Chapter 2. Sets

- **1. Sets**

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3. Functions

- 3.1 Definitions
- 3.2 Properties
- 3.3 Special Functions

3. Functions

3.1 Definitions

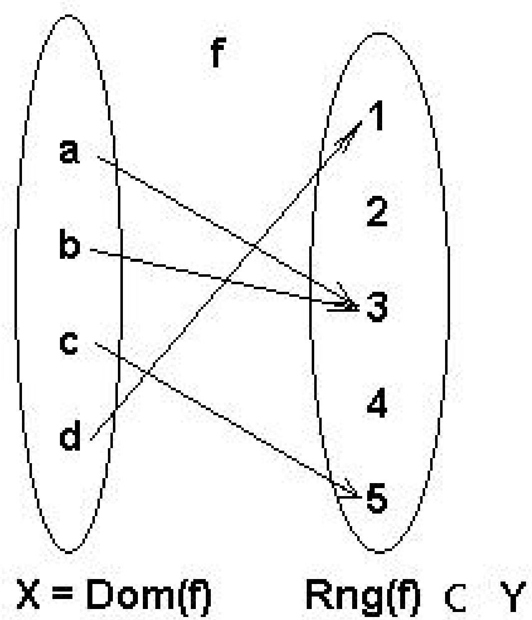
- A *function* f, from X to Y is a subset of Cartesian Product X x Y, satisfying the next two conditions (f: X-> Y)

1) $\forall x \in X, \exists y \in Y$ such that $(x,y) \in f$
2) $\forall x \in X, \exists y,z \in B$, (if $(x,y), (x,z) \in F$, then $y=z$)

- X is Domain, Y is Co-domain
- Range: 함수값 f(x)로 이루어진 B의 부분집합
  (range $\subseteq$ codomain)

Domain and Range

- *Domain* of f = X
- *Range* of f = $\{ y \mid y = f(x), \text{ for some } x \in X\}$
- A function $f : X \rightarrow Y$ assigns to each x in Dom(f) = X a unique element y in Range(f) $\subseteq$ Y.



Ex) Dom(f) = X = {a, b, c, d},
Range(f) = {1, 3, 5}

f(a) = f(b) = 3,
f(c) = 5, f(d) = 1.

3.2 Properties

- **Injective function (One-to-one, Injection)**

- A function $f : X \to Y$ is ***one-to-one*** $\Leftrightarrow$
  $\forall y \in Y$ there exists at most one $x \in X$ with $f(x) = y$.
- Alternative definition:
  $f : X \to Y$ is *one-to-one* $\Leftrightarrow$ for each pair of distinct elements
  $x_1, x_2 \in X$ there exist two distinct elements $y_1, y_2 \in Y$
  such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Ex) The function $f : R \to R$ defined by $\mathbf{f(x) = x^2}$ is <u>not</u> 1-to-1,
since for every real number x, $f(x) = f(-x)$.

Ex) $f=\{(1,b)(3,a)(2,c)\}$ $X=\{1, 2,3\}$ $Y=\{a,b,c,d\}$

Ex) $f=\{(1,a)(2,b)(3,a)\}$ $X=\{1,2,3\}$ $Y=\{a,b\}$

Surjective Function (Surjection, Onto)

A function $f : X \rightarrow Y$ is ***onto*** $\Leftrightarrow$

for each $y \in Y$ there exists at least one $x \in X$ with $f(x) = y$, i.e. Range(f) = Y.

- Ex: $f=\{(1,b)(2,a)(3,d)\}$, $X=\{1,2,3\}$ $Y=\{a,b,c,d\}$

sol) not ONTO

- Ex) $f=\{(1,a)(2,a)(3,b)(4,c)\}$, $X=\{1,2,3,4\}$ $Y=\{a,b,c\}$

sol) ONTO

Bijective function (Bijection)

A function f : X$\rightarrow$ Y is bijective $\Leftrightarrow$ f is one-to-one and onto

Ex) **f={(1,a)(2,c)(3,b)}, X={1,2,3}, Y={a,b,c}**

Ex) f: Z $\rightarrow$Z, f(n) = 2n+1,

g: Z $\rightarrow$Z, g(n) = n^2 $\forall n \in Z$ (set of all integers)

- Is f one-to-one?

(sol) Yes. Suppose n_1 and n_2 are integers, such that $f(n_1)= f(n_2)$ for some integers. We need to show that $n_1=n_2$. Then by def of f, $2n_1+1 = 2n_2+1 \Rightarrow n_1=n_2$. Therefore f is one-to-one

- Is g one-to-one?

More examples

Ex) Define h: $R \to R$, $h(x) = 4x-1$ $\forall x \in R$ (set of all real numbers)

f: $Z \to Z$, $f(x) = 4x-1$ $\forall x \in Z$

a) Is h onto? b) Is f onto?

Ex) Define f: $Z \to 3Z$ by $f(n) = 3n$, $\forall n \in Z$

Show f is bijection

Functions and Cardinality

- Thm:

  Let A and B be finite sets

  Let f:A->B be a function

- If f is an injection, then |A| <= |B|
- If f is an surjection, then |A| >= |B|
- If f is bijection, then |A| =|B|

3.3 Special Functions

* Inverse Function

- f:A->B is **bijection**, then there is a function f^{-1}: B->A

  f^{-1} is the set $\{(y, x) \mid y = f(x)\}$. **f^{-1} => inverse function for f**

Ex) Let A={1,2,3,4} B={a,b,c,d} f={(1,a)(2,a)(3,d)(4,c)}

Then $f^{-1} = \{(a, 1)(a, 2)(d, 3)(c, 4)\}$

- The f^{-1} is not a function from B to A, **since $f^{-1}(a) = \{1,2\}$**

- When f:A->B is a function, if f^{-1} : B-> A is a function then f is a bijection

ex) f:R->R, $f(x)=x^2$, is f inverse function?

Composition of functions(1)

- Def: f: X->Y, g:Y->Z with property that the co-domain of f is a subset of domain of g.

  $\forall x \in X$, $(g\bullet f)(x) = g(f(x))$, "g of f of x"

  → Function **g•f** is called **composition of 'f and g'**

Ex) Let A,B,C=Z. Let f:A->B, g:B->C be defined by

$f(a)=a+1$, $g(b)=2b$, Find g•f.

(sol) $(g\bullet f)(a)=g(f(a))=g(a+1)=2(a+1)$

Ex) $f(n)=n+1$(successor function), $g(n) = n^2$ (squaring function)

1) What is g•f? 2) What is f•g? 3) $g\bullet f = f\bullet g$?

Composition of functions(2)

Ex) X={1,2,3} Y={p, q} Z={a, b}

F: X->Y F={(1, p)(2, p)(3, q)}

G: Y->Z G={(p, b)(q, b)} what is g•f?

- (sol)

(g•f)(1)=g(f(1))=g(p)=b

(g•f)(2)=g(f(2))=g(p)=b

(g•f)(3)=g(f(3))=g(q)=b ➔ g•f= {(1 b)(2 b)(3 b)}

- Composition of functions is <u>associative</u>: $f \circ (g \circ h) = (f \circ g) \circ h$,
- But, in general, it is <u>not</u> <u>commutative</u>: $f \circ g \neq g \circ f$.

Special functions

- **특성 함수(characteristic function)**
  - 전체 집합 U에 속한 어떤 집합 A에 대해서 다음과 같은 성질을 만족할 때 이 함수를 A의 특성 함수(characteristic function)라 하고 f_A로 쓴다.
  - $\mathbf{f_A(x)=}$ $\begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$

  - **예)** 집합 $U = \{a, b, c\}$이고 $A = \{a\}$라 하면

    $f_A(a) = 1, \quad f_A(b) = 0, \quad f_A(c) = 0$ 이다.

특수함수

- **바닥함수(floor function)**
  - $x \in R$ 에 대해 x 와 같거나 x 보다 작은 수 중에서 x 에 가장 가까운 정수를 대응시키는 함수
  - $\lfloor x \rfloor$ 와 같이 나타냄

  최대정수함수(greatest integer function)
  - Ex) $\lfloor -3.4 \rfloor = -4$
- **천정함수(ceiling function)**
  - x 와 같거나 x 보다 큰 수 중에서 x 에 가장 가까운 정수를 대응시키는 함수
  - $\lceil x \rceil$ 와 같이 나타냄
  - 최소정수함수(least integer function)
  - ex) $\lceil \pi \rceil = 4$

4. Posets - Partial Ordered Set

- Def: A relation R on a set S is called a **partial order** on S if R is **Reflexive, Anti-symmetric and Transitive**

→ means not every pair of element of S must be related

ex) The relation of **set inclusion, '$\subseteq$'** is a partial order on S.

(sol) for $A \subseteq A$ for any sets in S => reflexive

if $A \subseteq B$ and $B \subseteq A$ then $A=B$ => antisymmetric

if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ => transitive

Ex) Consider the positive integers N. We say that "a divides b, written $a \mid b$, if there is an integer "c", such that $ac=b$.

(sol) For example, $2|4$, $3|12$, $7|21$,...

This relation of divisibility is a partial order on N

Poset notation

1) $a \prec b$ is read "a strictly precedes b" (a comes before b)

2) $a \preccurlyeq b$ is read "a precedes b"

3) $b \succ a$ is read " b strictly succeeds a,

4) $b \succcurlyeq a$ is read " b succeeds a"

- Comparable: Two elements a, b in Poset are **<u>Comparable</u>**

  if either $a \preccurlyeq b$ **or** $b \succcurlyeq a$,

If $x \not\preccurlyeq y$ and $y \not\succcurlyeq x$, INCOMPARABLE / NONCOMPARABLE

(ex. 3 divides 5 is incomparable since neither divides the other)

- If every pair of elements in Poset A are called **Comparabl**e then A is said to be **<u>TOTALLY ORDERED</u>** or **<u>LINEARLY ORDERED</u>**

exercise

ex) Positive integers N with “≤” is linearly ordered set.

예: 집합 A={1,2,3,4,5,6} 에서의 수의 대소관계는 comparable 하기 때문에

예) 양의 정수 N상에서의 나눗셈의 관계는 comparable한 관계도 존재하지만 (예: 3과 6), incomparable (예: 2와 3)한 관계도 존재하기 때문에 Total Order는 아니다.

Diagram of POSET (Hasse Diagram)

- Diagram of Poset S

  - directed graph (arrows always UPWARD)

  - vertices are the elements of S

  - there is an arc from a to b if $a \preccurlyeq b$ in S.

- Procedure:

  1) Eliminate Loops

  2) Eliminate transitive edges

  3) Initial vertex below of terminal vertex

  4) Remove all arrows

(Hasse Diagram)

ex) A = {1,2,3,4,12} be ordered by the relation "x divides y".

Draw Diagram for $(A, \preccurlyeq)$.

ex) A partition of positive integer 'm' is a set of positive integers whose sum is 'm'. For example, there are 7 partitions of m=5. (Partition: 5, 3-2, 2-2-1, 1-1-1-1-1, 4-1, 3-1-1, 2-1-1-1)

Maximal and Minimal Element

- Element is MAXIMAL (극대원소) - if no element succeeds in a POSET (if no a∈A above in HASSE diagram)

- Element is MINIMAL (극소원소) - if no element precedes b in a POSET.
- There can be more than one maximal and minimal

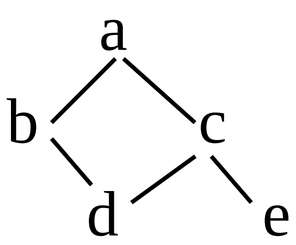
예) 집합 A에서 나눗셈(Divisibility)의 관계를 구하고, 극대원소와 극소원소를 찾으시오. 집합 A = {1 , 2,3,4, 6, 8, 9, 12 , 18, 24}

예) 정수의 집합에서 극대원소와 극소원소를 찾으시오.

Z = {...-2, -1, 0, 1, 2,...}

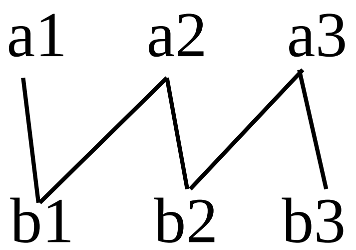
con’t

Ex)



Minimal = ?   Maximal = ?

Ex)



Minimal =?   Maximal = ?

Supremum and Infimum

* A is POSET and B is subset of A,

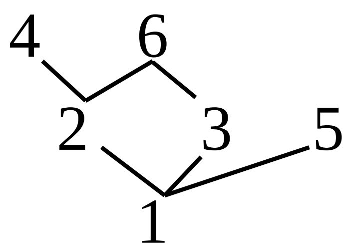
- a∈A가, ∀b∈B에 대하여, b≼a 이면, a는 B 의 UPPER BOUND (상계)
- a∈A가, ∀b∈B에 대하여, a≼b 이면, a는 B 의 LOWER BOUIND (하계)

- a∈A 가 B 의 upper bound 이고, a≼a' (a' 은 B 의 upper bound) 이면, a를 B 의 상한 (Least Upper Bound: SUPREMUM) 이라 한다. (LUB(B) i.e. 상한은 상계중 가장 작은것)

- \- a∈A 가 B 의 lower bound 이고, a≼a' (a' 은 B 의 lower bound) 이면, a 를 B 의 하한 (Greatest Lower Bound: INFIMUM)이라 한다. (GLB(B) i.e. 하한은 하계중 가장 큰것)

examples

Ex) A = {1,2,3,4,5,6}   Relation R = “|”



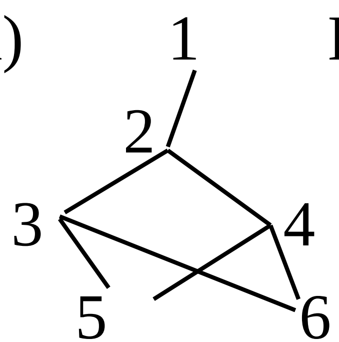
1) B = {2,3}   UB= 6,   LUB(B) = 6

LB = 1,   GLB(B) = 1

2) B={4,6}   UB = X   LB(B)= {1,2} GLB(B)=2

3) B={3,6}   UB = 6, LUB=6, LB(B)= 1,3, GLB(B)=3

Ex)

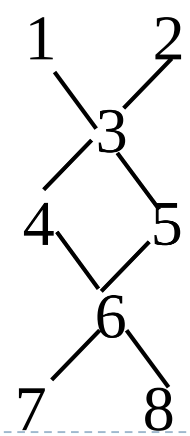


B = {2,3,4}

UB=1,2 LUB=2

LB=5,6   GLB=X

ex)



B={4,5}

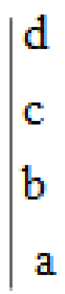
UB=1,2,3 LUB=3

LB=6,7,8 GLB=6

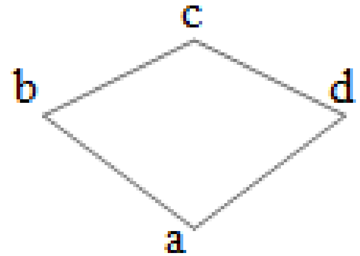
Lattice (격자)

- 정의: Poset (L, $\preccurlyeq$) 에서, 임의의 a,b $\in$L 에 대하여, 두개로 이루어진 모든 집합 {a,b}의 상한(LUB), 하한(GLB)이 하나씩 존재하면, 이를 LATTICE 라 한다

- LUB(a,b) 를 a$\vee$b 로 표기 => a 와 b의 JOIN 이라 한다
- GLB(a,b) 를 a$\wedge$b 로 표기 => a 와 b의 MEET 라 한다



(LATTICE)



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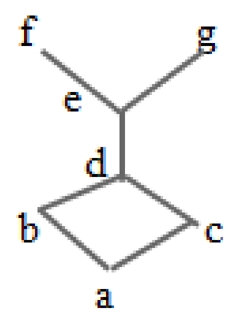
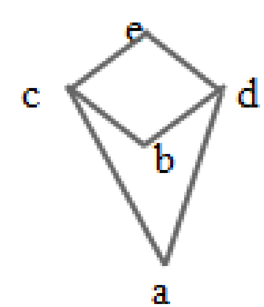


(f$\vee$g) 존재 않음



(c$\wedge$d) 존재 않음