Exercise Guide for *Analysis on Manifolds* by Michael Spivak

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About

> *"Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising from its own ashes, is its own derivative?"*
>
> \- Francois de Lionnais

This book is simply too famous and too short to not work through.

1 Functions on Euclidean Space

1.1 Norm and Inner Product

1.1.1 Exercise 1

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} |x_i|^2 \leq \left(\sum_{i=1}^{n} |x_i| \right)^2 \implies \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} |x_i|^2$$

1.1.2 Exercise 2

We need $\sum_{i=1}^{n} x_i y_i = |x||y|$. Linear dependence by itself is not enough, since if x and y are opposite sign we see that the lefthand sum will be negative. Thus, we need linear dependence as well as x and y having the same sign.

1.1.3 Exercise 3

The proof is identical to the $|x+y|$ case, except now we have a $-2\sum_{i=1}^{n} x_i y_i$ term. Thus, equality holds when x and y are linearly dependent and have opposite signs.

1.1.4 Exercise 4

$||x|-|y||^2 = |x|^2 + |y|^2 - 2|x||y|$. Since $|x||y| \geq \langle x, y \rangle$, we have the desired inequality.

1.1.5 Exercise 5

$$|z-x| = |z-y+y-x| \leq |z-y| + |y-x|$$

Geometrically, this is just the fact that any sidelength of a triangle must be bounded by the sum of the other two sidelengths.

1.1.6 Exercise 6

(a) We can proceed as Spivak hints by noting that for the $\int_a^b (f-\lambda g)^2 > 0$ case, the proof is identical to that of Theorem 1-1 (2). For the $\int_a^b (f-\lambda g)^2 = 0$ case (which is when equality is obtained), we can use the fact that $(f-\lambda g)^2 = 0$ almost everywhere.

However, I think it's a little smoother to handle both cases at once:

$$\int_a^b (f-\lambda g)^2 = \int_a^b f^2 - 2\lambda \int_a^b fg + \lambda^2 \int_a^b g^2 \geq 0$$
$$\int_a^b f^2 - \frac{2\left(\int_a^b fg\right)^2}{\int_a^b g^2} + \frac{\left(\int_a^b fg\right)^2}{\int_a^b g^2} \geq 0 \quad \text{for } \lambda = \frac{\int_a^b fg}{\int_a^b g^2}$$
$$\sqrt{\int_a^b f^2 \int_a^b g^2} \geq \left| \int_a^b fg \right|$$

(b) Equality does not necessarily imply that $f = \lambda g$, since we could consider f and g to be 0 everywhere except two points a and b such that $f(a) \neq \lambda g(a)$ and $f(b) \neq \lambda g(b)$. However, if f and g are continuous, then $\int_a^b (f-\lambda g)^2 = 0$ implies that $f = \lambda g$.

(c) Define $f(m) = x_i$ and $g(m) = y_i$ for $m \in [i, i+1)$. Then $\int_1^n fg = \sum_1^n x_i y_i$ (we can break up the integral at the points of discontinuity) and Theorem 1-1 (2) follows.

1.1.7 Exercise 7

(a) If T is inner product preserving then we have $\langle Tx, Tx\rangle = \langle x, x\rangle$, so T is norm preserving. If T is norm preserving, then

$$\langle T(x-y), T(x-y)\rangle = |T(x)|^2 + |T(y)|^2 - 2\langle T(x), T(y)\rangle$$
$$\langle x-y, x-y\rangle = |x|^2 + |y|^2 - 2\langle x, y\rangle$$
$$\implies \langle T(x), T(y)\rangle = \langle x, y\rangle$$

so T is inner product preserving.

(b) Since $Tx = Ty \implies T(x-y) = 0 \implies |x-y| = 0$, T is injective. Furthermore, $Tx = 0 \implies |x| = 0$, so the nullspace of T is trivial and T is thus surjective. Now if we consider $T^{-1}y = x$, then we have $\langle y, y\rangle = \langle TT^{-1}y, TT^{-1}y\rangle = \langle T^{-1}y, T^{-1}y\rangle$.

1.1.8 Exercise 10

Let $\|T\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m T_{ij}^2}$ (Frobenius norm). Then we have

$$|Th|^2 = \sum_{i=1}^n \left(\sum_{j=1}^m T_{ij} h_j\right)^2 \leq \sum_{i=1}^n \sum_{j=1}^m T_i j^2 \sum_{j=1}^m h_j^2 = \|T\|_F^2 |h|^2$$

so letting $M = \|T\|_F$ gives the desired inequality.

1.1.9 Exercise 12

Linearity and injectivity of T follow from linearity of inner product. To see surjectivity, we note that any element $f \in (\mathbb{R}^n)^*$ is determined entirely by $f(e_1), ..., f(e_n)$ due to linearity. Thus, $f(y) = \langle x, y \rangle$ for the unique x satisfying $x_i = f(e_i)$.

1.1.10 Exercise 13

Expanding $\langle x + y, x + y \rangle$ gives the desired result.

1.2 Subsets of Euclidean Space

1.2.1 Exercise 14

Any point a in the union is also in some set which contains an open set around a, and by the definition of union this open set is also in the union. For finite intersection, if a point is in two open sets, then both sets must contain some open rectangles around that point; the smaller of these rectangles is in the intersection of both sets. For infinite intersection, we can consider the intersection of $(-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$, as it consists only of the single point 0.

1.2.2 Exercise 17

The procedure hinted by Spivak seems to be to split the square into 4 quadrants and then select a point from each quadrant while satisfying the constraint, then repeat this procedure indefinitely (split the 4 quadrants into 16 quadrants, etc.). However, I'm not sure how to make this procedure more rigorous.

1.2.3 Exercise 19

We can use the facts that the rationals are dense in $\mathbb{R}$ and that closed sets contain all of their limit points (although neither fact is proved so far in this book) to immediately get the desired result.

1.2.4 Exercise 21

(a) Since A is closed, A^c is open, which means x has an open rectangle (and thus, an open ball) around it that is contained within A^c. Therefore, there exists $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.

(b) Suppose that for every $d > 0$, we could choose $y_i \in A$ and $x_i \in B$ such that $|y_i - x_i| < d$. This would imply that we could pick a sequence $\{y_n\} \in A$ and a sequence $\{x_n\} \in B$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$. Since B is compact, this limit must be a point in B, which contradicts the disjointness of A and B.

(c) Consider $A = \{(0, i) \mid i \in \mathbb{N}\}$ and $B = \{(0, i + \frac{1}{i}) \mid i \in \mathbb{N}\}$. For any d, we can choose i such that $\frac{1}{i} < d$.

1.2.5 Exercise 22

This is easier to argue with open/closed balls instead of open/closed rectangles (for me). Every point $c \in C$ must have an open ball $B_\delta(c) \subset U$ for some $\delta > 0$. For each such ball, consider instead $B_r(c)$ with $r < \delta$. The closure of $B_r(c)$ is a closed ball that is contained within U. Now, since C is compact, it can be covered by finitely many of these closed balls. Since closed sets are closed under finite union, we can take D to be the union of these balls to get a compact set whose interior contains C.