A NON-REFLEXIVE BANACH SPACE ISOMETRIC WITH ITS SECOND CONJUGATE SPACE

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A Banach space B is isometric with a subspace of its second conjugate space B^{**} under the "natural mapping" for which the element of B^{**} which corresponds to the element x_0 of B is the linear functional F_{x_0} defined by $F_{x_0}(f) = f(x_0)$ for each f of B^*. If every F of B^{**} is of this form, then B is said to be reflexive and B is isometric with B^{**} under this natural mapping. The purpose of this note is to show that B can be isometric with B^{**} without being reflexive. The example given to show this is a space isomorphic with a Banach space known to not be reflexive, but to be isomorphic with its second conjugate space.[1]

A sequence $\{x^n\}$ of elements of a Banach space B is said to be a *basis* for B if for each x of B there is a unique sequence of numbers $\{a_n\}$ such that $x = \sum_1^\infty a_i x^i$ in the sense that $\lim_{n\to\infty} \|x - \sum_1^n a_i x^i\| = 0$. A fundamental sequence $\{x^n\}$ is a basis if and only if there is a positive number ϵ such that $\|\sum_1^{n+p} a_i x^i\| \geq \epsilon \|\sum_1^n a_i x^i\|$ for any positive integers n and p and numbers $\{a_i\}$.[2] If $\epsilon = 1$, the basis will be called an *orthogonal basis*. But for any basis $\{x^n\}$, $|||x||| = \sup_n \|\sum_1^n a_i x^i\|$ for $x = \sum_1^\infty a_i x^i$ defines a new norm $|||\ \ |||$ which is equivalent to $\|\ \ \|$ and for which $\{x^n\}$ is an orthogonal basis.[3] Hence if B has a basis $\{x^n\}$ for which $\lim_{n\to\infty} \|f\|_n = 0$ for each f of B^*, where $\|f\|_n$ is the norm of f on $x^{n+1} \oplus x^{n+2} \oplus \ldots$, then the following theorem describes B^{**} completely if the basis is orthogonal and describes B^{**} to within an isomorphism if the basis is not an orthogonal basis.

Theorem. *Let B be a Banach space with an orthogonal basis $\{z^n\}$ for which $\lim_{n\to\infty} \|f\|_n = 0$ for each f of B^*, where $\|f\|_n$ is the norm of f on $z^{n+1} \oplus z^{n+2} \oplus \ldots$. Then $\{g^n\}$ is a basis for B^* if $g^n(z^m) = \delta^n_m$ for each n and m. If $F \in B^{**}$, then $\|F\| = \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$, where $F_i = F(g^i)$. If the sequence $\{F_n\}$ is such that $\lim_{n\to\infty} \|\sum_1^n F_i z^i\| < +\infty$, then $F \in B^{**}$ if one defines $F(f) = \sum_1^\infty F_i f_i$ for each $f = \sum_1^\infty f_i g^i$ of B^*.*

Proof: It has been previously known that $\{g^n\}$ is a basis for B^*.[4] It follows from this that $F(f) = \sum_1^\infty F_i f_i$ for each F of B^{**} and each $f = \sum_1^\infty f_i g^i$ of B^*, where $F_i = F(g^i)$. But, for each $f = \sum_1^\infty f_i g^i$, $|\sum_1^n F_i f_i| = |f(\sum_1^n F_i z^i)| \leq \|f\| \|\sum_1^n F_i z^i\|$. Thus $|\sum_1^\infty F_i f_i| \leq \|f\|(\lim_{n\to\infty} \|\sum_1^n F_i z^i\|)$, and $\|F\| \leq \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$. For a fixed n, let $u^n = \sum_1^n F_i z^i$. Define a linear functional h by $h(z^i) = 0$ for $i > n$ and $h(u^n) = \|u^n\|$. Then $|h(au^n + \sum_{n+1}^\infty a_i z^i)| = \|au^n\| \leq \|au^n + \sum_{n+1}^\infty a_i z^i\|$. Thus $\|h\| = 1$ on $u^n \oplus z^{n+1} \oplus z^{n+2} \oplus \ldots$. Extend h to all of B so that $\|h\| = 1$ on B. Then, for this h, $h = \sum_1^\infty h_i g^i$ with $h_i = 0$ for $i > n$, so that $|\sum_1^\infty F_i h_i| = |\sum_1^n F_i h_i| = |h(u^n)| = \|u^n\| \leq \|F\|$. Since this can be done for each n, it follows that $\|F\| \geq \|\sum_1^n F_i z^i\|$ for each n and $\|F\| \geq \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$. It has thus been shown that $\lim_{n\to\infty} \|\sum_1^n F_i z^i\| = \|F\|$ for each element $F = \{F_n\}$ of B^{**}. Now suppose that $\{F_n\}$ is a sequence such that $\lim_{n\to\infty} \|\sum_1^n F_i z^i\| = M < +\infty$. Then $\|\sum_n^{n+p} F_i z^i\| \leq 2M$. Thus for any fixed $f \in B^*$, $|\sum_n^{n+p} F_i f_i| = |f(\sum_n^{n+p} F_i z^i)| \leq \|f\|_n(2M)$, so that it follows from $\lim_{n\to\infty} \|f\|_n = 0$ that $\sum_1^\infty F_i f_i$ is convergent. Thus $F(f) = \sum_1^\infty F_i f_i$ is defined for each $f \in B^*$ and $\|F\| = \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$.

Example: For $x = (x_1, x_2, x_3, \ldots)$, *let*

$$\|x\| = l.\ u.\ b. \left[\sum_{i=1}^n (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_{n+1}} - x_{p_1})^2\right]^{1/2}, \qquad (1)$$

where the l. u. b. is over all positive integers n and all finite increasing sequences of at least two positive integers $p_1, p_2, \ldots, p_{n+1}$. *Let B be the Banach space of all x for which* $\|x\|$ *is finite and* $\lim_{n\to\infty} x_n = 0$. *Then B is isometric with* B^{**}, *but is isometric under the natural mapping with a closed maximal linear subspace of* B^{**}.

Proof: For $x = (x_1, x_2, \ldots)$, let

$$|||x||| = \text{l. u. b.} \left[\sum_{i=1}^n (x_{p_{2i-1}} - x_{p_{2i}})^2 + (x_{p_{2n+1}})^2\right]^{1/2}, \qquad (2)$$

where the l. u. b. is over all positive integers n and finite increasing sequences of positive integers $p_1, p_2, \ldots, p_{2n+1}$. It follows from $\lim_{n\to\infty} x_n = 0$ and $\|x\| \geq |x_n - x_p|$ that $\|x\| \geq |x_p|$ for each p. Clearly $|||x||| \geq |x_p|$ for each p. But by grouping alternating terms of (1) and isolating x_{p_1}, one gets $\|x\| \leq$ l. u. b. $\{|x_{p_1}| + [(x_{p_{n+1}})^2 + (x_{p_{n-1}} - x_{p_n})^2 + (x_{p_{n-3}} - x_{p_{n-2}})^2 + \ldots]^{1/2} + [(x_{p_n} - x_{p_{n+1}})^2 + (x_{p_{n-2}} - x_{p_{n-1}})^2 + \ldots]^{1/2}\} \leq 3|||x|||$. But extra terms can be introduced in (2) to give a sum of type (1), except for replacing $(x_{p_{2n+1}})$ by $(x_{p_{2n+1}} - x_{p_1})$. Thus $|||x||| \leq 2\|x\|$. Since $1/2|||x||| \leq \|x\| \leq 3|||x|||$, these two norms are equivalent. But the Banach space of all $x = (x_1, x_2, \ldots)$ for which $\lim_{n\to\infty} x_n = 0$ and $|||x|||$ is finite is known to not be reflexive, but to be isometric under the natural mapping with a closed maximal linear subspace of its second conjugate space.[1] Hence this is also true of the space B.

Let $z^n = (0, 0, \ldots, 0, 1, 0, \ldots)$ be the element of B whose components are all zero except for the nth component, which is 1. Then $z^1 \oplus z^2 \oplus \ldots = B$, so that $\{z^n\}$ is an orthogonal basis for B if $\|\sum_1^n a_i z^i + \sum_{n+1}^{n+p} b_i z^i\| \geq \|\sum_1^n a_i z^i\|$ for all numbers $\{a_i\}$ and $\{b_i\}$ and positive integers n and p. Since $\sum_1^n a_i z^i$ has only a finite number of non-zero components, a sequence $p_1, p_2, \ldots, p_{k+1}$ can be chosen so that

$$\|\sum_1^n a_i z^i\| = \left[\sum_{i=1}^{k} (a_{p_i} - a_{p_{i+1}})^2 + (a_{p_{k+1}} - a_{p_1})^2\right]^{1/2}, \tag{3}$$

where $a_r = 0$ if $r > n$. If $p_{k+1} \leq n$, then it is immediate from (1) and (3) that $\|\sum_1^n a_i z^i + \sum_{n+1}^{n+p} b_i z^i\| \geq \|\sum_1^n a_i z^i\|$. If $p_{k+1} > n$, then each p_i with $p_i > n$ can be replaced by some $p_i > n + p$ without changing the value of (3), since $a_r = 0$ if $r > n$. But it will then again follow from (1) and (3) that $\|\sum_1^n a_i z^i + \sum_{n+1}^{n+p} b_i z^i\| \geq \|\sum_1^n a_i z^i\|$. For B with the norm $|||\ \ |||$, and hence also for B with the norm $\|\ \ \|$, it is known that $\lim_{n\to\infty} \|f\|_n = 0$, where $\|f\|_n$ is the norm of f on $z^{n+1} \oplus z^{n+2} \oplus \ldots$.[1] Hence by Theorem 1 above, B^{**} is the space of all $F = (F_1, F_2, \ldots)$ for which $\|F\| = \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$ is finite. Thus for F to belong to B^{**}, it is necessary that $\lim_{n\to\infty} F_n$ exist. Consider the correspondence:

$$x = (x_1, x_2, \ldots) \longleftrightarrow (x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1, \ldots) = (F_1, F_2, \ldots) = F_x.$$

To show that $\|x\| = \|F_x\|$, first consider a sum $\left[\sum_{i=1}^{n} (x_{p_i} - x_{p_{i+1}})^2 + (x_{p_{n+1}} - x_{p_1})^2\right]$. If $p_1 \geq 2$, this is equal to $\left[\sum_{i=1}^{n} (F_{p_i-1} - F_{p_{i+1}-1})^2 + (F_{p_{n+1}-1} - F_{p_1-1})^2\right]$. If $p_1 = 1$, it is equal to $\left[\sum_{i=2}^{n} (F_{p_i-1} - F_{p_{i+1}-1})^2 + (F_{p_{n+1}-1} - F_N)^2 + (F_N - F_{p_2-1})^2\right]$, if $N > p_{n+1} - 1$ and F_N is replaced by zero. Since $\|\sum_1^n F_i z^i\|$ is a monotonically increasing function of n, it follows that $\|x\| \leq \|F_x\|$, where $\|F_x\| = \lim_{n\to\infty} \|\sum_1^n F_i z^i\|$. Now consider a sum $\left[\sum_{i=1}^{n} (F_{p_i} - F_{p_{i+1}})^2 + (F_{p_{n+1}} - F_{p_1})^2\right]$, formed for the element $\sum_1^m F_i z^i$, where F_p is to be replaced by 0 if $p > m$. If $p_{n+1} \leq m$, then this sum is equal to $\left[\sum_{i=1}^{n} (x_{p_i+1} - x_{p_{i+1}+1})^2 + (x_{p_{n+1}+1} - x_{p_1+1})^2\right]$. Now suppose that $p_{k+1} > m$, but $p_i \leq m$ if $i \leq k$. Then the sum becomes $\left[\sum_{i=1}^{k-1} (F_{p_i} - F_{p_{i+1}})^2 + (F_{p_k})^2 + (F_{p_1})^2\right] = \left[\sum_{i=1}^{k-1} (x_{p_i+1} - x_{p_{i+1}+1})^2 + (x_{p_k+1} - x_1)^2 + (x_1 - x_{p_1+1})^2\right]$. Thus $\|x\| \geq \|\sum_1^n F_i z^i\|$ for each n. Hence $\|x\| = \|F_x\|$ and $x \longleftrightarrow F_x$ is an isometry with domain equal to B. But if $F = (F_1, F_2, \ldots)$ is an element of B^{**}, and $\lim_{n\to\infty} F_n = L$, then $x_F = (-L, F_1 - L, F_2 - L, \ldots)$ is, by the above, an element of B for which $\|x_F\| = \|F\|$ and $x_F \longleftrightarrow F$. Thus the range of the isometry is B^{**}.

[1] James, R. C., "Bases and Reflexivity of Banach Spaces," *Ann. Math.*, **52**, 518–527 (1950).

[2] Grinblum, M. M., "Certain théorèmes sur la base dans un espace du type (B)," *C. R. (Doklady) Acad. Sci. URSS* (N. S.), **31**, 428–432 (1941).

[3] Banach, S., *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 111.

[4] James, *loc. cit.*, Theorem 3.