Schwartz 函数和傅里叶变换

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1 Schwartz 函数

下面给出 Schwartz 函数的定义.

Definition 1. A $C^\infty(\mathbb{R}^n)$ complex-valued function f on $\mathbb{R}^n$ is called a Schwartz function if for every pair of multi-indices α, $\beta \in \mathbb{Z}_+^n$,

$$\rho_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} \left| x^\alpha (\partial^\beta f)(x) \right| < \infty.$$

The set of all Schwartz functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$.

下面给出 Schwartz 函数的一个非常有用的等价定义.

Theorem 2. Let $f \in C^\infty(\mathbb{R}^n)$. Then $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if for any $\alpha \in \mathbb{Z}_+^n$ and $N \in \mathbb{N}$, there exists a positive constant C such that

$$|(\partial^\alpha f)(x)| \le C \frac{1}{(1+|x|)^N}.$$

证明. 先证 ” $\Longleftarrow$ ”. 对任意 α, $\beta \in \mathbb{Z}_+^n$,

$$\begin{aligned}\rho_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} \left| x^\alpha (\partial^\beta f)(x) \right| &\lesssim \sup_{x\in\mathbb{R}^n} |x|^{|\alpha|} \left| (\partial^\beta f)(x) \right| \\ &\lesssim \sup_{x\in\mathbb{R}^n} (1+|x|)^{|\alpha|} \left| (\partial^\beta f)(x) \right| < \infty,\end{aligned}$$

故 $f \in \mathcal{S}(\mathbb{R}^n)$.

再证 ” $\Longrightarrow$ ”. 对任意 $\beta \in \mathbb{Z}_+^n$ 和 $N \in \mathbb{N}$,

$$\sup_{x\in B(0,1)} (1+|x|)^N |(\partial^\alpha f)(x)| < \infty$$

且

$$
\begin{aligned}
&\sup_{x\in\mathbb{R}^n\setminus B(0,1)}(1+|x|)^N|(\partial^\alpha f)(x)| \\
&\quad\lesssim \sup_{x\in\mathbb{R}^n\setminus B(0,1)}|x|^N|(\partial^\beta f)(x)| \\
&\quad\lesssim \sup_{x\in\mathbb{R}^n\setminus B(0,1)}\left(\sum_{\alpha\in\mathbb{Z}_+^n,\,|\alpha|=N}|x^\alpha|\right)|(\partial^\beta f)(x)| \\
&\quad\lesssim \sum_{\alpha\in\mathbb{Z}_+^n,\,|\alpha|=N}\rho_{\alpha,\beta}(f)<\infty,
\end{aligned}
$$

故

$$
\sup_{x\in\mathbb{R}^n}(1+|x|)^N|(\partial^\alpha f)(x)|<\infty,
$$

定理 2 证毕. $\square$

下面给出 $\mathcal{S}(\mathbb{R}^n)$ 和其它空间的关系.

Proposition 3. 设 $p\in[1,\infty)$. 则有以下性质:

(i) $C_c^\infty(\mathbb{R}^n)\subsetneqq\mathcal{S}(\mathbb{R}^n)\subsetneqq L^p(\mathbb{R}^n)$;

(ii) $C_c^\infty(\mathbb{R}^n)$ 在 $L^p(\mathbb{R}^n)$ 中稠.

证明. 先证 (i). 容易看出 $C_c^\infty(\mathbb{R}^n)\subset\mathcal{S}(\mathbb{R}^n)$, 又 $e^{-|x|^2}\in\mathcal{S}(\mathbb{R}^n)$ 但无紧支集, 故 $C_c^\infty(\mathbb{R}^n)\subsetneqq\mathcal{S}(\mathbb{R}^n)$.

又由定理 2 知, 对任意 $f\in\mathcal{S}(\mathbb{R}^n)$ 和 $x\in\mathbb{R}^n$,

$$
|f(x)|\lesssim\frac{1}{(1+|x|)^{n+1}},
$$

因此

$$
\begin{aligned}
\|f\|_{L^p(\mathbb{R}^n)}^p&=\int_{\mathbb{R}^n}|f(x)|^p\,dx=\int_{\mathbb{R}^n}\frac{1}{(1+|x|)^{n+1}}\,dx \\
&\lesssim 1+\int_{|x|>1}\frac{1}{|x|^{n+1}}\,dx\sim 1+\int_1^\infty\frac{1}{r^2}\,dx\sim 1.
\end{aligned}
$$

故 $\mathcal{S}(\mathbb{R}^n)\subset L^p(\mathbb{R}^n)$. 又显然 $\mathcal{S}(\mathbb{R}^n)\neq L^p(\mathbb{R}^n)$, 从而 $\mathcal{S}(\mathbb{R}^n)\subsetneqq L^p(\mathbb{R}^n)$. (i) 证毕.

(ii) 是经典结论, 用卷积去逼近即可. $\square$

下面给出 Schwartz 函数意义下的收敛.

Definition 4. 设 $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ 且 $f \in \mathcal{S}(\mathbb{R}^n)$. 称 f_k 收敛到 f in $\mathcal{S}(\mathbb{R}^n)$, as $k \to \infty$, 若对任意 α, $\beta \in \mathbb{Z}_+^n$,

$$\rho_{\alpha,\beta}(f_k - f) \to 0, \quad \text{as } k \to \infty.$$

通过收敛可以定义闭集, 从而有了拓扑. 在此拓扑意义下, Schwartz 空间可赋度量:

$$d:\ \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to [0,\infty),\ (f,g) \mapsto \sum_{\alpha,\,\beta\in\mathbb{Z}_+^n} \frac{\rho_{\alpha,\beta}(f-g)}{1+\rho_{\alpha,\beta}(f-g)};$$

但是不能赋予范数.

下面给出 Schwartz 函数傅里叶变换的定义.

Definition 5. Let $f \in L^1(\mathbb{R}^n)$. For any $\xi \in \mathbb{R}^n$, define

$$\begin{aligned} \widehat{f}(\xi) &:= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx; \\ \widetilde{f}(\xi) &:= f(-\xi); \\ (\tau^y f)(\xi) &:= f(\xi - y), \quad \text{where } y \in \mathbb{R}^n. \end{aligned}$$

Remark 6. 傅里叶变换限制在 $\mathcal{S}(\mathbb{R}^n)$ 上, 是到自身的同胚.

Proposition 7. Let $f \in \mathcal{S}(\mathbb{R}^n)$, then

(i) $f^{\wedge\wedge}(\cdot) = f(-\cdot)$;

(ii) 对任意 $y \in \mathbb{R}^n$, $\widehat{\tau^y f}(\xi) = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi)$;

(iii) 对任意 $y \in \mathbb{R}^n$, $\tau^y(\widehat{f})(\xi) = [e^{-2\pi i x\cdot y} f(x)]^\wedge(\xi)$;

(iv) 平移变换不改变傅里叶变换的支集. 即对任意 $y \in \mathbb{R}^n$, $\operatorname{supp} \widehat{\tau^y f} = \operatorname{supp} \widehat{f}$.

证明. 现在证 (ii). 对任意 $\xi \in \mathbb{R}^n$,

$$\begin{aligned} \widehat{\tau^y f}(\xi) &= \int_{\mathbb{R}^n} (\tau^y f)(x) e^{-2\pi i x\cdot\xi}\, dx \\ &= \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i x\cdot\xi}\, dx \\ &= e^{-2\pi i y\cdot\xi} \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx \\ &= e^{-2\pi i y\cdot\xi}\widehat{f}(\xi), \end{aligned}$$

(ii) 得证.

(iv) 是 (ii) 的直接推论. $\square$

2 Schwartz 分布

下面给出 Schwartz 分布傅里叶变换的定义.

Definition 8. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, define

$$
\begin{aligned}
\widehat{f}:\ & \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n), \langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle; \\
\widetilde{f}:\ & \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n), \langle \widetilde{f}, \varphi \rangle := \langle f, \widetilde{\varphi} \rangle; \\
\tau^y f:\ & \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),\ \forall\, \varphi \in \mathcal{S}(\mathbb{R}^n), \langle \tau^y f, \varphi \rangle := \langle f, \tau^{-y} \varphi \rangle, \quad \text{where } y \in \mathbb{R}^n.
\end{aligned}
$$

Definition 9. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $u : \mathbb{R}^n \to \mathbb{C}$. We say that f coincides with the function u if, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$
\langle f, \varphi \rangle = \int_{\mathbb{R}^n} u(x) \varphi(x)\, dx.
$$

For any complex-valued function u, let

$$
\text{supp } u := \{x \in \mathbb{R}^n :\ u(x) \neq 0\}.
$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$
\begin{aligned}
\text{supp } f := \bigcap \{ & \text{closed set } K \subset \mathbb{R}^n : \\
& \langle f, \varphi \rangle = 0 \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ with } \text{supp } \varphi \subset \mathbb{R}^n \setminus K \}.
\end{aligned}
$$

Remark 10. If $f \in \mathcal{S}'(\mathbb{R}^n)$ coincides with the function u, then

$$
\widehat{f} = \widehat{u}, \quad \widetilde{f} = \widetilde{u}, \quad \tau^y f = \tau^y u,
$$

where the " $=$ " means "coincides with". 在我看来, 这也是 Definition 8 的由来.

若 u 是连续函数, 则

$$
\text{supp } f = \text{supp } u,
$$

其中 supp u 是函数意义下的支集.

证明. 由定义可直接得到

$$
\widehat{f} = \widehat{u}, \quad \widetilde{f} = \widetilde{u}, \quad \tau^y f = \tau^y u,
$$

下面只证 supp $f =$ supp u.

事实上, supp u 是闭集且对任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with supp $\varphi \subset \mathbb{R}^n \setminus$ supp u,

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} u(x)\varphi(x)\, dx = 0,$$

因此由分布支集定义知 supp $f \subset$ supp u. 再证 supp $u \subset$ supp f. 对任意 $x \in$ supp u, 有 $u(x) \neq 0$, 不妨设 $u(x) > 0$. 因为 u 连续, 故存在 $r_x \in (0, \infty)$ 使得 u 在 $B(x, r_x)$ 上恒大于 $u(x)/2$. 假设存在

$$\begin{aligned} K_x \in \{&\text{closed set } K \subset \mathbb{R}^n : \\ &\langle f, \varphi \rangle = 0 \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ with } \operatorname{supp} \varphi \subset \mathbb{R}^n \setminus K\} \end{aligned} \tag{1}$$

使得 $x \notin K_x$, 则存在 $\delta \in (0, r_x)$ 使得 $B(x, \delta) \cap K_x = \emptyset$. 令 $\varphi_x \in \mathcal{S}(\mathbb{R}^n)$ 满足

$$\varphi_x(t) := \begin{cases} 1, & t \in B(x, \delta/2), \\ \text{取值为 } [0,1], & t \in B(x, \delta) \setminus B(x, \delta/2), \\ 0, & t \in \mathbb{R}^n \setminus B(x, \delta). \end{cases}$$

则

$$\langle f, \varphi_x \rangle = \int_{\mathbb{R}^n} u(t)\varphi_x(t)\, dt = \int_{B(x,\delta)} u(t)\varphi_x(t)\, dt \geq \frac{u(x)}{2} \int_{B(x,\delta)} \varphi_x(t)\, dt > 0,$$

这与 (1) 相矛盾. 故

$$\begin{aligned} x \in \bigcap \{&\text{closed set } K \subset \mathbb{R}^n : \\ &\langle f, \varphi \rangle = 0 \text{ for any } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ with } \operatorname{supp} \varphi \subset \mathbb{R}^n \setminus K\} \\ = \operatorname{supp} f&. \end{aligned}$$

因此 supp $u \subset$ supp f, Remark 10 证毕. $\square$

Proposition 11. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then

(i) $f^{\wedge\wedge}(\cdot) = \widetilde{f}$;

(ii) 平移变换不改变傅里叶变换的支集. 即对任意 $y \in \mathbb{R}^n$, supp $\widehat{\tau^y f} =$ supp $\widehat{f}$.

证明. 现在证 (ii). 对任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\langle \widehat{\tau^y f}, \varphi \right\rangle = \langle \tau^y f, \widehat{\varphi} \rangle = \left\langle f, \tau^{-y} \widehat{\varphi} \right\rangle = \left\langle f, [e^{-2\pi i x \cdot y}\varphi(x)]^\wedge \right\rangle = \left\langle \widehat{f}, e^{-2\pi i x \cdot y}\varphi(x) \right\rangle,$$

故 supp $\widehat{\tau^y f} =$ supp $\widehat{f}$, (ii) 得证. $\square$