Equidistributed

The content is from [1, Chapter 4, Section 2].

Definition 1. *A sequence of number $\{\xi_n\}_{n\in\mathbb{N}} \subset [0,1)$ is said to be equidistributed if for any interval $(a,b) \subset [0,1)$,*

$$\lim_{N\to\infty} \frac{\#\{1 \leq n \leq N : \xi_n \in (a,b)\}}{N} = b - a.$$

where the $\#A$ denotes the cardinality of the finite set A.

Definition 2. *Let $[x]$ denote the greatest integer less than or equal to x and call $[x]$ the integer part of x. Let $\langle x \rangle := x - [x]$ and call $\langle x \rangle$ the fractional part of x.*

Theorem 1 (Weyl's criterion). *A sequence of number $\{\xi_n\}_{n\in\mathbb{N}} \subset [0,1)$ is equidistributed if and only if for any $k \in \mathbb{Z} \setminus \{0\}$,*

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0, \quad as \ N \to \infty.$$

From Weyl's criterion, we have following corollaries.

Corollary 1. *Let $P(x) : \mathbb{R} \to \mathbb{R}, x \mapsto c_m x^m + \cdots + c_0$, where $\{c_0, \ldots, c_m\} \subset \mathbb{R}$ and one of them is irrational number. Then $\{\langle P(n) \rangle\}_{n\in\mathbb{N}} \subset [0,1)$ is equidistributed.*

Corollary 2. *$\{\sin n\}_{n\in\mathbb{N}}$ is dense in $[0,1]$.*

References

[1] E. M. Stein and R. Shakarchi, Fourier Analysis: An Introduction, Princeton University Press, 2011.