A Really Simple Elementary Proof of the Uniform Boundedness Theorem

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Abstract. I give a proof of the uniform boundedness theorem that is elementary (i.e., does not use any version of the Baire category theorem) and also extremely simple.

One of the pillars of functional analysis is the uniform boundedness theorem:

Uniform Boundedness Theorem. *Let $\mathcal{F}$ be a family of bounded linear operators from a Banach space X to a normed linear space Y. If $\mathcal{F}$ is pointwise bounded (i.e., $\sup_{T\in\mathcal{F}} \|Tx\| < \infty$ for all $x \in X$), then $\mathcal{F}$ is norm-bounded (i.e., $\sup_{T\in\mathcal{F}} \|T\| < \infty$).*

The standard textbook proof (e.g., [**17**, p. 81]), which goes back to Stefan Banach, Hugo Steinhaus, and Stanisław Saks in 1927 [**3**], employs the Baire category theorem or some variant thereof.[1] This proof is very simple, but its reliance on the Baire category theorem makes it not completely elementary.

By contrast, the original proofs given by Hans Hahn [**7**] and Stefan Banach [**2**] in 1922 were quite different: they began from the assumption that $\sup_{T\in\mathcal{F}} \|T\| = \infty$ and used a "gliding hump" (also called "sliding hump") technique to construct a sequence (T_n) in $\mathcal{F}$ and a point $x \in X$ such that $\lim_{n\to\infty} \|T_n x\| = \infty$.[2] Variants of this proof were later given by T. H. Hildebrandt [**11**] and Felix Hausdorff [**9, 10**].[3] These proofs are elementary, but the details are a bit fiddly.

Here is a *really* simple proof along similar lines:

Lemma. *Let T be a bounded linear operator from a normed linear space X to a normed linear space Y. Then for any $x \in X$ and $r > 0$, we have*

$$\sup_{x'\in B(x,r)} \|Tx'\| \geq \|T\| r, \tag{1}$$

where $B(x,r) = \{x' \in X\colon\ \|x' - x\| < r\}$.

Proof. For $\xi \in X$ we have

$$\max\{\|T(x+\xi)\|, \|T(x-\xi)\|\} \geq \tfrac{1}{2}[\|T(x+\xi)\| + \|T(x-\xi)\|] \geq \|T\xi\|, \tag{2}$$

where the second $\geq$ uses the triangle inequality in the form $\|\alpha - \beta\| \leq \|\alpha\| + \|\beta\|$. Now take the supremum over $\xi \in B(0,r)$. ■

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[1] See [**4**, p. 319, note 67] concerning credit to Saks.

[2] Hahn's proof is discussed in at least two modern textbooks: see [**14**, Exercise 1.76, p. 49] and [**13**, Exercise 3.15, pp. 71–72].

[3] See also [**18**, pp. 63–64] and [**21**, pp. 74–75] for an elementary proof that is closely related to the standard "nested ball" proof of the Baire category theorem; and see [**8**, Problem 27, pp. 14–15 and 184] and [**12**] for elementary proofs in the special case of linear functionals on a Hilbert space.

Proof of the Uniform Boundedness Theorem. Suppose that $\sup_{T\in\mathcal{F}} \|T\| = \infty$, and choose $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\|T_n\| \ge 4^n$. Then set $x_0 = 0$, and for $n \ge 1$ use the lemma to choose inductively $x_n \in X$ such that $\|x_n - x_{n-1}\| \le 3^{-n}$ and $\|T_n x_n\| \ge \frac{2}{3}3^{-n}\|T_n\|$. The sequence (x_n) is Cauchy, hence convergent to some $x \in X$; and it is easy to see that $\|x - x_n\| \le \frac{1}{2}3^{-n}$ and hence $\|T_n x\| \ge \frac{1}{6}3^{-n}\|T_n\| \ge \frac{1}{6}(4/3)^n \to \infty$. ■

Remarks.

1. As just seen, this proof is most conveniently expressed in terms of a *sequence* (x_n) that converges to x. This contrasts with the earlier "gliding hump" proofs, which used a *series* that sums to x. Of course, sequences and series are equivalent, so each proof can be expressed in either language; it is a question of taste which formulation one finds simpler.
2. This proof is extremely wasteful from a quantitative point of view. A quantitatively sharp version of the uniform boundedness theorem follows from Ball's "plank theorem" [**1**]: namely, if $\sum_{n=1}^{\infty} \|T_n\|^{-1} < \infty$, then there exists $x \in X$ such that $\lim_{n\to\infty} \|T_n x\| = \infty$ (see also [**15**]).
3. A similar (but slightly more complicated) elementary proof of the uniform boundedness theorem can be found in [**6**, p. 83].
4. "Gliding hump" proofs continue to be useful in functional analysis: see [**20**] for a detailed survey.
5. The standard Baire category method yields a slightly stronger version of the uniform boundedness theorem than the one stated here, namely: if $\sup_{T\in\mathcal{F}} \|Tx\| < \infty$ for a *nonmeager* (i.e., second category) set of $x \in X$, then $\mathcal{F}$ is norm-bounded.
6. The uniform boundedness theorem has generalizations to suitable classes of non-normable and even non-metrizable topological vector spaces (see, e.g., [**19**, pp. 82–87]). I leave it to others to determine whether any ideas from this proof can be carried over to these more general settings.
7. More information on the history of the uniform boundedness theorem can be found in [**4**, pp. 302, 319n67], [**5**, pp. 138–142], and [**16**, pp. 21–22, 40–43].

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On the Group of Automorphisms of a Group

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Abstract. This note gives a generalization of the classical result asserting that if the center of a group G is trivial, then so is the center of its automorphism group $\mathrm{Aut}(G)$.

Let G be a group and let $Z(G)$ denote its center. We will denote the automorphism group of G by $\mathrm{Aut}(G)$. One of the classic results in group theory is the following theorem.

Theorem 1. *If* $Z(G) = 1$, *then* $Z(\mathrm{Aut}(G)) = 1$.

We use the notation $G = 1$ to mean that G is the appropriate trivial group, i.e., a group with one (necessarily trivial) element. This is a convenient abuse of notation, as the trivial element of $\mathrm{Aut}(G)$ is actually the identity automorphism Id_G.

Theorem 1 is surprising, being one of the first results establishing a relationship between the structure of a group and that of its automorphism group. It is a simple but powerful result, which was the starting point in H. Wielandt's proof of his elegant

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