

# Exercise Guide for *Principles of Mathematical Analysis (3rd Ed.)* by Walter Rudin

Muthu Chidambaram

Last Updated: June 25, 2019

## Contents

<b>1</b>	<b>The Real and Complex Number Systems</b>	<b>5</b>
1.1	Exercise 1 . . . . .	5
1.2	Exercise 2 . . . . .	5
1.3	Exercise 4 . . . . .	5
1.4	Exercise 5 . . . . .	5
1.5	Exercise 6 . . . . .	5
1.6	Exercise 7 . . . . .	5
1.7	Exercise 8 . . . . .	6
1.8	Exercise 9 . . . . .	6
1.9	Exercise 10 . . . . .	6
1.10	Exercise 11 . . . . .	6
1.11	Exercise 12 . . . . .	6
1.12	Exercise 13 . . . . .	6
<b>2</b>	<b>Basic Topology</b>	<b>7</b>
2.1	Exercise 1 . . . . .	7
2.2	Exercise 2 . . . . .	7
2.3	Exercise 3 . . . . .	7
2.4	Exercise 4 . . . . .	7
2.5	Exercise 5 . . . . .	7
2.6	Exercise 6 . . . . .	7
2.7	Exercise 7 . . . . .	7
2.8	Exercise 8 . . . . .	8
2.9	Exercise 10 . . . . .	8
2.10	Exercise 12 . . . . .	8
2.11	Exercise 13 . . . . .	8
2.12	Exercise 14 . . . . .	8
2.13	Exercise 15 . . . . .	8
2.14	Exercise 16 . . . . .	8

2.15	Exercise 17	8
2.16	Exercise 18	9
<b>3</b>	<b>Numerical Sequences and Series</b>	<b>10</b>
3.1	Exercise 1	11
3.2	Exercise 2	11
3.3	Exercise 3	11
3.4	Exercise 4	11
3.5	Exercise 5	11
3.6	Exercise 6	11
3.7	Exercise 7	12
3.8	Exercise 8	12
3.9	Exercise 9	12
3.10	Exercise 10	12
3.11	Exercise 15	12
<b>4</b>	<b>Continuity</b>	<b>14</b>
4.1	Exercise 1	14
4.2	Exercise 2	14
4.3	Exercise 3	14
4.4	Exercise 4	14
4.5	Exercise 5	14
4.6	Exercise 6	14
4.7	Exercise 7	15
4.8	Exercise 8	15
4.9	Exercise 9	15
4.10	Exercise 10	15
4.11	Exercise 11	16
4.12	Exercise 12	16
4.13	Exercise 13	16
4.14	Exercise 16	16
4.15	Exercise 17	16
4.16	Exercise 23	17
<b>5</b>	<b>Differentiation</b>	<b>18</b>
5.1	Exercise 1	18
5.2	Exercise 2	18
5.3	Exercise 3	18
5.4	Exercise 4	19
5.5	Exercise 5	19
5.6	Exercise 6	19
5.7	Exercise 7	19
5.8	Exercise 8	20
5.9	Exercise 9	20
5.10	Exercise 10	20
5.11	Exercise 11	20

5.12	Exercise 12	21
5.13	Exercise 14	21
5.14	Exercise 15	22
5.15	Exercise 18	22
<b>6</b>	<b>The Riemann-Stieltjes Integral</b>	<b>23</b>
6.1	Exercise 1	23
6.2	Exercise 2	23
6.3	Exercise 3	23
6.4	Exercise 4	24
6.5	Exercise 5	24

## About

*“The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics; and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.”* - John von Neumann

An actual solution manual for the book can be found [here](#). What follows are notes I took as I did exercises (they’re more like hints towards my thinking than solutions) while working through the book on my own.

# 1 The Real and Complex Number Systems

## 1.1 Exercise 1

If  $rx = q$  or  $r + x = q$  for some rational  $q$ , then subtracting  $r$  from  $q$  or dividing  $q$  by  $r$  yields  $x$  rational, which is a contradiction.

## 1.2 Exercise 2

We can first show that  $\sqrt{3}$  is irrational by seeing that  $\frac{a^2}{b^2} = 3 \implies 3|a, 3|b$ . Then, since  $12 = 3 * 2^2$ , we have that  $\sqrt{12}$  is irrational as well.

## 1.3 Exercise 4

If  $\alpha > \beta$  then  $\alpha$  would be an upper bound as well.

## 1.4 Exercise 5

$\forall x \in A, -x \leq \sup -A$  and  $\forall \epsilon \in \mathbb{R}, \exists x \in A | \sup -A + \epsilon < -x \leq \sup -A$ . Negating the last inequality gives  $\inf A = -\sup -A$ .

## 1.5 Exercise 6

(a) Follows from  $m = \frac{np}{q}$ .

(b) Put  $r = \frac{m}{n}, s = \frac{p}{q}$ . Then  $b^r b^s = b^{\frac{mq}{nq}} b^{\frac{np}{nq}}$ . Pulling out  $\frac{1}{nq}$  gives the desired result.

(c)  $b^r$  is an upper bound since  $b > 1$ , and if it were not the supremum we could choose  $t < r$  such that  $b^t > b^r$ . This is not possible since again,  $b > 1$ .

(d) Every element in  $B(x + y)$  can be expressed as  $b^{s+t} = b^s b^t$   $s \leq x, t \leq y$ . If  $\sup B(x + y) = \alpha < \sup B(x) \sup B(y)$ , then  $b^s b^t \leq \alpha \implies B(x) \leq \alpha b^{-t} \implies B(y) \leq \frac{\alpha}{B(x)} \implies B(x)B(y) \leq \alpha$ .

## 1.6 Exercise 7

(a)  $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + \dots + 1) \geq n(b - 1)$  since  $b > 1$ .

(b) Plug  $b^{\frac{1}{n}}$  into (a).

(c) Plug  $n > \frac{b-1}{t-1}$  into (b).

(d) Using (c) gives that we can choose  $n$  such that  $b^{\frac{1}{n}} < y b^{-w} \implies b^{w+\frac{1}{n}} < y$ .

(e) We can take the reciprocal of (c) and do the same as in (d).

(f) If  $b^x > y$  we can apply (e) for a contradiction, if  $b^x < y$  we can apply (d) for a contradiction.

(g) Supremum is unique.

### 1.7 Exercise 8

Suppose  $(0, 1) < (0, 0)$ . Then  $(0, -1) < (0, 0)$  after multiplying by  $(0, 1)$  twice yields a contradiction. Similarly, assuming the opposite yields  $(-1, 0) > (0, 0)$ .

### 1.8 Exercise 9

Does exhibit least upper-bound property since you can take  $(\sup a_i, \sup b_i)$ .

### 1.9 Exercise 10

Exception is 0.

### 1.10 Exercise 11

Take  $w = \frac{1}{|z|}z$  and  $r = |z|$  when  $|z| \neq 0$ .  $w$  and  $r$  are not uniquely determined; take  $z = 0$  for example.

### 1.11 Exercise 12

By strong induction:

$$\begin{aligned} |z_1 + \dots + z_{n+1}| &\leq |z_1 + \dots + z_n| + |z_{n+1}| \\ &\leq |z_1| + \dots + |z_{n+1}| \end{aligned}$$

### 1.12 Exercise 13

$$\begin{aligned} |x - y|^2 &= x\bar{x} - 2|x||y| + y\bar{y} \\ &\geq (|x| - |y|)^2 \end{aligned}$$

## 2 Basic Topology

### 2.1 Exercise 1

The empty set has no elements, so all of its elements are vacuously also elements of every set.

### 2.2 Exercise 2

The roots of complex polynomials with integer coefficients can be expressed as elements of the countable cross product of  $\mathbb{N}$  with itself (cross  $\mathbb{N}$  with itself  $n$  times for the coefficients, and then once more to indicate which root).

### 2.3 Exercise 3

If all real numbers were algebraic, then the set of algebraic numbers would be uncountable (thus contradicting Exercise 2).

### 2.4 Exercise 4

The set of irrational numbers is  $\mathbb{R}/\mathbb{Q}$ , which must be uncountable as otherwise  $\mathbb{R}$  would be countable.

### 2.5 Exercise 5

We can use  $\left(\frac{n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{2n}{n+1}\right)_{n \in \mathbb{N}} \cup \left(\frac{3n}{n+1}\right)_{n \in \mathbb{N}}$  to get the three limit points 1, 2, 3.

### 2.6 Exercise 6

If  $p$  is a limit point of  $E'$ , then every neighborhood of  $p$  contains a limit point  $q$  of  $E$ , and every neighborhood of  $q$  contains a point of  $E$  thereby implying that  $p$  is a limit point of  $E$ .  $E$  and  $E'$  do not need to have the same limit points, since  $E'$  could be finite and thus have no limit points.

### 2.7 Exercise 7

(a) If  $p$  is a limit point of  $\overline{B_n}$ , then every neighborhood of  $p$  contains a point  $q \in A_i$ . Since there are only finitely many  $A_i$ ,  $p$  must be a limit point for at least one of the  $A_i$ , as an infinite number of neighborhoods of  $p$  must have non-zero intersection with some of the  $A_i$ .

(b) If we take  $A_i = \left(\frac{in}{(i+1)n+1}\right)_{n \in \mathbb{N}}$ , then 1 is a limit point of  $B_n$  despite not being a limit point of any of the  $A_i$ .

## 2.8 Exercise 8

Every point of an open set in  $\mathbb{R}^2$  is by definition a limit point of the set, since the point must have a neighborhood contained in the set. The same is not true for closed sets, since we can just take a finite set.

## 2.9 Exercise 10

Every set in  $X$  is open, since any set containing  $p$  also contains  $N_r(p)$  for  $r < 1$ . No set in  $X$  is closed, since  $N_r(p) = p$  for  $r < 1$ . All infinite sets in  $X$  are not compact, since we can take balls of radius  $r < 1$  around each point as an open cover.

## 2.10 Exercise 12

Take any open cover of  $K$ . There must be some open set in this cover containing 0, which means that the same set contains all but a finite number of the elements of  $K$  (since 0 is the only limit point of  $K$ ). Take a union of this set as well as the finitely many other sets containing the aforementioned points to get a finite subcover.

## 2.11 Exercise 13

Take  $\cup_{k=1}^{\infty} \{0, (\frac{n}{kn+1})_{n \in \mathbb{N}}, \frac{1}{k}\}$ . This set is closed and bounded, so it is compact by Heine-Borel. Its limit points are 0 and  $(\frac{1}{k})_{n \in \mathbb{N}}$ .

## 2.12 Exercise 14

We can use  $\cup_{n \in \mathbb{N}} (0, \frac{n}{n+1})$ , which has no finite subcover (since we could choose  $x \in (0, 1)$  larger than the largest endpoint in the finite subcover).

## 2.13 Exercise 15

For closed, we can take  $K_i = \mathbb{N}/0, \dots, i-1$ , since any  $x \in K_i$  will not be in  $K_j$  if  $j > x$ . For bounded, we can take  $K_i = (0, \frac{1}{i})$ .

## 2.14 Exercise 16

$E$  is by definition bounded, and  $E$  is closed since  $q^2 \neq 3$  ( $q$  is rational), and  $q^2 > 3 \implies \exists \epsilon \mid p \in N_{\epsilon}(q) \implies p^2 > 3$ . Same logic gives that  $E$  is also open in  $\mathbb{Q}$ .  $E$  is, however, not compact, since we can construct an open cover consisting of  $G_n = \{x \mid 2 < x^2 < 2 + \frac{n}{n+1}\}$ .

## 2.15 Exercise 17

$E$  is not countable by diagonalization.  $E$  is not dense in  $[0, 1]$ , since  $E \cap [0, 0.1] = \emptyset$ .  $E$  is not perfect, consider  $N_{0.001}(0.77)$ .  $E$  is closed and therefore compact by



Heine-Borel. To see closed, suppose a limit point  $q$  had a non-4/7 digit in the  $i^{th}$  decimal spot. Then we could take a neighborhood of size  $10^{-(i+1)}$ .

### **2.16 Exercise 18**

Rationals are dense in  $\mathbb{R}$ , so no.

### 3 Numerical Sequences and Series

#### Definition 3.5

Since  $\{p_n\} \rightarrow p \implies \forall \epsilon, \exists N | n \geq N \implies |p_n - p| < \epsilon$ , we can choose  $k | n_k \geq N \implies \{p_{n_k}\} \rightarrow p$ . The reverse direction can be shown via contradiction of  $\{p_n\} \rightarrow p$ .

#### Examples 3.18

- (a) Density of rationals in reals.
- (b)  $|s_n| < 1$ , take  $n$  odd to get -1 and even to get 1.
- (c) Every subsequential limit has to converge to  $s$ .

#### Theorem 3.19

For all  $\{n_k\}$ , we have  $\exists K | k \geq K \implies n_k \geq N \implies \lim_{k \rightarrow \infty} t_{n_k} - s_{n_k} \geq 0$ .

#### Theorem 3.26

$$s_n = 1 + x + \dots + x^n \implies x s_n = x + x^2 + \dots + x^{n+1} \implies (1 - x)s_n = 1 - x^{n+1}.$$

#### Examples 3.40

- (a) Root test:  $n \rightarrow \infty$ .
- (b) Ratio test:  $\frac{1}{n+1} \rightarrow 0$ .
- (c)  $1 \rightarrow 1$ .
- (d) Ratio test:  $\frac{n}{n+1} \rightarrow 1$ .  $z = 1$  leads to harmonic series.
- (e) Ratio test:  $\frac{n^2}{(n+1)^2} \rightarrow 1$ .

#### Example 3.53

$\sum_{k=1}^{\infty} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} < \frac{5}{6} + \sum_{k=2}^{\infty} \frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k}$ . The RHS converges since  $\frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k} = \frac{1}{2k^2-2k}$ .

### 3.1 Exercise 1

All we need is the inequality  $|s_n - s| \geq ||s_n| - |s||$ . The converse is not true, since we can take  $s_n = (-1)^n$ .

### 3.2 Exercise 2

My original idea:  $\sqrt{(n+x)^2} - n = x$ . Setting  $(n+x)^2 \geq n^2 + n$  gives  $x^2 \geq (1-2x)n$ . The last inequality is only true for all  $n$  when  $x \geq \frac{1}{2}$ . This implies that  $\frac{1}{2}$  is the supremum of  $\sqrt{n^2+n} - n$ . Since  $\sqrt{n^2+n} - n$  is increasing, it converges to  $\frac{1}{2}$ .

Better:  $(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n) = n \implies \sqrt{n^2+n} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$ .

### 3.3 Exercise 3

Clearly  $s_{n+1} > s_n$ . We can see that  $s_n < 2$  by induction, since  $s_1 < 2$  and  $2 + \sqrt{s_n} < 4$ . This gives that  $s_n$  is monotone and bounded, implying it converges.

### 3.4 Exercise 4

$$s_{2m+1} = \sum_{i=1}^m \frac{1}{2^i}, \quad s_{2m} = \sum_{i=2}^m \frac{1}{2^i}$$
$$\implies \limsup_{n \rightarrow \infty} s_n = 1, \quad \liminf_{n \rightarrow \infty} s_n = \frac{1}{2}$$

### 3.5 Exercise 5

$$\begin{aligned} \limsup_{n \rightarrow \infty} (a_n + b_n) &= \sup_{\{k\}} \left\{ \lim_{k \rightarrow \infty} (a_{n_k} + b_{n_k}) \right\} \\ &= \sup_{\{k\}} \left\{ \lim_{k \rightarrow \infty} a_{n_k} + \lim_{k \rightarrow \infty} b_{n_k} \right\} \end{aligned}$$

### 3.6 Exercise 6

(a)  $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$  diverges from comparison to harmonic series (same technique as Exercise 2).

(b) Converges, by comparison to  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  for  $p = \frac{3}{2}$ .

(c) Converges by root test, since  $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ .

(d) Converges when  $|z| > 1$  and diverges otherwise. To see this, put  $z = |z|e^{i\theta}$  to get  $\lim_{n \rightarrow \infty} \frac{1}{1+|z|^n e^{in\theta}}$ .

### 3.7 Exercise 7

We proceed via the ratio test.

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{n}{n+1} * \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} &= \limsup_{n \rightarrow \infty} \frac{n}{n+1} \limsup_{n \rightarrow \infty} \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} \\ &= \sqrt{\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}} \\ &< 1 \end{aligned}$$

Since  $\sum a_n$  converges.

### 3.8 Exercise 8

Since  $b_n$  is monotonic and bounded,  $|b_n| \leq B$  for all  $n$ . Then we have that  $\sum a_n b_n$  converges by the comparison test, since  $|a_n b_n| \leq B|a_n|$  and  $B \sum a_n$  converges.

### 3.9 Exercise 9

(a) Applying the ratio test, we see that  $|z| \limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$  when  $|z| < 1$ . Thus  $\sum n^3 z^n$  has radius of convergence 1.

(b) Again, applying the ratio test, we see that  $2|z| \limsup_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0$ , implying  $R = +\infty$ .

(c) The ratio test is the only hammer we need:  $2|z| \limsup_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| < 1$  gives  $R = \frac{1}{2}$ .

(d) What are the other tests again?  $\frac{|z|}{3} \limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$  gives  $R = 3$ .

### 3.10 Exercise 10

The infinitely many non-zero  $a_n$  must satisfy  $|a_n| \geq 1$ . The radius of convergence of  $\sum a_n z^n$  will be maximized when  $|a_n|$  is minimized, so we can just consider the case where there are infinitely many  $|a_n| = 1$ . In this case, we can choose a subsequence  $a_{n_k}$  consisting only of 1. Applying the ratio test using this subsequence gives  $|z| < 1$ .

### 3.11 Exercise 15

Theorems 3.22, 3.23, and 3.25(a) require no changes in their proofs, since the Cauchy criterion is applicable for  $\mathbb{R}^k$ . Theorem 3.33(a, b) also require no changes once we have the comparison test for  $\mathbb{R}^k$ . For Theorem 3.33(c), we can take  $a \in \mathbb{R}^k$  such that all of its components are  $\frac{1}{n}$  or  $\frac{1}{n^2}$ .

Theorem 3.34(a, b) just need to be modified to use  $\frac{|a_{n+1}|}{|a_n|}$ . Theorem 3.42 needs to be modified to use the dot product, but then it follows from applying the  $\mathbb{R}$  version of 3.42 to the components of the dot product sum. Theorems 3.45, 3.47, and 3.55 require no changes to their proofs.

## 4 Continuity

### 4.1 Exercise 1

Continuity implies  $\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$ , since we can choose  $h$  to be within  $\delta$  of  $x$  such that  $|f(x+h) - f(x) + f(x) - f(x-h)| \leq |f(x+h) - f(x)| + |f(x-h) - f(x)| < \epsilon$ . However, the converse (as asked in the question) need not be true, since we don't have to have  $\lim_{h \rightarrow 0} f(x+h) = f(x) = \lim_{h \rightarrow 0} f(x-h)$ . For example, consider  $x \neq 0 \implies f(x) = \frac{1}{|x|}$ ,  $f(0) = 0$ .

### 4.2 Exercise 2

Suppose  $p$  is a limit point of  $E$ . Then there is a sequence  $(x_n) \in E \mid \lim_{n \rightarrow \infty} x_n = p$ . Since  $f$  is continuous, we have that  $\lim_{n \rightarrow \infty} f(x_n) = f(p)$ , which implies that  $f(p)$  is a limit point of  $f(E)$  giving us that  $f(\overline{E}) \subset \overline{f(E)}$ .

To see that  $f(\overline{E})$  can be a proper subset, consider  $f : \mathbb{Z}^+ \rightarrow \mathbb{Q}$  with  $f(x) = \frac{1}{x}$ . Then  $f$  is continuous and  $0 \notin f(\mathbb{Z}^+) = f(\mathbb{Z}^+)$ .

### 4.3 Exercise 3

Similar to Exercise 2: if  $p$  is a limit point of  $Z(f)$ , then there exists some sequence  $(x_n) \in E \mid \lim_{n \rightarrow \infty} x_n = p$ . Since  $f$  is continuous, we have that  $\lim_{n \rightarrow \infty} f(x_n) = f(p)$ . Then it follows that  $x_n \in Z(f) \implies f(x_n) = 0 \implies f(p) = 0$ .

### 4.4 Exercise 4

The fact that  $f(E)$  is dense in  $f(X)$  follows from Exercise 2, since  $X = \overline{E}$ . Similarly,  $\lim_{n \rightarrow \infty} g(p_n) = g(p) \implies \lim_{n \rightarrow \infty} f(p_n) = g(p)$  since  $p_n \in E$ . Thus,  $g(p) = f(p)$  for all  $p \in X$ .

### 4.5 Exercise 5

If  $f$  is defined on an open set in  $\mathbb{R}^1$ , then it need not be defined at its endpoints. For example, consider  $f(x) = \frac{1}{x}$  defined on  $(0, 1)$ . However, if  $f$  is defined on a closed subset  $E \subset \mathbb{R}^1$ , then  $E^c$  is an open set in  $\mathbb{R}$  and can thus be decomposed into the union of a countable number of open intervals  $(a_n, b_n)$ . We can thus take  $g$  to be  $g(x) = \frac{b_n - x}{b_n - a_n} f(a_n) + (1 - \frac{b_n - x}{b_n - a_n}) f(b_n)$  (the straight line interpolation between  $f(a_n)$  and  $f(b_n)$ ).

### 4.6 Exercise 6

$f$  is a bijection from  $E$  to its graph  $G(E)$ . If  $f$  is continuous, then we can take the inverse image of an open cover of  $G(E)$  to get an open cover of  $E$ . Since  $E$  is compact, this open cover must have a finite subcover whose image under  $f$  will be a finite subcover for  $G(E)$ , thereby giving the compactness of  $G(E)$ .

I looked up a hint on the reverse direction. Consider an infinite (finite case presents no issues) closed set  $V \subset G(E)$ . Take some arbitrary subsequence  $(x_k, f(x_k)) \in V$ . By the compactness of  $G(E)$ , this subsequence has a limit point  $(x, f(x)) \in G(E)$ , and this limit point is contained in  $V$  since  $V$  is closed. Thus,  $f^{-1}(V)$  also contains  $x_k \rightarrow x$ , implying that  $f^{-1}(V)$  contains all of its limit points and is therefore closed. This shows that  $f$  is continuous.

For what it's worth, I think this argument using projections is much nicer.

#### 4.7 Exercise 7

Suppose for any  $M$  that  $\exists x, y \mid f(x, y) > M$  (we consider only the case where  $x > 0$ , as the other case is identical). Then we can solve the resulting quadratic to see that, if such  $x$  and  $y$  exist, then  $x > \frac{y^2(1+\sqrt{1-4M^2})}{2M}$ . However,  $\sqrt{1-4M^2}$  is not defined in  $\mathbb{R}$  for  $M > \frac{1}{2}$ , so  $f$  must be bounded. Performing the same analysis for  $g$  yields  $x > \frac{y^2(1+\sqrt{1-4y^2M^2})}{2M}$ . Since  $y$  can be chosen to make the inequality for  $x$  have a solution in  $\mathbb{R}$ ,  $g$  is unbounded.

To show that  $f$  is discontinuous at  $(0, 0)$ , we need only consider the sequence consisting of  $(0, \frac{n}{n+1})$  to see that  $\lim_{n \rightarrow \infty} f(0, \frac{n}{n+1}) = 1 \neq 0$ . Plugging in  $y = ax + b$  leads to  $f$  and  $g$  being quotients of two polynomials with non-zero denominator, indicating that they're both continuous.

#### 4.8 Exercise 8

Suppose  $f$  is not bounded. Then there is a sequence  $f(x_n) \mid \forall N, \exists m, n \geq N \mid |f(x_n) - f(x_m)| > \epsilon$  for some  $\epsilon$ , since otherwise  $f(x_n)$  would converge to some point of  $\mathbb{R}$ . As  $f$  is uniformly continuous, this means that  $|x_n - x_m| > \delta$  for infinitely many  $n, m$ . However, that would then imply that  $E$  is not bounded, which is a contradiction. Thus,  $f$  is bounded on  $E$ .

If  $E$  is not bounded, we can just take  $f(x) = x$ .

#### 4.9 Exercise 9

Let  $E$  consist of all  $x, y \mid d_X(x, y) < \delta$ . Then  $\text{diam} E < \delta$ . Similarly, if  $\forall x, y \mid d_Y(f(x), f(y)) < \epsilon$ , then  $\text{diam} f(E) < \epsilon$ .

#### 4.10 Exercise 10

Suppose  $f$  is not uniformly continuous. Then there is a sequence  $x_n \in X \mid x_n \rightarrow x$ , but  $\forall N, \exists m, n \geq N \mid d_Y(f(x_n), f(x_m)) > \epsilon$  for some  $\epsilon > 0$ . This, however, makes  $f(x_n)$  an infinite subset of  $f(X)$  which does not have a limit point, thereby contradicting the fact that  $f(X)$  is compact.

### 4.11 Exercise 11

The first part of this exercise is basically what I was doing for Exercises 8 and 10. Since  $f$  is uniformly continuous,  $\exists \delta \mid d_X(x_n, x_m) < \delta \implies d_Y(f(x_n), f(x_m)) < \epsilon$ . Since  $(x_n)$  Cauchy converges, we can make  $d_X(x_n, x_m)$  arbitrarily small, which then implies that we can make  $d_Y(f(x_n), f(x_m))$  arbitrarily small, indicating that  $f(x_n)$  Cauchy converges as well.

### 4.12 Exercise 12

To state it more precisely: if  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are both uniformly continuous, then  $g \circ f$  is also uniformly continuous.

From uniform continuity of  $g$ ,  $\exists \delta \mid d_Y(y_1, y_2) < \delta \implies d_Z(g(y_1), g(y_2)) < \epsilon$ . Since  $f$  is uniformly continuous,  $\exists \delta' \mid d_X(x_1, x_2) < \delta' \implies d_Y(f(x_1), f(x_2)) < \delta$ . The existence of this  $\delta'$  gives us that  $g \circ f$  is uniformly continuous.

### 4.13 Exercise 13

Suppose  $p$  is a limit point of  $E$  and  $x_n \in E \mid x_n \rightarrow p$ . Then  $f(x_n)$  Cauchy converges to a point  $q$  in the codomain of  $f$ . We can simply take  $g(p) = q$  whenever  $p \notin E$  to get a continuous extension of  $f$ . Since this proof depends only on the convergence of the Cauchy sequence  $f(x_n)$  to a point in the codomain, it will hold for the codomain being any complete metric space.

### 4.14 Exercise 16

The function  $[x]$  has a simple discontinuity at every integer  $x$ , since the left-hand limit is  $x - 1$  and the right-hand limit is  $x$ . Similarly, the function  $(x)$  also has a simple discontinuity at every integer, since the left-hand limit is 1 and the right-hand limit is 0.

### 4.15 Exercise 17

We proceed as hinted in the text. The two types of simple discontinuity we need to consider are  $f(x-) \neq f(x+)$  and  $f(x-) = f(x+) \neq f(x)$ . For the first case, suppose (WLOG) that  $f(x-) < f(x+)$ . Then we can construct a rational triple  $(p, q, r)$  such that

$$\begin{aligned} f(x-) &< p < f(x+) \\ a < q < t < x &\implies f(t) < p \\ x < t < r < b &\implies f(t) > p \end{aligned}$$

To see that such a triple can only be associated with one such  $x$ , consider  $x' = x + \epsilon$  with  $\epsilon > 0$  (the other case is identical). Then we can choose  $t \in (x, x')$  with  $q < x < t < r < x'$ , which means  $t > q$  does not imply  $f(t) < p$ . This handles simple discontinuities of the form  $f(x-) \neq f(x+)$ .



We can similarly handle the case where  $f(x-) = f(x+) \neq f(x)$ . Suppose (WLOG) that  $f(x) > f(x+)$  ; we can then construct a rational triple  $(p, q, r)$  such that

$$\begin{aligned} f(x+) &< p < f(x) \\ a < q < t < x &\implies f(t) < p \\ x < t < r < b &\implies f(t) < p \end{aligned}$$

Again, such a triple can only be associated with a single  $x$ , since  $x \in (x, x+\epsilon)$  and  $f(x) > p$ . Therefore  $f$  has only countably many simple discontinuities.

#### 4.16 Exercise 23

From the definition of convexity, we have that

$$\begin{aligned} f(\lambda x + (1 - \lambda)p) &\leq \lambda f(x) + (1 - \lambda)f(p) \\ f(\lambda x + (1 - \lambda)p) - f(p) &\leq \lambda(f(x) - f(p)) \\ f(p) - f(\lambda x + (1 - \lambda)p) &\leq \lambda(f(p) - f(x)) \\ \implies \lim_{\lambda \rightarrow 0} f(\lambda x + (1 - \lambda)p) &= f(p) \end{aligned}$$

Since  $\lim_{\lambda \rightarrow 0} \lambda x + (1 - \lambda)p = p$  for all choices of  $x$ , we have that  $f$  is continuous.

## 5 Differentiation

### 5.1 Exercise 1

We have that

$$\begin{aligned} |f(x) - f(y)| &\leq (x - y)^2 = |x - y|^2 \\ \frac{|f(x) - f(y)|}{|x - y|} &\leq |x - y| \\ \implies f'(x) &= 0 \forall x \end{aligned}$$

Since we can write  $\frac{f(t) - f(x)}{t - x} = f'(x) + u(t)$  with  $\lim_{t \rightarrow x} u(t) \rightarrow 0$ , we have that

$$\begin{aligned} f(t) - f(x) &= (t - x)u(t), \quad f(t) - f(y) = (t - y)v(t) \\ f(y) - f(x) &= (y - x)u(y), \quad f(x) - f(y) = (x - y)v(x) \implies u(y) = v(x) \forall x, y \\ f(y) - f(x) &= 0 \implies f(x) = f(y) \forall x, y \end{aligned}$$

Whoops, I did this before reading the mean value theorem section - this problem follows immediately from applying the mean value theorem after showing  $f'(x) = 0$ .

### 5.2 Exercise 2

Take  $x, t \in (a, b)$  with  $t > x$ . Applying the mean value theorem to  $f$  on  $[x, t]$ , we get  $f(t) - f(x) = (t - x)f'(y)$  for some  $y \in (x, t)$ . Since  $f'(y) > 0 \implies f(t) - f(x) > 0$ ,  $f$  is strictly increasing on  $(a, b)$ . We can prove  $g = f^{-1}$  is differentiable directly

$$\begin{aligned} \lim_{t \rightarrow x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} &= \lim_{t \rightarrow x} \frac{t - x}{f(t) - f(x)} \\ &= \frac{1}{f'(x)} \end{aligned}$$

### 5.3 Exercise 3

Suppose (WLOG) that  $x_2 > x_1$  but  $f(x_2) = f(x_1)$ . Then we have that

$$\begin{aligned} x_2 + \epsilon g(x_2) &= x_1 + \epsilon g(x_1) \\ (x_2 - x_1) + \epsilon(g(x_2) - g(x_1)) &= 0 \\ 1 + \epsilon g'(x) &= 0 \quad x \in (x_1, x_2) \\ 1 + \epsilon g'(x) &\geq 1 - \epsilon |g'(x)| \\ &> 0 \quad \forall \epsilon < \frac{1}{M} \end{aligned}$$

Where the penultimate step follows from the mean value theorem. Thus, we can choose an  $\epsilon$  such that  $f(x_2) \neq f(x_1)$ , which means we can make  $f$  injective.

## 5.4 Exercise 4

Let  $f(x) = C_0 + C_1x + \dots + C_nx^n$  and  $g(x) = C_0x + \frac{C_1}{2}x^2 + \dots + \frac{C_n}{n+1}x^{n+1}$ . Then  $g'(x) = f(x)$ . Applying the mean value theorem to  $g(x)$  on  $[0, 1]$  yields that there is an  $x$  such that  $g'(x) = f(x) = g(1) - g(0) = 0$ , so  $f$  has a root in  $(0, 1)$ .

## 5.5 Exercise 5

It looks like the mean value theorem is this chapter's ratio test; you can guess how this will go. By the mean value theorem,  $f'(y) = f(x+1) - f(x)$  for  $y \in [x, x+1]$ . Thus we have  $\lim_{x \rightarrow \infty} g(x) = \lim_{y \rightarrow \infty} f'(y) = 0$ .

## 5.6 Exercise 6

Consider  $x > y > 0$ . By the mean value theorem (surprise), we have that

$$\begin{aligned} f(x) - f(y) &= f'(a)(x - y) \quad a \in (x, y) \\ &< f'(x)(x - y) \\ \lim_{y \rightarrow 0} \frac{f(x) - f(y)}{x - y} &< \lim_{y \rightarrow 0} f'(x) \\ \frac{f(x)}{x} &< f'(x) \end{aligned}$$

Differentiating  $g$ , we get

$$\begin{aligned} g'(x) &= \frac{xf'(x) - f(x)}{x^2} \\ f'(x) &> \frac{f(x)}{x} \implies g'(x) > 0 \end{aligned}$$

So  $g$  is monotonically increasing.

## 5.7 Exercise 7

If  $f$  and  $g$  are real, then the result follows immediately from L'Hopital's, since the existence of  $f'(x), g'(x)$  imply that  $f$  and  $g$  are continuous (so the requirement that  $f(x) \rightarrow 0$  and  $g(x) \rightarrow 0$  is satisfied). More generally, for complex functions, we have

$$\begin{aligned} \lim_{t \rightarrow x} \frac{f(t)}{g(t)} &= \lim_{t \rightarrow x} \frac{f_1(t) + f_2(t)i}{g_1(t) + g_2(t)i} \\ &= \lim_{t \rightarrow x} \frac{\frac{f_1(t) + f_2(t)i}{t-x}}{\frac{g_1(t) + g_2(t)i}{t-x}} \\ &= \frac{f'(x)}{g'(x)} \end{aligned}$$

### 5.8 Exercise 8

Exercise 7 was a brief detour, but we are now back to hammering away with the mean value theorem. We have

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$
$$|f'(y) - f'(x)| < \epsilon$$

Where  $y \in (x, t)$  or  $y \in (t, x)$ . Since  $f'$  is continuous, we can choose  $\delta$  such that  $|y - x| < \delta$  implies the above inequality, which means we can take  $|t - x| < \delta$  to get the desired result.

This does not seem to have to hold for vector-valued functions, since in that case we only have  $|f(t) - f(x)| \leq (t - x)|f'(y)|$ .

### 5.9 Exercise 9

Yes, since we can apply L'Hopital's to  $(0, +\infty)$  and  $(-\infty, 0)$

$$f'(0) = \lim_{t \rightarrow 0} \frac{f(t) - f(0)}{t}$$
$$= \lim_{t \rightarrow 0} f'(t) = 3$$

### 5.10 Exercise 10

We proceed exactly as directed by Rudin

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \left( \frac{f(x)}{x} - A \right) \frac{x}{g(x)} + \frac{Ax}{g(x)}$$
$$= \frac{\lim_{x \rightarrow 0} \frac{f(x) - Ax}{x}}{\lim_{x \rightarrow 0} \frac{g(x)}{x}} + \frac{A}{\lim_{x \rightarrow 0} \frac{g(x)}{x}}$$
$$= 0 + \frac{A}{B}$$

Where the final step follows from breaking up  $f, g, A$  into their real and imaginary components and applying L'Hopital's ( $\operatorname{Re} f'(x) - \operatorname{Re} A$  and  $\operatorname{Im} f'(x) - \operatorname{Im} A$  both go to 0).

### 5.11 Exercise 11

Double application of L'Hopital's

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h}$$
$$= \lim_{h \rightarrow 0} \frac{f''(x+h) + f''(x-h)}{2}$$
$$= f''(x)$$

If we let  $f(x) = x \sin \frac{1}{|x|}$  when  $x \neq 0$  and  $f(0) = 0$ , then  $f''(0)$  does not exist. However, plugging  $f$  into the above limit and setting  $x = 0$ , we see that the limit is 0.

### 5.12 Exercise 12

$$\begin{aligned} f(x) = |x|^3 &= \begin{cases} -x^3 & x < 0 \\ 0 & x = 0 \\ x^3 & x > 0 \end{cases} \implies f'(x) = \begin{cases} -3x^2 & x < 0 \\ 0 & x = 0 \\ 3x^2 & x > 0 \end{cases} \\ \implies f''(x) &= \begin{cases} -6x & x < 0 \\ 0 & x = 0 \\ 6x & x > 0 \end{cases} \implies f^{(3)}(x) = \begin{cases} -6 & x < 0 \\ 0 & x = 0 \\ 6 & x > 0 \end{cases} \end{aligned}$$

So the left and right limits of  $f^{(3)}(0)$  do not equal one another.

### 5.13 Exercise 14

Suppose  $f$  is a convex differentiable function. Then

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y) \\ f(\lambda x + (1 - \lambda)y) - f(y) &\leq \lambda(f(x) - f(y)) \\ \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)} &\geq \frac{f(y) - f(x)}{y - x} \\ \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)} &\geq \frac{f(y) - f(x)}{y - x} \end{aligned}$$

Since  $x$  was arbitrary, the last inequality implies that  $f'$  is monotonically increasing (because we can consider a new interval whose left endpoint is  $\lambda x + (1 - \lambda)y$ ).

For the other direction, we can modify the convexity condition until we get something that we can apply the mean value theorem to. Letting  $z = \lambda x + (1 - \lambda)y$ , we have

$$\begin{aligned} f(z) &\leq \lambda f(x) + (1 - \lambda)f(y) \\ \lambda(f(z) - f(x)) &\leq (1 - \lambda)(f(y) - f(z)) \\ \frac{f(z) - f(x)}{z - x} &\leq \frac{(1 - \lambda)(f(y) - f(z))}{\lambda(z - x)} \\ &\leq \frac{f(y) - f(z)}{y - z} \end{aligned}$$

The final inequality is true by monotonicity of  $f'$  and applying the mean value theorem to  $(x, z)$  and  $(z, y)$ . The final step follows from the fact that  $\lambda(z - x) = (1 - \lambda)(y - z)$ .

### 5.14 Exercise 15

Proceeding as directed by the hint, we have

$$\begin{aligned}
 f(x+2h) &= f(x) + 2hf'(x) + 2h^2f''(c) \\
 f'(x) &= \frac{1}{2h}(f(x+2h) - f(x)) - hf''(c) \\
 M_1 &\leq \frac{1}{2h}(|f(x+2h)| + |f(x)|) + h|f''(c)| \\
 &\leq \frac{M_0}{h} + hM_2 \\
 M_1^2 &\leq \frac{M_0^2}{h^2} + 2M_0M_2 + h^2M_2^2 \\
 &\leq 4M_0M_2 \quad h = \sqrt{\frac{M_0}{M_2}}
 \end{aligned}$$

### 5.15 Exercise 18

Differentiating  $f(t) - f(\beta) = (t - \beta)Q(t)$  and evaluating at  $\alpha$ , we have

$$\begin{aligned}
 f^{(k)}(\alpha) &= (\alpha - \beta)Q^{(k)}(\alpha) - kQ^{(k-1)}(\alpha) \\
 \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k &= -\frac{Q^{(k)}(\alpha)}{k!}(\beta - \alpha)^{k+1} + \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta - \alpha)^k \\
 \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k &= -\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n + (f(\beta) - f(\alpha)) \\
 P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n &= f(\beta)
 \end{aligned}$$

## 6 The Riemann-Stieltjes Integral

### 6.1 Exercise 1

We first note that  $\inf f(x) = 0$  on any interval  $[p_{i-1}, p_i] \subset [a, b]$ . Similarly,  $\sup f(x) = 0$  if  $x_0 \notin [p_{i-1}, p_i]$  and 1 otherwise. Thus, we proceed by constructing a partition  $P$  of  $[a, b]$  such that  $x_0 \in [p_{i-1}, p_i]$  and  $\alpha(p_i) - \alpha(p_{i-1}) < \epsilon$ . This is possible since  $\alpha$  is continuous at  $x_0$ , so there exists  $\delta$  such that choosing  $p_i - p_{i-1} < \delta$  gives us the previous inequality. We then have that  $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ , so  $f \in \mathcal{R}(\alpha)$ . Furthermore, since  $L(P, f, \alpha) = 0$  for all  $P$ , we have that  $\int_a^b f d\alpha = 0$ .

### 6.2 Exercise 2

Since we're given  $f(x) \geq 0$  and  $\int_a^b f(x) dx = 0$ , we have that  $m = \inf f(x) = 0$ . Letting  $m_i$  and  $M_i$  denote the infimum and supremum of  $f$  on the interval  $[x_{i-1}, x_i] \subset [a, b]$ , we further have that  $m_i \leq m \implies m_i = 0 \forall i$ . Additionally, since  $f$  is continuous on  $[a, b]$  and therefore uniformly continuous (from compactness), we can choose a  $\delta$  such that

$$\begin{aligned} |x_i - x_{i-1}| < \delta &\implies |f(x_i) - f(x_{i-1})| < \epsilon \\ &\implies |M_i - m_i| < \epsilon \implies M_i < \epsilon \end{aligned}$$

Thus, all of the  $M_i$  can be made arbitrarily small, implying that  $\sup f(x) < \epsilon$  for every positive  $\epsilon$ . This gives us that  $f(x) = 0$ .

### 6.3 Exercise 3

(a) Suppose  $f(0+) = f(0)$ . We consider the partition  $P = x_0, x_1, x_2, x_3$  where  $x_0 = -1, x_1 = 0, x_3 = 1$ . Then  $U(P, f, \beta_1) = M_2$  and  $L(P, f, \beta_1) = m_2$ , and we have that  $M_2 \rightarrow f(0)$  and  $m_2 \rightarrow f(0)$  as  $x_2 \rightarrow 0$ , so  $f \in \mathcal{R}(\beta_1)$ . For the other direction, if  $f \in \mathcal{R}(\beta_1)$ , then there exists  $P$  such that  $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$ . We can then consider the refinement  $P^*$  of  $P$  that contains an interval of the form  $[0, x_i]$ . Suppose now that  $|f(0+) - f(0)| = \epsilon > 0$ . Then

$$M_i - m_i \geq |f(0+) - f(0)| > \epsilon$$

which is a contradiction, so  $f(0+) = f(0)$ .

(b) Only difference from (a) is that we need  $f(0-) = f(0)$  instead of  $f(0+) = f(0)$ . The only changes that need to be made to the proof of (a) involve replacing  $[0, x_i]$  with  $[x_{i-1}, 0]$ .

(c) If  $f$  is continuous at 0, we can consider the partition  $P$  that contains  $x_{i-1} < x_i = 0 < x_{i+1}$ . Then  $U(P, f, \beta_2) = \frac{M_i + M_{i+1}}{2}$  and  $L(P, f, \beta_2) = \frac{m_i + m_{i+1}}{2}$ . Since  $f$  is continuous at 0,  $M_i, M_{i+1}, m_i, m_{i+1} \rightarrow f(0)$  as  $x_{i-1} \rightarrow 0$  and  $x_{i+1} \rightarrow 0$ , so  $f \in \mathcal{R}(\beta_2)$ . For the other direction, we proceed similarly to the proof of (a) and

see that

$$\begin{aligned}\frac{1}{2}(M_i + M_{i+1}) - \frac{1}{2}(m_i + m_{i+1}) &= \frac{1}{2}((M_i - m_i) + (M_{i+1} - m_{i+1})) \\ &\geq \frac{1}{2}(|f(0-) - f(0)| + |f(0+) - f(0)|) \\ &> \epsilon\end{aligned}$$

for some  $\epsilon > 0$  unless  $f(0-) = f(0+) = f(0)$ , so  $f$  must be continuous at 0 if  $f \in \mathcal{R}(\beta_2)$ .

(d) If  $f$  is continuous at 0 then we have  $f(0+) = f(0-) = f(0)$ , so we are done by parts (a)-(c).

#### 6.4 Exercise 4

From the density of the rationals and irrationals in the reals, we have that  $M_i - m_i = 1 \forall i$ , so  $f \notin \mathcal{R}$ .

#### 6.5 Exercise 5

If we consider  $f(x) = 1$  for all rational  $x$  and  $f(x) = -1$  for all irrational  $x$ , we have that  $f^2 \in \mathcal{R}$  but  $f \notin \mathcal{R}$ . However, if  $f^3 \in \mathcal{R}$ , then  $f \in \mathcal{R}$ . This is because  $m \leq f^3 \leq M$  (since  $f$  is bounded) and  $x^{\frac{1}{3}}$  is continuous on any  $[m, M]$ , so we can apply Theorem 6.11.