

“Sparknotes” for *Principles of Mathematical Analysis (3rd Ed.)* by Walter Rudin

Muthu Chidambaram

Last Updated: May 28, 2019

Contents

1	The Real and Complex Number Systems	4
1.1	Exercise 1	4
1.2	Exercise 2	4
1.3	Exercise 4	4
1.4	Exercise 5	4
1.5	Exercise 6	4
1.6	Exercise 7	4
1.7	Exercise 8	5
1.8	Exercise 9	5
1.9	Exercise 10	5
1.10	Exercise 11	5
1.11	Exercise 12	5
1.12	Exercise 13	5
2	Basic Topology	6
2.1	Exercise 1	6
2.2	Exercise 2	6
2.3	Exercise 3	6
2.4	Exercise 4	6
2.5	Exercise 5	6
2.6	Exercise 6	6
2.7	Exercise 7	6
2.8	Exercise 8	7
2.9	Exercise 10	7
2.10	Exercise 12	7
2.11	Exercise 13	7
2.12	Exercise 14	7
2.13	Exercise 15	7
2.14	Exercise 16	7

2.15	Exercise 17	7
2.16	Exercise 18	8
3	Numerical Sequences and Series	9
3.1	Exercise 1	10
3.2	Exercise 2	10
3.3	Exercise 3	10
3.4	Exercise 4	10
3.5	Exercise 5	10
3.6	Exercise 6	10
3.7	Exercise 7	11
3.8	Exercise 8	11
3.9	Exercise 9	11
3.10	Exercise 10	11
3.11	Exercise 15	11
4	Continuity	13
4.1	Exercise 1	13
4.2	Exercise 2	13
4.3	Exercise 3	13
4.4	Exercise 4	13
4.5	Exercise 5	13
4.6	Exercise 6	13
4.7	Exercise 7	14
4.8	Exercise 8	14

About

“A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.”

- Hermann Weyl

An actual solution manual for the book can be found [here](#). What follows are notes I took as I did exercises (they're more like hints towards my thinking than solutions) while working through the book on my own.

1 The Real and Complex Number Systems

1.1 Exercise 1

If $rx = q$ or $r + x = q$ for some rational q , then subtracting r from q or dividing q by r yields x rational, which is a contradiction.

1.2 Exercise 2

We can first show that $\sqrt{3}$ is irrational by seeing that $\frac{a^2}{b^2} = 3 \implies 3|a, 3|b$. Then, since $12 = 3 * 2^2$, we have that $\sqrt{12}$ is irrational as well.

1.3 Exercise 4

If $\alpha > \beta$ then α would be an upper bound as well.

1.4 Exercise 5

$\forall x \in A, -x \leq \sup -A$ and $\forall \epsilon \in \mathbb{R}, \exists x \in A | \sup -A + \epsilon < -x \leq \sup -A$. Negating the last inequality gives $\inf A = -\sup -A$.

1.5 Exercise 6

(a) Follows from $m = \frac{np}{q}$.

(b) Put $r = \frac{m}{n}, s = \frac{p}{q}$. Then $b^r b^s = b^{\frac{mq}{nq}} b^{\frac{np}{nq}}$. Pulling out $\frac{1}{nq}$ gives the desired result.

(c) b^r is an upper bound since $b > 1$, and if it were not the supremum we could choose $t < r$ such that $b^t > b^r$. This is not possible since again, $b > 1$.

(d) Every element in $B(x + y)$ can be expressed as $b^{s+t} = b^s b^t$ $s \leq x, t \leq y$. If $\sup B(x + y) = \alpha < \sup B(x) \sup B(y)$, then $b^s b^t \leq \alpha \implies B(x) \leq \alpha b^{-t} \implies B(y) \leq \frac{\alpha}{B(x)} \implies B(x)B(y) \leq \alpha$.

1.6 Exercise 7

(a) $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \dots + 1) \geq n(b - 1)$ since $b > 1$.

(b) Plug $b^{\frac{1}{n}}$ into (a).

(c) Plug $n > \frac{b-1}{t-1}$ into (b).

(d) Using (c) gives that we can choose n such that $b^{\frac{1}{n}} < y b^{-w} \implies b^{w+\frac{1}{n}} < y$.

(e) We can take the reciprocal of (c) and do the same as in (d).

(f) If $b^x > y$ we can apply (e) for a contradiction, if $b^x < y$ we can apply (d) for a contradiction.

(g) Supremum is unique.

1.7 Exercise 8

Suppose $(0, 1) < (0, 0)$. Then $(0, -1) < (0, 0)$ after multiplying by $(0, 1)$ twice yields a contradiction. Similarly, assuming the opposite yields $(-1, 0) > (0, 0)$.

1.8 Exercise 9

Does exhibit least upper-bound property since you can take $(\sup a_i, \sup b_i)$.

1.9 Exercise 10

Exception is 0.

1.10 Exercise 11

Take $w = \frac{1}{|z|}z$ and $r = |z|$ when $|z| \neq 0$. w and r are not uniquely determined; take $z = 0$ for example.

1.11 Exercise 12

By strong induction:

$$\begin{aligned} |z_1 + \dots + z_{n+1}| &\leq |z_1 + \dots + z_n| + |z_{n+1}| \\ &\leq |z_1| + \dots + |z_{n+1}| \end{aligned}$$

1.12 Exercise 13

$$\begin{aligned} |x - y|^2 &= x\bar{x} - 2|x||y| + y\bar{y} \\ &\geq (|x| - |y|)^2 \end{aligned}$$

2 Basic Topology

2.1 Exercise 1

The empty set has no elements, so all of its elements are vacuously also elements of every set.

2.2 Exercise 2

The roots of complex polynomials with integer coefficients can be expressed as elements of the countable cross product of \mathbb{N} with itself (cross \mathbb{N} with itself n times for the coefficients, and then once more to indicate which root).

2.3 Exercise 3

If all real numbers were algebraic, then the set of algebraic numbers would be uncountable (thus contradicting Exercise 2).

2.4 Exercise 4

The set of irrational numbers is \mathbb{R}/\mathbb{Q} , which must be uncountable as otherwise \mathbb{R} would be countable.

2.5 Exercise 5

We can use $\left(\frac{n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{2n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{3n}{n+1}\right)_{n \in \mathbb{N}}$ to get the three limit points 1, 2, 3.

2.6 Exercise 6

If p is a limit point of E' , then every neighborhood of p contains a limit point q of E , and every neighborhood of q contains a point of E thereby implying that p is a limit point of E . E and E' do not need to have the same limit points, since E' could be finite and thus have no limit points.

2.7 Exercise 7

(a) If p is a limit point of $\overline{B_n}$, then every neighborhood of p contains a point $q \in A_i$. Since there are only finitely many A_i , p must be a limit point for at least one of the A_i , as an infinite number of neighborhoods of p must have non-zero intersection with some of the A_i .

(b) If we take $A_i = \left(\frac{in}{(i+1)n+1}\right)_{n \in \mathbb{N}}$, then 1 is a limit point of B_n despite not being a limit point of any of the A_i .

2.8 Exercise 8

Every point of an open set in \mathbb{R}^2 is by definition a limit point of the set, since the point must have a neighborhood contained in the set. The same is not true for closed sets, since we can just take a finite set.

2.9 Exercise 10

Every set in X is open, since any set containing p also contains $N_r(p)$ for $r < 1$. No set in X is closed, since $N_r(p) = p$ for $r < 1$. All infinite sets in X are not compact, since we can take balls of radius $r < 1$ around each point as an open cover.

2.10 Exercise 12

Take any open cover of K . There must be some open set in this cover containing 0, which means that the same set contains all but a finite number of the elements of K (since 0 is the only limit point of K). Take a union of this set as well as the finitely many other sets containing the aforementioned points to get a finite subcover.

2.11 Exercise 13

Take $\cup_{k=1}^{\infty} \{0, (\frac{n}{kn+1})_{n \in \mathbb{N}}, \frac{1}{k}\}$. This set is closed and bounded, so it is compact by Heine-Borel. Its limit points are 0 and $(\frac{1}{k})_{n \in \mathbb{N}}$.

2.12 Exercise 14

We can use $\cup_{n \in \mathbb{N}} (0, \frac{n}{n+1})$, which has no finite subcover (since we could choose $x \in (0, 1)$ larger than the largest endpoint in the finite subcover).

2.13 Exercise 15

For closed, we can take $K_i = \mathbb{N}/0, \dots, i-1$, since any $x \in K_i$ will not be in K_j if $j > x$. For bounded, we can take $K_i = (0, \frac{1}{i})$.

2.14 Exercise 16

E is by definition bounded, and E is closed since $q^2 \neq 3$ (q is rational), and $q^2 > 3 \implies \exists \epsilon \mid p \in N_{\epsilon}(q) \implies p^2 > 3$. Same logic gives that E is also open in \mathbb{Q} . E is, however, not compact, since we can construct an open cover consisting of $G_n = \{x \mid 2 < x^2 < 2 + \frac{n}{n+1}\}$.

2.15 Exercise 17

E is not countable by diagonalization. E is not dense in $[0, 1]$, since $E \cap [0, 0.1] = \emptyset$. E is not perfect, consider $N_{0.001}(0.77)$. E is closed and therefore compact by

Heine-Borel. To see closed, suppose a limit point q had a non-4/7 digit in the i^{th} decimal spot. Then we could take a neighborhood of size $10^{-(i+1)}$.

2.16 Exercise 18

Rationals are dense in \mathbb{R} , so no.

3 Numerical Sequences and Series

Definition 3.5

Since $\{p_n\} \rightarrow p \implies \forall \epsilon, \exists N | n \geq N \implies |p_n - p| < \epsilon$, we can choose $k | n_k \geq N \implies \{p_{n_k}\} \rightarrow p$. The reverse direction can be shown via contradiction of $\{p_n\} \rightarrow p$.

Examples 3.18

- (a) Density of rationals in reals.
- (b) $|s_n| < 1$, take n odd to get -1 and even to get 1.
- (c) Every subsequential limit has to converge to s .

Theorem 3.19

For all $\{n_k\}$, we have $\exists K | k \geq K \implies n_k \geq N \implies \lim_{k \rightarrow \infty} t_{n_k} - s_{n_k} \geq 0$.

Theorem 3.26

$$s_n = 1 + x + \dots + x^n \implies x s_n = x + x^2 + \dots + x^{n+1} \implies (1 - x)s_n = 1 - x^{n+1}.$$

Examples 3.40

- (a) Root test: $n \rightarrow \infty$.
- (b) Ratio test: $\frac{1}{n+1} \rightarrow 0$.
- (c) $1 \rightarrow 1$.
- (d) Ratio test: $\frac{n}{n+1} \rightarrow 1$. $z = 1$ leads to harmonic series.
- (e) Ratio test: $\frac{n^2}{(n+1)^2} \rightarrow 1$.

Example 3.53

$\sum_{k=1}^{\infty} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} < \frac{5}{6} + \sum_{k=2}^{\infty} \frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k}$. The RHS converges since $\frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k} = \frac{1}{2k^2-2k}$.

3.1 Exercise 1

All we need is the inequality $|s_n - s| \geq ||s_n| - |s||$. The converse is not true, since we can take $s_n = (-1)^n$.

3.2 Exercise 2

My original idea: $\sqrt{(n+x)^2} - n = x$. Setting $(n+x)^2 \geq n^2 + n$ gives $x^2 \geq (1-2x)n$. The last inequality is only true for all n when $x \geq \frac{1}{2}$. This implies that $\frac{1}{2}$ is the supremum of $\sqrt{n^2+n} - n$. Since $\sqrt{n^2+n} - n$ is increasing, it converges to $\frac{1}{2}$.

Better: $(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n) = n \implies \sqrt{n^2+n} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$.

3.3 Exercise 3

Clearly $s_{n+1} > s_n$. We can see that $s_n < 2$ by induction, since $s_1 < 2$ and $2 + \sqrt{s_n} < 4$. This gives that s_n is monotone and bounded, implying it converges.

3.4 Exercise 4

$$s_{2m+1} = \sum_{i=1}^m \frac{1}{2^i}, \quad s_{2m} = \sum_{i=2}^m \frac{1}{2^i}$$

$$\implies \limsup_{n \rightarrow \infty} s_n = 1, \quad \liminf_{n \rightarrow \infty} s_n = \frac{1}{2}$$

3.5 Exercise 5

$$\begin{aligned} \limsup_{n \rightarrow \infty} (a_n + b_n) &= \sup_{\{k\}} \{ \lim_{k \rightarrow \infty} (a_{n_k} + b_{n_k}) \} \\ &= \sup_{\{k\}} \{ \lim_{k \rightarrow \infty} a_{n_k} + \lim_{k \rightarrow \infty} b_{n_k} \} \end{aligned}$$

3.6 Exercise 6

(a) $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$ diverges from comparison to harmonic series (same technique as Exercise 2).

(b) Converges, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p = \frac{3}{2}$.

(c) Converges by root test, since $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$.

(d) Converges when $|z| > 1$ and diverges otherwise. To see this, put $z = |z|e^{i\theta}$ to get $\lim_{n \rightarrow \infty} \frac{1}{1+|z|^n e^{in\theta}}$.

3.7 Exercise 7

We proceed via the ratio test.

$$\begin{aligned}\limsup_{n \rightarrow \infty} \frac{n}{n+1} * \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} &= \limsup_{n \rightarrow \infty} \frac{n}{n+1} \limsup_{n \rightarrow \infty} \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} \\ &= \sqrt{\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}} \\ &< 1\end{aligned}$$

Since $\sum a_n$ converges.

3.8 Exercise 8

Since b_n is monotonic and bounded, $|b_n| \leq B$ for all n . Then we have that $\sum a_n b_n$ converges by the comparison test, since $|a_n b_n| \leq B|a_n|$ and $B \sum a_n$ converges.

3.9 Exercise 9

(a) Applying the ratio test, we see that $|z| \limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$ when $|z| < 1$. Thus $\sum n^3 z^n$ has radius of convergence 1.

(b) Again, applying the ratio test, we see that $2|z| \limsup_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0$, implying $R = +\infty$.

(c) The ratio test is the only hammer we need: $2|z| \limsup_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| < 1$ gives $R = \frac{1}{2}$.

(d) What are the other tests again? $\frac{|z|}{3} \limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$ gives $R = 3$.

3.10 Exercise 10

The infinitely many non-zero a_n must satisfy $|a_n| \geq 1$. The radius of convergence of $\sum a_n z^n$ will be maximized when $|a_n|$ is minimized, so we can just consider the case where there are infinitely many $|a_n| = 1$. In this case, we can choose a subsequence a_{n_k} consisting only of 1. Applying the ratio test using this subsequence gives $|z| < 1$.

3.11 Exercise 15

Theorems 3.22, 3.23, and 3.25(a) require no changes in their proofs, since the Cauchy criterion is applicable for \mathbb{R}^k . Theorem 3.33(a, b) also require no changes once we have the comparison test for \mathbb{R}^k . For Theorem 3.33(c), we can take $a \in \mathbb{R}^k$ such that all of its components are $\frac{1}{n}$ or $\frac{1}{n^2}$.

Theorem 3.34(a, b) just need to be modified to use $\frac{|a_{n+1}|}{|a_n|}$. Theorem 3.42 needs to be modified to use the dot product, but then it follows from applying the \mathbb{R} version of 3.42 to the components of the dot product sum. Theorems 3.45, 3.47, and 3.55 require no changes to their proofs.

4 Continuity

4.1 Exercise 1

Continuity implies $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$, since we can choose h to be within δ of x such that $|f(x+h) - f(x) + f(x) - f(x-h)| \leq |f(x+h) - f(x)| + |f(x-h) - f(x)| < \epsilon$. However, the converse (as asked in the question) need not be true, since we don't have to have $\lim_{h \rightarrow 0} f(x+h) = f(x) = \lim_{h \rightarrow 0} f(x-h)$. For example, consider $x \neq 0 \implies f(x) = \frac{1}{|x|}$, $f(0) = 0$.

4.2 Exercise 2

Suppose p is a limit point of E . Then there is a sequence $(x_n) \in E \mid \lim_{n \rightarrow \infty} x_n = p$. Since f is continuous, we have that $\lim_{n \rightarrow \infty} f(x_n) = f(p)$, which implies that $f(p)$ is a limit point of $f(E)$ giving us that $f(E) \subset \overline{f(E)}$.

4.3 Exercise 3

Similar to Exercise 2: if p is a limit point of $Z(f)$, then there exists some sequence $(x_n) \in E \mid \lim_{n \rightarrow \infty} x_n = p$. Since f is continuous, we have that $\lim_{n \rightarrow \infty} f(x_n) = f(p)$. Then it follows that $x_n \in Z(f) \implies f(x_n) = 0 \implies f(p) = 0$.

4.4 Exercise 4

The fact that $f(E)$ is dense in $f(X)$ follows from Exercise 2, since $X = \overline{E}$. Similarly, $\lim_{n \rightarrow \infty} g(p_n) = g(p) \implies \lim_{n \rightarrow \infty} f(p_n) = g(p)$ since $p_n \in E$. Thus, $g(p) = f(p)$ for all $p \in X$.

4.5 Exercise 5

If f is defined on an open set in \mathbb{R}^1 , then it need not be defined at its endpoints. For example, consider $f(x) = \frac{1}{x}$ defined on $(0, 1)$. However, if f is defined on a closed subset $E \subset \mathbb{R}^1$, then we can take $g(x)$ to be $f(x)$ on E and straight line "continuations" everywhere else.

4.6 Exercise 6

f is a bijection from E to its graph $G(E)$. If f is continuous, then we can take the inverse image of an open cover of $G(E)$ to get an open cover of E . Since E is compact, this open cover must have a finite subcover whose image under f will be a finite subcover for $G(E)$, thereby giving the compactness of $G(E)$.

I looked up a hint on the reverse direction. Consider an infinite (finite case presents no issues) closed set $V \subset G(E)$. Take some arbitrary subsequence $(x_k, f(x_k)) \in V$. By the compactness of $G(E)$, this subsequence has a limit point $(x, f(x)) \in G(E)$, and this limit point is contained in V since V is closed.

Thus, $f^{-1}(V)$ also contains $x_k \rightarrow x$, implying that $f^{-1}(V)$ contains all of its limit points and is therefore closed. This shows that f is continuous.

For what it's worth, I think this argument using projections is much nicer.

4.7 Exercise 7

4.8 Exercise 8