I-AB 与I-BA

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Theorem 0.1. *Let* $A, B \in M_n(\mathbb{C})$.

$$I_n - AB \text{ is invertible} \Longleftrightarrow I_n - BA \text{ is invertible},$$

where $I_n \in M_n(\mathbb{C})$ *is identity matrix.*

The key of the proof of above theorem is the following equation:

$$(I_n - BA)(I + B(I - AB)^{-1}A) = I. \tag{1}$$

Now a problem is coming, how can we find (1). Next I will show a intersting idea from StackExchange.

Absolutely, our goal is using $I_n - AB$ and $(I_n - AB)^{-1}$ to construct $I_n - BA^{-1}$. We assume $(I_n - AB)^{-1}$ and $(I_n - BA)^{-1}$ are exist and can be expand series, then

$$\begin{aligned}(I_n - AB)^{-1} &= I_n + AB + (AB)^2 + \cdots, \\ (I_n - BA)^{-1} &= I_n + BA + (BA)^2 + \cdots.\end{aligned}$$

Then

$$\begin{aligned}B(I_n - AB)^{-1}A &= BA + (BA)^2 + (BA)^3 + \cdots, \\ &= (I_n - BA)^{-1} - I_n.\end{aligned}$$

From this we get (1). Of course, this isn't a proof, but the argument is heuristic. Next is a strict proof.

Proof of Theorem 0.1. We should only proof, if $I_n - AB$ is invertible, then $I_n - BA$ is invertible. In fact, assume $I_n - AB$ is invertible, then

$$\begin{aligned}&(I_n - BA)[I_n + B(I_n - AB)^{-1}A] \\ &\quad = I_n + B(I_n - AB)^{-1}A - BA - BAB(I_n - AB)^{-1}A \\ &\quad = I_n - BA + B(I_n - AB)(I_n - AB)^{-1}A = I_n.\end{aligned}$$

Similar we have

$$[I_n + B(I_n - AB)^{-1}A](I_n - BA) = I_n.$$

So $I_n - BA$ is invertible. This complete the proof of Theorem 0.1. $\square$