Problem Guide for *Fifty Challenging Problems in Probability with Solutions* by Mosteller

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About

> *"A man must love a thing very much if he not only practices it without any hope of fame or money, but even practices it without any hope of doing it well."* - G.K. Chesterton (Maybe)

Almost certainly my most useless set of notes, as the title of this book says *with Solutions*.

1 The Sock Drawer

Let the number of socks in the drawer be n, and let the number of red socks be k. Then we are given that $\binom{k}{2}/\binom{n}{2}$ is $\frac{1}{2}$. Thus, we can constrain our possibilities as follows

$$\begin{aligned}\binom{k}{2}+\binom{k}{1}\binom{n-k}{1}+\binom{n-k}{2}&=\binom{n}{2}\\ \binom{k}{1}\binom{n-k}{1}+\binom{n-k}{2}&=\frac{1}{2}\binom{n}{2}\\ n(n-1)&=2k(k-1)\end{aligned}$$

From the last equation, we have that $n=4, k=3$ is one solution. Additionally, we can note that $21*20=3*7*5*2^2=2*15*14$ to get a solution with an even number of black socks. Beyond that, I'll have to get back to you; I haven't started reading number theory books yet.

2 Successive Wins

Let the probability that Elmer beats his dad be p and let the probability that he beats the champion be q. We assume that these two probabilities are independent (which may not be safe, hot streaks are a thing). We are given that $p>q$. The probability that Elmer wins the prize in the dad-champ-dad series is $p^2q+2pq(1-p)=2pq-p^2q$ (either Elmer wins all 3, the first 2, or the last 2). Similarly, the probability for the other series is $2pq-pq^2$. Since $p^2q>pq^2$ from our assumption that $p>q$, Elmer should choose the champ-dad-champ series.

3 The Flippant Juror

We see that the probability that the three-man jury succeeds is $p(1-p)+p^2=p$, since either both non-coin-flipping jurors pick correctly or one of them picks incorrectly but the coin flip is correct. Thus, the one-man and three-man juries both have the same probability of being correct.

4 Trials until First Success

This question is the same as asking for the expected value of a geometric random variable with $p=\frac{1}{6}$, so the answer is 6.

5 Coin in Square

Consider a table consisting of just a single 1-by-1 square. The probability that the coin lands entirely within this square is the same as the probability of picking a point (x, y) with $\frac{3}{8} \leq x, y \leq \frac{5}{8}$ (assuming the position of the center of the coin is uniformly distributed). This is because we can consider the square with opposite corners $(0, 0)$ and $(1, 1)$ and note that the coin is only within this square if its center is at least $\frac{3}{8}$ (its radius) away from each of the boundaries. Since these constraints form a square with sidelength $\frac{1}{4}$, the probability the coin lands in a single 1-by-1 square is $\frac{1}{16}$. Since the squares are all 1-by-1, the total area of the table scales exactly as the probability, so the number of squares does not matter. Thus, the probability is just $\frac{1}{16}$.

6 Chuck-a-Luck

The player loses with probability $\left(\frac{5}{6}\right)^3$ and wins with the complement of that probability. When the player wins, their expected gain is $\mathrm{E}[X]$, where X is binomially distributed with $n = 3$ and $p = \frac{1}{6}$. Thus, the total change in the player's stake is $-\left(\frac{5}{6}\right)^3 + \frac{1}{2} = -\frac{17}{216} \approx -0.079$ (where we used the fact that $\mathrm{E}[X] = np$).

7 Curing the Compulsive Gambler

The cure is crushing Kind Friend's bank account; for Mr. Brown to not be ahead after 36 plays, he cannot win on even a single play (since then the worst he could do is lose the 35 other plays, which would bring him to net 0). The probability that Mr. Brown loses all 36 plays is $\left(\frac{37}{38}\right)^{36} \approx 0.383$, so Kind Friend is paying him quite often.

8 Perfect Bridge Hand

There are $\binom{52}{13}$ possible bridge hands, of which only 4 consist of 13 cards of the same suit. Thus, assuming each hand is equally likely, the chance of getting such a hand is basically 0 ($\approx 6.299 * 10^{-12}$).