Schwartz Space and Tempered Distributions

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在本文中,

$$\mathbb{N} := \{1, 2, \ldots\},$$
$$\mathbb{Z}_+ := \{0, 1, 2, \ldots\}.$$

对$\forall \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$,

$$|\alpha| := \sum_{k=1}^{n} \alpha_k.$$

对$\forall \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ 和$\forall f \in \mathcal{S}$,

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} f.$$

对$\forall \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ 和$\forall x := (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$x^\alpha := \prod_{k=1}^{n} x_k^{\alpha_k}.$$

$\mathcal{S}(\mathbb{R}^n)$ 代表Schwartz Space.

0.1 Schwartz Space

本section 的内容来自[1, 2.2].

Proposition 0.1. 设$f, g \in \mathcal{S}(\mathbb{R}^n)$, 则$fg, f * g \in \mathcal{S}(\mathbb{R}^n)$ 且对$\forall \alpha \in \mathbb{Z}_+^n$,

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g).$$

0.2 The Space of Tempered Distributions

本section 的内容来自[1, 2.3].

$\mathcal{S}'(\mathbb{R}^n)$ 中的傅里叶变换, 导数, 支集.

Definition 0.2 (傅里叶变换)**.** 设$T \in \mathcal{S}'(\mathbb{R}^n)$, 对$\forall f \in \mathcal{S}(\mathbb{R}^n)$, 定义

$$\widehat{T}(f) := T(\hat{f}).$$

Definition 0.3 (导数)**.** 设$T \in \mathcal{S}'(\mathbb{R}^n)$, 对$\forall f \in \mathcal{S}(\mathbb{R}^n)$ 和$\alpha \in \mathbb{Z}_+^n$, 定义

$$(\partial^\alpha T)(f) := (-1)^{|\alpha|} T(\partial^\alpha f).$$

Remark 0.4. $\widehat{T} \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 0.5 (支集)**.** 设$T \in \mathcal{S}'(\mathbb{R}^n)$, 定义

$$\operatorname{supp} T := \bigcap \{K: \text{ for any } f \in \mathcal{S}(\mathbb{R}^n) \text{ and } \operatorname{supp} f \subset \mathbb{R}^n \setminus K, T(f) = 0\}.$$

Remark 0.6. 称T is support in K, 若对$\forall f \in \mathcal{S}(\mathbb{R}^n)$ 且$\operatorname{supp} f \subset \mathbb{R}^n \setminus K$, $T(f) = 0$.

Definition 0.7. 设$T \in \mathcal{S}'(\mathbb{R}^n)$. 称$T$ 在分布意义下与$\mathbb{R}^n$ 上的可测函数h 是一致的, 若对$\forall f \in \mathcal{S}(\mathbb{R}^n)$, 有

$$T(f) = \int_{\mathbb{R}^n} h(x) f(x) dx.$$

Proposition 0.8 (Proposition 2.4.1)**.** 设$T \in \mathcal{S}'(\mathbb{R}^n)$. 若$T$ 支在单点集$\{x_0\}$ 上, 则存在$m \in \mathbb{Z}_+$, $\{c_\alpha\}_{|\alpha| \le m} \subset \mathbb{C}$, 使得

$$u = \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \le m} c_\alpha \partial^\alpha \delta_{x_0}.$$

0.3 Space of Tempered Distributions Modulo Polynomials

本section 的内容来自[2, 1.1.1]

首先介绍多项式空间$\mathcal{P}(\mathbb{R}^n)$.

Definition 0.9.

$$\mathcal{P}(\mathbb{R}^n) := \left\{ \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \le m} c_\alpha x^\alpha : m \in \mathbb{Z}_+, \text{ for any } \alpha \in \mathbb{Z}_+^n \text{ and } |\alpha| \le m, c_\alpha \in \mathbb{C} \right\}.$$

设$T_1, T_2 \in \mathcal{S}'(\mathbb{R}^n)$. 称$T_1 \equiv T_2$, 若存在$p \in \mathcal{P}(\mathbb{R}^n)$, 使得$T_1 - T_2$ 在分布意义下与p 一致, 则$\equiv$ 是个等价关系, 记$\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ 为该等价关系诱导的商空间.

Definition 0.10.

$$\mathcal{S}_0(\mathbb{R}^n) := \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \text{ for any } \alpha \in \mathbb{Z}_+^n, \int_{\mathbb{R}^n} x^\alpha f(x) dx = 0 \right\}.$$

$\mathcal{S}_0(\mathbb{R}^n)$ 是$\mathcal{S}(\mathbb{R}^n)$ 的子空间.

对$\forall \alpha \in \mathbb{Z}_+^n$ 和$f \in \mathcal{S}(\mathbb{R}^n)$, 由$\partial^\alpha(\hat{f}) = ((-2\pi i x)^\alpha f)^\wedge$ 知, $\partial^\alpha(\hat{f})(0) = 0$ 当且仅当

$$\int_{\mathbb{R}^n} x^\alpha f(x) dx = 0.$$

故

$$\mathcal{S}_0(\mathbb{R}^n) = \left\{ f \in \mathcal{S} : \text{ for any } \alpha \in \mathbb{Z}_+^n, \partial^\alpha(\hat{f})(0) = 0 \right\}.$$

Theorem 0.11 (Proposition 1.1.3)**.**

$$\mathcal{S}_0'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n).$$

Proof. 先证$\mathcal{S}_0'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. 对$\forall T \in \mathcal{S}'(\mathbb{R}^n)$, 定义

$$J(T) := T\big|_{\mathcal{S}_0(\mathbb{R}^n)}.$$

则J 是$\mathcal{S}'(\mathbb{R}^n)$ 到$\mathcal{S}_0'(\mathbb{R}^n)$ 的线性映射.

断言1: $\ker J = \mathcal{P}(\mathbb{R}^n)$. 事实上, 对$\forall p := \sum_{\alpha\in\mathbb{Z}_+^n, |\alpha|\le m} c_\alpha x^\alpha \in \mathcal{P}(\mathbb{R}^n)$ 和$\forall f \in \mathcal{S}_0(\mathbb{R}^n)$,

$$\langle p, f\rangle = \int_{\mathbb{R}^n} p(x)f(x)dx = \sum_{\alpha\in\mathbb{Z}_+^n, |\alpha|\le m} c_\alpha \int_{\mathbb{R}^n} x^\alpha f(x)dx = 0.$$

故$p = \theta$ 在$\mathcal{S}_0'(\mathbb{R}^n)$ 中成立, 即$p \in \ker J$. 因此

$$\mathcal{P}(\mathbb{R}^n) \subset \ker J.$$

另一方面, 对$\forall T \in \ker J$, $T|_{\mathcal{S}_0(\mathbb{R}^n)} = 0$, 即对$\forall f \in \mathcal{S}_0(\mathbb{R}^n)$,

$$\langle T, f\rangle = 0. \tag{$*$}$$

对$\forall g \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp} g \subset \mathbb{R}^n \setminus \{\vec{0}_n\}$, 且$\alpha \in \mathbb{Z}_+^n$,

$$\partial^\alpha[(\tilde{g}^\vee)^\wedge](0) = \partial^\alpha(\tilde{g})(0) = 0,$$

故$\tilde{g}^\vee \in \mathcal{S}_0(\mathbb{R}^n)$, 由此及$(*)$ 知,

$$\langle \widehat{T}, g\rangle = \langle \widehat{T}, \tilde{\tilde{g}}\rangle = \langle \widehat{T}, \tilde{g}^{\wedge\wedge}\rangle = \langle T, \tilde{g}^{\wedge\wedge\wedge}\rangle = \langle T, \tilde{g}^\vee\rangle = 0,$$

则$\widehat{T}$ 支在$\{\vec{0}_n\}$ 上. 由此及Proposition 2.4.1 知, $T \in \mathcal{P}(\mathbb{R}^n)$. 因此

$$\ker J \subset \mathcal{P}(\mathbb{R}^n).$$

从而断言1成立.

断言2: $R(J) = \mathcal{S}_0'(\mathbb{R}^n)$. 事实上, ...

由断言1和断言2知,

$$\mathcal{S}_0'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n).$$

□

References

[1] L. Grafakos, Classical Fourier Analysis, Third edition, Graduate Texts in Mathematics, 249, Springer, New York, 2014.

[2] L. Grafakos, Modern Fourier Analysis, Third edition, Graduate Texts in Mathematics, 250, Springer, New York, 2014.