Chapter 2. Sets

- **1. Sets**

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- **2. Relations**

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- **4. POSETS (Partial Ordered Sets)**

1.1 Set Definitions

- *Set* = a collection of distinct unordered objects, and represented as A,B,C. The members of a set are called *elements*

  ex) A={1,2,3,4}, Graph G={V, E} // set of vertices and edges

- How to define a set?
  - Listing: (원소 나열법) Example: $A = \{1,3,5,7\}$
  - Predicates:조건 제시법 (for large set or infinite set)
    - Example: $B = \{x \mid x = 2k + 1, 0 \leq k \leq 3\}$

      $A = \{x \mid x \text{ is a positive, even integer}\}$

  $S= \{1,2,3\} \rightarrow \{x \mid x \text{ is a positive integer less than } 4\}$

Kinds of Sets

- *Finite* sets
  - Ex: $A = \{1, 2, 3, 4\}$, $B = \{x \mid x \text{ is an integer}, 1 \leq x \leq 4\}$

- *Infinite* sets
  - Ex: $Z = \{\text{integers}\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

    $S = \{x \mid x \text{ is a real number and } 1 \leq x \leq 4\} = [0, 4]$

- The *empty* set (*null set*), $\varnothing$ has no elements. $\varnothing = \{\ \}$

- *Universal* set: the set of all elements about which we make assertions. (must be given explicitly, or inferred from context)

  Ex: $U = \{\text{all real numbers}\}$, $U = \{x \mid x \text{ is a natural number and } 1 \leq x \leq 10\}$

Cardinality

- Cardinality of a set A (in symbols $|A|$) is the number of elements in A, $|A| = n$으로 표기

  (a measure of how many different elements of S has)

- Examples:

  If $A = \{1, 2, 3\}$ then $|A| = 3$

  If $B = \{x \mid x \text{ is a natural number and } 1 \leq x \leq 9\}$ then $|B| = 9$

Subsets

- X is a ***subset*** of Y if every element of X is also contained in Y (in symbols $X \subseteq Y$)

- X is a ***proper subset*** of Y if $X \subseteq Y$ but $Y \not\subseteq X$
  - Observation: $\varnothing$ is a subset of every set
  - Ex) $X=\{1,2\}$ $Y=\{1,2,3\}$

- ***Equality***: $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$

  (two sets are Equal iff each is a SUBSET of the other)

  Ex) $A = \{x \mid x^2 + x - 6 = 0\}$, $B=\{2, -3\}$ then A=B.

Power set

- The *power set* of X is the set of all subsets of X, in symbols $\mathcal{P}(X)$, i.e. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$

Ex) if $X = \{1, 2, 3\}$,

then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Ex) given set S, $S=\{1,2\}$, $P(S) = \{\emptyset, \{1\},\{2\},\{1,2\}\}$

- Theorem: If $|X| = n$, then $|\mathcal{P}(X)| = 2^n$.

1.2 Set operations:

- The *union* of X and Y is defined as the set

$$X \cup Y = \{ x \mid x \in X \text{ or } x \in Y\}$$

- The *intersection* of X and Y is defined as the set

$$X \cap Y = \{ x \mid x \in X \text{ and } x \in Y\}$$

The *difference* of two sets $X - Y = \{ x \mid x \in X \text{ and } x \notin Y\}$

- *Symmetric difference* $X \odot Y = (X - Y) \cup (Y - X)$

Ex) X={a, b, c, e, f}, Y={b, d, r, s} $X \odot Y = \{a, c, e, f, d, r, s\}$

- **COMPLEMENT** : every element that not in A.

$$\tilde{A} = \{x \mid x \in U \text{ and } x \notin A\} = U\text{-}A,$$

ex) A={1,3,5}, U = {1,2,3,4,5}, $\tilde{A} = \{2,4\}$

Cartesian Product

- A x B : a set of all ordered pairs *(a,b)* , where
  $a \in A$ and $b \in B$

- **<u>'Ordered</u>**' => a = first element, and b = second element
  $A \times B \neq B \times A$, unless $A=\varnothing$ or $B=\varnothing$ or $A=B$
  (not commutative)

- ex) Let A={1,2,3}, B={a, b}. Find A X B
  sol) A X B= {(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)}

Partition

- Let S be a set, then a partition of S is a set

$$\Pi = \{A_1, A_2, \ldots A_i, \ldots A_k\}$$

(1) each A_i is a **nonempty** subset of S, i=1,...k,

(2) the A_i **cover** S, in that $S = A_1 \cup A_2 \cup \ldots. \cup A_k$

(3) the A_i are **MUTUALLY) DISJOINT** in that

If $i \neq j$, then $A_i \cap A_j = \varnothing$

Ex) Let $S = S_9$, Let $A_1 = \{1,5\}$, $A_2 = \{2\}$, $A_3 = \{7,8,9\}$, $A_4 = \{3,4,6\}$

show that $\Pi = \{A_1, A_2, A_3, A_4\}$ is a partition of S

1) each set is nonempty

2) 3)

Properties of set operations

<u>Theorem 2.1.10</u>: Let U be a universal set, and A, B and C subsets of U. The following properties hold:

a) Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$

b) Commutativity: $A \cup B = B \cup A$, $A \cap B = B \cap A$

c) Distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

d) Identity laws: $A \cap U = A$ $A \cup \varnothing = A$

e) Complement laws: $A \cup A^c = U$ $A \cap A^c = \varnothing$

f) Idempotent laws: $A \cup A = A$ $A \cap A = A$

g) Bound laws: $A \cup U = U$ $A \cap \varnothing = \varnothing$

h) Absorption laws: $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$

i) Involution law: $(A^c)^c = A$

j) De Morgan's laws for sets: $(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$

1.3 Sequences and strings

- A *sequence* is an **ordered list** of numbers, usually defined according to a formula:

  $$s_n = \text{a function of } n = 1, 2, 3, \ldots$$

- If s is a sequence $\{s_n | n = 1, 2, 3, \ldots\}$,
  - s_1 denotes the first element, s_2 the second element,…
  - s_n the n^{th} element…

- finite: stops after n steps - infinite : continues infinitely

- countable: finite sets

- uncountable : ex) real number between 0, 1

SEQUENCES & Subsequence

- **Increasing sequence**, if $S_n \leq S_{n+1}$, $\forall$ n, when n=1,2,3..

  Ex) $S_n = 2n-1$, is increasing: $S_n = 1,3,5,\ldots$

- **Decreasing sequence**, if $S_n \geq S_{n+1}$, $\forall$ n, when n=1,2,3..

  Ex) Sn = 4 – 2n, is decreasing:, 2, 0, -2, -4, -6,…

ex) If we let S denote this sequence, $s_1 = 2$, $s_2 = 4$, ... $s_n=2n$

=> { **infinite sequence, increasing sequence** }

- A **subsequence** of a sequences $S=\{S_n\}$ is a sequence $T = \{T_n\}$ that consists of certain elements of s retained in the original order they had in S.

Ex) let $S = \{S_n = n \mid n = 1, 2, 3,\ldots\}$ 1, 2, 3, 4, 5, 6, 7, 8,…

Let $T = \{T_n = 2n \mid n = 1, 2, 3,\ldots\}$ 2, 4, 6, 8, 10, 12, 14, 16,… **T is a subsequence of S**

Sigma & PI notation

If $\{a_n\}$ is a sequence, then the sum

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + \ldots + a_m$$

This is called the "**sigma notation**", where the Greek letter Σ indicates a sum of terms from the sequence

If $\{a_n\}$ is a sequence, then the product

$$\prod_{k=1}^{m} a_k = a_1 a_2 \ldots a_m$$

This is called the "**pi notation**", where the Greek letter Π indicates a product of terms of the sequence

Strings (finite sequence)

- Let X be a nonempty set. A *string over X* is a finite sequence of elements from X.

Ex) if $X = \{a, b, c\}$ Then α = bbaccc is a string over X

Notation: bbaccc = b^2ac^3

The **length** of a string α is the number of elements of α and is denoted by $|\alpha|$. If $\alpha = b^2ac^3$ then $|\alpha| = 6$.

- order is important baac $\neq$ abbc

- The *null string* is the string with no elements and is denoted by the Greek letter λ (lambda). It has length zero.

More on strings

- Let X^* = {all strings over X including λ}
- Let $X^+ = X^* - \{\lambda\}$, the set of all non-null strings

  ex) $X = \{a,b\}$ $X^* \Rightarrow \lambda$, a,b,abab, $b^{20}a^{50}ba$,...

- ***<u>Concatenation</u>*** of two strings α and β is the operation on strings consisting of writing α followed by β to produce a new string $\alpha\beta$

  - Ex: α = bbaccc and β = caaba,

  then $\alpha\beta$ = bbaccccaaba = $b^2ac^4a^2ba$

  Clearly, $|\alpha\beta| = |\alpha| + |\beta|$

2. Relations

2.1 Definitions

- What is a relation?

  ex) comparison between two objects (bigger, better, faster,...)

- relationship between objects belong to same set or different set
  ==> result is ORDERED PAIRS (순서쌍)

ex) S = {set of students}, C = {set of courses} ==>

ordered pairs (s,c) from S X C.

For each student there will be some pairs

Binary Relation

- Binary relation is relationship between elements of two sets using ordered pairs

- Let A, B be sets   A binary relation from A to B is subsets R of AXB

ex) Let A = {1,2,3}   B={a,b}   Define some relations?

(답)   R1 = {(1,a)(2,b)(3,a)(1,b)}

R2 = {(3,b)},   R3=AXB 전부,   R4 = $\varnothing$

- A relation R from a set X to a set Y is a subset of **<u>cartesian product</u>** X x Y.
- If $(x,y) \in R$, we write **xRy** and say that **<u>X IS RELATED TO Y</u>**.

More binary relations

- **Domain and Range**

* The set $\{x \in X \mid (x,y) \in R$ for some $y \in Y\}$=> DOMAIN of R

* The set $\{y \in Y \mid (x,y) \in R$ for some $x \in X\}$=> RANGE of R

ex) Let A = {1,2,3}. R be the relation on A consisting of ordered pairs (a,b). such that $a \geq b$. List elements of R (A X A)

Ex) Let X ={2,3,4} and Y ={3,4,5,6,7} If we define a relation R from X to Y by "$(x,y) \in R$ if x divides y (with zero remainder)", list elements of R.

Ex) Set arising from relations

A={1,2,3} B={r,s}, R = {(1,r)(2,s)(3,r)}

- (sol) Dom (R) = A Ran(R) = B

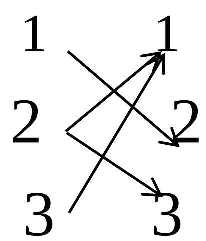
Relation Representation

1) English style representation: ex) {x is greater than y}

2) List **<u>ordered pairs (순서쌍)</u>** ex) A={1,2,3}, A x A,

R={(1,2)(2,1)(2,3)(3,1)}

3) Graph 표현



4) Matrix

Ex) Matrix representation

from \ To	c1	c2	c3
c1	0	140	100
c2	190	0	200
c3	110	180	0

Relation R on set of cities

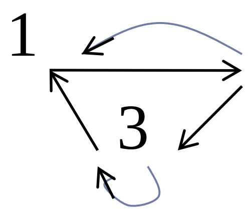
A = {c1,c2,c3}

Q)Find 'Ci R Cj' iff cost of going from Ci to Cj is less than $150

Graph representation (Directed Graph)

- Graph representation 은 two copies of set 을 표현
- Directed graph 는 one copy of set 으로 표현

ex)



R = {(1,2)(1,3),…}

- Degree (indegree : 들어오는 갯수 (내차수)

  (outdegree : 나가는 갯수 (외차수)

  정점 3 의 차수는=> (내차수 2, 외차수 2)

ex) Draw the directed graph representation of the relation "≤" on set S3 => S3 means (1,2,3)

(sol) R =

Matrix Representation (1)

- Let $A = \{a_1, a_2, \ldots, a_n\}$ $B = \{b_1, b_2, \ldots, b_m\}$, R be a relation between A and B

- Define n X m matrix M by $M(i,j) = \begin{cases} \text{false}, & \text{if } (a_i, b_j) \notin R \\ \text{true}, & \text{if } (a_i, b_j) \in R \end{cases}$

ex) R = {(1,2)(2,1)(2,3)(3,1)(3,3)}

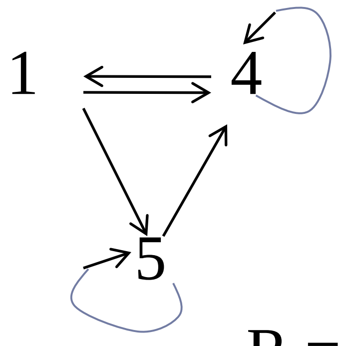
$$M = \begin{bmatrix} F & T & F \\ T & F & T \\ T & F & T \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M = \begin{matrix} F & T & F \\ T & F & T \\ T & T & T \end{matrix}$$

ex) Describe the relations corresponding to the matrix given above, using directed graph.

Matrix Representation (2)

ex) Let A ={1,4,5} and graph is given below. Find Mr and R



$$Mr = \begin{array}{c|ccc} & 1 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 5 & 0 & 1 & 1 \end{array}$$

R = {(1.4)(1.5)(4.1)(4.4)(5.4)(5.5)}

ex) Let A = {a,b,c,d}. Let R be a $Mr = \boxed{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}}$

Construct digraph of R, and list in-degree and out-degree of all vertices

Path

- R be a relation on A. a 에서 b 까지의 길이가 n 개인 path 는 a 부터 시작, b 에서 끝나는 유한 sequence (수열),
  $\prod = a, x_1, x_2,,\ldots.x_{n-1}, b$ 를 말하는데, 각 원소들은 aRx_1, $x_1Rx_2,\ldots..x_{n-1}Rb$ 이어야 한다.

(def)

xR^ny = if n is fixed positive integer, x 에서 시작 y 에서 끝나는 길이 n 인 통로가 있다는 의미

$xR^\infty y$ = x 에서 시작, y 에서 끝나는 어떤 통로가 있다는 의미

2.2 Properties of Relations

1) Reflexive relation, Irreflexive relation
2) Symmetric relation, Anti-symmetric relation
3) Transitive relation

1) **<u>Reflexive Relation (반사관계)</u>**

정의) A relation R on a set X is called **<u>REFLEXIVE</u>**

if $(x,x) \in R$, for every $x \in X$.

($\forall x \in X$, xRx => reflexive)

* if **for all** $x \in X$, $x \not{R} x$ 이면, **<u>irreflexive relation</u>**

Reflexive relations

Ex) X = {1,2,3,4}

R = {(1,1)(1,2)(1,3)(1,4)(2,2)(2,3)(2,4)(3,3)(3,4)(4,4)}

dom(R) = X ran(R) = X

(sol)

- R is reflexive, since each element $x \in X$, $(x,x) \in R$

  Specifically, (1,1)(2,2)(3,3)(4,4) are in R
- in Matrix representation, must have 1’s on main diagonal
- or, in Digraph, each vertex has a LOOP.

- Ex) A ={1,2,3}   R = {(1,1)(1,2)}

Symmetric relation (대칭관계)

def: A relation R on set X is symmetric, if $\forall$ x,y $\in$ X,

if (x, y) $\in$R, then (y, x) $\in$R

→ R = R^{-1}

→ an element is related to second element iff second element is related to first element. $V \rightleftarrows W$

- if for all x,y $\in$X, xRy = yRx,
- matrix representation

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

ex) X = {a,b,c,d} R = {(a,a)(b,c)(c,b)(d,d)}

ex) X = {1,2,3,4} R = {(1,1)(1,2)(1,3)(1,4)(2,2)(2,3)(2,4)(3,3)}

Anti-symmetric relation

Def: A relation R on a set X is called anti-symmetric,

1) if $\forall x,y \in X$, if $(x,y) \in R$ and $x \neq y$, then $(y,x) \notin R$.

2) if $(x,y) \in R$ & $(y,x) \in R$ and $x=y$, then anti-symmetric

- Property: The diagraph of an anti-symmetric relation has a property that between any two vertices there is at most one directed edge.

ex) X = {a,b,c,d} R = {(a,a)(b,c)(c,b)(d,d)}

(sol) R is not antisymmetric since there is (b,c) & (c,b)

ex) X = {1,2,3,4} R = {(1,1)(1,2)(1,3)(1,4)(2,2)(2,3)(2,4)(3,3)(3,4)(4,4)}

Con’t

ex) Let A = Z, R = {(a,b) ∈ A X A | a 〈 b}

(sol) If a 〈 b, then it’s not true for b〈a, so R is <u>not symmetric</u>

If a ≠ b then either a ≮ b or b ≮ a,   so R is <u>antisymmetric</u>

ex) Let A = {1,2,3,4} and   Let R = {(1,2)(2,2)(3,4)(4,1)}

ex) R = { (a, a)(b, b)(c, c)}

ex) M1 =

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}$$

M2 =

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Transitive relation (추이관계)

- Def: A relation R on a set X is called TRANSITIVE

  If $\forall$ x,y,z $\in$ X, if (x,y) and (y,z) $\in$ R, then (x,z) $\in$R.

  If $M_{ij}=1$, $M_{jk}=1$, then $M_{ik}=1$

ex) A = Z, R = $\{(a,b) \in A \times A \mid a \langle b\}$

(sol) aRb and bRc, $a \langle b \rightarrow b \langle c$, therefore, $a \langle c$ $\therefore$ aRc

therefore , R is <u>transitive</u>

ex) X = {a,b,c,d} R = {(a,a)(b,c)(c,b)(d,d)}

ex) A = {1, 2, 3, 4} R = {(1,1) (2, 2)(2, 3)(3, 2)(4, 2)(4, 4)}

ex) A = {0,1,2} R = {(0,0)(0,1)(0,2)(1,2)}

exercises

ex) $X = \{1,2,3,4\}$

$R = \{(1,1)(1,2)(1,3)(1,4)(2,2)(2,3)(2,4)(3,3)(3,4)(4,4)\}$

(sol) R is transitive

If x=y, then it is transitive since $\dfrac{(x,y)\ (y,z)\ (x,z)}{(1,1)\ (1,1)\ (1,1)}$

We just need to check for

(1,2)(2,3)(1,3),

(1,2)(2,4)(1,4),

(1,3)(3,4)(1,4),

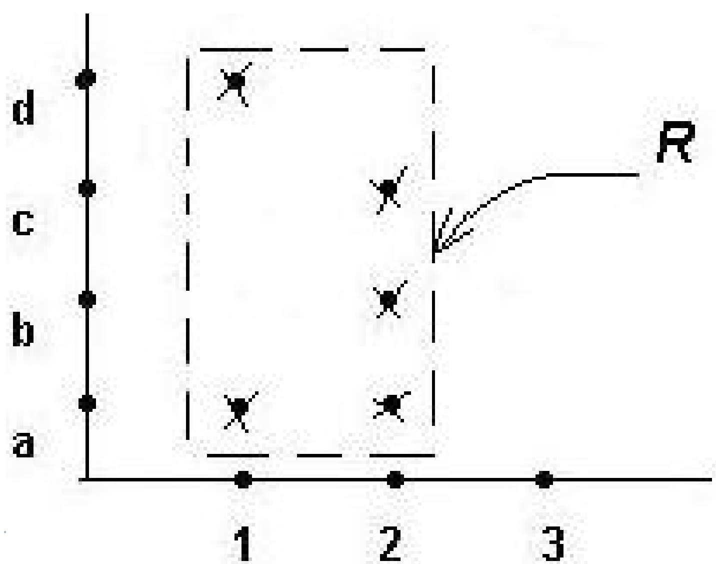
(2,3)(3,4)(2,4)

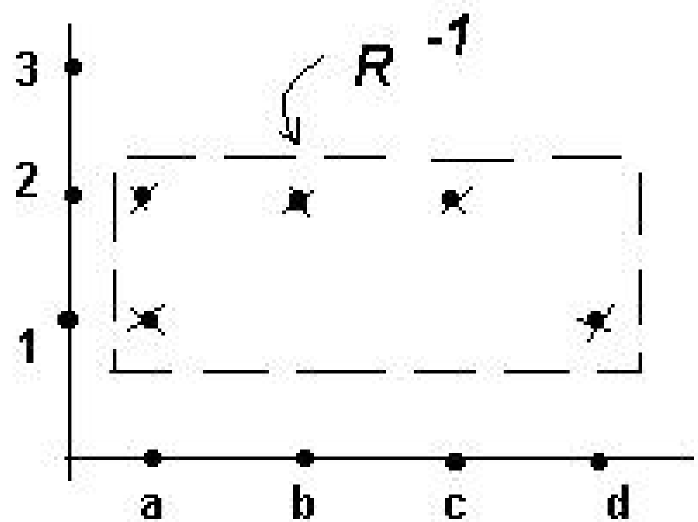
2.3 기타속성 - Inverse of a relation

Given a relation R from X to Y, its inverse R^{-1} is the relation from Y to X defined by $R^{-1} = \{ (y,x) \mid (x,y) \in R \}$

Ex) if $R = \{(1,a), (1,d), (2,a), (2,b), (2,c)\}$ then

$R^{-1} = \{(a,1), (d,1), (a,2), (b,2), (c,2)\}$

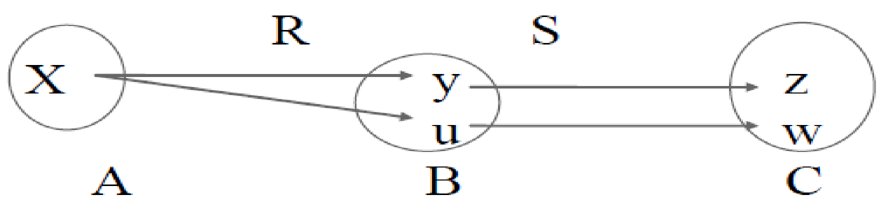




Composition of Relation

- Create new relations from existing ones. Let R: A x B, S: B x C
- Composition of relation R and S is the relation between A and C.

$$S.R = \{(x,z) \mid x \in A,\ z \in C \text{ and } \exists y \in B, \text{ such that } xRy \text{ and } ySz\}$$



ex) A = {a,b,c} B = {1,2} C = {5,9,10}
S (B X C) = {(1,5)(1,9)(2,9)}   R(AXB) = {(a,1)(c,2)}   S.R?

* **Composition with itself**
ex) R on set A = {1,2,3}   R = {(1,1)(1,2)(1,3)(3,2)}   R.R?

2.4 Equivalence relations

- THM1) Let Q be a partition of a set X. Define xRy to mean that for some set S in Q, both x and y belong to Q. Then R is reflexive, symmetric, transitive.

- Let X be a set and *R* a relation on X, *R* is an *equivalence relation* on X $\Leftrightarrow$ ***R* is reflexive, symmetric and transitive.**

Ex) Consider the partition Q={{1,3,5}, {2,6}, {4}} of X={1,2,3,4,5,6}.

The complete relation R is ={(1,1)(1,3)(1,5) (3,1)(3,3)(3,5) (5,1)(5,3)(5,5) (2,2)(2,6)(6,2)(6,6) (4,4)}

Con’t

ex)Let X = S4, R={(1,1)(1,2)(1,3)(1,4)(2,2)(2,3)(2,4)(3,3)(3,4) (4,4)}

The diagraph of R is reflexive ? symmetric ? transitive ?

ex) Let X = S5, R= {(1,1)(1,3)(1,5)(2,2)(2,4)(3,1)(3,3)(3,5)(4,2)(4,4)(5,1)(5,3) (5,5)} => E.R

ex) Let A = Z, R ={(a,b) $\in$ A X A | a $\leq$ b}

(sol) 1) since, a $\leq$ a , reflexive. 2) if a $\leq$b, b $\leq$a, not SYM

3) since a $\leq$ b and b $\leq$ c, transitive, NOT E.R.

- **Def: Equivalence Relation leads to Natural partitions of A into groups of like objects.**

Set of equivalence classes

def1: Let R be an equivalence relation on X. $\forall a \in X$,

let $[a] = \{x \in X \mid xRa\}$ then

$S = \{[a] \mid a \in X\}$ is a partition of X.

- Let R be a equivalence relation on set X. The sets [a] defined in def1 are called the equivalence Class of X given by relation R.

Ex) Let R={(1,1)(1,3)(1,5)(3,1)(3,3)(3,5)(5,1)(5,3)(5,5)(2,2) (2,6)(6,2) (6,6)(4,4)}

The equivalence class [1] containing 1 consists of all x , such that $(x,1) \in R$. Therefore, $[1] = \{1,3,5\}$

The remaining E.C are found similarly, $[3] = [5] = \{1,3,5\}$,

$[2] = [6] = \{2,6\}$ $[4] = \{4\}$

Exercises

ex) Let S=S6. For $x,y \in S$. Let xRy if x-y is divisible by 2.

- Show that R is an E.R. - Find E.C. relative to R.

(sol)

1) R is reflexive
2) R is SYM => since x-y / 2 이면 <=> y-x / 2 이기 때문에
3) Suppose, xRy and yRz, then there exist k and m, such that

   $x-y = 2k$ and $y-z = 2m$. // since it is divisible by 2

   $x-z = (x-y) + (y-z) = 2k+2m=2(k+m)$. so R is TRAN
4) by 1)2)3), it is E.R on S.
5) E.C: $[1] = \{x\in S6: x-1 \text{ is divisible by } 2\} = \{1,3,5\}$

   $[2] =\{x\in S6: x-2 \text{ is divisible by } 2\} =\{2,4,6\}$

Which leads to a natural partition of set S6 into [1] and [2], and $[1]\cup[2] = S6$.

More examples for E.C.

ex) Let $X = \{1,2,...10\}$. Define xRy to mean that 3 divides $x-y$.

ex) If $R = \{(1,1)(1,3)(1,5)(2,2)(2,4)(3,1)(3,3)(3,5)(4,2)\ (4,4)(5,1)$
$(5,3)\ (5,5)\}$

ex) If $R = \{(a,a)(b,b)(c,c)\}$

Partition and E.C.

- **We need to show to be E.C.**

  1) For all $x \in S$, $[x] \neq 0$ 2) if $[x] \neq [y]$, then $[x] \cap [y] = \varnothing$

  3) $\cup \{[x] \mid x \in S\} = S$

ex) When R = {(1,1)(1,2)(2,1)(2,2)(3,3)(3,4)(4,3)(4,4)}, Find E.C.

(sol) $[1] = \{1,2\}$, $[3] = \{3,4\}$ and $[1] \cap [3] = \varnothing$ 이므로,

[1]과[3] 은 partition 을 이룬다. Partition A ={ {1,2}, {3,4} }

ex) Let Set A = {1,2,3,4}. Partition of A, P = { {1,2,3}{4}} 일때,

P 에 의해 결정되는 Set A 의 Equivalence Relation R 을 구하라.

ex) S = {a,b,c,d,e,f} P = {{a,b}, {c,e,f}, {d} } .

- Find Equivalence Relation R.

2.5 Closure

Relation은 반사적, 대칭적, 추이적 성질을 가지고 있지 않을 수도 있다.

→ Relation 이 특정 성질을 가지고 있지 않으면, 그 성질을 같아질 때까지 Relation 에 관련된 순서쌍들을 첨가할 수 있다

→ 우리가 원하는 성질을 가지며, 관계 R을 포함하는, 최소의 관계 R1을 찾는 것이 필요하다 => CLOSURE

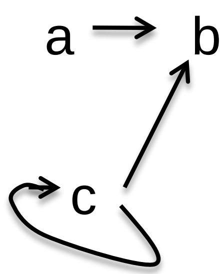
- 1) reflexive closure
- 2) symmetric closure
- 3) transitive closure

Reflexive closure

R1 = R∪∇: **smallest** reflexive relation on A containing R.

<u>Reflexive closure</u> of R is R∪∇

$$\text{ex)}\quad \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

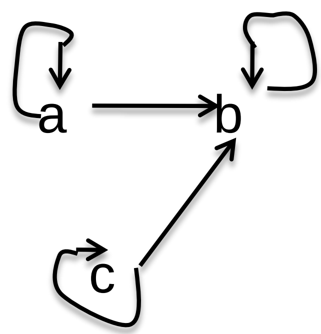


not reflexive

(sol)

$$\text{So,}\ \underset{R}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \cup \underset{\nabla}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{R1}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$

(identity matrix)



Symmetric Closure

- If R is relation on A which is not symmetric.

Then $\exists$ $(x,y) \in R$, such that $(y,x) \notin R^{-1}$. So, if R is not symmetric, then we must add all pairs from R^{-1}.

$\rightarrow$ Enlarge R to $R \cup R^{-1}$,

$\rightarrow$ $R \cup R^{-1}$ is smallest symmetric Closure containing R.

- ex) A={a,b,c,d}. R={(a,b)(b,c)(a,c)(c,d)}

Then R^{-1} = {(b,a)(c b)(c a)(d c)}

so symmetric closure of R is

$R \cup R^{-1}$ = {(a b)(b c)(a c)(c d)(b a)(c b)(c a)(d c)}

Transitive Closure

- Let R be a relation on A.   Then R^* is transitive closure of R.

ex) Let R be the relation $R=\{(a,b)(b,c)(c,a)\}$ on set $S=\{a, b, c\}$.
Find Transitive closure of R

(sol) 1) $(a,b)(b,c)(c,a) \in R^*$, since they are in R

2) Since aRb & $bRc \Rightarrow aR^*c$ similarly, bRc & $cRa \Rightarrow bR^*a$,

cRa & $aRb \Rightarrow cR^*b$

Transitive closure $R^* = \{(a,b)(b,c)(c,a)(a,c)(b,a)(c,b)\}$

ex) $A =\{1\ 2\ 3\ 4\}$ $R=\{(1\ 2)(2\ 3)(3\ 4)(2\ 1)\}$ Find Transitive closure of R

(sol) $R^* = R^1 \cup R^2 \cup R^3 \ldots \cup R^n$

$R^* = \{(1\ 1)(1\ 2)(1\ 3)(1\ 4)(2\ 1)(2\ 2)(2\ 3)(2\ 4)(3\ 4)\}$

(sol) matrix $Mr = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

boolean multiplication

$(Mr)\bullet^2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $(Mr)\bullet^3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ....

$(Mr)\bullet^n = \{ Mr:$ when $n=1$,
Mr^2 when $n=$ even,
Mr^3, when $n=$ odd$\}$

Therefore $Mr\bullet^* = Mr \cup Mr^2 \cup Mr^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Warshall's algorithm

- Graphic, Matrix: impractical for large sets, not systematic, inefficient
- Use WARSHALL's algorithm (find transitive closure for n x n matrix)

(Warshall's algorithm) :   FIND W_k from W_{k-1}

1) Move all 1's in W_{k-1} into W_k
2) W_{k-1}의 열(column)k 에서 값이 1인곳의 위치에 p1,p2,..를 기록하고
   W_{k-1}의 행(row)k에서 값이 1인곳의 위치에 q1,q2,...를 기록한다.
3) W_k 의 p_i, q_j의 위치에 1을 쓴다.

Ex) 위의 예제 이용, W_0= M**r** = 0 1 0 0 이고 n=4 이다

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

2.6 관계의 응용 - Relational databases

- A ***binary*** relation R is a relation among two sets X and Y, already defined as $R \subseteq X \times Y$.

- An ***n-ary* relation *R*** is a relation among n sets $X_1, X_2,,,, X_n$,

(i.e. a subset of the Cartesian product, $R \subseteq X_1 \times X_2 \times \ldots \times X_n$.)

- A ***database*** is a collection of records that are manipulated by a computer. They can be considered as n sets $X_1..X_n$, each of which contains a list of items with information.

- ***Database management systems* (DBMS)** are programs that help access and manipulate information stored in databases

Operators Examples

- The *selection* T1:=PLAYER[Position = Guard]
- The *projection* T2:=PLAYER[Name,ID]
- The *join* T3:=PLAYER[ID = PID] ASSIGNMENT

PLAYERS

ID	Name	Position	Age
23	M.Jordan	Guard	40
32	S. O'Neal	Center	33
3	A.Iverson	Guard	29
8	K.Bryant	Guard	26
33	G. Hill	Forward	32

ASSIGNMENT

PID	Team
32	Heats
3	76ers
8	Lakers

Operators Examples

T1:=PLAYER[Position = Guard]

T2:=PLAYER[Name,ID]

T3:=PLAYER[ID = PID] ASSIGNMENT

T1

ID	Name	Position	Age
23	M.Jordan	Guard	40
3	A.Iverson	Guard	29
8	K.Bryant	Guard	26

T2

ID	Name
23	M.Jordan
32	S. O'Neal
3	A.Iverson
8	K.Bryant
33	G. Hill

T3

ID	Name	Position	Age	Team
32	S. O'Neal	Center	33	Heats
3	A.Iverson	Guard	29	76ers
8	K.Bryant	Guard	26	Lakers