Superexponentiation and Fixed Points of Exponential and Logarithmic Functions

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1. Introduction

We shall investigate an application of "superexponentiation," an operation that we denote by $\Uparrow$ (following **[5]**) and define as follows:

$$b \Uparrow n := b\,\hat{}\,(b\,\hat{}\,(\cdots\hat{}\,b)\cdots) \quad (b>0,\ n=1,2,\ldots), \tag{1}$$

where exponentiation occurs n times. Superexponentiation simply continues the pattern of addition, multiplication, and exponentiation. It seems to have first appeared in the literature in **[4]**, for the purpose of exhibiting extremely large (albeit finite) numbers (its *implicit* use in **[7]**, an earlier work than **[4]**, is shown in **[5]**). Superexponentiation is used in **[5]** and **[6]** to examine the logical foundation of mathematical induction. In **[2]** superexponentiation and its inverse operation, iteration of logarithms, are used to analyze the running time of certain algorithms. A general discussion of superexponentiation is given in **[1]**.

For our purposes, superexponentiation naturally arises from analyzing fixed points of exponential functions (and hence of the corresponding logarithms). We approach this using orbit analysis as in **[3]**, i.e., by iterating the exponential function $F(x)=b^x$, starting with input $x=b$. For $b>1$ the resulting orbit clearly will be strictly increasing, and for bases such as 2 or 10 (those used in the applications cited above), the orbit will diverge rapidly to infinity. Will this divergence occur for all $b>1$? What kind of behavior is observed if $0<b<1$? Will the orbit converge to a single number for suitable values of b?

In Section 2 we discuss the context under which the author arrived at the problem. In Section 3 we answer the question of convergence, finding the set of b for which the orbit does indeed converge; this result is stated as a theorem at the end of the section. In Section 4 we offer several suggestions for further exploration.

2. Motivation

When introducing exponential and logarithmic functions,

$$F(x)=b^x \quad \text{and} \quad G(x)=\log_b x \quad (b>0, b\neq 1),$$

the instructor will inevitably display the graphical consequence of the fact that $F(x)$ and $G(x)$ are inverses. FIGURE 1 shows the case $b=\mathrm{e}$, surely the most popular case to present, given the importance of the functions e^x and $\ln x$.

For other bases, however, the graphical situation may not be as familiar. FIGURE 2 shows the case $b=\{1/2\}$. For $0<b<1$, b^x exhibits exponential decay rather than exponential growth. While this often may be presented, the corresponding logarithm

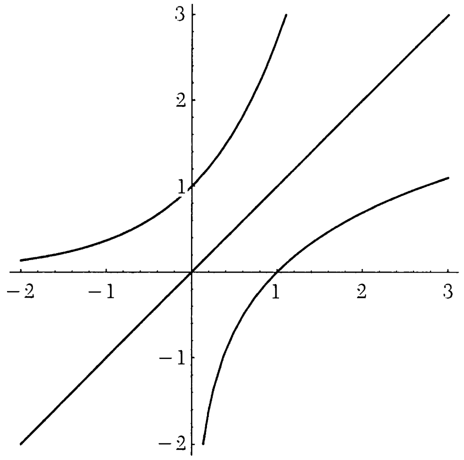

FIGURE 1
The familiar graphs of e^x, $\ln x$, and x.

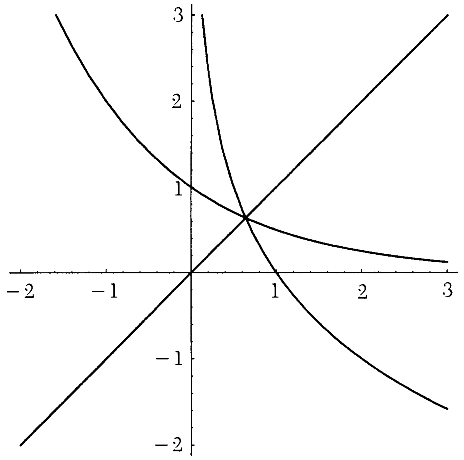

FIGURE 2
The graphs of $(1/2)^x$, $\log_{1/2} x$, and x.

may often be omitted (after all, the three most prevalent logarithmic bases, $b = 2$, $b = e$ and $b = 10$, all satisfy $b > 1$). Consider, however, that in FIGURE 2, unlike in FIGURE 1, the graphs of the exponential and logarithmic functions intersect (along the line $y = x$). Thus, there is some real number x_0, clearly between 0 and 1, for which

$$(1/2)^{x_0} = x_0 = \log_{1/2} x_0 .$$

In other words, x_0 is simultaneously a fixed point of the functions $f(x) = (1/2)^x$ and $g(x) = \log_{1/2} x$. How can we find the value of x_0? Trying to solve $(1/2)^x = x$ using the inverse function results in $x = \log_{1/2} x$ (and vice versa, of course), and the equation $(1/2)^x = \log_{1/2} x$ does not seem any more promising.

Let us try a recursive approach. Take the given equation, $x = 0.5^x$, and substitute it into itself:

$$x = 0.5^{(0.5^x)}.$$

(Note that, as indicated by the parentheses, this cannot be simplified using the familiar property $(b^m)^n = b^{mn}$. Unlike addition and multiplication, exponentiation is nonassociative; cf. **[1]**) Repeating this process ad infinitum leads to a formal expression for the solution:

$$x_0 = 0.5^{\left(0.5^{(\cdot^{\cdot^{\cdot}})}\right)}. \tag{2}$$

It remains to determine whether or not (2) converges. Numerical calculation suggests oscillatory convergence, as evidenced by the orbit of 0.5 under iteration of $f(x) = 0.5^x$ (rounded to three significant figures):

$$0.500, 0.707, 0.613, 0.654, 0.635, 0.643, 0.640, 0.641, \ldots.$$

The oscillation and convergence of this orbit can also be seen graphically in FIGURE 3, which stems from FIGURE 2. At least we can thus approximate the value of the fixed point x_0 to any desired degree of accuracy. In the next section, we shall generalize the above discussion to other bases and explore the behavior of the resulting family of functions.

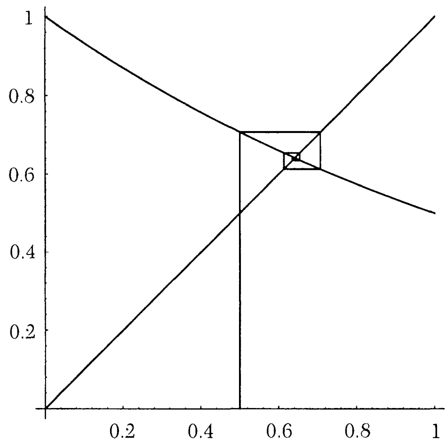

FIGURE 3

The graphs of $(1/2)^x$ and x, as well as the orbit of 1/2 under iteration of $f(x) = (1/2)^x$, approaching the limit of (2).

3 Generalization

Given a base $b > 0$, we wish to determine whether

$$b \Uparrow \infty := \lim_{n \to \infty} b \Uparrow n \tag{3}$$

exists as a real number, or is infinite, or does not exist. In case the limit exists as a number, which we denote by x, it follows that x is a solution to

$$b^x = x, \text{ or equivalently, } x = \log_b x. \tag{4}$$

We must first ask, for what $b > 0$ is it even possible to solve (4)? The case $b = 1$ is trivial: we can easily solve $1^x = x$ (but note that $\log_b x$ is not defined for $b = 1$). For $b \in (0, 1)$ the graphs of b^x, x, and $\log_b x$ will intersect, as in FIGURE 2, and so it is possible to solve (4). What of the case $b > 1$? For sufficiently large b (e.g., $b = \mathrm{e}$) the graphs of b^x, x, and $\log_b x$ do not intersect, and so (4) is impossible. Since, however, b^x does intersect the line $y = x$ when $0 < b \le 1$, (4) should be possible when $b > 1$ is sufficiently close to 1 as well.

We explore this by focusing on the exponential function (results for the logarithmic function would follow immediately). To this end, consider the one-parameter family of functions

$$F_b(x) = b^x \quad (b > 0), \tag{5}$$

parameterized by the base b. (Unlike with $F(x)$, we now explicitly indicate that the base b is a parameter.) What happens to the graph of $F_b(x)$ as $b > 0$ varies? We show five snapshots of $F_b(x)$ plotted in the same coordinate plane in FIGURE 4. Note that the family pivots about the point $(0,1)$ since $b^0 = 1$ for all $b > 0$. For b sufficiently close to 0, $F_b'(0)$ is a negative number of large magnitude. While $F_b'(0)$ stays negative for all $0 < b < 1$, its magnitude lessens as $b \to 1$. When $b = 1$, $F_b'(0) = 0$ (and the graph is simply a horizontal line). When b is just slightly larger than 1, then, $F_b'(0)$ is just slightly positive, and we see that this allows for *two* intersection points of $F_b(x)$ with x, i.e., two solutions of (4).

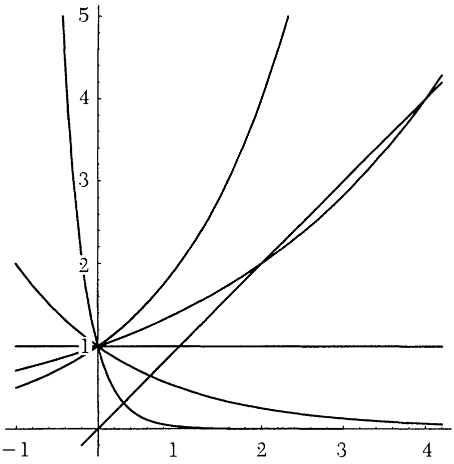

FIGURE 4

The graphs of $F_b(x) = b^x$, for $b = 1/40, 1/2, 1, \sqrt{2}$, and 2, along with the line $y = x$.

As $b > 1$ becomes larger, the value of $F_b'(0)$ becomes larger in magnitude, and eventually the graphs of $F_b(x)$ and x no longer intersect, so that (4) is no longer possible to solve. We can determine the unique value of $b > 1$ for which the graphs of $F_b(x)$ and x intersect at just one point by solving the system of two equations,

$$\{F_b(x) = x,\ F_b'(x) = 1\}, \tag{6}$$

in the two unknowns b and x; see FIGURE 5. These two equations are simply

$$\{b^x = x,\ b^x \ln b = 1\}, \tag{7}$$

from which it immediately follows that

$$1 = b^x \ln b = x \ln b = \ln(b^x) = \ln x,$$

providing the solution to (6): $x = \mathrm{e}$ and $b = \mathrm{e}^{1/\mathrm{e}}$, the latter of which may be called the e^{th} root of e!

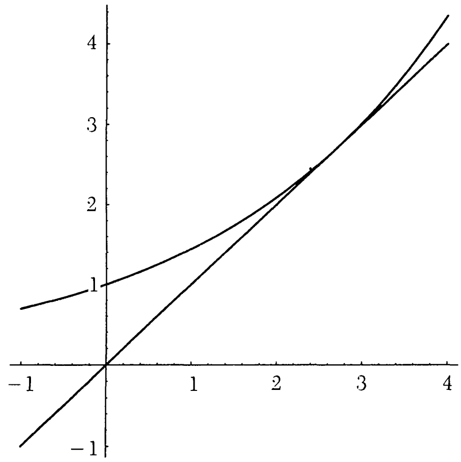

FIGURE 5
The graph of $F_{e^{1/e}}(x) = e^{x/e}$ tangentially intersecting the line $y = x$ at the point (e, e).

Therefore, we seek to solve (4) only for $0 < b < e^{1/e}$. Furthermore, while there may be other ways to solve this equation (see items i. and ii. in Section 4), we wish to use the iterative process that was outlined in Section 2 for the case $b = 1/2$. To do so, we employ the methods given in **[3]** (see especially Chapters 3–6, 12), which introduces the mathematics of chaos using a dynamical systems approach. This involves analyzing the evolution of fixed points of a family of functions as the parameter of the family varies. In fact, our present query is framed within the same context: our family is defined in (5), with parameter b. For a given $b \in (0, e^{1/e})$, we are attempting to find the fixed point of $F_b(x)$ by analyzing the orbit of the initial input (seed), b, under iteration of F_b. (While this seed is a natural choice for us, it is wise to choose seeds more deliberately in general; see Chapter 16 of **[3]**.)

We use the methods of **[3]** in two cases: $1 < b < e^{1/e}$ and $0 < b < 1$. For $b \in (1, e^{1/e})$, the graph of $F_b(x)$, being qualitatively the same as that of $F_{\sqrt{2}}(x)$ (see FIGURE 6), has two fixed points. At the lower fixed point $|F_b'(x)| < 1$, while at the

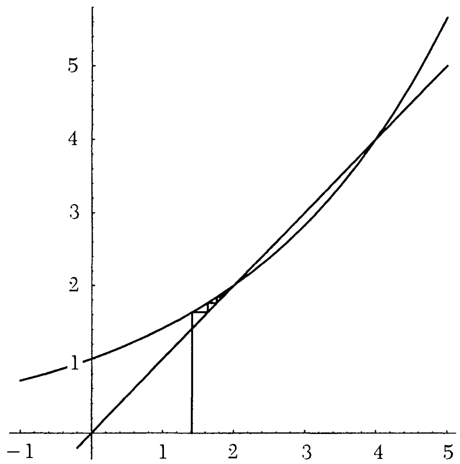

FIGURE 6
The graphs of $F_{\sqrt{2}}(x) = (\sqrt{2})^x$ and x, as well as the orbit of the seed $\sqrt{2}$ under iteration of $F_{\sqrt{2}}$, approaching the lower fixed point of $F_{\sqrt{2}}$, which is $x = 2$.

upper fixed point, $|F_b'(x)| > 1$ (this is clear since the slope of $y = x$ itself is 1). Therefore, the lower fixed point is attracting and the upper fixed point is repelling. Orbit analysis readily reveals that for any seed chosen less than the value of the upper fixed point, its orbit will be attracted to the lower fixed point. For any seed greater than the upper fixed point, however, its orbit will diverge to ∞. Luckily, our seed b is in the basin of attraction of the lower fixed point!

We can conclude, therefore, that for $b \in (1, e^{1/e})$, the limit $b \Uparrow \infty$ exists and yields the value of one of the two solutions to (4), namely the lower fixed point. Before we move on to the second case, consider the endpoints of the first case: $1 \Uparrow \infty = 1$ is trivial, and we have seen $b = e^{1/e}$ in FIGURE 5. In fact, we know that $F'_{e^{1/e}}(e) = 1$, so that a saddle-node bifurcation should occur at the parameter value $b = e^{1/e}$. Indeed, F_b has no fixed points for $b > e^{1/e}$, one fixed point for $b = e^{1/e}$, and two fixed points (one attracting and one repelling) for $1 < b < e^{1/e}$. We shall revisit this upper endpoint below.

To analyze the case $0 < b < 1$, let us get better acquainted with the family of functions (5) for such values of b; FIGURE 7 shows five members. Clearly, for $b \in (0, 1)$, the function F_b will have exactly one fixed point, which will necessarily be in the interval $0 < x < 1$. This fixed point will be attracting or repelling depending on whether the derivative F_b', evaluated at the fixed point, will be less than or greater than 1 in magnitude. Moreover, a period-doubling bifurcation will occur at the value of b for which the derivative is equal to -1. We can find this particular parameter value and corresponding fixed point by solving the system (6) again, except that we replace the 1 on the right side of the second equation with -1. Solving as before, it is straightforward to show that the solution is $b = (1/e)^e$ and $x = 1/e$. (The critical parameter values $b = e^{1/e}$ and $b = (1/e)^e$ were found only empirically in [**1**].)

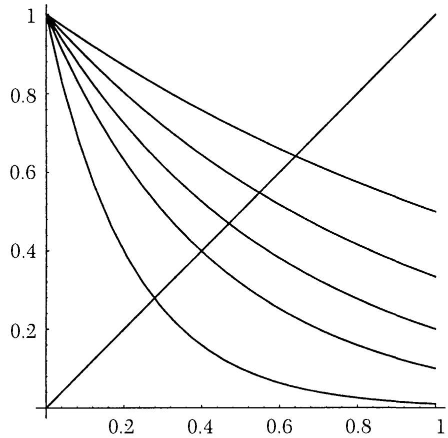

FIGURE 7

The graphs of $F_b(x) = b^x$, for $b = 1/40$, $1/10$, $1/5$, $1/3$, and $1/2$, along with the line $y = x$.

To better understand the saddle-node bifurcation at $b = e^{1/e} \approx 1.44$ and the period-doubling bifurcation at $b = (1/e)^e \approx .0660$, we plot two versions of a bifurcation diagram in FIGURE 8. Let us start in the upper right corner of the top diagram, at the point $(e^{1/e}, e)$. This point on the bifurcation diagram signifies that for the base value $b = e^{1/e}$, the corresponding function from the family, $F_{e^{1/e}}$, has a fixed point $x = e$. To the right of this point, i.e., for larger values of b, there are no fixed points. When b becomes smaller than $e^{1/e}$ (but larger than 1), we know that there are, in fact, two fixed points of F_b. The upper fixed point is repelling, however, and so the

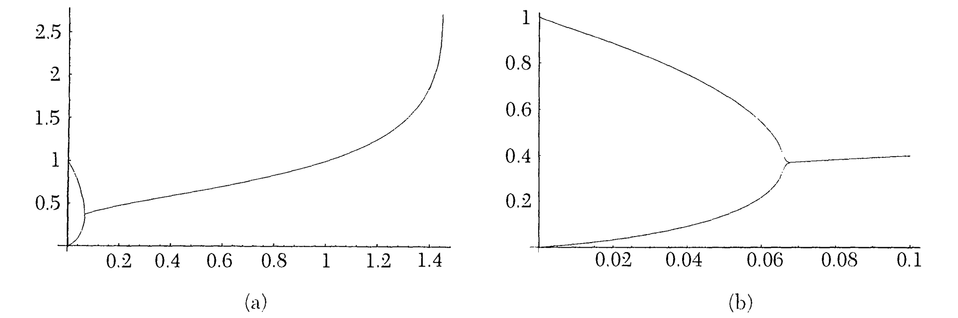

FIGURE 8

Two bifurcation diagrams for $F_b(x) = b^x$, the first for $0 < b < e^{1/e}$ and the second for $0 < b < 0.1$.

upper fixed point branch (which would trace a curve upwards and to the left of $(e^{1/e}, e)$) does not appear in this bifurcation diagram. Only the lower (attracting) fixed point branch appears.

Although it is not apparent in the bifurcation diagram, something special occurs at $b = 1$, for there is suddenly only one fixed point of F_b here. This would be seen in the repelling fixed point branch, if it were present. It would increase without bound, asymptotically approaching the line $b = 1$, since as b decreases to 1, the upper (repelling) fixed point increases to ∞. On the other hand, since the attracting fixed points change continuously as b decreases through 1, the attracting fixed point branch is uneventful at $(1, 1)$.

The single fixed point branch of the bifurcation diagram continues down to the point $((1/e)^e, 1/e)$, where the period-doubling bifurcation occurs (see the second graph in FIGURE 8). Indeed, as b decreases through $(1/e)^e$, the attracting fixed point branch bifurcates into an attracting 2-cycle branch. The remaining fixed points are repelling, and hence this fixed point branch for $b \in (0, (1/e)^e)$ does not appear in the diagram. For completeness we show, in FIGURE 9, the orbit analysis for a base less than $(1/e)^e$, namely $b = 1/40$. Of course, in this case the spiral approaches a 2-cycle rather than a fixed point (see FIGURE 3).

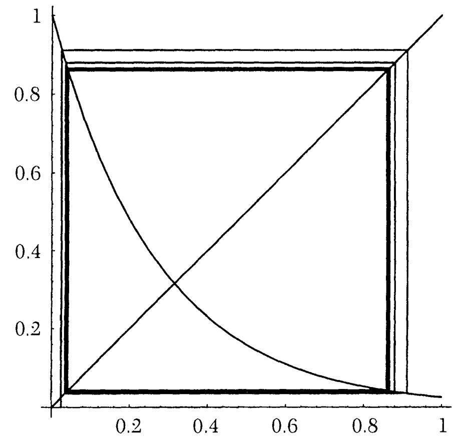

FIGURE 9

The graphs of $F_{1/40}(x) = (1/40)^x$ and x, as well as the orbit of the seed $1/40$ under iteration of $F_{1/40}$, approaching a 2-cycle.

Referring to Chapter 16 of [3], we can show that for any $b < e^{1/e}$, b is in the basin of attraction of the attractor of $F_b(x)$, which is simply the lower fixed point branch together with the 2-cycle branch (i.e., the first graph of FIGURE 8). Indeed, for $b < e^{1/e}$,

$$F_b'(b) = (\ln b)b^b < (\ln e^{1/e})b^b = (1/e)b^b < 1,$$

provided that $b^b < e$, or equivalently, $b \ln b < 1$. However, this also follows from the fact that $b < e^{1/e}$:

$$b \ln b < e^{1/e} \ln e^{1/e} = e^{(1/e)-1} < 1.$$

We can now state our main theorem:

THEOREM. *The limit $b \Uparrow \infty$, defined in (1) and (3), exists, and hence provides a solution to $b^x = x$, if and only if $b \in [(1/e)^e, e^{1/e}]$. For $b \in [(1/e)^e, 1]$, moreover, $b \Uparrow \infty$ is the unique solution to $b^x = x$.*

It remains only to verify the convergence of $b \Uparrow \infty$ when $b = (1/e)^e$ and when $b = e^{1/e}$. At both of these endpoints, the fixed points are neutral, since $|F'_{(1/e)^e}(1/e)| = |F'_{e^{1/e}}(e)| = 1$. Through orbit analysis, it is straightforward to show that the fixed point of $F_{(1/e)^e}(x)$, namely $1/e$, is weakly attracting. Essentially, this describes the fact that $((1/e)^e) \Uparrow \infty$ exists, but that the convergence of the orbit to $1/e$ is extremely slow. Similarly, the fixed point of $F_{e^{1/e}}(x)$, namely e, weakly attracts seeds less than e but weakly repels greater seeds. Since the seed we use in this case, namely the base $e^{1/e}$, is less than the fixed point e (lucky again!), we have $(e^{1/e}) \Uparrow \infty = e$.

4. Exploration

We propose six avenues for further investigation:

i. Equation (4) has at least one solution for all $b \in (0, e^{1/e}]$ (see FIGURES 4 and 5). As previously stated the solutions to (4) for $0 < b < (1/e)^e$, as well as the upper fixed points for $1 < b < e^{1/e}$, are precisely the repelling fixed points missing from the bifurcation diagrams of FIGURE 8. How can the repelling fixed points be found? What is the behavior of the fixed point as $b \to 0$, i.e., what does the repelling fixed point branch for $0 < b < (1/e)^e$ look like? A nice graphical project is to plot the repelling fixed point branches where they belong in the bifurcation diagram.

ii. Write $b = e^a$, so that the equation $b^x = x$ becomes $e^{ax} = x$. Solve this last equation for a as a function of x. Show that this function is strictly increasing for $0 < x < e$ with image $-\infty < a < 1/e$, and strictly decreasing for $x > e$ with image $1/e > a > 0$. Conclude that for each pair (x, a) the exponential function with base $b = e^a$ has a fixed point x. Note that for each $a \in (0, 1/e)$, i.e., for each $b \in (1, e^{1/e})$, there are two pairs (x_1, a) and (x_2, a), which correspond to the lower and upper fixed points! (This method gives us a closed form for the base given a desired fixed point, but we still do not have a closed form for the fixed point(s) given a desired base.)

iii. Provide a classical proof of convergence for $b \in [(1/e)^e, e^{1/e}]$ by considering the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = b \Uparrow n$. For $1 \le b \le e^{1/e}$, the proof is straightforward since the sequence is clearly monotone and easily bounded. For $(1/e)^e \le b < 1$, however, the situation is a bit more complicated since the sequence is not monotone. Considering $r \le b < 1$ with $r \ge (1/e)^e$, the difficulty of proof increases as r decreases to $(1/e)^e$!

iv. For $x > 0$, define a sequence of functions by $f_n(x) = x \Uparrow n$, so that $f_1(x) = x$, $f_2(x) = x^x, f_3(x) = x^{(x^x)}$, etc. Using a graphing calculator or a computer, plot $f_n(x)$ for several values of n. How does the $\lim_{x \to 0^+} f_n(x)$ depend on n? Why? Does the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ seem to converge pointwise to a limit function $f(x)$, at least for some interval of values for x? How does this relate to FIGURE 8?

v. The reader may have seen a teaser where one *begins* with superexponentiation: for example, solve

$$b^{b^{\cdot^{\cdot^{\cdot}}}} = 2, \tag{8}$$

i.e., $b \Uparrow \infty = 2$. Assuming the convergence of $b \Uparrow \infty$, show that $b = \sqrt{2}$ (cf. FIGURE 6). Try solving $b \Uparrow \infty = 4$ in the same way; what is amiss? How big can the right hand side of (8) be for the assumption of convergence of $b \Uparrow \infty$ to be valid? On the other hand, what information is obtained upon solving (8) if the right hand side is "too big"?!

vi. Consider the implications of the tautology $(x^{1/x})^x = x$ $(x > 0)$, and relate your findings to item ii.

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