Schwartz 函数和分布

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目录

1 Schwartz 函数

下面给出 Schwartz 函数的定义.

Definition 1.1. *A $C^\infty(\mathbb{R}^n)$ complex-valued function f on $\mathbb{R}^n$ is called a Schwartz function if for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$,*

$$\rho_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} \left| x^\alpha \left(\partial^\beta f\right)(x) \right| < \infty.$$

The set of all Schwartz functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$.

为了给出 Schwartz 函数的一个等价定义, 首先介绍来自 [2, (2.2.1),(2.2.2)] 的两个估计.

Lemma 1.2. *存在仅依赖于 n 和 k 的常数 C 使得, 对任意的 $x \in \mathbb{R}^n$,*

$$|x|^k \le C \sum_{|\beta|=k} |x^\beta| \tag{1}$$

且

$$(1+|x|)^k \le 2^k(1+C) \sum_{|\beta|\le k} |x^\beta|. \tag{2}$$

下面介绍 Schwartz 函数的一个非常有用的等价定义.

Theorem 1.3. *Let $f \in C^\infty(\mathbb{R}^n)$. Then $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if for any $\beta \in \mathbb{Z}_+^n$ and $N \in \mathbb{N}$, there exists a positive constant C such that*

$$|(\partial^\alpha f)(x)| \le C\frac{1}{(1+|x|)^N}.$$

在证明该结论前,

证明. 先证 "$\Longleftarrow$". 对任意 $\alpha, \beta \in \mathbb{Z}_+^n$,

$$\begin{aligned}
\rho_{\alpha,\beta}(f) &= \sup_{x\in\mathbb{R}^n} \left|x^\alpha(\partial^\beta f)(x)\right| \lesssim \sup_{x\in\mathbb{R}^n} |x|^{|\alpha|}\left|(\partial^\beta f)(x)\right| \\
&\lesssim \sup_{x\in\mathbb{R}^n}(1+|x|)^{|\alpha|}\left|(\partial^\beta f)(x)\right| < \infty,
\end{aligned}$$

故 $f \in \mathcal{S}(\mathbb{R}^n)$.

再证 "$\Longrightarrow$". 对任意 $N \in \mathbb{N}$ 和 $\beta \in \mathbb{Z}_+^n$,

$$\sup_{x\in B(0,1)} (1+|x|)^N|(\partial^\beta f)(x)| < \infty$$

且

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^n\setminus B(0,1)} (1+|x|)^N|(\partial^\beta f)(x)| \\
&\lesssim \sup_{x\in\mathbb{R}^n\setminus B(0,1)} |x|^N|(\partial^\beta f)(x)| \\
&\lesssim \sup_{x\in\mathbb{R}^n\setminus B(0,1)} \left(\sum_{\alpha\in\mathbb{Z}_+^n,\,|\alpha|=N} |x^\alpha|\right)|(\partial^\beta f)(x)| \\
&\lesssim \sum_{\alpha\in\mathbb{Z}_+^n,\,|\alpha|=N} \rho_{\alpha,\beta}(f) < \infty,
\end{aligned}$$

故

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^N|(\partial^\beta f)(x)| < \infty,$$

定理 1.3 证毕. □

Remark 1.4. 注意到对任意 $p \in (1, \infty)$ 和 $N \in (p/n, \infty)$,

$$\frac{1}{(1+|x|)^N} \in L^p(\mathbb{R}^n),$$

故对任意 $f \in \mathcal{S}(\mathbb{R}^n)$ 和 $\beta \in \mathbb{Z}_+^n$, $\partial^\beta f \in L^p(\mathbb{R}^n)$.

下面给出 $\mathcal{S}(\mathbb{R}^n)$ 和其它空间的关系.

Proposition 1.5. 设 $p \in [1, \infty)$. 则有以下性质:

(i) $C_c^\infty(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n) \subsetneq L^p(\mathbb{R}^n)$;

(ii) $C_c^\infty(\mathbb{R}^n)$ 在 $L^p(\mathbb{R}^n)$ 中稠.

证明. 先证 (i). 容易看出 $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, 又 $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$ 但无紧支集, 故 $C_c^\infty(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n)$.

又由定理 1.3 知, 对任意 $f \in \mathcal{S}(\mathbb{R}^n)$ 和 $x \in \mathbb{R}^n$,

$$|f(x)| \lesssim \frac{1}{(1+|x|)^{n+1}},$$

因此

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} \, dx \\ &\lesssim 1 + \int_{|x|>1} \frac{1}{|x|^{n+1}} \, dx \sim 1 + \int_1^\infty \frac{1}{r^2} \, dx \sim 1.\end{aligned}$$

故 $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. 又显然 $\mathcal{S}(\mathbb{R}^n) \neq L^p(\mathbb{R}^n)$, 从而 $\mathcal{S}(\mathbb{R}^n) \subsetneq L^p(\mathbb{R}^n)$. (i) 证毕.

(ii) 是经典结论, 用卷积去逼近即可. □

下面给出 Schwartz 函数意义下的收敛.

Definition 1.6. 设 $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ 且 $f \in \mathcal{S}(\mathbb{R}^n)$. 称 f_k 收敛到 f *in* $\mathcal{S}(\mathbb{R}^n)$, *as* $k \to \infty$, 若对任意 $\alpha, \beta \in \mathbb{Z}_+^n$,

$$\rho_{\alpha,\beta}(f_k - f) \to 0, \quad as\ k \to \infty.$$

通过收敛可以定义闭集, 从而有了拓扑. 在此拓扑意义下, Schwartz 空间可赋度量:

$$d: \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to [0, \infty), \ (f, g) \mapsto \sum_{\alpha, \beta \in \mathbb{Z}_+^n} \frac{\rho_{\alpha,\beta}(f-g)}{1+\rho_{\alpha,\beta}(f-g)};$$

但是不能赋予范数.

下面给出 Schwartz 函数傅里叶变换的定义.

Definition 1.7. *Let $f \in L^1(\mathbb{R}^n)$. For any $\xi \in \mathbb{R}^n$, define*

$$
\begin{aligned}
\widehat{f}(\xi) &:= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx; \\
\widetilde{f}(\xi) &:= f(-\xi); \\
(\tau^y f)(\xi) &:= f(\xi - y), \quad \text{where } y \in \mathbb{R}^n.
\end{aligned}
$$

Remark 1.8. 傅里叶变换限制在 $\mathcal{S}(\mathbb{R}^n)$ 上, 是到自身的同胚.

Proposition 1.9. *Let $f \in \mathcal{S}(\mathbb{R}^n)$, then*

(i) $f^{\wedge\wedge}(\cdot) = f(-\cdot)$;

(ii) 对任意 $y \in \mathbb{R}^n$, $\widehat{\tau^y f}(\xi) = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi)$;

(iii) 对任意 $y \in \mathbb{R}^n$, $\tau^y(\widehat{f})(\xi) = [e^{-2\pi i x\cdot y} f(x)]^\wedge(\xi)$;

(iv) 平移变换不改变傅里叶变换的支集. 即对任意 $y \in \mathbb{R}^n$, $\operatorname{supp}\widehat{\tau^y f} = \operatorname{supp}\widehat{f}$.

证明. 现在证 (ii). 对任意 $\xi \in \mathbb{R}^n$,

$$
\begin{aligned}
\widehat{\tau^y f}(\xi) &= \int_{\mathbb{R}^n} (\tau^y f)(x) e^{-2\pi i x\cdot\xi}\, dx \\
&= \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i x\cdot\xi}\, dx \\
&= e^{-2\pi i y\cdot\xi} \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx \\
&= e^{-2\pi i y\cdot\xi}\widehat{f}(\xi),
\end{aligned}
$$

(ii) 得证.

(iv) 是 (ii) 的直接推论. □

2 Schwartz 分布

下面给出 Schwartz 分布傅里叶变换的定义.

Definition 2.1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, define*

$$
\begin{aligned}
\widehat{f}:\ & \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \forall\varphi \in \mathcal{S}(\mathbb{R}^n), \langle \widehat{f}, \varphi\rangle := \langle f, \widehat{\varphi}\rangle; \\
\widetilde{f}:\ & \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \forall\varphi \in \mathcal{S}(\mathbb{R}^n), \langle \widetilde{f}, \varphi\rangle := \langle f, \widetilde{\varphi}\rangle; \\
\tau^y f:\ & \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \forall\varphi \in \mathcal{S}(\mathbb{R}^n), \langle \tau^y f, \varphi\rangle := \langle f, \tau^{-y}\varphi\rangle, \quad \text{where } y \in \mathbb{R}^n.
\end{aligned}
$$

Definition 2.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $u : \mathbb{R}^n \to \mathbb{C}$. We say that f coincides with the function u if, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,*

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} u(x)\varphi(x)\, dx.$$

For any complex-valued function u, let

$$\operatorname{supp} u := \{x \in \mathbb{R}^n : u(x) \neq 0\}.$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\operatorname{supp} f := \bigcap \{\text{closed set } K \subset \mathbb{R}^n : \\ \langle f, \varphi \rangle = 0 \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ with } \operatorname{supp} \varphi \subset \mathbb{R}^n \setminus K\}.$$

由定义知, $\operatorname{supp} f$ 是闭集.

Remark 2.3. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ coincides with the function u, then*

(i)
$$\widehat{f} = \widehat{u}, \quad \widetilde{f} = \widetilde{u}, \quad \tau^y f = \tau^y u,$$

where the '=' means 'coincides with' (这应该是 Definition 2.1 的由来);

(ii) $\operatorname{supp} f \subset \overline{\operatorname{supp} u}$;

(iii) 若 u 是连续函数, 则

$$\operatorname{supp} f = \operatorname{supp} u,$$

其中 $\operatorname{supp} u$ 是函数意义下的支集.

证明. 由定义可直接得到 (i). 下证 (ii) . $\overline{\operatorname{supp} u}$ 是闭集且对任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset \mathbb{R}^n \setminus \overline{\operatorname{supp} u}$,

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} u(x)\varphi(x)\, dx = 0,$$

因此由分布支集定义知 $\operatorname{supp} f \subset \overline{\operatorname{supp} u}$.

再证 (iii). 由 (ii) 和 u 连续知 $\operatorname{supp} f \subset \operatorname{supp} u$. 故只需证 $\operatorname{supp} u \subset \operatorname{supp} f$. 对任意 $x \in \operatorname{supp} u$, 有 $u(x) \neq 0$, 不妨设 $u(x) > 0$. 因为 u 连续, 故存在 $r_x \in (0, \infty)$ 使得 u 在 $B(x, r_x)$ 上恒大于 $u(x)/2$. 假设存在

$$K_x \in \{\text{closed set } K \subset \mathbb{R}^n : \\ \langle f, \varphi \rangle = 0 \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ with } \operatorname{supp} \varphi \subset \mathbb{R}^n \setminus K\} \tag{3}$$

使得 $x \notin K_x$, 则存在 $\delta \in (0, r_x)$ 使得 $B(x,\delta) \cap K_x = \emptyset$. 令 $\varphi_x \in \mathcal{S}(\mathbb{R}^n)$ 满足

$$\varphi_x(t) := \begin{cases} 1, & t \in B(x,\delta/2), \\ \text{取值为 } [0,1], & t \in B(x,\delta) \setminus B(x,\delta/2), \\ 0, & t \in \mathbb{R}^n \setminus B(x,\delta). \end{cases}$$

则

$$\langle f, \varphi_x \rangle = \int_{\mathbb{R}^n} u(t)\varphi_x(t)\,dt = \int_{B(x,\delta)} u(t)\varphi_x(t)\,dt \geq \frac{u(x)}{2} \int_{B(x,\delta)} \varphi_x(t)\,dt > 0,$$

这与 (3) 相矛盾. 故

$$\begin{aligned} x \in \bigcap \{&\text{closed set } K \subset \mathbb{R}^n : \\ &\langle f, \varphi \rangle = 0 \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ with } \operatorname{supp} \varphi \subset \mathbb{R}^n \setminus K\} \\ = &\operatorname{supp} f. \end{aligned}$$

因此 $\operatorname{supp} u \subset \operatorname{supp} f$, Remark 2.3 证毕. □

Proposition 2.4. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then*

(i) $f^{\wedge\wedge}(\cdot) = \widetilde{f}$;

(ii) 平移变换不改变傅里叶变换的支集. 即对任意 $y \in \mathbb{R}^n$, $\operatorname{supp} \widehat{\tau^y f} = \operatorname{supp} \widehat{f}$.

证明. 现在证 (ii). 对任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\langle \widehat{\tau^y f}, \varphi \right\rangle = \langle \tau^y f, \widehat{\varphi} \rangle = \langle f, \tau^{-y} \widehat{\varphi} \rangle = \left\langle f, [e^{-2\pi i x\cdot y}\varphi(x)]^{\wedge} \right\rangle = \left\langle \widehat{f}, e^{-2\pi i x\cdot y}\varphi(x) \right\rangle,$$

故 $\operatorname{supp} \widehat{\tau^y f} = \operatorname{supp} \widehat{f}$, (ii) 得证. □

3 再生公式

4 Schwartz 函数模多项式, $\mathcal{S}_\infty(\mathbb{R}^n)$

$$\mathcal{S}_\infty(\mathbb{R}^n) := \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\alpha f(x) = 0 \quad \text{for any } \alpha \in \mathbb{Z}_+^n \right\}.$$

Theorem 4.1.

$$\mathcal{S}(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) = \mathcal{S}_\infty(\mathbb{R}^n).$$

这个空间上的拓扑怎么构造?

下面给出判断函数属于 $\mathcal{S}_\infty(\mathbb{R}^n)$ 的一个方法.

Proposition 4.2. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. If $0 \notin \overline{\operatorname{supp} \widehat{f}}$, then $f \in \mathcal{S}_\infty(\mathbb{R}^n)$.*

证明. 关键是利用傅里叶变换的性质: 对任意 $\alpha \in \mathbb{Z}_+^n$ 和 $\xi \in \mathbb{R}^n$,

$$\left(\partial^\alpha \widehat{f}\right)(\xi) = [(-2\pi i x)^\alpha f(x)]^\wedge(\xi),$$

又因为 $\mathbf{0} \notin \overline{\operatorname{supp} \widehat{f}}$, 故 $(\partial^\alpha \widehat{f})(\mathbf{0}) = 0$, 从而

$$0 = [(-2\pi i x)^\alpha \partial^\alpha f(x)]^\wedge(\mathbf{0}) = \int_{\mathbb{R}^n} (-2\pi i x)^\alpha f(x)\, dx,$$

即

$$\int_{\mathbb{R}^n} x^\alpha f(x)\, dx = 0.$$

□

4.1 离散再生公式

Lemma 4.3. 设 $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ 满足

$$\sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi}(2^j\xi)}\widehat{\psi}(2^j\xi) = 1 \quad \text{if } \xi \neq 0$$

且 $\widehat{\varphi}$ 和 $\widehat{\psi}$ 支在一个环上, 即

$$\operatorname{supp} \widehat{\varphi},\ \operatorname{supp} \widehat{\psi} \subset \{\xi \in \mathbb{R}^n :\ a \le |\xi| \le b\}.$$

则

(i) 对任意 $f \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$f = \sum_{Q\in\mathcal{D}} \langle f, \varphi_Q\rangle \psi_Q \tag{4}$$

in $\mathcal{S}_\infty(\mathbb{R}^n)$;

(ii) 对任意 $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, (4) 在 $\mathcal{S}'_\infty(\mathbb{R}^n)$ 意义下收敛.

4.2 小波再生公式

Proposition 4.4. 设 $\{e_Q^{(i)}\}_{Q\in\mathcal{D}}$ 是 $L^2(\mathbb{R}^n)$ 的一组正交小波基. 则

(i) 对任意 $f \in L^2(\mathbb{R}^n)$

$$f = \sum_{Q\in\mathcal{D}} \langle f, e_Q\rangle \, e_Q \tag{5}$$

in $L^2(\mathbb{R}^n)$*;*

(ii) 对任意 $f \in \mathcal{S}_\infty(\mathbb{R}^n)$, (5) 在 $\mathcal{S}(\mathbb{R}^n)$ 意义下收敛*;*

(iii) 对任意 $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, (5) 在 $\mathcal{S}'_\infty(\mathbb{R}^n)$ 意义下收敛.

证明. 由基的定义和正交性, (i) 不难得到. 下证 (ii). 固定 $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$. 对任意 $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, 由 (i) 知

$$\varphi = \sum_{Q\in\mathcal{D}} \langle \varphi, e_Q\rangle \, e_Q$$

(ii) 要对小波加限制. □

5 特殊 Schwartz 函数的构造

很多时候, 我们需要构造一些特殊的 Schwartz 函数, 下面来举几个例子.

5.1 Pseudo-differential 算子

在证明 pseudo-differential 算子在 Besov 和 Triebel-Lizorkin 空间上有界性时, 需要用到 Calderón reproducing formulae 的特殊情况, 即让

$$f = \sum_Q \langle f, \varphi_Q\rangle \psi_Q$$

中的 $\varphi = \psi$.

参考 [1, Lemma (6.9)] 的证明可以构造满足上述条件的 φ.

Lemma 5.1. *There exists a* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *such that* $0 \notin \operatorname{supp} \widehat{\varphi}$ *and, for any* $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\sum_{j\in\mathbb{Z}} \left[\widehat{\varphi}\left(2^j \xi\right)\right]^2 = 1.$$

证明. 令 $h \in \mathcal{S}(\mathbb{R}^n)$ 满足

$$\begin{aligned} &\operatorname{supp} \widehat{h} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}, \\ &\widehat{h}(\xi) \geq c, \qquad \text{for any } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}. \end{aligned}$$

对任意 $\xi \in \mathbb{R}^n$, 令

$$\psi(\xi) := \left\{ \sum_{j\in\mathbb{Z}} \left[\widehat{h}\left(2^j\xi\right)\right]^2 \right\}^{1/2},$$

则对任意 $i \in \mathbb{Z}$, $\psi(2^i\xi) = \psi(\xi)$, 对任意 $\xi \in \mathbb{R}^n$, $\psi(\xi) \geq c$, 且 $\psi \in C^\infty(\mathbb{R}^n)$. 注意到

$$\widehat{h}/\psi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n) \overset{\text{Proposition 1.5(i)}}{\subset} \mathcal{S}(\mathbb{R}^n),$$

故可令 $\varphi := (\widehat{h}/\psi)^\vee$. 则 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ 且

$$\sum_{j\in\mathbb{Z}} \left[\widehat{\varphi}\left(2^j\xi\right)\right]^2 = \frac{\sum_{j\in\mathbb{Z}}[\widehat{h}(2^j\xi)]^2}{[\psi(\xi)]^2} = 1.$$

□

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[2] L. Grafakos, Classical Fourier Analysis, Third edition, Graduate Texts in Mathematics 249, Springer, New York, 2014.