Exercise Guide for *Analysis on Manifolds* by Michael Spivak

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About

> *"Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising from its own ashes, is its own derivative?"*
>
> \- Francois de Lionnais

This book is simply too famous and too short to not work through.

1 Functions on Euclidean Space

1.1 Norm and Inner Product

1.1.1 Exercise 1

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} |x_i|^2 \leq \left(\sum_{i=1}^{n} |x_i| \right)^2 \implies \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} |x_i|^2$$

1.1.2 Exercise 2

We need $\sum_{i=1}^{n} x_i y_i = |x||y|$. Linear dependence by itself is not enough, since if x and y are opposite sign we see that the lefthand sum will be negative. Thus, we need linear dependence as well as x and y having the same sign.

1.1.3 Exercise 3

The proof is identical to the $|x + y|$ case, except now we have a $-2\sum_{i=1}^{n} x_i y_i$ term. Thus, equality holds when x and y are linearly dependent and have opposite signs.

1.1.4 Exercise 4

$||x| - |y||^2 = |x|^2 + |y|^2 - 2|x||y|$. Since $|x||y| \geq \langle x, y \rangle$, we have the desired inequality.

1.1.5 Exercise 5

$$|z - x| = |z - y + y - x| \leq |z - y| + |y - x|$$

Geometrically, this is just the fact that any sidelength of a triangle must be bounded by the sum of the other two sidelengths.

1.1.6 Exercise 6

(a) We can proceed as Spivak hints by noting that for the $\int_a^b (f - \lambda g)^2 > 0$ case, the proof is identical to that of Theorem 1-1 (2). For the $\int_a^b (f - \lambda g)^2 = 0$ case (which is when equality is obtained), we can use the fact that $(f - \lambda g)^2 = 0$ almost everywhere.

However, I think it's a little smoother to handle both cases at once:

$$\begin{aligned} \int_a^b (f-\lambda g)^2 = \int_a^b f^2 - 2\lambda \int_a^b fg + \lambda^2 \int_a^b g^2 \geq 0 \\ \int_a^b f^2 - \frac{2\left(\int_a^b fg\right)^2}{\int_a^b g^2} + \frac{\left(\int_a^b fg\right)^2}{\int_a^b g^2} \geq 0 \quad \text{for } \lambda = \frac{\int_a^b fg}{\int_a^b g^2} \\ \sqrt{\int_a^b f^2 \int_a^b g^2} \geq \left| \int_a^b fg \right| \end{aligned}$$

(b) Equality does not necessarily imply that $f = \lambda g$, since we could consider f and g to be 0 everywhere except two points a and b such that $f(a) \neq \lambda g(a)$ and $f(b) \neq \lambda g(b)$. However, if f and g are continuous, then $\int_a^b (f-\lambda g)^2 = 0$ implies that $f = \lambda g$.

(c) Define $f(m) = x_i$ and $g(m) = y_i$ for $m \in [i, i+1)$. Then $\int_1^n fg = \sum_1^n x_i y_i$ (we can break up the integral at the points of discontinuity) and Theorem 1-1 (2) follows.

1.1.7 Exercise 7

(a) If T is inner product preserving then we have $\langle Tx, Tx \rangle = \langle x, x \rangle$, so T is norm preserving. If T is norm preserving, then

$$\begin{aligned} \langle T(x-y), T(x-y) \rangle &= |T(x)|^2 + |T(y)|^2 - 2\langle T(x), T(y) \rangle \\ \langle x-y, x-y \rangle &= |x|^2 + |y|^2 - 2\langle x, y \rangle \\ \implies \langle T(x), T(y) \rangle &= \langle x, y \rangle \end{aligned}$$

so T is inner product preserving.

(b) Since $Tx = Ty \implies T(x-y) = 0 \implies |x-y| = 0$, T is injective. Furthermore, $Tx = 0 \implies |x| = 0$, so the nullspace of T is trivial and T is thus surjective. Now if we consider $T^{-1}y = x$, then we have $\langle y, y \rangle = \langle TT^{-1}y, TT^{-1}y \rangle = \langle T^{-1}y, T^{-1}y \rangle$.

1.1.8 Exercise 10

Let $\|T\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m T_{ij}^2}$ (Frobenius norm). Then we have

$$|Th|^2 = \sum_{i=1}^n \left(\sum_{j=1}^m T_{ij} h_j \right)^2 \leq \sum_{i=1}^n \sum_{j=1}^m T_ij^2 \sum_{j=1}^m h_j^2 = \|T\|_F^2 |h|^2$$

so letting $M = \|T\|_F$ gives the desired inequality.

1.1.9 Exercise 12

Linearity and injectivity of T follow from linearity of inner product. To see surjectivity, we note that any element $f \in (\mathbb{R}^n)^*$ is determined entirely by $f(e_1), ..., f(e_n)$ due to linearity. Thus, $f(y) = \langle x, y \rangle$ for the unique x satisfying $x_i = f(e_i)$.

1.1.10 Exercise 13

Expanding $\langle x + y, x + y \rangle$ gives the desired result.