Cauchy-Schwarz Inequality

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一般书上讲的都是Hilbert 空间中的Cauchy–Schwarz inequality, 张恭庆的书上讲了个更一般的[p.55]. 而下面的Cauchy–Schwarz inequality, 是我见过的条件最弱的, 暂时还没找到该定理的出处.

本文中的X 均为线性空间.

Definition 0.1. 若二元函数$a : X \times X \to \mathbb{C}$ 满足: for any $\alpha \in \mathbb{C}$ and $x, y \in X$,

(i) $a(\alpha x, y) = \alpha a(x, y)$;

(ii) $a(x, \alpha y) = \overline{\alpha} a(x, y)$.

我们称$a(\cdot, \cdot)$ 为X 上的共轭双线性函数.

Theorem 0.2 (Cauchy–Schwarz inequality)**.** 设$a(\cdot, \cdot)$ 为X 上的共轭双线性函数且*for any* $x \in X$,

$$a(x, x) \geq 0,$$

then for any $x, y \in X$,

$$|a(x, y)|^2 \leq a(x, x)a(y, y). \tag{1}$$

Proof. For any $x, y \in X$, if $a(x, x) = a(y, y) = 0$, then

$$\begin{aligned}
0 &\leq a(x + y, x + y) = a(x, x) + a(y, y) + 2\mathrm{Re}a(x, y) = 2\mathrm{Re}a(x, y),\\
0 &\leq a(x - y, x - y) = a(x, x) + a(y, y) - 2\mathrm{Re}a(x, y) = -2\mathrm{Re}a(x, y),\\
0 &\leq a(x + iy, x + iy) = a(x, x) + a(y, y) + 2\mathrm{Im}a(x, y) = 2\mathrm{Im}a(x, y),\\
0 &\leq a(x - iy, x + iy) = a(x, x) + a(y, y) - 2\mathrm{Im}a(x, y) = -2\mathrm{Im}a(x, y),
\end{aligned}$$

So $\mathrm{Re}a(x, y) = 0 = \mathrm{Im}a(x, y)$, i.e $a(x, y) = 0$.

$$|a(x, y)|^2 = 0 = a(x, x)a(y, y).$$

Otherwise, we assume $a(y,y) \neq 0$, let $\lambda := a(x,y)/a(y,y)$, then

$$\begin{aligned}
0 &\leq a(x - \lambda y, x - \lambda y) \\
&= a(x,x) - \overline{\lambda} a(x,y) - \lambda a(y,x) + |\lambda|^2 a(y,y) \\
&= a(x,x) - \frac{|a(x,y)|^2}{a(y,y)}.
\end{aligned}$$

From this we have

$$|a(x,y)|^2 \leq a(x,x)a(y,y).$$

The proof of Cauchy-Schwarz inequality is finished. $\square$