Chapter 1. Logic and Proofs

- Logic

  - Propositional Logic

  - **Predicate Logic**

- Proofs
  - Direct proof
  - Indirect proof
  - Mathematical Induction
  - Resolution Proof

Limitations of the Propositional Logic

- Objects and properties are hidden in the statement, it is not possible to reason about them ex) Kim(object) is a student (property)

(1) Statements that must be repeated for many objects

– In propositional logic these must be exhaustively enumerated

• **Example:**

– If KIM is a CS KMU graduate then KIM has passed cs231

– If Ahn is a CS KMU graduate then Ahn has passed cs231

– If LEE is a CS KMU graduate then LEE has passed cs231

– **…** => **What is a more natural solution to express the above knowledge?**

- **Solution:** make statements with **variables**

– If "X" is a CS KMU graduate then "**X**" has passed cs231

Limitations of the Propositional Logic

(2) Statements that define the property of the group of objects

- **Example:**

– All new cars must be registered.

– Some of the CS graduates graduate with honors.

- **Solution:** make statements with **quantifiers**

– **Universal quantifier** –the property is satisfied by all members of the group

– **Existential quantifier –** at least one member of the group satisfy the property

- **Predicate Logic**

Remedies the limitations of the propositional logic

- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

2. Predicate Logic (술어논리)

- **Predicate Logic** is an extension of **Propositional logic** that permits reasoning about whole classes of entities.

- **Basic building blocks of the predicate logic:**

  1) constant (models of specific object ex. Kim, 2, ..) 2) Variable

  3) Predicate (predicate 란 한 객체의 성질이나, 객체와 객체간의 관계를 표현)

  ex) red(car12), student(x), married(Kim, Lee)

- Statements involving variables.

  ex) **X** is **greater than** 3

  variable predicate (객체의 성질, 객체간의 관계)

- Propositional logic does not express every statements.

  ex) P: “**x**” is an odd integer (P is either true/false, -it depends on the value of **x)**

Predicate logic (con't)

- 표현=> P(X) : (P → predicate, X → individual variable)

- P $(X_1, X_2, X_3, \ldots, X_n)$: (X_i = 개별변수, P는 n개의 인수를 갖는 predicate 이다. **Ex) P(X): X>3 what are the truth value of P(4), P(2) ?**

- Let P(n) be the statement "n is an odd integer" then P is a ***propositional function*** with domain D since for each **n** in D, - P(n) is a proposition, either true or false

  if n = 1, 1 is an odd integer => true

  if n = 2, 2 is an odd integer => false ………………

Ex) Assume a predicate P(X) that represents the statement: X is a prime number

P(2) T, P(3) T, P(4) F, P(5) T, P(6) F, P(7) T → propositions

Ex) Q(X,Y) is statement, X=Y+3, **what is Q(1,2), Q(3,0)?**

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

- **Example: • Older(John, Peter) denotes 'John is older than Peter'**
  – this is a proposition because it is either true or false

• **Older(x,y) -> 'x is older than y'**
  – not a proposition, but after the substitution, it becomes one

Ex) **Let Q(x,y) denote 'x+5 >y'**

– Is Q(x,y) a proposition? **No!** – Is Q(3,7) a proposition? **Yes.** It is true.

– What is the truth value of: – Q(3,7) T, – Q(1,6) F – Q(2,2) T

– Is Q(3,y) a proposition? **No!** We cannot say if it is true or false.

Compound statements in predicate logic

- **Compound statements are obtained via <u>logical connectives</u>**

Ex) Student(Ann) $\wedge$ Student(Jane)

- **Translation:** "Both Ann and Jane are students" • **Proposition:** yes.

Ex) Country(Sienna) v River(Sienna)

- **Translation:** "Sienna is a country or a river" • **Proposition:** yes.

- **Important:** • statement P(x) is **not a proposition** since there are more objects it can be applied to.

  • predicate logic allows us to explicitly manipulate and substitute for the objects

  • Predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

2.1 Quantifier (한정자)

- Propositional function으로부터 proposition을 만드는 방법은 **값을 할당하거나**, **한정자를 사용**하는 것이다.

- 하나의 predicate를 proposition 으로 바꾸기 위해서는 한정자를 사용하여 predicate 의 개별 변수를 한정 시켜야 한다.

- Two types of quantified statements (한정자):

  1)전체 수량자 (universal quantifier : $\forall$)

  Ex) ‘ all CS KMU graduates have to pass cs441”

  – the statement is true for all graduates

  2) 존재 수량자 (existential quantifier: $\exists$)

  Ex) ‘Some CS KMU students graduate with honor.’

  – the statement is true for some people

Universal quantifier (전체 수량자: ∀)

- One can write **P(x) *for every* x in a domain D**
  - In symbols**: ∀x P(x)**
  - (변수 x가 가질수 있는 모든 값에 대하여 p(x) 는 참이다)

- ∀ is called the ***universal quantifier****, and means*
  'for every, or for any"

Ex) Let P(x) : x > x - 1. What is the truth value of ∀ x P(x)?
Assume domain D of X is all real numbers

Ex) What is the truth value of ∀x ,P(x), where P(X) is the statement "X^2 < 10" and with the **positive integers** are not exceeding 4

Existential quantifier

- *For some* x ∈ D, P(x) is true if *there exists* an element x in the domain D for which P(x) is true. In symbols: **∃x, P(x)**
(어떤 x에 대하여 p(x)가 참인 x가 존재한다)

- Statements are true for some values of variables
  "for some x, P(x)", "for at least one x, P(x)",

- The symbol "∃" is called the *existential quantifier.*

Ex) Let T(x) denote x > 5 (and x is from Real numbers)

- What is the truth value of ∃ x T(x)?
- **Answer:** Since 10 > 5 is true. Therefore, it is **true that ∃ x T(x).**

Existential quantifier (con't)

ex) Let P(X) denote the statement "x>3".

What is the truth value of the quantification x, P(x) where the domain is the set of real numbers.

ex) What is the truth value of x, P(x) where P(X) is the statement "X^2 >10" and the domain is positive integers not exceeding 4.

ex) verify that "for every real number x, x^2>= 0"

ex) Q(x): x = x + 2 (x is real numbers) What is the truth value of ∃ x Q(x)?

- **Answer:** Since no real number is 2 larger than itself,

the truth value of ∃ **x Q(x) is false**.

Existential quantifier (con't)

ex) verify that "for every real number x, if $x>1$, then $x+1>1$"

(sol) since every real number either $x<=1$ or $x>1$

1) if $x>1$ is true

2) if $x>1$ is false

3) Therefore, if $x>1$ and $x+1>1$ is true and "for every real number x, if $x>1$ then $x+1>1$" is also true

Counter example

- The universal statement $\forall x\, P(x)$ is false if $\exists x \in D$ such that $P(x)$ is false.

(“for every x , P(x)” 에서는 만약 하나의 x 에 대해서 P(x) 가 false라면, 전체적으로 false 가 된다.)

- The value x that makes P(x) false is called a *counter example* to the statement $\forall x\, P(x)$.

Ex) P(x) = "every x is a prime number", for every integer x.

But if x = 4 (an integer), this x is not a primer number.

Then 4 is a counterexample to P(x) being true.

Counterexample

예) P(x): $x=x^2$, what is truth value of the following propositions

- for every x, P(x)

\- False 증명 위해=>단 한개의 counter example 찾음

\- True 증명 위해=> domain의 모든 원소가 true임을 증명

- 반대로, “for some x in D, P(x)” is true if P(x) is true for at least one x in D.

ex) for some real number x, $x/ (x^2 + 1) = 2/5$

Generalized De Morgan's laws for Logic

- If P(x) is a propositional function, then each pair of propositions in a) and b) below have the same truth values:

a) $\sim(\forall x \ P(x))$ and $\exists x: \sim P(x)$

"It is not true that for every x, P(x) holds" is equivalent to "There exists an x for which P(x) is not true“

b) $\sim(\exists x \ P(x))$ and $\forall x: \sim P(x)$

"It is not true that there exists an x for which P(x) is true" is equivalent to "For all x, P(x) is not true"

Summary of propositional logic

- In order to prove the universally quantified statement $\forall x\, P(x)$ is <u>true</u>
  - It is not enough to show $P(x)$ true for some $x \in D$
  - You must show $P(x)$ is true <u>for every</u> $x \in D$

- In order to prove the universally quantified statement $\forall x\, P(x)$ is <u>false</u>
  - It is enough to exhibit <u>some</u> $x \in D$ for which $P(x)$ is false
  - This x is called the **counterexample** to the statement $\forall x\, P(x)$ is true

3. Proofs

- Proof:

  An argument that establishes the truth of a theorem (sequence of argument/statements previously proved theorems, assumptions, hypothesis)

- Proof methods:

  1) Direct Proof

  2) Indirect Proof

  3) Resolution Proof

  4) Mathematical Induction

Definitions

- A ***definition*** is a proposition constructed from undefined terms and previously accepted concepts in order to create a new concept.

  - Ex) In Euclidean geometry the following are definitions:

- Two triangles are *congruent* if their vertices can be paired so that the corresponding sides are equal and so are the corresponding angles.

- Two angles are *supplementary* if the sum of their measures is 180 degrees.

Axioms

- An ***axiom*** is a proposition accepted as true without proof within the mathematical system.

- There are many examples of axioms in mathematics:
  - Ex) In Euclidean geometry the followings are axioms
    - Given two distinct points, there is exactly one line that contains them.
    - Given a line and a point not on the line, there is exactly one line through the point which is parallel to the line.

Theorems

- A ***theorem*** is a proposition of the form $p \rightarrow q$ which must be shown to be true by a sequence of logical steps that assume that p is true, and use definitions, axioms and previously proven theorems.

3.1 Direct Proof

- *Direct* proof: $p \rightarrow q$
  - Proving that a statement is true by a direct argument
  - Proving that a statement is false by finding a counterexample

- if P true then Q must be true

(“P”가 참이라고 하면, “P=> Q”가 참이기 위해서는 “Q”도 참 이어야 한다)

  - Assumes that $P(x_1,x_2,..)$ is true, and then using $P(x_1,x_2,..)$ as well as other axioms, definitions, other theorems, show directly that $Q(x_1,x_2,...)$ is true.

Example

- Prove that “If N odd, then N2 is odd”

- Prove that “If integers m and n are multiples of 3, then m+n is a multiple of 3”

- Prove that “For all real numbers d, d_1,d_2, x, If d= min{d_1, d_2} and x $\le$d, then x$\le d_1$ and x$\le d_2$

3.2 Indirect proof

- The method of ***proof by contradiction*** of a theorem $p \rightarrow q$ consists of the following steps:
    1. Assume p is true and q is false
    2. Show ~q results in ~p
    3. Then we have that p ^ (~p)
    4. But this is impossible, since the statement p ^ (~p) is always false. There is a contradiction!
    5. Therefore p->q is true.

- Direct Proof 와 contradiction Proof 의 차이점 => assume negated conclusion (~Q)

Example

- Prove that for all real numbers x and y, if $x+y>=2$, then either x>=1 or y>=1.
- Prove that "If $x^2 +x-2=0$, then $x\neq0$. "
- Prove that "Let n be a positive integer. Give a proof by contradiction that if n is a prime number different than 2, then n is odd."
- Prove that "if 3n+2 odd, then n odd"

Examples

- Sum of two odd integer is even

  --------------- -----

  P Q

  1) direct proof

  2) indirect proof (proof by contradiction)

3.3 Mathematical Induction

- Principles:

Let P(n) be a proposition that is valid for $n \geq k$, n,k integers

1) Basis Step: verify P(k) is true for “1” or initial value “k”,

2) Induction Step: Assume P(n) is true,

then prove $P(n) \rightarrow P(n+1)$

3) Conclusion: Therefore, P(n) is true $\forall$ positive

integers $n \geq k$.

Mathematical induction: examples

- Prove, by means of Math. Induction, that $\forall n \geq 1$,

  $1+2+3+.....+n= n(n+1)/2$ …(2.1)

(sol) hypothesis => LHS

Basis Step : P(1), $1= 1(1+1)/2$

Induction step : assume (2.1) holds for some n.

We need to show that, $1+2+3+....+n+(n+1) = (n+1)(n+2)/2$

LHS is,

$n(n+1)/2 +(n+1) = (n(n+1) +2(n+1))/2 = (n+1)(n+2)/2$ ==> RHS

즉,P(n) 이 참이면, P(n+1) 도 참이된다.

Therefore, we conclude that P(n) is true $\forall n \geq 1$

Example

- Prove 5^n-1 is divisible by 4 for all $n \geq 1$
- Prove "Sum first n odd integers is n^2"
- Prove that "$6.7^n - 2.3^n$ is divisible by 4"
- Prove that "$2n+1 \leq 2^n$ for $n=3,4,5\ldots$ "

Mathematical Induction 의 응용

- **Inductive Definition (Recursive Function) 수학적 재귀법**

1) The first element of the sequence is defined.

2) And then the n^{th} element is defined in terms of preceding elements (ex. factorial) (Assume P(n) is true => Prove P(n+1))

```
Function fac(n)
  if n=1  then  a = 1                /* basis step */
    else        a = n*fac(n-1)   /* induction step */
  endif
  return(a)
end
```

3.4 Resolution 증명

- ***Deductive reasoning***: the process of reaching a conclusion q from a sequence of propositions $p_1, p_2, \ldots, p_n$.
- The propositions $p_1, p_2, \ldots, p_n$ are called *premises* or *hypothesis*.
- The proposition q that is logically obtained through the process is called the *conclusion*.

Rules of inference

1\. ***modus ponens***

- $p \rightarrow q$
- p
- Therefore, q

2. ***Modus tollens***

- $p \rightarrow q$
- ~q
- Therefore, ~p

Resolution (Proof technique by Robinson, 1965)

- Resolution Rule: If $p \vee q$ and $\neg q \vee r$ are both true, then $p \vee r$ is true
- Hypotheses and conclusion: written as CLAUSES

  (a CLAUSE => terms separated by OR's)
- Clause Form 변환 (OR 형태로 변환)

1) $p \rightarrow q \Rightarrow \neg p \vee q$ 2) pq => $p \wedge q$

ex) $a \vee b \vee \neg c$ => clause

$xy \vee z$ => not clause, since xy is two variables

p -> q => not clause, since '->'

Resolution Examples

ex)

1. $a \vee b$
2. $\neg a \vee c$
3. $\neg c \vee d$

con: $b \vee d$

(Proof)

(1+2): 4. $b \vee c$

(3+4): 5. $b \vee d$,

Therefore, we proved the conclusion $b \vee d$

Ex)

1. a
2. $\neg a \vee c$
3. $\neg c \vee d$

con: d

(proof)

4. (1+2) = c
5. (3+4)=d,

which is conclusion

LOGIC 응용분야: 지식베이스 시스템

- **인공지능(Artificial intelligence)** – 인간과 같이 사고할 수 있는 컴퓨터 프로그램.
- **인공 지능 분야** – 로보틱스(robotics), 게임 놀이(game playing), 문제 해결(problem solving), 자연어 처리(natural language processing) 및 패턴 인식(pattern matching) 등.
- **전문가 시스템** – 전문가 시스템은 사실들과 추론 규칙, 추론 기관(inference engine) 등으로 구성되어 있다. 추론 기관은 전문가 시스템에게 문제를 풀 수 있는 능력을 부여한다.