Exercise Guide for *Statistical Inference (2nd Ed.)* by Casella and Berger

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About

> *"I believe that we do not know anything for certain, but everything probably."*
>
> \- Christiaan Huygens

At this point I'm probably just doing introductory probability exercises to feel like I'm doing something.

1 Probability

1.1 Exercise 1

If $x \in B_i$, then $x \in A_i$ and $x \notin A_j \forall j < i$, so $x \notin B_j \forall j < i$. Thus, all of the B_i are disjoint. Furthermore, if $x \in A_n$, then $x \in A_i$ for some $i \leq n$, so $x \in \cup_{i=1}^n B_i$. The reverse direction is also true by definition of B_i, so $\cup_{i=1}^n A_i = \cup_{i=1}^n B_i$.

To prove the monotonically decreasing case, we first note that $A_n \setminus A = \cup_{i=n}^\infty A_i \setminus A_{i+1}$. This can be seen from the fact that $x \in A_n \setminus A \implies x \notin \cap_{i=n}^\infty A_i$, so there exists i such that $x \in A_i$ and $x \notin A_{i+1}$, which gives $x \in A_i \setminus A_{i+1}$. The reverse direction follows from the definition of monotonically decreasing, so we have equality.

Since all of the $A_i \setminus A_{i+1}$ are disjoint, we have

$$\begin{aligned}
\lim_{n\to\infty} P(A_n) &= P(A) + \lim_{n\to\infty} P(A_n \setminus A) \\
&= P(A) + \lim_{n\to\infty} \sum_{i=n}^{\infty} P(A_i \setminus A_{i+1}) \\
&= P(A)
\end{aligned}$$

as desired.

1.2 Exercise 5

The sample space consists of all strings that end in an H and otherwise contain only Ts except for a single other H (i.e. "TTTTHTTTH"). If we stop at k tosses, then toss k must have been an H and there must also have been exactly one H in the first $k-1$ tosses. Since we have $k-1$ options for the other H, we have that the probability is $\frac{k-1}{2^k}$.

1.3 Exercise 6

Consider the partition of Ω into singletons. Assigning any non-zero probability to these singletons would cause the probability of Ω (which is the sum of the probabilities of the singletons) to diverge.

1.4 Exercise 8

Applying the results from exercises 4 and 7, we have

$$\begin{aligned} P\big(\cap_{i=1}^{\infty} A_i\big) &= P\Big(\big(\cup_{i=1}^{\infty} A_i^c\big)^c\Big) \\ &= 1 - P\big(\cup_{i=1}^{\infty} A_i^c\big) \\ &\geq 1 - \sum_{i=1}^{\infty} P(A_i^c) \\ &= 1 \end{aligned}$$

Where the last line follows from the fact that $P(A_i) = 1$ was given.

1.5 Exercise 10

We define the sample space Ω to consist of the doubles (A, B) where $A, B \in \{1, 2, 3\}$ and A, B correspond to the door containing the prize and the door shown by Monty respectively. We can then define the event that door i contains the prize as $X_i = \{(A, B) \mid A = i\}$. Similarly, we can define the event that door i is shown as $Y_i = \{(A, B) \mid B = i\}$.

Now we can compute the probability $P(X_2Y_3)$ using Bayes theorem as $P(X_2Y_3) = P(Y_3|X_2)P(X_2) = 1 * \frac{1}{3}$. The calculation for $P(X_3Y_2)$ is identical. Furthermore, $P(X_1Y_2) = P(X_1Y_3) = \frac{1}{6}$, so our likelihood of winning is always increased by switching.

1.6 Exercise 11

$$\begin{aligned} P(A^c)P(B^c) &= (1 - P(A))(1 - P(B)) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= 1 - P(A) - P(B) + P(AB) \\ &= 1 - P(A \cup B) \\ &= P(A^cB^c) \end{aligned}$$

1.7 Exercise 12

Let the sample space Ω consist of the doubles (C, S) where $C \in \{1, 2, 3\}$ (card number) and $S \in \{1, 2\}$ (card side). Let A denote the event that we see a green side, and let B denote the event that we see the double green card. Then $P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{3}/\frac{1}{2} = \frac{2}{3}$.

1.8 Exercise 14

If $P(A) = 0$, then we have that $P(AB) = 0 = P(A)P(B)$ for all B since $AB \subset A$. If $P(A) = 1$, then we have $P(A \cup B) = P(A) + P(B) - P(A \cap B) =$

$1 + P(B) - P(A \cap B)$ which implies $P(A \cap B) = P(B) = P(A)P(B)$. If A is independent of itself, then $P(A)^2 = P(A)$, which is only possible if $P(A) = 1$ or $P(A) = 0$.

1.9 Exercise 17

$P(ABC) = \frac{P(ABC)}{P(BC)} \frac{P(BC)}{P(C)} P(C) = P(A|BC)P(B|C)P(C)$.

1.10 Exercise 18

We have that $\sum_i P(B|A_i)P(A_i) = P(B)$. If $P(A_1|B) < P(A_1)$, then applying Bayes' Theorem gives $P(B|A_1) < P(B)$. Now assume $P(B|A_k) \leq P(B)$ for all $k > 1$. Then $\sum_i P(B|A_i)P(A_i) < \sum_i P(B)P(A_i) = P(B)$, which is a contradiction.