Cauchy-Schwarz inequality

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一般的书上都是 Hilbert 空间中的 Cauchy‐Schwarz inequality, 张恭庆的书上讲了个更一般的. 而下面的 Cauchy‐Schwarz inequality, 是我见过的条件最弱的, 暂时还没找到该定理的出处.

Definition 1. *If bilinear function $a : X \times X \to \mathbb{C}$ satisfied: for any $\alpha \in \mathbb{C}$ and $x, y \in X$,*

(i) $a(\alpha x, y) = \alpha a(x, y)$*;*

(ii) $a(x, \alpha y) = \overline{\alpha} a(x, y)$*.*

We call $a(\cdot, \cdot)$ is a conjugate bilinear function on linear space X.

Theorem 1 (Cauchy‐Schwarz inequality)**.** *Let $a(\cdot, \cdot)$ be a conjugate bilinear function on linear space X and for any $x \in X$,*

$$a(x, x) \geq 0,$$

then for any $x, y \in X$,

$$|a(x, y)|^2 \leq a(x, x)a(y, y). \tag{1}$$

证明. For any $x, y \in X$, if $a(x, x) = a(y, y) = 0$, then

$$0 \leq a(x + y, x + y) = a(x, x) + a(y, y) + 2\mathrm{Re}a(x, y) = 2\mathrm{Re}a(x, y),$$
$$0 \leq a(x - y, x - y) = a(x, x) + a(y, y) - 2\mathrm{Re}a(x, y) = -2\mathrm{Re}a(x, y),$$
$$0 \leq a(x + iy, x + iy) = a(x, x) + a(y, y) + 2\mathrm{Im}a(x, y) = 2\mathrm{Im}a(x, y),$$
$$0 \leq a(x - iy, x + iy) = a(x, x) + a(y, y) - 2\mathrm{Im}a(x, y) = -2\mathrm{Im}a(x, y),$$

So Re$a(x,y) = 0 =$ Im$a(x,y)$, i.e $a(x,y) = 0$.

$$|a(x,y)|^2 = 0 = a(x,x)a(y,y).$$

Otherwise, we assume $a(y,y) \neq 0$, let $\lambda := a(x,y)/a(y,y)$, then

$$\begin{aligned} 0 &\leq a(x-\lambda y, x-\lambda y) \\ &= a(x,x) - \overline{\lambda}a(x,y) - \lambda a(y,x) + |\lambda|^2 a(y,y) \\ &= a(x,x) - \frac{|a(x,y)|^2}{a(y,y)}. \end{aligned}$$

From this we have

$$|a(x,y)|^2 \leq a(x,x)a(y,y).$$

The proof of Cauchy-Schwarz inequality is finished. $\square$