TAUBERIAN THEOREMS CONCERNING POWER SERIES AND DIRICHLET'S SERIES WHOSE COEFFICIENTS ARE POSITIVE*

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1. The general nature of the theorems contained in this paper resembles that of the "Tauberian" theorems which we have proved in a series of recent papers.† They have, however, a character of their own, in that they are concerned primarily with series of positive terms.

Let
$$f(x) = \Sigma a_n x^n$$
be a power series convergent for $|x| < 1$. We shall consider only positive values of x less than 1.

Let
$$s_n = a_0 + a_1 + \ldots + a_n,$$
$$L(u) = (\log u)^{\alpha_1} (\log \log u)^{\alpha_2} \ \ldots,$$
where the α's are real. Then it is known that, *if*
$$s_n \sim A n^{\alpha} L(n),$$
where $A \neq 0$, *as* $n \to \infty$, *the indices* $\alpha, \alpha_1, \alpha_2, \ldots$ *being such that* $n^{\alpha} L(n)$ *tends to a positive limit or to infinity, then*
$$f(x) \sim A \frac{\Gamma(\alpha+1)}{(1-x)^{\alpha+1}} L\left(\frac{1}{1-x}\right),$$
as $x \to 1$.‡

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* A short abstract of some of the principal results of this paper was published under the title "Tauberian Theorems concerning Series of Positive Terms" in the *Messenger of Mathematics*, Vol. 42, pp. 191, 192.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320; Vol. 9, pp. 434–448; Vol. 11. pp. 1–16 and pp. 411–478.

‡ The first of the α's which is not zero must be positive. If $\alpha = \alpha_1 = \alpha_2 = \ldots = 0$, the theorem reduces to Abel's well known theorem. The special theorem in which $\alpha_1 = \alpha_2 = \ldots = 0$

The principal object of this paper is to prove that *the converse of the theorem is true when the coefficients a_n are positive.* We shall prove, for example, that *if* $a_n \geqslant 0$, *and*

$$f(x) = \Sigma a_n x^n \sim \frac{A}{(1-x)^a} \quad (A > 0,\ a > 0),$$

then

$$s_n \sim \frac{An^a}{\Gamma(1+a)}.$$

This result is a very curious one, largely because it lies much deeper and is much harder to prove than a first impression might tempt one to believe. Its appearance is that of a "special"* (or "o") Tauberian theorem. In reality, as will appear in the sequel, it is a theorem of the "general" (or "O") type, and left-handed"† in addition. It is, in fact, of the same order of difficulty as the theorem "*if* $a_n > -K/n$, *and* $f(x) \to A$, *then* Σa_n *converges to the sum* A."‡ The proof therefore naturally involves all the apparatus of repeated differentiation on which the proofs of such theorems ultimately depend.§

2. We begin by proving some subsidiary theorems which are interesting in themselves. It is hardly necessary to remark that all variables and functions considered in them are real. We suppose first that x is a variable which tends to infinity.

We shall begin by proving a theorem which is due to Landau, and on which nearly all our subsequent analysis depends. The theorem is of great interest in itself, inasmuch as its general character is that of an "O" Tauberian theorem, and it was the first theorem of this nature stated explicitly.

THEOREM 1.—*Suppose that* (i) $f(x)$ *is differentiable, and* (ii) $xf'(x)$

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was proved by Appell, *Comptes Rendus*, Vol. 87, p. 689. The substance of the general theorem is due to Lasker, *Phil. Trans. Roy. Soc.*, (A), Vol. 196, p. 444: Lasker does not actually state it, but it is a trivial deduction from the theorem which he proves. The theorem was first stated explicitly in the form we have adopted by Pringsheim, *Acta Mathematica*, Vol. 28, p. 29. Pringsheim, however, proves a more general theorem, inasmuch as he considers paths of approach of x to 1 other than the real axis.

* *Cf.* Hardy and Littlewood, *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 413.

† *L.c.*, p. 416.

‡ We stated this theorem without proof at the end of our paper already quoted (*l.c.*, p. 478).

§ Littlewood, "The Converse of Abel's Theorem," *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 434 *et seq.*

steadily increases. Then

$$f(x) \sim x,$$

involves

$$f'(x) \sim 1.^*$$

Suppose that the theorem is untrue. Then it must be possible to find a number h different from 1, and a sequence (x_ν) of values of x tending to infinity, such that

$$f'(x_\nu) \to h,$$

as $\nu \to \infty$. Let us suppose, for example, that $h > 1$; and let δ be a positive number. Then, as $\nu \to \infty$,

$$\frac{f(x_\nu + \delta x_\nu) - f(x_\nu)}{\delta x_\nu} = \frac{1}{\delta x_\nu} \int_{x_\nu}^{x_\nu + \delta x_\nu} f'(x)\, dx \geqslant \frac{x_\nu f'(x_\nu)}{\delta x_\nu} \int_{x_\nu}^{x_\nu + \delta x_\nu} \frac{dx}{x}$$

$$\sim \frac{h}{\delta} \int_{x_\nu}^{x_\nu + \delta x_\nu} \frac{dx}{x} = \frac{h \log(1+\delta)}{\delta}.$$

But, since $f(x) \sim x$,

$$\frac{f(x_\nu + \delta x_\nu) - f(x_\nu)}{\delta x_\nu} \sim 1;$$

and these two results are contradictory if δ is sufficiently small. The hypothesis that $h < 1$ may be disposed of similarly.

3. THEOREM 2.—*Let $\phi(x)$ be a function which tends to infinity with x and has a continuous and positive derivative, and suppose that*

$$\text{(i)} \quad \frac{\phi(x)}{\phi'(x)} \sim x,$$

$$\text{(ii)} \quad xf'(x) \text{ steadily increases.}$$

Then

$$f(x) \sim \phi(x)$$

involves

$$f'(x) \sim \phi'(x).$$

This theorem follows at once from Theorem 1 by means of the substitution

$$x = \phi(y).$$

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* The converse inference may, of course, *always* be made. Theorem 1 was proved by Landau (*Rendiconti di Palermo*, Vol. 26, p. 218). We insert a proof for the sake of completeness.

By means of one or other of the substitutions

$$x = \frac{1}{y-c}, \quad x = \frac{1}{c-y}$$

we deduce

THEOREM 2a.—*Let $\phi(x)$ be a function of x which tends to infinity as x tends to c from above or from below; and suppose that*

(i) $\dfrac{\phi(x)}{\phi'(x)} \sim -(x-c)$,

(ii) $(x-c)f'(x)$ *steadily decreases or increases.**

Then $f(x) \sim \phi(x)$ involves $f'(x) \sim \phi'(x)$.

Suppose, in particular, that

$$\phi(x) = \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right),$$

where $A > 0$, $a > 0$, and $x \to 1$ from below. Then $1-x \sim a\phi/\phi'$. Hence we have

THEOREM 3.—*If* $$f(x) \sim \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right),$$

where $A > 0$, $a > 0$, and $(1-x)f'(x)$ increases as $x \to 1$, then

$$f'(x) \sim \frac{aA}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

4. From Theorem 3 we can deduce a preliminary theorem concerning power series which seems of considerable interest in itself.

THEOREM 4.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive*† *coefficients, and*

$$f(x) \sim \frac{A}{(1-x)^a} L\left(\frac{1}{1-x}\right) \quad (A > 0,\ a > 0),$$

then $$f^{(r)}(x) \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(1-x)^{a+r}} L\left(\frac{1}{1-x}\right).$$

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* $(x-c)f'(x)$ must be an increasing or a decreasing function according as x tends to its limit from below or above.

† We use "positive" to include "zero."

We have

$$g(x) = \Sigma s_n x^n = \frac{f(x)}{1-x} \sim \frac{A}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

Also

$$(1-x)g'(x) = (1-x)\,\Sigma n s_n x^{n-1} = s_1 + (2s_2 - s_1)\,x + (3s_3 - 2s_2)x^2 + \ldots$$

has all its coefficients positive, since s_n increases steadily with n. Hence $(1-x)\,g'(x)$ increases with x, and so, by Theorem 3,

$$g'(x) \sim \frac{(a+1)A}{(1-x)^{a+2}} L\left(\frac{1}{1-x}\right),$$

$$f'(x) = (1-x)\,g'(x) - g(x) \sim \frac{aA}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

A repetition of the argument leads to a complete proof of the theorem.

It is important to observe that this process of differentiation is *not* legitimate when $a = 0$. Suppose, *e.g.*, that

$$f(x) \sim \log\left(\frac{1}{1-x}\right).$$

We cannot infer that
$$f'(x) \sim \frac{1}{1-x};$$
all that the argument leads to is

$$f'(x) = o\left\{\frac{1}{1-x}\log\left(\frac{1}{1-x}\right)\right\}.$$

We can show, moreover, by actual examples, that the suggested inference would be invalid. Suppose, for example, that
$$f(x) = \Sigma x^{a^n} \quad (a \geqslant 2).$$

Then it is easy to see that $s_n \sim \log_a n$, and so

$$f(x) \sim \frac{1}{\log a}\log\left(\frac{1}{1-x}\right).$$

But it is not true that $(1-x)f'(x)$ leads to a limit as $x \to 1$. This is most easily proved by means of Theorem 8 below. Since

$$xf'(x) = \Sigma\, a^n x^{a^n}$$

is a series of positive terms, $f'(x) \sim A/(1-x)$ would involve

$$t_\nu = \sum_{a^n \leqslant \nu} a^n \sim A\nu;$$

and this is obviously untrue, since whenever ν passes through a value equal to a power of a, a new term is introduced into t_ν which is greater than the sum of all which precede.*

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* See Hardy, *Quarterly Journal*, Vol. 38, pp. 279 *et seq.*, for analytical formulæ which show in an explicit manner the behaviour as $x \to 1$ of the series Σx^{a^n}, $\Sigma(-1)^n x^{a^n}$ and their derivatives.

In the sequel we shall use not Theorem 4 itself, but the theorem into which it is transformed by the substitutions

$$x = e^{-t}, \quad f(x) = F(t).$$

THEOREM 4a.—*If* $a_n \geqslant 0$, *and*

$$F(t) = \Sigma a_n e^{-nt} \sim A t^{-a} L\left(\frac{1}{t}\right),$$

as $t \to 0$, *then*

$$(-1)^{(r)} F^{(r)}(t) \sim \frac{\Gamma(a+r)}{\Gamma(a)} A t^{-a-r} L\left(\frac{1}{t}\right).$$

5. Theorem 4 is capable of various interesting generalisations.

THEOREM 5.—*The condition that* $a_n \geqslant 0$ *of Theorem* 4 *may be replaced by the more general condition that*

$$na_n = O_L\{n^a L(n)\},$$

i.e., that

$$na_n > -Kn^a L(n).$$

Let

$$g(x) = \Sigma b_n x^n = \Sigma \{a_n + Kn^{a-1} L(n)\} x^n.$$

Then $b_n > 0$, and

$$g(x) \sim \left\{A + \frac{K}{\Gamma(a)}\right\} \frac{1}{(1-x)^a} L\left(\frac{1}{1-x}\right).$$

Hence, by Theorem 4,

$$g'(x) \sim \left\{A + \frac{K}{\Gamma(a)}\right\} \frac{a}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right);$$

and so

$$f'(x) \sim \frac{Aa}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

THEOREM 6.—*The condition that* $a_n \geqslant 0$ *of Theorem* 4 *may also be replaced by* $s_n \geqslant 0$, *or by* $s_n^k \geqslant 0$, *where* s_n^k *is any one of Cesàro's means formed from the series* Σa_n; *or, more generally, by* $s_n = O_L\{n^a L(n)\}$ *or*

$$s_n^k = O_L\{n^{a+k} L(n)\}.$$

In the proof of Theorem 4, the condition $a_n \geqslant 0$ is used only to justify the differentiation of the asymptotic equality

$$g(x) = \Sigma s_n x^n \sim \frac{A}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

And the result of the theorem shows that $s_n \geqslant 0$ is a sufficient condition for this. Repeating this argument we see that $s_n^k \geqslant 0$ is a sufficient condition.

The more general results may be established in the same way, if we appeal at each stage to Theorem 5 instead of to Theorem 4.

6. Suppose that the conditions of Theorem 4 (or of one of its generalisations) are satisfied. Then

$$\Sigma na_n x^n = xf'(x) \sim \frac{Aa}{(1-x)^{a+1}} L\left(\frac{1}{1-x}\right).$$

Operating repeatedly in this manner, we see that

$$\Sigma n^r a_n x^n \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(1-x)^{a+r}} L\left(\frac{1}{1-x}\right)$$

for all positive integral values of r. We shall now show that *this result holds for all values of r greater than $-\alpha$.*

It is plainly enough to prove this when $r = -\beta$, $0 < \beta < \alpha$. We write

$$x = e^{-t}, \quad f(x) = F(t),$$

so that

$$F(t) \sim \frac{A}{t^{\alpha}} L\left(\frac{1}{t}\right),$$

as $t \to 0$. Then

$$\Sigma n^{-\beta} a_n e^{-nt} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} u^{\beta-1} \Sigma a_n e^{-n(t+u)}\, du = \frac{1}{\Gamma(\beta)} \int_0^{\infty} u^{\beta-1} F(t+u)\, du.$$

Also

$$\int_0^{\infty} u^{\beta-1} F(t+u)\, du \sim A \int_0^{\infty} \frac{u^{\beta-1}}{(t+u)^{\alpha}} L\left(\frac{1}{t+u}\right) du$$

$$\sim \frac{A\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\beta-\alpha} L\left(\frac{1}{t}\right).^{*}$$

The result is thus established for $-\alpha < r < 0$; the general result then follows by using the special result in which r is a positive integer. We have thus proved

THEOREM 7.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive coefficients (or subject to the more general conditions of Theorems 5 or 6), and*

$$f(x) \sim \frac{A}{(1-x)^{\alpha}} L\left(\frac{1}{1-x}\right) \quad (A > 0,\ \alpha > 0),$$

then

$$f_r(x) = \Sigma n^r a_n x^n \sim \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{A}{(1-x)^{\alpha+r}} L\left(\frac{1}{1-x}\right),$$

for any value of r, integral or not, greater than $-\alpha$.

7. We pass now to the proof of the principal theorem of the paper.

THEOREM 8.—*If $f(x) = \Sigma a_n x^n$ is a power series with positive coefficients, and*

$$f(x) \sim \frac{A}{(1-x)^{\alpha}} L\left(\frac{1}{1-x}\right),$$

where $A > 0$ and the indices $\alpha, \alpha_1, \alpha_2, \ldots$ are such that $n^{\alpha} L(n)$ tends to a positive limit or to infinity as $n \to \infty$, then

$$s_n \sim \frac{A}{\Gamma(\alpha+1)} n^{\alpha} L(n).$$

We suppose $A = 1$, and write $x = e^{-t}$, so that $t \to 0$, and

(1) $$f(x) = F(t) = \Sigma a_n e^{-nt} \sim t^{-\alpha} L(1/t).$$

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* These transformations, of course, merely express the general lines of a straightforward proof, the details of which will easily be supplied by anyone accustomed to work of this character.

In the first place, we have

$$s_n \leqslant e \sum_0^n a_\nu e^{-\nu/n} \leqslant eF(1/n),$$

and so, from (1),

(2) $$s_n = O\{n^a L(n)\}.$$

Next, we have

(3) $$\Sigma s_n e^{-nt} \sim t^{-a-1} L(1/t).$$

Differentiating this relation r times, as we may do in virtue of Theorem 4, since $a+1>0$, we obtain

(4) $$\Sigma s_n n^r e^{-nt} \sim \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right).$$

We shall now prove that, *if any positive ϵ is given, we can choose, first r and ζ, and then*

$$t_0 = t_0(\epsilon, r, \zeta),*$$

in such a way that

(5) $$\sum_{(1+\zeta)(a+r)/t}^{\infty} s_n n^r e^{-nt} < \epsilon \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

(6) $$\sum_{0}^{(1-\zeta)(a+r)/t} s_n n^r e^{-nt} < \epsilon \frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$.†

We shall suppose that r and ζ are functions of one another such that $\zeta^2 r \to \infty$ and $\zeta^3 r \to 0$ as $r \to \infty$ and $\zeta \to 0$. We may suppose, for example, that $\zeta^5 r^2 = 1$. The condition that $\zeta^3 r \to 0$ will not be used until § 9.

8. It follows from (2) that the left-hand side of (5) is of the form

$$O \sum_{(1+\zeta)(a+r)/t}^{\infty} n^{a+r} L(n) e^{-nt}.‡$$

The maximum of the function under the sign of summation, considered as

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* r will be large, ζ small, and t_0 small; and r and ζ will be connected in a way which will be defined precisely in a moment.

† The limits of summation are not in general integral. The extreme terms, considered as functions of t, are not of higher order than $t^{-a-r} L(1/t)$, and the argument is in no way affected by including or excluding an additional term or two.

‡ Here and in the sequel the constant implied by the O is independent of r, ζ, and t.

a function of n, occurs for a value of n given by an equation

$$a+r+\epsilon_n = nt,$$

where ϵ_n is a function of n only which tends to zero as $n \to \infty$. Hence the function decreases steadily throughout the limits of the summation, and

$$\sum_{(1+\zeta)(a+r)/t}^{\infty} n^{a+r} L(n)\, e^{-nt} < \nu^{a+r} L(\nu)\, e^{-\nu t} + \int_{(1+\zeta)(a+r)/t}^{\infty} u^{a+r} L(u)\, e^{-ut}\, du,$$

where $n = \nu$ corresponds to the first term of the sum. The isolated term may be neglected.*

The integral we write in the form

$$\int_{(1+\zeta)(a+r)/t}^{\infty} u^{a+r} L(u)\, e^{-ut/(1+\zeta)}\, e^{-\zeta ut/(1+\zeta)}\, du.$$

The maximum of the function $u^{a+r} L(u) e^{-ut/(1+\zeta)}$ is given by an equation of the form

$$(1+\zeta)(a+r+\epsilon_u) = ut.$$

Writing $(1+\zeta)(a+r+\epsilon_u)/t$ for u in the first three factors of the subject of integration, and observing that the functions

$$\left(1+\frac{\epsilon_u}{a+r}\right)^{a+r}, \qquad L\left\{\frac{(1+\zeta)(a+r+\epsilon_u)}{t}\right\}$$

are, when r is large enough and t small enough, certainly less than constant multiples of 1 and $L(1/t)$ respectively, we see that our integral is of the form

$$O\left[\left\{\frac{(1+\zeta)(a+r)}{t}\right\}^{a+r} L\left(\frac{1}{t}\right) e^{-a-r} \int_{(1+\zeta)(a+r)/t}^{\infty} e^{-\zeta ut/(1+\zeta)}\, du\right]$$

$$= O\left[\frac{1+\zeta}{\zeta}(a+r)^{a+r} e^{-(a+r)\{1+\zeta-\log(1+\zeta)\}} t^{-a-r-1} L\left(\frac{1}{t}\right)\right]$$

$$= O\left\{\frac{1+\zeta}{\zeta}(a+r)^{a+r} e^{-a-r} t^{-a-r-1} L\left(\frac{1}{t}\right)\right\},$$

since $\zeta - \log(1+\zeta) > 0$ when $\zeta > 0$. Our conclusion now follows from

* See the last footnote.

the facts that

$$\Gamma(a+r+1) \sim (a+r)^{a+r+\frac{1}{2}} e^{-a-r} \sqrt{(2\pi)},$$

as $r \to \infty$, and that $\zeta^2 r \to \infty$.

The inequality (6) may be established in the same way. We write

$$e^{-ut} = e^{-ut/(1-\zeta)} e^{\zeta ut/(1-\zeta)},$$

and have finally to observe that $\zeta + \log(1-\zeta) < 0$. Otherwise the argument is practically the same.

From (4), (5), and (6) it follows that *when ϵ is given, we can choose r, ζ, and $t_0(\epsilon, r, \zeta)$ in such a way that*

$$(7) \quad (1-\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right) < \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} s_n n^r e^{-nt} < (1+\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$.* Hence, as s_n is an increasing function of n, we obtain

$$(8) \quad s_{(1-\zeta)(a+r)/t} \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} n^r e^{-nt} < (1+\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right),$$

$$(9) \quad s_{(1+\zeta)(a+r)/t} \sum_{(1-\zeta)(a+r)/t}^{(1+\zeta)(a+r)/t} n^r e^{-nt} > (1-\epsilon)\frac{\Gamma(a+r+1)}{\Gamma(a+1)} t^{-a-r-1} L\left(\frac{1}{t}\right).\dagger$$

9. Now write
$$n = \frac{a+r}{t} + \lambda,$$

so that $|\lambda| < \zeta(a+r)/t$. The next step in the proof consists in showing that we may replace $n^r e^{-nt}$, in the inequalities (8) and (9), by

$$\left(\frac{a+r}{t}\right)^r \exp\left\{-a-r-\frac{r\lambda^2 t^2}{2(a+r)^2}\right\}.$$

We have here to make use of the second relation between r and ζ, namely that $\zeta^3 r$ is small. We have

$$n^r e^{-nt} = \left(\frac{a+r}{t}\right)^r \exp\left\{-a-r-\lambda t + r\log\left(1+\frac{\lambda t}{a+r}\right)\right\}.$$

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* We do not imply that r, ζ, t_0 have the *same* values as before.

† We may interpret s_x, when x is not integral, as meaning $s_{[x]}$. But the truth of the inequalities would not be affected by the inclusion or exclusion of an additional term or two.

Also, as $|\lambda| < \zeta(a+r)/t$, we have

$$r\log\left(1+\frac{\lambda t}{a+r}\right)-\lambda t = -\frac{r\lambda^2t^2}{2(a+r)^2}+\frac{r\lambda^3t^3}{3(a+r)^3}-\ldots-\frac{a\lambda t}{a+r}$$
$$= -\frac{r\lambda^2t^2}{2(a+r)^2}+O(\zeta^3 r)+O(\zeta);$$

and the factor
$$e^{O(\zeta^3 r)+O(\zeta)}$$

tends to 1 as r tends to infinity and ζ to zero. If now we make this substitution in (8) and (9), and also substitute for $\Gamma(a+r+1)$ its asymptotic equivalent given by Stirling's theorem, we arrive at the following conclusion. *Given any positive* ϵ, *it is possible to choose first* r *and* ζ, *and then* $t_0 = t_0(\epsilon, r, \zeta)$, *in such a way that*

$$(10)\qquad s_{(1-\zeta)(a+r)/t}\,\Sigma e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} < \frac{(1+\epsilon)\sqrt{(2\pi)}}{\Gamma(a+1)}(a+r)^{a+\frac{1}{2}}\,t^{-a-1}L\left(\frac{1}{t}\right),$$

$$(11)\qquad s_{(1+\zeta)(a+r)/t}\,\Sigma e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} > \frac{(1-\epsilon)\sqrt{(2\pi)}}{\Gamma(a+1)}(a+r)^{a+\frac{1}{2}}\,t^{-a-1}L\left(\frac{1}{t}\right),$$

for $0 < t \leqslant t_0$, *the values of* λ *included in the sums being those which differ from* $(a+r)/t$ *by an integer and are less in absolute value than* $\zeta(a+r)/t$.

10. In the inequalities (10) and (11) we may suppose that λ ranges from $-\infty$ to $+\infty$. For, when this is so,

$$(12)\qquad \Sigma e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} \sim \frac{a+r}{t}\sqrt{\left(\frac{2\pi}{r}\right)},$$

as $t \to 0$. On the other hand,

$$(13)\qquad \sum_{\lambda>\zeta(a+r)/t} e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2} = O(e^{-\frac{1}{2}\zeta^2 r})+\int_{\zeta(a+r)/t}^{\infty} e^{-\frac{1}{2}r\lambda^2t^2/(a+r)^2}\,d\lambda.$$

The integral is

$$\frac{\zeta(a+r)}{t}\int_1^\infty e^{-\frac{1}{2}\zeta^2\mu^2 r}\,d\mu = O\left\{\frac{a+r}{t\sqrt{r}}e^{-\frac{1}{2}\zeta^2 r}\right\}.$$

As $\zeta^2 r$ is large, the sum (13) is small compared with the sum (12).

A similar argument may, of course, be applied to the terms for which λ is large and negative. We may therefore suppose that λ ranges from $-\infty$ to $+\infty$ in (10) and (11).

We now use the asymptotic relation (12) to transform these inequalities. Observing that $r \sim a+r$ as $r \to \infty$, and that, when r is fixed,

$$L\left(\frac{1}{t}\right) \sim L\left(\frac{a+r}{t}\right)$$

as $t \to 0$, we see that *given ϵ it is possible to choose r, ζ, and $t_0(\epsilon, r, \zeta)$, so that*

$$s_{(1-\zeta)(a+r)t} < \frac{1+\epsilon}{\Gamma(a+1)} \left(\frac{a+r}{t}\right)^a L\left(\frac{a+r}{t}\right), \tag{13}$$

$$s_{(1+\zeta)(a+r)t} > \frac{1-\epsilon}{\Gamma(a+1)} \left(\frac{a+r}{t}\right)^a L\left(\frac{a+r}{t}\right), \tag{14}$$

for $0 < t \leqslant t_0$. Taking $n = (1-\zeta)(a+r)/t$ and $n = (1+\zeta)(a+r)/t$ in turn, and remembering that ζ is small, we see that *when ϵ is given it is possible to choose n_0 so that*

$$(1-\epsilon)\, n^a L(n) < \Gamma(1+a)\, s_n < (1+\epsilon)\, n^a L(n),$$

for $n \geqslant n_0$. Thus Theorem 8 is proved.

11. Theorem 8 has, as we remarked, in § 1, the appearance of a "special" (or "o") Tauberian theorem.* But we can at once deduce from it a theorem of an obviously "general" (or "O") character.

THEOREM 9.—*If we suppose, in Theorem 8, that $a > 0$, then the condition that $a_n \geqslant 0$ may be replaced by the condition that*

$$na_n = O_L\{n^a L(n)\};$$

i.e., that

$$na_n > -Kn^a L(n).$$

For let $\quad g(x) = \Sigma\{a_n + Kn^{a-1}L(n)\}\, x^n = \Sigma b_n x^n.$†

Then $b_n > 0$, and

$$g(x) \sim \frac{A + K\Gamma(a)}{(1-x)^a} L\left(\frac{1}{1-x}\right).$$

* When all the a's are zero, the theorem really is an "o" theorem. This may account for its having this appearance in general.

† We may suppress enough terms at the beginning to ensure that $L(n)$ is defined for all values of n in question.

Hence, by Theorem 8,

$$\sum^{n}\{a_\nu+K\nu^{a-1}L(\nu)\}\sim\frac{A+K\Gamma(a)}{\Gamma(a+1)}n^aL(n),$$

and so

$$s_n\sim\frac{A}{\Gamma(a+1)}n^aL(n).$$

It is also easy to see that, in all the theorems which we have been discussing, the function $L(u)$, instead of having the special form

$$(\log u)^{a_1}(\log\log u)^{a_2}\ldots,$$

may be any logarithmico-exponential function,* such that

$$u^{-\delta}\prec L(u)\prec u^{\delta}.$$

Hence we deduce

THEOREM 10.—*If* $a_n\geqslant 0$, *and*

$$f(x)\sim\mathfrak{L}\left(\frac{1}{1-x}\right),$$

where $\mathfrak{L}(u)$ *is any logarithmico-exponential function such that*

$$1\prec\mathfrak{L}(u)\prec u^{\Delta},$$

so that $\mathfrak{L}(u)=u^aL(u)$, *where* $a\geqslant 0$ *and* $u^{-\delta}\prec L(u)\prec u^{\delta}$, *then*

$$s_n\sim\frac{1}{\Gamma(a+1)}\mathfrak{L}(n).$$

12. Theorem 9 may be proved by another method which possesses considerable interest. It is less direct than that which we have followed, and involves an appeal to a theorem of which we have not published any proof. But it exhibits the relations between Theorem 9 and some of our former theorems in a very interesting light.

We suppose, for simplicity, that

$$a_1=a_2=\ldots=0.$$

It is easy to see that it is enough to prove that *if*

$$f(x)=o(1-x)^{-a},$$

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* For an explanation of the terminology and notation of the next few lines see Hardy, "Orders of infinity," *Camb. Math. Tracts*, No. 12.

and $$a_n = O_L(n^{\alpha-1}),$$

then $$s_n = o(n^{\alpha}).$$

We write $$g(x) = \Sigma(a_n + Kn^{\alpha-1})x^n = \Sigma b_n x^n.$$

Then $b_n > 0$ and $g(x) = O(1-x)^{-\alpha}$; and it follows, as at the beginning of § 7, that

$$\sum_0^n b_\nu = O(n^{\alpha}),$$

and so that $s_n = O(n^{\alpha})$. But, from the equations

$$s_n = O(n^{\alpha}), \quad \Sigma s_n x^n = o(1-x)^{-\alpha-1},$$

it follows, by Theorem 26 of our last paper,* that

$$\sigma_n = s_0 + s_1 + \ldots + s_n = o(n^{\alpha+1}).$$

Now we know that if $\phi(x)$ is a function of x which has a continuous second derivative $\phi''(x)$ for all sufficiently large values of x, and

$$\phi(x) = o(x^{\alpha+1}), \quad \phi''(x) = O(x^{\alpha-1}),$$

then $$\phi'(x) = o(x^{\alpha}).\dagger$$

It is possible to generalise this result by writing

$$\phi''(x) = O_L(x^{\alpha-1})$$

for $$\phi''(x) = O(x^{\alpha+1}).$$

And this generalised result possesses an analogue for series, namely: *if*

$$s_0 + s_1 + \ldots + s_n = o(n^{\alpha+1}), \quad a_n = O_L(n^{\alpha-1}),$$

then $$s_n = o(n^{\alpha}).$$

From this it is clear that we can deduce the result required. It must not be supposed, however, that in this proof we really dispense with the process of r-fold differentiation. This is involved in our former Theorem 26, by an appeal to which we covered the most difficult transition of our proof.

* *L.c.*, p. 443.

† This follows from Theorem 1 of our last paper if $\alpha \geqslant 1$, and from Theorem 5 in any case.

13. The analogue of Theorem 9, in the case in which

$$a = a_1 = a_2 = \dots = 0,$$

is as follows.

THEOREM 11.—*If $f(x) \to A$ as $x \to 1$, and $a_n > -K/n$, then Σa_n converges to the sum A.*

This theorem is true, and constitutes a very interesting extension of Littlewood's generalisation of Tauber's theorem. But a special proof is required.*

We have, as $x \to 1$, $$f(x) = A + o(1),$$
and $$f''(x) = \Sigma n(n-1) a_n x^{n-2} > -K \Sigma (n-1) x^{n-2} = O_L (1-x)^{-2},$$
From this it may be deduced that $f' = o(1-x)^{-1}$.

We write y for $1-x$, so that $y \to 0$, by positive values. The theorem we wish to prove is that *the equations* $$F(y) = A + o(1), \quad F''(y) = O_L(1/y)^2$$
imply $$F'(y) = o(1/y).$$

Suppose first that, if possible $$F'(y_s) > H/y_s \quad (H > 0),$$
for an infinity of values y_s of y whose limit is zero. We have also $F''(y) > -K/y^2$, and so, if $y > y_s$,

$$F'(y) = F'(y_s) + \int_{y_s}^{y} F''(u)\,du > \frac{H}{y_s} - \frac{K(y-y_s)}{y_s^2}.$$

It is clearly possible to choose a positive number δ, so that

$$F'(y) > \tfrac{1}{2}H/y_s,$$

for $$y_s \leqslant y \leqslant \eta_s = (1+\delta)\,y_s.$$

And then $$F(\eta_s) - F(y_s) = \int_{y_s}^{\eta_s} F'(y)\,dy > \tfrac{1}{2}H\frac{\eta_s - y_s}{y_s} = \tfrac{1}{2}\delta H,$$

which contradicts $$F(y) = A + o(1).$$

Similarly we can show that it is impossible that

$$F'(y_s) < -H/y_s \quad (H > 0).$$

In this case we start from the fact that, if $0 < y < y_s$,

$$F'(y) = F'(y_s) - \int_{y}^{y_s} F''(u)\,du < -\frac{H}{y_s} + \frac{K(y_s - y)}{y^2},$$

and argue in the same way. Hence $F'(y) = o(1/y)$.

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* If we write $$g(x) = \Sigma\left(a_n + \frac{K}{n}\right) x^n = \Sigma b_n x^n,$$
$g(x)$ is of higher order than $f(x)$. Hence we cannot prove that $s_n = O(1)$ in the way in which we proved $s_n = O(n^a)$ in § 12.

Hence $$\Sigma n a_n x^n = o\left(\frac{1}{1-x}\right),$$

and $$n a_n = O_L(1).$$

Hence, by Theorem 9, $$a_1 + 2a_2 + \ldots + n a_n = o(n);$$

and the convergence of Σa_n now follows from Pringsheim's generalisation of Tauber's theorem.*

14. In Theorem 9 we supposed that $\alpha > 0$. An argument similar to that of the last section enables us to remove this restriction.

THEOREM 12.—*The result of Theorem 9 holds even when* $\alpha = 0$.†

We have $$f(x) \sim L\left(\frac{1}{1-x}\right),$$

and $$f''(x) > -K\Sigma(n-1)L(n)x^{n-2} = O_L\left\{\frac{1}{(1-x)^2}L\left(\frac{1}{1-x}\right)\right\}.$$

From this we deduce‡ $$f'(x) = o\left\{\frac{1}{1-x}L\left(\frac{1}{1-x}\right)\right\}.$$

Hence $$\Sigma n a_n x^n = o\left\{\frac{1}{1-x}L\left(\frac{1}{1-x}\right)\right\};$$

and therefore, by Theorem 9, $a_1 + 2a_2 + \ldots + n a_n = o\{nL(n)\}$.

That $s_n \sim L(n)$ now follows from Theorem 45 of our last paper.§

15. Before leaving power series and passing on to Dirichlet's series we may add one further remark. The theorems which we have proved are all of what we have called an "Abel-Tauber" type; in all of them we start from (i) a hypothesis as to the behaviour of $f(x) = \Sigma a_n x^n$ as $x \to 1$, (ii) an inequality satisfied by a_n, and deduce information as to the behaviour of s_n. There are, of course, corresponding theorems of a "Cesàro-Tauber" type, in which the hypothesis (i) is replaced by a hypothesis as to the behaviour of one of Cesàro's means formed from Σa_n. These theorems are naturally easier to prove. We may content ourselves with enunciating the simplest analogue of Theorem 9, viz.,

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* See Bromwich, *Infinite Series*, p. 251, Ex. 28.

† This theorem contains Theorem 11 as a particular case.

‡ The proof is similar to that of § 13.

§ The argument by which Theorem 9 itself was proved would only lead to the result with an unnecessarily severe restriction on a_n, viz., that

$$a_n = O_L\{\psi(n)\},$$

where $$\int^n \psi(u)\,du \sim L(n).$$

Thus, if $L(u) = \log u$, this argument would require $a_n = O_L(1/n)$, whereas the real condition is $a_n = O_L(\log n/n)$. The reason why a more elaborate argument is needed when $\alpha = 0$ than when $\alpha > 0$ lies in the fact that

$$\int^n n^\alpha L(u)\frac{du}{u}$$

is of order $n^\alpha L(n)$ when $\alpha > 0$, but of higher order when $\alpha = 0$.

THEOREM 13.—*If* $(s_0+s_1+\ldots+s_n)/(n+1) \sim An^a \quad (A>0,\ a>0),$

and $na_n = O_L(n^a),$

then $s_n \sim An^a.$

It was substantially this theorem which was assumed at the end of § 12. The reader will have no difficulty in framing further theorems of this type and of a more general character.

16. We conclude by a brief statement of the analogues of the most important of the preceding theorems for ordinary Dirichlet's series. There are corresponding theorems, for Dirichlet's series of the general type $\Sigma a_n e^{-\lambda_n s}$, which it is our intention to publish elsewhere, and we shall therefore not enter into the details of the proofs.

THEOREM 14.—*If* $f(s) = \Sigma a_n n^{-s}$ *is an ordinary Dirichlet's series with positive coefficients, convergent for* $s > 1$, *and*

$$f(s) \sim \frac{A}{(s-1)^a} L\left(\frac{1}{s-1}\right) \quad (a>0),$$

as $s \to 1$, *then*

$$(-1)^r f^{(r)}(s) = \Sigma a_n (\log n)^r n^{-s} \sim \frac{\Gamma(a+r)}{\Gamma(a)} \frac{A}{(s-1)^{a+r}} L\left(\frac{1}{s-1}\right).$$

The argument by which we prove this theorem is substantially the same as that which we used in proving Theorem 4. We have only to observe that, if $g(s) = f(s)/(s-1)$, then

$$(s-1)\,g'(s) = f'(s) - \frac{f(s)}{s-1} = -\Sigma \log n\, a_n n^{-s} - \frac{1}{s-1}\Sigma a_n n^{-s}$$

steadily decreases as $s \to 0$.

Theorem 14, though interesting in itself, does not give us precisely what is required for the proof of the analogue of Theorem 8. This is contained in

THEOREM 15.—*A result similar to that of Theorem* 14 *holds for series of the form*

$$\Sigma a_n \left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} \quad (a_n \geqslant 0).$$

The proof of this theorem is very much the same as that of Theorem 14.

17. From Theorem 15 we can deduce the analogue of Theorem 8, viz.,

THEOREM 16.—*If* $f(s) = \Sigma a_n n^{-s}$ *is an ordinary Dirichlet's series with* *positive coefficients, convergent for* $s > 1$, *and*

$$f(s) \sim \frac{A}{(s-1)^a} L\left(\frac{1}{s-1}\right),$$

the indices $a, a_1, \ldots$ *being such that* $(\log n)^a L(\log n)$ *tends to a positive limit or to infinity as* $n \to \infty$, *then*

$$s_n = \frac{a_1}{1} + \frac{a_2}{2} + \ldots + \frac{a_n}{n} \sim \frac{A}{\Gamma(a+1)} (\log n)^a L(\log n).$$

We have first

$$s_n \leqslant e \sum_1^n \frac{a_\nu}{\nu} e^{-s \log \nu / \log n} < ef\left(\frac{1}{\log n}\right),$$

and so
$$s_n = O\{(\log n)^a L(\log n)\}.$$

Next
$$f(s) = \Sigma s_n \left\{\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\right\} \sim \frac{A}{(s-1)^a} L\left(\frac{1}{s-1}\right),$$

a relation which takes the place of (3) of § 7. We differentiate r times, as we are entitled to do in virtue of Theorem 15; and the rest of the argument follows the lines of the proof of Theorem 8, no new difficulty of principle occurring.

17. We do not propose to write this argument out at length, nor to discuss in detail the analogues of Theorems 9 *et seq.* It may, however, be observed that the left-handed condition which now occurs instead of

$$na_n = O_L\{n^a L(n)\}$$

is
$$\log n \, a_n = O_L\{(\log n)^a L(\log n)\}.$$

Further, the analogue of Theorem 11 deserves a separate statement. It is

THEOREM 17.—*If* $f(s) = \Sigma a_n n^{-s} \to A$ *as* $s \to 1$, *and* $a_n > -K/\log n$, *then* $\Sigma(a_n/n)$ *converges to the sum* A.

This is the "left-handed" form of Littlewood's* generalisation of Landau's† analogue of Tauber's theorem for ordinary Dirichlet's series.

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* Littlewood, *l.c.*, p. 438. As we are supposing s to tend to unity instead of to zero, our condition is $a_n > -K/\log n$ instead of $a_n > -K/n \log n$.

† Landau, *Monatshefte für Math.*, Vol. 18, p. 8.