

Finally, if both w and $v^{1-p'}$ satisfy the reverse doubling condition (RD), it is an easy matter to compute the integrals in (1.6) and (1.7). Assuming $(A_{p,q}^\alpha)$ we have

$$\begin{aligned}\int s_Q{}^q w &\le |Q|^{(\alpha/n-1)q}|Q|_w + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} s_Q{}^q w \\ &\le C \sum_{k=0}^{\infty} (2^k|Q|^{1/n})^{(\alpha-n)q}|2^{k+1}Q|_w \\ &\le C \sum_{k=0}^{\infty} |2^{k+1}Q|_\sigma^{-q/p'}, \qquad \text{by } (A_{p,q}^\alpha), \\ &\le C|Q|_\sigma^{-q/p'}, \qquad \text{since } \sigma \in (\mathrm{RD}).\end{aligned}$$

Thus (1.7) holds and a similar computation using $w \in (\mathrm{RD})$ shows that (1.6) holds. This completes the proof of Theorem 1.

Remark. If w satisfies a doubling condition of order β,

$$(\mathrm{D}_\beta) \qquad \frac{|RW|_w}{|Q|_w} \le CR^\beta, \qquad \text{for all cubes } Q \text{ and } R > 1,$$

for some $\beta < (n - \alpha)q$, then $\int s_Q{}^q w \le C|Q|^{(\alpha/n-1)q} \int_Q w$ and so $A_{p,q}^\alpha$ implies (1.7). Since doubling implies (RD), we also have (1.6) by the above. Thus if $w \in (D_\beta)$, $\beta < (n - \alpha)q$, and $1 < p < q < \infty$, then $A_{p,q}^\alpha$ implies (1.3). Similarly, if $\sigma \in (D_\beta)$, $\beta < (n - \alpha)p'$, and $1 < p < q < \infty$, then $A_{p,q}^\alpha$ implies (1.3). In the same way, the assumption at the end of Theorem 5 that $1 < p < q < \infty$ and $w \in (\mathrm{RD})$ can be replaced by the assumption that $1 < p < q < \infty$ and $\sigma \in (\mathrm{D}_\beta)$, $\beta < (n - 1)p'$.

While Theorem 1 in [S4] extends readily to the setting of a general homogeneous space (see Theorem 3(C)), the use of the Besicovitch covering lemma in [GK] provides an obstacle to a similar extension of the results there. We now give a modification of the argument in [GK] used to show (1.6) implies (2.12), which avoids the Besicovitch covering lemma and permits an extension to homogeneous spaces with an appropriate group structure, such as the Heisenberg group (see Theorem 4(B) and Section 4). In addition, this modification permits w to be replaced by any locally finite positive Borel measure—in [GK] it was assumed that $|B(x, r)|_w$ is a continuous function of r for all x. The idea is to use the analogue of Lemma 2.2 for weak type norms.

So suppose that (1.6) holds and that f is nonnegative with compact support. Using (2.3) and Minkowski's inequality for the weak type norm on the left side of (2.12) as in the proof of Lemma 2.2, we obtain

(2.13) $$\sup_{\lambda>0} \lambda \left|\{x : I_\alpha f(x) > \lambda\}\right|_w^{1/q} \leq C \sup_{z\in\mathbb{R}^n} \sup_{\lambda>0} \lambda \left|\{x : I^{dy}_{\alpha,z} f(x) > \lambda\}\right|_w^{1/q}$$

for $0 < \alpha < n$, $1 < q < \infty$ and $w(x) \geq 0$ on $\mathbb{R}^n$. Thus it suffices to prove

(2.14) $$\sup_{\lambda>0} \lambda \left|\{x : I^{dy}_\alpha f(x) > \lambda\}\right|_w^{1/q} \leq C\left(\int f^p v\right)^{1/p}$$

for all f nonnegative with compact support. Let Q_o be one of the maximal dyadic cubes (of the form $\Pi_{i=1}^n [y_i, y_i + 2^k)$, $y \in 2^k Z^n$) contained in $\{I^{dy}_\alpha f > \lambda/2\}$, and fix an x in $Q_o \cap \{I^{dy}_\alpha f > \lambda\}$. For any dyadic cube Q let Q^* denote the dyadic cube containing Q and having twice the side length. Then there is $z \in Q_o^*$ with $I^{dy}_\alpha f(z) \leq \lambda/2$ and so

(2.15) $$\sum_{Q\supset Q_o^*} |Q|^{\alpha/n-1} \int_Q f \leq \sum_{z\in Q} |Q|^{\alpha/n-1} \int_Q f = I^{dy}_\alpha f(z) \leq \lambda/2,$$

where Q is used here and for the remainder of this proof to denote only dyadic cubes. Now define a decreasing sequence of dyadic cubes $Q_o \supset Q_1 \supset Q_2 \supset \cdots$ each containing x by the properties

(2.16) $$\text{(i) } |Q_{k+1}|_w < \frac{1}{2}|Q_k|_w$$

$$\text{(ii) } |Q|_w \geq \frac{1}{2}|Q_k|_w \quad \text{if} \quad Q^*_{k+1} \subset Q \subset Q_k,$$

for $k \geq 0$. We can now obtain an analogue of the key inequality in the proof of Theorem 1.1 in [GK]:

(2.17) $$\begin{aligned} \lambda/2 &< I^{dy}_\alpha f(x) - \sum_{Q\supset Q_o^*} |Q|^{\alpha/n-1} \int_Q f, \qquad \text{by (2.15)} \\ &= \sum_{k\geq 0} \sum_{Q^*_{k+1}\subset Q\subset Q_k} |Q|^{\alpha/n-1} \int_Q f \\ &= \sum_{k\geq 0} \int \left[\sum_{Q^*_{k+1}\subset Q\subset Q_k} |Q|^{\alpha/n-1} \chi_Q(y)\right] f(y)dy \end{aligned}$$

$$\le C \sum_{k\ge 0} \int_{Q_k} s_{Q^*_{k+1}}(y) f(y)\,dy$$

$$\le C \sum_{k\ge 0} \left(\int s_{Q^*_{k+1}}{}^{p'} v^{1-p'} dy \right)^{1/p'} \left(\int_{Q_k} f^p v \right)^{1/p}$$

$$\le C \sum_{k\ge 0} |Q^*_{k+1}|_w^{-1/q} \left(\int_{Q_k} f^p v \right)^{1/p}, \qquad \text{by (1.6)}$$

$$\le C \sum_{k\ge 0} |Q_k|_w^{-1/q} \left(\int_{Q_k} f^p v \right)^{1/p}, \qquad \text{by (2.16)(ii)}$$

$$\le C \left[\sum_{k\ge 0} |Q_k|_w^{1/p-1/q} \right] \overline{M}{}^{dy}_{Q_o} f(x)$$

$$\le C |Q_o|_w^{1/p-1/q} \overline{M}{}^{dy}_{Q_o} f(x)$$

by (2.16)(i) and since $p < q$ and where

$$\overline{M}{}^{dy}_{Q_o} f(x) = \sup_{Q : x\in Q\subset Q_o} \left[\frac{\int_Q f^p v}{|Q|_w} \right]^{1/p}$$

is a localized dyadic version of the maximal operator $\overline{M}$ in [GK]. It follows from (2.17) that for each $x \in Q_o \cap \{I^{dy}_\alpha f > \lambda\}$, there is a dyadic cube $Q(x)$ contained in Q_o and containing x such that

$$\lambda < C |Q_o|_w^{1/p-1/q} \left[\frac{\int_{Q(x)} f^p v}{|Q(x)|_w} \right]^{1/p}. \tag{2.18}$$

Let $\{Q_i\}_i$ denote the maximal dyadic cubes in the collection $\{Q(x) : x \in$

$Q_o \cap \{I_\alpha^{dy} f > \lambda\}\}$. Then

$$\text{(2.19)}\quad \begin{aligned} \lambda^q |Q_o \cap \{I_\alpha^{dy} f > \lambda\}|_w &\leq \lambda^q \Big|\bigcup_i Q_i\Big|_w = \lambda^q \sum_i |Q_i|_w \\ &\leq C\lambda^{q-p} |Q_o|_w^{1-p/q} \sum_i \int_{Q_i} f^p v, \qquad \text{by (2.18)} \\ &\leq C\lambda^{q-p} |Q_o|_w^{1-p/q} \int_{Q_o} f^p v. \end{aligned}$$

Summing (2.19) over the collection $\mathscr{C}$ of all maximal dyadic cubes Q_o contained in $\{I_\alpha^{dy} f > \lambda/2\}$, we obtain

$$\text{(2.20)}\quad \begin{aligned} \lambda^q |\{I_\alpha^{dy} f > \lambda\}|_w &\leq C\lambda^{q-p} \sum_{Q_o \in \mathscr{C}} |Q_o|_w^{1-p/q} \int_{Q_o} f^p v \\ &\leq C\lambda^{q-p} \left[\sum_{Q_o \in \mathscr{C}} |Q_o|_w\right]^{1-p/q} \left[\sum_{Q_o \in \mathscr{C}} \left(\int_{Q_o} f^p v\right)^{q/p}\right]^{p/q} \\ &\leq C[\lambda^q |\{I_\alpha^{dy} f > \lambda/2\}|_w]^{1-p/q} \int f^p v, \end{aligned}$$

since the cubes in $\mathscr{C}$ are pairwise disjoint and $p < q$. It is the use of dyadic cubes in (2.19) and (2.20) that replaces the use of the Besicovitch covering lemma in [GK]. For $t > 0$ take the supremum over $0 < \lambda < t$ in (2.20) to get

$$\text{(2.21)}\quad \sup_{0<\lambda<t} \lambda^q |\{I_\alpha^{dy} f > \lambda\}|_w \leq C\left[\sup_{0<\lambda<t} \lambda^q |\{I_\alpha^{dy} f > \lambda\}|_w\right]^{1-p/q} \int f^p v.$$

Now divide both sides of (2.21) by the first factor on the right side and then let t tend to infinity to obtain (2.14) as desired. The only point remaining to be verified is that the factor we divided (2.21) by is finite. To see this we argue as on page 343 of [S3]. Suppose f is supported in a cube Q and let $\lambda > 0$. If $x \in \{I_\alpha^{dy} f > \lambda\} \backslash 2Q$, then $\lambda < I_\alpha^{dy} f(x) \leq C|x - x_Q|^{\alpha-n} \int_Q f$ where x_Q is the centre of Q. With $R = (C/\lambda \int_Q f)^{1/(n-\alpha)}$, we thus have

$$\lambda^q\left|\{I_\alpha^{dy}f > \lambda\}\backslash 2Q\right|_w \le \lambda^q\left|B(x_Q, R)\right|_w$$

$$= \left(CR^{\alpha-n}\int_Q f\right)^q \left|B(x_Q, R)\right|_w$$

$$\le CR^{(\alpha-n)q}\left|B(x_Q, R)\right|_w \left|B(x_Q, R)\right|_\sigma^{q/p'}\left(\int_Q f^p v\right)^{q/p}$$

$$\le C\left(\int_Q f^p v\right)^{q/p},$$

since (1.6) implies the $(A_{p,q}^\alpha)$ condition, and we are done since λ is at most t and $|2Q|_w$ is finite. This completes our modification of the proof in [GK] that (1.6) implies the weak type inequality (2.12). We will refer to this proof in proving part (B) of Theorem 4 in Section 4.

Proof of Theorem 2. We prove only the special case where both w and $v^{1-p'}$ satisfy A_∞^β, $\beta > n - \alpha$, and $A_{p,p}^\alpha$ holds. The general case is proved in the same way using $\mathcal{A}(Q)$ and $\mathcal{B}(Q)$ as in the proof of Theorem 1(A). It is enough to show (2.4), and for this we use (2.5) to write

(2.22)

$$\int_{\mathbb{R}^n} I_\alpha^{dy}(f\sigma)gw = \sum_Q |Q|^{\alpha/n-1}\left(\int_Q f\sigma\right)\left(\int_Q gw\right)$$
$$= \sum_Q \left(|Q|^\epsilon \int_Q f\sigma\right)\left(|Q|^{\alpha/n-1-\epsilon}\int_Q gw\right), \quad \text{for } \epsilon > 0.$$

We intend to modify the argument leading from (2.5) to (2.8) so as to replace the dyadic cubes Q in (2.22) by a Calderon-Zygmund collection $\{Q_j^k\}$ satisfying an even stronger condition than (2.8). For each k in $\mathbb{Z}$, let

$$\mathcal{C}^k = \{Q \text{ dyadic} : 2^{k(n-\alpha+\epsilon n)} < |Q|^{\alpha/n-1-\epsilon}\int_Q gw \le 2^{(k+1)(n-\alpha+\epsilon n)}\}$$

and let $\{Q_j^k\}_j$ denote the maximal dyadic cubes satisfying $|Q|^{\alpha/n-1-\epsilon}\cdot$

$\int_Q gw > 2^{k(n-\alpha+\epsilon n)}$. Then $Q_j^k \in \mathscr{C}^k$ and arguing as in (2.6), we obtain

$$\text{(2.23)}\quad \sum_Q \left(|Q|^\epsilon \int_Q f\sigma\right)\left(|Q|^{\alpha/n-1-\epsilon}\int_Q gw\right)$$
$$\leq C\sum_{k,j}\left(\sum_{Q\subset Q_j^k}|Q|^\epsilon\int_Q f\sigma\right)\left(|Q_j^k|^{\alpha/n-1-\epsilon}\int_{Q_j^k} gw\right)$$
$$\leq C\sum_{k,j}|Q_j^k|^{\alpha/n-1}\left(\int_{Q_j^k} f\sigma\right)\left(\int_{Q_j^k} gw\right),$$

but this time, instead of (2.8), we have gained the stronger inequality

$$\text{(2.24)}\quad \sum_{i:Q_i^\ell\subset Q_j^k}\left|Q_i^\ell\right|^{1-\alpha/n+\epsilon} \leq C2^{(k-\ell)(n-\alpha+\epsilon n)}\left|Q_j^k\right|^{1-\alpha/n+\epsilon},\quad \text{for all } k,j,\ell,$$

$$\text{and if } Q_i^\ell \subsetneqq Q_j^k \text{ then } \ell > k.$$

If we take $\epsilon = (\beta + \alpha - n)/n$, then

$$\text{(2.25)}\quad \begin{aligned}\sum_{i:Q_i^\ell\subset Q_j^k}|Q_i^\ell|_\sigma &\leq C\left(\frac{\left\|\bigcup_{i:Q_i^\ell\subset Q_j^k} Q_i^\ell\right\|_{\beta,Q_j^k}}{|Q_j^k|^{\beta/n}}\right)^\delta |Q_j^k|_\sigma \qquad \text{since } \sigma\in A_\infty^\beta,\\ &\leq C2^{(k-\ell)\beta\delta}|Q_j^k|_\sigma \qquad \text{by (2.24).}\end{aligned}$$

Now apply the $A_{p,p}^\alpha$ condition to the right side of (2.23) and then use Holder's inequality as in (2.9) to obtain

$$\text{(2.26)}\quad \sum_{k,j}|Q_j^k|^{\alpha/n-1}\left(\int_{Q_j^k} f\sigma\right)\left(\int_{Q_j^k} gw\right)$$
$$\leq C\sum_{k,j}|Q_j^k|_\sigma^{1/p}\left(\frac{1}{|Q_j^k|_\sigma}\int_{Q_j^k} f\sigma\right)|Q_j^k|_w^{1/p'}\left(\frac{1}{|Q_j^k|_w}\int_{Q_j^k} gw\right)$$
$$\leq C\left(\sum_{k,j}|Q_j^k|_\sigma\left(\frac{1}{|Q_j^k|_\sigma}\int_{Q_j^k} f\sigma\right)^p\right)^{1/p}\left(\sum_{k,j}|Q_j^k|_w\left(\frac{1}{|Q_j^k|_w}\int_{Q_j^k} gw\right)^{p'}\right)^{1/p'}.$$

By Lemma 2.10 with $\beta = 1$, $\{Q_i\}_{i\in I}$ replaced by $\{Q_j^k\}_{k,j}$ and a_i by $|Q_j^k|_\sigma$ (property (ii) of Lemma 2.10 then follows from (2.25)), the first factor on the right side of (2.26) is dominated by $C(\int f^p\sigma)^{1/p}$. Similarly, the second factor is dominated by $C(\int g^{p'}w)^{1/p'}$, and together with (2.22), (2.23) and (2.26), this establishes (2.4) as required.

Proof of Theorem 5. To prove (1.26), we repeat the proof of the case $\alpha = 1$ of Theorems 1 and 2, including the argument in [GK] or its modification in (2.13)–(2.21), for

$$I_1(|\nabla f|\chi_{Q_o})(x) = \int_{Q_o} |x - y|^{1-n}|\nabla f(y)|\,dy, \qquad x \in Q_o,$$

and note that the only points x, y which occur are in Q_o. The proof of Lemma 2.2 is then easily modified so that the only cubes Q which arise there are subcubes of $8Q_o$. These are then the only cubes arising in (2.5) for I_α^{dy} and its translates $I_{\alpha,z}^{dy}$. Consequently, we need only check the hypotheses of Theorems 1 and 2 over subcubes of $8Q_o$. For the case where $p < q$ and f is supported in Q_o, the result in [LN], together with (2.13) and the string of inequalities in (2.17), shows that we need only check (1.6) for $Q \subset 8Q_o$ and with the integration over $\mathbb{R}^n$ restricted to $8Q_o$. For the case where $p < q$ and f has mean value zero on Q_o, we use the following variant of the argument in [LN]. Let f be Lipschitz continuous on Q_o with $\int_{Q_o} f = 0$ and let $\Omega_k = \{x \in Q_o : 2^k < |f(x)| \leq 2^{k+1}\}$. Then

$$(2.27) \quad |f(x)| \leq CI_1(\chi_{\Omega_{k-1}}|\nabla f|)(x) + C|Q_o|^{1/n-1}\int_{Q_o}|\nabla f|, \qquad x \in \Omega_k.$$

To see (2.27), let $f_k = \varphi_k(|f|)$ where $\varphi_k(t) = 2^{k-1}\chi_{[0,2^{k-1}]}(t) + t\chi_{(2^{k-1},2^k)}(t) + 2^k\chi_{[2^k,\infty)}(t)$ as in [LN]. Then for $x \in \Omega_k$,

$$2^k = f_k(x) \leq \left| f_k(x) - \frac{1}{|Q_o|}\int_{Q_o} f_k \right| + \frac{1}{|Q_o|}\int_{Q_o} f_k$$

$$\leq CI_1(|\nabla f_k|)(x) + 2^{k-1} + \frac{1}{|Q_o|}\int_{Q_o}|f|$$

implies that

$$|f(x)| \le 2^{k+1} \le 4CI_1(|\nabla f_k|)(x) + \frac{4}{|Q_o|}\int_{Q_o}|f|$$

$$\le CI_1(|\nabla f_k|)(x) + C|Q_o|^{1/n-1}\int_{Q_o}|\nabla f|$$

by integrating (1.2) over Q_o, which holds since f has mean value zero on Q_o, and this proves (2.27). Now suppose $2^{N-1} < C|Q_o|^{1/n-1}\int_{Q_o}|\nabla f| \le 2^N$. Then for $k > N$, (2.27) shows that

$$\Omega_k \subset \{x \in Q_o : I_1(\chi_{\Omega_{k-1}}|\nabla f|)(x) > 2^k/2C\}$$

and so, if

$$A = \sup_{\substack{\lambda>0 \\ \text{supp } g\subset Q_o \\ \int |g|^p v\le 1}} \lambda|Q_o \cap \{I_1 g > \lambda\}|_w^{1/q},$$

then

$$\text{(2.28)} \quad \int_{Q_o}|f|^q w = \sum_{k\le N}\int_{\Omega_k}|f|^q w + \sum_{k>N}\int_{\Omega_k}|f|^q w$$

$$\le C_q 2^{Nq}|Q_o|_w + \sum_{k>N} 2^{(k+1)q}|Q_o \cap \{I_1(\chi_{\Omega_{k-1}}|\nabla f|) > 2^k/2C\}|_w$$

$$\le C_q\left(|Q_o|^{1/n-1}\int_{Q_o}|\nabla f|\right)^q|Q_o|_w + C_q A^q \sum_{k>N}\left[\int_{\Omega_{k-1}}|\nabla f|^p v\right]^{q/p}$$

$$\le C_q\left[\left[|Q_o|^{1/n-1}|Q_o|_w^{1/q}|Q_o|_\sigma^{1/p'}\right]^q + A^q\right]\left[\int_{Q_o}|\nabla f|^p v\right]^{q/p}$$

$$\le C_q A^q\left[\int_{Q_o}|\nabla f|^p v\right]^{q/p}.$$

From (2.28) we see that (1.26) holds provided the weak type inequality (2.12) (with $\alpha = 1$) holds for functions f supported in Q_o and with w replaced by $\chi_{Q_o}w$. As above, this requires only that we check (1.6) with $Q \subset 8Q_o$ and the integration over $\mathbb{R}^n$ restricted to $8Q_o$. This completes the proof of Theorem 5 in the case $\int_{Q_o} f = 0$. The case when $\int_{Q_o} fw = 0$ follows from this since we then have $(\int_{Q_o} |f|^q w)^{1/q} \leq 2(\int_{Q_o} |f - f_{Q_o}|^q w)^{1/q}$ with $f_{Q_o} = (1/|Q_o|) \int_{Q_o} f$.

We close this section by pointing out that if w and $\sigma = v^{1-p'}$ satisfy the A_∞ condition, then the $A^\alpha_{p,q}$ condition implies (1.4) and (1.5) as follows: A simple covering argument (see e.g. [M]) shows that $A^\alpha_{p,q}$ holds if and only if the fractional maximal operator defined by $M_\alpha f(x) = \sup_{Q:x\in Q} |Q|^{\alpha/n-1} \int_Q |f(y)|dy$ takes $L^p(v)$ to weak $L^q(w)$. If w satisfies A_∞, a good λ inequality argument (see e.g. [MW]) then shows that if we assume the $A^\alpha_{p,q}$ condition, I_α takes $L^p(v)$ to weak $L^q(w)$, and (1.5) follows (even for all measurable sets Q) by duality (see e.g. [S3]). Similarly, (1.4) follows from the embedding $M_\alpha : L^{q'}(w^{1-q'}) \to$ weak $L^{p'}(v^{1-p'})$ (which is also equivalent to $A^\alpha_{p,q}$) and the A_∞ condition on $\sigma = v^{1-p'}$. The above provides another proof (besides using Theorem 1(A)) of a result obtained independently by C. Perez [P]: namely, that if w, $v^{1-p'}$ satisfy A_∞, then (1.3) holds for $p = q$ if and only if $A^\alpha_{p,p}$ holds. Finally, we point out that this last result also follows from the equivalence of $\|I_\alpha f\|_{L^q(w)}$ and $\|M_\alpha f\|_{L^q(w)}$ when w satisfies A_∞ ([MW]) together with the fact that when $1 < p \leq q < \infty$ and $v^{1-p'}$ satisfies A_∞, then $M_\alpha : L^p(v) \to L^q(w)$ if and only if $A^\alpha_{p,q}$ holds ([S1]).

3. Operators on homogeneous spaces. Throughout this section, (X, d) will denote a quasi-metric space, i.e., $d : X \times X \to [0, \infty)$ satisfies

(3.1) (i) $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$ for all x, y in X

(iii) $d(x, y) \leq K[d(x, z) + d(z, y)]$ for all x, y, z in X, where $K \geq 1$ is a constant independent of x, y, z.

In addition, we suppose there is a nonnegative Borel measure μ on X satisfying the doubling condition,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \qquad \text{for all } x \in X,\ r > 0. \tag{3.2}$$

Here $B(x, r) = \{y \in X : d(x, y) < r\}$ denotes the open ball of radius r about x. The space X, equipped with a quasi-metric and a doubling measure as above is called a homogeneous space ([CoW]).

In order to transport the arguments of Section 2 to the setting of a homogeneous space (X, d, μ), where X, $d(x, y)$ and $d\mu(x)$ replace $\mathbb{R}^n$, $|x - y|^{n-\alpha}$ and dx respectively, we need the following covering lemma.

LEMMA 3.3. *Suppose (X, d, μ) is a homogeneous space and $\{B_\alpha\}_{\alpha\in A}$ is a family of balls contained in some fixed ball B of X. Then there is a countable subcollection $\{B_i\}_{i\in I}$ of these balls such that*

(3.4) (i) $B_i \cap B_j = \phi$ if $i \neq j$,

(ii) Every B_α is contained in some $\tilde{B}_i$ where $\tilde{B}_i$ is the ball concentric with B_i and radius $K + 4K^2$ times that of B_i,

(iii) $\mu(\cup_{\alpha\in A} B_\alpha) \leq C \,\Sigma_{i\in I}\, \mu(B_i)$,

and where C is a constant depending only on the quasi-metric constant K in (3.1)(iii) *and the doubling constant C for μ in* (3.2).

Proof of Lemma 3.3. We use the selection process of Vitali (see E. Stein [St]). Choose B_1 so that the radius $r(B_1)$ of B_1 is at least $\frac{1}{2}\sup\{r(B_\alpha) : \alpha \in A\}$. If $B_1, \ldots, B_k$ have been chosen, choose B_{k+1} disjoint from $B_1, \ldots, B_k$ so that $r(B_{k+1}) \geq \frac{1}{2}\sup\{r(B_\alpha) : B_\alpha$ is disjoint from $B_1, \ldots, B_k\}$. As we shall show, this construction yields a sequence $\{B_i\}_{i\in I}$ of pairwise disjoint balls with the property that for every $\alpha \in A$, there is some $i \in I$ such that $B_\alpha \cap B_i \neq \phi$ and the radius of B_i is at least half that of B_α. Indeed, the doubling condition (3.2) forces $\lim_{i\to\infty} r(B_i) = 0$ since if $r(B_i) \geq \delta$, $1 \leq i \leq N$, then $\mu(B) \geq \Sigma_{i=1}^N \mu(B_i) \geq C(\delta)\mu(B)N$ where $C(\delta) > 0$ depends only on δ, B and the doubling constant for μ. Now if we take the first k with $r(B_{k+1}) < \frac{1}{2}r(B_\alpha)$, then B_α must intersect one of the balls $B_1, \ldots, B_k$, say B_i, and since $i \leq k$, we have $r(B_i) \geq \frac{1}{2}r(B_\alpha)$. It follows that if $B_i = B(x_i, r_i)$ then $B_\alpha \subset B(x_i, (K + 4K^2)r_i)$ since, if $B_\alpha = B(x_\alpha, r_\alpha)$ and $z \in B_\alpha \cap B_i$ and $w \in B_\alpha$, then

$$d(x_i, w) \le K[d(x_i, z) + K[d(z, x_\alpha) + d(x_\alpha, w)]]$$

$$< K[r_i + K[r_\alpha + r_\alpha]] = K(r_i + 2Kr_\alpha) \le (K + 4K^2)r_i.$$

Thus (3.4)(ii) holds and

$$\mu\left(\bigcup_{\alpha\in A} B_\alpha\right) \le \mu\left(\bigcup_{i\in I} B(x_i, (K + 4K^2)r_i)\right)$$

$$\le \sum_{i\in I} \mu(B(x_i, (K + 4K^2)r_i)) \le C \sum_{i\in I} \mu(B_i)$$

by (3.2), and this gives (3.4)(iii) and completes the proof of the lemma.

Later in this section, we construct a grid of dyadic sets ("almost" balls) in X that can be used to prove part (C) of Theorem 3 and that provide a dyadic version T^{dy} of T, suitable for use in proving Theorem 4 in the presence of a (right or left) translation invariant quasi-metric d and measure μ. In general, however, there is no group structure on a homogeneous space and consequently no analogue of Lemma 2.2. Instead, we construct the following crude substitute for dyadic cubes. Set $\lambda = K + 2K^2$ where K is as in (3.1)(iii). For each k in $\mathbb{Z}$, let $\{\hat{B}_j^k\}_j$ be a sequence of balls of radius λ^{k-1}, maximal with respect to the property that $\hat{B}_i^k \cap \hat{B}_j^k = \phi$ for $i \ne j$. Set $B_j^k = B(x_j^k, \lambda^k)$ where x_j^k is the centre of $\hat{B}_j^k$. Then

(3.5) (i) Every ball of radius λ^{k-1} is contained in at least one of the balls B_j^k.

(ii) $\sum_j \chi_{B_j^k} \le M$, for all k in $\mathbb{Z}$, where M is a constant depending only on K in (3.1)(iii) and C in (3.2).

(iii) $\hat{B}_i^k \cap \hat{B}_j^k = \phi$ for $i \ne j$, $k \in \mathbb{Z}$.

Indeed, for x in X, $B(x, \lambda^{k-1})$ intersects at least one of the $\hat{B}_j^k$, say $z \in B(x, \lambda^{k-1}) \cap \hat{B}_j^k$, by the maximal property of $\{\hat{B}_j^k\}_j$. But then $B(x, \lambda^{k-1})$ is contained in B_j^k since if $w \in B(x, \lambda^{k-1})$,

$$d(x_j^k, w) \leq K[d(x_j^k, z) + K[d(z, x) + d(x, w)]]$$

$$< K[\lambda^{k-1} + K[\lambda^{k-1} + \lambda^{k-1}]] = (K + 2K^2)\lambda^{k-1} = \lambda^k,$$

and this proves (3.5)(i). To prove (3.5)(ii), suppose x is in B_j^k, $1 \leq j \leq N$. Then $B_j^k \subset B(x, 2K\lambda^k)$ for $1 \leq j \leq N$ by (3.1)(iii). It follows from the doubling condition (3.2) that $\mu(\hat{B}_j^k) \geq \epsilon\mu(B(x, 2K\lambda^k))$ for $1 \leq j \leq N$, where ϵ is a positive number which depends only on K and C in (3.2). Thus $N\epsilon\mu(B(x, 2K\lambda^k)) \leq \sum_{j=1}^N \mu(\hat{B}_j^k) = \mu(\cup_{j=1}^N \hat{B}_j^k) \leq \mu(B(x, 2K\lambda^k))$, so that $N\epsilon \leq 1$, and (3.5)(ii) follows with $M = \epsilon^{-1}$.

We refer to the balls B_j^k as dyadic balls. If B is a dyadic ball, say $B = B_j^k$, let $\hat{B} = \hat{B}_j^k$. We single out one further property of dyadic balls.

(3.6) Suppose $\{B_\alpha\}_{\alpha\in A}$ is a family of dyadic balls in X. If $\{B_j\}_{j\in J}$ is a collection of maximal (with respect to inclusion) balls in $\{B_\alpha\}_{\alpha\in A}$, then the balls $\{\hat{B}_j\}_{j\in J}$ are pairwise disjoint.

To see this, first recall from (3.5)(iii) that $\hat{B}_i^k \cap \hat{B}_j^k = \phi$ for $i \neq j$. Now suppose $z \in \hat{B}_j^k \cap \hat{B}_i^\ell$ where $\ell < k$. Then $B_i^\ell \subset B_j^k$ since if $w \in B_i^\ell$,

$$d(x_j^k, w) \leq K[d(x_j^k, z) + K[d(z, x_i^\ell) + d(x_i^\ell, w)]]$$

$$< K[\lambda^{k-1} + K[\lambda^{\ell-1} + \lambda^\ell]] \leq K[\lambda^{k-1} + 2K\lambda^{k-1}] = \lambda^k.$$

From this and the maximality of the balls, (3.6) follows.

Finally, $r(B)$ will denote the radius of B.

Proof of part (A) of Theorem 3. The necessity of condition (1.18) follows by setting $f = \chi_B v^{1-p'}$ in (1.17) and using (1.13) and (1.16). Indeed, suppose $x, y, x', y' \in B$ where $d(x, y) \geq c(K)r(B)$. Note by (1.14) that the distance between any two points in B is at most $2Kr(B)$. Since $c(K)r(B) \leq d(x, y) \leq K[d(x, x') + d(x', y)]$, we may assume, by renaming x and y if necessary, that $d(x', y) \geq (c(K)/2K)r(B)$. With $C_2 = 4K^2/c(K)$ we then have both $d(x', y') \leq 2Kr(B) \leq C_2 d(x', y)$ and $d(x', y) \leq 2Kr(B) \leq C_2 d(x, y)$. By (1.13) (see the comments following (1.16)) we have $\mathcal{K}(x, y) \leq C_1\mathcal{K}(x', y)$ and $\mathcal{K}(x', y) \leq C_1\mathcal{K}(x', y')$. Altogether we obtain $\varphi(B) \leq C_1^2\mathcal{K}(x', y')$ from (1.16) and so

$$Tf(x') = \int_B \mathcal{K}(x', y')v(y')^{1-p'}d\mu(y') \geq C_1^{-2}\varphi(B)\int_B v^{1-p'}d\mu,$$

for $x' \in B$. Using this in (1.17) we obtain

$$\varphi(B)\left(\int_B wd\mu\right)^{1/q}\left(\int_B v^{1-p'}d\mu\right) \leq C\left(\int_B [Tf]^q wd\mu\right)^{1/q}$$

$$\leq C\left(\int f^p vd\mu\right)^{1/p} = C\left(\int_B v^{1-p'}d\mu\right)^{1/p},$$

since $f^p v = \chi_B v^{(1-p')p+1} = \chi_B v^{1-p'}$. Dividing both sides by the right hand side yields (1.18)—if the right side is 0 or ∞, then simple arguments as in [M] show that $A^\alpha_{p,q}$ holds with the convention $0\cdot\infty = 0$.

Conversely, as in (2.4) in Section 2, (1.17) is equivalent by duality to

$$\int_X T(f\sigma)gwd\mu \leq C\left(\int_X f^p\sigma d\mu\right)^{1/p}\left(\int_X g^{q'}wd\mu\right)^{1/q'}, \tag{3.7}$$

for all $f, g \geq 0$ bounded with support in some ball and where $\sigma = v^{1-p'}$. To obtain an analogue of (2.5) we use the inequality

$$Tf(x) = \int_X \mathcal{K}(x, y)f(y)d\mu(y) \leq \sum_{B:x\in B \text{ dyadic}} \varphi(B)\int_B fd\mu. \tag{3.8}$$

This follows from the fact that if $\lambda^{k-1} \leq d(x, y) \leq \lambda^k$, then both x and y are in $B(x, \lambda^k)$ which is contained in some B_j^{k+1} by (3.5)(i). Thus $r(B_j^{k+1}) = \lambda^{k+1} \leq 9K^4\lambda^{k-1} \leq 9K^4 d(x, y)$ and so by the definition of $\varphi(B_j^{k+1})$,

$$\mathcal{K}(x, y) \leq \varphi(B_j^{k+1})\chi_{B_j^{k+1}}(x)\chi_{B_j^{k+1}}(y) \leq \sum_{B \text{ dyadic}} \varphi(B)\chi_B(x)\chi_B(y),$$

and (3.8) follows after multiplying through by $f(y)$ and integrating with respect to $d\mu(y)$.

Using (3.8) in the left side of (3.7) we obtain

$$\int_X T(f\sigma)gw\,d\mu \le \sum_{B \text{ dyadic}} \varphi(B)\left(\int_B f\sigma\,d\mu\right)\left(\int_B gw\,d\mu\right). \tag{3.9}$$

Following Section 2, let $\{Q_j^k\}_j$ be the collection of maximal dyadic balls over which the average of gw, with respect to $d\mu$, exceeds γ^k, k in $\mathbb{Z}$. Here γ is chosen so large, depending on the doubling constant of μ, that $(1/\mu(Q_j^k)) \int_{Q_j^k} gw\,d\mu \le \gamma^{k+1}$. To see that such γ exists, note that Q_j^k has radius λ^ℓ for some ℓ, and therefore by (3.5)(i), there is a ball $B_i^{\ell+1}$ of radius $\lambda^{\ell+1}$ such that $Q_j^k \subset B_i^{\ell+1}$. By doubling, there exists γ depending only on λ and the doubling constant of μ such that $\mu(B_i^{\ell+1}) \le \gamma\mu(Q_j^k)$. Moreover, by the maximality of Q_j^k, $(1/\mu(B_i^{\ell+1})) \int_{B_i^{\ell+1}} gw\,d\mu \le \gamma^k$, and consequently

$$\frac{1}{\mu(Q_j^k)}\int_{Q_j^k} gw\,d\mu \le \frac{\mu(B_i^{\ell+1})}{\mu(Q_j^k)}\left[\frac{1}{\mu(B_i^{\ell+1})}\int_{B_i^{\ell+1}} gw\,d\mu\right] \le \gamma\gamma^k = \gamma^{k+1},$$

as desired. Note also by (3.6) that $\{\hat{Q}_j^k\}_j$ are pairwise disjoint.

For each $k \in \mathbb{Z}$, let

$$\mathscr{C}^k = \left\{B \text{ dyadic} : \gamma^k < \frac{1}{\mu(B)}\int_B gw\,d\mu \le \gamma^{k+1}\right\}.$$

Note that every B (for which gw is not identically zero on B) belongs to exactly one $\mathscr{C}^k$. Note also that $Q_j^k \in \mathscr{C}^k$ by construction. Thus,

$$\frac{1}{\mu(B)}\int_B gw\,d\mu \le \gamma\,\frac{1}{\mu(Q_j^k)}\int_{Q_j^k} gw\,d\mu \quad \text{if} \quad B \in \mathscr{C}^k,\ \forall j.$$

If $B \in \mathscr{C}^k$ then since $1/\mu(B) \int_B gw\,d\mu > \gamma^k$, it follows that $B \subset Q_j^k$ for some j. The right side of (3.9) is

$$\sum_B \varphi(B)\left(\int_B f\sigma d\mu\right)\left(\int_B gwd\mu\right)$$

$$= \sum_k \sum_{B\in\mathscr{C}^k} \mu(B)\varphi(B)\left(\int_B f\sigma d\mu\right)\left(\frac{1}{\mu(B)}\int_B gwd\mu\right)$$

$$\leq \gamma \sum_k \sum_j \left[\sum_{B\subset Q_j^k} \mu(B)\varphi(B)\left(\int_B f\sigma d\mu\right)\right] \frac{1}{\mu(Q_j^k)}\int_{Q_j^k} gwd\mu$$

$$\leq MC_{\epsilon,\lambda}\gamma \sum_{k,j} \varphi(Q_j^k)\left(\int_{Q_j^k} f\sigma d\mu\right)\left(\int_{Q_j^k} gwd\mu\right),$$

where the last inequality follows from

$$\sum_{B\ \text{dyadic}\subset B_o} \mu(B)\varphi(B)\int_B f\sigma d\mu$$

$$\leq C_\epsilon \sum_{B\ \text{dyadic}\subset B_o} \left(\frac{r(B)}{r(B_o)}\right)^\epsilon \left(\int_B f\sigma d\mu\right)\mu(B_o)\varphi(B_o), \qquad \text{by (1.20)},$$

$$= C_\epsilon\mu(B_o)\varphi(B_o) \sum_{\ell=0}^\infty \sum_{\substack{B:B\subset B_o\\ r(B)=\lambda^{-\ell}r(B_o)}} \left(\frac{r(B)}{r(B_o)}\right)^\epsilon \int_B f\sigma d\mu$$

$$\leq C_\epsilon\mu(B_o)\varphi(B_o)\left(\sum_{\ell=0}^\infty \lambda^{-\epsilon\ell}\right)M\int_{B_o} f\sigma d\mu, \qquad \text{by (3.5)(ii)},$$

$$= C_{\epsilon,\lambda}\mu(B_o)\varphi(B_o)\int_{B_o} f\sigma d\mu$$

for any dyadic ball B_o. Combining the above inequalities we have

$$\text{(3.10)} \qquad \int_X T(f\sigma)gwd\mu \leq C\sum_{k,j} \varphi(Q_j^k)\left(\int_{Q_j^k} f\sigma d\mu\right)\left(\int_{Q_j^k} gwd\mu\right).$$

As in (2.8) of Section 2 let us show that

$$\sum_{i:Q_i^\ell \subset Q_j^k} \mu(Q_i^\ell) \le C\gamma^{1-(\ell-k)}\mu(Q_j^k), \qquad \text{for all } k, j, \ell, \tag{3.11}$$

where C depends only on K and the doubling constant for μ; we also have that $\ell > k$ if $Q_i^\ell \subsetneqq Q_j^k$. Indeed,

$$\sum_{i:Q_i^\ell \subset Q_j^k} \mu(Q_i^\ell) \le C \sum_{i:Q_i^\ell \subset Q_j^k} \mu(\hat{Q}_i^\ell), \qquad \text{by (3.2)},$$

$$\le C\mu\left(\bigcup_{i:Q_i^\ell \subset Q_j^k} Q_i^\ell\right), \qquad \text{by (3.6)}, \tag{3.12}$$

$$\le C \sum_{i\in\Gamma} \mu(Q_i^\ell)$$

for some pairwise disjoint subcollection $\{Q_i^\ell\}_{i\in\Gamma}$ by Lemma 3.3. Now note that $\mu(Q_i^\ell)^{-1} \int_{Q_i^\ell} gwd\mu > \gamma^\ell$ while $\mu(Q_j^k)^{-1} \int_{Q_j^k} gwd\mu \le \gamma^{k+1}$, so that the right side of (3.12) is at most

$$C \sum_{i\in\Gamma} \gamma^{-\ell} \int_{Q_i^\ell} gwd\mu \le C\gamma^{-\ell} \int_{Q_j^k} gwd\mu \le C\gamma^{1-(\ell-k)}\mu(Q_j^k).$$

The proof that $\ell > k$ if $Q_i^\ell \subsetneqq Q_j^k$ is similar to the proof of the corresponding fact in Section 2.

Now set

$$\mathcal{A}_j^k = \mu(Q_j^k)^{1/r'}\left(\int_{Q_j^k} \sigma^r d\mu\right)^{1/r}$$

and

$$\mathcal{B}_j^k = \mu(Q_j^k)^{1/r'}\left(\int_{Q_j^k} w^r d\mu\right)^{1/r}$$

where r is as in condition (1.19). Just as in (2.9) of Section 2, condition

(1.19) and Hölder's inequality imply

$$(3.13)\quad \sum_{k,j} \varphi(Q_j^k)\left(\int_{Q_j^k} f\sigma d\mu\right)\left(\int_{Q_j^k} gw d\mu\right)$$
$$\leq C\left(\sum_{k,j} \mathcal{A}_j^k\left(\frac{1}{\mathcal{A}_j^k}\int_{Q_j^k} f\sigma d\mu\right)^p\right)^{1/p}\left(\sum_{k,j} \mathcal{B}_j^k\left(\frac{1}{\mathcal{B}_j^k}\int_{Q_j^k} gw d\mu\right)^{q'}\right)^{1/q'}.$$

Also, as in (2.11) of Section 2, (3.11), together with Hölder's inequality, yields the Carleson condition

$$(3.14)\qquad \sum_{\ell,i:Q_i^\ell \subset Q_j^k} \mathcal{A}_i^\ell \leq C\mathcal{A}_j^k, \text{ for all } k, j.$$

For the part of the proof of this analogue of (2.11) which uses the inequality

$$\sum_{i:Q_i^\ell \subset Q_j^k} \int_{Q_i^\ell} \sigma^r d\mu \leq C \int_{Q_j^k} \sigma^r d\mu,$$

we will use the doubling property of $\sigma^r d\mu$. In fact, the sum on the left is then at most

$$C \sum_{i:Q_i^\ell \subset Q_j^k} \int_{\hat{Q}_i^\ell} \sigma^r d\mu \leq C \int_{Q_j^k} \sigma^r d\mu,$$

since by (3.6) the balls $\{\hat{Q}_i^\ell\}_{i:Q_i^\ell \subset Q_j^k}$ are pairwise disjoint. The analogue of (3.14) for $\mathcal{B}_j^k$ requires the doubling condition on $w^r d\mu$. The proof of part (A) of Theorem 3 is now completed as in Section 2 using the following analogue of Lemma 2.10 with $\beta = 1$, $a(B)$ replaced first by $\mathcal{A}(B) = \mu(B)^{1/r'}(\int_B \sigma^r d\mu)^{1/r}$ and then by $\mathcal{B}(B) = \mu(B)^{1/r'}(\int_B w^r d\mu)^{1/r}$, and with $\Gamma = \{Q_j^k\}_{j,k}$. We note that if $a(B)$ is defined in either of these ways, then it satisfies (3.16)(i), (ii) below by using the doubling of μ, $\sigma^r d\mu$ and $w^r d\mu$, and by using Hölder's inequality, respectively.

Lemma 3.15. *Suppose $a(B)$ is a nonnegative function, defined for all balls B, that satisfies,* (3.16) (i) (*doubling*) $a(B(x, 2r)) \leq Ca(B(x, r))$, *for all* $x \in X$, $r > 0$,

(ii) (*weak superadditivity*) $\sum_{B\in\Omega} a(B) \leq Ca(B_o)$ *whenever* Ω *is a collection of pairwise disjoint balls contained in a ball* B_o.

Suppose further that $u(x) \geq 0$ *on* X, $\beta \geq 1$, Γ *is a countable collection of dyadic balls in* X *and*

$$\text{(3.17)}\quad \begin{aligned} &\text{(i)} \int_B u\,d\mu \leq Ca(B), \qquad \textit{for all } B \in \Gamma,\\ &\text{(ii)} \sum_{\substack{B\in\Gamma\\ B\subset B_o}} a(B)^\beta \leq Ca(B_o)^\beta, \qquad \textit{for all } B_o \in \Gamma. \end{aligned}$$

Then

$$\left(\sum_{B\in\Gamma} a(B)^\beta\left(\frac{1}{a(B)}\int_B fu\,d\mu\right)^t\right)^{1/t} \leq C_{s,t}\left(\int_X f^s u\,d\mu\right)^{1/s},$$

for all $f \geq 0$ *on* X *and* $t = s\beta$, $1 < s < \infty$.

Proof of Lemma 3.15. As in the proof of Lemma 2.10, we show that the map $f \to (1/a(B)\int_B fu\,d\mu)_{B\in\Gamma}$ takes $L^\infty(X, u\,d\mu)$ to $\ell^\infty(\Gamma, a(B)^\beta)$ by (3.17)(i), and $L^1(X, u\,d\mu)$ to weak $\ell^\beta(\Gamma, a(B)^\beta)$ by (3.16) and (3.17)(ii). For f bounded with support in some ball and $\lambda > 0$, let $\{Q_j\}_{j\in J}$ be the maximal dyadic balls B in Γ such that $1/a(B)\int_B |f| u\,d\mu > \lambda$ (we may assume Γ is finite). Then

$$\text{(3.18)}\quad \begin{aligned} \sum_{B\in\Gamma:|1/a(B)\int_B fu\,d\mu|>\lambda} a(B)^\beta &\leq \sum_{j\in J}\sum_{B\subset Q_j} a(B)^\beta \\ &\leq C\sum_{j\in J} a(Q_j)^\beta, \qquad \text{by (3.17)(ii)} \\ &\leq C\left(\sum_{j\in J} a(\hat{Q}_j)\right)^\beta, \\ &\text{by (3.16)(i) and since } \beta \geq 1. \end{aligned}$$

Now by Lemma 3.3, there is a pairwise disjoint subcollection $\{Q_i\}_{i\in I}$ of the dyadic balls $\{Q_j\}_{j\in J}$ with the property that every Q_j, $j \in J$, is contained in some $\tilde{Q}_i$, $i \in I$, as in (3.4)(ii). Thus

$$\sum_{j\in J} a(\hat{Q}_j) \le \sum_{i\in I} \sum_{j\in J: Q_j \subset \tilde{Q}_i} a(\hat{Q}_j)$$

$$\le C \sum_{i\in I} a(\tilde{Q}_i), \qquad \text{by (3.16)(ii) and (3.6),}$$

$$\le C \sum_{i\in I} a(Q_i), \qquad \text{by (3.16)(i),}$$

$$\le \frac{C}{\lambda} \sum_{i\in I} \int_{Q_i} |f| u\, d\mu \le \frac{C}{\lambda} \int_X |f| u\, d\mu,$$

since the balls Q_i, $i \in I$, are pairwise disjoint. Combining this inequality with (3.18) shows the weak $(1, \beta)$ boundedness of the map $f \to (1/a(B) \int_B fu d\mu)_{B\in\Gamma}$. The (∞, ∞) boundedness is obvious from (3.17)(i) and interpolation completes the proof of the lemma.

Proof of part (B) of Theorem 3. Suppose that $p < q$ and that (1.18) holds. If suffices to show that (3.7) holds. Using (3.9) and (1.18) and letting $|B|_\sigma = \int_B \sigma d\mu$ and $|B|_w = \int_B w d\mu$, we obtain

$$\int_X T(f\sigma) gw\, d\mu \le \sum_{B \text{ dyadic}} \varphi(B) \left(\int_B f\sigma d\mu\right)\left(\int_B gw d\mu\right) \tag{3.19}$$

$$\le C \sum_{B \text{ dyadic}} |B|_\sigma^{1/p} \left(\frac{1}{|B|_\sigma} \int_B f\sigma d\mu\right) |B|_w^{1/q'} \left(\frac{1}{|B|_w} \int_B gw d\mu\right)$$

$$\le C \left(\sum_{B \text{ dyadic}} |B|_\sigma^{r/p} \left(\frac{1}{|B|_\sigma} \int_B f\sigma d\mu\right)^r\right)^{1/r} \left(\sum_{B \text{ dyadic}} |B|_w^{r'/q'} \left(\frac{1}{|B|_w} \int_B gw d\mu\right)^{r'}\right)^{1/r'}$$

by Hölder's inequality with exponents r and r' where r is chosen such that $p < r < q$ (so also $q' < r' < p'$). We now claim that

$$\sum_{B \text{ dyadic} \subset B_o} |B|_\sigma^\beta \le C_\beta |B_o|_\sigma^\beta, \text{ for all } \beta > 1 \text{ and dyadic balls } B_o. \tag{3.20}$$

In fact,

$$\sum_{B \text{ dyadic} \subset B_o} |B|_\sigma^\beta = \sum_{\ell=0}^{\infty} \sum_{\substack{B \text{ dyadic} \subset B_o \\ r(B)/r(B_o)=\lambda^{-\ell}}} |B|_\sigma^{\beta-1} |B|_\sigma$$

$$\leq \sum_{\ell=0}^{\infty} \sum_{\substack{B \text{ dyadic} \subset B_o \\ r(B)/r(B_o)=\lambda^{-\ell}}} (C\delta^\ell |B_o|_\sigma)^{\beta-1} |B|_\sigma,$$

for some $\delta < 1$ since $\sigma d\mu \in (D) \subset (RD)$,

$$\leq \sum_{\ell=0}^{\infty} (C\delta^\ell |B_o|_\sigma)^{\beta-1} M |B_o|_\sigma, \qquad \text{by (3.5)(ii)}$$

$$= C|B_o|_\sigma^\beta, \qquad \text{since } \beta > 1 \text{ and } \delta < 1.$$

Now set $\beta = r/p > 1$ and let $a(B) = |B|_\sigma$ for B a dyadic ball. Then (3.20) and Lemma 3.15 ((3.16)(i) holds since $\sigma d\mu \in (D)$) show that the first factor on the right side of (3.19) is dominated by $C(\int_X f^p \sigma d\mu)^{1/p}$. Similarly, the second factor is dominated by $C(\int_X g^{q'} w d\mu)^{1/q'}$, which proves (3.7) and completes the proof of part (B) of Theorem 3.

In order to prove part C of Theorem 3 (and Theorem 4), we need a more refined notion of dyadic grid than that given by the collection of dyadic balls used above. More specifically, we require that at each level, the "dyadic cubes" cover X and are pairwise disjoint. In addition, these "dyadic cubes" must look like balls and be nested in the sense that for any two dyadic cubes, either one is contained in the other, or they are pairwise disjoint. Our construction yields such an analogue for X, not of all dyadic cubes in X, but for dyadic cubes of side length at least 2^m, m arbitrarily close to minus infinity.

LEMMA 3.21. *Suppose (X, d) is a separable quasi-metric space, i.e.,* (3.1) *holds and X has a countable dense subset. Then there is $\lambda > 1$, depending only on K in* (3.1), *such that for every m in $\mathbb{Z}$, there are points x_j^k and Borel sets E_j^k, $1 \leq j < n_k$, $k \geq m$ (where $n_k \in \mathbb{N} \cup \{\infty\}$), such that*

(3.22) (i) $B(x_j^k, \lambda^k) \subset E_j^k \subset B(x_j^k, \lambda^{k+1})$, $1 \leq j < n_k$, $k \geq m$,

(ii) $X = \bigcup_j E_j^k$, $k \geq m$, *and* $E_j^k \cap E_i^k = \phi$ *if* $i \neq j$,

(iii) *Given* i, j, k, ℓ *with* $m \leq k < \ell$, *then either* $E_j^k \subset E_i^\ell$ *or* $E_j^k \cap E_i^\ell = \phi$.

Proof of Lemma 3.21. Set $\lambda = 8K^5$ where K is as in (3.1). For each k in $\mathbb{Z}$, choose $\{x_j^k\}_{1 \leq j < n_k}$ maximal with respect to the property that the balls $\{B(x_j^k, 3K^2\lambda^k)\}_{1 \leq j < n_k}$ are pairwise disjoint (the separability of X forces the cardinality of $\{x_j^k\}_j$ to be at most countable). Then, as in (3.5)(i), this maximal property together with (3.1)(iii) yields

$$X = \bigcup_j B(x_j^k, 6K^3\lambda^k). \tag{3.23}$$

In fact, given any $x \in X$, $B(x, 3K^2\lambda^k)$ intersects $B(x_i^k, 3K^2\lambda^k)$ for some i by maximality. If w is a common point then

$$d(x, x_i^k) \leq K[d(x, w) + d(w, x_i^k)] < K[3K^2\lambda^k + 3K^2\lambda^k] = 6K^3\lambda^k.$$

Thus $x \in B(x_i^k, 6K^3\lambda^k)$ and consequently (3.23) is valid.

Now fix m in $\mathbb{Z}$. Define

$$E_1^m = B(x_1^m, 6K^3\lambda^m) - \left[\bigcup_{i \neq 1} B(x_i^m, \lambda^m)\right]$$

$$E_2^m = B(x_2^m, 6K^3\lambda^m) - \left[\bigcup_{i \neq 2} B(x_i^m, \lambda^m)\right] - E_1^m$$

(3.24)

$$\vdots$$

$$E_j^m = B(x_j^m, 6K^3\lambda^m) - \left[\bigcup_{i \neq j} B(x_i^m, \lambda^m)\right] - \left[\bigcup_{i<j} E_i^m\right]$$

$$\vdots$$

Let us first show that properties (3.22)(i) and (ii) hold for $k = m$. The choice of λ makes it clear that $E_j^m \subset B(x_j^m, \lambda^{m+1})$. To see that $B(x_j^m, \lambda^m) \subset E_j^m$, first note that the balls $\{B(x_j^m, \lambda^m)\}_j$ are pairwise disjoint since in fact the larger balls $\{B(x_j^m, 3K^2\lambda^m)\}_j$ are pairwise disjoint. Thus

$$B(x_j^m, \lambda^m) \subset B(x_j^m, 6K^3\lambda^m)\setminus\bigcup_{i\neq j} B(x_i^m, \lambda^m).$$

Moreover, since $B(x_j^m, \lambda^m)$ is one of the balls subtracted in defining E_i^m for $i \neq j$, $B(x_j^m, \lambda^m)$ does not intersect $\cup_{i<j} E_i^m$. Thus $B(x_j^m, \lambda^m) \subset E_j^m$, which shows that (i) holds. For (ii), it is clear from (3.24) that the sets $\{E_j^m\}_j$ are pairwise disjoint. Let $x \in X$. If $x \in \cup_j B(x_j^m, \lambda^m)$ then $x \in \cup_j E_j^m$ by what has already been shown. If $x \notin \cup_j B(x_j^m, \lambda^m)$ and we use (3.23) for $k = m$ to pick the first j_o such that $x \in B(x_{j_o}^m, 6K^3\lambda^m)$, then $x \in E_{j_o}^m$ by (3.24). Thus we again see that $x \in \cup_j E_j^m$, so that $X = \cup_j E_j^m$.

We now construct $\{E_j^k\}$ for $k > m$ by induction on k. Suppose $\ell > m$ and that $\{E_j^k\}$ has been defined so as to satisfy (3.22) for $1 \leq j < n_k$, $m \leq k < \ell$. For any $R > 0$ and $1 \leq j < n_\ell$, set

$$\tilde{B}(x_j^\ell, R) = \cup\{E_i^{\ell-1} : E_i^{\ell-1} \cap B(x_j^\ell, R) \neq \phi\}.$$

Let us show that

$$B(x_j^\ell, R) \subset \tilde{B}(x_j^\ell, R) \subset B(x_j^\ell, K^2R + (K^2 + 1)\lambda^\ell), \qquad R > 0,\ 1 \leq j < n_k. \tag{3.25}$$

Indeed, the first inclusion follows from (3.22)(ii) with $k = \ell - 1$, and if $w \in E_i^{\ell-1}$ and $z \in E_i^{\ell-1} \cap B(x_j^\ell, R)$, then by (3.1)(iii) and (3.22)(i),

$$\begin{aligned} d(w, x_j^\ell) &\leq K[d(w, x_i^{\ell-1}) + K[d(x_i^{\ell-1}, z) + d(z, x_j^\ell)]] \\ &\leq K(\lambda^\ell + K(\lambda^\ell + R)) = K^2R + (K^2 + 1)\lambda^\ell. \end{aligned}$$

In analogy with (3.24), define

$$E_1^\ell = \tilde{B}(x_1^\ell, 6K^3\lambda^\ell) - \left[\bigcup_{i\neq 1} \tilde{B}(x_i^\ell, \lambda^\ell)\right]$$

$$E_2^\ell = \tilde{B}(x_2^\ell, 6K^3\lambda^\ell) - \left[\bigcup_{i\neq 2} \tilde{B}(x_i^\ell, \lambda^\ell)\right] - E_1^\ell$$

$$\vdots \tag{3.26}$$

$$E_j^\ell = \tilde{B}(x_j^\ell, 6K^3\lambda^\ell) - \left[\bigcup_{i\neq j} \tilde{B}(x_i^\ell, \lambda^\ell)\right] - \left[\bigcup_{i<j} E_i^\ell\right]$$

$$\vdots$$

Taking $R = 6K^3\lambda^\ell$ in (3.25) shows that

$$\begin{aligned} E_j^\ell &\subset B(x_j^\ell, K^2 6K^3\lambda^\ell + (K^2+1)\lambda^\ell) \\ &\subset B(x_j^\ell, \lambda^{\ell+1}), \qquad 1 \le j < n_\ell, \end{aligned} \tag{3.27}$$

since $6K^5 + K^2 + 1 \le 8K^5 = \lambda$, $K \ge 1$. On the other hand with $R = \lambda^\ell$ in (3.25), we have

$$\tilde{B}(x_i^\ell, \lambda^\ell) \subset B(x_i^\ell, (2K^2+1)\lambda^\ell) \subset B(x_i^\ell, 3K^2\lambda^\ell). \tag{3.28}$$

Now if $i \neq j$, then $B(x_i^\ell, 3K^2\lambda^\ell)$ and $B(x_j^\ell, 3K^2\lambda^\ell)$ are disjoint and it follows from (3.25), (3.28) and the definitions in (3.26) that

$$B(x_j^\ell, \lambda^\ell) \subset E_j^\ell, \qquad 1 \le j < n_\ell. \tag{3.29}$$

Inclusions (3.27) and (3.29) together with the induction assumption show that (3.22)(i) holds for $1 \le j < n_k$, $m \le k \le \ell$.

To verify property (3.22)(ii) we begin by noting that $X = \bigcup_j \tilde{B}(x_j^\ell, 6K^3\lambda^\ell)$ by (3.23) with $k = \ell$ and by the first inclusion in (3.25) with $R = 6K^3\lambda^\ell$. Also,

$$\tilde{B}(x_j^\ell, \lambda^\ell) \subset \tilde{B}(x_j^\ell, 6K^3\lambda^\ell) \setminus \bigcup_{i \neq j} \tilde{B}(x_i^\ell, \lambda^\ell)$$

by using (3.28) and the fact that the balls $B(x_i^\ell, 3K^2\lambda^\ell)$ are pairwise disjoint in i. Thus

$$\bigcup_j \tilde{B}(x_j^\ell, \lambda^\ell) \subset \bigcup_j E_j^\ell.$$

If $x \in X$, pick the first j_o such that $x \in \tilde{B}(x_{j_o}^\ell, 6K^3\lambda^\ell)$. To show that $x \in \bigcup_j E_j^\ell$, we may assume by the above that $x \notin \bigcup_j \tilde{B}(x_j^\ell, \lambda^\ell)$. Then $x \in E_{j_o}^\ell$ by (3.26).

Finally, to show (3.22)(iii), it is enough by the induction assumption to show that if $m \le k < \ell$ and $E_{i_o}^k$ intersects $E_{j_o}^\ell$, then $E_{i_o}^k \subset E_{j_0}^\ell$. Since each $\tilde{B}(x_j^\ell, R)$ is a union of certain $E_i^{\ell-1}$ and since the $E_i^{\ell-1}$ are pairwise disjoint in i, it follows that each E_j^ℓ is also a union of certain $E_i^{\ell-1}$ and that any $E_i^{\ell-1}$ which intersects E_j^ℓ must be contained in E_j^ℓ. Hence if $E_{i_o}^k \cap E_{j_o}^\ell \neq \phi$, then $E_{i_o}^k \cap E_i^{\ell-1} \neq \phi$ for some i such that $E_i^{\ell-1} \subset E_{j_o}^\ell$. Since $m \le k \le \ell - 1$, the induction hypothesis implies that $E_{i_o}^k \subset E_i^{\ell-1}$, and consequently that $E_{i_o}^k \subset E_{j_o}^\ell$. This completes the inductive step and the proof of Lemma 3.21.

Proof of Part (C) *of Theorem* 3. Condition (1.21) follows from setting $f = \chi_Q v^{1-p'}$ in (1.17) and condition (1.22) follows from setting $g = \chi_Q w$ in the inequality dual to (1.17),

$$\left(\int_X [T^*(gw)]^{p'} \sigma d\mu\right)^{1/p'} \le C\left(\int_X g^{q'} w d\mu\right)^{1/q'}, \qquad \text{for all } g \ge 0.$$

The proof of the converse for the case $X = \mathbb{R}^n$ given in [S4] applies almost verbatim in the present context once we have set up the appropriate Whitney decomposition for an open set in X. Lemma 3.21 is the key to accomplishing this as follows. Fix $m \in Z$ and denote by $\mathcal{D}_m$ the collection of sets $\{E_j^k : 1 \le j < n_k, k \ge m\}$ given in Lemma 3.21. We shall refer to these sets as dyadic cubes and use Q to denote them. Furthermore, if $Q = E_j^k$, let Q^* denote the containing ball $B(x_j^k, \lambda^{k+1})$ in (3.22)(i) where $\lambda = 8K^5$.

Now suppose (1.21) and (1.22) hold. In particular, they hold for $Q \in \mathfrak{D}_m$. Suppose also, without loss of generality, that f is nonnegative and bounded with support in some ball. Let $R = 3K^2$ and for each k in Z, let $\{Q_j^k\}_j$ denote the dyadic cubes maximal among those dyadic cubes Q with the property that RQ^* is contained in the open set $\Omega_k = \{x : T(f\sigma)(x) > 2^k\}$. Let $\Omega_{k,m}$ be the subset of Ω_k consisting of those x for which there exists $Q \in \mathfrak{D}_m$ with $x \in Q$ and $RQ^* \subset \Omega_k$. Let E^c denote the complement of a set E. We claim that

$$
(3.30)\quad
\begin{aligned}
&\text{(i) } \Omega_{k,m} = \bigcup_j Q_j^k \text{ and } Q_j^k \cap Q_i^k = \phi \text{ for } i \neq j.\\
&\text{(ii) } RQ_j^{k*} \subset \Omega_k \text{ and } 2KR\lambda Q_j^{k*} \cap {\Omega_k}^c \neq \phi, \text{ for all } k, j.\\
&\text{(iii) } \sum_j \chi_{2KQ_j^{k*}} \leq C\chi_{\Omega_k}, \text{ for all } k.\\
&\text{(iv) The number of cubes } Q_s^k \text{ intersecting a fixed}\\
&\qquad \text{ball } 2KQ_j^{k*} \text{ is at most } C.\\
&\text{(v) } Q_j^k \subsetneqq Q_i^\ell \text{ implies } k > \ell.
\end{aligned}
$$

Properties (i) and (v) of (3.30) as well as the first statement in (ii) follow immediately from the definitions. Suppose $Q_j^k = E_i^\ell$. By the maximality of Q_j^k, if $E_h^{\ell+1}$ is the unique dyadic cube containing E_i^ℓ, then $RB(x_h^{\ell+1}, \lambda^{\ell+2})$ intersects ${\Omega_k}^c$. If z is in this intersection, then

$$
d(z, x_i^\ell) \leq K[d(z, x_h^{\ell+1}) + d(x_h^{\ell+1}, x_i^\ell)] \leq K[R\lambda^{\ell+2} + \lambda^{\ell+2}] \leq 2KR\lambda^{\ell+2},
$$

which proves the second statement in (3.30)(ii). To prove (3.30)(iii), we adapt the packing argument on page 16 of [F1]. Fix z in Ω_k and suppose $z \in 2KQ_j^{k*}$ where $Q_j^k = E_i^\ell$. We claim that

$$
(3.31)\qquad K\lambda^{\ell+1} \leq d(z, {\Omega_k}^c) \leq 50K^9\lambda^{\ell+1}.
$$

Indeed, the first statement in (ii) shows that

$$3K^2\lambda^{\ell+1} = R\lambda^{\ell+1} \le d(x_i^\ell, \Omega_k{}^c) \le K[d(x_i^\ell, z) + d(z, \Omega_k{}^c)]$$

$$\le K[2K\lambda^{\ell+1} + d(z, \Omega_k{}^c)]$$

which proves the first inequality in (3.31). The second statement in (3.30)(ii) yields $d(x_i^\ell, \Omega_k{}^c) \le 2KR\lambda^{\ell+2}$ and so

$$d(z, \Omega_k{}^c) \le K[d(z, x_i^\ell) + d(x_i^\ell, \Omega_k{}^c)] \le K[2K\lambda^{\ell+1} + 2KR\lambda^{\ell+2}]$$

$$= 2K^2(\lambda R + 1)\lambda^{\ell+1} \le 50K^9\lambda^{\ell+1}$$

which proves the second inequality in (3.31).

Now set $s = d(z, \Omega_k{}^c)$. Then

$$Q_j^k = E_i^\ell \subset B(z, 3Ks), \tag{3.32}$$

since if $w \in E_i^\ell$, then

$$d(z, w) \le K[d(z, x_i^\ell) + d(x_i^\ell, w)]$$

$$< K[2K\lambda^{\ell+1} + \lambda^{\ell+1}] \le 3K^2\lambda^{\ell+1} \le 3Ks$$

by (3.31). We also have

$$B(z, 3Ks) \subset 1216\ K^{16}B(x_i^\ell, \lambda^\ell) \tag{3.33}$$

since if $w \in B(z, 3Ks)$, then

$$d(w, x_i^\ell) \le K[d(w, z) + d(z, x_i^\ell)] \le K[3Ks + 2K\lambda^{\ell+1}]$$

$$\le 152\ K^{11}\lambda^{\ell+1} = 1216\ K^{16}\lambda^\ell.$$

Since μ is doubling, there is thus a constant $C(K)$, depending only on K and the doubling constant for μ, such that

$$\begin{aligned} \mu(B(z, 3Ks)) &\le C(K)\mu(B(x_i^\ell, \lambda^\ell)) \\ &\le C(K)\mu(E_i^\ell) = C(K)\mu(Q_j^k). \end{aligned} \tag{3.34}$$

Altogether we have

$$\sum_j \chi_{2KQ_j^{k*}}(z) \le \frac{C(K)}{\mu(B(z, 3Ks))} \sum_j \mu(Q_j^k)\chi_{2KQ_j^{k*}}(z), \text{ by (3.34)}$$

$$= \frac{C(K)}{\mu(B(z, 3Ks))} \mu(\bigcup_j \{Q_j^k : z \in 2KQ_j^{k*}\}), \text{ by (3.30)(i)}$$

$$\le C(K)$$

by (3.32) and this proves (3.30)(iii).

Finally, to prove (3.30)(iv), suppose Q_s^k intersects $2KQ_j^{k*}$. Let z be in the intersection. If $Q_j^k = E_i^\ell$ and $Q_s^k = E_h^n$, then (3.31) holds both for ℓ and for n in place of ℓ. Thus $K\lambda^{n+1} \le d(z, \Omega_k{}^c) \le 50K^9\lambda^{\ell+1}$ or $\lambda^n \le 50K^8\lambda^\ell$. It follows that

$$Q_s^k = E_h^n \subset 102K^{10}B(x_i^\ell, \lambda^{\ell+1}) = 102K^{10}Q_j^{k*} \tag{3.35}$$

since if $w \in E_h^n$, then $d(w, z) \le 2K\lambda^{n+1} \le 100K^9\lambda^{\ell+1}$ and so

$$d(w, x_i^\ell) \le K[d(w, z) + d(z, x_i^\ell)]$$

$$\le K[100K^9\lambda^{\ell+1} + 2K\lambda^{\ell+1}] \le 102K^{10}\lambda^{\ell+1}.$$

Arguing much the same as in (3.33) and (3.34) we also have

$$\mu(102K^{10}Q_j^{k*}) \le C(K)\mu(Q_s^k) \tag{3.36}$$

for a constant $C(K)$ depending only on K and the doubling constant for μ. Altogether then

$$\#\{s : Q_s^k \cap 2KQ_j^{k*} \ne \phi\} \le \frac{C(K)}{\mu(102K^{10}Q_j^{k*})}$$

$$\cdot \sum_s \{\mu(Q_s^k) : Q_s^k \cap 2KQ_j^{k*} \ne \phi\}, \text{ by (3.36)}$$

$$\le \frac{C(K)}{\mu(102K^{10}Q_j^{k*})} \mu(\bigcup_s \{Q_s^k : Q_s^k \cap 2KQ_j^{k*} \ne \phi\})$$

$$\le C(K)$$

by (3.35) and this completes the proof of (3.30).

Now the proof proceeds exactly as in the proof of Theorem 1 of [S4], but with the measure μ and the cubes $3Q_j^k$ in [S4] replaced everywhere by the measure $\sigma d\mu = v^{1-p'}d\mu$ and the balls $2KQ_j^{k*}$ respectively. The maximum principle (see inequality (2.3) in [S4]),

$$T(\chi_{(2KQ_j^{k*})^c} f\sigma)(x) \leq C2^k, \qquad x \in Q_j^k, \tag{3.37}$$

follows from the argument in [S4] and the property (1.13)(i) of the kernel $\mathcal{K}$. Indeed, if $x \in Q_j^k = E_i^\ell$ and $y \notin 2KQ_j^{k*}$, then

$$\begin{aligned} 2K\lambda^{\ell+1} &\leq d(x_i^\ell, y) \leq K[d(x_i^\ell, x) + d(x, y)] \\ &\leq K[\lambda^{\ell+1} + d(x, y)], \end{aligned}$$

which yields $d(x, y) \geq \lambda^{\ell+1}$. If we now choose $x' \in 2KR\lambda Q_j^{k*} \cap \Omega_k{}^c$, which is possible by (3.30)(ii), then

$$\begin{aligned} d(x', y) &\leq K\{K[d(x', x_i^\ell) + d(x_i^\ell, x)] + d(x, y)\} \\ &\leq K\{K[2KR\lambda^{\ell+2} + \lambda^{\ell+1}] + d(x, y)\} \\ &\leq K\{K[2KR\lambda + 1] + 1\}d(x, y) = C_2 d(x, y). \end{aligned}$$

Condition (1.13)(i) now shows $\mathcal{K}(x, y) \leq C_1\mathcal{K}(x', y)$, and if we multiply this inequality by $f(y)\sigma(y)$ and then integrate over $(2KQ_j^{k*})^c$ with respect to $d\mu(y)$, we obtain (3.37).

The growth estimate (see the displayed inequality following (2.7) in [S4]),

$$\max_{x\in Q_i^{k+m}} \mathcal{K}^*(x, y) \leq C \min_{x\in Q_i^{k+m}} \mathcal{K}^*(x, y), \qquad y \notin 2KQ_i^{k+m*},$$

follows from property (1.13)(ii) of $\mathcal{K}$ since if $x, x' \in Q_i^{k+m} = E_h^\ell$ and $y \notin 2KQ_i^{k+m*}$, then as we shall see, $d(y, x') \leq C_2 d(y, x)$, and so $\mathcal{K}^*(x, y)$

$\leq C_1\mathcal{K}^*(x', y)$ by (1.13)(ii). To see that $d(y, x') \leq C_2 d(y, x)$, first note that

$$2K\lambda^{\ell+1} \leq d(y, x_h^\ell) \leq K[d(y, x) + d(x, x_h^\ell)]$$

$$\leq K[d(y, x) + \lambda^{\ell+1}],$$

so that $d(y, x) \geq \lambda^{\ell+1}$. Then

$$d(y, x') \leq K\{d(y, x) + K[d(x, x_h^\ell) + d(x_h^\ell, x')]\}$$

$$\leq K\{d(y, x) + 2K\lambda^{\ell+1}\}$$

$$\leq K\{d(y, x) + 2Kd(y, x)\} = C_2 d(y, x).$$

Finally, we correct a small error in the argument in [S4], deriving from the invalid assertion (see (2.2)(v) of [S4]) that $Q_j^k \subset Q_i^\ell$ implies $k \geq \ell$, used there in place of (3.30)(v). In the construction of the "principal" cubes following (2.10) in [S4], all indices (k, j) should in addition be restricted to lie in G, since for (k, j), $(\ell, i) \in G$, it is true that $Q_j^k \subset Q_i^\ell$ implies $k \geq \ell$. Indeed, if Q_j^k is properly contained in Q_i^ℓ, then $k > \ell$ while if $Q_j^k = Q_i^\ell$ with $\ell > k$, then $\ell \geq k + \mathfrak{m}$ (Since "m" is used here for a different purpose than in [S4], we use "$\mathfrak{m}$" to denote the "m" in [S4].) and so $E_j^k = \phi$, contradicting the fact that $|E_j^k|_\omega > 0$ for $(k, j) \in G$ (see (2.4) of [S4]). The statement, at the top of page 542 of [S4], that any fixed $Q_i^{k+\mathfrak{m}}$ occurs at most C times in a certain sum over $(k, j) \in G$, can now be verified. Using the notation of both this section and [S4], we give the details. The problem is to show that for a given dyadic cube Q, there are at most C pairs of indices (k, j) satisfying

(3.38) (i) $k \equiv 0 \pmod{\mathfrak{m}}$

(ii) The cube Q_j^k is not among the cubes $\{Q_i^{k+\mathfrak{m}}\}_i$

(iii) $Q = Q_i^{k+\mathfrak{m}}$ for some i such that $Q_i^{k+\mathfrak{m}} \cap 2KQ_j^{k*} \neq \phi$.

This will suffice because, as has already been observed, (3.38)(ii) holds for $(k, j) \in G$ since $E_j^k = Q_j^k \setminus \cup_i Q_i^{k+\mathfrak{m}}$ has positive ω-measure.

So fix a dyadic cube Q. Then there exists a sequence of consecutive multiples of $\mathfrak{m}$, say $k_1, k_2, \ldots, k_n$ with $n \geq 1$, and a corresponding sequence $i_1, i_2, \ldots, i_n$ such that $Q = Q_{i_\sigma}^{k_\sigma+m}$ for $\sigma = 1, 2, \ldots, n$ and such that $Q = Q_i^{k+m}$ implies $(k, i) = (k_\sigma, i_\sigma)$ for some σ. Now the finite overlap property (3.30)(iii) shows that there are at most C pairs (k, j) satisfying (3.38) with $k = k_1$. On the other hand, if (k, j) satisfies (3.38) with $k = k_\sigma$, $\sigma > 1$, then $Q = Q_{i_{\sigma-1}}^{k_{\sigma-1}+m} = Q_{i_\sigma-1}^{k_\sigma}$ is among the dyadic cubes $\{Q_s^k\}_s$. Since Q intersects $2KQ_j^{k*}$, (3.30)(ii) and (3.31) together show that the side length of the dyadic cube Q_j^k is bounded between constant multiples of the side length of Q. Now there are at most C dyadic cubes Q' whose side lengths are bounded between constant multiples of that of Q and such that $2KQ'^*$ intersects Q. By property (3.38)(ii), there is, for each such dyadic cube Q', at most one pair (k, j) with $Q_j^k = Q'$, and this completes the proof that there are at most C pairs (k, j) such that (3.38) holds.

The conclusion that now follows from the proof in [S4] is that, for some integer m depending only on the constant C in (3.37), we have

$$\sum_k (2^{k+m})^q |\Omega_{k+m-1,m} \backslash \Omega_{k+m,m}|_w \leq C\left(\int f^p \sigma d\mu\right)^{1/p} \tag{3.39}$$

with C independent of m. Since the sets $\Omega_{k,m}$ increase to Ω_k as m decreases to $-\infty$, we obtain

$$\begin{aligned} \int [T(f\sigma)]^q w d\mu &\leq \lim_{m\to-\infty} \sum_k (2^{k+m})^q |\Omega_{k+m-1,m}|_w \\ &= (1 - 2^{-q}) \lim_{m\to-\infty} \sum_k (2^{k+m})^q |\Omega_{k+m-1,m} \backslash \Omega_{k+m,m}|_w \leq C\left(\int f^p \sigma d\mu\right)^{1/p} \end{aligned} \tag{3.40}$$

by (3.39). If we replace f by $fv^{p'-1}$, then (3.40) becomes (1.17) and this completes the proof of Part (C) of Theorem 3.

4. Operators on homogeneous spaces with a group structure. The purpose of this section is to prove a version of Theorem 3(A) without any doubling assumptions on the weights, and also to extend Theorem 1(B)—the characterization of the weighted inequality for fractional integrals when $p < q$—to homogeneous spaces X. In the event that the Besicovitch covering lemma holds for X, then the characterization of the weak type inequality in [GK] also holds and, combined with Theorem 3(C), this extends Theorem 1(B) to X. However, the Besicovitch covering lemma typically fails if the family of balls in X become increasingly eccentric as they shrink, and tilt as their centres move. For example, we show below that the Besicovitch lemma fails on the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$ with

$$d((z, t), (w, s)) = |(w, s)^{-1}(z, t)| \tag{4.1}$$

where $(z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}(z_1\overline{z_2}))$ for $(z_i, t_i) \in \mathbb{C}^n \times \mathbb{R}$, and $|(z, t)| = (|z|^4 + t^2)^{1/4}$. Note that $(w, s)^{-1} = (-w, -s)$ and so the right side of (4.1) is $\{|z - w|^4 + [t - s + 2\,\mathrm{Im}(z\overline{w})]^2\}^{1/4}$. See Section 4 of [N] for a proof that d is a left invariant quasi-metric making $(\mathbb{C}^n \times \mathbb{R}, d, \mu)$, where μ denotes Lebesgue measure, into a homogeneous space. (In fact, d is a metric; see [C].)

Let us say that a quasi-metric space (X, d) satisfies the Besicovitch covering property (B.C.P.) if there is a positive integer M such that for every family $\mathcal{G}$ of balls in X whose union is a bounded set, there is a subfamily $\mathcal{F} \subset \mathcal{G}$ satisfying

$$\text{(4.2)}\qquad \begin{aligned} &\text{(i)}\ \cup\{x_B : B \in \mathcal{G}\} \subset \cup\{B : B \in \mathcal{F}\},\\ &\text{(ii)}\ \sum_{B\in\mathcal{F}} \chi_B(x) \le M \text{ for all } x \in X,\end{aligned}$$

where x_B denotes the centre of B. Then (cf. Lemma 1.2 of [dG]) (X, d) has the B.C.P. only if there is a positive integer M such that (4.2)(ii) holds for every family $\mathcal{F}$ of balls in X satisfying

$$x_B \notin B' \text{ whenever } B \text{ and } B' \text{ are distinct balls in } \mathcal{F}. \tag{4.3}$$

Indeed, if $\mathcal{F}$ satisfies (4.3), then no proper subfamily of $\mathcal{F}$ can cover $\cup \{x_B : B \in \mathcal{F}\}$ and so the B.C.P. with $\mathcal{G} = \mathcal{F}$ implies (4.2)(ii).

LEMMA 4.4. *The quasi-metric space* $(\mathbb{C}^n \times \mathbb{R}, d)$, $n \ge 1$, *where* d *is given by* (4.1), *fails to have the Besicovitch covering property.*

Proof. Suppose $n = 1$. We construct a sequence of balls $\{B_j\}_{j=4}^{\infty}$ in $\mathbb{C} \times \mathbb{R}$ with centres (z_j, t_j) satisfying

$$(4.5)\qquad \begin{aligned} &\text{(i) } (0,0) \in \overline{B_j} && \text{for all } j,\\ &\text{(ii) } (z_j, t_j) \notin \overline{B_k} && \text{for all } j \neq k. \end{aligned}$$

This shows that the B.C.P. fails since given any M, we can slightly enlarge the balls $B_4, \ldots, B_{M+5}$ so that $(0, 0) \in B_j$ for $4 \le j \le M + 5$ and yet (4.5)(ii) persists for $4 \le j \neq k \le M + 5$. Thus (4.3) holds and (4.2)(ii) fails for $\mathcal{F} = \{B_j\}_{j=4}^{M+5}$ as required.

To construct the sequence $\{B_j\}_{j=4}^{\infty}$, define $r_1 = 1/2$ and $r_j = 2^{-2j} r_{j-1}$ for $j > 1$ so that $r_j = 2^{-j(j+1)+1}$. Let $\theta_j = 2^{4-j}$, $t_j = 2^j r_j^2$ and $z_j = r_j e^{i\theta_j}$. Define B_j to be the ball centered at (z_j, t_j) with radius

$$R_j = d((z_j, t_j), (0, 0)) = (r_j^4 + t_j^2)^{1/4}.$$

Now (4.5)(i) holds by the choice of R_j, so we turn our attention to (4.5)(ii). Suppose first that $j > k \ge 4$. We have

$$\begin{aligned} d((z_k, t_k), (z_j, t_j))^4 &= |z_k - z_j|^4 + [t_k - t_j + 2\,\mathrm{Im}(z_k \overline{z_j})]^2 \\ &= [r_k^2 + r_j^2 - 2r_k r_j \cos(\theta_k - \theta_j)]^2 + [t_k - t_j + 2r_k r_j \sin(\theta_k - \theta_j)]^2 \\ &= (r_k^2 + r_j^2)^2 + (t_k - t_j)^2 + 4r_k^2 r_j^2 \\ &\quad - 4r_k r_j (r_k^2 + r_j^2)\cos(\theta_k - \theta_j) + 4r_k r_j (t_k - t_j)\sin(\theta_k - \theta_j) \\ &> R_k^4 + 4r_k r_j (t_k - t_j)\sin(\theta_k - \theta_j) - 4r_k r_j (r_k^2 + r_j^2) - 2t_k t_j. \end{aligned}$$

Thus it suffices to show that

$$(4.6)\qquad 4r_k r_j (t_k - t_j)\sin(\theta_k - \theta_j) \ge 4r_k r_j (r_k^2 + r_j^2) + 2t_k t_j,$$

for $j > k \ge 4$. But

$$\begin{aligned} 4r_k r_j (t_k - t_j)\sin(\theta_k - \theta_j) &= 4r_k r_j (2^k r_k^2 - 2^j r_j^2)\sin(2^{4-k} - 2^{4-j}) \\ &> 4r_k r_j \left(\frac{3}{4}\, 2^k r_k^2\right)\left(\frac{1}{4}\, 2^{4-k}\right) = 12 r_k^3 r_j, \end{aligned}$$

for $j > k \geq 4$, while

$$4r_k r_j(r_k^2 + r_j^2) \leq 8r_k^3 r_j$$

and

$$2t_k t_j = 2\, 2^k r_k^2 2^j r_j^2 = 2^{k+1-j} r_k^2 r_{j-1} r_j \leq r_k^3 r_j,$$

since $j > k$. This proves (4.6) and shows that (4.5)(ii) holds for $j > k \geq 4$.

Now suppose $k > j \geq 4$. We've just shown that $(z_k, t_k) \notin \overline{B_j}$, i.e. $d((z_k, t_k), (z_j, t_j)) > R_j$, and since $R_k < R_j$, we also have $(z_j, t_j) \notin \overline{B_k}$. This establishes (4.5) for $j, k \geq 4$ and completes the proof of Lemma 4.4 in the case $n = 1$. The case $n > 1$ is similar.

We remark that the B.C.P. need not be preserved by an equivalent quasi-metric. For example, the quasi-metric

$$d_1((x_1, y_1), (x_2, y_2)) = (x_1 - x_2)^4 + (y_1 - y_2)^2$$

on $\mathbb{R}^2$ has the B.C.P. but the equivalent quasi-metric

$$d_2((x_1, y_1), (x_2, y_2)) = (x_1 - x_2)^4 + (y_1 - y_2 - (x_1 - x_2)|x_1 - x_2|)^2$$

does not. In fact, if B_j denotes the d_2-ball with centre $P_j = (x_j, 3x_j^2)$ and radius $R_j = d_2(P_j, (0, 0)) = 5x_j^4$, and if the x_j decrease to zero fast enough, then an argument similar to that in the proof of Lemma 4.4 shows that $(0, 0) \in \overline{B_j}$ for all j and $P_j \notin \overline{B_k}$ for all $j \neq k$. The point here is that a d_2-ball of radius $\epsilon^4 \ll 1$ looks roughly like an $\epsilon \times \epsilon^2$ ellipse whose major axis has slope ϵ. Such balls are equivalent to untilted $\epsilon \times \epsilon^2$ ellipses, which are in turn equivalent to d_1-balls.

We suspect there is no quasi-metric on the Heisenberg group equivalent to d in (4.1) that satisfies the B.C.P. The simple device of untilting the balls, used in passing from d_2 to d_1 above, doesn't apply here since the balls given by (4.1) look like $\epsilon \times \epsilon \times \epsilon^2$ ellipsoids with slope depending not on ϵ, but on the centre of the ball (see the top of page 261 in [N]), and thus cannot be equivalent to the corresponding untilted ellipsoids.

Although the Besicovitch covering lemma fails for the Heisenberg group metric d in (4.1), it is possible to exploit the left translation invariance of d (obvious from (4.1)) to control an operator T in terms of dyadic operators $T^{dy}_{(z,t)}$ associated to translations of a dyadic grid of "cubes" as in Lemma 2.2 in Section 2. One can then obtain a version of Theorem 1(B) for operators whose kernel $\mathcal{K}$ is appropriately related to d (see Theorem 4(B)). The translation invariance of d can be used in a similar way to remove the doubling hypotheses on the weights in Theorem 3(A) (see Theorem 4(A)).

Now we suppose that the homogeneous space (X, d, μ) admits a group structure on X satisfying the translation invariance conditions (1.23). We show that in this setting there is an analogue of Lemma 2.2 for the operator T in (1.12) using the grid of "dyadic cubes" E_j^k given in Lemma 3.21. However, since this grid is only defined for sets E_j^k of diameter $\sim\lambda^k$, $k \geq m$, we must consider local truncations of the operator T given by

$$T_m f(x) = \int_{\{y\in X: d(x,y)>\lambda^m\}} \mathcal{K}(x, y) f(y) d\mu(y).$$

For m in Z, the truncated operator T_m will be controlled by dyadic operators based on translates of the grid $\mathcal{D}_m = \{E_j^k\}_{1\leq j<n_k, k\geq m}$ given in Lemma 3.21. We will henceforth refer to the sets E_j^k as dyadic cubes. Moreover, Q will now be used to denote only dyadic cubes and if $Q = E_j^k$, we set $s(Q) = \lambda^k$, which by (3.22)(i), should be thought of as the side length of the dyadic cube Q. Now define the dyadic operator

$$T_m^{dy} f(x) = \sum_{Q\in\mathcal{D}_m: x\in Q} \varphi(Q) \int_Q f(y) d\mu(y),$$

where $\varphi(Q)$ is defined to be $\varphi(B(x_j^k, \lambda^{k+1}))$ if $Q = E_j^k$ (see (3.22)(i)) and where $\varphi(B)$ is given by (1.16). We also consider the corresponding operators dyadic with respect to translates of $\mathcal{D}_m$ by z in X, namely

$$T_{m,z}^{dy} f(x) = \sum_{Q\in\mathcal{D}_m: x\in Q+z} \varphi(Q + z) \int_{Q+z} f(y) d\mu(y).$$

where $\varphi(Q + z)$ is defined to be $\varphi(B(x_j^k, \lambda^{k+1}) + z)$ if $Q = E_j^k$. Note that $B(x, r) + z = B(x + z, r)$ by (1.23)(i).

LEMMA 4.7. *Suppose T is an operator given by* (1.12) *where* (X, d, μ) *is a homogeneous space with a group structure on X satisfying* (1.23). *Then, for* $1 \le q \le \infty$ *and* $w(x) \ge 0$ *on* X,

$$\left(\int_X (T_m f)^q w d\mu\right)^{1/q} \le C \sup_{z \in X} \left(\int_X (T_{m,z}^{dy} f)^q w d\mu\right)^{1/q},$$

for all $f \ge 0$ *and* m *in* $\mathbb{Z}$, *and where C is independent of f and m.*

Proof of Lemma 4.7. We adapt the proof of Lemma 2.2. Momentarily fix m in $\mathbb{Z}$ and for k in $\mathbb{Z}$, $k > m$, define the truncation (at infinity this time) of the operator T_m by

$$T_m^k f(x) = \int_{\{y \in X : \lambda^m < d(x,y) \le \lambda^k\}} \mathcal{K}(x, y) f(y) d\mu(y).$$

Let B_k denote the ball of radius λ^k about the identity element in X. As in (2.3) of Section 2, we claim that

$$T_m^k f(x) \le C \frac{1}{\mu(B_{k+3})} \int_{B_{k+3}} T_{m,z}^{dy} f(x) d\mu(z), \qquad \text{for all } f \ge 0, \tag{4.8}$$

but this time for $x \in B_k$, $k > m$. For fixed $k > m$ and x in B_k we have

$$\frac{1}{\mu(B_{k+3})} \int_{B_{k+3}} T_{m,z}^{dy} f(x) d\mu(z)$$

$$= \int_X \left(\frac{1}{\mu(B_{k+3})} \int_{B_{k+3}} \left[\sum_{\substack{Q \in \mathfrak{D}_m \\ x,y \in Q+z}} \varphi(Q + z)\right] d\mu(z)\right) f(y) d\mu(y).$$

Momentarily fix y with $\lambda^m < d(x, y) \le \lambda^k$ and choose $\ell \in \mathbb{Z}$ so that $\lambda^{\ell-1} < d(x, y) \le \lambda^\ell$, $m < \ell \le k$. Let Ω consist of those z in B_{k+3} such that there is $Q \in \mathfrak{D}_m$ with $s(Q) = \lambda^{\ell+1}$ and with both x and y in $Q + z$. Note that for such a cube $Q = E_i^{\ell+1}$, $\varphi(Q + z) = \varphi(B(x_i^{\ell+1}, \lambda^{\ell+2}) + z) \ge \mathcal{K}(x, y)$ provided the constant $c(K)$ in (1.16) is at most $\lambda^{-3} = K^{-15}/512$. We claim

$$\mu(\Omega) \ge c\mu(B_{k+3}). \tag{4.9}$$

Assuming (4.9), we have

$$\frac{1}{\mu(B_{k+3})}\int_{B_{k+3}}\left[\sum_{\substack{Q\in\mathfrak{D}_m\\ x,y\in Q+z}}\varphi(Q+z)\right]d\mu(z)\geq c\mathcal{K}(x,y)\frac{\mu(\Omega)}{\mu(B_{k+3})}\geq c\mathcal{K}(x,y),$$

and using this estimate in the previous identity for $f \geq 0$ yields (4.8). Minkowski's inequality now shows that

$$\left[\int_{B_k}(T_m^k f)^q w\,d\mu\right]^{1/q} \leq C\frac{1}{\mu(B_{k+3})}\int_{B_{k+3}}\left[\int_X (T_{m,z}^{dy}f)^q w\,d\mu\right]^{1/q} d\mu(z)$$

$$\leq C\sup_{z\in B_{k+3}}\left[\int_X (T_{m,z}^{dy}f)^q w\,d\mu\right]^{1/q}$$

and letting $k\to\infty$ completes the proof of Lemma 4.7, leaving only (4.9) to be verified.

To prove (4.9), let $\Gamma = \{j : E_j^{\ell+1} \cap B(x, \lambda^{k+2}) \neq \phi\}$ where $\{E_j^i\}_{1\leq j<n_i, i\geq m}$ are the dyadic cubes in $\mathfrak{D}_m$ as given by Lemma 3.21. In particular, recall that by (3.22)(i), $B(x_j^{\ell+1}, \lambda^{\ell+1}) \subset E_j^{\ell+1} \subset B(x_j^{\ell+1}, \lambda^{\ell+2})$. Now suppose that z is in $-B(x_j^{\ell+1}, \lambda^{\ell}) + x$ where $-B$ denotes the set $\{w \in X : -w \in B\}$. We claim that both x and y lie in $E_j^{\ell+1} + z$. Indeed, $x - z \in B(x_j^{\ell+1}, \lambda^{\ell}) \subset B(x_j^{\ell+1}, \lambda^{\ell+1}) \subset E_j^{\ell+1}$, which shows that $x \in E_j^{\ell+1} + z$. Since $d(x - z, x_j^{\ell+1}) < \lambda^{\ell}$ by the above and $d(y - z, x - z) = d(y, x) \leq \lambda^{\ell}$ by (1.23)(i), we have

$$d(y - z, x_j^{\ell+1}) \leq K[d(y - z, x - z) + d(x - z, x_j^{\ell+1})]$$

$$< K[\lambda^{\ell} + \lambda^{\ell}] < \lambda^{\ell+1}.$$

Thus $y - z \in B(x_j^{\ell+1}, \lambda^{\ell+1}) \subset E_j^{\ell+1}$ and so $y \in E_j^{\ell+1} + z$ also.

Now if $j \in \Gamma$, then $E_j^{\ell+1} \cap B(x, \lambda^{k+2}) \neq \phi$ and we claim it follows that $-B(x_j^{\ell+1}, \lambda^{\ell}) + x \subset B_{k+3}$. Indeed, if $w \in B(x_j^{\ell+1}, \lambda^{\ell})$ and $u \in E_j^{\ell+1} \cap B(x, \lambda^{k+2})$, then

$$d(u, w) \leq K[d(u, x_j^{\ell+1}) + d(x_j^{\ell+1}, w)] < K[\lambda^{\ell+2} + \lambda^{\ell}] < 2K\lambda^{k+2},$$

implies

$$d(x, w) \leq K[d(x, u) + d(u, w)] < K[\lambda^{k+2} + 2K\lambda^{k+2}] < 3K^2\lambda^{k+2},$$

which implies

$$\begin{aligned} d(0, w) &\leq K[d(0, x) + d(x, w)] < K[\lambda^k + 3K^2\lambda^{k+2}] \\ &< 4K^3\lambda^{k+2}, \text{ since } x \in B_k, \end{aligned}$$

which finally implies

$$\begin{aligned} d(0, -w + x) &\leq K[d(0, x) + d(x, -w + x)] \\ &< K[\lambda^k + d(0, w)], \qquad \text{by (1.23)(i), (ii),} \\ &< K[\lambda^k + 4K^3\lambda^{k+2}] < 5K^4\lambda^{k+2} < \lambda^{k+3}, \end{aligned}$$

as desired, since $\lambda = 8K^5 > 5K^4$. Thus, since the sets $\{-B(x_j^{\ell+1}, \lambda^\ell) + x\}_{j\in\Gamma}$ are pairwise disjoint,

$$\begin{aligned} \mu(\Omega) &\geq \sum_{j\in\Gamma} \mu(-B(x_j^{\ell+1}, \lambda^\ell) + x) \\ &= \sum_{j\in\Gamma} \mu(B(x_j^{\ell+1}, \lambda^\ell)), \qquad \text{by (1.23)(iii), (iv)} \\ &\geq c\sum_{j\in\Gamma} \mu(E_j^{\ell+1}), \qquad \text{since } \mu \text{ is doubling.} \\ &\geq c\mu(B(x, \lambda^{k+2})), \qquad \text{by (3.22)(ii),} \\ &\geq c\mu(B(x, \lambda^{k+3})), \qquad \text{since } \mu \text{ is doubling,} \\ &= c\mu(B(0, \lambda^{k+3}) + x), \qquad \text{by (1.23)(i),} \\ &= c\mu(B(0, \lambda^{k+3})) = c\mu(B_{k+3}), \qquad \text{by (1.23)(iii),} \end{aligned}$$

as required. This completes the proof of Lemma 4.7.

Proof of Theorem 4. We begin with part (A). By Lemma 4.7, it is enough to prove (1.17) with T replaced by $T^{dy}_{m,z}$ and with a constant C independent of m in $\mathbb{Z}$ and z in X. We shall prove (1.17) for T^{dy}_m with a constant independent of m since it is an easy matter to verify that all of the following arguments hold, with the same constants, for the operators $T^{dy}_{m,z}$. As in (2.4) and (2.5) of Section 2, it is enough to prove

$$(4.10)\quad \sum_{Q\in\mathcal{D}_m} \varphi(Q)\left(\int_Q f\sigma d\mu\right)\left(\int_Q gwd\mu\right) \leq C\left(\int_X f^p\sigma d\mu\right)^{1/p}\left(\int_X g^{q'}wd\mu\right)^{1/q'},$$

for all $f, g \geq 0$ bounded with support in some ball and where $\sigma = v^{1-p'}$. If $Q = E^k_j$ and $B = B(x^k_j, \lambda^{k+1})$, then $\varphi(Q)$ is by definition $\varphi(B)$ and so conditions (1.19) and (1.20) for balls $B' \subset 2KB$ show, just as in the proof of Part A of Theorem 3 (see (3.9)–(3.13)), that the left side of (4.10) is dominated by

$$(4.11)\quad C\left(\sum_{k,j}\mathcal{A}^k_j\left(\frac{1}{\mathcal{A}^k_j}\int_{Q^k_j} f\sigma d\mu\right)^p\right)^{1/p}\left(\sum_{k,j}\mathcal{B}^k_j\left(\frac{1}{\mathcal{B}^k_j}\int_{Q^k_j} gwd\mu\right)^{q'}\right)^{1/q'}$$

where $\mathcal{A}^k_j = \mu(Q^k_j)^{1/r'}(\int_{Q^k_j}\sigma^r d\mu)^{1/r}$ and $\mathcal{B}^k_j = \mu(Q^k_j)^{1/r'}(\int_{Q^k_j} w^r d\mu)^{1/r}$, r is as in (1.19), and where the dyadic cubes $\{Q^k_j\}_j$ are the maximal dyadic cubes over which the average of gw, with respect to $d\mu$, exceeds γ^k, k in $\mathbb{Z}$. The reason (1.20) is used for $B' \subset 2KB$ is that if a dyadic cube E^ℓ_i is properly contained in a dyadic cube E^k_j, then $k > \ell$ by (3.22)(iii) and $B(x^\ell_i, \lambda^{\ell+1}) \subset 2KB(x^k_j, \lambda^{k+1})$ by the triangle inequality (3.1)(iii). These facts are needed in order to use (1.20), which is stated in terms of balls, to prove the following inequality for cubes:

$$\sum_{\substack{Q \text{ dyadic}\\ Q\subset Q^k_j}} \mu(Q)\varphi(Q)\int_Q f\sigma d\mu \leq C\mu(Q^k_j)\varphi(Q^k_j)\int_{Q^k_j} f\sigma d\mu,$$

with C independent of Q^k_j. This inequality is used in order to pass from the analogue of (3.9) to the analogue of (3.10). We also use the fact that the cubes $\{E_i^\ell\}_i$ are pairwise disjoint. The analogue of (3.11) is

$$\sum_{i:Q_i^\ell \subset Q_j^k} \mu(Q_i^\ell) \leq C\gamma^{1-(\ell-k)}\mu(Q_j^k), \tag{4.12}$$

$$\text{and if } Q_i^\ell \subsetneqq Q_j^k, \text{ then } \ell > k,$$

and this is proved by the same method used to prove (2.8), noting that the cubes $\{Q_i^\ell\}_i$ are pairwise disjoint.

To establish the Carleson condition,

$$\sum_{\ell,i:Q_i^\ell \subset Q_j^k} \mathcal{A}_i^\ell \leq C\mathcal{A}_j^k, \qquad \text{for all } k, j, \tag{4.13}$$

we will use property (3.22)(iii) of dyadic cubes in place of the doubling assumption on $\sigma^r d\mu$ that we used in proving (3.14) in Part A of Theorem 3. In fact, since (3.22)(iii) shows that for any fixed ℓ, the maximal cubes $\{Q_i^\ell\}_i$ are pairwise disjoint, the argument used to prove (2.11) in Section 2, together with (4.12), yields (4.13). The proof of Lemma 2.10 now shows that the first factor on the right side of (4.11) is dominated by $C(\int_X f^p \sigma d\mu)^{1/p}$. Similarly, the second factor is dominated by $C(\int_X g^{q'} w d\mu)^{1/q'}$ and this proves (4.10) and completes the proof of Part A of Theorem 4.

We now prove Part (B) of Theorem 4. The necessity of (1.25) in the case $\mathcal{K}(x, y) = d(x, y)^{-1}$ follows from the argument used to prove the necessity of (1.9) in Section 2 since $\varphi(B) \approx r(B)^{-1}$ in this case.

Conversely, since $\varphi(B) \leq C\mathcal{K}(x, y)$ for all $x, y \in B$ by (1.13) (see the first paragraph in the proof of Theorem 3(A)), (1.25) implies

$$\left(⨍_B w d\mu\right)^{1/q} \left(\int s_B{}^{*p'} v^{1-p'} d\mu\right)^{1/p'} \leq C, \qquad \text{for all balls } B \subset X. \tag{4.14}$$

Using the argument in Section 2 that showed (1.6) implies the weak type inequality (2.12), we obtain that (4.14) implies

$$\sup_{\lambda>0} \lambda \left|\{x : Tf(x) > \lambda\}\right|_w^{1/q} \leq C\left(\int_X f^p v d\mu\right)^{1/p}, \qquad \text{for all } f \geq 0. \tag{4.15}$$

Indeed, by the analogue of Lemma 4.7 for weak type norms, it is enough to show

$$(4.16)\qquad \sup_{\lambda>0} \lambda \left|\{x : T_m^{dy} f(x) > \lambda\}\right|_w^{1/q} \le C\left(\int_X f^p v\, d\mu\right)^{1/p}, \qquad \text{for all } f \ge 0,$$

with a constant C independent of m. The argument running from (2.15) through (2.21) applies to T_m^{dy} with only one change. In the course of proving the analogue of (2.17) we must show that for any y,

$$(4.17)\qquad \begin{aligned} \sum_{Q: Q_{k+1}^* \subset Q \subset Q_k} \varphi(Q)\chi_Q(y) &\le C s_{B_{k+1}^*}^*(y) \\ &= C \min\{\varphi(B_{k+1}^*), \mathcal{K}^*(x_{B_{k+1}^*}, y)\}, \end{aligned}$$

where B_{k+1}^* denotes the containing ball for the cube Q_{k+1}^*, i.e., $B(x_i^\ell, \lambda^{\ell+1})$ if $Q_{k+1}^* = E_i^\ell$ (see (3.22)(i)). Indeed, if Q is the smallest dyadic cube satisfying both $Q_{k+1}^* \subset Q \subset Q_k$ and $y \in Q$, then $\varphi(Q) \le C \min\{\varphi(B_{k+1}^*), \mathcal{K}^*(x_{B_{k+1}^*}, y)\}$ by (1.13), (1.24) and the fact that if Q' is a cube contained in Q, then their respective containing balls B'^*, B^* satisfy $B'^* \subset 2KB^*$. Condition (1.24) then yields (4.17). The result of the argument in Section 2 is that (4.16) holds. As in [S3], the weak type inequality (4.15) implies condition (1.22). Similarly (1.25) implies (1.21) and Theorem 3(C) now shows that (1.17) holds and this completes the proof of Theorem 4.

Added in proof. (i) The failure of the Besicovitch covering lemma on the Heisenberg group has been shown independently in a 1991 preprint of A. Koranyi and H. M. Reimann entitled "Foundations for the theory of quasiconformal mappings on the Heisenberg group."

(ii) Lemma 3.21 on the construction of dyadic cubes is contained in a 1990 preprint of M. Christ entitled "A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral." See also G. David, "Morceaux de graphes Lipschitziens et integrales singulières sur une surface," *Revista Mat. Iberoamericana* **4** (1988), 73–144.

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