

# “Sparknotes” for *Algebra* by MacLane and Birkhoff

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## Preface

“A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.” - Hermann Weyl

These notes contain short summaries of (my) proof ideas for exercises and some theorems from the book *Algebra* by MacLane and Birkhoff. I have tried to make the summaries as brief as possible, sometimes only one line or one equation. My hope is that the summaries will give enough information to reconstruct a full proof without bogging the reader down with details. In many cases, I am sure that I inadvertently sacrificed clarity in an attempt to obtain brevity, and would greatly appreciate any feedback.

Also, I like when people include (what they presume to be) relevant quotes from mathematicians of past generations in their notes, so I have to ask you to forgive my haughtiness in starting these notes with a quote from Hermann Weyl.

# 1 Sets, Functions, and Integers

## 1.1 Sets

### 1.1.1 Exercise 5

When constructing a subset, each element in the set can either be in or out (2 choices). Hence,  $2^n$ .

### 1.1.2 Exercise 6

There are  $n$  choices for the first element,  $n - 1$  choices for the second element, and so on up to  $n - m$ , hence dividing  $n!$  by  $(n - m)!$ . The order of these  $m$  selected elements doesn't matter, hence the division by  $m!$ .

## 1.2 Functions

### 1.2.1 Exercise 2

$h_g \circ h_f$ , where  $h$  corresponds to left-inverse.

### 1.2.2 Exercise 3

Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be surjections. Then  $g \circ f$  is surjective since  $\exists x \in B$  such that  $g(x) = y \quad \forall y \in C$ , and  $\exists x' \in A$  such that  $f(x') = x \quad \forall x \in B$  (from the surjectivity of  $f$  and  $g$ ). Proving injectivity follows similarly.

### 1.2.3 Exercise 4

The reverse direction follows from Exercise 3. If  $f \circ g$  is injective and  $g$  is not, we could choose two elements from the domain of  $g$  that map to the same element in the domain of  $f$  (contradiction). Surjectivity is a similar argument.

### 1.2.4 Exercise 5

$f$  has no right inverse since it is not surjective. There are infinitely many left inverses of  $f$ , two possibilities are mapping to square roots when possible and to 1 or 2 otherwise.

### 1.2.5 Exercise 6

Apply the left inverse of  $f$ .

### 1.2.6 Exercise 7

When surjective, use right inverse.

### 1.2.7 Exercise 8

Define  $h$  such that  $h(y) = x$  if  $\exists x \in S \mid f(x) = y$ , and  $h(y) = x'$  otherwise (axiom of choice necessary for choosing  $x$ ). If  $f$  is injective, there will only be one choice of  $x$ , and if  $f$  is surjective, there will be some  $x$  for every  $y$ .

### 1.2.8 Exercise 9

Unique right inverse indicates that every element in the range has only one choice to map back to in the domain, implying injectivity.

### 1.2.9 Exercise 10

If  $g$  is a bijection, then we can define  $f$  such that  $f(y) = x$  where  $g(x) = y$ .  $f$  is then a two-sided inverse. If  $f$  is a two-sided inverse of  $g$ , then every element of  $T$  maps to a unique element of  $S$  (from left inverse) and vice versa. Hence  $g$  is a bijection.

### 1.2.10 Exercise 11

Following the hint, we can see that  $f : U \rightarrow \mathcal{F}$  is surjective since  $S \in \mathcal{F} \implies S \neq \emptyset \implies \exists u \in S \implies u \in U \implies f(u) = S$ . The existence of the right inverse then gives us the axiom of choice.

## 1.3 Relations and Binary Operations

### 1.3.1 Exercise 2

Symmetry + transitivity imply circularity. For the other direction, we have  $xRy, yRy \implies yRx$ , which gives both symmetry and transitivity.

### 1.3.2 Exercise 3

This only implies reflexivity for the elements  $x, y \in X \mid (x, y) \in R$ , not  $\forall x \in X$ .

### 1.3.3 Exercise 4

If  $R$  is transitive  $T = R$ . Otherwise, start with  $T = R$  and add  $(x, z)$  to  $T$  whenever  $(x, y), (y, z) \in R$ . Repeat this process until there are no more pairs to add.

### 1.3.4 Exercise 5

Let  $R \subset X \times Y$ ,  $S \subset Y \times Z$ ,  $T \subset Z \times A$ .

$$\begin{aligned} xR \circ (S \circ T)a &\implies \exists y \in Y \mid xRy, y(S \circ T)a \\ &\implies \exists z \in Z \mid ySz, zTa \\ &\implies x(R \circ S)z \\ &\implies x(R \circ S) \circ Ta \end{aligned}$$

### 1.3.5 Exercise 6

Let  $R \subset X \times Y$ ,  $S \subset Y \times Z$ .

$$\begin{aligned} z(R \circ S)^\sim x &\implies x(R \circ S)z \\ &\implies \exists y \in Y \mid xRy, ySz \\ &\implies yR^\sim x, zS^\sim y \\ &\implies z(S^\sim \circ R^\sim)x \end{aligned}$$

### 1.3.6 Exercise 7

$$\begin{aligned} (x, z) \in G(g \circ f) &\implies \exists y \in Y \mid g(y) = z, f(x) = y \\ &\implies (x, y) \in G(f), (y, z) \in G(g) \\ &\implies (x, z) \in G(f) \circ G(g) \end{aligned}$$

### 1.3.7 Exercise 9

$$\begin{aligned} (x, y) \in G(f) &\implies \forall x \in X, \exists y \in Y \mid f(x) = y \\ &\implies \forall x \in X, (x, x) \in G(f) \circ G^\sim(f) \\ \text{and } \forall y \in \text{Im} f, &(y, y) \in G^\sim(f) \circ G(f) \end{aligned}$$

### 1.3.8 Exercise 10

$$\begin{aligned} x \square y &= u \square (x \square y) = (u \square y) \square x = y \square x \\ x \square (y \square z) &= x \square (z \square y) = (x \square y) \square z \end{aligned}$$

## 1.4 The Natural Numbers

### 1.4.1 Exercise 1

$f^0 = 1_X$  is trivially an injection. Suppose  $f^n$  is an injection for some  $n \in \mathbb{N}$ . Then  $f^{\sigma(n)} = f \circ f^n$  is a composition of injections and we are done.

### 1.4.2 Exercise 2

Same thing as Exercise 1.

### 1.4.3 Exercise 3

We have that  $\sigma^0(0) = 0$ . Now assuming  $\sigma^n(0) = n$  for some  $n \in \mathbb{N}$ , we have  $\sigma^{\sigma(n)}(0) = \sigma \circ \sigma^n(0) = \sigma(n) = n + 1$ .

### 1.4.4 Exercise 6

We can take  $\sigma^{-1}(n) = n - 1$  for  $n > 0$  and  $\sigma^{-1}(0) = 0, 1, 2$  to get 3 different left inverses.

### 1.4.5 Exercise 8

Let  $n \in U$  if the elements in all sets of size  $n$  are equal. Since we can construct a set with two different elements, we have that  $n = 1$  does not imply  $\sigma(n) \in U$ , and the induction axiom cannot be applied to  $U$ .

### 1.4.6 Exercise 9

(Property I, Property II): Take  $X = \mathbb{N}$  and  $\sigma(x) = x^2 + 1$ .

(Property I, Property III): Let  $X = \{0, 1\}$  and let  $\sigma(0) = 1$ ,  $\sigma(1) = 0$ . Then  $\sigma$  is clearly injective, and any subset of  $X$  that contains 0 and  $\sigma(0)$  is all of  $X$ .

(Property II, Property III): Again take  $X = \{0, 1\}$ , but this time let  $\sigma(0) = \sigma(1) = 1$ .

## 1.5 Addition and Multiplication

### 1.5.1 Exercise 1

$$\begin{aligned} n = 0 : (f^m)^0 &= 1 = f^0 = f^{(\sigma^m)^0(0)} = f^{m0} \\ \text{Assume } n : (f^m)^{(\sigma(n))} &= f^m \circ f^{mn} = f^{m(n+1)} \end{aligned}$$

### 1.5.2 Exercise 2

$$(a) \quad mn = (\sigma^m)^n(0) = \sigma^{mn}(0) = \sigma^{nm}(0) = nm.$$

$$(b) \quad \sigma(m)(n + n') = (\sigma^{\sigma(m)})^{n+n'}(0) = (\sigma^{\sigma(m)})^n(0) + (\sigma^{\sigma(m)})^{n'}(0).$$

### 1.5.3 Exercise 3

(a) To obtain a valid  $\tau$ , simply permute the first few mappings of  $\sigma$ . For example,  $\tau(0) = 2, \tau(1) = 3, \tau(2) = 1, n \geq 3 : \tau(n) = n + 1$ .

(b) Suppose  $\tau$  satisfies Peano. Then we can let  $\beta(0) = 0$  and  $\beta(n) = \tau(\beta(n - 1)) \forall n > 0$ .  $\beta$  is a bijection since  $\tau$  is injective and maps to all of  $\mathbb{N}/\{0\}$ . Furthermore,  $\beta\sigma(n) = \beta(n + 1) = \tau\beta(n)$ .

#### 1.5.4 Exercise 4

(a)

$$\begin{aligned}\phi(n) = m &\implies \sigma(\phi(n)) = m + 1 \\ &\implies \phi(\sigma(n)) = \phi(n + 1) = m + 1\end{aligned}$$

Thus, once we fix  $\phi(0)$ , we fix the rest of  $\phi$ .

(b) There is only one choice of  $\tau$  which satisfies Peano's Postulates:  $\tau(0) = 1$  with  $\tau$  satisfying the relation indicated in (a). This is exactly the successor function  $\sigma$ .

#### 1.5.5 Exercise 6

$k + n = \sigma^n(k) = \sigma^n(m) \implies k = m$  since a composition of injections is an injection.