Exercise Guide for *Principles of Mathematical Analysis (3rd Ed.)* by Walter Rudin

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About

> *"It is not possible to overstate how good this book is. I tried to give it uncountably many stars but they only have five. Five is an insult. I'm sorry Dr. Rudin..."* - Amazon Review

An actual solution manual for the book can be found here. What follows are notes I took as I did exercises (they're more like hints towards my thinking than solutions) while working through the book on my own.

1 The Real and Complex Number Systems

1.1 Exercise 1

If $rx = q$ or $r + x = q$ for some rational q, then substracting r from q or dividing q by r yields x rational, which is a contradiction.

1.2 Exercise 2

We can first show that $\sqrt{3}$ is irrational by seeing that $\frac{a^2}{b^2} = 3 \implies 3|a, 3|b$. Then, since $12 = 3 * 2^2$, we have that $\sqrt{12}$ is irrational as well.

1.3 Exercise 4

If $\alpha > \beta$ then α would be an upper bound as well.

1.4 Exercise 5

$\forall x \in A$, $-x \leq \sup -A$ and $\forall \epsilon \in \mathbb{R}$, $\exists x \in A \,|\, \sup -A + \epsilon < -x \leq \sup -A$. Negating the last inequality gives $\inf A = -\sup -A$.

1.5 Exercise 6

(a) Follows from $m = \frac{np}{q}$.

(b) Put $r = \frac{m}{n}$, $s = \frac{p}{q}$. Then $b^r b^s = b^{\frac{mq}{nq}} b^{\frac{np}{nq}}$. Pulling out $\frac{1}{nq}$ gives the desired result.

(c) b^r is an upper bound since $b > 1$, and if it were not the supremum we could choose $t < r$ such that $b^t > b^r$. This is not possible since again, $b > 1$.

(d) Every element in $B(x + y)$ can be expressed as $b^{s+t} = b^s b^t \; s \leq x$, $t \leq y$. If $\sup B(x + y) = \alpha < \sup B(x) \sup B(y)$, then $b^s b^t \leq \alpha \implies B(x) \leq \alpha b^{-t} \implies B(y) \leq \frac{\alpha}{B(x)} \implies B(x)B(y) \leq \alpha$.

1.6 Exercise 7

(a) $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + ... + 1) \geq n(b - 1)$ since $b > 1$.

(b) Plug $b^{\frac{1}{n}}$ into (a).

(c) Plug $n > \frac{b-1}{t-1}$ into (b).

(d) Using (c) gives that we can choose n such that $b^{\frac{1}{n}} < y\dot{b}^{-w} \implies b^{w+\frac{1}{n}} < y$.

(e) We can take the reciprocal of (c) and do the same as in (d).

(f) If $b^x > y$ we can apply (e) for a contradiction, if $b^x < y$ we can apply (d) for a contradiction.

(g) Supremum is unique.

1.7 Exercise 8

Suppose $(0, 1) < (0, 0)$. Then $(0, -1) < (0, 0)$ after multiplying by $(0, 1)$ twice yields a contradiction. Similarly, assuming the opposite yields $(-1, 0) > (0, 0)$.

1.8 Exercise 9

Does exhibit least upper-bound property since you can take $(\sup a_i, \sup b_i)$.

1.9 Exercise 10

Exception is 0.

1.10 Exercise 11

Take $w = \frac{1}{|z|}z$ and $r = |z|$ when $|z| \neq 0$. w and r are not uniquely determined; take $z = 0$ for example.

1.11 Exercise 12

By strong induction:

$$\begin{aligned} |z_1 + ... + z_{n+1}| &\leq |z_1 + ... + z_n| + |z_{n+1}| \\ &\leq |z_1| + ... + |z_{n+1}| \end{aligned}$$

1.12 Exercise 13

$$\begin{aligned} |x - y|^2 &= x\bar{x} - 2|x||y| + y\bar{y} \\ &\geq (|x| - |y|)^2 \end{aligned}$$

2 Basic Topology

2.1 Exercise 1

The empty set has no elements, so all of its elements are vacuously also elements of every set.

2.2 Exercise 2

The roots of complex polynomials with integer coefficients can be expressed as elements of the countable cross product of $\mathbb{N}$ with itself (cross $\mathbb{N}$ with itself n times for the coefficients, and then once more to indicate which root).

2.3 Exercise 3

If all real numbers were algebraic, then the set of algebraic numbers would be uncountable (thus contradicting Exercise 2).

2.4 Exercise 4

The set of irrational numbers is $\mathbb{R}/\mathbb{Q}$, which must be uncountable as otherwise $\mathbb{R}$ would be countable.

2.5 Exercise 5

We can use $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}} \cup \left(\frac{2n}{n+1}\right)_{n\in\mathbb{N}} \cup \left(\frac{3n}{n+1}\right)_{n\in\mathbb{N}}$ to get the three limit points $1, 2, 3$.

2.6 Exercise 6

If p is a limit point of E', then every neighborhood of p contains a limit point q of E, and every neighborhood of q contains a point of E thereby implying that p is a limit point of E. E and E' do not need to have the same limit points, since E' could be finite and thus have no limit points.

2.7 Exercise 7

(a) If p is a limit point of $\overline{B_n}$, then every neighborhood of p contains a point $q \in A_i$. Since there are only finitely many A_i, p must be a limit point for at least one of the A_i, as an infinite number of neighborhoods of p must have non-zero intersection with some of the A_i.

(b) If we take $A_i = \left(\frac{in}{(i+1)n+1}\right)_{n\in\mathbb{N}}$, then 1 is a limit point of B_n despite not being a limit point of any of the A_i.

2.8 Exercise 8

Every point of an open set in $\mathbb{R}^2$ is by definition a limit point of the set, since the point must have a neighborhood contained in the set. The same is not true for closed sets, since we can just take a finite set.

2.9 Exercise 10

Every set in X is open, since any set containing p also contains $N_r(p)$ for $r < 1$. No set in X is closed, since $N_r(p) = p$ for $r < 1$. All infinite sets in X are not compact, since we can take balls of radius $r < 1$ around each point as an open cover.

2.10 Exercise 12

Take any open cover of K. There must be some open set in this cover containing 0, which means that the same set contains all but a finite number of the elements of K (since 0 is the only limit point of K). Take a union of this set as well as the finitely many other sets containing the aforementioned points to get a finite subcover.

2.11 Exercise 13

Take $\cup_{k=1}^{\infty}\{0, \left(\frac{n}{kn+1}\right)_{n\in\mathbb{N}}, \frac{1}{k}\}$. This set is closed and bounded, so it is compact by Heine-Borel. Its limit points are 0 and $\left(\frac{1}{k}\right)_{n\in\mathbb{N}}$.

2.12 Exercise 14

We can use $\cup_{n\in\mathbb{N}}(0, \frac{n}{n+1})$, which has no finite subcover (since we could choose $x \in (0,1)$ larger than the largest endpoint in the finite subcover).

2.13 Exercise 15

For closed, we can take $K_i = \mathbb{N}/0, ..., i-1$, since any $x \in K_i$ will not be in K_j if $j > x$. For bounded, we can take $K_i = (0, \frac{1}{i})$.

2.14 Exercise 16

E is by definition bounded, and E is closed since $q^2 \neq 3$ (q is rational), and $q^2 > 3 \implies \exists\epsilon \mid p \in N_\epsilon(q) \implies p^2 > 3$. Same logic gives that E is also open in $\mathbb{Q}$. E is, however, not compact, since we can construct an open cover consisting of $G_n = \{x \mid 2 < x^2 < 2 + \frac{n}{n+1}\}$.

2.15 Exercise 17

E is not countable by diagonalization. E is not dense in $[0,1]$, since $E \cap [0, 0.1] = \emptyset$. E is not perfect, consider $N_{0.001}(0.77)$. E is closed and therefore compact by

Heine-Borel. To see closed, suppose a limit point q had a non-4/7 digit in the i^{th} decimal spot. Then we could take a neighborhood of size $10^{-(i+1)}$.

2.16 Exercise 18

Originally I thought this was no, but this can actually be done with a modified version of the Cantor set construction. See here for a discussion.

3 Numerical Sequences and Series

Definition 3.5

Since $\{p_n\} \to p \implies \forall \epsilon, \exists N | n \geq N \implies |p_n - p| < \epsilon$, we can choose $k | n_k \geq N \implies \{p_{n_k}\} \to p$. The reverse direction can be shown via contradiction of $\{p_n\} \to p$.

Examples 3.18

(a) Density of rationals in reals.

(b) $|s_n| < 1$, take n odd to get -1 and even to get 1.

(c) Every subsequential limit has to converge to s.

Theorem 3.19

For all $\{n_k\}$, we have $\exists K | k \geq K \implies n_k \geq N \implies \lim_{k\to\infty} t_{n_k} - s_{n_k} \geq 0$.

Theorem 3.26

$s_n = 1 + x + ... + x^n \implies xs_n = x + x^2 + ... + x^{n+1} \implies (1-x)s_n = 1 - x^{n+1}$.

Examples 3.40

(a) Root test: $n \to \infty$.

(b) Ratio test: $\frac{1}{n+1} \to 0$.

(c) $1 \to 1$.

(d) Ratio test: $\frac{n}{n+1} \to 1$. $z = 1$ leads to harmonic series.

(e) Ratio test: $\frac{n^2}{(n+1)^2} \to 1$.

Example 3.53

$\sum_{k=1}^{\infty} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} < \frac{5}{6} + \sum_{k=2}^{\infty} \frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k}$. The RHS converges since $\frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k} = \frac{1}{2k^2-2k}$.

3.1 Exercise 1

All we need is the inequality $|s_n - s| \geq ||s_n| - |s||$. The converse is not true, since we can take $s_n = (-1)^n$.

3.2 Exercise 2

My original idea: $\sqrt{(n+x)^2} - n = x$. Setting $(n+x)^2 \geq n^2 + n$ gives $x^2 \geq (1-2x)n$. The last inequality is only true for all n when $x \geq \frac{1}{2}$. This implies that $\frac{1}{2}$ is the supremum of $\sqrt{n^2+n} - n$. Since $\sqrt{n^2+n} - n$ is increasing, it converges to $\frac{1}{2}$.

Better: $(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n) = n \implies \sqrt{n^2+n} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$.

3.3 Exercise 3

Clearly $s_{n+1} > s_n$. We can see that $s_n < 2$ by induction, since $s_1 < 2$ and $2 + \sqrt{s_n} < 4$. This gives that s_n is monotone and bounded, implying it converges.

3.4 Exercise 4

$$s_{2m+1} = \sum_{i=1}^{m} \frac{1}{2^i},\ s_{2m} = \sum_{i=2}^{m} \frac{1}{2^i}$$
$$\implies \limsup_{n\to\infty} s_n = 1,\ \liminf_{n\to\infty} s_n = \frac{1}{2}$$

3.5 Exercise 5

$$\begin{aligned}\limsup_{n\to\infty}(a_n + b_n) &= \sup_{\{k\}}\{\lim_{k\to\infty}(a_{n_k} + b_{n_k})\} \\ &= \sup_{\{k\}}\{\lim_{k\to\infty} a_{n_k} + \lim_{k\to\infty} b_{n_k}\}\end{aligned}$$

3.6 Exercise 6

(a) $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges from comparison to harmonic series (same technique as Exercise 2).

(b) Converges, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p = \frac{3}{2}$.

(c) Converges by root test, since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(d) Converges when $|z| > 1$ and diverges otherwise. To see this, put $z = |z|e^{i\theta}$ to get $\lim_{n\to\infty} \frac{1}{1+|z|^n e^{ni\theta}}$.

3.7 Exercise 7

We proceed via the ratio test.

$$\begin{aligned}
\limsup_{n\to\infty} \frac{n}{n+1} * \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} &= \limsup_{n\to\infty} \frac{n}{n+1} \limsup_{n\to\infty} \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} \\
&= \sqrt{\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}} \\
&< 1
\end{aligned}$$

Since $\sum a_n$ converges.

3.8 Exercise 8

Since b_n is monotonic and bounded, $|b_n| \leq B$ for all n. Then we have that $\sum a_n b_n$ converges by the comparison test, since $|a_n b_n| \leq B|a_n|$ and $B \sum a_n$ converges.

3.9 Exercise 9

(a) Applying the ratio test, we see that $|z| \limsup_{n\to\infty} \left|\frac{(n+1)^3}{n^3}\right| < 1$ when $|z| < 1$. Thus $\sum n^3 z^n$ has radius of convergence 1.

(b) Again, applying the ratio test, we see that $2|z| \limsup_{n\to\infty} \left|\frac{1}{n+1}\right| = 0$, implying $R = +\infty$.

(c) The ratio test is the only hammer we need: $2|z| \limsup_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| < 1$ gives $R = \frac{1}{2}$.

(d) What are the other tests again? $\frac{|z|}{3} \limsup_{n\to\infty} \left|\frac{(n+1)^3}{n^3}\right| < 1$ gives $R = 3$.

3.10 Exercise 10

The infinitely many non-zero a_n must satisfy $|a_n| \geq 1$. The radius of convergence of $\sum a_n z^n$ will be maximized when $|a_n|$ is minimized, so we can just consider the case where there are infinitely many $|a_n| = 1$. In this case, we can choose a subsequence a_{n_k} consisting only of 1. Applying the ratio test using this subsequence gives $|z| < 1$.

3.11 Exercise 15

Theorems 3.22, 3.23, and 3.25(a) require no changes in their proofs, since the Cauchy criterion is applicable for $\mathbb{R}^k$. Theorem 3.33(a, b) also require no changes once we have the comparison test for $\mathbb{R}^k$. For Theorem 3.33(c), we can take $a \in \mathbb{R}^k$ such that all of its components are $\frac{1}{n}$ or $\frac{1}{n^2}$.

Theorem 3.34(a, b) just need to be modified to use $\frac{|a_{n+1}|}{|a_n|}$. Theorem 3.42 needs to be modified to use the dot product, but then it follows from applying the $\mathbb{R}$ version of 3.42 to the components of the dot product sum. Theorems 3.45, 3.47, and 3.55 require no changes to their proofs.

4 Continuity

4.1 Exercise 1

Continuity implies $\lim_{h\to 0}[f(x+h)-f(x-h)] = 0$, since we can choose h to be within δ of x such that $|f(x+h)-f(x)+f(x)-f(x-h)| \leq |f(x+h)-f(x)| + |f(x-h)-f(x)| < \epsilon$. However, the converse (as asked in the question) need not be true, since we don't have to have $\lim_{h\to 0} f(x+h) = f(x) = \lim_{h\to 0} f(x-h)$. For example, consider $x \neq 0 \implies f(x) = \frac{1}{|x|}$, $f(0) = 0$.

4.2 Exercise 2

Suppose p is a limit point of E. Then there is a sequence $(x_n) \in E |\lim_{n\to\infty} x_n = p$. Since f is continuous, we have that $lim_{n\to\infty} f(x_n) = f(p)$, which implies that $f(p)$ is a limit point of $f(E)$ giving us that $f(\overline{E}) \subset \overline{f(E)}$.

To see that $f(\overline{E})$ can be a proper subset, consider $f : \mathbb{Z}^+ \to \mathbb{Q}$ with $f(x) = \frac{1}{x}$. Then f is continuous and $0 \notin f(\overline{\mathbb{Z}^+}) = f(\mathbb{Z}^+)$.

4.3 Exercise 3

Similar to Exercise 2: if p is a limit point of $Z(f)$, then there exists some sequence $(x_n) \in E \mid \lim_{n\to\infty} x_n = p$. Since f is continuous, we have that $\lim_{n\to\infty} f(x_n) = f(p)$. Then it follows that $x_n \in Z(f) \implies f(x_n) = 0 \implies f(p) = 0$.

4.4 Exercise 4

The fact that $f(E)$ is dense in $f(X)$ follows from Exercise 2, since $X = \overline{E}$. Similarly, $\lim_{n\to\infty} g(p_n) = g(p) \implies \lim_{n\to\infty} f(p_n) = g(p)$ since $p_n \in E$. Thus, $g(p) = f(p)$ for all $p \in X$.

4.5 Exercise 5

If f is defined on an open set in $\mathbb{R}^1$, then it need not be defined at its endpoints. For example, consider $f(x) = \frac{1}{x}$ defined on $(0, 1)$. However, if f is defined on a closed subset $E \subset \mathbb{R}^1$, then E^c is an open set in $\mathbb{R}$ and can thus be decomposed into the union of a countable number of open intervals (a_n, b_n). We can thus take g to be $g(x) = \frac{b_n - x}{b_n - a_n} f(a_n) + (1 - \frac{b_n - x}{b_n - a_n}) f(b_n)$ (the straight line interpolation between $f(a_n)$ and $f(b_n)$).

4.6 Exercise 6

f is a bijection from E to its graph $G(E)$. If f is continuous, then we can take the inverse image of an open cover of $G(E)$ to get an open cover of E. Since E is compact, this open cover must have a finite subcover whose image under f will be a finite subcover for $G(E)$, thereby giving the compactness of $G(E)$.

I looked up a hint on the reverse direction. Consider an infinite (finite case presents no issues) closed set $V \subset G(E)$. Take some arbitrary subsequence $(x_k, f(x_k)) \in V$. By the compactness of $G(E)$, this subsequence has a limit point $(x, f(x)) \in G(E)$, and this limit point is contained in V since V is closed. Thus, $f^{-1}(V)$ also contains $x_k \to x$, implying that $f^{-1}(V)$ contains all of its limit points and is therefore closed. This shows that f is continuous.

For what it's worth, I think this argument using projections is much nicer.

4.7 Exercise 7

Suppose for any M that $\exists x, y \mid f(x, y) > M$ (we consider only the case where $x > 0$, as the other case is identical). Then we can solve the resulting quadratic to see that, if such x and y exist, then $x > \frac{y^2(1+\sqrt{1-4M^2})}{2M}$. However, $\sqrt{1-4M^2}$ is not defined in $\mathbb{R}$ for $M > \frac{1}{2}$, so f must be bounded. Performing the same analysis for g yields $x > \frac{y^2(1+\sqrt{1-4y^2M^2})}{2M}$. Since y can be chosen to make the inequality for x have a solution in $\mathbb{R}$, g is unbounded.

To show that f is discontinuous at $(0, 0)$, we need only consider the sequence consisting of $(0, \frac{n}{n+1})$ to see that $\lim_{n\to\infty} f(0, \frac{n}{n+1}) = 1 \neq 0$. Plugging in $y = ax + b$ leads to f and g being quotients of two polynomials with non-zero denominator, indicating that they're both continuous.

4.8 Exercise 8

Suppose f is not bounded. Then there is a sequence $f(x_n) \mid \forall N, \exists m, n \geq N\ |f(x_n) - f(x_m)| > \epsilon$ for some ϵ, since otherwise $f(x_n)$ would converge to some point of $\mathbb{R}$. As f is uniformly continuous, this means that $|x_n - x_m| > \delta$ for infinitely many n, m. However, that would then imply that E is not bounded, which is a contradiction. Thus, f is bounded on E.

If E is not bounded, we can just take $f(x) = x$.

4.9 Exercise 9

Let E consist of all $x, y \mid d_X(x, y) < \delta$. Then $\text{diam} E < \delta$. Similarly, if $\forall x, y\ d_Y(f(x), f(y)) < \epsilon$, then $\text{diam} f(E) < \epsilon$.

4.10 Exercise 10

Suppose f is not uniformly continuous. Then there is a sequence $x_n \in X \mid x_n \to x$, but $\forall N, \exists m, n \geq N \mid d_Y(f(x_n), f(x_m)) > \epsilon$ for some $\epsilon > 0$. This, however, makes $f(x_n)$ an infinite subset of $f(X)$ which does not have a limit point, thereby contradicting the fact that $f(X)$ is compact.

4.11 Exercise 11

The first part of this exercise is basically what I was doing for Exercises 8 and 10. Since f is uniformly continuous, $\exists \delta \,|\, d_X(x_n, x_m) < \delta \implies d_Y(f(x_n), f(x_m)) < \epsilon$. Since (x_n) Cauchy converges, we can make $d_X(x_n, x_m)$ arbitrarily small, which then implies that we can make $d_Y(f(x_n), f(x_m))$ arbitrarily small, indicating that $f(x_n)$ Cauchy converges as well.

4.12 Exercise 12

To state it more precisely: if $f : X \to Y$ and $g : Y \to Z$ are both uniformly continuous, then $g \circ f$ is also uniformly continuous.

From uniform continuity of g, $\exists \delta \mid d_Y(y_1, y_2) < \delta \implies d_Z(g(y_1), g(y_2)) < \epsilon$. Since f is uniformly continuous, $\exists \delta' \,|\, d_X(x_1, x_2) < \delta' \implies d_Y(f(x_1), f(x_2)) < \delta$. The existence of this δ' gives us that $g \circ f$ is uniformly continuous.

4.13 Exercise 13

Suppose p is a limit point of E and $x_n \in E \mid x_n \to p$. Then $f(x_n)$ Cauchy converges to a point q in the codomain of f. We can simply take $g(p) = q$ whenever $p \notin E$ to get a continuous extension of f. Since this proof depends only on the convergence of the Cauchy sequence $f(x_n)$ to a point in the codomain, it will hold for the codomain being any complete metric space.

4.14 Exercise 16

The function $[x]$ has a simple discontinuity at every integer x, since the left-hand limit is $x - 1$ and the right-hand limit is x. Similarly, the function (x) also has a simple discontinuity at every integer, since the left-hand limit is 1 and the right-hand limit is 0.

4.15 Exercise 17

We proceed as hinted in the text. The two types of simple discontinuity we need to consider are $f(x-) \neq f(x+)$ and $f(x-) = f(x+) \neq f(x)$. For the first case, suppose (WLOG) that $f(x-) < f(x+)$. Then we can construct a rational triple (p, q, r) such that

$$\begin{aligned}
&f(x-) < p < f(x+) \\
&a < q < t < x \implies f(t) < p \\
&x < t < r < b \implies f(t) > p
\end{aligned}$$

To see that such a triple can only be associated with one such x, consider $x' = x + \epsilon$ with $\epsilon > 0$ (the other case is identical). Then we can choose $t \in (x, x')$ with $q < x < t < r < x'$, which means $t > q$ does not imply $f(t) < p$. This handles simple discontinuities of the form $f(x-) \neq f(x+)$.

We can similarly handle the case where $f(x-) = f(x+) \neq f(x)$. Suppose (WLOG) that $f(x) > f(x+)$; we can then construct a rational triple (p, q, r) such that

$$
\begin{aligned}
&f(x+) < p < f(x) \\
&a < q < t < x \implies f(t) < p \\
&x < t < r < b \implies f(t) < p
\end{aligned}
$$

Again, such a triple can only be associated with a single x, since $x \in (x, x+\epsilon)$ and $f(x) > p$. Therefore f has only countably many simple discontinuities.

4.16 Exercise 23

From the definition of convexity, we have that

$$
\begin{aligned}
f(\lambda x + (1-\lambda)p) &\leq \lambda f(x) + (1-\lambda)f(p) \\
f(\lambda x + (1-\lambda)p) - f(p) &\leq \lambda(f(x) - f(p)) \\
f(p) - f(\lambda x + (1-\lambda)p) &\geq \lambda(f(p) - f(x)) \\
\implies \lim_{\lambda \to 0} f(\lambda x + (1-\lambda)p) &= f(p)
\end{aligned}
$$

Since $\lim_{\lambda \to 0} \lambda x + (1-\lambda)p = 0$ for all choices of x, we have that f is continuous.

5 Differentiation

5.1 Exercise 1

We have that

$$\begin{aligned}
|f(x) - f(y)| &\leq (x-y)^2 = |x-y|^2 \\
\frac{|f(x) - f(y)|}{|x-y|} &\leq |x-y| \\
&\implies f'(x) = 0 \, \forall x
\end{aligned}$$

Since we can write $\frac{f(t)-f(x)}{t-x} = f'(x) + u(t)$ with $\lim_{t \to x} u(t) \to 0$, we have that

$$\begin{gathered}
f(t) - f(x) = (t-x)u(t), \quad f(t) - f(y) = (t-y)v(t) \\
f(y) - f(x) = (y-x)u(y), \quad f(x) - f(y) = (x-y)v(x) \implies u(y) = v(x) \, \forall x, y \\
f(y) - f(x) = 0 \implies f(x) = f(y) \, \forall x, y
\end{gathered}$$

Whoops, I did this before reading the mean value theorem section - this problem follows immediately from applying the mean value theorem after showing $f'(x) = 0$.

5.2 Exercise 2

Take $x, t \in (a, b)$ with $t > x$. Applying the mean value theorem to f on $[x, t]$, we get $f(t) - f(x) = (t-x)f'(y)$ for some $y \in (x, t)$. Since $f'(y) > 0 \implies f(t) - f(x) > 0$, f is strictly increasing on (a, b). We can prove $g = f^{-1}$ is differentiable directly

$$\begin{aligned}
\lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} &= \lim_{t \to x} \frac{t-x}{f(t) - f(x)} \\
&= \frac{1}{f'(x)}
\end{aligned}$$

5.3 Exercise 3

Suppose (WLOG) that $x_2 > x_1$ but $f(x_2) = f(x_1)$. Then we have that

$$\begin{aligned}
x_2 + \epsilon g(x_2) &= x_1 + \epsilon g(x_1) \\
(x_2 - x_1) + \epsilon(g(x_2) - g(x_1)) &= 0 \\
1 + \epsilon g'(x) &= 0 \quad x \in (x_1, x_2) \\
1 + \epsilon g'(x) &\geq 1 - \epsilon |g'(x)| \\
&> 0 \quad \forall \epsilon < \frac{1}{M}
\end{aligned}$$

Where the penultimate step follows from the mean value theorem. Thus, we can choose an ϵ such that $f(x_2) \neq f(x_1)$, which means we can make f injective.

5.4 Exercise 4

Let $f(x) = C_0 + C_1 x + ... + C_n x^n$ and $g(x) = C_0 x + \frac{C_1}{2} x^2 + ... + \frac{C_n}{n+1} x^{n+1}$. Then $g'(x) = f(x)$. Applying the mean value theorem to $g(x)$ on $[0, 1]$ yields that there is an x such that $g'(x) = f(x) = g(1) - g(0) = 0$, so f has a root in $(0, 1)$.

5.5 Exercise 5

It looks like the mean value theorem is this chapter's ratio test; you can guess how this will go. By the mean value theorem, $f'(y) = f(x + 1) - f(x)$ for $y \in [x, x + 1]$. Thus we have $\lim_{x\to\infty} g(x) = \lim_{y\to\infty} f'(y) = 0$.

5.6 Exercise 6

Consider $x > y > 0$. By the mean value theorem (surprise), we have that

$$\begin{aligned}
f(x) - f(y) &= f'(a)(x - y) \quad a \in (x, y) \\
&< f'(x)(x - y) \\
\lim_{y\to 0} \frac{f(x) - f(y)}{x - y} &< \lim_{y\to 0} f'(x) \\
\frac{f(x)}{x} &< f'(x)
\end{aligned}$$

Differentiating g, we get

$$\begin{aligned}
g'(x) &= \frac{xf'(x) - f(x)}{x^2} \\
f'(x) > \frac{f(x)}{x} &\implies g'(x) > 0
\end{aligned}$$

So g is monotonically increasing.

5.7 Exercise 7

If f and g are real, then the result follows immediately from L'Hopital's, since the existence of $f'(x), g'(x)$ imply that f and g are continuous (so the requirement that $f(x) \to 0$ and $g(x) \to 0$ is satisfied). More generally, for complex functions, we have

$$\begin{aligned}
\lim_{t\to x} \frac{f(t)}{g(t)} &= \lim_{t\to x} \frac{f_1(t) + f_2(t)i}{g_1(t) + g_2(t)i} \\
&= \lim_{t\to x} \frac{\frac{f_1(t)+f_2(t)i}{t-x}}{\frac{g_1(t)+g_2(t)i}{t-x}} \\
&= \frac{f'(x)}{g'(x)}
\end{aligned}$$

5.8 Exercise 8

Exercise 7 was a brief detour, but we are now back to hammering away with the mean value theorem. We have

$$\begin{aligned}\left|\frac{f(t)-f(x)}{t-x}-f'(x)\right|&<\epsilon\\ |f'(y)-f'(x)|&<\epsilon\end{aligned}$$

Where $y \in (x,t)$ or $y \in (t,x)$. Since f' is continuous, we can choose δ such that $|y-x|<\delta$ implies the above inequality, which means we can take $|t-x|<\delta$ to get the desired result.

This does not seem to have to hold for vector-valued functions, since in that case we only have $|f(t)-f(x)| \le (t-x)|f'(y)|$.

5.9 Exercise 9

Yes, since we can apply L'Hopital's to $(0,+\infty)$ and $(-\infty,0)$

$$\begin{aligned}f'(0)&=\lim_{t\to 0}\frac{f(t)-f(0)}{t}\\ &=\lim_{t\to 0}f'(t)=3\end{aligned}$$

5.10 Exercise 10

We proceed exactly as directed by Rudin

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{g(x)}&=\lim_{x\to 0}\left(\frac{f(x)}{x}-A\right)\frac{x}{g(x)}+\frac{Ax}{g(x)}\\ &=\frac{\lim_{x\to 0}\frac{f(x)-Ax}{x}}{\lim_{x\to 0}\frac{g(x)}{x}}+\frac{A}{\lim_{x\to 0}\frac{g(x)}{x}}\\ &=0+\frac{A}{B}\end{aligned}$$

Where the final step follows from breaking up f, g, A into their real and imaginary components and applying L'Hopital's ($\operatorname{Re} f'(x) - \operatorname{Re} A$ and $\operatorname{Im} f'(x) - \operatorname{Im} A$ both go to 0).

5.11 Exercise 11

Double application of L'Hopital's

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}&=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\\ &=\lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}\\ &=f''(x)\end{aligned}$$

If we let $f(x) = x \sin \frac{1}{|x|}$ when $x \neq 0$ and $f(0) = 0$, then $f''(0)$ does not exist. However, plugging f into the above limit and setting $x = 0$, we see that the limit is 0.

5.12 Exercise 12

$$f(x) = |x|^3 = \begin{cases} -x^3 & x < 0 \\ 0 & x = 0 \\ x^3 & x > 0 \end{cases} \implies f'(x) = \begin{cases} -3x^2 & x < 0 \\ 0 & x = 0 \\ 3x^2 & x > 0 \end{cases}$$

$$\implies f''(x) = \begin{cases} -6x & x < 0 \\ 0 & x = 0 \\ 6x & x > 0 \end{cases} \implies f^{(3)}(x) = \begin{cases} -6 & x < 0 \\ 0 & x = 0 \\ 6 & x > 0 \end{cases}$$

So the left and right limits of $f^{(3)}(0)$ do not equal one another.

5.13 Exercise 14

Suppose f is a convex differentiable function. Then

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda)f(y) \\ f(\lambda x + (1-\lambda)y) - f(y) &\leq \lambda(f(x) - f(y)) \\ \frac{f(y) - f(\lambda x + (1-\lambda)y)}{\lambda(y-x)} &\geq \frac{f(y) - f(x)}{y - x} \\ \frac{f(y) - f(\lambda x + (1-\lambda)y)}{y - (\lambda x + (1-\lambda)y)} &\geq \frac{f(y) - f(x)}{y - x} \end{aligned}$$

Since x was arbitrary, the last inequality implies that f' is monotonically increasing (because we can consider a new interval whose left endpoint is $\lambda x + (1-\lambda)y$).

For the other direction, we can modify the convexity condition until we get something that we can apply the mean value theorem to. Letting $z = \lambda x + (1-\lambda)y$, we have

$$\begin{aligned} f(z) &\leq \lambda f(x) + (1-\lambda)f(y) \\ \lambda(f(z) - f(x)) &\leq (1-\lambda)(f(y) - f(z)) \\ \frac{f(z) - f(x)}{z - x} &\leq \frac{(1-\lambda)(f(y) - f(z))}{\lambda(z-x)} \\ &\leq \frac{f(y) - f(z)}{y - z} \end{aligned}$$

The final inequality is true by monotonicity of f' and applying the mean value theorem to (x, z) and (z, y). The final step follows from the fact that $\lambda(z - x) = (1-\lambda)(y - z)$.

5.14 Exercise 15

Proceeding as directed by the hint, we have

$$\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + 2h^2 f''(c) \\
f'(x) &= \frac{1}{2h}(f(x+2h) - f(x)) - hf''(c) \\
M_1 &\leq \frac{1}{2h}(|f(x+2h)| + |f(x)|) + h|f''(c)| \\
&\leq \frac{M_0}{h} + hM_2 \\
M_1^2 &\leq \frac{M_0^2}{h^2} + 2M_0M_2 + h^2M_2^2 \\
&\leq 4M_0M_2 \qquad h = \sqrt{\frac{M_0}{M_2}}
\end{aligned}$$

5.15 Exercise 18

Differentiating $f(t) - f(\beta) = (t-\beta)Q(t)$ and evaluating at α, we have

$$\begin{aligned}
f^{(k)}(\alpha) &= (\alpha - \beta)Q^{(k)}(\alpha) - kQ^{(k-1)}(\alpha) \\
\frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k &= -\frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1} + \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta-\alpha)^k \\
\sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k &= -\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n + (f(\beta) - f(\alpha))
\end{aligned}$$

$$P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n = f(\beta)$$

6 The Riemann-Stieltjes Integral

6.1 Exercise 1

We first note that $\inf f(x) = 0$ on any interval $[p_{i-1}, p_i] \subset [a, b]$. Similarly, $\sup f(x) = 0$ if $x_0 \notin [p_{i-1}, p_i]$ and 1 otherwise. Thus, we proceed by constructing a partition P of $[a, b]$ such that $x_0 \in [p_{i-1}, p_i]$ and $\alpha(p_i) - \alpha(p_{i-1}) < \epsilon$. This is possible since α is continuous at x_0, so there exists δ such that choosing $p_i - p_{i-1} < \delta$ gives us the previous inequality. We then have that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$, so $f \in \mathscr{R}(\alpha)$. Furthermore, since $L(P, f, \alpha) = 0$ for all P, we have that $\int_a^b f d\alpha = 0$.

6.2 Exercise 2

Since we're given $f(x) \geq 0$ and $\int_a^b f(x)dx = 0$, we have that $m = \inf f(x) = 0$. Letting m_i and M_i denote the infimum and supremum of f on the interval $[x_{i-1}, x_i] \subset [a, b]$, we further have that $m_i \leq m \implies m_i = 0 \, \forall i$. Additionally, since f is continuous on $[a, b]$ and therefore uniformly continuous (from compactness), we can choose a δ such that

$$\begin{aligned} |x_i - x_{i-1}| < \delta &\implies |f(x_i) - f(x_{i-1})| < \epsilon \\ &\implies |M_i - m_i| < \epsilon \implies M_i < \epsilon \end{aligned}$$

Thus, all of the M_i can be made arbitrarily small, implying that $\sup f(x) < \epsilon$ for every positive ϵ. This gives us that $f(x) = 0$.

6.3 Exercise 3

(a) Suppose $f(0+) = f(0)$. We consider the partition $P = x_0, x_1, x_2, x_3$ where $x_0 = -1, x_1 = 0, x_3 = 1$. Then $U(P, f, \beta_1) = M_2$ and $L(P, f, \beta_1) = m_2$, and we have that $M_2 \to f(0)$ and $m_2 \to f(0)$ as $x_2 \to 0$, so $f \in \mathscr{R}(\beta_1)$. For the other direction, if $f \in \mathscr{R}(\beta_1)$, then there exists P such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. We can then consider the refinement P^* of P that contains an interval of the form $[0, x_i]$. Suppose now that $|f(0+) - f(0)| = \epsilon > 0$. Then

$$M_i - m_i \geq |f(0+) - f(0)| > \epsilon$$

which is a contradiction, so $f(0+) = f(0)$.

(b) Only difference from (a) is that we need $f(0-) = f(0)$ instead of $f(0+) = f(0)$. The only changes that need to be made to the proof of (a) involve replacing $[0, x_i]$ with $[x_{i-1}, 0]$.

(c) If f is continuous at 0, we can consider the partition P that contains $x_{i-1} < x_i = 0 < x_{i+1}$. Then $U(P, f, \beta_2) = \frac{M_i + M_{i+1}}{2}$ and $L(P, f, \beta_2) = \frac{m_i + m_{i+1}}{2}$. Since f is continuous at 0, $M_i, M_{i+1}, m_i, m_{i+1} \to f(0)$ as $x_{i-1} \to 0$ and $x_{i+1} \to 0$, so $f \in \mathscr{R}(\beta_2)$. For the other direction, we proceed similarly to the proof of (a) and see that

$$\begin{aligned}\frac{1}{2}(M_i + M_{i+1}) - \frac{1}{2}(m_i + m_{i+1}) &= \frac{1}{2}((M_i - m_i) + (M_{i+1} - m_{i+1})) \\ &\geq \frac{1}{2}(|f(0-) - f(0)| + |f(0+) - f(0)|) \\ &> \epsilon\end{aligned}$$

for some $\epsilon > 0$ unless $f(0-) = f(0+) = f(0)$, so f must be continuous at 0 if $f \in \mathscr{R}(\beta_2)$.

(d) If f is continuous at 0 then we have $f(0+) = f(0-) = f(0)$, so we are done by parts (a)-(c).

6.4 Exercise 4

From the density of the rationals and irrationals in the reals, we have that $M_i - m_i = 1\,\forall i$, so $f \notin \mathscr{R}$.

6.5 Exercise 5

If we consider $f(x) = 1$ for all rational x and $f(x) = -1$ for all irrational x, we have that $f^2 \in \mathscr{R}$ but $f \notin \mathscr{R}$. However, if $f^3 \in \mathscr{R}$, then $f \in \mathscr{R}$. This is because $m \leq f^3 \leq M$ (since f is bounded) and $x^{\frac{1}{3}}$ is continuous on any $[m, M]$, so we can apply Theorem 6.11.

6.6 Exercise 6

P is compact, so the open cover consisting of neighborhoods around each point of P has a finite subcover. Thus, P can be covered by finitely many segments. Additionally, the total length of these segments can be made arbitrarily small, since P contains no segments itself. We can then proceed exactly as in the proof of Theorem 6.10 to get the desired result.

6.7 Exercise 7

(a) If $f \in \mathscr{R}$, then by Theorem 6.12 (c) we have

$$\begin{aligned}\left|\int_c^1 f dx - \int_0^1 f dx\right| &= \left|\int_0^c f dx\right| \\ &\leq cM < \epsilon \quad \text{for } c < \frac{\epsilon}{M}\end{aligned}$$

Where $M = \sup f$ on $[0, 1]$.

(b) Consider f such that

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & 0 < x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ -\frac{1}{x} & \frac{1}{2} < x \leq 1 \end{cases}$$

Then $\int_0^1 f dx = 0$, but $\int_0^1 |f| dx$ diverges.

6.8 Exercise 8

We see that the partition $P = 1, 2, ..., b+1$ of the interval $[1, b+1]$ corresponds to the upper sum $U(P, f) = \sum_{n=1}^{b} f(n)$, and $\int_1^{b+1} f(x)dx \leq U(P, f)$. Thus, if $\sum_{n=1}^{\infty} f(n)$ converges, then we have that $\int_1^{b+1} f(x)dx$ is bounded and monotonically increasing (since $f(x) \geq 0$), so $\int_1^{\infty} f(x)dx$ converges. For the other direction, we can take the same partition P and consider $L(P, f) = \sum_{n=1}^{b} f(n+1)$. Since $L(P, f) \leq \int_1^{b+1} f(x)dx$, if $\int_1^{\infty} f(x)dx$ converges then so does $\sum_{n=2}^{\infty} f(n)$, and we are done (as $f(1) < \infty$).

6.9 Exercise 9

Theorem: Let f and g be differentiable functions on $[0, \infty)$ such that $\lim_{x \to \infty} f(x)g(x) = 0$ and $f', g' \in \mathscr{R}$ for every $[0, b]$. Then we have that

$$\int_0^{\infty} f(x)g'(x)dx = -f(0)g(0) - \int_0^{\infty} f'(x)g(x)$$

if both improper integrals converge.

Proof: Take limits on both sides of the integration by parts formula.

Letting $f(x) = \frac{1}{1+x}$ and $g'(x) = cos(x)$ in the above formula shows that

$$\int_0^{\infty} \frac{cos(x)}{1+x} dx = \int_0^{\infty} \frac{sin(x)}{(1+x)^2} dx$$

We have that $|cos(n)| + |cos(n+1)| \geq c$ for some constant c since $cos(n)$ and $cos(n+1)$ cannot both be 0. As such, $\sum_{n=1}^{\infty} \frac{cos(n)}{1+n}$ diverges since the harmonic series diverges and therefore the integral on the left also diverges by the integral test.

6.10 Exercise 10

(a) We can use the convexity of e^x to show this. Let $\lambda = \frac{1}{p}$ and let $u^p = e^a, v^q = e^b$ for some a, b (this is possible because $u, v \geq 0$). Then we have

$$
\begin{aligned}
e^{\lambda a+(1-\lambda)b} &\leq \lambda u^p + (1-\lambda)v^q \\
e^{\lambda a}e^{(1-\lambda)b} &\leq \frac{u^p}{p} + \frac{v^q}{q} \\
uv &\leq \frac{u^p}{p} + \frac{v^q}{q}
\end{aligned}
$$

(b) From part (a) we have that

$$
\begin{aligned}
f(x)g(x) &\leq \frac{f^p(x)}{p} + \frac{g^q(x)}{q} \\
\int_a^b fgd\alpha &\leq 1
\end{aligned}
$$

(c) We can use part (a) with $u = \frac{|f(x)|}{\left(\int_a^b f^p d\alpha\right)^{\frac{1}{p}}}$ and $v = \frac{|g(x)|}{\left(\int_a^b g^q d\alpha\right)^{\frac{1}{q}}}$ to get the desired result.

(d) Assuming the limits exist, the inequality follows for imprompter integrals by taking limits on both sides and then using limit rules.

7 Sequences and Series of Functions

7.1 Exercise 1

We have that $|f_n(x)| \leq M_n$. From uniform convergence, we get that there exists N such that

$$n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon \implies |M_n - M_m| < \epsilon$$

So $\lim_{n\to\infty} M_n = M$ for some M. Thus, we can choose N such that $n \geq N \implies M_n < M+1$, so we can set $M_u = \max\{M_1, M_2, ..., M_{N-1}, M+1\}$. By construction, M_u must be a uniform bound for all of the f_n.

7.2 Exercise 2

Since f_n, g_n are both uniformly convergent, we can choose N_1, N_2 such that

$$n, m \geq \max\{N_1, N_2\} \implies |f_n(x) - f_m(x)| < \frac{\epsilon}{2},\ |g_n(x) - g_m(x)| < \frac{\epsilon}{2}$$
$$|f_n(x) + g_n(x) - f_m(x) - g_m(x)| \leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)| < \epsilon$$

So $f_n + g_n$ converges uniformly as well.

Suppose $f_n \to f$ and $g_n \to g$, with $f_n(x) \leq M_f$ and $g_n(x) \leq M_g$. Then $g(x)f_n(x)$ converges uniformly since $M_g f_n(x)$ converges uniformly, and likewise for $g_n(x)f(x)$. Thus,

$$\begin{aligned}
\frac{\epsilon}{3} &> |(f_n(x) - f(x))(g_n(x) - g(x))| \\
&> |f_n(x)g_n(x) - f_n(x)g(x) - f(x)g_n(x) + f(x)g(x)| \\
&> |f_n(x)g_n(x) - f(x)g(x)| - |f(x)g(x) - f_n(x)g(x)| - |f(x)g(x) - f(x)g_n(x)| \\
&> |f_n(x)g_n(x) - f(x)g(x)| - \frac{\epsilon}{3} - \frac{\epsilon}{3} \\
\implies \epsilon &> |f_n(x)g_n(x) - f(x)g(x)|
\end{aligned}$$

So $f_n g_n \to fg$ uniformly.

7.3 Exercise 3

Let $f_n(x) = \frac{1}{x} + \frac{1}{n}$ and $g_n(x) = \frac{1}{x}$ on $E = (0, 1)$. Both f_n and g_n converge uniformly, and $f_n(x)g_n(x)$ converges pointwise to $\frac{1}{x^2}$. However,

$$\left| f_n(x)g_n(x) - \frac{1}{x^2} \right| = \left| \frac{1}{xn} \right|$$

So the convergence of $f_n(x)g_n(x)$ is not uniform on E.

7.4 Exercise 4

Since $1 + n^2 x = 0$ whenever $x = -\frac{1}{n^2}$, $f(x)$ is not defined/does not converge for $x = 0, -\frac{1}{n^2}$. For all other values of x, however, $f(x)$ converges absolutely since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. $f(x)$ converges uniformly on any intervals of the form $(a, -1)$ and (b, c) with $b > 0$, since such intervals do not contain points of the form $-\frac{1}{n^2}$ and $\frac{1}{x}$ is bounded on all such intervals. f fails to converge uniformly on all other intervals. $f(x)$ is continuous on the intervals that it converges uniformly on. f is not bounded since $\frac{1}{x}$ can be made arbitrarily large.

7.5 Exercise 5

For every positive real number x, there exists N such that $\frac{1}{N} < x$ (Archimedean property). Hence, choosing $n \geq N \implies f_n(x) = 0$. Since every non-positive real number is less than $\frac{1}{n+1}$ for all $n \in \mathbb{N}$, we have that f_n converges pointwise to the function $f(x) = 0$. As such, it is clear that $\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{N} |f_n(x)|$ converges for all x , and yet we do not have uniform convergence.

7.6 Exercise 6

We can rewrite the series as

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} + \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}$$

so the series converges uniformly since the alternating harmonic series converges and the series $\sum_{n=1}^{\infty} \frac{M}{n^2}$ converges (where $x^2 \leq M$, which is possible on every bounded interval). However, this series also clearly does not converge absolutely for any value of x since the harmonic series does not converge.

7.7 Exercise 7

We posit that f_n converges uniformly to the function $f(x) = 0$. For this to be true, there needs to be a single N such that $n \geq N$ gives

$$\left| \frac{x}{1 + nx^2} \right| < \epsilon \implies \frac{|x| - \epsilon}{\epsilon x^2} < n$$

for all x. In other words, the lefthand side of the last expression above must have a maximum. Since the lefthand side is differentiable for all $x \neq 0$, we can differentiate and equate to 0 to find that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{x - \epsilon}{\epsilon x^2} = \frac{2\epsilon^2 x - \epsilon x^2}{(\epsilon x^2)^2}$$
$$2\epsilon^2 x - \epsilon x^2 = 0 \implies x = 2\epsilon$$

giving that $n > \frac{1}{4\epsilon^2}$ is sufficient for all x. Additionally, since $f_n(0) = 0$ for all n, we have that $f_n \to f$ uniformly. Differentiating f_n also shows that $f_n'(x) \to 0$ for all $x \neq 0$ and that $f_n'(0) = 1 \neq f'(0)$.

7.8 Exercise 8

Since $\sum |c_n|$ converges, there exists N such that $n, m \geq N$ gives

$$\epsilon > \sum_{i=n}^{m} |c_i| \geq \sum_{i=n}^{m} |c_i I(x - x_i)| \geq \left| \sum_{i=n}^{m} c_i I(x - x_i) \right|$$

so $f(x)$ converges uniformly as x was arbitrary ($0 \leq I(x - x_i) \leq 1$ for all x). If $x \neq x_n$ and x is not a limit point of x_n, then $f(x)$ is clearly continuous since there is a neighborhood $B_\delta(x)$ that contains none of the x_n which implies that $f(x)$ is constant in this neighborhood. If x is a limit point of the sequence x_n, then for any $\epsilon > 0$ we can choose N such that $n \geq N \implies |c_n| < \epsilon$, so f is continuous at x.

7.9 Exercise 9

Since f_n is a sequence of continuous functions that converges uniformly to f, f is continuous. By uniform convergence of f_n, we have that there exists $N_1 \mid n \geq N_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2}$. By continuity of f, we have that there exists $N_2 \mid n \geq N_2 \implies |f(x_n) - f(x)| < \frac{\epsilon}{2}$. Choosing $n \geq \max(N_1, N_2)$ then gives

$$\begin{aligned} |f_n(x_n) - f(x)| &= |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \\ &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \epsilon \end{aligned}$$

so $\lim_{n\to\infty} f_n(x_n) = f(x)$. The converse need not be true. Consider

$$f_n(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n} \\ 0 & x \neq \frac{1}{n} \end{cases}$$

which satisfies $\lim_{n\to\infty} f_n(x_n) = 0$ for all sequences x_n despite none of the f_n being continuous.

7.10 Exercise 10

$f(x)$ converges uniformly since $0 \leq (nx) < 1$. Since (nx) is discontinuous only when nx is an integer, $f(x)$ is discontinuous only at rational values of x (because we can choose $n = b$ so that $n\frac{a}{b} = a$). The continuity of $f(x)$ at irrational values of x comes from the fact that the partial sums are continuous at all such x and converge uniformly to $f(x)$. Since $f_n(x) = \sum_{i=1}^{n} \frac{(ix)}{i^2}$ contains only finitely many discontinuities in any bounded interval (there are only finitely many rational $x = \frac{z}{i}$ such that $z \in [a, b]$), we have that $f_n \in \mathscr{R}$. Thus, applying Theorem 7.16, we get that $f \in \mathscr{R}$.