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Note on the class $L\log L$

by

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1. The purpose of this note is to prove two theorems. Each of these incidentally characterizes the class $L\log L$ in terms of the converse of some well-known inequality.

In Theorem 1 the setting is $\boldsymbol{R}^n$, and for a given integrable function $f(x)$, we define the maximal function $Mf(x)$ by

$$(1)\qquad (Mf)(x) = \sup_{r>0} \frac{1}{m\big(B(x,r)\big)} \int\limits_{B(x,r)} |f(y)|\,dy,$$

where $B(x,r)$ denotes the ball of radius r centered at x and $m\big(B(x,r)\big)$ is its Lebesgue measure.

THEOREM 1. *Suppose that f is integrable and is supported on some finite ball B. Then $\int\limits_B Mf\,dx<\infty$ if and only if*

$$\int\limits_B |f|\log^+|f|\,dx < \infty.$$

One direction, that $f\in L\log L$ implies $Mf\in L$, is very well known; but the converse although not really deeper, seems to have been overlooked all these years.

We shall also obtain a consequence of this result dealing with the Hilbert transform and its n-dimensional generalization, the Riesz transforms. The most appropriate setting for this will be periodic functions, i.e. those that satisfy $f(x+m) = f(x)$, where $m = (m_1, m_2, \ldots, m_n)$ is any vector with integral coordinates. We denote by Q the "fundamental cube" $-\frac{1}{2} < x_j \leqslant \frac{1}{2}$, $j = 1, \ldots, n$. Let

$$(2)\qquad f \sim \sum_m a_m e^{2\pi i m\cdot x}$$

be the Fourier series of a periodic function integrable over Q, and let its Riesz transforms be given by

$$(3)\qquad R_k(f) \sim i \sum_m{}' \frac{m_k}{|m|} a_m e^{2\pi i m\cdot x}, \qquad k = 1, \ldots, n.$$

When $n = 1$, R_1 is the usual (periodic) Hilbert transform.

THEOREM 2. *Suppose $f \geqslant 0$, and f is integrable. Then all the $R_k f$ are also integrable* [1] *if and only if*

$$\int_Q f \log^+ f\, dx < \infty. \tag{4}$$

Our main interest here, as above, is in the direction $R_k f \in L^1$ implies $f \in L \log L$. The converse was known previously (see Calderón and Zygmund [3]). So was the other direction when $n = 1$, but this was by an argument which seems to be restricted to that case. The argument given below has the additional virtue of showing Theorem 1 and Theorem 2 are really closely linked together.

2. Proof of Theorem 1. Suppose $f \in L^1(\mathbf{R}^n)$. Then it is possible to estimate the distribution function of Mf as follows. For any $a > 0$

$$m\{x: Mf(x) > a\} \leqslant \frac{A}{a} \int_{|f| > a/2} |f|\, dx. \tag{5}$$

This known inequality may be found in Weiner [5].

Our basic observation is that the estimate (5) may also be reversed. In fact, if we apply to $|f|$ and a the lemma of Calderón and Zygmund ([2], p. 91), we obtain a sequence $\{Q_j\}$ of disjoint cubes so that $|f(x)| \leqslant a$, if $x \notin UQ_j$; also

$$a < \frac{1}{m(Q_j)} \int_{Q_j} |f|\, dx \leqslant 2^n a.$$

Because of this if $x \in Q_j$, $Mf(x) > c^{-1} a$, with $c = n^{n/2} \times$ volume of the unit ball. So

$$\{x: Mf(x) > c^{-1} a\} \supset U_j Q_j,$$

and therefore

$$m\{x: Mf(x) > c^{-1} a\} \geqslant \sum m(Q_j) \geqslant (2^{-n}/a) \int_{U_j Q_j} |f|\, dx \geqslant (2^{-n}/a) \int_{|f| > a} |f|\, dx,$$

since $x \in UQ_j$, if $|f(x)| > a$. Passing from a to ca then gives

$$m\{x: Mf(x) > a\} \geqslant \frac{2^{-n} c^{-1}}{a} \int_{|f| > ca} |f|\, dx. \tag{6}$$

3. Let now B be any fixed finite ball and B' the ball with the same center as B with twice the diameter. We observe that if f is supported in B and Mf is integrable over B, then Mf is also integrable over B'. In

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[1] More precisely, the series (3) are Fourier series of integrable functions.

fact, suppose for simplicity that $B = \{x: |x| \leqslant 1\}$, and $B' = \{x: |x| \leqslant 2\}$. Then if $x \in B' - B$ (i.e. $1 < |x| \leqslant 2$), we can easily verify that $Mf(x) \leqslant cMf(\bar{x})$, where $\bar{x} = x/|x|^2$, which proves that Mf is also integrable over B'.

However if we are outside B', we are at a positive distance from B and there Mf is bounded, and in fact decreases to zero as $|x| \to \infty$ (again since f is supported in B). These observations can be summarized in the following lemma:

LEMMA 1. *Suppose f is supported in a finite ball B and Mf is integrable over that ball. Then for any fixed a_0, $a_0 > 0$, Mf is integrable over the set where $(Mf)(x) > a_0$.*

According to the lemma then

$$\int_{a_0}^{\infty} m\{x: Mf > a\}\, da < \infty$$

and by (6) this shows that $\int_B |f| \log^+ |f|\, dx < \infty$ which proves the theorem.

4. Proof of Theorem 2. For the proof of Theorem 2 we shall need some facts from the n-dimensional theory of H^p-spaces as developed by G. Weiss and the author [4], but in the periodic setting. The periodic analogues can be given very similar proofs as in the non-periodic case treated in [4] and so we shall content ourselves with the bare statement of the needed results.

Suppose both (2) and (3) are Fourier series of integrable functions. We form the $n+1$ harmonic functions $u_j(x, t)$ given by

$$u_0(x, t) = \sum a_m e^{-2\pi|m|t} e^{2\pi i m\cdot x}, \tag{7}$$

$$u_k(x, t) = i \sum{}' a_m \frac{m_k}{|m|} e^{-2\pi|m|t} e^{2\pi i m\cdot x}, \qquad k = 1, \ldots, n. \tag{8}$$

The $u_j(x, t)$, $j = 0, \ldots, n$, satisfy the generalized Cauchy-Riemann equations

$$\sum_{j=0}^{n} \frac{\partial u_j(x, t)}{\partial x_j} = 0 \qquad (x_0 = t),$$

$$\frac{\partial u_k(x, t)}{\partial x_j} = \frac{\partial u_j(x, t)}{\partial x_t}, \qquad 0 \leqslant k,\ j \leqslant n.$$

Let

$$F(x, t) = \big(u_0(x, t), u_1(x, t), \ldots, u_n(x, t)\big).$$

Then we say that $F \in H^1$ if

$$\sup_{t>0} \int_Q |F(x, t)|\, dx \leqslant M < \infty.$$

The result we shall use is the following

LEMMA 2. *Suppose* $F \in H^1$. *Then*

$$\int_Q \sup_{t>0} |F(x,t)|\,dx \leqslant A \sup_{t>0} \int_Q |F(x,t)|\,dx.$$

(For the non-periodic analogue see [4], pp. 46 and 47).

5. We shall also need to make a few remarks about the Poisson kernel. Let

$$P(x,t) = \frac{c_n}{(|x|^2+t^2)^{(n+1)/2}}, \qquad c_n = \frac{\Gamma\big((n+1)/2\big)}{\pi^{(n+1)/2}}$$

be the n-dimensional (non-periodic) Poisson kernel. Its periodic analogue $\mathscr{P}(x,t)$ is

$$\mathscr{P}(x,t) = \sum_m P(x+m,t).$$

It can be seen that

(9a) $$\mathscr{P}(x,t) \geqslant 0, \quad \text{in fact } \mathscr{P}(x,t) \geqslant P(x+m,t);$$

(9b) $$\int_Q \mathscr{P}(x,t)\,dx = 1;$$

(9c) $$\mathscr{P}(x,t) = \sum_m e^{-2\pi|m|t} e^{2\pi i m\cdot x}.$$

The only statement that really needs proof is (9c), but this is a consequence of the Poisson summation formula (see e.g. [1], p. 32).

6. We shall now prove that part of the theorem that concerns us. Suppose that f and the $R_k(f)$ given by (3) are in $L^1(Q)$. Since

$$u_0(x,t) = \int_Q f(y)\,\mathscr{P}(x-y,t)\,dy$$

and

$$u_k(x,t) = \int_Q (R_k f)(y)\,\mathscr{P}(x-y,t)\,dy,$$

we see that $F = (u_0, u_1, \ldots, u_n) \in H^1$.

Next observe that if g is any *non-negative* function on $\boldsymbol{R}^n$ which is integrable, then

(10) $$\sup_{t>0} \int_{\boldsymbol{R}^n} g(y)P(x-y,t)\,dy \geqslant \sup_{t>0} \int_{|x-y|\leqslant t} g(y)P(x-y,t)\,dy$$
$$\geqslant \sup_{t>0} bt^{-n} \int_{|x-y|<t} g(y)\,dy = b' M(g)(x).$$

Now we have already noted the fact that $f, R_k(f)\in L'(Q)$ implies that $F\in H^1$. Thus by Lemma 2

$$\int\limits_Q \sup_{t>0} |F(x,t)|\,dx < \infty$$

and so since $f \geqslant 0$ and $\mathscr{P}(x,t) \geqslant P(x,t)$, we get by (10)

$$\int\limits_Q Mg(x)\,dx < \infty,$$

where $g = f$ for $x\in Q$, and $g = 0$, $x\notin Q$. Similarly, since $\mathscr{P}(x,t) \geqslant P(x+m,t)$, we also have

$$\int\limits_{Q+m} Mg\,dx < \infty$$

for any vector m with integral components. Finally, suppose B is a fixed ball which contains Q. Then B is contained in the union of finitely many integral translates of Q. So

$$\int\limits_B Mg\,dx < \infty,$$

which by Theorem 1 implies that

$$\int\limits_B g\log^+ g\,dx = \int\limits_Q f\log^+ f\,dx < \infty.$$

7. Addendum. The following non-periodic analogue of theorem 2, is stated without proof. Suppose $f\in L^p(\boldsymbol{R}^n)$ for some p, $1 \leqslant p < \infty$. Then the non-periodic Riesz transform R_k is defined by

$$R_k(f)(x) = \lim_{\varepsilon\to 0} c \int\limits_{|y|\geqslant\varepsilon} f(x-y)\frac{y_k}{|y|^{n+1}}\,dy.$$

The limit exists almost everywhere, and if in addition $p > 1$, it also exists in L^p-norm.

Suppose now B_1 and B_2 are a pair of finite open balls so that $\bar{B}_1 \subset B_2$.

THEOREM 3. (a) *Suppose* $\int\limits_{B_2} |f|\log^+|f|\,dx < \infty$. *Then*

$$\int\limits_{B_1} |R_k f|\,dx < \infty.$$

(b) *Conversely, suppose* $\int\limits_{B_2} |R_k f|\,dx < \infty$, $k = 1, \ldots, n$, *and* $f \geqslant 0$ *on* B_2. *Then*

$$\int\limits_{B_1} |f|\log^+|f|\,dx < \infty.$$

Part (a) was known previously, see [3]. Part (b) is proved by an adaptation of the argument used for the proof of the periodic case, and may be left as an exercise to the interested reader.

The reason that this non-periodic version seems ackward in comparison to its periodic analogue is the fact that if $f \geqslant 0$ and $f \not\equiv 0$, $f \in L^1(R^n)$, then $R_k(f)$ *never* belongs to $L^1(R^n)$.

The author profitted from several valuable conversations with A. Zygmund on the contents of this note.

References

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