

Equidistributed

The content is from [1, Chapter 4, Section 2].

Definition 1. A sequence of number $\{\xi_n\}_{n \in \mathbb{N}} \subset [0, 1)$ is said to be equidistributed if for any interval $(a, b) \subset [0, 1)$,

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \xi_n \in (a, b)\}}{N} = b - a.$$

where the $\#A$ denotes the cardinality of the finite set A .

Definition 2. Let $[x]$ denote the greatest integer less than or equal to x and call $[x]$ the integer part of x . Let $\langle x \rangle := x - [x]$ and call $\langle x \rangle$ the fractional part of x .

Theorem 1 (Weyl's criterion). A sequence of number $\{\xi_n\}_{n \in \mathbb{N}} \subset [0, 1)$ is equidistributed if and only if for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{1}{N} \sum_{n=1}^N e^{2\pi i k \xi_n} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

From Weyl's criterion, we have following corollaries.

Corollary 1. Let $P(n) := c_n x^n + \cdots + c_0$, for any $k \in \{0, \dots, n\}$, $c_n \in \mathbb{R}$ and one of them is irrational number. Then $\{\langle P(n) \rangle\}_{n \in \mathbb{N}} \subset [0, 1)$ is equidistributed.

Corollary 2. $\{\sin n\}_{n \in \mathbb{N}}$ is dense in $[0, 1]$.

References

- [1] E. M. Stein and R. Shakarchi, Fourier Analysis: An Introduction, Princeton University Press, 2011.