# “Sparknotes” for *Linear Algebra* by Peter Lax

Muthu Chidambaram

Last Updated: May 23, 2019

## Contents

# About

> *"A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details."*
> - Hermann Weyl

These notes contain short summaries of (my) proof ideas for exercises and some theorems from the book *Linear Algebra* by Peter Lax. I have tried to make the summaries as brief as possible, sometimes only one line or one equation. My hope is that the summaries will give enough information to reconstruct a full proof without bogging the reader down with details. In many cases, I am sure that I inadvertently sacrificed clarity in an attempt to obtain brevity, and would greatly appreciate any feedback.

Also, I like when people include (what they presume to be) relevant quotes in their notes, so I have to ask you to forgive my haughtiness in starting these notes with a quote from Hermann Weyl.

# 1 Fundamentals

## 1.1 Exercise 1

$x + z = x = x + z' \implies z = z'$.

## 1.2 Exercise 2

$0x + x = (0 + 1)x = x$.

## 1.3 Exercise 3

Coefficients can be represented as row vectors.

## 1.4 Exercise 4

Function can be represented as row vector by letting $a_i = f(s_i)$ for each $s_i \in S$.

## 1.5 Exercise 5

Follows from exercises 3 and 4.

## 1.6 Exercise 6

$y_1 + z_1 + y_2 + z_2 = (y_1 + y_2) + (z_1 + z_2)$ and $k(y_1 + z_1) = ky_1 + kz_1$.

## 1.7 Exercise 7

$a \in Y \cap Z \implies ka \in Y, ka \in Z \implies ka \in Y \cap Z$.

## 1.8 Exercise 8

$k0 = 0, 0 + 0 = 0$.

## 1.9 Exercise 9

If $S$ contains $x_i$ then it must contain $kx_i$.

## 1.10 Exercise 10

If $x_i = 0$, $k_i$ can be anything.

## 1.11 Exercise 11

$x = \sum_{i=1}^{m} \sum_{j=1}^{\dim Y_i} y_j^{(i)}$.

### 1.12 Exercise 12

Complete basis for $W$ to $U$ and $V$. Use $W$ basis vectors and additional $U$ and $V$ basis vectors to get $\dim X = \dim U - \dim W + \dim V - \dim W + \dim W$.

### 1.13 Exercise 13

Send $i^{\text{th}}$ basis vector to $e_i$, where $e_i$ is vector of all zeroes except a one in the $i^{\text{th}}$ place. Can permute mapping to get different isomorphisms.

### 1.14 Exercise 14

$x_1 - x_2 + x_2 - x_3 = x_1 - x_3$.

### 1.15 Exercise 15

$x' = x + z_x, y' = y + z_y \implies x' + y' = x + y + (z_x + z_y)$.

### 1.16 Exercise 16

$x \in X_1 \bigoplus X_2 \implies x = (x_1, x_2) = (x_1, 0) + (0, x_2)$.

### 1.17 Exercise 17

Construct a basis for $X$ from $Y$: $y_1, ..., y_j, x_{j+1}, ..., x_n$.
Then $X/Y = \text{span}\{x_{j+1}, ..., x_n\}$.

# 2 Duality

### Theorem 1

$x = \sum_{i=1}^{n} a_i x_i \implies k_i(x) = a_i$.

## 2.1 Exercise 1

$l_1, l_2 \in Y^\perp \implies l_1(y) + l_2(y) = 0 = (l_1 + l_2)(y)$.

## 2.2 Exercise 2

$\forall \xi \in Y^{\perp\perp} \implies \forall l \in Y^\perp,\ \xi(l) = 0 = l(y)\ \forall y \in Y$.

# 3 Linear Mappings

## 3.1 Exercise 1

(a) $x \in X \implies x = \sum_{i=1}^{n} k_i x_i \implies T(x) = \sum_{i=1}^{n} k_i T(x_i) \in U$.

(b) $T(x), T(y) \in U \implies T(x+y) \in U \implies x+y \in X$.

### Theorem 1

$x \in X, y \in N_T \implies T(x+y) = T(x) + T(y) = T(x)$.

## 3.2 Exercise 2

(a) Differentiation constant and sum rules imply linearity, and multiplication by $s$ is distributive. Take $p(s) = 1$ to see that $ST \neq TS$.

(b) Rotation by 90 degrees amounts to swapping and negating coordinates, which is linear. Take $p = (1, 1, 0)$ to see that $ST \neq TS$.

## 3.3 Exercise 3

(i) $T^{-1}(T(a+b)) = T^{-1}(T(a)+T(b)) = a+b = T^{-1}(T(a)) + T^{-1}(T(b))$.

(ii) Composition of isomorphisms is an isomorphism, hence $ST$ is invertible.

## 3.4 Exercise 4

(i) Let $T : X \to U$, $S : U \to V$ and $l_v \in V'$. Then $(ST)'(l_v) = l_v(ST) = (l_v S)T = (S' l_v)T = T'S'l_v$, since $S'l_v \in U'$.

(ii) Follows from linearity of transpose (definition).

(iii) Let $T : X \to U$ be an isomorphism. Then $l_x = l_u T \implies l_x T^{-1} = l_u$ for $l_u \in U'$, $l_x \in X'$.

## 3.5 Exercise 5

$T''(l_{x'}) = l_{x'} T'$ where $l_{x'} \in X''$ and $l_{x'} T' \in U''$. Since we can identify elements in $X''$ and $U''$ with elements in $X$ and $U$ respectively, we have that $T''$ assigns elements of $U$ to $X$.

### Theorem 2'

Since $T' : U' \to X'$ we have $l_u \in N_{T'} \implies T'(l_u) = l_u T = 0$. $N_{T'}^{\perp}$ consists of elements $l_{u'} | l_{u'}(l_u) = 0$. From $l_u T x = 0$ we have that each $l_{u'}$ is identified with a $u \in R_T$.

## 3.6 Exercise 6

The first two elements of $x$ are already 0 after applying $P$, so $P^2 = P$. Linearity follows from linearity of vector addition.

## 3.7 Exercise 7

$P$ is linear since function addition is linear. $P^2 f = \frac{f(x)+f(-x)}{4} + \frac{f(x)+f(-x)}{4} = Pf$.

# 4 Matrices

## 4.1 Exercise 1

$(P+T)_{ij} = ((P+T)e_j)_i = (Pe_j + Te_j)_i = P_{ij} + T_{ij}$.

## 4.2 Exercise 2

Represent $A$ as a column of row vectors $A_i$ and $B$ as a row of column vectors $B_i$. Denote blocks by parenthesized subscripts. Then the first block of $AB$ looks like:

$$\begin{aligned}
(AB)_{(11)} &= \begin{pmatrix} A_1 B_1 & & \\ & \ddots & \\ & & A_k B_k \end{pmatrix} \\
&= \begin{pmatrix} A_{1,:(k+1)} B_{1,:(k+1)} & & \\ & \ddots & \\ & & A_{k,:(k+1)} B_{k,:(k+1)} \end{pmatrix} \\
&+ \begin{pmatrix} A_{1,(k+1):} B_{1,(k+1):} & & \\ & \ddots & \\ & & A_{k,(k+1):} B_{k,(k+1):} \end{pmatrix} \\
&= A_{(11)} B_{(11)} + A_{(12)} B_{(21)}
\end{aligned}$$

Where:

$$\begin{aligned}
A_{i,:(k+1)} B_{i,:(k+1)} &= \sum_{j=1}^{k} A_{i,j} B_{i,j} \\
A_{i,(k+1):} B_{i,(k+1):} &= \sum_{j=k+1}^{n} A_{i,j} B_{i,j}
\end{aligned}$$

The rest follow similarly.

# 5 Determinant and Trace

## 5.1 Exercise 1

(a) The discriminant already has ordered versions of all the $(i, j)$ difference terms. Applying a permutation only changes the signs of some of the difference terms, hence $\sigma(p) = 1, -1$.

(b) $\sigma(p_1 \circ p_2) = \text{sign}(P(p_1 \circ p_2(x_1, ..., x_n))) = \sigma(p_1)\text{sign}(P(p_2(x_1, ..., x_n)))$.

## 5.2 Exercise 2

(c) A transposition swaps two indices, and hence flips the sign of their associated difference term in the discriminant.

(d) If $p(i) = j$, then we can start with the permutation $(i\ j)$. Next, if $p(j) = k$, we can compose with $(i\ k)$ to get $(i\ k) \circ (i\ j)$. We can do this until we have completely reconstructed the permutation using transpositions.

## 5.3 Exercise 3

By starting with a different $i$ in Exercise 2 (d), we can obtain a different decomposition of transpositions. However, the parity of the decomposition must be the same, as otherwise $\sigma(p)$ will take on two different values for the same $p$.

## 5.4 Exercise 4

(Property II): Each term in $D(a_1, ..., a_n)$ contains exactly one element from each of the $a_i$. Thus, scaling any of the $a_i$ by $k$ scales the entire determinant by $k$. Similar logic for vector addition.

(Property III): The only non-zero term in $D(e_1, ..., e_n)$ is associated with the identity permutation, hence $D(e_1, ..., e_n) = 1$.

(Property IV): Swapping two arguments is the same as applying a transposition to each of the terms in $D(a_1, ..., a_n)$, which flips the sign of $D$.

## 5.5 Exercise 5

Suppose $a_1 = a_2$. Then:

$$\begin{aligned} D(a_1, a_2, ..., a_n) &= -D(a_2, a_1, ..., a_n) \\ D(a_1, a_2, ..., a_n) + D(a_1, a_2, ..., a_n) &= 0 \end{aligned}$$

## 5.6 Exercise 6

We can swap rows and columns until $A$ is in the same form as in Lemma 2. Since each row and column swap is equivalent to applying a transposition, we get that $\det A = (-1)^{i+j} \det A_{ij}$.

## 5.7 Exercise 7

Each term in the sum $D(a_1, ..., a_n) = \sum \sigma(p) a_{p_1 1} ... a_{p_n n}$ consists of exactly one element from each column and each row; swapping rows and columns does not change the terms in the sum. However, the permutation associated with each term is changed. The permutation $p$ that sends $1 \to p_1$ becomes $p'$ sending $p_1 \to 1$. This $p'$ is exactly $p^{-1}$. Since $\sigma(1) = \sigma(p^{-1} \circ p) = \sigma(p^{-1})\sigma(p)$, $\sigma(p^{-1}) = \sigma(p)$ we are done.

## 5.8 Exercise 8

$P$ is the linear transformation such that $P(e_j) = e_i$; in other words, $P$ rearranges the representation of $x$ by applying $p$ to the components of $x$. We also have that $PQx = Pq(x) = p \circ q(x)$, since $Qx$ permutes the components of $x$ to produce $q(x)$, and $Pq(x)$ permutes the components of $q(x)$ to produce $p \circ q(x)$.

## 5.9 Exercise 9

$$\begin{aligned}
\operatorname{Tr} AB &= \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji} \\
\operatorname{Tr} BA &= \sum_{j=1}^{n} (BA)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij}
\end{aligned}$$

## 5.10 Exercise 10

$$\begin{aligned}
\operatorname{Tr} AA^\top &= \sum (AA^\top)_{ii} = \sum \sum a_{ij} a_{ji}^\top \\
&= \sum \sum a_{ij}^2
\end{aligned}$$