for all x sufficiently close to a. The result analogous to equation (12) reads

$$F(x, \xi) = g(x) - h(\xi) = 0, \tag{15}$$

say, where

$$g(x) = \sum_{k=0}^{\infty} \frac{(x-a)^{k+1}}{k!} \sum_{i=0}^{\infty} \binom{1/p}{i} i!\, \mathbf{B}_{k,i}\,[f_\nu],$$

$$h(\xi) = \binom{n+p}{n}^{1/p} \sum_{j=0}^{\infty} \frac{(\xi-a)^{j+1}}{j!} \sum_{i=0}^{j} \binom{1/p}{i} i!\, \mathbf{B}_{j,i}\,[\widetilde{f_\nu}].$$

Since the partial derivative $F_\xi(a, a) = -h'(a) = -\binom{n+p}{n}^{1/p} \neq 0$, the implicit function theorem guarantees the existence of an interval $(a-\delta, a+\delta)$ with $\delta > 0$ on which (15) can be solved explicitly for ξ in terms of x, yielding a unique solution. Therefore ξ can be uniquely expressed in the neighborhood of a by a function $\xi = \xi(x)$. Moreover, the condition $h'(a) \neq 0$ implies that h is a one-to-one function on some disk with center a and possesses an analytic inverse h^{-1} in a (complex) neighborhood of $h(a) = 0$. Since $\lim_{x\to a} g(x) = g(a) = 0$, we can choose a positive constant r such that $\xi(x) = h^{-1}(g(x))$ for all x in $(a-r, a+r)$, where $h^{-1} \circ g$ is analytic in the disk $D_r(a) = \{z : |z-a| < r\}$. Thus, $\xi(x)$ can be expanded in a power series (9) converging for all x in $D_r(a)$. Obviously, the coefficients are the same as in the asymptotic expansion (4). ∎

REFERENCES

1. A. G. Azpeitia, On the Lagrange remainder of the Taylor formula, this MONTHLY **89** (1982) 311–312.
2. Z. Bao-lin, A note on the mean value theorem for integrals, this MONTHLY **104** (1997) 561–562.
3. L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
4. B. Jacobson, On the mean value theorem for integrals, this MONTHLY **89** (1982) 300–301.
5. R. Mera, On the determination of the intermediate point in Taylor's theorem, this MONTHLY **99** (1992) 56–58.
6. P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore, 1998.
7. W. J. Schwind, J. Ji, and D. E. Koditschek, A physically motivated further note on the mean value theorem for integrals, this MONTHLY **106** (1999) 559–564.

Fachbereich MND, Fachhochschule Giessen–Friedberg, University of Applied Sciences, Wilhelm–Leuschner–Straße 13, 61169 Friedberg, Germany
Ulrich.Abel@mnd.fh-friedberg.de

A Proof of the Mazur-Ulam Theorem

Jussi Väisälä

Throughout this note we let E and F denote real normed spaces. A map $f\colon E \to F$ is an *isometry* if $\|fx - fy\| = \|x - y\|$ for all x and y in E, and f is *affine* if

$$f((1-t)a + tb) = (1-t)fa + tfb \tag{1}$$

for all a and b in E and for all real numbers t (for this it actually suffices that (1) hold for $0 \le t \le 1$). Equivalently, f is affine if the map $T: E \to F$ defined by $Tx = fx - f0$ is linear.

An isometry need not be affine. To see this, let E be the real line $\mathbf{R}$, let F be the plane $\mathbf{R}^2$ with the norm $\|x\| = \max(|x_1|, |x_2|)$, and let $\varphi: \mathbf{R} \to \mathbf{R}$ be any function such that $|\varphi(s) - \varphi(t)| \le |s - t|$ for all s and t; e.g., $\varphi(t) = |t|$ or $\varphi(t) = \sin t$. Setting $f(s) = (s, \varphi(s))$ we get an isometry $f: E \to F$, which is affine only if its range is a straight line.

An isometry $f: E \to F$ is affine if

$$f((a+b)/2) = (fa + fb)/2 \tag{2}$$

for all a and b in E, that is, if f preserves midpoints of line segments. Indeed, iteration of (2) gives (1) for all dyadic rational t between 0 and 1. Since an isometry is continuous, we obtain (1) for all t in $[0, 1]$.

There are two important cases when every isometry is affine:

(i) F is strictly convex, that is, no sphere in F contains a line segment. Then (2) follows from the fact that the spheres with centers at fa and fb with radii $\|a - b\|/2$ meet only at $(fa + fb)/2$. Every inner product space is strictly convex, and so are the spaces l_p for $1 < p < \infty$.

(ii) f is bijective (equivalently, surjective). This result was proved by S. Mazur and S. Ulam [**3**] in 1932, and their proof is also given in the books [**1**, p. 166] and [**2**, 14.1].

The purpose of this note is to give a simple proof for the theorem of Mazur and Ulam. It is based on the ideas of A. Vogt [**4**], and it makes use of reflections in points.

For a point z of E, the *reflection of E in z* is the map $\psi: E \to E$ defined by $\psi x = 2z - x$. Then $\psi\psi$ is the identity, and hence ψ is bijective with $\psi^{-1} = \psi$. Moreover, ψ is an isometry, and z is the only fixed point of ψ. The equations

$$\|\psi x - z\| = \|x - z\|, \qquad \|\psi x - x\| = 2\|x - z\| \tag{3}$$

hold for all x in E.

Theorem of Mazur-Ulam. *Every bijective isometry $f: E \to F$ between normed spaces is affine.*

Proof. Let a and b belong to E and set

$$z = (a+b)/2.$$

Let W be the family of all bijective isometries $g: E \to E$ keeping the points a and b fixed, and set

$$\lambda = \sup\left\{\|gz - z\| : g \in W\right\} \in [0, \infty].$$

For g in W we have $\|gz - a\| = \|gz - ga\| = \|z - a\|$, so

$$\|gz - z\| \le \|gz - a\| + \|a - z\| = 2\|a - z\|,$$

which yields $\lambda < \infty$.

Let ψ be the reflection of E in z. If g is a member of W, then so also is $g^* = \psi g^{-1} \psi g$, and therefore $\|g^* z - z\| \leq \lambda$. Since g^{-1} is an isometry, this fact and (3) imply that

$$2\|gz - z\| = \|\psi gz - gz\| = \|g^{-1}\psi gz - z\| = \|\psi g^{-1}\psi gz - z\| = \|g^* z - z\| \leq \lambda$$

for all g in W, showing that $2\lambda \leq \lambda$. Thus $\lambda = 0$, which means that $gz = z$ for all g in W.

Let $f\colon E \to F$ be a bijective isometry. Set $z' = (fa + fb)/2$. To verify (2) we must show that $fz = z'$. Let ψ' be the reflection of F in z'. Then the map $h = \psi f^{-1} \psi' f$ is in W, whence $hz = z$. This implies that $\psi' fz = fz$. Since z' is the only fixed point of ψ', we obtain $fz = z'$, as desired. ■

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, Subwencji Funduszu Kultury Narodwej, Warsaw, 1932.
2. Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis I*, AMS Colloquium Publications 48, American Mathematical Society, Providence, 2000.
3. S. Mazur and S. Ulam, Sur les transformationes isométriques d'espaces vectoriels normés, *C. R. Acad. Sci. Paris* **194** (1932) 946–948.
4. A. Vogt, Maps which preserve equality of distance, *Studia Math.* **45** (1973) 43–48.

Matematiikan laitos, PL 4, Hallituskatu 5, 00014 Helsingin yliopisto, Finland
jvaisala@cc.helsinki.fi