

Exercise Guide for *Principles of Mathematical Analysis (3rd Ed.)* by Walter Rudin

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About

“It is not possible to overstate how good this book is. I tried to give it uncountably many stars but they only have five. Five is an insult. I’m sorry Dr. Rudin...” - Amazon Review

An actual solution manual for the book can be found [here](#). What follows are notes I took as I did exercises (they’re more like hints towards my thinking than solutions) while working through the book on my own.

1 The Real and Complex Number Systems

1.1 Exercise 1

If $rx = q$ or $r + x = q$ for some rational q , then subtracting r from q or dividing q by r yields x rational, which is a contradiction.

1.2 Exercise 2

We can first show that $\sqrt{3}$ is irrational by seeing that $\frac{a^2}{b^2} = 3 \implies 3|a, 3|b$. Then, since $12 = 3 * 2^2$, we have that $\sqrt{12}$ is irrational as well.

1.3 Exercise 4

If $\alpha > \beta$ then α would be an upper bound as well.

1.4 Exercise 5

$\forall x \in A, -x \leq \sup -A$ and $\forall \epsilon \in \mathbb{R}, \exists x \in A | \sup -A + \epsilon < -x \leq \sup -A$. Negating the last inequality gives $\inf A = -\sup -A$.

1.5 Exercise 6

(a) Follows from $m = \frac{np}{q}$.

(b) Put $r = \frac{m}{n}, s = \frac{p}{q}$. Then $b^r b^s = b^{\frac{mq}{nq}} b^{\frac{np}{nq}}$. Pulling out $\frac{1}{nq}$ gives the desired result.

(c) b^r is an upper bound since $b > 1$, and if it were not the supremum we could choose $t < r$ such that $b^t > b^r$. This is not possible since again, $b > 1$.

(d) Every element in $B(x + y)$ can be expressed as $b^{s+t} = b^s b^t$ $s \leq x, t \leq y$. If $\sup B(x + y) = \alpha < \sup B(x) \sup B(y)$, then $b^s b^t \leq \alpha \implies B(x) \leq \alpha b^{-t} \implies B(y) \leq \frac{\alpha}{B(x)} \implies B(x)B(y) \leq \alpha$.

1.6 Exercise 7

(a) $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \dots + 1) \geq n(b - 1)$ since $b > 1$.

(b) Plug $b^{\frac{1}{n}}$ into (a).

(c) Plug $n > \frac{b-1}{t-1}$ into (b).

(d) Using (c) gives that we can choose n such that $b^{\frac{1}{n}} < y b^{-w} \implies b^{w+\frac{1}{n}} < y$.

(e) We can take the reciprocal of (c) and do the same as in (d).

(f) If $b^x > y$ we can apply (e) for a contradiction, if $b^x < y$ we can apply (d) for a contradiction.

(g) Supremum is unique.

1.7 Exercise 8

Suppose $(0, 1) < (0, 0)$. Then $(0, -1) < (0, 0)$ after multiplying by $(0, 1)$ twice yields a contradiction. Similarly, assuming the opposite yields $(-1, 0) > (0, 0)$.

1.8 Exercise 9

Does exhibit least upper-bound property since you can take $(\sup a_i, \sup b_i)$.

1.9 Exercise 10

Exception is 0.

1.10 Exercise 11

Take $w = \frac{1}{|z|}z$ and $r = |z|$ when $|z| \neq 0$. w and r are not uniquely determined; take $z = 0$ for example.

1.11 Exercise 12

By strong induction:

$$\begin{aligned} |z_1 + \dots + z_{n+1}| &\leq |z_1 + \dots + z_n| + |z_{n+1}| \\ &\leq |z_1| + \dots + |z_{n+1}| \end{aligned}$$

1.12 Exercise 13

$$\begin{aligned} |x - y|^2 &= x\bar{x} - 2|x||y| + y\bar{y} \\ &\geq (|x| - |y|)^2 \end{aligned}$$

2 Basic Topology

2.1 Exercise 1

The empty set has no elements, so all of its elements are vacuously also elements of every set.

2.2 Exercise 2

The roots of complex polynomials with integer coefficients can be expressed as elements of the countable cross product of \mathbb{N} with itself (cross \mathbb{N} with itself n times for the coefficients, and then once more to indicate which root).

2.3 Exercise 3

If all real numbers were algebraic, then the set of algebraic numbers would be uncountable (thus contradicting Exercise 2).

2.4 Exercise 4

The set of irrational numbers is \mathbb{R}/\mathbb{Q} , which must be uncountable as otherwise \mathbb{R} would be countable.

2.5 Exercise 5

We can use $\left(\frac{n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{2n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{3n}{n+1}\right)_{n \in \mathbb{N}}$ to get the three limit points 1, 2, 3.

2.6 Exercise 6

If p is a limit point of E' , then every neighborhood of p contains a limit point q of E , and every neighborhood of q contains a point of E thereby implying that p is a limit point of E . E and E' do not need to have the same limit points, since E' could be finite and thus have no limit points.

2.7 Exercise 7

(a) If p is a limit point of $\overline{B_n}$, then every neighborhood of p contains a point $q \in A_i$. Since there are only finitely many A_i , p must be a limit point for at least one of the A_i , as an infinite number of neighborhoods of p must have non-zero intersection with some of the A_i .

(b) If we take $A_i = \left(\frac{in}{(i+1)n+1}\right)_{n \in \mathbb{N}}$, then 1 is a limit point of B_n despite not being a limit point of any of the A_i .

2.8 Exercise 8

Every point of an open set in \mathbb{R}^2 is by definition a limit point of the set, since the point must have a neighborhood contained in the set. The same is not true for closed sets, since we can just take a finite set.

2.9 Exercise 10

Every set in X is open, since any set containing p also contains $N_r(p)$ for $r < 1$. No set in X is closed, since $N_r(p) = p$ for $r < 1$. All infinite sets in X are not compact, since we can take balls of radius $r < 1$ around each point as an open cover.

2.10 Exercise 12

Take any open cover of K . There must be some open set in this cover containing 0, which means that the same set contains all but a finite number of the elements of K (since 0 is the only limit point of K). Take a union of this set as well as the finitely many other sets containing the aforementioned points to get a finite subcover.

2.11 Exercise 13

Take $\cup_{k=1}^{\infty} \{0, (\frac{n}{kn+1})_{n \in \mathbb{N}}, \frac{1}{k}\}$. This set is closed and bounded, so it is compact by Heine-Borel. Its limit points are 0 and $(\frac{1}{k})_{n \in \mathbb{N}}$.

2.12 Exercise 14

We can use $\cup_{n \in \mathbb{N}} (0, \frac{n}{n+1})$, which has no finite subcover (since we could choose $x \in (0, 1)$ larger than the largest endpoint in the finite subcover).

2.13 Exercise 15

For closed, we can take $K_i = \mathbb{N}/0, \dots, i-1$, since any $x \in K_i$ will not be in K_j if $j > x$. For bounded, we can take $K_i = (0, \frac{1}{i})$.

2.14 Exercise 16

E is by definition bounded, and E is closed since $q^2 \neq 3$ (q is rational), and $q^2 > 3 \implies \exists \epsilon | p \in N_{\epsilon}(q) \implies p^2 > 3$. Same logic gives that E is also open in \mathbb{Q} . E is, however, not compact, since we can construct an open cover consisting of $G_n = \{x | 2 < x^2 < 2 + \frac{n}{n+1}\}$.

2.15 Exercise 17

E is not countable by diagonalization. E is not dense in $[0, 1]$, since $E \cap [0, 0.1] = \emptyset$. E is not perfect, consider $N_{0.001}(0.77)$. E is closed and therefore compact by

Heine-Borel. To see closed, suppose a limit point q had a non-4/7 digit in the i^{th} decimal spot. Then we could take a neighborhood of size $10^{-(i+1)}$.

2.16 Exercise 18

Originally I thought this was no, but this can actually be done with a modified version of the Cantor set construction. See [here](#) for a discussion.

3 Numerical Sequences and Series

Definition 3.5

Since $\{p_n\} \rightarrow p \implies \forall \epsilon, \exists N | n \geq N \implies |p_n - p| < \epsilon$, we can choose $k | n_k \geq N \implies \{p_{n_k}\} \rightarrow p$. The reverse direction can be shown via contradiction of $\{p_n\} \rightarrow p$.

Examples 3.18

- (a) Density of rationals in reals.
- (b) $|s_n| < 1$, take n odd to get -1 and even to get 1.
- (c) Every subsequential limit has to converge to s .

Theorem 3.19

For all $\{n_k\}$, we have $\exists K | k \geq K \implies n_k \geq N \implies \lim_{k \rightarrow \infty} t_{n_k} - s_{n_k} \geq 0$.

Theorem 3.26

$$s_n = 1 + x + \dots + x^n \implies x s_n = x + x^2 + \dots + x^{n+1} \implies (1 - x)s_n = 1 - x^{n+1}.$$

Examples 3.40

- (a) Root test: $n \rightarrow \infty$.
- (b) Ratio test: $\frac{1}{n+1} \rightarrow 0$.
- (c) $1 \rightarrow 1$.
- (d) Ratio test: $\frac{n}{n+1} \rightarrow 1$. $z = 1$ leads to harmonic series.
- (e) Ratio test: $\frac{n^2}{(n+1)^2} \rightarrow 1$.

Example 3.53

$\sum_{k=1}^{\infty} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} < \frac{5}{6} + \sum_{k=2}^{\infty} \frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k}$. The RHS converges since $\frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k} = \frac{1}{2k^2-2k}$.

3.1 Exercise 1

All we need is the inequality $|s_n - s| \geq ||s_n| - |s||$. The converse is not true, since we can take $s_n = (-1)^n$.

3.2 Exercise 2

My original idea: $\sqrt{(n+x)^2} - n = x$. Setting $(n+x)^2 \geq n^2 + n$ gives $x^2 \geq (1-2x)n$. The last inequality is only true for all n when $x \geq \frac{1}{2}$. This implies that $\frac{1}{2}$ is the supremum of $\sqrt{n^2+n} - n$. Since $\sqrt{n^2+n} - n$ is increasing, it converges to $\frac{1}{2}$.

Better: $(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n) = n \implies \sqrt{n^2+n} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$.

3.3 Exercise 3

Clearly $s_{n+1} > s_n$. We can see that $s_n < 2$ by induction, since $s_1 < 2$ and $2 + \sqrt{s_n} < 4$. This gives that s_n is monotone and bounded, implying it converges.

3.4 Exercise 4

$$s_{2m+1} = \sum_{i=1}^m \frac{1}{2^i}, \quad s_{2m} = \sum_{i=2}^m \frac{1}{2^i} \\ \implies \limsup_{n \rightarrow \infty} s_n = 1, \quad \liminf_{n \rightarrow \infty} s_n = \frac{1}{2}$$

3.5 Exercise 5

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \sup_{\{k\}} \{ \lim_{k \rightarrow \infty} (a_{n_k} + b_{n_k}) \} \\ = \sup_{\{k\}} \{ \lim_{k \rightarrow \infty} a_{n_k} + \lim_{k \rightarrow \infty} b_{n_k} \}$$

3.6 Exercise 6

(a) $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$ diverges from comparison to harmonic series (same technique as Exercise 2).

(b) Converges, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p = \frac{3}{2}$.

(c) Converges by root test, since $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$.

(d) Converges when $|z| > 1$ and diverges otherwise. To see this, put $z = |z|e^{i\theta}$ to get $\lim_{n \rightarrow \infty} \frac{1}{1+|z|^n e^{in\theta}}$.

3.7 Exercise 7

We proceed via the ratio test.

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{n}{n+1} * \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} &= \limsup_{n \rightarrow \infty} \frac{n}{n+1} \limsup_{n \rightarrow \infty} \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} \\ &= \sqrt{\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}} \\ &< 1 \end{aligned}$$

Since $\sum a_n$ converges.

3.8 Exercise 8

Since b_n is monotonic and bounded, $|b_n| \leq B$ for all n . Then we have that $\sum a_n b_n$ converges by the comparison test, since $|a_n b_n| \leq B|a_n|$ and $B \sum a_n$ converges.

3.9 Exercise 9

(a) Applying the ratio test, we see that $|z| \limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$ when $|z| < 1$. Thus $\sum n^3 z^n$ has radius of convergence 1.

(b) Again, applying the ratio test, we see that $2|z| \limsup_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0$, implying $R = +\infty$.

(c) The ratio test is the only hammer we need: $2|z| \limsup_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| < 1$ gives $R = \frac{1}{2}$.

(d) What are the other tests again? $\frac{|z|}{3} \limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$ gives $R = 3$.

3.10 Exercise 10

The infinitely many non-zero a_n must satisfy $|a_n| \geq 1$. The radius of convergence of $\sum a_n z^n$ will be maximized when $|a_n|$ is minimized, so we can just consider the case where there are infinitely many $|a_n| = 1$. In this case, we can choose a subsequence a_{n_k} consisting only of 1. Applying the ratio test using this subsequence gives $|z| < 1$.

3.11 Exercise 15

Theorems 3.22, 3.23, and 3.25(a) require no changes in their proofs, since the Cauchy criterion is applicable for \mathbb{R}^k . Theorem 3.33(a, b) also require no changes once we have the comparison test for \mathbb{R}^k . For Theorem 3.33(c), we can take $a \in \mathbb{R}^k$ such that all of its components are $\frac{1}{n}$ or $\frac{1}{n^2}$.

Theorem 3.34(a, b) just need to be modified to use $\frac{|a_{n+1}|}{|a_n|}$. Theorem 3.42 needs to be modified to use the dot product, but then it follows from applying the \mathbb{R} version of 3.42 to the components of the dot product sum. Theorems 3.45, 3.47, and 3.55 require no changes to their proofs.

4 Continuity

4.1 Exercise 1

Continuity implies $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$, since we can choose h to be within δ of x such that $|f(x+h) - f(x) + f(x) - f(x-h)| \leq |f(x+h) - f(x)| + |f(x-h) - f(x)| < \epsilon$. However, the converse (as asked in the question) need not be true, since we don't have to have $\lim_{h \rightarrow 0} f(x+h) = f(x) = \lim_{h \rightarrow 0} f(x-h)$. For example, consider $x \neq 0 \implies f(x) = \frac{1}{|x|}$, $f(0) = 0$.

4.2 Exercise 2

Suppose p is a limit point of E . Then there is a sequence $(x_n) \in E \mid \lim_{n \rightarrow \infty} x_n = p$. Since f is continuous, we have that $\lim_{n \rightarrow \infty} f(x_n) = f(p)$, which implies that $f(p)$ is a limit point of $f(E)$ giving us that $f(\overline{E}) \subset \overline{f(E)}$.

To see that $f(\overline{E})$ can be a proper subset, consider $f : \mathbb{Z}^+ \rightarrow \mathbb{Q}$ with $f(x) = \frac{1}{x}$. Then f is continuous and $0 \notin f(\mathbb{Z}^+) = f(\mathbb{Z}^+)$.

4.3 Exercise 3

Similar to Exercise 2: if p is a limit point of $Z(f)$, then there exists some sequence $(x_n) \in E \mid \lim_{n \rightarrow \infty} x_n = p$. Since f is continuous, we have that $\lim_{n \rightarrow \infty} f(x_n) = f(p)$. Then it follows that $x_n \in Z(f) \implies f(x_n) = 0 \implies f(p) = 0$.

4.4 Exercise 4

The fact that $f(E)$ is dense in $f(X)$ follows from Exercise 2, since $X = \overline{E}$. Similarly, $\lim_{n \rightarrow \infty} g(p_n) = g(p) \implies \lim_{n \rightarrow \infty} f(p_n) = g(p)$ since $p_n \in E$. Thus, $g(p) = f(p)$ for all $p \in X$.

4.5 Exercise 5

If f is defined on an open set in \mathbb{R}^1 , then it need not be defined at its endpoints. For example, consider $f(x) = \frac{1}{x}$ defined on $(0, 1)$. However, if f is defined on a closed subset $E \subset \mathbb{R}^1$, then E^c is an open set in \mathbb{R} and can thus be decomposed into the union of a countable number of open intervals (a_n, b_n) . We can thus take g to be $g(x) = \frac{b_n - x}{b_n - a_n} f(a_n) + (1 - \frac{b_n - x}{b_n - a_n}) f(b_n)$ (the straight line interpolation between $f(a_n)$ and $f(b_n)$).

4.6 Exercise 6

f is a bijection from E to its graph $G(E)$. If f is continuous, then we can take the inverse image of an open cover of $G(E)$ to get an open cover of E . Since E is compact, this open cover must have a finite subcover whose image under f will be a finite subcover for $G(E)$, thereby giving the compactness of $G(E)$.

I looked up a hint on the reverse direction. Consider an infinite (finite case presents no issues) closed set $V \subset G(E)$. Take some arbitrary subsequence $(x_k, f(x_k)) \in V$. By the compactness of $G(E)$, this subsequence has a limit point $(x, f(x)) \in G(E)$, and this limit point is contained in V since V is closed. Thus, $f^{-1}(V)$ also contains $x_k \rightarrow x$, implying that $f^{-1}(V)$ contains all of its limit points and is therefore closed. This shows that f is continuous.

For what it's worth, I think this argument using projections is much nicer.

4.7 Exercise 7

Suppose for any M that $\exists x, y \mid f(x, y) > M$ (we consider only the case where $x > 0$, as the other case is identical). Then we can solve the resulting quadratic to see that, if such x and y exist, then $x > \frac{y^2(1+\sqrt{1-4M^2})}{2M}$. However, $\sqrt{1-4M^2}$ is not defined in \mathbb{R} for $M > \frac{1}{2}$, so f must be bounded. Performing the same analysis for g yields $x > \frac{y^2(1+\sqrt{1-4y^2M^2})}{2M}$. Since y can be chosen to make the inequality for x have a solution in \mathbb{R} , g is unbounded.

To show that f is discontinuous at $(0, 0)$, we need only consider the sequence consisting of $(0, \frac{n}{n+1})$ to see that $\lim_{n \rightarrow \infty} f(0, \frac{n}{n+1}) = 1 \neq 0$. Plugging in $y = ax + b$ leads to f and g being quotients of two polynomials with non-zero denominator, indicating that they're both continuous.

4.8 Exercise 8

Suppose f is not bounded. Then there is a sequence $f(x_n) \mid \forall N, \exists m, n \geq N \mid |f(x_n) - f(x_m)| > \epsilon$ for some ϵ , since otherwise $f(x_n)$ would converge to some point of \mathbb{R} . As f is uniformly continuous, this means that $|x_n - x_m| > \delta$ for infinitely many n, m . However, that would then imply that E is not bounded, which is a contradiction. Thus, f is bounded on E .

If E is not bounded, we can just take $f(x) = x$.

4.9 Exercise 9

Let E consist of all $x, y \mid d_X(x, y) < \delta$. Then $\text{diam} E < \delta$. Similarly, if $\forall x, y \mid d_Y(f(x), f(y)) < \epsilon$, then $\text{diam} f(E) < \epsilon$.

4.10 Exercise 10

Suppose f is not uniformly continuous. Then there is a sequence $x_n \in X \mid x_n \rightarrow x$, but $\forall N, \exists m, n \geq N \mid d_Y(f(x_n), f(x_m)) > \epsilon$ for some $\epsilon > 0$. This, however, makes $f(x_n)$ an infinite subset of $f(X)$ which does not have a limit point, thereby contradicting the fact that $f(X)$ is compact.

4.11 Exercise 11

The first part of this exercise is basically what I was doing for Exercises 8 and 10. Since f is uniformly continuous, $\exists \delta \mid d_X(x_n, x_m) < \delta \implies d_Y(f(x_n), f(x_m)) < \epsilon$. Since (x_n) Cauchy converges, we can make $d_X(x_n, x_m)$ arbitrarily small, which then implies that we can make $d_Y(f(x_n), f(x_m))$ arbitrarily small, indicating that $f(x_n)$ Cauchy converges as well.

4.12 Exercise 12

To state it more precisely: if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both uniformly continuous, then $g \circ f$ is also uniformly continuous.

From uniform continuity of g , $\exists \delta \mid d_Y(y_1, y_2) < \delta \implies d_Z(g(y_1), g(y_2)) < \epsilon$. Since f is uniformly continuous, $\exists \delta' \mid d_X(x_1, x_2) < \delta' \implies d_Y(f(x_1), f(x_2)) < \delta$. The existence of this δ' gives us that $g \circ f$ is uniformly continuous.

4.13 Exercise 13

Suppose p is a limit point of E and $x_n \in E \mid x_n \rightarrow p$. Then $f(x_n)$ Cauchy converges to a point q in the codomain of f . We can simply take $g(p) = q$ whenever $p \notin E$ to get a continuous extension of f . Since this proof depends only on the convergence of the Cauchy sequence $f(x_n)$ to a point in the codomain, it will hold for the codomain being any complete metric space.

4.14 Exercise 16

The function $[x]$ has a simple discontinuity at every integer x , since the left-hand limit is $x - 1$ and the right-hand limit is x . Similarly, the function (x) also has a simple discontinuity at every integer, since the left-hand limit is 1 and the right-hand limit is 0.

4.15 Exercise 17

We proceed as hinted in the text. The two types of simple discontinuity we need to consider are $f(x-) \neq f(x+)$ and $f(x-) = f(x+) \neq f(x)$. For the first case, suppose (WLOG) that $f(x-) < f(x+)$. Then we can construct a rational triple (p, q, r) such that

$$\begin{aligned} f(x-) &< p < f(x+) \\ a < q < t < x &\implies f(t) < p \\ x < t < r < b &\implies f(t) > p \end{aligned}$$

To see that such a triple can only be associated with one such x , consider $x' = x + \epsilon$ with $\epsilon > 0$ (the other case is identical). Then we can choose $t \in (x, x')$ with $q < x < t < r < x'$, which means $t > q$ does not imply $f(t) < p$. This handles simple discontinuities of the form $f(x-) \neq f(x+)$.

We can similarly handle the case where $f(x-) = f(x+) \neq f(x)$. Suppose (WLOG) that $f(x) > f(x+)$; we can then construct a rational triple (p, q, r) such that

$$\begin{aligned} f(x+) &< p < f(x) \\ a < q < t < x &\implies f(t) < p \\ x < t < r < b &\implies f(t) < p \end{aligned}$$

Again, such a triple can only be associated with a single x , since $x \in (x, x+\epsilon)$ and $f(x) > p$. Therefore f has only countably many simple discontinuities.

4.16 Exercise 23

From the definition of convexity, we have that

$$\begin{aligned} f(\lambda x + (1 - \lambda)p) &\leq \lambda f(x) + (1 - \lambda)f(p) \\ f(\lambda x + (1 - \lambda)p) - f(p) &\leq \lambda(f(x) - f(p)) \\ f(p) - f(\lambda x + (1 - \lambda)p) &\leq \lambda(f(p) - f(x)) \\ \implies \lim_{\lambda \rightarrow 0} f(\lambda x + (1 - \lambda)p) &= f(p) \end{aligned}$$

Since $\lim_{\lambda \rightarrow 0} \lambda x + (1 - \lambda)p = p$ for all choices of x , we have that f is continuous.

5 Differentiation

5.1 Exercise 1

We have that

$$\begin{aligned} |f(x) - f(y)| &\leq (x - y)^2 = |x - y|^2 \\ \frac{|f(x) - f(y)|}{|x - y|} &\leq |x - y| \\ \implies f'(x) &= 0 \forall x \end{aligned}$$

Since we can write $\frac{f(t) - f(x)}{t - x} = f'(x) + u(t)$ with $\lim_{t \rightarrow x} u(t) \rightarrow 0$, we have that

$$\begin{aligned} f(t) - f(x) &= (t - x)u(t), \quad f(t) - f(y) = (t - y)v(t) \\ f(y) - f(x) &= (y - x)u(y), \quad f(x) - f(y) = (x - y)v(x) \implies u(y) = v(x) \forall x, y \\ f(y) - f(x) &= 0 \implies f(x) = f(y) \forall x, y \end{aligned}$$

Whoops, I did this before reading the mean value theorem section - this problem follows immediately from applying the mean value theorem after showing $f'(x) = 0$.

5.2 Exercise 2

Take $x, t \in (a, b)$ with $t > x$. Applying the mean value theorem to f on $[x, t]$, we get $f(t) - f(x) = (t - x)f'(y)$ for some $y \in (x, t)$. Since $f'(y) > 0 \implies f(t) - f(x) > 0$, f is strictly increasing on (a, b) . We can prove $g = f^{-1}$ is differentiable directly

$$\begin{aligned} \lim_{t \rightarrow x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} &= \lim_{t \rightarrow x} \frac{t - x}{f(t) - f(x)} \\ &= \frac{1}{f'(x)} \end{aligned}$$

5.3 Exercise 3

Suppose (WLOG) that $x_2 > x_1$ but $f(x_2) = f(x_1)$. Then we have that

$$\begin{aligned} x_2 + \epsilon g(x_2) &= x_1 + \epsilon g(x_1) \\ (x_2 - x_1) + \epsilon(g(x_2) - g(x_1)) &= 0 \\ 1 + \epsilon g'(x) &= 0 \quad x \in (x_1, x_2) \\ 1 + \epsilon g'(x) &\geq 1 - \epsilon |g'(x)| \\ &> 0 \quad \forall \epsilon < \frac{1}{M} \end{aligned}$$

Where the penultimate step follows from the mean value theorem. Thus, we can choose an ϵ such that $f(x_2) \neq f(x_1)$, which means we can make f injective.

5.4 Exercise 4

Let $f(x) = C_0 + C_1x + \dots + C_nx^n$ and $g(x) = C_0x + \frac{C_1}{2}x^2 + \dots + \frac{C_n}{n+1}x^{n+1}$. Then $g'(x) = f(x)$. Applying the mean value theorem to $g(x)$ on $[0, 1]$ yields that there is an x such that $g'(x) = f(x) = g(1) - g(0) = 0$, so f has a root in $(0, 1)$.

5.5 Exercise 5

It looks like the mean value theorem is this chapter's ratio test; you can guess how this will go. By the mean value theorem, $f'(y) = f(x+1) - f(x)$ for $y \in [x, x+1]$. Thus we have $\lim_{x \rightarrow \infty} g(x) = \lim_{y \rightarrow \infty} f'(y) = 0$.

5.6 Exercise 6

Consider $x > y > 0$. By the mean value theorem (surprise), we have that

$$\begin{aligned} f(x) - f(y) &= f'(a)(x - y) \quad a \in (x, y) \\ &< f'(x)(x - y) \\ \lim_{y \rightarrow 0} \frac{f(x) - f(y)}{x - y} &< \lim_{y \rightarrow 0} f'(x) \\ \frac{f(x)}{x} &< f'(x) \end{aligned}$$

Differentiating g , we get

$$\begin{aligned} g'(x) &= \frac{xf'(x) - f(x)}{x^2} \\ f'(x) &> \frac{f(x)}{x} \implies g'(x) > 0 \end{aligned}$$

So g is monotonically increasing.

5.7 Exercise 7

If f and g are real, then the result follows immediately from L'Hopital's, since the existence of $f'(x), g'(x)$ imply that f and g are continuous (so the requirement that $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ is satisfied). More generally, for complex functions, we have

$$\begin{aligned} \lim_{t \rightarrow x} \frac{f(t)}{g(t)} &= \lim_{t \rightarrow x} \frac{f_1(t) + f_2(t)i}{g_1(t) + g_2(t)i} \\ &= \lim_{t \rightarrow x} \frac{\frac{f_1(t) + f_2(t)i}{t-x}}{\frac{g_1(t) + g_2(t)i}{t-x}} \\ &= \frac{f'(x)}{g'(x)} \end{aligned}$$

5.8 Exercise 8

Exercise 7 was a brief detour, but we are now back to hammering away with the mean value theorem. We have

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

$$|f'(y) - f'(x)| < \epsilon$$

Where $y \in (x, t)$ or $y \in (t, x)$. Since f' is continuous, we can choose δ such that $|y - x| < \delta$ implies the above inequality, which means we can take $|t - x| < \delta$ to get the desired result.

This does not seem to have to hold for vector-valued functions, since in that case we only have $|f(t) - f(x)| \leq (t - x)|f'(y)|$.

5.9 Exercise 9

Yes, since we can apply L'Hopital's to $(0, +\infty)$ and $(-\infty, 0)$

$$f'(0) = \lim_{t \rightarrow 0} \frac{f(t) - f(0)}{t}$$

$$= \lim_{t \rightarrow 0} f'(t) = 3$$

5.10 Exercise 10

We proceed exactly as directed by Rudin

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \left(\frac{f(x)}{x} - A \right) \frac{x}{g(x)} + \frac{Ax}{g(x)}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{f(x) - Ax}{x}}{\lim_{x \rightarrow 0} \frac{g(x)}{x}} + \frac{A}{\lim_{x \rightarrow 0} \frac{g(x)}{x}}$$

$$= 0 + \frac{A}{B}$$

Where the final step follows from breaking up f, g, A into their real and imaginary components and applying L'Hopital's ($\operatorname{Re} f'(x) - \operatorname{Re} A$ and $\operatorname{Im} f'(x) - \operatorname{Im} A$ both go to 0).

5.11 Exercise 11

Double application of L'Hopital's

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

$$= \lim_{h \rightarrow 0} \frac{f''(x+h) + f''(x-h)}{2}$$

$$= f''(x)$$

If we let $f(x) = x \sin \frac{1}{|x|}$ when $x \neq 0$ and $f(0) = 0$, then $f''(0)$ does not exist. However, plugging f into the above limit and setting $x = 0$, we see that the limit is 0.

5.12 Exercise 12

$$\begin{aligned} f(x) = |x|^3 &= \begin{cases} -x^3 & x < 0 \\ 0 & x = 0 \\ x^3 & x > 0 \end{cases} \implies f'(x) = \begin{cases} -3x^2 & x < 0 \\ 0 & x = 0 \\ 3x^2 & x > 0 \end{cases} \\ \implies f''(x) &= \begin{cases} -6x & x < 0 \\ 0 & x = 0 \\ 6x & x > 0 \end{cases} \implies f^{(3)}(x) = \begin{cases} -6 & x < 0 \\ 0 & x = 0 \\ 6 & x > 0 \end{cases} \end{aligned}$$

So the left and right limits of $f^{(3)}(0)$ do not equal one another.

5.13 Exercise 14

Suppose f is a convex differentiable function. Then

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y) \\ f(\lambda x + (1 - \lambda)y) - f(y) &\leq \lambda(f(x) - f(y)) \\ \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)} &\geq \frac{f(y) - f(x)}{y - x} \\ \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)} &\geq \frac{f(y) - f(x)}{y - x} \end{aligned}$$

Since x was arbitrary, the last inequality implies that f' is monotonically increasing (because we can consider a new interval whose left endpoint is $\lambda x + (1 - \lambda)y$).

For the other direction, we can modify the convexity condition until we get something that we can apply the mean value theorem to. Letting $z = \lambda x + (1 - \lambda)y$, we have

$$\begin{aligned} f(z) &\leq \lambda f(x) + (1 - \lambda)f(y) \\ \lambda(f(z) - f(x)) &\leq (1 - \lambda)(f(y) - f(z)) \\ \frac{f(z) - f(x)}{z - x} &\leq \frac{(1 - \lambda)(f(y) - f(z))}{\lambda(z - x)} \\ &\leq \frac{f(y) - f(z)}{y - z} \end{aligned}$$

The final inequality is true by monotonicity of f' and applying the mean value theorem to (x, z) and (z, y) . The final step follows from the fact that $\lambda(z - x) = (1 - \lambda)(y - z)$.

5.14 Exercise 15

Proceeding as directed by the hint, we have

$$\begin{aligned}
 f(x+2h) &= f(x) + 2hf'(x) + 2h^2f''(c) \\
 f'(x) &= \frac{1}{2h}(f(x+2h) - f(x)) - hf''(c) \\
 M_1 &\leq \frac{1}{2h}(|f(x+2h)| + |f(x)|) + h|f''(c)| \\
 &\leq \frac{M_0}{h} + hM_2 \\
 M_1^2 &\leq \frac{M_0^2}{h^2} + 2M_0M_2 + h^2M_2^2 \\
 &\leq 4M_0M_2 \quad h = \sqrt{\frac{M_0}{M_2}}
 \end{aligned}$$

5.15 Exercise 18

Differentiating $f(t) - f(\beta) = (t - \beta)Q(t)$ and evaluating at α , we have

$$\begin{aligned}
 f^{(k)}(\alpha) &= (\alpha - \beta)Q^{(k)}(\alpha) - kQ^{(k-1)}(\alpha) \\
 \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k &= -\frac{Q^{(k)}(\alpha)}{k!}(\beta - \alpha)^{k+1} + \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta - \alpha)^k \\
 \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k &= -\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n + (f(\beta) - f(\alpha)) \\
 P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n &= f(\beta)
 \end{aligned}$$

6 The Riemann-Stieltjes Integral

6.1 Exercise 1

We first note that $\inf f(x) = 0$ on any interval $[p_{i-1}, p_i] \subset [a, b]$. Similarly, $\sup f(x) = 0$ if $x_0 \notin [p_{i-1}, p_i]$ and 1 otherwise. Thus, we proceed by constructing a partition P of $[a, b]$ such that $x_0 \in [p_{i-1}, p_i]$ and $\alpha(p_i) - \alpha(p_{i-1}) < \epsilon$. This is possible since α is continuous at x_0 , so there exists δ such that choosing $p_i - p_{i-1} < \delta$ gives us the previous inequality. We then have that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$, so $f \in \mathcal{R}(\alpha)$. Furthermore, since $L(P, f, \alpha) = 0$ for all P , we have that $\int_a^b f d\alpha = 0$.

6.2 Exercise 2

Since we're given $f(x) \geq 0$ and $\int_a^b f(x) dx = 0$, we have that $m = \inf f(x) = 0$. Letting m_i and M_i denote the infimum and supremum of f on the interval $[x_{i-1}, x_i] \subset [a, b]$, we further have that $m_i \leq m \implies m_i = 0 \forall i$. Additionally, since f is continuous on $[a, b]$ and therefore uniformly continuous (from compactness), we can choose a δ such that

$$\begin{aligned} |x_i - x_{i-1}| < \delta &\implies |f(x_i) - f(x_{i-1})| < \epsilon \\ &\implies |M_i - m_i| < \epsilon \implies M_i < \epsilon \end{aligned}$$

Thus, all of the M_i can be made arbitrarily small, implying that $\sup f(x) < \epsilon$ for every positive ϵ . This gives us that $f(x) = 0$.

6.3 Exercise 3

(a) Suppose $f(0+) = f(0)$. We consider the partition $P = x_0, x_1, x_2, x_3$ where $x_0 = -1, x_1 = 0, x_3 = 1$. Then $U(P, f, \beta_1) = M_2$ and $L(P, f, \beta_1) = m_2$, and we have that $M_2 \rightarrow f(0)$ and $m_2 \rightarrow f(0)$ as $x_2 \rightarrow 0$, so $f \in \mathcal{R}(\beta_1)$. For the other direction, if $f \in \mathcal{R}(\beta_1)$, then there exists P such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. We can then consider the refinement P^* of P that contains an interval of the form $[0, x_i]$. Suppose now that $|f(0+) - f(0)| = \epsilon > 0$. Then

$$M_i - m_i \geq |f(0+) - f(0)| > \epsilon$$

which is a contradiction, so $f(0+) = f(0)$.

(b) Only difference from (a) is that we need $f(0-) = f(0)$ instead of $f(0+) = f(0)$. The only changes that need to be made to the proof of (a) involve replacing $[0, x_i]$ with $[x_{i-1}, 0]$.

(c) If f is continuous at 0, we can consider the partition P that contains $x_{i-1} < x_i = 0 < x_{i+1}$. Then $U(P, f, \beta_2) = \frac{M_i + M_{i+1}}{2}$ and $L(P, f, \beta_2) = \frac{m_i + m_{i+1}}{2}$. Since f is continuous at 0, $M_i, M_{i+1}, m_i, m_{i+1} \rightarrow f(0)$ as $x_{i-1} \rightarrow 0$ and $x_{i+1} \rightarrow 0$, so $f \in \mathcal{R}(\beta_2)$. For the other direction, we proceed similarly to the proof of (a) and

see that

$$\begin{aligned}\frac{1}{2}(M_i + M_{i+1}) - \frac{1}{2}(m_i + m_{i+1}) &= \frac{1}{2}((M_i - m_i) + (M_{i+1} - m_{i+1})) \\ &\geq \frac{1}{2}(|f(0-) - f(0)| + |f(0+) - f(0)|) \\ &> \epsilon\end{aligned}$$

for some $\epsilon > 0$ unless $f(0-) = f(0+) = f(0)$, so f must be continuous at 0 if $f \in \mathcal{R}(\beta_2)$.

(d) If f is continuous at 0 then we have $f(0+) = f(0-) = f(0)$, so we are done by parts (a)-(c).

6.4 Exercise 4

From the density of the rationals and irrationals in the reals, we have that $M_i - m_i = 1 \forall i$, so $f \notin \mathcal{R}$.

6.5 Exercise 5

If we consider $f(x) = 1$ for all rational x and $f(x) = -1$ for all irrational x , we have that $f^2 \in \mathcal{R}$ but $f \notin \mathcal{R}$. However, if $f^3 \in \mathcal{R}$, then $f \in \mathcal{R}$. This is because $m \leq f^3 \leq M$ (since f is bounded) and $x^{\frac{1}{3}}$ is continuous on any $[m, M]$, so we can apply Theorem 6.11.

6.6 Exercise 6

P is compact, so the open cover consisting of neighborhoods around each point of P has a finite subcover. Thus, P can be covered by finitely many segments. Additionally, the total length of these segments can be made arbitrarily small, since P contains no segments itself. We can then proceed exactly as in the proof of Theorem 6.10 to get the desired result.

6.7 Exercise 7

(a) If $f \in \mathcal{R}$, then by Theorem 6.12 (c) we have

$$\begin{aligned}\left| \int_c^1 f dx - \int_0^1 f dx \right| &= \left| \int_0^c f dx \right| \\ &\leq cM < \epsilon \quad \text{for } c < \frac{\epsilon}{M}\end{aligned}$$

Where $M = \sup f$ on $[0, 1]$.

(b) Consider f such that

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & 0 < x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ -\frac{1}{x} & \frac{1}{2} < x \leq 1 \end{cases}$$

Then $\int_0^1 f dx = 0$, but $\int_0^1 |f| dx$ diverges.

6.8 Exercise 8

We see that the partition $P = 1, 2, \dots, b+1$ of the interval $[1, b+1]$ corresponds to the upper sum $U(P, f) = \sum_{n=1}^b f(n)$, and $\int_1^{b+1} f(x) dx \leq U(P, f)$. Thus, if $\sum_{n=1}^{\infty} f(n)$ converges, then we have that $\int_1^{b+1} f(x) dx$ is bounded and monotonically increasing (since $f(x) \geq 0$), so $\int_1^{\infty} f(x) dx$ converges. For the other direction, we can take the same partition P and consider $L(P, f) = \sum_{n=1}^b f(n+1)$. Since $L(P, f) \leq \int_1^{b+1} f(x) dx$, if $\int_1^{\infty} f(x) dx$ converges then so does $\sum_{n=2}^{\infty} f(n)$, and we are done (as $f(1) < \infty$).

6.9 Exercise 9

Theorem: Let f and g be differentiable functions on $[0, \infty)$ such that $\lim_{x \rightarrow \infty} f(x)g(x) = 0$ and $f', g' \in \mathcal{R}$ for every $[0, b]$. Then we have that

$$\int_0^{\infty} f(x)g'(x) dx = -f(0)g(0) - \int_0^{\infty} f'(x)g(x) dx$$

if both improper integrals converge.

Proof: Take limits on both sides of the integration by parts formula.

Letting $f(x) = \frac{1}{1+x}$ and $g'(x) = \cos(x)$ in the above formula shows that

$$\int_0^{\infty} \frac{\cos(x)}{1+x} dx = \int_0^{\infty} \frac{\sin(x)}{(1+x)^2} dx$$

We have that $|\cos(n)| + |\cos(n+1)| \geq c$ for some constant c since $\cos(n)$ and $\cos(n+1)$ cannot both be 0. As such, $\sum_{n=1}^{\infty} \frac{\cos(n)}{1+n}$ diverges since the harmonic series diverges and therefore the integral on the left also diverges by the integral test.

6.10 Exercise 10

(a) We can use the convexity of e^x to show this. Let $\lambda = \frac{1}{p}$ and let $u^p = e^a, v^q = e^b$ for some a, b (this is possible because $u, v \geq 0$). Then we have

$$\begin{aligned} e^{\lambda a + (1-\lambda)b} &\leq \lambda u^p + (1-\lambda)v^q \\ e^{\lambda a} e^{(1-\lambda)b} &\leq \frac{u^p}{p} + \frac{v^q}{q} \\ uv &\leq \frac{u^p}{p} + \frac{v^q}{q} \end{aligned}$$

(b) From part (a) we have that

$$\begin{aligned} f(x)g(x) &\leq \frac{f^p(x)}{p} + \frac{g^q(x)}{q} \\ \int_a^b fg d\alpha &\leq 1 \end{aligned}$$

(c) We can use part (a) with $u = \frac{|f(x)|}{\left(\int_a^b f^p d\alpha\right)^{\frac{1}{p}}}$ and $v = \frac{|g(x)|}{\left(\int_a^b g^q d\alpha\right)^{\frac{1}{q}}}$ to get the

desired result.

(d) Assuming the limits exist, the inequality follows for improper integrals by taking limits on both sides and then using limit rules.

7 Sequences and Series of Functions

7.1 Exercise 1

We have that $|f_n(x)| \leq M_n$. From uniform convergence, we get that there exists N such that

$$n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon \implies |M_n - M_m| < \epsilon$$

So $\lim_{n \rightarrow \infty} M_n = M$ for some M . Thus, we can choose N such that $n \geq N \implies M_n < M + 1$, so we can set $M_u = \max\{M_1, M_2, \dots, M_{N-1}, M + 1\}$. By construction, M_u must be a uniform bound for all of the f_n .

7.2 Exercise 2

Since f_n, g_n are both uniformly convergent, we can choose N_1, N_2 such that

$$\begin{aligned} n, m \geq \max\{N_1, N_2\} &\implies |f_n(x) - f_m(x)| < \frac{\epsilon}{2}, |g_n(x) - g_m(x)| < \frac{\epsilon}{2} \\ |f_n(x) + g_n(x) - f_m(x) - g_m(x)| &\leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| < \epsilon \end{aligned}$$

So $f_n + g_n$ converges uniformly as well.

Suppose $f_n \rightarrow f$ and $g_n \rightarrow g$, with $f_n(x) \leq M_f$ and $g_n(x) \leq M_g$. Then $g(x)f_n(x)$ converges uniformly since $M_g f_n(x)$ converges uniformly, and likewise for $g_n(x)f(x)$. Thus,

$$\begin{aligned} \frac{\epsilon}{3} &> |(f_n(x) - f(x))(g_n(x) - g(x))| \\ &> |f_n(x)g_n(x) - f_n(x)g(x) - f(x)g_n(x) + f(x)g(x)| \\ &> |f_n(x)g_n(x) - f(x)g(x)| - |f(x)g(x) - f_n(x)g(x)| - |f(x)g(x) - f(x)g_n(x)| \\ &> |f_n(x)g_n(x) - f(x)g(x)| - \frac{\epsilon}{3} - \frac{\epsilon}{3} \\ \implies \epsilon &> |f_n(x)g_n(x) - f(x)g(x)| \end{aligned}$$

So $f_n g_n \rightarrow f g$ uniformly.

7.3 Exercise 3

Let $f_n(x) = \frac{1}{x} + \frac{1}{n}$ and $g_n(x) = \frac{1}{x}$ on $E = (0, 1)$. Both f_n and g_n converge uniformly, and $f_n(x)g_n(x)$ converges pointwise to $\frac{1}{x^2}$. However,

$$\left| f_n(x)g_n(x) - \frac{1}{x^2} \right| = \left| \frac{1}{xn} \right|$$

So the convergence of $f_n(x)g_n(x)$ is not uniform on E .

7.4 Exercise 4

Since $1 + n^2x = 0$ whenever $x = -\frac{1}{n^2}$, $f(x)$ is not defined/does not converge for $x = 0, -\frac{1}{n^2}$. For all other values of x , however, $f(x)$ converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. $f(x)$ converges uniformly on any intervals of the form $(a, -1)$ and (b, c) with $b > 0$, since such intervals do not contain points of the form $-\frac{1}{n^2}$ and $\frac{1}{x}$ is bounded on all such intervals. f fails to converge uniformly on all other intervals. $f(x)$ is continuous on the intervals that it converges uniformly on. f is not bounded since $\frac{1}{x}$ can be made arbitrarily large.

7.5 Exercise 5

For every positive real number x , there exists N such that $\frac{1}{N} < x$ (Archimedean property). Hence, choosing $n \geq N \implies f_n(x) = 0$. Since every non-positive real number is less than $\frac{1}{n+1}$ for all $n \in \mathbb{N}$, we have that f_n converges pointwise to the function $f(x) = 0$. As such, it is clear that $\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^N |f_n(x)|$ converges for all x , and yet we do not have uniform convergence.

7.6 Exercise 6

We can rewrite the series as

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} + \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}$$

so the series converges uniformly since the alternating harmonic series converges and the series $\sum_{n=1}^{\infty} \frac{M}{n^2}$ converges (where $x^2 \leq M$, which is possible on every bounded interval). However, this series also clearly does not converge absolutely for any value of x since the harmonic series does not converge.

7.7 Exercise 7

We posit that f_n converges uniformly to the function $f(x) = 0$. For this to be true, there needs to be a single N such that $n \geq N$ gives

$$\left| \frac{x}{1 + nx^2} \right| < \epsilon \implies \frac{|x| - \epsilon}{\epsilon x^2} < n$$

for all x . In other words, the lefthand side of the last expression above must have a maximum. Since the lefthand side is differentiable for all $x \neq 0$, we can differentiate and equate to 0 to find that

$$\begin{aligned} \frac{d}{dx} \frac{x - \epsilon}{\epsilon x^2} &= \frac{2\epsilon^2 x - \epsilon x^2}{(\epsilon x^2)^2} \\ 2\epsilon^2 x - \epsilon x^2 &= 0 \implies x = 2\epsilon \end{aligned}$$

giving that $n > \frac{1}{4\epsilon^2}$ is sufficient for all x . Additionally, since $f_n(0) = 0$ for all n , we have that $f_n \rightarrow f$ uniformly. Differentiating f_n also shows that $f'_n(x) \rightarrow 0$ for all $x \neq 0$ and that $f'_n(0) = 1 \neq f'(0)$.