“Sparknotes” for *Algebra* by MacLane and Birkhoff

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Preface

> “A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.” - Hermann Weyl

These notes contain short summaries of (my) proof ideas for exercises and some theorems from the book *Algebra* by MacLane and Birkhoff. I have tried to make the summaries as brief as possible, sometimes only one line or one equation. My hope is that the summaries will give enough information to reconstruct a full proof without bogging the reader down with details. In many cases, I am sure that I inadvertently sacrificed clarity in an attempt to obtain brevity, and would greatly appreciate any feedback.

Also, I like when people include (what they presume to be) relevant quotes from mathematicians of past generations in their notes, so I have to ask you to forgive my haughtiness in starting these notes with a quote from Hermann Weyl.

1 Sets, Functions, and Integers

1.1 Sets

1.1.1 Exercise 5

When constructing a subset, each element in the set can either be in or out (2 choices). Hence, 2^n.

1.1.2 Exercise 6

There are n choices for the first element, $n-1$ choices for the second element, and so on up to $n-m$, hence dividing $n!$ by $(n-m)!$. The order of these m selected elements doesn't matter, hence the division by $m!$.

1.2 Functions

1.2.1 Exercise 2

$h_g \circ h_f$, where h corresponds to left-inverse.

1.2.2 Exercise 3

Let $f : A \to B$ and $g : B \to C$ be surjections. Then $g \circ f$ is surjective since $\exists x \in B$ such that $g(x) = y \quad \forall y \in C$, and $\exists x' \in A$ such that $f(x') = x \quad \forall x \in B$ (from the surjectivity of f and g). Proving injectivity follows similarly.

1.2.3 Exercise 4

The reverse direction follows from Exercise 3. If $f \circ g$ is injective and g is not, we could choose two elements from the domain of g that map to the same element in the domain of f (contradiction). Surjectivity is a similar argument.

1.2.4 Exercise 5

f has no right inverse since it is not surjective. There are infinitely many left inverses of f, two possibilities are mapping to square roots when possible and to 1 or 2 otherwise.

1.2.5 Exercise 6

Apply the left inverse of f.

1.2.6 Exercise 7

When surjective, use right inverse.

1.2.7 Exercise 8

Define h such that $h(y) = x$ if $\exists x \in S \,|\, f(x) = y$, and $h(y) = x'$ otherwise (axiom of choice necessary for choosing x). If f is injective, there will only be one choice of x, and if f is surjective, there will be some x for every y.

1.2.8 Exercise 9

Unique right inverse indicates that every element in the range has only one choice to map back to in the domain, implying injectivity.

1.2.9 Exercise 10

If g is a bijection, then we can define f such that $f(y) = x$ where $g(x) = y$. f is then a two-sided inverse. If f is a two-sided inverse of g, then every element of T maps to a unique element of S (from left inverse) and vice versa. Hence g is a bijection.

1.2.10 Exercise 11

Following the hint, we can see that $f : U \to \mathcal{F}$ is surjective since $S \in \mathcal{F} \implies S \neq \emptyset \implies \exists u \in S \implies u \in U \implies f(u) = S$. The existence of the right inverse then gives us the axiom of choice.