

“Sparknotes” for *Principles of Mathematical Analysis* by Walter Rudin

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About

“A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.”

- Hermann Weyl

These notes contain short summaries of (my) proof ideas for exercises and some theorems from the book *Principles of Mathematical Analysis* by Walter Rudin. I have tried to make the summaries as brief as possible, sometimes only one line or one equation. My hope is that the summaries will give enough information to reconstruct a full proof without bogging the reader down with details. In many cases, I am sure that I inadvertently sacrificed clarity in an attempt to obtain brevity, and would greatly appreciate any feedback.

Also, I like when people include (what they presume to be) relevant quotes in their notes, so I have to ask you to forgive my haughtiness in starting these notes with a quote from Hermann Weyl.

1 The Real and Complex Number Systems

1.1 Exercise 1

If $rx = q$ or $r + x = q$ for some rational q , then subtracting r from q or dividing q by r yields x rational, which is a contradiction.

1.2 Exercise 2

We can first show that $\sqrt{3}$ is irrational by seeing that $\frac{a^2}{b^2} = 3 \implies 3|a, 3|b$. Then, since $12 = 3 * 2^2$, we have that $\sqrt{12}$ is irrational as well.

1.3 Exercise 4

If $\alpha > \beta$ then α would be an upper bound as well.

1.4 Exercise 5

$\forall x \in A, -x \leq \sup -A$ and $\forall \epsilon \in \mathbb{R}, \exists x \in A | \sup -A + \epsilon < -x \leq \sup -A$. Negating the last inequality gives $\inf A = -\sup -A$.

1.5 Exercise 6

(a) Follows from $m = \frac{np}{q}$.

(b) Put $r = \frac{m}{n}, s = \frac{p}{q}$. Then $b^r b^s = b^{\frac{mq}{nq}} b^{\frac{np}{nq}}$. Pulling out $\frac{1}{nq}$ gives the desired result.

(c) b^r is an upper bound since $b > 1$, and if it were not the supremum we could choose $t < r$ such that $b^t > b^r$. This is not possible since again, $b > 1$.

(d) Every element in $B(x + y)$ can be expressed as $b^{s+t} = b^s b^t$ $s \leq x, t \leq y$. If $\sup B(x + y) = \alpha < \sup B(x) \sup B(y)$, then $b^s b^t \leq \alpha \implies B(x) \leq \alpha b^{-t} \implies B(y) \leq \frac{\alpha}{B(x)} \implies B(x)B(y) \leq \alpha$.

1.6 Exercise 7

(a) $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \dots + 1) \geq n(b - 1)$ since $b > 1$.

(b) Plug $b^{\frac{1}{n}}$ into (a).

(c) Plug $n > \frac{b-1}{t-1}$ into (b).

(d) Using (c) gives that we can choose n such that $b^{\frac{1}{n}} < y b^{-w} \implies b^{w+\frac{1}{n}} < y$.

(e) We can take the reciprocal of (c) and do the same as in (d).

(f) If $b^x > y$ we can apply (e) for a contradiction, if $b^x < y$ we can apply (d) for a contradiction.

(g) Supremum is unique.

1.7 Exercise 8

Suppose $(0, 1) < (0, 0)$. Then $(0, -1) < (0, 0)$ after multiplying by $(0, 1)$ twice yields a contradiction. Similarly, assuming the opposite yields $(-1, 0) > (0, 0)$.

1.8 Exercise 9

Does exhibit least upper-bound property since you can take $(\sup a_i, \sup b_i)$.

1.9 Exercise 10

Exception is 0.

1.10 Exercise 11

Take $w = \frac{1}{|z|}z$ and $r = |z|$ when $|z| \neq 0$. w and r are not uniquely determined; take $z = 0$ for example.

1.11 Exercise 12

By strong induction:

$$\begin{aligned} |z_1 + \dots + z_{n+1}| &\leq |z_1 + \dots + z_n| + |z_{n+1}| \\ &\leq |z_1| + \dots + |z_{n+1}| \end{aligned}$$

1.12 Exercise 13

$$\begin{aligned} |x - y|^2 &= x\bar{x} - 2|x||y| + y\bar{y} \\ &\geq (|x| - |y|)^2 \end{aligned}$$

2 Basic Topology

2.1 Exercise 1

The empty set has no elements, so all of its elements are vacuously also elements of every set.

2.2 Exercise 2

The roots of complex polynomials with integer coefficients can be expressed as elements of the countable cross product of \mathbb{N} with itself (cross \mathbb{N} with itself n times for the coefficients, and then once more to indicate which root).

2.3 Exercise 3

If all real numbers were algebraic, then the set of algebraic numbers would be uncountable (thus contradicting Exercise 2).

2.4 Exercise 4

The set of irrational numbers is \mathbb{R}/\mathbb{Q} , which must be uncountable as otherwise \mathbb{R} would be countable.

2.5 Exercise 5

We can use $\left(\frac{n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{2n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{3n}{n+1}\right)_{n \in \mathbb{N}}$ to get the three limit points 1, 2, 3.

2.6 Exercise 6

If p is a limit point of E' , then every neighborhood of p contains a limit point q of E , and every neighborhood of q contains a point of E thereby implying that p is a limit point of E . E and E' do not need to have the same limit points, since E' could be finite and thus have no limit points.

2.7 Exercise 7

(a) If p is a limit point of $\overline{B_n}$, then every neighborhood of p contains a point $q \in A_i$. Since there are only finitely many A_i , p must be a limit point for at least one of the A_i , as an infinite number of neighborhoods of p must have non-zero intersection with some of the A_i .

(b) If we take $A_i = \left(\frac{in}{(i+1)n+1}\right)_{n \in \mathbb{N}}$, then 1 is a limit point of B_n despite not being a limit point of any of the A_i .

2.8 Exercise 8

Every point of an open set in \mathbb{R}^2 is by definition a limit point of the set, since the point must have a neighborhood contained in the set. The same is not true for closed sets, since we can just take a finite set.

2.9 Exercise 10

Every set in X is open, since any set containing p also contains $N_r(p)$ for $r < 1$. No set in X is closed, since $N_r(p) = p$ for $r < 1$. All infinite sets in X are not compact, since we can take balls of radius $r < 1$ around each point as an open cover.

2.10 Exercise 12

Take any open cover of K . There must be some open set in this cover containing 0, which means that the same set contains all but a finite number of the elements of K (since 0 is the only limit point of K). Take a union of this set as well as the finitely many other sets containing the aforementioned points to get a finite subcover.

2.11 Exercise 13

Take $\cup_{k=1}^{\infty} \{0, (\frac{n}{kn+1})_{n \in \mathbb{N}}, \frac{1}{k}\}$. This set is closed and bounded, so it is compact by Heine-Borel. Its limit points are 0 and $(\frac{1}{k})_{n \in \mathbb{N}}$.

2.12 Exercise 14

We can use $\cup_{n \in \mathbb{N}} (0, \frac{n}{n+1})$, which has no finite subcover (since we could choose $x \in (0, 1)$ larger than the largest endpoint in the finite subcover).

2.13 Exercise 15

For closed, we can take $K_i = \mathbb{N}/0, \dots, i-1$, since any $x \in K_i$ will not be in K_j if $j > x$. For bounded, we can take $K_i = (0, \frac{1}{i})$.

2.14 Exercise 16

E is by definition bounded, and E is closed since $q^2 \neq 3$ (q is rational), and $q^2 > 3 \implies \exists \epsilon \mid p \in N_{\epsilon}(q) \implies p^2 > 3$. Same logic gives that E is also open in \mathbb{Q} . E is, however, not compact, since we can construct an open cover consisting of $G_n = \{x \mid 2 < x^2 < 2 + \frac{n}{n+1}\}$.

2.15 Exercise 17

E is not countable by diagonalization. E is not dense in $[0, 1]$, since $E \cap [0, 0.1] = \emptyset$. E is not perfect, consider $N_{0.001}(0.77)$. E is closed and therefore compact by

Heine-Borel. To see closed, suppose a limit point q had a non-4/7 digit in the i^{th} decimal spot. Then we could take a neighborhood of size $10^{-(i+1)}$.

2.16 Exercise 18

Rationals are dense in \mathbb{R} , so no.

3 Numerical Sequences and Series

Definition 3.5

Since $\{p_n\} \rightarrow p \implies \forall \epsilon, \exists N | n \geq N \implies |p_n - p| < \epsilon$, we can choose $k | n_k \geq N \implies \{p_{n_k}\} \rightarrow p$. The reverse direction can be shown via contradiction of $\{p_n\} \rightarrow p$.

Examples 3.18

- (a) Density of rationals in reals.
- (b) $|s_n| < 1$, take n odd to get -1 and even to get 1.
- (c) Every subsequential limit has to converge to s .

Theorem 3.19

For all $\{n_k\}$, we have $\exists K | k \geq K \implies n_k \geq N \implies \lim_{k \rightarrow \infty} t_{n_k} - s_{n_k} \geq 0$.

Theorem 3.26

$$s_n = 1 + x + \dots + x^n \implies x s_n = x + x^2 + \dots + x^{n+1} \implies (1 - x)s_n = 1 - x^{n+1}.$$

Examples 3.40

- (a) Root test: $n \rightarrow \infty$.
- (b) Ratio test: $\frac{1}{n+1} \rightarrow 0$.
- (c) $1 \rightarrow 1$.
- (d) Ratio test: $\frac{n}{n+1} \rightarrow 1$. $z = 1$ leads to harmonic series.
- (e) Ratio test: $\frac{n^2}{(n+1)^2} \rightarrow 1$.

Example 3.53

$\sum_{k=1}^{\infty} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} < \frac{5}{6} + \sum_{k=2}^{\infty} \frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k}$. The RHS converges since $\frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k} = \frac{1}{2k^2-2k}$.