“Sparknotes” for *Principles of Mathematical Analysis* by Walter Rudin

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About

> *"A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details."*
> - Hermann Weyl

These notes contain short summaries of (my) proof ideas for exercises and some theorems from the book *Principles of Mathematical Analysis* by Walter Rudin. I have tried to make the summaries as brief as possible, sometimes only one line or one equation. My hope is that the summaries will give enough information to reconstruct a full proof without bogging the reader down with details. In many cases, I am sure that I inadvertently sacrificed clarity in an attempt to obtain brevity, and would greatly appreciate any feedback.

Also, I like when people include (what they presume to be) relevant quotes in their notes, so I have to ask you to forgive my haughtiness in starting these notes with a quote from Hermann Weyl.

1 The Real and Complex Number Systems

1.1 Exercise 1

If $rx = q$ or $r + x = q$ for some rational q, then substracting r from q or dividing q by r yields x rational, which is a contradiction.

1.2 Exercise 2

We can first show that $\sqrt{3}$ is irrational by seeing that $\frac{a^2}{b^2} = 3 \implies 3|a, 3|b$. Then, since $12 = 3 * 2^2$, we have that $\sqrt{12}$ is irrational as well.

1.3 Exercise 4

If $\alpha > \beta$ then α would be an upper bound as well.

1.4 Exercise 5

$\forall x \in A$, $-x \leq \sup -A$ and $\forall \epsilon \in \mathbb{R}$, $\exists x \in A \,|\, \sup -A + \epsilon < -x \leq \sup -A$. Negating the last inequality gives $\inf A = -\sup -A$.

1.5 Exercise 6

(a) Follows from $m = \frac{np}{q}$.

(b) Put $r = \frac{m}{n}$, $s = \frac{p}{q}$. Then $b^r b^s = b^{\frac{mq}{nq}} b^{\frac{np}{nq}}$. Pulling out $\frac{1}{nq}$ gives the desired result.

(c) b^r is an upper bound since $b > 1$, and if it were not the supremum we could choose $t < r$ such that $b^t > b^r$. This is not possible since again, $b > 1$.

(d) Every element in $B(x + y)$ can be expressed as $b^{s+t} = b^s b^t$ $s \leq x$, $t \leq y$. If $\sup B(x + y) = \alpha < \sup B(x) \sup B(y)$, then $b^s b^t \leq \alpha \implies B(x) \leq \alpha b^{-t} \implies B(y) \leq \frac{\alpha}{B(x)} \implies B(x)B(y) \leq \alpha$.

1.6 Exercise 7

(a) $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + ... + 1) \geq n(b - 1)$ since $b > 1$.

(b) Plug $b^{\frac{1}{n}}$ into (a).

(c) Plug $n > \frac{b-1}{t-1}$ into (b).

(d) Using (c) gives that we can choose n such that $b^{\frac{1}{n}} < y\dot{b}^{-w} \implies b^{w+\frac{1}{n}} < y$.

(e) We can take the reciprocal of (c) and do the same as in (d).

(f) If $b^x > y$ we can apply (e) for a contradiction, if $b^x < y$ we can apply (d) for a contradiction.

(g) Supremum is unique.

1.7 Exercise 8

Suppose $(0,1) < (0,0)$. Then $(0,-1) < (0,0)$ after multiplying by $(0,1)$ twice yields a contradiction. Similarly, assuming the opposite yields $(-1,0) > (0,0)$.

1.8 Exercise 9

Does exhibit least upper-bound property since you can take $(\sup a_i, \sup b_i)$.

1.9 Exercise 10

Exception is 0.

1.10 Exercise 11

Take $w = \frac{1}{|z|}z$ and $r = |z|$ when $|z| \neq 0$. w and r are not uniquely determined; take $z = 0$ for example.

1.11 Exercise 12

By strong induction:

$$\begin{aligned} |z_1 + ... + z_{n+1}| &\leq |z_1 + ... + z_n| + |z_{n+1}| \\ &\leq |z_1| + ... + |z_{n+1}| \end{aligned}$$

1.12 Exercise 13

$$\begin{aligned} |x - y|^2 &= x\bar{x} - 2|x||y| + y\bar{y} \\ &\geq (|x| - |y|)^2 \end{aligned}$$

2 Numerical Sequences and Series

Definition 3.5

Since $\{p_n\} \to p \implies \forall \epsilon, \exists N | n \geq N \implies |p_n - p| < \epsilon$, we can choose $k | n_k \geq N \implies \{p_{n_k}\} \to p$. The reverse direction can be shown via contradiction of $\{p_n\} \to p$.

Examples 3.18

(a) Density of rationals in reals.

(b) $|s_n| < 1$, take n odd to get -1 and even to get 1.

(c) Every subsequential limit has to converge to s.

Theorem 3.19

For all $\{n_k\}$, we have $\exists K | k \geq K \implies n_k \geq N \implies \lim_{k\to\infty} t_{n_k} - s_{n_k} \geq 0$.

Theorem 3.26

$s_n = 1 + x + ... + x^n \implies xs_n = x + x^2 + ... + x^{n+1} \implies (1-x)s_n = 1 - x^{n+1}$.

Examples 3.40

(a) Root test: $n \to \infty$.

(b) Ratio test: $\frac{1}{n+1} \to 0$.

(c) $1 \to 1$.

(d) Ratio test: $\frac{n}{n+1} \to 1$. $z = 1$ leads to harmonic series.

(e) Ratio test: $\frac{n^2}{(n+1)^2} \to 1$.

Example 3.53

$\sum_{k=1}^{\infty} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} < \frac{5}{6} + \sum_{k=2}^{\infty} \frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k}$. The RHS converges since $\frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k} = \frac{1}{2k^2-2k}$.