Chapter 4. Counting methods & Recurrence Relation

- 1. Counting Methods
  - 1.1 Sampling
  - 1.2 Permutation
  - 1.3 Combination
  - 1.4 Pigeon Principle

- 2. Recurrence Relations
  - 2.1 Definitions
  - 2.2 Solving Recurrence Relation

1.1 Counting

Counting Definition:

How many ways are there to…

How many operations does this algorithm perform?

ex) Number of elements in a set,..

ex) S = {x,y,z,w} for k= 3. *Consider Repetition, Order.*

sampling = xxx, xxy, xxz, xzx,... = 64 (4 x 4 x 4)

permutation= 4x3x2 = 24

combination = xyz, xyw, xzw, yzw = 4

Sampling

* K- Sampling (select k elements from a set S)
allows repetition & ordering

1) ***Multiplication principle***

- If an activity can be performed in k successive steps,
  - Step 1 can be done in n_1 ways
  - Step 2 can be done in n_2 ways
  - …
  - Step k can be done in n_k ways

Then: the entire activity can be performed in $\mathbf{n_1 n_2 \ldots n_k}$ ways

Ex) A PIN (personal identification number) is a sequence of any 4 symbols chosen from the 26 letters in the alphabet and 10 digits, <u>with repetition allowed</u>.

How many different PINs are possible?

$36 \cdot 36 \cdot 36 \cdot 36 = 36^4$

Examples

ex) How many strings of length 4 can be formed using the letters ABCDE, if repetitions are not allowed?

- How many strings begin with letter B
- How many strings do not **begin** with letter B

ex) 26 alphabet에서 3 char 사용하여 code만들때, how many different code words are there (ex. CRE, TIL, ABC,...) (repetition allowed)

- by product rule, there are 26 x 26 x 26 code words
- if without repetition, then 26 x 25 x 24

ex) 자동차 번호판의 구성이 3 개의 문자와 3개의 숫자의 순서로 구성된다면 얼마나 많은 방법으로 번호판을 구성할 수 있는가?

2) Addition principle – when multiplication rule is difficult or impossible to apply

A finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots A_k$, then, $\boldsymbol{n(A) = n(A_1) + n(A_2) + \ldots + n(A_k)}$

(if the first task can be done in n1 ways and

second task can be done in n2 ways and

and the task cannot be done at the same time, then these are

$n_1 + n_2 + ,,,,, +n_k$ ways to do

ex) 5로 나누어지는 수의 합은? (100 - 999 중에서)

ex) 6 person committee composed of A, B, C, D, E, F is to select a **president, secretary, treasurer.**

- How many ways can this be done if either A or B must be chairperson?
- How many ways can this be done if E must hold one of the positions?

ex) Basic variables - (if length 1 -> 영문자 & if length 2 -> 영문자+숫자)

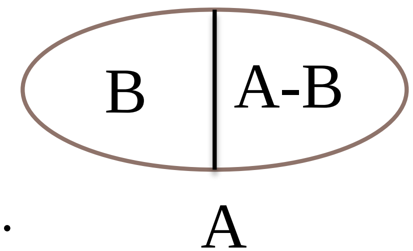
Other principle

3) Difference Principle

. If A is a finite set and B is a subset of A. Then $n(A-B) = n(A) - n(B)$

$B \cup (A-B) = A$ and B and A-B are disjoint sets

So, $n(B) + n(A-B) = n(A)$, $n(A-B) = n(A) - n(B)$



ex) 3 letter computer access code with repetitions allowed.

How many code words contain repeated letters?

4) Inclusion/Exclusion Rule (for two or more sets)

. If A, B and C are finite sets. Then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ and

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

ex) 1 ~ 1000에서 3의 multiple or 5 의 multiple의 합은?

1.2 Permutations – allows order & no repetition

A ***permutation*** of n distinct elements $x_1, x_2, \ldots, x_n$ is an ordering of the n elements. There are n! permutations of n elements.

. Product Rule => n.(n-1).(n-2)....2.1=n!

Ex) Set A ={a, b, c}. there are 3! = 6 permutations of three elements a, b, c:

abc bac cab acb bca 총

ex) C, O, M, P, U, T, E, R => How many ways this can be arranged in a row

- Def: An **K-permutation** of set n elements is an ordered selection of k elements from the set of n elements , $P(n,k) = n! / (n-k)!$, $k \leq n$

$$P(n,k) = n(n-1)(n-2)\ldots(n-k+1)$$

ex) how many ways can we select a **chairperson, vice-chairperson, secretary, treasurer** from 10 person?

1.3 Combinations - no repetition & no ordering

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set containing n distinct elements

- An k-*combination* of X is an **unordered selection of k elements** of X, for $k \leq n$
- $C(n,k) = n!/k!(n-k)! = P(n,k)/ k! = n! / k!(n-k)!$ (ordered selection: k-permutation)

- **<u>Binomial Coefficient : C(n,k)</u>**
- **<u>Binomial Theorem:</u>** $(a+b)^n = \Sigma\, C(n,k)\, a^{n-k} b^k$

$$(a+b)^n = \Sigma\, C(n,k)\, a^{n-k} b^k = C(n,0)a^n b^0+C(n,1)a^{n1}b^1+ \ldots.+C(n,n)a^1 b^{n-1}+C(n, n)a^0 b^n$$

* Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

ex) Team consists of 5 men and 7 women.

1) consists of 3 man (from 5) and 2 women(from 7)

2) at least one man : [x]= [total] – [no man]

3) at most one man : [x]=[no man]+[one man]

ex)12명중 5명으로 팀을 구성할 때, C와 D는 함께 구성되지 않는 방법?

Catalan numbers

- Belgian mathematician, 1814-1894
- Catalan numbers can also be generated by the formula:

  $C_n = C(2n,n)-C(2n,n-1)= C(2n,n) / (n+1) = (2n)! / (n+1)!\, n!$ for $n \geq 0$

Ex) 각각 n 개의 왼쪽, 오른쪽 괄호가 있을 때 괄호의 짝을 맞춰서 식을 만드는 경우의 수는?

(sol) n = 3 인 경우, ((())), ()(()), (())(), (()()), ()()()

The first few *Catalan numbers* are:

n	0	1	2	3	4	5	6	7	8	9	10	11
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786

1.4 The pigeonhole principle

- THM: |Dom(f)|=n |Ran(f)|=m

If **<u>n pigeons</u>** are assigned to **<u>m pigeonsholes,</u>** (if n>m) then at least one pigeonhole contains two or more pigeons

- If X and Y are finite sets with |X| > |Y| and f:X→Y is a function, then f(x1) = f(x2) for some x1,x2 ∈X, x1≠x2.

ex) 8명이 있을때, 그들중 2명은 (적어도) 같은 요일에 태어남

ex) 1 부터 8 까지의 수로부터 임의의 다섯개의 숫자를 고르면, 그중 한쌍의 수의 합이 9가 될 수 있는가? 또한 4개의 숫자를 고른다면 9가 될 수 있는가?
pigeonhole -> pigeons ->

Ex) 151개의 교과목이 있고, 각 교과목의 학수 번호가 1~300 사이로 한정되어 있다면, 적어도 두개의 교과목은 연속적일 것이다.

2. Recurrence Relation (점화관계)

- A sequence can be defined in a variety of different ways
- Informal way: with expectation, but may misleading.

  ex) 2,5,7,.. next term may be 9 (odd), or 11 (prime)

- **Recurrence Relation**: defines a sequence by giving nth value in terms of certain of its predecessors (relates later terms to earlier terms)

- **specification**: initial condition (first few terms of sequence)

ex) start with 5, and add 3 to get the next term 5,8,11,14,....

- if rephrase, $a_1=5$, (initial condition) $a_n=a_{n-1} + 3$ $(n>=2)$ (recurrence relation)

Ex) Define sequence $c_0, c_1, c_2,...$ recursively as follows $\forall k \geq 2$. Find c_2, c_3?

1) $c_k = c_{k-1} + k.c_{k-2} +1$ (recurrence relation) 2) $c_0 = 1$, $c_1 = 2$ (initial condition)

Ex) Compound interest

- Given : P = initial amount (principal) n = number of years

  r = annual interest rate A = amount of money at the end of n years

At the end of:

- 1 year: $A_1 = P + rP = P(1+r)$, 2 years: $A_2 = A_1 (1+r) = P(1+r)^2$
- Obtain the formula $\mathbf{A_n = P (1 + r)^n}$

ex) A person invests \$1000 at 12% compounded annually. If An represents the amount at the end of n years, find a recurrence relation and initial conditions that define the sequence $\{A_n\}$.

$A_n = A_{n-1} + (0.12)A_{n-1} = (1.12) A_{n-1}$ ➔ recurrence relation

If the initial amount is $A_0 = \$1000$ ➔ initial condition

$\mathbf{A_n = (1.12)^n(1000)}$

- **<u>Recursive Algorithm</u>**

```
Procedure compound-interest (n)
    if n=0 then return (1000)
    return (1.12*compound-interest(n-1))
end
```

Towers of Hanoi (E. Lucas)

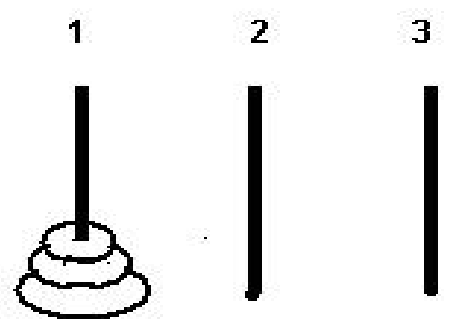
Start with three pegs numbered 1, 2 and 3 mounted on a board, n disks of different sizes.

(1) can move one disk at a time

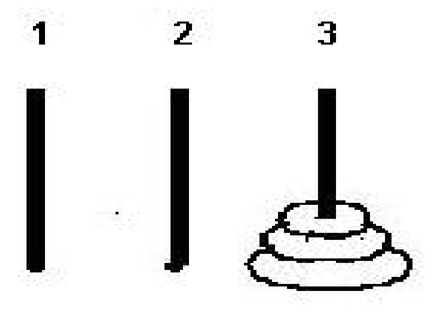
(2) Only a disk of smaller diameter can be placed on top of another disk

- Object of the game: find the minimum number of moves needed to have all n disks stacked in the same order in peg number 3.
- $M_k = 2M_{k-1} + 1 \quad (k>1)$, $M_1 = 1$ Then $M_n = 2^n - 1$ for $n \geq 1$

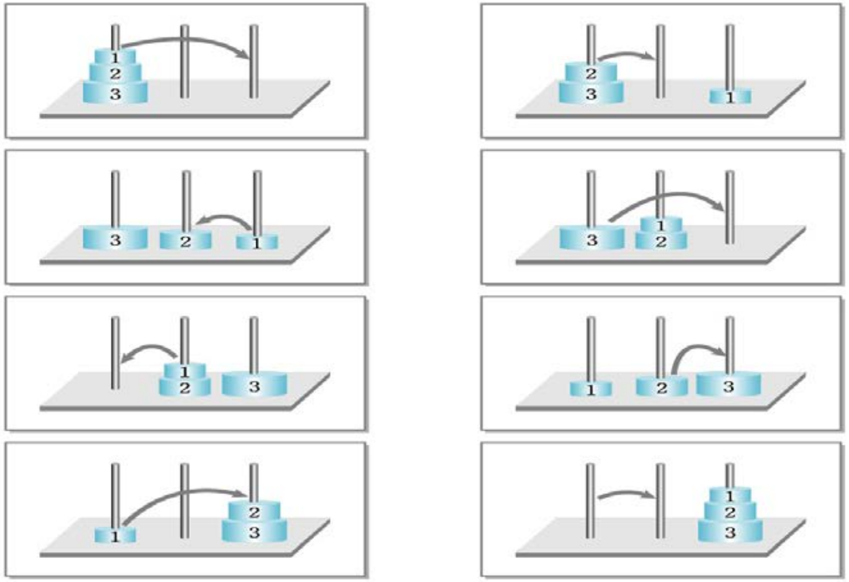
Start



End



Ex) 3 disks



Ex) Fibonacci

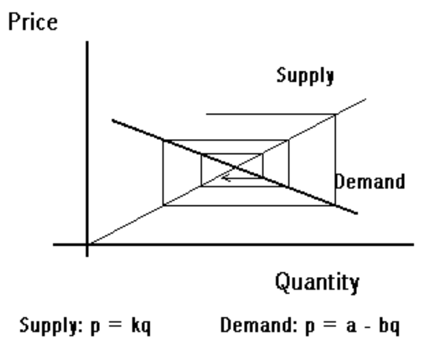
Initial conditions: $f_1 = 1, f_2 = 2$

Recursive formula: $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$

Ex) A problem in Economics -cobweb (p: price, q: quantity, a,b: constant)

- Demand equation: $p = a - bq$ Supply equation: $p = kq$

  Given: $p_n = a - bq_n$ (demand) $p_n = kq_{n+1}$ (supply)
- The recurrence relation obtained is: $p_{n+1} = a - bp_n /k$



(sol) $p_n = a - b/k \,.\, p_{n-1}$ (let $s= -b/k$)

$p_n = a + s.p_{n-1} = a + s(a+ sp_{n-2}) = a + as+ s^2p_{n-2}$

$= a + as+ s^2(a+ sp_{n-3})= \ldots = a + as + as^2+ \ldots as^{n-1} + s^np_0$

$= (a-as^n)/(1-s) + s^np_0 = (-b/k)^n(-ak/(k+b) + p_0)$

- Recurrence relation – uses prior values in a sequence to compute the current value
- Recursive algorithm – uses smaller instances of current input to process the current input
- Mathematical induction – assumes prior instances of the statement to prove the truth of the current statement

2.2 Solving recurrence relations

- Solving Recurrence relation is to find an **explicit formula** for the general term.
- Method for finding explicit formula: **iteration / linear method,**
- Problem: Given a recursive expression with **initial conditions** a_0, a_1 and try to **express a_n**

procedure power(x, n)

if (n == 0) return 1; $\quad$ T(0) = c1 for some constant c1

else return x * power(x, n-1); $\quad$ T(n) = c2 + T(n − 1) for some constant c2

If we know T(n-1), we could solve T(n)

$T(n) = T(n-1) + c2 \qquad T(n-1) = T(n-2) + c2$

$= T(n-2) + c2 + c2 = T(n-2) + 2c2 = \ldots \ldots = T(n-k) + k{*}c2$

- If we set k=n, $T(n) = T(n-n) + nc2 = T(0) + nc2 \Rightarrow \theta(n)$

examples

ex) $a_n = a_{n-1} + 3$, $a_1 = 2$ (Using iteration method)

(sol) $a_n = a_{n-1} + 3$

$= a_{n-2} + 3 + 3$ (since $a_{n-1} = a_{n-2} + 3$)

$= a_{n-3} + 3 + 3 + 3$ (since $a_{n-2} = a_{n-3} + 3$)

....

$= a_{n-k} + k.3$

if we set $k=n-1$, we have, $a_n = a_1 + (n-1).3$

since $a_1=2$, we obtain explicit formula => $\mathbf{a_n = 2 + 3(n-1)}$

More examples

ex) $\mathbf{a_k = a_{k-1} + 2}$ **(R.R)** $\mathbf{a_0 = 1}$ **(I.C)**

(sol) $a_0 = 1$ ---> $1 + 0.2$

$a_1 = a_0+2 =1+2$ ---> $1+ 1.2$

$a_2 = a_1+2 = 1+2+2$ --> $1+ 2.2$

....

$a_n = a_{n-1} + 2 = 1+(2++....+2)$ ---> $1+n.2$ Therefore, $a_n= 1+2n$

ex) Find explicit formula for C_n. $C_n = 2C_{n-1} + 1$ (R.R), $C_1 = 1$ (I.C)

(sol) $C_n = 2C_{n-1} + 1$

$= 2 (2.C_{n-2}+1) + 1$ $= 2^2.C_{n-2} + 2 + 1$

$= 2 (2. (2. Cn-3 + 1) + 1) + 1 = 2^3.C_{n-3} + 2^2+2+1$

....

$= 2^{n-1}. C_1 + 2^{n-2} + 2^{n-3} + ... + 2 + 1 = 2^{n-1} + 2^{n-2} + 2^{n-3} + ... + 2 + 1 = \mathbf{2^n -1}$

{because $1. (2^{n-1+1} - 1) / (2 -1) = 2^n -1$ }

Ex) Find explicit formula from the following complete graphs. Use the relation for the edges of the graphs. (K1, K2,..)

Solving Linear Homogeneous recurrences

Examples)

- $a_n = (1.11)\, a_{n-1}$: a linear homogeneous recurrence relation of degree 1
- $a_n = a_{n-1} + a_{n-2}$: a linear homogeneous recurrence relation of degree 2
- $a_n = a_{n-6}$: a linear homogeneous recurrence relation of degree 6

- $a_n = a_{n-1} + \mathbf{a^2}_{n-2}$: not linear
- $a_n = 2a_{n-1}+1$: not homogeneous

*Solving **Linear Homogeneous recurrences**

1) Guess and Verify using Mathematical Induction
2) Look for the pattern (polynomial sequence, **geometric sequence**,..)
3) Repeated Substitution
4) Using Linear Recurrences (one term is a linear function of earlier terms)

Solving Linear Homogeneous Recurrences (LHR)

- Recurrence relation $\mathbf{a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}}$
- Suppose we have solution in **<u>geometric sequence</u>** $\mathbf{a_n = r^n}$ for some r.
- Try to find a solution of form $\mathbf{r^n}$

  $\mathbf{r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}}$ => $\mathbf{r^n - c_1r^{n-1} - c_2r^{n-2} - \ldots - c_kr^{n-k} = 0}$

  $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$ (This equation is called **<u>characteristic equation</u>**)

  ➔ r^n is solution of $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$

Ex) Fibonacci recurrence is $F_n = F_{n-1} + F_{n-2}$ Its characteristic equation is $r^2 - r - 1 = 0$

- Thm: consider characteristic equation $\mathbf{r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0}$ and relation $\mathbf{a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}}$. Let α_1, α_2 .. be any constants

  So, $\mathbf{a_n = \alpha_1r_1^n + \alpha_2r_2^n + \ldots + \alpha_mr_m^n}$ satisfies the recurrence for some constant $\boldsymbol{\alpha}$

Examples using characteristic equation $\mathbf{a_n = r^n}$

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} \Rightarrow a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n} + \ldots + \alpha_m r_m^{\,n}$

Ex) What is the solution of the recurrence relation (

$$\mathbf{a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0=2 \text{ and } a_1=7}$$

(sol) find characteristic equation, $r^2-r-2=0$, $(r+1)(r-2)=0$

$r_1=2$ and $r_2= -1$. By the Theorem, $a_n = \boldsymbol{\alpha_1 2^n + \alpha_2(-1)^n}$

n=0: $\boldsymbol{\alpha_1 + \alpha_2 = 2}$, $(a_0=2)$

n=1: $\boldsymbol{2\alpha_1 + (-1)\alpha_2 = 7}$ $(a_1=7)$ => $\boldsymbol{\alpha_1 = 3}$ $\boldsymbol{\alpha_2 = -1}$ $\underline{\mathbf{a_n = 3.2^n - (-1)^n}}$

Ex) $\boldsymbol{a_n=2a_{n-1}+3a_{n-2}}$**, with** $\boldsymbol{a_0=3}$ **and** $\boldsymbol{a_1=5}$**?**

(sol) $r^2-2r-3 = (r-3)(r+1)$, r1= 3, r2=-1 $a_n = \boldsymbol{\alpha}_1 3^n + \boldsymbol{\alpha}_2(-1)^n$

n=0: $a_0 = \boldsymbol{\alpha}_1 3^0 + \boldsymbol{\alpha}_2(-1)^0$ => $\boldsymbol{\alpha}1 + \boldsymbol{\alpha}2 = 3$

n=1: $a_1 = \boldsymbol{\alpha}_1 3^1 + \boldsymbol{\alpha}_2(-1)^1$ => $3\boldsymbol{\alpha}1 - \boldsymbol{\alpha}2 = 5$

$4\boldsymbol{\alpha}1= 8$ $\boldsymbol{\alpha}1=2$, $\boldsymbol{\alpha}2=1$ Therefore, $\underline{\mathbf{a_n = 2.3^n + (-1)^n}}$

More examples on LHR

Ex) $\boldsymbol{a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}}$ **with** $\boldsymbol{a_0=8, a_1=6, a_2=26}$**?**

$r^3+r^2-4r-4 = 0$, => $(r+1)(r+2)(r-2)$, $r1=-1, r2=2, r3=-2$ $a_n = \boldsymbol{\alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3(2)^n}$

n=0: $\alpha_1 + \alpha_2 + \alpha_3 = 8$, **n=1:** $-\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$ **n=2:** $\alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$

➔ $\boldsymbol{\alpha_1 = 2 \quad \alpha_2 = 1 \quad \alpha_3 = 5}$

$a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3(2)^n$ **=>** $\underline{2\boldsymbol{(-1)^n + (-2)^n + 5(2)^n}}$

THM: Assume r is solution of C.E with **multiplicity**, then $\underline{\boldsymbol{n^m r^n}}$ is a solution

$\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_m r_m^n}$ ➔

$\boldsymbol{a_n = (\alpha_{10} + \alpha_{11} n + \ldots \alpha_{1,m1-1} n^{m1-1})r_1^n + (\alpha_{20} + \alpha_{21} n + \ldots \alpha_{2,m2-1} n^{m2-1})\, r_2^n + ..(\ )\, r_t^n}$

Ex) $\boldsymbol{a_n = 6a_{n-1} - 9a_{n-2}}$ **with** $\boldsymbol{a_0=1, a_1=6}$

$r^2 - 6r + 9 = 0$ $(r-3)^2 = 0$, $r=3$ $\underline{a_n = \boldsymbol{(\alpha_{10} + \alpha_{11} n)(3)^n}}$

n=0: $\boldsymbol{(\alpha_{10} + 0)(3)^0 = 1}$**,** $\boldsymbol{\alpha_{10} = 1}$**,** **n=1:** $\boldsymbol{3\alpha_{10} + 3\alpha_{11} = 6}$**,** $\boldsymbol{\alpha_{11} = 1}$

$\underline{\boldsymbol{a_n = 1.3^n + n.3^n}}$

Ex) $\boldsymbol{a_n = 8a_{n-2} - 16a_{n-4}}$ **with** $\boldsymbol{n \geq 4}$**,** $\boldsymbol{a_0=1, a_1=4, a_2=28, a_3=32}$ $\underline{\boldsymbol{a_n = 2^n + 2n2^n + n(-2)^n}}$

Solving Linear NON-Homogeneous Recurrences (LNHR)

- **NLHR example**: $a_{n-} = a_{n-1} + 2^n$, $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$, $an = a_{n-1} + n2^n$,,,,

  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$, $c_1 \ldots . c_k$: constants f(n): function

NLHR의 Solution: $a_n^{(p)} + b_n$ ($a_n^{(h)}$: LHR 의 solution, b_n : LNHR 의 solution)

Ex) $a_n = 2a_{n-1} - a_{n-2} + 2^n$ $n \geq 2$, **with** $a_0 = 1$, $a_1 = 2$ $(f(n) = c.2^n + d)$

1) Let $b_n \Rightarrow 2b_{n-1} - b_{n-2} + 2^n$,

$$c.2^n + d = 2(c2^{n-1} + d) - (c2^{n-2} + d) + 2^n = c2^n + 2d - c2^{n-2} - d + 2^n$$

$$c2^n + 2d - c2^{n-2} - d + 2^n - c.2^n + d = 0$$

$$2^{n-2}(-4c + 4c - c + 4) + (-d + 2d - d) = 0 \quad (c=4, d=0) \qquad b_n = 4.2^n$$

2) $a_n = a_n^{(p)} + b_n$

$a_n^{(p)} \Rightarrow a_n = 2a_{n-1} - a_{n-2} = r^2 - 2r + 1 = (r-1)^2$, $r = 1$

$a_n^{(p)} = (\alpha_1 + \alpha_2 n)(1)^n$, $a_n = a_n^{(p)} + b_n = 4.2^n + \alpha_1 + \alpha_2 n$ is a solution.

With initial condition: $a_0 = 4 + \alpha_1 = 1$, $\alpha_1 = -3$ $a_1 = 8 + \alpha_1 + \alpha_2 n = 2$, $\alpha_2 = -3$

$$a_n = 4.2^n + \alpha_1 + \alpha_2 n = 4.2^n - 3n - 3$$

Checking correctness of formula by Math Induction

ex) Tower of Hanoi

$\mathbf{M_k = 2M_{k-1} + 1 \quad (k \geq 2)}$, $\mathbf{M_1 = 1}$ Then $\mathbf{M_n = 2^n - 1}$ for n>= 1

(pf) $m_1 = 1$

$m_2 = 2m_1 + 1 = 3$, $m_3 = 2m_2 + 1 = 7$ ....

1) for n=1 , $m_1 = 2^1 - 1$ ok

2) Suppose n=k true, then $\mathbf{M_k = 2^k - 1}$, for k>= 1 is true

3) We need to prove for $\mathbf{M_{k+1} = 2^{k+1} - 1}$

$M_{k+1} = 2M_{(k+1)-1} + 1$, by Recurrence Relation

$= 2\mathbf{M_k} + 1$

$= 2. (2^k - 1) + 1$ by sub to hypothesis

$= 2^{k+1} - 2 + 1$

$= \mathbf{2^{k+1} - 1}$ Therefore True