# Cauchy-Schwarz Inequality

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一般书上讲的都是Hilbert 空间中的Cauchy–Schwarz inequality, 而下面的Cauchy–Schwarz inequality, 是我见过的条件最弱的, 暂时还没找到该定理的出处.

本文中的$X$ 均为线性空间.

## 1 复数的Cauchy–Schwarz inequality

**Definition 1.1.** 若二元函数$a : X \times X \to \mathbb{C}$ 满足: 对任意$\forall \alpha_1,\ \alpha_2 \in \mathbb{C}$ 和$\forall x,\ x_1,\ x_2,\ y,\ y_1,\ y_2 \in X$,

(i) $a(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 a(x_1, y) + \alpha_2 a(x_2, y)$;

(ii) $a(x, \alpha_1 y_1 + \alpha_2 y_2) = \overline{\alpha_1} a(x, y_1) + \overline{\alpha_2} a(x, y_2)$.

我们称$a(\cdot, \cdot)$ 为$X$ 上的共轭双线性函数.

**Theorem 1.2** (Cauchy–Schwarz inequality)**.** 设$a(\cdot, \cdot)$ 为$X$ 上的共轭双线性函数且*for any* $x \in X$*,*

$$a(x, x) \geq 0,$$

*then for any* $x, y \in X$*,*

$$|a(x, y)|^2 \leq a(x, x) a(y, y). \tag{1}$$

注意, $a(x, x) \geq 0$ 能推出$a(x, y) = \overline{a(y, x)}$, 这个结论在证明中会用到. 此外, 由于该Cauchy–Schwarz inequality 减弱了条件, 取等条件变得复杂了许多.

*Proof.* For any $x, y \in X$, if $a(x, x) = a(y, y) = 0$, then

$$\begin{aligned}
0 &\leq a(x + y, x + y) = a(x, x) + a(y, y) + 2\mathrm{Re}a(x, y) = 2\mathrm{Re}a(x, y),\\
0 &\leq a(x - y, x - y) = a(x, x) + a(y, y) - 2\mathrm{Re}a(x, y) = -2\mathrm{Re}a(x, y),\\
0 &\leq a(x + iy, x + iy) = a(x, x) + a(y, y) + 2\mathrm{Im}a(x, y) = 2\mathrm{Im}a(x, y),\\
0 &\leq a(x - iy, x + iy) = a(x, x) + a(y, y) - 2\mathrm{Im}a(x, y) = -2\mathrm{Im}a(x, y),
\end{aligned}$$

So Re$a(x,y) = 0 =$ Im$a(x,y)$, i.e $a(x,y)=0$.

$$|a(x,y)|^2 = 0 = a(x,x)a(y,y).$$

Otherwise, we assume $a(y,y)\neq 0$, let $\lambda := a(x,y)/a(y,y)$, then

$$\begin{aligned}
0 &\le a(x-\lambda y, x-\lambda y)\\
&= a(x,x) - \overline{\lambda}a(x,y) - \lambda a(y,x) + |\lambda|^2 a(y,y)\\
&= a(x,x) - \frac{|a(x,y)|^2}{a(y,y)}.
\end{aligned}$$

From this we have

$$|a(x,y)|^2 \le a(x,x)a(y,y).$$

The proof of Cauchy-Schwarz inequality is finished. □

## 2 实数的Cauchy–Schwarz inequality

**Definition 2.1.** 若二元函数$a : X\times X \to \mathbb{R}$ 满足: 对任意$\forall \alpha_1,\ \alpha_2\in\mathbb{R}$ 和$\forall x,\ x_1,\ x_2,\ y,\ y_1,\ y_2\in X$,

(i) $a(\alpha_1x_1+\alpha_2x_2, y) = \alpha_1 a(x_1,y)+\alpha_2 a(x_2,y)$;

(ii) $a(x,\alpha_1y_1+\alpha_2y_2) = \alpha_1 a(x,y_1)+\alpha_2 a(x,y_2)$.

我们称$a(\cdot,\cdot)$ 为$X$ 上的共轭双线性函数.

**Theorem 2.2** (Cauchy–Schwarz inequality)**.** 设$a(\cdot,\cdot)$ 为$X$ 上的双线性函数, 对$\forall x,\ y\in X$, $a(x,y)=a(y,x)$, 且对$\forall x\in X$,

$$a(x,x)\ge 0.$$

则对$\forall x,y\in X$,

$$|a(x,y)|^2 \le a(x,x)a(y,y).$$

证明类似.