“Sparknotes” for *Principles of Mathematical Analysis* by Walter Rudin

Muthu Chidambaram

Last Updated: May 28, 2019

Contents

About

> *"A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details."*
> \- Hermann Weyl

These notes contain short summaries of (my) proof ideas for exercises and some theorems from the book *Principles of Mathematical Analysis* by Walter Rudin. I have tried to make the summaries as brief as possible, sometimes only one line or one equation. My hope is that the summaries will give enough information to reconstruct a full proof without bogging the reader down with details. In many cases, I am sure that I inadvertently sacrificed clarity in an attempt to obtain brevity, and would greatly appreciate any feedback.

Also, I like when people include (what they presume to be) relevant quotes in their notes, so I have to ask you to forgive my haughtiness in starting these notes with a quote from Hermann Weyl.

1 The Real and Complex Number Systems

1.1 Exercise 1

If $rx = q$ or $r + x = q$ for some rational q, then substracting r from q or dividing q by r yields x rational, which is a contradiction.

1.2 Exercise 2

We can first show that $\sqrt{3}$ is irrational by seeing that $\frac{a^2}{b^2} = 3 \implies 3|a, 3|b$. Then, since $12 = 3 * 2^2$, we have that $\sqrt{12}$ is irrational as well.

1.3 Exercise 4

If $\alpha > \beta$ then α would be an upper bound as well.

1.4 Exercise 5

$\forall x \in A$, $-x \leq \sup -A$ and $\forall \epsilon \in \mathbb{R}$, $\exists x \in A \mid \sup -A + \epsilon < -x \leq \sup -A$. Negating the last inequality gives $\inf A = -\sup -A$.

1.5 Exercise 6

(a) Follows from $m = \frac{np}{q}$.

(b) Put $r = \frac{m}{n}$, $s = \frac{p}{q}$. Then $b^r b^s = b^{\frac{mq}{nq}} b^{\frac{np}{nq}}$. Pulling out $\frac{1}{nq}$ gives the desired result.

(c) b^r is an upper bound since $b > 1$, and if it were not the supremum we could choose $t < r$ such that $b^t > b^r$. This is not possible since again, $b > 1$.

(d) Every element in $B(x + y)$ can be expressed as $b^{s+t} = b^s b^t$ $s \leq x$, $t \leq y$. If $\sup B(x + y) = \alpha < \sup B(x) \sup B(y)$, then $b^s b^t \leq \alpha \implies B(x) \leq \alpha b^{-t} \implies B(y) \leq \frac{\alpha}{B(x)} \implies B(x)B(y) \leq \alpha$.

1.6 Exercise 7

(a) $b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + ... + 1) \geq n(b - 1)$ since $b > 1$.

(b) Plug $b^{\frac{1}{n}}$ into (a).

(c) Plug $n > \frac{b-1}{t-1}$ into (b).

(d) Using (c) gives that we can choose n such that $b^{\frac{1}{n}} < y\dot{b}^{-w} \implies b^{w+\frac{1}{n}} < y$.

(e) We can take the reciprocal of (c) and do the same as in (d).

(f) If $b^x > y$ we can apply (e) for a contradiction, if $b^x < y$ we can apply (d) for a contradiction.

(g) Supremum is unique.

1.7 Exercise 8

Suppose $(0,1) < (0,0)$. Then $(0,-1) < (0,0)$ after multiplying by $(0,1)$ twice yields a contradiction. Similarly, assuming the opposite yields $(-1,0) > (0,0)$.

1.8 Exercise 9

Does exhibit least upper-bound property since you can take $(\sup a_i, \sup b_i)$.

1.9 Exercise 10

Exception is 0.

1.10 Exercise 11

Take $w = \frac{1}{|z|}z$ and $r = |z|$ when $|z| \neq 0$. w and r are not uniquely determined; take $z = 0$ for example.

1.11 Exercise 12

By strong induction:

$$\begin{aligned} |z_1 + ... + z_{n+1}| &\leq |z_1 + ... + z_n| + |z_{n+1}| \\ &\leq |z_1| + ... + |z_{n+1}| \end{aligned}$$

1.12 Exercise 13

$$\begin{aligned} |x-y|^2 &= x\bar{x} - 2|x||y| + y\bar{y} \\ &\geq (|x| - |y|)^2 \end{aligned}$$

2 Basic Topology

2.1 Exercise 1

The empty set has no elements, so all of its elements are vacuously also elements of every set.

2.2 Exercise 2

The roots of complex polynomials with integer coefficients can be expressed as elements of the countable cross product of $\mathbb{N}$ with itself (cross $\mathbb{N}$ with itself n times for the coefficients, and then once more to indicate which root).

2.3 Exercise 3

If all real numbers were algebraic, then the set of algebraic numbers would be uncountable (thus contradicting Exercise 2).

2.4 Exercise 4

The set of irrational numbers is $\mathbb{R}/\mathbb{Q}$, which must be uncountable as otherwise $\mathbb{R}$ would be countable.

2.5 Exercise 5

We can use $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}} \cup \left(\frac{2n}{n+1}\right)_{n\in\mathbb{N}} \cup \left(\frac{3n}{n+1}\right)_{n\in\mathbb{N}}$ to get the three limit points $1, 2, 3$.

2.6 Exercise 6

If p is a limit point of E', then every neighborhood of p contains a limit point q of E, and every neighborhood of q contains a point of E thereby implying that p is a limit point of E. E and E' do not need to have the same limit points, since E' could be finite and thus have no limit points.

2.7 Exercise 7

(a) If p is a limit point of $\overline{B_n}$, then every neighborhood of p contains a point $q \in A_i$. Since there are only finitely many A_i, p must be a limit point for at least one of the A_i, as an infinite number of neighborhoods of p must have non-zero intersection with some of the A_i.

(b) If we take $A_i = \left(\frac{in}{(i+1)n+1}\right)_{n\in\mathbb{N}}$, then 1 is a limit point of B_n despite not being a limit point of any of the A_i.

2.8 Exercise 8

Every point of an open set in $\mathbb{R}^2$ is by definition a limit point of the set, since the point must have a neighborhood contained in the set. The same is not true for closed sets, since we can just take a finite set.

2.9 Exercise 10

Every set in X is open, since any set containing p also contains $N_r(p)$ for $r < 1$. No set in X is closed, since $N_r(p) = p$ for $r < 1$. All infinite sets in X are not compact, since we can take balls of radius $r < 1$ around each point as an open cover.

2.10 Exercise 12

Take any open cover of K. There must be some open set in this cover containing 0, which means that the same set contains all but a finite number of the elements of K (since 0 is the only limit point of K). Take a union of this set as well as the finitely many other sets containing the aforementioned points to get a finite subcover.

2.11 Exercise 13

Take $\cup_{k=1}^{\infty}\{0, \left(\frac{n}{kn+1}\right)_{n\in\mathbb{N}}, \frac{1}{k}\}$. This set is closed and bounded, so it is compact by Heine-Borel. Its limit points are 0 and $\left(\frac{1}{k}\right)_{n\in\mathbb{N}}$.

2.12 Exercise 14

We can use $\cup_{n\in\mathbb{N}}(0, \frac{n}{n+1})$, which has no finite subcover (since we could choose $x \in (0,1)$ larger than the largest endpoint in the finite subcover).

2.13 Exercise 15

For closed, we can take $K_i = \mathbb{N}/0, ..., i-1$, since any $x \in K_i$ will not be in K_j if $j > x$. For bounded, we can take $K_i = (0, \frac{1}{i})$.

2.14 Exercise 16

E is by definition bounded, and E is closed since $q^2 \neq 3$ (q is rational), and $q^2 > 3 \implies \exists\epsilon \mid p \in N_\epsilon(q) \implies p^2 > 3$. Same logic gives that E is also open in $\mathbb{Q}$. E is, however, not compact, since we can construct an open cover consisting of $G_n = \{x \mid 2 < x^2 < 2 + \frac{n}{n+1}\}$.

2.15 Exercise 17

E is not countable by diagonalization. E is not dense in $[0,1]$, since $E \cap [0, 0.1] = \emptyset$. E is not perfect, consider $N_{0.001}(0.77)$. E is closed and therefore compact by

Heine-Borel. To see closed, suppose a limit point q had a non-4/7 digit in the i^{th} decimal spot. Then we could take a neighborhood of size $10^{-(i+1)}$.

2.16 Exercise 18

Rationals are dense in $\mathbb{R}$, so no.

3 Numerical Sequences and Series

Definition 3.5

Since $\{p_n\} \to p \implies \forall \epsilon,\ \exists N | n \geq N \implies |p_n - p| < \epsilon$, we can choose $k | n_k \geq N \implies \{p_{n_k}\} \to p$. The reverse direction can be shown via contradiction of $\{p_n\} \to p$.

Examples 3.18

(a) Density of rationals in reals.

(b) $|s_n| < 1$, take n odd to get -1 and even to get 1.

(c) Every subsequential limit has to converge to s.

Theorem 3.19

For all $\{n_k\}$, we have $\exists K | k \geq K \implies n_k \geq N \implies \lim_{k \to \infty} t_{n_k} - s_{n_k} \geq 0$.

Theorem 3.26

$s_n = 1 + x + ... + x^n \implies x s_n = x + x^2 + ... + x^{n+1} \implies (1-x) s_n = 1 - x^{n+1}$.

Examples 3.40

(a) Root test: $n \to \infty$.

(b) Ratio test: $\frac{1}{n+1} \to 0$.

(c) $1 \to 1$.

(d) Ratio test: $\frac{n}{n+1} \to 1$. $z = 1$ leads to harmonic series.

(e) Ratio test: $\frac{n^2}{(n+1)^2} \to 1$.

Example 3.53

$\sum_{k=1}^{\infty} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} < \frac{5}{6} + \sum_{k=2}^{\infty} \frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k}$. The RHS converges since $\frac{1}{4k-4} + \frac{1}{4k-4} - \frac{1}{2k} = \frac{1}{2k^2-2k}$.

3.1 Exercise 1

All we need is the inequality $|s_n - s| \geq ||s_n| - |s||$. The converse is not true, since we can take $s_n = (-1)^n$.

3.2 Exercise 2

My original idea: $\sqrt{(n+x)^2} - n = x$. Setting $(n+x)^2 \geq n^2 + n$ gives $x^2 \geq (1-2x)n$. The last inequality is only true for all n when $x \geq \frac{1}{2}$. This implies that $\frac{1}{2}$ is the supremum of $\sqrt{n^2+n} - n$. Since $\sqrt{n^2+n} - n$ is increasing, it converges to $\frac{1}{2}$.

Better: $(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n) = n \implies \sqrt{n^2+n} - n = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$.

3.3 Exercise 3

Clearly $s_{n+1} > s_n$. We can see that $s_n < 2$ by induction, since $s_1 < 2$ and $2 + \sqrt{s_n} < 4$. This gives that s_n is monotone and bounded, implying it converges.

3.4 Exercise 4

$$s_{2m+1} = \sum_{i=1}^{m} \frac{1}{2^i},\ s_{2m} = \sum_{i=2}^{m} \frac{1}{2^i}$$
$$\implies \limsup_{n\to\infty} s_n = 1,\ \liminf_{n\to\infty} s_n = \frac{1}{2}$$

3.5 Exercise 5

$$\begin{aligned}\limsup_{n\to\infty}(a_n + b_n) &= \sup_{\{k\}}\{\lim_{k\to\infty}(a_{n_k} + b_{n_k})\} \\ &= \sup_{\{k\}}\{\lim_{k\to\infty} a_{n_k} + \lim_{k\to\infty} b_{n_k}\}\end{aligned}$$

3.6 Exercise 6

(a) $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges from comparison to harmonic series (same technique as Exercise 2).

(b) Converges, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p = \frac{3}{2}$.

(c) Converges by root test, since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(d) Converges when $|z| > 1$ and diverges otherwise. To see this, put $z = |z|e^{i\theta}$ to get $\lim_{n\to\infty} \frac{1}{1+|z|^n e^{ni\theta}}$.

3.7 Exercise 7

We proceed via the ratio test.

$$\begin{aligned}
\limsup_{n\to\infty} \frac{n}{n+1} * \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} &= \limsup_{n\to\infty} \frac{n}{n+1} \limsup_{n\to\infty} \frac{\sqrt{a_{n+1}}}{\sqrt{a_n}} \\
&= \sqrt{\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}} \\
&< 1
\end{aligned}$$

Since $\sum a_n$ converges.

3.8 Exercise 8

Since b_n is monotonic and bounded, $|b_n| \leq B$ for all n. Then we have that $\sum a_n b_n$ converges by the comparison test, since $|a_n b_n| \leq B|a_n|$ and $B \sum a_n$ converges.

3.9 Exercise 9

(a) Applying the ratio test, we see that $|z| \limsup_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$ when $|z| < 1$. Thus $\sum n^3 z^n$ has radius of convergence 1.

(b) Again, applying the ratio test, we see that $2|z| \limsup_{n\to\infty} \left| \frac{1}{n+1} \right| = 0$, implying $R = +\infty$.

(c) The ratio test is the only hammer we need: $2|z| \limsup_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right| < 1$ gives $R = \frac{1}{2}$.

(d) What are the other tests again? $\frac{|z|}{3} \limsup_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \right| < 1$ gives $R = 3$.

3.10 Exercise 10

The infinitely many non-zero a_n must satisfy $|a_n| \geq 1$. The radius of convergence of $\sum a_n z^n$ will be maximized when $|a_n|$ is minimized, so we can just consider the case where there are infinitely many $|a_n| = 1$. In this case, we can choose a subsequence a_{n_k} consisting only of 1. Applying the ratio test using this subsequence gives $|z| < 1$.

3.11 Exercise 15

Theorems 3.22, 3.23, and 3.25(a) require no changes in their proofs, since the Cauchy criterion is applicable for $\mathbb{R}^k$. Theorem 3.33(a, b) also require no changes once we have the comparison test for $\mathbb{R}^k$. For Theorem 3.33(c), we can take $a \in \mathbb{R}^k$ such that all of its components are $\frac{1}{n}$ or $\frac{1}{n^2}$.

Theorem 3.34(a, b) just need to be modified to use $\frac{|a_{n+1}|}{|a_n|}$. Theorem 3.42 needs to be modified to use the dot product, but then it follows from applying the $\mathbb{R}$ version of 3.42 to the components of the dot product sum. Theorems 3.45, 3.47, and 3.55 require no changes to their proofs.