Exercise Guide for *Algebra (2nd Edition)* by MacLane and Birkhoff

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Contents

About

> *"Groups, as men, will be known by their actions."* - Guillermo Moreno

What follows are short summaries of my solution ideas (most of them aren't really proofs) to exercises from the book *Algebra* (2nd Edition) by Saunders MacLane and Garrett Birkhoff. I used the 2nd Edition due to having access to a hard copy; the exercises/exposition through the majority of the 2nd and 3rd Editions are identical as far as I can tell.

1 Sets, Functions, and Integers

1.1 Sets

1.1.1 Exercise 5

When constructing a subset, each element in the set can either be in or out (2 choices). Hence, 2^n.

1.1.2 Exercise 6

There are n choices for the first element, $n-1$ choices for the second element, and so on up to $n-m$, hence dividing $n!$ by $(n-m)!$. The order of these m selected elements doesn't matter, hence the division by $m!$.

1.2 Functions

1.2.1 Exercise 2

$h_g \circ h_f$, where h corresponds to left-inverse.

1.2.2 Exercise 3

Let $f : A \to B$ and $g : B \to C$ be surjections. Then $g \circ f$ is surjective since $\exists x \in B$ such that $g(x) = y \quad \forall y \in C$, and $\exists x' \in A$ such that $f(x') = x \quad \forall x \in B$ (from the surjectivity of f and g). Proving injectivity follows similarly.

1.2.3 Exercise 4

The reverse direction follows from Exercise 3. If $f \circ g$ is injective and g is not, we could choose two elements from the domain of g that map to the same element in the domain of f (contradiction). Surjectivity is a similar argument.

1.2.4 Exercise 5

f has no right inverse since it is not surjective. There are infinitely many left inverses of f, two possibilities are mapping to square roots when possible and to 1 or 2 otherwise.

1.2.5 Exercise 6

Apply the left inverse of f.

1.2.6 Exercise 7

When surjective, use right inverse.

1.2.7 Exercise 8

Define h such that $h(y) = x$ if $\exists x \in S \mid f(x) = y$, and $h(y) = x'$ otherwise (axiom of choice necessary for choosing x). If f is injective, there will only be one choice of x, and if f is surjective, there will be some x for every y.

1.2.8 Exercise 9

Unique right inverse indicates that every element in the range has only one choice to map back to in the domain, implying injectivity.

1.2.9 Exercise 10

If g is a bijection, then we can define f such that $f(y) = x$ where $g(x) = y$. f is then a two-sided inverse. If f is a two-sided inverse of g, then every element of T maps to a unique element of S (from left inverse) and vice versa. Hence g is a bijection.

1.2.10 Exercise 11

Following the hint, we can see that $f : U \to \mathcal{F}$ is surjective since $S \in \mathcal{F} \implies S \neq \emptyset \implies \exists u \in S \implies u \in U \implies f(u) = S$. The existence of the right inverse then gives us the axiom of choice.

1.3 Relations and Binary Operations

1.3.1 Exercise 2

Symmetry + transitivity imply circularity. For the other direction, we have xRy, $yRy \implies yRx$, which gives both symmetry and transitivity.

1.3.2 Exercise 3

This only implies reflexivity for the elements $x, y \in X \mid (x, y) \in R$, not $\forall x \in X$.

1.3.3 Exercise 4

If R is transitive $T = R$. Otherwise, start with $T = R$ and add (x, z) to T whenever $(x, y), (y, z) \in R$. Repeat this process until there are no more pairs to add.

1.3.4 Exercise 5

Let $R \subset X \times Y$, $S \subset Y \times Z$, $T \subset Z \times A$.

$$\begin{aligned} xR \circ (S \circ T)a &\implies \exists y \in Y \mid xRy, y(S \circ T)a \\ &\implies \exists z \in Z \mid ySz, zTa \\ &\implies x(R \circ S)z \\ &\implies x(R \circ S) \circ Ta \end{aligned}$$

1.3.5 Exercise 6

Let $R \subset X \times Y$, $S \subset Y \times Z$.

$$\begin{aligned} z(R \circ S)^{\smile} x &\implies x(R \circ S)z \\ &\implies \exists y \in Y \mid xRy, ySz \\ &\implies yR^{\smile}x,\ zS^{\smile}y \\ &\implies z(S^{\smile} \circ R^{\smile})x \end{aligned}$$

1.3.6 Exercise 7

$$\begin{aligned} (x, z) \in G(g \circ f) &\implies \exists y \in Y \mid g(y) = z,\ f(x) = y \\ &\implies (x, y) \in G(f),\ (y, z) \in G(g) \\ &\implies (x, z) \in G(f) \circ G(g) \end{aligned}$$

1.3.7 Exercise 9

$$\begin{aligned} (x, y) \in G(f) &\implies \forall x \in X,\ \exists y \in Y \mid f(x) = y \\ &\implies \forall x \in X,\ (x, x) \in G(f) \circ G^{\smile}(f) \\ \text{and} &\quad \forall y \in \mathrm{Im} f,\ (y, y) \in G^{\smile}(f) \circ G(f) \end{aligned}$$

1.3.8 Exercise 10

$$\begin{aligned} x\Box y = u\Box(x\Box y) = (u\Box y)\Box x = y\Box x \\ x\Box(y\Box z) = x\Box(z\Box y) = (x\Box y)\Box z \end{aligned}$$

1.4 The Natural Numbers

1.4.1 Exercise 1

$f^0 = 1_X$ is trivially an injection. Suppose f^n is an injection for some $n \in \mathbb{N}$. Then $f^{\sigma(n)} = f \circ f^n$ is a composition of injections and we are done.

1.4.2 Exercise 2

Same thing as Exercise 1.

1.4.3 Exercise 3

We have that $\sigma^0(0) = 0$. Now assuming $\sigma^n(0) = n$ for some $n \in \mathbb{N}$, we have $\sigma^{\sigma(n)}(0) = \sigma \circ \sigma^n(0) = \sigma(n) = n + 1$.

1.4.4 Exercise 6

We can take $\sigma^{-1}(n) = n - 1$ for $n > 0$ and $\sigma^{-1}(0) = 0, 1, 2$ to get 3 different left inverses.

1.4.5 Exercise 8

Let $n \in U$ if the elements in all sets of size n are equal. Since we can construct a set with two different elements, we have that $n = 1$ does not imply $\sigma(n) \in U$, and the induction axiom cannot be applied to U.

1.4.6 Exercise 9

(Property I, Property II): Take $X = \mathbb{N}$ and $\sigma(x) = x^2 + 1$.

(Property I, Property III): Let $X = \{0, 1\}$ and let $\sigma(0) = 1$, $\sigma(1) = 0$. Then σ is clearly injective, and any subset of X that contains 0 and $\sigma(0)$ is all of X.

(Property II, Property III): Again take $X = \{0, 1\}$, but this time let $\sigma(0) = \sigma(1) = 1$.

1.5 Addition and Multiplication

1.5.1 Exercise 1

$$\begin{aligned} n = 0: \ & (f^m)^0 = 1 = f^0 = f^{(\sigma^m)^0(0)} = f^{m0} \\ \text{Assume n}: \ & (f^m)^{(\sigma(n))} = f^m \circ f^{mn} = f^{m(n+1)} \end{aligned}$$

1.5.2 Exercise 2

(a) $mn = (\sigma^m)^n(0) = \sigma^{mn}(0) = \sigma^{nm}(0) = nm$.

(b) $\sigma(m)(n + n') = (\sigma^{\sigma(m)})^{n+n'}(0) = (\sigma^{\sigma(m)})^n(0) + (\sigma^{\sigma(m)})^{n'}(0)$.

1.5.3 Exercise 3

(a) To obtain a valid τ, simply permute the first few mappings of σ. For example, $\tau(0) = 2, \tau(1) = 3, \tau(2) = 1, n \geq 3: \ \tau(n) = n + 1$.

(b) Suppose τ satisfies Peano. Then we can let $\beta(0) = 0$ and $\beta(n) = \tau(\beta(n-1))\ \forall n > 0$. β is a bijection since τ is injective and maps to all of $\mathbb{N}/\{0\}$. Furthermore, $\beta\sigma(n) = \beta(n+1) = \tau\beta(n)$.

1.5.4 Exercise 4

(a)

$$\begin{aligned}\phi(n) = m &\implies \sigma(\phi(n)) = m+1 \\ &\implies \phi(\sigma(n)) = \phi(n+1) = m+1\end{aligned}$$

Thus, once we fix $\phi(0)$, we fix the rest of ϕ.

(b) There is only one choice of τ which satisfies Peano's Postulates: $\tau(0) = 1$ with τ satisfying the relation indicated in (a). This is exactly the successor function σ.

1.5.5 Exercise 6

$k + n = \sigma^n(k) = \sigma^n(m) \implies k = m$ since a composition of injections is an injection.

1.6 Inequalities

1.6.1 Exercise 1

Since $x = x$ we have reflexivity of $\leq$. Since $x \leq y \implies x + a = y$ and $y \leq z \implies y + b = z$, we have $x + a + b = z$ giving transitivity.

1.6.2 Exercise 2

$$\begin{aligned}m < n &\implies m + x = n \\ &\implies m + x + k = n + k \\ &\implies m + k < n + k\end{aligned}$$

Multiplication is also isotonic since it's just iterated addition.

1.6.3 Exercise 3

Suppose $0 \in U$, $n \in U \implies \sigma(n) \in U$ and $U \neq \mathbb{N}$. Then from well-ordering, we have that $\mathbb{N}/U$ has a first element f such that $m < f \implies m \in U$. However, this gives us that $\exists m \in U \mid \sigma(m) = f$ which leads to a contradiction.

1.6.4 Exercise 4

Suppose S is well-ordered with first element f but $U \subset S$ is not. Then $V \subset U \mid V \neq \emptyset$ and V has no first element. However, since $V \subset S$, we have a contradiction, since well-ordering implies that every subset of S has a first element.

1.6.5 Exercise 6

The subset consisting of that infinite descending sequence would contain no first element.

1.7 The Integers

1.7.1 Exercise 1

Let $u = sdu + u_0$ and let $v = sdv + v_0$.

$$\begin{aligned} uv &= (sdu)(sdv) + (sdu)(v_0) + (u_0)(sdv) + u_0 v_0 \\ d(uv) &= d((sdu)(sdv)) + 0 + 0 + 0 \\ &= (du)(dv) \end{aligned}$$

1.7.2 Exercise 3

Follows from the steps of lemma, since we have that $du \oplus' dv = d(u+v) = d(sdu + sdv) = du \oplus dv$.

1.7.3 Exercise 4

Suppose $a \oplus x_1 = a \oplus x_2$. Then $a' \oplus (a \oplus x_1) = a' \oplus (a \oplus x_2)$, which gives $x_1 = x_2$.

1.7.4 Exercise 5

Same logic as Exercise 3, except using the result of Exercise 1.

1.8 The Integers Modulo N

1.8.1 Exercise 3

$$\begin{aligned} h - k \in n\mathbb{Z},\ r - s \in n\mathbb{Z} &\implies (h-k) + (r-s) \in n\mathbb{Z} \\ &\implies (h+r) - (k+s) \in n\mathbb{Z} \\ h(r-s) \in n\mathbb{Z},\ s(h-k) \in n\mathbb{Z} &\implies h(r-s) + s(h-k) \in n\mathbb{Z} \\ &\implies hr - ks \in n\mathbb{Z} \end{aligned}$$

1.8.2 Exercise 4

Just check the squares of $0, ..., 7$ mod 8 to get the desired result.

1.8.3 Exercise 5

7 cannot be decomposed into a sum of 3 integers from the set $\{0, 1, 4\}$.

1.8.4 Exercise 6

One of the three consecutive integers must be divisible by 3; let the remainder of this integer mod 9 be k. Then, WLOG, we can let the other two integers be $k-1$ and $k+1$ mod 9. We then have that $(k-1)^3 + k^3 + (k+1)^3 = 3k^3 + 6k$, which is divisible by 9 since k is divisible by 3.

1.9 Equivalence Relations and Quotient Sets

1.9.1 Exercise 1

The quotient T/S consists of the set of all possible equivalence classes of triangles based on the relation of triangle similarity. Thus, each element of T/S corresponds to a different kind of triangle similarity, or "shape".

1.9.2 Exercise 2

$p \times p$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$. Furthermore, $(p \times p)(x, y) = (p \times p)(x', y') \implies p(x+y) = p(x'+y')$. Then by Theorem 19, we can define addition of cosets of two integers as the function that commutes with the coset of the sum of the integers.

1.9.3 Exercise 3

Reflexivity and symmetry are clear; transitivity follows from the fact that if $(x_1, y_1)E(x_2, y_2)$, $(x_2, y_2)E(x_3, y_3)$, then $x_3 - x_1 = x_3 - x_2 + x_2 - x_1$ which is the sum of two integers and therefore an integer.

1.10 Morphisms

1.10.1 Exercise 1

The additive endomorphisms of $\mathbb{Z}$ are completely determined by the value they map 1 to. Thus, they are all functions of the form $f(z) = cz$ for some constant $c \in \mathbb{Z}$.

1.10.2 Exercise 2

Every additive morphism from $\mathbb{Z}_n$ to $\mathbb{Z}_m$ is of the form $f(z) = p_m(cz)$ where $p_m : \mathbb{Z} \to \mathbb{Z}_m$ maps elements of $\mathbb{Z}$ to their remainders mod m and $c \in \mathbb{Z}_m$.

1.10.3 Exercise 3

Follows the structure indicated in Exercise 2.

1.10.4 Exercise 4

Each rotation of the square can be decomposed into clockwise rotations. If we label the vertices of the square as $0, 1, 2, 3$, then a clockwise rotation can be thought of as adding 1 mod 4. Thus, the isomorphisms between $(\mathbb{Z}_4, +)$ and $(Q, \circ)$ are exactly the additive isomorphisms between $\mathbb{Z}_4$ and itself. There are only 2 such isomorphisms: $f(1) = 1$ and $f(1) = 3$.

1.10.5 Exercise 5

Follows from left inverse for injectivity and right inverse for surjectivity.

1.10.6 Exercise 7

Any morphism $f : (\mathbb{R}, \times) \to (\mathbb{R}, +)$ satisfies

$$
\begin{aligned}
f(1 * 1) = f(1) + f(1) &\implies f(1) = 0 \\
f(0 * 0) = f(0) + f(0) &\implies f(0) = 0
\end{aligned}
$$

Which means f cannot be an isomorphism.

1.11 Semigroups and Monoids

1.11.1 Exercise 1

If u and u' are both units, then $u \Box u' = u' = u$.

1.11.2 Exercise 2

The terms $a_1, ..., a_m$ and $a_{m+1}, ..., a_{m+n}$ together give $a_1, ..., a_{m+n}$.

1.11.3 Exercise 3

As stated in the text, follows from induction on n (the proofs can be found in previous sections).

1.11.4 Exercise 4

Due to commutativity, we can rearrange the terms in the double sum as we like, thereby allowing us to swap sums.

1.11.5 Exercise 5

Let $f : (\mathbb{N}, +) \to (\mathbb{N}, \times)$ be such that $f(n) = 0 \, \forall n \in \mathbb{N}$. Then f is a morphism that does not map the additive unit 0 to the multiplicative unit 1.

2 Groups

2.1 Groups and Symmetry

2.1.1 Exercise 2

Map each element $x \in \mathbb{Z}_6$ to the pair $(p_2(x), p_3(x))$. This is an isomorphism, since the projections $\mathbb{Z}_6 \to \mathbb{Z}_3$ and $\mathbb{Z}_6 \to \mathbb{Z}_3$ are both group morphisms, and the mapping itself is a bijection.

2.1.2 Exercise 3

To see that there is no isomorphism $f : \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$, consider $f(1)$ and $f(3)$. We have that $f(0) = f(1+3) = f(1) + f(3)$ which is not possible since $f(0) = (0,0)$ (has to be the case since $f(x) = f(0) + f(x)$).

Rotations do not preserve symmetry for rectangles, since distances between adjacent vertices change. The only transformations that preserve symmetry are reflections across the vertical and horizontal axes, giving 4 possible transformations. We can then map $(0,0)$ to the identity, $(0,1)$ to a vertical reflection, $(1,0)$ to a horizontal reflection, and $(1,1)$ to a vertical + horizontal reflection.

2.1.3 Exercise 4

2.1.4 Exercise 5

2.1.5 Exercise 6

2.1.6 Exercise 10

The set of these permutations has identity $(1,0)$, and any permutation (a,b) has inverse $(\frac{1}{a}, -\frac{b}{a})$. Furthermore, $(a_2, b_2) \circ (a_1, b_1) = (a_1 a_2, a_2 b_1 + b_2)$, which is associative since multiplication and addition are both associative.

2.1.7 Exercise 11

(a) To show that the given function is a permutation on $\mathbb{R} \cup \infty$, we need to show that it is a bijection from $\mathbb{R} \cup \infty \to \mathbb{R} \cup \infty$. Suppose $f(x_1) = f(x_2)$. Then

$$\frac{ax_1 + b}{cx_1 + d} = \frac{ax_2 + b}{cx_2 + d}$$
$$(ad - bc)x_1 = (ad - bc)x_2 \implies x_1 = x_2$$

So f is an injection from $\mathbb{R} \cup \infty \to \mathbb{R} \cup \infty$. Furthermore, if we set $f(x) = y$, we can solve for x, which gives us that f is also a surjection.

(b) I'm sure an inverse can be found, but it's tedious... Associativity then follows again from associativity of multiplication and addition.

2.1.8 Exercise 12

2.1.9 Exercise 13

(a) Any automorphism of $\mathbb{Z}_3$ has to fix 0. Thus, the only two automorphisms are the identity and the automorphism that swaps 1 and 2.

(b) Fixing $(0, 0)$, we see that we can permute the remaining three elements as we want, giving the isomorphism to S_3.

(c)

2.2 Rules of Calculation

2.2.1 Exercise 1

(a) Multiply by inverse and use associativity.

(b) Associativity.

(c) Associativity and then inverse of product.

2.2.2 Exercise 2

Multiply by a^{-1}.

2.2.3 Exercise 3

Since the unit is its own inverse, we're left with $2n-1$ elements that need to be paired with one another. Since $2n-1$ is odd, we have that one of the elements must be its own inverse.

2.2.4 Exercise 4

Any group with 3 elements must be of the form $1, a, a^{-1}$. Thus, each of these groups is clearly isomorphic to the others.

2.2.5 Exercise 5

I struggled to untie the ideas of cancellation and inverse, so I ended up looking up a hint for this one. To see that an infinite set with cancellation does not need to be a group, consider $(\mathbb{N}, +)$. This is a monoid that was proven to have cancellation in chapter 1, but does not contain inverses.

For the case of a finite set G, we can use the fact that $f(x) = ax$ is an injection for any $a \in G$, since $ax = ay \implies x = y$ by cancellation. Since G is finite, f is also a surjection. Therefore, $\exists a \mid ax = 1$ which gives us that there is a left inverse. Applying the same logic using $f(x) = xa$ gives a right inverse, which completes the proof since these inverses must be equal.

2.2.6 Exercise 6

Left cancellation is possible due to left inverse and left unit. Furthermore, $uu = u \implies (a'a)u = a'a \implies au = a$ by left cancellation, indicating that u is also a right unit. Then we have that $ua' = a'u \implies a'aa' = a'u \implies aa' = u$, and a' is also a right inverse. This proves that X is a group.

2.2.7 Exercise 7

We proceed as directed in the hint. Since the equation $ua = a$ has solution u, and any b can be written as $b = ay$, we have $ub = u(ay) = ay = b$. Thus, u is a left unit. Since the equation $a'a = u$ also has a solution a', we are done by Exercise 6.

2.2.8 Exercise 10

Since each element of G has a unique inverse, $f(a) = a^{-1}$ is a bijection. Additionally, $f(ab) = (ab)^{-1} = b^{-1}a^{-1} = f(b)f(a) = f(a)\Box^{\text{op}} f(b)$.

2.2.9 Exercise 11

Associativity of $\Box$ immediately follows from the associativity of G 's binary operation and the fact that p is a morphism. Additionally, since p is an epimorphism, $\forall x$, $\exists g \mid x = p(g)$. Since $ug = gu$, $p(u)$ is then the unit for X. Similarly, $p(g')$ is the inverse of x, thus making X a group.

2.2.10 Exercise 12

$bb_R = u \implies b_L bb_R = b_L \implies b_R b = b_L b = u$.

2.3 Cyclic Groups

We first show that $\mathbb{Z}_n$ is generated only by those c that are coprime to n. If c is coprime to n, then $ac = 0$ only when $a = n$ since c and n share no prime factors. Thus, the subgroup generated by c has order n and is therefore all of $\mathbb{Z}_n$. Similarly, if c is a generator of $\mathbb{Z}_n$, then c has order n and must therefore be coprime to n.

2.3.1 Exercise 1

The only possible generators are 1 and 5, since those are the only elements of $\mathbb{Z}_6$ that are coprime to 6.

2.3.2 Exercise 2

The endomorphisms of $\mathbb{Z}_n$ are completely determined by the mapping of 1, so there are only n such endomorphisms.

2.3.3 Exercise 3

5 is prime, so all elements of $\mathbb{Z}_5$ other than 0 are coprime to it.

2.3.4 Exercise 4

14 has 6 positive integers less than it that are coprime to it (3, 5, 7, 9, 11, 13).

2.3.5 Exercise 5

The two generators of $\mathbb{Z}$ are 1 and -1, as elements of $\mathbb{Z}$ can be written as $-m$ or m.

2.3.6 Exercise 7

If G is abelian, then $(g_1 g_2)^m$ can be rearranged to $g_1^m g_2^m$. If $(g_1 g_2)^m = g_1^m g_2^m$. The reverse direction follows from the $m = 2$ case, $g_1 g_2 g_1 g_2 = g_1^2 g_2^2$.

2.3.7 Exercise 8

$(g_1 g_2)(g_1 g_2) = 1 \implies g_1 g_2 = g_2 g_1$.

2.3.8 Exercise 9

The automorphisms are all determined by the mappings of the generators; the isomorphisms follow from the number of generators of each group.

2.3.9 Exercise 10

2.4 Subgroups

2.4.1 Exercise 1

The subgroup mapping a given diagonal to itself consists of $\{1, R^3, D, D'\}$, where R^3 is 3 clockwise rotations, D is reflection across the given diagonal, and D' is reflection across the diagonal perpendicular to the given. Mapping those elements to $\{(0,0),(1,1),(0,1),(1,0)\}$ (in order) is an isomorphism.

2.4.2 Exercise 4

If S is closed under product and inverse, then it contains the identity and is thus a subgroup.

2.4.3 Exercise 5

We have that $(s,t)(t,s) = st^{-1}ts^{-1}$, so S contains the identity. Then $(1,s) = s^{-1}$ and $(s,t^{-1}) = st$, so S is closed under products and inverses as well, thus making it a subgroup.

2.4.4 Exercise 6

(a) The identity has order 1. Additionally, if a has finite order, so does a^{-1}. Finally, $a^n = 1,\ b^k = 1 \implies (ab)^{nk} = a^{nk}b^{nk} = 1$.

(b) If non-abelian, we do not necessarily have $(ab)^{nk} = a^{nk}b^{nk}$.

2.4.5 Exercise 7

If G has no proper subgroups, then it is generated by all of its non-identity elements. This is only possible if G has order 1 (vacuously true), or if G is a cyclic group of prime order (as was shown in the beginning of the previous section).

2.4.6 Exercise 8

(a) If a has order n, so does a^{-1}. Additionally, $(ab)^n = a^n b^n = 1$, making all elements that satisfy $a^n = 1$ a subgroup of A. To see that this is not true for non-abelian groups, consider S_3. The elements (12) and (23) are both of order 2, but $(12)(23) = (123)$ is of order 3.

(b) That the n^{th} powers form a subgroup follows from $a^n a^{-n} = 1$ and $a^n b^n = (ab)^n$.

2.4.7 Exercise 9

If T is a submonoid of S, then $i : T \to S$ is a morphism of monoids, so T must necessarily be closed under products and identity. For the reverse direction, if T is closed under products and identity, then the insertion i is a morphism of monoids and T is a submonoid of S.

2.5 Defining Relations

2.5.1 Exercise 3

The subgroup of rotations is isomorphic to $\mathbb{Z}_5$, so each element other than the identity has order 5. The element D has order 2. Furthermore, we have from the generator relations that

$$DR = R^{n-1}D \implies DR^i = R^{n-1}DR^{i-1} = DR^i = R^{i(n-1)}D = R^{n-i}D$$

So all elements of the form DR^i also have order 2.

2.5.2 Exercise 4

I believe the inclusion diagram looks like a tree with Δ_5 as the root, and the subgroups generated by R and each of the DR^i as leaves (they don't contain one another).

2.5.3 Exercise 5

After reflecting, it takes $2(i-1)$ rotations to get vertex i back to its original place. Thus, reflection through vertex i can be expressed as $DR^{2(i-1)}$.

2.5.4 Exercise 6

The two groups are the same order, so we just need to identify two elements of $S_3 \times S_2$ with R and D and show that these two elements satisfy the generator relations. Let $x = ((123),(12))$ and $y = ((13),1)$. Then $x^6 = (1,1)$ since (123) has order 3 and (12) has order 2. Similarly, $y^2 = (1,1)$ and $yx = x^{n-1}y$, so we have an isomorphism.

2.5.5 Exercise 8

(a) From $a^4 = 1$ we get that a is an element of order 4. From $b^2 = a^2$, we get that only b, b^3, ab, and ab^3, are distinct from the a^i. Hence, there are 8 distinct elements.

(b) There is no isomorphism to Δ_8, since the only element of order 2 is $b^2 = a^2$.

2.5.6 Exercise 10

We let $\psi((b,c)) = bc$. This is a morphism, since $\psi((b,c)(b',c')) = uu'vv' = uvu'v'$. Additionally, ψ sends $(b,1)$ to u and $(1,c)$ to v. To see that ψ is unique, we note that

$$\psi'((b,1)) = u,\ \psi'((1,c)) = v \implies \psi'((b,c)) = uv$$

if ψ' is a morphism.

2.6 Symmetric and Alternating Groups

2.6.1 Exercise 3

Since $(123)(12) = (13)$ and $(12)(123) = (23)$ we have that S_3 is not abelian, and therefore S_n with $n \geq 3$ is non-abelian ($S_3 \subset S_{n\geq 3}$). S_1 and S_2 are cylic and thus abelian. As for the alternating groups, it is again straightforward to see that A_2 is abelian (it only consists of the identity). A_3 is also abelian, since the only even permutations are $1, (123), (132)$, all of which commute. For $n \geq 4$ though, we have that $(123)(234) \neq (234)(123)$, so $A_{n\geq 4}$ is non-abelian.

2.6.2 Exercise 4

(a) The 4-element subgroup consisting of $1, (12), (34), (12)(34)$.

(b) The 6-element subgroup consisting of all of the elements in (a), plus $(13)(24), (14)(23)$.

2.6.3 Exercise 5

There are $\binom{4}{3} = 4$ ways of choosing 3 elements in S_4. The subgroup generated by the transpositions of the 3 selected elements is isomorphic to S_3. There are $\binom{4}{2} = 6$ ways of choosing 2 elements in S_4. The subgroup generated by the transposition of these two elements is ismorphic to S_2. Additionally, any such transposition can be paired with the transposition of the remaining two elements (e.g. $(12)(34)$) to produce another subgroup isomorphic to S_2, giving 9 such subgroups.

2.6.4 Exercise 6

The idea is the same as the second part of Exercise 5. We have $\binom{6}{3} = 20$ ways to pick 3 elements in S_6, and the transpositions of these elements can then be paired with the transpositions of the remaining 3 elements to produce at least another 10 subgroups isomorphic to S_3 (the pairing order can be changed to produce more).

2.6.5 Exercise 7

The fact that σ and $\tau\sigma\tau^{-1}$ have the same parity follows immediately from Proposition 15 ($(x_1 \ldots x_k) \rightarrow (\tau(x_1) \ldots \tau(x_k))$). That the two need not have the same number of inversions can be seen from looking at $(123)(12)(132) = (13)$. The permutation (12) only inverts $(1,2)$, whereas (13) inverts $(1,2), (1,3), (2,3)$.

2.6.6 Exercise 8

Any cycle of length $2m$ has parity $2m-1$ (Theorem 17, Corollary 2), and is thus odd. Therefore the product of not necessarily disjoint cycles being even implies that the product contains an even number of even length cycles. Noting that odd cycles have even parity gives the reverse direction.

2.6.7 Exercise 9

A permutation of order 14 must consist of either a single cycle of order 14, two cycles of orders 2 and 7, or a combination of both (since the order is the LCM of the disjoint cycle decomposition lengths). However, since we are considering permutations on 10 letters, only the case of 2 and 7 is possible, which means any such permutation must be odd.

2.6.8 Exercise 10

We first show that any odd length cycle can be written as a product of length 3 cycles. Let $\sigma = (x_1 \ldots x_{2n+1})$. Then σ can be decomposed as $(x_1 x_{2n} x_{2n+1}) \ldots (x_1 x_2 x_3)$, or, in other words, the product of 3-cycles consisting of its first element x_1 paired with consecutive pairs x_{2k}, x_{2k+1}. Next, we show that a product of two disjoint even length cycles $\sigma_1 = (x_1 \ldots x_{2n})$ and $\sigma_2 = (y_1 \ldots y_{2m})$ can be rewritten as the product of two odd length cycles. To do so, we modify σ_1 to be

$\sigma_1' = (x_1 \ldots x_{2n} y_1)$ and modify σ_2 to be $\sigma_2' = (y_1 \ldots y_{2m} x_{2n})$. We can then verify that $\sigma_1' \circ \sigma_2' = \sigma_1 \circ \sigma_2$. Thus, since any even permutation must have a disjoint cycle decomposition consisting of an even number of even length cycles (since they have odd parity), an even permutation can be written as the product of 3-cycles.

2.6.9 Exercise 11

2.6.10 Exercise 12

2.6.11 Exercise 13

That $(12), (23), \ldots\ (n-1n)$ are generators for S_n follows immediately from Exercise 11 and the fact that $(12 \ldots n-1) = (n-2n-1)\ldots(12)$.

2.7 Transformation Groups

2.7.1 Exercise 2

The left regular representation of S_3 is the function that assigns each element $\sigma \in S_3$ to $f_\sigma(x) = \sigma \circ x$ where $f_\sigma : S_3 \to S_3$.

2.7.2 Exercise 4

The isotropy subgroup of a single vertex is just the group of permutations that leave the given vertex fixed and permute the other seven vertices (isomorphic to S_7).

2.7.3 Exercise 5

The isotropy subgroups are all isomorphic to S_{n-1}, as they consist of all permutations that leave a single element fixed. To see that these subgroups are conjugate to one another, let G_i be the isotropy subgroup of i. Then we have that for $g \in G_i$, $(ij)g(ij) \in G_j$, since (ij) maps j to i and then back to j again.

2.7.4 Exercise 6

From Proposition 15, we have that cyles of the same length are conjugate to one another. Furthermore, we know that every element of S_n has a unique disjoint cycle decomposition. The different possible length cycle decompositions form unique conjugacy classes (from Proposition 15), so the number of conjugacy classes for S_n is just the number of partitions of n. For S_3 this is 3 and for S_4 this is 5.

2.7.5 Exercise 7

The left regular representation of the additive group $\mathbb{R}$ assigns to each element $z \in \mathbb{R}$ the function $f_z(x) = x + z$, which is exactly a translation by z of the real line. Similarly, the left regular representation of the additive group $\mathbb{R} \times \mathbb{R}$ corresponds to a translation of (z_1, z_2) in the cartesian plane.

2.7.6 Exercise 8

Since G acts transitively on x, there exists g such that $gx = y$. Let z be an element that fixes x. Then we have that $gzg^{-1}y = gzx = gx = y$, so gzg^{-1} fixes y as desired.

2.7.7 Exercise 9

For any subgroup S of Δ_4, we can consider any action of Δ_4 on the square that has each element of an equivalence class of Δ_4/S act the same way on an element of the square (not exactly sure how an "element" of the square should be defined, I suppose a point). Such an action necessarily fixes the subgroup S.

I was a bit confused by this question and found some more discussion here.

2.7.8 Exercise 10

Let I be an invariant subset containing x. Then $gx \in I$ for all $g \in G$, implying that $\text{Orb}(x) \subset I$. Since by definition $\text{Orb}(x)$ is invariant, it must be the smallest such set. By the previous logic, we further have that $I = \cup_{x \in G} \text{Orb}(x)$, which can be reduced to a union of disjoint orbits (since one element appearing in another's orbit means their orbits are the same).

2.8 Cosets

2.8.1 Exercise 1

The image of a right coset $aS = \{as \mid s \in S\}$ under the bijection $a \mapsto a^{-1}$ is the set $\{s^{-1}a^{-1} \mid s \in S\}$, which is the left coset Sa^{-1}.

2.8.2 Exercise 2

The indicated subgroup S is just $D, 1$. Thus, the left cosets consist of reflections followed by rotations while the right cosets consist of rotations followed by reflections.

2.8.3 Exercise 5

Consider $x \in S/S \cap T$ and $y \in T$. By the definition of join, the product xy must be in $S \vee T$. Since $S/S \cap T$ and T are disjoint, this means that $[S : S \cap T][T : 1] \leq [S \vee T : 1]$ (since $S \vee T$ contains every such product xy). Using the fact that $[S : S \cap T] = \frac{[S:1]}{[S \cap T:1]}$, we then have the desired inequality.

2.8.4 Exercise 6

We use the result of Exercise 8 to get that a group with 6 elements is either isomorphic to $\mathbb{Z}_6$ or to a group with 3 elements of order 2 and 2 elements of order 3. Since S_3 has 3 elements of order 2 and 2 elements of order 3, there is an isomorphism between the latter category of order 6 groups and S_3.

2.8.5 Exercise 7

Again we rely on Exercise 8. Since Δ_5 contains 5 elements of order 2 and 4 elements of order 5, any group of order 10 without an order 10 element is isomorphic to Δ_5 by exercise 8.

2.8.6 Exercise 8

If a group of order $2p$ contains an element of order $2p$, then it is cyclic and thus isomorphic to $\mathbb{Z}_{2p}$. If a group of order $2p$ does not contain an element of order $2p$, then we will show the following:

- It can have at most one subgroup of order p.
- It must have at least one subgroup or order p.

To see the first, we appeal to the inequality from Exercise 5. If there were two distinct subgroups of order p, then their join would consist of at least $p^2 > 2p$ elements (for $p > 2$). To see the second, we consider the case where all $2p - 1$ non-identity elements have order 2. Such a group must be abelian, since we have $abab = 1 \implies ab = b^{-1}a^{-1} = ba$. Taking the join of the groups generated by a and b would then give us a subgroup of order 4, which is not possible since 4 does not divide $2p$. Thus, there must be an element (and hence a generated subgroup) of order p. Combining these two results gives that a group of order $2p$ that does not contain an element of order $2p$ must have $p - 1$ elements of order p and p elements of order 2.

2.8.7 Exercise 9

(a) Suppose $sat = s'bt'$. Then we would have $a = s^{-1}s'bt't^{-1}$, so $a \in SbT$ and the double cosets SaT and SbT are equal. The alternative is that $sat \neq s'bt'$ for all s, s', t, t', which would imply that the double cosets are disjoint. Thus, the double cosets of G form a partition of G, so their union is G.

(b)

2.9 Kernel and Image

2.9.1 Exercise 2

If $N \triangleleft S_3$, then all elements in N must have the same sign (since $\operatorname{sgn}(ana^{-1}) = \operatorname{sgn}(n)$). As there are no subgroups consisting of only odd permutations, we need only consider subgroups consisting of even permutations. The only such proper subgroup of S_3 is A_3.

2.9.2 Exercise 3

Since $DR^i = R^{n-i}D \implies DR^iD^{-1} = R^{n-i}$, we have that $\{R^i\} \triangleleft \Delta_p$. The only other proper subgroups of Δ_p are the order 2 groups generated by DR^i. However, none of these subgroups are normal since $DDR^iD = DR^{n-i}$, which is clearly not DR^i or 1 for all $0 < i < n$. Thus, the subgroup of rotations is the only normal subgroup of Δ_p.

2.9.3 Exercise 4

We first note that $R^jDR^iR^{-j} = DR^{i-2j}$, and $DR^{i-2j} = DR^i$ only when $j = 0$ or $j = \frac{n}{2}$. Thus, none of the subgroups generated by elements of the form DR^i are normal, and we only need to consider subgroups of the rotation subgroup. Any such subgroup is normal, since i divides $n - i$, so $\{R^i\}, \{R^{2i}\} \triangleleft \Delta_4$ and $\{R^i\}, \{R^{2i}\}, \{R^{3i}\} \triangleleft \Delta_6$.

2.9.4 Exercise 5

(a) $ax = xa \implies x^{-1}ax = a$, so $Z(G)$ is normal.

(b) From Exercises 3 and 4, we can see that $DR^i \notin Z(\Delta_n)$. Furthermore, $R^i \in Z(\Delta_n)$ only if $R^i = R^{n-i}$, which is only possible if n is even. Thus, $Z(\Delta_n)$ is either 1 or the subgroup generated by $R^{\frac{n}{2}}$, with the latter being isomorphic to $\mathbb{Z}_2$.

(c) From Proposition 15, we have that $\tau\sigma\tau^{-1} = \sigma$ only if τ and σ commute or if $\tau = 1$. For $n > 2$, S_n is not commutative, so $Z(S_n) = 1$.

2.9.5 Exercise 6

Let A and B be two normal subgroups of G. Then for $a \in A$, $b \in B$, and $g \in G$, we have $gag^{-1}gbg^{-1} = gabg^{-1} = a'b'$ for some $a' \in A$ and $b' \in B$, so $A \vee B$ is normal in G. Additionally, if $a \in A \cap B$, then we have that $gag^{-1} \in A$ and $gag^{-1} \in B$ by normality of A and B, so $A \cap B \triangleleft G$.

2.9.6 Exercise 7

(a) Let $f_g(x) = gxg^{-1}$. Then $f_{g'} \circ f_g(x) = g'gxg^{-1}g'^{-1} = f_{g'g}(x)$ so $\mathrm{In}(G)$ is a group under composition.

(b) Let $h \in \mathrm{Aut}(G)$. Then $h \circ f_g \circ h^{-1}(x) = h(gh^{-1}(x)g^{-1}) = h(g)xh(g)^{-1}$, so $\mathrm{In}(G) \triangleleft \mathrm{Aut}(G)$.

2.9.7 Exercise 12

Since $g \to ag$ is a permutation on G, $gT \to agT$ is a permutation on G/T. We also have that $h_a \circ h_b(gT) = a(bgT) = (ab)gT = h_{ab}(gT)$ so $h : G \to S(G/T)$ is a morphism.

2.10 Quotient Groups

2.10.1 Exercise 1

Let ϕ be a morphism with $\phi(R) = 0$ and $\phi(D) = 1$. Then ϕ is an epimorphism with kernel S, so $\Delta_n/S \cong \mathbb{Z}_2$.

2.10.2 Exercise 2

The quotient groups of Δ_4 correspond to its normal subgroups, which are:

- $\{1\}, \{R\}, \{R^2\}$
- $\{R^2, D\}, \{R^2, RD\}$

2.10.3 Exercise 3

(a) Let $\phi : 4\mathbb{Z} \to \mathbb{Z}_5$ be defined as $\phi(x) = x \mod 5$. Then ϕ is an epimorphism with kernel $20\mathbb{Z}$, so $4\mathbb{Z}/20\mathbb{Z} \cong \mathbb{Z}_5$. Similarly, we can also define $\phi : \mathbb{Z}_6 \to \mathbb{Z}_3$ as $\phi(x) = x \mod 3$. This is another epimorphism with kernel $3\mathbb{Z}_6$, so $\mathbb{Z}_6/3\mathbb{Z}_6 \cong \mathbb{Z}_3$.

(b) More generally, we can define $\phi : k\mathbb{Z} \to \mathbb{Z}_m$ as $\phi(x) = \frac{x}{k} \mod m$, which is an epimorphism with kernel $mk\mathbb{Z}$. Thus, $k\mathbb{Z}/mk\mathbb{Z} \cong \mathbb{Z}_m$. From this result, we get that $(\mathbb{Z}/mk\mathbb{Z})/(k\mathbb{Z}/mk\mathbb{Z}) \cong \mathbb{Z}_{mk}/\mathbb{Z}_m$, which is in turn isomorphic to $\mathbb{Z}_k$.

2.10.4 Exercise 4

I'm not really sure what counts as a "familiar group here", but we can consider $x \to |x|$ again to see that $Q^*/\{\pm 1\}$ is isomorphic to the multplicative group of positive rationals.

2.10.5 Exercise 6

Consider $\phi : G \to \text{In}G$ where $\phi(g) = \gamma_g$ with $\gamma_g(x) = gxg^{-1}$. By construction, ϕ is an epimorphism, and $\gamma_g(x) = x$ iff $g \in Z$. Thus, $G/Z \cong \text{In}G$.

2.10.6 Exercise 7

(a) Since $[xgx^{-1}, xhx^{-1}] = xgx^{-1}xhx^{-1}xg^{-1}x^{-1}xh^{-1}x^{-1} = x[g,h]x^{-1}$, we have that $[G,G] \triangleleft G$.

(b) Since $[G,G][x,y] = [G,G] \implies [G,G]xy = [G,G]yx$, $G/[G,G]$ is abelian. Any morphism $\phi : G \to A$ carries all of the elements of the coset $[G,G]x$ to the single element $\phi(x)$. Thus, we can factor ϕ as $\phi' \circ p$, where $\phi' : G/[G,G] \to A$ with $\phi'([G,G]x) = \phi(x)$.

2.10.7 Exercise 8

Since $G/N \cong \mathbb{Z}_5$, G must be a group of order 10. This implies that G is either Δ_5 or $\mathbb{Z}_10$ (see exercise 7 in section 8). However, Δ_5 does not have a normal subgroup of order 2, so $G \cong Z_{10}$.

2.10.8 Exercise 9

Since $N \subset M$, we can define an epimorphism $\phi : G/N \to G/M$ that maps the coset Ng to the coset Mg. The kernel of this epimorphism is Nm for all $m \in M$, which is exactly M/N. Thus, we get that $(G/N)/(M/N) \cong G/M$.