Exercise Guide for *Complex Analysis* by Lars Ahlfors

Muthu Chidambaram

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About

> *"One of the jewels in the crown of mathematics is complex analysis..."* - Tim Gowers, *The Princeton Companion to Mathematics*

I believe some solutions for this book are available here. What follows are some of my solutions to some exercises in the book.

1 Complex Numbers

1.1 Arithmetic Operations

1.1.1 Exercise 3

This exercise is much easier if we represent the complex numbers in polar form (which has not been introduced yet). We see that

$$\frac{-1 \pm i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$$
$$\frac{\pm 1 \pm i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}},\ e^{i\frac{\pi}{3}},\ e^{i\frac{5\pi}{3}}$$

which gives us the desired equality since $e^{i2\pi} = 1$.

1.2 Square Roots

Once again, all of these computational exercises are made much easier with polar form. They seem more tedious than instructive.

1.3 Justification

Going to come back to this after finishing chapter 3 of Birkhoff/MacLane (which I've been neglecting...).

1.4 Conjugation, Absolute Value

1.4.1 Exercise 3

We can manipulate the equality to get

$$\left\| \frac{a-b}{1-\bar{a}b} \right\| = 1$$
$$\|a-b\|^2 = \|1-\bar{a}b\|^2$$
$$\|a\|^2 + \|b\|^2 - 2\,\mathrm{Re}(a\bar{b}) = 1 + \|a\|^2\|b\|^2 - 2\,\mathrm{Re}(a\bar{b})$$

Thus we see that equality holds if either $\|a\| = 1$ or $\|b\| = 1$, excepting the case where $a = b = 1$ as that makes the denominator in the equality 0.

1.4.2 Exercise 4

Let $z = \alpha + \beta i$. Then we have

$$\big((a+b)\alpha + c\big) + (a-b)\beta i = 0$$
$$\implies (a+b)\alpha + c = 0$$
$$\implies (a-b)\beta = 0$$

If $a-b=0$, β could be anything, so we must have $a-b\neq 0$ for the solution to be unique. Similarly, if $a+b=0$ then α can either be anything or there is no solution for α (if $c\neq 0$). Thus, the two conditions we need are $a+b\neq 0$ and $a-b\neq 0$.

1.4.3 Exercise 5

We can write $\left|\sum_{i=1}^n a_i b_i\right|^2$ as $(\sum_{i=1}^n a_i b_i)(\sum_{j=1}^n \overline{a_j b_j})$ to see that it can be expanded as a sum whose terms consist of $|a_i|^2|b_i|^2$ and $a_i b_i \overline{a_j b_j}$ over all $1\leq i,j\leq n$. The $a_i b_i \overline{a_j b_j}$ terms can be paired with the $a_j b_j \overline{a_i b_i}$ terms to get that

$$\begin{aligned}\left|\sum_{i=1}^n a_i b_i\right|^2 &= \sum_{i=1}^n |a_i|^2|b_i|^2 + \sum_{1\leq i<j\leq n} 2\operatorname{Re} a_i b_i \overline{a_j b_j} \\ &= \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 - \sum_{1\leq i<j\leq n} \left|a_i \bar{b}_j - a_j \bar{b}_i\right|^2\end{aligned}$$