

Equidistributed

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The content is from [1, Chapter 4, Section 2].

Definition 1. A sequence of number $\{\xi_n\}_{n \in \mathbb{N}} \subset [0, 1)$ is said to be equidistributed if for any interval $(a, b) \subset [0, 1)$,

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \xi_n \in (a, b)\}}{N} = b - a.$$

where the $\#A$ denotes the cardinality of the finite set A .

Definition 2. Let $[x]$ denote the greatest integer less than or equal to x and call $[x]$ the integer part of x . Let $\langle x \rangle := x - [x]$ and call $\langle x \rangle$ the fractional part of x .

Theorem 1 (Weyl's criterion). A sequence of number $\{\xi_n\}_{n \in \mathbb{N}} \subset [0, 1)$ is equidistributed if and only if for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{1}{N} \sum_{n=1}^N e^{2\pi i k \xi_n} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

From Weyl's criterion, we have following corollaries.

Corollary 1. Let $P(x) : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto c_m x^m + \cdots + c_0$, where $\{c_0, \dots, c_m\} \subset \mathbb{R}$ and one of them is irrational number. Then $\{\langle P(n) \rangle\}_{n \in \mathbb{N}} \subset [0, 1)$ is equidistributed.

Proposition 2. 设 $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. 则 $\{\langle n\alpha \rangle\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 中稠密.

证法1. By Corollary 1 with $m = 1, c_0 = 0, c_1 = \alpha$, we find that

$$\{\langle n\alpha \rangle\}_{n \in \mathbb{N}} = \{\langle P(n) \rangle\}_{n \in \mathbb{N}} \subset [0, 1)$$

is equidistributed, and hence $\{\langle n\alpha \rangle\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 中稠密. □

证法2(来自 lzc). 欲证 $\{\langle n\alpha \rangle\}_{n \in \mathbb{N}}$ 在 $[0, 1]$ 中稠密, 即证对任意 $\varepsilon \in (0, 1)$ 和 $x \in [0, 1]$, 存在 $n \in \mathbb{N}$ 使得 $|\langle n\alpha \rangle - x| < \varepsilon$. 固定 $k \in \mathbb{N}$ 和 $x \in [0, 1]$, 下面来构造 n . 因为 $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, 故 $\{\langle n\alpha \rangle\}_{n \in \mathbb{N}}$ 两两不等, 从而存在 $n, m \in \mathbb{N}$ 使得

$$\langle n\alpha \rangle - \langle m\alpha \rangle \in (0, \varepsilon).$$

注意到

$$\begin{aligned} na &= [n\alpha] + \langle n\alpha \rangle \\ ma &= [m\alpha] + \langle m\alpha \rangle \end{aligned}$$

故

$$\langle (n - m)a \rangle = \langle n\alpha \rangle - \langle m\alpha \rangle \in (0, \varepsilon).$$

取 $p := [x / \langle (n - m)a \rangle]$, 则

$$\langle p(n - m)a \rangle = p\langle (n - m)a \rangle \in (x - \langle (n - m)a \rangle, x]$$

故

$$\langle p(n - m)a \rangle - x \in (-\langle (n - m)a \rangle, 0] \subset (-\varepsilon, 0]$$

□

Corollary 2. $\{\sin n\}_{n \in \mathbb{N}}$ is dense in $[0, 1]$.

参考文献

- [1] E. M. Stein and R. Shakarchi, Fourier Analysis: An Introduction, Princeton University Press, 2011.