Exercise Guide for *Analysis on Manifolds* by Michael Spivak

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About

> *"Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising from its own ashes, is its own derivative?"*
>
> - Francois de Lionnais

This book is simply too famous and too short to not work through.

1 Functions on Euclidean Space

NOTE: My notes differ from Spivak's text in that I use subscripts to denote components instead of superscripts.

1.1 Norm and Inner Product

1.1.1 Exercise 1

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} |x_i|^2 \leq \left(\sum_{i=1}^{n} |x_i| \right)^2 \implies \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} |x_i|^2$$

1.1.2 Exercise 2

We need $\sum_{i=1}^{n} x_i y_i = |x||y|$. Linear dependence by itself is not enough, since if x and y are opposite sign we see that the lefthand sum will be negative. Thus, we need linear dependence as well as x and y having the same sign.

1.1.3 Exercise 3

The proof is identical to the $|x+y|$ case, except now we have a $-2\sum_{i=1}^{n} x_i y_i$ term. Thus, equality holds when x and y are linearly dependent and have opposite signs.

1.1.4 Exercise 4

$||x| - |y||^2 = |x|^2 + |y|^2 - 2|x||y|$. Since $|x||y| \geq \langle x, y \rangle$, we have the desired inequality.

1.1.5 Exercise 5

$$|z - x| = |z - y + y - x| \leq |z - y| + |y - x|$$

Geometrically, this is just the fact that any sidelength of a triangle must be bounded by the sum of the other two sidelengths.

1.1.6 Exercise 6

(a) We can proceed as Spivak hints by noting that for the $\int_a^b (f - \lambda g)^2 > 0$ case, the proof is identical to that of Theorem 1-1 (2). For the $\int_a^b (f - \lambda g)^2 = 0$ case (which is when equality is obtained), we can use the fact that $(f - \lambda g)^2 = 0$ almost everywhere.

However, I think it's a little smoother to handle both cases at once:

$$\int_a^b (f-\lambda g)^2 = \int_a^b f^2 - 2\lambda \int_a^b fg + \lambda^2 \int_a^b g^2 \geq 0$$

$$\int_a^b f^2 - \frac{2\left(\int_a^b fg\right)^2}{\int_a^b g^2} + \frac{\left(\int_a^b fg\right)^2}{\int_a^b g^2} \geq 0 \quad \text{for } \lambda = \frac{\int_a^b fg}{\int_a^b g^2}$$

$$\sqrt{\int_a^b f^2 \int_a^b g^2} \geq \left|\int_a^b fg\right|$$

(b) Equality does not necessarily imply that $f = \lambda g$, since we could consider f and g to be 0 everywhere except two points a and b such that $f(a) \neq \lambda g(a)$ and $f(b) \neq \lambda g(b)$. However, if f and g are continuous, then $\int_a^b (f-\lambda g)^2 = 0$ implies that $f = \lambda g$.

(c) Define $f(m) = x_i$ and $g(m) = y_i$ for $m \in [i, i+1)$. Then $\int_1^n fg = \sum_1^n x_i y_i$ (we can break up the integral at the points of discontinuity) and Theorem 1-1 (2) follows.

1.1.7 Exercise 7

(a) If T is inner product preserving then we have $\langle Tx, Tx\rangle = \langle x, x\rangle$, so T is norm preserving. If T is norm preserving, then

$$\begin{aligned}
\langle T(x-y), T(x-y)\rangle &= |T(x)|^2 + |T(y)|^2 - 2\langle T(x), T(y)\rangle \\
\langle x-y, x-y\rangle &= |x|^2 + |y|^2 - 2\langle x, y\rangle \\
\implies \langle T(x), T(y)\rangle &= \langle x, y\rangle
\end{aligned}$$

so T is inner product preserving.

(b) Since $Tx = Ty \implies T(x-y) = 0 \implies |x-y| = 0$, T is injective. Furthermore, $Tx = 0 \implies |x| = 0$, so the nullspace of T is trivial and T is thus surjective. Now if we consider $T^{-1}y = x$, then we have $\langle y, y\rangle = \langle TT^{-1}y, TT^{-1}y\rangle = \langle T^{-1}y, T^{-1}y\rangle$.

1.1.8 Exercise 10

Let $\|T\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m T_{ij}^2}$ (Frobenius norm). Then we have

$$|Th|^2 = \sum_{i=1}^n \left(\sum_{j=1}^m T_{ij} h_j\right)^2 \leq \sum_{i=1}^n \sum_{j=1}^m T_i j^2 \sum_{j=1}^m h_j^2 = \|T\|_F^2 |h|^2$$

so letting $M = \|T\|_F$ gives the desired inequality.

1.1.9 Exercise 12

Linearity and injectivity of T follow from linearity of inner product. To see surjectivity, we note that any element $f \in (\mathbb{R}^n)^*$ is determined entirely by $f(e_1), ..., f(e_n)$ due to linearity. Thus, $f(y) = \langle x, y\rangle$ for the unique x satisfying $x_i = f(e_i)$.

1.1.10 Exercise 13

Expanding $\langle x + y, x + y\rangle$ gives the desired result.

1.2 Subsets of Euclidean Space

1.2.1 Exercise 14

Any point a in the union is also in some set which contains an open set around a, and by the definition of union this open set is also in the union. For finite intersection, if a point is in two open sets, then both sets must contain some open rectangles around that point; the smaller of these rectangles is in the intersection of both sets. For infinite intersection, we can consider the intersection of $(-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$, as it consists only of the single point 0.

1.2.2 Exercise 17

The procedure hinted by Spivak seems to be to split the square into 4 quadrants and then select a point from each quadrant while satisfying the constraint, then repeat this procedure indefinitely (split the 4 quadrants into 16 quadrants, etc.). However, I'm not sure how to make this procedure more rigorous.

1.2.3 Exercise 19

We can use the facts that the rationals are dense in $\mathbb{R}$ and that closed sets contain all of their limit points (although neither fact is proved so far in this book) to immediately get the desired result.

1.2.4 Exercise 21

(a) Since A is closed, A^c is open, which means x has an open rectangle (and thus, an open ball) around it that is contained within A^c. Therefore, there exists $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.

(b) Suppose that for every $d > 0$, we could choose $y_i \in A$ and $x_i \in B$ such that $|y_i - x_i| < d$. This would imply that we could pick a sequence $\{y_n\} \in A$ and a sequence $\{x_n\} \in B$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$. Since B is compact, this limit must be a point in B, which contradicts the disjointness of A and B.

(c) Consider $A = \{(0, i) \mid i \in \mathbb{N}\}$ and $B = \{(0, i + \frac{1}{i}) \mid i \in \mathbb{N}\}$. For any d, we can choose i such that $\frac{1}{i} < d$.

1.2.5 Exercise 22

This is easier to argue with open/closed balls instead of open/closed rectangles (for me). Every point $c \in C$ must have an open ball $B_\delta(c) \subset U$ for some $\delta > 0$. For each such ball, consider instead $B_r(c)$ with $r < \delta$. The closure of $B_r(c)$ is a closed ball that is contained within U. Now, since C is compact, it can be covered by finitely many of these closed balls. Since closed sets are closed under finite union, we can take D to be the union of these balls to get a compact set whose interior contains C.

1.3 Functions and Continuity

1.3.1 Exercise 23

Suppose $\lim_{x \to a} f(x) = b$. Then we can choose δ such that $|x - a| < \delta$ implies $|f(x) - b| < \epsilon$ for any $\epsilon > 0$. Rewriting $|f(x) - b|$ as $\sqrt{\sum_{i=1}^m (f_i(x) - b_i)^2}$ gives $|f_i(x) - b_i| < \epsilon$ for the same δ, so we get $\lim_{x \to a} f_i(x) = b_i$. The other direction reverses the steps after choosing $|f_i(x) - b_i| < \frac{\epsilon}{\sqrt{m}}$.

1.3.2 Exercise 24

This is basically the same as the previous exercise.

1.3.3 Exercise 25

From Exercise 1-10 we have that there exists M such that $|T(x)| \leq M|x|$. Thus, for any $\epsilon > 0$ and $a \in \mathbb{R}^n$, we can choose $\delta = \frac{\epsilon}{M}$. Using this δ, we get $|x - a| < \delta$ implies $|T(x) - T(a)| = |T(x - a)| < \epsilon$, so T is continuous.

1.3.4 Exercise 28

Choose any x on the boundary of A and let $f(y) = \frac{1}{|y-x|}$. Since f is the quotient of two continuous functions ($g(y) = 1$ and $h(y) = |y - x|$) with non-zero denonimator, f is continuous. Furthermore, by construction we can choose y arbitrarily close to x, so f is unbounded.

1.3.5 Exercise 29

Since f is continuous, $f(A)$ is compact. By Heine-Borel, $f(A) \subset \mathbb{R}$ is closed and bounded, so it contains its maximum and minimum.

1.3.6 Exercise 30

Let $x_0 = a$. We have that

$$\begin{aligned}
\sum_{i=1}^{n} o(f, x_i) &= \lim_{\delta \to 0} \sum_{i=1}^{n} [M(f, x_i, \delta) - m(f, x_i, \delta)] \\
&\leq \lim_{\delta \to 0} \sum_{i=1}^{n} [M(f, x_i, \delta) - M(f, x_{i-1}, \delta)] \\
&\leq \lim_{\delta \to 0} M(f, x_n, \delta) - m(f, a, \delta) \\
&\leq f(b) - f(a)
\end{aligned}$$

Where we used the fact that f is increasing to get $m(f, x_i, \delta) \geq M(f, x_{i-1}, \delta)$.

2 Differentiation

2.1 Basic Definitions

2.1.1 Exercise 1

Let the derivative of f at a be the linear operator λ. Then, by problem $1 - 10$, we have that $|\lambda(h)| \leq M|h|$ for some $M \in \mathbb{R}$. Thus,

$$\begin{aligned}
|(f(a+h) - f(a)| &= \frac{|f(a+h) - f(a)|}{|h|} * |h| \\
&\leq \frac{|f(a+h) - f(a) - \lambda(h)| + |\lambda(h)|}{|h|} * |h| \\
\implies \lim_{h \to 0} |f(a+h) - f(a)| &\leq \lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)| + |\lambda(h)|}{|h|} \lim_{h \to 0} |h| \\
&\leq M * 0 = 0
\end{aligned}$$

2.1.2 Exercise 2

If f is independent, we can define $g = f(x, y_0)$ for some arbitrary y_0. If there exists g such that $f(x, y) = g(x)$ for all x, y, then we have $f(x, y_1) = g(x) = f(x, y_2)$ so f is independent of the second variable. In this case, $f'(a, b) = g'(a)$.

2.1.3 Exercise 3

A function that is independent of both variables must, by construction, be constant.

2.1.4 Exercise 4

(a) When $t < 0$, we have

$$h(t) = -t|x|g\left(\frac{tx}{-t|x|}\right) = t|x|g\left(\frac{x}{|x|}\right)$$

which is the same as $h(t)$ when $t > 0$, so $h(t)$ is differentiable with derivative $f(x)$.

(b) Since $g(0, 1) = g(1, 0) = 0$, we have that $f(h, 0) = f(0, k) = 0$. Letting $\lambda = Df(0, 0)$, we have that

$$\begin{aligned}
\lim_{(h,0) \to 0} \frac{|\lambda(h, 0)|}{|h|} &= \lim_{(h,0) \to 0} \frac{|h|\lambda(1, 0)}{|h|} \\
&= \lambda(1, 0) = 0
\end{aligned}$$

if λ exists. Similarly, considering $(0,k) \to 0$ gives $\lambda(0,1) = 0$, so $\lambda = 0$. As a result, we have that f is differentiable at $(0,0)$ only if

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{|(h,k)|} = \lim_{(h,k)\to(0,0)} \left| g\left(\frac{(h,k)}{|(h,k)|} \right) \right| = 0$$

which implies that $g = 0$ (we can consider $t(x,y)$ as $t \to 0$ for every (x,y) on the unit circle).

2.1.5 Exercise 5

Apply the previous exercise with $g(x,y) = x|y|$.

2.1.6 Exercise 8

If f is differentiable, then

$$\begin{aligned}
\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} &= 0 \\
&= \lim_{h\to 0} \frac{|(f_1(a+h) - f_1(a) - \lambda_1(h), f_2(a+h) - f_2(a) - \lambda_2(h))|}{|h|} \\
&\geq \lim_{h\to 0} \frac{|f_1(a+h) - f_1(a) - \lambda_1(h)|}{|h|}
\end{aligned}$$

so f_1 and f_2 are both differentiable. The reverse direction is a straightforward application of the triangle inequality.

2.1.7 Exercise 9

(a) If f is differentiable at a, then we can let $g(x) = f(a) + f'(a)(x-a)$. For the other direction,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - a_0 - a_1 h}{h} &= 0 \\
\implies \lim_{h\to 0} \frac{f(a+h) - a_0}{h} &= a_1 \\
\implies a_0 &= f(a)
\end{aligned}$$

so f is differentiable at a.

(b) We can break up the n^{th} order limit into

$$\begin{aligned}
\lim_{x\to a} \frac{f(x) - g(x)}{(x-a)^n} &= \lim_{x\to a} \frac{f(x) - \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i}{(x-a)^n} + \frac{f^{(n)}(a)}{n!} \\
&= \frac{f^{(n)}(a)}{n!} - \frac{f^{(n)}(a)}{n!} = 0
\end{aligned}$$

where the last step follows from repeated application of L'Hopital's.

2.2 Basic Theorems

2.2.1 Exercise 11

(a) Let G be a function such that $G' = g$. Then

$$\begin{aligned} f'(x,y) &= g(x+y)(\pi_1'(x,y) + \pi_2'(x,y)) \\ &= g(x+y)((1,0)+(0,1)) = (g(x+y), g(x+y)) \end{aligned}$$

(b) Following the setup of (a), we have

$$\begin{aligned} f'(x,y) &= g(xy)(y\pi_1'(x,y) + x\pi_2'(x,y)) \\ &= g(xy)(y,x) \end{aligned}$$

2.2.2 Exercise 12

(a) Since $f(h,k) = hkf(1,1)$, the desired result follows immediately from $\lim_{(h,k)\to 0} \frac{|hk|}{|(h,k)|} = 0$.

(b) We have that $f(a+h, b+k) - f(a,b) = f(a,k) + f(h,b) + f(h,k)$ from bilinearity, so $Df(a,b)(x,y) = f(a,y) + f(x,b)$ as desired.

(c) The function $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ from Theorem 2-3 is bilinear, so it is just a special case of (b).

2.2.3 Exercise 14

(a) We can bound the multilinear case with the bilinear case:

$$\begin{aligned} \lim_{h\to 0} \frac{|f(a_1,...,h_i,...,h_j,...,a_k)|}{|h|} &\leq \lim_{h\to 0} \frac{|f(a_1,...,h_i,...,h_j,...,a_k)|}{|(h_i,h_j)|} \\ &\leq \lim_{(h_i,h_j)\to 0} \frac{|f(a_1,...,h_i,...,h_j,...,a_k)|}{|(h_i,h_j)|} \\ &= 0 \end{aligned}$$

(b) Expanding $f(a_1+h_1,...,a_n+h_n) - f(a_1,...,a_n) - Df(a_1,...,a_n)(h_1,...,h_n)$ leaves only terms that are at least bilinear, so by part (a) we are done.

2.2.4 Exercise 16

Using the fact that $Dx = x$ (from Theorem 2-3), we have that

$$\begin{aligned} & f'(f^{-1}) \circ f^{-1\prime}(x) = x \\ \implies & f^{-1\prime}(x) = [f'(f^{-1}(x))]^{-1} \end{aligned}$$

2.3 Partial Derivatives

2.3.1 Exercise 19

Since $f(1,y) = 1$, $D_2 f(1,y) = 0$.

2.3.2 Exercise 21

(a) The first term of f is independent of y, so we need not consider it. Letting G_2 be the antiderivative of g_2, we have that the second term is equivalent to $G_2(x, y) - G_2(x, 0)$. Differentiating this with respect to y yields $g_2(x, y)$ as desired.

(b) Replacing the first integrand with $g_1(t, y)$ and the second integrand with $g_2(0, t)$ will give $D_2 f(x, y) = g_1(x, y)$.

(c) The functions $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$ and $f(x, y) = xy$ will work (obtained by integrating the desired partial derivatives with respect to x and y).

2.3.3 Exercise 22

Applying the mean value theorem to each variable gives the desired results.

2.3.4 Exercise 23

(a) Consider (x_1, y_1) and (x_2, y_2). Since $D_1 f = D_2 f = 0$, we have that f is constant along the lines from (x_1, y_1) to $(-\epsilon, y_1)$, $(-\epsilon, y_1)$ to $(-\epsilon, y_2)$, and $(-\epsilon, y_2)$ to (x_2, y_2) by the logic of the previous exercise (where $\epsilon > 0$). Thus, $f(x_1, y_1) = f(x_2, y_2)$ for all (x_1, y_1) and (x_2, y_2).

(b) Take the following function:

$$f(x, y) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0, y > 0 \\ -1 & x \geq 0, y < 0 \end{cases}$$

This function is not independent of y, since $f(0, 1) \neq f(0, -1)$. However, $D_2 f = 0$ (due to $x \geq 0, y = 0$ not being in A).

2.3.5 Exercise 25

Since $\exp(-x^{-2})$ is C^∞ for all $x \neq 0$, f is also C^∞ for all such x. For $x = 0$, we can proceed via L'Hopital's as suggested:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{\exp(-h^{-2})}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{h}}{\exp(h^{-2})} \\ &= \lim_{h \to 0} \frac{h}{2\exp(h^{-2})} \\ &= 0 \end{aligned}$$

Repeating this procedure indefinitely shows that $f^{(i)}(0) = 0$ and that f is C^∞.

2.3.6 Exercise 26

(a) The strategy from the previous exercise should work.

(b) Consider the following function:

$$f(x) = \begin{cases} \exp\bigl(-x^{-2}\bigr)\exp\bigl(-(x-\epsilon)^{-2}\bigr) & x \in (0,\epsilon) \\ 0 & x \notin (0,\epsilon) \end{cases}$$

Now we can define $g(x) = \int_0^x f / \int_0^\epsilon f$ as suggested. Since $f(x) = 0$ for all $x \geq \epsilon$, we have that $\int_0^x f = \int_0^\epsilon f$. Similarly, $\int_0^x f = 0$ for $x \leq 0$.

(c) The fact that g is C^∞ follows from f being C^∞. Similarly, plugging $\frac{x_i - a_i}{\epsilon}$ into f gives $\exp\Bigl(-\bigl(\frac{x_i-(a_i+\epsilon)}{\epsilon}\bigr)^{-2}\Bigr)\exp\Bigl(-\bigl(\frac{x_i-(a_i-\epsilon)}{\epsilon}\bigr)^{-2}\Bigr)$, from which it can be seen that f is positive only on $(a_1-\epsilon, a_1+\epsilon)\times ... \times(a_n-\epsilon, a_n+\epsilon)$ and 0 everywhere else.

(d) Since C is compact and in $\mathbb{R}^n$, we can choose finitely many closed rectangles around points in C such that these rectangles are contained in A and their union covers C. For each such rectangle, we can then mimic the style of the function g in part (c) to produce a C^∞ function that is non-zero only in that rectangle. Finally, we can take the sum of these functions to get a function that is C^∞ and positive on C while being 0 outside of a closed set contained in A.

(e) The function described in (d) satisfies $f(x) \geq \epsilon$ (for some $\epsilon > 0$) for all $x \in C$ since C is compact. Thus, we can consider $g \circ f$ with the aforementioned ϵ (g is described in (b)) to get the desired function.