Discrete Mathematics

Chapter 1. Logic and proofs

1.1 Propositional Logic (명제논리)

1.2 Predicate Logic (술어논리)

1.3 Proof(증명)

Logic

- Logic = the study of correct reasoning
- Use of logic
  - In mathematics: to prove theorems
  - In computer science:
    - to **prove that programs** do what they are supposed to do
    - to the **queries** to databases & search engines
    - To **design** of digital logic circuits
    - To be used in AI (expert systems, automatic theorem provers,..)

  - It includes
    - **명제논리 (propositional logic)**
    - **술어논리 (predicate logic)**

Examples of Logic

Ex) Assume a sentence:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

1) Parse: **If (** you are older than 13 **or** you are with your parents **) then**
(you can attend a PG-13 movie**)**

2) Atomic (elementary) propositions:

– A= you are older than 13 – B= you are with your parents

– C=you can attend a PG-13 movie

- **Translation: A $\vee$ B → C**

Ex) <u>You can have free coffee </u>if <u>you are senior citiz</u>en and <u>it is a Tuesday</u>

a b c

rewrite the sentence in propositional logic : **b ^ c → a**

Propositional Logic

- **Propositional logic:** a formal language for representing knowledge and for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Section 1. Propositional Logic

- **Propositional logic:** a formal language for representing knowledge and for making logical inferences
- A ***proposition*** is a statement or sentence that can be determined to be either **true** or **false**. (**Assertion**: a declarative statement)

- Examples:
  - “John is a programmer" is a proposition(T)
  - “I wish I were wise” is not a proposition(F)

- Exercise

  a. 100>99 b. 3 is not an even integer

  c. 3-x = 5 d. Take two aspirins

  e. Is he a CS major?

1.1 Logical Operations

- 주어진 명제를 논리 연산자(logical operators)를 사용, 합성하여 새명제를 만들고, 거기에 진리값을 부여함

- Simple proposition(단순명제): 하나의 문장이나 식으로 구성, 참이나 거짓을 구분하는것

- 논리연산자 (logical operators, Connectives): 단순명제들을 연결시켜주는 역할
- 합성명제 (Compound propositions) : 여러 개의 단순명제가 연결되어 만들어진 명제
- 진리표(Truth Table): 단순명제들의 진리값에서 부터 논리연산자에 따라 단계적으로 진리값을 나타내는 표.

논리연산자 (Connectives)

If p and q are propositions, new ***compound*** **propositions** can be formed by using ***connectives***

- Most common connectives:
  - Conjunction AND. Symbol ^ (P and Q)
  - (Inclusive) disjunction OR Symbol v (P OR Q)
  - Exclusive disjunction OR Symbol $\underline{\vee}$, $\oplus$
  - Negation Symbol ~, $\neg$
  - Implication Symbol $\rightarrow$, (if..then)
  - Double implication Symbol $\leftrightarrow$, (if and only if)

Truth table of conjunction

- The truth values of compound propositions can be described by *truth tables*.

- Truth table of *conjunction*

p	**q**	**p ^ q**
T	T	T
T	F	F
F	T	F
F	F	F

- p ^ q is true only when both p and q are true.

Example

- Let p = “Tigers are wild animals”
- Let q = “Seoul is the capital of KangwonDo”

- p ^ q = "Tigers are wild animals and Seoul is the capital of KangwonDo“

- p ^ q is false. Why?

Truth table of disjunction

- The truth table of (inclusive) *disjunction* is

p	q	p v q
T	T	T
T	F	T
F	T	T
F	F	F

- $p \vee q$ is false only when both p and q are false

Ex) p= "It rains outside", q = "2 is a prime number"

pvq = It rains outside or 2 is a prime number

Exclusive disjunction (XOR)

- "Either p or q" (but not both), in symbols $p \oplus q$

p	q	p $\underline{\vee}$ q
T	T	F
T	F	T
F	T	T
F	F	F

- $p \oplus q$ is true only when p is true and q is false, or p is false and q is true.
  - Example: p = "today is March 1st ", q = "today is March 2nd."
  - $p \oplus q$ = "Either today is March 1st or today is March 1st "

Negation

- Negation of p: in symbols ~p

p	**~p**
T	F
F	T

- ~p is false when p is true, ~p is true when p is false
  - Example: p = “ 5 is a prime number"
  - ~p = "It is not true that 5 is a prime number"

More compound statements

- Let p, q, r be simple statements
- We can form other compound statements, such as
  - $(p \vee q) \wedge r$
  - $p \vee (q \wedge r)$
  - $(\sim p) \vee (\sim q)$
  - $(p \vee q) \wedge (\sim r)$
  - and many others…

Example: truth table of $(p \vee q) \wedge r$

p	**q**	**r**	$(p \vee q) \wedge r$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Conditional propositions and logical equivalence

- A *conditional* proposition is of the form "If p then q"
- In symbols: $p \to q$

- Example: p = " John is a good swimmer" q = " He can cross the river"
  - $p \to q$ = "If John is a good swimmer then He can cross the river"

- Example: p: you drive over 65 mph q: you get a speeding ticket

1) $(p \to q)$ you will get a speeding ticket if you drive over 65 mph.
2) $(p \wedge \neg q)$ you drive over 65 mph, but you don't get a speeding ticket.
3) $(\neg p \to \neg q)$ if you do not drive over 65 mph then you will not get a speeding ticket
4) $(q \wedge \neg p)$ you get a speeding ticket, but you do not drive over 65mph.
5) $(\neg q \to \neg p)$ if you don't get a speeding ticket, then you don't drive over 65mph

Truth table of $p \rightarrow q$

p	**q**	**$p \rightarrow q$**
T	T	T
T	F	F
F	T	T
F	F	T

- Implication: $p \rightarrow q$, if P then Q, P implies Q

- **<u>False hypothesis</u>** implies any conclusion

Hypothesis and conclusion

- $p \to q$ is true, when both p and q are true or when p is false

  - If p is false, then truth value of $p \to q$, does not depend on the truth value of q (always true)

ex) P: 1>2, Q: 3<6

- $p \to q$ is true, since p is false, q does not matter
- $q \to p$ is false, since q is true, p is false

- In a conditional proposition $p \to q$,

  p is called the *hypothesis (antecedents)*

  q is called the *conclusion (consequences)*

Necessary and sufficient

- A *necessary* condition is expressed by the conclusion.
- A *sufficient* condition is expressed by the hypothesis.

  - Example:

  "*If* John is a good swimmer *then* he can cross the river."

  - Necessary condition: "John can cross the river."
  - Sufficient condition: "John is a good swimmer"

Converse

- The *converse* of $p \rightarrow q$ is $q \rightarrow p$

p	**q**	**$p \rightarrow q$**	**$q \rightarrow p$**
T	T	T	T
T	F	**F**	**T**
F	T	**T**	**F**
F	F	T	T

These two propositions are <u>not</u> logically equivalent

Contrapositive

- The *contrapositive* of the proposition $p \to q$ is $\sim q \to \sim p$.

p	**q**	**$p \to q$**	**$\sim q \to \sim p$**
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

They are logically equivalent.

Implication

- The converse of $p \rightarrow q$ is $q \rightarrow p$.
- The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$
- Examples: $p \rightarrow q$
  - (P)If it snows, (q) the traffic moves slowly.

- The converse: If the traffic moves slowly then it snows. ($q \rightarrow p$)
- The contrapositive: If the traffic does not move slowly, then it does not snow. ($\neg q \rightarrow \neg p$)
- The inverse: If it does not snow the traffic does not move slowly. ($\neg p \rightarrow \neg q$)

Further implication

Ex) Give converse and contrapositive statement

“if it is raining, then I get wet

- converse: if I get wet, then it is raining
- contrapositive: if I don’t get wet, then it is not raining

Ex) Assume p is true, q is false. Find truth value of the following propositions

1) $p \wedge q \rightarrow r$: true
2) $p \vee q \rightarrow \sim r$: false
3) $p \rightarrow (q \rightarrow r)$: true
4) $p \wedge (q \rightarrow r)$: true

Double implication (equivalence)

- The *double implication* “p if and only if q” is defined in symbols as $p \leftrightarrow q$

p	q	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

$p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q)\wedge(q \rightarrow p)$

Tautology & Contradiction

- A proposition is a ***tautology*** if its truth table contains only **true** values for every case
  - Example: $P \vee \neg P$

p	$p \vee (\neg p)$
T	T
F	T

- A proposition is a **Contradiction** if its truth table contains only **false** values for every case
  - Example: $p \wedge \neg p$

p	$p \wedge (\neg p)$
T	F
F	F

De Morgan's laws for logic

- The following pairs of propositions are logically equivalent:
  - $\sim (p \wedge q) \leftrightarrow (\sim p) \vee (\sim q)$
  - $\sim (p \vee q) \leftrightarrow (\sim p) \wedge (\sim q)$

Logical Equivalence

- compound proposition that always have the same truth value called logically equivalent

Ex)

P	¬ P	¬(¬ P)
T	F	T
F	T	F

=> So, proposition “P” is logically equivalent to the proposition “not(not P)”

Logical Equivalences

- Identity Law $P \wedge T \leftrightarrow P$, $P \vee F \leftrightarrow P$
- Domination Law $P \vee T \leftrightarrow T$, $P \wedge F \leftrightarrow F$
- Idempotent $P \vee P \leftrightarrow P$, $P \wedge P \leftrightarrow P$
- Double negation $\neg(\neg P) \leftrightarrow P$,
- Commutative $P \vee Q \leftrightarrow Q \vee P$, $P \wedge Q \leftrightarrow Q \wedge P$
- Associative $(P \vee Q) \vee R \leftrightarrow P \vee (Q \vee R)$
  $(P \wedge Q) \wedge R \leftrightarrow P \wedge (Q \wedge R)$
- Distributive $P \vee (Q \wedge R) \leftrightarrow (P \vee Q) \wedge (P \vee R)$
  $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$
- Demorgan's $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$
  $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$

Proof of Logical Equivalence

- Proving Method 1) with truth table 2) without truth table

Ex) Verify that the propositions **$R = \sim(P \wedge Q)$ and $S = (\sim P) \vee (\sim Q)$** are logically equivalent.

(proof) ➔ **with truth table**

Table for R

P	Q	$P \wedge Q$	R
T	T	T	**F**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Table for S

P	Q	$\neg P$	$\neg Q$	R
T	T	F	F	**F**
T	F	F	T	**T**
F	T	T	F	**T**
F	F	T	T	**T**

Proof of Logical Equivalence

Ex) Show that $\sim(P \vee (\sim P \wedge Q))$ and $\sim P \wedge \sim Q$

(proof) ➔ **without truth table**

$\sim(P \vee (\sim P \wedge Q)) \Leftrightarrow \sim P \wedge \sim(\sim P \wedge Q)$ (by 1st Demorgan's)

$\Leftrightarrow \sim P \wedge [\sim(\sim P) \vee \sim Q]$ (by 2nd Demorgan's)

$\Leftrightarrow \sim P \wedge (P \vee \sim Q)$ (from double negation)

$\Leftrightarrow (\sim P \wedge P) \vee (\sim P \wedge \sim Q)$ (from distributive law)

$\Leftrightarrow \sim P \wedge \sim Q$ (Since, $\sim P \wedge P \Leftrightarrow F$)

Therefore, $\sim(P \vee (\sim P \wedge Q))$ and $\sim P \wedge \sim Q$ are logically equivalent

Proof of Logical Equivalence

- **Example: Show $(p \wedge q) \rightarrow p$ is a tautology**.

- **Proof:** (we must show $(p \wedge q) \rightarrow p \iff T$)

$(p \wedge q) \rightarrow p \iff \neg(p \wedge q) \vee p$ (from logical inference rules)

$\iff [\neg p \vee \neg q] \vee p$ by DeMorgan's rule

$\iff [\neg q \vee \neg p] \vee p$ by Commutative rule

$\iff \neg q \vee [\neg p \vee p]$ by Associative rule

$\iff \neg q \vee [T]$ by tautology

$\iff T$ by Domination rule

- Proof by using truth tables

P	Q	$P \wedge Q$	$(P \wedge Q) \rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Application: inference

- **Assume we know the following sentences are true:**

If you are older than 13 or you are with your parents then you can attend a PG-13 movie. You are older than 13. (– A= you are older than 13. – B= you are with your parents. – C=you can attend a PG-13 movie)

- **$(A \vee B \rightarrow C) \wedge A$ is true**

- **With the help of the logic, we can infer the following statement (proposition):**

– **Y**ou can attend a PG-13 movie or C **is true**

- **The field of Artificial Intelligence:**

- Builds programs that act intelligently. • Programs often rely on symbolic manipulations

- **Expert systems:**

- Encode knowledge about the world in logic. • Support inferences where new facts are inferred from existing facts following the semantics of logic

- **Theorem provers:**

- Encode existing knowledge (e.g. about math) using logic. • Show some hypothesis is true