CONCRETE BRIDGE PRACTICE

Analysis, Design and Economics

SECOND EDITION

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Tata McGraw-Hill Publishing Company Limited

New Delhi

McGraw-Hill Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Guatemala Hamburg Lisbon London Madrid Mexico Milan Montreal Panama Paris San Juan São Paulo Singapore Sydney Tokyo Toronto

ISBN 0-07-462362-1

Published by Tata McGraw-Hill Publishing Company Limited,
4/12 Asaf Ali Road, New Delhi 110 002, lasertypeset at Laser Words,
13, Prithvi Avenue, Madras 600 018 and printed at Rajkamal Electric Press,
B-35/9 G T Kamal Road, Delhi 110 033

RCARRRZJDQDYX

. . . for the development of the so-called
THIRD WORLD COUNTRIES
many of which, in the past were
physically conquered,
politically subjugated,
culturally oppressed,
and
economically exploited.

I dedicate this work to

THE INTERNATIONAL BANK FOR RECONSTRUCTION AND DEVELOPMENT

and

THE UNITED NATIONS ORGANISATION

for their pious and commendable efforts

. . . whether 'they' be the 'wealthy' poor, the 'well-to-do' poor, or the 'rock-bottom' poor, their continued dependence on alien ownership of technology must be checked so that these peoples are reasonably protected against commercial exploitation. There is no alternative for these countries but to search from within to strengthen their own infrastructures! We, as today's well-wishers, have an urgent job to do. We must take a leap-of-faith and remember that the only thing dark about these countries is our own ignorance about them! Technologies must be bent to suit the indigenously available manpower, materials and equipment as far as viable, and simultaneously the infrastructure must be strengthened by massive manpower training programmes . . .

Foreword

The twentieth century heralded a new era in bridge building concepts with large improvements in materials and methods. Structural steel and reinforced concrete began to be used extensively. Rapid developments in the theory of structures along with the advent of the computer made it possible to pioneer innovative designs. Sophisticated mathematical and model analysis was increasingly used to predict the behaviour of structures. With the massive knowledge explosion and the eagerness of creative men to develop large and more daring spans, there have been many pioneering achievements in the USA and Europe, which are indeed marvels of engineering.

The design of bridge structures has become intricate with the changeover from the conventional girder slab bridges to the complex interchanges requiring curved units or cable stays or suspended units. The analysis of such structures, having different forms and shapes, requires ingenuity of a high order as research may lag behind practical possibilities. How then can we, builders of bridges, calculate and design those daring structures to safely support the loads of railway trains or heavy vehicles and to withstand the often unpredictable forces of wind and water.

The emphasis on theory and too little consideration for structural detailing and on-site realities have resulted in bridge collapses in the not-too-distant past. To desist from using new methods and materials seems to be a wise way of reducing the risk of error and consequent failures. Thanks to the dynamism of the professionals, bridge builders continue to build spans larger or attempt forms of construction more elegant and more economical than ever before. In the words of Paul Bonatz and Fritz Leonard, "Every new difficulty challenges the human spirit to think of new solutions which in turn push forward the threshold of what can be done."

Undoubtedly, the public and even some engineers believe that the ultimate in bridge design and construction has been reached for the present. Of course, this remains to be seen; history would indicate otherwise.

The engineering work on large bridge projects of today is so complex that many engineers are involved in the design and construction activities. The credit cannot go primarily to one person as it did in the past. Large firms of specialists in many areas are involved. Nevertheless, one man has to make the final decision. These decisions, of which there may be several, call for a wealth of technical knowledge and sound judgement based on many years of experience.

This book by Dr V K Raina—*Concrete Bridge Practice: Analysis, Design and Economics*, the fourth book in his series of six books treating various facets of bridge engineering—provides a comprehensive coverage on the subject for both the designer and the constructor. The book is like a programmed text giving discrete steps to decision making. It is based on the author's experience both in India and abroad. His present assignment is as United Nations Expert in Saudi Arabia. He says, "It is no exaggeration to state that more bridges of many varieties, including some flyovers exceeding 10 km in length each, have been designed and built in Saudi Arabia in the late seventies and eighties than anywhere else in the world. If there ever existed in the world a feast of designing and building prestressed and reinforced concrete bridges, it was in Saudi Arabia during this period and it was not for nothing that so many internationally operating consultants and contractors converged in."

It is this setting that inspired Dr Raina to write this fantastic book providing an integrated coverage of the structural analysis and design of both conventional and modern bridges. It was his passion for the bridge art, which led him to take up this study. Dr Raina has written in the language of the engineer. The reader, on going through this book, will acquire a far-reaching insight into design procedures and methods and the interpretation and use of design specifications. ***It furnishes the practising engineer with much needed data to meet the challenges in his work life. It is an excellent source of reference.***

Dr T N Subba Rao
Managing Director
Gammon India Limited, Bombay

A Word to the Reader

Talking alone never pulled out a stump! Many try to throw about the weight of their purely academic degrees, non-productive publications, classroom or staid-office experiences, and even the thunder of their committee-memberships. In the end, only those that have finally actually been moulded on the professional anvil, are of real value—those that have had prolonged but successful exposures to furiously result-oriented and profit-bearing competitive practical commercial experiences where the next month's survival depends on the previous month's turnover.

My aim in writing this book is two-fold. One is to benefit those who may wish to receive exposure to actual professional practice from the 'scene of action' standpoint as distinct from a 'theoretical' classroom hyperbole that belongs to an almost imaginary 'air-conditioned' world that is far and remote from the sleeves up workman-like life-size actuality! Second is to try and 'talk' to the engineer in short straight steps, explaining the subtleties en route, in the vein of a story narrated informally, without 'mystifying' him with exotica. Descriptions have been written with clarity and brevity so that the engineer is neither overawed nor bored with jargon that is either too theoretical or oozing with impressive looking useless detail. This book takes the reader by the finger through the labyrinths of the subject in a workman-like manner, and thus caters for the contractor, the client and the practice-oriented engineer student alike!

Of the numerous works that have been written upon the subject of bridge analysis and design, many are excellent examples of mathematical gymnastics rather than of engineering application!

In this book the steps of the reader are guided in paths often trodden by and therefore familiar to the author, who, thereby, is able to recommend a straight course without the designer having to waste time in search for a route. If I have succeeded in some measure, it is not only by being encyclopaedic, but because the presentation is fresh in treatment, and, above all, easy to study and follow. It concisely provides what the designer/engineer wants, without making demands on his energy. However, the subject being what it is, and the work involved being awesome (as suggested by the title of the book), I have had to presume that the reader already has some experience in the analysis, design and detailing of concrete bridges, with a reasonable exposure to competitive professional practice.

Engineering is not just doing theoretical sums, nor is it a matter of blind adherence to graphs and formulae. One can run the danger of becoming too concerned with 'learning' and not be concerned enough with 'practical realities.' ***It is more meaningful to have an approximate solution to an exact problem than an exact solution to an approximated problem. A useful book does not have to be the graveyard of dead Ph Ds!*** As a prolific practitioner who has operated in so many countries and has worked with myriads of contractors and consultants, I am disturbed if a book purports to be 'practical' when it is packed with pages of iterative empirics and impressive looking graphs that are only of very restricted use and, worse still, is written by someone who has never stayed, survived and surfaced in the merciless world of competitive practice in construction and design. That is where all the fun is and where one grapples with the survival situations that can cause ulcers! ***A good musician is far superior to a music-critic!*** Practical engineers must be conceptual more than perceptual, creative more than analytical and more visual than merely mathematical. They have to have a wide breadth of experience rather than an isolated narrow specialisation alone. Originality comes out of understanding, and understanding comes out of relentless practice, not from mere information. ***Last but not the least, good judgement comes out of experience and experience often comes out of bad judgement!***

Dr V K Raina

Acknowledgements

As acknowledged in my other books, one of the prices a professional practitioner has to pay is that he, ***unlike those involved in research and laboratory work, classroom lectures, or staid-office work,*** hardly has time to write. A chronic practitioner would rather spend the time in designing (and still more designing) and constructing (and still more constructing) than just writing! But of course it would be very useful ***if such a real-life practising-professional, who has his fingers on the pulse of practice and in fact has a lot to*** write about, could squeeze time in order to 'also write' for the profession, however hard it might be for him to find that time! ***It would be even more meaningful if he, additionally, had a practical research background that would help him sift grain from husk.***

Fired by this feeling, I took up writing the present book in the humble hope that it may provide an amalgam of practice and theory, with the former subordinating the latter in order that the book be of gainful use to the practising engineer.

With the mind boggling and unparalleled spree of fast growth of world-class super-expressway and highway networks in Saudi Arabia, I, seconded by the United Nations (Department of Technical Cooperation for Development) as the in-house adviser to the Ministry of Communications of the Saudi Government, was involved first-hand with optimised design and construction of several bridges in many interchanges, crossings and flyovers, of various types, spans, skews and curves, with many consultants and contractors. (It is no exaggeration to state here that more bridges of many varieties, including some flyovers exceeding 10 km in length each, have been designed and built in Saudi Arabia in the late seventies and the eighties than anywhere else in the world! If there ever existed in the world a feast of designing and building prestressed and reinforced concrete bridges, it was in Saudi Arabia during this period and it was not for nothing that so many internationally operating consultants and contractors converged in.) I am indeed grateful both to the United Nations and to the Saudi Government for this challenging responsibility and the additional satisfying and revealing practical experience this opportunity afforded. In turn, I gratefully acknowledge the trust, confidence and appreciation shown by them. If I have acquitted myself creditably in their eyes, the credit is all theirs. It is this additional important experience that further prompted me to write this book.

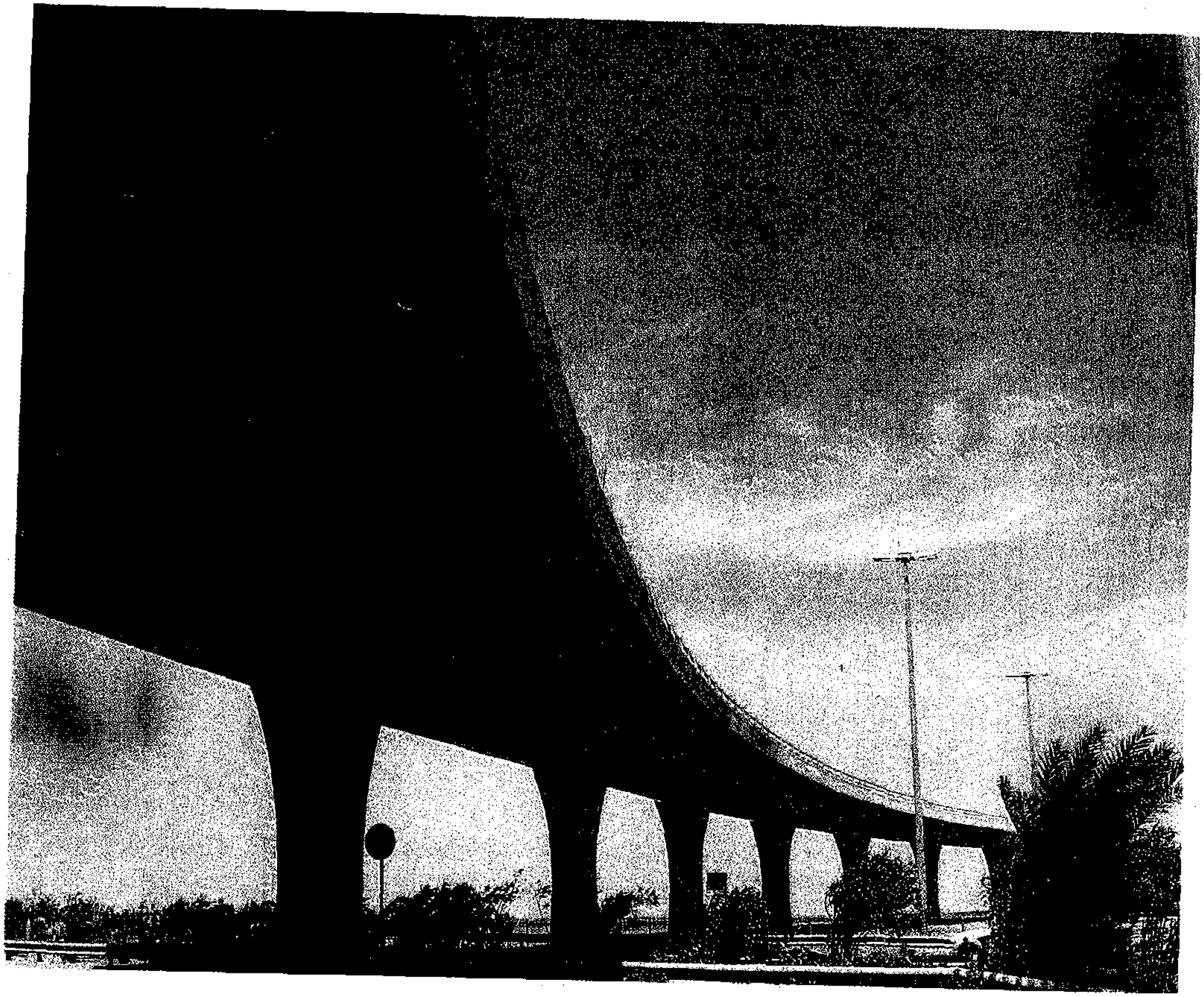
Apart from drawing upon my own experience and interaction with others, in preparing this book, I have also drawn on some of the material published by the British Standards, the Indian Standards, the American Concrete Institute, the American Association of State Highway and Transportation Officials, the C&CA London, the FIP, CEB, the Indian National Group of the IABSE (Zurich), the Indian Roads Congress, New Delhi, the *British Steel Designer's Manual*, CS Reynolds, AH Allen, AA Witecki, B Richardson et al., GH Ryder, Podolny & Muller, Freyssinet International, Paris, PK Thomas, GN Smith and EN Pole, E Pennels, MJ Tomlinson, PW Ables et al., Fisher Cassie, W Teng, and various proprietary firms, to all of whom I am greatly indebted and owe grateful thanks.

I am thankful to various consulting engineering firms and contractors (Dar Al-Handasah, London, UK), Wilson Murrow (Salina, USA), Ital-Consult (Rome, Italy), Sauti-Renardet (Rome, Italy), Technic (Rome, Italy), Saudi-Consult Riyadh (KSA), R. Travers Morgan and Ptns, (London, UK), Arabian Engg: Bureau (Riyadh, KSA), Rhein-Ruhr Ingenieur (Dortmund, W. Germany), Dar Al-Riyadh (KSA), Doxiadis (Athens, Greece), Ove—Arup (London, UK), Gilcon PS Ltd. (New Delhi, India) Gammon India Ltd. (Bombay, India), US Dugal & Co. P. Ltd. (New Delhi, India), CCC (Lebanon), Al-Mashrik Contracting (KSA), Naser Haza & Bros. (KSA), Han Yang (S Korea), Edok-Eter (Athens, Greece), Tanmia (Riyadh, KSA) and J & P (Cyprus), to name only some), with whom I worked in different capacities in different countries. I also wish to acknowledge the opportunity I got of designing some of the first class curved and skew continuous prestressed concrete bridges in Canada while working with the Ontario Department of Transportation (previously, The Department of Highways), Toronto. These experiences I assiduously sifted and stored over the long years with a view to sharing them with others through this book.

Last but certainly not the least, I wish to express my heartfelt gratitude to Vinita, my dear wife, for her limit state endurance. While we both tried to serve the underserved through our respective professions, engineering and medicine (she has a Doctorate from London University in Bone Pathology), but perhaps we did this too devotedly...since this led to the neglect of priorities on our domestic front. Perhaps the cost to us in terms of common and worldly-mundane equations of understanding has been high, ***but it took us strength to stand the oneness of self-inflicted individual solitudes and solitude is always an exercise in agony!*** I can never pardon myself enough for causing to my wife (and to some extent to my parents) ***silent suffering and loneliness, with all its consequences and resulting despatch from (what to most people are) essential worldly norms though in reality only transitory, indeed illusory, in the ultimate mortal sense***! This was due to my professional commitments taking me away often-times, often to different lands, for long periods of time for years not just months, while her own professional commitments kept her tied back. The only solace was that in the resulting void I invested the time for five years in writing this book and my other four books (and various papers), utilising literally each available minute every single day (outside my crowded official work schedule), shunning the time-consuming and generally frivolous social get-togethers, and denying myself even the minor indiscretions of relaxation. (Only a bit of yoga kept me going.) ***Judgement is left to posterity***.

DR V K RAINA

Contents



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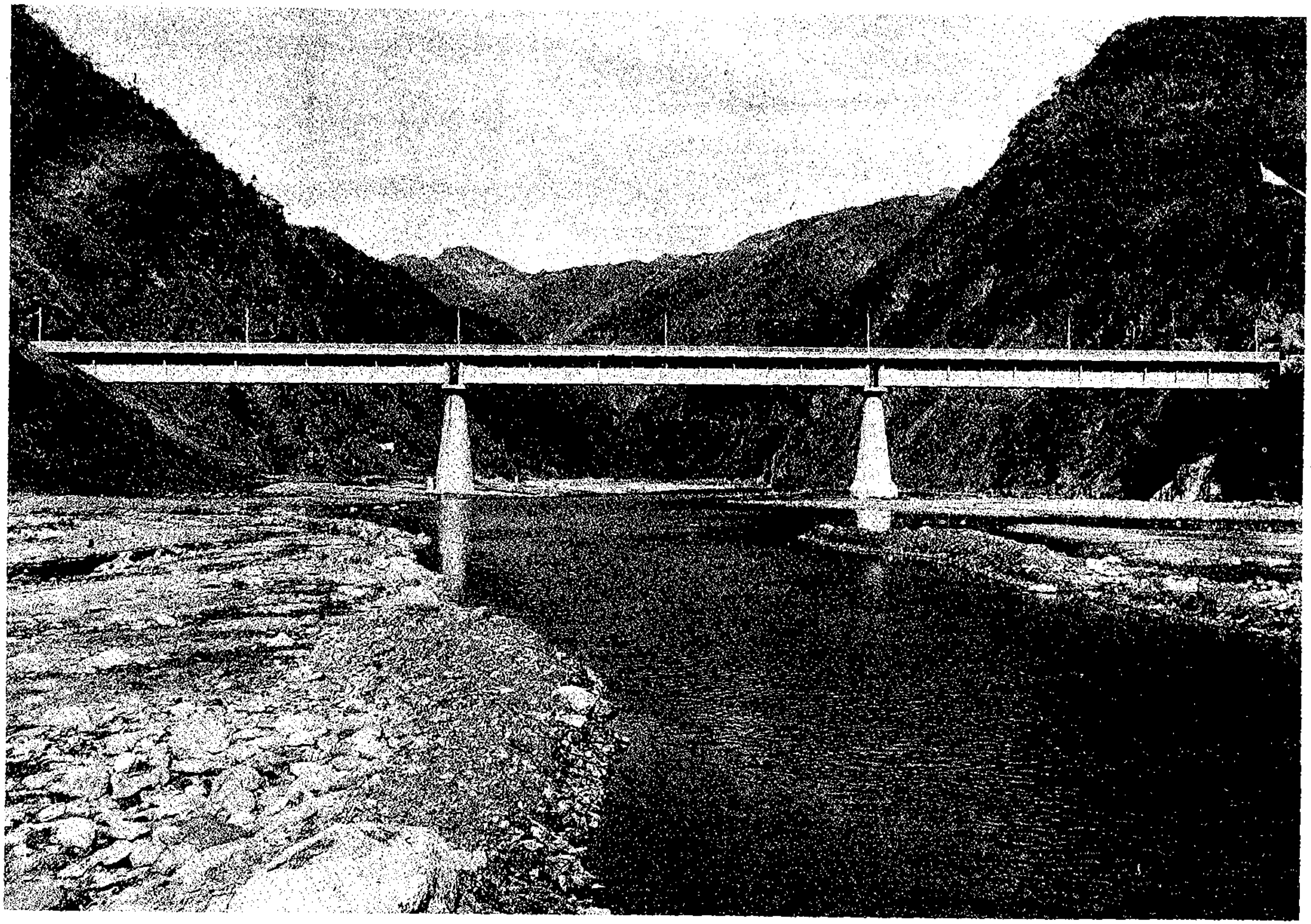

Giri Bridge (near Paonta, H.P., India)

CHAPTER 1

The Basic Principle of Practical Structural Analysis and Design

1.1 INTRODUCTION

After deciding the type of bridge, span arrangement and span lengths, assume suitable first-trial cross-sections of foundations and deck in concert with the method of construction. Hence, establish the loading sequence. For each load in the sequence see what it acts on, or what span or spans it acts on and under what end-conditions. From this find out what moments, etc. it causes at various sections, and which of these act on what section properties at those sections, and hence cause what stresses. The resultant stresses at every load stage at each section must not exceed their permissible values that are set out in the relevant Code of Practice (the design specification). This, in a nutshell, is the essence of structural analysis and design.

A good design can be produced only if developed along with scaled sketches and drawings with an eye for practical detail. Indeed design is guided by drawings made in parallel. For a regular workman-like practical design, the designer must not lose sight of the overall requirement which is, 'to produce a workable and practical structure in a limited amount of time at the minimum cost' keeping in mind the recent state of art and the contractually binding specifications. There is a large gap between a purely theoretical approach and a down-to-earth competitive practising professional's approach. It is more useful to carry out a practical design and produce a workman-like detailing in an execution drawing, rather than, for instance, merely be able to lecture on the 'ultimate strength of a nut' for hours! One has to appreciate the difference between husk and the grain, between the so-called 'coach' and the actual player, between the music critic sitting on the sidelines and the actual performing musician, between the class-room teacher and the practising professional working in a commercial, result- and profit-oriented scene of action. At the end of the day it is more important to have an approximate solution to an exact problem rather than try for an exact solution to an approximated problem. Very few theoreticians, lecturers, and those in staid office services have ever produced competitive practical structures themselves. Indeed, it should not be surprising if some of them, left to themselves, may even find the existing structures unsafe! In fact merely writing of papers and books, based on little practical experience and with no step-by-step, tool-kit applicability for a competitive practical end-product, often times is a manifestation of frustrated academics, and is no substitute for intelligent, long and hard drawn, cold, commercial practical experience resulting from swimming against the current covering the whole gamut of work from reconnaissance and feasibility through alternative working designs and actual construction and maintenance. Creativity cannot be taught. It is an experience one must live through to learn with one's own two hands in a cold competitive practice. (For more on this subject, see the 'Reflections' on 'Design Education' in Ch. 46 of this book.)

1.2 SUMMARY OF THE TOOL-KIT APPROACH FOR ANALYSIS AND DESIGN OF A USUAL TYPE OF BRIDGE

General Steps

Step 1 Knowing the required road-formation level, establish the permissible structural deck-depth (after allowing for (i) the minimum vertical clearance needed between the affluxed high flood level and deck, soffit, and (ii) the wearing-coat thickness below the road-formation level).

Step 2 Depending upon the depth of foundations, the height of deck above bed level (and above low water level), average depth of standing water during construction season, method of construction adequately suited to the site and the construction expertise available, decide: the type of bridge, span-lengths and arrangement, the type of foundations, the type and cross-section of the deck, method of construction and the loading sequence in the entire construction. (Considerations described in Chs. 7, 18 and 42 have direct bearing here.) Finally, the optimum type of bridge may well have to be decided by weighing between relevant alternatives.

Step 3 Decide the first-trial cross-sections and sizes of various elements of the substructure and superstructure, draw these to scale and establish the Preliminary General Arrangement Drawing (PGAD) of the bridge. (Some sizes and proportions, when seen to scale, will attract modifications and will be decided better through such scaled drawings. This is necessary so as to 'feel' the order in the transmission of forces and moments and the flow of stress trajectories that are to be surrounded by elegant enveloping proportions by practical detailing. Various trials lead to a structural form with optimum placement of its load-masses. Relative proportions and approximate sizes of certain members as well as their shapes will be best decided only through these scaled sketches, provided they are drawn by an experienced practical designer with an eye for detail.) Decide the type of bearings to be used and their locations (fixed, free, etc.).

Establish the preliminary member sections and sizes of various structural elements from a quick preliminary analysis and design. This is necessary for the subsequent detailed analysis and design work.

Substructure Design Steps

Step 4 Establish deck dead load reaction and the maximum and minimum live load reactions on the pier/abutment under consideration. Also estimate the co-existing moments due to these loads, about the transverse and longitudinal axes of the bridge due to the maximum possible eccentric transfer of these dead and live load reactions, as also the braking and temperature forces.

Step 5 Estimate total vertical load at base of foundation (at soffit of pile cap in case of piles) under maximum, minimum and no live load conditions, taking into account the upward buoyancy force equal to 100 or 50% weight of water in volume equal to that of the submerged mass (100% in case of saturated soil and fissured or weak rock, 50% in case of good rock).

Step 6 Estimate total moment about the bridge's longitudinal axis and horizontal force in the transverse direction, at various levels and at the base of the foundation, due to possible eccentricity of live and dead loads, flood water force and afflux if any (usually 10–15 cm).

Step 7 Estimate total moment about the bridge's transverse axis and horizontal force in longitudinal direction, at various levels and at the base of the foundation, due to possible eccentricity of live and dead loads, braking and temperature forces, and flood water force (or alternatively in case of an unerodible bed, the cross current effect of 25 cm static head difference across pier thickness, if this is greater than the flood water effect). (i) In case of simply supported spans on rocker/roller bearings, braking force from the live load on one complete span may be assumed to go to its rocker bearing alone so that the foundation under it will take either 'braking-temperature'* or 'half of braking + temperature'*, the latter usually applying to an abutment or a pier supporting unequal spans. (ii) In case of simple spans with identical neoprene bearings under each end of an individual span, the foundation will take the sum of half of braking from the live loads on each of the two spans supported by it and the horizontal temperature force equal to $(S_L m_L - S_R \cdot m_R)$, where S_L is the shear-rating of the neoprene bearings supporting the left side span, S_R that of those supporting the right side span, and m_L and m_R the deck movements above them, respectively. (iii) Refer to Chs. 8 and 9 for the method of distributing the externally applied as well as the self-induced horizontal forces among various bridge supports with different types of bearings (taking into account both the shear-rating of each support as well as the location of the zero-movement-point in the deck) the deck being continuous, or curved and/or skewed (simply supported or continuous), respectively.

Step 8 Estimate the wind force in the transverse direction that can be attracted by the exposed surface area of the bridge with or without the live load on deck. Generally, it is enough to consider wind on the deck surface area between its soffit and top of the solid parapet or up to mid-height of parapet in case of an open-type parapet, and that on the body of live load at the rate of 300 kg/m length of live load under maximum and minimum live load conditions. Wind pressure on deck surface area depends on the height of the centroid of its exposed surface area (indicated above) above the *mean retarding surface*, i.e. above the high flood level or the bed level, as the case may be. With live load on bridge, a wind force of 450 kg/m length of deck alone (ignoring that on live load) is also considered an alternative to 'wind on the deck exposed area and on the body of the live load'. Under 'no live load' conditions directly the effect of a wind pressure of 240 kg/m^2 on the exposed area of the deck is

* Temperature force here will be equal to $\mu(V - V')$, where μ = coeff. of friction at the roller (or sliding) bearing, and V and V' are the 'dead + live' load reactions at the two roller (sliding) bearings on the two sides of the rocker bearing on the foundation under consideration.

considered, assuming no live loads would ply under such a wind. (In coastal and certain specific areas, higher wind pressures (generally 100% extra or as pertinent) have to be considered.)

Since wind can also hit the bridge obliquely, therefore, as an alternative to the above-mentioned purely transverse wind condition, a combination of simultaneous wind forces in transverse and longitudinal directions, in magnitudes respectively equal to 67 and 33% of the said purely transverse wind force, should also be considered.

Step 9 (i) Estimate the static equivalent of the horizontal seismic force as can be attracted by the mass of the structure above the *embedment level* (maximum scour level in case of hydraulic bridges). Earthquake force is based on the full weight, even of the submerged portions of the structure (so long as they are above the embedment level).

(ii) Horizontal seismic force on a mass may be taken as a certain fraction of its weight, acting through its centroid. This fraction may vary between 0.10 for severe seismic zones to 0.05 for moderate seismic zones to zero for non-seismic zones, depending on the seismicity of the area. In addition, a vertical seismic force, upward or downward, equal to half the aforementioned horizontal value, can also co-exist and should be catered for—particularly when maximum or minimum base pressures are critical.

(iii) *If the earthquake is in bridge transverse direction*, then, as far as the contributions of the live load and the deck dead load to it are concerned, the aforementioned fraction may be applied on the magnitude of their reactions on the support under consideration, acting respectively at 1.2 m height above road surface and at the deck centroid level. (The lines of bearings in this direction effectively act as one fixed bearing.)

However, if the *earthquake is in bridge longitudinal direction*, then, the contribution of live load to it can be ignored since braking force is already considered and any further longitudinal horizontal force on live load will only cause skidding of its wheels. As for the longitudinal seismic force coming on the foundation from the weight of the deck and footpath live load, it will depend on whether the bearings are rocker and roller-rocker (sliding) type or shearing (elastomeric) type. In the former case the seismic fraction may be applied on the entire weight of a simple span deck on the rocker (fixed) bearing and the footpath live load on it, and this be assumed to go to the rocker bearing and to the foundation under it (roller-rocker bearing only takes the temperature force). However, in the latter case the fraction may be applied on the sum of the deck dead load and footpath live load reactions from the two simple spans sitting on the foundation under consideration. In each case, the point of action will be the bearing level.

NOTE that for distribution of externally applied longitudinal horizontal forces (e.g., seismic, wind and braking) in straight, simple or continuous decks and in curved and skewed (simple or continuous) decks, reference may be made to Chs. 8 and 9 as indicated earlier.

Step 10 Estimate the 'active' earth pressure force and moment (at various levels and at the base of foundation) on account of the retained fill above the soffit level of footing/pile cap. Passive relief from the front fill is generally to be ignored, but if it is well protected dependably then a fraction of it may be taken as dependable (but accounting for the negative surcharge angle effect if sloping downwards). This depends on the actual conditions *in situ*, case by case, the fraction may be such as to limit the magnitude of passive coefficient equal to the active coefficient.

If part of the foundation has been taken down well into permanent and unexcavated soil* (e.g., a caisson taken below the maximum scour level), estimate and take into account the net 'passive less active' earth pressure relief (force and moment) from such assisting soil grip. For this purpose, reference may be made to Ch. 13.

Step 11 (i) Summarise the net vertical load, the net horizontal forces in the two orthogonal directions and the net moments about these two directions, at the base of the foundation (at soffit of pile cap in case of piles), under each critical load combination. In other than piled foundations, establish the base pressures and the safety factors for stability against overturning and sliding and ensure that the requirements are satisfied, and if necessary redesign with revised dimensions. (Guidance on substrata bearing capacity... from Ch. 12.)

(ii) In case of piled foundations estimate the maximum and minimum axial loads in the piles by the traditional rivet-group approach (taking account of rakes if piles are raked). Ensure that no pile is

* For minimum depth of foundation refer to the book *Consultancy and Construction Agreements for Bridges including Field Investigations* by the author.

in tension unless tension-piles are provided. Ensure the structural design of the pile-section for the incumbent load and moment combinations on it (i.e. the moment resulting from the multilegged frame-action of the pile group under the action of the orthogonal horizontal forces, with the top and bottom fixity points in pile defined).

Check for the adequacy of soil resistance around an individual pile, the block failure and group action of the pile group. (Reference may be made to Ch. 11 for more details on this subject.)

(iii) Having then ensured the stability of the foundation, work out vertical load and moments at various intermediate critical sections (including at the maximum bending moment section in a caisson) in the foundation structure (this includes whole pier/abutment structure), and then structurally design these sections. Also design various other structural elements of the foundation structure (as they exist), taking due account of the critical load combinations incumbent for them.

Step 12 Estimate the vertical loads on each bearing under various load combinations, together with the co-existing orthogonal horizontal forces and rotations. Design the bearings for these effects. Use may be made of the standard manufacturers' catalogues for certain standard bearings. For details regarding bearings refer to Ch. 17 which also gives design procedure for elastomeric neoprene bearings and concrete hinge bearings.

Superstructure Design Steps

NOTE that for butterfly (i.e., double cantilever) decks, constructed essentially in free cantilever, as also for various other design considerations, refer to Chs. 18 and 37 and App. 6.

Step 13 Analyse and design the transverse-deck-slab and its cantilever wings, unless the superstructure is a purely longitudinally reinforced solid slab with no cantilevering wings. This is necessary as this decides the top flange thickness of the deck section which is essential to know in order to work out the deck section properties for the subsequent longitudinal design work.

Step 14 (i) Work out dead load and live load bending moments at each critical section (e.g. $^1/_2$, $^1/_4$ and $^1/_8$th span sections in simply supported spans up to about 30 m — one-tenth span sections for longer spans particularly in continuous and balanced cantilever decks, in the longitudinals of the deck).

(ii) In order to know the maximum and minimum live load effects that a particular longitudinal can receive, carry out the transverse load distribution for live load placed in various lanes (maximum, minimum and most eccentric placements). This* may be done by the simple *wheel-spacing method* given in the 1983 AASHTO specifications or by Dr Courbone's method (essentially if span to width ratio is between 2 and 4) or by Little and Morice's method, which basically is the Guyon-Massonet method. (A somewhat similar method suggested by Hendry and Jaeger somehow has not been commonly used.) Alternatively, use may be made of the *Plane-Grid method* which involves using one of the many standard computer programs (e.g., STRUDL program which is extremely powerful). The Plane Grid method is basically a *finite element method*. Though time consuming in writing the input data, it is nevertheless very powerful as it can easily take account of any skew effect and even the effect of curve in plan. (The curve is broken down into a series of jointed straight chord members.) In relatively narrow box section decks with full-depth end cross girders (and if their spacing exceeds about 45 m, then intermediate cross girders as well) the entire load effect may be assumed to be taken by almost full section evenly, owing to almost complete maturation of the section, ignoring only outer parts of the deck slab cantilevers (beyond a distance equal to six times the average deck slab thickness from the outer faces of the box section) due to shear-lag effect. For wide and multi-cell boxes the transverse live load distribution may be studied by the finite element method but it is time consuming. Alternatively, the total live load effect on them may be suitably increased (generally by 8–15%, depending on width and number of cells) and taken as acting on the modified box section properties mentioned above. The 1983 AASHTO specifications are much simpler to apply even for boxes.

(iii) Design against bending the aforementioned critical sections, in reinforced or in prestressed concrete as the case may be. In case of reinforced concrete — first a quick approximate design and detailing may be made and then the stresses and crack-widths checked accurately and then the ultimate moment capacities ensured, modifying the detailing appropriately. In case of prestressed concrete — basically, the relevant steps indicated in Ch. 18 and App. 6 may be followed. Also

* See more details in Ch. 19. Also refer Ch. 31 for transverse deck section design of certain special types of box sections.

include the effect of temperature stresses, as discussed in Ch. 30.

Step 15 (i) Work out dead load and live load shear forces at each critical section [e.g., at face of end-block (which in reinforced concrete deck may be taken at a distance equal to effective depth from centre of bearing), at $^1/_8$th and $^1/_4$th span sections and sections near intermediate supports] in the longitudinals of the deck. (Transverse live load distribution as in Step 14.)

(ii) Design the sections and reinforcements for shear on load factor basis adopting the cracked section approach discussed in Ch. 24.

Step 16 Design the reinforcement and its detailing in the anchorage zones (in case of prestressed concrete) and also above and below the bearings. Now design the remaining structural elements of the deck (e.g. cross girders, brackets, parapets, footpath-slabs, road-kerbs, etc.). Design the expansion joints for the appropriate deck movement (for some details see Ch. 32).

Step 17 Ensure the vibration characteristics of the deck (natural frequency and amplitude of vibration) so that it preferably falls below the strongly perceptible limit in Lenzen's criteria. Refer Ch. 39. However, it is to be noted that vibration is generally never a problem in concrete bridges, and even if the aforementioned vibration characteristic falls above the strongly perceptible limit, this alone is of no discomfiture to the driving public—unless either it is a pedestrian bridge or it has cycle tracks and footpaths with significant foot-movement traffic.

CHAPTER 2

Forces to be Considered in the Analysis for the Design of a Bridge

2.1 MAIN FORCES

The following is a list of the main forces whose effects should be analysed to estimate the load-effects (moments, shears, etc.) at all critical sections in the structure. Only then the structure should be designed for these load-effects to decide the section size, reinforcements, prestress, etc., so as to resist these forces at the specified stress levels, and serviceability criteria (crack-widths, deflections, etc.):

- Dead load of the structure (self-weight may come in stages)
- Live load (on roadway, cycle tracks and footpaths)
- Impact effect of moving live load
- Braking force (generated by the application of brakes on the live load)
- Wind load
- Earthquake force
- Lateral horizontal loads on parapets and kerbs
- Centrifugal force in horizontally curved decks
- Flood water current force in the bridge longitudinal and transverse directions (which is different from the static water pressure)
- Effect of afflux head (created locally on a pier as it obstructs the flood flow)
- Effect of cross-current force in bridge longitudinal direction (the effect of about 25 cm static head of water across the pier, looking at the bridge length), where the river bed is unerodible (rock) at scour level. (This force is an alternative to the flood water force in bridge longitudinal direction.)
- Buoyancy (floatation)
- Earth pressure
- Self-induced horizontal force caused at bearings by movement/rotation of deck due to temperature variation, creep and shrinkage of deck concrete, elastic-shortening of deck due to prestress, etc.)
- Thermal effect (comprising (i) the effect of non-linear distribution of temperature through the deck depth (leading to eigen stress resulting from the difference between final linear thermal strain gradient and the unrestrained non-linear thermal strain gradient), and (ii) the indeterminacy effect (comprising (a) effect of restraint to change in body mean temperature, and (b) effect of restraint to angular movements at the supports))
- Secondary effects (e.g., effect of eccentric connections, and shrinkage and creep of deck concrete)
- Effect of possible differential settlement of supports
- Loads resulting from temporary erection conditions and partial span-dislodgement conditions.

2.2 SOME RELEVANT CONSIDERATIONS

The following recommendations are intended to be used in conjunction with conditions prevailing at the sites and design loads and criteria (i.e. the particular codes of practice) specified by the organizations sponsoring the projects.

Dead Loads

(i) *Structure dead loads*: Structure dead loads are loads imposed on a member by its own weight and the weight of other structural elements that it supports including rails, sidewalks, slabs, and beams.
(ii) *Superimposed dead loads*: In addition to the structure dead loads, members should be designed to support the weight of superimposed dead loads including footpaths, earth-fill, wearing course, stay-in-place forms, ballast, water-proofing, signs, architectural ornamentation, pipes, conduits, cables and any other immovable appurtenances installed on the structure.

Construction, Handling, and Erection Loads

Consideration should be given to the effect of temporary loads imposed by sequence of construction stages, forming, falsework and construction equipment and the stresses created by lifting or placing precast members. Whereas a construction scheme should be considered for the feasibility of the project, it should be recognized that the contractor should be left to his own ingenuity for developing the construction procedure. The stability of precast members during and after construction should be investigated and provisions made for stability as needed. Effects of member-shortening and redistribution of loads during prestressing should be considered. Owing to long term creep effect the distribution of moments, shears, etc. in a continuous structure initially built in parts with different span arrangement for each casting becomes nearly same as if the whole structure was cast monolithically in one go. This calls for a workman-like understanding of true stress build up as against a mere numerical superimposition of stresses.

Shrinkage and Creep

Shrinkage of unreinforced concrete varies according to age, moisture conditions, temperature, water-cement ratio of mix, size and quality of aggregate and chemical composition of cement. Because a major portion of the shrinkage occurs soon after the concrete sets, shrinkage stresses of restrained concrete structures may be reduced by scheduling the placement of concrete. For example, alternate sections of continuous members may be placed and allowed to shrink prior to placing concrete between the sections to complete the continuity. Forces or displacements due to shrinkage of conventionally reinforced concrete may be evaluated assuming a strain of 0.0002.

Because shrinkage is not effectively restricted by the reinforcement in members such as arch ribs or prestressed beams that are subjected primarily to compression, the shrinkage strain can be significantly greater. For design of such members a shrinkage coefficient equivalent to a drop in temperature of about 30 to 80°F, and even that corresponding to strains of 0.0018 to 0.0048 has been suggested by some.

Creep may be assumed proportional to the sustained stress. For conventionally reinforced concrete members the ratio of creep deformation to the instantaneous deformation is also dependent upon the shape of the cross-section and the amount of compression reinforcement.

The average creep deformation due to sustained load at the end of 20 years is approximately two to three times the instantaneous elastic deformation. The approximate percentage of this strain occurring after application of sustained load are as follows:

25%	2 weeks
50%	2 months
75%	1 year
100%	20 years

More accurate prediction of creep and shrinkage values accounting for the shape of the cross-section, properties of concrete, its age at loading, the time under loading and the environmental conditions, can be made by the use of standard references. Also see Ch. 26 ahead.

Thermal Effects

Refer Ch. 30 of this book.

Earth Pressure Effects

Refer Chs. 11, 13, 14 of this book.

Bed Scour and Minimum Founding Level

Reference may be made to the author's book—*Consultancy and Construction Agreements for Bridges including Field Investigations.*

Differential Settlement of Supports

Its magnitude should be estimated from the considerations reported in the soil investigation report for the project site and then the effect of settlement of each individual support may be computed by the method explained in Ch. 22. Thereafter, the worst combinations of simultaneous settlement of different supports can be picked up for maximum effects at individual sections. (However, long term creep reduces this effect in the long run.)

Wind, Earthquake, Buoyancy and Vibration Effects

Reference may be made to the relevant mandatory codes of practice and Ch. 39 of this book.

Effects of Horizontal Forces—Externally Applied and Self Induced

For the effects of the externally applied horizontal forces (e.g. braking, seismic and wind) and self induced horizontal forces (caused by movement/rotation of the deck due to temperature variation, shrinkage and creep of deck concrete and due to elastic shortening of the deck due to prestress), reference may be made to Chs. 8 and 9. Here one finds the method for evaluating the horizontal forces caused at the various bridge supports as a result of the distribution of these forces as effected by the type of bearings, shear ratings of the supports, and location of the zero movement point in the deck.

Flood Water Force on Substructure

The flood current, flowing at a maximum velocity V m/s (or a maximum mean velocity v m/s) may be assumed to strike the substructure at $\theta = 20°$ to the normal flow direction, so that flood strikes the width of pier @ $V \cos\theta$ m/s and the length of pier @ $V \sin\theta$ m/s.

The resulting dynamic stream pressures on the substructure in the two orthogonal directions, respectively, may be assumed to vary linearly from zero at maximum scour level to the maxima of $52\ K(V\cos\theta)^2$ and $52\ K(V\sin\theta)^2$ kg/m^2 at the high flood level, where K = a coefficient, taken as 1.4 for square ended (i.e., square nosed) substructure, 0.5 if the noses include an angle of 30° or less and 0.67 for noses that are circular. V^2 is taken equal to $2v^2$,

V and v in m/s. (Note that this dynamic water pressure triangle is unlike the static pressure triangle which varies from zero at the surface to maximum at the bottom of water depth.)

Ice Pressures and Associated Forces

Reference may be made to the latest AASHTO design specifications.

Traffic Lanes

Lane-loading or standard-truck are generally considered to have a width of 1.8 m centres of wheels and 3.00 m overall. These loads should be placed in the design traffic lanes, spaced across the entire bridge roadway-width, in numbers and positions required to produce the maximum stress in the member under consideration. Roadway-width should be the distance between kerbs. Fractional parts of design-lanes should not be used. Roadway-width in metres, should be divided by 3.65 and the quotient approximated to the nearest whole number which then represents the Number of Design Traffic Lanes (NDTL), i.e. the number of lanes that can be assumed to be loaded for the purposes of bridge design. The said roadway-width should include shoulders also.

The lane-loadings or standard-trucks should be assumed to occupy any position within their individual design traffic lane-width, which will produce the maximum stress, maintaining generally 3 m as-centre to centre distance between adjacent trucks in cross section.

Reduction in Live Load Intensity

Where maximum stresses are produced in any member by loading more than one traffic lane simultaneously (as is generally the case), the following percentages of the resultant live load effects should be used in view of improbably coincident number of lanes loaded:

One or two lanes	100%
Three lanes	90%
Four lanes or more	75%

The reduction in intensity of floor beam loads should be determined as in the case of main trusses or girders, using the width of roadway which must be loaded to produce maximum stresses in the floor beam.

CHAPTER 3

Live Load on Road Bridges

3.1 GENERAL BACKGROUND

Loading specifications have a history which goes back to well before the general use of methods of structural analysis capable of simulating plate action, and some loading specifications were drawn up to enable the strength requirements for slab elements within steel girder bridges to be evaluated by means of simple hand calculations. However, methods of analysis which evaluate the design moments due to complex loading cases are now in widespread use, and it is therefore no longer considered necessary to enhance the distributed loading on short spans to give the appropriate design forces, because these can be evaluated directly from the concentrated wheel loads.

For major roads, and those giving access to certain types of industrial installations, provision has to be made for moving abnormal loads.

Abnormal loading has to be accommodated on all motorways and trunk roads.

The application of the normal *line load* is taken parallel to the supports. This is because the knife-edge load does not specifically represent an axle but is a load which, when combined with the distributed loads specified for the span effect, gives rise to design forces appropriate to the strength requirement for an element of a deck structure.

For the design of local structural elements within a bridge deck, such as the slabs spanning between longitudinal members, the requirement of a single wheel-axle load is common. Concentrated loads govern for short spans.

The distribution of the moments arising from concentrated wheel loads at the edges of a slab which is fixed along the line of support (as is commonly the case in a cellular or box deck) is subject to sharp peaks. Plotting the design bending moment along the length of a support will show the peak and illustrate how the area over which the load is applied becomes significant in evaluating design moments. It is therefore relevant to take the thickness of finishes and the wheel contact area into account in evaluating the load dispersion dimensions.

Simple types of bridge deck rarely produce stability problems but where narrow pier arrangements are used, as is frequently the case with lengths of elevated roadway, it becomes important to check that a structure remains stable with heavy vehicles on the outer extremities of the deck. Another form of instability is for uplift to develop at the bearings under some loading conditions where there are marked differences in the span on each side of a support.

The value of accuracy and refinement in design methods is very much diminished if it is not matched by similar qualities in the assessment of design live loads.

The early loading standards in some countries were not applied nationally; they were generally specified by local authorities who took into consideration the traffic which was likely to use the bridge concerned. These loadings often consisted of steam-rollers or some form of traction-engine.

The needs of military transport and its heavy equipment caused consideration to be given to the specification of a loading train for bridges representative of the actual and envisaged vehicles they would have to carry. This resulted in the introduction of the (then) Ministry of Transport's first 'standard loading train' in the UK in 1922, and the original loading standards of many other European countries at about the same time. This trend in the introduction of standard highway bridge loadings was almost a universal trend arising out of the technological advancements and industrial developments that were taking place at a fast rate all over the world.[1]

It was, however, the 1930s which witnessed the introduction of the most standard bridge loadings applied nationally and on a more scientific basis than hitherto. For example, in the UK the equivalent 'Ministry of Transport standard loading curve' was introduced in 1932, this was a new approach which proved to be very popular. Contemporary with this there was the British Standards (BS) train of loading. These two standards formed the basis of the present type HA loading of BS:153. In the USA, a loading standard consisting of truck-trains and equivalent loads was introduced by the American Association of State Highway Officials (AASHO)* in 1935. It is significant that even in some of the developing countries, like India, loading standards were introduced nationally during this period (in 1937).

* This name and abbreviation is now changed to the Association of American State Highway and Transportation Officials (AASHTO).

The highway bridge loading standards of most countries have developed gradually often with little regard to the standards prevalent in other countries. This has resulted in wide variations in loading standards, even among neighbouring countries, which is inhibiting the proper development of 'through' road transport all over the world. Therefore, there is a fundamental need to know how the loading standards of one country differ from those of others. Galambos[2], in a recent study made for the International Road Federation, has reported that many countries are currently engaged in revising their highway bridge loading standards.

Comparative studies reported[3-4] at ACI's second international symposium on *Concrete Bridge Design* (1969) were based on the loadings of different countries. However, in at least one of them, the dynamic effects were not considered. As the impact allowance varies considerably in different countries, it has to be added to the basic values for a realistic comparison. In one of them there are some obvious anomalies in the results. For example, the loading in the UK is shown to be more severe than that of West Germany and in fact, the heaviest, which is not the case.

3.2 LOADINGS OF DIFFERENT COUNTRIES

AASHTO Loadings

The Association of American State Highway and Transportation Officials (AASHTO), Washington DC, specified heaviest loading, designated as HS 20-44, comprises a tractor truck with a semi-trailer having a total load of 320.3 kN or the corresponding lane loading. The lane loading is made up of a ud load of 9.3 kN/m and a knife edge load of 80 kN for bending moment and 115.7 kN for shear. Impact is to be added in both the cases as per the formula given in the AASHTO specifications.

For the design of bridges, both the truck and lane loading are to be considered and the one which gives the worst effect is to be adopted. With the truck loading, only one truck is considered for each traffic lane for the whole of its length. There is no reduction in load intensity for up to two lanes of traffic loaded.

BS Loadings

The British Standards (BS), London specify two types of loading known as the type HA and type HB loadings. The HA type is also followed in Malaysia, Sri Lanka, Kenya, Zambia, Zimbabwe, etc. (with minor changes in some cases). The HA type is the normal design loading and consists of a uniformly distributed lane loading varying from 318.6 kN/m for 1 m loaded length (span) to 5.8 kN/m for 900 m loaded length (span), and a knife-edge load of 120 kN per lane. The values given are inclusive of impact. There is no reduction in the intensity of HA loading for up to two lanes of traffic loaded. An alternative axle load is also specified in this chapter on which impact must be considered.

The HB type is an abnormal unit loading. The number of units per axle (four axles in all) specified in the UK for bridges carrying the heaviest class of load is 45, amounting to a total load of 1800 kN. This is an idealised load on four axles which allows for the weight of the tractors accompanying trailers. With this loading, an overstress of 25% is allowed. No allowance is to be made for impact. Only one lane is to be loaded with type HB loading, all other lanes being considered as occupied by one-third full lane HA loading (latter only if its presence gives worst effect).

The type HA and HB loadings are currently under revision. In the UK, the Department of Environment has already effected certain changes in the type HA loading.

IRC Loadings

The Indian Roads Congress (IRC) specifies three classes of loads, designated as Class 70-R, Class AA and Class A for the design of permanent bridges, and all of them are followed in India. Pakistan has adopted Class AA* and Class A loadings for the design of bridges.

The Class 70-R and Class AA are of two types each. The first is a 700 kN tracked vehicle which is common to both the classes; the only difference is in the loaded length, which is slightly more for the Class 70-R. The second, which is of the wheeled type is a 1000 kN train of vehicles on seven axles for the Class 70-R, and a 400 kN vehicle on two closely spaced axles for the class AA. The Class A loading is a 554 kN train of wheeled vehicles on eight axles. Impact is to be allowed for in all the loadings as per the formulae given. The formulae are different for steel and concrete bridges.

All the three classes of loads are to be separately considered in the design and the worst effect is to be taken. For the design of two-lane bridges, only one lane of Class 70-R or Class AA load is considered, whereas both the lanes are assumed to be occupied by Class A loading if that gives worst effects.

Loadings of France

There are two normal systems of loads known as the System A and System B loads. The System A loading consists of a ud load which varies from 18.7 kN/m^2 for 10 m loaded length to 4 kN/m^2 for 199 m loaded length. This load, which is inclusive of impact, is given by a formula in terms of the loaded length. The System B comprises three types known as Systems B_c, B_r and B_t. While System B_c consists

* Tracked vehicle only.

of two trucks of 300 kN each per lane, B_r is a single wheel load of 100 kN, and B_t, a tandem axle of 320 kN. Impact is to be added to these according to a formula which takes into consideration the dead load of the structure as well.

In the design of bridges, systems A and B are to be considered successively and the worst effect is to be taken. There is no reduction in load intensity for up to two lanes of traffic loaded.

Loadings of West Germany

For federal autobahns (expressways), federal highways and rural highways of the first order, designated as Class 60, the loading per lane consists of a 600 kN vehicle on three axles and a ud load of 5 kN/m^2 in the remaining portion of the carriageway. Allowance for impact is to be made as per the formula given.

An equivalent ud load is also given as a substitute for the design vehicle.

Loadings of Japan

The live load specified for the design of main girders of the first class of bridges is known as the L-20 loading. There is a corresponding truck loading called T-20 loading for the design of floor systems. For a lane width of 5.5 m or less, the L-20 loading consists of a knife-edge load of 50 kN/m and a ud load of 3.5 kN/m^2 for spans up to 80 m reducing to 3 kN/m^2 for greater span lengths. For bridges having a width of more than 5.5 m, the knife edge load and the ud load are assumed to be reduced by one half on the portion of the roadway in excess of 5.5 m width. Impact is to be added as per the formula given.

Loadings of New Zealand

The HS 20-44 truck and the lane loading of the AASHTO and another truck loading designated H20-S16-T16 design vehicle are specified. The latter is the same as the HS 20-44 truck with a 142 kN trailer attached to it. Both the loadings are to be considered in the design and the one which gives the worst effect is to be taken. Impact is to be allowed as per the formula given; but for shear force, it is taken as 30% for all the spans. There is no reduction in load intensity for up to two lanes of traffic loaded.

Loadings of Sweden

Sweden specifies two types of loads. One is a lane loading, consisting of a 140 kN axle plus a ud load varying from 24 kN/m for 10 m span and less to 11 kN/m for a 90 m span and above. The other is a 1000 kN single truck on five axles. Impact is to be added only to the axle load of the lane loading. For the design of two lane bridges, either the lane loading in both the lanes or the single truck loading is considered.

NOTE: More details of the above loads, as also of certain other National (Highway Bridge) loads are given ahead.

3.3 SOME INTERESTING COMPARISONS[1] IN THE DIFFERENT TYPES OF LOADINGS

While many other countries specify the same ud load for bending and shear, Italy gives different values, those for shear being more than those for bending. This is understandable as there is no knife-edge load or axle load with it. It is, however, significant to note that France and West Germany do not distinguish between bending and shear in their equivalent ud load values, although they have no knife-edge load or axle load with the ud load.

Unlike other countries which specify ud loads for the full width of the traffic lane, Finland and Sweden specify it as a strip load in two strips of 0.6 m each running for the entire loaded length. As an alternative, Sweden allows the ud load to be applied uniformly over a width of 2.4 m.

The countries which specify a knife-edge load in combination with a ud load fall into two groups. While the HA type group gives the same value of knife-edge load for both bending and shear, the AASHTO type group, Iran and New Zealand, specify different values. In the latter case, the knife-edge load for shear is always more than that for bending.

With the exception of the HA type group, all countries which have an equivalent ud load system, have at least an alternative truck loading which is also to be considered in the design. Even in the HA type group, designs are generally to be checked for the type HB loading, although the number of units of the HB vehicle to be taken may vary from country to country. For example, the specifications of Kenya require the designs to be checked for 25 and 30 units respectively of the HB vehicle as against the UK practice of checking the designs of all the important bridges for 45 units for trunk roads and 37 1/2 units for principal roads. BS:153 permits an overstress of 25% with the type HB loading. There is, however, an ambiguity in the present provision in the code permitting this overstress, as it is not related to the number of units of the HB vehicle.

Italy, like some other countries, has separate civil and military loadings and all its important bridges are designed for the latter, which is heavier. Though not explicitly, military loadings are however covered in the standards of many other countries. In this category comes the IRC Class 70-R of India, the Caterpillar of Austria and the NK-80 loading of the USSR.

Lane Width

The design lane width of 5.5 m followed in Japan is an unusual one. Except for it, the lane width lies in the range 2.5–4.0 m, the most common value being 3 m. Norway specifies a range for the design lane widths and Sweden gives different widths for different types of loading. These practices do not conform to the ideal concept of a 'standard design lane width' and 'lane loading'. In countries like India, Pakistan and USSR, where there is no standard lane loading, only the minimum widths of carriageways for different number of lanes are specified. Therefore, they do not have any standard design lane width as such.

Impact Allowance

(i) BS:153 specifies an impact allowance of 25% to be added to the axle load (the pair of adjacent wheels) if it produces the greatest bending moment or shear, in the HA load case. The stipulations in the Norwegian standard are similar; the only difference is that instead of 25%, 38.5% impact is added to the heaviest axle load.

(ii) In the majority of countries, impact is related to the loaded length (span length) although the exact relationship varies considerably from country to country.

(iii) Some countries like Austria and India specify different impact factors for concrete and steel bridges, the factor for steel being more than that for concrete. This apparently is based on the principle that a lighter structure will be subjected to a more dynamic effect. Finland too has a similar approach, but it distinguishes only timber bridges from the others by specifying a lower impact factor for them presumably on account of the damping effect of timber.

(iv) At least West Germany and Italy ignore impact when the span length exceeds 50 and 100 m, respectively. But even for longer spans, countries like Australia and India specify certain minimum values of impact.

(v) Various standards give an upper limit for the impact allowance either with respect to the type of vehicle (tracked or wheeled) or in relation to the type of bridge (concrete, steel or timber) and the value of this varies from 25 to 64% in different standards.

(vi) Unlike other countries, Belgium and France relate the impact factor to the dead load of the bridge structure. The principle behind this is, however, implicit in the impact formulae being used by many other countries which relate impact to the type of bridge and length of span. The impact formula of Belgium is further complicated by including the speed of the vehicle in it.

(vii) Austria specifies different impact factors for the directly loaded and indirectly loaded main girders of concrete bridges, the factors for the former being more than that for the latter. Again in the case of steel bridges, it distinguishes between the first and second lanes of steel bridges, specifying higher impact factors for the first than the second and allowing no impact for lanes in excess of two.

Quantitative Comparison of Road Live Load from Various Countries

The following are the bases of comparison[1] of road live load:

(i) For comparing the loadings from the quantitative point of view, the maximum bending moment and shear force that would be caused by them in simply supported spans are taken as the basis. Simple spans were chosen since it was presumed that they are more common and are also indicative of what is likely to happen in other types of construction. A span range of 5–100 m was expected to cover the great majority of simple span bridges.

(ii) As the impact allowance varies considerably for different loadings, it is added to the calculated values of bending moment and shear force, and a comparison is made of the total values. Wherever the same standard gives different impact formulae for steel and concrete bridges, the one which gives the higher value is taken.

Computation of Bending Moment and Shear Force (Comparative Picture)

On the basis of the above assumptions, the values of the maximum bending moment and shear force, including impact, were calculated for each loading separately, for single and double lanes, for spans up to 100 m in 5 m increments. The results obtained are given in Tables 3.1 to 3.4[1].

Considering the predominant range of spans, the following general observations can, however, be made from the results of Tables 3.1 to 3.4:

(i) Although the IRC loadings appear to be the heaviest for a single lane, they are lighter than the French, West German, Japanese and Type HA loadings when two lanes are considered.

(ii) The West German loading, which is lighter than the IRC and Japanese loadings for a single lane is the heaviest when two lanes are considered.

(iii) For both single and double lanes, the AASHTO loading gives the minimum effect in bending and shear, being only about one half of that given by the West German loading.

(iv) The Type HA and French loadings are almost identical in effect for spans up to about 50 m, beyond that, the latter gives slightly higher values.

(v) In the higher span ranges, the global effect of Type

Table 3.1 *Simply Supported Span Versus Maximum Bending Moment for One Lane*

Span (m)	Maximum bending moment for one lane, including impact allowance (kN-m)								
	Loadings of New Zealand	L-20 Loading of Japan	Loadings of France	Class 60 Loading of W. Germany	IRC Loadings	HS 20-44 Loading of AASHTO	BS Loadings		Loadings of Sweden†
							Type HA	Type HB**	
5	231	551	390‡	612	687	231@	243	756	450
10	573	1237	816	1624	1548	573@	694	1863	1300
15	1137	2057	1539	2690	2725	1073@	1336	3331	2550
20	1797	3006	2371	3804	4198	1552@	2175	5654	3800
25	2485	4083	3290	4952	5680	2022@	3156	7862	5050
30	3157	5285	4280	6125	7058	2481@	4151	10085	6300
35	3817	6612	5338	7310	8412	2935@	5184	12315	7550
40	4465	8065	6454	8497	9739	3379@	6340	14550	8800
45	5107	9647	7637	9674	11059	3863	7501	16788	10050
50	5738	11344	8870	10830	12496	4597	8656	19029	11300
55	6367	13162	10151	12452	13933	5391	9780	21271	12550
60	6990	15115	11498	14168	15371	6243	11070	23515	13800
65	7607	17182	12884	15977	16808	7155	12354	25759	15050
70	8299	19376	14342	17880	18245	8125	13738	28005	16300
75	9155*	21682	15848	19877	19683	9155	15117	30251	17550
80	10244*	24119	17388	21968	22339	10244	16560	32497	18800
85	11392*	26387	18997	24152	25138	11392	17993	34744	20050
90	12631*	28714	20660	26430	27945	12631	19508	36992	21300
95	13872*	31080	22348	28802	30759	13872	21069	39239	22550
100	15184*	33494	24106	31268	33580	15184	22875	41487	23800

* Based on lane loading. Otherwise H20-S16-T16 truck loading governs
** 45 units. An overstress of 25% is permissible under this loading
† Truck loading governs throughout
‡ Based on System B_c loading. Otherwise System A loading governs
@ Based on standard HS truck loading. Otherwise standard lane loading governs

Table 3.2 *Simply Supported Span Versus Maximum Shear Force for One Lane*

Span (m)	Maximum shear force for one lane, including impact allowance (kN)								
	Loadings of New Zealand	L-20 Loading of Japan	Loadings of France	Class 60 Loading of W. Germany	IRC Loadings	HS 20-44 Loading of AASHTO	BS Loadings		Loadings of Sweden†
							Type HA	Type HB**	
5	212	441	359‡	571	549	212@	199	738	420
10	304	495	369‡	689	633	298@	278	927	590
15	375	549	454‡	743	820	337@	356	1218	700
20	420	601	523‡	779	927	358@	435	1364	775
25	447	653	569‡	807	978	369@	505	1451	820
30	465	705	592‡	828	997	375@	554	1509	850
35	478	756	610	845	1009	379@	592	1551	871
40	488	807	645	858	1015	387@	634	1582	888
45	495	858	679	867	1027	390@	667	1606	900
50	501	908	710	872	1090	410	693	1625	910
55	509	957	738	912	1181	433	711	1641	918
60	518	1008	767	950	1272	457	738	1655	925
65	545	1057	793	988	1351	481	760	1666	931
70	575*	1107	820	1026	1419	505	785	1675	936
75	606*	1156	845	1064	1477	529	806	1684	940
80	636*	1206	869	1102	1529	552	828	1691	944
85	666*	1242	894	1140	1574	576	847	1697	947
90	697*	1276	918	1178	1625	600	867	1703	950
95	727*	1309	941	1216	1695	623	887	1708	953
100	758*	1340	964	1254	1776	647	915	1713	955

* Based on lane loading. Otherwise H20-S16-T16 truck loading governs
** 45 units. An overstress of 25% is permissible under this loading
† Truck loading governs throughout
‡ Based on System B_c loading. Otherwise System A loading governs
@ Based on standard HS truck loading. Otherwise standard lane loading governs

Table 3.3 *Simply Supported Span Versus Maximum Bending Moment for Two Lanes*

Span (m)	Maximum bending moment for two lanes, including impact allowance (kN-m)								
	Loadings of New Zealand	L-20 Loading of Japan	Loadings of France	Class 60 Loading of W. Germany	IRC Loadings	HS 20-44 Loading of AASHTO	BS Loadings		Loadings of Sweden†
							Type HA	Type HB**	
5	462	827	780‡	1224	687	462@	488	838	640
10	1146	1856	1632	3248	1548	1146@	1388	2095	1580
15	2274	3086	3078	5380	2725	2146@	2672	3776	2775
20	3594	4509	4742	7608	4198	3104@	4350	6379	4198
25	4970	6125	6580	9904	5680	4044@	6312	8914	5819
30	6314	7928	8560	12250	7058	4962@	8302	11468	7609
35	7634	9918	10676	14620	8412	5870@	10368	14043	9537
40	8930	12098	12098	16994	9739	6758@	12680	16663	11572
45	10214	14471	15274	19348	11059	7726	15002	19288	13679
50	11476	17016	17740	21660	12496	9194	17312	21914	15838
55	12734	19743	20302	24904	14232	10782	19560	24531	18012
60	13980	22672	22996	28336	15598	12486	22140	27205	20172
65	15214	25773	25768	31954	17166	14310	24708	29877	22277
70	16598	29064	28684	35760	20046	16250	27476	32584	24316
75	18310*	32523	31696	39754	23486	18310	30234	35290	26250
80	20488*	36178	34776	43936	26650	20488	33120	38017	28048
85	22784*	39580	37994	48304	29908	22784	35986	40742	29662
90	25262*	43071	41320	52860	33620	25262	39016	43494	31095
95	27744*	46621	44696	57604	38050	27744	42138	46262	34129
100	30368*	50241	48212	62536	42880	30368	45750	49112	37300

* Based on lane loading. Otherwise H-20-S16-T16 truck loading governs
** One lane Type HB full and the other lane 1/3 Type HA as per the Code. An overstress of 25% is permissible under this loading
† Lane-loading governs throughout
‡ Based on System B_c loading. Otherwise System A loading governs
@ Based on standard HS truck loading. Otherwise standard lane loading governs

Table 3.4 *Simply Supported Span Versus Maximum Shear Force for Two Lanes*

Span (m)	Maximum shear force for two lanes, including impact allowance (kN)								
	Loadings of New Zealand	L-20 Loading of Japan	Loadings of France	Class 60 Loading of W. Germany	IRC Loadings	HS 20-44 Loading of AASHTO	BS Loadings		Loadings of Sweden†
							Type HA	Type HB**	
5	424	661	718‡	1142	594	424@	398	804	512
10	608	742	738‡	1378	692	596@	556	1020	632
15	750	823	908‡	1486	820	674@	712	1337	740
20	840	902	1046‡	1558	927	716@	870	1509	840
25	894	980	1138‡	1614	978	738@	1010	1619	931
30	930	1057	1184‡	1656	997	750@	1108	1694	1015
35	956	1134	1220	1690	1009	758@	1184	1748	1090
40	976	1210	1290	1716	1015	774@	1268	1793	1157
45	990	1286	1358	1734	1082	780@	1334	1828	1207
50	1002	1361	1420	1744	1174	820	1386	1856	1267
55	1018	1436	1476	1824	1274	866	1422	1878	1300
60	1036	1512	1534	1900	1368	914	1476	1901	1345
65	1090	1586	1586	1976	1446	962	1520	1919	1371
70	1150*	1661	1640	2052	1512	1010	1570	1937	1390
75	1212*	1735	1690	2128	1572	1058	1612	1952	1400
80	1272*	1809	1738	2204	1642	1104	1656	1967	1402
85	1332*	1863	1788	2280	1728	1152	1694	1980	1402
90	1394*	1914	1836	2356	1816	1200	1734	1992	1402
95	1454*	1963	1882	2432	1908	1246	1774	2004	1437
100	1516*	2010	1928	2508	1996	1294	1830	2018	1492

* Based on lane loading. Otherwise H20-S16-T16 truck loading governs
** One lane Type HB full and the other lane 1/3 Type HA as per BS 153. An overstress of 25% is permissible under this loading
† Lane loading governs throughout
‡ Based on System B_c loading. Otherwise System A loading governs
@ Based on standard HS truck loading. Otherwise standard lane loading governs

HB loading (consisting of 45 units of the HB vehicle in one lane and one-third of type HA in the second) is less than that of full type HA in both the lanes, when the former is adjusted for the permissible increase in stress of 25%.

(vi) The New Zealand loading is somewhat heavier than that of AASHTO for spans up to about 70 m. Beyond that, it gives the same values of bending moment as for the AASHTO loading; but in the case of shear force, it gives higher values throughout.

The loadings of different countries vary considerably both qualitatively and quantitatively. While the qualitative differences are understandable, it is difficult to see that such wide variations in intensity are warranted. This brings out the need for systematic surveys of vehicular loads on bridges. Apart from the intensity of traffic, the aspect of safety is closely linked with design loading and needs due consideration.

Of the different types of loadings, the equivalent ud load system appears to be the most popular one, possibly because it is simpler to apply. This explains the adoption of the Type HA and AASHTO loadings by many countries.

There are basic differences in the approach of different countries for assessing the dynamic effect of live loads on the bridge structure. The impact allowance formulae specified by some are unnecessarily complicated and not fully justified, particularly since the effect of live load on the bridge is comparatively less than that of dead load for span lengths of approximately 25 m and above. Thus there is need for more research in this field to ascertain the actual behaviour of bridge structures under dynamic loads and to evolve suitable design procedures (refer Ch. 39).

From a consideration of the simplicity of loading and the ease of its application in design, the Type HA loading appears to be the most favourable.

There is wide variation in the highway bridge loading standards of different countries. The extreme example is that of the USA where, for the range of spans covered, the effect of the design live load is only about one half of that in West Germany. (In fact, the loading of the Netherlands is even more severe than that of West Germany.) The loadings of the other countries generally fall in between the AASHTO and West German loadings.

3.4 DETAILS OF SOME NATIONAL (HIGHWAY) BRIDGE LOADINGS

AASHTO Loadings (Figs. 3.1 and 3.2)

(USA, Australia, Bangladesh, Canada, Ethiopia, Philippines, Turkey*)

* Axle loads followed in Turkey are 4 and 16 t in place of 8000 and 32,000 lbs respectively.

Truck Loading

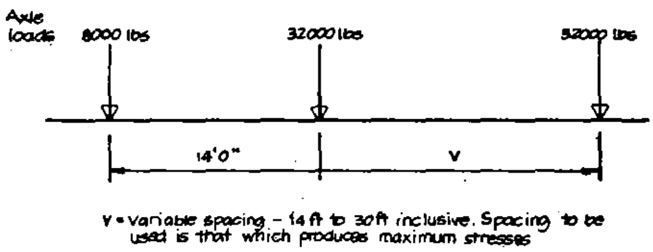

Fig. 3.1 *Standard HS 20-44 truck*

Impact Allowance

Impact allowance** is $50/(L+125)$ where L is the length in feet of the portion of the span to produce the maximum stress in the member. Maximum† impact allowed is 30%. For shear due to truck loads, L is taken as the loaded part of the span from the point being considered to the reaction, except for cantilever arms where the impact allowance is 30%.

Lane Loading

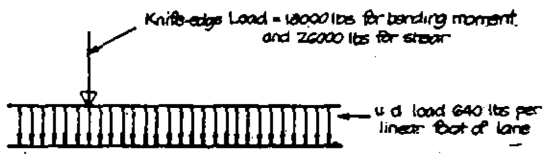

Fig. 3.2 *Standard HS 20-44 lane loading*

BS Loadings (Figs. 3.3 and 3.4)

(UK, Malaysia‡, Sri Lanka‡, Kenya‡§, Rhodesia‡)

Type HA Loading

The Type HA loading consists of:

(i) A ud lane loading as per the loading curve given plus a knife-edge load of 120 kN uniformly distributed across the width of the traffic lane, or

(ii) Two wheel loads§ each of 112 kN in line transversely to the direction of traffic flow, spaced at 0.9 m.

The ud load has a constant value of 31.5 kN per metre run of one lane for loaded lengths from 6.5 m to 23.0 m. For spans below 6.5 m, BS:153 gives separate curves for the ud load.

** Turkey specifies the impact allowance as $15/(L+37)$, where L is the span length in metres.

† Australia specifies also a minimum impact allowance of 10%.

‡ For Type HA loading only.

§ In Kenya, the wheel loads are specified as 40 kN each at the spacing of 1.0 m.

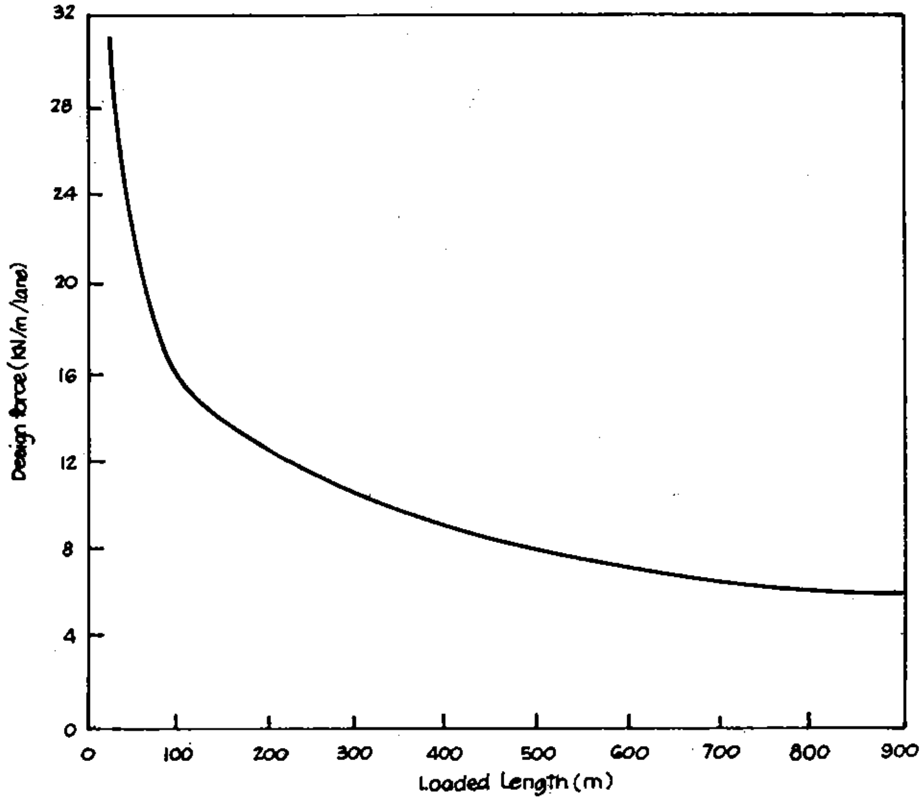

Fig. 3.3 *Loading curve for type HA loading*

Two lanes are always considered as occupied by full type HA loading while all other lanes in excess of two are considered as occupied by one-third the full lane loading.

The standard design lane width is 3.0 m. The number of traffic lanes to be considered for different widths of carriageway are specified.

In considering the effects of the 112 kN wheel loads, an overstress of 25% is permitted.

Impact Allowance

Type HB loading: no allowance is to be made for impact. Type HA loading incorporates an impact allowance of 25% on the heaviest axle in the train of vehicles from which the loading has been derived.

Type HB Loading

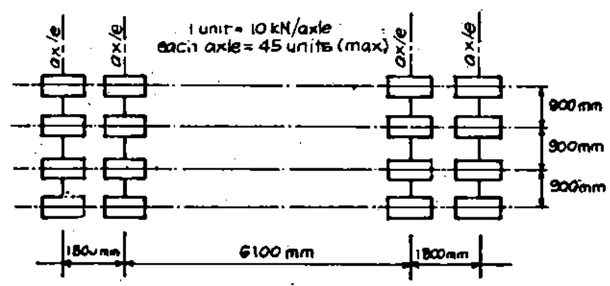

Fig. 3.4 *Plan view of type HB loading*

IRC Loadings (India and Pakistan*) (Figs. 3.5–3.10)

Class 70-R Loading

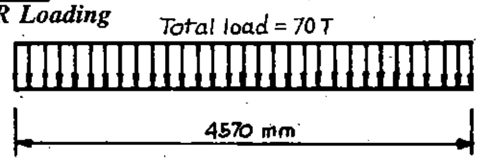

Fig. 3.5 *Class 70-R 'Tracked' loading*

* For the Class AA tracked and Class A loadings only.

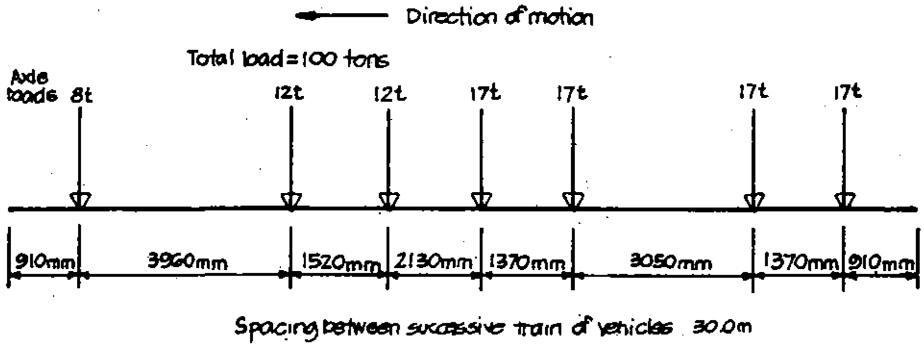

Fig. 3.6 *Class 70-R 'Wheeled' loading*

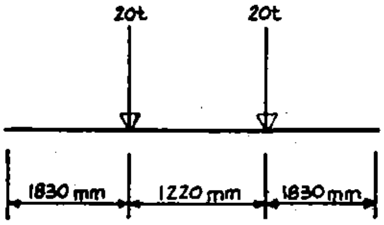

Fig. 3.7 *Class 70-R 'Bogie' loading*

Class AA Loading

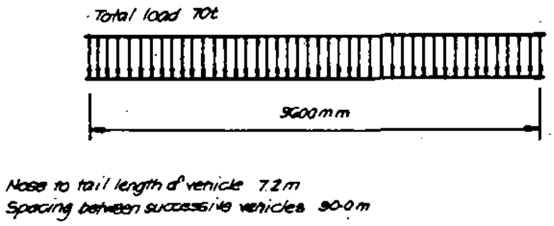

Fig. 3.8 *Class AA 'Tracked' loading*

Impact Allowance

When L is the length of span in metres, impact allowance for concrete bridges is equal to $4.5/(6 + L)$ subject to a maximum of 50% and minimum of 8.8%.

The impact allowance for steel bridges is $9/(13.5 + L)$ subject to a maximum of 54.5% and minimum of 15.4% with the following exceptions in the case of Class 70-R loading and Class AA loading:

(i) *For spans less than 9 m:*

- Tracked vehicles — 25% for spans up to 5 m reducing to 10% for spans of 9 m.
- Wheeled vehicles — 25%

(ii) *For spans of 9 m or more:*

- Tracked vehicles on concrete bridges — 10% up to a span of 40 m
- Wheeled vehicles on concrete bridges — 25% for spans up to 12 m

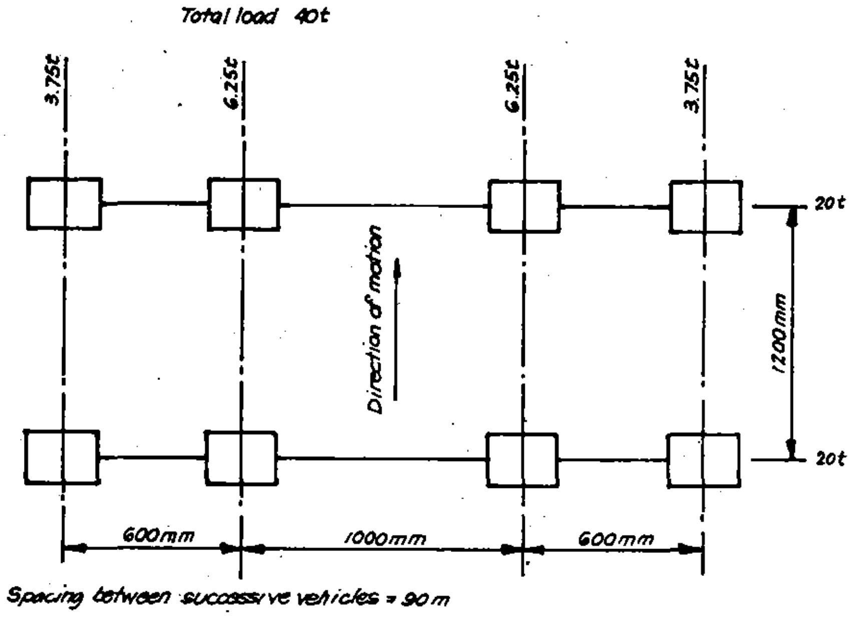

Fig. 3.9 *Plan view of the class AA 'Wheeled' vehicle*

Class A Loading

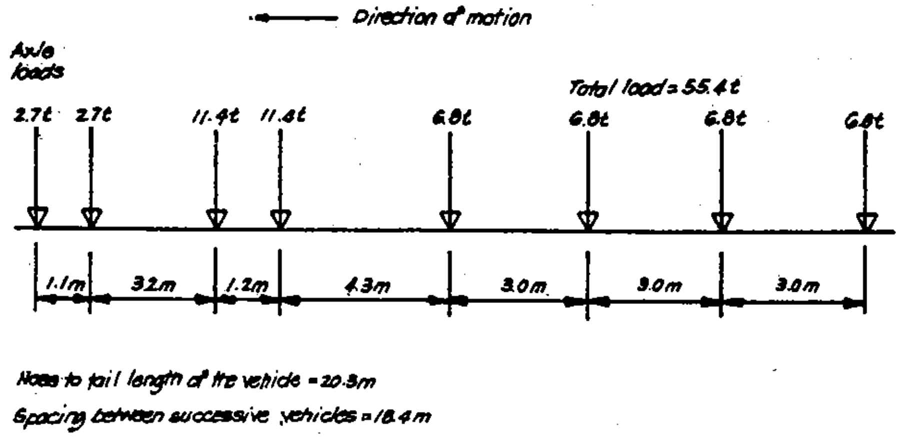

Fig. 3.10 *Class A train of vehicles*

• Tracked vehicles on steel bridges	10% for all spans
• Wheeled vehicles on steel bridges	25% for spans up to 23 m

Loadings of France (Figs. 3.11 and 3.12)

System A Loads

$$\text{ud load } A(l) = 230 + \frac{36,000}{l + 12} \text{ kg/m}^2$$

where l is the loaded length in metres.

The ud load $A(l)$ obtained from the above formula is to be multiplied by a coefficient a_1 whose value is 1.0 up to two lanes and then reduces with an increase in the number of lanes.

$a_1 A(l)$ is not to be less than $(400 - 0.2\ l)$ kg/m^2.

For class 1 roads, if the lane width is different from the standard lane width of 3.50 m, the value of $A(l)$ is to be multiplied by a coefficient a_2 also, so as to keep the total load per linear metre of lane unaltered for any loaded length.

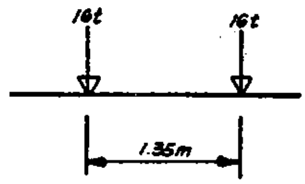

Fig. 3.12 *System B_t tandem axle loading*

System B Loads

System B_r is a single wheel load of 10 t.

Impact Allowance

System A loading is inclusive of impact. For System B loading, the impact factor, δ, is given by the formula

$$\delta = 1 + \frac{0.4}{1 + 0.2L} + \frac{0.6}{1 + 4\frac{G}{s}}$$

where L = length of the element in metres

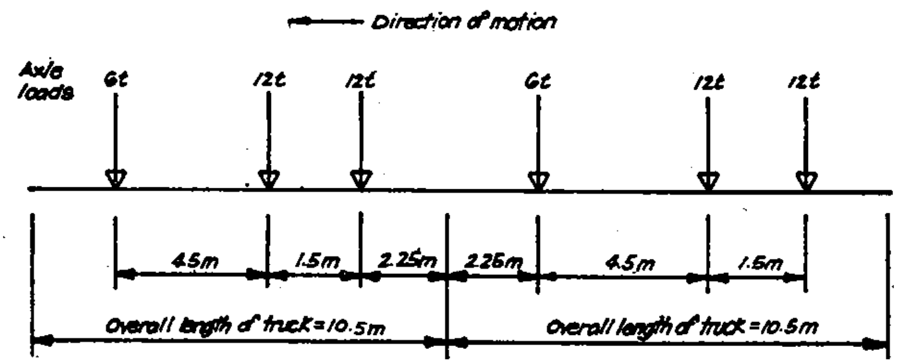

Fig. 3.11 *System B_c truck loading*

G = permanent weight of the bridge
s = maximum load of the truck

Loadings of West Germany (Fig. 3.13)

Class 60 Loading

The Class 60 loading consists of a 60 t heavy truck and a ud load of 0.5 t/m^2 in the portion of the lane not occupied by the truck. The substitute ud load for the 60 t heavy truck is 3.33 t/m^2.

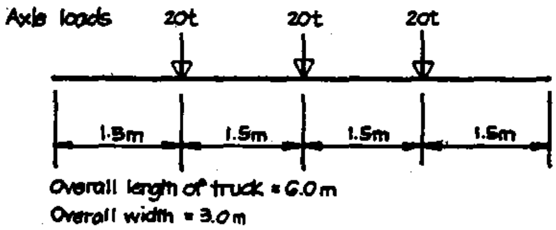

Fig. 3.13 *60 t heavy truck (SLW)*

The standard design lane width is 3.0 m. There is no reduction in intensity of load for up to two lanes of traffic. A ud load of 0.3 t/m^2 is specified in the area outside the main lanes.

Impact Allowance

The impact factor ϕ with which the live load values are to be multiplied is given by the formula

$$\phi = 1.4 - 0.008\, l_{\phi} \text{ but } \geq 1.0$$

where l_{ϕ} is the governing length in metres.

Loadings of Japan (Fig. 3.14)

L-20 Loading

For a lane width of 5.5 m or less, the L-20 loading consists of a knife-edge (line) load, P, of 5000 kg/m and a ud load, p, which has the following values,

For $l < 80$ m, $p = 350$ kg/m^2
For $l > 80$ m, $p = 430 - l$ but ≥ 300 kg/m^2

For bridges with a width of more than 5.5 m, the values of P and p are to be reduced by one-half on the portion of the roadway in excess of 5.5 m.

The full values of P and p are known as 'main loads' and the reduced values (50% of the main loads) are known as 'subloads'. The main loads are to be so placed on a 5.5 m wide part of the roadway and the subloads in the remaining part of the roadway as to produce maximum stresses.

In the expressions for the ud load p, l denotes the span length in metres.

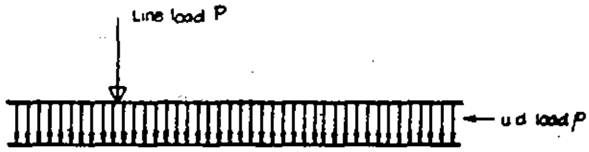

Fig. 3.14 *Composition of L-20 loadin~*

Impact Allowance

Impact allowance i is determined by the formula,

$$i = \frac{20}{50 + l}$$

where l is the length of the element in metres.

Loadings of New Zealand (Fig. 3.15)

Design Load

The design load per lane consists of the HS 20-44 truck and lane loading of the AASHTO or the H20-S16-T16 truck loading whichever gives the worst effects. The standard design lane width is 10 ft. (Fig. 3.15).

Impact Allowance

The impact allowance specified is the same as that given by the AASHTO, but there is, however, no upper limit to it.

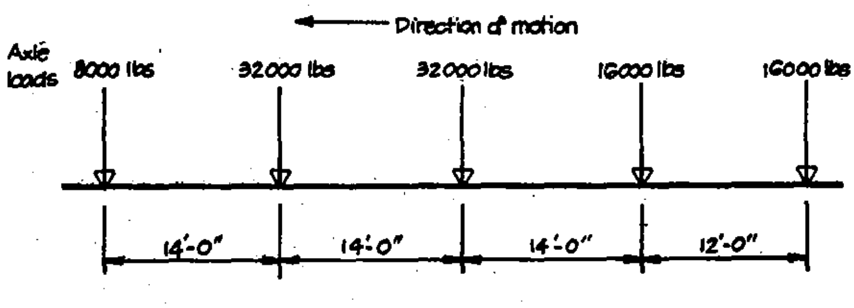

Fig. 3.15 *H20-S16-T16 truck*

Loadings of Sweden (Figs. 3.16 and 3.17)

Lane Loading

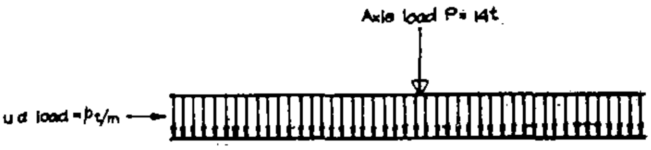

Fig. 3.16 *Composition of lane loading*

$$p = 2.4 \text{ t/m for } L < 10 \text{ m}$$

$$p = 2.4 - \frac{1.3(L-10)}{80} \text{ t/m for } 10 < L < 90 \text{ m}$$

$$p = 1.1 \text{ t/m for } L > 90 \text{ m}$$

where L = loaded length in metres
Design lane width = 3.0 m

Truck Loading (see Fig. 3.17)

Two lane bridges are designed with lane loading in both the lanes or only with single truck loading whichever gives the worst results. For continuous structures, there is a separate loading consisting of two axle loads and a ud load.

Impact Allowance

The axle load, P, of the lane loading is to be increased by 40% for impact effects. No allowance for impact is to be made for the ud load and for the single truck loading.

Loadings of Austria (Figs. 3.18 and 3.19)

Tracked Loading

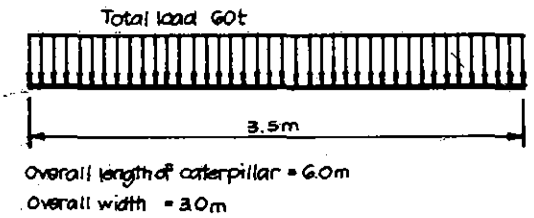

Fig. 3.18 *60 t caterpillar*

Truck Loading

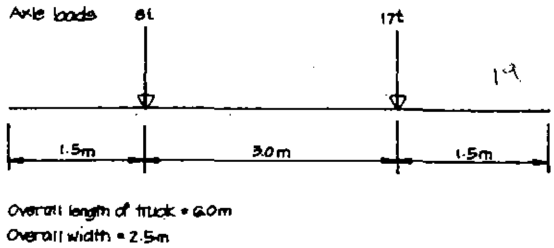

Fig. 3.19 *25 t truck*

A ud load of 0.5 t/m^2 is to be assumed on the portion of the lane not occupied by the truck.

On two adjoining lanes, the truck loading is assumed in both the lanes, the portion of the carriageway not occupied by the trucks being assumed to be carrying a ud load of 0.5 t/m^2. With the tracked loading, however, only one caterpillar is to be assumed for the whole carriageway and there will be no ud load with it. Both the cases are to be tried and the worst effect taken in the design.

The specifications also give the following equivalent weights of the caterpillar and truck loading which are to be used for the design of spans more than 30 m.

60 t Caterpillar	3.33 t/m^2
25 t Truck	1.67 t/m^2

Impact Allowance

The following are the impact factors given:
(i) Concrete bridges

Impact factor for different spans

Span of structural part (m)	0	10	30	50	70
Direct loaded main girder	1.40	1.30	1.20	1.10	1.00
Indirect loaded main girder	1.40	1.25	1.10	1.00	1.00

Impact factor for floor slab = 1.4

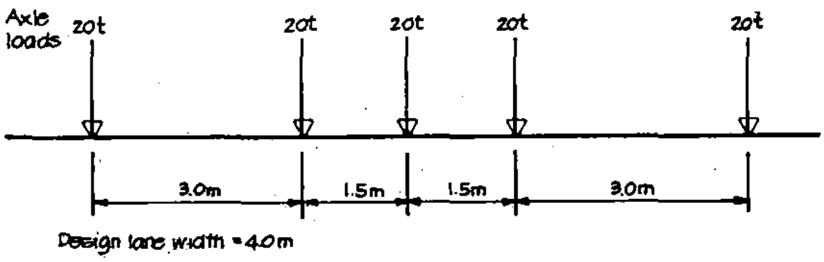

Fig. 3.17 *100 t single truck*

(ii) Steel bridges

Impact factor for different spans

Span of structural part (m)	2	6	10	20	40	60	80	100
Lane I	1.64	1.41	1.30	1.18	1.10	1.07	1.05	1.04
Lane II	1.32	1.20	1.15	1.09	1.05	1.03	1.02	1.02

For all remaining lanes impact factor is 1.00.

Loadings of Belgium (Figs. 3.20 and 3.21)

Normal Truck Loading

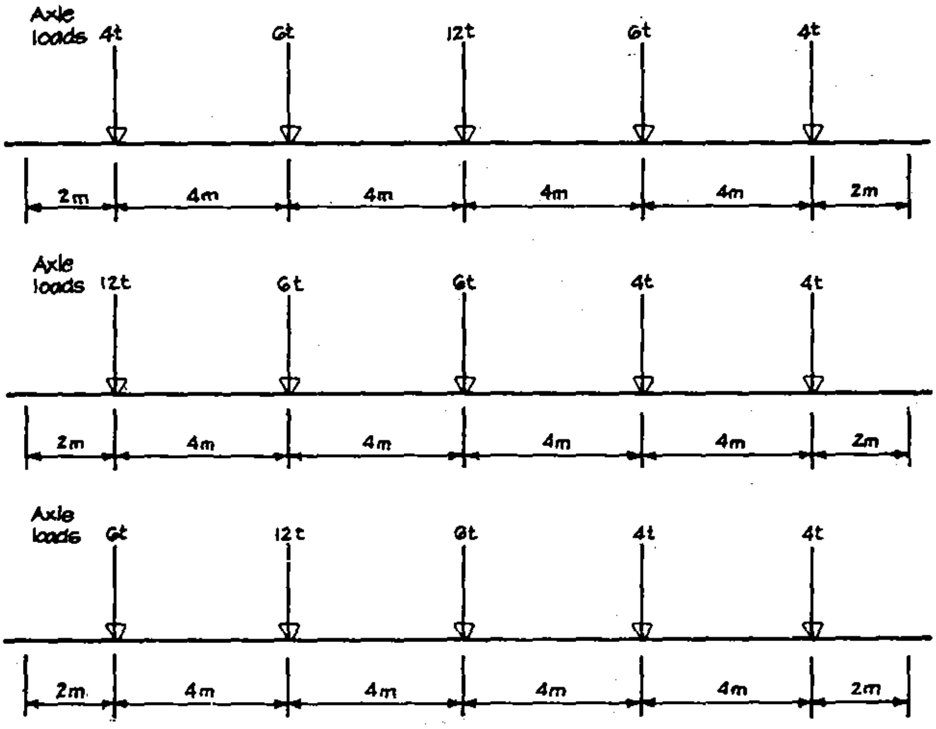

Fig. 3.20 *Different combinations of 32 t normal truck*

Heavy Truck Loading

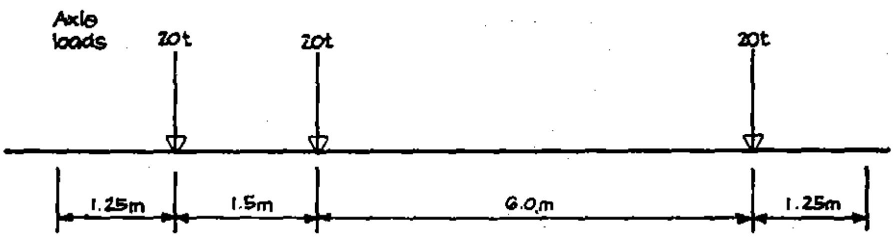

Fig. 3.21 *60 t heavy truck*

For two lane bridges on important roads, the loading consists of a 60 t truck in one lane in combination with a 32 t truck plus a ud load of 400 kg/m^2 in the other lane.

Impact Allowance

The impact factor, ϕ, is given by the formula:

$$\phi = 1 + \frac{0.377v}{\sqrt{l\alpha}} \cdot \sqrt{1 + \frac{2Q}{P}}$$

where v = the speed in km per hour (always greater than 60)

l = the distance between supports in metres

$\alpha = \dfrac{l}{fs}$

fs = static deflection in metres due to dead weight

Q = moving loads on the bridge deck in tonnes

P = dead weight of the bridge in tonnes

Loadings of Italy (Figs. 3.22, 3.23 and 3.24)

Design Loading

For the design of category 1 bridges, any one of the following three types of loads, flanked by one or more trains of 12 t trucks, producing the worst effect is to be taken:

The width of the three types of loadings is 3.5 m. An equivalent ud load having different values for bending moment and shear is also specified in tabular form.

Impact Allowance

For spans up to 100 m, the impact factor ϕ is given by the formula,

$$\phi = 1 + \frac{(100 - L)^2}{100(250 - L)}$$

where L is the span of the bridge in metres.

For spans exceeding 100 m, ϕ is assumed to be unity.

Loadings of Netherlands (Fig. 3.25)

Class 60 Loading

This is the highest class of loading and it consists of a 600 kN vehicle on three axles of 200 kN each plus a ud load as shown in Fig. 3.25. The standard design lane width for this loading is 3.0 m.

Impact Allowance

The magnitude of the impact coefficient, S, for bridges carrying normal traffic is given by the formula,

$$S = 1 + \frac{40}{100 + L}$$

where L is the span in metres.

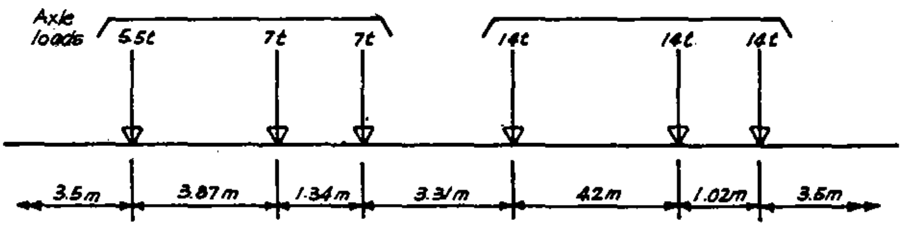

Fig. 3.22 *Continuous train of military load of 61.5 t*

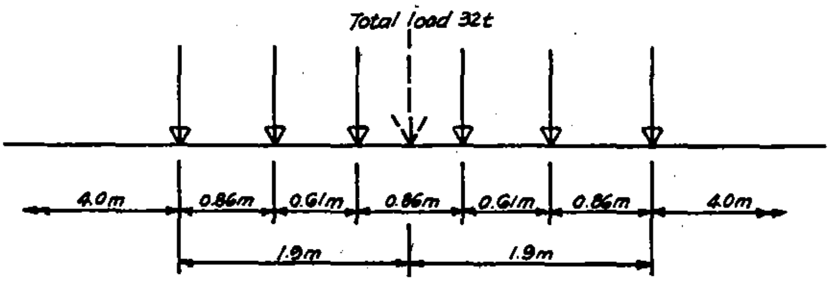

Fig. 3.23 *Continuous train of military load of 32 t*

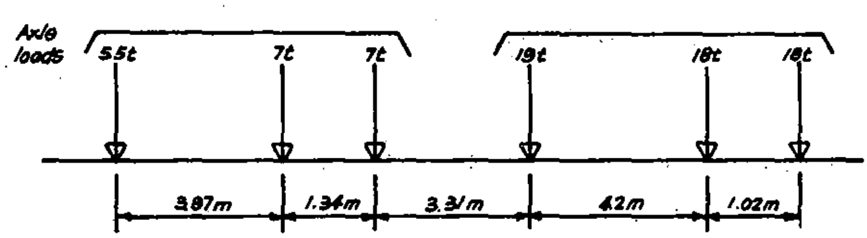

Fig. 3.24 *Single military load of 74.5 t*

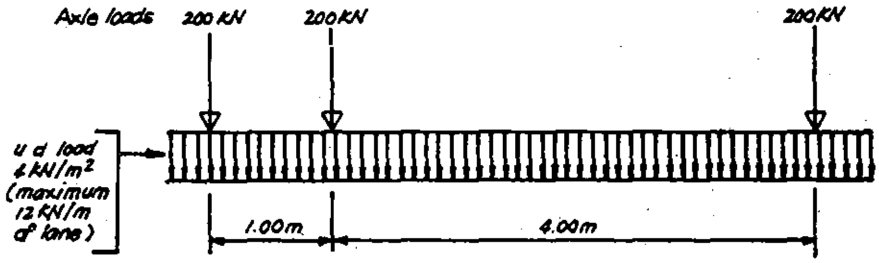

Fig. 3.25 *Class 60 loading*

Loadings of Norway (Fig. 3.26)

Lane Loading

The equivalent lane loading per lane for Class 1 bridges consists of a knife-edge load, A, and a ud load, p, as shown in Fig. 3.26.

$$A = 12 + \frac{8x}{L}\,\text{t}$$

$$p = 0.5 + \frac{35}{L+5}\ \text{t/m of lane}$$

L = Actual loaded length of lane in metres

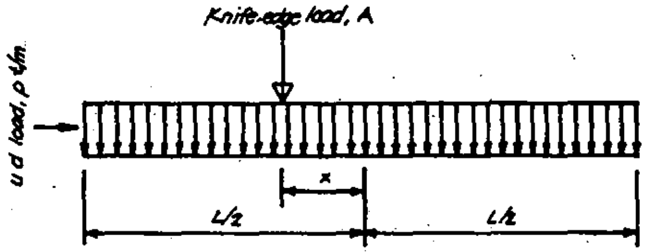

Fig. 3.26 *Equivalent lane loading per lane*

The above lane loadings are normally considered over lane widths from 3.0 to 3.75 m. For two lane bridges, the full equivalent loading is assumed in both the lanes. Besides the lane loading, the structure is designed for a local loading of two axles, each of 13 t.

Impact Allowance

It is assumed that 38.5% impact is to be added to the heaviest axle and it is unnecessary to add any impact to the remaining axles. The values of the knife-edge load, A and ud load, p, are inclusive of impact calculated on this basis. An impact of 38.5% is to be added to the axle loads.

Loadings of the USSR (Figs. 3.27 and 3.28)

Wheel Loadings

There are three types of wheeled loadings (Fig. 3.27).

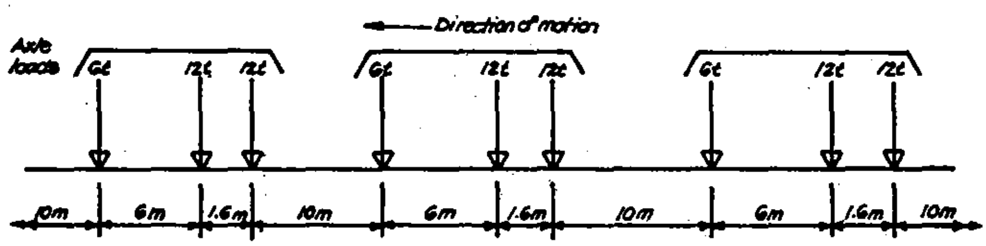

Fig. 3.27(a) *N-30 loading*

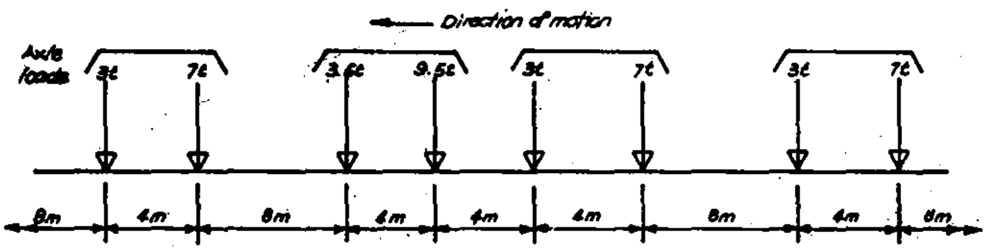

Fig. 3.27(b) *N-10 loading*

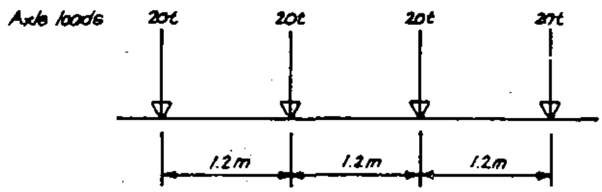

Fig. 3.27(c) *NK-80 loading*

Tracked Loading

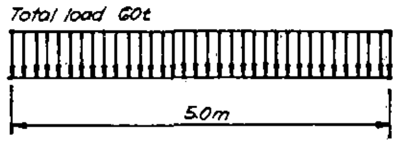

Fig. 3.28 *NG-60 caterpillar loading*

Saudi-Arabian Highway Bridge Loading (Fig. 3.29)

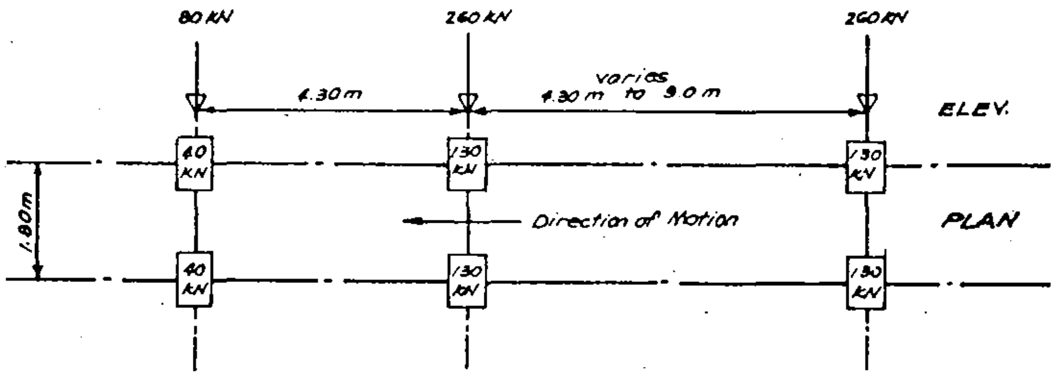

Fig. 3.29(a) *600 kN Truck (each lane can be loaded by a truck but only one truck/lane longitudinally)*

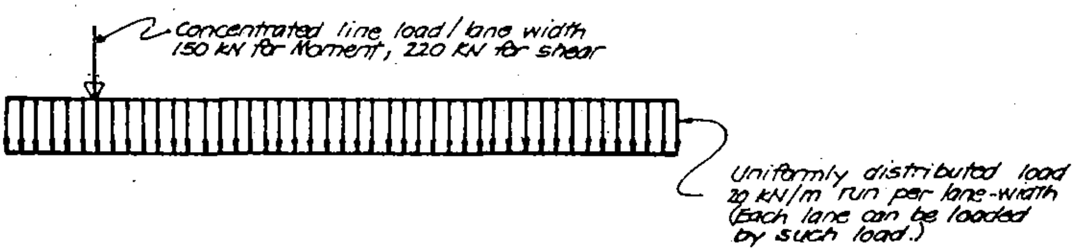

Fig. 3.29(b) *Uniformly distributed lane load plus one line load*

NOTE

(i) Design Load either (a) or (b), whichever produces the maximum effect.

(ii) Impact allowance as per AASHTO specifications.

(iii) For more than two lanes loaded, reduction factor on live load effect as per AASHTO specifications.

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CHAPTER 4

Structural Concrete

4.1 CONCRETE FOR CONSTRUCTION

This topic has been dealt with in detail in the author's book *Concrete for Construction—Facts and Practice*. Therefore, only a list of the chapter contents of this book is given here and reference may be made to the book itself. (The latter part of the present chapter contains some rough-and-ready information.)

The Concrete Mix

• Historical note • Specifications for concrete • Coarse aggregate size • Air entrainment in the concrete mix • Structural requirements of concrete • Portland cements (details, types, properties, manufacture) • Aggregates (coarse, fine, types, choice, tests, grading, sieve analysis, practical gradings) • Water • Admixtures, as an aid to making concrete • Selecting mix proportions • Batching, mixing, transporting, placing and curing of concrete including vibrating, and finishing the concrete • Control of surface evaporation and its importance • Construction, expansion and cold joints • Depositing concrete under water • Inspecting, sampling and testing for concrete • Curing and protection of concrete.

Strength and Properties of Concrete

• General considerations • Effect of mix proportions, water-cement ratio and curing • Effect of cement content and type • Effect of aggregate types and characteristics • Effect of admixtures • Bleeding and 'setting shrinkage' • Cement hydration • Effect of continuous moist storage • Effect of curing method • Effect of thermal changes • Effect of age at test • Relations between various types of concrete strength • Effect of materials on concrete properties • Effect of construction practices on concrete properties • Effect of sampling and testing procedure on observed concrete properties • Effect of type of compression specimen: cylinder or cube? • Effect of form and size of specimen • Effect of moisture content in test specimen at the time of testing • Effect of end bearing conditions of test specimens on observed concrete strength • Effect of lateral restraint on crushing strength • Effect of temperature at time of test • Effect of rate of loading while testing • Predicting 28-day strength of concrete • Causes of low rate of strength-gain • How to accelerate strength-gain of concrete • Plastic shrinkage • Drying shrinkage (details, effects) • Carbonation shrinkage • Creep (details, effects, etc.) • Modulus of elasticity of concrete (general, its determination, effects of variables, relationship with compressive strength) • Poisson's ratio of concrete • Cracking of concrete • Fatigue strength of concrete • Toughness (shock resistance) of concrete • Non-destructive tests on concrete ~

> Schmidt hammer (indentation rebound) test; sonic tests (for vibration characteristics for dynamic modulus of elasticity); pulse-transmission tests at sonic and ultrasonic frequencies; radioactive tests (for density and presence of reinforcement, etc.); penetration probe tests (for compressive strength)

• Tests on composition of hardened concrete (determination of cement content; determination of original water-cement ratio) • Non-destructive methods of detecting defects in concrete structures • Rebound and penetration methods; stress-wave method; • Magnetic methods (for locating reinforcements); Electrical methods (for resistivity of concrete and corrosion of steel); Chemical methods (for depth of carbonation and amount of chloride ion); Infrared thermography (to detect delaminations/discontinuities to identify deteriorated areas); Radiographic technique with gamma rays • Thermal properties (and heat resistance) of concrete.

Deterioration and Durability of Concrete

• Effect of weathering on concrete • Effect of entrained air on freezing and thawing characteristic of concrete • Effect of de-icing agents on concrete • Sulphate attack and effect of sewage on concrete (and protective measures) • Effect of acids on concrete • Effect of leaching (by water and particularly snow-water) on concrete (by the dissolution of soluble calcium hydroxide) • Effect of sulphates on concrete • Effect of sea-water on concrete • Deterioration of concrete through rusting of steel • Deterioration of concrete from the actions of aluminium, copper, lead and zinc with cement • Corrosion of steel in reinforced concrete structures in marine environment and coastal regions • Effect of carbonation of concrete • Effect of free calcium oxide and magnesium oxide on concrete • Effect of reactive aggregates (alkali-aggregate reaction) • Efflorescence in concrete • Rocks and minerals which may be deleterious in concrete aggregates • Unduly large deflections (due to restrained shrinkage) in reinforced concrete • Surface cracking and crazing • Crumbling of concretes and plasters • Abrasion resistance (wear) of concrete • Effects of various substances on concrete and recommended protective treatments and precautions ~

Magnesium fluosilicate or zinc fluosilicate; Acids; Alkalies; Salts; Petroleum oils; Coal-tar distillates; Vegetable oils; Fats and fatty acids: Sodium silicate; Drying oils; Cumar; Varnishes and paints; Bituminous or Coal-tar paints, tar and pitches; Bituminous enamel; Bituminous mastic; Vitrified brick or tile; Glass, Lead; Synthetic resin, rubber and synthetic rubber.

Concrete in Desert Regions with Hot-Dry Humid Climates and Concrete in Sea-Water Surroundings

• Introduction • Causes of rapid deteriorative damage affecting concrete structures in hot climates • Influence of hot-climate on making concrete ~

Influence of temperature on destructive processes that affect hardened concrete; Influence of moisture on destructive processes that affect hardened concrete; Effect of surrounding water; Effect of biological attack; Effect of erosion; Effect of sea water and marine environment; Conclusion

- Influence of cement choice on durability ~
  Types of cement used; Durability problems in relation to choice of cement; Corrosion of reinforcement due to chloride penetration; Sulphate attack; Alkali-aggregate reactivity; Thermal deformation; Storage of cement; Conclusion
- Suitability of aggregates ~
  Types of rock for aggregates; Alkali-silica reaction; Alkali-carbonate reaction; Contamination by salts; Contamination by clay, mica and dust; Grading and particle shape of aggregates; Other durability aspects; Conclusions
- Admixtures, as an aid to making concrete ~
  Principal types, their characteristics; Experiences (in various middle eastern countries on different types of projects); Concluding remarks
- Effect of climate and working conditions (and various ways in which steel can pick up corrosion; concrete production vis-a-vis temperatures of atmosphere, materials; and precautions)
- Curing (special aspects)
- Recommendations for: Type of cement to be used (and the applicable requirements); (requirements of) coarse and fine aggregates; Water for the mix and for curing; Limitations on water-cement ratio, temperature of concrete mix, chloride-content, sulphate-content and alkali-content in concrete; Admixtures (to be used); Ensuring low permeability of concrete; Grading, measuring (batching), mixing, transporting, placing, compacting, finishing and curing of concrete; Concrete cover to reinforcement bars; Criterion for acceptability of test results

Design of a Concrete Mix and Statistical Control of Concrete Quality

• Introduction • Average and minimum strengths • Degree of workability of concrete • Grading of aggregates • Method of combining aggregates • Design of a concrete mix • Examples of high-strength concrete mix design • Statistical control of concrete quality ~

The normal curve (standard deviation, coefficient of variation, average strength, minimum strength); Standard deviation, for different degrees of quality control; Computation of the correct average or mean strength from 'works test results', and their standard deviation; Saving in cement through quality control; Minimum concrete strength; Factors effecting standard deviation and concrete strength

Pumped Concrete and Mortar, 'Pneumatically Sprayed' Concrete or Mortar, 'Non-Shrink' Concrete or Mortar

• Pumped concrete and pumped mortar versus shotcrete and gunite • Pumped concrete and mortar (equipment, pipes, mixes, pumping) • 'Pneumatically applied' concrete and mortar • Non-shrink or expansive concrete and mortar (prepacked concrete, expansive materials).

Control of General Cracking in Concrete by Controlling the Mix and the Making of Concrete

• Cure against cracking of concrete—not by calculations alone (various considerations from the standpoints of adequacy of materials—quality and grading; correctness of construction practices with regard to making, placing and curing the concrete, etc.).

Cracks in Concrete Due to Other than Structural Loading (Types, Causes, Prevention and Remedies)

• Introduction (intrinsic cracking, structural cracks) • Intrinsic cracks • Factors affecting cracking (water, cement, aggregates, admixture, bleeding, placing, curing, temperature, exposure, restraint, etc.) • Types of intrinsic cracks (and their times of appearance) • Frequency of occurrence of various intrinsic cracks • Causes and remedies of plastic cracks • Early thermal contraction cracks • Long-

term drying shrinkage cracks • Crazing type of intrinsic cracking • Other types of intrinsic cracks • Calcium chloride and corrosion of reinforcement; Sulphate-attack cracks; Alkali-aggregate reaction cracks
• Repair Techniques of cracks • Classification of cracks; Materials for repairs (rigid fillers for dormant cracks, and flexible-fillers for live cracks); Repairs to dormant cracks (fine dormant cracks, wide dormant cracks, dormant fractures, dormant multiple cracks); Repairs to live cracks (mastics, thermoplastics, elastomers, surface sealing, membrane sealing)
• References (from ACI and ASTM).

Crack and Wound Repair in Concrete by Epoxy-adhesive Injection and Mortars

• Introduction (and Caution) • Development • Structural applications • Product range (some epoxy preparations in INSTRUCTION SHEET format) • Supplementary data on long-term behaviour • Adhesion of new concrete to old (EMPA-test) • Heat resistance • Repair by shotcrete and gunite.

Repair and Strengthening of Concrete-structures by Externally Bonded Steel Plates

• Introduction • Information from some tests conducted at EMPA • Some interesting examples of repair strengthening works actually carried out (building floor slabs, reinforced concrete frames, bridges) • Some comments on execution of bonding work (steel, concrete, pretreatment, bonding procedures, adhesive, maturity of bond, testing of adhesive, work to be done after bonding) • Butt joints in tension-carrying bonding plates • Recommendations regarding: bonding adhesive, anti-corrosive primer, anti-weathering paint • Some considerations affecting strength of repair (effect of surface conditions on bond, effect of adhesive thickness) • Background information on the adhesives • Some cautions • References.

4.2 SOME ROUGH-AND-READY INFORMATION

Properties of Concrete

Compressive Strength

The compressive cube strength f_{cu} of concretes made with the same cement and cured and tested under the same conditions have been shown by Feret to comply approximately with the expression,

$$f_{cu} = \alpha_1 \left(\frac{B_c}{B_c + B_w + B_v} \right)^2$$

in which B_c, B_w and B_v are respectively the net volumes of cement, water and voids in a unit volume of mixed concrete, i.e., if B_a is the net volume of aggregate in a unit volume of mixed concrete $B_c + B_a + B_w + B_v = 1$. The coefficient α_1 is a numerical factor determined from tests and depends on the nature of the materials.

Tensile Strength

The ratio of the direct tensile strength f_{cut} to the compressive strength f_{cu} varies from 0.05 to more than 0.1, the relation being approximately of the form $f_{cut} = \alpha_2 (f_{cu})^{\beta}$, where β is between 0.5 and 1.0. A formula derived by Feret is $f_{cut} = \alpha_3 \sqrt{f_{cu}} - \alpha_4$ (N/mm²). Coefficients α_3 and α_4 are obtained by testing and depend

Table 4.1 *Some Mix Proportions*

Prescribed mixes: proportions per 50 kg bag of cement

Concrete grade*	Nom. Max. agg. size (mm)	40		20		14		10	
	Workability	medium	high	medium	high	medium	high	medium	high
20	Total aggregate (kg)	305	270	280	250	255	220	240	200
	Sand : Zone 1 (%)	35	40	40	45	45	50	50	55
	Zone 2 (%)	30	35	35	40	40	45	45	50
	Zone 3 (%)	30	30	30	35	35	40	40	45
	Vol. of finished concrete (m³)	0.165	0.155	0.156	0.143	0.146	0.130	0.137	0.121
25	Total aggregate (kg)	265	240	240	215	220	195	210	175
	Sand : Zone 1 (%)	35	40	40	45	45	50	50	55
	Zone 2 (%)	30	35	35	40	40	45	45	50
	Zone 3 (%)	30	30	30	35	35	40	40	45
	Vol. of finished concrete (m³)	0.147	0.137	0.137	0.127	0.130	0.118	0.124	0.110
30	Total aggregate (kg)	235	215	210	190	195	170	180	150
	Sand : Zone 1 (%)	35	40	40	45	45	50	50	55
	Zone 2 (%)	30	35	35	40	40	45	45	50
	Zone 3 (%)	30	30	30	35	35	40	40	45
	Vol. of finished concrete (m³)	0.134	0.127	0.124	0.115	0.115	0.106	0.109	0.097

Percentages are by weight of fine aggregate in weight of total dry aggregates. Volumes of finished concrete are approximate only.
* Grade represents standard cube crushing strength f_{cu} (N/mm²) at 28 days.

on the nature of the cement, the type, grading and maximum size of aggregate, the amount of water, the conditions of curing and the method of testing employed.

The British draft Unified Code gives values of indirect tensile strength at 28 days for concretes having various crushing strengths. These correspond approximately to the expression:

$$\text{Indirect tensile strength} = \frac{2}{3} + \frac{1}{15}f_{cu} - \frac{1}{2600}f_{cu}^2 \text{ (N/mm}^2\text{)}$$

A very simple approximate relationship is,

$$\text{Indirect tensile strength} = \frac{1}{2}\sqrt{f_{cu}} \text{ (N/mm}^2\text{)}$$

Note: f_{cu} in N/mm^2.

Flexural Strength (Modulus of Rupture)

For the values obtained for various concretes to be comparable, the test pieces must be of standard dimensions, say 100 × 100 × 400 mm or 4 × 4 × 16 in. If such a specimen is supported over a span of 300 mm or 12 in with a centrally-applied load, the modulus of rupture $f_{cur} = 0.00425$ FN/mm^2 or 0.28125 Flb/in^2, where F is the load in kg or lb that causes the test piece to break. These expressions comply with CP 114 (BS). Factors that contribute to a high compressive strength f_{cu} also lead to an increase in the modulus of rupture. The relation of f_{cur} to f_{cu} is given by Feret as approximately.

$$f_{cur} = \alpha_5\sqrt{f_{cu}} - \alpha_6 \text{ (N/mm}^2\text{)}$$

where α_5 and α_6 are parameters affected by the same conditions that affect α_3 and α_4 for direct tensile strength.

Data for estimating the flexural strength f_{cuf} at 28 days from a specified crushing strength given in the British draft Unified Code correspond closely to the relationship.

$$f_{cuf} = \frac{1}{2} + \frac{1}{10}f_{cu} - \frac{1}{2000}f_{cu}^2 \text{(N/mm}^2\text{)}$$

The flexural strength of a concrete is about 1 1/2 times the cylinder splitting strength.

Modulus of Elasticity

The modulus of elasticity of concrete E_c increases with increases in cement content, age, repetition of stress, and various other factors, actual values ranging between 21 and 28 kN/mm^2

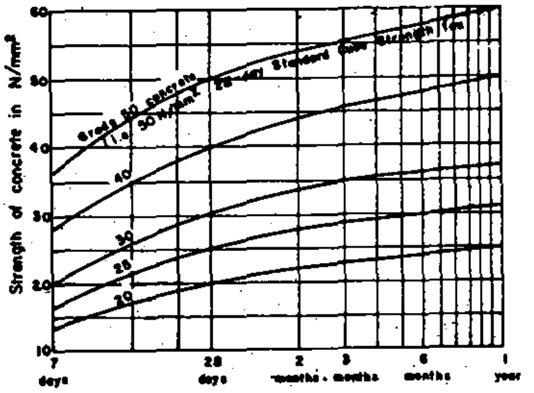


Fig. 4.1 *Concrete properties per CP 110. Increase of strength with time*

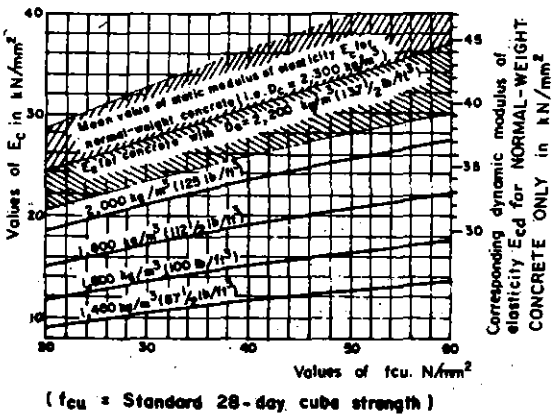


Fig. 4.2 *Variation of modulus of elasticity with strength and density of concrete*

Table 4.2 *Concrete: Properties and Stresses: Metric Values per CP 114 (BS)*

Mix	Type of cement	Nominal proportions or grade	Quantities of aggregates* per 50 kg of cement: Fine (dry)	Quantities of aggregates* per 50 kg of cement: Coarse (19 mm)	Strength N/mm²: Crushing: Age in days	Strength N/mm²: Crushing: Works cubes f_{cu}‡	Strength N/mm²: Modulus of rupture	Permissible stress (N/mm²): Compression: Bending f_{cr}	Permissible stress (N/mm²): Compression: Direct f_{cr}	Permissible stress (N/mm²): Shear v_d	Bond: Anchorage: Plain bars	Bond: Anchorage: High-bond bars	Bond: Local: Plain bars	Bond: Local: High-bond bars
		1	2	3	4	5	6	7	8	9	10	11	12	13	14
Nominal	Portland per BS 12 or BS 146	1 : 2 : 4	cu.m 0.07	cu.m 0.14	3 7 28	— 14 21	1.7 2.4 —	7	5.3	0.7	0.83	1.16	1.25	1.75
Nominal	Portland per BS 12 or BS 146	1 : 1½ : 3	0.05	0.10	3 7 28	— 17 25.5	1.9 2.75	8.5	6.5	0.8	0.93	1.3	1.4	1.96
Nominal	High** alumina	1 : 1 : 2	0.035	0.07	3 7 28	— 14 30	2.05 3.10 —	10	7.6	0.9	1.0	1.4	1.5	2.1
Nominal	High** alumina	1 : 2 : 4	0.07	0.14	1 3 7	40 — —	— 3.10 3.45	10	7.6	0.9	1.0	1.4	1.5	2.1
Standard†	Portland per BS 12 or BS 146	Grade A; Work-ability Low / Med. / High; Slump mm 12–25 / 25–50 / 50–125	kg 90 90 90	kg 190 155 135	28	21	—	7.7	5.8	0.7	0.83	1.16	1.25	1.75
Standard†	Portland per BS 12 or BS 146	Grade B; Low / Med. / High; 12–25 / 25–50 / 50–125	80 80 80	165 135 110	28	25.5	—	9.3	7.0	0.8	0.93	1.3	1.4	1.96
Standard†	Portland per BS 12 or BS 146	Grade C; Low / Med. / High; 12–25 / 25–50 / 50–125	65 65 65	145 110 90	28	30	—	11.0	8.2	0.9	1.0	1.4	1.5	2.1
Designed	Portland per BS 12 or BS 146	A B C	Proportions determined by trial mixes (or otherwise). Cement content ≮ 240 ≯ 540 kg/m³ of compacted concrete		As for standard mixes									
Designed	Portland per BS 12 or BS 146	D E			28 28	40 50	— —	14.7 15.3	11.0 13.7	0.9 0.9	1.0 1.0	1.4 1.4	1.5 1.5	2.1 2.1
Designed	Portland per BS 12 or BS 146	Inter-mediate			28	f_{cu}	—	$\frac{f_{cu}}{2.73}$	$\frac{f_{cu}}{3.65}$	By linear proportions with respect to f_{cu}				
Designed	High** alumina	Ditto But cement content ≮ 270 ≯ 420 kg/m³			1	40 50	— —	14.7 18.3	11.0 13.7	0.9 0.9	1.0 1.0	1.4 1.4	1.5 1.5	2.1 2.1

Modifications to tabulated stresses

Narrow beams	Age of loading	Months					Wind:
$\frac{\text{span}}{\text{breadth}} : 30 < \frac{l}{b} \not> 60$ Reduced bending stress $= (1\frac{3}{4} - \frac{l}{40b}) f_{cr}$	(Portland-cement concrete only)	1	2	3	6	12	If bending moments and forces include effects of wind increase stresses by 25% (Normal stresses apply for wind effects excluded.)
	Increase compressive@ stresses f_{cr} and f_{ce} by	0	10	16	20	24	
		per cent					

* Aggregates conforming to BS 882 or BS 1047.
** Approval for use of high-alumina cement in structural concrete has at present been withdrawn from CP 114.
† Maximum size of aggregates 19 mm and standard deviation 7 N/mm²
‡ f_{cu} at 28 days.
@ Not true under certain weather and exposure conditions.

Table 4.3 *Weights of Concrete*

				kN/m³	lb/ft³
Ordinary concrete (dense aggregates)	Non-reinforced plain or mass concrete	Nominal weight		22.6	144
		Aggregate: limestone		21.2-23.6	135-150
		gravel		22.0-23.6	140-150
		broken brick		19.6 (av.)	125 (av.)
		other crushed stone		22.8-24.4	145-155
	Reinforced concrete	Nominal weight		23.6	150
		Reinforcement: 1%		22.6-24.2	144-154
		2%		23.1-24.7	147-157
		4%		24.0-25.6	153-163
		Solid slabs	Thickness	kN/m²	lb/ft²
		(floors, walls	75 mm or 3 in	1.80	37.5
		etc.)	100 mm or 4 in	2.40	50
			150 mm or 6 in	3.6	75
			250 mm or 10 in	6.00	125
			300 mm or 12 in	7.20	150
		Ribbed slabs	125 mm or 5 in	2.00	42
			150 mm or 6 in	2.15	45
			225 mm or 9 in	2.75	57
			300 mm or 12 in	3.35	70
Lightweight concrete		Compressive strength			
	Aggregate or type	N/mm²	lb/in²	kN/m³	lb/ft³
	Clinker (1 : 8)	2.1-6.2	300-900	10.2-14.9	65-95
	Pumice (1 : 6 semi-dry)	1.4-3.8	200-550	7.1-11.0	45-70
	Formed blast-furnace slag	1.4-5.5	200-800	9.4-14.9	60-95
	ditto structural	13.8-34.5	2000-5000	16.5-20.4	105-130
	Expanded clay or shale	5.6-8.4	800-1200	9.4-11.8	60-75
	ditto structural	13.8-34.5	2000-5000	13.4-18.1	85-115
	Vermiculite (expanded mica)	0.5-3.5	70-500	3.9-11.0	25-70
	Pulverized fuel ash (sintered)	2.8-6.9	400-1000	11.0-12.6	70-80
	ditto structural	13.8-34.5	2000-5000	13.4-17.3	85-110
	No-fines (gravel)	—	—	15.7-18.9	100-120
	Cellular (aerated or gas concrete)	1.4	200	3.9 (min.)	25 (min.)
	ditto structural	10.3-15.5	1500-2250	14.1-15.7	90-100
Special concretes, etc.	Heavy concrete	Aggregates: barytes, magnatite,		31.5 (min.)	200 (min.)
		steel shot punchings		51.8	330
	Lean mixes	Dry-lean (gravel aggregate)		22.0	140
		Soil-cement (normal mix)		15.7	100
	Finishes, etc.	Rendering, screed, etc. } Granolithic, terrazzo		N/m² per mm thick	lb/ft² per in thick
				18.9 to 23.6	10 to 12.5
		Glass-block (hollow) concrete		17.0 (approx.)	9 (approx.)
	Prestressed concrete	Weights as for reinforced concrete (upper limits)			
	Air-entrained concrete	Weights as for plain or reinforced concrete			
Construction with concrete products	Concrete block and brick walls	Blockwork: 200 mm or 8 in thick		kN/m²	lb/ft²
		Stone aggregates: solid		4.31	90
		hollow		2.87	60
		Lightweight aggregates: solid		2.63	55
		hollow		2.15	45
	Weights of walls of other thickness pro rata	Cellular (aerated gas)		1.15-1.53	24-32
		Brickwork: 120 mm or 4 1/2 in (nominal)		2.6	54
	Other products	Paving slabs (flags) 50 mm or 2 in thick		1.15	24
		Roofing tiles: plain		0.6-0.9	12.5-19
		interlocking		0.6	12.5

To convert values in kN to values in kg multiply by 102.

CHAPTER 5

Details of Structural Reinforcement Bars and Mesh Fabrics

(i) Practical details of reinforcement for design and construction purposes, as per ACI practice, are given at the end of App. 6.

(ii) Properties of and normal stresses allowed in reinforcement bars as per BS are presented in Tables 5.1 and 5.2.

(iii) Design aids (e.g. section areas, perimeters, weights, section areas for various bar arrangements, etc.) pertinent to reinforcement bars and mesh-fabrics are presented in Tables 5.3 to 5.9.

Table 5.1 *Reinforcement: Properties and Stresses*

Type of reinforcement				Minimum tensile properties per specified BS		Normal permissible stresses per CP 114 (BS)			
Type of bar			Size	Yield or characteristic strength f_y		Tensile		Compressive	
				N/mm^2	lb/in^2	N/mm^2	lb/in^2	N/mm^2	lb/in^2
Hot rolled bars	Plain round mild-steel bars	Metric BS 4449	mm						
			$\ngtr$ 40	250	—	140	—	125	—
			> 40	250	—	125	—	110	—
		Imperial BS 785	in						
			$\ngtr 1\frac{1}{2}$	—	36,000	—	20,000	—	18,000
			$> 1\frac{1}{2}$	—	33,000	—	18,000	—	16,000
	Deformed high-yield bars	Metric BS 4449	mm						
			$\ngtr$ 16	460	—	230*	—	175*	—
			16 to 20	425	—	230*	—	—	—
			> 20	425	—	210*	—	175*	—
		Imperial BS 785	in						
			$\ngtr {}^7/_8$	—	60,000	—	33,000*	—	25,000*
			$> {}^7/_8$	—	60,000	—	30,000*	—	25,000*
Cold worked bars (including twisted square bars)		Metric BS 4461	mm						
			$\ngtr$	460	—	230*	—	175*	—
			> 16 to 20	425	—	230*	—	175*	—
			> 20	425	—	210*	—	175*	—
		Imperial BS 1144	in						
			$\ngtr {}^5/_8$	—	66,000*	—	33,000*	—	25,000*
			${}^5/_8$ to ${}^7/_8$	—	60,000	—	33,000*	—	25,000*
			$> {}^7/_8$	—	60,000	—	30,000*	—	25,000*
Hard drawn mild-steel wire		Metric BS 4482	$\ngtr$ 12 mm	485	—	230	—	—	—
		Imperial BS 785	all sizes	—	70,000	—	—	—	—

* $\ngtr 0.55 f_y$

Table 5.2 *Reinforcement Bond BS CP 110 Requirements (Ultimate Values)*

Anchorage bond: minimum lengths in millimetres for normal-weight concrete

Dia of bar mm	Minimum* 12ϕ mm	Minimum Lap† Tension mm	Minimum Lap† Comp. mm	Type of hook		f_{cu} = 20 N/mm²			f_{cu} = 25 N/mm²			f_{cu} = 30 N/mm²			f_{cu} = 40 N/mm²		
						$f_y = 250$	f_y = 425/460 Type 1	f_y = 425/460 Type 2	$f_y = 250$	f_y = 425/460 Type 1	f_y = 425/460 Type 2	$f_y = 250$	f_y = 425/460 Type 1	f_y = 425/460 Type 2	$f_y = 250$	f_y = 425/460 Type 1	f_y = 425/460 Type 2
6	75	300	270	Tension	0	275	355	275	235	320	245	220	275	210	175	235	180
					90°	225	285	200	185	245	175	170	205	140	125	160	110
					180°	180	210	130	140	175	100	125	130	70	80	90	35
				Compression		200	240	185	175	210	165	155	190	145	130	160	125
8	100	350	310	Tension	0	365	475	365	315	425	325	290	365	280	230	310	240
					90°	300	375	270	250	330	230	230	270	185	165	215	140
					180°	235	280	170	185	230	135	165	175	90	105	120	45
				Compression		265	320	245	235	280	215	210	250	190	175	210	165
10	120	400	350	Tension	0	455	590	455	390	530	405	365	455	350	290	385	300
					90°	375	470	335	310	410	285	285	335	230	210	265	180
					180°	295	350	215	230	290	165	205	215	110	130	145	60
				Compression		330	400	310	290	350	270	260	310	240	215	265	205
12	145	450	390	Tension	0	545	710	545	470	635	490	435	550	420	345	465	360
					90°	450	565	400	370	490	345	340	405	280	250	320	215
					180°	355	420	255	275	345	200	245	260	135	155	175	70
				Compression		395	480	370	350	420	325	310	375	285	260	315	245
16	195	550	470	Tension	0	725	945	725	625	845	650	580	730	560	460	620	475
					90°	600	750	535	495	655	460	455	540	370	330	425	285
					180°	470	560	340	370	460	265	325	345	180	205	235	90
				Compression		525	635	490	465	560	430	415	495	380	345	420	325
20	240	650	550	Tension	0	910	1.090	840	780	975	750	725	840	650	575	715	550
					90°	750	850	600	620	735	510	565	600	410	415	475	310
					180°	590	610	360	460	495	270	405	360	170	255	235	70
				Compression		655	745	575	580	650	500	520	580	445	430	490	375
25	300	775	650	Tension	0	1.135	1.360	1.050	975	1.220	940	910	1.050	810	720	890	685
					90°	935	1.060	750	775	920	640	710	750	510	520	590	385
					180°	735	760	450	575	620	340	510	450	210	320	290	85
				Compression		820	930	715	725	815	625	645	725	560	535	610	470
32	385	950	790	Tension	0	1.450	1.740	1.340	1.245	1.560	1.200	1.160	1.345	1.035	920	1.140	875
					90°	1.195	1.360	955	990	1.175	815	905	960	650	660	755	495
					180°	940	975	570	735	790	430	650	580	270	405	370	110
				Compression		1.050	1.190	915	925	1.040	800	830	925	715	685	780	600

* Minimum stopping-off length = 12ϕ or d whichever is greater.

† Minimum lap in tension: The greater of $25\phi + 150$ mm or anchorage length of smaller bar (mild steel) or $1\frac{1}{2}$ times anchorage length of smaller bar (high-yield steel).
Minimum lap in compression: The greater of $20\phi + 150$ mm *or* anchorage length of smaller bar.

NOTE

1. $f_y = 250$ indicates mild steel. $f_y = 425/460$ indicates high yield bars.
2. All lengths rounded to 5 mm value above exact figure.
3. Values for hooks correspond to internal radius of 2 ϕ for mild steel bars and 3 ϕ for high-yield steel bars.
4. Bar must extend a minimum distance of 4 ϕ beyond bend.
5. Lengths given correspond to maximum design stresses in steel of $0.87f_y$ in tension and $2{,}000f_y/(2{,}300+f_y)$ in compression. For lower design stresses at point beyond which anchorage is to be provided, determine length required from 'no hook' value on pro rata basis. Then if hook is provided, subtract length equal to difference between appropriate values given in table.

Table 5.3 *Reinforcement: Weights at Specified Spacings and unit Weights*

Weights of metric (millimetre) bars in kilograms per square metre

Size (mm)	Weight per m (kg)	Length per tonne (m)	Spacing of bars in millimetres									
			75	100	125	150	175	200	225	250	275	300
6	0.222	4,505	2.960	2.220	1.776	1.480	1.269	1.110	0.987	0.888	0.807	0.740
8	0.395	2,532	5.267	3.950	3.160	2.633	2.257	1.975	1.756	1.580	1.436	1.317
10	0.616	1,623	8.213	6.160	4.928	4.107	3.520	3.080	2.738	2.464	2.240	2.053
12	0.888	1,126	11.84	8.880	7.104	5.920	5.074	4.440	3.947	3.552	3.229	2.960
16	1.579	633	21.05	15.79	12.63	10.53	9.023	7.895	7.018	6.316	5.742	5.263
20	2.466	406	32.88	24.66	19.73	16.44	14.09	12.33	10.96	9.864	8.967	8.220
25	3.854	259	51.39	38.54	30.83	25.69	22.02	19.27	17.13	15.42	14.01	12.85
32	6.313	158	—	63.13	50.50	42.09	36.07	31.57	28.06	25.25	22.96	21.04
40	9.864	101	—	—	78.91	65.76	56.37	49.32	43.84	39.46	35.87	32.88

Basic weight = 0.00785 kg/mm^2/m
Weight per metre = 0.006165 ϕ^2 (kg)
Weight per mm^2 at spacing s (mm) = $6.165\phi^2/s$ (kg)
ϕ = diameter of bar in millimetres

Weights of imperial (inch) bars in pounds per square foot

Size (in)	Weight per foot (lb)	Length per ton (ft)	Spacing of bars in inches												
			3	3½	4	4½	5	5½	6	7	7½	8	9	10½	12
1/4	0.1669	13,421	0.688	0.572	0.501	0.445	0.401	0.364	0.334	0.286	0.267	0.250	0.223	0.191	0.167
5/16	0.2608	8,590	1.043	0.894	0.782	0.695	0.626	0.569	0.522	0.447	0.417	0.391	0.348	0.298	0.261
3/8	0.3755	5,965	1.502	1.287	1.127	1.001	0.901	0.819	0.751	0.644	0.601	0.563	0.501	0.429	0.376
7/16	0.5111	4,383	2.044	1.752	1.533	1.363	1.227	1.115	1.022	0.876	0.818	0.767	0.681	0.584	0.511
1/2	0.6676	3,355	2.670	2.289	2.003	1.780	1.602	1.457	1.335	1.144	1.068	1.001	0.890	0.763	0.668
5/8	1.0431	2,147	4.712	3.576	3.129	2.782	2.503	2.276	2.086	1.788	1.669	1.565	1.391	1.192	1.043
3/4	1.5021	1,491	6.008	5.150	4.506	4.006	3.605	3.277	3.004	2.575	2.403	2.253	2.003	1.717	1.502
7/8	2.0445	1,096	8.178	7.010	6.133	5.452	4.907	4.461	4.089	3.505	3.271	3.067	2.726	2.337	2.044
1	2.6704	839	10.68	9.155	8.011	7.121	6.409	5.826	5.341	4.578	4.273	4.006	3.560	3.052	2.670
1⅛	3.3797	663	—	11.59	10.14	9.012	8.111	7.374	6.759	5.794	5.407	5.069	4.506	3.862	3.380
1¼	4.1724	537	—	—	12.52	11.13	10.01	9.103	8.345	7.153	6.676	6.259	5.563	4.768	4.172
1½	6.0083	373	—	—	—	16.02	14.42	13.11	12.02	10.30	9.613	9.012	8.011	6.867	6.008

Basic weight = 3.4 lb/in^2/ft
Weight per foot = 2.6704 ϕ^2 (lb)
Weight per ft^2 at spacing s (in) = $32.044\phi^2/s$ (lb)
ϕ = diameter of bar (in inches)

Plain round bars The weights tabulated are basically for plain round bars.

Deformed (high-bond bars) The weights tabulated apply to deformed (high-bond) bars on uniform cross-sectional area if the specified size (effective diameter) of the bar is the diameter of a circle of the same cross-sectional area.

Twisted square bars The weights tabulated apply to small non-chamfered and larger chamfered twisted square bars if the specified size is based on 'round area' but do not apply if based on 'square area'.

Table 5.4 *Reinforcement: Combinations of Metric Bars at Specific Spacings*

Cross sectional area	Bar arrangement
94	6 @ 300
102	6 @ 275
113	6 @ 250
125	6 @ 225
130	6/8 @ 300
141	6 @ 200
142	6/8 @ 275
157	6/8 @ 250
161	6 @ 175
167	8 @ 300
174	6/8 @ 225
178	6/10 @ 300
182	8 @ 275
188	6 @ 150
194	6/10 @ 275
196	6/8 @ 200
201	8 @ 250
213	6/10 @ 250
214	8/10 @ 300
223	8 @ 225
224	6/8 @ 175
226	6 @ 125
234	8/10 @ 275
237	6/10 @ 225
251	8 @ 200
257	8/10 @ 250
261	6/8 @ 150 10 @ 300
267	6/10 @ 200
272	8/12 @ 300
282	6 @ 100
285	10 @ 275
286	8/10 @ 225
287	8 @ 175
297	8/12 @ 275
305	6/10 @ 175
314	6/8 @ 125 10 @ 250
319	10/12 @ 300
322	8/10 @ 200
326	8/12 @ 250
335	8 @ 150
348	10/12 @ 275
349	10 @ 225
356	6/10 @ 150
363	8/12 @ 225
368	8/10 @ 175
376	6 @ 75 12 @ 300
383	10/12 @ 250
392	6/8 @ 100 10 @ 200
402	8 @ 125
408	8/12 @ 200
411	12 @ 275

Cross sectional area	Bar arrangement
425	10/12 @ 225
427	6/10 @ 125
429	8/10 @ 150
448	10 @ 175
452	12 @ 250
466	8/12 @ 175 10/16 @ 300
479	10/12 @ 200
502	8 @ 100 12 @ 225
508	10/16 @ 275
515	8/10 @ 125
523	6/8 @ 75 12/16 @ 300 10 @ 150
534	6/10 @ 100
544	8/12 @ 150
547	10/12 @ 175
559	10/16 @ 250
565	12 @ 200
571	12/16 @ 275
621	10/16 @ 225
628	12/16 @ 250 10 @ 125
638	10/12 @ 150
644	8/10 @ 100
646	12 @ 175
653	8/12 @ 125
670	8 @ 75 16 @ 300
698	12/16 @ 225
699	10/16 @ 200
712	6/10 @ 75 12/20 @ 300
731	16 @ 275
753	12 @ 150
766	10/12 @ 125
776	12/20 @ 275
785	12/16 @ 200 10 @ 100
798	10/16 @ 175
804	16 @ 250
816	8/12 @ 100
854	12/20 @ 250
858	8/10 @ 75
858	16/20 @ 300
893	16 @ 225
897	12/16 @ 175
904	12 @ 125
932	10/16 @ 150
936	16/20 @ 275
949	12/20 @ 225
958	10/12 @ 100
1,005	16 @ 200
1,030	16/20 @ 250

Cross sectional area	Bar arrangement
1,047	12/16 @ 150 10 @ 75 20 @ 300
1,068	12/20 @ 200
1,089	8/12 @ 75
1,118	10/16 @ 125
1,130	12 @ 100
1,142	20 @ 275
1,144	16/20 @ 225
1,148	16 @ 175
1,153	16/25 @ 300
1,220	12/20 @ 175
1,256	12/16 @ 125 20 @ 250
1,258	16/25 @ 275
1,277	10/12 @ 75
1,288	16/20 @ 200
1,340	16 @ 150
1,341	20/25 @ 300
1,383	16/25 @ 250
1,396	20 @ 225
1,398	10/16 @ 100
1,424	12/20 @ 150
1,463	20/25 @ 275
1,472	16/20 @ 175
1,507	12 @ 75
1,537	16/25 @ 225
1,570	12/16 @ 100 20 @ 200
1,608	16 @ 125
1,610	20/25 @ 250
1,636	25 @ 300
1,709	12/20 @ 125
1,717	16/20 @ 150
1,729	16/25 @ 200
1,784	25 @ 275
1,788	20/25 @ 225
1,795	20 @ 175
1,864	10/16 @ 75 20/32 @ 300
1,963	20 @ 250
1,976	16/25 @ 175
2,010	16 @ 100
2,012	20/25 @ 200
2,033	20/32 @ 275
2,060	16/20 @ 125
2,094	12/16 @ 75 20 @ 150
2,136	12/20 @ 100
2,158	25/32 @ 300
2,181	25 @ 225
2,236	20/32 @ 250
2,300	20/25 @ 175
2,306	16/25 @ 150
2,354	25/32 @ 275

Cross sectional area	Bar arrangement
2,454	25 @ 200
2,485	20/32 @ 225
2,513	20 @ 125
2,576	16/20 @ 100
2,590	25/32 @ 250
2,680	16 @ 75 32 @ 300
2,683	20/25 @ 150
2,767	16/25 @ 125
2,796	20/32 @ 200
2,804	25 @ 175
2,848	12/20 @ 75
2,878	25/32 @ 225
2,924	32 @ 275
3,141	20 @ 100
3,195	20/32 @ 175
3,216	32 @ 250
3,220	20/25 @ 125
3,237	25/32 @ 200
3,272	25 @ 150
4,473	16/20 @ 75
3,459	16/25 @ 100
3,574	32 @ 225
3,700	25/32 @ 175
3,728	20/32 @ 150
3,926	25 @ 125
4,021	32 @ 200
4,025	20/25 @ 100
4,188	20 @ 75
4,317	25/32 @ 150
4,473	20/32 @ 125
4,595	32 @ 175
4,612	16/25 @ 75
4,908	25 @ 100
5,180	25/32 @ 125
5,361	32 @ 150
5,366	20/25 @ 75
5,592	20/32 @ 100
6,433	32 @ 125
6,475	25/32 @ 100
6,544	25 @ 75
7,456	20/32 @ 75
8,042	32 @ 100

NOTE Cross-sectional areas of metric bars in mm^2 per m width. 10 at the rate of 75, etc., denotes 10 mm bars at 75 mm centres, etc., 10/16 at the rate of 75, etc., denotes 10 mm and 16 mm bars alternately at 75 mm centres, etc. Only combinations of bars not differing by more than two sizes and spaced at multiples of 25 mm are tabulated. All areas are rounded to value in mm^2 below exact value.

Table 5.5 *Reinforcement: Areas of Combinations of Metric Bars*

Cross sectional area	Bar arrangement
113	1/12
201	1/16
226	2/12
314	1/20 1/12 + 1/16
339	3/12
402	2/16
427	1/12 + 1/20 2/12 + 1/16
452	4/12
490	1/25
515	1/12 + 2/16 1/16 + 1/20
540	2/12 + 1/20 3/12 + 1/16
565	5/12
603	3/16
628	2/12 + 2/16 2/20
653	3/12 + 1/20 4/12 + 1/16
678	6/12
691	1/16 + 1/25
716	1/12 + 3/16 2/16 + 1/20
741	1/12 + 2/20 3/12 + 2/16
766	4/12 + 1/20 5/12 + 1/16
791	7/12
804	4/16 1/32
805	1/20 + 1/25
829	1/16 + 2/20 2/12 + 3/16
854	2/12 + 2/20 4/12 + 2/16
879	5/12 + 1/20
892	2/16 + 1/25
904	8/12
917	1/12 + 4/16 3/16 + 1/20
942	3/12 + 3/16 3/20
967	5/12 + 2/16 3/12 + 2/20
981	2/25
1,005	5/16
1,017	9/12
1,030	2/12 + 4/16 2/16 + 2/20
1,055	1/12 + 3/20 4/12 + 3/16
1,080	4/12 + 2/20
1,094	3/16 + 1/25

Cross sectional area	Bar arrangement
1,118	1/12 + 5/16 4/16 + 1/20 1/20 + 1/32
1,119	2/20 + 1/25
1,130	10/12
1,143	3/12 + 4/16 1/16 + 3/20
1,168	5/12 + 3/16 2/12 + 3/20
1,182	1/16 + 2/25
1,193	5/12 + 2/20
1,206	6/16
1,231	2/12 + 5/16 3/16 + 2/20
1,256	4/12 + 4/16 4/20
1,281	3/12 + 3/20
1,295	4/16 + 1/25 1/20 + 2/25 1/25 + 1/32
1,319	5/16 + 1/20
1,344	3/12 + 5/16 2/16 + 3/20
1,369	1/12 + 4/20 5/12 + 4/16
1,383	2/16 + 2/25
1,394	4/12 + 3/20
1,407	7/16
1,432	4/16 + 2/20 2/20 + 1/32
1,433	3/20 + 1/25
1,457	1/16 + 4/20 4/12 + 5/16
1,472	3/25
1,482	2/12 + 4/20
1,496	5/16 + 1/25
1,507	5/12 + 3/20
1,545	3/16 + 3/20
1,570	5/20 5/12 + 5/16
1,584	3/16 + 2/25
1,595	3/12 + 4/20
1,608	8/16 2/32
1,610	2/20 + 2/25
1,633	5/16 + 2/20
1,658	2/16 + 4/20
1,673	1/16 + 3/25
1,683	1/12 + 5/20
1,709	4/12 + 4/20
1,746	3/20 + 1/32 4/16 + 3/20
1,747	4/20 + 1/25
1,771	1/16 + 5/20
1,785	4/16 + 2/25 2/25 + 1/32
1,795	1/20 + 3/25
1,796	2/12 + 5/20

Cross sectional area	Bar arrangement
1,809	9/16
1,822	5/12 + 4/20
1,859	3/16 + 4/20
1,874	2/16 + 3/25
1,884	6/20
1,910	3/12 + 5/20
1,922	1/20 + 2/32
1,924	3/20 + 2/25
1,947	5/16 + 3/20
1,963	4/25
1,972	2/16 + 5/20
1,987	5/16 + 2/25
2,010	10/16
2,023	4/12 + 5/20
2,060	4/16 + 4/20 4/20 + 1/32
2,061	5/20 + 1/25
2,075	3/16 + 3/25
2,099	1/25 + 2/32
2,100	2/20 + 3/25
2,136	5/12 + 5/20
2,164	1/16 + 4/25
2,173	3/16 + 5/20
2,199	7/20
2,236	2/20 + 2/32
2,238	4/20 + 2/25
2,261	5/16 + 4/20
2,276	3/25 + 1/32 4/16 + 3/25
2,277	1/20 + 4/25
2,365	2/16 + 4/25
2,375	4/16 + 5/20 5/20 + 1/32
2,412	3/32
2,415	3/20 + 3/25
2,454	5/25
2,477	5/16 + 3/25
2,513	8/20
2,550	3/20 + 2/32
2,552	5/20 + 2/25
2,566	3/16 + 4/25
2,576	5/16 + 5/20
2,590	2/25 + 2/32
2,591	2/20 + 4/25
2,655	1/16 + 5/25
2,726	1/20 + 3/32
2,729	4/20 + 3/25
2,767	4/25 + 1/32 4/16 + 4/25
2,768	1/20 + 5/25
2,827	9/20
2,856	2/16 + 5/25
2,865	4/20 + 2/32
2,903	1/25 + 3/32
2,905	3/20 + 4/25
2,945	6/25
2,968	5/16 + 4/25

Cross sectional area	Bar arrangement
3,041	2/20 + 3/32
3,043	5/20 + 3/25
3,057	3/16 + 5/25
3,081	3/25 + 2/32
3,082	2/20 + 5/25
3,141	10/20
3,179	5/20 + 2/32
3,216	4/32
3,220	4/20 + 4/25
3,258	5/25 + 1/32 4/16 + 5/25
3,355	3/20 + 3/32
3,394	2/25 + 3/32
3,436	7/25
3,459	5/16 + 5/25
3,531	1/20 + 4/32
3,534	5/20 + 4/25
3,571	4/25 + 2/32
3,669	4/20 + 3/32
3,707	1/25 + 4/32
3,711	4/20 + 5/25
3,845	2/20 + 4/32
3,885	3/25 + 3/32
3,926	8/25
3,983	5/20 + 3/32
4,021	5/32
4,025	5/20 + 5/25
4,062	5/25 + 2/32
4,159	3/20 + 4/32
4,198	2/25 + 4/32
4,335	1/20 + 5/32
4,376	4/25 + 3/32
4,417	9/25
4,473	4/20 + 4/32
4,512	1/25 + 5/32
4,649	2/20 + 5/32
4,689	3/25 + 4/32
4,787	5/20 + 4/32
4,825	6/32
4,867	5/25 + 3/32
4,908	10/25
4,963	3/20 + 5/32
5,002	2/25 + 5/32
5,180	4/25 + 4/32
5,277	4/20 + 5/32
5,493	3/25 + 5/32
5,592	5/20 + 5/32
5,629	7/32
5,671	5/25 + 4/32
5,984	4/25 + 5/32
6,433	8/32
6,475	5/25 + 5/32
7,238	9/32
8,042	10/32

NOTE Cross-sectional areas of metric bars in mm^2. 4/16 + 3/25, etc. denotes combination of four 16 mm bars plus three 25 mm bars, etc. Only combinations of up to five bars of two diameters differing by not more than two sizes (or ten bars of a single size) are considered. All areas are rounded to value in mm^2 below exact value.

Table 5.6 *Reinforcement: Metric Bar Data*

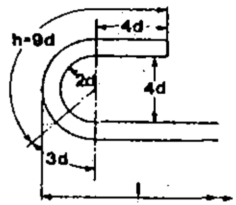
			Bar size in millimetres									
			6	8	10	12	16	20	25	32	40	50
Cross sectional area of bars/m at specific spacings	Bar spacing in millimetres (non-preferred spacings shown in italics)	*75*	*376*	*670*	*1,047*	*1,507*	*2,680*	*4,188*	*6,544*	—	—	—
		80	*353*	*628*	*981*	*1,413*	*2,513*	*3,926*	*6,135*	—	—	—
		90	*314*	*558*	*872*	*1,256*	*2,234*	*3,490*	*5,454*	—	—	—
		100	282	502	785	1,130	2,010	3,141	4,908	8,042	—	—
		110	*257*	*456*	*713*	*1,028*	*1,827*	*2,855*	*4,462*	*7,311*	—	—
		120	*235*	*418*	*654*	*942*	*1,675*	*2,617*	*4,090*	*6,702*	*10,471*	—
		125	226	402	628	904	1,608	2,513	3,926	6,433	10,053	—
		130	*217*	*386*	*604*	*869*	*1,546*	*2,416*	*3,775*	*6,186*	*9,666*	—
		140	*201*	*359*	*560*	*807*	*1,436*	*2,243*	*3,506*	*5,744*	*8,975*	—
		150	188	335	523	753	1,340	2,094	3,272	5,361	8,377	13,090
		160	*176*	*314*	*490*	*706*	*1,256*	*1,963*	*3,067*	*5,026*	*7,853*	*12,272*
		175	161	287	448	646	1,148	1,795	2,804	4,595	7,180	11,220
		180	*157*	*279*	*436*	*628*	*1,117*	*1,745*	*2,727*	*4,468*	*6,981*	*10,908*
		200	141	251	392	565	1,005	1,570	2,454	4,021	6,283	9,817
		220	*128*	*228*	*356*	*514*	*913*	*1,427*	*2,231*	*3,655*	*5,711*	*8,925*
		225	125	223	349	502	893	1,396	2,181	3,574	5,585	8,727
		240	*117*	*209*	*327*	*471*	*837*	*1,308*	*2,045*	*3,351*	*5,235*	*8,181*
		250	113	201	314	452	804	1,256	1,963	3,216	5,026	7,854
		275	102	182	285	411	731	1,142	1,784	2,924	4,569	7,140
		300	94	167	261	376	670	1,047	1,636	2,680	4,188	6,545
	Number of bars	1	28.3	50.3	78.5	113.1	201.1	314.2	490.9	804.2	1,257	1,963
		2	56.5	100.5	157.1	226.2	402.1	628.3	981.7	1,608	2,513	3,927
		3	84.8	150.8	235.6	339.3	603.2	942.5	1,473	2,413	3,770	5,890
		4	113.1	201.1	314.2	452.4	804.2	1,257	1,963	3,217	5,027	7,854
		5	141.4	251.3	392.7	565.5	1,005	1,571	2,454	4,021	6,283	9,817
		6	169.6	301.6	471.2	678.6	1,206	1,885	2,945	4,825	7,540	11,781
		7	197.9	351.9	549.8	791.7	1,407	2,199	3,436	5,630	8,796	13,744
		8	226.2	402.1	628.3	904.8	1,608	2,513	3,927	6,434	10,053	15,708
		9	254.5	452.4	706.9	1,018	1,810	2,827	4,418	7,238	11,310	17,671
		10	282.7	502.7	785.4	1,131	2,011	3,142	4,909	8,042	12,566	19,635
		11	311.0	552.9	863.9	1,244	2,212	3,456	5,400	8,847	13,823	21,598
		12	339.3	603.2	942.5	1,357	2,413	3,770	5,890	9,651	15,080	23,562
		13	367.6	653.5	1,021	1,470	2,614	4,084	6,381	10,455	16,336	25,525
		14	395.8	703.7	1,100	1,583	2,815	4,398	6,872	11,259	17,593	27,489
		15	424.1	754.0	1,178	1,696	3,016	4,712	7,363	12,064	18,850	29,452
		16	452.4	804.2	1,257	1,810	3,217	5,027	7,854	12,868	20,106	31,416
		17	480.7	854.5	1,335	1,923	3,418	5,341	8,345	13,672	21,363	33,379
		18	508.9	904.8	1,414	2,036	3,619	5,655	8,836	14,476	22,619	35,343
		19	537.2	955.0	1,492	2,149	3,820	5,969	9,327	15,281	23,876	37,306
		20	565.5	1,005	1,571	2,262	4,021	6,283	9,817	16,085	25,133	39,270
Perimeters of specific numbers of bars	Number of bars	1	18.8	25.1	31.4	37.6	50.2	62.8	78.5	100.5	125.6	157.1
		2	37.6	50.2	62.8	75.3	100.5	125.6	157.0	201.0	251.3	314.2
		3	56.5	75.3	94.2	113.0	150.7	188.4	235.6	301.5	376.9	471.2
		4	75.3	100.5	125.6	150.7	201.0	251.3	314.1	402.1	502.6	628.3
		5	94.2	125.6	157.0	188.4	251.3	314.1	392.6	502.6	628.3	785.4
		6	113.0	150.7	188.4	226.1	301.5	376.9	471.2	603.1	753.9	942.5
		7	131.9	175.9	219.9	263.8	351.8	439.8	549.7	703.7	879.6	1,100
		8	150.7	201.0	251.3	301.5	402.1	502.6	628.3	804.2	1,005	1,257
		9	169.6	226.1	282.7	339.2	452.3	565.4	706.8	904.7	1,130	1,414
		10	188.4	251.3	314.1	376.9	502.6	628.3	785.3	1,005	1,256	1,571

NOTE Areas are given in square millimetres, perimeters in millimetres

Table 5.7 *BS4466 Preferred shapes*

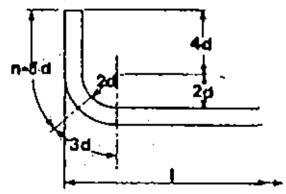
MINIMUM HOOK AND BEND ALLOWANCES FOR MILD STEEL BARS TO BE BS4449*

Semi-circular hooks for use with shape codes 32, 33 and 72 only



h = hook allowance = $9d$ (min.) taken to the nearest 10 mm over, or not less than 100 mm, to be added to dimension L. Hook length (min.) = $h + 3d$.

Bends forming end anchorages for use with shape codes 34, 35 and 42 only



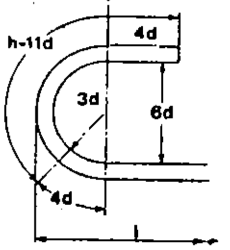
n = bend allowance = $5d$ (min.) taken to the nearest 10 mm over or not less than 100 mm, to be added to dimension L.

Bar size (mm)	d	6	8	10	12	16	20	25	32	40
Hook allowance (mm)	h	100	100	100	110	150	180	230	290	360
Bend allowance (mm)	n	100	100	100	100	100	100	130	160	200

NOTE. For intermediate sizes the dimensions and radii for the next larger size should be used.

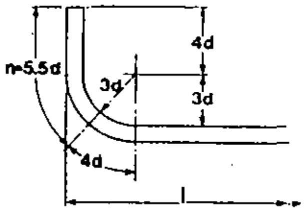
MINIMUM HOOK AND BEND ALLOWANCES FOR HOT ROLLED HIGH YIELD BARS COMPLYING WITH BS4449* AND COLD WORKED HIGH YIELD BARS COMPLYING WITH BS4461†

Semi-circular hooks for use with shape codes 32, 33 and 72 only



h = hook allowance = $11d$ (min.) taken to the nearest 10 mm over, or not less than 100 mm to be added to dimension L. Hook length (min.) = $h + 4d$.

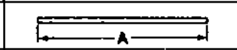
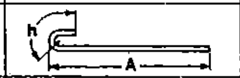
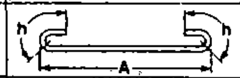
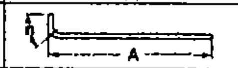
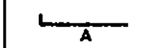
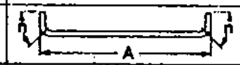
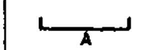
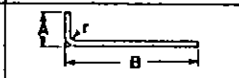
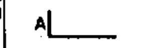
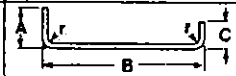
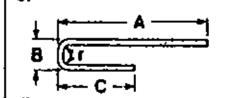
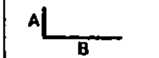
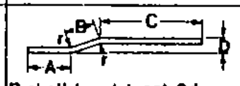
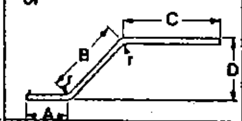
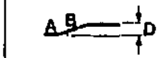
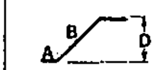
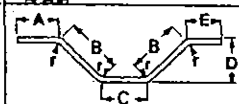
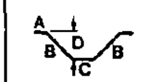
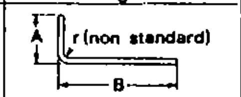
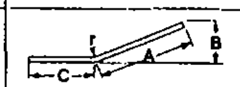
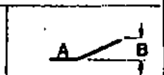
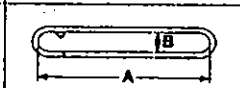
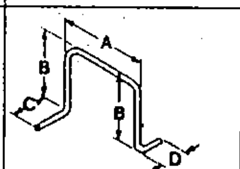
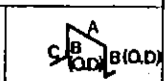
Bends forming end anchorages for use with shape codes 34, 35 and 42 only



n = bend allowance = $5.5d$ (min.) taken to the nearest 10 mm over, or not less than 100 mm to be added to dimension L.

Bar size (mm)	d	6	8	10	12	16	20	25	32	40
Hook allowance (mm)	h	100	100	110	140	180	220	280	360	440
Bend allowance (mm)	n	100	100	100	100	100	110	140	180	220

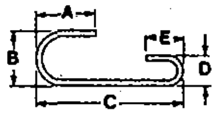
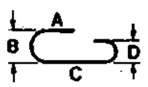
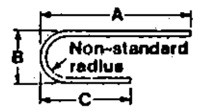
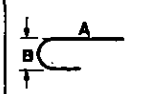
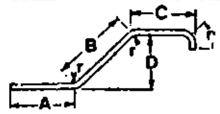
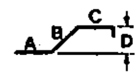
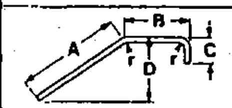
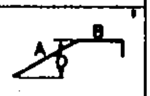
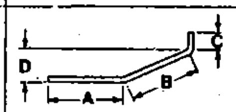
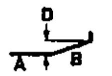
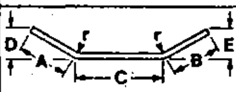
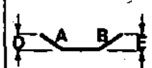
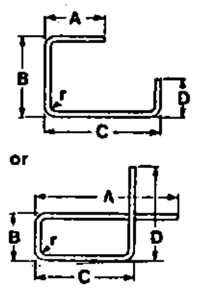
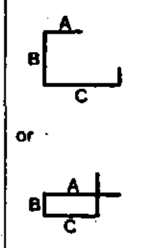
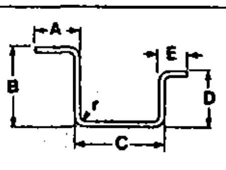
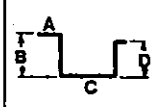
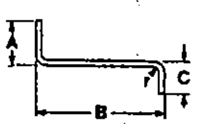
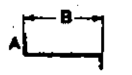
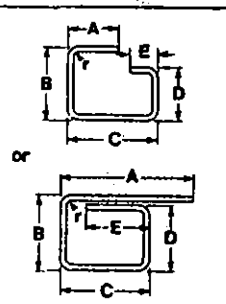
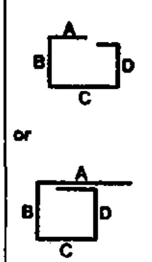
NOTE. For intermediate sizes the dimensions and radii for the next larger size should be used.

Shape code	Method of measurement of bending dimensions	Total length of bar (L) measured along centre line	Dimensions to be given in schedule
20		A	Straight
32		A+h	
33		A+2h	
34		A+n	
35		A+2n	
37		If r is non-standard use shape code 51 A+B−½r−d	
38	 or 	A+B+C−r−2d A+B+C−r−2d	 or 
41	 D shall be at least 2d or 	A+B+C	 or 
43		If angle with horizontal is 45° or less A+2B+C+E	
51		If r is standard use shape code 37 A+B−½r−d	
60		2(A+B)+20d	
62		If angle with horizontal is 45° or less A+C	
81		2A+3B+22d	
83	 Isometric view	A+2B+C+D−2r−4d	 Isometric view

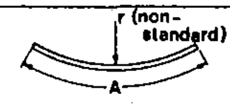
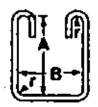
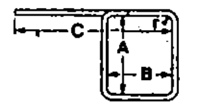
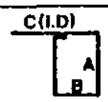
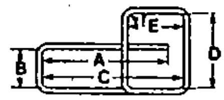
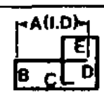
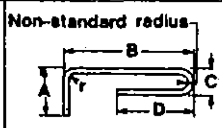
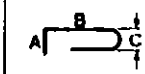
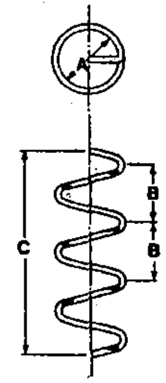
INSTRUCTION. Generally an inside or outside dimension shall be indicated by the position of the bending dimensions in the sketch. If the form of the bar is such that there may be doubt as to which is the inside of the bar, arrows should be shown on the bending schedule or the dimension stated with the suffix O.D. or I.D. (outside or inside dimension).

* BS4449, "Hot rolled steel bars for the reinforcement of concrete". (Metric units).
† BS4461, "Cold worked steel bars for the reinforcement of concrete". (Metric units)

Table 5.8 *BS4466 Other shapes*

Shape code	Method of measurement of bending dimensions	Total length of bar (L) measured along centre line	Dimensions to be given in schedule
36		(A+C+E)+0.57(B+D)−3.14d where d = size of bar	
39		A+0.57B+C−1.57d	
42		If angle with horizontal is 45° or less A+B+C+n	
45		A+B+C−½r−d	
48		A+B+C	
49		If angle with horizontal is 45° or less A+B+C	
52		A+B+C+D−1½r−3d	
53		A+B+C+D+E−2r−4d	
54		A+B+C−r−2d	
58		A+B+C+D+E−2r−4d	

r = standard radius of bend unless otherwise stated

Shape code	Method of measurement of bending dimensions	Total length of bar (L) measured along centre line	Dimensions to be given in schedule
65		A	
72	 These dimensions shall be assumed to be internal unless noted otherwise by the use of arrows or the suffix "O.D.".	2A+B+20d	
73	 These dimensions shall be assumed to be internal unless noted otherwise by the use of arrows or the suffix "O.D.". Generally an inside or outside dimension shall be indicated by the position of the bending dimensions in the sketch. If the form of the bar is such that there may be doubt as to which is the inside of the bar, arrows should be shown on the bending schedule or the dimension stated with the suffix O.D. or I.D. (outside or inside dimension).	2A+B+C+10d	
74	 The dimensions shall be assumed to be internal unless noted otherwise by the use of arrows or the suffix "O.D.".	2A+3B+20d	
75	 These dimensions shall be assumed to be internal unless noted otherwise by the use of arrows or the suffix "O.D.". Generally an inside or outside dimension shall be indicated by the position of the bending dimension in the sketch. If the form of the bar is such that there may be doubt as to which is the inside of the bar, arrows should be shown on the bending schedule or the dimension stated with the suffix O.D. or I.D. (outside or inside dimension).	A+B+C+2D+E+10d	
85		A+B+0.57C+D−½r−2.57d	
86		Where B is not greater than A/5 $\frac{C}{B}\pi(A+d)+8d$ Where d = size of bar	Helix A = Internal dia B = Pitch of helix C = Overall height of helix Dimensions (mm)
99	All other shapes		A dimensioned sketch of the shape shall be given on the schedule

r = standard radius of bend unless otherwise stated

Table 5.9 *Reinforcement: Fabrics and Wire*

	Type of fabric	Size of mesh (mm× mm)	BS ref. no.	Weight (kg/m²)	Main wires: Size (mm)	Main wires: Cross-sectional area (mm²/m)	Transverse wires: Size (mm)	Transverse wires: Cross-sectional area (mm²/m)	Notes
Fabrics	Square mesh	200 × 200	A98†	1.54	5	98	5	98	Fabrics made in either hard-drawn wire (BS 4482) or cold-worked bars (BS 4461) *Exceptions*: A98 and all long-mesh fabrics and main wires of B196: plain hard drawn wire only. *Preferred sizes* Sheets: 2.4 m wide 4.8 m long Rolls: 2.4 m wide 48 m long (indicated thus*) or 72 m long (indicated thus†)
			A142*	2.22	6	142	6	142	
			A193*	3.02	7	193	7	193	
			A252	3.95	8	252	8	252	
			A393	6.16	10	393	10	393	
	Structural	100 × 200	B196†	3.05	5	196	7	193	
			B283*	3.73	6	283			
			B385*	4.53	7	385			
			B503	5.93	8	503	8	252	
			B785	8.14	10	785			
			B1131	10.90	12	1131			
	Long mesh	100 × 400	C283	2.61	6	283	5	49	
			C385*	3.41	7	385			
			C503	4.34	8	503			
			C636	5.55	9	636	6	71	
			C785	6.72	10	785			
	Wrapping	100 × 100	D31	0.49	2	31	2	31	Stock sheets 2.4 m × 1.2 m
			D49	0.77	2.5	49	2.5	49	

Wire		SWG. No	6g	5g	4g	3g	2g	1g	1/0g	2/0g	3/0g	4/0g	5/0g
	Size	in	0.192	0.212	0.232	0.252	0.276	0.300	0.324	0.348	0.372	0.400	0.432
		mm	4.9	5.4	5.9	6.4	7.0	7.6	8.2	8.8	9.5	10.2	11.0
	Cross sectional area	in²	0.029	0.035	0.042	0.050	0.060	0.071	0.082	0.095	0.109	0.126	0.146
		mm²	19	23	27	32	39	46	53	61	70	81	95

CHAPTER 6

Details of Prestressing-steel, Tendons and Anchorages*

6.1 TYPES OF STEEL

Steel for prestressed concrete must have high tensile strength and adequate ductility. These qualities are found in

(i) carbon or alloy steel, hot rolled, but otherwise untreated,
(ii) cold worked steel, which is drawn or deformed, and preferably tempered, and
(iii) hot rolled and tempered steel.

Carbon or alloy steel has a carbon content not greater than 1% and this is mainly responsible for its high tensile strength. Alloying elements such as manganese, nickel and chromium may also be added to improve the mechanical properties of the steel, and various heat treatments have a beneficial effect. These treatments make use of the fact that if steel is heated to a temperature higher than about 850°C (1550°F) (termed the *transformation temperature*), its final structure and the extent to which its properties are improved depend on the rate of cooling. If the steel is cooled slowly from its transformation temperature, the treatment is termed *annealing*: if it is allowed to cool from the transformation temperature at its normal rate, the treatment is termed *normalizing*. If, on the other hand, steel is suddenly cooled from above the transformation temperature to room temperature by immersion or *quenching* in oil, its hardness and brittleness are appreciably increased. Quenching is usually followed by a tempering process in which the steel is reheated to about 400°C (750°F) and allowed to cool in air. This reduces the brittleness of the steel. If the steel is rapidly cooled from above the transformation temperature to about 450°C (850°F) and then allowed to cool slowly to room temperature the process is termed *patenting*, and has an effect similar to that of quenching and tempering. The term *stress relieving* is used to describe heat treatment for a prolonged period at about 260°C (500°F) or a short period at about 500°C (950°F). The term *stabilizing* denotes heat treatment at about 400°C (750°F) combined with a tensile stress of about 65% of the ultimate strength of the steel.

Cold working of steel increases its strength, and is mainly carried out by drawing wire through a series of dies, with progressive reductions in the diameter of each die, and consequently of the wire. Rolling may also be used to produce the same result. Rolling, whether hot or cold, enables the steel to be deformed or indented, if required.

It is usual to apply heat treatment to all prestressing steels except those of natural hardness, and secret or proprietary processes are often used. Ordinary oil quenching is generally considered to be unsatisfactory. Other methods, termed *mar tempering* are used and cooling is often carried out in lead, salt or oil baths.

Although the effects of the foregoing processes are known qualitatively, the actual properties of any steel can be determined only by tests. It is essential that sufficient satisfactory data should be available before any type of steel is used for prestressing.

Strength of Prestressing Steel

The practice of specifying a minimum strength for prestressing steel has been superseded, in recent British Standards and Codes of Practice, by the concept of a *characteristic strength* defined as that value below which not more than 5% of test results fall. Typical values are 210 000–240 000 lbf/in^2 (14 800–16 900 kgf/cm^2; 1450–1660 N/mm^2) for wire; 240 000–260 000 lbf/in^2 (16 900–18 300 kgf/cm^2; 1660–1800 N/mm^2) for strand; and 150 000 lbf/in^2 (10 550 kgf/cm^2; 1040 N/mm^2) for alloy bar.

Stress-Strain Relationship

An ideal stress-strain diagram for prestressing steel is shown in Fig. 6.1 which meets the following requirements:

(i) It is imperative to have a high tensile stress which must be accompanied by only a small amount of creep. This is achieved if the permanent elongation at the working stress is small, and the type of steel for which the stress-strain diagram is linear for a large proportion of the ultimate load is used. This property is measured by the *proof stress* which

* The author wishes to gratefully acknowledge with thanks P.W. Ables, B. Roy et al. from whose monumental book on *Prestressed Concrete Design* the material in this chapter has been taken for compact presentation, material that is otherwise well known and exists already in standard references on the subject.

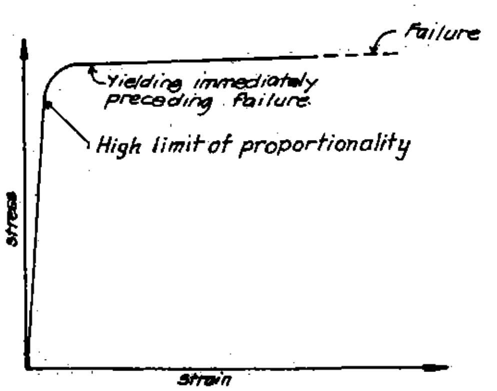
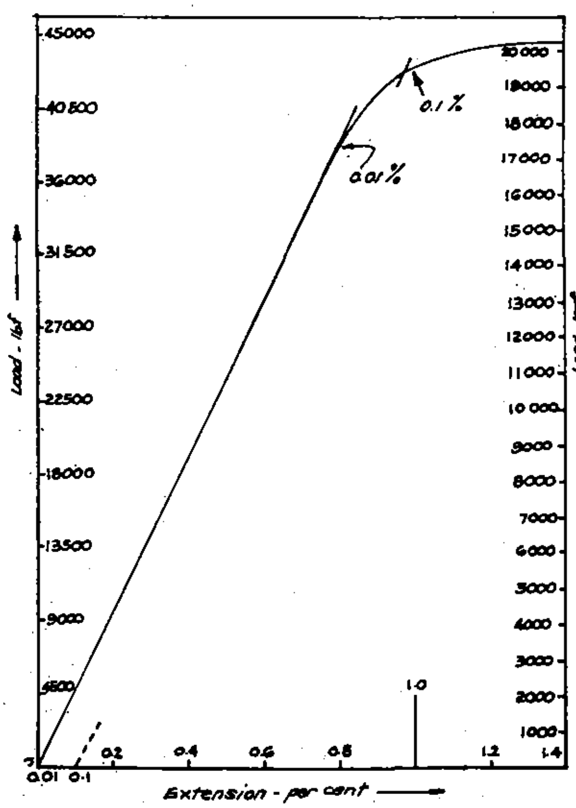


Fig. 6.1 *Ideal stress-strain diagram for prestressing steel*

is defined as the stress which produces a certain permanent deformation (usually 0.2% but sometimes 0.1%) on first loading, and a steel which is suitable for prestressing should have a high proof stress (Fig. 6.2).



0.5 in (12.70 mm) diameter Dyform strand; 0.01% off-set 38.250 lbf (17.350 kgf); 0.1% off-set 42,800 lbf (19,414 kgf); Load at 1% ext. 43.100 lbf (19,550 kgf); Breaking load 48,000 lbf (21,772 kgf); Modulus of elasticity; 27.88 × 10^6 lbf/in^2 (19,602 kgf/mm^2) Area = 0.174 in^2 (112.25 mm^2)

Fig. 6.2 *Load extension diagram (proof load)*

(ii) It is also most desirable that an ultimate elongation of appreciable magnitude should be obtained in order to reduce as much as possible the chance of sudden fracture; this may occur, for example, with piano wire, which has a very small elongation at failure. Prestressing wire and strand have a minimum elongation of between 3 and 5%, which is quite sufficient with satisfactory bond; the value for alloy bars is about 10%.

The stress-strain diagrams for various types of steel in Fig. 6.3 indicate that the ultimate elongation tends to decrease as the ultimate strength increases. It is clear, therefore, that piano wire is not entirely suitable for prestressing, despite its high proof stress, as its ultimate elongation is very limited. On the other hand, mild steel and deformed bars, which have a large ultimate elongation, are unsuitable because of their low yield point or proof stress. Figure 6.3 also shows that a distinct yield point occurs in low-alloy bars, and this influences the ultimate strength of structures with bonded steel in which the steel is the weaker part and failure is initiated by its excessive deformation in some cases by its fracture.

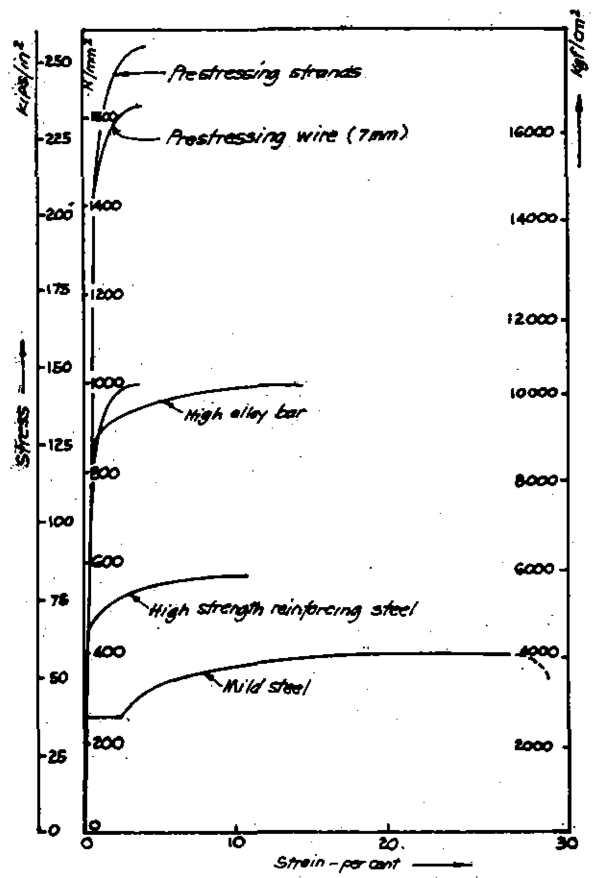


Fig. 6.3 *Stress-strain diagram for various steels*

The modulus of elasticity for prestressing steel depends on the type of steel employed, and values should be obtained from the supplier of the steel. Typical values are 25×10^6 lbf/in^2 (1.76×10^6 kgf/cm^2; 0.173×10^6 N/mm^2) for low-alloy bars, 28×10^6 lbf/in^2 (1.97×10^6 kgf/cm^2; 0.194×10^6 N/mm^2) for carbon steel wires, and between 23.5×10^6 and 29×10^6 lbf/in^2 (1.65×10^6 kgf/cm^2; $0.163 \times 10^6 - 0.2 \times 10^6$ N/mm^2) for strands. A typical load-extension curve for strands remains linear for only about a half of its length, and a typical 0.2 per cent proof stress is between 85% and 95% of the breaking load.

6.2 PRESTRESSING TENDONS

Prestressing tendons normally take the form of separate wires, wires spun together helically to form strands, or bars. For pre-tensioned steel, wires, strands, and occasionally bars are used singly, to permit the concrete to bond directly to them; when post-tensioning is used, it is common practice to group the separate tendons together, so as to reduce the number of anchorages and ducts required to accommodate them. When grouped in this way, the tendons in each duct are usually termed a *cable*.

British Standards for Tendons

The requirements for prestressing tendons are set out in the following British Standards:

BS 2691:1969: Steel wire for prestressed concrete
BS 3617:1971: Stress relieved seven-wire steel strand for prestressed concrete
BS 4757:1971: Nineteen-wire steel strand for prestressed concrete
BS 4486:1969: Cold worked high tensile alloy steel bars for prestressed concrete

Wires and Strands

The wire is required to be cold drawn from plain carbon steel (BS 2691) or patented plain carbon steel (BS 3617 and 4486); the chemical composition is shown in Table 6.1.

Table 6.1 *Chemical Composition of Alloying Elements of Prestressing Wires and Strands*

Element	Minimum %	Maximum %
Carbon	0.60	0.90
Silicon	0.10	0.35
Manganese	0.50	0.90
Sulphur	—	0.05
Phosphorus	—	0.05

The drawn wire is to be free from surface or other defects, and the finished wire or strand must be free from oil and grease unless otherwise specified by the purchaser. Superficial rusting is allowed, provided that there is no visible pitting of the surface. In the case of *wire (BS 2691)*, the coils supplied to the purchaser must not include welds; except that, by agreement between the buyer and the supplier, special lengths may be drawn from rods welded before the patenting process is applied. For *strands*, no length of strand may be joined to another by any method, though separate wires within the strand may be welded together prior to patenting. No welding is allowed after patenting or during or after wire drawing. If special lengths of seven-wire strand (BS 3617) are required, and provided the user is made fully aware of the reduced mechanical properties involved. not more than one wire in any 40 m (130 ft) may be welded after patenting or drawing; this relaxation applies to seven-wire strand only.

The tolerance on the nominal diameter of *prestressing wire (BS 2691)* is ±0.025 mm (±0.001 in.) for wires under 2.5 mm (0.104 in.) in diameter, and ±0.050 mm (±0.002 in.) for wires of 2.5 mm (0.104 in.) or more in diameter. If the wire is to pay out straight from the coil, the internal diameter of the coil shall not be less than 1.8 m (6 ft) for wires of 7 mm (0.276 in.) diameter or greater, or 1.2 m (4 ft) for wires less than 7 mm (0.276 in.) in diameter. In *seven-wire strands (BS 3617)* the nominal diameter of the centre wire is to be at least 2% greater than that of the surrounding wires; after heat treatment, it is to be wound onto coils of such a size [and not less than 600 mm (2 ft) in any case] that it pays off reasonably straight. The tolerances on the nominal diameter of the finished strand are + 0.4 mm (+ 0.016 in.) and −0.2 mm (− 0.008 in.) in all cases. For *nineteen-wire strand (BS 4757)*, different requirements are laid down for treated and 'as-spun' strands. A treated strand has a nominal diameter of 18 mm (0.725 in.) with tolerances on diameter of ±0.5 mm (±0.02 in.) and −0.25 mm (−0.01 in.) The treatment comprises low-temperature heating as a continuous linear process, after which it is to be wound onto coils with a minimum diameter of 900 mm (3 ft), from which it pays off 'substantially straight'. 'As-spun' strands, with nominal diameters of 25.4 mm (1 in.), 28.6 mm (1.125 in.) and 31.8 mm (1.25 in.) have tolerances on diameter of + 0.6 mm (+ 0.024 in.) and −0.25 mm (− 0.01 in.) in all cases; no heat treatment is required, and the minimum coil diameter is 1.5 m (5 ft).

Testing

In the case of *wire (BS 2691)* the manufacturer is required to provide one load/extension curve for each parcel of wire, a parcel being defined as any quantity of finished wire presented for testing at any one time. Tests are to be made on samples taken from the end of one coil in every five within the parcel, but the results of these are only required to be kept available for inspection by the purchaser.

Table 6.2 *Mechanical Properties of Wires (BS 2691)*

Nominal wire diameter		Specified characteristic strength			Reverse bend radius		Conditions in which wire supplied (see below)
mm	in.	N/mm^2	kgf/mm^2	lbf/in^2	mm	in.	
7	0.276	1470	150	214000	20	0.8	1,2
*7	0.276	1570	160	228000	20	0.8	1,2
*5	0.197	1570	160	228000	15	0.6	1,2,3
5	0.197	1720	175	248000	15	0.6	1,2,3
4.5	0.177	1620	165	235000	15	0.6	2,3,
*4	0.1575	1720	175	248000	12.5	0.5	1,2,3
*3.25	0.128	1720	175	248000	10	0.4	3
3.25	0.128	1870	190	270000	10	0.4	3
*3	0.118	1720	175	248000	10	0.4	3
2.65	0.104	1870	190	270000	7.5	0.3	3
2	0.079	2020	205	291000	5	0.2	3

Conditions in which wire is supplied:

Description	Number		
	1	2	3
	Cold drawn, pre-straightened, normal relaxation	Cold drawn, pre-straightened, low relaxation	Cold drawn
0.2% Proof stress, as percentage of specified characteristic strength	85	90	75
Maximum relaxation after 1000 hours from:			
70% Initial stress	5%	2%	—
80% Initial stress	8.5%	3%	—

* Preferred specified characteristic strengths.

Specimens are tested for characteristic strength, proof stress, and reverse bend tests; relaxation tests may also be called for. The specified values are shown in Table 6.2; the wire is deemed to comply with the requirements for specified characteristic strength provided that not more than two of any 40 consecutive results fall below the specified value, no results are less than 95% of the specified value, and none are more than 230 N/mm^2 (24 kgf/mm^2; 33 600 lbf/in^2) above it.

For Strands

The manufacturer is required to provide dated test certificates prepared from the relevant test results. Tests are to be made on samples cut from each coil; they comprise a tensile test, an elongation test, a proof-load test (for seven-wire strand only), and, if required, relaxation test results. For seven-wire strand, proof-load tests and load-extension curves are called for only for one test piece in every five; for nineteen-wire strands a proof-load test is required only for treated strands; for these, the test and the plotting of a load-extension curve, are specified for one test piece in every three. For 'as-spun' strand, load-extension curves are to be plotted for every test piece. The values specified are summarized in Table 6.3; the specified characteristic strength is defined in the same way as that for wire, except that no upper limiting value is imposed. The minimum elongation at failure is specified as 3.5%, except for 'as-spun' nineteen-wire strand; no value is specified for this.

All three specifications included provisions for re-testing, in the event of failure of a sample to meet the requirements.

Bars

No chemical composition is given for the steel, except that sulphur and phosphorus must not exceed 0.05%, but the manufacturer is required to provide the chemical analysis on request. Threads, if provided, are to be cold-rolled; no welds are permitted, and the bars are to be protected at all times from the effects of local heat. Tolerances are specified only on the mass; on the basis that the density of the steel is 7850 kg/m^3 (0.283 lb/in^3) the variations permitted are + 4% and −2% for a batch (defined as a number of lengths of one size from one cast) and + 6% and −2% for any one bar.

From the purchaser's viewpoint, the requirements for testing are less satisfactory than those included in the standards for wires and strands. The manufacturer is not required to provide any documentary evidence of the test results obtained, though the records of the tests must be 'available for inspection by the purchaser or his representative.' Further, unlike the standards for wires and strands, no option of independent testing before delivery

Table 6.3 *Mechanical Properties of Strands (BS 3617, 4757)*

Nominal diameter of strand		Nominal area of steel		Specified characteristic load			Conditions in which strand supplied	BS No.
(mm)	(in.)	(mm^2)	(in^2)	(kN)	(kgf)	(lbf)		
6.4	0.253	24.5	0.038	44.5	4540	10000	1,2	3617 (7-wire strand)
7.9	0.312	37.4	0.058	69.0	7040	15500	1,2	
9.3	0.366	52.3	0.083	93.5	9530	21000	1,2	
10.9	0.430	71.0	0.110	125.0	12750	28100	1,2	
12.5	0.492	94.2	0.146	165.0	16820	37100	1,2	
15.2	0.600	138.7	0.216	227.0	23150	51000	1,2	
18	0.7	210	0.325	370	37730	83180	1,2	4757 (19-wire strand)
25.4	1.0	423	0.656	659	67200	148150	3	
28.6	1 1/8	535	0.830	823	83920	185000	3	
31.8	1 1/4	660	1.020	979	99830	220100	3	

Condition in which strand is supplied:

Description	Number 1	Number 2	Number 3
	Normal relaxation heat treated	Low relaxation heat treated	As spun
0.2% proof stress as percentage of specific characteristic strength	85	90	—
Maximum relaxation after 1000 hours from:			
70% Initial stress	7%	2.5%	9%
80% Initial stress	12%	3.5%	14%

is available to the purchaser. The routine tests comprise a tensile test, a proof-load test and a minimum elongation (of 6%); one sample is to be taken from each 5 metric tonnes within a batch. The samples may be cut from the ends of processed tendons, or from off-cuts produced during processing. The specified characteristic load and the 0.2% proof load (defined as the load at 0.7% total strain) are given in Table 6.4; the breaking load for the thread is required to be at least 95% of that for the bar. The modulus of elasticity is to be determined from the test readings and recorded. The breaking load is required to be not less than 95% of the specified characteristic load, and not more than two out of the last 40 test results may be less than the specified values; no upper limiting value is specified.

Provision is made in the standard whereby a purchaser may, if he wishes, check that a batch attains the specified characteristic load. After delivery, ten bars are selected at random, and test pieces are cut from one end of each bar; if one should fail at less than 95% of the specified value, that bar is rejected. If two fall below this value, the whole batch is deemed not to comply with the requirements of the standard.

The purchaser may require the maker to provide evidence of the relaxation properties of the tendon. The standard also includes provisions for re-testing, if samples tested by the maker should fail to meet the requirements.

Wires Strands and Bars for Pre-Tensioning

It was thought at one time that a satisfactory bond between tensioned steel and concrete could be obtained only by the use of wires of small diameter, and piano wire of 2 mm (0.08 in.) diameter is still used occasionally. This type of wire has a smooth hard surface which for large

Table 6.4 *Properties of Cold-Worked High-Tensile Alloy Steel Bars (BS 4486)*

Nominal size		Specified characteristic load			Minimum 0.2% proof load		
(mm)	(in.)	(kN)	(kgf)	(lbf)	(kN)	(kgf)	(lbf)
*20	0.78	325	32850	73100	275	27750	61900
22	0.87	375	37900	84400	325	32850	73100
*25	0.985	500	50600	112500	425	42900	95800
28	1.11	625	63000	140600	525	53000	118100
*32	1.26	800	80900	180200	675	68050	152500
*40	1.57	1250	126200	281500	1050	106000	236500

* Preferred sizes.

wires prevents the development of good bond, and its unsatisfactory behaviour at ultimate load has been described earlier.

These disadvantages led to the use of indented wires. Single smooth wires of 0.2 in. and 0.276 in. (5 and 7 mm) diameters were introduced in Britain in 1939 and 1952 respectively and have proved satisfactory for pretensioning since their surface conditions are such as to ensure good bond. This is due to a very slight corrosion of the surface of the wire, such that no peeling of the surface is likely. Indented wires also provide a good bond, but the indentations must not be so large as to reduce appreciably the cross-sectional area of the wire or cause fatigue failure at the notches. Seven-wire strands are also widely used for pretensioning; in addition to the normal surface bond they provide a mechanical bond with the concrete, because of the configuration of the wires comprising the strand.

With pre-tensioned steel, a certain minimum embedded length, termed the *transmission length,* is necessary, along which the force is gradually developed in the concrete by bond. The transmission length required increases when the diameter of the wire increases and also to some extent when the strength of the concrete is reduced. With small wires the prestress in the concrete is developed over a very short length, but with larger sizes the required length may be 2 to 3 ft (0.67 to 0.9 m). It should be noted that the rate of transmission is not uniform. More than half of the prestressing force is transferred to the concrete in the first quarter of the transmission length and up to 85% may be transferred in the first half. In CP 110, it is noted that the transmission length for wire may vary between 50 and 160 diameters, and the following general recommendations are given:

Plain or lightly crimped wire:	100 diameters; 80% transfer in first 70 diameters
Heavily-crimped wire:	65 diameters; 80% transfer in first 54 diameters
Strand, 9.3 mm (0.366 in.) diameter:	200 mm ± 25 mm (8 in. ± 1 in.)
Strand, 12.5 mm (0.492 in.) diameter:	330 mm ± 25 mm (13 in. ± 1 in.)
Strand, 17.8 mm (0.7 in.) diameter:	500 mm ± 50 mm (20 in. ± 2 in.)

A special strand, known as Dyform, is made by British Ropes Ltd. The strand is first formed in the normal way and is then compacted to form the cross-section shown in Fig. 6.4. In this case, the objective is primarily to increase the force which a strand of given diameter can apply; there is no gain in transmission properties, but space taken is reduced.

High-alloy steel bars with special indentations have also been used for pretensioning. In this case the bar is tensioned in a manner similar to that described ahead for post-tensioned alloy bars, but after the concrete hardens, the end anchors are removed and the prestressing force is transmitted solely by bond, as for any pre-tensioning. When selecting a suitable size of wire it is desirable to ensure that the number of wires is sufficient to distribute the compressive stress uniformly over the concrete but not so great as to impede the placing of the concrete. In general it appears desirable to provide at least six wires in the tensile zone, but to avoid the adoption of a multitude of wires. If the number of wires is less than six, the failure of one would greatly reduce the factor of safety of the member. For this reason, the use of only two or three wires in a member should be avoided where possible.

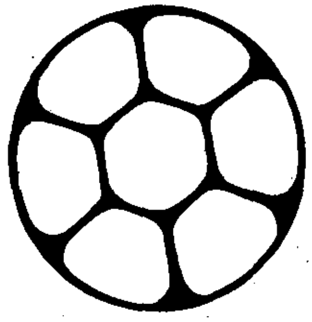


Fig. 6.4 *Cross-section of dyform strand*

Details of prestressing wires, strands and bars are given ahead.

Tendons for Post-Tensioning

Many cables with different arrangements of wires and strands and different methods of anchorage are available for post-tensioning. The main types are briefly described here; more data is given ahead.

Historically, the two basic types are represented first by the Freyssinet cable and later by the Magnel cable. In the Freyssinet cable, the wires, which usually number twelve, are closely spaced around a central spring, or core helix, thus forming a cable of annular cross-section. The cables may be very small and special care is then necessary to ensure satisfactory grouting, neat cement and water being used for the purpose. The spacing of cables should also be considered. In the Magnel cable, multiples of four or eight wires are provided in horizontal layers and the wires are well separated by spacers which allow easy grouting with cement mortar. With this type of cable a considerable prestressing force may be concentrated in a single cable. The Magnel system itself is no longer commercially very much in use now.

Many other types are available, including CCL Systems and PSC (Great Britain) and Leoba, Holzmann, and Beton-und-Monier Bau Gesellschaft (Germany), which more or less follow the Magnel principle with regard to the use of spacers; Systems Franki-Smet (Belgium), Morandi (Italy), Hochtief, and Grun and Bilfinger (Germany), which more or less follow the Freyssinet principle. Other early types like Gifford-Udall and Gifford-Burrow systems are no longer commercially available. BBRV and Losinger VSL (Switzerland), Pl and Prescon (USA), represent a type intermediate between Magnel and Freyssinet. In some systems of this type, the wires may be distanced by spacers, but are not necessarily separated by them. In the latter case, if the wires are bent up they touch each other, forming a group into which the grout cannot be inserted.

In some types of cable, spacing may be obtained automatically if (Neptun or Sigma) wires with diagonal cross ribs are used instead of round wires. The cross ribs of adjacent wires touch each other at points only, since the ribs on opposite sides of a wire slope in opposite directions, and sufficient space is available between the wires to allow the easy admission of grout.

The cables may be inserted into holes formed in the concrete or placed in ducts (tubes or sheaths). The Baur-Leonhardt cable (Germany) may also be placed around the outside of the concrete, forming closed loops. This cable consists of closely spaced stranded wires, and is therefore of the Magnel type.

Cables comprising single or multiple strands, which can be inserted in holes or placed in tubes, have been introduced by Anderson and Roebling (USA), Rheinhausen (Germany), Freyssinet and SEEE (France), PSC, Stress Block and CCL Systems (Great Britain), and most suppliers of post-tensioning systems now provide components for anchoring strands. In order to obtain the greatest possible strength of a large stranded cable it is necessary that the outer wires should be stranded in the same direction as those forming the inner core; such strands are described as *parallel lay*. As a consequence, an untwisting torque may occur during tensioning in systems in which the jack is restrained from rotating (though most strand jacks now permit rotation during stressing). At transfer, this torque may be transmitted to the prestressed unit, and it may occasionally be large enough to warrant consideration in the design!

In addition to the medium and large cables already described, there are several types of cables with two, three, four, six and eight wires which are used to provide smaller prestressing forces. In all these cables, wires of 0.2 and 0.276 in. (5 and 7 mm) diameter are generally used. The two-wire and four-wire ducts of the PSC system are the smallest in size as no spacers are used, the arrangement of the wires being such that the space available for grout is not less than that obtained with spacers.

Bars for Post-Tensioning

Two types of steel bar have been developed for use in post-tensioning. They are used in the Macalloy system (Britain), Stressteel in the USA and the Dywidag system (Germany). In the Macalloy system, high-alloy steel bars from 18 mm (3/4 in.) up to 40 mm (1-5/8 in.) diameter are used. In the Dywidag system, the bars are of low-alloy steel of natural hardness, but with a definite yield point and are usually 25 mm (1 in.) in diameter although bars of lesser diameter are also available. In the Dywidag system high-alloy bars with greater strength have also been introduced.

The bars may be inserted into holes or placed within tubes or sheaths in the concrete, in the same way as cables; the Macalloy system also includes a four-bar tendon. It is possible to obtain a good bond if the grouting is carefully done with neat cement and water. The bars may be placed relatively close to one another, in the same way as Freyssinet cables.

Anchoring Prestressing Steel

There are four basic methods of anchoring the steel after it has been tensioned. Three of these are represented by the Freyssinet, Magnel and Macalloy methods respectively; the fourth is represented by CCL, PSC and BBRV systems.

In the Freyssinet system all the wires or strands of the cable are wedged (locked) between a cylinder which is embedded in the concrete and a cone which is inserted therein. In the Magnel system pairs of wires are anchored by flat wedges to plates, termed *sandwich plates*, which in turn transfer the prestressing force to the concrete through an anchor-plate; as previously mentioned, this system is no longer available. In the Macalloy system, the prestressing force is also transferred to the concrete through an anchor-plate, by means of a nut tightened on a thread, rolled on to the end of the bar. With SEEE system, soft steel cylinders, containing the strands are pushed through a die; threads are rolled onto the swaging cylinders, and nuts are tightened to anchor the tendons. In the Dywidag system, bars with threads throughout their length are available.

In the fourth method, single wires or strands are secured to cylindrical grips by means of one or more wedges, or alternatively by button heads formed on the wire (BBRV and Prescon). The wedge system has also been adopted for bars (Stressteel and Macalloy alternative anchorage) and for cables of single or multiple strands (CCL Systems, PSC, Stress Block, and Anderson.)

When the wires are secured by wedges, whether they be concrete or steel cones or steel wedges, some slipping is unavoidable when the pull on the prestressing steel is relaxed. This may affect the tensioning stress substantially if the prestressing tendon is short.* When the steel is secured by a nut the process is simple and no slip occurs during transfer. Moreover, no difficulty is experienced in regulating the prestressing force at any time. Because of the many advantages of this method of anchoring, it is employed in several of the cable systems previously mentioned, including some systems which use strand. In several systems, the separate wires of the cable are secured to a threaded anchor-head before tensioning. The wires are then tensioned simultaneously and anchored by means of a nut. The wires are connected to the anchor-head by upsetting and enlarging the ends of the wires (BBRV, Prescon), by concreting the wires into the head (Beton-und-Monier Bau, Holzmann A.G.), or by looping them around a cross-bar with a threaded hole (Leoba). In some of these systems it is also possible to employ a temporary anchor-plate and dispense with it as soon as cement mortar or concrete, which is inserted round the anchor-head, hardens and secures the head to the concrete. The anchor-plate is then removed. The anchor-head is usually conical. The advantage of a positive anchorage which will not slip is thereby retained without the cost of a permanent anchor-plate.

In the original Holzmann large-cable system (Germany), the cable comprises layers of four oval-shaped well-spaced wires, and is secured by means of a wedge to a large prefabricated member (corresponding to a female cone or large grip) which also forms the anchor-plate. In the improved Holzmann KA system (Klem Anker, i.e., clamp anchor system), now being used, up to forty oval-shaped or rectangular wires with diagonal ribs are clamped by means of transverse bolts and nuts which are tightened against the outer plates.

In the Losinger VSL system (Switzerland), up to 36 wires are secured to an anchor head by means of a single conical wedge with circumferential grooves in which the wires are housed.

A continuous cable is used in the Baur-Leonhardt system, in which the tensioning is done by jacking apart two parts of the structure around which the cables are looped. Alternatively, separate cables may be used, one end being anchored in the structure and the other in a movable anchor-block to which the jacks are applied.

Another post-tensioning system, which is in fact the oldest, is that developed by Coyne, and used mainly for retaining walls, dams, and barrages. It comprises a straight cable of 600 to 800 wires of 0.2 in. (5 mm) diameter, strapped together to form a bundle. One end is embedded in a bulkhead of concrete and held by bond, and the other end is fixed to a large steel drum filled with cement mortar, thereby forming an anchor head to which the jacking force is applied. This system was used to prestress the first prestressed concrete pressure vessels for nuclear reactors, at Marcoule (France).

Cables may have the same type of anchorage at both ends, or the wires may be embedded in the concrete at one end before they are tensioned; loops or other shapes which ensure a satisfactory anchorage may be formed at the dead end of the cable.

The systems described in the foregoing can also be used for prestressing circular containers or pipes by arranging the cables in overlapping arcs; special anchors are available with some systems to simplify the work. Circular structures can also be prestressed by means of wire under tension wound around them in the form of a continuous helix (Preload, BBRV and Dywidag systems). The British contractors Taylor Woodrow Ltd., have also developed a wire-winding system for large pressure vessels.

Relaxation of Stress in Steel

When a high tensile steel wire is stretched and maintained at a constant strain, the initial force in the wire does not remain constant but decreases with time. The decrease of stress in steel at constant strain is termed as *relaxation*. In a prestressed member, the high tensile steel between the anchorages is more or less in a state of constant strain. However, the actual relaxation will be rather less than that indicated by a test of a tendon at constant length, as there will be a shortening of the member due to other causes. With the high tensile steels at elevated stresses the relaxation of stress has been observed and it increases with the magnitude of initial stress. If the stress is maintained constantly, the material exhibits a plastic strain over and above the initial elastic strain, generally referred to as *creep*.

The cold drawn steels creep more than heat treated or tempered steels due to their lower magnitude of proof stress. The phenomenon of creep is influenced by the chemical composition. Micro-structure, grain size and variables in the manufacturing process, which results in changes in the internal crystal structure. Several hypotheses for explaining the mechanism of creep in steel are presented by several investigators.

The steel in a prestressed concrete member strictly does not remain under a constant condition of either stress or strain. The most severe condition generally occurs at the stage of initial stressing; subsequently, the strain in the steel reduces as the concrete deforms under the prestressing force.

The code provision for the relaxation of stress in steel is based on the results of the 1000 hours relaxation test on specimens. Experience has shown that the loss recorded over a period of about 1000 hours from an initial stress of 70% of the tensile strength is about the same as the loss experienced

* In a short tendon the extension may not be very much more than the slip!

over a period of four years from an initial stress of 60% of the tensile strength. According to Stussi the relaxation curves obtained over 1000 hours can be extrapolated by a logarithmic plot. The Indian Standard specification I.S. 1785 prescribes the 1000-hour relaxation test with a relaxation of stress not exceeding 70 N/mm^2 for cold drawn stress relieved wires. In the absence of this, the 100-hour relaxation test is also provided for with a limiting value of relaxation stress of 46.7 N/mm^2.

Experiments have shown that a reduction in relaxation stress is possible by preliminary overstressing! A preliminary overstress of 5–10% maintained for two or three minutes results in a considerable reduction in the magnitude of relaxation. Some of the codes permit temporary overstressing with correspondingly lower magnitudes of relaxation stress.

Stress Corrosion

The phenomenon of stress corrosion in steel is particularly dangerous since it results in sudden brittle fractures. Stress corrosion cracking results from the combined action of corrosion and static tensile stress, which may be either residual or externally applied. This type of attack in alloys is due to the internal metallurgical structure which is influenced by composition, heat treatment and mechanical processing. The causes of the susceptibility of high tensile steels to stress corrosion are manifold. Schwier has reported that heat treated wires are specially prone to stress corrosion fractures when compared to drawn wires. If the ducts of post-tensioned members are not quickly grouted, there is the possibility of stress corrosion leading to a catastrophic failure of the structure.

There are other common types of corrosion frequently encountered in prestressed concrete constructions such as *pitting corrosion* and *chloride corrosion.* A critical review of the different types of corrosion of high tensile steel in structural concrete is reported elsewhere. Some of the important protective measures to prevent stress corrosion include protection from chemical contamination, protective coatings for high tensile steel and grouting of ducts immediately after prestressing operations.

Hydrogen Embrittlement

Atomic hydrogen is liberated due to the action of acids on high tensile steels. This penetrates into the steel surface making it brittle and resulting in fractures on being subjected to tensile stress. Even small amounts of hydrogen are sufficient to cause considerable deterioration in the tensile strength of high tensile steel wires.

Use of high alumina cement, blast furnace slag cement which is rich in sulphides, when used to make prestressed concrete is likely to give rise to hydrogen embrittlement. Use of dissimilar metals such as aluminium and zinc for sheaths to house high tensile steel wires also results in hydrogen embrittlement. Minute traces of sulphur which come in contact with high tensile steel wires in the presence of moisture results in reduction in the strength due to hydrogen embrittlement.

Table 6.5 *Strand Data*

BS3617 Normal-relaxation strand

Nominal diameter of strand	Nominal area of steel	Nominal mass per 1000 m run	Specified characteristic load	Minimum 0.2% proof load	Minimum elongation	Maximum relaxation after 1000 hours from initial load of 70% of the specified characteristic load	initial load of 80% of the specified characteristic load
mm	mm^2	kg	kN	kN	%	%	%
9.3	52.3	411	93.5	79.5	3.5	7	12
10.9	71.0	564	125.0	106.3			
12.5	94.2	744	165.0	140.3			
15.2	138.7	1101	227.0	193.0			

BS3617 Low-relaxation strand

Nominal diameter of strand	Nominal area of steel	Nominal mass per 1000 m run	Specified characteristic load	Minimum 0.2% proof load	Minimum elongation	Maximum relaxation after 1000 hours from initial load of 70% of the specified characteristic load	initial load of 80% of the specified characteristic load
mm	mm^2	kg	kN	kN	%	%	%
9.3	52.3	411	93.5	84.1	3.5	2.5	3.5
10.9	71.0	564	125.0	112.5			
12.5	94.2	744	165.0	148.5			
15.2	138.7	1101	227.0	204.3			

(Contd.)

Table 6.5 *(Contd.)*

Dyform L-R prestressing strand

Nominal diameter of strand mm	Nominal area of steel mm²	Nominal mass per 1000 m run kg	Specified characteristic load kN	Specified characteristic load lbf	Minimum load at 1% relaxation: Normal-relaxation strand kN	Minimum load at 1% relaxation: Low-relaxation strand kN
12.7	112.0	890	209.0	4.6985	—	181
15.2	165.0	1300	300.0	6.7443	—	260
18.0	223.0	1750	380.0	8.5427	—	330

Bridon SUPA-7 prestressing strand

Nominal diameter of strand mm	Nominal area of steel mm²	Nominal mass per 1000 m run kg	Specified characteristic load kN	Specified characteristic load lbf	Minimum load at 1% relaxation: Normal-relaxation strand kN	Minimum load at 1% relaxation: Low-relaxation strand kN
9.6	56.0	440	102.5	23.043	87.1	92.3
11.3	76.0	600	138.0	31.024	117.3	124.2
12.9	100.5	800	184.0	41.365	156.4	165.6
15.4	143.2	1130	250.0	56.202	212.5	225.0

In order to prevent hydrogen embrittlement, it is essential that steel is properly protected from the action of acids. Protective coverings like bituminous crepe paper covering during transport reduces the chances of contamination. The steel should be protected from rain and excessive humidity by storing it in dry conditions.

Strand Data

There are several grades of prestressing strand available. All prestressing strands are stress relieved, but further processes are often employed to reduce the losses arising from the relaxation of the steel. These processes involve a combination of applied heat and stress, carried out under such varying trade names as thermalising, normalising, etc.

Compact strand is pulled through a die after being spun as a stranded cable, which not only physically modifies the cross-sectional shape, but also enhances the strength characteristics of the stranded cable as a result of the further cold working.

Table 6.6 *Forces in Different Types, Numbers and Sizes of Strands*

Tendon: Strand size mm	Tendon: Type	Number of strands	Cross-sectional area (mm²)	Force: Specified characteristic load in kN 100%	Force: Specified characteristic load in kN 70%
12.7	DYF	7	784	1463	1024
18.0	DYF	4	892	1520	1064
15.2	STD	7	970	1589	1112
15.4	SUPA	7	1002	1750	1225
12.5	STD	12	1130	1980	1386
15.2	DYF	7	1155	2100	1470
12.9	SUPA	12	1206	2208	1545
12.7	DYF	12	1344	2508	1755
18.0	DYF	7	1561	2660	1862
15.2	STD	12	1664	2724	1906
15.4	SUPA	12	1718	3000	2100
12.5	STD	19	1789	3135	2194
15.2	STD	15	2080	3405	2383
12.9	SUPA	19	1909	3496	2447
15.2	DYF	12	1980	3600	2520
15.2	DYF	13	2145	3900	2730
12.5	STD	25	2355	4125	2887
15.2	STD	19	2635	4313	3019
12.9	SUPA	25	2512	4600	3220
15.4	SUPA	19	2720	4750	3325
12.5	STD	31	2920	5115	3580
15.2	DYF	19	3135	5700	3990
12.9	SUPA	31	3115	5704	3992
18.0	DYF	19	4237	7220	5054

For, the purposes of assessing prestressing strand extensions, calculations should be based on values of E taken from tests on specimens of the actual strand used. For design purposes a figure of 200 kN/mm^2 may be used.

Couplers An economic range of couplers has been designed for simple assembly on site. The first-stage tendon is stressed and anchored in the normal way using standard equipment and the dead-end of the second tendon is assembled around it, using swaged grips on each strand to afford maximum security.

The coupler assembly is enclosed with a conical cover which has a grout access point for second stage grouting.

'Freyssinet' Multi-Wire 12/7 mm and 12/8 mm
Prestressing Anchorages (All Tendons Tensioned Together)

Internal female cones				
Size	Diameter		Length	
	mm	in.	mm	in
12/7	120	$4^3/_4$	125	5
12/8	150	6	125	5

External female cones

Size	Diameter		Length	
	mm	in.	mm	in
12/7	140	$5^1/_2$	125	5
12/8	150	6	125	5
External cones should be specially ordered.				

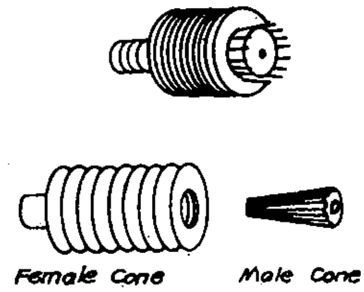
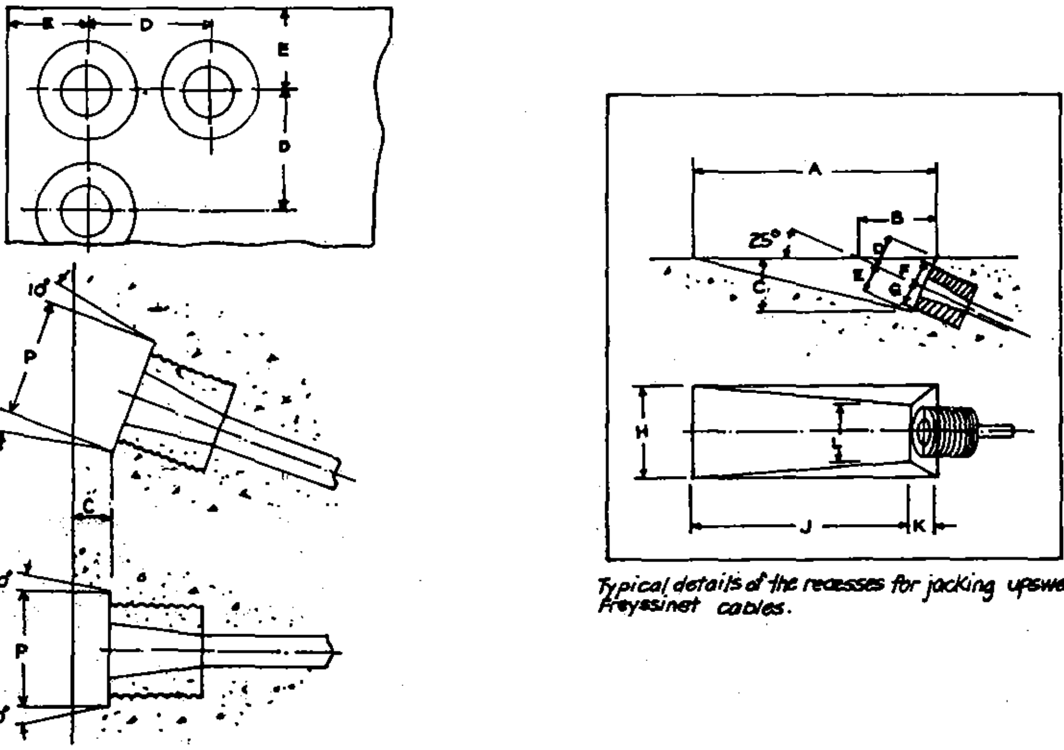


Fig. 6.5

Cone size		A	B	C	D	E	F	G	H	J	K	L
12/7	mm	600	200	140	85	70	60	60	220	550	65	140
	in.	24	$7^7/_8$	$5^1/_2$	$3^1/_4$	$2^3/_4$	$2^3/_8$	$2^3/_8$	$8^3/_4$	$21^1/_2$	$2^1/_2$	$5^1/_2$
12/8	mm	650	275	200	115	100	75	75	300	550	90	200
	in.	25	$10^3/_4$	$7^3/_4$	$4^1/_2$	4	3	3	12	$21^1/_2$	$3^1/_2$	8



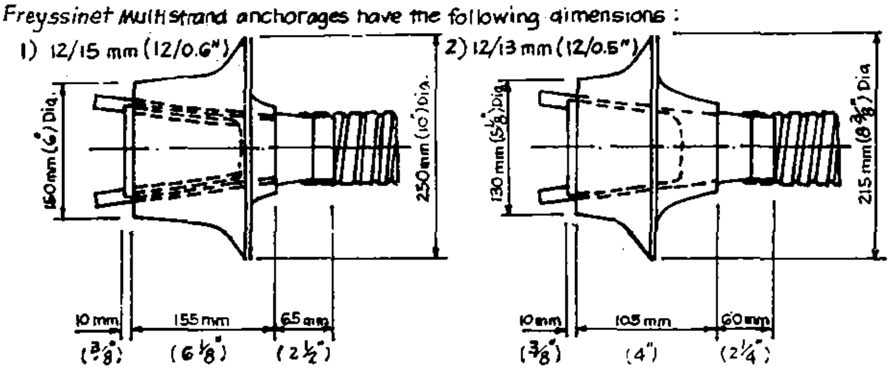
Typical details of the recesses for jacking upswept Freyssinet cables.

Fig. 6.6

'Freyssinet' Multistrand Anchorage

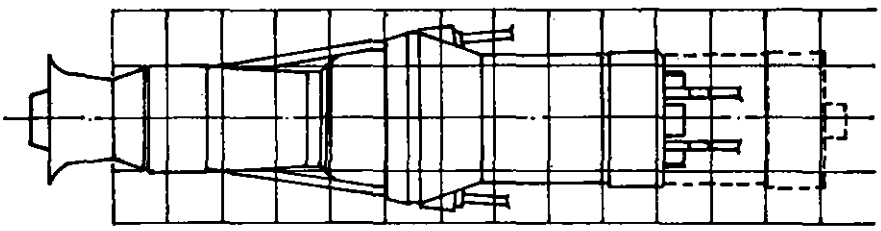
Cable Characteristics

Cable Type		Cable Diameter		Initial Design Force (80% ult.)		Initial Design Force (70% ult.)	
mm	in.	mm	in.	kN	kip force	kN	kip force
12/15	12/0.6	62	$2^1/_2$	2 180	490	1 907	428
12/13	12/0.5	52	$2^1/_8$	1 584	355	1 386	311



Anchorage Centres and Edge Distances

Anchorage size		Centre to Centre		Centre to Edge	
mm	in.	mm	in.	mm	in.
12/15	12/0.6	325	13	200	8
12/13	12/0.5	270	$10^1/_2$	150	6



JACK CLEARANCE DIAGRAM
SCALE : ONE SQUARE REPRESENTS 100 mm (4 in)

Fig. 6.7 *Anchorage detailing*

Blind-End Anchorages The normal anchorage can be used in accessible dead-end positions, but for situations where the anchorages are to be cast into the concrete, or are inaccessible, a range of blind-end anchorages is provided.

Looped Anchorages This is the preferred solution where a bond length is available, but due allowance has to be made in the overall design for the slow build-up of stress along the length of strand cast into the concrete. This anchorage is only suitable for small cables.

A saddle is fixed in position to space the strands and to assist in the distribution of the load to the concrete. A grout access point must be provided near to the sheath termination.

Swaged Anchorages This anchorage permits a rapid build-up of stress behind the guide. Swaged grips are used to ensure maximum security and the whole assembly is completed with a cast-iron cap containing a grout access point. This type of anchorage can be used for any size of cable.

Swaged Grips The swaged grip consists of a hardened steel inner coil over which a ductile steel body is compressed by drawing it through a die using a light, portable and robust jack designed for continuous site work. These swaged grips are used in couplers and blind end anchorages.

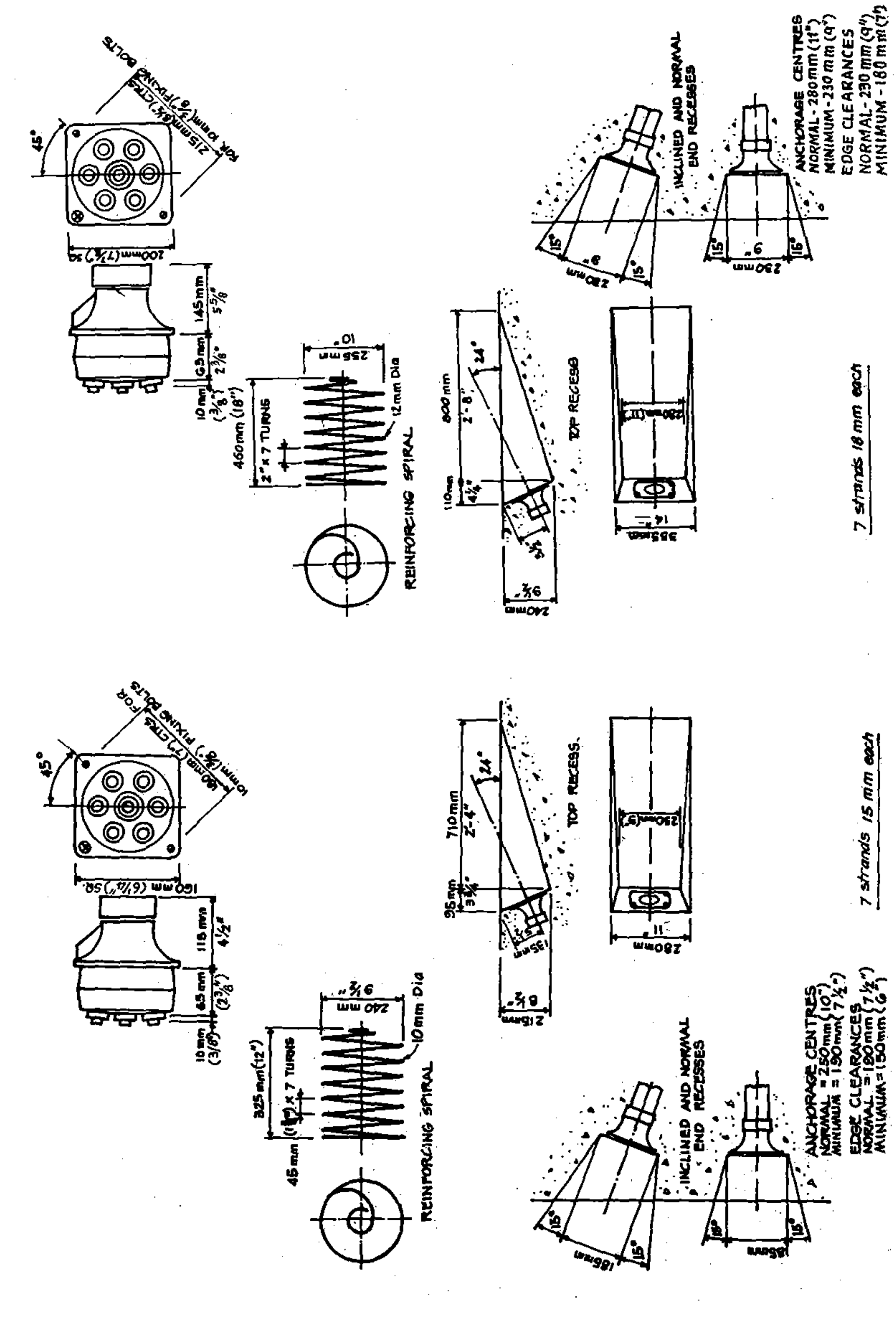


Fig. 6.8 'Freyssinet' monostrand anchorages (single tendon tensioning)

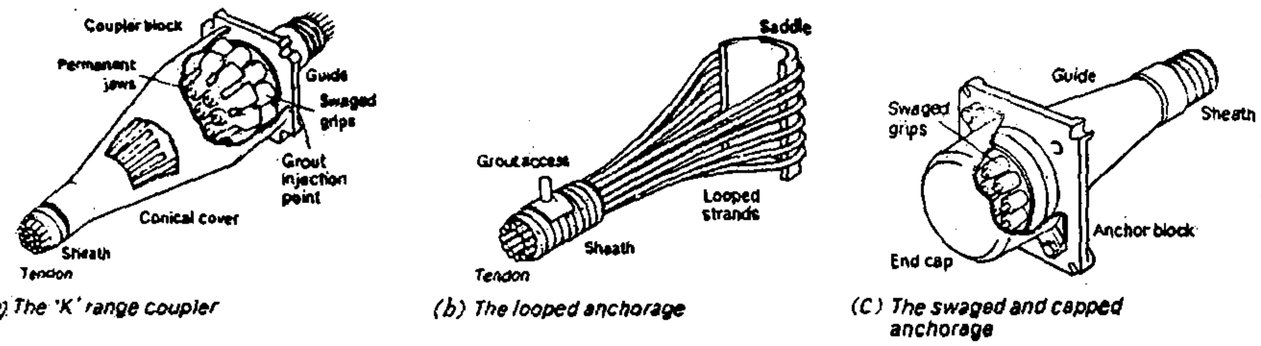


Fig. 6.9(a), (b), (c) *Freyssinet system*

External Prestressing

Removable External Prestressing

Placing of prestressing cables on the outside of structural concrete is by no means a new idea. Many applications of this type over the past fortyfive years, in various countries of the world, are based on this idea.

The use of external prestressing has proved to be of particular interest for strengthening of structures, whether for the purpose of adapting them to new loading regulations, or in order to make them comply with new design regulations and to completely restore their capability of resisting applied loads.

It was however, in France, in the course of the past fifteen years and at the instigation of SETRA* that external prestressing was given pride of place among the modern techniques used in the construction of new structures.

Freyssinet has designed and developed a new range of anchorages which takes into account dismantling requirements, safety in placing and under load and reinforced corrosion protection which ensures the durability of the prestressing.

A New Range of Anchorages—Two Systems of Cables

This new range of external prestressing anchorages offered by FREYSSINET INTERNATIONAL allows a choice between two systems of cables.

System 1: Ordinary Strand Cables (normal or super grade)

The cable is formed of ordinary strands threaded in the traditional manner into a thick, high density polyethylene (HDPE) sheath.

* French Ministerial Service for Technical Design of Roads and Motorways.

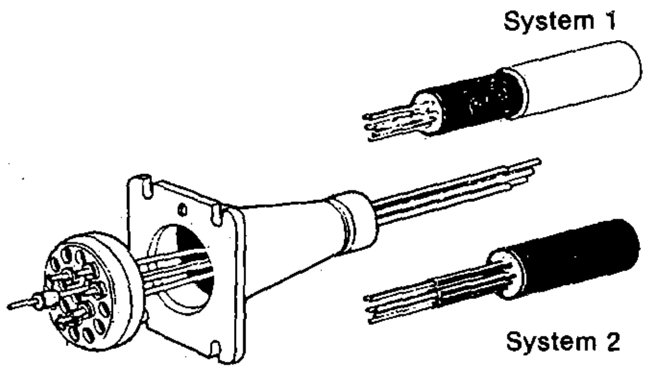


Fig. 6.10 *12 K 15 External prestressing anchorage (Freyssinet)*

The duct, which is continuous from one anchorage to the other, passes freely through the intermediate concrete cross-beams by means of sleeves, generally of metal, cast into the concrete of the structure during pouring.

These sleeves may also act as deviators.

Stressing and grouting, with cement grout, are then carried out in the traditional manner with standard equipment.

This method offers numerous advantages:

- As the duct is external to the structure, the quality of sheath placing and its watertightness may be checked at any moment.
- The cable/sheath friction coefficients are low and the high coefficient of transmission results in an appreciable improvement in the efficiency of the prestressing.
- The system is easily dismantable and allows replacement of the cable if necessary.

System 2: Plastic Coated, Greased Strand Cables

The cable is formed of greased strands, individually coated with a layer of high density, heat-extruded polyethylene and grouped together inside a thick HDPE sheath.

When the structure is cast *in situ*, the duct, which is continuous from one anchorage to another, passes through the deviation cross-beams of which it forms an integral part.

When the structure is precast special arrangements must be envisaged and defined case by case.

The originality of this method, which is proposed and patented by Freyssinet International, resides in the fact that the sheath is injected with cement grout prior to stressing, which prevents all interaction between strands during stressing and avoids damage to their individual protection system (grease + plastic coating).

Very low coefficients of friction and safety against corrosion are thereby guaranteed.

Stressing is then carried out strand by strand, in stages using a monostrand jack or in the traditional manner with a multistrand jack.

There are multiple advantages to this method:

- Strand/sheath friction coefficients are extremely low and the high value of the coefficient of transmission of the cable results in highly efficient prestressing.
- Stressing strand by strand allows the use of lighter, and therefore, more easily handled stressing equipment. The size of the jack does not constitute an obstacle to the instalment of large prestressing tendons.

  The use of a monostrand jack — less bulky than a multistrand jack — allows positioning of the axis of the cable closer to the wall of the structure, which is particularly advantageous when the cables are anchored in internal blisters (either cast-in-place or bolted on).
- Fourfold protection (HDPE sheath; cement grout; polyethylene coating and grease) guarantees a very high level of safety against corrosion.
- Ulterior adjustment of the prestress, during the life of the structure, is always possible — provided that the jacking lengths of strand, allowing gripping by the jack, are left uncut after the initial stressing operation.

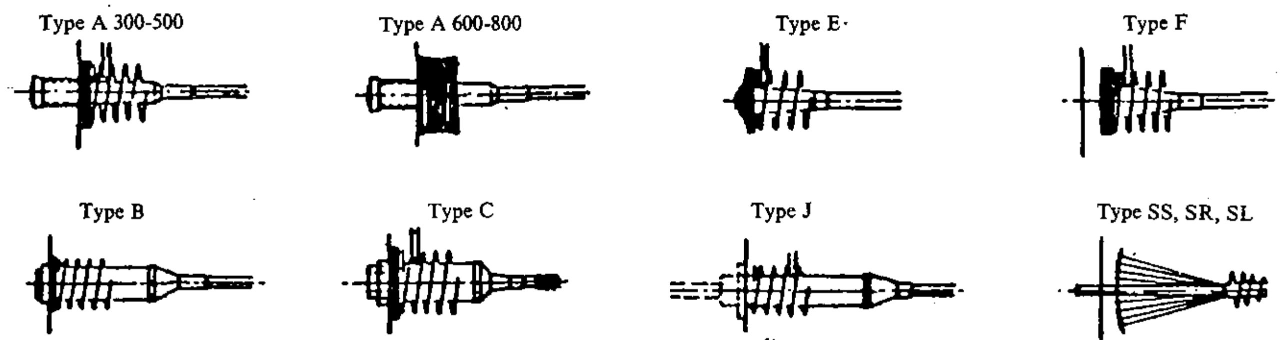
'B B R V' System

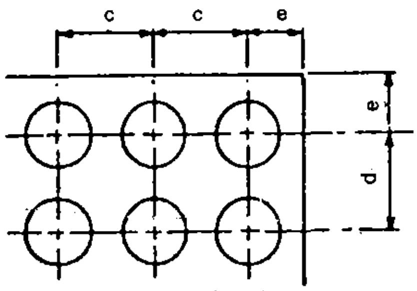
Small capacity tendons up to 2658 kN

Maximum working load (80% f_u)	Number of 7 mm dia. wires	Anchorages: Stressing	Anchorages: Fixed	Anchorages: Nominal reference	Standard single bearing plate	Combined bearing plate: Minimum centres of anchorages	Combined bearing plate: Minimum concrete edge distance	Sim-Tube sheathing internal dia.	Stressing equipment clearance dimensions with dynamometer	without dynamometer
kN	No.	Type	Type	Ref. No.	mm	mm	mm	mm	mm	mm
					$a \times b$	$c \times d$	e		$f \times g$ dia	$h \times g$ dia
387	8	B J	F SR SL	32	150 × 150 150 × 150 120 × 120 120 × 150 60 × 300	138 × 138	76	30	1524 × 310	1344 × 310
773	16	B J	F SR SL	64	175 × 175 175 × 175 160 × 160 150 × 220 80 × 400	171 × 171	89	40	1524 × 310	1344 × 310
1160	24	B J	F SS SR SL	100	200 × 200 200 × 200 220 × 220 220 × 220 160 × 300 80 × 560	197 × 197	108	50	1524 × 310	1344 × 310
1498	31	C	E SS SR SL	130	250 × 250 235 dia. 260 × 260 180 × 360 120 × 560	235 × 235	127	55	1524 × 340	1324 × 340
1643	34	B J	F SS SR SL	138	250 × 250 250 × 250 260 × 260 260 × 260 180 × 360 120 × 560	235 × 235	127	55	1524 × 340	1324 × 340
2029	42	C	E SS SR SL	170	280 × 280 270 dia. 300 × 300 200 × 450 140 × 650	241 × 267	152	65	1880 × 440	1580 × 440
2658	55	C	E SS SR SL	220	300 × 300 300 dia. 340 × 340 220 × 500 160 × 700	267 × 305	152	75	1880 × 440	1580 × 440

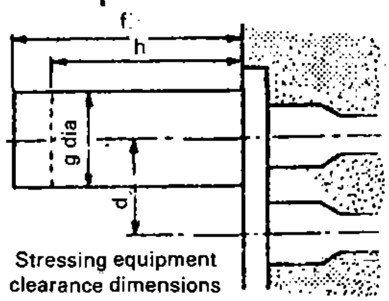
*Large capacity tendons up to 9036 kN Type A**

Maximum working load (80% f_u)	Number of 7 mm dia. wires	Anchorage nominal reference	Standard single bearing plate	Sheating internal diameter	Stressing equipment clearance dimensions with dynamometer	Preferred sizes
kN	No.	Ref. No.	mm	mm	mm	
			$a \times b$		$f \times a$ dia.	
3237	67	300	435 × 435	90	2500 × 550	
3817	79	350	435 × 435	95	2500 × 550	←
4397	91	400	435 × 435	105	2500 × 550	
4977	103	450	435 × 435	105	2500 × 550	←
5557	115	500	640 × 640	120	2600 × 800	
6716	139	600	640 × 640	130	2600 × 800	
7876	163	700	640 × 640	140	2600 × 800	←
9036	187	800	640 × 640	150	2600 × 800	



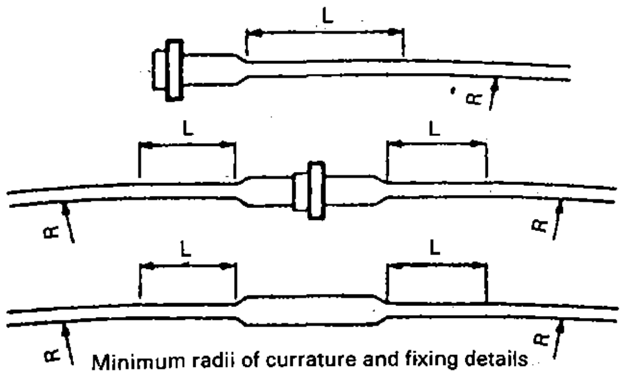


Combined bearing plate dimentions



Stressing equipment clearance dimensions

Nom. Ref.	Length L min.	Radius R min.
No.	mm	mm
32	480	1980
64	700	3050
100	920	3200
130	1090	3960
138	1090	3960
170	1320	4500
220	1520	5030
300	1550	5500
350	1650	5900
400	1750	6100
450	1800	6500
500	1900	7000
600	2000	7100
700	2100	7500
800	2200	8000



Minimum radii of currature and fixing details

Fig. 6.11

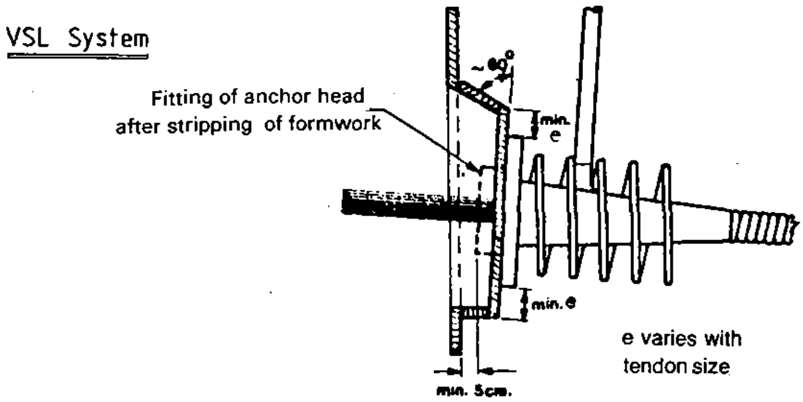


Fig. 6.12 *VSL system*

(Refer to Table 6.7)

Table 6.7

*13 mm (0.5 in.) STRAND										
Unit	No. of Strands	Duct Diameter mm		Standard fu = 165 kN		Super fu = 184 kN		Dyform Compact 209 kN		Jack Used
				A = 94.2	W = 0.744	A = 100.5	W = 0.80	A = 112.0	W = 0.89	
		Factory Assembled	Pull-through	0.7 fu kN	0.8 fu kN	0.7 fu kN	0.8 fu kN	0.7 fu kN	0.8 fu kN	
5-1	1	18/20	20/25	115	132	128	147	146	167	ZPE-30
5-3	2	30/35	30/35	231	264	257	294	292	334	ZPE-60
	3	35/40	35/40	346	396	386	441	438	501	
5-7	4	40/45	40/45	462	528	515	588	585	668	
	5	40/45	45/50	577	660	644	736	731	836	ZPE-100
	6	45/50	50/55	693	792	772	883	877	1003	
	7	50/55	55/60	808	924	901	1030	1024	1170	
5-12	8	50/55	60/67	924	1056	1030	1177	1170	1337	
	9	55/60	60/67	1039	1188	1159	1324	1316	1504	ZPE-200
	10	60/67	65/72	1155	1320	1288	1472	1463	1672	
	11	60/67	65/72	1270	1452	1416	1619	1609	1839	
	12	65/72	70/77	1386	1584	1545	1766	1755	2006	
5-19	13	65/72	70/77	1501	1716	1674	1913	1901	2173	
	14	70/77	75/82	1617	1848	1803	2060	2048	2340	
	15	70/77	75/82	1732	1980	1932	2208	2194	2508	
	16	75/82	80/87	1848	2112	2060	2355	2340	2675	ZPE-340
	17	75/82	80/87	1963	2244	2189	2502	2487	2842	
	18	80/87	85/92	2079	2376	2318	2649	2633	3009	
	19	80/87	85/92	2194	2508	2447	2796	2779	3176	
5-22	20	80/87	90/97	2310	2640	2576	2944	2926	3344	
	21	85/92	90/97	2425	2772	2704	3091	3072	3511	
	22	85/92	95/102	2541	2904	2833	3238	3218	3678	
5-31	23	85/92	95/102	2656	3036	2962	3385	3364	3845	ZPE-500
	24	90/97	100/107	2772	3168	3091	3532	3511	4012	
	25	90/97	100/107	2887	3300	3220	3680	3657	4180	
	26	95/102	100/107	3003	3432	3448	3827	3803	4347	
	27	95/102	110/117	3118	3564	3477	3974	3950	4514	
	28	95/102	110/117	3234	3696	3606	4121	4096	4681	
	29	100/107	110/117	3349	3828	3735	4268	4242	4848	
	30	100/107	110/117	3465	3960	3864	4416	4389	5016	
	31	100/107	110/117	3580	4092	3992	4563	4535	5183	
5-42	32	110/117	120/127	3696	4224	4121	4710	4681	5350	
	33	110/117	120/127	3811	4356	4250	4857	4827	5417	
	34	110/117	120/127	3927	4488	4379	5004	4974	5684	
	35	110/117	120/127	4042	4620	4508	5152	5120	5872	
	36	110/117	120/127	4158	4752	4636	5299	5266	6019	ZPE-1000
	37	120/127	120/127	4273	4884	4765	5446	5413	6186	
	38	120/127	130/137	4389	5016	4894	5593	5559	6353	
	39	120/127	130/137	4504	5148	5023	5740	5705	6520	
	40	120/127	130/137	4620	5280	5152	5888	5852	6688	
	41	120/127	130/137	4735	5412	5280	6035	5998	6855	
	42	120/127	130/137	4851	5544	5409	6182	6144	6922	
5-55	55	140/150	150/160	6352	7260	7084	8096	8046	9196	

A = nominal steel area per strand (mm²)
W = nominal weight per strand (kg/m)
} These values are approximate only and may vary slightly depending on the source of supply.

*Similar tables exist for 15 and 18 mm strands. For particulars see firm's catalogue.

CHAPTER 7

The Substructure

7.1 INTRODUCTION

Usually *substructure* of a bridge refers to that part of it which supports the structure that carries the roadway (called *superstructure*). Thus the substructure covers pier and abutment bodies together with their foundations, and also the arrangements above the piers and abutments through which the superstructure sits, i.e., bears on the substructure. The latter are called the bearings. These have been dealt with separately.

The more usual types of foundation for substructure are briefly discussed below:

Shallow Type

These are foundations generally placed after open excavation, and are called *open foundations*. Examples of such foundations are isolated footing, combined footing, strip footing, raft and cellular monolith. The latter is constructed at or near the ground level by building up its body in portions (lifts) and sinking down the monolith by openly excavating from within its cells and finally sealing or plugging its bottom and capping its top effectively to support the pier or abutment body on it.

Deep Type

These are constructed by various special means. Deep foundations are *piles* and *caissons* (or *wells*). 'Piles' are essentially giant-sized nails (of concrete, steel or timber) that are either driven into the subsoil (in which case they *displace* the soil in their place) or are placed-in after boring holes in subsoil (in which case they *replace* the soil in their place). These giant-sized 'nails' can be square, rectangular, H-shaped or circular in section (20 to 200 cm or more in diameter), and can range in length from about 4 to 40 m or more. A group of piles is capped together at top (usually by a reinforced concrete cap) to support the pier or abutment body above.

'Caisson' is basically constructed at the open surface level in portions and sunk downwards by essentially mechanically excavating soil from within its dredge-hole all the way till its cutting edge reaches the desired founding level, after which the well is effectively sealed (i.e., plugged) at bottom, then filled by sand (at least partly), and then capped at or near the surface level. The pier or abutment body is then constructed on the cap.

Owing to interaction between the bridge deck and its supporting structure, it is essential that the two be considered together in formulating the overall proposal. Ground conditions with significant differential foundation settlement possibility may rule out the use of structural forms involving continuous spans.

Soil investigation should be concluded by means of adequate boreholes, penetration tests, and complementary tests on appropriate soil samples. Investigation carried out without proper supervision and understanding may be of little value, and can even be misleading and may give rise to problems during and even after construction.

7.2 IMPORTANT DEFINITIONS

Abutments

These are the first and last supports (i.e., the end supports) of a bridge. Mass concrete construction is generally economical for small heights, but is not competitive with other available alternatives in reinforced concrete for taller heights. Upstanding cantilevered reinforced concrete walls are probably the most widely used form of construction for typical highway bridge abutments. For tall heights it is more economical to shape the plan-section of the wall-stem into a series of T-junctions. This allows use of wall panels of the minimum practical thickness in combination with cantilevered upstanding T-beams to act as counterforts.

For large abutments where the ground is rising away from the bridge, there can be advantages in using a *hollow abutment*. This consists of four walls forming a box in plan and supporting a deck of simple cast *in situ* reinforced concrete beam-and-slab construction. The front and side walls simply act as supports to the deck, while the rear wall retains the earthfill of the approach embankment. The potential advantage of this arrangement is that the height of the retaining wall at the rear of the hollow abutment is much less than would be required if the retaining wall were the front wall of the abutment.

Piers

The bridge-supports inbetween the abutment-supports are referred to as *piers*. The choice of construction of the bridge deck will dictate the choice of type of pier. If support is required at intervals across the full width of the bridge deck, then some form of supporting wall or portal frame is made for the pier. However, where a deck has some capacity within itself to span transversely at intermediate-support positions by means of a diaphragm within the depth of the deck, there is a wider choice available for the type of pier.

Simplicity in the formation of a pier not only has the merit of providing easier and more economical construction, but is also likely to produce a more attractive result. But for special cases, complex shapes may be adopted. In this case the bearings are placed at the heads or feet of the piers. A monolithic connection between the head of a pier and bridge deck looks clean, but bearings at the foot of a pier require a chamber, and may cause drainage problems which could create additional expenses. There are also problems of providing stability to the pier during construction. That is why bearings are usually preferred at the heads of piers.

Bank Seat (Dwarf Abutment Seat)

At the end of a riding span (the short-end span generally about 5 m.) of the bridge which is supported at the head of a slope formed by a cutting or embankment, the foundation may be a strip footing, a buried skeletal abutment, or a piled bank seat, depending upon the level of suitable founding strata. It is possible to detail this foundation in such a way that it enables the deck profile to continue into the earthworks without the supporting foundation being visible.

Piling

It may become necessary to employ piled foundations for bridge works where ground nearer the surface is too soft to sustain spread (acceptable sized) footings and is hence susceptible to substantial settlement. In addition to providing a means of supporting the foundation loads, the use of piling can make it possible for other groundworks (such as the construction of pile caps in place of spread footings) to be carried out at levels higher than might otherwise be possible. This can be beneficial where the foundation is to be built adjacent to a waterway or in waterlogged ground.

The choice of type of pile to be used is influenced by the ground conditions. Where rock or some other hard-bearing stratum occurs at an accessible depth, preformed piles driven to provide an end-bearing is an attractive proposition. Steel H-piles are more easily driven, cut, and extended, than reinforced concrete precast piles. However, it is self-evident that reinforced concrete is a more suitable material when corrosive conditions exist. Preformed piles can be driven at a rake of up to 1 : 4, thereby better absorbing the horizontal forces.

Piles of a large diameter are normally installed vertically. But it is still possible to absorb horizontal loads in this position, though this gives rise to bending in the piles. Methods of assessing the horizontal-load capacity of large-diameter piles have been developed and these utilize subgrade resistance in combination with stiffness of the pile. The techniques of constructing large-diameter bored piles are best suited to cohesive soils. Granular layers near to the surface can be successfully dealt with, but at greater depths the risks of the shaft-sides collapsing, become greater.

Piling adds to the cost of a bridge, so that the practicability of providing traditional footings always merits careful investigation. Even where the soil will only permit low-bearing pressures, it is cheaper to provide extensive spread footings than to employ piles. But individual cases differ and settlement problems may not permit footings.

For more details on piles see Sec. 7.4 and Sec. 7.5 of this chapter. This is also discussed in Ch. 11 of this book.

Under-Reamed Small-Diameter (up to 50 cm diameter) Bored Piles

Under-reamed small-diameter piles are bored cast *in situ* concrete piles, having one or more bulbs formed near the bottom by enlarging the bore hole for the pile-stem by an under-reaming tool. These piles find applications in certain land situations in different types of soils, where foundations are required to be taken down to a certain depth to avoid the undesirable effect of seasonal moisture changes (as in expansive soils, e.g., black cotton soils) or to reach strata or obtain adequate capacity for downward, upward and lateral loads. In expansive soils the pile cap should be cast 2 to 5 cm above the finished soil level.

A pile with only one bulb is known as single under-reamed pile, while one with more than one bulb is known as multi under-reamed pile. Generally, the diameter of under-reamed bulbs is kept equal to 2.5 times the diameter of pile-stem. However, it may vary from 2 to 3 times the stem diameter, if required, depending upon the design requirements and feasibility of construction.

Under-Reamed Large-Diameter (more than 100 cm diameter) Bored Piles (or Drilled Caissons)

Urban area flyovers sometimes demand large-diameter bored piles, with or without enlarged or 'belled-out' bases. The various aspects of design, construction, testing, and economics of large-diameter bored piles are basically same as those for other piles.

The savings in cost incurred by the use of under-reamed large-diameter bored piles are mainly due to savings in material excavated from the pile borehole, and in concrete used to replace the excavated material. However, in difficult ground such as boulder clays containing lenses of silt or sand, or in cohesionless soil, it is difficult to form a large diameter under-ream. Also, because of the longer time taken to form an under-ream by mechanical means or hand excavation (compared to the time taken to drill straight-sided piles), the economic advantages of the under-reamed large-diameter bored pile are somewhat marginal even in favourable ground such as clay.

Excavation for the under-ream is achieved by a belling bucket rotated by drill rods. Two types of belling buckets are used. The one generally favoured has arms which are hinged at the top of the bucket (Fig. 7.1) and are actuated by drill rods. The arms are provided with cutting teeth and the soil excavated is removed with the bucket. This type cuts to a conical shape, which has the advantage of maintaining stability in fissured soils. Besides, the arms are forced back into the bucket when it is raised from the hole.

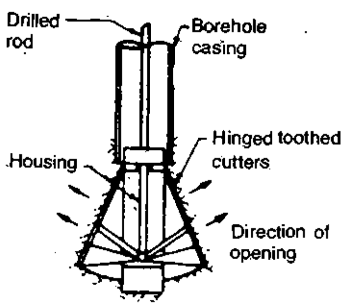


Fig. 7.1 *Top-hinged belling bucket*

The other type (Fig. 7.2) has arms hinged at the bottom of the bucket. This type has the advantage of being capable of cutting a larger bell than the top-hinged type; and because the action is always on the base of the hole, it produces a cleaner base with less loose and softened material. However, the hemispherical upper surface of the bell is less stable than the conical surface, and the bottom-hinged arms have a tendency to jam in the hole when raising the bucket.

Belling buckets normally cut to base diameters of up to 3700 mm, although diameters of as much as 7300 mm, are possible with special equipment. It is not usually practicable to form bells on piles having shafts of less than 760 mm diameter. Although the base of a mechanically under-reamed pile can be cleaned by specially designed mechanical tools, this is a somewhat tedious operation, and it is generally preferable to clean out the base by hand so that all soil crumbs and softened material are removed.

Enlarged bases can be formed in stable and relatively dry soils or rocks by hand excavation. This requires some form of support to the roof of the bell, to ensure safety of the

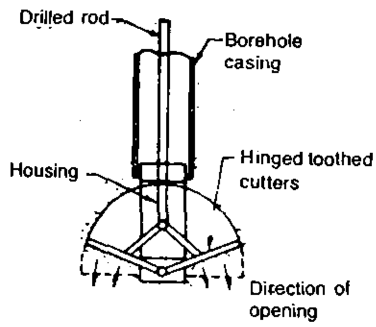


Fig. 7.2 *Bottom-hinged belling bucket*

workmen. One method of support which has been used is in the form of a "spider" consisting of a number of hinged steel ribs. The assembly is lowered down the borehole and the ribs are then expanded to force them into contact with the roof of the bell. If very large base areas are required, tunnels can be driven to connect the bells, and the whole base area can then be filled with concrete, which is suitably reinforced to achieve the required beam action.

It is often impossible to predict from ordinary site investigation boreholes, all the difficulties which may be encountered in attempting to form under-reamed bases on large-diameter piles. For this reason, it is a good practice to include an item in piling contracts for drilling a trial pile borehole in advance of the main piling contract. This item can be expensive as the selected piling contractor must bring his men and equipment onto the site and take them away again. But this procedure can often save a considerable amount of money at the main piling stage, since difficulties can be foreseen and any modifications to the pile design can be made, and if necessary, the idea of under-reaming the piles can be abandoned in favour of adopting deeper straight-sided piles. Test loading to check design assumptions can also be made at this preliminary trial stage.

It is desirable to make a close inspection of the base of all boreholes, with or without bells, to ensure that they are clean and free of softened material, and that the walls of the shafts are in a stable condition. Large-diameter bored piles are frequently used on a 'one-column-one-pile' basis, and it is unusual to provide more than three piles to a column. In these conditions, failure, even of a single pile due to faulty construction methods would have disastrous consequences. Therefore, during site supervision of piling contracts, the engineer should treat each pile in the same way as he would an ordinary pad foundation. That means the piling contractor must not be permitted to place concrete until the resident engineer or clerk of works has satisfied himself that the soil is not weaker than that taken as the basis for the pile design and that the hole is in a fit condition to receive concrete.

The following precautions are recommended in the construction and subsequent inspection of bored piles:

(i) The pile shaft should be supported by casing through soft or loose soils to prevent the walls of the shaft from collapsing. This casing may be extracted after concreting the shaft, depending upon whether or not the soil is too soft.

(ii) Casing should be provided to seal off soft water-bearing soil layers. Any soil, adhering to the inside of the casing, should be cleaned off, before inserting the casing in the borehole, and again before placing the concrete. The casing should be drilled into an impervious soil layer beneath the water-bearing layer to a sufficient depth to maintain the seal effect until the remainder of the borehole is completed, and until the concrete is brought up above ground-water level.

(iii) Soil or rock cuttings removed from the pile borehole should be compared and correlated with the descriptions stated on the site investigation borehole records, as otherwise more investigation could be warranted.

(iv) Shear-strength tests should be made when necessary on undisturbed soil samples taken from the bottom of selected piles, as a check on design data.

(v) Bores for shallow piles, if too small to enable a man to go down and inspect the base, should be inspected by shining a light down the shaft. Any loose crumbs or lumps of soil should be cleaned out before the concrete is placed.

(vi) All deep pile holes should be 'plumbed' to the bottom immediately before concreting by lowering a cage to the full depth. The plumbed depth should be compared to the depth accurately measured immediately after completion of drilling. This check will ensure that no soil has collapsed into the borehole.

(vii) All piles having a shaft large enough to admit a man should be inspected immediately before placing concrete. Any loose soil adhering to sides should be cleaned off.

Caisson (or Well) Foundations

Such foundations are advisable in rivers where a heavy scour at flood time would otherwise bare the piles and lead to buckling. In such conditions, it could also be possible for the floating debris to get entangled inbetween the naked pile shafts which can exert unpredictable forces in them. Caissons are relatively easy to construct, and other things being equal, could more than compete with piled solutions. Their stability essentially derives from the considerable passive resistance mobilised from the soil grip below the maximum scour level and the huge end-bearing resistance.

Diaphragm Walls

For vertical cuttings, such as those required for lengths of sunken road, the work of excavation can often be minimized by using such constructional techniques as *contiguous bored piling* or *diaphragm-wall construction*, in place of conventional retaining walls. Since these techniques are usually associated with particularly difficult ground conditions, such as those arising with over-consolidated clays, the design approach will involve consultation with authoritative experts.

The construction of a diaphragm wall requires excavation of a deep trench in short lengths, using a bentonite slurry to support the faces of the excavation where necessary. A prefabricated cage of reinforcement is lowered into the excavation, and concrete is placed by tremie. Each short length forms a panel, and the joints between panels introduce some measure of structural discontinuity into the wall.

Reinforced Earth Construction for Abutments and Retaining Walls

A rapidly constructed and lighter form of retaining wall construction has been developed in recent years. This is based on the use of facing panels that are stacked without any attempt to provide fixity or bond with adjacent units, but each panel is tied back to the earth-fill, by straps that are buried in the retained embankment during construction. The facing to a reinforced earth wall consists of precast concrete panels. In addition to giving a lighter wall than could be achieved in traditional reinforced concrete construction, this technique has the merit of allowing construction to proceed on ground which may not be suitable to form the foundation for a conventional wall. Joints between the facing panels are usually made to accept movements which may arise due to settlement. Such a flexibility of the finished construction makes it highly tolerant to differential settlement without affecting its structural integrity. The technique has been used for bridge abutments as well as free-standing walls. Some settlement is likely to occur, although this is nominal where the ground conditions are firm. In circumstances where the use of conventional abutments would involve extensive groundworks associated with foundations, it may be found that the use of reinforced earth could provide a solution which makes substantial savings by eliminating much of the groundwork. However, since there are only a few agencies who own patents for this type of construction, the cost may not always be as low as it should be relatively.

Analysis and design of Reinforced Earth Retaining Walls and Abutments is discussed in Ch. 16 of this book.

7.3 OPEN FOUNDATIONS

These foundations are generally of the following type:

(a) Isolated, combined, and strip footings.

(b) Raft foundations.

The design of the open foundations is based on complete subsoil investigations. But in the case of low safe-bearing capacity of soil, such foundations may have to be disallowed, owing to the possibility of problems of long-term settlement. However, some of the important features are given below for guidance:

(i) The selection of the appropriate type of open foundation will normally depend upon the magnitude and disposition of structural loads, requirements of structures (settlement characteristics, etc.), type of soil or rock encountered, allowable bearing pressures, etc. Where rocky stratum is encountered at shallow depths, it may be preferable to adopt open foundations because of its advantage in permitting proper seating over rock and speed of construction work.

(ii) In case the two adjoining foundations are at different levels, the horizontal distance between the foundations should be kept sufficiently large depending upon the type of soil on which the foundations are resting, to avoid overlapping of foundation pressure zones and consequent distress during construction.

(iii) For footings and raft foundations in erodible strata, protection against scour should be provided by means of suitably designed flooring, cut-off walls and launching aprons.

(iv) In the case of plain concrete, brick, or stone masonry footing, the load from the pier or column should be taken as dispersed through the footing at an angle of less than 45°C.

(v) For reinforced concrete footings, the design should satisfy the relevant provisions of standard design method.

(vi) The design of raft should be based on the assumption that it is resting on elastic soil medium, and guidance may be taken from standard literature on the subject.

(vii) The minimum thickness of the footing should not be less than 300 mm.

(viii) The protective works should be completed before the floods so that the foundation does not get undermined.

(ix) Excavation for open foundations should be done after taking the necessary precautions.

(x) Where blasting is required to be done for carrying out excavation in rock and there is likelihood of danger to the adjoining foundations or other structures, necessary precautions should be taken to prevent any such damage.

(xi) For laying the foundation concrete after de-watering, either of the following procedures should be adopted:

(a) A pit, moat, or trench, deeper than the foundation level as necessary, may be dug around the foundation pit so that the water is kept below the foundation level until after the concrete has suitably set.

(b) The water-table is depressed by *well-point* or other methods, until after the concrete has suitably set.

(xii) For laying the foundation concrete under water, if the percolation of water is heavy, it is advisable to lay the foundation concrete by skip boxes or tremie pipe. In the case of flowing water or artesian conditions, the flow should be stopped or reduced as far as possible during placing of concrete until after it has set. No pumping out of water should be permitted from the time of placing of concrete up to at least 24 hours after placement, for it may blow up due to the removal of balancing pressure against upward water pressure.

(xiii) All spaces excavated and not occupied by abutments, piers, or other permanent works, should be refilled with earth up to the surface of the surrounding ground with sufficient allowance for settlement. All backfill should be thoroughly compacted and in general, its top surface should be neatly graded.

(xiv) In case of excavation in rock, the annular space around the footing should be filled up to the rock surface with lean concrete of 1:3:6 mix.

7.4 PILE FOUNDATIONS

Pile foundations are suited for adoption in the following situations:

(i) Availability of good founding strata below large depth of soft soil.

(ii) Need to have very deep foundations beyond the limit of pneumatic operations (usually depth beyond 35 m or so).

(iii) Founding strata underlying deep 'standing' water, the strata being very hard not permitting easy sinking of wells.

(iv) Economic factors deciding the use of piles as compared to wells.

Classification of Piles

(i) Precast driven piles (soil displacement type)
(ii) Driven cast *in situ* piles (soil displacement type)
(iii) Bored cast *in situ* piles (soil replacement type)
(iv) Bored precast piles (soil replacement type)
(v) Driven steel piles (soil displacement type).

Precast Driven Piles

Advantages

(i) ensured quality of concrete
(ii) correct disposition of reinforcement
(iii) least deviations in structural dimensions
(iv) fast pace of construction.

Disadvantages

(i) not suitable for depths greater than about 24 m, as lengthening of piles is difficult and may involve proprietary systems for jointing
(ii) difficult to be driven through stiff layers, or layers containing, boulders
(iii) unsuitable if piles are to be anchored or keyed to rocky strata
(iv) limited load carrying capacity on account of relatively small structural size.

Driven Cast in situ Piles

Advantages

(i) suitable for larger depths of the order of 50 m or so
(ii) can penetrate a harder strata by virtue of the steel shoe at the end.

Disadvantages

(i) in the cast *in situ* concreting operations, there is a likelihood of segregation of concrete due to dropping it from height and interference by the reinforcement
(ii) in uncased piles 'necking' takes place each time the casing is lifted, while concreting is being done to fill up the hole
(iii) in softer soils with uncased piles, the caving in of the soil, as also waving, is possible
(iv) these piles are also generally of small diameter (under 100 cm) and therefore may not be very competitive particularly when negative friction is expected to develop and/or keying with rock is required.

Bored Cast in situ Piles

Advantages

(i) very large depths can be achieved
(ii) larger diameter piles are possible, achieving higher load carrying capacity
(iii) no disturbance of the surrounding mass of soil and hence no reduction in soil resistance
(iv) specially suited where harder and stiffer strata have to be penetrated (e.g., stiff clay)
(v) easy for keying into rock
(vi) in view of larger load carrying capacity the number of piles can be reduced under each foundation, which may reduce construction time.

Disadvantages and difficulties

(i) larger equipment cost
(ii) precise constructional control on concreting procedure, concrete mix, etc.
(iii) maintenance of correct specific gravity of bentonite slurry at about 1.1 to 1.2, otherwise sides may collapse and cave in
(iv) reduced skin-friction (since the pile is not driven into the soil).

Safe Load Carrying Capacity

It shall be lesser of the following two values:

(a) ultimate load carrying capacity based on the soil parameters surrounding the pile divided by a suitable factor of safety.
(b) structural strength of the pile.

These shall be assessed as given here.

Ultimate Load Carrying Capacity Based on Soil Parameters

Static Formula (based on soil data)

$$R_u = R_b + R_f - W - R'_f$$

where R_u = ultimate load carrying capacity
R_b = ultimate base resistance
R_f = ultimate positive skin resistance
W = self weight of pile
R'_f = ultimate negative skin friction
$R_b = A_b \times f_{bu}$
where A_b = plan area of base of pile
f_{bu} = ultimate bearing capacity of the soil at the pile base
and $R_f = A_s \times f_s$
$R'_f = A'_s \times f'_s$
A_s and A'_s = surface areas of pile in positive and negative skin friction zones, respectively (latter, for instance, exists from normal bed level down to max. scour level).
f_s and f'_s = average skin friction or 'adhesion factor times cohesion value' per unit area of the pile surface (depending on soil type) in positive and negative skin friction zones, respectively.

For working out safe load carrying capacity of the pile, a factor of safety of 2.5 may be adopted.

In addition, 'block failure' and 'group action' must be considered. For details see Ch. 11 ahead.

Structural Strength of Pile

The structural strength of a pile section shall be assessed based on its axial load and moment (caused by lateral loads) based on short or long column action, depending on the location of upper and lower points of fixity in it. (See the earlier mentioned reference for more details, including the effects of pile cap movement, defective construction, buckling effect, etc.)

In addition to the above points the following general features shall also be satisfied in design and construction of piles.

Spacing of Piles

(a) Friction piles—spacing, centre to centre, not less than perimeter of the pile.
(b) End bearing piles—spacing, centre to centre, not less than twice the least width of pile.
(c) Generally—2.5 times the bigger dimension of pile section in plan.

Size of Concrete Piles

Not less than 0.75 m diameter or equivalent section area for bridge foundations in major river, and not less than 0.4 m diameter or equivalent section for other locations, viz., wing walls, foundations for flyovers, etc.

Rake in Piles

The maximum rake normally should not be more flat than the following:

(i) 1 in 8 for pile diameter 0.75 m and above
(ii) 1 in 5 for smaller diameter bored piles
(iii) 1 in 4 for smaller diameter driven piles.

Tolerance in Pile Alignment

(i) for vertical piles: 75 mm at piling platform level and tilt not exceeding 1 in 50
(ii) for raking piles: tolerance of up to 5% in rake and up to 75 mm in position at platform level
(iii) where raking piles are installed from a level significantly different/higher than the ground level, then the 75 mm lateral tolerance mentioned above shall be suitably increased due to the effect of error in the rake.

Pile Cap

All pile caps shall be in reinforced concrete, and their sizes fixed, taking into consideration, the allowable tolerances mentioned above. A minimum off-set of 150 mm shall be provided beyond the outer faces of the outermost piles in the group. For pile caps resting on earth a levelling course of minimum 80 mm thickness on lean concrete shall be provided. The attachment of the pile-head to the cap shall be adequate for transmission of loads and forces. Concrete piles are stripped off at the top so that their reinforcement is exposed and anchored into the cap. About 50 to 100 mm of the pile itself should project into the pile cap, and the bottom reinforcement in the cap suitably adjusted. In marine conditions, or in areas exposed to the action of harmful chemicals, etc., apart from the use of densely compacted concrete of a higher grade, the pile cap and the piles up to about 600 mm above low water or tide level should be protected with a suitable anti-corrosive paint. High alumina cement, i.e., quick-setting cement shall not be used in marine construction, instead blast furnace slag cement is preferable. Pile caps should be designed against shear as well as bending due to pile forces. Critical section for shear is where a 45° plane from column face meets the mid depth plane of cap. Piles falling 150 mm or more inside such section may be ignored in contributing shear at this section.

Mix of Concrete

The concrete used in the piles shall not be leaner than 1 : $1^1/_2$: 3 or equivalent controlled concrete with cement content not less than 350 kg/cu.m of concrete from durability considerations.

Driven Precast Concrete Piles

Longitudinal reinforcement The section area of longitudinal reinforcement shall be based on the actual design but shall not be less than the following percentages of the cross-sectional area of the piles:

(i) for piles with a length less than 30 times the least width, 1.25%
(ii) for piles with a length of 30 to 40 times the least width, 1.5%
(iii) for piles with a length greater than 40 times the least width, 2%

The curtailment of the reinforcement along the length of pile shall avoid sudden discontinuity which may cause cracks during heavy driving.

Lateral reinforcement The lateral reinforcement is of particular value in restraining driving stresses and should be in the form of hoops, spirals or closed links. The minimum diameter of bars for this purpose shall not be less than 6 mm. The lateral reinforcement in the pile from each end for a distance of about 3 times the least width or diameter shall not be less than 0.6% of the gross volume, and in the body of the pile not less than 0.2% of the gross volume.

Driven Cast in situ Concrete Piles

(i) Reinforcement in the pile should preferably be provided in the entire length and shall be based on design requirements. However, longitudinal reinforcement within the pile shaft shall not be less than 0.4% of the cross-sectional area.

(ii) Where the casing pipe is withdrawn for the formation of cast *in situ* piles, the concreting should be done with necessary precaution to minimise the softening of the soil by excess water. Cast *in situ* piles shall not be allowed where mud flow conditions exist.

Bored Cast in situ Piles

(a) Reinforcement in the pile shall be provided in the entire length depending upon the manner of transmission of the load by the pile to the soil, and shall not be less than that specified above for driven cast *in situ* concrete piles.

(b) In soils which are stable it may often be possible to drill an unlined hole and place the concrete without having a casing. In such cases even if 1 : $1\frac{1}{2}$: 3 concrete is used, the permissible stresses shall be limited to 90% of the design value. In cases in which side soil can fall into the hole, it is necessary to stabilize the sides of the bore hole with drilling mud, e.g., bentonite, and where possible a suitable steel lining may be used. The liner may be withdrawn when the concrete is poured in or it may be left in position permanently especially in cases where the aggressive action of the ground water is to be avoided or in the case of piles built in marine/muddy conditions.

Precautions for Concreting under Water*

(i) The concreting of the pile must be completed in one continuous operation, using tremie method.

(ii) The concrete should be easily workable, rich in cement (not less than 370 kg/m^3) and of slump not less than 150 mm.

(iii) When concreting is being carried out under water, a temporary casing should be installed to the full depth of soil, except the portion in rock, in order the fragments of soil cannot drop from the sides of the hole into the concrete as it is placed.

(iv) The tremie pipe will have to be large enough with due regard to the size of aggregate. For 20 mm aggregate the tremie pipe should be of diameter not less than 150 mm, and for larger aggregate larger diameter tremie pipes are required.

(v) The first charge of concrete should be placed with a sliding plug pushed down the tube ahead of concrete, so as to prevent mixing of concrete with water in pipe.

(vi) The tremie pipe should always penetrate well into the placed concrete, with an adequate margin against accidental withdrawal if the pipe is surged to discharge the concrete.

(vii) The pile should be concreted wholly by tremie and the method of deposition should not be changed partway of the pile (to prevent the laitance from being entrapped within the pile).

(viii) All tremie tubes should be scrupulously cleaned after use (and in any case before any concrete in them sets).

(ix) The top of concrete in a pile shall be brought above the cut-off level to permit removal of all laitance and weak concrete before pile cap is laid. This will ensure good concrete at the cut-off level after stripping open its bars for embedment into cap.

Load Tests on Piles

Ultimate and working load tests shall be carried out as outlined in various standards as applicable to the particular contract in question.

7.5 SMALL DIAMETER SINGLE AND DOUBLE UNDER-REAMED AND RELATIVELY SHORT BORED PILES (. . . I.S.2911 — PART III, 1973)

Construction of Small Diameter Under-reamed Piles

The various stages involved in the construction of small diameter short under-reamed piles are as given here:

(a) Boring by augers
(b) Under-reaming by under-reamer
(c) Placing reinforcement cage in position
(d) Concreting of pile
(e) Concreting of pile caps.

One of the equipments used for boring and under-reaming has been developed by the Central Building Research Institute, Roorkee, India. It is covered by an Indian patent and is licensed to M/s MSJ (Engineers) and Co., Roorkee. Boring is done with the help of a spiral auger. The use of a boring guide is essential to keep the bore holes vertical and in position. Each guide is provided with a circular collar and four arms. The collar is fixed to the boring guide on the lower side and it does not allow the mouth of the bore hole to widen due to frequent insertion and removal of the auger and other boring tools. After setting the guide assembly in position, the spiral auger is introduced into the circular collar of the guide by opening out the two sets of flaps of the guide assembly. The auger is pressed down and rotated manually until the spirals are half full of earth. The auger is then taken out and earth removed. The auger is again introduced and the boring process repeated till the required depth is reached.

Under-reaming or locally bulbing the stem of bore hole at the required depth is achieved by means of the under-reamer consisting of an assembly of two collapsible blades

* Also refer to the author's book "Concrete for Construction — Facts and Practice" for more details regarding 'under water concreting by tremie'.

fixed around the central shaft with a detachable bucket for receiving the cut soil. The equipment is attached to extension rods and lowered down the hole (which has already been bored to the required depth) until the bucket rests at the bottom of the bore hole. The guide flaps are then closed. The tool is pressed down constantly, and at the right elevation, rotated slowly. The cutting blades of the tool open out and start cutting the sides of hole. The loose earth is collected in the bucket at the bottom. When the bucket is full, the assembly is pulled out and the bucket is emptied. The depth of the bore hole is checked each time before insertion of the under-reamer so that any loose earth spilled from the bucket is removed (otherwise the bucket position will get shifted upwards due to loose soil lying at the bottom, and this will shift the position of the bulb). The under-reamer is then lowered into the bore hole and the process repeated until the cutting blades have expanded fully and no further earth is cut by the blades. Generally, removal of about eight buckets full of earth is required for completion of one under-ream for an average sized bulb (pile stem diameter up to 50 cm).

In the case of double under-reamed pile, further boring is done, after the first bulb is formed. After boring to the required depth, under-reaming for the second bulb is carried out. The dimensions of the bulb can be checked by means of a graduated GI pipe assembly. After the bore hole and under-ream are checked, the reinforcement cage, already fabricated, is lowered into the hole. Concreting of the pile is carried out through a concreting funnel placed at the mouth of the bore hole and care is taken to see that the top of the pile shaft is 5 cm higher than the bottom of the pile cap to be cast on it. Care must be taken against possibility of segregation in poured concrete.

Details of Pile and Under-reamed Bulb

In deep layers of expansive soil a minimum pile length of 3.5 m is recommended. Where the ground movements become negligible, in shallow depths of expansive soils, and other poor soils, the length may be reduced depending upon the load requirements, and the piles taken-down to at least 50 cm in the stable zone (i.e., a zone where there are no ground movements due to seasonal moisture changes). The pile length may be increased for higher loads.

The diameter of manually bored piles ranges from 20 to 37.5 cm. The spacing of the piles should be considered in relation to the nature of the ground, the type of piles, and the manner in which the piles transfer loads to the ground. Generally, the centre to centre spacing for under-reamed piles should not be less than 2 D_u (where D_u is the under-reamed diameter). It may be reduced to 1.5 D_u when a reduction in load carrying capacity of 10% is allowed. For the spacing of 2 D_u the bearing capacity of pile group may be taken equal to the number of piles multiplied by the bearing capacity of the individual pile. If the adjacent piles are of different diameters, an average value for spacing should be taken. The maximum spacing of the under-reamed piles should not normally exceed 2 $^1/_2$ metres so as to avoid heavy caps.

In double under-reamed piles of stem diameter less than 30 cm, the center-to-center vertical spacing between the two under-reams may be kept equal to 1.5 D_u, while for piles of 30 cm and more, this distance may be reduced to 1.25 D_u. The upper bulb should not be placed too close to the ground. The minimum desirable depth of the centre of this bulb is 1.5 m or 2 D_u, whichever is greater.

Load Carrying Capacity of Small Diameter Under-reamed Piles Based on Soil Properties

The under-reamed pile is nominally reinforced with longitudinal bars of 10 to 12 mm diameters and 6 mm diameter rings. The details of the carrying capacity and *minimum reinforcement* are shown in Table 7.1, but structural design of the stem section should be carried out for actual stresses in order to decide the amount of steel and grade of concrete. A clear cover of 4 cm is generally provided to the reinforcement.

Load Test on Small Diameter Under-reamed Piles

Piles may be tested for determining their load carrying capacity in compression, tension and lateral loading. Two categories of tests are conducted,

(i) initial test
(ii) routine tests

Initial tests should be carried out on test piles, not on working piles. In case the initial tests show consistently higher or lower values than the estimated safe allowable loads on piles, designs should be re-examined and necessary modifications should be made. Routine tests are carried out as checks on working piles.

Procedure for Initial Test (Axial Compression)

Following are the recommendations of Indian Standard IS:2911 (Part III) - 1973.

1. The test shall be carried out by applying a series of loads to the pile unaided by any other support. Pile groups may be tested as free-standing piles or piled foundations, as specified.

   The load shall preferably be applied by means of a hydraulic jack reacting against a loaded platform, or rolled steel joists or suitable load frame held down by soil anchors and piles or other anchorage. The anchor piles may also be working piles, but they shall be sufficient in number and adequately reinforced to take the full tension with a proper factor of safety. The reaction available for loading should not be less

Table 7.1 *Safe loads for vertical under-reamed piles in sandy and clayey soils including black cotton soils as recommended in the IS:2911 Part III-1973*

		'Minm.' Reinforcement* (See Notes 3 and 13 below)			Safe Loads									
		Longitudinal		Spacing	Bearing Resistance (Compn.)				Uplift Resistance (Tension)				Lateral thrust	
Dia. of pile (D)	Under-ream Dia (D_u)	No. of bars	Dia.	of 6 mm dia. rings	Single under-reamed	Double under-reamed	Increase per 30 cm length	Decrease per 30 cm length	Single under-reamed	Double under-reamed	Increase per 30 cm length	Decrease per 30 cm length	Single under-reamed	Double under-reamed
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
cm	cm		mm	cm	t	t	t	t	t	t	t	t	t	t
20	50	3	10	18	8	12	0.9	0.7	4	6	0.65	0.55	1.0	1.2
25	62.5	4	10	22	12	18	1.15	0.9	6	9	0.85	0.70	1.5	1.8
30	75	4	12	25	16	24	1.4	1.1	8	12	1.05	0.85	2.0	2.4
37.5	94	5	12	30	24	36	1.8	1.4	12	18	1.35	1.10	3.0	3.6
40	100	6	12	30	28	42	1.9	1.5	14	21	1.45	1.15	3.4	4.0
45	112.5	7	12	30	35	52.5	2.15	1.7	17.5	25.75	1.60	1.30	4.0	4.8
50	125	9	12	30	42	63	2.4	1.9	21	31.5	1.80	1.45	4.5	5.4

* Mild Steel

NOTE

1. The value of bearing resistance, uplift resistance and lateral thrust, given in the table are for a minimum pile length of 3.5 m, except in double under-reamed piles of more than 30 cm diameter. In such double-reamed piles, the minimum recommended lengths for 37.5, 40, 45 and 50 cm piles will normally be 3.75, 4.0, 4.5 and 5.0 m, respectively, so as to suitably accommodate the bulbs at specified distances.
2. Longitudinal bars may be curtailed or eliminated towards the toe depending upon the stresses in pile section.
3. For under-reamed piles subjected to a pull and/or lateral thrust, the requisite amount of steel should be provided as per proper design.
4. Values given in columns 14 and 15 for lateral thrust may not be reduced for changes in pile lengths and are fairly conservative. Higher values may be adopted after conducting lateral load tests on single or group of piles for surety.
5. In 25 and 30 cm dia., normal under-reamed piles when concreting is done by tremie, equivalent reinforcement in shape of single angle iron piece placed centrally may also be considered.
6. When a pile designed for a certain safe load is found to be just short of the load required to be carried by it, an over load of up to 10% may be allowed on it.
7. For working out the safe load for a group of the above piles, the safe load of individual piles is multiplied with the number of piles in the group. This would be applicable for piles taking lateral thrusts also.
8. Only 75% of the above safe loads should be taken for piles in which the bore holes are full of sub-soil water during concreting. When water is confined to the bucket portion only, no such reduction need be made.
9. In sandy soils when boring and under-reaming under water, minimum size of pile recommended is 25 cm.
10. In multi-under-reamed piles the depth of the center of upper bulb below ground level shall be kept a minimum of two times the diameter of the under-ream bulbs.
11. The values given above may be increased by 50% for broken wire condition in the design of transmission line tower footings.
12. Safe loads for multi-under-reamed piles may be worked out from the table by allowing additional 50% of the load as per col. (6) for each additional bulb. Increase in capacity due to increase in length will be as per col. (8).
13. Based on actual structural design, either the pile shaft should be suitably increased in diameter and/or additional reinforcement provided (treating it as a short column) to control the stresses in section design. The indicated reinforcement is only the minimum but actual amount must be structurally designed, to suit.
14. The load carrying capacity of an under-reamed pile may be determined from load test. In the absence of actual tests, the safe loads allowed on piles under-reamed to 2.5 D may be taken from above table (IS:2911, part III-1973). The safe loads given in the table apply to both medium compact sandy soil and clayey soils of medium consistency. For dense sandy ($N \geq 30$) and stiff clayey ($N \geq 8$) soils, the loads may be increased by 25%. However, the values of the lateral thrust should not be increased unless stability of the top soil (i.e., strata to a depth of about three times the stem diameter) is ascertained. On the other hand, a 25% reduction should be made in case of loose sandy ($N \leq 10$) and soft clayey ($N \leq 4$) soils. However, these are only rough guidelines to arrive at a first shot design which must be substantiated by actual pile load tests which alone can decide the acceptable design.

than three times the estimated safe load carrying capacity of the pile. The jack should be of adequate capacity, preferably with a remote control pump, and should have a pressure gauge or other suitable device for reading the applied loads.

2. Readings of settlement and rebound shall be recorded with the help of at least three dial gauges of 0.02 mm sensitivity, positioned at equal distances around the pile. The dial gauges shall be fixed to datum bars resting on non-movable supports at least $5D$ (subject to a maximum of 2.5 m) away from the piles where D is the pile stem diameter.
3. The test load shall be applied in increments of about 1/5 of the estimated safe load. At each stage of

loading/unloading, the load shall be maintained till the rate of movement of the pile top is not more than about 0.02 mm per hour.

4. Loading shall generally be continued up to $2\frac{1}{2}$ times the estimated safe load or to a settlement of 7.5% of the bulb diameter, whichever is earlier.
5. The safe load on the pile shall be the least of the following:
   - (i) Two-thirds of the final load at which the total settlement attains a value of 12 mm, unless it is established that a total settlement different from 12 mm is permissible in a given case on the basis of nature and type of the structure; in the latter case the actual total settlement permissible shall be used for assessing the safe load instead of 12 mm.
   - (ii) Fifty per cent of the final load at which the total settlement equals 7.5% of the bulb diameter.

Procedure for Routine Test (Axial Compression)

Loading shall be carried out up to $1\frac{1}{2}$ times the working safe load. The procedure followed for the test and determination of the allowable load shall be the same as per initial test excluding item (ii) above.

7.6 CAISSONS OR 'WELL' FOUNDATIONS

Suitability

Caisson construction is almost restricted to major foundation works, where other types of foundations cannot satisfy the requirements economically. Usually a caisson is advantageous as opposed to other types of deep foundations when some of the following conditions exist:

- (i) A massive substructure is required to extend to well below the river bed in order to attract necessary net soil resistance against overturning, heavy scour, rolling boulders, and floating debris. Under such conditions piles would obviously be unsuitable.
- (ii) The substrata contains large boulders which obstruct penetration of piles.
- (iii) The foundation is subjected to large lateral forces.

Historical Note

The essential feature of caissons is that they are constructed above the ground or water level and then sunk as a single unit to the required depth, and also that this unit forms part of the permanent works. Because extensive temporary works, such as sheet piled cofferdams, are not required, they are specially suited to work in deep and fast-flowing waterways. Open well-type caissons were used by the inhabitants of India, Burma (now Myanmar) and Egypt for many centuries for the foundations of river bridges. The masonry of the wells was built on timber curbs, and the caissons sunk by hand excavation from within the wells. Skin divers doing the excavation could not work deeper than 6 m, which limited the usefulness of caisson foundations of this type to sites where a firm inerodible stratum could be reached within this depth. However, the engineers who were responsible for bridging over mighty rivers in India in the late nineteenth and early twentieth centuries, adopted the methods of using grabs and sand-pumps for underwater excavation in the wells. By these methods they were able to sink caissons to depths of more than 30 m. A notable example of this construction was the caissons for the Hardinge Bridge over the Lower Ganges, where the river piers were sunk to depths varying between 32 and 36 m below river bed level.

Compressed air was first used in bridge caissons by John Wright in 1851 for the piers of Rochester Bridge, and a few years later by Isambard Brunel at Saltash Bridge. Its first use for the foundations of very large bridges was by James B Eads for the St Louis Bridge over the river Mississippi, commenced in 1869. The two river piers were sunk under compressed air to depths of about 30 m, which was a notable achievement, since the physiological effects of working under high air pressures were more or less unknown at that time. The sinking methods devised by Eads have only been changed in matters of detail up to the present day. A new development in caisson construction known as the floatation caisson principle was introduced in 1936 by Daniel E Moran for the San Francisco-Oakland Bay bridge.

Usually the limiting depth of cofferdams is about 20 m. Caissons are therefore essential for constructing foundations through water or through unstable shifting ground to depths greater than this.

Caisson Construction and Sinking Methods (also refer to the author's other Book: "Concrete Bridge Practice... Constructions, Maintenance and Rehabilitation".)

Construction of Well Curb (Shoe)

The normal practice in caisson construction is to build the 'shoe' on land and slide or lower it into the water for floating out to the site, or to construct it in a dry dock which is subsequently flooded to float out the shoe. Land caissons are of course constructed directly in their final position. Caisson shoes, constructed on the bank of a river or other waterway, are slid down launching ways into the water, or rolled out on a horizontal track and then lowered vertically by a system of jacks and suspended links. Gently sloping banks on a waterway with a high tidal range favour construction on sloping launching ways, whereas steep banks either in tidal or non-tidal conditions, usually require construction by rolling out on a horizontal track.

Care must be taken to avoid distortion of the shoe during construction. On poor ground the usual practice is to lay a thick blanket of crushed stone or brick rubble over the building site and to support the launching ways on timber or steel piles.

Economy in temporary works is given by constructing caissons in their final position. This can be done for land caissons, and for river work by constructing caissons on the dry river bed or on sand-islands (in up to 6 m of water). This is only advisable when the low water periods can be predicted reasonably accurately, and there is no risk of sudden 'flash' floods.

Towing a Floating Caisson to Sinking Site

The operation of towing a floated caisson from the construction site to its final location must be carefully planned. Soundings must be taken along the route to ensure an adequate depth of water at the particular state of tide or river stage at which the towing is planned to take place. An essential requisite of the launching, towing, and sinking programme is a 'stability diagram' for the caisson. This shows the draught at each stage of construction. In these diagrams the draught is plotted against the weight of the caisson for various conditions of free floating or floating with compressed air in the working chamber. The weight of each strake of skin plating and concrete within its walls to be added to give a desired draught can be read from the diagram. Also, the air pressures in the working chamber required to give any desired internal water level can be read off the appropriate lines.

Bed Preparation

The first operation is to take soundings over the sinking location to determine whether any dredging or filling is required to give a level bed for the caisson. A study should be made of the regime of the waterway to determine whether any bed movement is caused by vagaries of current. Such movement can cause difficulty in keeping a caisson plumb when landing it on the bottom, especially at the last stages when increased velocity below the cutting edge may cause non-uniform scour. Difficulties with bed movement can be overcome by sinking flexible mattresses of crushed stone on the sinking site, and sometimes by constructing control-dykes (or spurs) upstream and downstream that can alter the bed level considerably and also even it up.

Sometimes constructing a dyke (normal or inclined to the river bank) upstream of the site can silt up the area and very significantly reduce the water depth and velocity, which can be of immense help.

Supporting Structures

The various methods used to hold a caisson in position during its lowering include

(a) an enclosure formed by piling
(b) dolphins formed from groups of piles or circular sheet pile cells
(c) sinking through a sand island
(d) wire cables attached to submerged anchors.

The choice of method depends on the size of caisson, the depth of water, and particularly on the stability of the bed of the waterway. River bed conditions at the site of the Mackinac Bridge, USA, were favourable for the construction of a piled enclosure for the circular caissons on two pier sites. The 4.6 m wide space between the two steel shells was divided radially into eight watertight compartments. Four steel tubular towers were spaced at equal distances around the caissons. The towers were prefabricated and taken by barge to the site where they were lowered onto the river bed. Then steel H-beam piles were lowered down each of the pipe piles and driven to refusal. The space between the pipes and piles was grouted. Three towers were constructed in this way and connected by horizontal box-type trusses. After floating in the caisson the fourth tower was constructed and the enclosure completed by additional connecting trusses. A clearance of 30 cm was provided between the caisson and the towers.

The 67 m by 29 m caisson for the West anchorage of the Delaware Memorial Bridge, USA, was enclosed by a rectangular pen formed by two 9 m diameter sheet pile cells filled with sand on each long side of the caisson, and two steel pile dolphins formed from three vertical and three battered piles on the short (shoreward) side. Fendering spanned between the cells and dolphins to give a 1.2 m clearance around the caisson. After towing in the caisson another pair of dolphins was driven to complete the fourth side of the enclosure.

Sand islands were used for four of the caisson piers of the Baton Rouge bridge over the Mississippi river. The fast flowing river was known to cause deep scour, and bed protection was given to the sites of the two deepest piers in the form of 137 m by 76 m woven board mattresses. The islands were 34 and 37 m, respectively, in diameter, and were formed by steel plate sheets filled with sand. The shells were surrounded by a double row of piles. The sand islands narrowed the waterway, and this caused deep scour which the mattresses did little to prevent. The scour at Pier 3 was 12 m deep, and a similar depth of scour at Pier 4 caused the whole of the sand filling in the island to disappear in 2 to 3 minutes!

The external water pressure on the shell then pushed in the 9.5 mm plating which was torn apart. The caisson, which at that time had only penetrated 4.6 m into the river bed, tilted by 2.1 m in line with the bridge and 0.6 m in the other direction, and was plumbed with great difficulty. These experiences emphasize the hazards resulting from obstructions to flow caused by substantial temporary works in a river with an erodible bed.

The minimum of temporary construction and the lowest risk of bed erosion is given by the method of securing a floating caisson to submerged anchors; the caisson being moored between floating pontoons. This method is particularly suited to a multi-span structure when the high capital cost of an elaborate pontoon-mounted sinking plant can be spread over a number of caissons; whereas, if fixed stagings are provided for the piers of a multi-span structure, much time will be spent in driving and extracting piles for construction of the stagings at each pier site, with inevitable damage due to repeated re-use. A floating plant is highly mobile, and can be rapidly switched from one pier site to another to suit changing conditions of river level and accessibility at low water stages. It is advantageous in these conditions to design the floating plant to be adaptable to working in the dry.

Lowering Caissons

Four main methods are used for maintaining position and verticality of caissons during sinking. These are

(a) free sinking, using guides between caissons, and fixed stagings or floating plant
(b) lowering by block and tackle from piled stagings or floating plant
(c) lowering by suspension links and jacks from piled stagings or floating plant
(d) lowering without guides but controlling verticality by use of air domes.

Sinking Open Well Caissons

When a caisson reaches the stage where concrete has to be added to maintain downward movement, the rate of sinking should be governed by a fixed cycle of operations. The usual procedure is to maintain a 24-hour cycle comprising excavation from the wells, erecting steel plating or formwork in the walls (steining) and concreting a 1.2 to 1.5 m lift of the walls.

In floated caissons, the top of the skin plating should always be maintained at least about 1 m above water level to guard against an unexpected rise in level. However, the freeboard should not be so much that the centre of gravity is too high to give proper control of verticality.

Control of verticality can be achieved by one or a combination of the following methods:

(a) adding concrete on one side or the other
(b) differential dredging from beneath the cutting edge
(c) pulling by block and tackle to anchorages
(d) jetting under the cutting edge on the 'handing' side
(e) placing kentledge on one side or the other.

More details of this have been discussed in Ch. 7 of the author's other book, *Concrete Bridge Practice — Construction, Maintenance and Rehabilitation*, under 'construction considerations'.

Excavation Method

Grabbing is the most commonly used method of excavating from the open wells although ejectors, operated by compressed air or water pressure, have been used in sandy soils.

De-watering for Freeing a 'Hanging' Caisson, and the Phenomena of Sand Blow

If excavation becomes difficult, a caisson can be partially de-watered or pumped out. This increases its 'effective' weight, so increasing the sinking effort. The procedure may be dangerous where the cutting edge has only just penetrated a clay stratum overlain by waterbearing sand. The water in the sand may then force its way in through a limited thickness of the clay, causing a localized *blow* in and up the caisson, followed by tilting of the caisson which is not easy to rectify. In the process the caisson actually hangs on one side, which may even lead to cracking.

Explosives fired in the wells can be used to cause a jerk, creating a temporary breakdown in skin friction; but they are rarely effective in breaking down stiff material from beneath the cutting edge. Explosive charges, carefully placed by divers, can be used to break up boulders or other obstructions to sinking.

Water jetting is not usually effective in freeing hanging caissons, since 'sticky' sinking conditions usually occur in stiff clays or boulder clays which are not amenable to removal by jetting.

Jetting and Lubrication

Sometimes, to facilitate the sinking, a film of grease is applied to the exterior surface of the caisson, and/or water jetting is used. Jet pipes, $1^1/_2$ to 2 inches diameter, with nozzles, are cast in the concrete, usually one series of jet pipes is provided on the sloping surface immediately above the cutting edge, and one or two series on the periphery of the caisson at several meters above the bottom of the cutting edge. All jets are arranged symmetrically to induce straight sinking. Since fixed jet pipes can readily become plugged, movable jets have been found more efficient. Eight inch diameter pipes may be cast in the concrete for inserting movable jet pipes for inside jetting.

Rectifying TILT in Wells

Caissons can never be sunk perfectly in plumb and true to position. A certain amount of deviation from the planned location should be allowed for in design and permitted. For a deep caisson, the actual center may even be 30 cm or more from the required location. It is important to keep the caisson in the vertical position during the entire process of sinking. As soon as it tilts corrective measures should be taken. These can be:

(i) Excavating on the high side ahead of the low side, but not stopping excavation on the low side
(ii) Dredging on the outside of the high side
(iii) Jetting on the outside and inside of the high side
(iv) Pulling the caisson by wire ropes (using timber sleeper-packings)—attach cables to a deadman or dolphin, and apply tension as the sinking proceeds
(v) Blocking under the cutting edge on the low side; this can be done more readily in pneumatic sinking.

It should be noted that it is impossible to plumb a caisson without lowering it as a whole, and this may also lower the final founding level for no fault of the client who otherwise has to pay for increased sinking.

Skin Friction in Caissons

A conservative approach of ignoring skin friction should be used, when assessing the contribution of skin friction to the carrying capacity of a caisson in service, but when considering the dead weight to be provided to sink the caisson, the skin friction should not be under-estimated. Some caisson details incorporate water and air jets near and above cutting edge level to reduce skin friction during sinking. The main disadvantage of external jet pipes is that they readily become clogged, especially when sinking through tight ground.

Generally engineers view built-in jetting arrangements with suspicion, doubting their effectiveness, and many hold the view that independently operated external jet pipes worked down the outside of the caisson wall is the only sure method. Present day practice is to inject thixotropic clay slurries (e.g., bentonite) above the cutting edge or shoe level, thus providing a membrane of slurry around the walls. Successive injections are made as the caisson is sunk to its final level.

This reduces the skin friction very considerably and enables the dead weight of the caisson to be reduced, with the likely elimination of the need for kentledge to assist sinking. Pipes in caisson walls used for slurry injection or for air/water jetting should be interconnected by a header at shoe level because individual vertical pipes may become damaged or blocked as the walls are built up. Problems can arise if the circulation of bentonite up the outside of the caisson is interrupted, for example, by erosion of the river bed around the caisson. If the circulation cannot be restored, the slurry will lose its effectiveness as a means of reducing skin friction.

Design Features of Open Caissons and Monoliths

The principal design features of open caissons and monoliths are shown in Fig. 7.3. The cutting edge forms the lowermost extremity of the shoe (or well curb). The latter usually has vertical steel outer skin plates and sloping inner steel haunch plates (or cant plates). The skin plates (if tall enough) are braced internally with steel trusses, members, or heavy reinforcement, in vertical and horizontal planes. The trusses prevent distortion of the shoe during fabrication, towing to site, and the early stages of sinking. As soon as possible before or after the initial sinking, the space between the skin plates is filled with concrete (steining). When the structure has attained sufficient rigidity by reason of the concrete filling, the skin plating can be terminated and the steining carried up in reinforced concrete placed between formwork. At or below low water level in a bridge the caisson proper is completed and capped and then the pier/abutment carried up in concrete masonry, or brickwork. If the water level rises above the top of the caisson at its finally sunk level, a cofferdam or temporary false steining is constructed above the caisson for ease of operations. The dredging space within the walls may form a single dredging well or it may be divided by cross walls into a number of wells.

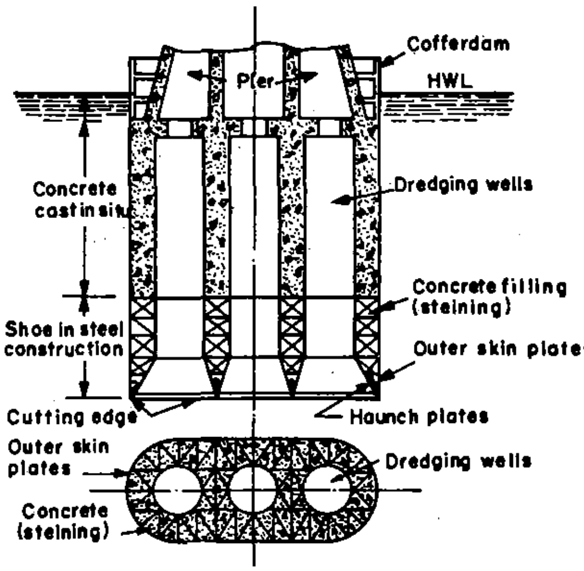


Fig. 7.3 *Design features of open well caisson*

In the following comments on the design of open caissons it must be realized that in most practical cases there is no ideal solution to the problem, and the final design is usually a compromise brought about by a number of conflicting requirements. For example, thick heavy walls may be desirable to provide maximum weight for easier sinking through stiff ground, but thick walls add to cost and may mean small dredge holes and then the grabs may not reach beneath the haunch plates to remove the stiff ground. Lightness of weight is desirable in the first stages if floating out the caisson, but this can only be obtained at the expense of rigidity and sinking effort which are so essential at the second stage of sinking through the upper layers of soil when the caisson is semi-buoyant and may not have uniform bearing, and when stresses due to sagging of the structure consequent on differential dredging levels may be critical. Maximum height of skin plates is desirable when sinking caissons in a waterway where there is a high tidal range, but the extra height of plates may mean excessive draught for towing to site, not to mention increased cost of plates. The shape of a caisson will, in most cases, be dictated by the requirements of the superstructure. The ideal shape for ease in sinking is circular in plan, since this gives the minimum surface area in skin friction for a given base area. However, the structural function of the caisson is, in most cases, the deciding factor.

The size and layout of the dredging wells is dependent mainly on the type of soil. For sinking through dense sands, or firm to stiff clays, the number and thickness of the cross walls, and the thickness of the outer walls, should be kept to a minimum consistent with the need for weight to aid in sinking and for rigidity against distortion. Grabs can excavate close to the cutting edge in caissons that have thin walls, assuming the outer dimension as unchanged. This is important in firm or stiff clays, since these soils do not easily slump towards the centre of a dredging well; whereas in sands and soft silts, grabbing below cutting edge level will cause the ground to readily slump away from the haunch plates towards the deepest part of the excavation sump, especially if assisted by water jetting. However, as already noted, thin walls mean reduced sinking effort and it is inconvenient to have to take kentledge on and off the top of the steining (for each lift of concrete that is placed) for assisting sinking.

Control of verticality in large caissons is facilitated by the provision of a number of dredge-holes. To give control in two directions at right angles to one another they should be disposed on both sides of the centre lines (Fig. 7.4), but for a narrow caisson there may be only room for one row, since sufficient width must be provided for a grab to work. Heavy monoliths sunk through soft material onto but barely into a firm or hard stratum need only have small dredge holes. Experiences in sinking open caissons for the piers of the Lower Zambezi Bridge showed that straight walls were preferable to circular walls when sinking through stiff clay, since with circular walls there was a tendency for the clay to arch and wedge itself around the cutting edge, rather than be forced towards the centre of the well.

If occasional obstructions in sinking are encountered

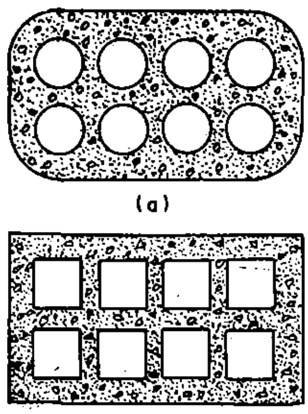


Fig. 7.4 *Layout of dredging wells in caisson (a) Circular wells (b) Square wells*

(e.g., old logs, sunken barges, boulders, etc.) then, down to about 20 m, workers wearing air-lock helmets connected to compressed air can dive down and break the obstructions underwater, using pneumatic tools.

Materials for Caissons and 'Sinking Cycle'

The desirable material for caisson construction is steel in the form of a double skin of plating which is subsequently filled with concrete. The concrete in the lower part of the shoe should be of high quality since it is required to develop high early strength to resist stresses developed in the 'tender' early stages of sinking. The cement content, however, should be sufficiently high to give it resistance to attack by sea or river water and the all important denseness.

Reinforced concrete has been used for caisson shoes but it has the disadvantage of being too heavy at the early stages of construction where lightness is needed for floatation and handling. The quantity of concrete-filling in a steel caisson may be somewhat greater than the volume of concrete in a reinforced concrete caisson, but due to the use of a slightly leaner mix, the ease in placing and the elimination of formwork, the unit cost of the concrete-filling in a steel caisson is appreciably lesser. Steel caisson is essential where it has to be floated out; however steel plating may be discontinued after grounding.

Reinforced concrete caissons utilize concrete to provide the structural strength as well as the weight for sinking. They are often more economical than the steel caissons. However, concrete caissons must be poured in sections (lifts), and the sinking operation must be interrupted while pouring each lift and while waiting for the concrete to mature. Every time the sinking is started from a stationary position, additional effort is required to overcome the static friction. Furthermore, the cyclic operation of stops and starts takes a long time to sink the caisson, and unless extreme scientific care is taken, the successive concrete lifts may inadvertently get cast vertically individually with the previous one off-plumb owing to well-tilt, thus giving a slightly zig-zag shaped well in elevation, which, with all its consequent 'necks', can prove difficult for sinking. Each lift should be cast in line with the previous lift even if the line is interimly tilted and the tilt should be controlled as a whole.

General Arrangement of Pneumatically Sunk Caissons

Pneumatic caissons are used in preference to open-well caissons in situations where dredging from open-wells would cause loss of ground around the caisson resulting in settlement of adjacent structures, or when sinking through variable ground or through ground containing obstructions (very hard lenses, conglomerates, etc.) where an open caisson would tend to tilt or refuse further sinking. Pneumatic caissons have the advantage that excavation can be carried out by hand in the 'dry' working chamber, and obstructions such as tree trunks or boulders can be broken out from beneath the cutting edge. Also, the soil at the foundation level can be inspected and, if necessary, bearing tests made directly upon it. The foundation concrete is placed under ideal conditions in the dry, whereas with open-well caissons the final excavation and sealing (plugging) with concrete is almost always carried out under water.

Pneumatic caissons have the disadvantage, compared with open-well caissons, of requiring more plant and labour for their sinking, and the rate of sinking is much lower. There is also the important limitation that men cannot work in air pressures much higher than 3.5 kg/cm^2, which limits the depth of sinking to about 35 m below the water-table, unless some form of ground water lowering is used outside the caisson. If such methods are used to reduce air pressures in the working chamber they must be entirely reliable, and the de-watering wells must be placed at a sufficient distance from the caisson to be unaffected by the ground movement caused by caisson sinking.

The development of large diameter cylindrical foundations installed by rotary drilling and the limitation in sinking depth due to considerations of limiting air pressure that the human lungs can accept, means that pneumatic caissons are only rarely used.

Design Features of Pneumatically Sunk Caissons

When caissons are designed to be sunk wholly under compressed air, it is usual to provide a single large working chamber (Fig. 7.5) instead of having a number of separate working chambers separated by cross walls. The single chamber is a convenient arrangement for minimising bearing resistance to sinking, since this resistance is then only given by the outer walls. Control of sinking by differential excavation from a number of cells is not necessary since control of position and verticality can readily be achieved by other means, for example by the use of shores and wedges beneath the cutting edge, or by differential excavation beneath the cutting edge.

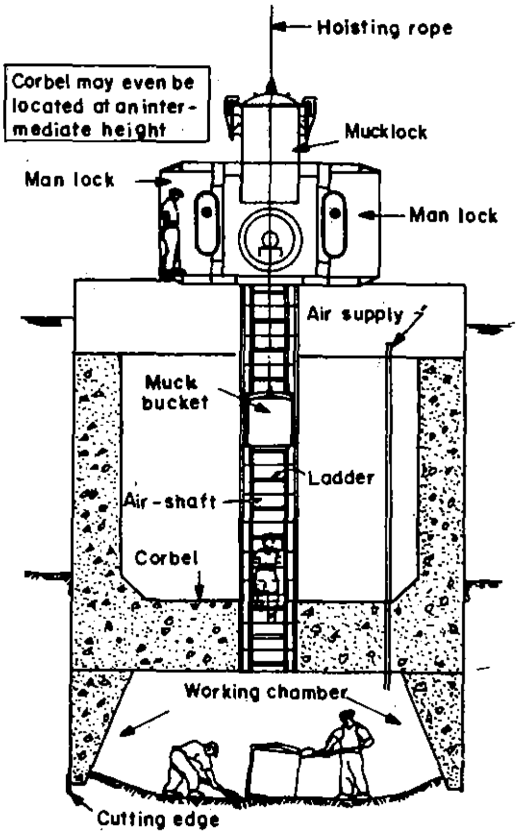


Fig. 7.5 *General arrangement of pneumatic caisson*

The working chamber is usually 2.5 to 10 m high, although where the caisson chamber is sunk to a limited penetration the height may be somewhat smaller. The roof of the working chamber (called *corbel*), must be strongly built as it may have to resist high air pressures over a wide span.

Access to the working chamber is through shafts. Since all excavated material must be lifted through the shafts, the shafts must have adequate capacity in size and numbers to pass the required quantity of spoil in buckets through the air-locks to meet the programmed rate of sinking. The air-shaft is usually oval in plan and is divided into two compartments by a vertical ladder. One compartment is used for hoisting and lowering spoil buckets and the other is for the workmen. The shaft is built up in 1.5 or 3 m lengths to permit its heightening as the caisson sinks down. The air-lock is mounted on top of the shaft, and it is essential for the safety of the workmen to ensure that the lock is always above the highest tidal or river flood levels, with sufficient safety margin to allow for unexpected rapidity in sinking of the caisson. Alternatively, the air-locks can be protected against flooding by building up the skin plating or providing a cofferdam around the top of the caisson to the required height.

Design Details for Compressed Air Sinking

Air-locks The number of air-locks required in a caisson depends on the number of men employed in any one working chamber in the caisson. The size of the air-locks and air-shafts is governed largely by the quantity of material to be excavated, i.e., by the size of the 'muck bucket'.

(1) For excavating in hard material, one man can be effectively employed in about 3 m^2 of working area of a caisson, but in loose material such as sand or gravel one man may be allowed in 6 to 7 m^2 area.
(2) Generally, the number of men in a caisson per shaft will range from 5 to 10 except in the smallest caissons. The optimum number is 10 men per shaft.
(3) Sufficient air-locks should be provided to allow the whole shaft to pass out of the caisson in reasonable time. This depends on the working pressure. For moderate pressures (under 2.5 kg/cm^2), an air-lock should be provided for every 90-100 m^2 of base area. For high pressures two locks would be considered for 90-100 m^2 of base area, because of the longer time required in 'locking' each shaft in and out of the caisson.
(4) The size of the lock is governed by the rate of excavation and the number of men to be accommodated. Thus the main chamber (or muck lock) has to accommodate a skip of sufficient size to pass through it the excavated material at the programmed rate. For example, with a base area of about 90 m^2 per lock, a large air-lock can deal with about 9 m^3 of spoil per hour; for continuous shift working at say 2.0 kg/cm^2, 180 m^3 of material would be passed through in 20 hours of 'effective' working. This gives about 100 m^3 of material 'in the solid' or 1.14 m of sinking per effective 3-shift day. With a smaller air-lock under similar conditions the output would be about 6 m^3/h or 120 m^3 of material in 20 effective hours, corresponding to a sinking rate of about 0.70 m per effective 3-shift day. The plant for the production and supply of compressed air to the working chamber shall deliver a supply sufficient to provide at the pressure in the chamber 0.30 m^3 of fresh air per minute per person for the time being in the chamber. BS CP 2004 recommends that whenever work involving compressed air at pressures greater than 1 kg/cm^2 above atmospheric pressure is undertaken, the Medical Research Council's Decompression Sickness Panel should be consulted for advice on decompression rates.

If the air supplied in accordance with the above rule is more than the amount lost under the cutting edge and through the air-locks, the surplus should be exhausted from the caisson through a control valve.

Compressors for air supply are usually stationary types. Ideally, they should be driven by variable speed motors to enable the supply to be progressively increased as the caisson sinks deeper. The type of plant in general use is a twin-cylinder single stage piston compressor of 8.5 m^3/min capacity motor-driven through a vee belt. Rotary compressors can be used for supplying low-pressure air. At least 50% spare compressor capacity should be provided for emergency purposes. Consideration should be given to alternative means of power supply, for example diesel generators for electrically-driven compressors normally supplied from the mains system, or a standby steam plant. The total available air supply may require to be twice the actual requirement if failure is liable to cause danger to life or property. However, a standby supply need not be provided if the loss of air pressure will not endanger the workmen: for example, if the caisson is being sunk on to a hard stratum which will remain stable, and if workmen have ample time to escape from the working chamber.

Air treatment Improved working conditions and greater immunity to caisson sickness is given by treatment of the air supply. The air-conditioning plant should aim to remove moisture and oil, and to warm the air for cold weather working, or to cool it for working in hot climates. The need to supply cool dry air is especially important for compressed air work in hot and humid climates. In cool climates it is advantageous to provide heating in man-locks since the cooling of the air which always takes place during decompression can cause discomfort to the occupants.

*Pneumatic Sinking of Caissons**

Control of position and verticality of pneumatic-caissons is more readily attainable than with open-well caissons. It is possible to maintain control by careful adjustments of the excavation beneath the cutting edge, and if this is insufficient, raking shores can be used in the working chamber, or the caisson can be moved bodily at early stages of sinking by placing sliding wedges or 'kickers' beneath the cutting edge.

Excavation in the working chamber is usually undertaken by men hand-shovelling into crane skips, compressed-air

* For estimation of bursting tension in well-steining see App. 5 of this book.

tools such as clay spades or breakers being used in stiff clays or boulder clays. When excavations are in sands or gravels, hand-held water jets can be used to sluice the material into a sump from where it is raised to the surface through a shaft. The latter can also be an open-ended pipe with its lower end dipping into water in the sump. By opening a valve on this 'snorer' pipe, the water and soil are forced by the air pressure in the working chamber out of the caisson. The snorer also performs a useful function in clearing water from the floor of the excavation if the soil is too impermeable for the air pressure to easily drive the water down into it.

The compressed air supply must be regulated to provide adequate ventilation for the workmen. In permeable ground this is readily attained by allowing it to escape through the soil and beneath the cutting edge. However, when sinking in impermeable clays and silts, ventilation must be maintained by opening a valve to allow air to escape through the caisson roof. Careful regulation of air pressure is necessary when sinking in ground affected by changes in tidal water levels.

Smoking or naked lights should not be permitted in the working chamber because of the risk of encountering explosive gases, e.g., methane (marsh gas), during sinking. A careful watch should be maintained in neighbouring excavations. Accidents have been known to happen by compressed air passing through beds of peat and becoming deprived of oxygen due to oxidation of the peat. The escape of this oxygen-deficient air into the confined spaces of excavations has caused asphyxiation of the workmen in them.

In very permeable ground the escape of air may be so great as to overtax the compressor plant. The quantity escaping can be greatly reduced by pre-grouting the well steining and even the ground, with cement or clay. However, each site has its own problems which must be looked into carefully in advance.

Blowing Down a Pneumatically Sunk Caisson

If a pneumatic caisson stops sinking due to build-up of skin friction, it can be induced to move by the process known as 'blowing down'. This involves reducing the air pressure to increase the effective weight of the caisson, so increasing the sinking effort. The process is ineffective if the ground is so permeable that air escapes from beneath the cutting edge at a faster rate than can be achieved by opening a valve and thus buoyancy-creating water is let in.

The procedure in blowing down a caisson is first to remove the men from the working chamber. The control valve is then opened and the caisson should soon begin to move. If it does not do so, the skin friction is too high, and either kentledge must be added or further excavation should be done below the cutting edge.

Careful control should be exercised when blowing down in ground containing boulders, or when blowing down a caisson to land it on an uneven rock bed. In some circumstances it may be necessary to excavate high spots in the rock and fill them with clay and then blow the caisson down into the clay.

An example of difficulties in sinking on to rock is the pneumatic caisson pier for a pipe bridge over the Mississippi River at Grand Tower, Illinois, which has been described by Newell.

If at all possible, a caisson should not be blown down in soft or loose ground, as this might result in soil surging into the working chamber, so increasing the quantity to be excavated. There is also the risk of penetration of ground into the working chamber, causing settlement of adjacent structures. It must be remembered that pneumatic caissons are, in many instances, used as a safeguard against such settlement. Blowing down, if properly controlled, is a safe procedure in a stiff clay.

If a caisson is sinking freely without the need for blowing down, measures must be taken to arrest the sinking on reaching founding level. This can be achieved by casting concrete blocks in pits excavated at each corner of the working chamber at such a level that the caisson comes to rest on the blocks at the desired founding level.

Safety Problems

For the safety and welfare of workmen, the following precautions should be exercised:

1. *Accurate control of air pressure* A gauge-tender should watch the pressure gauge constantly, and the gauge should be accurate, regularly calibrated, and in good working condition.
2. *Sufficient air circulation* To avoid the air in the working chamber becoming stale, fresh air must be circulated into the working chamber constantly. This may be done by opening a valve in the air lock. In granular soils, where certain amount of leakage takes place below the cutting edge and through the soil, the air is automatically circulated.
3. *Slow decompression* Men working under compressed air must be decompressed slowly. If coming out too fast, they are subjected to caisson disease. This disease is due to air bubbles formed in the blood and body tissues which are compressed while working under pressure. A period of about one-half hour is necessary for decompression from a pressure of 3.5 kg/cm^2.
4. *Duplicate and spare equipment* A spare or duplicate set of air compressors and other equipment for pneumatic operation should be provided in case of contingency.
5. *A medical doctor* should be available at site all

the time. All the workers must be medically fit to work inside a pneumatic caisson. No worker should work inside it for longer than about 4 hours per 24 hours, no more than 2 hours at a stretch. The figures reduce as the depth increases. Extreme care has to be taken against the caisson disease (called 'the bends') because of which the workers can loose control over their joints, vomit blood or suffer from nasal bleeding, paralysis, 'bubbles in skin', and even death. This calls for very controlled acclimatisation of the workers in the air-locks both for compression as well as for decompression.

Structural Design of Steining

In addition to the usual design forces for which the caisson must be designed for 'bridge in service' condition, the caisson, during its construction, is likely to be subjected to certain odd loading conditions, the effects of which may not be easily controllable. Some of these are outlined below:

(a) The caisson is hung up from near the top by skin friction necking. The lower portion of the caisson is then subjected to tension. Sufficient strength should be provided in the caisson to carry the weight of its lower portion (vague loading condition);

(b) The caisson is held on one side only or on two opposite points only over some length. This can lead to vertical cracks and, in the limit, may require refilling the cracked caisson and then sinking a new caisson inside it;

(c) The caisson is subjected to unbalanced earth pressures;

(d) The caisson has to be pulled into its correct position if it has tilted during construction. Large racking force and earth pressure would be introduced by pulling; and

(e) The caisson is 'dropped' suddenly during sinking, sometime owing to massive sand blow, and sometimes due to sudden break in skin friction while the sump is deep.

In addition, where the caisson is likely to be sunk pneumatically, additional hoop reinforcement may be required in the steining to withstand the bursting ring tension. The details of its calculation are dealt with in App. 5 of this book.

Some Considerations in the Dimensioning and Design of Wells

Some of the important features are given below for guidance:

(i) The minimum dimension of any dredge hole shall not be less than 2 metres.

(ii) For plain concrete wells, the mix of concrete in the steining shall not be leaner than 1 : 3 : 6. For wells located in marine areas or other similar adverse conditions of exposure, the concrete in steining shall not be leaner than 1 : 2 : 4 with cement content not less than 350 kg/m^3 of concrete and water-cement ratio not more than 0.45. In the case of plain and reinforced concrete single circular wells, the diameter of wells shall not normally exceed 12 metres.

(iii) The external diameter of brick masonry wells shall not exceed 6 m and such wells shall not be used for depths exceeding 20 m. For brick masonry wells, bricks of highest quality shall be used in cement mortar not leaner than 1 : 3 for steining.

(iv) Steining with two or more shells of different materials should not be permitted as experience has shown that these develop splitting cracks (differential shrinkage, etc.).

(v) The minimum thickness of well steining shall not be less than 500 mm and should satisfy the following relationship:

$$h = Kd\sqrt{l}$$

h = minimum thickness of steining in meters.

d = external diameter of circular well or dumb-bell shaped well or the smaller dimension in plan of twin 'D' well, in metres.

l = depth of well in metres below LWL or ground level whichever is higher.

K = a coefficient, with following values:

(1) Single circular or dumb-bell shaped well in cement concrete
K = 0.030 for predominantly sandy strata
K = 0.033 for predominantly clayey strata

(2) Twin-D well in cement concrete
K = 0.039 for predominantly sandy strata
K = 0.043 for predominantly clayey strata

(3) Single circular or dumb-bell shaped well in brick masonry
K = 0.047 for predominantly sandy strata
K = 0.052 for predominantly clayey strata

(4) Twin-D well in brick masonry
K = 0.062 for predominantly sandy strata
K = 0.068 for predominantly clayey strata

NOTE (a) For bouldery strata or for wells resting on rock where blasting may be involved, higher thickness of steining, better grade of concrete, heavier reinforcement, use of steel strake plates in the lower portions, etc., are advisable.

(b) For wells passing through very soft clayey strata, the steining thickness may be reduced based on local experience and in accordance with the decision of the engineer-in-charge to prevent the well from sinking by its own weight. In such cases, the steining may require relatively more reinforcement.

(vi) For plain concrete wells, the minimum reinforcement shall not be less than as indicated below:
Vertical reinforcements: 0.12% of gross sectional area of actual steining provided, distributed equally on both faces of the steining.
Hoop reinforcement: 0.04% of the volume per unit length of steining.

(vii) RC wells shall be designed as RC columns for combined axial load and bending. The reinforcements arrived at shall, however, not be less than as indicated below:
Vertical reinforcement: 0.2% (for either mild steel or deformed bars) of actual gross cross-sectional area of steining of which at least one-third shall be on inner face.
Hoop reinforcement: 0.04% of the volume per unit length of steining.

(viii) For brick masonry wells, the reinforcement shall not be less than as indicated below:
Vertical bond rods: 0.10% of cross-sectional area and shall be encased into cement concrete 1 : 2 : 4 bands of size 150 mm × 150 mm.
Hoop reinforcement: 0.04% of volume per unit length, provided in a concrete ring band and at spacing of 4 times the thickness of steining or 3 m, whichever is less (for details see Fig. 7.6).

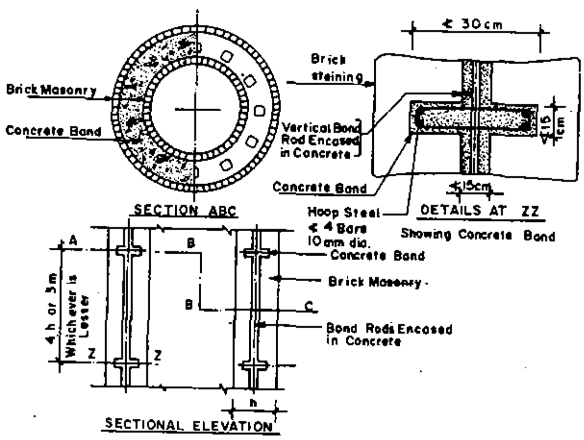


Fig. 7.6 *Details of brick masonry well steining.*

NOTE The horizontal RC ring bands shall not be less than 300 mm wide and 150 mm high reinforced with bars of diameter not less than 10 mm placed at the corners and tied with 6 mm dia stirrups at 300 mm centres.

(ix) Mild steel cutting edge of weight not less than 40 kg/m shall be provided properly anchored to the well curb. In case of wells with two or more compartments, the lower end of the cutting edge of middle stems shall be kept 300 mm above that of the outer walls to prevent rocking. Heavier cutting edge (85 kg/m or more) is required in hard and bouldery sub-strata.

(x) The well curb shall always be of reinforced concrete of mix not leaner than 1 : 1.5 : 3, with minimum reinforcement of 72 kg/m^3 excluding the bond rods of steining. A typical arrangement is shown in Fig. 7.7.

(xi) The angle contained by the vertical and the inclined surfaces of the well-curb should be around 30° and the curb height about 1.8 times the steining thickness.

(xii) Where blasting is anticipated, the outer faces of the well curb shall be protected with suitable plates of thickness not less than 6 mm up to half the height of the well curb and inner face with plate not less than 10 mm thick up to top of well curb and 6 mm to a height of 3 metres above top of the well curb. The steel plates shall be properly anchored to the curb and steining. The well curb shall be provided with additional 10 mm diameter hoop reinforcement at 150 mm centres up to a height of 3 m into the well steining in which portion the mix of concrete shall also be not leaner than 1 : 1.5 : 3 (by volume).

(xiii) Bottom plug shall be provided in all wells, and shall extend up to 300 mm above the well curb. The concrete mix shall be 1 : 2 : 4 with a minimum cement content of 330 kg/m^3 and a slump of about 150 mm. Where grouted concrete (colcrete) is used, the grout mix shall not be leaner than 1 : 2.

(xiv) Well dredge-hole shall be filled up to the top plug (placed at maximum scour level) with sand or excavated material free of organic matter.

(xv) A top plug 500 mm thick in c.c. 1 : 3 : 6 shall be provided over the dredge-hole filling.

(xvi) All wells shall be provided with a RC well cap with its bottom surface preferably at LWL.

(xvii) All structural design shall in advance account for a tilt of at least 1 in 80 and shift of at least 150 mm (in the resultant direction) in the body of well. Final founding level shall be decided after taking into account the moments due to actual values of tilts and shifts in the two orthogonal directions.

Bottom Plugging

The sealing (bottom-plugging) concrete in open-wells is placed by tremie pipe or bottom-opening skip to the required thickness and sometimes roughly levelled by a diver (after

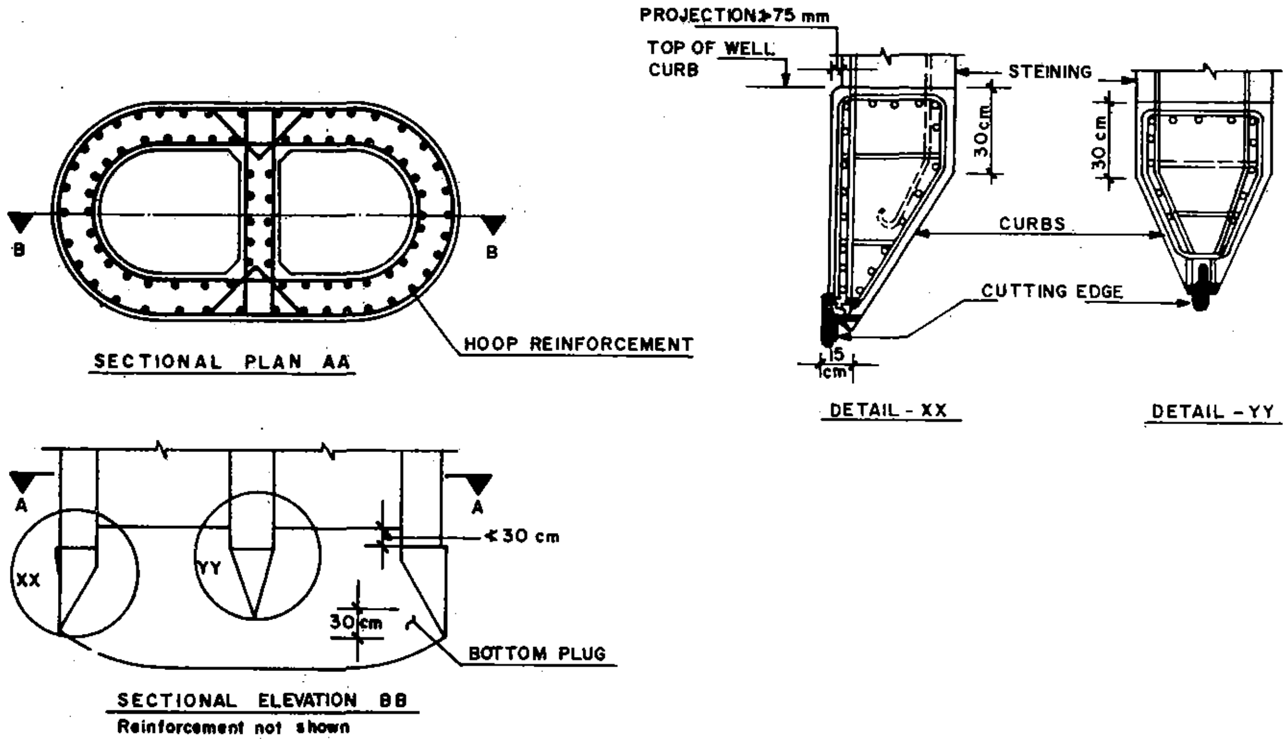


Fig. 7.7 *Details of well curb and cutting edge*

which sand is poured in to fill the remaining space in the wells up to the design level).

7.7 PIER AND ABUTMENT

Pier and abutment bodies can be constructed in plain concrete, reinforced concrete or in masonry (brick, stone or block), and in some special cases in timber, steel or prestressed concrete. Selection of a particular type depends upon the span and type of superstructure, height of substructure, the magnitude of loads and forces to be transmitted, availability of type of construction material and construction equipment at site, time for construction, and minimal cost.

In general, the shape of piers and abutments should be such as to cause minimum obstruction to the flow of water. For fast flowing rivers, highly surcharged with particles of abrasive nature, special precautions like masonry jacketing or steel lining to a suitable height above the normal scour level or bed level may have to be adopted.

Piers and abutments shall be designed to be safe under the worst combination of loads and forces during construction and service.

While finalising the designs of the substructure, however, the following general considerations shall also be satisfied.

(i) In case of plain concrete substructure, on exposed faces a surface reinforcement at the rate of 5 kg/m^2 shall be provided. This total reinforcement shall be made of small size bars spaced equally, both horizontally and vertically or orthogonally, in equal proportion. The spacing shall not exceed 200 mm.

(ii) The width of abutment-cap and pier-cap shall be sufficient to accommodate not only the bearings but also an off-set of 150 mm beyond the edges of the bearings and, in case of abutment, to carry dirt wall also. The thickness of such caps shall not be less than 225 mm up to a span of 25 m and 300 mm for longer spans. In addition, they should afford enough room and strength to accommodate lifting jacks, etc., for attending to the replacement of bearings and any future lifting of deck, etc.

(iii) Suitably designed cut and ease waters should be provided in piers up to affluxed HFL or higher (from consideration of waves) in order to reduce the current forces.

(iv) Abutment-piers may have to be provided at locations where there may be need of increasing the waterway subsequently. The design of such an abutment-pier shall be such that it should be possible to convert it to the similar shape as piers in the active channel as well as should be safe subsequently as a pier.

(v) Piers may be rigid or flexible. In case of framed type connections for pier, the base of such frames should be above the HFL. For bridges having multicolumn piers across rivers carrying floating trees or timber, they shall be braced by means of diaphragm walls of thickness not less than 150 mm, extending at least up to HFL.

(vi) The outer diameter of a single hollow circular pier should not be less than 2.75 m. It should be at least of reinforced concrete of strength not less than 200 kg/cm^2. Preferably the thickness of shell shall not be less than 600 mm and vertical reinforcement not less than 0.4% of the cross-sectional area of pier shaft.

The lateral reinforcement in the walls of hollow RC pier should not be less than 0.3% of the sectional area of the wall of the pier. This lateral reinforcement shall be distributed on both faces, 60% on the outer face and 40% on the inner face.

(vii) Where supports are made with 2 or more piles, or the columns are spaced closer than 2 metres across the direction of flow, the group should be treated as a solid pier of the same overall width, and value of shape factor (K) taken as 1.25 for working out the intensity of water pressure due to flood.

(viii) Spill-through type abutments should be provided with adequately designed pitched protection in front with apron/toe wall such that the entire system is safe from considerations of slipping, undermining, etc. Usually the pitching is provided in a minimum slope of $1\frac{1}{2}$H : 1V. The screen or cut-off wall should extend at least down to 500 mm depth into the fill so as to help prevent depression and squeeze-forward of the backfill—particularly during rains. Spill-through type abutments may be provided only in case solid abutments are not necessary. It is always preferable to adopt solid abutments, unless their cost is prohibitive and spill-through type is acceptable.

(ix) The top of wing/return walls shall preferably be carried 100 mm above the top of the slope of embankment to prevent any soil from being blown or washed away by rain.

(x) Cantilever-returns, where adopted, shall preferably not be of length more than 4 metres.

(xi) All abutments should be designed for a live load surcharge equivalent to 1.2 m height of earthfill unless a suitable RC approach slab at least 3.5 m long is provided with its one end sitting on the abutment and its remaining length resting evenly on compacted approach fill.

(xii) All wing walls and/or return walls provided for full height of approaches should be designed to withstand a live load surcharge equivalent to 0.6 m height of earthfill.

(xiii) In the case of spill-through type abutments, the active earth pressure calculated on the width of columns should be increased by 100% (to cater for the effect of arching action between columns).

(xiv) Structures designed to retain earthfill should be proportioned to withstand earth pressure calculated in accordance with any rational theory. Coulomb's theory shall be acceptable, subject to the modification that the centre of pressure exerted by the backfill is located at an elevation of 0.42 of the height of the wall above the base instead of 0.33 of that height. No structure shall however be designed to withstand a horizontal pressure less than that exerted by a fluid weighing 480 kg/m^3.

(xv) The fill behind abutments, wing walls, and return walls should conform to the standard specifications, and adequate weep-holes shall be provided in these structures for draining out static water pressure.

(xvi) In skew bridges where bearings are placed at right angles to the longitudinal axis of the bridge, the top width of the piers/abutments has to be more compared to right bridges in order to have a clear distance of 150 mm beyond the edge of the bearings.

(xvii) In the case of navigational streams, the effect of barge impact has to be considered in the design of piers and suitable fenders may have to be provided around the piers.

(xviii) For spill-through type of abutments, having columns and the cap beam supporting the deck, and dirt wall and the fly wings, it may be desirable to have the fly-wings directly resting on two extra columns in line with the wings, instead of connecting the fly wings to the cantilever projections of the abutment cap. This is to avoid unnecessary twist in the abutment cap. Further, for spill-through type of abutments, which are more prone to undergo deflections at top towards the river side due to earth pressure, the position of bearings has to be fixed adequately taking this probable deflection into account.

(xix) For skew bridges provided with square returns parallel to the bridge axis, the length of the wings at the obtuse angled corners will have to be more than at the acute corners to ensure that the revetment does not protrude into the waterway.

(xx) Piers should be designed for the possibility of one side span collapse (or unconstructed) case, if the structural system be such.

Achieving Economy in Design

The following measures often help in achieving economy in the design:

(i) For heights up to 5 to 6 m and spans up to 20 m, usually solid plain mass concrete or masonry piers may be suitable.

(ii) For heights above 6 m and spans beyond 20 m, RC piers are suitable.

(iii) For skew crossings, adoption of circular pier with a skew cap reduces the adverse effects of eddies, etc. In case the heights are large, hollow sections become economical.

(iv) At abutment locations, provision of sliding bearings or roller-cum-rocker bearings instead of fixed bearings helps in reducing the horizontal force (at bearings) on the abutment which is already under large horizontal earth pressure force.

(v) For abutments located on wells or pile foundations, the general practice is to keep them eccentric towards the backfill so that stabilizing moment increases.

(vi) Use of relieving shelves, etc., may be resorted to for reducing the effect of earth pressure in the design of the abutments.

(vii) In case of rivers hugging the banks, and in rivers with large depth of flow, it will be desirable to provide solid abutments with solid wings or box returns. However, where channels have shallow depth of flow near the banks and height of abutments is not much, cantilever returns offer cheaper solutions.

CHAPTER 8

Distribution of Externally-Applied and Self-Induced Horizontal Forces among Bridge-Supports in Straight-Decks

- Simple Spans on Unyielding Supports
- Simple Spans and Continuous Spans on Unyielding or Flexible Supports

8.1 INTRODUCTION

(i) The distribution of longitudinal horizontal forces among bridge supports is effected by the horizontal deformation of bearings, flexing of the supports and rotation of the foundations.

(ii) In simple non-skew straight decks, resting on similar stiff supports, the distribution of forces among supports may be assumed as indicated in 8.2 below.

(iii) In simple and continuous decks on flexible (or stiff) supports the distribution of longitudinal horizontal forces (e.g. the 'self-induced' horizontal forces due to change of temperature, shrinkage, creep, and elastic shortening of deck, and the 'applied' horizontal forces such as braking, earthquake and wind) among bridge supports shall be adequately estimated after taking due account of deformation of bearings, flexing of piers and abutments and rotation of foundations, as well as the location of the Zero-Movement Point (Z.M.P.) of the deck as explained in 8.3 ahead. However, in the case of simple spans on stiff (i.e. unyielding) supports, the simplified method outlined in 8.2 below is accurate enough and that the refined method outlined in 8.3 is not called for in such cases. (In curved and skewed decks, in addition, the effect of in-plan meander may also have to be considered; see Chapter 9 for curved/skew decks.)

8.2 SIMPLY SUPPORTED NON-SKEW STRAIGHT DECKS ON UNYIELDING SUPPORTS

(i) For a simply supported span with fixed (Rocker) and free (Roller-Rocker or sliding plate) bearings (i.e. not elastomeric type) on stiff supports, horizontal forces at the deck-ends, in the bridge longitudinal direction, shall be as follows:

at the FIXED BEARINGS	*at the FREE BEARINGS*
$\{F_h - \mu(R_g + R_q)\}$ or $\left\{\frac{F_h}{2} + \mu(R_g + R_q)\right\}$	$\mu(R_g + R_q)$

whichever greater.

Where:

F_h = Applied horizontal force on the Deck on the span under consideration (e.g. Braking, Earthquake, wind)

R_g = Reaction at the free end due to dead load

R_q = Reaction at free end due to live load

and μ = Coefficient of friction at the free bearing which shall be assumed to have the following values:

(a) For steel roller bearings 0.03*

(b) For concrete roller bearings 0.05

(c) For sliding bearings:

- Steel on cast iron or steel on steel 0.50
- Grey cast iron on grey cast iron (Mehanite) 0.40
- Concrete over concrete with bitumen layer in between 0.60
- Teflon on stainless steel 0.05

(ii) For simply supported reinforced concrete and prestressed concrete superstructures, the span up to which plate bearings can be used shall be limited to 15 m.

(iii) In case of simply supported small spans up to 7.5 meters resting on unyielding supports and where no bearings** are provided, horizontal force in the longitudinal direction at each deck-end shall be taken as

$$\frac{F_h}{2} \text{ or } (\mu \cdot R_g), \text{ whichever greater.}$$

(iv) For a simply supported span on identical elastomeric bearings at each end, resting on unyielding supports, the longitudinal horizontal force at each deck-end shall be taken as:

$$\frac{F_h}{2} + V \cdot \Delta$$

* 0.05 if more than two rollers.

** It is usual to provide felt-pad or thick bitumen-impregnated paper layers for bearings in such spans.

where:

V = sum of the shear ratings of all the elastomeric bearings at one end,

and Δ = movement of deck above the bearings other than that due to 'applied loads' (moving length being only half of the span length assuming the Zero Movement Point is at midspan in the present case).

FORCE TRANSMITTED TO A SUPPORT

Indicated in items (i), (iii) and (iv) above are the forces caused at different deck-ends under simply supported decks. The forces transmitted to the supporting PIERS and ABUTMENTS should thereafter be calculated appropriately. *As an example*, if an intermediate pier supports simple spans from its right (r) and left (l) sides through 'Rocker' and 'Rocker-Roller' type of Bearings, respectively, then the longitudinal horizontal force transmitted to this pier shall be taken as the greater of

$$\{F_{h_r} - \mu(R_{g_r} + R_{q_r} - R_{g_l} - R_{q_l})\}$$

and

$$\left\{\frac{F_{h_r}}{2} + \mu(R_{g_r} + R_{q_r} - R_{g_l} - R_{q_l})\right\}$$

F_{h_r} being the Applied Horizontal Force on the deck of right hand span, suffixes 'r' and 'l' refer to the values at 'free' Bearings in the right and left spans. (Since the left hand span sits on this pier through a 'Rocker-Roller' Bearing — which can transmit only a μR type of force, F_{h_l} is assumed not to be transmitted to this pier.)

However, if the Bearings are of Elastomeric type, then the longitudinal force transmitted to this pier would be:

$$\left(\frac{F_{h_r}}{2}\right) + \left(\frac{F_{h_l}}{2}\right) + \{(V_r \Delta_r) - (V_l \cdot \Delta_l)\}$$

where:

V_r and V_l are the values of V of the right-hand and left-hand span Bearings on this pier, respectively,

and Δ_r and Δ_l are the values of deck movements above the right hand and left hand span Bearings on this pier, respectively.

NOTE If the elastomeric bearings at the two ends are not identical such that a very thin elastomeric pad (6–8 mm) is provided at one end (which acts like a rocker/fixed bearing as its shear rating is consequently enormous), then such a case is akin to case (i) above only that $\mu \cdot (R_g + R_q)$ should be replaced by $V \cdot \Delta$ where V = 'shear rating' of elastomeric bearings on the 'free' side and Δ = movement of full span length since the null-point will be assumed at the fixed end in this case.

8.3 DISTRIBUTION OF LONGITUDINAL HORIZONTAL FORCES AMONG BRIDGE-SUPPORTS IN STRAIGHT-DECKS IN:

- SIMPLE-SPANS AND CONTINUOUS-SPANS ON UNYIELDING OR FLEXIBLE-SUPPORTS

A method[1] is presented for estimating the distribution of longitudinal horizontal forces (acting on the bridge superstructure) among the bridge supports, based on the effect of the location of the point of zero-movement of the Deck, which does not necessarily coincide with the position of the fixed bearing. The position of this point of zero-movement has a direct influence on the magnitude of horizontal forces induced at the top of the supports. The method analyzed takes into account all possible external and internal forces, e.g. temperature, creep, shrinkage, elastic shortening, braking forces, wind forces, etc., acting on the superstructure and their transmission to the bridge supports.

The distribution of longitudinal horizontal forces among bridge supports is affected by the horizontal deformation of bearings, bending of support shafts and rotation of foundations. The problem of distribution becomes all the more acute when different supports have different stiffnesses, as may well be the case in view of the increasing number of multilevel interchanges where all piers may not be identical because of minimum sight distance and clearance restrictions.

Some supports may have to be single columns while others could be double columns or shafts. This difference in the type of supports may cause bearings of different types to be used, e.g., rocking-sliding bearings on heavily loaded single-column piers and elastomeric rubber bearings on double-column or shaft piers. Moreover, different supports may have different types of foundations (spread footings, piles, etc.) dictated by differing subsoil conditions. The analysis of the problem takes into account all these characteristics by considering the shear-rating of each support.

Notation

Type A support = support with a bearing which produces μR type force

Type B support = support with a elastomeric bearing

L_n = distance from extreme left-hand support, '0' to any support 'n'

Δ_n = movement of the deck over the support n relative to the point of zero-movement.

x = distance from support '0' to the point of zero-movement.

c = movement coefficient due to temperature, shrinkage, creep, and elastic shortening of deck

s = shear rating of a Type B support, i.e., horizontal force required to move the top of a Type B support through a unit distance, taking into account horizontal deformation of elastomeric bearings, bending of support shaft and rotation of foundation

μ = coefficient of friction in a bearing used on Type A supports

R = dead and live load reaction on a support

with subscripts l and r to denote, respectively, left and right.

Analysis

Consider any continuous-beam bridge deck as shown in Fig. 8.1 and let the point of zero-movement be at a distance x from support '0'. Let n be the total number of supports.

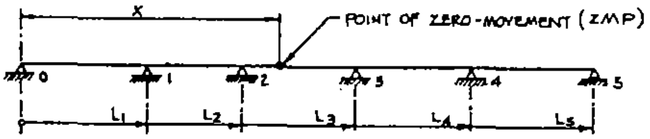


Fig. 8.1 *Continuous-Beam bridge deck showing point of zero longitudinal movement*

Therefore,

$$n = \Sigma \text{ Type } A + \Sigma \text{ type } B \tag{8.1}$$

Then deck movement at any support n to the left of the point of zero-movement is

$$\Delta_{ln} = c(x - L_{ln}) \tag{8.2}$$

and deck movement at any support n to the right of the point of zero-movement is

$$\Delta_{rn} = c(L_{rn} - x) \tag{8.3}$$

Noting also that at 'Type A supports' μR type forces are produced, while at 'Type B supports' $s\Delta_n$ type forces are produced, and that the total force to the left of the zero-movement point must balance the total force to its right, the equilibrium equation can be written as follows:

$$[\Sigma\mu R + \Sigma s\Delta_n]_l = [\Sigma\mu R + \Sigma s\Delta_n]_r \tag{8.4}$$

where suffixes l and r refer to left and right of the zero-movement point.

Substituting for Δ_n,

$$[\Sigma\mu R + \Sigma sc(x - L_{ln})]_l = [\Sigma\mu R + \Sigma sc(L_{rn} - x)]_r \tag{8.5}$$

Transposing and noting that the sum of similar forces on left and right of the zero-movement point equals the overall total of such forces,

$$\Sigma sc(x - L_n) = \Sigma\mu R \tag{8.6}$$

which gives the zero-movement point location:

$$x = \frac{c\Sigma sL_n + \Sigma\mu R}{c\Sigma s} \tag{8.7}$$

where

ΣsL_n = arithmetic sum of products of shear rating s and distance from extreme left hand support, for all of the Type B supports

Σs = arithmetic sum of shear ratings of all of the Type B supports

$\Sigma\mu R$ = algebraic sum of μR forces at all of the Type A supports; + sign for those on right of point of zero-movement; − sign for those on left of point of zero-movement

The above formula gives the distance of the point of zero-movement from the extreme left-hand support. The actual movement of the deck at any support ($c \times$ distance of the support from the point of zero-movement) and also the actual force over each Type B support ($s \times$ actual movement) can therefore be calculated. As for any Type A support, it is assumed that the deck movement is at least equal to or more than the deflection of the support due to μR so that the limiting force of friction μR is mobilized.

There are, in fact, two types of longitudinal forces, viz., that caused by the deck movement due to temperature shrinkage, creep and elastic shortening, and that applied externally (e.g., braking force, longitudinal wind on deck, longitudinal wind on live load, etc.). Distribution of the former depends both on the actual location of the point of zero-movement as well as on the shear ratings of the type B supports, while distribution of the latter depends only on the shear ratings of type B supports. Consequently, the second type of longitudinal force is distributed among Type B supports only, because Type A supports are assumed to have reached their maximum capacity through deck movement as explained earlier.

8.4 APPLICATION

Step 1 Find s for each type B support system as follows:

(a) Assume a unit horizontal force at the top of the support system. If the support shaft and foundation are assumed restrained against any movement, then only the elastomeric bearing is deformed horizontally by a distance m_1. Releasing the support shaft and allowing it to deflect, its top will move further by a distance m_2 under the action of the applied unit force transferred through the elastomeric bearing.

(b) Now, allowing the foundation to rotate under the effect of existing unit force at the top of the support system, the top of the support shaft will move further horizontally by a distance m_3.

(c) Thus, application of unit horizontal force at the top of the support system will cause a total movement of $m_1 + m_2 + m_3$. It is this total movement that will affect the distribution of the horizontal forces. The shear rating is then given by:

$$s = \frac{1}{m_1 + m_2 + m_3} \tag{8.8}$$

Step 2 Assume (by judgement) the span in which the point of zero-movement lies.

Step 3 Find movement coefficient c.

Step 4 Calculate μR values for each Type A support.

Step 5 Work out x in the tabular manner shown in Table 8.1 and proceed to Step 6 if assumption made in Step 2 holds; otherwise, make a fresh assumption and work out a new x until the assumption proves correct. The point of zero-movement location is defined by Eq. (8.7).

Table 8.1 *Summation of factors for calculation of zero-movement point*

Support	Type A or B	μR	s	L_n	sL_n
0					
1					
2					
3					
...					
...					
	Sum	$\Sigma\mu R$	Σs		ΣsL_n

Step 6 Calculate the forces at various supports as follows:

(i) Horizontal force produced at the top of Type A support = μR

(ii) Horizontal force produced at the top of Type B support = $s \times c \times$ distance from zero-movement point.

Check $\Sigma\mu R + \Sigma sc$ (distance from point of zero-movement) = 0

Caution regarding sign of μR forces.

Step 7 Calculate any externally applied longitudinal force, H on the superstructure, e.g., braking force, longitudinal wind on superstructure, longitudinal wind on live load. Knowing Σs and s values from earlier steps (Table 8.1), distribute the force H among Type B supports in the tabular manner shown in Table 8.2. These longitudinal forces, H, are not shared by Type A supports since these supports are already assumed to have reached their limiting capacity μR as a result of deck movement (Steps 1 to 6).

Table 8.2 *Distribution of externally applied horizontal forces*

Support	Type A or B	Type B Supports Only		
		Shear Rating	Total Force	Distributed Force
		s	H	$H \times \frac{s}{\Sigma s}$
0				
1				
2				
3				
...				
...				
		Σs		

Step 8 Add the results of Step 6 and Step 7, at corresponding supports, to obtain total distributed longitudinal horizontal force at each support.

Step 9 Design the necessary reinforcement in the supporting structure after combining the above calculated forces with those resulting from any agencies other than the ones described here.

Numerical Examples

EXAMPLE 1 The structure is a three-span continuous bridge, post-tensioned prestressed concrete slab superstructure with circular voids (Fig. 8.2). Supports 1 and 2 are circular reinforced concrete single columns (Fig. 8.3). Supports 0 and 3 are very rigid abutments. Elastomeric rubber bearings on supports 0 and 3; Rotaflon rocking sliding bearing on support 2; fixed bearing on support 1. Supports 1 and 2 founded on spread footings on hard strata. Supports 0, 1 and 3 are Type B, and support 2 is Type A.

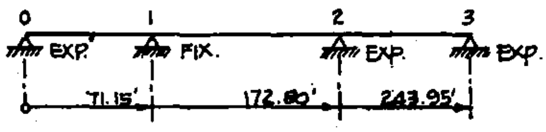


Fig. 8.2 *Structure of Example 1 (Three-span continuous bridge)*

Step 1 Determine shear rating s for each support. Assuming a unit horizontal force at the top of the support system (Fig. 8.3) compute:

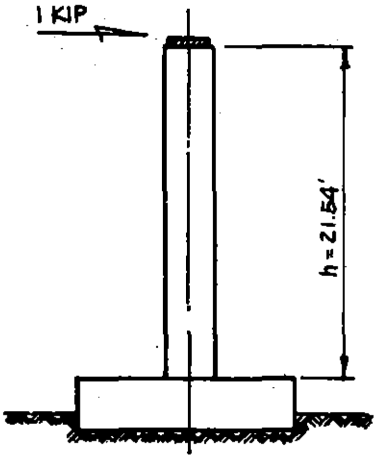


Fig. 8.3 *Intermediate support of bridge Example 1*

(a) *Horizontal deformation of bearings — supports 0 and 3*
If p = number of elastomeric bearings on a support,
q = shear stiffness of each bearing in kips/in.,
then

$$m_1 = \frac{1}{pq} \text{ (in.)}$$

(b) *Horizontal movement of top of column due to deflection of pier — support 1* For a column section having an $I = 13.41$ ft^4 and an $E = 619,000$ ksf, when acted upon by the applied unit kip force transferred through the fixed bearing, the top of the pier column moves horizontally by:

$$m_2 = \frac{h^3}{3EI} = \frac{21.54^3(12)}{3(619,000)(13.41)} = 0.0048 \text{ in.}$$

(c) *Horizontal movement of top of column due to rotation of pier foundation — support 1*
Since the foundation is on solid rock, there is no rotation of the footing and hence no related movement at the top of pier column. Thus,

$$m_3 = 0$$

The shear rating is then given by:

$$s = \frac{1}{m_1 + m_2 + m_3} = \frac{1}{\dfrac{1}{pq} + 0.0048} \text{ (kips/in.)}$$

Work out shear rating values s in the following tabular manner:

Support	p	q	s
0	2	25.6	51.2
1	—	—	208.5
2	—	—	—
3	2	15.5	31.0

Step 2 Assume that the point of zero-movement lies in the central span.

Step 3 Determine movement coefficient c.

$$c = at + \epsilon_s + \epsilon_c + \epsilon_{es}$$

where

a = coefficient of linear expansion or contraction $= 6 \times 10^{-6}$ per °F
t = change in temperature = 45°F
ϵ_s = shrinkage strain
ϵ_c = creep strain
ϵ_{es} = strain due to elastic shortening under prestress force

From the relevant bridge calculations:

$$\epsilon_s + \epsilon_c + \epsilon_{es} = 0.00064$$

Substituting these values:

$$c = 0.00091$$

Step 4 For support 2, which is the only one of Type A, μR = 22.3 kips from the relevant bridge calculations.

Step 5 Calculate position of zero-movement point.

Support	Type A or B	μR	s	L_n	sL_n
0	B	—	51.2	0	0
1	B	—	208.5	71.15	14834.78
2	A	22.3	—	172.80	—
3	B	—	31.0	243.95	7562.45
Sum		22.3	290.7	—	22397.23 × 12

$$x = \frac{0.00091(22397.23)(12) + 22.30}{0.00091(290.7)} = 84.08(12) \text{ in.}$$

i.e., x = 84.08 ft which means the point of zero-movement lies in the central span as assumed in Step 2.

Step 6 Calculate horizontal force produced by deck movement (for $c = 0.00091$).

Support	Type A or B	μR (1)	s (2)	Distance from point of zero-movement (3)	Distributed force, kips (1) or (2) × (3) × c
0	B	—	51.2	84.08 × 12	− 47.00
1	B	—	208.5	12.93 × 12	− 29.44
2	A	+ 22.3	—	88.72 × 12	+ 22.30
3	B	—	31.0	159.87 × 12	+ 54.12
				Total	0.00 OK

Step 7 Total externally applied longitudinal force due to braking action is 13.4 kips (from the actual calculation pertaining to this particular bridge).
Horizontal force produced by above braking force:

Support	s	H	Distributed force, kips
0	51.2	13.4	2.37
1	208.5		9.61
2	—		—
3	31.0		1.42
		Total	13.40 OK

Step 8 Add the results of Step 6 and Step 7 to obtain total distributed longitudinal horizontal force at each support.

Support	Horizontal distributed force, kips		Total force, kips
	Due to temp., shrinkage, creep	Due to braking	
0	47.00	2.37	49.37
1	29.44	9.61	39.05
2	22.30	—	22.30
3	54.12	1.42	55.54

Step 9 Design each support to resist the corresponding force calculated above.

EXAMPLE 2 The structure is a six-span semi-continuous bridge with prestressed pre-tensioned AASHO beams supporting a 7-in, reinforced concrete deck slab (Fig. 8.4). Supports 1 to 5 are triangulated piles (Fig. 8.5). Supports 0 and 6 are very rigid abutments. Elastomeric rubber bearings on all supports except on support 3, which carries a "fixed" bearing.

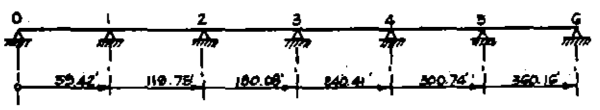


Fig. 8.4 *Structure of Example 2 (six-span semi-continuous bridge)*

Step 1 Determine shear rating s for each support. Assuming a unit horizontal force at the top of the support system (Fig. 8.5) compute:

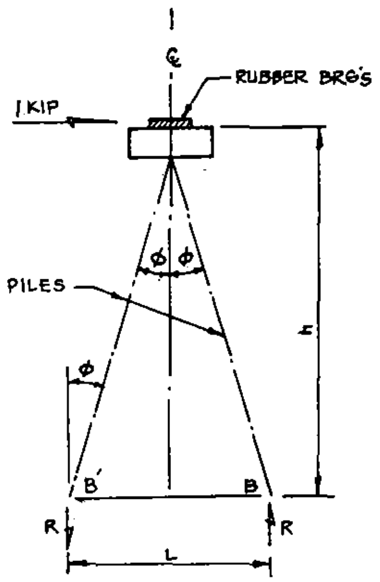


Fig. 8.5(a) *Intermediate support of bridge Example 2*

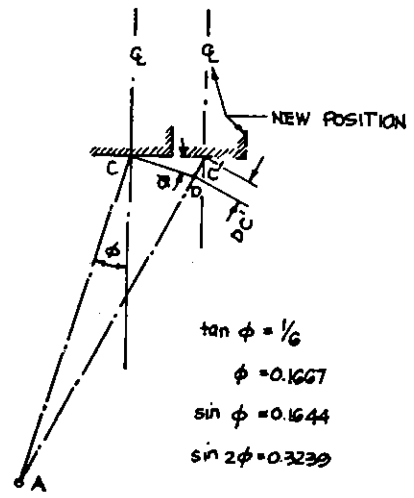


Fig. 8.5(b) *Horizontal movement due to pile deformation (bridge Example 2)*

(a) *Horizontal deformation of bearings*
If p = number of elastomeric bearings on a support,
q = shear stiffness of each bearing in kips/in.,
then

$$m_1 = \frac{1}{pq} \text{ (in.)}$$

(b) *Horizontal movement of top of pier due to deformation of piles*
Acted upon by the applied unit force transferred through the rubber bearings, the top of the pier

moves horizontally by:

$$m_2 = CC' = DC'/\sin\phi$$

The above relationship is easily established from Fig. 8.5(b). The small error due to the assumption of angle ϕ is negligible. If

$$DC' = \frac{L}{2\sin\phi}\times\frac{F}{AE} \text{ (i.e. original length × strain)}$$

and

$$F = \frac{R}{\cos\phi} = \frac{h}{L\cos\phi} \text{ (axial force in pile)}$$

Then

$$DC' = \frac{L}{2\sin\phi}\times\frac{h}{AEL\cos\phi} = \frac{h}{AE\sin 2\phi}$$

Therefore

$$m_2 = \frac{h}{AE\sin\phi\sin 2\phi}$$

where
A = section area of pile = 493 in^2.
E = mod. of elasticity of pile material = 3300 ksi

$$m_2 = \frac{h}{493(3300)(0.1644)(0.3239)} = \frac{h}{86,600} \text{ (in.)}$$

The shear rating is then given by:

$$s = \frac{1}{m_1 + m_2} = \frac{1}{\dfrac{1}{pq} + \dfrac{1}{86,600}} \text{ (kips/in.)}$$

Work out shear rating values s in the following tabular manner:

Support	p	q	h	s
0, 6	5	4.3	very rigid	21.5
1, 5	10	6.0	480	45.0
2, 4	10	6.0	600	42.4
3	10	∞	600	144.4

Step 2 Because of complete symmetry about the 'fixed' support 3, it is evident that the point of zero-movement coincides with it. However, this will be confirmed numerically in Step 5.

Step 3 Determine movement coefficient c.

$$c = at + \epsilon_s + \epsilon_c$$

where

a = coefficient of linear expansion or contraction = 6×10^{-6} per °F.
t = change in temperature = 45°F
ϵ_s = shrinkage strain
ϵ_c = creep strain

From the actual calculation pertaining to this particular bridge:

$$\epsilon_s + \epsilon_c = 0.00025$$

Substituting these values:

$$c = 0.000,006(45) + 0.00025 = 0.00052$$

Step 4 μR values are zero because no support is of Type A.

Step 5 Calculate position of zero-movement point.

Support	Type A or B	μR	s	L_n	sL_n
0	B	—	21.5	0	0
1	B	—	45.0	59.42	2670
2	B	—	42.4	119.75	5070
3	B	—	144.4	180.08	26000
4	B	—	42.4	240.41	10200
5	B	—	45.0	300.74	13500
6	B	—	21.5	360.16	7760
	Sum	—	362.4	—	65200 × 12

$$x = \frac{0.00052(65,200)(12)}{0.00052(362.4)} = 180.08(12) \text{ in.}$$

Step 6 Calculate horizontal force produced by deck movement (for $c = 0.00052$).

Support	s	Distance from point of zero-movement	Distributed force, kips
	(1)	(2)	(1) × (2) × c
0, 6	21.5	180.08(12)	24.1
1, 5	45.0	120.66(12)	33.8
2, 4	42.4	60.33(12)	16.0
3	144.4	0.00	0.0

Step 7 Total externally applied longitudinal force due to braking action is 25.65 kips, from relevant bridge calculation.

Horizontal force produced by above braking force:

Support	s	H	Distributed force, kips
0,6	21.5		1.52 each
1,5	45.0		3.18 each
2,4	42.4	25.65	3.00 each
3	144.4		10.25
sum	362.4	25.65	25.65

Step 8 Add the results of Step 6 and Step 7 to obtain total distributed longitudinal horizontal force at each support.

Support	Horizontal distributed force, kips		Total force kips
	Due to temp., shrinkage, creep	Due to braking	
0,6	24.1	1.52	25.62
1,5	33.8	3.18	36.98
2,4	16.0	3.00	19.00
3	0.00	10.25	10.25

Step 9 Design each support to resist the corresponding force calculated above.

EXAMPLE 3 (General) This example shows how to compute m_3, the movement at the top of a support due to rotation of: (A) spread-footing foundation, and (B) piled foundation.

(A) Case of Spread-footing foundation

If the footing rotates through an angle ϕ under the bending moment created by the unit horizontal force applied as shown in Fig. 8.6, then $m_3 = h\phi$, approximately, and, in the absence of a more accurate formula, ϕ can be estimated

$$\tan\phi = \frac{12M}{ab^3c}$$

where

$M = 1.00h = h$

c = coefficient of subgrade reaction of soil

a = plan dimension of footing normal to bridge

b = plan dimension of footing parallel to bridge

(B) Case of a piled foundation (Fig. 8.7)

If the axial force in the pile that is distant x_n from the pier centerline is

$$F = \frac{Mx_n}{I} = \frac{hx_n}{\Sigma x^2}$$

then the corresponding axial deformation of the pile is,

$$y = d\frac{F}{AE} = \frac{hx_nd}{AE\Sigma x^2}$$

so that: $m_3 = h\phi$

where:

$\phi = \dfrac{y}{x_n}$

A = area of pile section

E = modulus of elasticity of pile material

d = length of pile

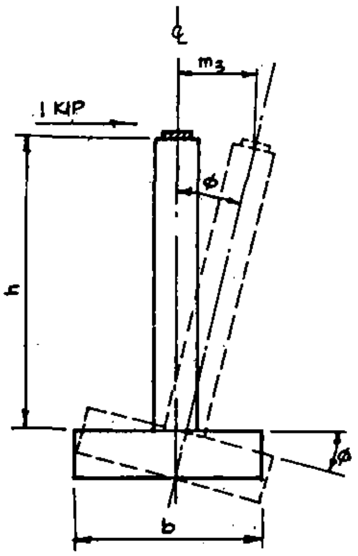


Fig. 8.6 *Rotation of footing caused by horizontal force at top of support (Example 3A)*

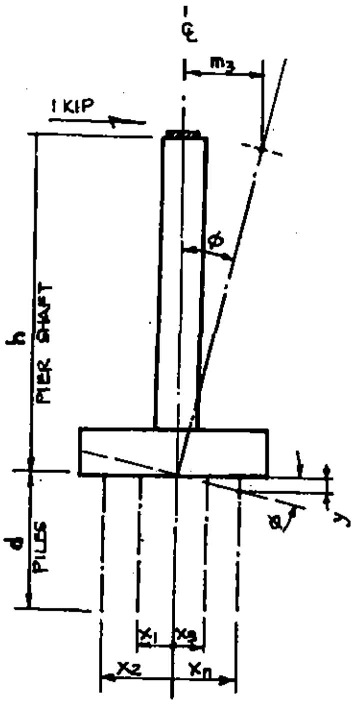


Fig. 8.7 *Axial deformation of pile caused by rotation of pier (Example 3B)*

8.5 CONCLUSION

In designing the bridge supports, it is very useful to distribute the longitudinal horizontal forces by taking into account the actual location of the zero-movement point and the shear ratings of all support systems. This is so because this realistically reduces the burden on the support with fixed bearing and thereby avoids its being overdesigned unnecessarily, while preventing the other supports from being underdesigned innocently.

REFERENCE

1. A.W. Witecki and V.K. Raina, "Distribution of Longitudinal Horizontal Forces among Bridge Supports (taking into account Deformation of Bearings, Flexing of Piers and Rotation of Foundations)", *1st Intnl. Symp. on Concrete Bridge Design*, A.C.I. April 1967 (Toronto), ACI Spl. Publication No. 23.

CHAPTER 9

Distribution of Externally Applied and Self-induced Horizontal Forces among Bridge-Supports in Curved and/or Skewed Decks (Simple or Continuous Spans)

Synopsis

The phenomenon of superstructure movement in a curved and/or skewed bridge deck, as affected by shrinkage, creep and elastic-shortening of concrete and any temperature variations, is studied and its effect on distribution of the horizontal forces it causes among the supports is analysed[1]. The analysis considers the actual stiffnesses of individual supports and their influence on the location of the zero-movement point, which does not necessarily coincide with the position of any of the supports. The concept of in-plan meandering of deck, as caused by differing shear ratings of any support in two orthogonal directions, is introduced in computing the location of zero-movement point. At the end a practical example is included in an endeavour to illustrate the application.

Notation

Type A support = support with a bearing which produces μR type force.

Type B support = support either with a bearing which produces Δs type force or a support which is monolithic with superstructure,

c = movement coefficient due to temperature change and shrinkage, creep and elastic-shortening of concrete.

Δ = deck movement,

s = shear rating of a Type B support, i.e., horizontal force required to move the top of a Type B support through a unit distance, taking into account horizontal deformation of its bearing, bending of pier shaft or column and rotation of its foundation,

μ = coefficient of friction of a bearing used on Type A support.

R = dead and live load reaction on a support

F = horizontal force caused at a Type B support

t = torsion rating of a support, i.e., twisting moment required to cause a unit radian twist in-plan in the support.

Support U = any arbitrarily chosen support, as a centre of (a, b) coordinate system, with respect to which the coordinates (a, b) of all other supports are known

x, y = coordinates of any support with respect to point of zero-movement; x and y being the coordinate axes through the point of zero-movement

$x_0 y_0$ = coordinates of support U with respect to point of zero-movement (along x and y axes)

ϕ = in-plan meander of superstructure, i.e., the angle between the two sets of coordinate axes (a, b) and (x, y).

T = in-plan torque at a support

M = in-plan moment

θ = angle used in derivation of Δ_x and Δ_y, see Fig. 9.1

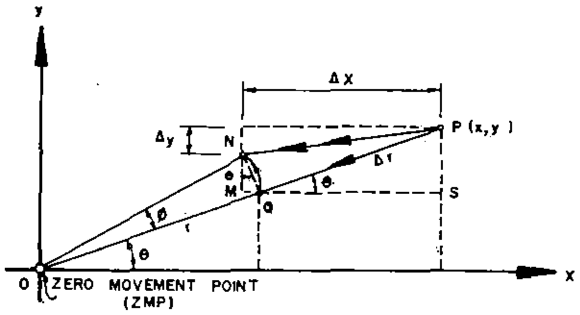
ZMP = zero-movement point,

r and Δ_r = a radial distance and a radial movement respectively as shown in Fig. 9.1

NOTE Suffixes x and y represent component effects in x and y directions, respectively.

9.1 INTRODUCTION

Shrinkage, creep and elastic shortening of concrete, as well as temperature variation, cause volume change in a bridge superstructure, resulting in its deformation in various directions and consequently affecting the distribution of horizontal forces among its supports. Similar volume change effects can occur among the supports of industrial concrete slab structures. The only significant deformations are in-plan movements. These are translatory (linear) in the case of straight bridges, but translatory and rotatory (curvilinear) in the case of curved bridges. The zero-movement point in



NQ ASSUMED NORMAL TO OP (ϕ SMALL)
$PQ = \Delta r$
$OQ = r$

Fig. 9.1 *Movement of support due to translation and deck meander*

the former case is consequently a full-width section normal to the bridge length, as against truly a point in the latter case. Thus, locating the point of zero-movement in a curved bridge is much more involved than in case of a straight bridge. It not only involves balancing the horizontal forces in two orthogonal directions but also determining the in-plan meandering twist in the deck resulting from the imbalance in the shear ratings of the supports in the two orthogonal directions.

The distribution of horizontal forces among bridge supports is affected by the horizontal deformation of bearings, bending of pier shafts or columns and rotation of their foundations. Externally applied horizontal forces like braking force, earthquake force, wind force, etc., may be distributed among the supports in proportion to their indirection shear ratings. However, the horizontal forces, induced by volume change due to shrinkage, creep, elastic shortening and temperature, are of a type, the distribution of which depends not only on shear ratings of individual supports but also on the location of zero-movement point. The resultant movement in a curved and/or skewed superstructure at any support takes place along the vector joining that support with the zero-movement point, in case of no in-plan deck meander. In case of straight bridge, the volume change effect on distribution of horizontal forces has already been investigated, analysed and reported by the Author and Witecki in the *First International Symposium on Concrete Bridge Design* published by American Concrete Institute (also refer Ch. 8). The present discussion investigates and analyses the above-mentioned effect in the case of a curved and/or skewed bridge, the most general case.

9.2 ANALYSIS

Consider a curved bridge superstructure subjected to temperature change and shrinking, creeping and elastic shortening of concrete, such that a support $P(x, y)$ moves to $N(x - \Delta_x, y - \Delta_y)$—due to translation Δ_r and in-plan deck meander of ϕ, as shown in Fig. 9.1. Generally angle ϕ is small enough to assume NQ to be normal to OP, so that,

$$\angle POX = \angle PQS = \theta \approx \angle QNM$$

and $$\text{chord } NQ = \text{arc } NQ = r\phi$$

Further the radial movement PQ is equal to the product of movement coefficient c and the distance OP, i.e.,

$$\Delta_r = c(r + \Delta_r)$$

and neglecting the second order term $c\Delta_r$,

$$\Delta_r = cr$$

From Fig. 9.1, the x-direction movement is,

$$\Delta_x = SQ + QM$$

Incorporating the above substitutions:

$$\Delta_x = cr\cos\theta + \phi r\sin\theta$$

and finally because,

$$r\cos\theta = x - QS = x - \Delta_r\cos\theta$$

and $$r\sin\theta = y - PS = y - \Delta_r\sin\theta$$

the above equation yields,

$$\Delta_x = cx - c\Delta_r\cos\theta + \phi y - \phi\Delta_r\sin\theta$$

However, knowing that the products, $c\Delta_r$ and $\phi\Delta_r$ are negligible as c, ϕ and Δ_r are individually very small magnitudes, the final movement of the point $P(x, y)$ in the x-direction is,

$$\Delta_x = cx + \phi y \tag{9.1}$$

Again from Fig. 9.1, the y-direction movement is,

$$\Delta_y = PS - NM$$

Incorporating the various substitutions described earlier and proceeding on similar lines as in the case of Δx, the above equation yields,

$$\Delta_y = cy - c\Delta_r\sin\theta - \phi x + \phi\Delta_r\cos\theta$$

and neglecting the second order terms as before, the final movement of the point $P(x, y)$ in the y-direction is,

$$\Delta_y = cy - \phi x \tag{9.2}$$

Due to deck movement of Δ_x and Δ_y at a (Type B) support $P(x, y)$, the horizontal forces generated are,

in x-direction

$$F_x = s_x \Delta_x = s_x(cx + \phi y) \tag{9.3}$$

in y-direction

$$F_y = s_y \Delta_y = s_y(cy - \phi x) \tag{9.4}$$

and resultant vectorial horizontal force equals

$$\sqrt{(F_x)^2 + (F_y)^2}$$

The x and y coordinates of all the supports with respect to ZMP are unknown, as well as different for different supports. In order to reduce the number of these unknowns to a minimum, it is advantageous to:

(i) establish, and therefore, know the coordinates of all the supports with respect to any arbitrarily chosen support U (0, 0) in the a, b coordinate system.
(ii) relate this support U to the ZMP by its unknown coordinates x_0, y_0 in the x, y coordinate system.
(iii) and finally relate each support to the ZMP by the principle of transformation of axes in terms of known a and b values and unknowns x_0 and y_0.

From the principle of transformation of coordinate axes, if (a, b) are the coordinates of a support $P(a, b)$ with respect to support U (0, 0), and (x, y) are coordinates with respect to ZMP, then, from Fig. 9.2,

$$x = x_0 + a\cos\phi + b\sin\phi \tag{9.5}$$

and

$$y = y_0 + b\cos\phi - a\sin\phi \tag{9.6}$$

It should be noted that Eqs. (9.5) and (9.6) are derived from Fig. 9.2 on the arbitrary assumption that the ZMP lies to the left of support U and below it.

The horizontal forces in the x and y directions, as given by Eqs. (9.3) and (9.4) can then be expressed as follows,

$$F_x = s_x[c(x_0 + a\cos\phi + b\sin\phi) + \phi(y_0 + b\cos\phi - a\sin\phi)] \tag{9.7}$$

$$F_y = s_y[c(y_0 + b\cos\phi - a\sin\phi) - \phi(x_0 + a\cos\phi + b\sin\phi)] \tag{9.8}$$

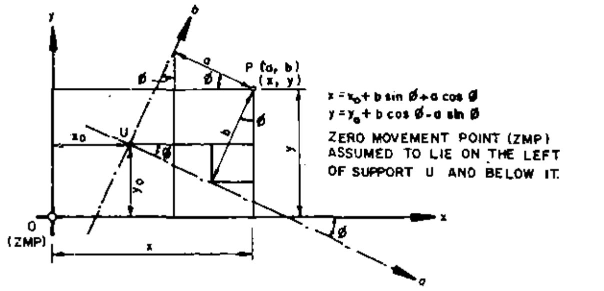


Fig. 9.2 *Relationship between the coordinates of a support P with reference to support U and ZMP*

Further neglecting the second order terms. Eqs. (9.7) and (9.8) may be reduced and re-written in the final form as follows,

$$F_x = s_x[cx_0 + \phi y_0 + (ca + \phi b)\cos\phi] \tag{9.9}$$

and

$$F_y = s_y[cy_0 - \phi x_0 + (cb - \phi a)\cos\phi] \tag{9.10}$$

The three unknowns x_0, y_0 and ϕ locate the ZMP. For locating it, it is necessary to consider the equilibrium of the in-plan horizontal forces in the two orthogonal directions x and y, together with the in-plan moment-equilibrium, viz.

$$\Sigma(F + \mu R)_x = 0 \tag{9.11}$$

$$\Sigma(F + \mu R)_y = 0 \tag{9.12}$$

and

$$\Sigma M = 0 \tag{9.13}$$

Equations (9.11) and (9.12) yield,

$$\Sigma\mu R_x + \Sigma s_x[cx_0 + \phi y_0 + (ca + \phi b)\cos\phi] = 0 \tag{9.14}$$

and

$$\Sigma\mu R_y + \Sigma s_y[cy_0 - \phi x_0 + (cb - \phi a)\cos\phi] = 0 \tag{9.15}$$

where

$$\mu R_x = \mu R\left[1 + \left(\frac{\Delta_y}{\Delta_x}\right)^2\right]^{-1/2} \tag{9.16}$$

$$\mu R_y = \mu R\left[1 + \left(\frac{\Delta_x}{\Delta_y}\right)^2\right]^{-1/2} \tag{9.17}$$

$$\Delta_x = cx_0 + \phi y_0 + (ca + \phi b)\cos\phi \tag{9.18}$$

(from Eqs. (9.1), (9.5) and (9.6))*

and

$$\Delta_y = cy_0 - \phi x_0 + (cb - \phi a)\cos\phi \tag{9.19}$$

(from Eqs. (9.2), (9.5) and (9.6))*

*Neglecting the second order terms.

Note that Eqs. (9.16) and (9.17) are derived from the relation between the force and displacement vector diagrams.

Taking moments about ZMP, Eq. (9.13) yields,

$$\Sigma F_x y + \Sigma F_y x + \Sigma t\phi + \Sigma \mu R_x y + \Sigma \mu R_y x = 0 \quad (9.20)$$

Substituting for the various terms in Eqs. (9.14), (9.15) and (9.20), and neglecting the second order terms, the following three simultaneous equations of equilibrium for evaluating x_0, y_0 and ϕ are arrived at,

$$\Sigma \mu R \left\{ 1 + \left[\frac{cy_0 - \phi x_0 + (cb - \phi a)\cos\phi}{cx_0 + \phi y_0 + (ca + \phi b)\cos\phi} \right]^2 \right\}^{-1/2} + \Sigma s_x [cx_0 + \phi y_0 + (ca + \phi b)\cos\phi] = 0 \quad (9.21)$$

$$\Sigma \mu R \left\{ 1 + \left[\frac{cx_0 + \phi y_0 + (ca + \phi b)\cos\phi}{cy_0 - \phi x_0 + (cb - \phi a)\cos\phi} \right]^2 \right\}^{-1/2} + \Sigma s_y [cy_0 - \phi x_0 + (cb - \phi a)\cos\phi] = 0 \quad (9.22)$$

and

$$\begin{aligned} &\Sigma s_x (y_0 + b\cos\phi)[cx_0 + \phi y_0 + (ca + \phi b)\cos\phi] + \\ &\Sigma s_y (x_0 + a\cos\phi)[cy_0 - \phi x_0 + (cb - \phi a)\cos\phi] + \\ &\Sigma \mu R \left\{ 1 + \left[\frac{cy_0 - \phi x_0 + (cb - \phi a)\cos\phi}{cx_0 + \phi y_0 + (ca + \phi b)\cos\phi} \right]^2 \right\}^{-1/2} \\ &\qquad \times (y_0 + b\cos\phi - a\sin\phi) + \\ &\Sigma \mu R \left\{ 1 + \left[\frac{cx_0 + \phi y_0 + (ca + \phi b)\cos\phi}{cy_0 - \phi x_0 + (cb - \phi a)\cos\phi} \right]^2 \right\}^{-1/2} \\ &\qquad \times (x_0 + a\cos\phi + b\sin\phi) + \Sigma t\phi = 0 \quad (9.23) \end{aligned}$$

These three equations are by no means simple to tackle in an average bridge design office. Fortunately they reduce to a simple set of two Eqs. (9.28) and (9.29) derived ahead in most practical bridge cases where ϕ equals zero. There are two specific cases, presented below, where ϕ equals zero and above-mentioned simplification in Eqs. (9.21), (9.22) and (9.23) takes place.

Case 1

The in-plan meander of deck is zero if its each support individually has its shear ratings in x and y directions equal, i.e.,

$\phi = 0$ if $s_x = s_y$ for every support individually.

Generally this is the case in majority of the bridges because their supports are usually of the following types:

(i) Deck monolithic with circular and/or square column supports.
(ii) Deck sitting on circular and/or square columns through bearings which absorb same effort in deforming equally in all directions.
(iii) Deck sitting on very stiff abutments through bearings which absorb same effort in deforming equally in all directions.

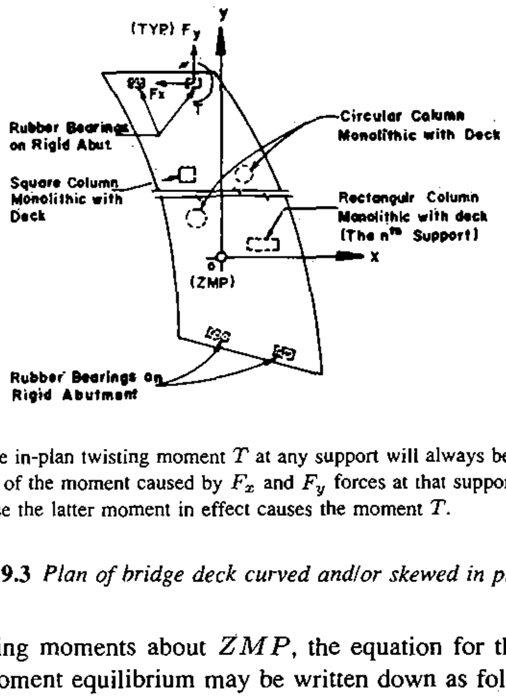
Case 2

The in-plan meander of deck is zero even if there are some supports whose shear ratings are different in different directions, but are so located that at least one of their coordinates with respect to ZMP is zero.

The statements in both these cases can be mathematically proved as follows:

Consider any curved and/or skewed bridge deck on n supports, in general, as shown in the plan in Fig. 9.3. Assume that the shear ratings in the x and y directions of all but the n-th support are equal. i.e.,

$s_x = s_y$ for each of the supports 1 to $n - 1$, equals s (say), and $s_x \neq s_y$ for the n-th support.



NOTE The in-plan twisting moment T at any support will always be in the direction of the moment caused by F_x and F_y forces at that support. This is because the latter moment in effect causes the moment T.

Fig. 9.3 *Plan of bridge deck curved and/or skewed in plan*

Taking moments about ZMP, the equation for the in-plan moment equilibrium may be written down as follows,

$$\sum_1^n [F_x y - F_y x + T] = 0$$

Noting that,

$$F_x = s_x(cx + \phi y)$$
$$F_y = s_y(cy - \phi x)$$

and

$$T = t\phi$$

then the above equation may be re-written as,

$$\sum_{1}^{n}\Sigma[s_x(cx+\phi y)y - s_y(cy-\phi x)x + t\phi] = 0$$

Considering the supports 1 to $n-1$ as a separate group from the n-th support, then the equation may be split up and re-written as follows,

$$\sum_{1}^{n-1}[s_x(cx+\phi y)y - s_y(cy-\phi x)x + t\phi] +$$
$$[s_x(cx+\phi y)y - s_y(cy-\phi x)x + t\phi]n\text{th} = 0$$

Noting that $s_x = s_y = s$ for each of the supports 1 to $n-1$, the above equation simplifies into,

$$\sum_{1}^{n-1}\Sigma[s(x^2+y^2)\phi + t\phi] + [x(s_x yc + s_y x\phi) +$$
$$y(s_x y\phi - s_y xc) + t\phi]n\text{th} = 0 \qquad (9.24)$$

If x-coordinate of the n-th support equals zero, then Eq. (9.24) reduces to

$$\phi\left\{\sum_{1}^{n-1}[s(x^2+y^2)+t] + (s_x y^2 + t)n\text{th}\right\} = 0 \quad (9.25)$$

If y-coordinate of the n-th support equals zero, then Eq. (9.24) reduces to,

$$\phi\left\{\sum_{1}^{n-1}[s(x^2+y^2)+t] + (s_y x^2 + t)n\text{th}\right\} = 0 \quad (9.26)$$

And if both x and y coordinates of n-th support equal zero, then Eq. (9.24) reduces to,

$$\phi\left\{\sum_{1}^{n-1}[s(x^2+y^2)+t] + (t)n\text{th}\right\} = 0 \qquad (9.27)$$

Since all the terms within the parentheses in Eqs. (9.25), (9.26) and (9.27) are always positive and finite, therefore, in each of them, it is ϕ which must equal zero.

In other words ϕ equals zero either if each support has its s_x equal to its s_y, or even if only some supports are of this type but the rest are so located as to fall at least on one coordinate axis through the ZMP.

Thus, with ϕ equal to zero only the two Eqs. (9.21) and (9.22) need be considered in order to solve remaining two unknowns x_0 and y_0. After the proper substitution and simplification, the two final simultaneous equations, as a solution to the problem of locating ZMP, are as follows,

$$\Sigma\mu R\left\{1+\left[\frac{y_0+b}{x_0+a}\right]^2\right\}^{-1/2} + \Sigma s_x(cx_0+ca) = 0 \qquad (9.28)$$

and

$$\Sigma\mu R\left\{1+\left[\frac{x_0+a}{y_0+b}\right]^2\right\}^{-1/2} + \Sigma s_y(cy_0+cb) = 0 \qquad (9.29)$$

For the particular case of a straight bridge, where always not only ϕ equals zero but also only one coordinate axis, viz., x (same as a) axis, need be considered, Eq. (9.28) yields,

$$\Sigma\mu R + \Sigma s_x c(x_0+a) = 0$$
$$\therefore \quad x_0 = -\frac{\Sigma\mu R + c\Sigma s_x a}{c\Sigma s_x} \qquad (9.30)$$

The $-$ve sign in Eq. (9.30) indicates that the ZMP lies in a direction opposite to that assumed in Fig. 9.2, i.e., to the right of the support U. In the case of a straight bridge if the extreme left-hand support is taken as the support U, then all the other supports lie to its right and obviously the ZMP must also lie to its right. This is pointed out by the $-$ve sign in Eq. (9.30).

It is interesting to note the location of the ZMP as given by the formula in Eq. (9.30) above, is exactly what had already been arrived at for the case of straight bridge in Ch. 8 of this book.

9.3 APPLICATION

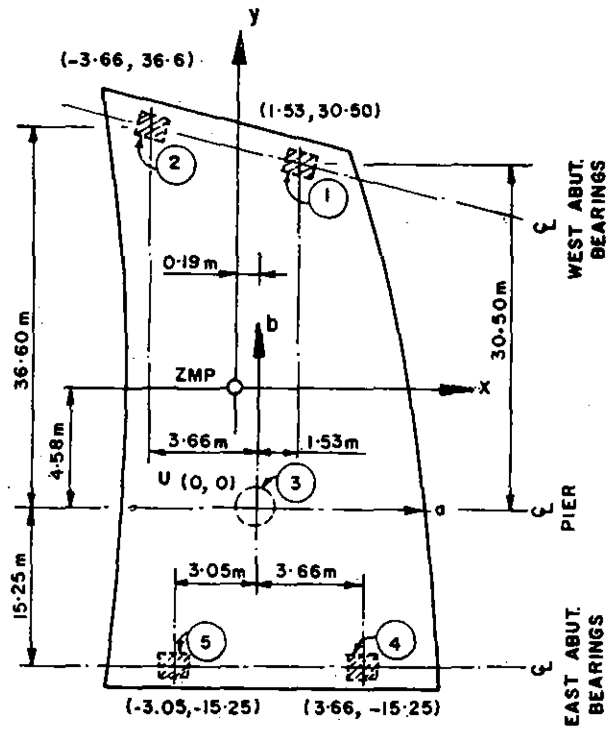
In order to analyse the distribution of horizontal forces that are induced in bridge-supports by the volume changes in a curved and/or skewed concrete bridge superstructure, proceed step by step as follows:

(i) Calculate the shear ratings in x and y directions of each Type B support (same procedure may be followed as explained earlier in Ch. 8).
(ii) Locate the ZMP using Eqs. (9.21), (9.22) and (9.23), unless ϕ equals zero, in which case use Eqs. (9.28) and (9.29) instead.
(iii) Calculate the horizontal forces F_x and F_y in the Type B supports using Eqs. (9.9) and (9.10).
(iv) Calculate the horizontal forces μR_x and μR_y in the Type A supports using Eqs. (9.16) and (9.17) with Δ_x and Δ_y from Eqs. (9.18) and (9.19).
(v) Check in-plan equilibrium by using Eqs. (9.11) and (9.12).

Numerical Example

This example is primarily intended to illustrate the procedure in a practical case.

The structure is a two-span, post-tensioned prestressed concrete bridge deck, curved and skewed in plan, as shown in Fig. 9.4. The superstructure is monolithic with the circular concrete column (support 3), and at the ends rests on rigid abutments, through rubber-steel elastomeric bearings (supports 1, 2, 4 and 5). Hence all supports are of Type B, and as it happens, every support has its s_x equal to its s_y. This means that the in-plan deck meander ϕ is zero.



1, 2, 4 & 5 are Elastomeric (Rubber-Steel) Bearings on Rigid Abutments.
3 is a Circular Column-Pier, Monolithic with Deck. Bridge is Multi-curved and Multiskewed Type.

Fig. 9.4 *Plan of a curved skewed bridge deck*

From the relevant bridge-design calculations,

- for supports 1, 2, 4 and 5, each, $s_x = s_y = 3.57$ T/cm.
- for support 3 $s_x = s_y = 14.28$ T/cm.
- movement coefficient, based on full shrinkage, creep and elastic-shortening of concrete, and a 7.2°C variation in surrounding temperature, $c = 0.001$ cm/cm.

Assume the support 3 to be the arbitrarily chosen support U, the centre of (a, b) coordinate system. From the known geometry of the bridge, the coordinates (a, b) to the supports 1, 2, 4 and 5, are then worked out. These are marked as shown in Fig. 9.4.

Proceeding step-by-step as explained in the beginning of this section, the horizontal forces at various supports as caused by volume change in the deck, may be evaluated as follows,

Step 1 Shear ratings of supports—as detailed above.

Step 2 Locating ZMP: use Eqs. (9.28) and (9.29) since ϕ equals zero. Noting that there are no μR type forces here because there are no Type A supports, and calling s_x and s_y simply as s since s_x equals s_y for every support, Eqs. (9.28) and (9.29) reduce to,

$$\sum_1^5 s\ c(x_0 + a) = 0$$

and
$$\sum_1^5 s\ c(y_0 + b) = 0$$

The summations on the left-hand sides of these equations may be worked out in a tabular manner as shown in Table 9.1.

Table 9.1

Support no:	s T/cm	a cm	b cm	$s.c.(x_0 + a)$ kg	$s.c.(y_0 + b)$ kg
1	3.57	153	3050	$3.57\ (x_0 + 153)$	$3.57\ (x_0 + 3050)$
2	3.57	−366	3660	$3.57\ (x_0 - 366)$	$3.57\ (y_0 + 3660)$
3	14.28	0	0	$14.28\ x_0$	$14.28\ y_0$
4	3.57	366	−1525	$3.57\ (x_0 + 366)$	$3.57\ (y_0 - 1525)$
5	3.57	− 305	−1525	$3.57\ (x_0 + 305)$	$3.57\ (y_0 - 1525)$
Note: c = 0.001 cm/cm			Total	$(28.56\ x_0 - 543)$	$(28.56\ y_0 + 13300)$

Then $\quad 28.56x_0 - 543 = 0$

and $\quad 28.56y_0 + 13330 = 0$

giving $x_0 = 543/28.56 = 19.1$ cm; i.e., 0.191 m

and $\quad y_0 = -13300/28.56 = -458$ cm, i.e., −4.58 m

Remembering that (x_0, y_0) are the coordinates of the support U with respect to ZMP on x, y coordinate system (Fig. 9.2), then, conversely, $(-x_0 - y_0)$ are the coordinates of the ZMP with respect to the support U, since, for ϕ equal to zero, (a, b) and (x, y) are parallel coordinate systems.

Step 3 Horizontal forces F_x and F_y may now be calculated at each Type B support using Eqs. (9.9) and (9.10). As ϕ equals zero, these equations reduce to,

$$F_x = s\ c(x_0 + a)$$

and
$$F_y = s\ c(y_0 + b)$$

These forces are worked out in a tabular manner in Table 9.2.

Table 9.2

c = 0.001 cm/cm, x_0 = 0.191m, y_0 = −4.58 m							
Support No	s T/cm	a cm	b cm	$(x_0 + a)$ cm	$(y_0 + b)$ cm	$F_x = sc(x_0 + a)$ kg	$F_y = sc(y_0 + b)$ kg
1	3.57	153	3050	172	2522	613	9250
2	3.57	−366	3660	−347	3202	−1239	11450
3	14.28	0	0	19	− 458	272	− 6550
4	3.57	366	−1525	385	− 1983	1372	− 7075
5	3.57	−305	−1525	−286	−1983	−1018	− 7075
					Total	0.00 O.K.	0.00 O.K.

Step 4 Horizontal forces μR_x and μR_y do not exist in the present case as there is no Type A support.

Step 5 The in-plan equilibrium is obviously existing since the summations in Table 9.2 show that,

$$\Sigma F_x = 0$$

and $$\Sigma F_y = 0$$

9.4 CONCLUSION

In view of the increasing number of highway bridges and particularly the multilevel interchanges, where not only all piers may not be identical because of minimum sight distance and clearance requirements but also the geometry often requires curved and skewed superstructures, the problem of distribution of horizontal forces among the supports due to vectorial deck movements attains all the more importance. In designing these supports it is therefore very useful to understand the true phenomenon of force distribution and accordingly design for the actual forces. This realistically reduces the burden on some supports and thereby avoids their being 'overdesigned' unnecessarily, while preventing the other supports from being 'underdesigned' innocently.

The concept outlined in this chapter is not restricted to bridges only. It can be used as a powerful tool in analysing the volume change stresses in any given set of supports under a concrete slab of any shape. Multilevel car parks and buildings and auditoria are only a few examples.

REFERENCE

1. V.K. Raina and A.W. Witecki, "Volume Change Effect on Distribution of Horizontal Forces among Supports in Curved and Skewed Concrete Bridges", *The Bridge and Structural Engineer*, The Indian National Group of the I.A.B.S.E. (Zurich), 1972.

CHAPTER 10

Estimation of 'Design Values' of Axial Load and Bending Moment in a Tall Slender Bridge Support—Guarding against Buckling Effect

When a gap has to be bridged at a high elevation with bridge supports rising vertically from the ground below, be it a crossing over a huge body of water with navigational requirements, an elevated highway over a valley or an elevated flyover over a part of a township—aesthetics, and perhaps economy (depending on the method of construction), may demand slenderlooking tall supports.

In the case of 'short' columns, where the ratio of effective length to least gyrational radius is under about 50, the sections may be designed for values of M (bending moment) and P (vertical load) as worked out from normal simple theory of the first order, i.e., by ignoring the effect of buckling deflections. However, in the case of so-called 'long' columns, where the above ratio is exceeded, the effect of buckling deflections may be significant. In such cases, the sections may be designed, using normal working stresses, either for increased M and P equal to those obtained as in above but divided by appropriate reduction factors described in the relevant design specifications, or for the actual value of P and an exact value of M obtained according to the theory of the second order.[1]

Since the application of the theory of the second order is relatively time consuming, for relatively lower ratios of effective length to least gyrational radius, resort may conveniently be taken to the practice of increasing the simple M and P values by dividing them by the appropriate reduction factors. For relatively higher ratios, however, it may be well worth to go into the intricacies of buckling analysis, because, beyond a certain range, the said factors may yield unrealistically high values of the increased M and P. In fact, in the case of the slender pier-columns of the Tasman Bridge at Hobart,[2] the ratio of effective length to least gyrational radius is so high that according to the appropriate reduction factors it would have been almost impossible to get away with such slender supports.

The Approach

The suggested approach for establishing the design values of bending moment M and axial load P in concrete columns subjected to bending and axial thrust is as follows:

Step 1 *Determine l_e/r Value (ratio of 'effective length' to least radius of gyration)*

- Establish effective length l_e (see following table)

Type of Column	l_e
Properly restrained at both ends in position and direction	0.75 L
Properly restrained at both ends in position and imperfectly restrained in direction at one or both ends	a value between 0.75 L and 1.00 L, depending on the efficiency of the directional restraint.
Properly restrained at one end in position and direction and imperfectly restrained in both position and direction at the other end	a value between 1.00 L and 2.00 L depending upon the efficiency of the imperfect restraint (l_e = 2.00 L for free cantilever)

NOTE L = actual length of column

- Compute least radius of gyration r of the section, $= \sqrt{\frac{I_{min}}{A}}$, where I_{min} is the minimum second moment of area of full section area A
- Compute l_e/r value

Step 2 *Procedure to Establish the Design Values of M and P*

(i) If $l_e/r \leq 50$, then buckling effect is insignificant. Therefore find M and P from simple 1st order theory of bending.

(ii) If $l_e/r > 50$ but ≤ 100, then buckling effect can be significant. Therefore establish design values of M and P by any of the following two approaches (second one is more accurate and more realistic)

(a) *Method 1*: Divide the 1st order theory values of M and P by reduction factor ϕ, where

$$\phi = (1.5 - l_e/100\ r)$$

(b) *Method 2*: For the given values of P, the lateral load, the movement at column top, the amount of out of plumb construction, and the amount of built-in sinusoidal curvature, compute M from the 2nd order theory taking account of buckling (the increase of deflection owing to increasing eccentricity between load line and section cg line).

(iii) If $l_e/r > 100$, then either revise the section to control l_e/r to less than 100 and then proceed as in method (1) or otherwise proceed directly as per the 2nd order theory referred to in method (2).

10.1 2ND ORDER THEORY

This section analyses a tall bridge support according to the theory of second order, taking into account:

(i) Vertical load
(ii) Uniformly distributed lateral wind.
(iii) Horizontal movement at the top of the support owing to deck movement (assuming a rocker connection between the deck and the support)
(iv) Defective construction of the support resulting in its being out of plumb along a curve.
(v) Defective construction of the support resulting in a built-in initial curvature.

Notation

EI = modulus of rigidity
w = uniformly distributed wind load
x = distance measured along length direction, top downwards.
l = length of bridge support,
M = bending moment
P = vertical load
V = support reaction
A, B = constants of integration
CF = complimentary function in the solution of a differential equation
PI = particular integral in the solution of a differential equation
CS = complete solution of a differential equation
y = deflection
$\dfrac{dy}{dx}$ = slope
D^2y or $\dfrac{d^2y}{dx^2}$ = curvature,
$\alpha^2 = \dfrac{P}{EI}$
a = deck movement at top of support
b = out of plumb offset at top of support.
c = maximum offset in the built-in sinusoidal curvature

Analysis

In an initially curvatured axially compressed column-member, which in addition is subjected to a certain lateral loading, the total deflection may be obtained by superimposing on the deflections due to initial curvature and compression the deflection due to lateral loading calculated as for an initially straight compressed column.[3] Such a superimposition is valid because the effect of an initial curvature can be replaced by the effect of an equivalent lateral load. However, it is essential in calculating the deflections produced by each kind of lateral loading, to assume the presence of the compressive force P with each.

On the basis of this principle of superimposition, the combined effect of the actions (i) to (v), listed earlier, is therefore, treated as a superimposition of the following four cases, taken one at a time:

Case (A) Axial compression and lateral wind
Case (B) Axial compression and initial curvature due to deck movement
Case (C) Axial compression and initial curvature due to out of plumb defective construction
Case (D) Axial compression and initial curvature due to built-in curvature due to defective construction.

NOTE For each of these cases (A to D) we will assume that the bridge support is fixed at base and pin* supporting the deck above.

Case (A)

See Fig. 10.1. According to the differential equation of flexure,

$$EI\frac{d^2y}{dx^2} = -Py - Vx + \frac{wx^2}{2} \qquad (10.1)$$

or
$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{Vx}{EI} + \frac{wx^2}{2EI}$$

or
$$(D^2 + \alpha^2)y = -\frac{Vx}{EI} + \frac{wx^2}{2EI}$$

$$CF \text{ is } y = A\sin\alpha x + B\cos\alpha x$$

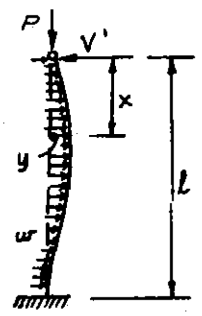


Fig. 10.1

* If the pier is monolithic with deck (i.e., fixed at top also), then an additional unknown moment M is introduced into the analysis. 2nd order theory analysis of such a column (subjected to axial load, horizontal force at top, a given deck movement at top, out of plumb construction, built-in curvature and transverse udl) can be carried out similarly as has been exemplified in the case of tall (slender) exposed piles in Chapter 11 in this book.

$$PI \text{ is } y = \frac{1}{(D^2+\alpha^2)}\left\{-\frac{Vx}{EI}+\frac{wx^2}{2EI}\right\}$$

$$= -\frac{V}{EI}\left\{\frac{1}{\alpha^2\left(1+\frac{D^2}{\alpha^2}\right)}x\right\} + \frac{w}{2EI}\left\{\frac{1}{\alpha^2\left(1+\frac{D^2}{\alpha^2}\right)}x^2\right\}$$

$$= -\frac{V}{\alpha^2 EI}\left\{\left(1+\frac{D^2}{\alpha^2}\right)^{-1}x\right\}$$

$$+\frac{w}{2\alpha^2 EI}\left\{\left(1+\frac{D^2}{\alpha^2}\right)^{-1}x^2\right\}$$

$$= -\frac{V}{P}\left\{\left(1-\frac{D^2}{\alpha^2}+\cdots\right)x\right\}$$

$$+\frac{w}{2P}\left\{\left(1-\frac{D^2}{\alpha^2}+\cdots\right)x^2\right\}$$

$$= -\frac{V}{P}\{x-0\}+\frac{w}{2P}\left\{x^2-\frac{2}{\alpha^2}\right\}$$

$$= -\frac{Vx}{P}+\frac{wx^2}{2P}-\frac{w}{\alpha^2 P}$$

$$CS \text{ is } y = A\sin\alpha x + B\cos\alpha x + \frac{wx^2}{2P} - \frac{Vx}{P} - \frac{w}{\alpha^2 P} \tag{10.2}$$

at $x = 0$, $y = 0$,

$$\therefore \text{ from Eq. (10.2) } B = \frac{w}{\alpha^2 P} \tag{10.3}$$

at $x = l$, $y = 0$, $\therefore$ from Eq. (10.2)

$$0 = A\sin\alpha l + B\cos\alpha l + \frac{w}{2P}l^2 - \frac{V}{P}l - \frac{w}{\alpha^2 P} \tag{10.4}$$

Also, from Eq. (10.2)

$$\frac{dy}{dx} = \alpha A\cos\alpha x - \alpha B\sin\alpha x + \frac{wx}{P} - \frac{V}{P}$$

at $x = l$, $\frac{dy}{dx} = 0$ so that,

$$0 = \alpha A\cos\alpha l - \alpha B\sin\alpha l + \frac{wl}{P} - \frac{V}{P} \tag{10.5}$$

Solving Eqs. (10.5) and (10.4) simultaneously, we obtain:

$$A = \frac{w}{\alpha^2 P}\left\{\frac{l^2\alpha^2+2(1-\cos\alpha l-\alpha l\sin\alpha l)}{2(\sin\alpha l-\alpha l\cos\alpha l)}\right\} \tag{10.6}$$

and $$V = \frac{w}{\alpha}\left\{\frac{l^2\alpha^2+2(1-\cos\alpha l-\alpha l\sin\alpha l)}{2(\tan\alpha l-\alpha l)}\right\}$$

$$-\frac{w}{\alpha}\sin\alpha l + wl \tag{10.7}$$

Thus knowing A, B and V, deflection can be evaluated from Eq. (10.2) and then bending moment from Eq. (10.1), at any section.

Greatest M occurs at base ($x = l$ and $y = 0$) and, after a lengthy simplification, it comes to

$$M_{\text{base}} = \frac{wl}{\alpha}\left\{\frac{\left(\frac{\alpha l}{2}-\tan\frac{\alpha l}{2}\right)\tan\alpha l}{(\tan\alpha l-\alpha l)}\right\} \tag{10.8}$$

Case (B)

See Fig. 10.2. After the deck has moved by an amount a the top of the support will be prevented by it from further movement due to P. So an unknown reaction V will be called into play.

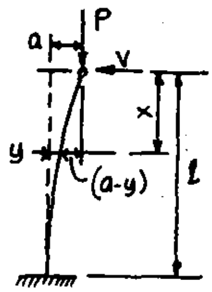


Fig. 10.2

According to the differential equation of flexure,

$$EI\frac{d^2y}{dx^2} = P(a-y) - Vx \tag{10.9}$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{P}{EI}a - \frac{V}{EI}x$$

or $$(D^2+\alpha^2)y = -\frac{V}{EI}x + \alpha^2 a$$

$$CF \text{ is } y = A\sin\alpha x + B\cos\alpha x$$

$$PI \text{ is } y = \frac{1}{(D^2+\alpha^2)}\left\{-\frac{V}{EI}x+\alpha^2 a\right\}$$

Proceeding as in Case (A), this gives PI as $y = a - \frac{Vx}{P}$

$$\therefore CS \text{ is } y = A\sin\alpha x + B\cos\alpha x - \frac{Vx}{P} + a \tag{10.10}$$

at $x = 0$, $y = a$, $\therefore$ from Eq. (10.10)

$$B = 0 \tag{10.11}$$

at $x = l$, $y = 0$, $\therefore$ from Eq. (10.10)

$$0 = A\sin\alpha l - \frac{Vl}{P} + a \qquad (10.12)$$

Also, from Eq. (10.10)

$$\frac{dy}{dx} = \alpha A\cos\alpha x - \frac{V}{P} \qquad (10.13)$$

at $x = l$, $\dfrac{dy}{dx} = 0$, $\therefore$ from Eq. (10.13)

$$0 = \alpha A\cos\alpha l - \frac{V}{P} \qquad (10.14)$$

Solving Eqs. (10.14) and (10.12) simultaneously, we obtain,

$$A = \frac{a\sec\alpha l}{\alpha l - \tan\alpha l} \qquad (10.15)$$

and

$$V = \frac{a\alpha P}{\alpha l - \tan\alpha l} \qquad (10.16)$$

Thus knowing A, B and V, deflection can be evaluated from Eq. (10.10) and then bending moment from Eq. (10.9), at any section.

Greatest M occurs at base ($x = l$, $y = 0$) and comes to

$$M_{\text{base}} = \frac{P_a}{(1 - \alpha l\cot\alpha l)} \qquad (10.17)$$

Case (C)

This case may be analysed on the same lines as case (B) above, only replacing 'a' by 'b', the out of plumb offset at top of support.

Case (D)

See Fig. 10.3. Assume a built-in curvature of sinusoidal form $c\sin\dfrac{\pi x}{l}$, having a maximum offset of c, and a built-in slope of $-\dfrac{c\pi}{l}$

From the differential equation of flexure

$$EI\frac{d^2}{dx^2}\left(y - c\sin\frac{\pi x}{l}\right) = -Py - Vx \qquad (10.18)$$

or $$\frac{d^2y}{dx^2} + c\left(\frac{\pi}{l}\right)^2\sin\frac{\pi x}{l} + \frac{P}{EI}y = -\frac{Vx}{EI}$$

or $$(D^2 + \alpha^2)y = -\frac{Vx}{EI} - c\left(\frac{\pi}{l}\right)^2\sin\frac{\pi x}{l}$$

CF is $y = A\sin\alpha x + B\cos\alpha x$

$$PI \text{ is } y = \frac{1}{(D^2+\alpha^2)}\left\{-\frac{Vx}{EI} - c\left(\frac{\pi}{l}\right)^2\sin\frac{\pi x}{l}\right\}$$

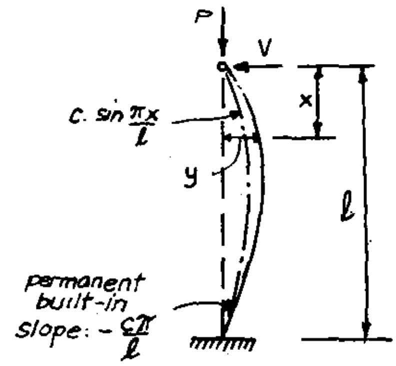


Fig. 10.3

$$= -\frac{V}{EI}\left\{\frac{1}{\alpha^2\left(1 + \dfrac{D^2}{\alpha^2}\right)}\cdot x\right\}$$

$$- c\left(\frac{\pi}{l}\right)\left\{\frac{1}{(D^2+\alpha^2)}\cdot\sin\frac{\pi x}{l}\right\}$$

$$= -\frac{V}{P}\left\{\left(1 + \frac{D^2}{\alpha^2}\right)^{-1}\cdot x\right\}$$

$$- c\left(\frac{\pi}{l}\right)^2\cdot\left\{\frac{\sin\dfrac{\pi x}{l}}{-\left(\dfrac{\pi}{l}\right)^2 + \alpha^2}\right\}.$$

$$= -\frac{V}{P}\left\{\left(1 - \frac{D^2}{\alpha^2} + \ldots\right)x\right\} + \frac{c\sin\dfrac{\pi x}{l}}{1 - \left(\dfrac{\alpha l}{\pi}\right)^2}$$

$$= -\frac{Vx}{P} + k\sin\frac{\pi x}{l}$$

where $$k = \frac{c}{1 - \left(\dfrac{\alpha l}{\pi}\right)^2} \qquad (10.19)$$

$$\therefore CS \text{ is } y = A\sin\alpha x + B\cos\alpha x - \frac{Vx}{P} + k\sin\frac{\pi x}{l} \qquad (10.20)$$

at $x = 0$, $y = 0$, $\therefore$ from Eq. (10.20),

$$B = 0 \qquad (10.21)$$

at $x = l$, $y = 0$, $\therefore$ from Eq. (10.20),

$$0 = A\sin\alpha l - \frac{Vl}{P} \qquad (10.22)$$

Also, from Eq. (10.20)

$$\frac{dy}{dx} = \alpha A \cos \alpha x - \frac{V}{P} + k\left(\frac{\pi}{l}\right)\cos\frac{\pi}{l}x \qquad (10.23)$$

at $x = l$, $\frac{dy}{dx} = -\frac{c\pi}{l}$, $\therefore$ from Eq. (10.23)

$$-\frac{c\pi}{l} = \alpha A \cos \alpha l - \frac{V}{P} - k\left(\frac{\pi}{l}\right) \qquad (10.24)$$

Solving Eqs. (10.24) and (10.22) simultaneously, we obtain,

$$A = \frac{\pi(c-k)}{(\sin \alpha l - \alpha l \cos \alpha l)} \qquad (10.25)$$

and

$$V = \frac{\pi(c-k)P}{l(1 - \alpha l \cot \alpha l)} \qquad (10.26)$$

Thus knowing A, B and V, deflection can be evaluated from Eq. (10.20), and then bending moment from Eq. (10.18), at any section.

Bending moment at base ($x = l$, $y = 0$) comes to

$$M_{\text{base}} = \frac{\pi(c-k)P}{(1 - \alpha l \cot \alpha l)} \qquad (10.27)$$

where k is as defined in Eq. (10.19).

Numerical Example

Consider a 110 ft tall concrete bridge support of constant rectangular section, 7 ft wide × 5 ft deep, subjected to a total vertical load of 1610 kips, a lateral wind of constant intensity 30 lb/s ft, a deck movement of 5 in., an out of plumb offset of 1 in. at top, and a sinusoidal curvature having a maximum offset of 4 in.

$P = 1.61 \times 10^6$ lb., $w = 30 \times 7 \times \frac{1}{12} = 17.5$ lb/in. run, $l = 110 \times 12 = 1320$ in.

Concrete is such that $EI = 6.20 \times 10^{12}$ lb in.^{2}

$$\alpha = \sqrt{\frac{P}{EI}} = 0.509 \times 10^{-3} \text{ in.}^{-1}, \alpha l = 0.6719 \text{ radian},$$

$$\left(\frac{\alpha l}{\pi}\right)^2 = .0456,$$

$$a = 5'', b = 1'', c = 4'', k = \frac{c}{1 - \left(\frac{\alpha l}{\pi}\right)^2} = 4.2 \text{ in. units.}$$

Case (A)

Evaluating A, B and V from Eqs. (10.6), (10.3) and (10.7), we get,

$$A = 10.80, \quad B = 41.9541 \text{ and } V = 8618.97$$

Then deflection and bending moment are evaluated from Eqs. (10.2) and (10.1) for various values of x. These values are recorded in Table 10.1.

Case (B)

Evaluating A and V from Eqs. (10.15) and (10.16) and noting that $B = 0$ in this case, we get:

$$A = -51.889, \quad V = -33285.6$$

Then deflection and bending moment are evaluated from Eqs. (10.10) and (10.9) for various values of x. These values are recorded in Table 10.1.

Case (C)

As $b/a = 1/5$, the deflections and bending moments in this case are simply 1/5th of those in Case (B) above.

Case (D)

Evaluating A and V from Eqs. (10.25) and (10.26) and

Table 10.1

	x (ft)	Deflection (in)					Bending Moment ($\times 10^6$ lb. in)				
		Case A	Case B	Case C	Case D	Total ...	Case A	Case B	Case C	Case D	Total ...
Top	0	0	5.00	1.00	0.00	6.000	0	0	0	0	0
↓	10	.015	4.32	.864	1.15	6.349	−.93	5.10	1.02	−1.26	3.93
	20	.030	3.64	.728	2.21	6.608	−1.62	10.18	2.03	−2.37	8.22
	30	.044	2.99	.598	3.09	6.722	−2.04	15.22	3.04	−3.20	13.02
	40	.049	2.37	.474	3.72	6.613	−2.20	20.21	4.04	−3.62	18.42
	50	.051	1.80	.360	4.03	6.241	−2.10	25.12	5.02	−3.52	24.52
	60	.048	1.34	.268	4.04	5.696	−1.75	29.86	5.97	−2.95	31.13
	70	.041	.85	.170	3.72	4.781	−1.13	34.65	6.93	−1.84	38.61
	80	.028	.48	.096	3.07	3.674	−.26	39.23	7.85	−.20	46.62
	90	.016	.22	.044	2.20	2.480	.87	43.65	8.73	1.80	55.05
	100	.012	.05	.010	1.15	1.222	2.24	47.92	9.58	4.08	63.82
Base	110	0	0	0	0	0	3.87	52.00	10.40	6.52	72.79

noting that $B = 0$ in this case, we get:

$$A = -6.51, \quad V = -4940$$

Then deflection and bending moment are evaluated from Eqs. (10.20) and (10.18) for various values of x. These values are recorded in Table 10.1.

10.2 CONCLUSION

The usefulness of this analysis can immediately be concluded by comparing the M and P values, say at the base section as obtained in the above examples, with their design values as obtained by the first order theory modified by the appropriate reduction factors.

Treating the pier as a propped cantilever subjected to a lateral wind of (30 × 7 =) 210 lb/ft run, a prop-sinking of (5 + 1 =)6 in. due to deck movement and out of plumb construction, and a longitudinal (vertical) load of 1610 kips causing bending moment at the base owing to prop sinking, then,

$$M_{\text{base}} = \left(\frac{3EIa}{l^2}\right) + P\cdot a + \left(\frac{wl^2}{8}\right)$$

$$= \left(\frac{3 \times 6.20 \times 10^{12} \times 6}{1320 \times 1320}\right) + (1610 \times 10^3 \times 6) + \left(\frac{210 \times 110 \times 1320}{8}\right) = 77.58 \times 10.6 \text{ lb in.}$$

For 7 ft × 5 ft section:

$$\frac{I_{\text{min}}}{A} = \frac{1}{12} \times 7 \times 5^3 \times \frac{1}{7 \times 5} = 2.08 \text{ ft}^2\text{, so that}$$

$$r = \sqrt{2.08} = 1.44 \text{ ft}$$

$$\therefore \frac{l_{\text{eff}}}{r} = \frac{0.9 \times 110}{1.44} = 68.75$$

which is > 50, hence reduction factor to be applied. Reduction Factor

$$\phi = 1.5 - l_{\text{eff}}/100\,r = 1.5 - \frac{0.9 \times 110}{100 \times 1.44}$$
$$= 0.8125$$

∴ design values of M and P, at base-section are,

$$M = \left(\frac{77.58 \times 10^6}{0.8125}\right) = 95.38 \times 10^6 \text{ lb in.}$$

and $$P = \left[\frac{1.610 \times 10^6}{0.8125}\right] = 1.98 \times 10^6 \text{ lb}$$

as compared to $(72.79 \times 10^6 - 6.52 \times 10^6 =)\ 66.27 \times 10^6$ lb in. and 1.61×10^6 lb respectively from the second order theory analysis.

Therefore, it can be seen that even in the present case of not too slender a pier ($\frac{l_{\text{eff}}}{r}$ only 68.75) the second order theory analysis can save about 44% on M and 23% on P, which is some saving—thanks to the 2nd order theory.

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CHAPTER 11

Analysis and Design of Slender Exposed Piles in a Group

Synopsis

Most of the methods available for analysing slender exposed (reinforced concrete) piles subjected to axial thrust, sway and lateral forces, are at best terribly theoretical and at worst inaccurate. This leaves a practising designer to the mercy of established and conservative methods whose scope is limited — leaving little room for bold engineering design. Since computerised calculations cannot be performed for every case by every designer, there is obviously need for a relatively easily applicable method, which simultaneously is also reasonably accurate. This chapter very simply derives the general expression for bending effects in long slender piles using second order theory for taking account of buckling deflections for various combinations of axial thrust and lateral forces. As well as this, it presents a general appreciation of various parameters enveloping the field of analysis and design of RC piles in scourable soil medium, a step by step application for analysis and design for RC piles and various recommendations. This is followed by a practical numerical example illustrating the entire procedure.

11.1 INTRODUCTION

Exposed piles are relatively long and slender structural members used for transmitting loads and lateral forces to deeper and more dependable subsoil strata. RC piles can either be precast and then driven into the ground, or driven and cast *in situ* or bored and cast *in situ*.

Piles in a group may either be all vertical or some vertical and some raked or all raked — either in opposite directions or in the same direction, depending on the functional requirements.

Owing to the slender columnar behaviour of exposed piles (and therefore being prone to buckling), the established and conservative method is to affix an effective length L to the column on the basis of its assumed end conditions, calculate the reduction factor, and then design the section using normal working stresses for increased bending moment and direct load equal to those obtained from first order analysis divided by the reduction factor. However, where the pile is exposed for a considerable length, the above referred reduction factor may be so low that it may almost be ridiculous if not impossible to design the pile section by this approach. Moreover, at any rate, affixing the effective length value to the column may in itself be ambiguous and unrealistic. All this may cumulatively lead to gross overdesign.

So long as the longitudinal reinforcement of piles is adequately embedded into the pile cap, the pile may be assumed to be fixed at top. The lower fixity point of the pile may be assumed at a distance below ground (or scour level) equal to either 10% of its exposed length or half the depth of soft strata whichever is greater. However, even in the case of soft marine clay, the lower fixity point does not lie more than 3–4 m below the ground (or scour) level.

The practice of designing pile sections for axial load alone so long as the net resultant lateral force on the pile group (after taking into account horizontal resistance of raked piles if any) does not exceed 5% of the total vertical load on the pile group, appears to be a rather crude way of terminating the process of design. This is not to say that the buried portions of the piles will not mobilise a dependable passive resistance equal to at least 5% of the total vertical load (in fact in most cases it is even more), but it is just that moments are set in the piles in the process of transmitting the horizontal forces through their exposed body, the effect of which should not be ignored in the design of the pile section before being optimistic about the subsequent horizontal soil-resistance.

Structural Strength of Pile

In the analysis of exposed pile groups, often the apparent mathematical exactitude displayed through the so-called sophisticated methods of analysis is diffused through the indefiniteness of the very assumptions on which they are based. This is particularly so when dealing with the abstruse. Apportioning of vertical loads among piles (resulting from direct load and orthogonal bending moments on the pile group) by the simple rivet group analysis, in the limit, is no more inaccurate than most other methods. It is the study of the combined effect of the thus calculated axial load on a pile and the lateral forces on it (latter estimated on the basis of all piles sharing the total lateral force equally owing to the relatively extremely stiff pile cap transom), that merits an in-depth mathematical investigation. The pile section should be adequately designed for the bending moment and direct load resulting from such analysis based on the second order theory (which takes into account the effect of buckling deflections), using normal working stresses. This saves having to sermonise in the field of ambiguity so inherent in the process of trying to affix effective lengths and reduction factors for really slender exposed piles.

Soil Strength

Ultimate soil resistance around an individual pile comprises the end bearing capacity and the shaft resistance due to friction or cohesion depending on the type of soil. However, the negative skin friction due to fill above ground (or scour level) and the functional self weight of pile should be deducted from it.

Ultimate soil resistance around a pile group against block failure can be estimated as in the case of an individual pile as described in the previous section but considering the whole group as one big pile of perimeter encircling all the piles in the group. From this block failure consideration, the ultimate soil resistance around an individual pile may be taken as an nth of the above, n being the total number of piles in the group.

In cohesive soils, since the ultimate soil resistance of the pile group of n piles is always lesser than n times the ultimate soil resistance around an individual pile, it is necessary in such soils to also ensure that the total vertical load on the pile group is lesser than n times the individual soil resistance around a pile multiplied by an efficiency factor[1] whose value ranges from 0.7 to 0.9, depending on the spacing of piles within the group. In cohesionless soils, the efficiency factor is more than unity owing to higher compaction.

The dependable carrying capacity of a pile may be taken as the least of that given in the above two paras divided by a factor of safety of 2.5, and this, in turn, should not be less than the maximum axial load actually coming on the pile.

However, the dependable soil resistance around a pile should not be taken to be more than the minimum working load carrying capacity of the pile as ascertained from load tests.

Where piles are founded on rock, their load carrying capacity should generally be based only on end bearing resistance even if they are surrounded by nonscourable soil. In fact, the surrounding soil mass, owing to its continued consolidation activity, loads the pile (negative skin friction) instead of supporting it. This is because the pile itself cannot move downwards relative to soil once it is seated on rock. In the absence of nonscourable soil surrounding these piles, the lower fixity point may be assumed at the foot of the pile provided the pile base is suitably anchored to rock, but this needs a very careful study for fear of adopting unrealistic moments.

11.2 ANALYSIS FOR COLUMN ACTION BY SECOND ORDER THEORY

In an initially curvatured axially compressed column member, which, in addition is subjected to certain lateral loading, the total deflection may be obtained by superimposing on the deflections due to initial curvature and compression the deflection due to lateral loading calculated as for an initially straight compressed column. Such a superimposition is valid because the effect on initial curvature can be replaced by the effect of equivalent lateral load. However, it is essential in calculating the deflection produced by each system of lateral loading to assume the presence of the axial compressive force.[2] On the basis of the principle of this superimposition, the combined effect of axial compression, lateral load and initial curvature, can therefore be treated as the superimposition of the following cases:

Case A: Axial compression P and lateral force H at the end of the pile.

Case B: Axial compression P and initial curvature due to possible inaccuracies in construction.

Case C: Axial compression P and uniform transverse load w (if there is any).

Case A

See Fig. 11.1. According to the differential equation of flexure,

$$EI\frac{d^2y}{dx^2} = P(q-y) - Hx - M \qquad (11.1)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{P}{EI}(q-y) - \frac{H}{EI}x - \frac{M}{EI}$$

If $\frac{P}{EI} = \alpha^2$, then $\frac{d^2y}{dx^2} = \alpha^2(q-y) - \alpha^2\frac{H}{P}x - \alpha^2\frac{M}{P}$

or $\quad D^2y = \alpha^2(q-y) - \alpha^2\frac{H}{P}x - \alpha^2\frac{M}{P}$

or $\quad (D^2+\alpha^2)y = \alpha^2\left\{q - \frac{H}{P}x - \frac{M}{P}\right\}$

Solving this differential equation, complimentary function (CF) is $y = A\sin\alpha x + B\cos\alpha x$ and particular integral (PI) is

$$y = \frac{1}{(D^2+\alpha^2)}\alpha^2\left[q - \frac{H}{P}x - \frac{M}{P}\right]$$

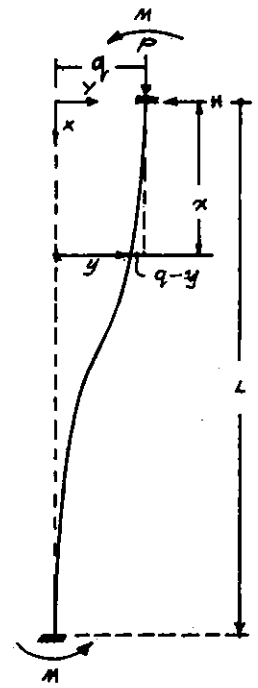


Fig. 11.1

$$= \frac{1}{\alpha^2\left(1+\frac{D^2}{\alpha^2}\right)}\alpha^2\left[q-\frac{H}{P}x-\frac{M}{P}\right]$$
$$= \left(1-\frac{D^2}{\alpha^2}+\ldots\right)\left[q-\frac{H}{P}x-\frac{M}{P}\right]$$
$$= \left[q-\frac{H}{P}x-\frac{M}{P}\right]$$

$\therefore$ Complete solution is $CF + PI$, i.e.,

$$y = A\sin\alpha x + B\cos\alpha x + q - \frac{H}{P}x - \frac{M}{P} \quad (11.2)$$

and
$$\frac{dy}{dx} = \alpha A\cos\alpha x - \alpha B\sin\alpha x - \frac{H}{P} \quad (11.3)$$

The four unknowns A, B, q and M are worked out from the above two equations for the following end conditions:

$$\text{at } x=0,\ \ y=q \text{ and } \frac{dy}{dx}=0$$
$$\text{at } x=L,\ \ y=0 \text{ and } \frac{dy}{dx}=0$$

giving $A = \dfrac{H}{\alpha P}$, $B = -\dfrac{H}{\alpha P}\tan\alpha L/2$,

$$q = -\frac{H}{\alpha P}(\alpha L - 2\tan\alpha L/2) \text{ and } M = -\frac{H}{\alpha}\tan\alpha L/2$$

Hence deflection y and BM: $EI\dfrac{d^2y}{dx^2}$ at any section can be evaluated from Eqs. (11.2) and (11.1).

Maximum BM occurs at ends and is given by;

$$\left[EI\frac{d^2y}{dx^2}\right]_{\substack{x=L\\y=0}} = Pq - HL - M$$
$$= -\frac{H}{\alpha}\tan\alpha L/2 \quad (11.4)$$

Case B

See Fig. 11.2. Assume a built-in curvature (due to possible inaccuracies in construction) of sinusoidal form $c\sin\dfrac{\pi x}{L}$ having a maximum offset c and a built-in slope $-\dfrac{c\pi}{L}$ at the lower end (at $x = L$) and $\dfrac{c\pi}{L}$ at the upper end (at $x = 0$).

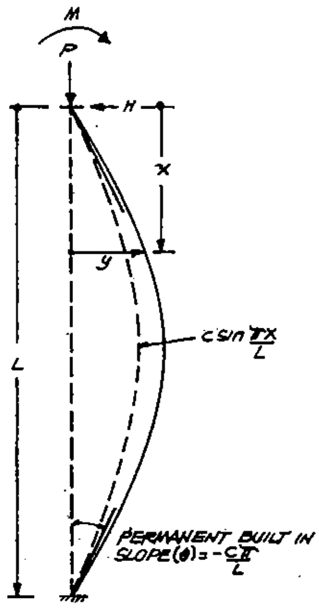


Fig. 11.2

From the differential equation of flexure:

$$EI\frac{d^2}{dx^2}\left(y - c\sin\frac{\pi x}{L}\right) = -Py - Hx + M \quad (11.5)$$

or
$$\frac{d^2}{dx^2}\left(y - c\sin\frac{\pi x}{L}\right) = -\frac{P}{EI}y - \frac{H}{EI}x + \frac{M}{EI}$$

If $\dfrac{P}{EI} = \alpha^2$, then $D^2\left(y - c\sin\dfrac{\pi x}{L}\right) = -\alpha^2 y - \alpha^2\dfrac{H}{P}x + \alpha^2\dfrac{M}{P}$

or
$$D^2y + c\left(\frac{\pi}{L}\right)^2\sin\frac{\pi x}{L} = -\alpha^2 y + \alpha^2\left[\frac{M}{P} - \frac{H}{P}x\right]$$

i.e., $(D^2+\alpha^2)y = \alpha^2\left[\frac{M}{P}-\frac{H}{P}x\right] - c\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L}$

Solving this differential equation,
CF is $y = A\sin\alpha x + B\cos\alpha x$,

$$\text{and } PI \text{ is } y = \frac{1}{(D^2+\alpha^2)}\left[\alpha^2\left(\frac{M}{P}-\frac{H}{P}x\right) - c\left(\frac{\pi}{L}\right)^2\sin\frac{\pi x}{L}\right]$$

$$= \frac{1}{\alpha^2\left(1+\frac{D^2}{\alpha^2}\right)}\alpha^2\left(\frac{M}{P}-\frac{H}{P}x\right) - c\left(\frac{\pi}{L}\right)^2\frac{1}{(D^2+\alpha^2)}\sin\frac{\pi x}{L}$$

$$= \left(1-\frac{D^2}{\alpha^2}+\ldots\right)\left(\frac{M}{P}-\frac{H}{P}x\right) + c\left(\frac{\pi}{L}\right)^2\frac{\sin\frac{\pi x}{L}}{(\pi/L)^2-\alpha^2}$$

$$= \frac{M}{P}-\frac{H}{P}x+\frac{c\sin\pi x/L}{1-(\alpha L/\pi)^2}$$

$$= \frac{M}{P}-\frac{H}{P}x+k\sin\frac{\pi x}{L}$$

$$\text{where } k = \frac{c}{1-(\alpha L/\pi)^2}$$

Complete solution is $CF + PI$, i.e.,

$$y = A\sin\alpha x + B\cos\alpha x + \frac{M}{P} - \frac{H}{P}x + k\sin\frac{\pi x}{L} \quad (11.6)$$

$$\frac{dy}{dx} = \alpha A\cos\alpha x - \alpha B\sin\alpha x - \frac{H}{P} + \frac{k\pi}{L}\cos\frac{\pi x}{L} \quad (11.7)$$

The four unknowns A, B, H and M are worked out from the above two equations for the following four end conditions:

$$\text{at } x=0,\quad y=0 \text{ and } \frac{dy}{dx}=\frac{c\pi}{L}$$

$$\text{at } x=L,\quad y=0 \text{ and } \frac{dy}{dx}=-\frac{c\pi}{L}$$

$$\text{giving } A=\frac{\pi(c-k)}{\alpha L},\quad B=-\frac{\pi(k-c)}{\alpha L}\cot\frac{\alpha L}{2}$$

$$H = P\left[\alpha A + \frac{\pi}{L}(k-c)\right], = 0$$

$$\text{and}\quad M = \frac{P\pi(k-c)}{\alpha L}\cot\alpha L/2$$

Hence deflection y and BM: $EI\frac{d^2}{dx^2}\left(y - c\sin\frac{\pi x}{L}\right)$ at any section can be evaluated from Eqs. (11.6) and (11.5).

BM at each end,

$$M_{\substack{x=0\\x=L}} = \frac{P\pi(k-c)}{\alpha L}\cot\alpha L/2 \quad (11.8)$$

Case C

See Fig. 11.3. Proceeding as in previous cases, maximum BM (which occurs at ends) is given by,

$$M = \frac{w}{\alpha^2}\left[1-\frac{\alpha L/2}{\tan\alpha L/2}\right] \quad (11.9)$$

w being the uniformly distributed lateral loading and $\alpha^2 = \frac{P}{EI}$ as before.

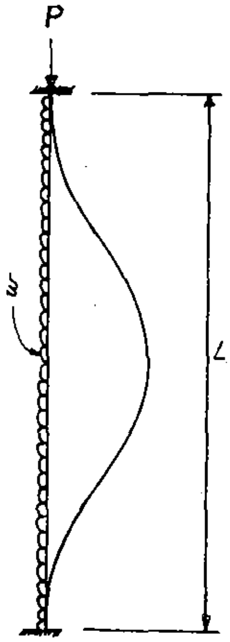


Fig. 11.3

11.3 APPLICATION

In order to analyse the pile group and design the pile, proceed step by step as follows,

Step 1 Estimate at the soffit of pile cap the total load W, total moments ΣM_T and ΣM_L about transverse and longitudinal directions of the structure and horizontal forces ΣH_L and ΣH_T along these directions.

Step 2 (a) Calculate the vertical load (and hence the axial load) in the extreme piles as per rivet group analysis. Axial load in a vertical pile is,

$$P = \frac{W}{n} \pm \frac{\Sigma M_T}{Z_T} \pm \frac{\Sigma M_L}{Z_L}$$

where Z_T and Z_L are section moduli $(\Sigma x^2/x_i)$ to the pile under consideration about T and L directions. (b) Calculate H_T $(= \Sigma H_T/n)$ and H_L $(= \Sigma H_L/n)$ per pile.

Step 3 Establish L i.e., the length of pile between its fixity points, assuming upper fixity point at the soffit of pile cap and lower fixity point as explained earlier.

Step 4 Calculate the maximum bending moments M_T and M_L in the pile section from Eqs. (11.4), (11.8) and (11.9).

Step 5 Design the reinforced concrete pile section for direct compression P and bending moments M_T and M_L calculated above.

Step 6 Ensure that the dependable load carrying capacity of the pile from the considerations of soil resistance around individual pile and block failure is not less than the maximum axial load actually coming on the pile (see under section 'Soil Strength' earlier).

Step 7 Ensure that the total vertical load coming on the pile group is not more than n times the individual soil resistance around a pile multiplied by the relevant efficiency factor as described under 'Soil Strength' earlier.

Step 8 Check for the long term settlement of the pile group by any of the standard methods.

11.4 RECOMMENDATIONS

Types of RC Piles

RC piles may usually be of the following types:

(a) Precast in full and then driven
(b) Precast in reasonable lengths and jointed[3] while driving
(c) Partly precast and partly cast *in situ* during driving
(d) Casing driven, pile cast *in situ* and casing withdrawn
(e) Bored and cast *in situ*.

Types (a), (b) and (c) are recommended in loose soil and in standing water. Type (d) is recommended for structures on land in loose soil. Type (e) piles are recommended in stiff clay or on rock and are convenient when piling has to be done in the proximity of existing and heavily loaded structures. Piling plant is less cumbersome and more easily manoeuvrable in the case of bored piles.

For piles cast *in situ*, in standing water or slushy conditions, a 3 mm thick mild steel liner should be used for encasing the concrete for protection.

In precast piles, the precast length is usually governed by the structural strength of pile section to resist handling and hoisting stresses. The precast pile unit can be handled by supporting it from single point or two point suspension, but while hoisting, it is usually suspended by a single point system only. Since a precast piece may require handling quite a few times (depending on the distance between precasting yard and location of pile foundation) it is recommended to allow only normal stresses in concrete and steel during handling. However, since hoisting is expected to be done only once and this operation lasts for a relatively much shorter duration, the stresses in concrete and steel may be allowed to be 25% higher than the normal permissible stresses during hoisting.

Also refer to Annexure (i) at the end of this chapter.

Detailing

Concrete for cast *in situ* piles (bored or casing driven type) should be cast by Tremie, and can be of 1 : 2 : 4 grade but the mix should be richened to preferably 1 : $1\frac{1}{2}$: 3 grade when such piling is done in water-logged and slushy conditions. Concrete in precast piles should preferably be not leaner than M250 grade. Concrete must be very dense (use of microsilica or pozolona or Portland Blast Furnace Slag Cement deserves a serious consideration). Minimum cement content should be 350 to 450 kg/cubic meter of concrete, depending on ground conditions and agressiveness of the surrounding environment.

The longitudinal reinforcement in a cast *in situ* pile should preferably be not less than 0.8% of the gross sectional area of concrete and this need not extend very much beyond the lower fixity point. This longitudinal steel should preferably be detailed as a six or eight bar arrangement. The transverse reinforcement (links or hoops) in a cast *in situ* pile should be provided on the same basis as for a regular RC column.

The longitudinal reinforcement in a precast pile should preferably be not less than $1\frac{1}{2}$% of the gross sectional area of concrete. Additional reinforcement may be required in critical zones from the considerations of handling and hoisting. This longitudinal steel in square section piles should preferably be detailed as a twelve bar arrangement with sturdy bars at the four corners. In order to utilise these bars to the full during handling and hoisting, the precast unit should be so handled as to cause bending about a face than about a diagonal. The volume of transverse reinforcement in the body of the precast pile should not be less than 0.2% of the gross volume of the pile. This should be increased to at least 0.6% at its ends for a length of about three times the lateral dimension of pile to account for driving stresses. The longitudinal corner bars should be held apart by diagonal fork shaped bars at about 2 m centres. (For more details see Ch. 7 earlier.)

All driven piles should be provided with a steel shoe at the bottom end to protect the pile-toe as well as to ease the driving.

While the longitudinal bars coming out from the pile head should be adequately anchored into the pile cap, the embedment of pile head into the pile cap should be between 50–75 mm. Deeper embedment can be dangerous as the same can function as a crack former.

Driving and Boring

A pile driving plant essentially involves a pile frame (raised from ground or mounted on a pontoon), winches and a driving hammer (monkey). The pile frame is of steel construction. The hammer can be activated by simple gravity, steam, compressed air or diesel. For a more efficient transfer of the driving shock wave, it is preferable to use a heavier hammer and a shorter drop and not the converse. The usual weight of hammer is between 3.5 to 5.0 tonnes and the usual drop is around 1.2 m. However, the weight of the hammer should preferably be not less than half the weight of the pile or casing being driven. To protect the pile head against the hammer hit, a cast steel helmet along with dolly and timber packing should be used during driving. Concrete in the top 60–90 cm, portion of the pile being driven, should be chipped off after driving it as this portion generally shows distress due to spalling effects.

For bored piles the hole in the ground is formed either by screwing an auger or by chiselling. In either case, generally, the sides of the hole are prevented from caving in either by a retractable mild steel casing or by the use of activated bentonite slurry. Bentonite slurry is basically a high density thixotropic fluidy mud. After the reinforcement cage is lowered into the hole, concrete should be placed by tremie beneath the bentonite ensuring that the delivery end of the tremie pipe always remains slightly beneath the top of freshly poured column of concrete.

Investigation for Soil Characteristics

The Dutch Cone Penetrometer test is highly recommended for estimating the soil strength around piles. The apparatus for this has been developed by the Government Soil Mechanics Laboratory, Delft. It essentially consists of a standard steel cone attached to rods which are protected by a sleeve. As the cone is driven into the soil under hydraulic pressure, the thrust on the rods and on the sleeve is measured separately. Readings are taken at regular intervals and this way a continuous serrated graph is drawn showing the end bearing and frictional resistance of the soil at various depths. The ultimate bearing capacity of soil at any depth may be taken equal to the cone resistance at that depth. However, Van der Veen recommends that the cone resistance at any depth should be taken as the average value over a depth equal to three times the lateral dimension of the pile above the pile founding level and one lateral dimension of the pile below the pile founding level.

The skin friction on the pile shaft in cohesive soil is not equal to the cohesion of the soil since the driving or boring into the cohesive medium relieves its adhesion characteristics. The reduction in adhesion factor is, however, less in the case of driven piles than in the case of bored piles. The actual skin friction mobilised around the pile shaft is expressed as the product of cohesion of soil and the adhesion factor and the latter varies almost inversely with the former. For cohesion values of 2.5 to 10 t/m^2, the adhesion factor varies from 0.9 to 0.5 for driven piles.[1] In the case of bored and cast *in situ* piles in clay, the adhesion factor may be as low as 0.45.

In piles through compressible fill, the consolidation of fill causes additional load on piles as the latter remain stationary relative to the consolidating fill. This additional load transmitted to the piles by reverse friction, termed negative skin friction, is a fraction of the cohesion, because of the sensitivity factor of the consolidating soil. This sensitivity factor generally varies between 2 and 4 for most clays.[4]

Since it is not practical to test load every working pile, it is a common practice to test load at least one to two per cent of the working piles (routine vertical test load). Such working piles to be tested may be loaded to 1.5 times their maximum working load they have to carry. However, the safe load carrying capacity of the pile should be taken as only two-thirds of the load at which the pile settles by 12 mm. It is also essential that a few piles are driven at representative locations but outside the working piles and are tested for ultimate load. Such ultimate load tests are essential in order to ensure the true safe load carrying capacity of the piles. That vertical load at which the pile settles by 10% of its lateral dimension may be taken as the ultimate load capacity of the pile. The safe working load capacity of the pile may be taken as 40% of the ultimate load capacity corresponding to a safety factor of 2.5.

In the case of driven piles (be they casing-driven and then cast *in situ* or precast and driven), in addition to the above-mentioned ultimate and routine load tests it is also possible to have an approximate idea about the load carrying capacity from the amount the empty casing or the precast pile (as the case may be) penetrates into ground under the last few hammer blows. This penetration or 'set' is governed by the dynamic impact of the elastic bodies. Based on equating the energy of the hammer blow to the work done in overcoming the ground resistance and penetration of the pile and allowing for losses of energy due to elastic contraction of the casing (or the precast pile as the case may be), its dolly arrangement and the subsoil, and taking due account of the inertia of the pile mass, a general formula of the following type can be derived:

$$U = \frac{k \times W \times h}{p + c}$$

where U = ultimate resistance of pile

c and k = empirical coefficients depending on the system of piling, pile driving plant, total number of blows imparted to the pile, inertia of pile mass, efficiency of blow, units of various parameters in the formula, etc.

W = weight of hammer

h = height of drop

p = average penetration per blow during the last few blows.

NOTE Also see Annexures (ii) and (iii) at the end of this chapter.

Duplex Driving for Consolidation Piling

Where the soil is loosely compacted, the shaft resistance as well as the end bearing resistance of casing driven piles can be improved by the *Duplex redriving* process. After driving the casing together with its shoe as deep as possible, the hole is filled by lean concrete while the casing is continuously withdrawn in jerks leaving the shoe behind. Another shoe is then placed on top of this freshly poured column of lean concrete and the casing is redriven together with the new shoe through this column of concrete, thereby forcing this concrete sideways and down until an acceptable set is obtained. Thereafter, the operations of placing the reinforcement, concreting and pulling out the casing afresh are executed as in a usual driven and cast *in situ* pile.

Settlement

The estimation of the total settlement of a pile group, comprising of immediate and long term settlements, is very complex as it depends on a multitude of parameters whose accurate prediction is almost impossible. Some of these parameters are pile shortening, relative movement between piles and surrounding soil, yielding of soil below the pile base, depth of fill undergoing settlement, void ratio and compression index of soil, true shape of load spread from piles into soil, unequal loading on piles, overlapping of pile pressure bulbs, etc. Therefore, it can be readily inferred that even if some values are assigned to these parameters, the magnitude of the long term settlement of a pile group so estimated should not be viewed with any great concern. Teng[5] recommends the method given by Peck wherein the load is assumed to spread out at one horizontal to two vertical from a depth of two-thirds the total depth of grip. The long-term settlement may then be estimated as per Terzaghi's theory of consolidation as further modified by Skempton and Bjerrum.

11.5 NUMERICAL EXAMPLE

Step 1 Consider the pile foundation as shown in Fig. 11.4 under a bridge pier in scourable cohesive soil. The layout of piles in plan and the values of direct load, moments and lateral forces at the soffit of pile cap are as follows:

Total vertical load	=	592.9 t
Total Moment about LL axis	=	109.2 mt
Total Moment about TT axis	=	53.0 mt
Total Horizontal force along LL axis	=	8.3 t
Total Horizontal force along TT axis	=	17.2 t

Step 2 max./min. vertical load on a pile

$$= \frac{592.9}{20} \pm \frac{53.0 \times 2.9}{94.0} \pm \frac{109.2 \times 3.8}{144.4}$$
$$\doteq 29.65 \pm 1.64 \pm 2.88$$
$$= 34.17 \text{ t}/25.13 \text{ t}$$

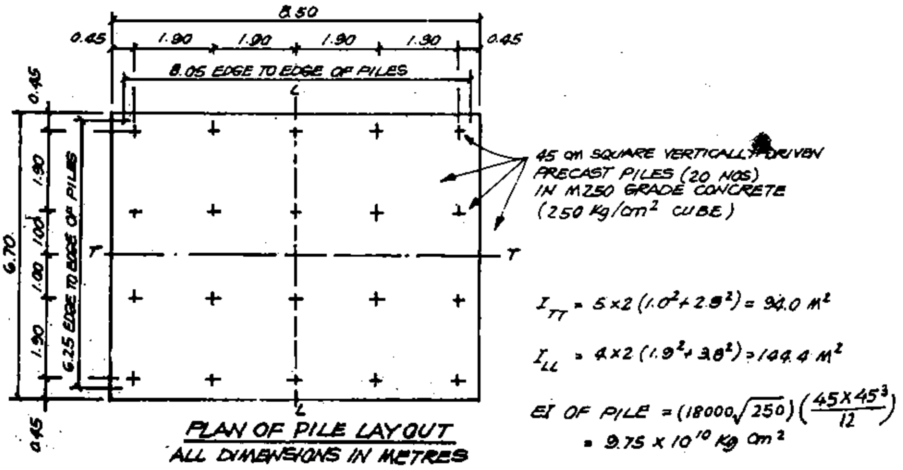


Fig. 11.4

H_T per pile $= \dfrac{17.2}{20} = 0.86$ t

H_L per pile $= \dfrac{8.3}{20} = 0.42$ t

Step 3 RL of soffit of pile cap = + 1.75 m.
Maximum scour level = − 10.80 m.
∴ Exposed length of pile = 12.55 m.

10% of exposed pile length = 1.255 m; half the depth of soft strata below scour level in the present case = 6 m. Hence assume lower fixity point to be at 4 m below scour level.

∴ RL of lower fixity point $= -10.80 - 4.00$
$= -14.80$ m

Length of pile between fixity points = L
$= 14.80 + 1.75 = 16.55$ m

Step 4 Considering the minimum loaded pile:

(i) $\alpha = \sqrt{\dfrac{P}{EI}} = \sqrt{\dfrac{25130}{9.75 \times 10^{10}}}$
$= 50.7 \times 10^{-5}\ \text{cm}^{-1}$, i.e., $50.7 \times 10^{-3}\ \text{m}^{-1}$

(ii) M_T per pile due to $H_L = (H_L/\alpha)\tan \alpha L/2$

$$= 0.42 \times \frac{\tan(50.7 \times 10^{-3} \times 16.55/2)}{50.7 \times 10^{-3}}$$
$= 3.66$ mt

(iii) M_L per pile due to $H_T = (H_T/\alpha)\tan \alpha L/2$

$$= 0.86 \times \frac{\tan(50.7 \times 10^{-3} \times 16.55/2)}{50.7 \times 10^{-3}}$$
$= 7.58$ mt

(iv) Effect of flood water force

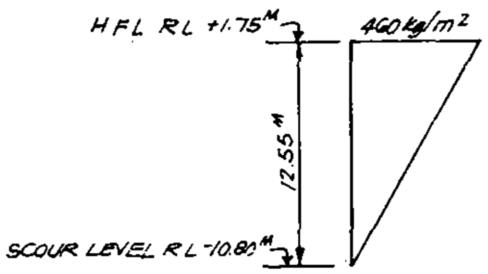


Fig. 11.5

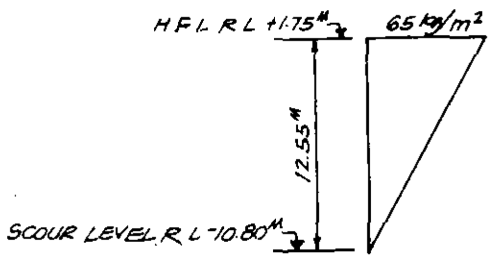


Fig. 11.6

In the present case, the max. mean velocity of flow at HFL (RL + 1.75) = 2.0 m/s
Component of max. velocity along TT direction
$= \sqrt{2} \times 2.0 \times \cos 20°$
$= 2.65$ m/s
Pressure at HFL $= 52 \times 1.25 \times 2.65^2$
$= 460\ \text{kg/m}^2$
(refer to clauses 213.2 and 213.7 of IRC Specifications—Section II)
∴ Total force on the pile group along TT axis
$= 1/2 \times 460 \times 12.55 \times 6.25 \times 10^{-3}$
$= 18.0$ t

∴ per pile $= \dfrac{18.0}{20} = 0.9$ t

Component of max. velocity along LL direction.
$= \sqrt{2} \times 2.0 \times \sin 20°$
$= 1.00$ m/s.
Pressure at HFL $= 52 \times 1.25 \times 1.0^2$
$= 65\ \text{kg/m}^2$
∴ Total force on the pile group along LL axis
$= 1/2 \times 65 \times 12.55 \times 8.05 \times 10^{-3}$
$= 3.20$ t

∴ per pile $= \dfrac{3.20}{20} = 0.16$ t

Assuming these forces as uniformly distributed along the height of pile (the effects as can be seen ahead are too small, hence this simplification),

$$M \text{ per pile} = (w/\alpha^2)\left[1 - \frac{\alpha L/2}{\tan \alpha L/2}\right]$$

$$\therefore M_L \text{ per pile} = \frac{0.90}{16.55 \times (50.7 \times 10^{-3})^2} \times \left[1 - \frac{50.7 \times 10^{-3} \times 16.55/2}{2\tan(50.7 \times 10^{-3} \times 16.55/2)}\right] = 1.36 \text{ mt}$$

and M_T per pile $= \dfrac{0.16}{0.90} \times 1.36 = 0.24$ mt (by proportion)

(v) Moment per pile due to built-in curvature: assuming $c = 5$ cm,

$$k = \frac{c}{1 - (\alpha L/\pi)^2}$$
$$= \frac{0.05}{1 - \left[\dfrac{50.7 \times 10^{-3} \times 16.55}{\pi}\right]^2}$$
$= 0.0539$ m

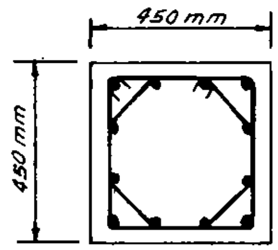
$$M = \frac{P\pi(k - c)}{\alpha L}\cot(\alpha L/2)$$
$$= \frac{25.13 \times \pi(0.0539 - 0.05)}{50.7 \times 10^{-3} \times 16.55}\cot(50.7 \times 10^{-3} \times 16.55/2)$$
$= 0.82$ mt

This moment may be assumed to act about LL axis for worst effects.

(vi) Total M_L per pile $= 7.58 + 1.36 + 0.82$
$= 9.76$ mt

Total M_T per pile $= 3.66 + 0.24$
$= 3.90$ mt

Step 5 Designing[6] the pile section for combined axial load of $P = 25.13$ t and biaxial bending of $M_T = 3.90$ mt and $M_L = 9.76$ mt, the stresses in reinforcement and concrete shown in Fig. 11.7 are found to be within permissible limits.



CONCRETE = M 250 GRADE
LONGITUDINAL REINFORCEMENT
OF 4 NOS. ϕ 40mm BARS AT CORNERS
+ 8 NOS. ϕ 25mm BARS AT FACES.

Fig. 11.7

Step 6 Approximate RL of bed $= -1.30$ m
Pile founding level $= -40.80$ m
Ultimate cohesion $= 2.1$ t/m^2 and adhesion factor $= 0.90$
Ultimate bearing capacity of soil at founding level $= 450$ t/m^2
Sensitivity factor for the cohesive soil $= 3$

Soil resistance around a single pile

Ultimate shaft resistance from maximum scour level to founding level

$= 2.1 \times 0.9 \times (4 \times 0.45) \times (40.8 - 10.8)$
$= 102.0$ t

Ultimate end bearing resistance $= 0.45 \times 0.45 \times 450$
$= 91.1$ t

Ultimate negative skin friction between bed level and scour level $= \dfrac{2.1}{3} \times 0.9 \times (4 \times 0.45) \times (10.8 - 1.3)$
$= 10.8$ t

Buoyant weight of the pile $= 0.45 \times 0.45 \times 42.55 \times (2.4 - 1.0)$
$= 12.1$ t

$\therefore$ Ultimate soil resistance around a pile $= 102.0 + 91.1 - 10.8 - 12.1$
$= 170.29$ t

$\therefore$ Net working load carrying capacity of the pile $= \dfrac{170.2}{2.5} = 68.1$ t,
> 34.17 t, ok

Check for block failure

Ultimate shaft resistance around pile group $= 2.1 \times 0.9 \times (8.05 + 6.25)2 \times (40.8 - 10.8)$
$= 1620$ t

Ultimate end bearing resistance $= 8.05 \times 6.25 \times 450$
$= 22650$ t

Buoyant weight of soil between founding level and bed level $= (8.05 \times 6.25 - 0.45 \times 0.45 \times 20) \times (40.8 - 1.3) \times (1.6 - 1.0)$
$= 1095.5$ t

Ultimate negative skin friction from bed level to scour level $= \dfrac{2.1}{3} \times 0.9 \times (8.05 + 6.25)2 \times (10.8 - 1.3)$
$= 171$ t

Buoyant weight of piles $= 12.1 \times 20 = 242$ t

$\therefore$ Ultimate soil strength around pile group $= 1620 + 22650 - 1095.5 - 171 - 242 = 22761.5$ t

$\therefore$ Working load carrying capacity of a pile in the group $= \dfrac{22761.5}{20 \times 2.5}$
$= 454.5$ t, > 34.17 t, ok

Step 7 Assuming an efficiency factor of 0.7

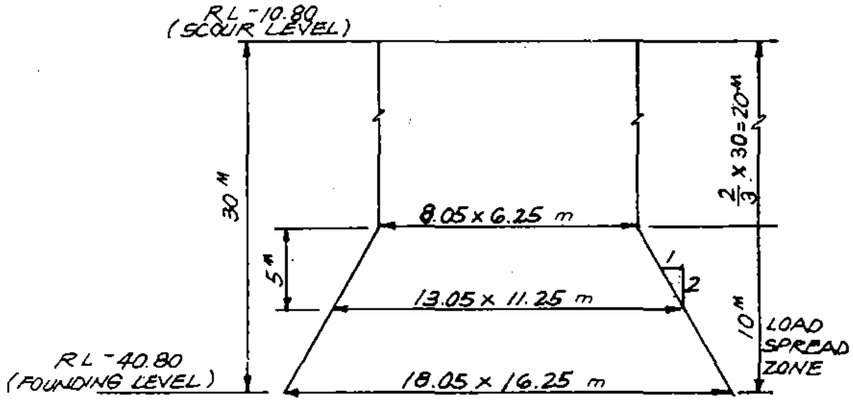


Fig. 11.8

Vertical load capacity of
pile group $= 68.1 \times 0.7 \times 20 = 957$ t
> 592.9 t, ok

Step 8 Assuming compression index of soil (C_s) = 1.12, void ratio (e_0) = 2.30, coeff. $k = 0.3$ (heavily over consolidated clay), and submerged soil density (γ_{sub}) = 0.6 t/m^3

Vertical pressure due to weight of soil at mid depth of load spread zone $p_0 = (30.0 - 10.0/2)0.6$
$= 15$ t/m^2

Overburden pressure due to permanent dead load of bridge $\Delta p = \dfrac{592.9}{13.05 \times 11.25}$
$= 4.03$ t/m^2

∴ Long term settlement

$$S = \frac{C_s H}{1 + e_0} \log_{10}\left(\frac{p_0 + \Delta p}{p_0}\right) k$$

$$= 1.12 \times \frac{10 \times 100}{1 + 2.3} \log_{10}\left(\frac{15 + 4.03}{15}\right) \times 0.3$$

$$= 10.6 \text{ cm}$$

NOTE Estimation of settlement in cohesionless soils may be made from the following formula:

$$S = \frac{H}{C} \log_e\left(\frac{p_0 + \Delta p}{p_0}\right)$$

where

S = final consolidation settlement of a layer of thickness H.

C = constant of compressibility

$= 1.5\dfrac{C_{kd}}{p_0}$ (where C_{kd} = Static Cone Resistance) other symbols as defined earlier.

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ANNEXURES

(i) Usual Pile-Types

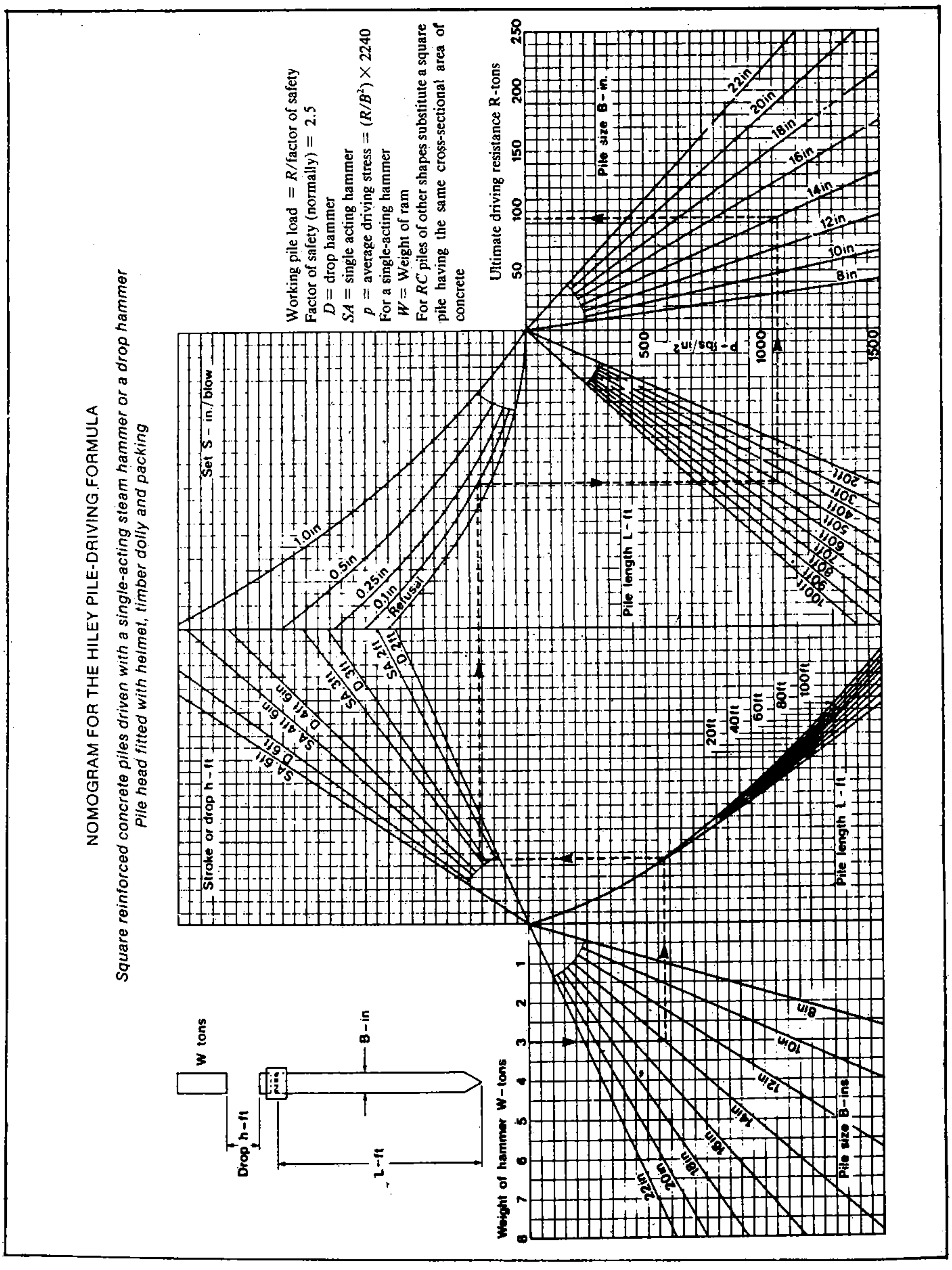
(a) Displacement piles (Driven piles)

Pile type			Normal range of sizes available: Cross-section	Normal range of sizes available: Length	Normal range of load
Preformed	Timber		Up to 400 mm× 400 mm	Up to 20 m	Up to 600 kN
	Concrete	Normal reinforced	Up to 450 × 400 mm	Up to 27 m	Up to 1000 kN
		Prestressed	Up to 400 mm square Up to 750 mm dia. hollow	Up to 27 m	Up to 1000 kN
	Steel	Box	Rendhex standard Frodingham octagonal Sheet pile fabrication	Up to 36 m	Up to 1500 kN
		Tubular	Heavy gauge up to 900 mm dia.	Up to 36 m	Up to 1500 kN
		H-section	200 mm × 200 mm to 300 mm × 300 mm	Up to 36 m	Up to 1700 kN
		Screw	600 to 2400 mm dia. helices	Up to 24 m	Up to 2500 kN
Partially Preformed	Precast and cast *in situ* concrete		450 to 600 mm dia.	Up to 50 m	Up to 2000 kN
	Steel and cast *in situ* concrete		250 to 500 mm dia.	Up to 18 m	Up to 800 kN
Driven *in situ*	Concrete		250 to 600 mm dia.	Up to 24 m	Up to 1500 kN

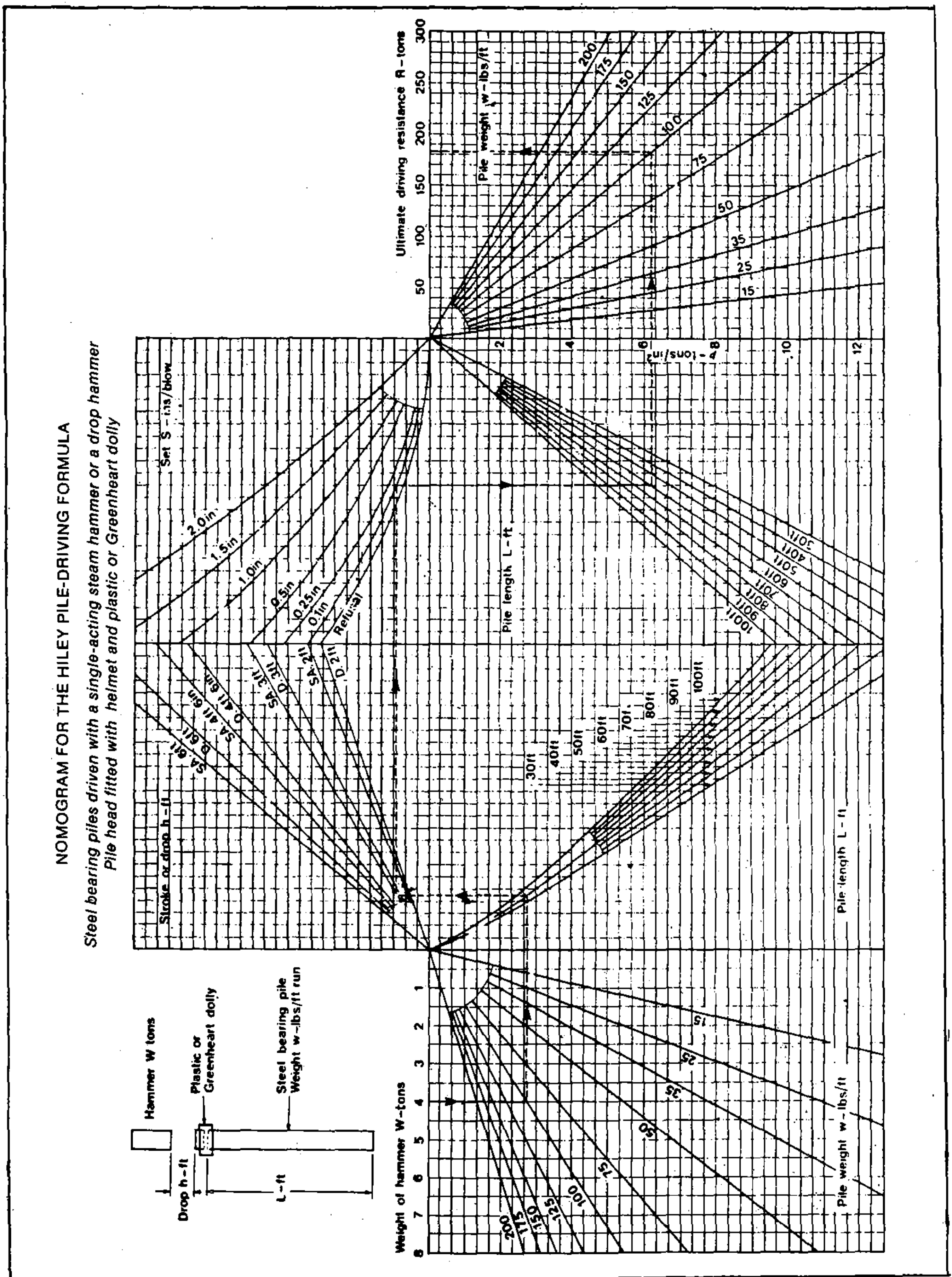
(b) Replacement piles (Bored piles)

Pile type			Normal range of sizes available: Cross-section	Normal range of sizes available: Length	Normal range of load
Percussion bored	Small diameter		450 to 600 mm dia.	Up to 24 m	Up to 1200 kN
Flush bored	Large diameter		600 mm dia. and over	Up to 45 m	Up to 10000 kN
Rotary bored	Large diameter	Straight shaft	600 to 1800 mm dia.	Up to 45 m	Up to 10000 kN
		Under-reamed base	As above with bell up to 3 times shaft diameter	Up to 45 m	Very high loads possible
	Small diameter		225 to 550 mm dia.	Up to 36 m	Up to 1000 kN

(ii) Precast Concrete Piles



(iii) Steel Bearing Piles



CHAPTER 12

A Practitioner's Guide to ~ Estimating Safe Bearing Capacity of Soils for Footings, Caissons and Piles

Synopsis

This chapter is deliberately presented in two parts. PART-I describes the actual step by step method for estimating the bearing capacity of soils (for footings, caissons, and piles) in a workman-like manner using a tool-kit-approach. ***(It is hoped that this clear-cut approach will help the engineer to 'stay on course for reaching the end of the problem' without feeling lost in the academic plethora that this subject is heaped in.)*** PART-II then separately tries to explain various subsidiary items (e.g. 'how to improve the bearing capacity of soils', 'various *in situ* penetration tests employed in the estimation of sub-strata bearing capacity', 'plate-load-bearing tests', 'bearing capacity of rocky SUBSTRATA', and a host of 'typical soil parameters') useful to a practising professional designer. ***The engineer thus has a clear choice of separating the presentation given in PART-I in order to identify the 'steps' involved in the chain of command for estimating the bearing capacity.***

12.1 INTRODUCTION

The bearing capacity of a soil depends upon the physical characteristics of the soil particles (i.e., size, shape, cohesive properties, frictional resistance and the power to retain moisture, etc.), moisture content and the changes brought in by the atmospheric influences such as heat, rain, etc. The finer the soil particles, the more variable are the cohesive and frictional properties of the soil under field conditions. In general, the heavier the unit weight of soil the greater the strength, and also the lesser the voids, the greater the strength.

With structures built on sands and gravels the settlement is likely to be practically completed at the end of construction, but when the site is underlain by clays or silts, settlement is likely to continue for a long time after construction and cracks may appear many years after completion.

All foundations settle under load and the general tendency is for some parts of a structure to settle more than others causing relative movement. The critical factor in the settlement of a structure is not the amount of settlement but the differential settlement between the different parts of a structure itself. Excessive pressure is a comparatively uncommon cause of settlement.

In clays, the ultimate bearing capacity under spread foundations is calculated using total stress parameters. This gives the end-of-construction case, which is the worst condition, and allows the design to be based on 'undrained' shear strength test, which are quick and inexpensive. For granular soils, where the dissipation of pore water pressures is rapid, 'effective' shear strength is used and, because of the difficulty in obtaining undisturbed samples, strength parameters are usually estimated from standard penetration test results.

PART I *(Workman-Like Approach)*

Contents ~

(A) Various PRELIMINARIES, and Obtaining 'Quickly' a Rough-estimate of the Safe Bearing Capacity of soil (S.B.C.) under a 'Footing' or a 'Caisson'

Step 1 Conduct Standard Penetration Tests (SPT) at short intervals (1 to 2 m) till about 10 m below a 'high-enough N-value zone (of say $N > 30$ to 50 if possible, for instance)' [where foundation is subject to scour, the founding level will have to be at least $x/3$ below the max. scour level (MSL) where x = depth from High Flood Level (HFL) to MSL; in such case conduct the SPT between MSL and a level about 10 to 15 m below min. founding-level].

Step 2 At the particular depth under consideration, note the SPT value and correct it for depth-effect as per Fig. 12.1 and further correct it in case of 'silts and fine-sands below water table' from the formula: Corrected $N = 15 + 1/2$ ("above corrected SPT value" -15) when the "above corrected SPT value" is greater than 15.

Step 3 Upon inspection of the soil samples retrieved by the Split-spoon-sampler from the successive SPTs decide whether soil is predominantly 'cohesive' type or 'non-cohesive' type.

Step 4 (a) If the soil is predominantly 'cohesive':
Find S.B.C. from Table 12.1 for the corrected N-value.

(b) If the soil is predominantly 'non-cohesive':
Find S.B.C. from Fig. 12.2 for the corrected N-value for an assumed approx. width of foundation.

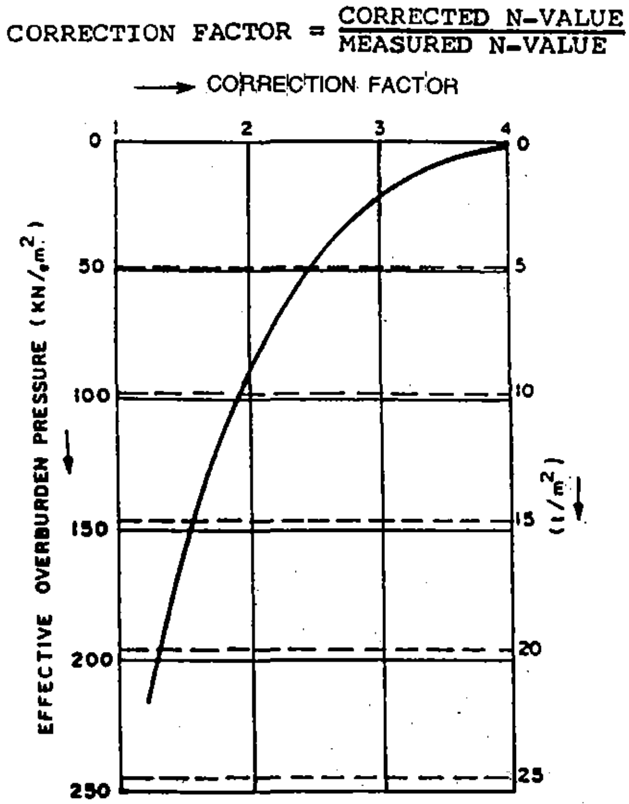


Fig. 12.1 *Depth correction factors for the measured SPT N-value (after Gibbs and Holtz)*

Figure 12.2 relates the settlement of foundations on sand to relative density, as determined by the standard penetration test. It shows the bearing pressure which will produce 25 mm of settlement for a given width of foundation. The relationship shown were established by Terzaghi and Peck from field observations and are intended only as a

Table 12.1 *Suggested allowable bearing values for clay*

N: Number of blows per 300 mm in standard penetration test.
c_u: Unconfined compressive strength.
q_d: Ultimate bearing capacity of continuous footing.
q_{ds}: Ultimate bearing capacity of square footing or circular footing.
q_a: Proposed 'allowable' bearing value (where $G_s = 3$).
G_s: Factor of safety with respect to base failure.

Description of clay	N (Corrected)	c_u	q_d	q_{ds}	q_a*,** Square $1.2\,c_u$ or circular	Continuous $0.9c_u$
Very soft	< 2	< 25	< 75	< 100	< 30	< 20
Soft	2 to 4	25 to 50	75 to 150	100 to 200	30 to 60	20 to 45
Medium	4 to 8	50 to 100	150 to 300	200 to 400	60 to 120	45 to 90
Stiff	8 to 15	100 to 200	300 to 600	400 to 800	120 to 240	90 to 180
Very stiff	15 to 30	200 to 400	600 to 1200	800 to 1600	240 to 480	180 to 360
Hard		> 400	> 1200	> 1600	> 480	> 360

All values in kN/m² (c_u, q_d, q_{ds}, q_a)

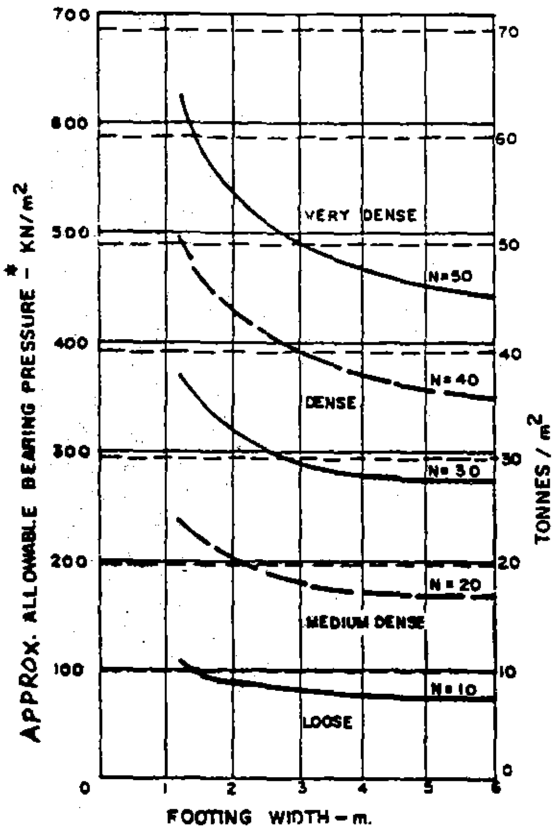
*q_a for square circular footings $= \dfrac{7.4c}{G_s} = \dfrac{3.7c_u}{G_s} = 1.2c_u \quad (G_s = 3)$

**q_a for continuous footings $= \dfrac{5.7c}{G_s} = \dfrac{2.85c_u}{G_s} = 0.9c_u \cdot (G_s = 3)$

Cohesion c being equal to $1/2\ c_u$.

Also, the 'ultimate' bearing capacity of a rectangular or an oblong footing of width B and length L, is approximately $= 2.85\ c_u \times (1 + 0.3\ B/L)$, giving an 'allowable' value of $0.9\ c_u\ (1 + 0.3\ B/L)$.

NOTE. Where the soil can get saturated (flooded), divide the above q_a values by 2 (i.e. a total safety factor of 6).



* Subject to corrections stated here.

Fig. 12.2 *Chart for estimating allowable bearing pressure based on standard penetration test results. Continuous lines are based on the original chart given by Terzaghi and Peck; broken lines are interpolations*

Ref.: Soil Mechanics in Engineering Practice by Terzaghi and Peck, and Foundation Design and Construction by M.J. Tomlinson rough guide. Some workers consider the bearing pressures obtained by using this chart to be too low, particularly for wider foundations.

• ***Corrections***

1. N-values: For foundations on clean, dry sands, the N-value obtained from the standard penetration test is used directly (see 2 below) to obtain the bearing pressure on a strip or pad foundation which will cause 25 mm settlement. The settlement for a different bearing pressure can be obtained on a prorata basis, provided the bearing pressure is well within the bearing capacity of the sand.

2. Corrections to N-values: Corrections may need to be carried out to the N-value to allow for 'depth' of test and the presence of silt or fine sand. These corrections are described in Step A-2. The corrected N-value is then used when reading the chart.

3. Corrections for water table: If the water table is at least one foundation width beneath the base of the foundation then no correction is required. However, if the water table is close to or at the foundation level, then, for a shallow foundation, the bearing pressure which will give 25 mm settlement is half the value read from the chart.

Alternatively, the bearing pressure read from the chart will produce 50 mm settlement.

4. Corrections for large rigid foundations: The rigidity of rigid raft or deep pier foundations results in smaller settlements and half the settlement estimated from the chart is to be expected. Thus, for 25 mm settlement, twice the bearing pressure read from the chart can be used where the water table is low and the actual values read off can be used where it is near or above the underside of the foundation.

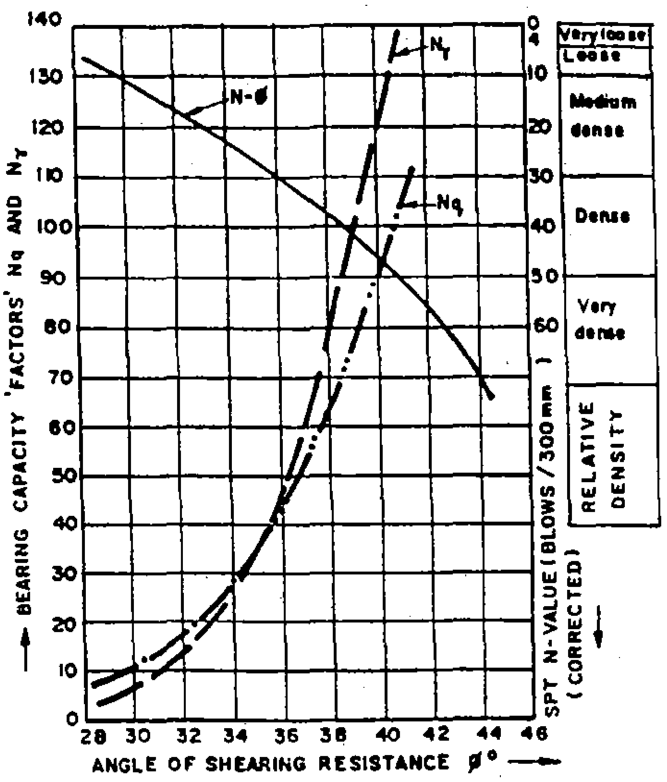


Fig. 12.3 *Correlation of values of ϕ, N_q and N_γ with SPT N-values (After Peck, Hanson and Thornburn)*

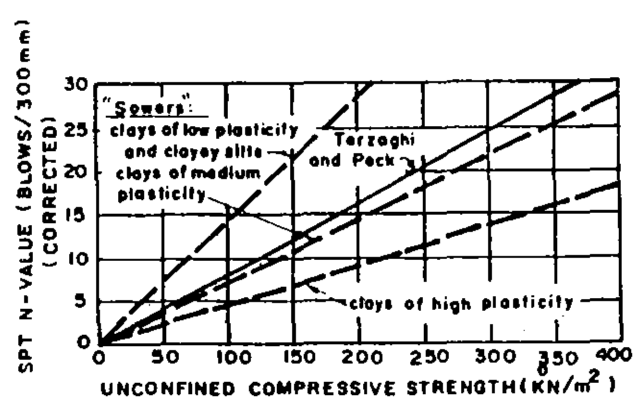


Fig. 12.4 *Approximate correlations between undrained shear strength c_u and SPT N-values, for clays. (Cohesion $c = \frac{1}{2}c_u$, c_u being the unconfined compressive strength of the soil)*

(B) More Accurate Estimation of S.B.C. of Soil under a 'Footing' or a 'Caisson'

Step 1. Same as Steps 1 and 3 of A above.

Step 2. *a) If the soil is found predominantly 'cohesive'*, then, using standard SHELBY's tubes, collect adequate number of 'undisturbed' soil-samples from the soil at each test-level, subject them to tri-axial compression test in the laboratory and thereby determine the values of cohesion c and angle of internal friction ϕ at the test-levels.

b) If the soil is found predominantly 'non-cohesive', then follow Step 2 of A above, and establish the corrected SPT N-value for each test-level. For this N-value (at any level) read off values of ϕ and c_u from Figs. 12.3 and 12.4 respectively. Hence ϕ and c are established (noting that cohesion $c = 1/2\ c_u$, where c_u is the unconfined compressive strength read off from Fig. 12.4).

Step 3. Estimate "ultimate" bearing capacity of soil at the concerned-level from *TERZAGHI'S* as well as *MEYERHOFF'S* approaches (explained ahead under I and II). Divide the smaller of these values by suitable safety-factor (generally 3 for 'dry' conditions and 6 if saturation possible) to obtain the S.B.C. i.e. the safe (permissible) bearing capacity value [but see '4' below].

Step 4. Also establish the S.B.C. from the "TOLERABLE SETTLEMENT approach" (explained ahead under III).

Step 5. Lesser of the two S.B.C. values, established in Steps '3' and '4' above, may then be taken as the acceptable S.B.C. value at the concerned test-level.

NOTE: In the case of a caisson it is usual to ignore the effects of skin-friction on its sides (because so much is unknown about the actual soil-characteristics that a conservative approach is preferred for *CAISSONS*).

... ***I.—Terzaghi's approach—***

The ultimate 'net' bearing capacity, p_{nu}, of a shallow foundation is given by:

(a) Strip Foundation:

$$p_{nu} = cN_c + p_o(N_q - 1) + \frac{1}{2}\Upsilon BN_\Upsilon$$

(b) Square or Circular Foundation:

$$p_{nu} = 1.2cN_c + p_o(N_q - 1) + 0.4\Upsilon BN_\Upsilon$$

where: Υ is the bulk density of the soil 'below' the foundation level*

c is the shear strength of the soil (cohesion).

p_o is the 'effective', overburden pressure at foundation level ($= \Upsilon' \cdot D$).

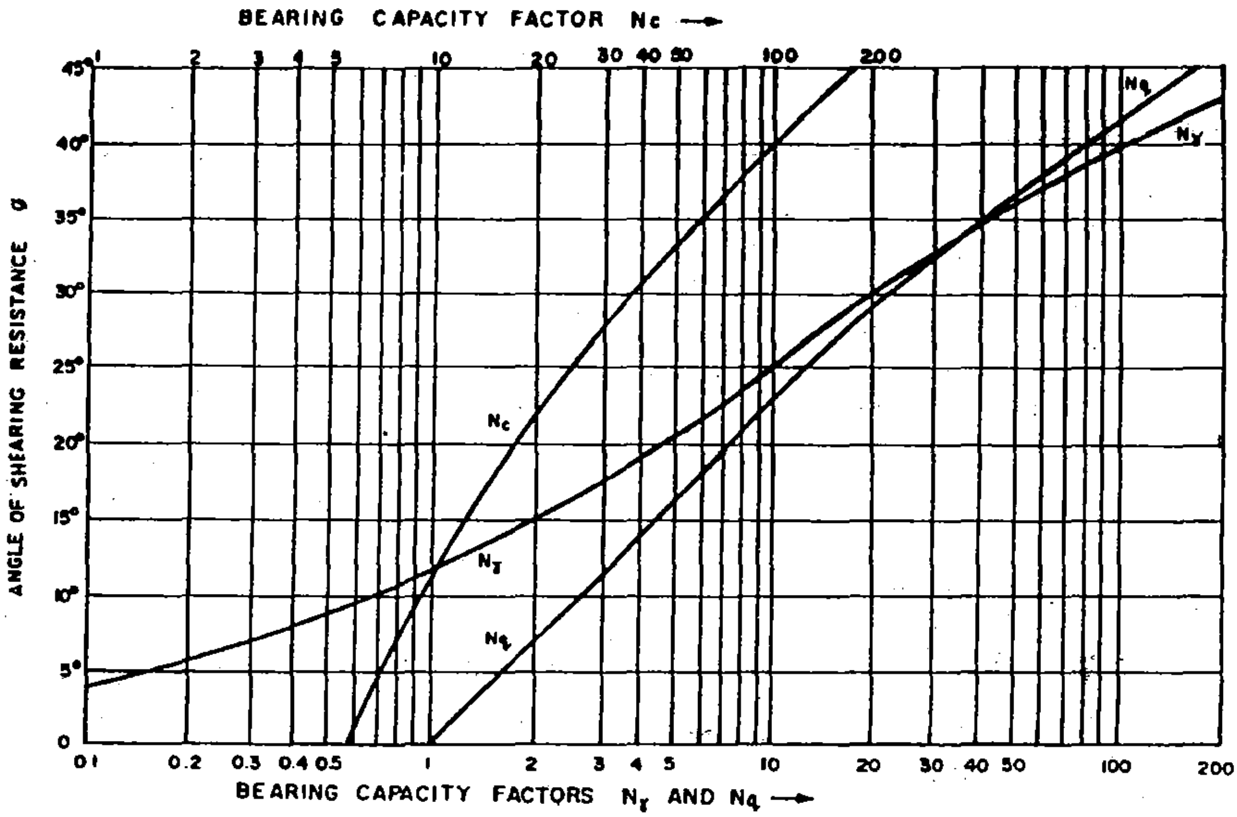


Fig. 12.5 *Terzaghi's bearing capacity factors*

Notes

(a) For clays, $\phi = 0$, giving $N_\Upsilon = 0$ $\quad N_q = 1$ $\quad N_c = 2 + \pi = 5.14$

(b) For sands, $c = 0$ so that the first term in the bearing capacity equations is zero and the value of N_c is not required. The value of ϕ may be obtained by direct testing but is usually estimated from SPT values.

* If the water table is at or above the founding level then the value of Υ used in the bearing capacity calculations is the submerged density. Also an allowance must be made for the height of the water table above the founding level when calculating p_0.

B is the foundation width (or diameter).
γ' = appropriate density of soil 'within' depth D
N_c, N_q and N_γ *are Terzaghi's bearing capacity factors, obtained from Fig. 12.5.*
D is depth of foundation.

The ultimate bearing capacity is given by

$$p_u = p_{nu} + p$$

where p is the 'total' overburden pressure at the founding level ($= \gamma' \cdot D$ approximately).

... ***II—Meyerhoff's approach—***

The ultimate 'net' bearing capacity under a *spread* foundation is given by:

$$p_{nu} = cN_c + p_o(N_q - 1) + \frac{1}{2}\gamma BN_\gamma$$

where N_c, N_q and N_γ are Meyerhoff's bearing capacity factors, obtained from Figs. 12.6, 12.7, 12.8 and 12.9.

It can be seen that this has the same form as Terzaghi's equation for a strip foundation. However, Meyerhoff's values of bearing capacity factors depend on the shape and depth of the foundation and on the roughness of the base.

• Meyerhoff's Bearing Capacity Factors—Figs. 12.6 to 12.9.

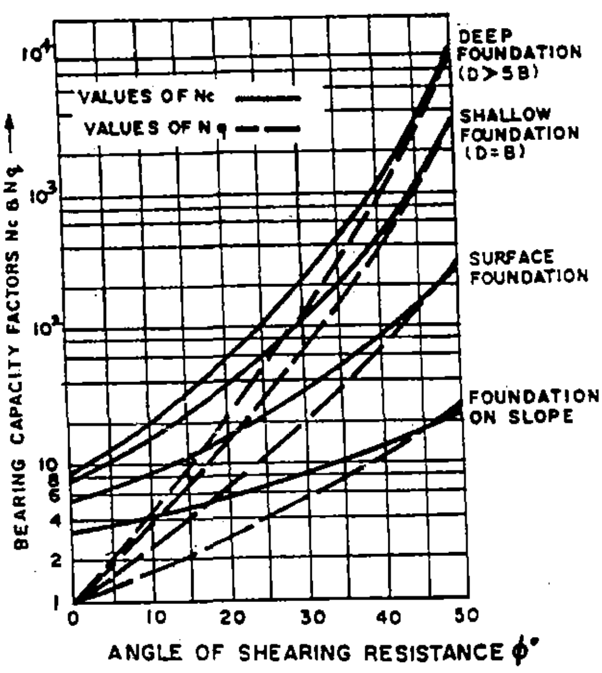


Fig. 12.6 *Values of N_c and N_q for a strip foundation —Meyerhoff's factors*

Values of N_c, N_q and N_γ for *strip* foundation are obtained using Figs. 12.6 and 12.7.

For *rectangular, square* and *circular* foundations, the values obtained from these graphs must be multiplied by a factor, λ, which depends on the shape and depth of the foundation, on the soil properties, and on the method of construction. Values of λ are obtained from Fig. 12.8.

If the *water table* is at or above the founding level, γ is replaced by γ_{sub} and the value of p_o is obtained as described for Terzaghi's equations.

Clay Soils (Meyerhoff) Using the usual procedure of total stress analysis, $\phi = 0$ so that $N_q = N_\gamma = 0$ and the bearing capacity equation reduces to:

$$p_{nu} = cN_c$$

For foundations on clay soils it is usual to use Meyerhoff's values for factor N_c and, for the special case of *purely cohesive* soils, values may be obtained from Fig. 12.9.

For *rectangular* foundations, values of N_c for a *strip* should usually be used.

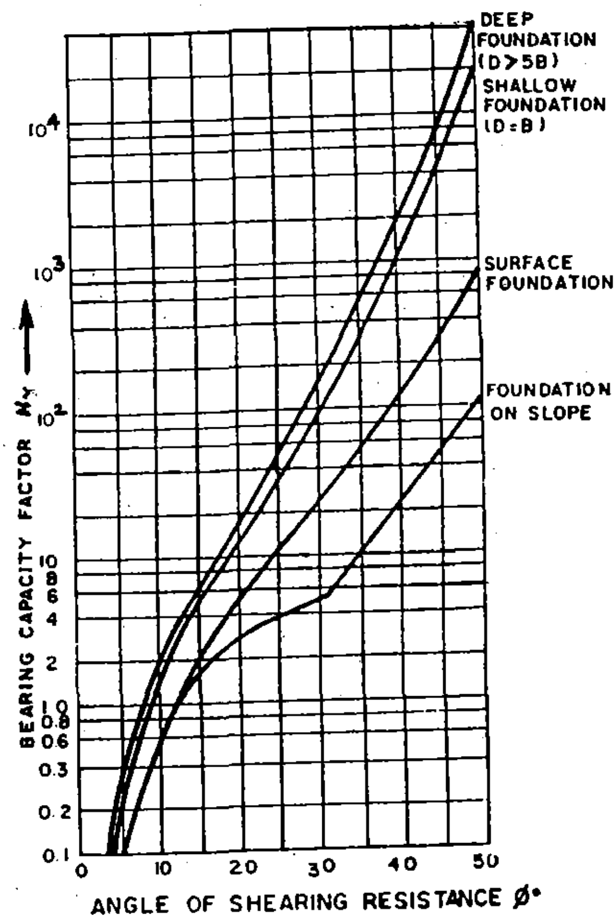


Fig. 12.7 *Values of N_γ for a strip foundation —Meyerhoff's factors*

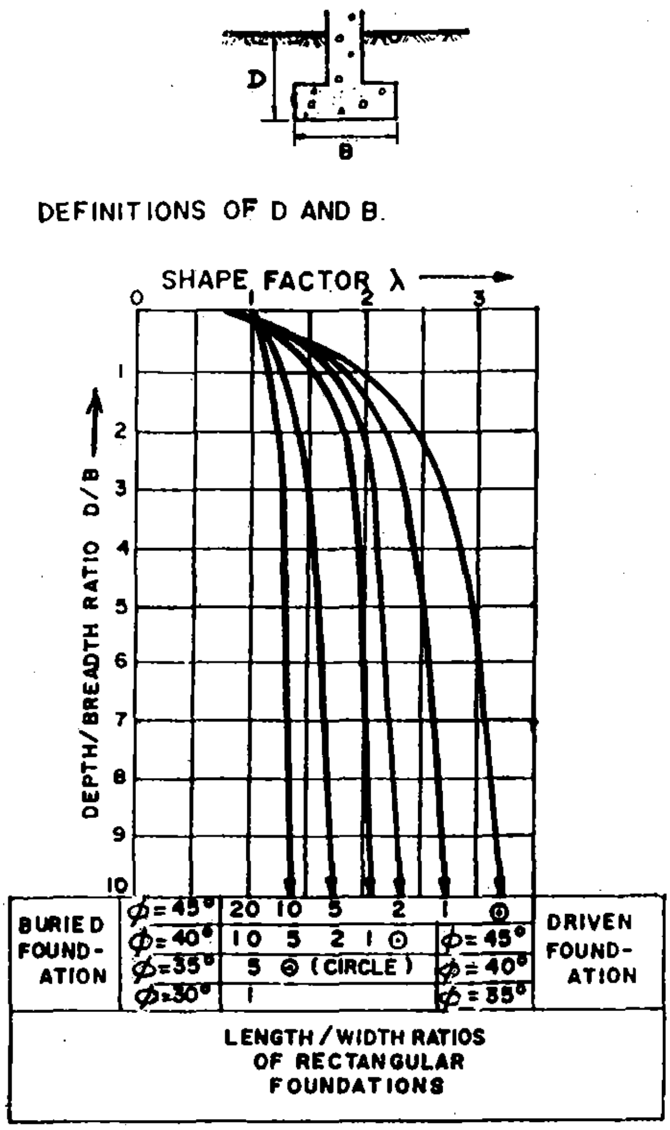


Fig. 12.8 *Values of shape factor λ for a rectangular foundations —Meyerhoff's factors*

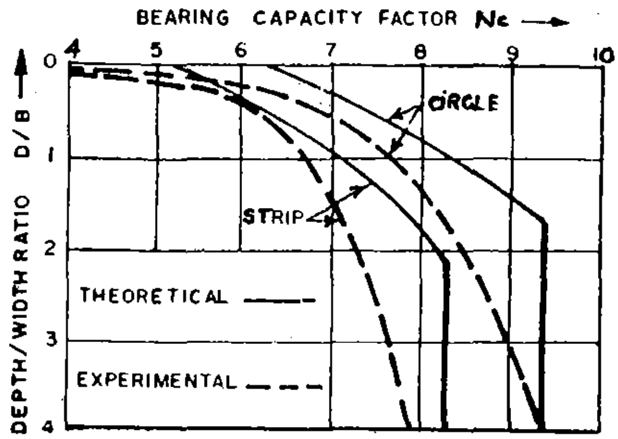


Fig. 12.9 *Values of N_c for foundations on a purely cohesive soil —Meyerhoff's factors*

. . . *III—"TOLERABLE SETTLEMENT" approach*—

This 'allowable' bearing pressure has been established empirically (Terzaghi and Peck, 1948), and may be expressed by the equation.

$$p_s = 3.5(N - 3)\left(\frac{B + 0.3}{2B}\right)\alpha \cdot \beta + W$$

where p_s = safe bearing capacity of soil in tonne/m^2 at the 'level at which the SPT is done', for a maximum tolerable settlement of 2.5 cm.

α = 0.5, if this level is submersible.

= 1.0, if water level is always below this level by at least a depth in magnitude equal to dimension B defined earlier (values may be interpolated in between these limits).

$\beta = \left(1 + \frac{D}{5B}\right)$, but $\not>$ 1.20

B and D. . . as defined earlier [D being the founding depth below the ground level (maximum scour level) at which SPT is done and the N-value established]

N = SPT value at the level considered (Standard Penetration Test).

W = Weight of the soil above the level at which N has been established in tonne/m^2. (For estimating this, use dry density for soil above water level and submerged density below water level, as in the case of p_o mentioned in I—earlier).

GENERAL CAUTION

The bearing capacity under a footing is largely affected by the characteristics of the volume of soil within a depth equal to about 1 to $1\frac{1}{2}$ times the width of the footing. Unless the soil possesses some cohesion, the upper layer of one to two metres can be easily disturbed and loosened by construction operation. *Therefore, it is not advisable to use large bearing capacity for small or narrow footings such as those supporting continuous walls, even if the natural soil is very compact.*

(C) Soil Resistance to a PILE

1) Ultimate Value

(i) In 'cohesive' soil: The ultimate bearing capacity Q_u is made up of *adhesion* Q_s and *end bearing* Q_b, less *negative skin* friction Q_n (Adhesion, often called skin friction, is usually much greater than end bearing in clays.) Thus:

$$Q_u = Q_s + Q_b - Q_n$$

- The *adhesion* on a pile is given by

$$Q_s = \alpha \cdot \bar{c} \cdot A_s$$

Where A_s is the embedded surface area of the pile
$\bar{c}$ is the 'average' undrained shear strength of the clay along the sides of the pile.
α is an adhesion factor.

Researchers have found that the value of α can vary widely so that it is difficult to allocate a value to it. For 'driven' piles, values obtained by Nordlund, given in Table 12.2, are usually used. Values for 'bored' piles, discussed by Tomlinson, are also given in Table 12.2.

- The *end bearing* is obtained from Meyerhoff's equation for the bearing capacity of cohesive soils:

$$Q_b = c \cdot N_c \cdot A_b$$

where c is the undisturbed shear strength at the base of the pile.
A_b is the pile base area (thus for a circular pile, radius R, $A_b = \pi R^2$).
N_c is Meyerhoff's bearing capacity factor, usually taken as 9.

- The *negative skin friction*, Q_n resulting from the tendency of the fill material (or compressible soil)

Table 12.2

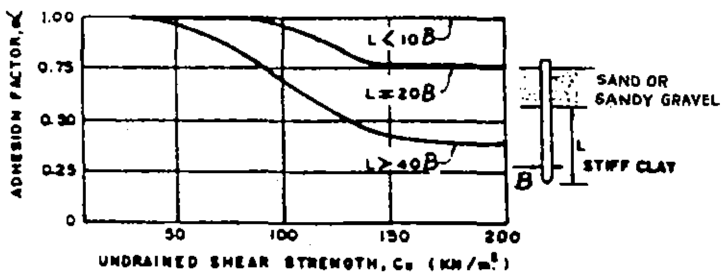
Adhesion factors for 'bored' piles (After Tomlinson)

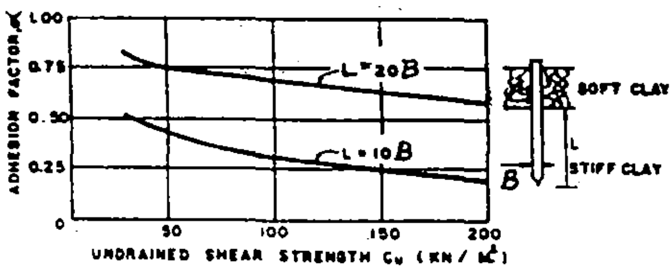
An adhesion factor α of 0.45 is used for bored piles in many clays, including London clay, although for short bored piles in London clay, where it may be heavily fissured, a value of 0.3 is more usual. Values of 0.49 to 0.52 have been reported for California clays but for hard lias clays α may be as low as 0.1.

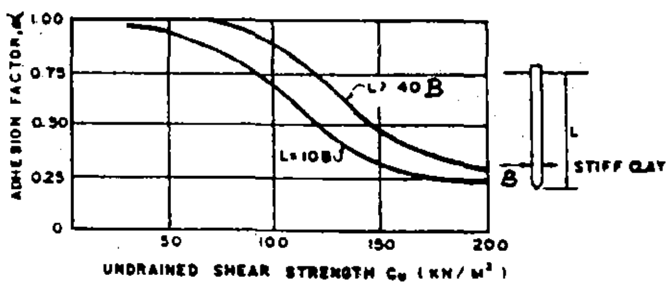
Tomlinson recommends that, where there is no previous experience with a particular clay, a value of 0.45 should be adopted, up to a maximum adhesion value of 100 kN/m^2.

This may be conservative for soft clays but optimistic for very stiff fissured clays.

Adhesion factors for 'driven' piles (After Nordlund)







in the upper reaches of the pile to settle (or travel) down along the surface of the pile exposed to such material, is given by:

$$Q_n = 0.90 \cdot c' \cdot H \cdot S$$

where c' is the 'average' undrained shear strength of the clay (i.e. the compressible fill material) existing in the depth H, and, S the circumference of the pile. [For instance, where the pile is in a scourable medium, then H = depth between normal bed level and the max. scour level, and, for a circular sectioned pile of radius r, $S = 2\pi r$.] Note that the value 0.90 is an approximated magnitude of the product of the adhesion factor and the averaging factor of distribution of c'.

(ii) In 'non-cohesive' (or 'Granular') Soil: The ultimate bearing capacity Q_u is made up of *skin friction* Q_s and end bearing Q_b less *negative skin friction*, Q_n (end bearing is usually much greater than skin friction in granular soils). Thus:

$$Q_u = Q_s + Q_b - Q_n$$

- *Skin Friction* f, at depth z, is given by

$$f = K_s p_d \cdot \tan \delta$$

where K_s is an earth pressure coefficient; the ratio of the lateral to vertical earth pressure at the sides of the pile (higher than K_a but lower than K_p values).
p_d is the overburden pressure at depth z. Generally $P_d = \Sigma \gamma z$ where γ is the bulk density for strata above the water table and the submerged density below the water table as for spread foundation.
δ is the angle of wall friction (between the pile and the soil).

For a pile surrounded by granular soil between depths z_1 and z_2, the total skin friction is

$$Q_s = \frac{1}{2} K_s \gamma (z_1 + z_2) \tan \delta \cdot A_s$$

where A_s is the embedded area from z_1 to z_2 (thus, for a circular pile of radius R, $A_s = 2\pi R(z_2 - z_1)$. If the pile is partly submerged then contributions from above and below the water table must be calculated separately.

Values of K_s and δ, obtained by Broms, are given in Table 12.3. This is valid up to a skin friction value

Table 12.3 *Values of K_s and δ for driven piles*

Pile Material	δ	K_s	
		Low Rel. Density ($\phi \leq 35°$)	High Rel. Density ($\phi > 35°$)
Steel	20°	0.5	1.0
Concrete	3/4ϕ	1.0	2.0
Wood	2/3ϕ	1.5	4.0

of 110 kN/m^2 which is the maximum skin friction value which should be used for any straight-sided pile.

- In calculating *end resistance*, the third term (relating to base friction) in Meyerhoff's equation is relatively small for long slender piles and is usually ignored. Thus, base resistance is given by

$$Q_b = p_{ob} \cdot (N_q - 1) \cdot A_b$$

where p_{ob} is the effective overburden pressure at the pile 'base' level.
A_b is the area of the pile base (thus, for a circular pile of radius R, $A_b = \pi R^2$).

Meyerhoff's value of N_q (given in Fig. 12.6) tend to be unrealistically high for piled foundations, when compared with actual failures, and values obtained by Berezantsev, given in Fig. 12.10, are more suitable. The maximum value of end bearing which should be used in 1100 kN/m^2.

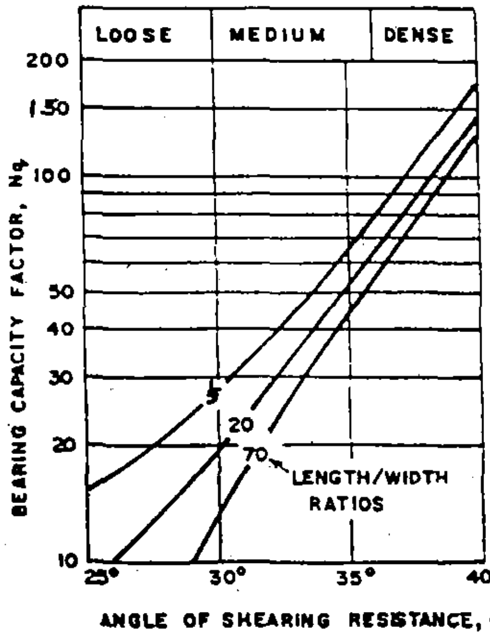


Fig. 12.10 *Berezantsev's bearing capacity factor N_q*

NOTE: When calculating both skin friction and end resistance of 'bored' piles in granular soil, a low relative density should always be assumed, whatever the initial state of the soil.

- The negative skin friction, Q_n, is as explained earlier, but, here, for non-cohesive fill material case, is given by:

$$Q_n = \left[\frac{1}{2} \cdot K_s \cdot \gamma \cdot z \cdot \tan\delta\right] \cdot H \cdot S$$

where: K_s is as explained earlier.

γ = bulk density of strata causing negative skin friction.

$H = z$ = depth of this strata.

(Note: $\gamma \cdot z$ may be treated as $\Sigma\gamma \cdot z$ where γ is the bulk density for strata above the water table and submerged density below the water table, as explained earlier.)

δ = angle of wall friction.

s = perimeter of the pile, as explained earlier.

2. Safe Value

The acceptable working-load value of the soil resistance to a pile, i.e. the SAFE value, is obtained by dividing the ULTIMATE value (calculated in (1) above) by a safety factor. While different authorities may recommend different values for this factor, a commonly adopted value is 2.5 or 3.0.

NOTE: Pile bearing capacity formulas should not be expected to give more than a rough indication of the ultimate load capacity of a pile and, except where piles are driven to refusal, it is usual to load test at least one pile at each site. Special test piles may be driven ahead of the main construction program and tested to failure. As a result of these tests, the engineer may decide to modify the pile lengths required.

— It is preferable to delay testing a pile for as long as possible after it has been driven to allow it to 'settle down'. This is not so important with piles in coarse granular soils, where time-dependant effects are negligible, but in silts and silty sands the ultimate capacity of a pile may be much higher immediately after driving than after it has been installed for a month or so. In clays, the reverse is usually (but not always) true; the carrying capacity increasing with time, particularly in soft or sensitive clays.

— Where piles are driven to refusal in rock or in very dense sands or gravels, the maximum allowable load is usually limited by the structural strength of the pile section rather than the support of the soil.

— Where piles pass through bands of different materials, the skin friction may be calculated for each band and the total skin friction taken as the sum of these values, unless very compressible layers are present. When calculating end bearing, care must be taken to check that weak material is not likely to occur near the tip, which would result in a decrease in end bearing capacity. If this is a possibility, it must be allowed for (by using a reduced value of N_c and N_q) when calculating end bearing.

— In $c-\phi$ soils, skin friction may be taken as the sum of friction and adhesion and end bearing may be taken as the sum of end bearing due to both cohesion and internal friction. However, results should be viewed with some scepticism because very little information is available about the behaviour of piles in $c-\phi$ soils.

(D) Soil Resistance to a 'Group' of Piles

Requirement # 1

Safe value of the soil-resistance to an individual pile (estimated as explained in (C) earlier) shall not be less than the max. axial compression load in the highest loaded pile in the group.

Requirement # 2

Block Failure The group of piles should also be assumed to act (hypothetically) as "one single large pile" of plan dimensions of the group, and behaving like a 'block'. Soil resistance (end bearing plus skin resistance less negative skin friction less self weight of piles less weight of soil within the group) should be estimated for this "block" or "large hypothetical pile" and this should not be less than greatest vertical load on a pile × no. of piles × factor of safety.

Requirement # 3

Group Action $\sim V \cdot nE_f \not< P$, where:

V = Safe value of the soil-resistance to an individual pile (esitmated as explained in (C) earlier).

n = No. of piles in the group.

E_f = Group Efficiency Factor (depending on type of soil, spacing between piles, etc.), as explained ahead*

P = Total vertical load on the group (working load value).

**Values of* E_f*: In Granular Soils*— 'Driven' piles compact the surrounding soil, increasing its bearing capacity, and model tests have shown that 'group efficiency' ratios of driven piles in sand can be as high as 2. With bored piles, the action of boring tends to reduce rather than increase compaction so that the group efficiency ratio of bored pile groups is unlikely to exceed 1.

For design purposes, a group efficiency ratio of 1 is typically used for all kinds of piles in granular soils, that is, the effects of the group are ignored when predicting bearing capacities. However, bored piles should not be spaced closer than 3 diameters (centre to centre).

In other words, *there is at least no reduction in the group load capacity* even if the pressure bulbs of individual piles may slightly overlap so long as piles are spaced at 3 diameters in 'bored' case and (even) 2.5 diameters in the 'driven' case, in non-cohesive soils.

In Cohesive Soils—In these soils the group efficiency factor is always less than 1.0. Figure 12.11 gives the results of model tests by Whitaker for 3 × 3 and 9 × 9 groups. It can be used as a method to account for the efficiency of group in cohesive soils.

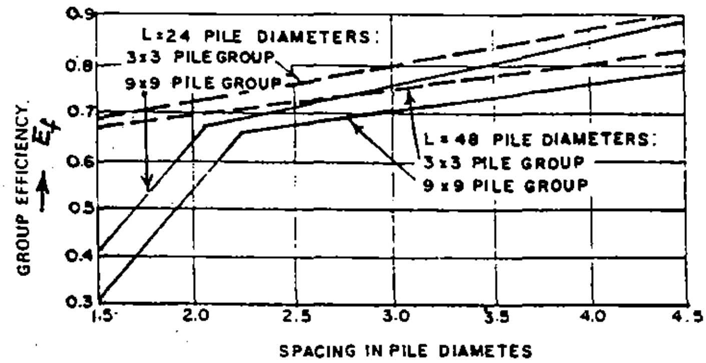


Fig. 12.11 *Group efficiencies for pile groups in 'cohesive' soil. (In grateful acknowledgement to the Controller of Her Majesty's Stationary Office, U.K.)*

NOTE: For a group of piles end-bearing on rock (either 'bored' into rock or 'driven-in' to refusal), the above-mentioned Requirement # 3 may be of no significance since the end-bearing value may be excessively high.

Requirement # 4

This requirement is only for pile-groups which are underlain by weak clay layers in which overstressing can occur. Hence a check should be made for this stress on such a weak layer as follows:

(a) Friction Piles in Clay: The load is assumed to spread out as shown in Fig. 12.12 from 2/3 of the way down the length of the pile embedded in the bearing stratum.

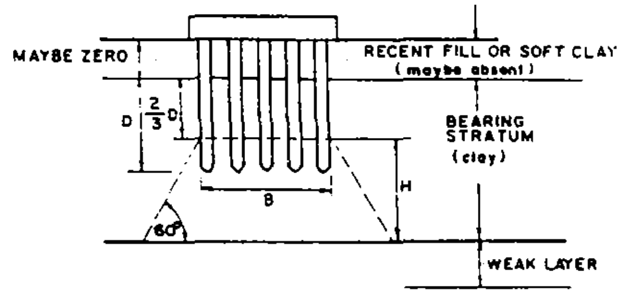


Fig. 12.12 *Spread of load for friction piles*

(b) End Bearing Piles in Sand or Gravel: The load is assumed to spread out as shown in Fig. 12.13 from the base of the piles.

... In each case above, for a pile group measuring L by B, in plan, supporting a load Q, the stressed area of weak material will be

$$(B + 2H \tan 30°) \cdot (L + 2H \tan 30°) = (B + 1.15H)(L + 1.5H)$$

Thus, the stress at the top of the weak layer will be

$$\frac{Q}{(B + 1.15H)(L + 1.15H)},$$ which should be ensured to be allowable there.

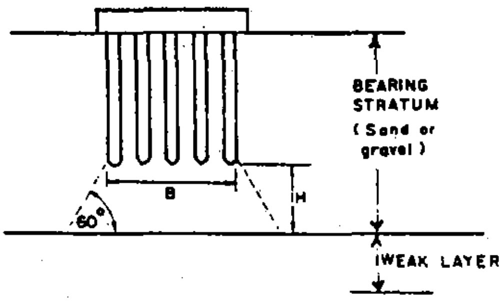


Fig. 12.13 *Spread of load for end-bearing piles*

Lengths of Closely-Spaced Piles

As far as possible, all piles on a pile cap should be of approximately equal length and capacity.

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PART II *(Some Relevant Details)*...

Contents ~

(I) Improving the Bearing Capacity of Soil and Making Foundations on Weak Soils

If the foundations are left open for one rainy season it will enable the soil to settle down, and it will also be known whether the natural movements of the soil below, due to increment of moisture, are likely to cause any damage. Foundations in poor soils can be improved by:

(a) Increasing the depth of the foundation except when the material grows wetter as the depth increases.
(b) Compacting the soil by ramming.
(c) Ramming-in sand, gravels, moorum, broken stone or brick bats *in situ* between the foundation concrete and soil. This is useful for silt or black cotton soils and also clayey soils.
(d) Removing the poor soil and filling the gap with sand, rubble stone, gravel or other hard material. This will increase the bearing power to about twice its original value. In this method the foundation trenches are excavated for a depth of about 2 m and 1 m wider, and filled with the hard material to a thickness of about 0.5 m and heavily rammed with water so as to force the hard material in the soft soil. If the filled material is burried completely, then another layer of the hard material may be filled into a depth of about 0.2 m and well rammed. This method is especially useful for black cotton soils. (See ahead under "Foundations in Black-cotton Soils"). Cement grouting the rammed materials will make the foundations much harder.
(e) Draining out water from wet foundations (Well-point system).
(f) Driving piles, either of wood or concrete, or driving and withdrawing piles and filling the holes with sand or concrete. This will increase the density of the soil. (The method has been explained in detail ahead).
(g) Artificial Stabilization can be used to seal off permeable strata for deep excavations, or to give soft soils additional strength if they are likely to flow.

Cement grout: Water-bearing gravel and coarse sand can be made very much less permeable by pumping cement grout into them. The process is successful only on coarse sands and gravels where the grout can fill up the voids; finer sands necessitate some form of chemical or bituminous emulsion treatment. Grouting is of much use for deep excavations, such as tunnels.

• ***Making foundations on Weak Soils***

a) Grillage footings Consist of single or double tiers of steel beams or rails. The top tier is laid at right angles to the bottom tier. The beams are held in position by spacers placed between them 1 to 1.5 m apart. The stanchion is usually bolted to the top tier and the entire footing is filled solidly with concrete and encased in concrete with a minimum cover of 10 cm; a layer of concrete 20 cm in thickness is placed under the lower beams. The maximum spacing of beams should not be more than 0.5 m centre to centre. Overhang ends are designed as cantilevers subject to an upward uniform load equal to the pressure on the foundation. The working stresses of the uncased beams may be increased by $33\frac{1}{2}$ per cent. It is necessary with grillage beams to check the strength of the web for resistance to buckling, and also shear strength for short spans. This type of foundation is generally suitable for single column loads. Steel grillage footings have been largely replaced by reinforced concrete footings known as "Mat Foundations."

b) Column footings For light loads the column footings may be of plain concrete but most column footings are reinforced concrete footings with two-way reinforcing. Small-diameter closely spaced bars with hooked ends, should be used.

c) Raft or Mat Foundation These usually consist of either (1) thick reinforced concrete slabs covering the entire area occupied by the building and reinforced with layers of bars running at right angles to each other a few cm below the top surface of the mat, and another layer a few cm above the bottom, or (2) inverted T-beams of reinforced concrete, with the slab covering the entire foundation area. The beams run under both directions and intersect under columns and support wall loads, if any. Slab and beams are formed into a monolithic structure and act as a unit. Reinforcement is provided in the beams to support walls, if necessary. The basement floor is placed over the beams. Before the basement floor is placed, the space between these beams may be filled with cinders or some other material. This kind of foundations are used on soft natural ground or fill where the capacity of the soil is very low and where piles cannot be used advantageously. A raft should be so shaped and proportioned that the centre of area of the ground-bearing should, where practicable, be vertically under the centre of gravity of its imposed load.

Raft foundations are stable so long as the underground conditions are undisturbed. Rise and fall of the underground water-table is dangerous for such type of foundations. Where ground water pressure is likely to occur, relief holes should be left in the mat to relieve the water pressure.

Foundations of the above type are sometimes called *floating foundations* and the term is applied where the earth excavated to a depth that will make the weight of the earth removed about equal to the building load. The total vertical pressure on the soil under the building is about the same after the building is completed as it was before the site was excavated and the settlement is reduced to a minimum.

d) Piles: Piles have been described in detail in the author's other book, Analysis, Design and Economics and also ahead. As far as possible, a structure should be erected in such a manner that its whole weight is evenly distributed over the solid foundation below to avoid unequal settlement of the sub-soil. All settlement cannot be eliminated because there is a tendency for the central portion of the building to settle more than the outer portion. In order to reduce differential or uneven settlement to a minimum, foundations must be made very rigid. Heavily loaded parts of a building should be separated from the rest, and the higher and heavier parts treated as separate units with independent foundations fitting in such a manner that the whole structure will have equal settlement. The foundations have also to be separated if the soil underneath is of varying nature and different bearing capacities.

Where possible, the axis of the loads of a unit, i.e. the vertical line passing through the centre of gravity of the weight of the whole unit structure, should coincide with centroid of the area of the foundation of the unit. If there is an eccentricity, the intensity of pressure becomes uneven at the two ends producing more compression at one end and less at the other (or even tension and lifting up of the structure) and the structure, thus, assumes an inclined members such as cantilevers, thrust from an arch wind pressure, earthquake, etc.

- ***Sand Piling Under Foundations***

If the foundation soil is unsatisfactory it can be improved sand piling. Holes are made in the foundation soil with wooden pegs 15 cm in diameter and 1–1.5 m long driven 0.6 to 1.0 m into the ground. These holes are filled with saturated sand. The holes are spaced diagonally so that each hole is nearly 0.6 m apart from those adjacent to it. Work should proceed from the centre of the trench outwards. Sand-piling must never be resorted to in foundations subject to occasional floods, or in foundations where water is met within the course of excavation or bottom of driven pegs.

For big structures, holes are made about 0.3 m diameter and 3 m deep which are filled with sand. The spacing may be about 3 m according to the arrangements of the columns of the structure. The filled-in sand is thoroughly consolidated and a concrete slab laid on top of the piles. The concrete is also let into the pile holes for about 0.15 to 0.3 m so as to be monolithic with the slab.

On shrinkable clays, it may be more economical to use short bored piles and beam foundations to support the external walls.

- ***Shallow Foundations in Black Cotton Soils***

The following methods are generally adopted to meet the characteristics of this soil:

(i) Foundations loads are limited to 5 T/m^2 if water finds access to the foundations, otherwise it may be about 10 T/m^2.

(ii) Foundations are taken down to such depths to which the cracks do not extend.

(iii) Trenches are dug on the side of the foundations and filled with sand or other material to prevent intimate contact of the black cotton soil with the concrete and masonry of the foundations.
If the thickness of the black soil is only 1 to 1.5 m, it should be completely removed and foundation laid on the soil below.

(iv) For important buildings, raft foundations of reinforced concrete are provided.

For ordinary buildings, the foundation trench should be about 1.5 m wide and taken down to at least 0.2 m below the depth at which the cracks cease. The bottom of the trench should be well watered and thoroughly rammed with heavy rammers. On the rammed bed a 30 cm layer of good hard moorum or other such soil is spread in 15 cm layers, well watered and rammed. On top of the moorum about 0.5 m of sand is spread. Before spreading the sand and in order to keep it from running, when dry, into the cracks in the black cotton soil, a half-brick wall of mud or a thin skin of stone masonry is built along both sides of the trench.

On top of this sand the concrete foundation of the building is laid, the masonry to start 15 cm below ground level. Or alternatively, boulder filling may be done underneath the foundation concrete and sides filled with sand. Sand filled around the foundations is about 15 cm for compound walls and unimportant buildings and 0.5 m to 0.6 m for main walls.

Another method similar to the above is when trenches are excavated to a depth of about 2 m and width greater than the width of the bottom of footings by 0.5 m. Cement concrete is filled into a thickness of 25 cm on the sides of the trench bottom for a width of 25 cm on either side, thus leaving a space equal to the width of the bottom of the masonry and 25 cm high which is filled with sand. On the top of this (for full width of the trench) RC slab is built 15 cm thick. Masonry (foundation footings) is built on the RC slab and 25 cm space left on both sides of the foundation masonry is filled with sand. A vertical pipe of 75 mm diameter is passed through the plinth masonry to the sand under the RC slab (through the masonry and the slab) which is kept filled with sand. The sand in the tube will fill up the hollows created at the bottom. Such tubes can be built from 1.5 to 2 m apart and inspected at every change of season and filled up with sand if required.

Black cotton soil can be improved by blending it with granular material, or white clay and coarse sand in equal proportions, which is spread on top and rolled.

(II) Various *in-situ* Penetration Tests Employed in the Estimation of Sub-strata Bearing Capacity

Estimation of bearing capacity of sub-strata involves certain *in situ* operations of 'physically attempting to penetrate the strata'. These are different from the laboratory test* that may subsequently be done on the samples of soil collected from various depths from the said penetration tests.

These penetration tests may be classified as follows:

a) Standard Penetration Test (SPT)
b) Static Cone Penetration Test (SCPT)
c) Dynamic Cone Penetration Test (DCPT)
d) Plate Load Bearing Test (PLBT)

Out of the four tests mentioned above, the SPT is the most commonly employed. The SCPT (also referred to as the 'Dutch Cone Penetrometer Test'), developed by the Dutch Government Soil Mechanics Laboratory, Delft, is applied more to non-cohesive soils, and has been used mostly in Europe only. The DCPT is the least commonly employed penetration test. It has no direct application to shallow foundation design nor does it have a recognized correlation between dynamic probe resistance and bearing capacity or *in situ* density, etc. The PLBT is reasonably reliable only if the sub-strata is of uniform formation in extent and depth but even then it cannot take into effect the soil conditions beyond about 1.5 to 2.0 m below the level at which it is carried out.

a) Standard Penetration Test (SPT)

It is a very useful means of determining the approximate *in situ* density of non-cohesive soils, and to a considerable extent, of most of the regular engineering soils which are $C - \phi$ type.

This test, at a particular depth, is made inside the borehole that is first drilled to that depth. After making the necessary borehole to the depth at which SPT is to be conducted, the SPT apparatus is then lowered through the bore to its bottom. The apparatus essentially comprises a 51 mm diameter annular metallic bottom cutter piece. In gravels, where no soil sample need be 'spooned' out, it can be solid and 60° conical in shape. To this bottom piece is screwed on a 609 mm tall metallic tube made up of two semi-circular halves called the 'split' spoon (tube) sampler. To its upper end is attached a metallic top piece called the adaptor, to which is connected the 'connection rod' (built up of a series of extension rods) which provides the connection to the head piece at ground level.

Penetration of the bottom piece and the split tube sampler into the soil is achieved by repeated 'blows' of a 63.5 kg 'monkey' weight freely falling on the head piece through a 760 mm height. The number of blows required for a total penetration of 450 mm is recorded and out of these, the number of blows required for penetrating the final 300 mm (one foot) is noted and this represents N', the standard penetration resistance at that level. (N' = Number of blows for the last 30 cm penetration.)

This N' value is then corrected for the overburden (i.e. depth) effect to obtain the standard SPT N-value. (Corrected N is higher nearer surface, and the correction-multiplier reduces as depth increases, see Fig. 12.1). In case of dense silty sands, this N-value is further corrected if it exceeds 15, as explained ahead.

It is this corrected N-value which is used in all the (empirical) correlations with the angle of internal friction ϕ and cohesion (or unconfined compression c_u) in finally estimating the bearing capacity, etc. (Figs. 12.3 and 12.4). The same corrected N-value is used in Table 12.1.

It is standard practice to count the number of blows for every 75 mm of penetration in the full 450 mm of driving. By this means the depth of any disturbed soil in the bottom of the borehole can be assessed and the level at which any obstructions to driving such as cobbles, large gravel or cemented layers are met can be noted. Normally

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* For example, Triaxial Compression Test (for determining the magnitudes of cohesion c and angle of internal friction ϕ), Consolidation Test (for estimating the magnitude and rate of consolidation settlement of soil beneath foundation), Permeability Test, Unconfined Compression Test, Particle-size distribution Test, etc.

not more than 50 blows (including the number of blows required to seat the sampler below the disturbed zone) are made in the test. If the full 300 mm penetration below the initial seating drive is not achieved, i.e. when 50 blows have been made before full 300 mm penetration is achieved, then both the depth at the start of the test and the depth at which it is concluded must be given in the borehole record, suitable symbols being used to denote whether the test was concluded within or below the initial seating drive. After withdrawal from the borehole the split tube is taken apart for examination of the soil contents, and although the sample is in a disturbed state it is often sufficiently intact to be able to see laminations or similar features.

The standard penetration test is used to determine the relative density of granular soils. Before starting the test, the bottom of the borehole must be carefully cleaned out to remove any disturbed material. In sands below the water-table, the sand may have a tendency to flow up into the borehole if the casing is not sufficiently far advanced, giving an unrealistically low value of density. If the casing is advanced too far, the sand below the borehole may get compacted. Thus, unless boring and testing are carried out carefully, the results of the test may be very misleading.

A standard split-spoon sampler, shown in Fig. 12.14, is driven 450 mm into the soil by repeated blows from a monkey of standard dimensions. The arrangement of monkey and sampler is shown in Fig. 12.15. The blows required to produce the first 150 mm penetration (termed the seating blows) are usually ignored and the number of blows required to drive the sampler a further 300 mm is recorded as the N-value.

Interpretation of the test is based on experience and on correlations carried out by various researchers. The various corrections and correlations commonly used when interpreting the test are given ahead next. When using the test in cemented soils, sands or gravels or weak rocks, these correlations are no longer valid and interpretation must be made with caution.

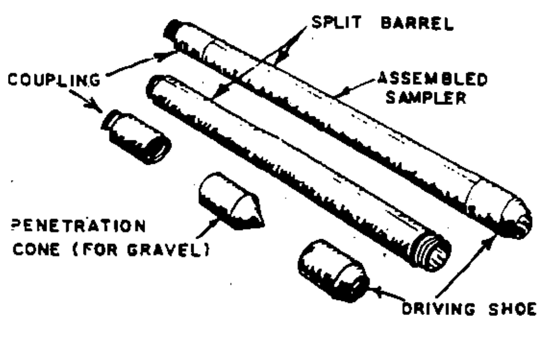


Fig. 12.14 *Split-spoon sampler*

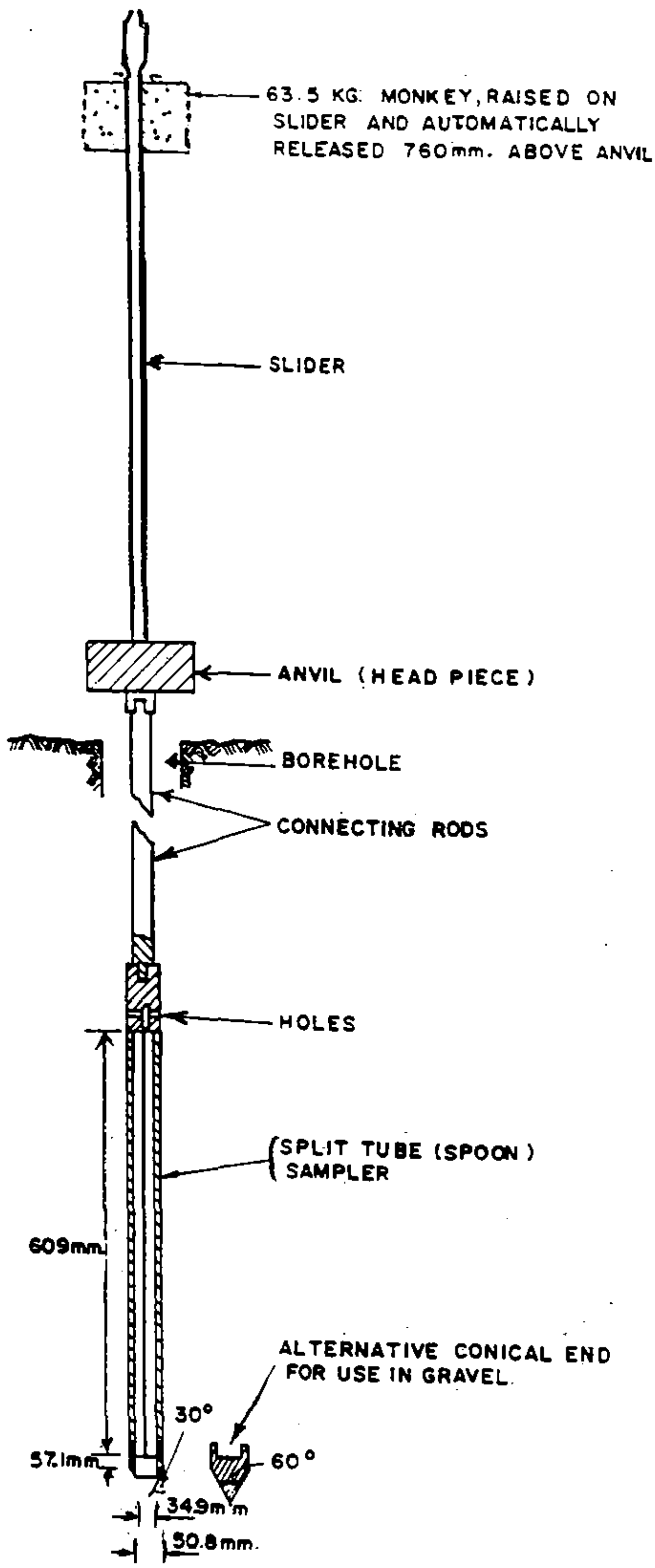


Fig. 12.15 *Schematic arrangement of standard penetration test equipment in borehole*

—Interpretation of Standard Penetration Test Results

SPT results are used primarily to assess the relative density of sands and gravels using a correlation by Terzaghi and Peck. They may also be used to estimate the angle of shearing resistance, θ, and Terzaghi's bearing capacity factors N_q and N_r, from correlations obtained by Peck, Hanson and Thornburn. All of these correlations are given in Fig. 12.3. In addition, SPT results may be used to give a rough indication of the settlement of spread foundations in sands, as indicated in Fig. 12.2.

Approximate relationships exist between the undrained shear strength of clays and N-values (Fig. 12.4) but it is usually better to obtain shear strengths by direct measurement (laboratory tests, e.g. triaxial compression test).

—Corrections in SPT Value

1. Depth effect: The standard penetration test tends to give underestimates of relative density near the surface and a correction factor is usually applied to measured N-value at shallow depths. Depth corrections obtained by Gibbs and Holtz are given in Fig. 12.1.

2. Silts and silty fine sands: In saturated dense silty sands the standard penetration test usually gives an over-estimate of relative density and it is usual to base relative density estimates on modified N-value, N_m, obtained from the expression—

$$N_m = 15 + 1/2(N - 15) \text{ if } N > 15$$

If N is less than 15, no correction is required.

b) Static Cone Penetration Test (SCPT)

As pointed out earlier, this test was developed by the Dutch, and they seem to use it the most. Long experience of its use and familiarity with their soil stratification have enabled the Dutch largely to dispense with conventional borings in their own country, and they rely almost entirely on this cone test for foundation design. The static cone test is also a valuable method of recording variations in the *in situ* density of loose sandy soils or laminated sands and clays in conditions where the *in situ* density is disturbed by boring operations, which tend to make the standard penetration test unreliable in evaluation. Only limited penetration can be achieved by the cone in coarse gravelly soils. The relationship between static cone tests and the standard penetration test shows that there is no unique relationship between them, but it appears to be related to particle size.

In the SCPT, a solid metallic cone is gradually 'pushed' down into the soil hydraulically (not by hammering, hence the word 'static' instead of 'dynamic'), without the need of any borehole to be made in advance (as required in the case of SPT). Hence, it provides a much more accurate and detailed record of the variation in the sub-strata.

Essentially the apparatus comprises a solid metallic cone (the bottom piece) which is connected through metallic extension 'sleeves' (tubes) to a head piece at ground level. The head piece is operated hydraulically, pressing the cone and the connecting tubes down, and additional sleeves are connected for making up the extension. The cone registers the end bearing resistance and the increasing length of sleeve tubes registers the skin friction! There are variations in the apparatus but the one most in use—the electrical cone type was developed by Fugro N.V. in Holland. In this, both the cone and the sleeve-tubes are jacked down together and continuously. The thrust on the cone-end and on a 120 mm length of the cylindrical sleeve are measured separately by electrical load-cells (strain gauges) installed at the lower end of the penetrometer. In modified versions, sensors in the cone read for bearing resistance and those in the tubes read for skin friction. The two values are then correlated and used to estimate the bearing capacity at various levels. However, the skin friction component may not be very reliable.

c) Dynamic Cone Penetration Test (DCPT)

This involves 'hammering' (and hence the word 'dynamic', not 'static') a penetrating metallic cone into sub-strata by the blows of a drop hammer on top of the connecting (extension) rods. The number of blows for a given distance of penetration is recorded. *The apparatus is like a mini pile-driver and is, in fact, used more by the piling contractors as a means of predicting empirically the driving resistance and hence the bearing capacity* (i.e. carrying capacity) *of piles.* It is, therefore, sometimes referred to as *'pre-piling test' apparatus.*

~ Static and Dynamic "Cones"

• *Basic principle of dynamic cone:* Like the standard penetration test, dynamic cones are driven into the ground using a standard falling hammer. However, friction between the drilling rods and soil limits the depth to which the standard sampler or cone can be driven and makes interpretation of results unreliable.

• *Example of a static cone:* The cone illustrated in Fig. 12.16 is pushed into the ground at a constant rate. Strain gauges 3a and 3b allow continuous separate measurements of sleeve friction and cone resistance, respectively.

• *Interpretation of results DCPT vs SPT:* Dynamic cone test results are usually converted to equivalent N-values of the standard penetration test. The relationship between the cone penetration test value N_c and the SPT N-value depends on the equipment used. For typical 60° push-fit cones, the relationship $N_c = 1.5\ N$ has often

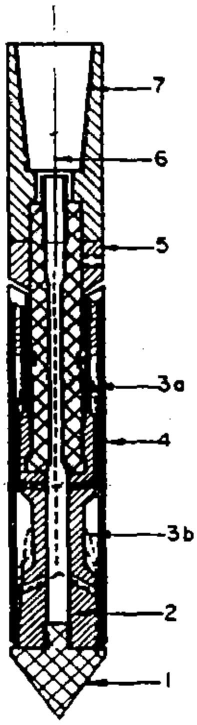


Fig. 12.16 *Example of a static cone: 1. Conical point 2. Load cell 3. Strain gauges 4. Friction sleeve 5. Adjustable ring 6. Cable 7. Connection with rods*

been quoted, for depths up to about 9 m, but it is not reliable.

d) Plate Load Bearing Test (PLBT)

Plate Load Bearing Tests give reliable results only when the soil condition is uniform from the bottom of the footing to a depth at least equal to the width of the largest footing, since settlement in cohesive and partially cohesive soils takes place in a long period of time, load bearing tests on such soils are not very practical. Fortunately, the bearing capacity and the settlement characteristics of such soils can be readily determined by laboratory tests on the relatively undisturbed sample.

The results of plate load bearing tests on soils are useful provided that the test is made with extreme care. The following are some of the factors that should be considered:

1. The test should be made on the loosest area contemplated to support any foundation.
2. The depth of ground water in the test case and in the actual cases should be comparable. Avoid making tests on a layer affected by capillary water.
3. In the 'maintained load test' each load increment is maintained until no further settlement of significant magnitude takes place, and only then the next loading is applied.
4. The ground is not frozen during the test.

There are many other factors which influence the test results. It is advisable to follow the standard test procedure of ASTM Designation D 1194.

The results of load bearing tests should be plotted in a graph similar to one shown in Fig. 12.17.

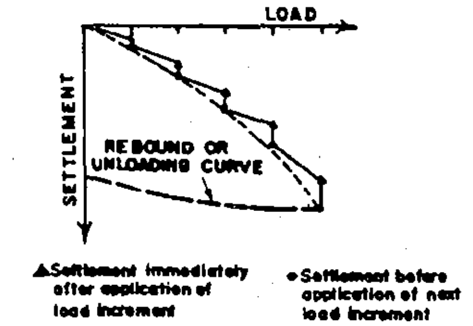


Fig. 12.17 *Graphical presentation of results of load bearing test*

More than one plate load bearing test should be made at the concerned foundation site. Because of variation in soil characteristics and other factors, two tests made under indentical conditions on a presumably uniform soil often show considerably different load-settlement curves. Therefore, results of load bearing tests require careful and expert interpretation.

Plate bearing tests are made by excavating a pit to the predetermined foundation level or other suitable depth below ground level, and then applying a static load to a plate set at the bottom of the pit. The load is applied in successive increments until failure of the ground in shear is attained or, more usually, until the bearing pressure on the plate reaches some multiple, say two or three, of the bearing pressure proposed for the full-scale foundations. The magnitude and rate of settlement under each increment of load is measured. After the maximum load is reached the pressure on the plate is reduced in successive decrements and the recovery of the plate is recorded at each stage of unloading.

This procedure is known as the maintained load test and is used to obtain the deformation characteristics of the ground. Alternatively, the load can be applied at a continuous and controlled rate to give a penetration of the plate of 2.5 mm/min. This is known as the constant rate of penetration test and is applicable to soils where the failure of the ground in undrained shear is required, as defined by gross settlement of the plate; or where there is no clear indication of failure with increasing load, the ultimate bearing capacity is defined by the load causing a settlement of 15 per cent of the plate diameter.

Although such tests appear to answer all the requirements of foundation design, the method is subject to serious limitations and in certain cases the information given by the tests can be wildly misleading! In the first place it is essential to have the bearing plate of a size which will take account of the effects of fissures or other discontinuities in the soil or rock, in plan area as well as depth.

A 300 mm plate is the minimum size which should be used which is suitable for obtaining the undrained shear strength of stiff fissured clays. If deformation characteristics are required from these soils, a *750 mm plate* should be provided in conjunction with the maintained load procedure. It is essential to make the plate tests in soil or rock of the same characteristics as will be stressed by the full-scale foundation. Misleading information will be given if, for examples, the tests are made in the stiff crust of weathered clay overlying a soft clay.

A 1000 mm plate is generally the economic limit, since a 1000 mm plate loaded say to 80 t/m^2 will require some 63 t of kentledge, which is expensive to hire including the costs of transport and handling. *The cost of a single plate-bearing test with a 300-600 mm plate with 50 t of kentledge is about three times the cost of a 12 m deep borehole (in soft ground) complete with in situ and laboratory testing.* A single plate bearing test on a site is, in any case, far from sufficient since the ground is generally variable in its characteristics both in depth and laterally. At least three tests, and preferably more, are required to obtain representative results.

Economies in plate bearing tests on rock can be made by jacking against cable or rod anchorages grouted into drill holes in the rock, instead of using kentledge. Even single anchors have been used successfully. The anchor cable, which is not bonded to the rock over its upper part, is passed through a hole drilled in the centre of the test plate. A test of this type can be made at the bottom of a borehole (pit).

The level of the water-table has an important effect on the bearing capacity and settlement of sands. Thus a plate bearing test made some distance above the water-table will indicate much more favourable results than will be felt by the large full-scale foundation which transmits stresses to the ground below the water-table. The plate bearing test gives no information whereby the magnitude and rate of long term consolidation settlement in clays may be calculated.

In spite of these drawbacks, the plate bearing test cannot be ruled out as ameans of site investigation.

Plate loading tests are best suited to investigating weak jointed rocks or soils containing large gravel or boulders in which *in situ* penetration tests cannot be made.

- ***Plate Bearing Test***

A square or circular plate is seated on the stratum to be tested, usually at the bottom of a trial pit and loaded. The load is maintained until full consolidation settlement has taken place. The test is continued with further increments of load. A plot of settlement against load-intensity (to logarithmic scales) allows a zero correction to be made and sometimes allows the yield point of the soil to be determined. Another indication of failure of the soil is *steadily increasing settlement with time at a constant load, with no tendency to reach a limiting value!*

The settlement of a 'square' plate at a given load can be related to the settlement of a square footing by the following formula, proposed by Terzaghi (1948):

$$S_2 = S_1 \left(\frac{2B}{1+B}\right)^2$$

where S_1 is the settlement of the footing, of width B ft, and S_2 is the settlement of a 1 ft square plate.

Although this refers to a standard plate, 1 ft square, the expression can be used for any sized plate, measured in any units, if B is taken as the ratio of footing width to plate width. This relationship usually leads to an underestimate of settlement for large footings and Menard and Rousseau proposed the following relationship:

$$\frac{S_1}{S_2} = \left(\frac{B_2}{B_1}\right)^\alpha$$

where S_2 and S_1 are settlements of 'plate' and 'footing', B_2 and B_1 are their respective widths, and α is an exponent which depends on soil type. Values of α are:

sands and gravels	: 1/2 to 1/3
saturated silts	: 1/2
clays and dry silts	: 2/3 to 1/2
compacted fill	: 1

The ultimate bearing capacity of the footing can be assessed from that of the plate by using standard bearing capacity formulas. The ultimate bearing capacity of a foundation in clay is approximately equal to the ultimate bearing capacity of the plate, with no allowance being made for scale. In granular soils ultimate bearing capacity is proportional to width and, approximately.

$$\frac{Q_2}{Q_1} = \frac{B_2}{B_1}$$

where Q_2 and Q_1 are the ultimate bearing capacities under foundation and plate, respectively, and B_2 and B_1 are their respective widths.

Plate loading tests are particularly suitable for coarse granular materials which cannot be tested by normal laboratory means or by a penetration test. The main pitfall in predicting settlement from these tests is that the zone of stressed soil beneath the plate is much smaller than that beneath the larger foundation; so will be unaffected by deeper strata whose load bearing and settlement characteristics may critically affect the behaviour of the actual foundation. With clays, tests do not usually continue for long enough for consolidation to be completed so settlement cannot be predicted. In order to obtain reliable results, plates should be as large as possible and should never be less than 0.3 m wide.

• ***Relative Density and the 'Standard Penetration Test'***

The relative density of a granular soil is defined in terms of the loosest and densest states of compaction which can be achieved in the laboratory with that soil, although no specific laboratory tests are universally agreed. It is defined in terms of the voids ratio of the soil but can also be expressed in terms of dry densities. Thus:

$$D = \frac{e_{max} - e}{e_{max} - e_{min}} = \frac{\gamma_{max}}{\gamma} \cdot \frac{\gamma - \gamma_{min}}{\gamma_{max} - \gamma_{min}}$$

where D is the relative density,

γ, γ_{max} and γ_{min} are the dry densities in the field and at the densest and loosest states of compaction, respectively, and

e, e_{min} and e_{max} are the corresponding voids ratios, respectively.

Relative density in the field is usually assessed from standard penetration test results using N-values.

However, SPT results tend to overestimate relative densities in silts and fine sands below the water-table and the relative density is usually assessed using a modified N-value, N_m, obtained from the expression.

$$N_m = 15 + 1/2(N - 15) \text{ if } N > 15$$

If N is less than 15, no correction is required.

Descriptive terms for relative density and equivalent N-values are given in Table 12.4.

Table 12.4 *Descriptive terms for relative density and equivalent SPT N-value*

Relative Density (%)	Descriptive Term	N-values
0–15	Very loose	0–4
15–35	Loose	4–10
35–65	Medium dense	10–30
65–85	Dense	30–50
85–100	Very dense	50+

When relative densities are given in soil descriptions, it should always be made clear whether they are based on SPT results or assessments using a pick and shovel or driven stake.

(III) Safe Bearing Capacity of 'Rocky' Substrata

In case of rock, first the ultimate bearing capacity may be determined by testing a number of its 'representative' core specimens in a compression testing machine, under saturated soaked condition, and then the safe bearing capacity estimated by dividing the average ultimate value by a factor ranging in value between 5 and 20—depending on the *in situ* characteristics of the rock and the combined effects of long term saturation, permanent eccentric compression, possibility of leaching away of its softer constituents from within its vein-matrix, and the abrasion from the flowing sand. An engineering geologist's opinion should also be sought. A plate load test can also be used to determine the allowable bearing capacity of rocky strata. For some typical values see ahead. Many rocks can withstand a compressive stress higher than normally used concrete. Some of the exceptions are given below:

1. Limestones with cavities and fissures which may be filled with clay or silt.
2. Rocks with bedding planes, folds, faults or joints at an angle with the bottom of footing.
3. Soft rocks often reduce their strength after wetting. Weathered rocks can be very treacherous. Shales may become clay or silt in a matter of hours of soaking!

The common sandstones and limestones have modulus of elasticity varying from that of a poor concrete to that of high strength concrete. Very hard igneous and metamorphic rocks exhibit considerably greater value of modulus of elasticity.

Some Terms used in Rock Descriptions: (Based on B.S. 5930, 1981)

Texture and Structure: 'Texture' refers to the physical appearance of the rock crystals or grains. Terms include crystalline, cryptocrystalline, porphyritic, granular, amorphous and glassy.

'Structure' is usually applied to the rock mass and refers mainly to the arrangement of structural features and discontinuities. Structural features include bedding planes, laminations, foliations, flow-banding and cleavages. Discontinuities include joints, fissures, faults and shear planes. Where discontinuities occur in three dimensions, rock blocks are formed. Descriptive terms used for the frequency of discontinuities and structural features age given in Table 12.5.

Rock Strength: This refers to the strength of the rock material and is based on the unconfined compressive strength. Descriptive terms used and the corresponding strengths are given in Table 12.6. Strengths depend on the moisture content of the specimen at the time of test, anisotropic features in the specimen and the test procedure,

Table 12.5 *Terms used to describe the spacing of discontinuities in rock masses*

Spacing	Structural Features	Discontinuities in One Dimension	Discontinuities in Three Dimensions*
> 2 m	Very thick	Very widely spaced	Very large
600 mm–2 m	Thick	Widely spaced	Large
200–600 mm	Medium	Medium spaced	Medium
60–200 mm	Thin	Closely spaced	Small
20–60 mm	Very thin	Very closely spaced	Very small
6–20 mm	Thickly laminated (sedimentary) Narrow (metamorphic and igneous)	Extremely closely spaced	*Second term* Blocky-equidimensional Tabular—thickness much less than length or width
< 6 mm	Thinly laminated (sedimentary) Very narrow (metamorphic and igneous)		Columnar—height much greater than cross-section

*Relates to spacing of maximum dimension.

Table 12.6 *Descriptive terms of the compressive strength of rock material*

Compr. Strength (MN/m^2)	Description
< 1.25	Very weak
1.25–5.0	Weak
5.0–12.5	Moderately weak
12.5–50.0	Moderately strong
50–100	Strong
100–200	Very strong
> 200	Extremely strong

and are generally of little use in assessing the strength of the rock mass.

Grain Size and Rock Name: For engineering purposes, broad classifications are usually sufficient; detailed geological names are not necessary.

Recovery and Rock Quality Description (RQD): These terms refer to rock cores and are defined as follows:

$$\text{Recovery (\%)} = \frac{\text{length of core recovered}}{\text{length of core run (length drilled)}} \times 100$$

$$\text{RQD (\%)} = \frac{\begin{array}{c}\text{length of rock recovered in sound}\\ \text{lengths of 100 mm or more}\end{array}}{\text{length of core run}} \times 100$$

They both give an indication of the strength of the rock mass but can be used only as rough guide because values obtained also depend on the diameter of the core, the method of drilling and the skill of the driller. Descriptive terms for RQD values are given in Table 12.7.

Table 12.7 *Descriptive terms for RQD† values*

RQD (%)	Description
0–25	Very poor
25–50	Poor
50–75	Fair
75–90	Good
90–100	Excellent

† Rock Quality Description

(IV) Soil Parameters: Some Typical Values

(i) Shearing Resistance of Cohensionless Soils: Estimates of the angle of shearing resistance can be obtained from Table 12.8 which gives typical values suggested by Terzaghi and Peck.

Table 12.8 *Typical values of the shearing resistance of cohesionless soils*

Material	ϕ-Degrees Loose	ϕ-Degrees Dense
Uniform sand, round grains	27	34
Well graded sand, angular grains	33	45
Sandy gravels	35	50
Silty sand	27–33	30–34
Inorganic silt	27–30	30–35

(ii) The Inter-Relationship between 'Cohesion', 'Internal Friction' and 'Stability': The unit shear resistance is composed of two parts, that furnished by the resistance of soil-grains to sliding over each other and that furnished by the cohesion existing between the soil particles. By experiments the cohesion c in lbs/sq ft and ϕ have been ascertained as given in Table 12.9.

Table 12.9 *Approx. values*

	Soil	c*	ϕ Deg.
• cohesive	Clay liquid	100	0
	Clay liquid very soft	200	2
	Clay liquid soft	400	4
	Clay liquid fairly stiff	1000	6
	Clay liquid stiff	1500	8
	Clay liquid very stiff	2000	12
	Silt	0	20
	Sand wet	0	10
	Sand wet dry or unmoved	0	34
	Sand wet predominating with some clay	400	30
	Cemented sand and gravel, wet	500	34
	Sand-gravel mixture cemented with clay, dry	1000	34

(Contd).

	Soil	Dry Density**	ϕ Deg.
• non-cohesive	Compact well-graded sands and gravel-sand mixtures	110–120	40–45
	Dense well-graded gravel	115–125	45–50
	Loose well-graded sands and gravel sand mixtures	100–110	35–40
	Dense sands	105–115	35–46
	Compact uniform sands	100–110	35–40
	Loose uniform sands	90–100	30–35
	Loose fine sands	90–100	28–34

*Cohesion lbs/sq.ft

**Dry density in lbs/c ft.

(iii) Friction and Adhesion at Interfaces

Table 12.10 *Typical values of friction and adhesion at interfaces*

Materials	Friction tan δ (δ = angle of wall-friction)	Adhesion C_s (kN/m^2)
• *Mass concrete or masonry against rocks and soils*		
Clean sound rock	0.7	
Clean gravel, gravel-sand mixtures, coarse sand	0.55–0.60	
Clean fine to medium sand, silty medium to coarse sand, silty or clayey gravel	0.45–0.55	
Clean fine sand, silty or clayey fine to medium sand	0.35–0.45	
Fine sandy silt, non-plastic silt	0.30–0.35	
Very stiff and hard residual or overconsolidated clay	0.40–0.50	
Stiff clay and silty clay	0.30–0.35	
• *Steel sheet piles against soils*		
Clean gravel, gravel-sand mixtures, well-graded rock fill	0.40	
Clean sand, silty sand-gravel mixtures, single size hard rock fill	0.30	
Silty sand, gravel or sand mixed with silt or clay	0.25	
Fine sandy silt, non-plastic silt	0.20	
Soft clay and clayey silt		5–30
Stiff and hard clay and clayey silt		30–60
• *Formed concrete or concrete sheet piling against soils*		
Clean gravel, gravel-sand mixtures, well-graded rock fill	0.40–0.50	
Clean sand, silty sand-gravel mixtures, single size hard rock fill	0.30–0.40	
Silty sand, gravel or sand mixed with silt or clay	0.30	
Fine sandy silt, non-plastic silt	0.25	
Soft clay and clayey silt		10–35
Stiff and hard clay and clayey silt		35–60
• *Various structural materials*		
Masonry or masonry, igneous and metamorphic rocks:		
dressed soft rock on dressed soft rock	0.70	
dressed hard rock on dressed soft rock	0.65	
dressed hard rock on dressed hard rock	0.55	
Masonry on wood (cross grain)	0.50	
Steel on steel at sheet pile interlocks	0.30	

(Contd.)

NOTE: The numbers are ultimate values and require sufficient movement for failure to occur. Where friction value alone is shown, the effect of adhesion is included in the friction factor. For data on adhesion on bearing piles, see Table 12.2.

• Typical Values of Allowable Bearing Pressure

Table 12.11 gives 'typical' values of allowable bearing pressures for shallow spread foundations subjected to vertical static loading. It is presumed that the founding level is at least 1 m deep in soils and that the ground surface is fairly level.

Table 12.11 *Typical values of allowable bearing pressure:*

	Type of Bearing Material	Allowable Bearing Pressure (kN/m^2)
Rocks	Massive hard crystalline igneous and metamorphic rocks	6000–10000
	Massive hard crystalline limestones; thoroughly-cemented sandstones and conglometers	4000–6000
	Unweathered schists and slates	3000–4000
	Hard shales and mudstones, moderate and weakly-cemented sandstones; hard unweathered marl or chalk	1500–2500
	Weathered and broken bedrock; clayey shales and soft mudstones	800–1200
Clays	Hard clays; cohesive strength > 300 kN/m^2	600
	Very stiff clays; cohesive strength 150–300 kN/m^2	300–600
	Stiff clays; cohesive strength 75–150 kN/m^2	150–300
	Firm clays; cohesive strength 35–75 kN/m^2	70–150
	Soft and very soft clays; cohesive strength < 35 kN/m^2	Negligible
Gravels	Very dense sands and gravels; SPT N-value > 50	400
	Dense sands and gravels; SPT N-value 30–50	300–400
	Medium dense sands and gravels; SPT N-value 10–30	100–300
	Loose sands and gravels; SPT N-value 5–10	50–100

• NOTES:

(1) *Approximate conversion:*
kN/m^2 to t/m^2, divide by 10
kN/m^2 to kg/cm^2 divide by 100
kN/m^2 to tons/ft^2 divide by 100

(2) *Clays:* A useful rule of thumb when dealing with shallow foundations for light structures is:

*under dry conditions: allowable bearing pressure = 2c = unconfined compressive strength, c_u.

*under saturated conditions: half of this value. The unconfined compressive strength can be estimated on site by taking numerous pocket penetrometer readings for borehole samples or in the sides of pits. Since most pocket penetrometers are graduated in terms of compressive strength, they give allowable bearing pressures directly. This convenient relationship arises because, from Meyerhoff's bearing capacity formula, net ultimate bearing capacity $p_{nu} = cN_c = 6c$ for shallow foundations. For dry conditions taking a factor of safety of 3, the allowable bearing capacity is $p_a = 1/3.\ p_{nu} = 2c$. For saturated conditions $p_a = 1/6.\ p_{nu} = c$.

(3) *Sands and gravels:*

(a) If SPT N-values are not available, the density of the sand and gravel may be judged by pick and shovel; if it can be excavated by shovel, it is loose; if a pick is required to excavate the material, it is compact. For compact sands and gravels, tested in this way, it is usual to make the conservative assumption that they are medium dense.

(b) The above bearing capacities are for foundations at least 1 m wide on dry sand; for saturated or submerged conditions, halve the values.

- **Broad Classification of Clays and Sands** (see Table 12.12)

Table 12.12

	Shear Strength/Relative Density			Structural Weathering	Colour	Particle Shape/ Composition/ Plasticity	Type of Particle			Inclusions
	Clay	*Cohesion* $c(kN/m^2)$	*Characteristics*				*Particle*		*Size (mm)*	
• Clays	Very soft	< 20	Exudes between fingers when squeezed	Intact	Grey	Angular	Clay		< 0.002	.. with shells.
				Fissured	Brown	Subangular	Silt	fine	0.002–0.006	.. with scattered cobbles and boulders.
						Subrounded				
	Soft	20–40	Moulded by light finger pressure.	Stratified	Blue-Gray	Rounded				
						Flat	Silt	medium	0.006–0.2	
	Firm	40–75	Moulded by strong finger pressure.	Laminated	Mottled yellow and brown	Elongated				.. with layers or lenses of fine sand.
						Irregular	Silt	coarse	0.02–0.06	
	Stiff	75–150	Can be indented by thumb.	Heterogeneous		Rough	Sand	fine	0.06–0.2	.. with some shell fragments.
						Smooth				
	Very stiff	150–300	Can be indented by thumb nail	Fibrous	Dark green	Polished				
							Sand	medium	0.2–0.6	
	Hard	> 300		Weathered	Yellowish	Sandstone				
						Limestone	Sand	coarse	0.6–2.0	
• Sands	*Sands*	*SPT**		etc.	etc.	Granite				etc.
	Very loose	< 4				Highly plastic	Gravel	fine	2–6	
	Loose	4–10	Can be dug by spade 50 mm peg easily driven.			Non-plastic				
							Gravel	medium	6–20	
	Medium dense	10–30				etc.				
							Gravel	coarse	20–60	
	Dense	30–50	Needs pick for excavation 50 mm				Cobbles		60–200	
	Very dense	> 50	peg hard to drive.				Boulders		> 200	
	Standard penetration test *N-values*									

Ref: B.S. 5930 (1981) for Site Investigations.

CHAPTER 13

Estimating the Net Dependable Passive Less Active' Earth-Pressure Relief from Undisturbed Soil Mass Gripping the Foundation-Bulkhead between the Maximum-Scour Level and the Founding-Level

NOTE: For details regarding the following:

(i) Minimum depth of foundations
(ii) Estimation of scour depth for design of piers and abutments
(iii) Estimation of linear waterway under a river bridge,

refer to the relevant ***data sheets*** in the author's book: *Consultancy and Construction Agreements for Bridges, including Field Investigations.*

The foundation (bulkhead) could be a very large diameter pile or a caisson. In order that the moments resulting from:

(i) its self weight acting in a slightly 'tilted' and possibly shifted condition (tilt and shift caused during actual construction of the bulkhead), and

(ii) those caused by possible local flexing of its body, do not enter the earth pressure computations, the bulkhead may be assumed weightless and rigid for the purpose of the calculations here. However, the tilt and shift moments and those due to any flexing must be computed separately and included along with the effect of other forces.

Two distinct cases are possible:

- *Case A:* Bulkhead resting on soil, with undisturbed soil mass gripping it below the maximum scour level.
- *Case B:* Bulkhead resting on rock, with undisturbed soil mass gripping it below the maximum scour level.

In case A, the soil under the foundation-base may be assumed to yield and give so that the foundation-base tends to move along an arc and kicks back into the soil. Thus the bulkhead tends to rotate about a point above its base.

In case B, if the rock under the foundation-base is assumed not to yield, then the bulkhead does not kick back into the soil. In such a case the base is the tending point of rotation, though not actually hinging physically at base.

The earth pressure distributions between these two cases differ vastly, and their pertinent details are briefly given in the following sections.

13.1 CASE A

Bulkhead rests on soil and tends to rotate about a point which lies above the base

Assumptions

(i) Bulkhead tends to rotate about a point which lies above its base.
(ii) Soil may be non-cohesive or cohesive, but the backfill (behind the abutment) above the MSL (maximum scour level) is always non-cohesive.
(iii) Consider the net passive-less-active earth pressure ordinate at each level above and below the point of rotation.
(iv) Horizontal forces (applied and passive-less-active pressure ones) are balanced, and the correspondingly generated net passive-less-active earth pressure moment relief is considered in the stability of the bulkhead.
(v) Ignore vertical components of passive and active earth pressure forces as also the skin friction and the force of friction generated at base.
(vi) Earth pressure on the portion of the structure above the MSL on account of the backfill earth mass above that level shall be considered separately (as another external force) but its surcharge effect shall be included in the present calculations. (See definition of H_R ahead, under 'Symbols'.)

Calculation of relief offered by the net passive-less-active earth pressure of soil, within the effective soil-grip around the bulkhead, is computed on the assumption that the bulkhead is a rigid body that derives its stability from the net earth pressure from the opposite sides owing to its tendency to rotate about a point slightly above its base. The point of rotation may be found from statical equilibrium between the external resultant horizontal force H_R at MSL and the net passive-less-active earth pressure force as worked out from the net pressure distribution in the relevant case. This is explained diagrammatically in Figs. 13.1 to 13.7 ahead (cases 1 to 6). After locating the point of rotation, the relieving net passive-less-active earth pressure moment at base (and at various intermediate levels) can then easily be worked out and the overall stability (as well as the material stresses at various sections) checked.

Symbols

K_a = horizontal component of active earth pressure coefficient of the soil below MSL

$$= \frac{\sin^2(\alpha+\phi)\cos\delta}{\sin^2\alpha\sin(\alpha-\delta)\left[1+\sqrt{\dfrac{\sin(\phi+\delta)\sin(\phi-\beta)}{\sin(\alpha-\delta)\sin(\alpha+\beta)}}\right]^2}$$

(K_{a1} and K_{a2} pertain to the dry and submerged portions of the backfill)

K_p = horizontal component of passive earth pressure coefficient of the soil below MSL

$$= \frac{\sin^2(\alpha-\phi)\cos\delta}{\sin^2\alpha\sin(\alpha+\delta)\left[1-\sqrt{\dfrac{\sin(\phi+\delta)\sin(\phi+\beta)}{\sin(\alpha+\delta)\sin(\alpha+\beta)}}\right]^2}$$

(K_{p1} and K_{p2} pertain to the dry and submerged portions of the backfill)

P_a and P_p = active and passive earth pressure forces.
α = angle of inclination of wall with horizontal (= 90° for a vertical wall)
β = angle of inclination of top surface of soil with horizontal (i.e., surcharge angle, = 0 for horizontal bed)
δ = angle of wall friction
ϕ = angle of repose of soil (internal friction)
c = cohesion value of soil below MSL
γ = submerged density of soil below MSL
γ_2 = submerged density of backfill soil
γ_1 = dry density of backfill soil
MSL = maximum scour level
HFL = high flood level
FL = founding level
H_R = Σ of all the external horizontal forces at the MSL (including the total P_a from backfill above MSL in case of abutments) expressed per unit width of bulk head,. so that if η be the diameter of the bulk head then $H_R\eta$ represents the full total value of all the externally applied horizontal forces at MSL (including the total P_a from backfill above MSL in case of abutments) on the full width of the bulk head.

Method (Cases 1–6)

(i) Plot $(P_p - P_a)$ ordinate at various depths.
(ii) then,
- H_R per unit width of bulk head = area A − area B
- Hence find d (ref. Fig. 13.1)
- Then passive-less active moment relief, for example at base, = (moment of area A − moment of area B) about base.

(iii) Maximum bending moment occurs at a depth h below MSL where algebraic sum of the horizontal forces (H_R and part of A, or H_R and A and part of B, as the case may be—to be found by trial) is zero. For instance if $h < (H-d)$ in Fig. 13.1 then,
$H_R \equiv (K_p - K_a)\gamma hh/2$
from where h can be found, and then the bending moment at h below MSL, i.e. the max. B.M., will be $M_0 + [H_R h - (K_p - K_a)\gamma h^3/6]\eta$

where M_0 = external BM at MSL (total)
η = diameter of bulk head

- ***Case 1***

Pier in non-cohesive soil (Fig. 13.1).

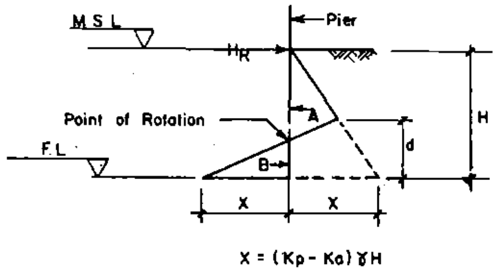


Fig. 13.1

- ***Case 2***

Pier in cohesive soil (Figs. 13.2 and 13.3).
(i) if depth of tension crack $d_c \geq H$ Figs. 13.2

i.e., $$\frac{2c}{\gamma\sqrt{K_a}} \geq H$$
$$X = 2c\sqrt{K_p}$$
$$Y = X + K_p\gamma H$$

(ii) If depth of tension crack $d_c < H$ Fig. 13.3

i.e., $$\frac{2c}{\gamma\sqrt{K_a}} < H$$
$$X = 2c\sqrt{K_p}$$
$$Y = X + K_p\gamma H - Z$$
$$Z = K_a\gamma H - 2c\sqrt{K_a}$$

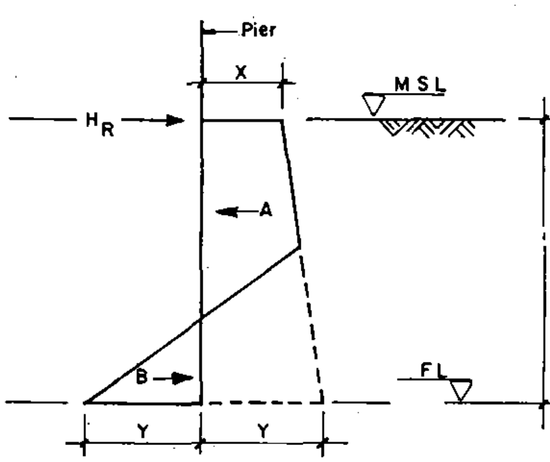


Fig. 13.2

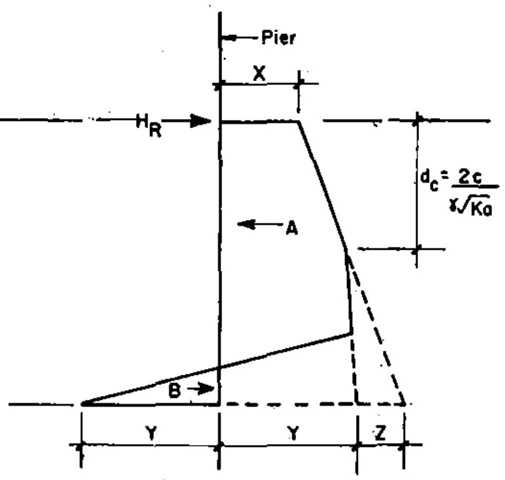


Fig. 13.3

• *Case 3*

Abutment in non-cohesive soil (Fig. 13.4).

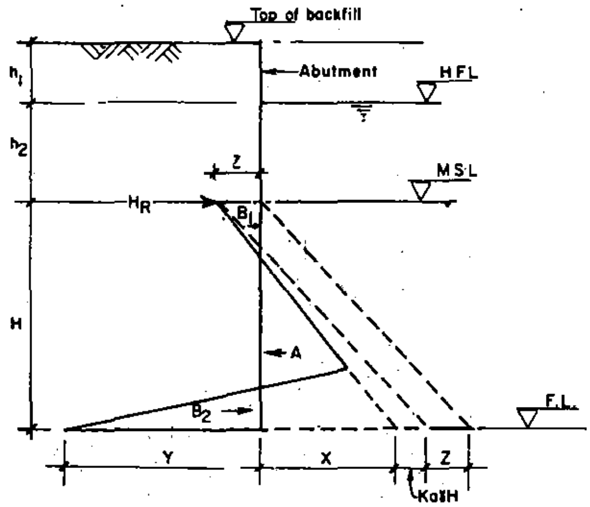


Fig. 13.4

$$Z = K_a(\gamma_1 h_1 + \gamma_2 h_2)$$
$$X = (K_p - K_a)\gamma H - Z$$
$$Y = (K_p - K_a)\gamma H + K_p(\gamma_1 h_1 + \gamma_2 h_2)$$

NOTE In case of abutments, in each case, the active earth pressure effects from the backfill above MSL shall be considered separately as if an additional external force. Its force-effect is already considered in the value of H_R, defined earlier. Also note that the backfill soil should always be non-cohesive.

• *Case 4*

Abutment in cohesive soil, with depth of tension crack $d_c \geq H$ (Fig. 13.5).

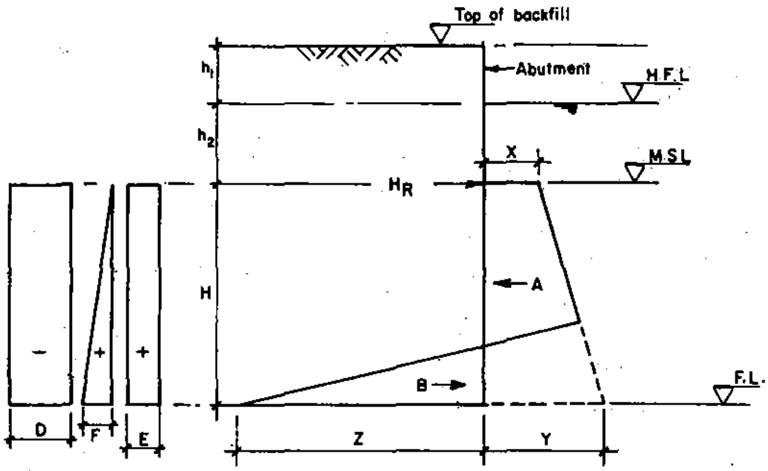


Fig. 13.5

$D = 2c\sqrt{K_a}$, $F = K_a\gamma H$, $E = Ka(\gamma_1 h_1 + \gamma_2 h_2)$

This means $D \geq (F + E)$, i.e.,

$2C\sqrt{K_a} \geq \{K_a\gamma H + K_a(\gamma_1 h_1 + \gamma_2 h_2)\}$

In such a case,

P_a ordinate below MSL is zero so that $(P_p - P_a)$ ordinate is only the P_p ordinate at any depth.

$$X = 2c\sqrt{K_p}$$
$$Y = X + K_p\gamma H$$

$$Z = Y + K_p(\gamma_1 h_1 + \gamma_2 h_2)$$

- ***Case 5***

Abutment in cohesive soil, with depth of tension crack $d_c < H$ (Fig. 13.6).

In such a case d_c the depth of tension crack is estimated from

$$2c\sqrt{K_a} \equiv (K_a \gamma d_c + K_a(\gamma_1 h_1 + \gamma_2 h_2)$$

giving
$$d_c = \frac{2c}{\gamma\sqrt{K_a}} - \left\{\frac{(\gamma_1 h_1 + \gamma_2 h_2)}{\gamma}\right\}$$

and also that, $D > E$ but $< (E + F)$,

i.e., $\quad 2c\sqrt{K_a} > K_a(\gamma_1 h_1 + \gamma_2 h_2)$

but $\quad < (K_a(\gamma_1 h_1 + \gamma_2 h_2) + K_a \gamma H)$

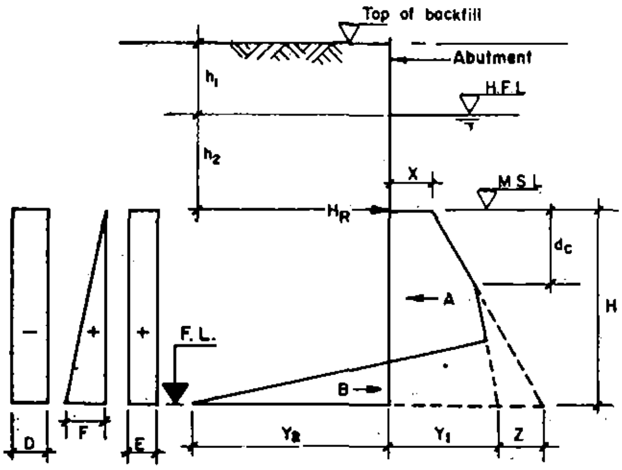


Fig. 13.6

$$D = 2c\sqrt{K_a}, \quad E = K_a(\gamma_1 h_1 + \gamma_2 h_2), \quad F = K_a \gamma H$$
$$X = 2c\sqrt{K_p}$$
$$Y_1 = X + K_p \gamma H - Z$$
$$Z = K_a(\gamma_1 h_1 + \gamma_2 h_2) + K_a \gamma H - 2c\sqrt{K_a}$$
$$Y_2 = K_p(\gamma_1 h_1 + \gamma_2 h_2) + 2c\sqrt{K_p} + K_p \gamma H - \underbrace{(K_a \gamma H - 2c\sqrt{K_a})}_{\text{ignore if } -\text{ve}}$$

- ***Case 6***

Abutment in cohesive soil, with no tension crack (Fig. 13.7).

This means $D < E$, i.e., $2c\sqrt{K_a} < K_a(\gamma_1 h_1 + \gamma_2 h_2)$, so that net P_a is a trapezium from the back.

$$D = 2c\sqrt{K_a}, \quad E = K_a(\gamma_1 h_1 + \gamma_2 h_2), \quad F = K_a \gamma H$$
$$X = 2c(\sqrt{K_p} + \sqrt{K_a}) - K_a(\gamma_1 h_1 + \gamma_2 h_2)$$
$$Y = X + K_p \gamma H - K_a \gamma H$$
$$Z = K_p(\gamma_1 h_1 + \gamma_2 h_2) + K_p \gamma H + 2c\sqrt{K_p} - \underbrace{(K_a \gamma H - 2c\sqrt{K_a})}_{\text{ignore if } -\text{ve}}$$

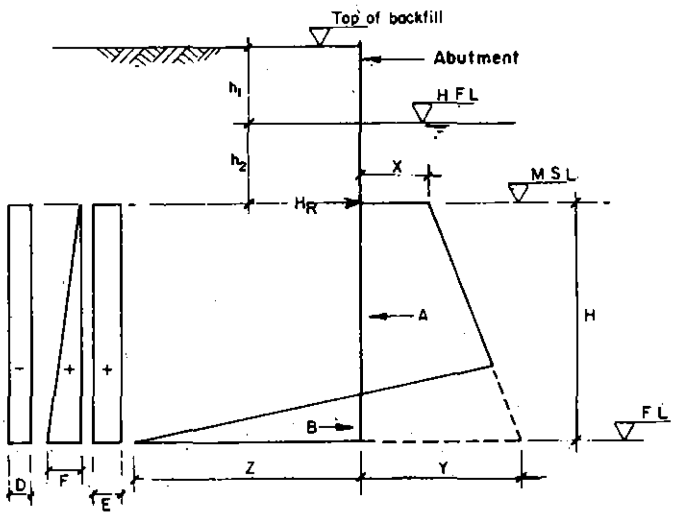


Fig. 13.7

13.2 CASE B

Bulkhead rests on rock* and therefore, it only tends to rotate about its base.

Assumptions

(i) Bulkhead tends to rotate about its base.

(ii) Soil may be non-cohesive or cohesive, but the backfill (behind the abutment) above the MSL is always non-cohesive.

(iii) Consider the passive and active earth pressure diagrams from opposite sides, with their maximum ordinates at base.

(iv) Since actual earth pressure distributions may well be slightly different, therefore, instead of attempting a statical equilibrium between the external resultant horizontal force H_R at MSL and the net passive-less-active earth pressure force, directly a dependable

* It should however be noted that this case B is applied sometimes even when the bulkhead is resting on soil (instead of rock), although not recommended.

value of net passive-less-active effects is assumed by dividing the net value by a factor of 3.0 in case of cohesive soils and 2.0 in case of non-cohesive soils under normal conditions and by 2.4 and 1.6, respectively, under seismic or wind or any other temporary condition.

(v) Ignore vertical components of passive and active earth pressure forces, as also the skin friction.

(vi) Earth pressure on the portion of the structure above the MSL on account of the backfill earth mass above that level shall be considered separately (as another external force) but its surcharge effect shall be included in the present calculations. (See definition of H_R earlier under 'Symbols'.)

Calculation of relief offered by the net passive-less-active earth pressure of soil, within the effective soil-grip around the bulkhead, is computed on the assumption that the bulkhead is a rigid body that derives its stability from the net earth pressure-distribution shown diagrammatically in Figs 13.8 to 13.11 ahead for the four individual cases (Cases 7 to 10), and its dependable value is estimated by dividing by the appropriate factor mentioned in (iv) above. Where this dependable moment is higher than the external moment at base level, revised value of the factor is computed by dividing the $P_p - P_a$ moment by the external moment, and this revised (higher) factor is used for subsequent calculations, and then the resultant moment transferred to base still is zero.

• ***Case 7***

Pier in non-cohesive soil (Fig. 13.8).

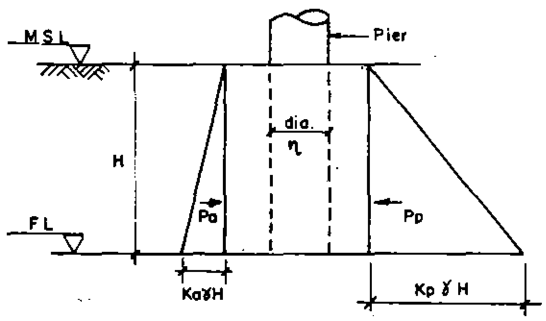


Fig. 13.8

• $M_{\text{dependable}}$ at base = dependable value of net passive-less-active earth pressure moment at base

$$= \frac{1}{f}\left\{\frac{\eta}{6}\gamma H^3(K_p - K_a)\right\}$$

• Factor f as per (iv) above.

and η = diameter (or width) of the bulkhead.

• maximum bending moment section and value of this moment can be found as explained in case 1.

• ***Case 8***

Pier in cohesive soil (Fig. 13.9).

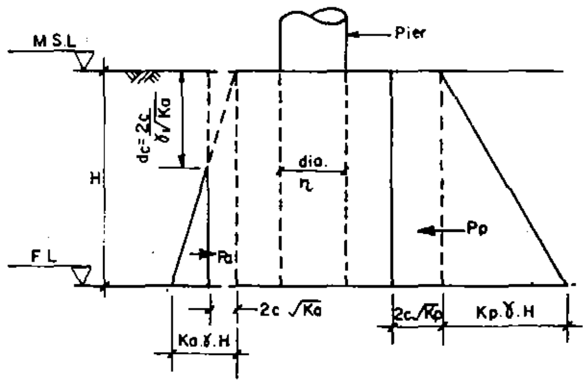


Fig. 13.9

• *Subcase (i)*

If depth of tension crack $\geq H$, i.e.

$$d_c\left(=\frac{2c}{\gamma\sqrt{K_a}}\right) \geq H,$$

then, $M_{\text{dependable}}$ at base

$$= \frac{1}{f}\left\{\frac{1}{6}K_p\gamma H^3 + c\sqrt{K_p}H^2\right\}\eta$$

• Factor 'f' as per (iv) above.

• *Subcase (ii)*

If $d_c\left(=\dfrac{2c}{\gamma\sqrt{K_a}}\right)$ is $< H$,

then, $M_{\text{dependable}}$ at base (approx.)

$$= \frac{1}{f}\left\{\frac{1}{6}\gamma H^3(K_p - K_a) + cH^2(\sqrt{K_p} + \sqrt{K_a})\right\}\eta$$

• Factor f as per (iv) above.

• ***Case 9***

Abutment in non-cohesive soil (Fig. 13.10).

• $M_{\text{dependable}}$ at base

$$= \frac{1}{f}\left\{\frac{1}{6}\gamma H^3(K_p - K_a) - p_o\frac{H^2}{2}\right\}\eta$$

• Factor f from (iv) above.

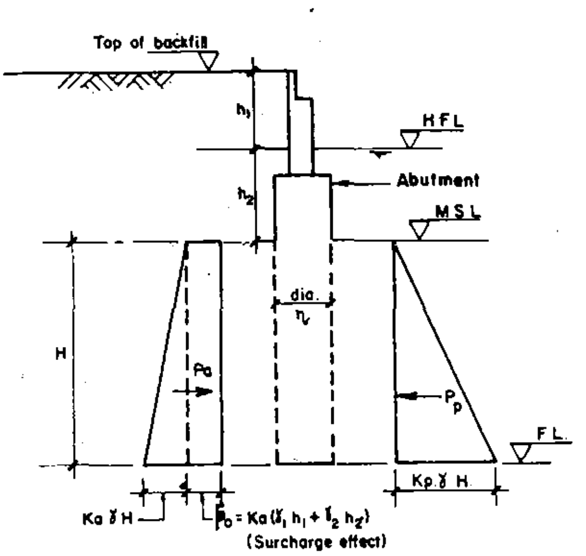


Fig. 13.10

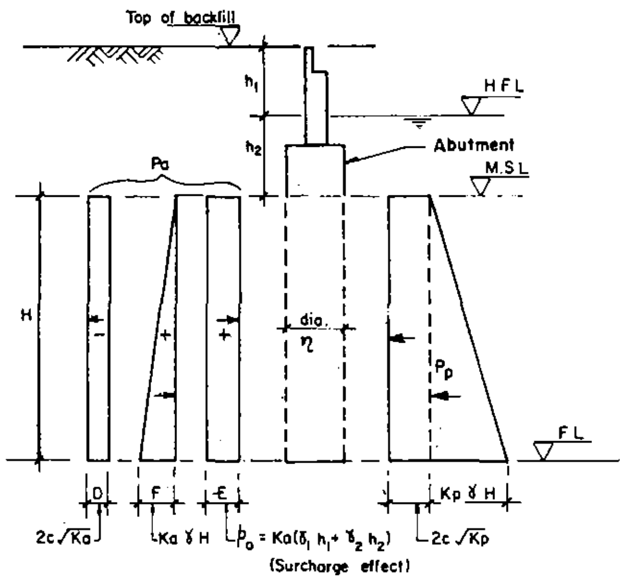


Fig. 13.11

• ***Case 10***

Abutment in cohesive soil (Fig. 13.11).

Tension crack extends down to a depth d_c where net $P_a \equiv 0$, i.e., where

$$E + F = D$$

i.e. $K_a(\gamma_1 h_1 + \gamma_2 h_2) + K_a \gamma d_c \equiv 2c\sqrt{K_a}$

giving $d_c = \dfrac{2c}{\gamma\sqrt{K_a}} - \left\{\dfrac{(\gamma_1 h_1 + \gamma_2 h_2)}{\gamma}\right\}$

• *Subcase (i)*

If $d_c \geq H$, i.e., if $D \geq E + F$,
then P_a is zero, so that

$$M_{\text{dependable}} \text{ at base } = \frac{1}{f}\left\{\frac{1}{6}K_p\gamma H^3 + c\sqrt{K_p}H^2\right\}\eta$$

• *Subcase (ii)*

If $d_c < H$, i.e., $D > E$ but $< E + F$, then P_a exists below d_c, so that,

$M_{\text{dependable}}$ at base (approx.)

$$= \frac{1}{f}\left\{\frac{1}{6}\gamma H^3(K_p - K_a) + cH^2(\sqrt{K_p} + \sqrt{K_a}) - p_o\frac{H^2}{2}\right\}\eta$$

• Factor 'f' from (iv) above.

• *Subcase (iii)*

If no tension crack, i.e., $D < E$, then net P_a is trapezoidal, so that $M_{\text{dependable}}$ at base

$$= \frac{1}{f}\left\{\frac{1}{6}\gamma H^3(K_p - K_a) + cH^2(\sqrt{K_p} + \sqrt{K_a}) - p_o\frac{H^2}{2}\right\}\eta$$

• Factor f from (iv) above.

13.3 ACTIVE EARTH-PRESSURE ON ABUTMENT OR ON RETAINING-WALL, FOUNDED ON FOOTING OR PILES

For active earth pressure on abutments on footings or on small diameter piles (diameter generally less than 1.2 m*) Coulomb's theory may be adopted subject to the modification that the centroid of the pressure exerted by the backfill, when considered dry, is located at an elevation of 0.42 of the height of the abutment above the soffit of footing/pile cap instead of at 0.333 of that height. In this dry condition this modification nearly approximates to the actual state that is between the earth pressure at 'rest' and the limiting 'active' pressure.

The structure shall, however, be designed to withstand a horizontal pressure not less than that exerted by a fluid weighing 480 kg/m³.

* For diameters bigger than about 1.20 m the pile shaft may begin to act more like an earth-pressure-attracting-bulkhead, in which case that might cause flexing of the pile system, thereby altering the state of earth pressure acting on the structure above the piles.

CHAPTER 14

Evaluation of Base-Pressure and Contact-area under Foundations Subjected to Direct Load and Any-Axis Bending

A base, subjected to a direct load Q and moments M_L and M_B about two of its orthogonal axes L–L and B–B, suffers variable base-pressures. The pressure distribution is assumed linear in regular practical design. If there is no uplifting tensile base-pressure, then full base-area remains in contact with foundation-strata underlying the foundation; otherwise, if part base-pressure is of uplifting type and no balancing is done against this tendency, the base-pressures will redistribute on the 'net' contact area (i.e., on the portion of the full base area which still remains in contact with substrata) and laborious calculations may be involved in order to estimate first the net contact area and then the redistributed maximum base pressure on it.

Cases of rectangular (or square) and circular in plan bases have been worked out and results, as an aid to quicker analysis, are presented below. Areas shown shaded in the corresponding figures represent the *kern limits*, so that if the resultant eccentricity of the load Q lies within it, there is no uplift, and the full base-area remains in contact with the substrata.

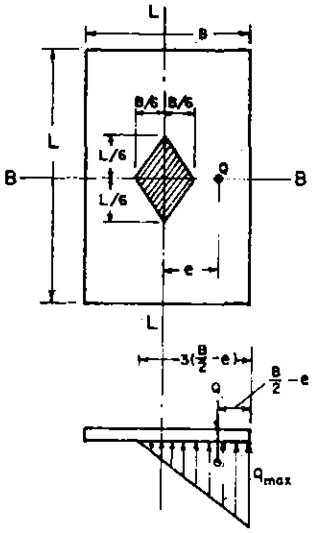
The figures and working details are self-explanatory as are the symbols. However, it may be noted that L–L axis is parallel to dimension L, B–B axis is parallel to dimension B, M_L represents moment about L–L axis and M_B represents moment about B–B axis. Accordingly,

$$e_l = \frac{M_B}{Q} \quad \text{and} \quad e_b = \frac{M_L}{Q}$$

where e_l and e_b represent the eccentricities along L–L and B–B axes if the load Q itself were placed eccentrically to cause the applied moments M_B and M_L, respectively.

Case of Rectangular (or Square) footing with moment applied about only one axis (see Fig. 14.1)

Load Q applied eccentric along B–B axis or footing subjected to M_L (moment about L–L axis). (Same procedure is used when Q applied eccentric along L–L axis or footing subjected to M_B. Only L replaces B in the formulae.)



$A = B.L.$

For $e \leq \frac{B}{6}$, $q = \frac{Q}{A}[1 \pm 6\frac{e}{B}]$

For $e > \frac{B}{6}$, $q_{max} = \frac{Q}{A}\left[\frac{4B}{3B-6e}\right]$

$q_{min.} = 0$ at a distance of $3\left(\frac{B}{2}-e\right)$ from compressed edge of footing

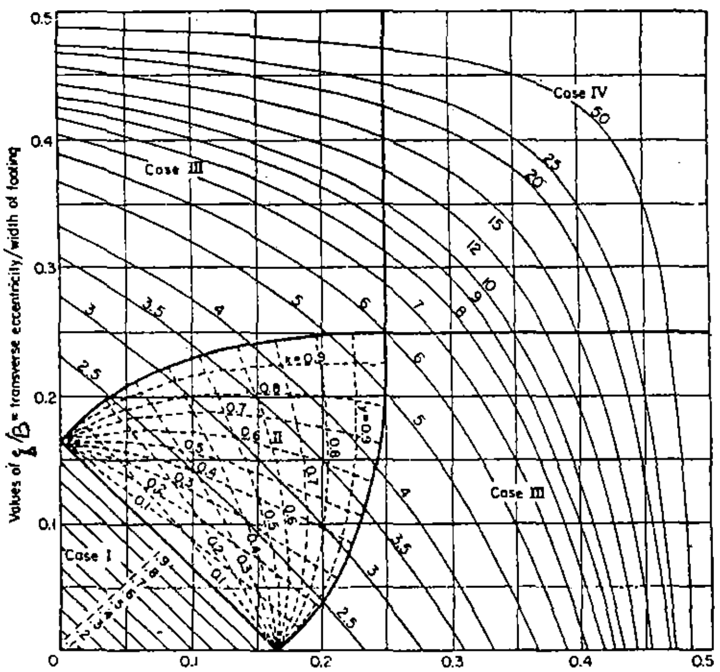
$e = \frac{M_L}{Q}$, i.e., $e = e_b$

Fig. 14.1 *Rectangular footing, load on one of the center lines of footing*

Case of Rectangular (or Square) Footing with Moments Applied about both its Orthogonal Axes (refer to Figs. 14.2 and 14.3)

Case of a Circular Footing with Any-Axis Moment (Vectorial Resultant Moment) (Fig. 14.4)

In case $e > \frac{r}{4}$, so that there is uplifting base pressure (and consequently only part of the full circle in plan is in contact with the substrata) then the net base-area in contact with substrata, a sector, and its properties (e.g., area, position of its centroid, and its second moment of area about an axis parallel to the resultant bending axis and passing through its centroid) can be quickly evaluated from the data given below in Fig. 14.5.



Values of e_l/L = *Longitudinal eccentricity/length of footing* ⟶
Solid curves give values of K
Maximum pressure $P_{max} = K \times Q/BL$
Q = concentrated load on footing

Fig. 14.2

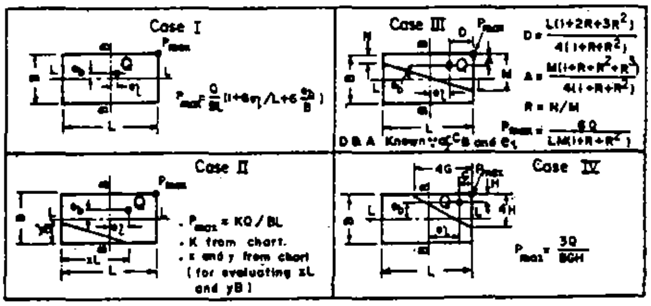
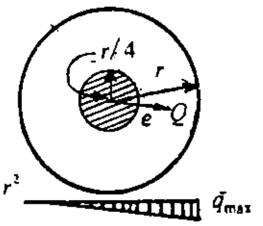


Fig. 14.3 *Rectangular footing, double eccentricity*

For $e \leq r/4, q = \frac{Q}{A}[1 \pm 4\,e/r]$
For $e > r/4, q_{max} = k\frac{Q}{A}$

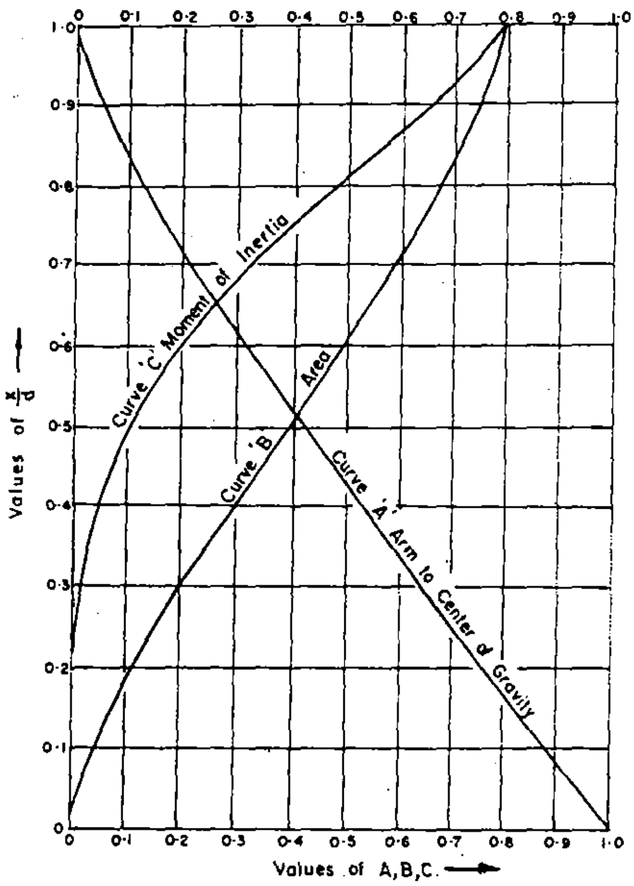


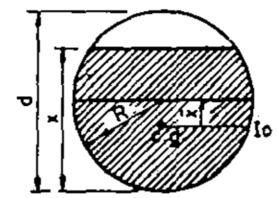
$A = \pi r^2$

k values are tabulated below.

e/r = 0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.90
k = 2.00	2.20	2.43	2.70	3.10	3.55	4.22	4.92	5.90	7.20	9.20	13.0	80.0

Fig. 14.4 *Circular footing*





PROPERTIES OF SHADED SEGMENT

$\bar{x} = AR$

AREA = Bd^2 VALUES OF A, B, C TAKEN

$I_0 = CR^4$ FROM ABOVE CHART

Fig. 14.5

CHAPTER 15

Friction Slab for Stabilising Abutments and Retaining Walls

In principle a friction slab is a mere slab of plain concrete attached to the earth retaining structure on the backfill side by tension rods, and has earth fill on top of it so that any tendency of the structure to move (slide, rotate) would tend to drag this slab along which would immediately generate horizontal frictional force on its top as well as bottom surfaces since the slab is sitting on soil carrying the fill on it. This resisting frictional force would help stabilise the structure against sliding, and even against overturning if the friction slab is at a higher elevation than the foundation base.

15.1 PROCEDURE

Step 1 Decide the elevation at which the friction slab should be attached to the structure. Owing to the tendency of sliding of the earth wedge above the plane of rupture (inclined to the horizontal at $45° + \phi/2$, where ϕ is the angle of repose of the backfill material), the weight on the slab of only that part of soil mass which falls below and beyond this plane of rupture should be considered in evaluating the frictional force generated by the friction slab. Hence higher the elevation of the slab, longer will the slab have to be in order for it to catch the soil mass below and beyond the plane of rupture. Therefore, draw to scale the structure in elevation and the said plane (i.e., line) of rupture behind it, starting this line from the centre of the width of the foundation at the maximum scour level if the soil mass surrounds the structure below that level or from where the back-plane of the main back-wall meets the founding level if the entire soil mass causing the earth pressure is all a backfill and is kept in place by wing walls.

Step 2 Evaluate the weight on the slab of the earth-mass between the plane of rupture and the top of slab, and add to it the self weight of slab. This total load is R.

Step 3 Evaluate the total horizontal resisting frictional force, H, generated by the friction slab.

(i) If the fill above and below the slab is noncohesive then,

$$H = 2(\mu R), \quad \text{where } \mu = \tan\left(\frac{2}{3}\phi\right)$$

(ii) If the soil below the slab is cohesive but the backfill above the slab is non-cohesive (backfill should be a non-cohesive material), then,

$$H = \mu R' + 0.7\ Ac,$$

where A = plan area of the portion of friction slab below and beyond the plane of rupture

R' = R—weight of slab

c = cohesion value of the cohesive soil

$\mu = \tan\left(\frac{2}{3}\phi\right)$

Even so it is advisable to fill at least a meter depth below the slab with noncohesive soil, and saturate-compact it thoroughly, in 15 cm layers, prior to casting the slab on it.

However, only 70% of the above calculated horizontal resisting frictional force H, acting towards the backfill, may be used as the dependable value, and this be introduced as a stabilising horizontal force in the stability and stress-check calculations of the structure.

Step 4 Design the tension reinforcement (that passes the full length of the slab and is well anchored into the main structure) for the above calculated horizontal force H, allowing in it a tensile stress of only 70% of the normal working stress value. (This stress reduction is advisable owing to a slight chance of long term corrosion of the steel under the circumstances.)

Step 5 Detailing.

(i) Since the slab body may bend and crack in case the soil below it should differentially settle, it is advisable to make the friction slab with built-in partial-depth V-grooves (deliberate partial depth breaks) in the two orthogonal directions, sealed with bitumen filler. This gives the slab a checkered appearance, as these grooves break it into a rectangular grid. The main tension reinforcement goes under the grooves normal to it, and additional local parallel bars are placed at the grooves, extending about 60 bar-diameters on either side of a groove. These bars should be provided in level with the main tension bars, one for one, and of same diameter. The main tension reinforcement should be placed at middle of slab thickness.

(ii) As for the secondary reinforcement in the individual checkered panels, placed normal to the main tension reinforcement, under the particular circumstances of embedment, it is almost unnecessary. However, it may be provided at the rate of 25 kg/m^2 of plan area of the panel, in one layer, above or below the main tension reinforcement.

(iii) A slab thickness of 15 cm, cast with concrete of 250 kg/cm^2 standard 28-day cube crushing strength, is ample for the slab. However, the concrete should be thoroughly vibrated and should be dense. The aforementioned partial depth V-grooves may be 6 mm wide and 3 cm deep, from top downwards and from bottom upwards, sealed with bitumen filler/sheet.

CHAPTER 16

Reinforced Earth Structures

Reinforced earth 'wall' is a composite engineering 'mass' consisting of compacted soil, horizontal layers of reinforcement, and a form of facing to prevent erosion of the soil.

The increased use of this material is primarily due to its versatility, cost effectiveness and ease of construction.

Reinforced earth is a relatively new civil engineering material which has been used commercially for the past twenty years or so. Its main use has been in the construction of earth-retaining structures and bridge abutments. But now it is adopted into the field of foundation stabilisation, and its possible use in the future might even include the strengthening of cuttings.

The *external stability* of a reinforced soil mass (called wall) is easily investigated by assuming that it behaves as a rigid gravity mass-structure and conforms to the simple laws of statics. The analysis of *internal stability* is essentially one of designing the reinforcement against tension failure and ensuring that it has a sufficient anchorage length into the stable soil mass.

In 1966 Henri Vidal, a French architect, published a paper entitled *Terre-Armee*. This heralded an intensive programme of research, development and construction, which changed the image of reinforced earth from that of an engineering novelty to a construction technique of major significance. As the name implies, reinforced earth is a composite material which combines soil and strong (comparatively inextensible) tensile reinforcement to produce a mass having superior properties to soil alone. At its inception it was conceived as a construction material having a versatility on par with reinforced concrete, with Vidal illustrating the potentital use of reinforced earth over a wide spectrum of structures including beams, arches, tunnels and dams, as well as various forms of earth-retaining structures.

A simple form of reinforced earth wall is illustrated in Fig. 16.1. Brief descriptions of components listed in the figure are set in reference 1 given at the end of this chapter, from which material has been taken with grateful thanks.

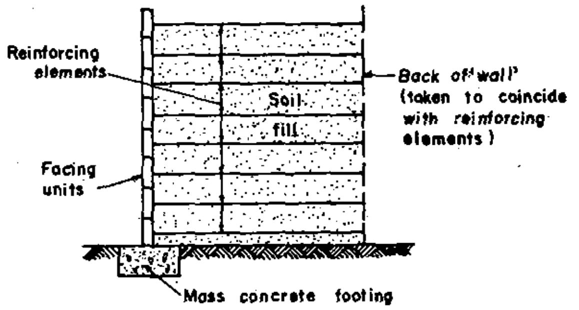


Fig. 16.1 *Reinforced earth retaining wall*

Soil Fill

Three types of soil fill are available.

Frictional fill Most practical reinforced earth structures that have been built have employed frictional or granular soil. The reasons are not far to seek as granular soil not only has good frictional resistance but is also free-draining and generally less corrosive than cohesive soil.

Cohesive frictional soil The UK department of transport memorandum (1978) permits the use of silty soils as reinforced earth-fill, subject to certain conditions listed in it.

Cohesive soil Research is being carried out into the possibility of using clay as a fill for reinforced earth. As clay is probably the most common soil, encouraging results from such research would be of interest, but any potential benefits arising from the local availability of such soil could be outweighed by the penalties that might arise, such as difficulty in handling, development of pore water pressures and a greater risk of corrosion.

Reinforcing Elements

Most of the reinforced earth structures that have been built so far have employed metallic reinforcing elements, the most common being galvanised steel. Each element is a thin strip of metal, typically 50–100 mm wide and up to 9 mm thick and several metres in length.

Other metals from which reinforcement strips have been prepared include stainless steel, aluminium, aluminium alloy and copper. Reinforcing elements made from reinforced concrete in the form of thin prestressed planks have been used occasionally.

Plastic materials show promise for the future, although more research is necessary. Two main types are presently available;

- *fibre reinforced plastic*, consisting of glass filaments *embedded in polyester resin.*
- *paraweb*, polyester filaments embedded in polyethylene.

Facing Units

At a free boundary of a reinforced earth structure it is necessary to provide some form of barrier so that the soil is contained. This 'skin' can be either flexible or stiff, but it must be strong enough to hold back the local soil and allow fastenings for the tension reinforcement to be attached.

The facing of a reinforced earth structure is usually prefabricated from units which are small and light enough to be manhandled for quick and easy construction. The units are generally made from steel, aluminium, reinforced concrete or plastic. Reinforced concrete is more common.

The facing units require a small foundation from which they can be built, generally consisting of a trench filled with mass concrete, giving a footing similar to those used in domestic housing.

16.1 PRINCIPLE OF REINFORCED EARTH

Consider a semi-infinite mass of cohesionless soil at rest. If the surface of the soil is horizontal then, at depth h below the surface,

$$\text{Vertical Stress} = \gamma h$$
$$\text{Lateral Stress} = K_0 \gamma h$$

where K_0 = coefficient of earth pressure at rest,
γ = unit weight of the soil.

According to Jaky (1944), for both normally consolidated clays and compacted soils, $K_0 \simeq (1 - \sin\phi)$ where ϕ = the angle of internal friction of the soil.

If the soil is allowed to expand laterally, the horizontal stress, $K_0\gamma h$, reduces to a limiting (or failure) value, $K_a\gamma h$, where K_a = coefficient of active earth pressure.

$$K_a = \frac{1 - \sin\phi}{1 + \sin\phi} = \tan^2(45° - \phi/2)$$

(ignoring the angle of wall friction)

If the soil is compressed laterally, the horizontal stress increases to the limiting value, $K_p\gamma h$,

$$\text{where } K_p = \text{coefficient of passive earth pressure}$$
$$= \frac{1 + \sin\phi}{1 - \sin\phi} = \tan^2(45° + \phi/2)$$

(ignoring the angle of wall friction)

Now reconsider the soil mass with horizontal reinforcement strips embedded within it (Fig. 16.2).

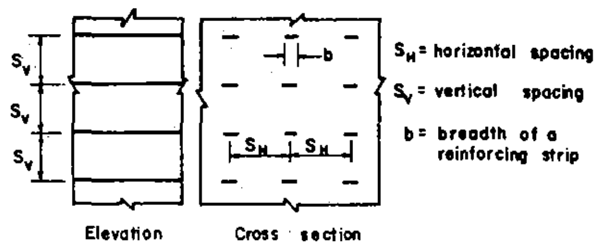


Fig. 16.2 *Arrangement of reinforcement strips*

Consider a soil laye between two reinforcement strips one above other. If enough friction is developed, the top and bottom of the layer will be attached to the reinforcements. If the strips are close enough then the whole soil layer will be more or less constrained and the maximum strain that it can experience in the direction of the reinforcements will be of the order of the strain in the reinforcements.

The types of reinforcement material available are discussed later but all have a Young's modulus much greater than that for the soil so that the resulting strains in the soil will be so small that the soil is essentially at rest and the lateral pressure within it can be assumed to be equal to $K_0\gamma h$. This fundamental idea of reinforced earth, was developed by Vidal in 1966. Reinforced earth, therefore, is a combination of soil which is weak in tension, and reinforcing elements which can carry the tensile forces transmitted from the soil. The composite material is strong in vertical compression and has tensile strength in the direction of the reinforcements. In this respect it is somewhat analogous to reinforced concrete.

It must be remembered that the tensile strength in reinforced earth is directional. If the soil is compressed laterally (instead of vertically), the horizontal reinforcement would have no effect.

The Rankine theory considers the equilibrium of an element in the soil mass and deduces that a network of potential failure planes at $(\phi/2 + 45°)$ to the horizontal, exist behind the wall [Fig. 16.3(a)], for a wall supporting dry sand with a horizontal surface.

The Coulomb assumption considers the whole of the soil mass retained and assumes that failure will occur by a wedge of soil sliding down a failure-surface. For practical purposes the failure surface can be considered as a straight line and, for sand behind a wall, is inclined at $(\phi/2 + 45°)$ to the horizontal [Fig. 16.3(b)].

The active pressure distribution behind a retaining wall approximates to either the Rankine or the Coulomb condition, depending upon the amount and type of yield

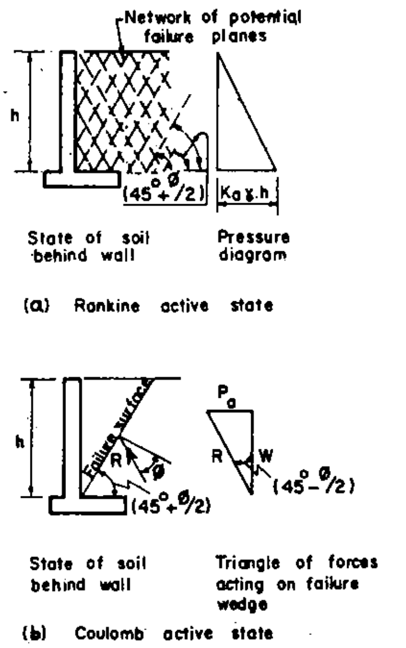


Fig. 16.3 *Rankine and Coulomb active states*

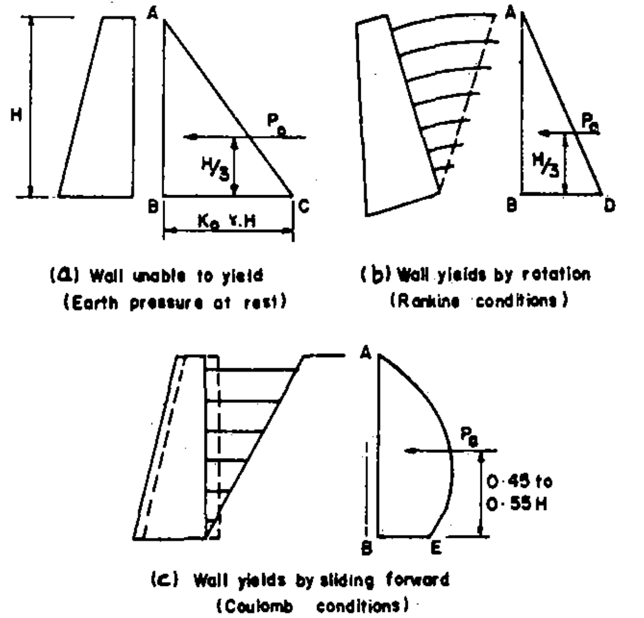


Fig. 16.4 *Influence of wall yield on pressure distributions behind a retaining wall*

that the wall has experienced.[1]

A wall can yield in one of the two ways—either by rotation about its lower edge or by sliding forward. Provided it yields sufficiently, a state of active pressure is reached and the total thrust on the back of the wall is P_a. The distribution that gives this P_a value can be very different and depends upon the way in which the wall has yielded.

Consider first a wall that is unable to yield [Fig. 16.4(a)]. The soil is at rest and pressure distribution is represented by line AC.

Consider now that the wall fails by rotation about its lower edge until the total active thrust is P_a [Fig. 16.4(b)]. This results in conditions approximating to the Rankine theory and is known as the totally active case (K_a applies instead of K_0).

Suppose, however, that the wall yields by sliding forward until active thrust conditions are achieved and the total thrust again equals P_a. A forward displacement hardly disturbs the upper layers of soil so that the top of the pressure diagram is similar to the earth pressure at rest diagram. As the total thrust on the wall is the same as for rotational yield, the pressure distribution must be similar to $AEBA$ in Fig. 16.4 (c). These conditions correspond to the Coulomb theory.

If we consider a frictional fill, angle of friction ϕ, unit weight γ, and assume a frictionless wall of height H, then both theories provide the same value for the total active thrust P_a, provided that the surface of the fill is horizontal and parallel with the back of the wall.

$$\text{Rankine } K_a = \tan^2\left(45° - \frac{\phi}{2}\right)$$

$$P_a = \text{area of pressure distribution diagram}$$

$$= \frac{1}{2}K_a\gamma H^2 = \frac{1}{2}\gamma H^2 \tan^2\left(45° - \frac{\phi}{2}\right)$$

Coulomb Area of the sliding triangular wedge

$$= \frac{1}{2}H^2 \tan\left(45° - \frac{\phi}{2}\right)$$

Weight of sliding wedge, W

$$= \frac{1}{2}\gamma H^2 \tan\left(45° - \frac{\phi}{2}\right)$$

From triangle of forces, $P_a = W\tan\left(45° - \dfrac{\phi}{2}\right)$

$$= \frac{1}{2}\gamma H^2 \tan^2\left(45° - \frac{\phi}{2}\right)$$

Tension Forces in Reinforcement Strips

Whether or not the theories of Rankine and Coulomb should be used in an unmodified form in the analysis of reinforced earth has been argued amongst researchers for several years. However, at this stage of development, there is little choice for designers but to use the Rankine and Coulomb assumptions. It should be remembered that thousands of reinforced earth structures designed with these theories have been constructed and have proved satisfactory. (It should be noted that there are other approaches, a summary of which has been prepared by Symons). In Fig. 16.5: $T = 1/2k_a\gamma H^2$ per m length of wall, where T = total of tensions in all the n tension strips holding the wall.

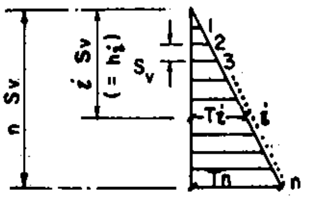


Fig. 16.5

This is the resultant of all the forces in the strips and must be distributed amongst them. This is achieved by assuming a triangular distribution, the forces increasing with depth, Fig. 16.5.

Consider strip number i counting down from top, we have from Fig. 16.5

$$\frac{T_i}{iSv} = \frac{T_n}{nSv}$$

$$\therefore \quad T_i = \frac{T_n}{n} i$$

Now, $T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n = T$

$$\therefore \frac{T_n}{n} + \frac{2T_n}{n} + \frac{3T_n}{n} + \cdots + \frac{(n-1)T_n}{n} + \frac{nT_n}{n} = T$$

$$\therefore \quad T_n(1 + 2 + 3 + \ldots + (n-1) + n) = nT$$

or $$T_n \sum n = nT$$

Now, $$\sum n = \frac{n(n+1)}{2}$$

$$\therefore \quad T_n = \frac{2nT}{n(n+1)} = \frac{2T}{n+1}$$

$$\therefore \quad T_i = \frac{2Ti}{n(n+1)} = \frac{i}{n(n+1)} K_a \gamma H^2$$

For large n, $T_i = iK_a\gamma(H/n)^2 = iK_a\gamma S_v^2$ (16.1)

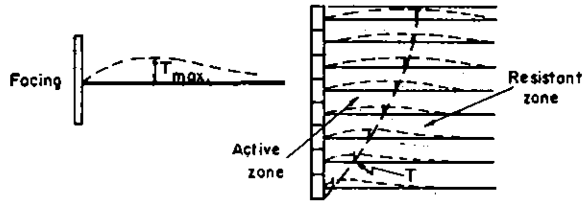
It can also be written as,

$$T_i = K_a S_v(\gamma i \cdot S_v)$$
$$= K_a S_v \sigma_v \qquad (16.2)$$

where $\sigma_v = \gamma i S_v$ = vertical stress at level i where depth is iS_v.

Maximum Tension Line

It has been found, from both model tests and measurements on constructed works, that the tensile force in a reinforcement strip varies. It generally has a low (even a zero) value at the facing unit, reaches a maximum value at a short distance from the facing, and then tends towards zero at the unattached end, Fig. 16.6(a).



(a) Variation of tension along strip (b) Maximum tension line

Fig. 16.6 *Maximum tension line*

If the points of maximum tensile force in various strips are joined, the imaginary line so formed is known as the maximum tension line, Fig. 16.6(b). It extends in a curve from the facing at the base of the wall and cuts the surface of the fill at some distance back from the facing. It is assumed that this line divides the reinforced fill into two zones, one on either side of it.

- *Active zone* in which the shearing stresses from the soil, on to the strips, act towards the facing, i.e., there is a tendency for the reinforcement to be pulled forward towards the facing.
- *Resistant zone* in which the shear stresses act away from the facing and tend to hold the reinforcement in the soil.

T_i as obtained from Eq. (16.1) is assumed to be the maximum tensile force in the strip, and therefore, occurs at some distance back from the facing.

Failure of a Reinforcing Strip

The tensile force in a reinforcing strip will tend to cause failure in one of the following two ways:

(i) Tensile failure—snapping of the strip.
(ii) Bond failure—slipping of the strip from within its surrounding soil.

Tensile Failure

The ultimate resistance of a reinforcing element to an axial tensile stress is equal to the ultimate axial tensile stress that the material can withstand ($p_{ult.}$) times the cross-section area of the strip,

$$R_t = p_{ult.} \times b \times t$$

where b = width of reinforcing element
t = thickness of reinforcing element.

Dividing p_{ult} by a suitable factor of safety gives the permissible axial tensile stress p_{at}.
Hence, when considering tensile failure effects, $T_i \ngtr p_{at}bt$.

Typical p_{at} values are,

	N/mm^2
Aluminium	120
Galvanised mild steel	120–190
Copper alloy	170
Stainless steel	120–220

Bond Failure

Just as a reinforcing bar in concrete requires a certain minimum bond length, so also a reinforcing element in reinforced earth must be long enough to prevent its slipping and pull out.

For a reinforcing element, the bond resistance, between it and the soil around it will be provided by one of the following three means, depending upon the type of soil fill,

(i) Frictional fill — by frictional resistance
(ii) Cohesive fill — by adhesion
(iii) Cohesive-frictional fill — both by frictional resistance and adhesion

(i) *Frictional resistance* Consider a strip at depth h_i and assume that the coefficient of friction between the soil and the reinforcing elements is μ.

$$\text{Normal stress acting on strip} = \text{vertical stress} = \gamma h_i$$
$$\therefore \text{Normal force acting on strip} = \text{normal stress} \times \text{area} = \gamma h_i bL$$
$$\therefore \text{Total frictional resistance available from strip} = 2\gamma h_i bL\mu$$

NOTE Just as the total peripheral area of a reinforcing bar in concrete is considered to provide bond resistance, so also the total area of the tension strip, i.e., its upper and lower surfaces are considered to provide bond resistance.

(ii) *Cohesion* Total resistance available from cohesion in a cohesive soil = $2cbL$.
where c = unit cohesion available along the length of the reinforcing element

NOTE μ and c values are obtained from shear box tests but μ is typically approximately equal to $\frac{1}{2}\tan\phi$ (some authorities take it as $\tan\left(\frac{2}{3}\phi\right)$, and c will lie somewhere between $\frac{1}{2}c_u$ and $1.0\,c_u$(c_u = ultimate value of cohesion).

(iii) *Friction and adhesion* Generally for a $c-\phi$ fill: Bond resistance of a reinforcing element = $2(\gamma h_i\mu + c)bL$.

16.2 CURRENT DESIGN AND CONSTRUCTION SYSTEMS

Three proprietary systems are currently used in the UK. These are the reinforced earth system (due to Vidal), the DoE or York system (due to Jones), and the Websol system (due to Price). The former system is used universally, with over 3500 structures having been erected by Vidal's licence. In comparison, the latter systems have to date, been used largely mainly in the UK where approximately 200 structures have been built. Both the Vidal and the York systems have been bolstered by extensive research and development programmes culminating in the issue of formal design directives'. The basic requirements are the same in all three systems with the need for a facing unit to prevent surface erosion, a series of reinforcing strips or sheets, and a suitable backfill. Additionally, it is necessary to incorporate a mechanism which permits the fill and associated reinforcing strips to settle without inducing unacceptable stresses in the facing units and the reinforcing strips. With the advent of the UK department of transport technical memorandum, a number of ad hoc systems have appeared. However, these tend to lack the finesse of the proprietary systems.

Vidal developed two systems for constructing reinforced earth walls. The first comprises semi-elliptical cross-section facing units, typically 250 mm high, which have a locating slot formed along the bottom edge. Reinforcing strips are connected to the units by bolts passing through the strip and the interlocking edges of the facing units. The standard units are straight, measure up to 10 m long and weigh 115 kg. Shorter units and specials are supplied to form corners. Mild steel and galvanised mild steel are standard construction materials, these being typically 1.5 mm to 3.0 mm thick. These thicknesses are consistent with a vertical unit stiffness that allows flexure under vertical load. If the backfill and reinforcing strips suffer internal settlement, this vertical movement is reflected in the facing units which compress like a bellows, so obviating high stresses at the reinforcement connections that would otherwise be induced by differential settlement between the fill and the facing units.

The metal facing unit has now been largely superseded by a more substantial precast concrete unit which is cruciform shaped in front elevation. Standard units weigh approximately one tonne and are 1.5 m by 1.5 m with a total thickness of 180 mm. All edges of the unit are rebated to obviate any straight through joints, with the rebates doubling as guide rails to facilitate alignment of the units during construction. A further aid to alignment is in the form of a dowel bar extending from the upper and lower edge of one arm of the cruciform. These dowels are also used as pivot points for the construction of curved walls. Each unit is furnished with four steel lugs, cast *in situ* during manufacture. These lugs, which are usually at 1 m horizontal and 0.75 m vertical centres, are drilled to take the reinforcing strip connecting bolts. During construction a strip of compressible filler, such as cork board, is laid on the back edge of the horizontal joints before the next unit is placed. Frequent use is made of temporary wedges to form an open joint and aid vertical alignment. These construction techniques allow the facing unit to compress vertically in sympathy with any internal settlement of the fill.

The reinforcing strips are almost exclusively metal, usually galvanised steel. Unit 1975 plain strips, 60 mm or 80 mm wide and 3 mm thick were in common use. These were subsequently superseded by ribbed strips, 40 mm or 60 mm wide and 5 mm thick. In extremely corrosive fill environments, stainless steel may be used. The effect and rates of corrosion are still not totally predictable, however, recent research work suggests that the 5 mm thick galvanised steel strips offer a service life in excess of 100 years in all but the most aggressive environments, Darbin Etal (1978). Suitable fill material is generally of a granular nature with a limit of no more than 15% finer than 80 microns. The maximum particle size is restricted to 350 mm with no more than 25% of the fill being coarser than 150 mm, Long (1977).

The current design methods adopted by the Reinforced Earth Company have been set out in the papers presented at the 1978 Sydney conference by both Schlosser and Mckittrick.

The York System, which was largely developed by Jones, initially used facing units made of glass reinforced cement, providing a very light weight of 18 kg per unit. The units take the form of a hexagon-based pyramid 225 mm deep and 600 mm across the flats. One pair of diametrically opposite flanges on each unit is provided with a pair of large diameter holes which allow the units to be threaded onto vertical guide poles. These poles which serve as face reinforcement, are made up of short lengths of 35 mm diameter PVC tubing with spigot and socket connections. In the finished structure the poles are reinforced with mild steel bars, grouted *in situ* to make them rigid. The earth reinforcement, in the form of strips or plastic grids, is attached to the vertical pole reinforcement at any required vertical spacing. When any settlement occurs in the fill containing the reinforcement, the reinforcement simply slides down the vertical pole obviating any settlement-induced stresses at these connections.

Recent structures using the York sliding system have used full height facing units formed from double T bridge or flooring beams, standing on edge. The use of the vertical plastic poles can be retained; alternatively a short slot or pin is used to provide a critical movement connection for the reinforcement.

All aspects of the design and specifications for the component parts of reinforced earth walls are clearly set out in the UK department of transport document—Technical Memorandum (Bridges) BE3/78. Permitted reinforcing materials include aluminium alloy, galvanised carbon steel, copper and proprietary material awarded the UK Agreement Board Certificate. Two reinforcing strips falling into this latter category are *Fibretain* (a glass-fibre reinforced plastic) and *Paraweb* (a linear composite of Terylene fibre cores in an alkathene sheath). Allowance made for corrosion of the metallic reinforcement during the specified 120 year design life is dependent on the class of backfill used: Table 16.1.

Table 16.1 *Corrosion Allowances (Sacrificial Thickness)*

	Thickness to be allowed for on each surface exposed to corrosion (mm)	
Material	Frictional fill	Cohesive frictional fill
Aluminium alloy	0.15	0.30
Copper	0.15	0.30
Galvanised steel	0.75	1.25
Stainless steel	0.10	0.20

Both frictional and cohesive-frictional fill are limited to a maximum particle size of 125 mm. However, it is specified that frictional fill shall not contain more than 10% passing the 63 micron sieve. Conversely, the so-called cohesive-frictional fill may contain more than 10% finer than this size provided the liquid limit and plasticity index do not exceed 45% and 20% respectively. However, the clay fraction, i.e., 2 microns and finer, is limited to a maximum of 10%. The coefficient of friction between the soil and the reinforcement may either be measured directly using the shear-box or taken from the expression, $\mu = \alpha \tan \phi$, where α is in the range 0.45–0.50. The lengths of the various reinforcements are determined by calculation, however, in no circumstance is the reinforcement length to be less than the greater of 0.8 H or 5 m, H being the wall height.

The Websol *Paraweb* system differs from the Vidal and the York systems described previously in as much as it incorporates non-metallic reinforcement and thus eliminates the problem of corrosion as such. Each layer of reinforcement is a composite comprising a continuous layer of permeable non-woven fabric combined with strips of *Paraweb* which are placed doubled so as to form a loop at the wall end, with the loop secured by a short toggle bar passing through a pair of metal eyes cast into the back of the facing unit. The reinforcing strips are made from high tenacity synthetic fibres encased in a durable polyethylene sheath with the strength of the strip being developed in the fibres whilst the sheath gives the strip its required form, shape and protection. The fabric employed is one of a range of geotextiles having well-defined strengths and durability properties. The system has been approved by the UK Agreement Board and thus should conform to the UK department of transport design specifications which require a 120 years design life. (For author's comments see ahead). The facing panel is of precast concrete, 120 mm thick and is T-shaped in its front elevation. These units are slightly larger than those in the Vidal system with a face area of 3.2 m^2, as opposed to 2.2 m^2. Settlement-induced stresses between the facing panel and the reinforcement are again relieved by the provision of compressible cork packing along the horizontal panel joints.

Although the paraweb tension strips, which are polyester filament protected in polyethylene sheathing, have a certificate from the Agreement Board in UK and have built some reinforced earth structures, the performance of such strips over long periods of time under continuous moist conditions is still questionable, despite their claims to the contrary. Moisture is reported to seep through 1 mm thickness of polyethylene in about 6 months time (Ref: Study of Tergal Webbing at Beaulieu near Poitiers, France, *Terre Armee Report* 1, Feb. 1982). Thus the filament does get moist and tensile strength seems to deteriorate badly when moist. The above report says, "the strength is reduced by about 50% in just 10 years". For this purpose although *sealing* of the ends of the strip is done, apart from the fact that moisture seeps through polyethylene anyway, direct water ingress is possible through pores caused by pitting due to penetration of sharp soil particles during compaction (and the consequent biodegradation). Hence the use of plastics is not entirely unquestionable. Heavily galavanised (1000 gm/m^2) mild steel strips (with increased sacrificial thickness) are preferable, depending on individual merits (e.g., location, subsoil water level, proximity of sea water, atmospheric conditions, etc.).

Design Criteria

Basically the design of reinforced earth type walls and abutments can be done as per method outlined in:

(i) the French Ministry of Transport Recommendations and Rules of the *Art of Reinforced Earth Structures*, (Reference 3).

(ii) the UK Dept. of Transport Technical Memorandum, BE 3/78 (Reference 4).

Two major aspects must be considered when designing a reinforced earth retaining wall:

(i) *Internal Stability* within the reinforced earth monolith,

- (a) Tensile resistance
- (b) Bond resistance
- (c) Pressure on facing.

(ii) *External Stability*, i.e., stability of the reinforced earth monolith,

- (a) Rotational type failure
- (b) Failure by sliding as a rigid body
- (c) Bearing failure of supporting soil under the rigid body.

Internal Stability

There are two major approaches by which internal stability can be checked:

(i) Analysis involving the local stability of individual reinforcing elements,

(ii) Analysis involving the overall wedge-block stability of the reinforced earth monolith.

Method (i) has the advantage of being quick to use and the general practice appears to be to design a reinforced earth retaining wall by that method and then to check the result by method (ii), modifying the initial design if necessary.

• *Check for internal stability by method (i), i.e., by the local internal stability check method* For a wall such as illustrated in Fig. 16.1, with a granular fill, the maximum tensile force in the reinforcing elements at level i is obtained from Eq. (16.1) assuming S_v as the element height:
$T_i = iK_a\gamma S_v^2$/m run of wall (minus $2cS_v\sqrt{K_a}$ if cohesive frictional fill).
This T_i has to include the effect of at least the following (a), (b) and (c):

(a) *Treatment of 'uniform surcharge' on top of wall.* If the wall is uniformly loaded by a surcharge of q/m^2, there will be an increase in T_i due to the uniform pressure distribution K_aq induced within the soil.
Then $T_i = K_aS_v(\gamma iS_v + q)$/m length of wall (minus $2cS_v\sqrt{K_a}$ if cohesive frictional fill).

(b) *Treatment of 'line load' on top of wall.* Schlosser and Long carried out measurements in reinforced earth fill materials, both on models and on an experimental wall, to guage the effect of a line load acting on the surface of the fill and found that the load spreads through the reinforced earth at a slope (V/H) between 2/1 and 1/1. They proposed the following design assumptions:

If line load = S_L per unit length along wall, and its point of application is distance d back from the facing, then (referring to Fig. 16.7),

Vertical stress at level i (at depth h_i) due to $S_L = \dfrac{S_L}{d + 1/2h_i}$

Hence, increase in T_i due to $S_L = K_aS_v\dfrac{S_L}{(d + 1/2h_i)}$

(per unit length along the wall), taking $d \not> 0.5h_i$

The method becomes conservative when, as is usually the case, the line load can be considered as being applied through a continuous footing or the equivalent, e.g., railway track.

If the load is applied concentrically, then the spread width D_i at level i will be $(d + B + 1/2h_i)$ where

B = width of footing (Fig. 16.7b).

If S_L is applied eccentrically to the footing, it can be assumed that the bearing pressure distribution beneath it is trapezoidal. Then, for simplicity, one can assume that the maximum value of bearing pressure applies uniformly beneath the footing, i.e., that a uniform vertical pressure of,

$$p_{\max} = \frac{S_L}{B}\left(1 + \frac{6e}{B}\right)$$

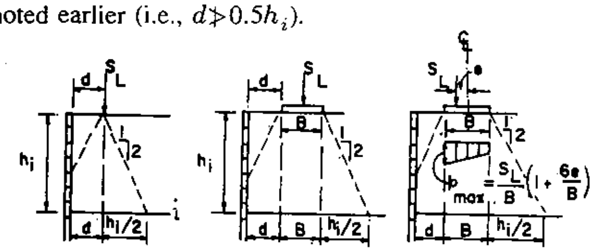
acts at the top of the wall over the distance B [Fig. 16.7(c)]. At depth h_i, the spread D_i, is again $(d + B + 1/2h_i)$ and the vertical pressure, σV_L can be taken as $[p_{\max}.(B/D_i)]$

or $$\sigma V_L = \frac{S_L}{D_i}\left(1 + \frac{6e}{B}\right)$$

Hence, increase in T_i due to S_L (over element height S_v)

$$= K_a S_v \frac{S_L}{D_i}\left(1 + \frac{6e}{B}\right)$$

where d in the expression for D_i has the same limitation as noted earlier (i.e., $d \not> 0.5h_i$).



(a) Unspread concentrated load (b) Concentric load (c) Eccentric load

Fig. 16.7 *Treatment of line load*

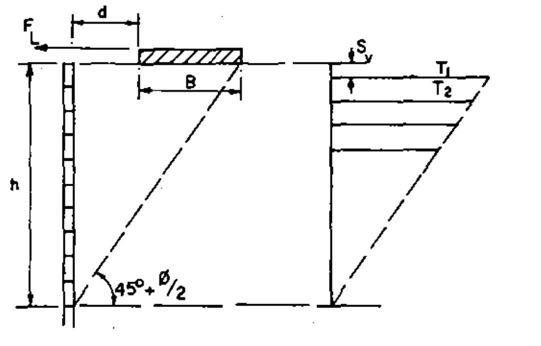
(c) *Treatment of horizontal force at top of wall.* Quite often the traction forces of machinery running along the top of the wall can induce a horizontal shear force, F_L, which is applied through some form of foundation of width B, and may be regarded as continuous along the length of the wall (Fig. 16.8).

Treatment is to assume that the force F_L is carried by the tension elements which fall within the Coulomb wedge passing through the edge of the width B [Fig. 16.8(a)].

It is assumed that there is a triangular distribution of forces, decreasing with depth.

If h = height of Coulomb wedge, then

$$h = \frac{(d + B)}{\tan(45° - 1/2\phi)}$$



(a) Coulomb wedge (b) Triangular distribution of tensile forces

Fig. 16.8 *Treatment of horizontal force*

As tensile force distribution is assumed to be triangular [Fig. 16.8(b)].

$$\frac{T_1}{h - S_v} = \frac{T_i}{h - iS_v}$$

$$\therefore \quad T_i = \frac{T_1(h - iS_v)}{(h - S_v)} \tag{16.3}$$

Now $$T_1 + T_2 + T_3 + \ldots + T_n = F_L$$

$$\therefore \quad T_1[(h - S_v) + (h - 2S_v) + (h - 3S_v) + \cdots + (h - nS_v)] = F_L(h - S_v)$$

or $$T_1\left[nh - \frac{S_v}{2}n(n + 1)\right] = F_L(h - S_v)$$

Since $n = \dfrac{h}{S_v}$, and if this substitution is made, the expression for T_1 becomes,

$$T_1 = \frac{2S_v F_L}{h}$$

and, substituting this in Eq. (16.3), we obtain,

$$T_i = \frac{2S_v F_L}{h}\left(\frac{h - iS_v}{h - S_v}\right) = \frac{2S_v F_L}{h}\left(\frac{h - h_i}{h - S_v}\right)$$

The S_v term in the denominator can be neglected with little loss in accuracy, hence,

$$T_i = \frac{2S_v F_L}{h}\left(1 - \frac{h_i}{h}\right), \text{ due to horizontal force } F_L.$$

Thus having evaluated total T_i (i.e., basic value, duly increased by the effects (a), (b) and (c) detailed above), a check is made to ensure against tensile failure and pull-out failure for each strip i. The required reinforcement perimeter and length per metre run of wall is then calculated.

• *Check for internal stability by method (ii), i.e., by the wedge block stability check method* As has been discussed earlier, the maximum tension line more or less marks the division of a reinforced earth fill into an active zone and a resistant zone. The block stability analysis assumes that the active zone consists of a failure wedge tending to pull away from the rest of the fill, whereas the resistant zone by gripping the ends of the reinforcing elements in the fill is anchoring the active zone in position.

Form of the failure wedge (or active zone): As a result of extensive observations by many research workers on both models and actual structures, it is now generally accepted that just before failure, a reinforced earth retaining wall develops some form of *failure wedge* and that the failure surface tends to follow the maximum tension line.

Obviously, if the exact shape of the failure wedge were known, it would lead to a relatively straightforward design method. However, as Schlosser points out, the boundary between the active and the resistant zones is variable, and depends upon the geometry, stress value, settlements within the subgrade and, possibly the most significant of all: the value of the factor of safety for the particular structure.

It must be remembered that the maximum tension line in a reinforced earth structure does not have a unique position; it varies, daily with the loading and the state of the weather.

For the almost theoretical case of a wall similar to that in Fig. 16.1, i.e., with an unloaded horizontal surface, the maximum tension line is nearly vertical for about one half the height of the wall, Fig. 16.9(b), an idealised form of which, suitable for design purposes, is shown in Fig. 16.9(c).

As can be seen, the volume of soil in the active zone of such a wall is considerably less than that contained in the Coulomb wedge [Fig. 16.9(a)]. Obviously, if the reinforcing elements are all of the same length, a much larger aggregate length is available for bond if the active zone is as in Fig. 16.9(c) than if the active zone approximates to the Coulomb wedge of Fig. 16.9(a). The question is whether it is safe to use the assumption of Fig. 16.9(c) for all the loadings that can be applied to a reinforced earth-retaining wall.

In view of Schlosser's findings such an assumption could be wrong. Murray observes that for the more complex situations involving concentrated loads and sloping backfill, a plane surface of failure is unlikely—it is more likely to be curved. Murray advocates, in view of the uncertainties involved, that the most direct approach to a solution would involve the analysis of a number of trial surfaces in order to determine the maximum thrust, as in normal retaining wall design.

Although the trial failure surfaces can be of any shape, the use of straight lines makes calculations easier.

Once the stability of each and every layer of

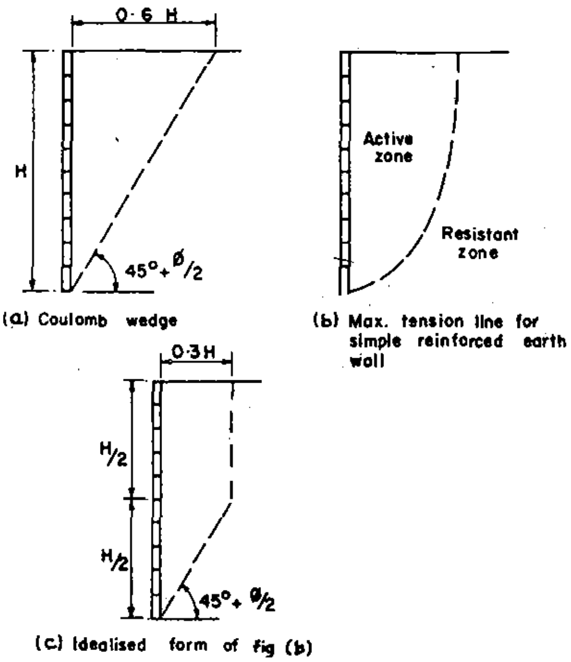


Fig. 16.9 *Form of failure wedge*

reinforcement has been checked as shown earlier, the overall stability of several trial wedges is actually checked using a *graphical method.* The proposed technique is illustrated in Fig. 16.10 for a simple wall loaded by a uniform surcharge q and the self-weight of the backfill. As can be seen, a family of potential failure surfaces are assumed to originate from the face of the wall at a depth h_i. Several inclinations are assumed for the failure surface i.e., β_1, β_2, etc. For each value of β a triangle of forces is drawn to determine T the total tensile force to be resisted by the reinforcement cut by the failure plane under consideration. By evaluating T for several trial values of β it is possible to determine a critical value of β associated with a maximum value of T. This maximum value of T is compared with both allowable *tensile* and allowable *pull-out* resistance of the reinforcing strips within the said depth h_i. In this case, the length of each reinforcement considered is the effective bond length extending beyond the potential failure plane under consideration.

Satisfied that the stability is ensured at depth h_i, a further family of potential failure planes is investigated for another value of h_i. In fact, the BE 3/78 memorandum implies that up to five locations be checked down the face of the wall.

External Stability

The reinforced earth mass (behind the facing units) is assumed to act as one rigid monolith on which act its self weight, the applied loads, and the horizontal earth pressure

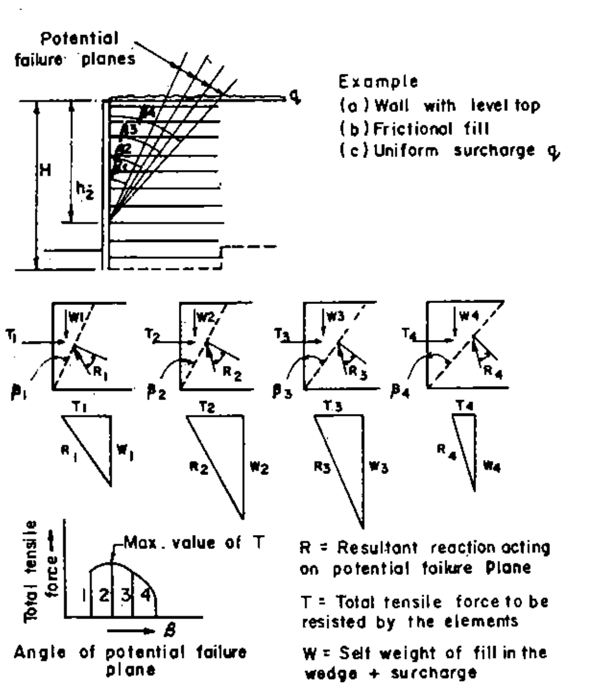


Fig. 16.10 *Wedge stability check calculation (Taken from UK Dot BE 3/78)*

force at its rear-end (from the soil there-beyond). Its stability against sliding, against base pressures and against overturning has to be computed and ensured in the same manner as in the case of regular rigid reinforced concrete wall. The following may be noted:

- Factor of safety against sliding, $\mu W/H'$, should not be less than 2.0 (where W = total weight of the reinforced earth mass including any loads on it, μ = coeff. of friction against its sliding at its base, generally 0.5 and H' = total horizontal thrust on the back of the wall due to the retained material beyond and behind it.
- As per British practice the base pressures are calculated without allowing any uplift (i.e., tension), and the distribution is assumed linear (trapezoidal generally, but in the limit, triangular on full contact area). French practice recognises that since a reinforced earth wall is essentially flexible, the assumption of rigidity in bearing capacity calculations, described above, could be very unrealistic, particularly with weak subgrades where a more uniform bearing pressure distribution could be expected, justifying Meyerhoff's approach of partial uplift and redistributed rectangular pressure distribution. (Thus, there is a clear difference here between the British and the French practices.)
- Factor of safety against overturning about toe should also not be less than 2.0. Where topography so demands, check may be made against rotational type failure by the usual slip circle methodology.

Design Calculation

Design of Tension Strips and Stability of Reinforced Earth

Although fairly simple in principle, the actual arithmetics can be very lengthy, depending on the number of tension strips and the geometry, etc. This is, therefore, best done by using the commercially available computer programs, some of them are already run at some universities (e.g., Edinburgh, UK). However, for the sake of getting a feel of the mensuration, reference may be made to the numerical example given in Reference 1. Following may be noted:

- In these design calculations it may be noted that the bond resistance requirements in the upper reaches of the wall necessitate longer lengths of reinforcement strips than are required further down.
- It is apparent that the design can be made sophisticated by designing the wall in sections and using shorter reinforcement strips nearer the base, thus effecting a saving in reinforcement strips. But whether this effort would be worthwhile, bearing in mind the extra site supervision that would be required, is a matter for the designer's/contractor's understanding.
- Obviously the design procedure outlined above is very much on the safe side. However, this is not necessarily bad when one considers that the calculations cannot take account of the possible horizontal stress increases (possibly far above 'active' values) that can be induced by the compaction taking place during construction.
- For a frictional fill, with its immediate drainage, the long term and short term stability values are the same and the ϕ value used in calculations will be with respect to effective stresses.
- If a cohesive frictional fill is to be used, the ϕ and c value in the calculations must be with respect to effective stresses. This will give the long term stability of the wall but it may be necessary to check the possibility of pore water pressures affecting the short term properties of the soil fill.

Design of Facing Units

It is impossible to assess accurately the values of the true horizontal pressure acting on the back of the facing units.

As has been indicated earlier, the tensile force in a reinforcing element at the facing of a wall can have a low or a high magnitude (a state of affairs related to the relative length of the element).

A further indeterminate factor is the magnitude of the horizontal stresses that are induced during construction compaction.

With the present state of knowledge a designer has little option but to design the facings to withstand full active pressure values, i.e., to act as an *anchor plate* for the maximum tensile force in the reinforcement elements fastened to it.

It should be noted that a facing unit must also be substantial enough to withstand the weight of other units placed above it and that there is always the possibility of localised bending and shear effects if the fill settles down, relative to the facing.

This latter effect can be alleviated somewhat by the provision of some form of sliding joint, which illustrates a further advantage of the use of metallic strips. They are so much simpler to attach to the facing units than most other forms of tension strip reinforcement.

16.3 A NOTE ABOUT COSTING, CONSTRUCTION TIME, AND CARE NEEDED DURING CONSTRUCTION

Costing

Cost of a reinforced earth retaining wall, compared to that of a regular rigid RC wall, can be 20% to 30% cheaper provided the considerations of corrosion and availability of the appropriate fill material are not prohibitive. It is wrong to generalise that such structures are always cheaper. This is simply not true. Depending on the site and space clearances, such a structure, in the long run in certain cases may prove costlier in view of corrosion and the chances of dislodgement in the event of hits from trucks.

In fact the actual cost can well prove high unless the costs of both the alternative structures (reinforced earth as well as the rigid RC wall) are sought before the award of work and analysed. With increased facing area, the cost could actually come down, but the contractor may cleverly so apportion the cost of just even the moulds (for the facia units) that the final bill proves costlier. Hence a thorough cost analysis check is called for at the time of award of work.

Normally, the cost of reinforced earth structure involves the following:

(i) Detailed design and drawings (preparation of)
(ii) Anchor bars and dowels for shoulder connections
(iii) Closed cell polyethylene foam for vertical joints
(iv) Resin bonded cork for horizontal joints
(v) Moulds for precasting facing panels ($\approx$ 1 no. per 1000 m^2 facing area)
(vi) Lifting anchors (2 per panel)
(vii) Toggles and loops or tie strips, nuts and bolts
(viii) Facing units
(ix) Tension strips (reinforcements)
(x) The fill material (earth)
(xi) Compaction
(xii) Expert supervision, including various quality control checks.

In 1984 the cost, for instance in Riyadh area, for about 6.0 m tall reinforced earth abutments, designed to the French Practice, worked out to approximately SR 300/m^2 of facing area. It was also stipulated as follows,

SR	90/m^2	of facing elements, plus
SR	17/m	of 60 mm wide strip*, plus
SR	11/m	of 40 mm wide strip*

Construction Time

This is obviously lesser than that for the regular reinforced concrete rigid structure, but depends on,

- rate of fill delivery (depends on the lead)
- rate of placement
- rate of compaction

Care Needed During Construction

Considerable care has to be taken, during execution, about the following points:

- quality of fill material (clean/frictional, or as specified in design/preferably free draining type/should not contain more than permissible amounts of any salts, acids, alkalies, mineral oils, fungus, microbes, moisture and any other deleterious matters/and should be of specified pH value and specified resistivity)
- method of compaction
- thickness of soil layers
- construction procedure
- proximity of compactors to facing wall
- connection of strips to facing elements
- breakage and mishandling of elements, connections and strips
- deliberate and adequate backleaning of facing panels prior to compaction
- proper use of horizontal jointing (resin bonded cork) and vertical jointing (closed cell polyethelene foam) materials
- need for separation between the reinforced fill and the natural fill behind it in case the latter contains such of the impurities as may leach into the reinforced fill by capillary action and excite corrosion of tension strips.
- removal of slack from the tension strips.

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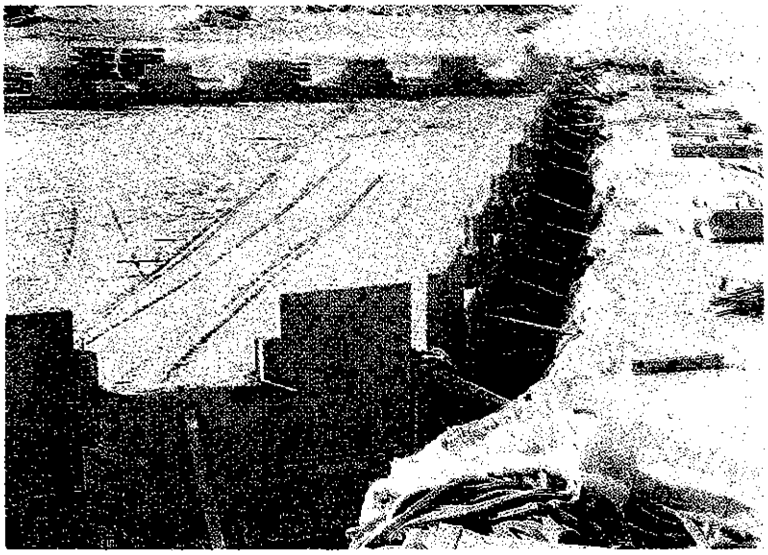
* 5 mm thick galvanised mild steel strip.

Placing the
precast facing units



Placing the
precast facing units

Abutment
nearing completion

16.4 THE BRITISH VERSUS THE FRENCH CODES OF PRACTICE

The British code BE 3/78 of the UK DOE (DOT) is based on the use of working loads and permissible stresses, and requires the actual measured values of soil parameters (density, internal friction angle ϕ and cohesion c) to be used in the design calculations. Moreover, design to BE 3/78 is for the ultimate condition in that the coefficient of active soil pressure is used throughout in the calculations of horizontal thrust due to the fill material. It provides, as it admits, a conservative approach to the design of reinforced earth structures.

The French code does not require the actual measured values of soil parameters to be used in the design calculations. Also, design to the French code is for the working condition in so far as the horizontal thrust due to the fill material is calculated as it uses a coefficient which varies between the coefficient of earth pressure at rest and the active soil pressure coefficient, depending on the depth of soil being considered. The French code also requires that a design angle of internal friction, $\phi = 36°$, be used in the design calculations, subject to certain specified particle size distribution criteria being met and subject to the compaction of the reinforced earthfill being carried out in accordance with the stipulations in the French code requirements. If the criteria are not met the fill material is deemed to be unsuitable for use. Figure 16.11 describes the mechanical criteria for the backfill material.

A Brief Comparison Between the British and the French Codes The British practice seeks a greater 'sacrificial thickness' for the tension strip, requires 1000 g/m^2 of galvanisation of mild steel tension strips (cf. 500 g/m^2 by the French practice); assumes rigid body behaviour of the reinforced earth fill, allowing no tension in base pressure (cf. uplift allowed and distribution as per Meyerhoff by the French); assumes 'working loads' and 'allowable stress' (cf. limit state analysis by the French); assumes active pressure coefficient for the earth pressure thrust (cf. the French practice of adopting earth pressure between that 'at rest' and the 'active' value, depending on the depth at which calculation is being made); uses the soil properties of the soil actually used at site (cf. the French practice of directly assuming the friction angle as 36° but specifying the particle sizes and their distribution and the compaction requirements).

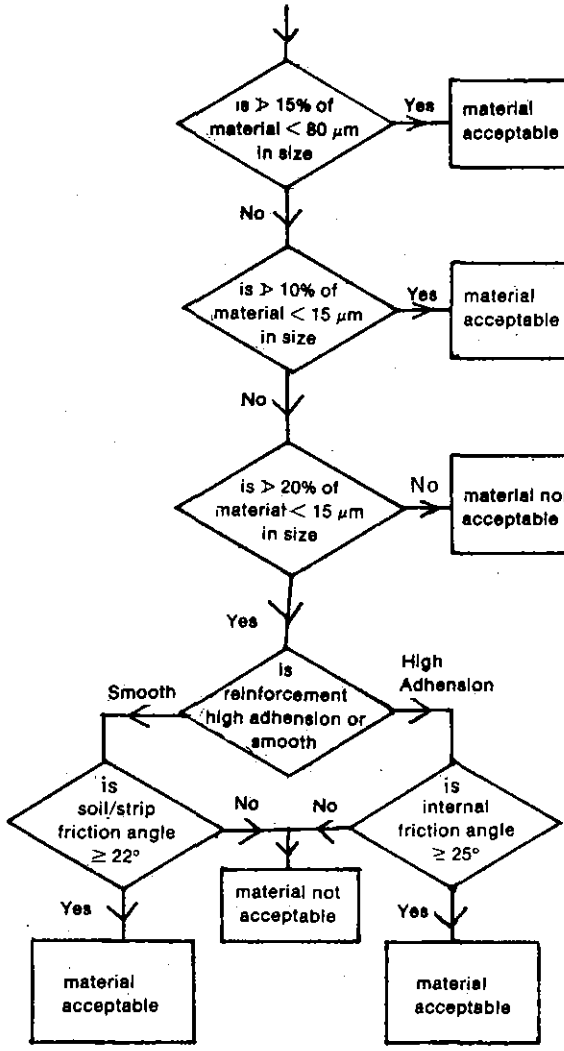


Fig. 16.11 *Mechanical criteria for acceptability of backfill material using French Code*

16.5 TESTING OF FILL MATERIAL

'Frictional' Soil
(ϕ-soil, c = 0)

Determination of Angle of Friction, ϕ

Standard practice is to determine the angle of friction by means of the shear box test. The plan of the shear box should be 300 mm × 300 mm and have a depth of not less than 150 mm. Generally the soil sample should fill the box.

The maximum particle size of the soil should not exceed 1/8 of the height of the test sample which should be prepared to a dry density of 92% ± 2% of the maximum dry density as determined from the vibrating hammer test, described in BS 1377 (1975).

Determination of Coefficient of Friction, μ, between Reinforcement and the Soil

This is achieved by testing, also using the 300 mm shear box. A piece of the reinforcement material is carefully cut so that it fits exactly the inside of the shear box. By means of packing the material is arranged to be exactly level with the surface of the lower half of the box. The soil is placed in the upper half of the box and the maximum shearing force corresponding to a specific normal load is determined. The test is carried out three times at least and the average ratio of maximum shear stress to normal stress is taken as μ. The rate of shearing should not exceed 1.2 mm/minute. The normal load used in the test should correspond to the maximum vertical pressure that will operate within the fill, obtained from the design calculations.

> NOTE *The above is an abstract of the requirements of the UK DOT Memorandum (1978)* which also states that frictional fill should have the following properties if it is to be acceptable for reinforced earth. It should be well graded (uniformity coefficient $\nless 5$) with a maximum particle size of 125 mm and with not more than 10% fines (i.e., material passing the 0.063 mm sieve), with an angle of friction of not less than 25°

Cohesive-Frictional Soil
(c − ϕ soil)

Determination of Effective Angle of Friction and Effective Cohesion

The test is carried out in the (300 mm) shear box with the sample compacted to a dry density of 92% + 2% of the maximum dry density value obtained from the laboratory compaction test (using the 4.5 kg rammer) BS 1377 (1975). The sample should fill the whole box and the maximum particle size should not be greater than 0.125 × the height of the sample.

A minimum of three samples are tested, first having allowed to soften under water for twenty-four hours, followed by twenty-four hours consolidation under the test normal stress. Normal stress values should roughly correspond to the vertical pressure in the fill of the completed structure, at the base, at quarter height and at half height.

The rate of shear should be slow enough to allow the dissipation of any pore water pressure so that the applied stresses are also effective stresses.

The results from the shear box tests are taken to give a measurement of the properties of the fill. For quality control, results from shear box tests using the 60 mm × 60 mm box are acceptable.

Determination of Friction and Adhesion between Reinforcement Strip and the Soil

The procedure is as for frictional fill, except for a possibly lower shearing rate. The adhesion is taken to be the intercept on the shear stress axis of the 'normal stress to shear stress' plot, and the angle of friction is taken as the angle of slope of the plot.

> NOTE Other requirements for a satisfactory cohesive-frictional soil fill for reinforced earth are: Liquid limit $\ngtr$ 45%; Plasticity index $\ngtr$ 20%; not more than 10% of the soil to have particle sizes smaller than 0.002 mm, and the angle of internal friction, with respect to effective stresses, not to be less than 20°.

Other Tests for Fill Material

Other necessary tests are chloride-ion content, sulphate content, pH value, and resistivity.

It should be noted that the maximum and minimum allowable values for these variables are affected by the type of reinforcement used in the structure.

> *Caution*: The use of chalk, unburnt colliery shale, pulverised fuel ash and other unsuitable material is not permitted for reinforced earth fills.

16.6 THE PROBLEM OF CORROSION

"... it appears that most engineers, either through their work or educational experience, have concluded that metal, buried in the earth, will corrode in a time period inversely proportional to their years of experience". Mckittrick (1979).

What is the definition of corrosion? What does it mean when it is said that a certain something has corroded?

There is little doubt that if corrosion creates a pin hole in an off-shore oil pipe the Government may be shaken, whereas if corrosion creates a pin hole in a reinforcing element of a reinforced earth structure the event is of much less consequence.

Perhaps if the tension strips were somehow eaten away, one would say that the event was of consequence. Pitting of the surface of a tension strip due to some years burial, might even improve the bond resistive qualities of the material, but of course the tension capacity will be effected.

However, Darbin *et al.* (1978), have illustrated that even in an aggressive soil environment, the type of galvanised steel reinforcement strips presently being used in construction will have a useful life of at least 120 years.

The UK DOT Memorandum (1978), gives guidance on the sacrificial thickness that should be added to the designed thickness of the metallic tension strips if full allowance is to be made for corrosive effects. This is indicated in the following table:

Reinforcement material	Total increase in strip thickness (mm)	
	Frictional fill	Cohesive frictional fill
Aluminium	0.3	0.6
Galvanised steel	1.5	2.5
Copper	0.3	0.6
Stainless steel	0.3	0.4

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CHAPTER 17

Bearings for Bridges

17.1 BEARINGS

Basic Definitions

A *Fixed Bearing* is a point of connection between a structure and its support, designed to transmit vertical and horizontal loads and allow for rotations. A *Moveable Bearing*, in addition, allows for horizontal movements of the structure.

- *Fixed* bearings can be made of steel plates—sometimes flat and pinned, and sometimes curved to provide for rotation, sometimes of two curved plates with rocking lubricated surfaces, and sometimes of elastomeric material—with or without dowel pins through the bearing or preferably keys.
- *Sliding type* moveable bearings can be made of steel plates with lubricated surfaces, or a steel plate with special low friction material (e.g., PTFE) to enhance the sliding ability.
- *Rolling type* moveable bearings are usually made of hardened steel curved surfaces which roll on other hardened steel surfaces.

Limitations of Different Types of Bearings

- Fixed bearings allow negligible relative movement between the superstructure and substructure at the bearing location and hence may generate large forces at this point, resulting in costly details, anchor-piers or anchor-abutments.
- Sliding bearings, when they work satisfactorily, transmit a horizontal force which is dependent on the coefficient of sliding friction of the bearing and the vertical load on it at the time. This horizontal force may have little relation to the capacity of the pier to accept the force, and in some cases the horizontal force developed from the pier flexibility may be insufficient to cause sliding to take place. The designer only has limited control over the coefficient of friction which may vary widely in practice.
- Rolling bearings are limited to an even greater extent than sliding bearings in their ability to transmit horizontal forces as they are usually designed to have a very low rolling resistance.
- Elastomeric bearings also have limitations such as size, and in some cases problems due to co-existing small vertical loads, but they are the only type of bridge bearing where the designer can readily 'manipulate' the force which will be transmitted to the pier through the bearing. These bearings usually have a stiffness in shear which is about 10% of their stiffness in compression. This method allows the bridge designer to adjust the 'horizontal bearing stiffness' on a given pier to suit the capacity of the pier to take horizontal load or movement.

17.2 BRIDGE SUPERSTRUCTURE MOVEMENTS

The horizontal movements of a bridge superstructure are due to:

- shrinkage of concrete
- elastic shortening of concrete due to prestressing
- creep of concrete under prestress and other permanent loads
- temperature expansion and contraction
- movements due to induced external loads (e.g., earthquake, wind, vehicular braking etc.).

Shrinkage of Concrete

This is a shortening of the bridge superstructure which depends on a number of factors—concrete type and quality, size of the member, relative humidity and time after casting. Sometimes for simplicity of calculation, this movement is treated as a negative temperature effect.

Elastic Shortening

Elastic shortening of a prestressed concrete deck occurs during the process of prestressing. The amount of this shortening, which has to be accommodated by the bearing and pier, will depend on the stage at which the superstructure is placed on the bearings and also at what stage it is prestressed. At times, partial or complete prestressing may be carried out before the superstructure is placed on the bearings thus eliminating this shortening from affecting the bearings, at least partly.

Creep Shortening

Creep of concrete under prestressing and other permanent loads must be allowed for, and again, this is a shortening of the superstructure. It is a time-dependent effect which is readily calculable when all the variables which affect its value are determined. Again, for simplification, creep can also be treated as a negative temperature effect.

Temperature

Temperature variations cause expansion and shortening of the superstructure and need to be estimated as a plus and minus range about a mean structure temperature which occurs when the superstructure is placed on the bearings. Depending on the time of the year, this may or may not be the mean temperature for the locality. When construction occurs during the extreme temperature seasons, these two values may be many temperature degrees apart.

Temperature differentials will also occur in the superstructure namely from top to bottom and from one side to the other of the superstructure. Temperature differentials through the depth of the superstructure will usually have little effect on the bearings and piers but temperature differentials from one side of the superstructure to the other will cause the superstructure to bend in plan which will cause horizontal forces on the bearings and piers which may be of magnitude.

Load Movements

Movements due to applied horizontal loads on the superstructure can be either transverse or longitudinal or both, with respect to the bridge centre line.

(a) Loads transverse to bridge centre line: Wind or earthquake on both the bridge superstructure and on the traffic using the bridge, and also the centrifugal force from traffic using the bridge if curved.

(b) Loads parallel to the bridge centre line: Apart from earthquake or wind on the superstructure, there is the braking force (often as a percentage of the live load on the bridge).

17.3 DEVELOPMENT OF BEARINGS

Until the end of the 18th century all structures of any appreciable size were built of stone, brick or mixed masonry. These structures, generally massive, are little affected by environmental changes and any slight movements which may occur are compensated either by deformation of the constituent materials or by small displacement of the supports.

The 19th century, and the introduction of cast-iron and steel materials having the advantage of resisting tension, considerably enlarged the possibilities available to builders. An increase in structural spans is generally accompanied by a corresponding lightening of the structure which becomes more slender and flexible and loses some of the thermal inertia. They must then be fitted at their support points with simple devices, called *bearings*, which can withstand movement and, more particularly, the expansion or contraction due to temperature changes.

The first devices used at that time, consisted of either metal plates sliding one on the other or, of rollers or, of a combination of both. These appliances; gradually improved upon by the incorporation of swivel arrangements in order to provide rotational movement, were in general use on steel structures for more than a century.

The 20th century saw the introduction of reinforced concrete but, as was the case with masonry structures, the early structures built with this material were so massive that the support movements were of little significance. At the end of the Second World War the necessity of rebuilding rapidly the structures which had been destroyed, and also, no doubt, the shortage of steel for constructional purposes, favoured the rapid development of reinforced concrete and, even more so, of prestressed concrete.

With the advent of structures which were more and more slender, came the necessity to incorporate bearing devices to allow for movement and rotation. As a result, systems of entirely new conception, using the elastic properties of rubber and steel, appeared on the market. The use of this material in civil engineering was not new. As early as 1830, British railways, placed rubber shock absorbing pads between rails and sleepers. The idea was again taken up a century later by the French railways, first in 1932 when elastic pads were placed underneath the steel bearings of certain structures in order to absorb vibration, and again in 1936 when chief engineer Valette adopted the use of rubber pads as bearings underneath the steel deck of a railway bridge at La Plaine St Denis in the Paris suburbs, and finally in 1948 when the SNCF decided to lay the rails of certain tracks on rubber *sole-plates*.

It was Eugene Freyssinet who first had the excellent idea of making general use of rubber pads as bearing devices by combining steel and elastomer in a single product in order to improve their individual performances as a combined product.

As early as 1952 the first rubber bearings, consisting of a stack of elastomer layers and sheets of tinned metal grillage were manufactured by Freyssinet. A first patent was applied for in 1954 by Eugene Freyssinet, but it quickly became apparent that *grillage* bearings had a limited use because of their very low resistance in compression and too much permanent set. Therefore, in 1956, these bearings were abandoned and the metal grillage replaced by steel plates adhering to the rubber by vulcanization. Thus the first laminated elastomeric bearings were born.

At the request of Freyssinet International, Francois Conversy drew up a method of design and dimensioning, which was the subject of a report, in 1958 to the French Committee for Bridges and Structural Engineering and which was confirmed by a series of tests carried out under the sponsorship of the International Union of Railways between 1961 and 1965. This theory was to serve as a basis for most national regulations.

At the same time, industrialisation of the manufacturing process led to rapid improvement in the product which, since then, has never ceased to develop, thanks to discovering of new elastomer compounds (neoprene) allowing the use of the bearings in special climatic environments, the use of stainless steel reinforcing plates in order to resist corrosion in aggressive environments, and combining laminated elastomeric bearings with polytetrafluorethylene (PTFE) sliding sheets in order to allow large movements. The development of neoprene bearings and the realisation of PTFE as a very low friction element, which further led to more sophisticated bearing systems, is explained ahead in the chapter.

In Britain, bridgeworks carried out under the auspices of the Department of Transport (now called the Dept of Environment, DOE) have to comply with technical instructions issued by the Department Memorandum 557 issued in 1945(!). These stated that it was unnecessary to provide for expansion (or contraction) for bridges less than 30 ft span and recommended *roller* and *rocker-roller* bearings for spans above 50 ft (with *pin* or *rocker* bearings for the fixed end). Memorandum 802, published in 1962, gave provisional rules for the use of rubber bearings which were coming into general use at this time. Memorandum 557/1, published in 1966, gave rules for the design and use of Freyssinet concrete hinges in highway structures. These were updated by Technical Memorandum BE1/76 and 5/75, respectively. Interim Memorandum IM11 in 1970 dealt with PTFE in bridge bearings.

Up to the middle of this century, bridges relied on steel roller, rocker or metal sliding bearings to permit movement. With more advanced designs to make full use of the materials employed and increased use of skewed and curved bridges to carry modern high speed roads over obstructions, the need arose for bearings to take movement in more than one direction. New types of bearings have been developed taking advantage of the new materials arising from improved technology.

Because of the high friction values associated with metal to metal sliding surfaces and complete seizure if not kept lubricated or protected from corrosion, modern sliding bearings usually rely on PTFE or similar low friction non-corroding synthetic materials to allow assured very low friction in sliding.

In the business of bearings, rubber means either the natural product or a synthetic material with rubber-like characteristics. Some countries (e.g., Germany) do not permit the use of natural rubber because of its unfavourable ageing behaviour. A *laminated bearing* is a bearing consisting of some rubber slabs bonded to steel plates in between, so as to form a sandwich arrangement. A *rubber bearing* pad is a single unreinforced rubber slab. A laminated rubber or neoprene bearing has impregnated in it, during its vulcanization, the restraining steel plates which form the reinforcement.

Natural Rubber (NR) is one of the polymers reported to have almost all the properties needed to meet the requirements of bridge bearings except poorer ageing performance. Although synthetic rubbers like polychloroprene (CR, termed Neoprene) have found wide acceptance for making bearings for concrete structures, it is expected that in countries where natural rubber is abundantly available, the bearings made out of suitably compounded NR will be more economical, even though NR is relatively poor against ageing and may call for earlier replacement.

17.4 TYPES OF BEARINGS RECOMMENDED FOR VARIOUS SPAN-LENGTHS AND SUPPORT-FLEXIBILITY CONDITIONS

Bridge bearings must be designed to transmit all vertical loads and appropriate horizontal forces (depending on the functionality of the individual bearing).

(i) Generally where a simple span deck is supported over rigid supports (unlike the adjoining cantilever tips in segmental free-cantilever construction, which are flexible) and the span is less than about 7.5 m, no special bearing devices are necessary — only tar paper or a felt layer is adequate (after the mating concrete surfaces are smoothened by carborundum stone).

(ii) For spans between about 7.5 m to about 15 m, mild steel plate bearings (with top plate slightly curved), sliding-type over free supports and rocking type (with a rocking pin) over the fixed support, may be used. Alternatively, neoprene bearings (pads or strips or laminated type) may also be used.

(iii) For spans in excess of about 15 m, metallic rocker type bearing (top-plate appropriately curved and a rocker-pin placed between the top and bottom plates) is provided over the fixed support and a roller-cum-rocker type bearing (one roller between top and bottom plates, or, if the load is high so that more rollers are needed, then a curved-soffit top-plate (called saddle plate) placed on a flat intermediate-plate (with rocking pin in between) seated on rollers that rest on a flat bottom-plate) is provided over the free supports. Alternatively, suitably designed laminated neoprene bearings may be provided.

NOTE Even for short simple span decks, placed across flexible cantilever tips mentioned in (i) above, full-fledged rocker and roller-rocker metallic bearings [described in (iii) above] are advisable because of considerable deck-movement and deflection (and consequent rotation) at the tips of the long cantilevers. Alternative types of bearings may be used, but these normally would not include the laminated neoprene bearings if the vertical reaction is low and deformations high.

Types of Bearings

Basically two types of bearings are used. *Rocker* (i.e., hinge or fixed) type which permits only rotation—acting as a moment release. *Rocker-cum-roller* (i.e., free) type which permits rotation as well as translation-acting as a moment as well as a thrust release. From the materials point of view, these bearings can be made from metal, rubber, metal and elastomer and even concrete.

Metallic Bearings

These are usually made of mild steel and cast steel. Leadsheets, phosphorbronze and stainless steel are used sometimes for their respective special merits. Metallic free bearings can have the advantage of generating small to very small horizontal force because of low friction coefficient against sliding and very low against rolling. When the sliding or rolling surfaces are chemically coated with teflon (a rubber synthetic), the friction coefficient reduces even further. *Meehanite* bearings are made of cast steel of special tensile strength with special resistance to vibration and wear. Armoured steel bearings are basically of mild steel but specially treated so as to harden the surface to a depth of about 10 mm. With this costly treatment, the load carrying capacity of a roller is simultaneously improved. Load, in tons, carried by a forged steel roller placed between plain surfaces, as per Hertz's formula, approximately works out to 6.25 times the product of its length and diameter in inches. This carrying capacity reduces with increase in the number of rollers in the assembly. Hi-Load bearings, marketed initially in the UK, are made of a special high quality steel and can carry a load about 8 times that of the usual forged steel roller of the same diameter and length. (In mild steel bearings the roller is generally made of forged railway axles.)

Rubber Bearings

These are of natural rubber or synthetic rubber (neoprene), reinforced by impregnated mild steel plates, and have been widely used as bridge bearings. These elastomeric bearings are cheap, easy to maintain, and are good in shear under compression. However, on their own, they are not suitable where compressive load is low while rotation is high (e.g., under high torsion conditions) because they accept rotation only under significant load.

Working Principle of Neoprene Bearings Reinforced With Restraining Steel Plates (Laminated Neoprene Bearings)

When subjected to loads and/or displacement, an individual elastomer layer distorts (Fig. 17.1), tangential shear stresses occur in the plane of contact of the layer with its support, which depend not only on the magnitude of the loads applied but also on the shape of the layer—plan surface and lateral exposed surface, on the nature of the elastomer, the rate at which the loads are applied, and on the temperature. These tangential stresses are manifested by a tendency for inter-granular sliding, which is opposed by the adherence to the elastomer layer of the steel plates forming the reinforcing laminations. With equal thicknesses of elastomer, these plates have the further advantage of either reducing the set due to the normal load or increasing the admissible load.

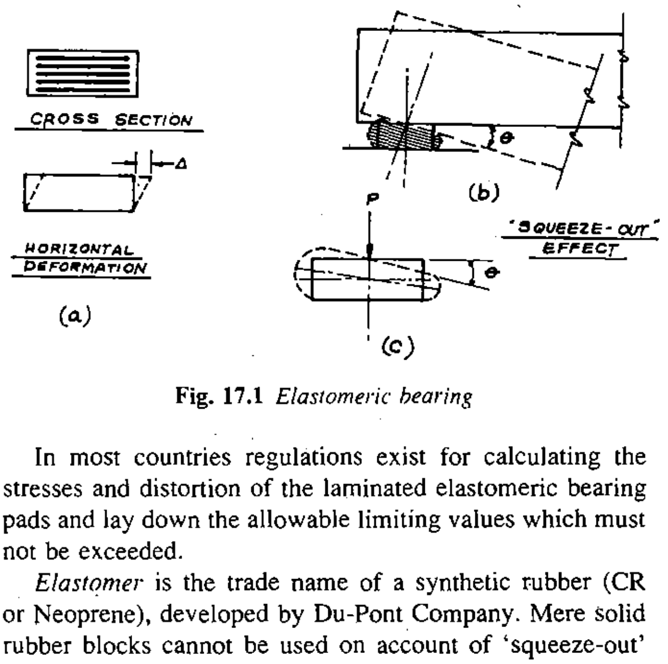

Fig. 17.1 *Elastomeric bearing*

In most countries regulations exist for calculating the stresses and distortion of the laminated elastomeric bearing pads and lay down the allowable limiting values which must not be exceeded.

Elastomer is the trade name of a synthetic rubber (CR or Neoprene), developed by Du-Pont Company. Mere solid rubber blocks cannot be used on account of 'squeeze-out' effect, shear failure and relatively early ageing problems. Interleaved layers of neoprene and steel-plates, bonded together, overcome these disadvantages. These bearings can be designed to be sufficiently soft horizontally so as to impose relatively small horizontal forces on the supports (= shear rating of the bearing × deck movement above it), but sufficiently stiff vertically to prevent appreciable changes in their height under variable loads. Rubber (and neoprene) are specified in terms of its 'shore' hardness. This is a measure of the penetration into the rubber by an indenting apparatus under a specified load and is, therefore, related to the elastic modulus of the rubber. For instance, a 60 shore hardness can correspond to an elastic modulus of 44.5 kg/cm^2, a shear modulus of 10.6 kg/cm^2, a bulk modulus of 20,000 kg/cm^2, and a maximum permissible shear strain of 0.7. The stiffness of rubber in compression depends on the *shape-factor* of its individual layer. This is the ratio of one loaded area to the total force-free (i.e., free-to-bulge) area of the individual layer. For stability, the least plan dimension should not be less than about four times the thickness of the bearing. Owing to the possibility of their deterioration with age and ozone attack, provision should be made in the bridge structure for replacement of the elastomeric bearings after about 25 years. Often it is advantageous to relieve these bearings after a few months of installation (or at least immediately after prestressing of a prestressed concrete deck) by momentarily jacking-up the deck so that the pads rebound back to their unstrained condition, then lowering the deck back on them. In this way they need be designed only for the balance movement (horizontal shear deformation) for permanent condition.

Elastomer and the Reinforcing Steel Plates in a Restrained (i.e., Laminated) Neoprene Bearing or a Rubber Bearing

- *Elastomer* The elastomer used for the manufacture of standard neoprene bearing pads is polychloroprene. This product was chosen, not only for its remarkable resistance to ageing, but also for its excellent behaviour in the presence of aggressive agents such as ozone, mineral oils or petrol, solvents and ultraviolet rays.

  Nevertheless, in cases where a particular specification so requires, natural rubber can be substituted for polychloroprene, and the inclusion, before vulcanisation, of certain additives, such as anti-oxidizing and anti-ozonizing agents, improves its resistance to ageing. It should be noted that several national regulations specify the use of one or the other or both these elastomers as basic components for the mixture used in the manufacture of these bearings.

  Design and production of special elastomer mixes can also be envisaged in order to meet the particular requirements of certain specifications—increased resistance to cold, and/or hydrocarbons, improved mechanical characteristics, etc.
- *Reinforcement Plates* The steel reinforcement sheets are of mild steel, conforming to current international regulations and norms. In the case of specific requirements, the thickness of the plates can be adapted to the required values.

  When used in an aggressive environment—maritime zones, corrosive chemical environments, etc., it is strongly recommended to use either fully-embedded bearings, or semi-embedded bearings with stainless steel reinforcement plates. The former are formed by vulcanizing the complete assembly of all the layers of neoprene and steel laminates in one single operation. Bonding together of individually vulcanized elements (in order later to form the total sandwich assembly) does not give as dependable a bearing as by the fully embedded method, and should not be permitted anymore.

Metal, rubber/elastomer and PTFE bearings combine all the advantages of each component. The so-called *pot* bearings (for example, the early German *Rota* and *Rotaflon* bearings, and Freyssinet's *Tetron disc* bearings D3 Series),

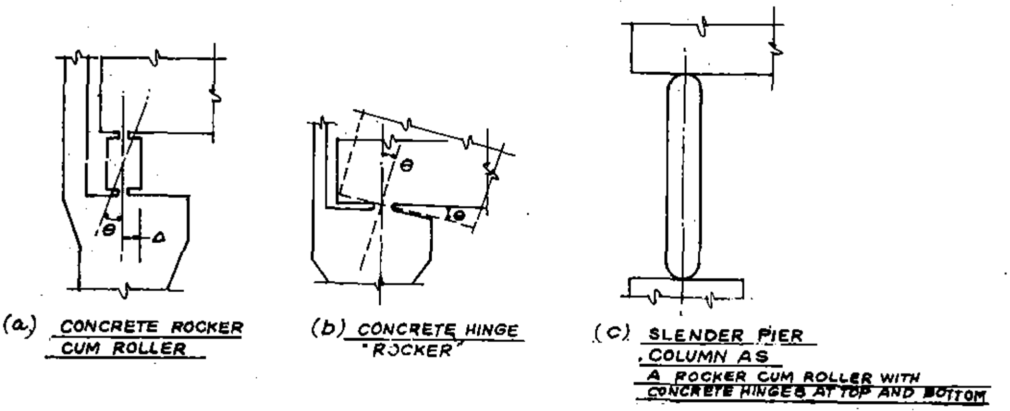

Fig. 17.2 *RC bearing*

essentially comprise of a circular steel pot, topped by a plunger steel plate, enclosing a round layer of high grade rubber (elastomer) in between. While rotation (i.e., rocking or moment-release) is attained by squeeze-deformation of the entrapped rubber medium that permits the plunger to rock on it in the pot, the movement in any direction is attained by low-friction-sliding between the plunger and the top plate on it. Three piece spherical PTFE bearings (Freyssinet's spherical *Tetron S3 series* bearings) allow movement in any direction and rotation about any axis through low friction contact surfaces. Three piece cylindrical PTFE bearing allows movement in any direction and rotation about the cylinder axis through low friction contact surfaces. Combined PTFE and laminated (i.e., plate-reinforced) neoprene bearings have been developed to allow high translation (through low friction sliding) and the required rotation (through deformation of plate-reinforced neoprene).

Concrete bearings and hinges are sometimes used for economy, but can be tricky in the execution of their construction. These (see Fig. 17.2) are usually in the form of shallow-necked hinges acting as rockers. Free bearings have two narrow throats (i.e., necks) and sometimes are of the full height of the column.

The advantages of concrete bearings are: low cost and little maintenance. The disadvantages are that they call for extreme care in construction, obstruct free movement of the deck, and create secondary stresses due to stiffness, require more room to construct, and suffer from an almost unknown internal stress distribution. The throat-width is designed so that under maximum working load the maximum compressive concrete stress can reach many times its 28-day crushing strength.

For fixed bearings, only one throat is provided. Guidelines for design of circular concrete hinges have been published by the UK DOE (as mentioned earlier). Reference may also be made to 'Design of Concrete Hinges' by Sims and Bridle in *Concrete and Constructional Engineering*, August, 1964. Details for design of linear concrete hinges are given ahead in the present chapter.

17.5 PRACTICAL CONSIDERATIONS IN THE SPECIFICATION, DESIGN, MANUFACTURE AND QUALITY CONTROL OF MECHANICAL BRIDGE BEARINGS

The requirement to build structures more economically led not only to improved techniques in the construction and design of bridges, but also to more sophisticated and advanced designs of bearings. Whilst some of the more simple bearings continue to be manufactured by general fabricators, most bearings are now supplied by specialist manufacturers.

Bearings constitute only a small fraction of the total cost of most structures, but the costs of malfunction and any subsequent rectification are high. It is therefore critical that serious attention is given to their selection and use. It is important for bridge designers and contractors to appreciate the extent to which bearing designs have advanced over the past 20 to 30 years. The principal reasons for the advances are listed below:

(i) Increasingly longer bridges and spans, often continuous, resulting in increased bearing loadings and movements.
(ii) The increasing occurrence of curved and skewed structures resulting in more complex movement and load distribution.
(iii) The availability of new materials such as polytetrafluoroethylene (PTFE).

Developments

The four developments which have had the greatest influence in the improvement of bearing designs since the Second World War have been:

(i) PTFE
(ii) Rubber pot and spherical bearings
(iii) High strength roller steels, and
(iv) Laminated elastomeric bearings.

PTFE

The copolymer polytetrafluoroethylene, PTFE, has excellent low frictional properties when properly designed into a sliding bearing assembly. In laboratory tests, coefficient of friction values as low as 0.002 have been achieved, which is 1/100th of the value which can be expected from lubricated bronze against steel. This compares well with roller bearings but with the added advantage that movement in any horizontal direction is possible.

Three basic principles should be observed when utilising PTFE:

- The mating surface must be smooth and flat or of the same curvature as the PTFE—and remain so in service. For this reason, polished stainless steel is the most commonly used mating surface. However, anodised aluminium, hard chromium plate (totally free of porosity) and acetal resin have all proven satisfactory. PTFE on PTFE is unsuitable.
- The PTFE should be mechanically retained on a rigid backing plate. Bonding PTFE to a backing plate is totally inadequate for most practical purposes. The PTFE sheet should be retained in a recess. When correctly designed, the recess will: (a) prevent excessive creep of the PTFE, and (b) secure the sheet and prevent its being inched out of the bearing due to structural movement.

- A self-aligning feature should be provided within the bearing assembly. Some misalignment will inevitably occur during service as a result of construction tolerances, settlement, creep, shrinkage, live loading, etc. Whilst PTFE has a small degree of elasticity, this is normally insufficient to prevent excessive edge loading due to misalignment. The alignment device should be such as to avoid high edge stresses on the internal (e.g., PTFE) and external (e.g., concrete) members.

Rubber Pot and Spherical Bearings

Typical rubber pot and spherical bearings are shown in Figs. 17.3 and 17.4.

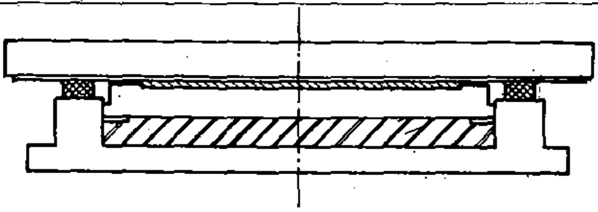

Fig. 17.3 *Rubber 'Pot' bearing (disc bearing)*

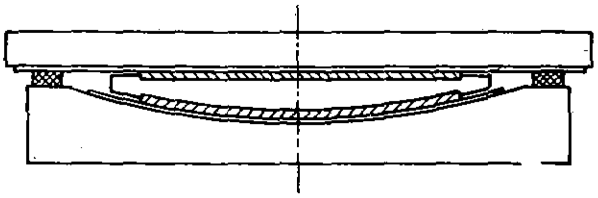

Fig. 17.4 *Spherical bearing*

In a rubber pot bearing the elastomer within the pot can be assumed to act as a fluid to accommodate rotation. The spherical bearing, utilising PTFE convex or concave surfaces, is self-aligning on the ball and socket principle. Both types of bearing have proven very successful throughout the world. The only differences in performance are with regard to torsional resistance under rotation and the degree of rotation available. The rubber pot bearing has a lower resistance to rotation for very small angles (say up to approximately 0.01 radians) but becomes increasingly more resistant with increasing rotation angles. The shperical bearing has a constant resistance to the friction in the sliding surfaces and will permit relatively large rotation angles.

High Strength Corrosion Resistant Roller Steels

Long term problems through fatigue and corrosion of rollers and plates have rendered many roller bearings ineffective. The commonly used materials have been forged railway axle steel, cast steel and mild steel, which would be designed in accordance with established codes. In practical terms, the load capacity of these bearings is limited to approximately 4 MN (900 kips). With the alloy used (for example) in the Glacier Cygnus roller bearings, the dimensions reduce considerably. The dimensions of a 4 MN (900 kips) roller would be:

	Mild Steel	Glacier Cygnus
Roller diameter (mm)	340	120
Roller length (mm)	1530	540

In this example, the volume of the mild steel roller is approximately 23 times that of the proprietary type. The cost savings result primarily from reduced machining times and the ease of handling of smaller components. Care must be taken in the selection of special steels and their heat treatment as embrittlement of very hard rollers has caused problems in service in the past.

Laminated Neoprene Bearings

See details ahead.

Specifying Bearing Requirements

To ensure a satisfactory proposal a bridge designer should specify clearly what his requirements are. Table 17.1 shows a typical bridge bearing performance schedule stating the important data required by the bearing manufacturer to select the most suitable bearing. This provides a basis for specification, but any special requirements should be indicated separately, e.g., abnormal high or low temperatures, abnormal construction loadings, etc. It is most important to give the bearing manufacturer drawings or sketches showing a plan view of the structure plus salient points such as space limitations, skew angles, etc.

Table 17.1 *Bridge Bearing Performance Schedule*

Bearing design data		
Bearing reference		
Quantity		
		Units
Load per bearing	$*V_{max}$	kN
	$*H_x$	kN
	$*H_y$	kN
	V_{perm}	kN
	H_x	kN
	H_y	kN
	V_{min}	kN
	H_x	kN
	H_y	kN
Movements	$\pm ex$	mm
	$\pm ey$	mm
	$\pm \alpha x$	rads.
	$\pm \alpha y$	rads.

(Contd.)

Table 17.1 *(Contd.)*

Bearing interfaces

Maximum dimensions	Upper	mm		
	Lower	mm		
	Height	mm		
	Upper material			
	Lower material			
Allowable contact stress	Upper		Mean	N/mm²
			Peak	N/mm²
	Lower		Mean	N/mm²
			Peak	N/mm²
Fixing method	Upper**			
	Lower**			

* Serviceability and ultimate limit states should be quoted together with any special considerations, e.g. earthquake.

** e.g., bolts in sockets, friction only, etc.

Symbols used

V_{max} = maximum vertical load
$V_{perm.}$ = permanent vertical load
V_{min} = minimum vertical load
H_x = longitudinal horizontal load
H_y = transverse horizontal load
ex = longitudinal movement
ey = transverse movement .
αx = rotation about a longitudinal axis
αy = rotation about a transverse axis

NOTE The following details should also be included in this table:

(i) Drawings showing space available, skew angles, etc.
(ii) Bearing pre-set as appropriate
(iii) Any special considerations affecting bearing designs.

17.6 LESSONS FROM SOME ACTUAL DISTRESS EXPERIENCES[1]

(a) Bearing failure can result from a number of causes (e.g., damage or displacement following an accident, attack by chemicals, fire, and corrosion of contact surfaces) but probably the greatest cause of bearing malfunction, particularly of modern bearings, is due to inadequate or improper installation. It is not unknown for such simple looking bearings to be installed 90° out of phase or even upside down ! It cannot be stressed too strongly that care in the installation of bearings is of the utmost importance.

(b) Bridges are usually designed with an expected life in excess of 70 years. Modern bearings and bearing materials have not been proved in service for this length of time so it is advisable to make provision in the design of bridges for bearing replacement, should this be found to be necessary. Facilities for correcting the effects of differential settlements, etc., should be provided unless the structure has been designed to accommodate such effects.

(c) Regular inspection of the bearings should be made so that any potential trouble is detected before serious damage is done to the structure. There should be adequate space around bearings to allow for inspection and maintenance in service. In certain circumstances, such as when piers or abutments are high or over water, it may be advisable to incorporate some form of travelling staging in the bridge design to facilitate inspection.

(d) Elastomeric bearings will take a considerable amount of maltreatment before failure unless grossly inferior materials are used. However, localised overloading due, for example, to uneven seating can cause breakdown of the bond between elastomer and steel reinforcing plates. Unreinforced elastomer strips can squeeze or work their way out under certain circumstances. Small seating plinths can disintegrate under shear forces generated by elastomer bearings (the seatings should extend at least 50 mm beyond the edge of the bearing, preferably 100 mm).

(e) Disintegration of poorly prepared bearing seatings (pedestals) is one of the most common causes of bearing failure. This problem has recently been highlighted at the Gravelly Hill motorway inter-change outside Birmingham, England. Here, the bearing seatings have disintegrated and allowed the deck-support beams to drop, causing tension cracks in the locally unsupported deck-slab above.

(f) At another project, it is thought that incorrectly proportioned constituents (too much hardener plus a small quantity of water in the aggregate) led to the failure of 2″ high epoxy resin bearing plinths when the precast concrete beams were lowered onto the bearings.

(g) Incorrect installation procedures led to the failure of bearings supporting a viaduct over a river estuary. Here, large mechanical bearings were to be set on 12 mm thick pads of polyester resin mortar with a sheet of polythene placed on top of the mortar bed to break the bond between the bearing and the mortar. The mortar was domed, the intention being that surplus material would squeeze out when the fixing bolts were tightened down. In practice, the large quantity of resin mortar needed for each bearing required that it be made up in a number of mixes, and consequently the material could not be considered as entirely homogeneous. On removing the damaged bearings it was found that the polythene sheeting had unevenly curled. Both these results led to a non-uniform support to the bearing, causing failure.

(h) Leaking expansion joints can lead to corrosion of metal bearings. Unsuitable materials can give rise to problems. Many of the 18500 sliding rocker bearings installed in the Midland Links viaduct (UK) are not functioning as they should. The bearings are made of three rolled steel plates, the middle one heavily chamfered to allow the top plate to rotate. The steel deck beams rest directly on the top plate with no special sliding medium at the steel to steel interface apart from an initial coating of molybdenum disulphide. Some of the bearings have seized and those that still slide do so very reluctantly. Attempts to introduce lubricant between the sliding surfaces have proved ineffective.[2] This malfunction of rockers not rocking and rollers not rolling is a common feature in such steel bearings.

(i) In a similar manner, the steel deck beams of Vauxhall Bridge over the River Thames in London, built about 1906, rested directly on steel plates bedded on cill stones. Over the years these corroded and seized to the beams. Movement of the deck caused the front of the cill stones to break away. In 1976 the steel bearing plates were replaced by laminated rubber bearings set on new precast concrete bed stones.

(j) The abutment bearings of Wandsworth Bridge over the River Thames in London consisted of large knuckle leaf bearings supported on a bank of four flat sided forged steel (cut) rollers tied together with side bars bolted to each roller. The rollers ran on a bottom casting. As the bearings were subject to uplift, the lower casting of the leaf bearing was tied down to the bottom casting by four 1 1/2" diameter bolts which passed through slotted holes in the middle, or lower leaf bearing, casting. The bottom casting in turn was bolted down to the concrete abutment bearing shelf. The bridge was built in the late thirties and inspection of the bearings in 1973 indicated that although the main castings and forged steel knuckle pins were in good condition, the forged steel rollers were badly corroded with no sign of any lubrication having been applied or any protection against the entry of dirt or moisture. Several of the side bars had come adrift due to corrosion of the fixing bolts and a number of the tie-down bolts had broken or bent due to the heads binding on the intermediate casting. The bearings have subsequently been replaced by steel rocker bearings incorporating a PTFE/stainless steel sliding element. These have been set on new bearing plinths. No provision had been made for an expansion joint in the deck surfacing, which consequently cracked at the abutment, allowing water to penetrate down to the bearings.

(k) Other problems that have come to light include roller bearings which have overrun their design travel so that the gear pinions ran off the end of the guidance rack and were sheared off when trying to re-engage on their return; end flanges sheared off rollers due to insufficient allowance for side thrust on these bearings. Compatibility of steelwork fabrication with the drawings is necessary if the bearings are to function in accordance with the design.

(l) Replacing bearings can be a very difficult operation unless suitable provision has been made in the design of the bridge structure for proper access to the bearings and for jacking of the bridge-deck to be undertaken. Long[3] has dealt with the problems of replacing bridge bearings.

17.7 STRUCTURAL DESIGN OF VARIOUS TYPES OF BEARINGS

(based on BS 5400, Section 9.1, 1983)

Definitions

Refer Fig. 17.5.

Elastomer A compound containing natural or chloroprene rubber with properties similar to those of rubber.

Roller bearing A bearing consisting essentially of one or more steel rollers between parallel upper and lower steel plates [see Figs. 17.5(a), (b)].

Rocker bearing A bearing consisting essentially of a curved surface in contact with a flat or curved surface and constrained to prevent relative horizontal movement. The curved surfaces may be cylindrical or spherical. [Figs. 17.5(c), and (d)]. Rocker bearings permit rotation by rolling of one part on another.

Knuckle bearing A bearing consisting essentially of two or more members with mating curved surfaces. The curved surfaces may be cylindrical or spherical. [Figs. 17.5(e), (g) and (h)]. Knuckle bearings permit rotation by sliding of one part on another.

Leaf bearing A bearing consisting essentially of a pin passing through a number of interleaved plates fixed alternately to the upper and lower outer bearing plates. [Fig. 17.5(f)].

Sliding bearing A bearing consisting essentially of two surfaces sliding one on the other. [Fig. 17.5(i)].

Elastomeric bearing A bearing comprising of a block of elastomer that may be reinforced internally with steel plates (steel laminates) — which make it a laminated or restrained elastomeric bearing.

Laminated bearing An elastomeric bearing reinforced with steel plates [Fig. 17.5(j)].

Plain pad bearing An unreinforced elastomeric bearing.

Strip bearing A plain pad bearing for which the length is at least ten times the width.

Pot bearing A bearing consisting essentially of a metal piston supported by a disc of unreinforced elastomer that is confined within a metal cylinder or 'pot'. [Fig. 17.5(k)].

Symbols

The symbols used are as follows:

- A overall plan area of elastomeric bearing
- A_e effective plan area of elastomeric bearing
- A_l reduced effective plan area of elastomeric bearing
- b overall width of bearing (the shorter dimension of a rectangular bearing)
- b_c effective width of elastomeric bearing
- E modulus of elasticity
- E_b bulk modulus of elastomer
- G shear modulus of elastomer
- H horizontal force
- k a factor
- l overall length of bearing (the longer dimension of a rectangular bearing)

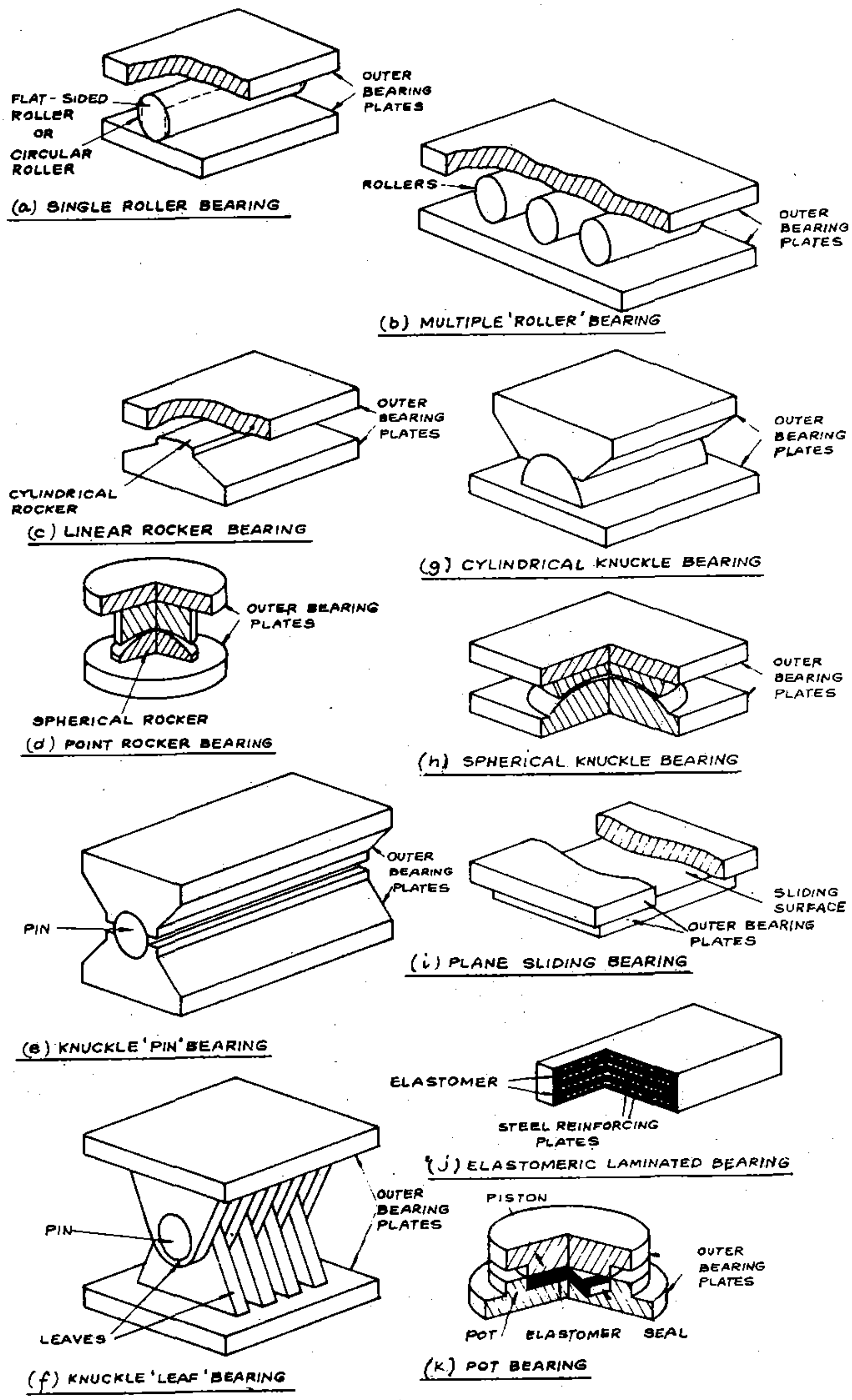

Fig. 17.5 *Types of bearings*

l_e effective length of elastomeric bearing
l_p force-free perimeter of elastomeric bearing
Q^* design loads
R radius of cylinder or sphere or convex surface
R_1 radius of concave surface
S shape factor
S' shape factor of thickest elastomer layer
S^* design load effects
T minimum shade air temperature
t thickness of a plain pad or strip bearing
t_1, t_2 thickness of adjacent elastomer layers
t_e effective thickness of elastomer in compression
t_i thickness of an individual elastomer layer in a laminated bearing
t_q total thickness of elastomer in shear
V vertical design load effect
α_b angular rotation across width b of bearing
α_l angular rotation across length l of bearing
Δ total vertical deflection
δ vertical deflection of individual elastomer layer
δ_b maximum horizontal relative displacement of parts of bearing in the direction of dimension b of the bearing
δ_l maximum horizontal relative displacement of parts of bearing in the direction of dimension 'l' of the bearing
δ_r maximum resultant horizontal relative displacement of parts of bearing obtained by vectorial addition of δ_b and δ_l.
ϵ_c nominal strain in elastomer slab due to compressive loads
ϵ_q shear strain in elastomer slab due to translational movement
ϵ_t total nominal strain in elastomer slab
ϵ_α nominal strain in elastomer due to angular rotation
σ_u nominal ultimate tensile strength of material
σ_s stress in steel

NOTE It is essential that the units used for these symbols in the formulae are compatible with each other.

Function of Bearings

The function of bearings, as explained earlier, is to provide a connection to control the interaction of loadings and movements between parts of a structure, usually between superstructure and substructure.

A guide to the suitability of various types of bearings for different functions is given in Table 17.2. To achieve the required degree of freedom it may be necessary to combine the characteristics of different types of bearings, the resultant bearing as a whole providing the required movements and load resistance, e.g., a plane sliding bearing to allow translation with a pot bearing to provide for rotation. The basic features of the various types of bearings have been illustrated in Fig. 17.5.

Table 17.2 *Bearing Function*

Types of bearing	Translation permitted		Rotation permitted			Loading resisted		
	Longitudinal	Transverse	Longitudinal*	Transverse†	Plan	Vertical	Longitudinal	Transverse
Roller								
single cylindrical	√	×	√	×	×	√	×	S
multiple cylindrical	√	×	×	×	×	√	×	S
non-cylindrical	√	×	√	×	×	√	×	S
Rocker								
linear	×	×	√	×	×	√	√	S
point	×	×	√	√	√	√	√	√
Knuckle								
pin	×	×	√	×	×	√	√	S
leaf	×	×	√	×	×	√	√	√
cylindrical	×	S	√	×	×	√	√	S
spherical	×	×	√	√	√	√	√	√
Plane *sliding*	√	√	×	×	√	√	S	S
Elastomeric								
unreinforced	√	√	√	√	√	√	√	√
laminated	√	√	√	√	√	√	√	√
Pot	×	×	√	√	S	√	√	√
Guide								
longitudinal	√	×	√	S	×	×	×	√
transverse	×	√	S	√	×	×	√	×

* Rotation about transverse axis.
† Rotation about longitudinal axis.

Key
√ suitable
× not suitable
S special consideration required

Provision for Handling

Where necessary, suitable handling attachments should be provided on bearings to facilitate their handling.

Movement Restraint

Where restraints are required to restrict the transitional movement of a structure, either totally, partially, or in a selected direction, they may be provided as part of or separate from the bearings and normally take the form of dowels, keys or side restraints.

In each case the restraints should still allow freedom of movement in the desired direction(s). The forces generated by the restraints should be considered in the design of the bearings and their connections and in the design of the structure. Where reliance is placed on friction to resist these forces, the lower bound value of friction coefficients obtained from available test data (appropriate to the surface condition in service) should be assumed.

Where bearing replacement may be required during the life of a structure, the provision of a restraint (e.g., dowels) through the bearing may cause difficulties, and alternative location of the restraints should be considered.

Uplift

If uplift can occur, bearings and their fixings should be designed to limit separation of the parts to a value agreed with the Engineer and to resist the consequent forces and actions.

Outer Bearing Plates or Spreader Plates

The outer plates of bearings should be so proportioned that concentrated loads are sufficiently distributed (or dispersed) to ensure that the permissible pressures on the adjacent bridge structure are not exceeded. The effective area for distributing a load may be taken as the contact area of the bearing member communicating the load to the plate plus the area within the uninterrupted dispersal lines drawn at a maximum of 60° to the line of application of the bearing reaction from the bearing contact area (Fig. 17.6). Where the adjacent structure is liable to deform significantly under load, the interaction of the structure and the bearing should be considered in the design of both.

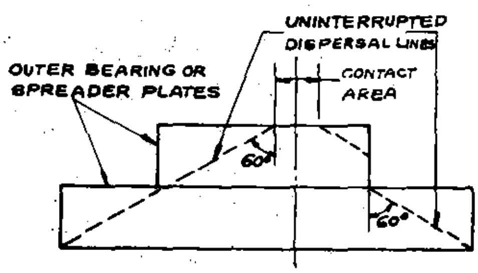

Fig. 17.6 *Load distribution through the metallic bearing plates*

Coefficient of Friction for Roller Bearings

For design purposes, the coefficient of friction for roller bearings should preferably be as given in Table 17.3(a).

Coefficient of Friction for Sliding Bearings

Recommended design coefficients of friction for bearings with stainless steel sliding on pure PTFE continuously lubricated are given in Table 17.3(b).

NOTE PTFE with lubricant contained in lubrication cavities (dimples) can be considered as continuously lubricated for the purposes of Table 17.3(b).

Table 17.3 (a) *Coefficient of friction of roller bearings*

	Type of roller bearing	Coefficient of friction
(a)	Roller bearing with one or two rollers in steel complying with BS 4360, or cast iron complying with BS 2789 with a hardness of 110 HB to 240 HB	0.03
(b)	Roller bearings as (a) but with more than two rollers	0.05
(c)	Single roller bearings with hardened steel contact surfaces with a hardness not less than 500 HB	0.02
(d)	Multiple roller bearings as (c)	0.03
(e)	Single roller bearings and bearing plates in special high tensile corrosion resistant steel hardened throughout with finely ground finish with a hardness not less than 350 HB	0.01
(f)	Multiple roller bearings as (e)	0.015

NOTE Values of hardness given in the above table are in accordance with BS 240:Part 1.

Table 17.3 (b) *Coefficient of friction for stainless steel sliding on pure PTFE continuously lubricated*

Bearing stress N/mm^2	Coefficient of friction
5	0.08
10	0.06
20	0.04
30 and over	0.03

NOTE

- Linear interpolation may be used for intermediate values. The load used for calculating the bearing stress should be that with which the coefficient of friction is being used.
- In the absence of test data, for design purposes, the coefficient of friction of pure unlubricated PTFE on stainless steel should be taken as twice the values given in Table 17.3. For PTFE sliding on any surface other than stainless steel, the coefficient of friction should be based on test data.

- The values given in Table 17.3(b) may be used for air temperatures down to $-24°C$.

Particular Recommendations for Roller and Rocker Bearings

(a) *Function* Roller bearings provide for transition in the direction of rolling only. Single rollers and rockers permit rotation about the line of contact, but multiple rollers require another element to provide for rotation, for example, a rocking saddle plate on top.

(b) *Curved surfaces* Any individual curved contact surface should have only one radius.

(c) *Surfaces in contract* Surfaces in contact should have the same nominal strength and hardness.

(d) *Length of rollers* The length of a roller should not be less than its diameter.

(e) *Guidance of rollers* Mechanical guidance should be provided to ensure that the axis of rolling is maintained in the desired orientation. Where gearing is used, the pitch circle diameter of the gear teeth should be the same as that of the rollers.

Allowable Loads on Steel and Cast Steel Roller and Rocker Bearings

(a) *Cylinder on Curved Surface* The design load effect per unit length on a cylinder of radius R running in a concave seating of redius R_1 should not exceed.

$$\frac{18\sigma_u^2}{E}\left(\frac{R_1 R}{R_1 - R}\right)$$

where, σ_u is the nominal ultimate tensile strength of the material

E is the modulus of elasticity of the material.

(b) *Cylinder on Flat Surface* The design load effect per unit length on a cylinder of radius R in contact with a flat surface should not exceed,

$$18R\sigma_u^2/E$$

where σ_u and E are as defined in (a) above.

(c) *Sphere in Spherical Seating* The vertical design load effect on a spherical surface of radius R in the concave seating of radius R_1 should not exceed,

$$\frac{170\sigma_u^3}{E^2}\left(\frac{R_1 R}{R_1 - R}\right)^2$$

where σ_u and E are as defined in (a) above.

(d) *Sphere on Flat Surface* The vertical design load effect on a spherical surface of radius R in contact with a flat surface should not exceed,

$$170R^2\sigma_u^3/E^2$$

where σ_u and E are as defined in (a) above.

Flat-sided (i.e., Cut) Rollers

If movement requirements permit, flat-sided rollers may be used. Such rollers should be symmetrical about the vertical plane passing through the centre. The minimum width should not be less than one-third of the diameter or such that the bearing contact does not fall outside the middle third of the rolling surface when the roller is at the extremes of movements. Flat-sided rollers can be mounted at closer centres than the circular rollers of the same load capacity, resulting in more compact bearings. However, they are not recommended in seismic areas.

Non-cylindrical Rollers

A single roller type of bearing with differing radii for the upper and lower curved surfaces of the roller can be designed using the appropriate expression given in (a) and (b) above. In all such designs, careful consideration should be given to the overall stability of the bearing. In particular, where the movement of the structure causes the line joining the upper and lower bearing contact points to depart from the vertical, a check should be made to ensure that the resulting horizontal force is resisted. Where the design of the bearing is such that horizontal movement is accompanied by a small vertical movement, the vertical movement should always be upward for horizontal movement on either side of the central position to ensure stability of the structure.

Multiple Rollers

For bearings having more than two rollers, the limiting values of design load effect should be taken as two-thirds of the value given by the expression in (b) above. This is so because more than two rollers may not be as fully effective as one or at best two in tandem might.

Particular Recommendations for Plane Sliding Bearings

(a) *Function* Plane sliding bearings normally provide for translation only; rotation can be permitted only with adequate provisions/incorporations.

(b) *Arrangement of Sliding Surfaces* Whenever possible, sliding bearings should have the larger of the sliding surfaces positioned above the smaller so that the sliding surfaces are kept clean.

(c) *Prevention of Rotation* Flat sliding surfaces should not be used to accommodate rotation other than about an axis perpendicular to the plane of sliding (e.g., in-plan rotation). Other provisions should be made for rotation about an axis in the plane of sliding.

Particular Recommendations for Sliding Elements with PTFE

(a) *Surfaces Mating with PTFE* Surfaces mating with PTFE should normally be stainless steel or hard anodized aluminium alloy; in all cases they should be harder than the PTFE and be corrosion resistant. The mating surface should normally form the upper component and overlap the PTFE at the extremes of movement.

(b) *Location of PTFE*

(i) *General* PTFE should be located either by confinement or by bonding. In either case it is essential that it is backed by a metal plate. The rigidity of this plate should be such that the plate retains its unloaded shape and resists shear forces under all loading conditions. The PTFE should be bonded or mechanically restrained in situations where the sliding surfaces can separate.

(ii) *Confined PTFE* Confined PTFE should be recessed into the metal backing plate. The shoulders of the recess should be sharp and square to restrict the flow of PTFE. The thickness of the PTFE and its protrusion from the recess should be related to its maximum plan dimension in accordance with Table 17.4.

Table 17.4 *Dimensions of confined PTFE*

Maximum dimension of PTFE (diameter or diagonal) (mm)	Minimum Th ckness (mm)	Maximum projection above races (mm)
≤ 600	4.5	2.0
> 600, ≤ 1200	5.0	2.5
> 1200, ≤ 1500	6.0	3.0

(iii) *Bonded PTFE* The thickness of the bonded PTFE sheet should be related to its maximum plan dimension in accordance with Table 17.5.

Table 17.5 *Thickness of bonded PTFE*

Maximum dimension of PTFE (diameter or diagonal) (mm)	Minimum thickness (mm)
≤ 600	1.0
> 600, ≤ 1200 (max.)	1.5

(c) *Allowable Sliding Bearing Pressures for Pure PTFE*

(i) *Maximum sliding contact pressures* For pure PTFE in bearings, the average pressure and the extreme fibre pressure should not exceed the values given in Table 17.6 ahead.

(ii) *Contact area* For calculation of pressures, the contact surface may be taken as the gross area of

Table 17.6 *Allowable sliding bearing pressures for pure PTFE*

Design load effects	Maximum average Contact pressure		Maximum extreme fibre pressure	
	Bonded PTFE (N/mm^2)	Confined PTFE (N/mm^2)	Bonded PTFE (N/mm^2)	Confined PTFE (N/mm^2)
Permanent design load effects	20	30	25	37.5
All design load effects	30	45	37.5	55

the PTFE without deduction for the area occupied by lubrication cavities. In the case of curved surfaces, the gross area should be taken as the projected area of the contact surface.

(d) *Thickness of Stainless Steel Sliding Surfaces* The thickness of the stainless steel sheet should be related to the difference between the PTFE and stainless steel dimension in the direction of movement in accordance with Table 17.7

Table 17.7 *Thickness of stainless steel sheet*

Dimensional difference between PTFE and stainless steel (mm)	Minimum thickness of stainless steel (mm)
≤ 300	1.5
> 300, ≤ 500	2.0
> 500, ≤ 1500	3.0

NOTE A dimensional difference in excess of 1500 mm requires special consideration.

(e) *Fixing of Stainless Steel Sheet*

(i) Stainless steel sheet should be attached to its backing plate by continuous welding along the edges or by fasteners supplemented by either peripheral sealing or full area bonding. It is essential that the method adopted ensures that the stainless steel sheet remains flat throughout its service life and interface corrosion cannot occur. The method of attachment should be capable of resisting the frictional forces set up in the bearing in the serviceability limit state.

(ii) *Attachment by welding* The backing plate should extend beyond the edges of the stainless steel sheet to accommodate the weld and the two should be attached by a continuous fillet weld along the edges. The weld should not be proud of the stainless steel sheet.

(iii) *Attachment by fasteners* Corrosion resistant fastenings, compatible with the stainless steel, should be used for securing the edges of the stainless steel sheet. They should be provided at all corners and along the outside edge, outside the area of contact with the PTFE sliding surface, with a maximum spacing of,

150 mm, for sheet 1.5 mm thick
300 mm, for sheet 2.0 mm thick
600 mm, for sheet 3.0 mm thick

Particular Recommendations for Elastomeric Bearings

(i) *Function* Elastomeric bearings can accommodate translational movements in any direction and rotational movements about any axis by elastic deformation. They should not be used in tension or when rotation is high and vertical load small. (Design stipulations, given ahead, will automatically draw the line for defining the latter while trying to satisfy the constraints.)

(ii) *Basis of design* The basis of the design is that the elastomer is an elastic material, the deflection of which under a compressive load is influenced by its shape (shape factor). Where reinforcing plates are included in the bearing, they should be bonded to the elastomer to prevent any relative movement at the steel/elastomer interface. Shape factor refers to an elastomer layer, not to the total assembly.

(iii) *Design recommendations* The design of elastomeric bearings should be such that:

- Their geometry satisfies the following conditions:
  - (i) The maximum strain of the elastomer due to translational movement does not exceed the limits given in the para ahead on Shear Strain.
  - (ii) The thickness of plain pad or strip bearings should not be less then 9 mm, to cater for irregularities in the seating surface.
  - (iii) The cover of elastomer to the steel interleaving plates in laminated bearings should be a minimum of 4.5 mm to all edges that would otherwise be exposed and a minimum of 2 mm to the contact surfaces, these values may need to be increased if there is a possibility of serious biological or chemical attack.
- They can resist the applied loads without exceeding:
  - (i) the mean pressure on plain pad or strip bearings (given ahead).
  - (ii) the maximum strain at any point in laminated bearings (given ahead).
  - (iii) the tensile stresses in the reinforcing plates (given ahead).
  - (iv) the stability criteria (given ahead).
- Their design movements satisfy the following conditions:
  - (i) the vertical deflection (calculated ahead) does not exceed the value specified by the engineer (i.e., the design authority).
  - (ii) the rotation of the bearing does not allow separation at the contact surfaces between the bearing and the structure; this may be deemed to be satisfied if the recommendations given ahead are met.
  - (iii) the force exerted on the structure by the bearing (resisting translational movement), calculated as follows, does not exceed the value specified by the engineer.

    Shear resistance of elastomeric bearings: For elastomeric bearings where horizontal movement is accommodated by shear in the elastomer, the nominal horizontal force H due to expansion or contraction is given by the expression,

$$H = AG\delta_r/t_q$$

where
A is the actual plan area of the individual elastomer slabs
G is the shear modulus of the elastomer
δ_r is the maximum resultant horizontal relative displacement of parts of the bearing.
t_q is the total thickness of elastomer in shear.
Typical values of G are given in Table 17.8. An allowance of ±20% should be made in the calculated value of H to give the most adverse effect.

Table 17.8 *Typical elastomer moduli*

Nominal hardness*	Shear modulus, G (N/mm^2)	Bulk modulus, E_b (N/mm^2)
IRHD		
50	0.6	2000
60	0.9	2000
70	1.2	2000

* Values of hardness in the above table are in accordance with BS 903: Part A26.

For movements due to live load effects on railway bridges, the value of G should be doubled (rapid deformation). Due allowance should be made in the value of G for temperature variation.

- Either they do not slip under the applied forces when checked in accordance with the data given ahead under 'Fixing of bearings' or they are mechanically fixed to the structure above and below.

(iv) *Design limit state* Elastomeric bearings should be designed to meet the provisions at the serviceability limit state only.

Shear Strain

The shear strain ϵ_q of the elastomer due to translational movement should not exceed 0.7, as given by the expression,

$$\epsilon_q = \delta_r/t_q$$

where δ_r is the maximum resultant horizontal relative displacement of parts of the bearing obtained by vectorial addition of δ_b and δ_l

δ_b is the maximum horizontal relative displacement of parts of the bearing in the direction of dimension b of the bearing due to all design load effects (Fig. 17.7)

δ_l is the maximum horizontal relative displacement of parts of the bearing in the direction of dimension l of the bearing due to all design load effects (Fig. 17.7).

t_q is the total thickness of the elastomer in shear.

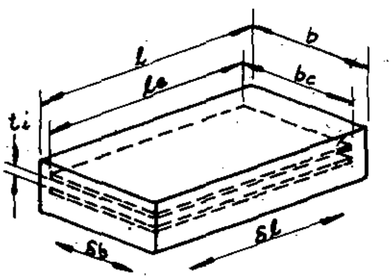

Fig. 17.7 *Elastomeric laminated bearing*

Shape Factor (S): Individual Layer of Elastomer

(i) The shape factor S is a means of taking account of the shape of the elastomer in strength and deflection calculations. It is the ratio of the effective plan area of an elastomeric layer to its force-free surface area (free to bulge) and is calculated as follows in (ii) to (iv).

Note that the factors associated with the effective thickness of the elastomer t_c in the expressions given in (ii) to (iv), allow for the fact that some slip will take place on faces restrained by friction only.

(ii) *Plain pad bearings* For plain pad bearings,

$$S = A/l_p t_c$$

where A is the overall plan area of the bearing

l_p is the force-free perimeter of the bearing, excluding that of any holes if these are not later effectively plugged

t_c is the effective thickness of elastomer in compression, which is taken as 1.8 t

t is the actual thickness of elastomer

Note that for a rectangular bearing without holes,

$$l_p = 2(l + b)$$

where l is the overall length of the bearing

b is the overall width of the bearing.

(iii) *Strip bearings* For strip bearings,

$$S = b/2t_e$$

where b and t_e are as defined in (ii) above.

(iv) *Laminated bearings* (Fig. 17.7) For laminated bearings, the shape factor S for each individual elastomer layer is given by the expression,

$$S = A_e/l_p t_e$$

where A_e is the effective plan area of the bearing, i.e., the plan area common to elastomer and steel plate, excluding the area of any holes if these are not later effectively plugged

l_p is as defined in (ii) above.

t_e is the effective thickness of an individual elastomer lamination in compression; it is taken as the actual thickness, t_i for inner layer, and $1.4t_i$ for outer layers.

t_i is the thickness of an individual elastomer layer

Note that for a rectangular bearing without holes,

$$A_e = l_e b_e$$
$$l_p = 2(l_e + b_e)$$

where l_e is the effective length of the bearing (= length of reinforcing plates)

b_e is the effective width of the bearing (= width of reinforcing plates)

Moduli of Elastomer

The shear modulus G should normally be obtained experimentally. Table 17.8 gives typical values of G and also an appropriate value for the bulk modulus E_b.

The variation of the shear modulus with low temperatures should be established by testing. For temperatures below 0°C, the values of G may, in the absence of test data, be taken as equal to the values in Table 17.8 multiplied by,

$$1 - \frac{T}{25}$$

where T is the minimum shade air temperature (in °C).

NOTE T is negative for temperatures below 0°C; the increased value of G applies only when variations in load and displacement take place at low temperature.

Design Pressure on Plain Pad and Strip Bearings

The mean design pressure (i.e., V/A) on a plain pad or strip bearing should not exceed GS or $5G$, whichever is the lesser,

where V is the vertical design load effect
A is the overall plan area of the bearing
G is the shear modulus of the elastomer
S is the shape factor of the elastomer slab (i.e., individual layer)

Maximum Design Strain in Laminated Bearings

At any point in the bearing the sum of the nominal strains due to all load effects, ϵ_t as given by the expression:

$$\epsilon_t = k(\epsilon_c + \epsilon_q + \epsilon_\alpha)$$

should not exceed 5.0 (see note at the end).

where k is a factor equal to 1.5 for live load effects 1.0 for all other effects (including wind and temperature),
ϵ_c is the nominal strain due to compressive loads where ϵ_c is given by the expression

$$\epsilon_c = 1.5V/GA_1S$$

ϵ_q is the shear strain due to translational movements as defined earlier.
ϵ_α is the nominal strain due to angular rotation, where ϵ_α is given by the expression,

$$\epsilon_\alpha = (b_e^2\alpha_b + l_e^2 \cdot \alpha_l)/2t_i\Sigma t_i$$

V and G are as defined earlier.
A_1 is the reduced effective plan area due to the loading effects, where A_1 is given by the expression

$$A_1 = A_e\left(1.0 - \frac{\delta_b}{b_e} - \frac{\delta_l}{l_e}\right)$$

A_e is as defined earlier
δ_b and δ_l are as defined earlier
b_e is the effective width of the bearing (Fig. 17.7)
l_e is the effective length of the bearing (Fig. 17.7)
S is the shape factor
α_b is the angle of rotation across the width b of the bearing (in radians)
α_l is the angle of rotation (if any) across the length l of the bearing (in radians)
t_i is the thickness of the individual layer of elastomer being checked
Σt_i is the total thickness of elastomer in the bearing

NOTE Value of 5.0 is an empirical value which has been found from fatigue tests on three types of elastomeric bearing to best fit the limiting criterion for a strain calculated by the method given here. It should not be taken to reflect the ultimate strain of the material.

Reinforcing Plate Thickness

To resist induced tensile stresses under load, the minimum thickness of the steel plates in a laminated bearing should be:

$$1.3V(t_1 + t_2)/A_1\sigma_s \text{ but not less than 2 mm}$$

where V and A_1 are as defined earlier.
t_1 and t_2 are the thickness of elastomer on either side of the plate
σ_s is the stress in the steel, which should be taken as not greater than the yield stress, nor greater than,
120 N/mm^2, for plates with holes
290 N/mm^2, for plates without holes

Stability

Elastomeric bearings will be stable if the recommendations of (i) and (ii) below are satisfied.

(i) *Plain pad and strip bearings.* For plain pad and strip bearings, the thickness should not exceed one-quarter of the least lateral dimension.

(ii) *Laminated bearings.* For laminated bearings the pressure V/A_1 should be less than $\{2b_eGS'/3\Sigma t_i\}$ This criterion will be satisfied automatically if

$$\Sigma t_i < b_e/4$$

where V, b_e, G, A_1 and Σt_i are as defined previously. S' is the shape factor for the thickest elastomer layer.

Vertical Deflection

(i) The vertical deflection of elastomeric bearings should be estimated from the expressions given ahead in (ii) to (iv). These expressions may be used to estimate the change in deflection between one-third of the total load and full load, with an accuracy of the order of $\pm 25\%$.

NOTE (a) The actual deflection of bearing may include an initial bedding-down phase that can produce a deflection of approximately 2 mm. Thereafter, the stiffness of the bearing increases with increasing load. Where the vertical deflection under load is critical to the design of the structure, the stiffness of the bearing should be ascertained by tests. However, a variation of as much as $\pm 20\%$

from the observed mean value may still occur. When a number of similar bearings are used at a support and the differential stiffness between the bearings is critical for the structure, a variation of compressive stiffness should be allowed for in the design, equal to either ±15% of the value estimated from (ii) to (iv) or ±15% of the mean value observed in tests.

(b) The calculations for the deflection of plain pad and strip bearings are likely to underestimate the deflection under permanent load and overestimate the deflection under transient loads.

(ii) *Plain pad bearings* The total vertical deflection of a plain pad bearing, Δ, is given by the expression,

$$\Delta = \frac{Vt}{5AGS^2} + \frac{Vt}{AE_b}$$

where V, G and S are as defined earlier
t and A are as defined earlier under 'Plain pad bearings'.
E_b is the bulk modulus of the elastomer

(iii) *Strip bearings* The total vertical deflection of a strip bearing, Δ, is given by the expression,

$$\Delta = Vt/5AGS^2$$

where V, G and S are as defined earlier
t and A are as defined earlier under 'Plain pad bearings'

(iv) *Laminated bearings* The total vertical deflection of a laminated bearing, Δ, is given by the expression,

$$\Delta = \Sigma\delta$$

$$\text{where } \delta = \frac{Vt_i}{5A_eGS^2} + \frac{Vt_i}{A_eE_b}$$

where
δ is the vertical deflection of an individual layer of elastomer
V, t_i, G and S are as defined earlier under 'Maximum design strain in laminated bearings'.
A_e is as defined earlier under 'Laminated bearings'.
E_b is the bulk modulus of the elastomer.

Rotational Limitation

The rotational limitation is satisfied if the recommendations (i) and (ii) given below are satisfied.

(i) *Plain pad and laminated bearings* For plain pad and laminated bearings, the total vertical deflection, Δ, should satisfy the expression:

$$\Delta > (b_e\alpha_b + l_e\alpha_l)/3$$

where b_e, α_b, l_e and α_l are as defined earlier.

(ii) *Strip bearings* For strip bearings, the total vertical deflection, Δ, should satisfy the expression:

$$\Delta > b_e\alpha_b/3$$

where b_e and α_b are as defined earlier.

Fixing of Bearings

If there is insufficient friction to prevent relative movement between the bearing and the structure under the most adverse loading conditions, positive means of location should be provided. Friction may be considered adequate if; under all loading conditions: numerically, $H < 0.1(V + 2A_1)$ and under permanent loads, $V/A_1 > (1 + b/l)$ for plain pad and strip bearings, and $V/A_1 > 2.0$ for laminated bearings.
Here all the terms and their units are as follows:

H is the design force exerted by the bearing to resist translational movement (in N)
V is the vertical design load effect (in N)
A_1 is the reduced effective plan area as defined earlier (in mm^2)
b is the overall width of the bearing (in mm) (Fig. 17.7)
l is the overall length of the bearing (in mm) (Fig. 17.7)

NOTE Positive means of location may limit the depth available for shear. This should be considered in the design of the bearing.

Particular Recommendations for Pot Bearings

(i) *Function* Pot bearings in themselves provide for rotational movements.

(ii) *Design* The stress in the elastomer in pot bearings due to the design load effects is limited by the effectiveness of the seal preventing it from extruding between the piston and the pot-wall, but it should not exceed 40 N/mm^2 at the serviceability limit state. The lateral pressure exerted on the confining cylinder walls resulting from vertical loading on the confined elastomer disc can be considered to be that produced by the disc acting as a fluid. Because details of pot bearings vary considerably and stress analysis is complex, their design should be verified by testing.

(iii) *Rotation* The rotation of pot bearings about a horizontal axis should be limited so that the vertical strain induced at the perimeter of the elastomeric pad, at the serviceability limit state, does not exceed 0.15.

NOTE The thickness and hardness of the elastomer have a direct relationship with the resistance of pot bearings to rotation, as does the friction between the piston and the pot. The latter is increased by increased force acting on the bearing. Sufficient test results should be available for a given elastomer stress, hardness and thickness to enable the resistance of the bearing to rotation to be calculated; otherwise, prototype tests should be made.

(iv) *Seal* A sealing device should be provided to prevent the elastomer extruding between the piston and the pot-wall. This seal should be effective under serviceability limit state loadings.

Particular Recommendations for Guides

1. *Function* Guides are used to constrain the movement of structures in a particular direction. They may be included in an independent guide bearing or may form a part of a bearing performing other functions.

2. *Sliding surfaces for guides*

(i) *PTFE Facing* Guides used for lateral restraint may be faced with unfilled or filled PTFE provided the frictional resistance to movement at the guides is either significantly smaller than that of the main bearing or the resulting frictional effects are taken into account. Commonly used materials for facing-guides are,

- unfilled PTFE
- PTFE filled with up to 25% by mass of glass fibres
- lead filled PTFE in a bronze matrix
- PTFE reinforced with a metal mesh

(a) *Lubrication* For this application (i.e., use in guides) lubrication of PTFE should not be considered to reduce friction.

(b) *Attachment* It is essential that all PTFE should be securely attached to the guides: reliance should not be placed on bonding alone for pure PTFE.

(ii) *Unfaced surfaces* For surfaces not intended to be in permanent contact, metal-to-metal contact may be permitted. The metal should be corrosion resistant.

3. *Allowable bearing pressure on guides*

(i) *PTFE* Under all serviceability design load effects, the average pressure on glass filled PTFE in guides should not exceed 45 N/mm^2, and on PTFE in a metal matrix 60 N/mm^2. Permissible values for other PTFE materials should be established by tests. In the absence of test data, the values for PTFE should be used.

For calculation of pressures, the contact surface may be taken as the gross area of the PTFE without deduction for the area occupied by any lubrication cavities (dimples).

(ii) *Bronze* At the serviceability limit state, the contact bearing stress for bronze should not exceed 30 N/mm^2.

17.8 DESIGN OF LINEAR CONCRETE HINGE (ROCKER) BEARING

Concrete hinges are simple and cheap to produce but require extremely careful construction and detail. They permit large rotations if they are constructed properly and accurately with correct dimensions. Such hinges do not require any corrosion protection and have the same life as that of the structure into which and along with which they are built. Such hinge bearings require no maintenance.

The design rules stipulated ahead are based on the experiments conducted at Stuttgart, West Germany, and the further work done at the EMPA (the Swiss Federal Laboratory for Testing Materials, etc.), Zurich.*

The functional form of a linear concrete hinge capable of rocking (rotating) about its length-axis, the directions of the three types of bursting forces (F_1, F_2 and F_3) set up in it, and the appropriate reinforcement details to resist these forces, are diagrammatically shown in the sketches in Fig. 17.8. The throat of the hinge (dimension a) should be narrow (generally 15 cm) so as to develop little resisting moment (M_{TT}) to rotation. For this purpose, the reinforcements against the three bursting forces should be adequate so as to permit a narrow throat of the hinge. A narrow throat and very low height of the hinge (generally about 2 cm), with circular curved faces all around, ensures an almost three dimensional confinement of hinge-concrete. This allows the hinge-concrete to accept even up to about 6 times its (standard 28-day cube) crushing compressive stress. Herein lies the secret of the working of a concrete hinge. Most of the cracking owing to rotation of permanent type and of the oscillating transient type heals itself owing to creep under the incumbent high compressive stress. Such hinges rarely require any reinforcements crossing the neck (along its longitudinal centrêline) unless horizontal shear exceeds about one-eighth of the co-existing vertical load and/or longitudinal moment (M_{LL}) on the hinge causes tension after allowing for compression due to vertical load. In fact, ensuring accurate placement and positioning of such bars is not easy and it is advisable to create conditions whereby such reinforcement can be avoided. (Usually this is the case.)

It is reported that such hinges can rotate in opposite directions (oscillating rotation) without compromising safety. In such cases the entire throat cracks in effect, with part of its remaining closed at any one time. The joint surfaces which re-close again and again, remain fully effective. In the EMPA tests on concrete hinges of a large railway bridge (working load up to 450 tonnes on a throat plan-section of 15 cm × 70 cm) the hinge tolerated rotations of up to 0.012 radian millions of times. In all, 37 million rotational loadings of varying magnitudes were undertaken. After all that, the joint, under static load tests, did not show any sign of cracking even with a 900 tonne load (twice the working load) coupled with a rotation of 0.006 radian. The joint was brought to breaking point under an exceptional

* *References*: Betongelenke, F. Leonhardt: Vorlesungen-uber Massivbau Zweiter Teil. Sondefalle der Bemessung im Stahlbetonbau: F. Leonhardt and E. Monnig: Zweite Auflage, Springer-Verlag, Berlin-Heidelberg-New York, 1975; and the EMPA report on the subject.

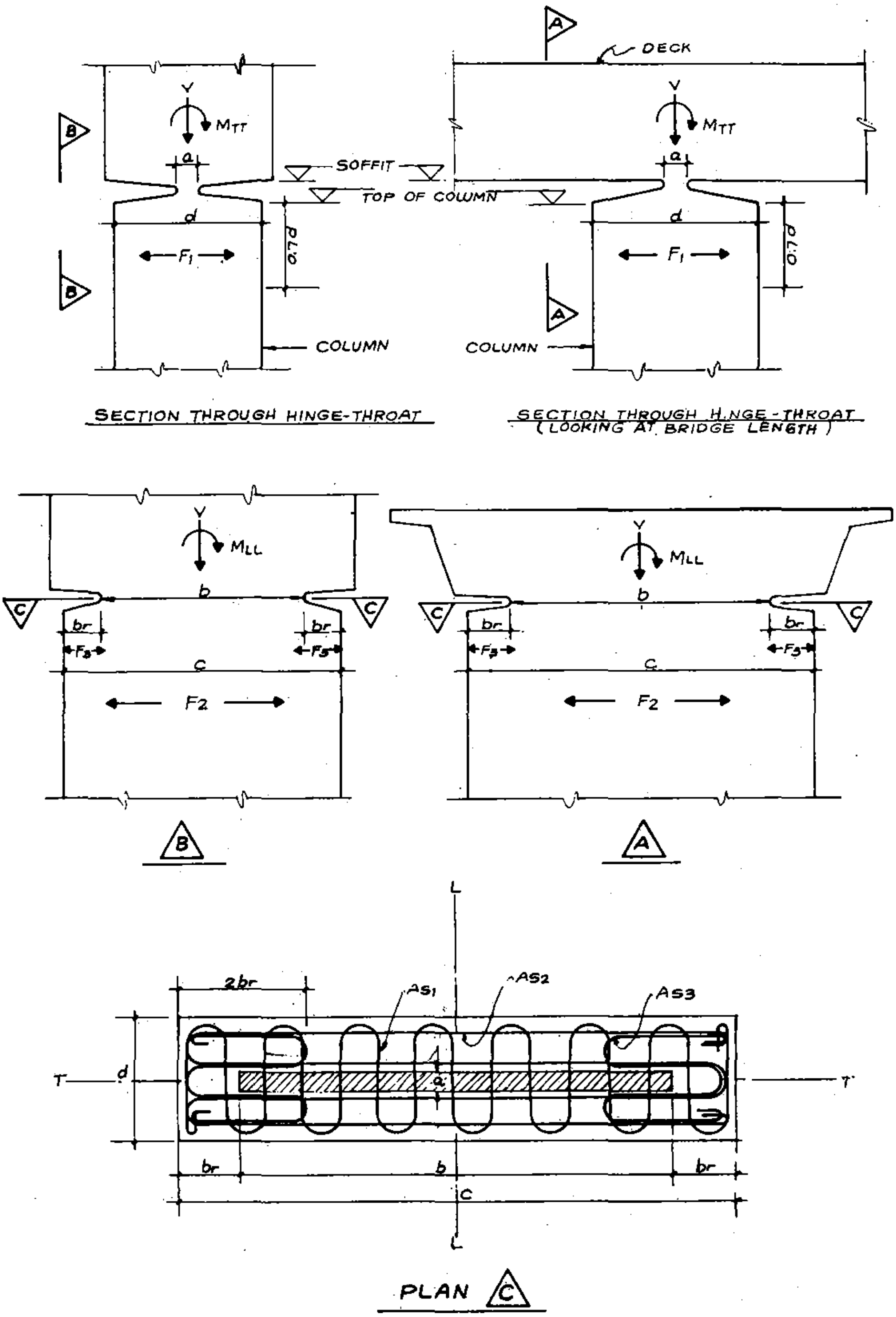
DECK
V
M_{TT}
a
SOFFIT
TOP OF COLUMN
d
F_1
0.7d
COLUMN
SECTION THROUGH HINGE-THROAT
SECTION THROUGH HINGE-THROAT
(LOOKING AT BRIDGE LENGTH)
M_{LL}
b
br
F_3
c
F_2
L
2br
A_{S1}
A_{S2}
A_{S3}
T
PLAN

Fig. 17.8

rotation of 0.500 radian (coupled with a load of 250 tonnes). The experiments, therefore, indicate that such joints are capable of resisting large dynamic loadings coexisting with large rotations.

Geometry of the Hinge Throat

Following dimensional constraints are stipulated (see Figs. 17.8 and 17.9)

- $a \leq 0.3d$, generally 15 cm.
- $b_r \geq 0.7a$, but never less than 5 cm.
- $t \leq 0.2a$, but never more than 2 cm.
- $r = 0.5t$
- Tangent $\beta \leq 0.1$
- For details see Figs. 17.8 and 17.9.

Design Steps

Step 1 *Estimate*: V_p, V_t, V_{max}, V_{min}, M_{LL}, θ_p, θ_t, H_L and H_T at the hinge,

where V_p = permanent vertical load on hinge (dead load, secondary prestress, settlement effect, etc.)

V_t = transient vertical load on hinge (live load, temperature, etc.), various cases

V_{max}, and V_{min} = maximum and minimum vertical loads on hinge, various cases

M_{LL} = moment on hinge, about bridge longitudinal axis, various cases

θ_p = permanent rotation at the hinge (due to dead load, prestress, creep, shrinkage, settlement, etc.)

θ_t = transient rotation at the hinge (due to live load, temperature, braking force, etc.), various cases

H_L, H_T = horizontal forces at the hinge, along the bridge longitudinal and transverse axes, under various cases

Step 2 Decide the 28-day cube crushing strength u of the concrete in the hinge. This should be same as for the deck with which it will be cast, and it is preferable to have same strength concrete at least in the top one meter height of the column below the hinge, u shall not be less than 250 kg/cm²

Step 3 Select the various hinge-dimensions and its construction-joint detail. (Refer to Geometry of the Hinge-Throat discussed earlier, and decide the dimensions b, c and d, see Figs. 17.8 and 17.9.)

Step 4 Compute the plan section-area of the hinge-throat (ignoring any reinforcement passing through the hinge) $A = ab$, and ensure that this A is not more than A_{max} and not less than A_{min},

where $$A_{max} = \frac{V_p}{1.25\theta\sqrt{u}}$$

$$A_{min} = \frac{V_{max}}{0.85u\left\{1 + \lambda\left(1 - \dfrac{1.47\theta\eta}{\sqrt{u}}\right)\right\}}$$

noting that

- A_{max} and A_{min} are in cm²
- V_p and V_{max} are in kg ($V_p \leq 1.5V_{min}$)
- u is in kg/cm² (cube strength at 28 days)

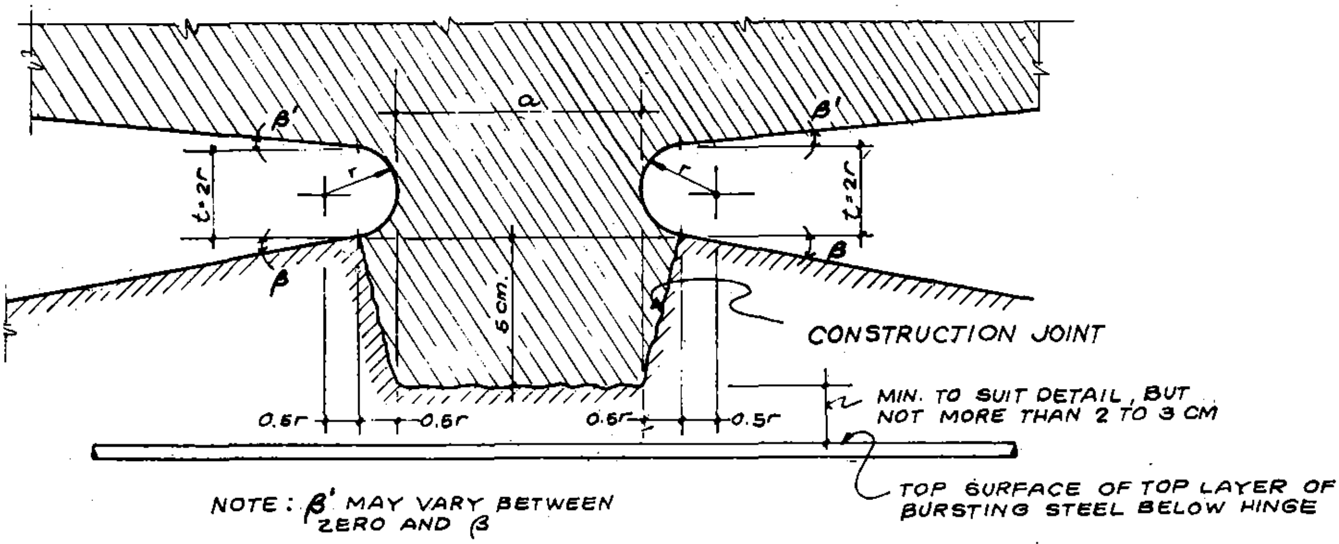

Fig. 17.9

- $\theta = (0.5\theta_p + \theta_t)$, in radians
- $\lambda = (1.2 - 4a/d)$, but never more than 0.8
- $\eta = \frac{V_{\max}}{V_p}$, but never less than 1.0

Step 5 Estimate allowable rotation, θ_a and ensure that $(\theta_p + \theta_t)$ does not exceed it, noting that

$\theta_a = \pm\frac{0.8V}{A\sqrt{u}}$, but $\not> \pm$ 0.015 rad. ($\pm$ means 'allowable in either direction'),

where V = total vertical load on hinge under the loading condition considered for evaluating θ_p and θ_t (V may be taken as V_p, on the safer side, if V is greater than V_p).

Units: V (and V_p) in kg, A in cm^2, u in kg/cm^2, θ_a in radian

Step 6 Reinforcement in the column immediately below the hinge:

Estimate the steel reinforcements A_{s1}, A_{s2}, and A_{s3}, against the bursting forces F_1, F_2 and F_3 respectively, as follows:

(i) $F_1 = 0.3V_{\max.}$, $\therefore A_{s1} = F_1/f_s$

(ii) $F_2 = 0.3\left(1 - \frac{b}{c}\right)V_{\max.}$, $\therefore A_{s2} = F_2/f_s$

(iii) $F_3 = 0.3V_{\max.}a/b$, $\therefore A_{s3} = F_3/f_s$

Note that,

- f_s = permissible normal working tensile stress in the steel reinforcement, which should not exceed 1800 kg/cm^2 in order that the cracks remain finer.
- while it is preferable to adopt smaller bar diameters to spread the splitting cracks (i.e., allow more but finer cracks), the detailing may be done using 16 mm bars for A_{s1} and 20 to 25 mm bars for A_{s2}, for ease in concreting. This is because F_1 being the biggest bursting force, A_{s1} controls the major bursting tendency and hence the above bar-diameter consideration is more strictly required for A_{s1} (i.e., smaller diameter for A_{s1}).
- for A_{s1} detailing, provide its each horizontal layer in the shape as indicated in Fig. 17.8, providing the necessary number of layers as required, but accommodating them all within a depth of 0.7 d below the top of the column.
- for A_{s2} detailing, the shape in each layer should be as indicated in Fig. 17.8, and the remaining details as for A_{s1} described above.
- detailing of A_{s3} presents no problems as its amount in relatively small, shape is as indicated in Fig. 17.8, and its amount can be easily accommodated within even less than 0.7 d below the top of the column.

Step 7 Estimate the rotational-resistance-moment, M_{TT}, of the hinge (since it is not actually a knife-edge rocker). This is obtained from the hinge-geometry, its concrete strength, and the rotation imposed on it. It is preferable to have $M_{TT}/V_{\max.}$ not greater than $a/6$ to avoid tension due to this moment. Also, as this much moment will travel down into the column and the foundation, their respective designs should account for it.

$$M_{TT} = maV_{\max.} \text{ kg cm}$$

where $m = \left(0.5 - \frac{1}{9\sqrt{\phi\theta}}\right)$, and preferably this

$m \not> \frac{1}{6}$

$\theta = (0.5\theta_p + \theta_t)$ as defined earlier

$\phi = \frac{ab\sqrt{u}}{V_{\max}}$

Units: a and b in cm, u in kg/cm^2, $V_{\max.}$ in kg and θ in radian.

Note that this rotational-resistance-moment reduces with time owing to concrete creep.

Step 8 Check for horizontal shear in the hinge.

H_R is resultant horizontal force $(= \sqrt{H_L^2 + H_T^2})$ through the hinge.

(i) If $H_R \not< 0.125V_{\max.}$ then no need to provide any vertical dowel bars through the hinge.

(ii) If $H_R > 0.125V_{\max.}$ but $\leq 0.25V_{\max.}$, then provide vertical dowel bars through the hinge along its linear axis, their section area in cm^2 being

$$A_s = \frac{H_R}{800} \qquad (H_R \text{ in kg})$$

(iii) If $H_R > 0.25V_{\max.}$, it is preferable to revise the whole design and change over to other forms of rocker bearings. However, reference may also be made to the EMPA tests, referred to earlier.

Step 9 Check against tension in the hinge under M_{LL}. It is strongly recommended to ensure that the ratio of M_{LL} to the coexisting vertical load V on the hinge does not exceed $b/6$ so that there is no tension caused in the hinge. Generally this is the case. (If necessary, the hinge dimension b may be suitably altered, taking care then to review the Steps 1 to 8 for meeting their checks.) However, if tension cannot be avoided, then large diameter vertical dowel bars (armour bars) will have to be provided through the hinge near the ends of its dimension b at the middle of its throat-width, so that owing to the lever arm between them, they can take the whole of M_{LL}. Tensile stress in these armour bars may be restricted to 70% of their normal working stress value. These armour bars must be well anchored into the concrete above and below the hinge, but extreme care must be taken in ensuring their positions accurately. Therefore, their lengths should be short.

For this purpose, coarse threads may be cut at their ends and nuts screwed on them. However, the bonding zones of these armour bars should be assumed to commence only after a length equal to throat-width dimension a above and below the hinge-throat. The provision of the screwed-on nuts enhances bonding force, thereby permitting shorter armour bar lengths which facilitates their handling for accurate placement while casting the hinge concrete. The bonding zones should be secured with helical steel binders against local spalling. If large rotation coexists with M_{LL} then the above armour bars should additionally be sleeved with suitable plastic tubing up to the start of the helical binders to maintain them free from bending in order that they can rotate with the deck.

Step 10 Reinforcement immediately above the hinge:

(i) If the structure above the hinge is similar to that below it, then provide same reinforcement and detailing as that below the hinge, as in Step 6.

(ii) If the structure above the hinge is a deck, much wider than the hinge-length b, and relatively infinitely longer than the column dimension d then the area ab may be taken as if it is a standard bearing plate area (loading plate) on the deck and the necessary reinforcement against bursting be calculated as in a traditional prestressing end-block. Approximately, the bursting force is

$$F = 0.3V_{\text{max}}.\left(1 - \frac{b}{2w}\right)$$

where w = deck-width at soffit

If six reinforcement grills are provided, the lowest at a distance x above the soffit and the remaining five respectively at x, $2x$, $3x$, $4x$, and $5x$ above the lowest grill (such that their centroid above the soffit falls at a height equal to $\frac{w}{4}$, giving $x = \frac{w}{14}$, but actual value to be adjusted to suit detail), then the effective area of bars in each of the two orthogonal directions in a grill may be taken as $\frac{F}{6f_{s_t}}$ where f_s is the normal working tensile stress in the steel (not exceeding 1800 kg/cm^2, as explained earlier).

Care in Constructing the Hinge

(i) Concrete in about one meter height at the top of the column (pier) below the hinge should preferably be of the same grade as that of the concrete in the hinge itself.

(ii) The hinge should be cast along with the deck, i.e., the structure above it, and the two should have the same grade of concrete (which should not be less than 250 kg/cm^2, standard cube strength nor less than that required by design of the deck).

(iii) Construction joint detail should preferably be as shown in Fig. 17.9

(iv) When concrete in the top portion of the column has set but not sufficiently hardened, the surface which will subsequently receive the *in situ* hinge and deck concrete should be brushed with a stiff wire brush (just sufficiently enough to remove the mortar skin and loose aggregates, exposing the larger aggregates without disturbing them) and the surface should be cleaned. Later, just prior to pouring concrete on it at the time of casting the hinge and the deck, this surface should again be thoroughly cleaned and wetcured with water for about two hours, thereafter blowing away the water collected in the recess with compressed air.

(v) Care should be taken not to disturb the hinge concrete and not to subject it to any kind of vibration, deformation or movement until after the deck is structurally ready for taking finishings and live load.

(vi) In its form-work and shuttering, the geometry of the concrete hinge should be followed strictly as per design details shown in the approved execution drawing. The form-work and shuttering for producing the hinge in all its details of dimensions, circular face, profiles, slopes, etc., should be almost 'sculpted' to give its exact geometry.

(vii) The temporary forms used to give the exact geometry of the concrete hinge should be made of a material such that:

  (a) the forms do not deform in any manner under the weight of wet concrete above, taking account of vibrations and moving about of workmen,

  (b) the forms can be removed reasonably easily without damage to the surrounded and the surrounding concrete.

(viii) The forms for the hinge that are sandwiched between the deck and the column should be carefully loosened at the time of prestressing of the deck only to the extent that the hinge is free to rotate during the prestressing operations.

Qualitative Comparison of Concrete Hinge with Dowelled Elastomeric Bearing and Fixed Pot Bearing

This is shown in the following Table 17.9, for a single bearing carrying approximately an average vertical load of 1600 tonnes.

17.9 DETAILS OF LAMINATED NEOPRENE BEARINGS

Some useful details about laminated (i.e., restrained neoprene bearings that are formed by vulcanizing all the restraining steel laminates and all the elastomer layers in one single operation altogether):

(i) *Description* The elastomeric bearings consist of rubber layers with steel plates vulcanized altogether in order to ensure a bond on their connection faces. The plates are completely embedded in the rubber to prevent any corrosion.

(ii) *Material specifications* See Tables 17.10(a) and 17.10(b).

(iii) *Handling and storage on site* Bearings should be stored under cover, away from sunlight, heat, oil and chemicals.
They should always be handled and stacked carefully. Damaged bearings, for example, with bent steel interleaving plates or partially debonded layers, should never be installed for use.

(iv) *Installation*

(a) *Seating* Where the support is concrete, its cast surface usually being irregular, the bearings should be placed on it accurately to line and level on a 6 mm thick bedding of stiff mortar. This can be ordinary sand/cement with a low water/cement ratio, or a mortar of fine dry sharp sand and chemical resin and hardener. In either case, the cube crushing strength of the mortar should be at least 20 N/mm^2. Where the support is steel, a rolled surface may be suitable for use directly, provided that it is

Table 17.9

Qualitative Comparisons	Concrete Hinge	Dowelled Elastomeric bearing	Fixed Pot Bearing
1. Vertical load carrying capacity	Good	Only very few elastomeric bearings have elastomers of strength to carry such heavy loads. Probably a special bearing is called for, unless numbers increase.	Good
2. Horizontal load carrying capacity	Good	Limited by crushing of concrete locally in pier and deck by local bearing of dowels against concrete.	Good
3. Appearance in elevation	Very small gap between top of pier column and deck gives the best detail in elevations.	Gap is deeper than in the case of hinge but the fine line between deck and column still maintained.	Fixed pot bearing itself requires a gap significantly deeper than elastomeric, but guided-bearings need 50% more gap (nearly)
4. Cost	Cheapest by far. Almost nil.	A little less expensive than the pot bearings but not by much.	Most expensive.
5. Delivery	No delivery problems for cast-in-place concrete.	Imported from special either one or at most two suppliers (high load). Hence supply and paperwork problems.	As in case of elastomeric bearings, but a slightly easier problem as pot bearings are manufactured by many suppliers.
6. Ease of construction	If the performed throat formwork is planned and made correctly and held in place during concreting, the hinge poured with deck becomes normal concreting.	Fixing dowels into top of column needs care and attention. Fixing bearings is a 2-stage operation. (a) grout-in dowels (b) bed-bearing slipping over top of dowels.	Mechanical fixings into top of column but, as deck is wide, then free sliding or guided sliding bearings are also required. Alignment of guides requires care.
7. Maintenance	Maintenance free.	If installed correctly, requires very little maintenance.	PTFE Sliding faces of sliding bearings have not yet proved themselves in terms of design life when fine airborne dust is considered.
8. Replacement	If abutment elastomeric bearings are to be replaced then each end of deck slightly jacked up with hinge holding down at the pier.	Producer as with pot bearings except jacking needs to be higher to clear dowels which sometimes stick—generally very inconvenient.	Whole deck is jacked up off central pier and abutments. Bearings removed and replaced.

reasonably smooth and true to level, but otherwise some surface preparation will be needed.

Trowelling often seems to produce a bedding that is slightly rounded on the top surface, and it is preferable to screed off or cast against a flat plate.

(b) *In situ superstructure* When the superstructure is to be in concrete which is cast *in situ*, the spaces around and between the bearings can be filled with expanded polystyrene or wellrammed damp sand covered with an impervious membrane such as polythene sheet. Extreme care must be taken not to disturb the bearings during casting, and a temporary bond to the substructure with an impact adhesive will help. After curing of the superstructure, the sand-infill can be washed away from around the bearings or the polystyrene can be broken up and blown out with compressed air. (It should not be dissolved, because the solvent may attack the elastomer in the bearings.)

(c) *Precast concrete superstructures* Where precast concrete beams are being used they should be lowered on to a mortar skim (2–3 mm), on the top of the bearings, to eliminate soffit irregularities and twist in the beams. The bearings should be so seated as to accommodate the rotation due to the precamber of the beams at this very low level of vertical loading, or the beams should be suitably propped near the bearings until the mortar skim has hardened into a wedge, so that the bearings are not rotated at this stage.

(v) Some suitable laminated (i.e., restrained) Neoprene bearings for different capacities (typical, for general guidance):

- The Tables 17.11 to 17.14 ahead are given for guidance only and allow the designer to make a preliminary selection within the range of Freyssinet International's standard bearings.
- For each table, the thickness of individual elastomer inner layer, and the associated thickness of the reinforcing steel plates are uniform as shown.
- The thickness of each outer elastomer layer is equal to half the thickness of the inner elastomer layer.
- The elastomer edge cover e is in every case equal to 5 mm.
- The values given in the tables are calculated in accordance with BS 5400:Section 9.1:1983, taking into account the following parameters and assumptions:
  - — Shear modulus of elastomer = G = 0.9 N/mm^2
  - — Bulk modulus of elastomer = E_b = 2000 N/mm^2
  - — Live load effect is similar to dead load effect (for load and rotation)
  - — Horizontal force on the bearing causes shear in the direction of the width of the bearing,
  - — Rotation is only across the width of the bearing (i.e., $\alpha_l = 0$)
  - The most important design criterion, and the one that normally governs, used for the calculations is:

    $k(\epsilon_c + \epsilon_q + \epsilon_\alpha) \leq 5.0$ (Symbols as explained earlier) where

    k is a factor equal to 1.5 for live load effects and 1.0 for all other effects.
- The maximum allowable load is given for each bearing for four cases, viz.,

  Case 1 No rotation, no shear
  $\epsilon_\alpha = 0$ and $\epsilon_q = 0$
  Case 2 No rotation, maximum shear
  $\epsilon_\alpha = 0$ and $\epsilon_q = 0.7$
  Case 3 Maximum rotation, no shear
  $\epsilon_\alpha = b_e^2\alpha_b/2t_i\Sigma t_i$ and $\epsilon_q = 0$
  Case 4 Maximum rotation, maximum shear
  ϵ_α = same as case 3 and $\epsilon_q = 0.7$
- The other two vertical-load design-limits are given by:

  Criterion of minimum thickness of the steel reinforcing plates, which is satisfied for the Freyssinet standard range of bearings

  Criterion of stability which is automatically satisfied for bearings for which $\Sigma t_i \leq b_e/4$ (symbols as explained earlier).
- The maximum value of shear movement is given by:

$$\epsilon_q = \frac{\delta_r}{t_q} = 0.7$$

- The shear-stiffness is calculated from the formula: $\dfrac{H}{\delta_r} = \dfrac{AG}{t_q}$, where G = 0.9 N/mm^2 (Also see the specification stated earlier for railway bridges.)
- The maximum allowable rotation, shown in Tables 17.11 to 17.14 has been calculated (following the expressions given in the specifications earlier) assuming V = 100 kN, so that actual max. allowable rotation will be equal to

$$\left\{ \text{tabulated value} \times \left[\frac{\text{actual vertical load (kN)}}{100}\right]\right\}$$

in ($\times 10^{-3}$ radian) units.

Table 17.10(a) *Elastomer*

Property	Test standard	Specified value			
Raw polymer		natural rubber (NR) or chloroprene rubber (CR)			
Hardness (IRHD)	B.S. 903: Part A26 (Method N)	NR		(CR)	
		60 ± 5	70 ± 5	60 ± 5	70 ± 5
Shear modulus G(N/mm^2)	B.S. 903 : Part A14 (Shear strain = 0.25)	0.90 ± 0.14	1.20 ± 0.18	0.90 ± 0.14	1.20 ± 0.18
Tensile strength R(N/mm^2)	B.S. 903 : Part A2	min 15.5			
Elongation at break A (%)	B.S. 903 : Part A2	min. 300	min. 300	min. 350	min. 30
Compression set (%)	B.S. 903 : Part A6	(24 hrs. at 70°C) max. 30		(24 hrs. at 100°C) max. 35	
Ageing resistance ~	B.S. 903 : Part A19	(7 days at 70°C)		(3 days at 100°C)	
• change in hardness-(IRHD)		max. 10		max. 15	
• change in tensile-strength (%)		max. 15		max. 15	
• change in elongation at break (%)		max. 20		max. 40	
Ozone resistance	B.S. 903 : Part A43	No cracks (25 pphm/20% strain 96 hrs. at 30°C)			
Low temperature resistance					
• brittleness	B.S. 903 : Part A25	— brittleness temperature = max. − 25°C			
• stiffening	B.S. 5400 : 9.2	— change in hardness = max. 15			
• crystallization	B.S. 903: Part A39	— compression set = max. 65%			
Bonding of elastomer to metal	B.S. 903 : Part A21	bond peel strength = min. 7 N/mm^2			

Table 17.10(b) *Reinforcing Plates*

Property	Standard	Specified value
Raw material	B.S. 1449 : Part 1	Rolled mild steel shee
Tensile strength (N/mm^2)		min. 400
Yield strength (N/mm^2)		min. 300
Elongation at break (%)		min. 20

Table 17.11 • *Elastomer inner layer thickness* $t_i = 8$ *mm*
• *Reinforcing steel plate thickness* $t_s = 2$ *mm*

Plan size (mm)	Overall height (mm)	Maximum dead load + live load (kN): No rotation, No shear	No rotation, maximum shear	Maximum rotation, No shear	Maximum rotation, Maximum shear	Shear movement (mm)	Shear stiffness (kN/mm)	Max. allowable rotation about longer axis (10^{-1} rad/100 kN
1	2	3	4	5	6	7	8	9
150 × 100	10	104	80	75	60	5.6	1.69	20.714
150 × 100	20	104	75	60	47	11.2	0.84	61.886
150 × 100	30	97	70	57	43	16.8	0.56	103.059
150 × 100	40	73	55	55	40	22.4	0.42	144.231
150 × 100	50	58	40	54	38	28.0	0.34	185.403
200 × 100	10	157	121	119	94	5.6	2.25	12.434
200 × 100	20	157	113	97	75	11.2	1.13	36.997
200 × 100	30	147	105	91	69	16.8	0.75	61.561
200 × 100	40	110	83	89	65	22.4	0.56	86.124
200 × 100	50	88	61	87	61	28.0	0.45	110.688
200 × 150	10	322	255	220	178	5.6	3.38	3.087
200 × 150	20	322	244	177	142	11.2	1.69	9.051
200 × 150	30	322	234	166	131	16.8	1.13	15.016
200 × 150	40	322	223	162	125	22.4	0.84	20.980
200 × 150	50	281	212	159	121	28.0	0.68	26.945
200 × 150	60	235	178	157	117	33.6	0.56	32.909
250 × 150	10	446	353	317	256	5.6	4.22	2.074
250 × 150	20	446	338	258	205	11.2	2.11	6.041
250 × 150	30	446	324	242	190	16.8	1.41	10.008
250 × 150	40	446	309	236	182	22.4	1.05	13.976
250 × 150	50	390	294	232	175	28.0	0.84	17.943
250 × 150	60	325	274	229	170	33.6	0.70	21.910

(Contd.)

Table 17.11 *(Contd).*

1	2	3	4	5	6	7	8	9
300 × 150	10	575	455	419	338	5.6	5.06	1.531
300 × 150	20	575	436	343	273	11.2	2.53	4.437
300 × 150	30	575	410	324	253	16.8	1.69	7.342
300 × 150	40	575	399	315	242	22.4	1.27	10.247
300 × 150	50	503	380	310	233	28.0	1.01	13.152
300 × 150	60	419	319	306	226	33.6	0.84	16.058
250 × 200	20	725	563	380	306	11.2	2.81	2.365
250 × 200	30	725	546	357	285	16.8	1.88	3.904
250 × 200	40	725	528	347	274	22.4	1.41	5.444
250 × 200	50	725	510	341	266	28.0	1.13	6.984
250 × 200	60	718	493	337	260	33.6	0.94	8.523
250 × 200	70	615	475	334	254	39.2	0.80	10.063
250 × 200	80	538	411	332	249	44.8	0.70	11.603
300 × 200	20	949	737	516	416	11.2	3.38	1.704
300 × 200	30	949	714	486	388	16.8	2.25	2.808
300 × 200	40	949	690	472	372	22.4	1.69	3.912
300 × 200	50	949	667	465	362	28.0	1.35	5.017
300 × 200	60	939	644	460	353	33.6	1.13	6.121
300 × 200	70	805	621	456	346	39.2	0.96	7.225
300 × 200	80	704	538	453	339	44.8	0.84	8.329
350 × 200	20	1181	919	659	531	11.2	3.94	1.311
350 × 200	30	1181	888	622	496	16.8	2.63	2.158
350 × 200	40	1181	860	606	477	22.4	1.97	3.004
350 × 200	50	1181	831	596	463	28.0	1.58	3.850
350 × 200	60	1169	802	589	452	33.6	1.31	4.697
350 × 200	70	1002	773	585	443	39.2	1.13	5.543
350 × 200	80	877	670	582	434	44.8	0.98	6.390
400 × 200	20	1420	1103	808	651	11.2	4.50	1.056
400 × 200	30	1420	1068	764	608	16.8	3.00	1.735
400 × 200	40	1420	1033	744	585	22.4	2.25	2.414
400 × 200	50	1420	999	732	569	28.0	1.80	3.094
400 × 200	60	1405	964	725	555	33.6	1.50	3.773
400 × 200	70	1205	930	720	544	39.2	1.29	4.452
400 × 200	80	1054	805	716	532	44.8	1.13	5.132

Table 17.12 • *Elastomer inner layer thickness t_i = 10 mm*
• *Reinforcing steel plate thickness t_s = 3 mm*

Plan size (mm)	Overall height (mm)	Maximum dead load + live load (kN)				Shear movement (mm)	Shear stiffness (kN/mm)	Max. allowable rotation about longer axis (10^{-3} rad/100 kN)
		No rotation		Maximum rotation				
		No shear	maximum shear	No shear	Maximum shear			
1	2	3	4	5	6	7	8	9
300 × 250	26	1097	852	568	458	14.0	3.38	1.559
300 × 250	39	1097	826	534	426	21.0	2.25	2.574
300 × 250	52	1097	799	518	410	28.0	1.69	3.590
300 × 250	65	1097	773	509	398	35.0	1.35	4.605
300 × 250	78	1097	747	503	389	42.0	1.13	5.621
300 × 250	91	940	720	499	381	49.0	0.96	6.636
400 × 250	26	1669	1296	915	738	14.0	4.50	0.935
400 × 250	39	1669	1256	863	688	21.0	3.00	1.539
400 × 250	52	1669	1216	840	662	28.0	2.25	2.144
400 × 250	65	1669	1176	826	643	35.0	1.80	2.748
400 × 250	78	1669	1136	817	628	42.0	1.50	3.353
400 × 250	91	1430	1096	811	615	49.0	1.29	3.957
400 × 300	26	2257	1772	1154	935	14.0	5.40	0.529
400 × 300	39	2257	1728	1088	875	21.0	3.60	0.869
400 × 300	52	2257	1683	1058	843	28.0	2.70	1.209
400 × 300	65	2257	1638	1041	822	35.0	2.16	1.548
400 × 300	78	2257	1593	1029	806	42.0	1.80	1.888
400 × 300	91	2257	1548	1021	792	49.0	1.54	2.227
400 × 300	104	2046	1503	1016	779	56.0	1.35	2.567
400 × 300	117	1818	1423	1011	767	63.0	1.20	2.907

(Contd.)

Table 17.12

1	2	3	4	5	6	7	8	9
500 × 300	26	3107	2439	1647	1334	14.0	6.75	0.363
500 × 300	39	3107	2377	1558	1251	21.0	4.50	0.595
500 × 300	52	3107	2315	1516	1207	28.0	3.38	0.826
500 × 300	65	3107	2254	1493	1177	35.0	2.70	1.057
500 × 300	78	3107	2192	1477	1154	42.0	2.25	1.289
500 × 300	91	3107	2130	1467	1134	49.0	1.93	1.520
500 × 300	104	2815	2068	1459	1116	56.0	1.69	1.751
500 × 300	117	2503	1959	1452	1099	63.0	1.50	1.983
600 × 300	26	3992	3135	2167	1754	14.0	8.10	0.272
600 × 300	39	3992	3055	2054	1648	21.0	5.40	0.445
600 × 300	52	3992	2976	2001	1592	28.0	4.05	0.617
600 × 300	65	3992	2896	1971	1553	35.0	3.24	0.789
600 × 300	78	3992	2817	1952	1522	42.0	2.70	0.962
600 × 300	91	3992	2736	1938	1496	49.0	2.31	1.134
600 × 300	104	3618	2658	1928	1472	56.0	2.03	1.307
600 × 300	117	3216	2517	1920	1449	63.0	1.80	1.479

Table 17.13 • *Elastomer inner layer thickness $t_i = 12$ mm*
• *Reinforcing steel plate thickness $t_s = 3$ mm*

Plan size (mm)	Overall height (mm)	Maximum dead load + live load (kN)				Shear movement (mm)	Shear stiffness (kN/mm)	Max. allowable rotation about longer axis (10^{-3} rad/100 kN)
		No rotation		Maximum rotation				
		No shear	Maximum shear	No shear	Maximum shear			
450 × 350	30	2869	2250	1464	1186	16.8	5.91	0.438
450 × 350	45	2869	2192	1379	1108	25.2	3.94	0.719
450 × 350	60	2869	2133	1340	1067	33.6	2.95	1.001
450 × 350	75	2869	2075	1318	1040	42.0	2.36	1.283
450 × 350	90	2869	2016	1304	1019	50.4	1.97	1.564
450 × 350	105	2869	1958	1293	1001	58.8	1.69	1.846
450 × 350	120	2541	1899	1286	985	67.2	1.48	2.123
450 × 350	135	2258	1756	1280	970	75.6	1.31	2.409
500 × 400	45	4150	3203	1924	1552	25.2	5.00	0.399
500 × 400	60	4150	3129	1871	1499	33.6	3.75	0.554
500 × 400	75	4150	3055	1841	1464	42.0	3.00	0.709
500 × 400	90	4150	2981	1821	1437	50.4	2.50	0.865
500 × 400	105	4150	2908	1807	1414	58.8	2.14	1.020
500 × 400	120	4150	2834	1797	1394	67.2	1.88	1.175
500 × 400	135	3747	2760	1789	1376	75.6	1.67	1.330
600 × 400	45	5403	4169	2586	2086	25.2	6.00	0.292
600 × 400	60	5403	4073	2518	2016	33.6	4.50	0.405
600 × 400	75	5403	3977	2479	1969	42.0	3.60	0.518
600 × 400	90	5403	3881	2454	1933	50.4	3.00	0.632
600 × 400	105	5403	3785	2436	1903	58.8	2.57	0.745
600 × 400	120	5403	3689	2423	1876	67.2	2.25	0.858
600 × 400	135	4877	3593	2412	1851	75.6	2.00	0.971
600 × 450	45	6543	5039	2967	2401	25.2	6.75	0.205
600 × 450	60	6543	4986	2889	2324	33.6	5.06	0.285
600 × 450	75	6543	4883	2844	2273	42.0	4.05	0.864
600 × 450	90	6543	4780	2815	2235	50.4	3.38	0.443
600 × 450	105	6543	4677	2795	2204	58.8	2.89	0.522
600 × 450	120	6543	4574	2780	2177	67.2	2.53	0.602
600 × 450	135	6543	4471	2768	2151	75.6	2.25	0.681
600 × 450	150	5998	4367	2759	2128	84.0	2.03	0.760
600 × 450	165	5453	4264	2751	2105	92.4	1.84	0.839
600 × 500	45	7739	6056	3328	2702	25.2	7.50	0.151
600 × 500	60	7739	5947	3242	2618	33.6	5.63	0.209
600 × 500	75	7739	5837	3192	2564	42.0	4.50	0.267
600 × 500	90	7739	5728	3159	2524	50.4	3.75	0.325
600 × 500	105	7739	5618	3136	2491	58.8	3.21	0.384
600 × 500	120	7739	5509	3120	2463	67.2	2.81	0.442
600 × 500	135	7739	5400	3107	2438	75.6	2.50	0.500
600 × 500	150	7739	5290	3096	2415	84.0	2.25	0.558
600 × 500	165	7182	5181	3088	2392	92.4	2.05	0.616

17.10 SOME OF THE VERSATILE AND MODERN LARGER CAPACITY BEARINGS

The case may however arise where due to heavy loads, large displacements or rotation, laminated elastomeric bearing pads may not be suitable. In this case other devices must be adopted.

In order to satisfy such a requirement. Freyssinet (and indeed other specialist manufacturers too) manufacture and market other types of bearing devices:

- Sliding bearings: Neoflon
- Mechanical bearings (spherical type and pot or disc type): Tetron

Some Useful Details About Some Freyssinet Spherical Bearings (Tetron Type S3 Range)

Material Specifications

- Tetron 'S3' bases and rockers use maintenance-free aluminium alloy
- Sliding plates are made of mild steel, faced with high quality stainless steel
- Sliding surfaces are lined with pure PTFE, to BS. 3784
- Pins for side restraints are special spring-steel with minimum yield strength 1100 N/mm^2.
- All permanently exposed steel surfaces are corrosion protected with a metallic zinc rich epoxy coating, followed by chlorinated rubber paint.
- Full details of material specifications are available from the specialist manufacturers.
- *Contact stress* The average base contact stress of the bearings, illustrated ahead, approaches 17.5 N/mm^2. Direct contact between the aluminium parts of the bearings and dissimilar metals must be avoided.

Table 17.14 • *Elastomer inner layer thickness t_i = 12 mm*
• *Reinforcing steel plate thickness t_s = 3 mm*

Plan size (mm)	Overall height (mm)	Maximum dead load + live load (kN): No rotation, No shear	No rotation, Maximum shear	Maximum rotation, No shear	Maximum rotation, Maximum shear	Shear movement (mm)	Shear stiffness (kN/mm)	Max. allowable rotation about longer axis (10^{-3} rad/100 kN)
600 × 600	57	8215	6416	3458	2806	31.5	7.20	0.158
600 × 600	76	8215	6295	3362	2714	42.0	5.40	0.220
600 × 600	95	8215	6174	3306	2655	52.5	4.32	0.281
600 × 600	114	8215	6054	3271	2611	63.0	3.60	0.342
600 × 600	133	8215	5933	3245	2576	73.5	3.09	0.403
600 × 600	152	8215	5813	3227	2546	84.0	2.70	0.465
600 × 600	171	8215	5692	3213	2519	94.5	2.40	0.526
600 × 600	190	8078	5571	3201	2493	105.0	2.16	0.587
700 × 600	57	10358	8089	4490	3642	31.5	8.40	0.120
700 × 600	76	10358	7937	4371	3526	42.0	6.30	0.167
700 × 600	95	10358	7785	4302	3451	52.5	5.04	0.213
700 × 600	114	10358	7633	4257	3396	63.0	4.20	0.260
700 × 600	133	10358	7481	4226	3350	73.5	3.60	0.306
700 × 600	152	10358	7329	4203	3311	84.0	3.15	0.352
700 × 600	171	10358	7177	4185	3276	94.5	2.80	0.399
700 × 600	190	10186	7025	4171	3243	105.0	2.52	0.445
700 × 700	57	13140	10346	5248	4272	31.5	9.80	0.078
700 × 700	76	13140	10181	5109	4142	42.0	7.35	0.108
700 × 700	95	13140	10016	5029	4060	52.5	5.88	0.138
700 × 700	114	13140	9851	4978	4000	63.0	4.90	0.167
700 × 700	133	13140	9686	4941	3953	73.5	4.20	0.197
700 × 700	152	13140	9521	4914	3912	84.0	3.68	0.227
700 × 700	171	13140	9356	4894	3877	94.5	3.27	0.257
700 × 700	190	13140	9191	4877	3844	105.0	2.94	0.287
800 × 800	57	18557	15622	7437	6069	31.5	12.80	0.043
800 × 800	76	18557	15405	7251	5897	42.0	9.60	0.059
800 × 800	95	18557	15189	7144	5789	52.5	7.68	0.075
800 × 800	114	18557	14973	7074	5712	63.0	6.40	0.092
800 × 800	133	18557	14757	7025	5651	73.5	5.49	0.108
800 × 800	152	18557	14540	6989	5600	84.0	4.80	0.124
800 × 800	171	18557	14324	6961	5556	94.5	4.27	0.140
800 × 800	190	18557	14108	6939	5516	105.0	3.84	0.157

(a) Tetron S3T: Fixed (Table 17.15 & Fig. 17.10)

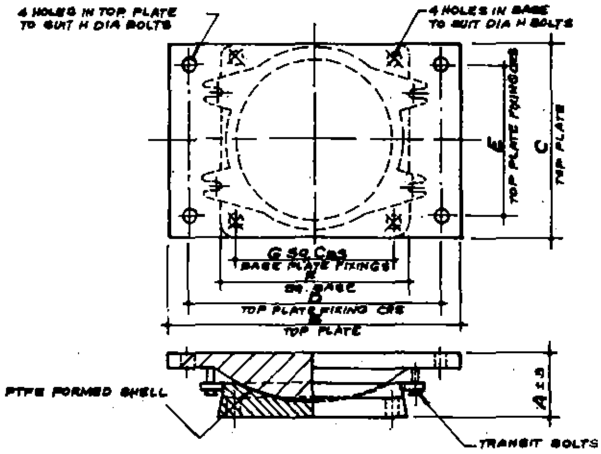

Fig. 17.10 *Tetron S3 T fixed*

Table 17.15

Bearing type	Principal dimensions (mm)								'Working stress' design (kN)		BS 5400: Section 9.1 Design Load effects (kN)			
									Max. load Vertical	Max. load Horizontal	Serviceability Limit State		Ultimate Limit State	
↓	A	B	C	D	E	F	G	H			Vertical	Horizontal	Vertical*	Horizontal
S3T70	70	400	215	340	145	230	177	M12	700	65	700	65	900	110
S3T150	100	475	305	410	235	305	254	M20	1500	150	1500	150	2000	210
S3T250	105	510	360	435	280	375	297	M20	2500	250	2500	250	3000	300
S3T300	115	570	400	490	310	420	336	M24	3000	280	3000	280	3800	400
S3T350	130	600	440	515	335	465	364	M24	3500	300	3500	330	4500	500
S3T400	140	660	485	565	355	500	396	M30	4000	360	4000	360	5000	600
S3T500	145	685	510	585	375	535	417	M30	5000	450	5000	450	6300	650
S3T600	155	760	565	640	430	600	463	M30	6000	500	6000	500	7500	710
S3T750	180	930	660	795	500	705	548	M30	7500	600	7500	600	9500	900
S3T1000	195	970	710	830	535	755	583	M30	10000	750	10000	750	12500	1100
S3T1200	215	1080	790	925	595	840	654	M30	12000	900	12000	900	15000	1300

* Assuming $\gamma_m = 1.2$ (partial material factor B S 5400, 9.1, 1983); NOTE: 1. 'Bearing type' column indicates maximum vertical design load in tonnes for 'Working Stress' design or B S 5400 serviceability limit state. 2. Larger bearings are also available.

(b) Tetron S3F sliding guided (Table 17.16 & Fig. 17.11)

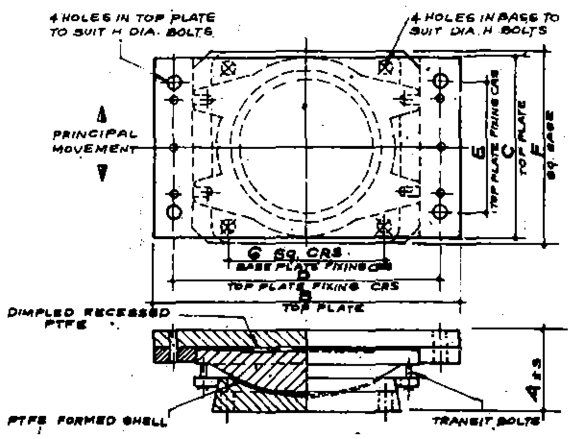

Fig. 17.11 *Tetron S 3 F sliding guided*

Table 17.16

Bearing type	Principal dimensions (mm)								'Working stress' design (kN)		BS 5400: Section 9.1 Design load effects (kN)			
									Max. load	Max. load	Serviceability Limit State		Ultimate Limit State	
↓	A	B	C	D	E	F	G	H	Vertical	Horizontal	Vertical	Horizontal	Vertical*	Horizontal
S3F70	100	351	270	286	190	230	177	M12	700	65	700	65	900	110
S3F150	130	497	360	432	280	305	254	M20	1500	150	1500	150	2000	210
S3F250	135	541	420	476	340	375	297	M20	2500	250	2500	250	3000	300
S3F300	145	636	460	546	360	420	336	M24	3000	280	3000	280	3800	400
S3F350	155	680	480	590	400	465	364	M24	3500	300	3500	300	4500	500
S3F400	170	718	540	628	460	500	396	M24	4000	360	4000	360	5000	600
S3F500	175	750	570	660	470	535	417	M30	5000	450	5000	450	6300	650
S3F600	190	814	625	724	525	600	463	M30	6000	500	6000	500	7500	710
S3F750	225	922	720	832	620	705	548	M30	7500	600	7500	600	9500	900
S3F1000	240	971	770	881	670	755	583	M30	10000	750	10000	750	12500	1100
S3F1200	265	1055	860	965	760	840	654	M30	12000	900	12000	900	15000	1300

* Assuming $\gamma_m = 1.2$ (partial material factor, BS 5400, 9.1, 1983)

NOTE

(i) Basic bearing as tabulated is for zero movement, and is specified typically as S3F250/00. For additional movements increase C and E dimensions by 100 mm per increment when bearing is described in a code, e.g. S3F250/10 (movement in cm)

(ii) 'Bearing type' column indicates maximum vertical design load in tonnes for 'working stress' design or BS 5400 serviceability limit state.

(iii) Larger bearings are also available.

(iv) ±3° (0.052 radian) rotation in any plane.

(c) Tetron S3E Free Sliding **(Table 17.17 & Fig. 17.12 a, b)**

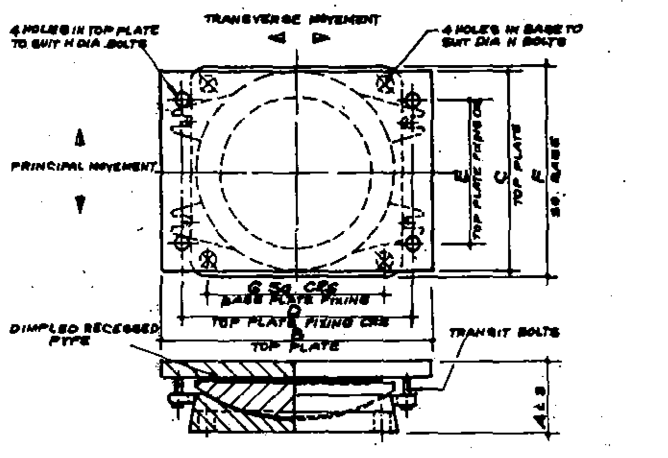

Fig. 17.12(a) *Tetron S3E free sliding*

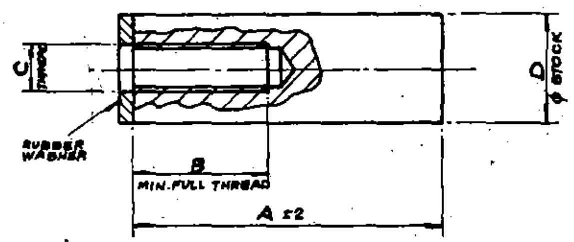

Fig. 17.12(b)

Standard socket detail

A	B	C	D
100	30	M12	40
100	50	M20	40
160	60	M24	40
220	75	M30	50

Fixing All bearings have provision for Grade 8.8 fixing bolts which are designed to cater for the horizontal force to which each specific type may be subjected, with some assistance from friction due to the minimum vertical load normally present in service. For ease of installation and to provide complete removability, bearing should be secured to cast-in sockets where possible.

Installation of Freyssinet Tetron Spherical (Type S3 Range) Bearings

In a bearing that is free sliding in all directions (i.e., the S3E range) positive fixing to the main structure may not be required if the bearing is always subjected to adequate vertical loading, because horizontal movement will occur on the plane of least resistance which is of course the bearing sliding surface. Nevertheless, in some cases it is prudent to provide fixings to guard against displacement during installation, impact, vibration and accidental unloading.

Most cases of bearing malfunction are attributable to faulty installation; and almost all bearing damage occurs during installation, or even earlier during handling and storage. Careless handling on site, and the ingress of dirt, can easily lead to abnormally high frictional resistance. Tetron 'S' bearings are normally delivered in a condition to discourage unnecessary dismantling, and bolts are used to connect together the upper and lower parts of the bearings. These temporary fixings, as well as excluding dirt during installation, prevent accidental relative displacement

Table 17.17

Bearing type	Principal dimensions (mm)								'Working stress' design (kN)	BS 5400: Section 9.1 Design Load (kN)	
									Max. load Vertical	Serviceability Limit State	Ultimate Limit State
	A	B	C	D	E	F	G	H		Vertical	Vertical*
S3E70	90	340	265	280	205	230	177	M12	700	700	900
S3E150	105	395	325	335	265	305	254	M20	1500	1500	2000
S3E250	120	485	385	410	315	375	297	M20	2500	2500	3000
S3E300	125	555	420	495	360	420	336	M20	3000	3000	3800
S3E350	140	620	460	560	400	465	364	M20	3500	3500	4500
S3E400	145	675	490	615	430	500	396	M20	4000	4000	5000
S3E500	150	710	520	650	460	535	417	M20	5000	5000	6300
S3E600	160	740	575	680	515	600	463	M20	6000	6000	7500
S3E750	190	815	660	755	600	705	548	M20	7500	7500	9500
S3E1000	200	910	720	840	650	755	583	M20	10000	10000	12500
S3E1200	220	1000	800	925	725	840	654	M20	12000	12000	15000

* Assuming $\gamma_m = 1.2$ (partial material factor, B S 5400, 9.1, 1983)

NOTE

(i) Basic bearing as tabulated is for zero movement in both the principal and transverse directions and is specified typically as S3E 150/0000.

(ii) For additional movement C and E increase by 100 mm per increment of principal movement and/or B and D increase by 100 mm per increment of transverse movement when bearing is described in a code. e.g.,

Transverse Movement (cm.)

S3E 150/10 10

Principal Movement (cm.)

(iii) 'Bearing type' column indicates maximum vertical design load in tonnes for 'Working stress' design or BS 5400 Serviceability Limit State.

(iv) Larger bearings are also available.

(v) ±3° (0.052 radian) rotation in any plane.

between the parts of the bearings, but they must be removed before the bearings are called upon to slide or rotate. However, they are not structural fixings and should be supplemented, for example, by wedges during installation.

Bearings should be clearly identified, by marking them with such details as type and location. Bearings should be transported and unloaded carefully and then stored under cover in clean, dry conditions until required.

As much preparatory work as possible should be carried out before bringing a bearing to its actual location. The seating should be level, and this usually necessitates the use of a mortar bedding, composed of sand and either cement, polyester resin or epoxy resin, with a cube crushing strength of at least 35 N/mm^2.

Base Fixings

An 'E' type bearing is free sliding in all directions, with a low coefficient of friction, so that the base may not require positive fixing, but see note above. All other types of bearing can resist horizontal loading and so their bases should be fixed.

There are several ways of achieving this. A recess can be left in the pier/abutment, into which the bearing base is placed bodily, bedded upon and surrounded with resin mortar (see Fig. 17.13). The recess must be correctly reinforced on all sides. Alternatively, small pockets can be left for dowels or bolts, which are then set accurately to position and level in grout using timber or light steel jig. The bearing base is then lowered over the dowels or bolts, on to a bed of wet mortar (Fig. 17.14).

Fig. 17.13

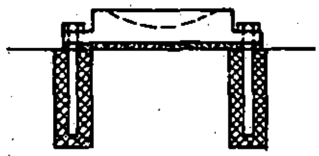

Fig. 17.14

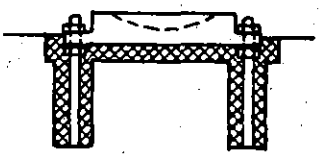

Fig. 17.15

NOTE
(i) Bearings should not be dismantled on site because the effects of dirt on the sliding and rotating surfaces are highly deleterious.

(ii) In all cases the transit bolts must be removed after the mortar has set and before the bearing is called upon to rotate or slide. These bolts are identified by metal labels.

It is possible to cast-in the fixing bolts and position the bearing in one operation, in which case the bearing is supported on rubber washers, over levelling nuts on the bolts, and a chemical resin grout is then poured around the bolts and under the bearing (Fig. 17.15). The fixings should not be over-tightened, when the grout has set, otherwise the bearing may be distorted due to compression in the washers. Nevertheless, rubber washers must be used, to prevent the fixing bolts carrying vertical loading.

Top Fixings

For similar reasons, top fixings must be provided for all bearings (optional for the E type).

A completed precast concrete structure may be lowered on to a skim of mortar on the top of the bearing, to eliminate soffit irregularities. The mortar mix needs careful control to ensure that it is not totally squeezed out by the weight of the superstructure, which must be supported until it has set. The fixing sockets for the top of the bearing should be ready cast into the soffit, but this requires accurate casting, using a jig drilled mould insert to match the bolting arrangement in the bearing. Alternatively, the sockets may be replaced by a single plate, cast in flush with the soffit, and tapped to receive the bearing fixing bolts.

Where a concrete superstructure is cast *in situ*, the bearing top plates, with dowels, can be built into the soffit shuttering. The area around the mould cut out must be carefully sealed to ensure that concrete mortar does not leak into the working parts of the bearing during placing, and all sliding plates must be propped to prevent them distorting under the weight of wet concrete.

Great care must be exercised to ensure that F range lateral restraint bearings are correctly oriented.

Some Useful Details About Freyssinet Pot Bearings (Tetron Disc Type, D3 Range)

Material Specification of Disc Bearings

Rubber disc Natural Rubber to BS1154 (+ anti-ozonants)
Ring High grade heat treated steel
Side Restraints (in D3F type only) Mild steel
Pins for side restraint Special spring-steel, minimum yield strength 1100 N/mm^2.
Seal and sliding surface High grade Stainless steel.
Top plate, rocker and base plate Mild Steel. PTFE BS.3784
Socket head cap screws BS.4168 Grade 12.9
Recommended HD bolt BS.3692 Grade 8.8 (zinc plated)
Cast-in sockets Mild Steel

All permanently exposed steel parts are corrosion protected with two-component metallic zinc rich epoxy coating followed by chlorinated rubber paint. Full technical data can be obtained from manufacturers.

Components

- Tetron Disc bases and rockers, and all sliding plates, are made of corrosion protected mild steel.
- Sliding plates are faced with a smooth surface of high quality stainless steel.
- Sliding surfaces are pure 'dimpled' PTFE, incorporating grease pockets which allows a permanent reservoir of lubricant.
- Elastomer used for rotational purposes in Tetron Disc bearings is high grade natural rubber to BS 1154.
- A mastic seal is provided around the rocker to prevent ingress of moisture into the base unit.
- These bearings can be removed from the structure with minimum of jacking when used in conjunction with special cast-in sockets or similar devices.

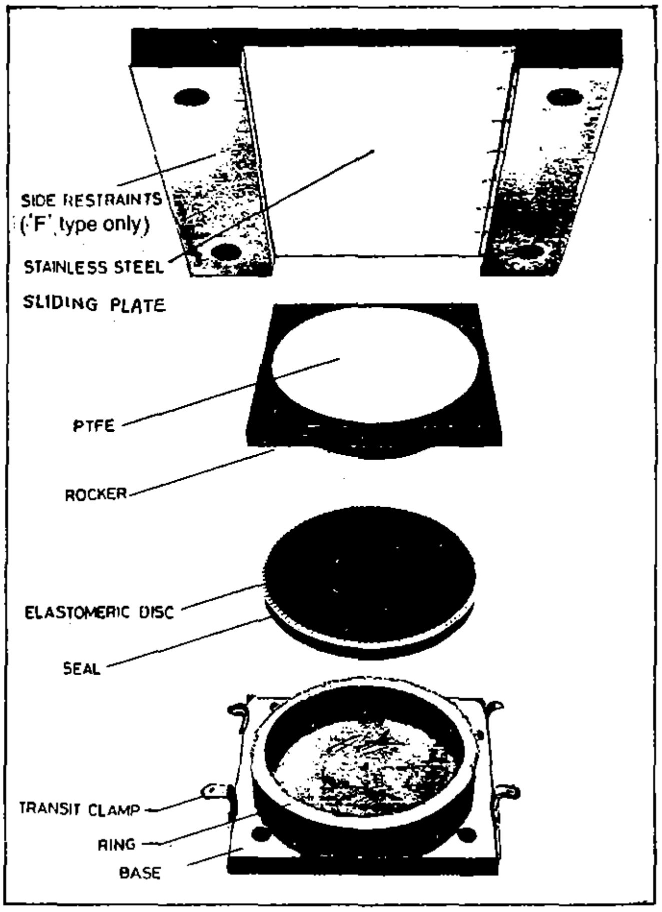

Fig. 17.16 *Tetron disc (pot) bearing*

(a) Tetron D3T Fixed (Table 17.18)

Fixed in all directions, free to rotate in all directions.

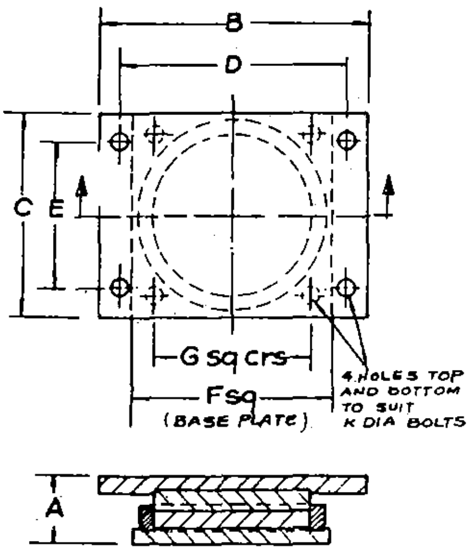

Fig. 17.17 *Tetron D3T fixed*

Table 17.18

Bearing type	Principal dimensions								Max. load vertical (kN)	Max. load horizontal (kN)
	A	B	C	D	E	F	G	K		
D3T 50	58	235	170	195	120	170	130	M12	500	100
D3T 80	75	340	235	280	155	235	175	M20	800	150
D3T 100	80	355	250	295	170	250	190	M20	1000	150
D3T 125	83	375	270	315	190	270	210	M20	1250	190
D3T 160	92	395	290	335	210	290	230	M20	1600	220
D3T 200	97	460	335	385	240	335	260	M20	2000	250
D3T 250	97	485	360	410	270	360	285	M20	2500	280
D3T 325	116	575	410	475	280	410	310	M20	3250	300
D3T 400	127	615	450	515	330	450	350	M24	4000	360
D3T 500	132	680	515	580	390	515	410	M30	5000	500
D3T 650	141	770	570	645	420	570	440	M30	6500	600
D3T 800	156	835	635	710	490	635	510	M30	8000	650
D3T 1000	175	950	710	805	540	710	560	M30	10000	700
D3T 1250	179	1015	785	870	620	785	640	M30	12500	900
D3T 1600	203	1140	870	970	680	870	700	M30	16000	1000
D3T 2000	203	1260	985	1090	780	985	800	M36	20000	1300
D3T 2500	232	1425	1100	1220	875	1100	895	M42	25000	1600
D3T 3000	257	1550	1230	1350	1000	1230	1030	M42	30000	2000

Rotation as for D3E type (see ahead Table 17.19)

(See notes at the end of Table 17.20)

(b) Tetron D3E Free Sliding (Table 17.19)

Free sliding in all directions, free to rotate in all directions.

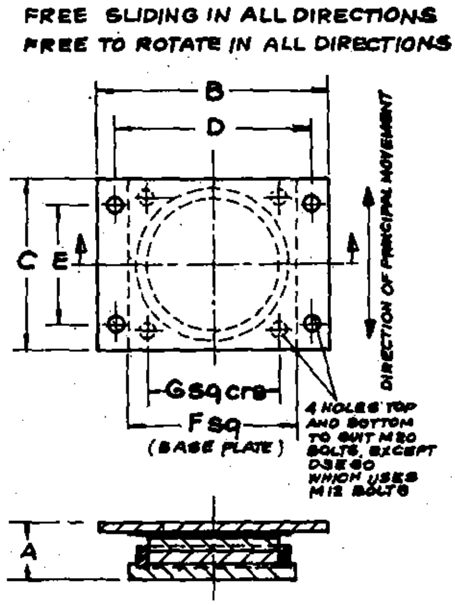

Fig. 17.18 *Tetron D3E free sliding*

Table 17.19

Bearing type	Principal dimensions							Max. load vertical	Normal rotation
	A	B*	C*	D*	E*	F	G	(kN)	(Radian)
D3E 50	60	235	170	195	120	170	130	500	0.026
D3E 80	73	340	235	280	155	235	175	800	0.026
D3E 100	78	355	250	295	170	250	190	1000	0.026
D3E 125	82	375	270	315	190	270	210	1250	0.026
D3E 160	86	395	290	335	210	290	230	1600	0.026
D3E 200	92	460	335	385	240	335	260	2000	0.026
D3E 250	99	485	360	410	270	360	285	2500	0.024
D3E 325	112	515	375	455	280	410	310	3250	0.022
D3E 400	128	555	420	495	330	450	350	4000	0.022
D3E 500	128	620	465	560	390	515	410	5000	0.020
D3E 650	137	675	510	615	420	570	440	6500	0.018
D3E 800	147	740	575	680	490	635	510	8000	0.016
D3E 1000	162	815	635	755	540	710	560	10000	0.016
D3E 1250	168	910	700	840	620	785	640	12500	0.014
D3E 1600	183	1000	780	925	680	870	700	16000	0.012
D3E 2000	193	1155	875	1055	770	985	800	20000	0.012
D3E 2500	213	1270	970	1170	865	1100	895	25000	0.012
D3E 3000	228	1440	1080	1310	950	1230	1030	30000	0.012

* Dimensions B, C, D and E are for zero movement, and specified movement has to be added to above in increments of 100 mm

(See notes at the end of Table 17.20)

(c) Tetron D3F Sliding guided (Table 17.20)

Sliding guided in one direction, free to rotate in all directions.

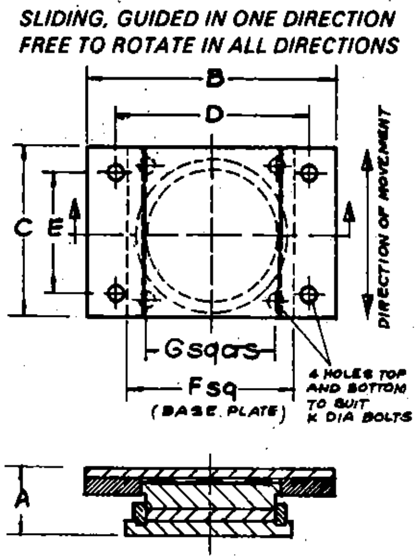

Fig. 17.19 *Tetron D3F sliding guided*

Table 17.20 *(for Tetron D3F sliding guided)*

Bearing type	Principal dimensions								Max. load vertical	Max. load horizontal
↓	A	B*	C*	D	E*	F	G	K	(kN)	(kN)
D3F30	75	225	160	180	110	160	120	M12	300	80
D3F50	79	260	170	195	120	170	130	M12	500	100
D3F80	103	345	235	280	155	235	175	M20	800	150
D3F100	108	360	250	295	170	250	190	M20	1000	150
D3F125	117	405	270	315	190	270	210	M20	1250	190
D3F160	121	425	290	335	210	290	230	M20	1600	220
D3F200	129	475	335	385	240	335	260	M20	2000	250
D3F250	129	500	360	410	270	360	285	M20	2500	280
D3F325	138	565	410	475	280	410	310	M20	3250	300
D3F400	158	605	450	515	330	450	350	M24	4000	360
D3F500	158	680	515	580	390	515	410	M30	5000	500
D3F650	167	745	570	645	420	570	440	M30	6500	600
D3F800	177	810	635	710	490	635	510	M30	8000	650
D3F1000	192	905	710	805	540	710	560	M30	10000	700
D3F1250	198	970	785	870	620	785	640	M30	12500	900
D3F1600	213	1070	870	970	680	870	700	M30	16000	1000
D3F2000	227	1215	985	1090	780	985	800	M36	20000	1300
D3F2500	267	1360	1100	1220	875	1100	895	M42	25000	1600
D3F3000	282	1490	1230	1350	1000	1230	1030	M42	30000	2000

Rotation as for D3E type (see Table 17.19)

* Dimensions B, C ~~and E~~ are for zero movement, and specified movement has to be added to above in increments of 100 mm.

NOTES For Tables 17.18 to 17.20

(i) Base contact stress, of the bearings illustrated, approaches 20 N/mm^2.

(ii) Sliding plate dimensions shown are for zero movement. Add to these the amount of sliding required in increments of 100 mm. The bearings may then be described in a code, for example, thus:

BEARING LATERALMOVEMENT (cm) BEARING

↓ ↙ ↙

D3E 650 / 10 10 OR D3F 650 / 10

↑ ↗

PRINCIPAL MOVEMENT (cm) MOVEMENT (cm)

(iii) The height A is nominal; manufacturing tolerances give a variation of $\pm$ 3 mm on tabulated figure.

(iv) The size of fixing bolts listed in the tables assumes assistance from friction due to the minimum vertical load normally present in service.

(v) Larger capacity bearings are also available.

Small Bearings—Simplified Fixing (Freyssinet Tetron D3M Range)

- A simplified fixing method is often preferred for the smaller sizes of disc bearing where cast-in sockets are not specified.
- Both D3M and D3MT types are smaller overall than the corresponding D3E and D3T equivalent but are built to a similar specification.

(a) Tetron D3M Free Sliding (Fig. 17.20 and Table 17.21)

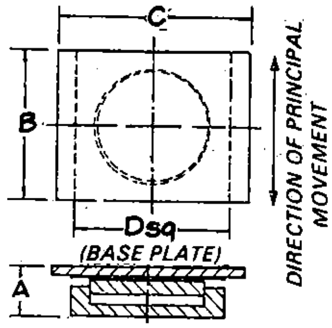

Fig. 17.20 *Tetron D3M free sliding*

Table 17.21

Bearing type ↓	Principal dimensions A	B*	C*	D	Max. load vertical (kN)	Max. rotation (Radian)
D3M 30	53	160	200	160	300	0.028
D3M 50	56	170	220	170	500	0.028
D3M 80	69	235	290	235	800	0.026
D3M 100	79	250	305	250	1000	0.026
D3M 125	83	270	325	270	1250	0.026
D3M 160	87	290	345	290	1600	0.026
D3M 200	96	335	390	335	2000	0.026
D3M 250	100	360	435	360	2500	0.024

- The D3M free sliding bearing is intended to locate by friction only, so adequate vertical loading must always be present to prevent slip.

* Dimensions B and C are for zero movement, and specified movement has to be added to above in increments of 50 mm.

(b) Tetron D3MT Fixed (Fig. 17.21 and Table 17.22)

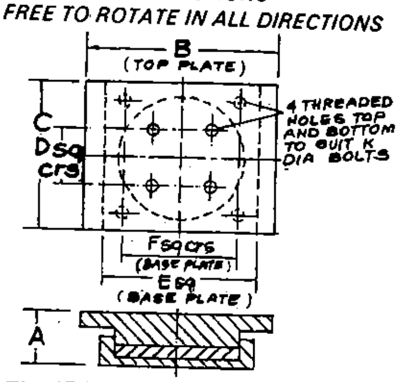

Fig. 17.21 *Tetron D3MT fixed*

Table 17.22

Bearing type ↓	Principal dimensions A	B	C	D	E	F	K	Max. load horizontal (kN)
D3MT 30	57	200	160	60	160	120	M12	80
D3MT 50	61	210	170	60	170	130	M12	80
D3MT 80	68	275	235	80	235	175	M16	110
D3MT 100	73	290	250	95	250	190	M16	120
D3MT 125	77	310	270	110	270	210	M16	140
D3MT 160	86	330	290	125	290	230	M16	160
D3MT 200	97	375	335	145	335	260	M20	210
D3MT 250	99	400	360	165	360	285	M20	250

- 'Rotation' and 'vertical-load' as for D3M type in Table 17.21.
- The D3MT fixed bearing is provided with threaded holes in top and base plates for simple bolt/dowel fixings. The size of fixing assumes assistance from friction due to vertical loading.

NOTES For Tables 17.21 to 17.22

1. Sliding plate dimensions shown are for zero movement. Add to these the amount of sliding required in increments of 50 mm. The bearing may then be described in a code, for example, thus
2. Base contact stresses are approximately 20 N/mm² at maximum rated load.

BEARING LATERAL MOVEMENT (cm)

D3M 80/ 05 05

PRINCIPAL MOVEMENT (cm)

3. The height *A* is nominal; manufacturing tolerances give a variation of ±3 mm on tabulated figures.

(c) Tetron D3MF Sliding guided

Where a sliding guided bearing is required use a D3F, Table 17.20.

Installation of Freyssinet Pot Bearings (Tetron Disc Type, D3 range)

In a bearing that is free sliding in all directions (e.g., the D3M and D3E ranges) positive fixing to the main structure may not be required at all because, if the bearing is always subjected to adequate vertical loading, horizontal movement will occur on the plane of least resistance which is, of course, the bearing sliding surface. Nevertheless, in some cases it is prudent to provide fixings to guard against displacement during installation, impact, vibration and accidental unloading. In a light structure, the bearing fixings must be vibration resistant, otherwise they may work loose.

Consideration should always be given to the practicability of removing and replacing the bearings, should this prove to be necessary.

Most cases of bearing malfunction (as indicated earlier) are attributable to faulty installation, and almost all bearing damage occurs during installation, or even earlier, during handling and storage. Careless handling on site, and the ingress of dirt, can easily lead to abnormally high frictional resistance. Tetron bearings are normally delivered in a condition to discourage unnecessary dismantling, and bolts or straps are used to connect together the upper and lower parts of the bearings. These temporary fixings, as well as excluding dirt during installation, prevent accidental relative displacement between the parts of the bearings, but they must be removed before the bearings are called upon to slide or rotate. However, as explained earlier, they are not structural fixings and should be supplemented, for example, by wedges during installation.

Bearings should be clearly identified by marking them with such details as type, and location. Marking is particularly important where the top plate is to be offset. Bearings should be transported and unloaded carefully and then stored under cover in clean, dry conditions until required. An inspection should be carried out shortly before installation, and bearings that have been damaged in store should not be accepted — almost certainly they will not work correctly, unless they are repaired.

As much preparatory work as possible should be carried out before bringing a bearing to its actual location. The seating should be level, and this usually necessitates the use of a mortar bedding composed of sand and either cement, polyester resin or epoxy resin, with a cube crushing strength of at least 35 N/mm^2, as stated earlier. If the bearing is located directly on steelwork, the seating area should be machined.

Base Fixings An E or M type bearing is free sliding in all directions, with a low coefficient of friction, so that theoretically the base does not require positive fixing, but see note above. All other types of bearing can resist horizontal loading and so their bases should be fixed.

There are several ways of achieving this. With some bearings a large recess can be left in the pier/abutment, into which the bearing base is placed bodily, bedded upon and surrounded with cement/sand or epoxy resin mortar (see Fig. 17.22). The recess must be correctly reinforced on all sides. Alternatively, small pockets can be left for dowels or bolts, which are then set accurately to position and level in grout using timber or light steel jig, made up from the bearing itself. The bearing base is then lowered over the dowels or bolts, on to a bed of wet mortar (Fig. 17.23).

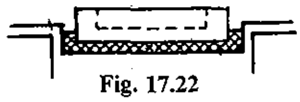

Fig. 17.22

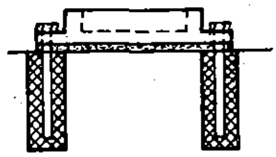

Fig. 17.23

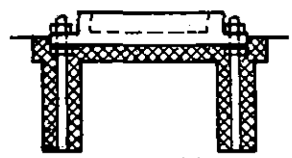

Fig. 17.24

It is possible to cast-in the fixing bolts and position the bearing in one operation, in which case the bearing is supported on rubber washers, over levelling nuts on the bolts, and a chemical resin grout is then poured around the bolts and under the bearing (Fig. 17.24). The fixing should not be over-tightened, when the grout has set, otherwise the bearing may be distorted, due to compression in the washers. Nevertheless, rubber washers must be used, to prevent the fixing bolts carrying vertical loading.

Sockets provide the easiest cast-in fixings with removal in view, for all types of bearing.

Top Fixings For similar reasons, top fixings must be provided for all bearings (optional for the E and M types). Fixed end bearings of the D3MT type are intended for use with steel or *in situ* concrete construction and have holes for dowels projecting into the rocker, whereas the D3T type has larger top plate with external holes for bolts, and is intended primarily for use with precast concrete, or steel structures, and to facilitate removal.

A completed precast concrete structure may be lowered on to a skim of mortar on the top of the bearing, to eliminate soffit irregulations. The mortar mix needs careful control to ensure that it is not totally squeezed out by the weight of the superstructure, which must be supported until it has set. The fixing sockets for the top of the bearing should be ready cast into the soffit, but this requires accurate casting, using a jig drilled mould insert to match the bolting arrangement in the bearing. Alternatively, the sockets may be replaced by a single plate, cast in flush with the soffit, and tapped to receive the bearing fixing bolts.

Where a concrete superstructure is cast *in situ*, the bearing top plates, with dowels, can be built into the soffit shuttering. The area around the mould cut out must be carefully sealed to ensure that concrete mortar does not leak into the working parts of the bearing during placing, and all sliding plates must be propped to prevent them distorting under the weight of wet concrete. A disc type bearing will compress by perhaps 2% of its total height during the casting operation, which must be provided for, to facilitate the removal of the shuttering.

Subsidiary spreader plates, with a special taper, may be required to accommodate a superstructure with an inclination or cross fall, if the bearings themselves are not capable of rotating to accept the requisite slope. When intermediate tapered plates are used, the plane of movement in sliding bearings may not coincide with the plane of the superstructure.

Great care must be exercised to ensure that *F* range lateral restraint bearings are correctly oriented.

NOTE

(i) It cannot be stressed too strongly that bearings should not be dismantled on site because the effects of dirt on the sliding and rotating surfaces are highly deleterious. Where appropriate, bearings should be requested with top plates and rockers preset to accommodate anticipated movements to prevent unnecessary dismantling on site.

(ii) In all cases the transit bolts connecting the bearing rocker to its base must be removed after the mortar has set and before the bearing is called upon to rotate or slide. These bolts are identified by metal labels.

17.11 ARTICULATION SYSTEMS

Choosing the type of bearing-connection (e.g., whether to provide fixed type, pinned type, sliding type, or elastomeric type, etc.) in order to attain an ideal Articulation system for different structural schemes of superstructures, keeping in view the effect of the location of the deck's zero-movement point is discussed here. Reference may be made to Chs. 8 and 9 for the mechanics of distribution of the applied and the self-induced horizontal forces among bridge supports.

~Articulation by Elastic Restraint Throughout

This type of articulation uses elastomeric bearings throughout and makes use of the elastic properties of elastomeric bearings to restrain the superstructure adequately and yet share the horizontal loads and movements between the piers and abutments according to their individual abilities (shear ratings, i.e., shear stiffnesses) to take these loads and movements. There are no rigid connections between the bridge superstructure and substructure in such an elastically restrained bridge deck as there is no need for this type of connection here.

The system has been found to be particularly effective on long continuous bridges but can also be used on simply supported bridges to economic advantage.

• Articulation Systems for Continuous Superstructures

Case 1 Superstructure pinned at one support and sliding bearings used on the remaining supports

This type of articulation allows great distances between expansion joints in the superstructure. The sliding bearing limits the force that can be transmitted to the pier/abutment and hence any excess force has to be taken by the pinned pier/abutment. These often need special strengthening when the bridge is long. It is usually best to arrange for the pinned pier/abutment to be near the zero-movement point of the deck. When this is done, the reactions (from shortening and lengthening) from the piers with sliding bearings on either side of the pinned pier are in opposite directions and hence tend to cancel each other.

Case 2 Superstructure monolithic at one support and sliding bearings on the other supports

This type of articulation produces similar results to that in Case 1. The system can be made efficient by altering the coefficient of friction of the sliding bearings.

Case 3 Superstructure monolithic at one support and elastomeric bearings on the others

More than one pier can be made monolithic with the superstructure in special cases such as short bridges, or bridges with high (slender) piers. The articulation is quite efficient because all piers share the horizontal loads and all piers are subjected to temperature and other superstructure movements. With flexible piers, these movements can be accommodated for considereable distances between joints. However, as the piers get shorter (and stiffer), the forces induced in the piers and superstructure can become considerable and the effort to retain this type of articulation becomes uneconomical in such cases.

With this articulation, the monolithic pier tends to carry a considerable proportion of the total longitudinal force if it is much stiffer than the other piers.

Case 4 Superstructure pinned at one support and elastomeric bearings on the others

This type of articulation has advantages over the monolithic type because the pinned pier has a lower stiffness than the monolithic pier and hence attracts less horizontal force, thereby throwing more horizontal force into other supports. This results in a more even distribution of the horizontal forces. The system is very efficient particularly if the pinned pier is placed nearest to the deck's zero movement point. While the load sharing of this articulation system is particularly good, the only real disadvantage is the necessity to design a pin detail and its possible construction difficulties. This can be improved by installing a rocker type bearing in place of the pinning arrangement.

Case 5 Superstructure with elastomeric bearings on all supports

This is the articulation system particularly suited to long continuous structures; and has been described earlier.

• Articulation System for Simply-Supported Superstructures

Case 1 Span pinned at one end, sliding-bearing installed at the other end

This type of articulation usually transmits most of the applied longitudinal force to the pinned support. Some degree of control of the force transmitted to the sliding bearing pier can be obtained by varying the coefficient of friction of the sliding material. This is usually limited in its extent and not particularly accurate.

Some care must be taken in this type of design to analyse the system thoroughly to see if the sliding bearing will infact actually slide at all. If placed on a tall slender pier with low stiffness which cannot develop sufficient horizontal reaction to overcome the friction force, then the bearing may never slide.

Case 2 Span pinned at one end, and elastomeric bearings installed at the other end

This type of articulation is common and easy to construct. Characteristically the pinned pier usually takes most of the longitudinal force because it is usually stiffer than the combined stiffness of the pier and elastomeric bearing at the other end. However, if needed for pier design, by reducing the bearing thickness the shear rating of the elastomeric bearing can be increased to attract more horizontal force to its pier.

Shortening and lengthening movements produce pier forces which depend on the combined stiffness of the elastomeric bearing and that of the pier under it.

Case 3 Span with elastomeric bearings at both ends

The characteristics of this type of articulation are different from those already considered. Longitudinal forces on a span are shared by the piers at each end of the span in proportion to the combined stiffness of the pier and its bearing.

Shortening and lengthening movements apply forces to the supports in proportion to the combined support-bearing system stiffnesses. These forces tend to cancel if adjacent spans are similar, but no such cancellation can occur on the end-pier (abutment) which may have to be designed to transmit the greatest horizontal force or alternatively may have to be designed with a bearing of less stiffness than the other piers. As for an intermediate pier, if it is assumed rigid (as is many times the case), the deck-movements above it from the adjoining simple spans will cause in it a net force of only $(S_1\Delta_1 - S_2\Delta_2)$ where S_1 and S_2 are the shear ratings of the elastomeric bearings on the two sides of the pier and Δ_1 and Δ_2 the deck-movements above them. (S_1 represents the sum of shear ratings of all the bearings under the deck-end on one side of the pier, and S_2 that for the bearings under the other deck-end on the pier.)

• Conclusions Regarding Articulation Systems

The following conclusions can therefore be drawn from above:

(i) That, the procedure of using bearings stiffness as a variable in the design process (in order to match pier horizontal loads to pier capacities) can produce savings.

(ii) That, the elastomeric bearings are superior in achieving the most economic longitudinal load sharing amongst bridge supports in both continuous and simply supported bridges.

(iii) That, completely elastically restrained structures allow the greatest degree of freedom to the designer to achieve the best longitudinal load sharing between supports, hence the most economic bridge substructure of any of the bridge articulation systems.

(iv) That, improvements in longitudinal load sharing between supports can be achieved with sliding bearings if the coefficient of friction is treated as a controllable variable.

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CHAPTER 18

The Superstructure

18.1 INTRODUCTION

The basic function of a bridge superstructure is to permit uninterrupted smooth passage of traffic over it and to transmit the loads and forces to the substructure safely through the bearings. Although it is difficult to stipulate the aesthetic requirements, it should, however, be ensured that the type of supersturcture adopted is simple, pleasing to the eye, and blends with the environment. No hard and fast rules can be laid regarding the economy in cost. The designer should, however, be able to evolve the most economical type of superstructure based on his judgement and experience given the particular conditions prevailing at the particular site at the particular time. (Also see Ch. 42.)

Materials Used for Constructing Superstructure

The following materials are generally adopted for constructing permanent bridge decks:

(a) Reinforced concrete
(b) Prestressed concrete
(c) Steel
(d) Masonry
(e) Composite construction using steel and reinforced concrete, or reinforced and prestressed concrete.

Reinforced Concrete Superstructure

The usual types of deck sections in this category are,

(a) Solid slab
(b) Slab and girder (T-beam)
(c) Hollow box girder.

The deck arrangement can be simple or continuous spans, or frame, arch, balanced cantilever or bow string type. Some components may also be precast.

Prestressed Concrete Superstructure

The usual types of deck sections in this category are:

(a) Voided slab
(b) Slab and girder (T-beam)
(c) Hollow box girder.

The deck arrangement can be simple or continuous spans, or frame, balanced cantilever, free cantilever: cast *in situ* precast or segmental.

Composite and Steel Superstructure

The usual types of deck in this category are:

(a) Longitudinal and transverse beams, with concrete slab
(b) Longitudinal plate girders, transverse beams and concrete slab
(c) Longitudinal box girder with concrete slab
(d) Steel truss.

The deck arrangement can be simple or continous spans, or arch or frame. The slab can be non-composite or can be made composite with the longitudinal beams by shear connectors.

Special Superstructures

(a) Orthotropic deck
(b) Cable stayed deck
(c) Suspension type deck.

Age old masonry superstructures are generally of arch type.

The composite type of decks generally involve combination of two dissimilar structural elements. Precast or prefabricated beams are connected to RC slab by special structural elements called shear connectors, acting together as a unit. The beams may be either:

(a) Prefabricated structural steel beams/girders, or
(b) Precast reinforced concrete, or precast prestressed concrete beams.

Basic Principle

The superstructure of any bridge must be designed such that it satisfies the geometric and load-carrying requirements set forth by its owner. (In distinct comparison, apart from any navigational requirements, the owner does not have to set forth any special requirements for the substructure.) The geometric requirements depend on the number and widths of traffic lanes and footpaths (and cycle tracks, if any) that have to be carried across. They also depend on the overall alignment and various horizontal and vertical clearances required above and below the roadway. Once the geometric considerations are decided, the superstructure has to be designed to meet various structural design requirements. These include considerations of strength, stiffness and stability. This first requires estimation of internal forces and moments and displacements which the externally applied forces will cause in the selected scheme and form of the structure (based on assumed first-trial dimensions of its elements), and then deciding the final section sizes, and reinforcements (and prestressing, if the structure is prestressed). The former process constitutes 'analysis' and the latter process constitutes 'design'. The analysis is generally done based on elastic behaviour of the structure. The design is then done either on elastic (i.e., working) strength basis or on load-factored (i.e., ultimate) strength basis, also ensuring the serviceability criteria (like limiting the flexural crack-widths, deflections and vibrations).

Some structural analysis work is done using empirical formulae. Sometimes, if the structure is very complicated, model analysis is resorted to. However, with the modern day analytical techniques, aided by fast computers, very complicated structural analyses can be accomplished. (Writing and perfecting the appropriate computer program for a particular analysis may consume time, and even preparing the input data for various load cases may take some time, but the actual analysis can be carried out relatively quickly. Indeed, nowadays ready-to-use programs are available in program libraries for a very large number of structural analysis (and even design) problems. This can simplify the work and make the drudgery more bearable.)

Geometrical Alignment

The horizontal and vertical alignment of a bridge should be governed by the geometrics of the highway, roadway or channel, it is crossing. A bridge may either be right or skewed, straight or curved, or any combination of these.

For girder type bridges, the girders may either be curved or straight, and may be aligned on chords between supports with the deck slab built on the curve. The following points require close examination when girders are aligned on a chord:

(a) Non-symmetrical deck cross-section
(b) Deck finish of the 'warped' surface
(c) Vertical alignment of the curbs and railings, to preclude visible discontinuities
(d) Proper development of super-elevation (cant).

Lighting

The lighting of the bridges should be in accordance with the provisions of the authority having jurisdiction on that area.

Drainage

The transverse drainage of the roadway should be accomplished by providing a suitable crown in the roadway surface, and the longitudinal drainage should be accomplished by camber or gradient. Water flowing downgrade in a gutter section, should be intercepted and not permitted to run onto the bridge. Short continuous span bridges, particularly over-passes, may be built without drain inlets and the water from the bridge surface carried off the bridge and downslope by open or closed chutes near the ends of the bridge structure. Special attention should be given to ensure that water coming off the end of the bridge is directed away from the structure without eroding the approach embankments. Such erosion has been a source of significant maintenance costs.

Longitudinal drainage on long bridges is accomplished by providing a longitudinal slope of the gutter (min. of 0.5% preferred), and draining to scuppers or inlets which should be of a size and number so as to drain the gutters adequately. At a minimum, scuppers should be located near each roadway joint. Downspouts, where required, should be of rigid corrosion resistant material not less than 100 mm (and preferably 150 mm) in the least dimension. The details of deck drains should be such as to prevent the discharge of drainage water against any portion of the structure and to prevent erosion at the outlet of the downspout. Overhanding portions of concrete deck should be provided with a drip bead or notch.

Traffic Lane Width, Road Width, Footpaths, and Clearance for Vehicles/Boats

Traffic lane width and design lane width have been discussed in Ch. 2 of this book. Road width, the distance between roadside faces of the kerbs, depends on the number (and width) of traffic lanes and the widths of the bounding hard shoulders.

Footpaths (i.e., walkways) are provided where pedestrian traffic is anticipated, but not on major arteries or in countrysides. Where provided, its width is 1.5 m generally, but may be as narrow as 0.60 m and as wide as 2.50 m depending on the requirements.

Minimum vertical clearance below a bridge deck, for vehicles to pass under it, is generally 5.50 m. Horizontal and vertical navigational clearances, under a bridge deck crossing a navigation channel, depend on the type of navigation, size of boats/ships expected to cross under, and high flood or high tide level, as the case may be. Horizontal clearance of 80 m with a vertical clearance of 20 m in the central half of the navigation spans is not unusual in some river crossings. However, actual values for a particular case have to be decided with the concerned waterway authorities.

Road Kerb, Crash-barrier, Parapet and Hand Rail

The road kerb is either 'surmountable' type (in which case the portion of the deck beyond it, e.g., footpath, is designed for an overriding point load of 4 to 5 tonnes, distributed over a 300 mm diameter contact area, allowing 25% overstress) or 'insurmountable' type. The latter is generally at least 225 mm high. In the absence of walkways, a road kerb is combined with the parapet. Parapets can be of many shapes and of variable sturdiness. They should be designed to prevent a fast moving vehicle of a given mass from shooting off the roadway (in the event of an accidental hit). Their height varies, but it should be at least 700 mm. They should be mounted by metal hand rail, about 350 mm high. Their road side face is best double-sloped, as shown in Fig. 18.1. This, sometimes referred to as the New Jersey barrier, has demonstrated superior safety aspects, and is presently being adopted for use by many highway authorities. Road kerbs, footpath slabs (and their supports), and the parapets should normally be provided with deliberate vertical cuts or other suitable details of discontinuity to prevent them from acting monolithically with the deck section, so that they do not share the deck moments and crack open over piers or spall in the midspan zones.

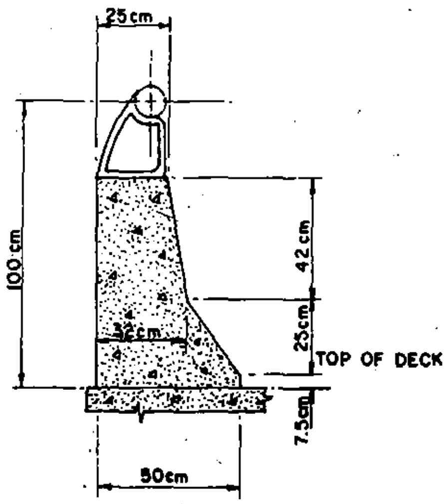


Fig. 18.1 *Parapet curb*

Sometimes walkways are protected from the erring vehicular traffic by crash barriers which act as insurmountable kerbs and deflect the hitting vehicles back into the traffic lane. A crash barrier essentially consists of corrugated or pressed metal sheet spanning horizontally between posts.

Expansion and Roadway Joints

To provide for expansion and contraction, joints should be provided at the movement end of spans, and at other points, where they may be desirable. Joints should preferably be sealed to prevent erosion and filling of debris. Presented below are some general details. More details are given in Ch. 32 of this book.

(a) *Field moulded joints (fillers)*

(i) *Mastics* The extension-compression range of these materials is approximately ±3%. These are used only where very small movements (±2 mm) are anticipated.

(ii) *Thermoplastics* The extension-compression range of these materials is generally ±5%. These are used only in horizontal joints where small movements (±4 mm) are anticipated and where first cost is a major factor.

(iii) *Thermosetting plastics* Sealants in this class are either one or two component systems, which cure by chemical reaction or by the release of solvent, changing from a liquid form into a solid state. They include polysulfide, silicone, urethane, chloro sulfonated polyethylene, butyl, neoprene and epoxy based materials. These materials have an extension-compression range of ±7% to ±25%, at a temperature range of −40° to 80°C. Their abrasion and indentation resistances are low. Joints are normally limited to 5 cm in width and must be protected if exposed to vehicular traffic.

(b) *Compression seals*

Compartmentalized neoprene, extruded to the desired configuration, is used for most compression seals. Neoprene compression seals used singly or as components in combined systems, can prove effective over wide ranges of movement. They must retain at least 15% compression at the widest opening. The allowable movement is approximately 40% of the uncompressed seal width. The maximum size of single neoprene compression seals is about 150 mm.

(c) *Compression-tension seals*

Neoprene expansive elements can be combined with encased steel-bearing plates and anchorage angles, to form a tension-compression joint. Such devices can be used for a range of movements from 30 to 700 mm. These joints, bolted directly to the bridge decks, transmit the wheel loads on the joint surface and seal the joint. Neoprene and metal combination joints can perform the expansion flexibility functions and simultaneously seal the joint effectively.

(d) *Steel plates and finger joints*

Steel joints of the sliding plate and finger conformation, commonly used for longer spans, are gradually being replaced by compression seals in combination with compression-tension seals.

To summarize, type (a) joints provide below average service and present excessive maintenance problems; type (b) joint should give good service; type (c) joints require careful consideration of thermal thrust transmitted by the joint; type (d) joints are inordinately costly and difficult to maintain. This is discussed in more detail in Ch. 32 of this book.

Medians

On expressways and freeways, the opposing traffic flows should be separated by median strips. Where possible, the lanes carrying opposing flows should be separated completely onto two distinct structures. However, where width limitations force the utilization of traffic separators (less than 1.2 m wide), the following median sections may be used:

(a) Parapet sections 300 to 700 mm in height, either integral or with a rail section. The bridge and approach parapets should have similar sections.
(b) Low rolled curb sections or double curb units with some form of paved surface in between.
(c) A series of transverse bars, rounded large blocks or buttons, placed on the deck along the centerline, as recommended by public agencies.

Super-elevation

The super-elevation of the surface of a bridge on a horizontal curve should be provided in accordance with the applicable standard for the highway. This should preferably not exceed 0.06 meter per meter, and never exceed 0.08 meter per meter.

Basic Structural Schemes

These are described in some detail in the remaining part of this chapter.

18.2 GENERAL COMPARISON OF REINFORCED CONCRETE AND PRESTRESSED CONCRETE SUPERSTRUCTURES

In recent years a great majority of bridges have been constructed in prestressed concrete. However, RC bridges have not been eliminated, and in certain circumstances still, are more economical.

Prestressed concrete bridges offer a high degree of freedom from cracks, low maintenance, long life; and they are particularly suitable for dynamic loads and vibrations, and offer great resistance to fatigue arising from repetitions of live loading. (Also see Ch. 37 of this book.)

In RC construction, live load normally approximately doubles the stress in steel, while in prestressed concrete this increase is very little. Position of neutral axis in RC (about which moments of areas from tension and compression side balance) remains fixed, but of that in prestressed concrete shifts with variation in applied load.

The greatest advantage of prestressed concrete for bridges is reduction in weight, slender appearance, possibility of precasting in factory conditions either in segments or even in whole spans, possibility of larger spans and freedom from tension and cracking. For long spans, prestressed concrete decks can be nearly comparable in weight with steel decks.

It is not possible to quote a minimum prestressed concrete span which can be most economical, it depends on local conditions.

The real meaning of economy of a bridge must include not only its initial cost of materials and labour, but also cost of maintenance, the life of the bridge, its depreciation, cost of attending to bearings and piers, sometimes adjustment of thrust, atmospheric deterioration, etc. The aesthetics of a bridge, however, cannot be measured in a similar way.

For short bridges the advantage of using smaller amount of concrete because of prestressing may be unimportant since there is always a certain basic cost which cannot be changed by the use of prestressing. In such cases the usual reinforced concrete construction may be used with equal advantage.

18.3 SLAB TYPE SUPERSTRUCTURE (SOLID OR VOIDED), STATICALLY DETERMINATE OR INDETERMINATE

Slab bridges are frequently used for small spans, they require more concrete and steel than girder bridges of the same spans, but construction cost is usually lower and their

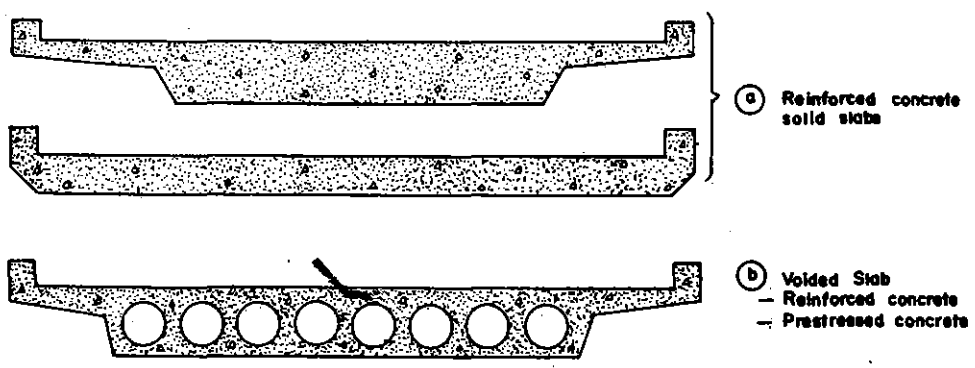


Fig. 18.2

formwork is simpler and less expensive.

The limit of span of slab bridges depends on the magnitude of load and the relative costs of formwork, materials and labour. The small headroom under slab bridges can also have some bearing on economy through cheaper formwork.

Slab bridges may be divided into three classes:

- slabs cast *in situ*,
- slabs built-up of precast elements, and
- composite slabs; in which precast elements are used in combination with *in situ* concrete filling.

Cast *in situ* slab bridges may be adopted from 6 to 30 m spans. For spans up to about 8 m, solid reinforced concrete slabs, with depths up to 60 cm, may be adopted. Voided RC slabs, with depths up to 80 cm, may be adopted for spans of about 8 to 15 m. However, for spans of 15 to 30 m, voided prestressed concrete slabs, of depths up to about 1.20 m, are cheaper.

Solid composite slab decks shown in Fig. 18.3 may be adopted for 8 to 15 m spans. These consist of precast units of various shapes. The units are either of inverted T or symmetrical I section, placed side by side, and stressed together transversely after the *in situ* filling or topping (Fig. 18.3). Shear connectors are used to achieve composite action between the precast elements and in filling concrete or top slab, as the case may be.

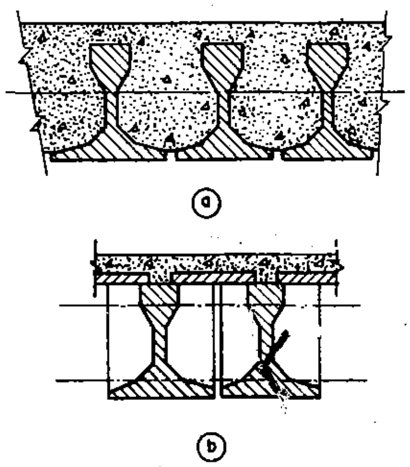


Fig. 18.3

Slab system is usually adopted when the erection of formwork presents no great difficulty. Cables or reinforcements are then placed in position, and after concrete is set, cables are stressed and anchored. Some cables are usually bent up, to reduce shear stresses and also for a convenient distribution of end anchorages. The advantage of this form is that the structure is monolithic and the stress distribution can be calculated. Disadvantages, however, are that the prestressing of a slab is generally uneconomical, the form of construction is often heavy, and the span lengths are limited.

The use of precast elements is of advantage when there is difficulty in supporting the formwork. A quick erection can be possible and very economical if large number of units is required.

Precast and prestressed units (forming a part of the total deck) can be temporarily used as formwork for *in situ* concrete. Additional cables are then often used and subsequently stressed on the composite section. Transverse prestressing is then often used to achieve transverse rigidity and unison of the whole section.

The main problems in the design of slab bridges are—the choice of maximum economical span, minimum depth, the choice whether to precast or cast *in situ*, the type of cross-section to be used, transverse load distribution, deflection and vibrations, and finally the choice of the method of erection.

18.4 BEAM-AND-SLAB AND BOX-TYPE SUPERSTRUCTURES

Cross-Section of Beam-and-Slab Decks

The type of cross-section of a bridge often governs the weight, maximum span, and cost of the bridge. No definite rules can be laid regarding the type of cross-section to be selected. Only a comparison of a few designs may supply the answer in a given case.

In recent years the beam-spacing has increased, and the number of beams in a cross-section has reduced. This is due to the availability of higher quality materials, better experience, and more exact methods of analysis. It is also due to more effective forms of beams developed.

However, this reduction of the number of beams and dead weight in relation to the live load has its limitations due to the following reasons:

- the deflections and vibrations of a light deck structure under live load may become unacceptable,
- demand on increase in permissible stresses may become too high, particularly compression and shear in concrete,
- the total amount of prestressing tendons for a given load, span and width of a bridge, seem to depend little on the number of beams used, and room has to be found to accommodate them,
- the minimum practical thickness of a web based on the practicability of placing concrete through it in the presence of congested reinforcement and tendons in it,
- transport and capacity of cranes sometimes can set limits to the size of the beams,
- the cost of shuttering and transverse stressing can have influence on type of cross-section,

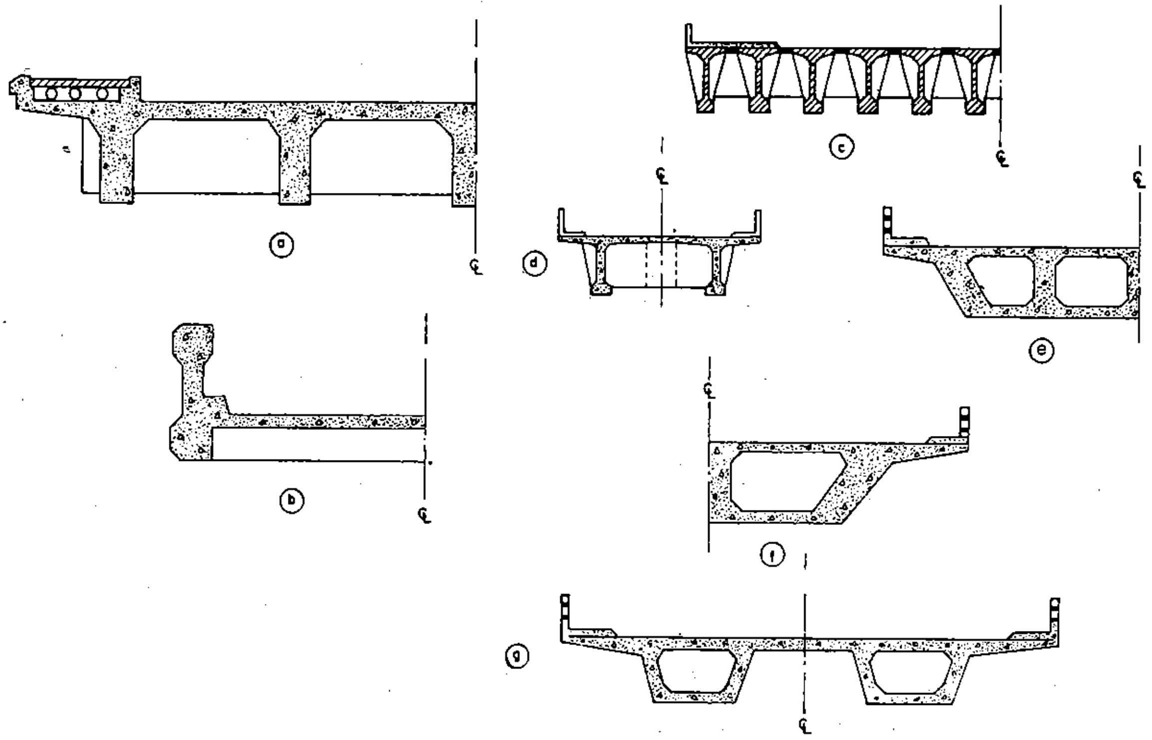


Fig. 18.4

(a) Multi-beam cross-section of the type shown in Fig. 18.4(a) is typical for many RC decks of moderate spans. Wide webs of these beams may not be suitable for pre-stressing as they will dilute the pre-stressing force.

(b) Upstanding beams shown in Fig. 18.4(b) are used when headroom is restricted and spans are small, for larger spans buckling of compression flange sets the limit on the span.

(c) Multi-beam pre-stressed concrete cross-sections of the types shown in Figs. 18.4(c) and 18.4(d) are economical for spans up to approximately 40 to 50 m in the very limit.

(d) Box type cross-section [Figs. 18.4(e), 18.4(f) and 18.4(g)] is uneconomical for simply-supported spans unless the span has necessarily to be large (45 m or more) and/or the construction depth is very limited. Its main advantage is that it facilitates placement of cables with maximum eccentricity, offers resolute section property for sagging as well as hogging moments, and is rigid for efficient transverse load distribution and torsion. For continuous decks however, box section is regarded as very good both from the elastic and safety points of view. If cables are not bonded (i.e., external cables), the ease of placing the cables, their bending up and stressing, are all in favour of box section.

Vertical prestressing of webs is sometimes resorted to on long spans in order to reduce shear stresses in webs.

18.5 ECONOMIC SPACING BETWEEN BEAMS

No definite rules can be given for the most economic spacing of beams and their number in the cross-section.

As a general guide however, and for preliminary design only, the following rules can be quoted (Fig. 18.5).

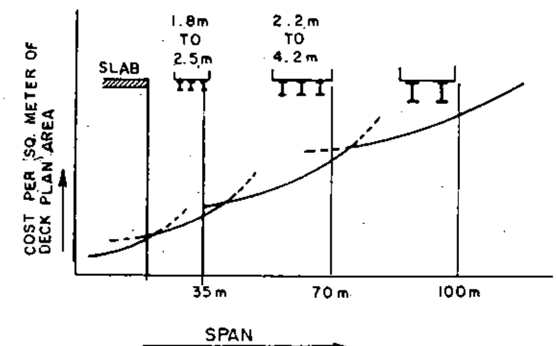


Fig. 18.5

- For small spans of about 9 to 15 m. particularly where headroom is restricted, a solid slab deck with precast U,T or I beams (infilled/topped) is probably the most economic solution.
- I-beams laid side by side are usually economical in the range of about 15 to 20 m. For larger spans up to about 30 m, the beams are usually spaced 1.8 to 2.5 m apart.
- For longer spans, three beams (even two beams) in the cross-section provide the cheapest solution, provided there is sufficient construction depth available.

18.6 BALANCED CANTILEVER TYPE SUPERSTRUCTURES

Balanced cantilever type decks offer some advantages when compared with simply supported decks, and compare well with fully continuous decks.

Hinges are usually positioned in such a way that the cantilevered span alternates with a simply-supported span as illustrated in Figs. 18.6(a) and 18.6(b). Occasionally (and very rarely only) one hinge per span is used [Fig. 18.6(c)], or a simply-supported span is supported by cantilevers from counterweighted abutments (either in mass concrete or in RC box filled with sand or gravel [Fig. 18.6(d)].

Hinges are positioned in the vicinity of low and zero bending moment (point of contraflexure) under dead load (usually at a distance of 0.18 to 0.20 units of the span).

By convenient location of hinges, the distribution of dead load bending moments can be made almost identical to that in continuous decks of the same shape and similar loading. However, the variation of live load bending moments in cantilever decks is not as favourable as in continuous decks in which even spans away from the live load contribute in carrying this load.

In the case of longer spans, however, it is not the variation, but maximum value of moments which matters. For long balanced cantilever spans the general arrangement is almost the same as for fully continuous decks, and hence so also the quantities of materials. (This is so because in long spans, dead load effect is more than live load effect.)

Other advantages of such cantilevered superstructures are: these type of bridges are unaffected by differential settlement of supports, by hogging or sagging due to difference in temperature between extreme fibres and by shrinkage or creep of deck concrete. They develop no parasitic prestress reactions, they conveniently allow segmental construction in certain zones which can require less formwork and scaffolding, and thus can be almost equally economical as continuous bridges, even though they may not have as good a riding quality because of more expansion joints.

The disadvantages could be summarized as follows: variation of bending moments is less favourable than in continuous spans; require more bearings, anchorages and expansion joints; shear stresses can be very high at hinges; hinges are very congested with steel and anchorages, and the joints are generally not pleasing to eye.

Cantilever decks are sometimes built determinate for dead load, and then are subsequently made fully continuous by means of stressing by cap cables or splice cables.

18.7 CONTINUOUS TYPE SUPERSTRUCTURES

Continuous girder decks have the following advantages over simply supported decks:

- longer spans are possible because of lesser span-moments,
- require much smaller ratio of the deck depth at the

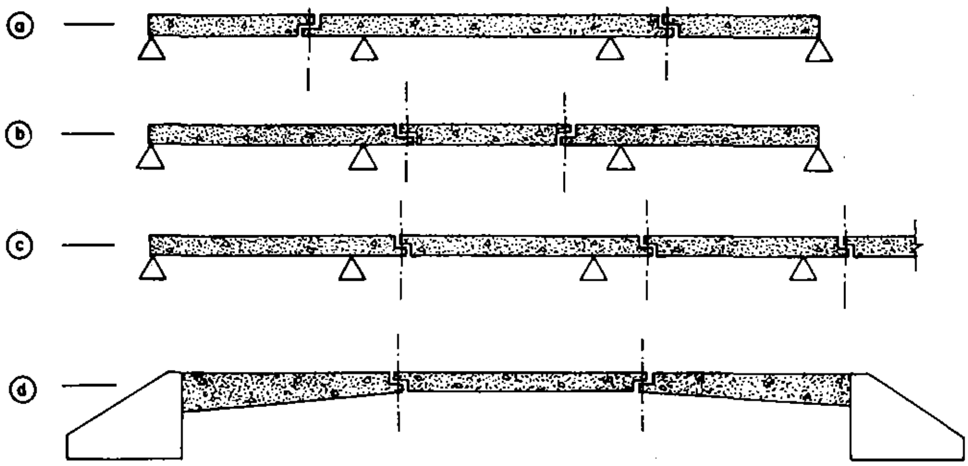


Fig. 18.6 *Position of hinges in balanced cantilever bridges*

centre of span to the span length,

- require fewer number of piers, bearings and expansion joints,
- have smaller deflections and vibrations,
- have a better riding quality, particularly because of lesser joints.

Disadvantages of continuous bridges are:

- sensitivity to the secondary stresses (settlement of supports, differential temperature, shrinkage and creep effects),
- prestressed concrete beams must either be designed with concordant cables or else the effect of lack of concordance must be carefully taken into account,
- continuous bridges require much greater skill in construction. (To facilitate construction, these type of decks are sometimes built as statically determinate and full continuity is restored subsequently.)

Continuity affords advantages for spans over about 35 m, particularly if a long and flexible span follows a short and stiff one. This ratio of short span to long span should ideally be about 0.3.

The number of spans between two consecutive expansion joints may be up to about 4 for a total length of about 150 m if all the cables have to be stressed (from both ends) in a single operation. Otherwise if couplers are used and maximum length of cable stressed at any one time is limited to about 35 to 45 m, the number of spans between two consecutive expansion joints will be limited only by the capacity and cost of the expansion joint.

18.8 SEGMENTAL DECK CONSTRUCTION

Bridge deck construction by free cantilever method, i.e., a butterfly deck built up segment by segment from both sides of the pier is discussed here (Fig. 18.7). Refer Chs. 36 and 37 of this book for more details.

General

Where conditions at the bridge site prohibit the erection of scaffolding and centering on riverbed and long spans are to be constructed to compensate for the high cost of tall piers and deep foundations, cantilever construction is elegantly convenient and competitive.

Cantilever construction is a method of progressive construction of a cantilever in segments and stitching them to the segments already completed, by prestressing. The cantilevering segments are constructed/erected from the pier outwards, one on either side, and stitched back simultaneously. The segments, normally 2.5 to 3.0 m long, can be either cast *in situ* on travelling gantries, or can be precast in yard and erected by launching truss or floating cranes. *In situ* construction is economical only in the case of a bridge having fewer spans. Usually it takes about 4 months to complete a 120 m 'butterfly' by cast *in situ* method. Hence for bridges where many long spans are involved, precasting can speed up the progress of work.

Though precasting involves additional investment on plant, machinery and organisation, for a longer bridge this investment ultimately proves economical over the cost of time saved. Usually in bridges where the foundations are deep, the superstructure is made monolithic with piers and at midspans either a short riding span or a hinge (connecting the two cantilevers) is provided. In bridges where the foundations rest on rock, the butterfly decks could instead be seated on bearings over piers and the cantilevers made continuous at midspan for superimposed deadloads and live loads. Many times two temporary supports are required, one on either side of a pier, to stabilise the deck during free cantilevering.

The *in situ* construction is done by a pair of travelling gantries, each weighing about 40 tonnes (for casting 2.5 to 3.0 m segments of a 2 lane deck). After constructing the pierhead unit (i.e., the portion of deck immediately above pier), a pair of gantry systems is erected on top, one on either side of the pier. The gantries project beyond the pierhead to support the hanging shuttering required for casting the next segment on either side. The external shuttering of the box section deck is supported directly from the gantry system. The internal shuttering is supported on a gantry-girder running inside the box along the length of bridge, which in turn is supported at its forward-end by the previously completed decking. Each travelling gantry system is counter weighted for supporting the shuttering system when it is moving from completed section to forward

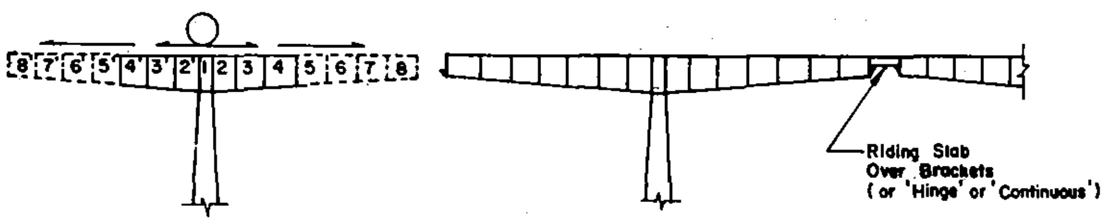


Fig. 18.7 *Segmental construction*

section. In addition to this, the reaction required to take up the weight and constructional loads of the unit to be cast by the cantilever shuttering is realised by means of suspenders passed through the decking and the bottom slab and anchored at the base of the previous unit. The gantry systems proceed in a systematic manner from section to section on either side of pier after the prestressing of the segments last cast. The gantry system also supports a suspended scaffolding for constructional convenience and labour safety.

Construction Cycle for in situ Working

Usually the construction cycle for *in situ* construction is a 10-day cycle, as summarised below:

(i) Shifting of travelling gantry system	1 day
(ii) Completion of entire shuttering and placing of reinforcement bars and prestressing cables for the segment	3 days
(iii) Casting of the segment	1 day
(iv) Curing	4 days
(v) Stressing and miscellaneous	1 day
Total	10 days

i.e., casting on Day-5, prestressing on Day-10, so that concrete in the segment is 5 days old when stressed by the cables anchored at its end. This periodicity is very important to know for the design calculations. However, the above construction cycle can be compressed into about 5 to 6 days as has been proved at various highly mechanized sites.

Choice Between Continuous, Hinge and Simple 'Riding Span'

Where differential settlement between foundations is not expected, the butterfly decks (constructed by free cantilever method) can either be made continuous (but calculations for continuous deck take longer), or a 'hinge' can be introduced between cantilever tips for shear transfer. Both these solutions result in a better riding quality. However, where significant differential settlement is possible, and/or contractor's construction experience is limited, a simply supported riding span may instead be provided between the adjacent cantilever tips. Despite the best construction care the adjacent cantilever tips generally do not 'mate', and, in order to safely 'hide' their level differences, a riding span is a simple alternative, as otherwise only additional stresses will be built in if the two cantilever tips are 'forced' to 'mate' while rendering continuity or building-in a hinge. However, where the bridge is long and precast construction is resorted to, at least a hinge solution may assist for rapid transportation of precast elements from one butterfly to the other (which otherwise is delayed if, instead, the casting of a riding span is involved).

Precast Segmental Construction for Long-span Bridges

The use of precast concrete segments offer the following advantages:

- control of high quality concrete
- manufacture of segments at a plant site instead of at site where the quality controls may not be as good
- lend themselves to new techniques and speed of erection
- great accuracy of section and profile can be obtained and problem of deflection during construction can be better overcome
- shrinkage can be practically eliminated and creep reduced by high quality of concrete.

The efficient construction of long-span prestressed concrete decks depends on:

(a) basic design scheme
(b) erection method
(c) suitable technique of jointing, stressing, etc.

Careful consideration must be given to ensure monolithic behaviour of the completed structure as well as flexibility of the structure during erection.

The weight of segments may vary from 5 to 100 t or more, and the length from 2.5 to 3.0 m, or more. Segments may be reinforced with mild steel along the bridge axis*, and are designed to be connected by post-tensioning after erection. Segments can also be prestressed in themselves, either by temporary cables to aid erection, or cables can be so arranged that the prestressing is efficient in the final condition as well. Transverse prestressing can also be applied to the segments at manufacture stage itself.

Short segments can be cast in end-on vertical position. Usually segments are match-cast. Steam curing is often used to attain high strength earlier and to reduce shrinkage. Strengths of 450 to 550 kg/cm^2 can be obtained.

Joints are of great importance in connecting the precast units, and to ensure monolithic behaviour. Very wide joints, 20 to 60 cm wide, have been used with reinforcement projecting to be lapped or welded. High strength concrete is then placed and well consolidated, and sometimes steam cured. This, however, is not done anymore.

Deep indentation shear keys are employed when high shear stresses must be transmitted. They are preferable even otherwise.

Normally, the ends of the segments are constructed as vertical planes with roughened surfaces. Shear keys, i.e., indentations, should preferably be formed in them. Tests show that reinforcing bars across the joints are of little

* With high yield deformed bars normal to traffic.

benefit to the ultimate strength, and shallow shear keys are also of no value.

Poured concrete joints: 7 to 10 cm wide joints have been found to be successful. Rich mixes using pea gravel for coarse aggregate, and high strength cement, are used.

Dry mortar packed joints: 3 cm wide, have been tried but it is extremely difficult to ensure their satisfactory performance.

Epoxy coatings have proved beneficial, and are extensively used as the jointing agents as an aid to bond the concrete joint.

Dry joints: where the segments but directly against each other, shear keys and epoxy painting can be resorted to. This requires special techniques and great care both in manufacture by match-casting of the consecutive segments and epoxy jointing with precisely time-controlled prestressing. This makes possible speedy and economical assembly on the site and assures a near perfect joint. The proven method, till now, is to cast each unit against the preceding segment. This generally requires a second handling of each segment in the casting yard and match-marking for erection.

All concreted joints must assure continuity of cable ducts across the joint (except when external cables are used). To form the duct in the joint, inflatable formers and plastic or metal sleeves have been used. While grouting the post-tensioned tendons, grout may escape at the joint and block the adjoining ducts. Grouting then is best postponed until all tendons are stressed but this can be dangerous since grouting should not be delayed.

Cables and Their Profiling

It is preferable to use bigger cables, e.g., a tendon of 12 strands (each strand 15 mm dia.) rather than the small 12/7 mm Freyssi wire type, as otherwise their number becomes inconveniently large which poses problems in profiling and handling. However, very big cables are not preferable as there should be at least one cable per web (preferably two) to stitch a segment.

Generally cables to be anchored at the end of a particular segment should be anchored at about three-quarters the depth of the segment below its top. Behind their anchorages these cables should rise 'straight inclined' through two or three segments to a plane which is convenient but a small depth s below the horizontal plane in which they travel from above the pier. This transition through the vertical drop of s should be accomplished preferably through one segment only. The value of s is adjustable—about 0.30 to 0.40 m for the above type of cables. This should permit about twice the minimum bending radius specified for the cables used (assuming this transition were a pure circle). The cable profile through the entire transition may in fact actually be specified as two 'inclined straights'.

Such a profile lends itself to easier construction as also to easier alterations, if required, while finalising the cable profiles in the design stage, and also permits easier calculation of angle-change. Beyond this transition, unless the cable has to remain in the vertical plane of its bending in web, it has to be suitably bent in plan while travelling in the top slab. This plan-bending (within the horizontal plane) may also be accomplished through a 'series of straights', each straight extended over at least one or more segments. The elevation and plan bending of cables in terms of straights makes not only the calculation and drawing easier but also the laying out at site.

Deck Section

A box section is the most ideal deck section for cantilever construction from the points of view of both construction and flow of stresses. Unlike a steel box, a well detailed concrete box is a 'thick walled' tube, has a tremendous torsional resistance and does not accept any significant distortion and warping. Depending, of course, on the section shape and the extent of cantilevering deck slab, almost a full cross-section of the concrete box of the more usual shape is effective for resisting the load effect. The more usual concrete box section may be assumed to permit complete maturation, provided it has full depth cross-girders at support points and at intermediate locations* such that the spacing between cross girders does not exceed about 45 to 55 m. In addition, the live load is placed away from the section at which its maximum bending effect is being considered, so that the load-effect trajectories get ample chance to flow into the full section. The torsional rigidity of the box greatly adds to this behaviour.

Soffit Surface

Soffit surface is generally a series of 'straights', segment to segment, while the soffit points at segment-ends may lie along a parabola if desired.

Instead of threading-in the longer cables right from the first segment onwards and keeping them dangling, it is enough to provide a cable-duct alone (sometimes filled with sand) for these cables all the way until their profile commences bending. It is only then that these cables need to be threaded-in (as subsequent cable-threading may be difficult through a bent-profile). However, the maximum dangling projection of a cable at any one time may be limited to about 10 to 15 m ahead of the preceding segment.

Air vents should be provided in the cable ducts over the pier and at points of change of curvature and at intervals

* Intermediate cross girders may be avoided if live and dead load effects are investigated in detail, e.g., by finite element method or by a model test.

of about 30 m. Neat cement grout (sometimes mixed with very fine sand and aluminium powder to give body and expansion to the grout) with about 0.5 w/c ratio should be injected simultaneously from both ends of a cable, against gravity, under about 7 kg/cm^2 pressure. The vents should be plugged when full grout issues through them. Care should be taken to see that grout from either side reaches the central vent by alternate pumping. (For more details for grouting operation, refer Ch. 7, 'Construction Considerations' of the author's book, *Concrete Bridge Practice—Construction, Maintenance and Rehabilitation*.)

Deflection and Pre-camber

Generally the order of deflection of a well-designed long span prestressed cantilever is about 1 in 600, which is very low.

To counter the long term creep 'dipping' effect of deflection under all dead loads and final prestress, it is necessary to calculate the deflection at various sections under this loading (based on the reduced modulus of elasticity of concrete due to creep) and give equal and opposite cambers at the soffit at these sections at the very time of construction. With this, it is hoped, that the structure will, within about 3 years (by when majority of losses due to creep would have occurred), finally take the intended profile, thus countering the dipping effect. These pre-cambers to be given during construction should be specified at the soffit level at each section. (Theoretically the variability of the atmospheric history during the entire period of construction will render it almost impossible to calculate these deflections correctly, but it is practically good enough to calculate these deflections taking E_c as half the instantaneous value.) Also refer to relevant details in Ch. 37.

The cambers may be laid out over the decking with theodolite located over the piers always at the same time immediately after the day-break, if only to somewhat offset the effect of temperature difference through the decking.

Expansion Joint

In order to admit relatively large longitudinal and vertical movements and rotation, usually a tongue and groove type steel joint or multi-grooved or layered elastomeric type expansion joint is adopted at expansion joint locations. Refer Ch. 32 of this book for relevant details.

Bearings

Bearings under the riding spans have to permit a large horizontal and vertical movement coupled with very significant rotation. For this purpose metallic rocker and full roller type bearings must be used. These bearings should be tied to the structure in order to prevent their dislocation since the relatively short riding spans may try to jump as the live load traverses from cantilever tip to it and vice versa. For this purpose, seismic attachment type tying arrangement may be adopted. Refer Ch. 17 of this book for further details.

Aesthetics

"The simplicity of form and proportion, with the load masses disposed precisely in locations where they are most effective, the long sweeps of the structure and the elegant arch of the soffit, all make the cantilever bridge deck a beautiful form of function, all well blended to give a pleasing architectural appearance."

Summary of Steps for Design of Butterfly Deck with Riding Spans

(i) Decide first-trial butterfly deck cross-section and sizes.

(ii) Design the cantilevering footpaths and do a quick design of top slab considering the box section action, taking the box section to be of an average depth (assuming approximate thicknesses for the webs and bottom slab).

(iii) Demarcate various segments and section numbers (starting from the cantilever-tip towards the root).

(iv) Decide arrangements of riding span and supporting brackets at the cantilever tip.

(v) Work out section properties at various sections.

(vi) Calculate weights of various segments and BMs and SFs at various sections due to self dead loads of the segments, the weight of finishings and due to dead load reaction at cantilever tip from riding span; calculate corresponding flexural stresses at various sections.

(vii) Calculate the governing live load BMs and SFs at various sections and the corresponding flexural stresses at these sections.

(viii) Decide the prestressing system; particulars of tendons and anchorages; minimum distance between cables, minimum distance from edge of concrete to centre of cable and to centre of anchorage; maximum jacking force allowable; grade of concrete and permissible flexural compressive and tensile stresses during construction and during service; estimation of losses due to shrinkage, creep and elastic shortening of concrete and relaxation of steel, coefficients of curvature and wobble friction; number and placement of cables above centre line of pier; profiling of cables, etc. (For convenience serial number of a cable may be taken as serial number of the section at which it is anchored plus one, e.g., cable anchored at Section 11 may be called cable number 12, and so on). For each cable calculate its eccentricity at various sections, cumulative angle turned by it in elevation and plan from jacking point up to each section. For each cable decide first trial anchorage force, calculate for friction, plot P_x values, estimate slips, extensions and jacking forces, read off P_i values at various sections, estimate values of '$P_i \cos \theta$', '$P_i \cos \theta e$' and '$P_i \sin \theta$'; calculate flexural stresses at various sections due to each cable; compose final flexural stress summaries at various sections and check against permissible flexural stresses, revise section and/or cables as necessary.

(ix) Check against ultimate moment of resistance at each section.

(x) Design against shear (revise section and/or cables if necessary, in which case recheck flexural stresses).

(xi) Transverse analysis of box at a few representative sections and design of reinforcement around the box.

(xii) Decide reinforcement detailing in box section vis-a-vis requirements from shear and transverse analysis and minimum reinforcements codally required.

(xiii) Anchorage zone reinforcement and detailing at various anchorages.

(xiv) Design of cross girders.

(xv) Design of brackets supporting riding span.

(xvi) Design of expansion joint.

(xvii) Design of riding span and its bearings.

(xviii) Design of parapet wall.

(xix) Calculation of pre-cambers to be given at various sections in the butterfly during construction; estimation of maximum tip deflection under service live load.

(xx) Ensure vibration characteristics under dynamic loading (details dealt in Ch. 39 in this book).

18.9 FRAME BRIDGES

In recent years there has been some revival of interest in this form of construction when applied to large span bridges, mainly because of the introduction of prestressing.

A portal frame is essentially an arch but of a shape very much different from the line of pressure, which is an ideal characteristic of an arch. Prestressing offers some advantages for these types of structures (see Fig. 18.8).

The main advantages of frame bridges in comparison with continuous beams can be summarised as follows:

(a) By prestressing the frames, the line of pressure can be kept within the middle third of the section.
(b) Frame bridges do not require expensive bearings at the supports.
(c) The stability of the supports is much greater than in the case of independent piers.
(d) Very large spans are possible (up to 100 m or more).
(e) The depth to span ratio can be as low as 1/50 at the crown.
(f) Frame bridges lend themselves to the use of jacks for correction of thrust and flat jacks for prestressing.
(g) Quantities of steel and concrete for long spans are relatively low in comparison with continuous beams.
(h) Can be even more economical than arches.
(i) Have pleasing appearance from aesthetic angle.
(j) Prestressed frames have greatly reduced horizontal thrust.

The disadvantages of frame bridges are:

(a) their sensitivity to secondary stresses, particularly horizontal and angular settlement of supports
(b) require greater skill in construction and high quality of materials
(c) stress distribution at corners is complex and may lead to serious redistribution of moments

It has been explained in a later chapter that the rate at which quantities increase with the span in the long spans is less for portal frames than for continuous bridges. In fact, concrete and steel quantities for portal frames theoretically need not necessarily increase with the span and this may be achieved by increase in horizontal thrust, which however is usually expensive to provide for.

Arch bridges are usually economical because their equilibrium can be maintained mainly by direct forces rather than by bending. But when comparing minimum construction depth, maximum headroom throughout the span, or general economy in the combined cost of substructure and superstructure, the frames may show advantages over arches.

For maximum efficiency the depth over the supports should be as much as even 3 times the depth at mid-span, although this is not always possible because of the

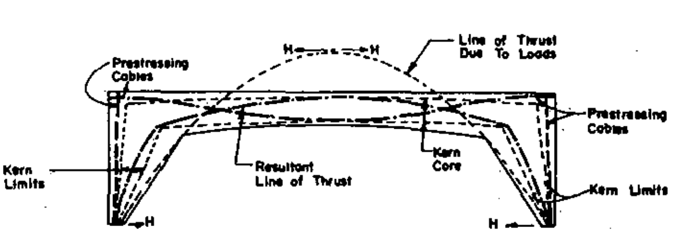


Fig. 18.8 *Prestressed portal*

appearance and constraints of headroom. In addition, at knees the total cable force would sometimes need to be increased to a value in excess of that required at mid-span. Alternatively, the increased prestressing force must be provided at these knee junction regions by cap cables.

The introduction of prestressing to frame bridges removed their main disadvantage of the line of pressure being otherwise far away from the centre line and reduced their sensitivity to settlement of supports. Also the greatly reduced horizontal thrust makes this form of construction economical.

The essential function of prestressing in frames is to superimpose a line of thrust upon the external load thrust line in such a way that the resultant thrust line is brought back into the section within certain limits.

At the corners, the leg cables and beam cables must cross, since the resultant thrust line must abruptly change its direction to pass from the narrow gap between the limit lines in the beam to that in the leg. This change of direction requires the action of a concentrated force, exerted by the resultant of anchorage forces at the corner. See Fig. 18.9.

For large spans, this crossing of two systems of cables causes problems of location of cables and anchorages. The outward thrust at this change of direction is balanced by the resultant of anchorage forces, and it is necessary to reinforce the corner by a diagonal rib to accommodate this balancing force. See Fig. 18.9.

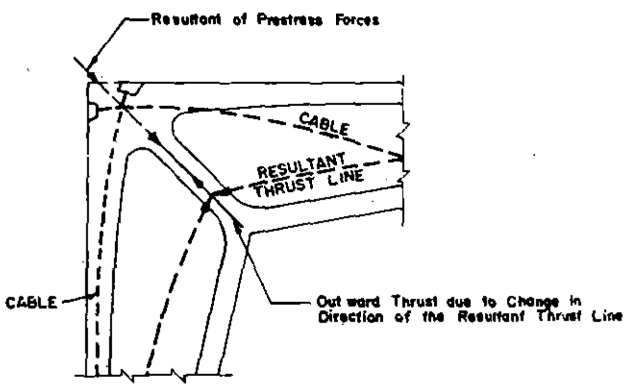


Fig. 18.9 *Forces in the portal knee*

However, the main disadvantages of frames lie in their sensitivity to differential settlement, particularly to a horizontal yielding of foundations under the action of horizontal thrust. Prestress greatly reduces the magnitude of horizontal thrust and therefore the horizontal settlement. To overcome the horizontal settlement, counterweights sometimes are used to minimize the horizontal thrust. However, heavy abutments should not be monolithically cast with the transome (Fig. 18.10). The ideal arrangement is to

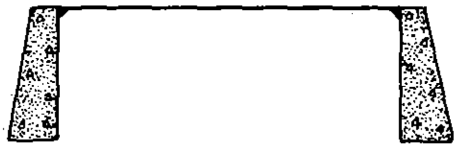


Fig. 18.10 *Wrong arrangement*

provide the tie if this is possible. Then only the vertical reactions are transferred to the ground, Fig. 18.11.

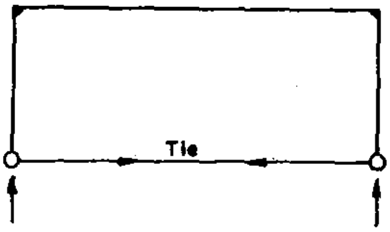


Fig. 18.11

If ties are not acceptable or possible, then the abutments should be disguised as separate units from the frame columns, thus counter-balancing horizontal thrust and making the systems statically clear and dependable, Fig. 18.12.

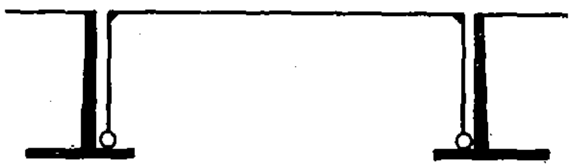


Fig. 18.12

As an alternative, counterweights have been used to reduce horizontal thrust, Fig. 18.13. The disadvantage however of this solution lies in great vibrations of weight as well as the main span. There are also difficulties in providing good joints between the ends of the counterweights and the roadway.

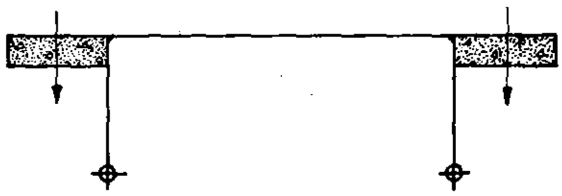


Fig. 18.13

Similar effect of reducing the horizontal thrust can be obtained by cantilevers at the level of the hinges. Dead weights of the fill above these cantilevers reduces the thrust so that essentially vertical component may be transferred to the foundations at normal working conditions, Figs. 18.14 and 18.15.

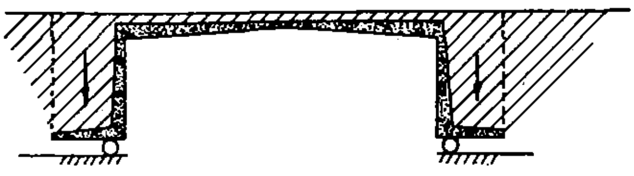


Fig. 18.14

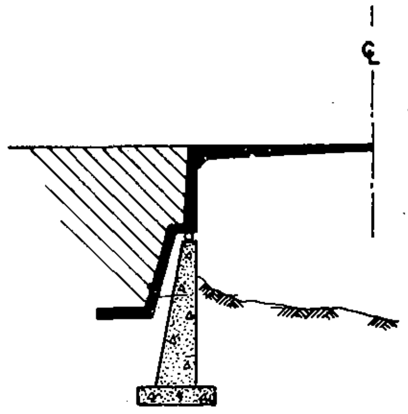


Fig. 18.15

Alternatively, hinges can be set inside, Fig. 18.16, with a similar effect of reducing the horizontal thrust.

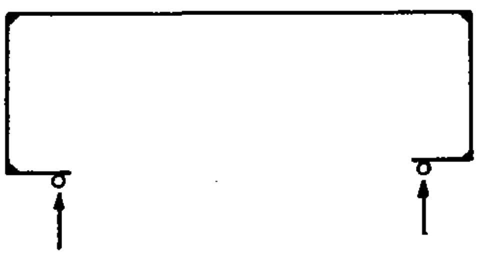


Fig. 18.16

Figures 18.17(a) to (e) show typical arrangements of abutments and columns in frame bridges.

Frames with Inclined Legs

Frames with inclined columns (Figs. 18.18 and 18.19) are used to improve visibility under the bridge, to reduce the main span, and to improve the appearance. These are usually in the form of a single-span bridge or three-span bridge. With certain arrangement of piers, horizontal thrust at ground level can also be greatly reduced.

One of the major factors in the cost of a portal frame is the wide gap between the funicular line of loads and the centroid lines of the frame. This gap is greatest at the corners of the frame and consequently requires great depths above the posts and very high prestressing forces. Reduction of that gap is most easily effected by sloping the legs of the portal outwards, providing that the specified headroom can still be maintained. Using a suitable slope of leg and adopting a convenient ratio of depth at mid-span to depth at the corner, well balanced designs may be obtained and materials (concrete and steel) can be utilised with the same efficiency in the two critical sections of the beam. There is still, in each case, an optimum value of the ratio of beam to leg stiffness, which is necessary to obtain the desired rise of the pressure line under external loads.

The length of the central span is usually about 0.7 of the distance between hinges. The ratio of depth at corner to the depth at mid-span is usually 2 to $2^1/_2$, Fig. 18.20.

Cantilevers are sometimes tied to the bottom of the leg or to the foundation. The problem of crossing of cables is much easier with inclined legs than with vertical legs, since the anchorages of the beam are at the ends of cantilevers, and the thrust line in the cantilevers intersects the thrust line of the leg some distance below the leg-cable anchorages (Fig. 18.21).

Frames with Triangulated Supports

Sharp corners inside the knees create a zone of extremely heavy and complicated compression stresses, which, in that locality in a redundant frame, can result in excessive strain and redistribution of bending moments, significantly different from the one calculated. The introduction of a sufficiently heavy triangulation strut can considerably alleviate this danger.

Another method frequently used is to split the column into two, forming a triangulated support. One column is then working as a strut and the other as a tie. High negative bending moments can be resisted by such system of supports.

Inclined columns essentially resist the horizontal and vertical forces by direct compression and tension only, and the bending moments in columns are either eliminated or are small, even in the case of complete continuity at the top.

Tension column is often pre-stressed.

Various arrangements of this type of bridges are shown diagrammatically in Fig. 18.22.

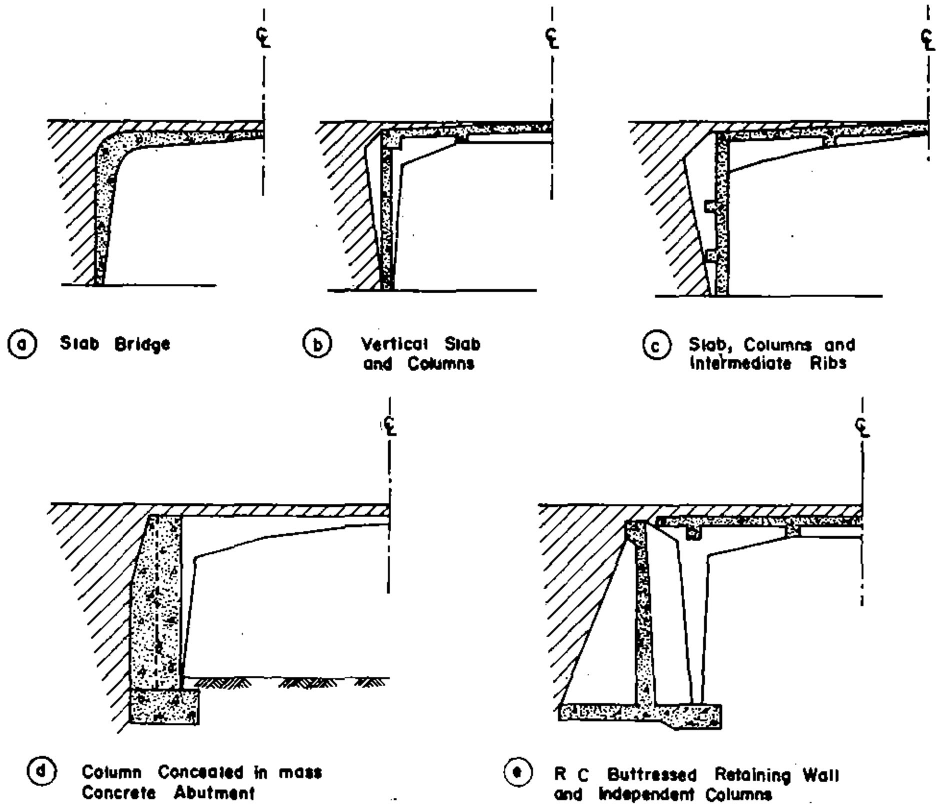
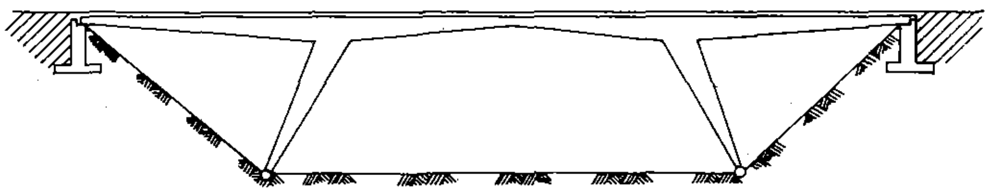


Fig. 18.17



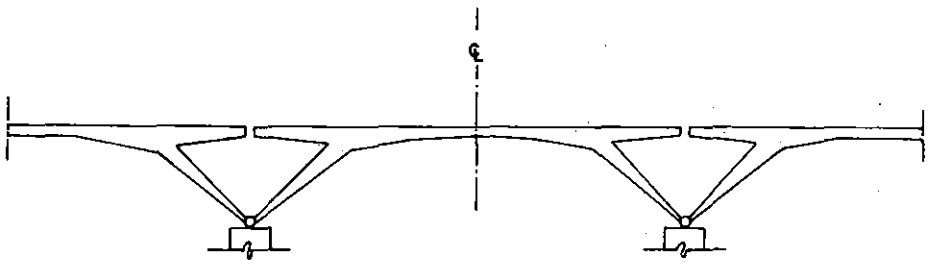


Fig. 18.18

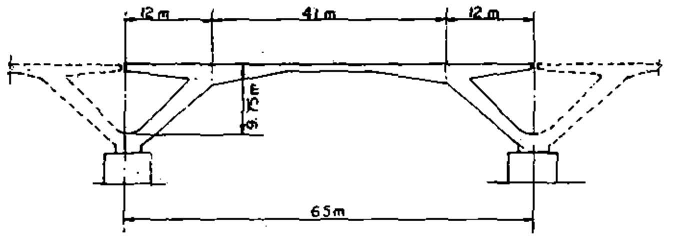


Fig. 18.19 *St. Michel Bridge at Toulouse (five spans of 65 m each)*

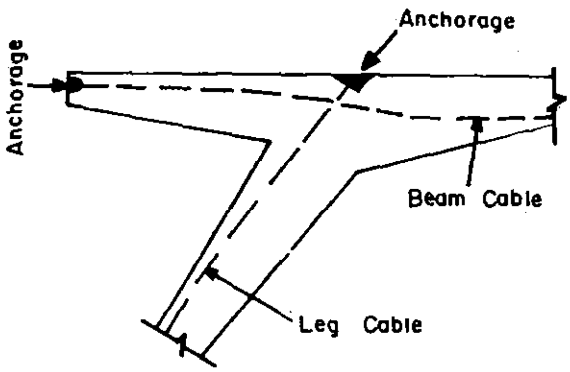


Fig. 18.21

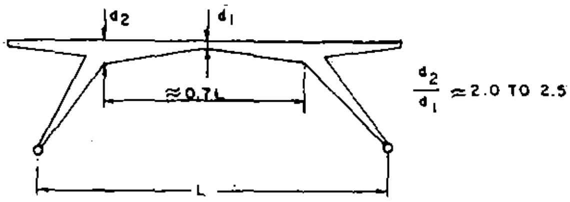


Fig. 18.20

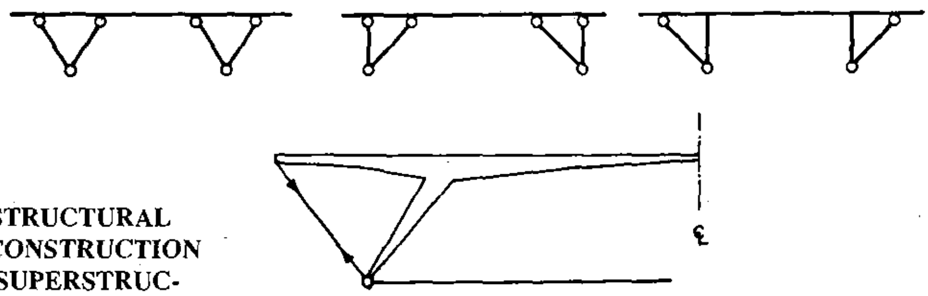


Fig. 18.22

18.10 BRIEF CHECK-LIST FOR STRUCTURAL ANALYSIS, DESIGN AND CONSTRUCTION FOR VARIOUS TYPES OF SUPERSTRUCTURES

Slab Bridges

In this simplest type of bridge superstructure, the deck slab also serves as the principal load carrying element. The concrete slab, which may be solid, voided, or ribbed, is supported directly on the substructures.

Analysis Method

Reinforced concrete bridge decks should be designed in accordance with recognised elastic analysis, assisted by certain empirical analysis. The analysis should take into consideration the configuration of the slab, the supporting conditions, and the effects of cracking on the slab stiffness.

Types of Construction

(i) *Cast in Place Slab Decks* Concrete slab decks may be cast monolithically together with the supporting elements. Alternatively, they may be cast separately but effectively bonded to achieve composite action under future loads.

(ii) *Precast Deck System* Precast concrete deck systems may be constructed in several different ways. These include:

(a) Precast panels extending the full width of the bridge attached to supporting members, without composite action.

(b) Precast panels extending the full width of the bridge and effectively bonded to the supporting members to develop composite action.

T-Beam and Box-Girder Decks

These types of superstructures consist of a deck slab supported by longitudinal beams or girders. The longitudinal beams or girders may in turn be supported by abutments, piers, bents or floor beams. Transverse intermediate diaphragms are used as may be required.

The longitudinal beams or girders may have any cross-section, with I, T or box sections being the most common, and they may have a prismatic or variable section along their span. Simple span or continuous structures may be used.

Either reinforced or prestressed concrete, or combinations thereof may be utilised. Also, the structures may be entirely cast in place, or constructed using precast beams or girders with a cast in place composite deck slab.

Analysis

The analysis of bridge superstructures should be based on elastic, empirical, or model analysis methods. All sections should be proportioned to resist the forces determined from the results of the analysis. The analysis should recognise stresses occurring from temperature, shrinkage, creep, transport and handling of units, prestress and other conditions which may affect the design. Torsional stresses should be considered in the design.

(i) *Elastic and Model Methods* The forces present in the component parts of the superstructure should be established in accordance with a recognised elastic analysis (analytical) or representative model analysis, except when an empirical method of analysis may be used.

(ii) *Empirical Methods* The empirical methods referred to here are those conforming with the recommendations presented in the "Standard Specifications for Highway Bridges," published by the American Association of State Highway and Transportation Officials.

Design Considerations

• *Simply Supported Spans with No Skew or Moderate Skew (less than* 20°*)* The loads and forces acting on simple spans include:

(a) Self weight (dead load),
(b) Superimposed dead load,
(c) Prestressing-Pretensioning and/or Post-tensioning,
(d) Live loads,
(e) Shrinkage, creep, and (for composite construction) differential shrinkage and differential creep.

In general, different combinations of the above loads act at various stages of construction and throughout the service life. If shoring for precast girders is used during construction, the effect of removal of shoring should be considered as another specific load combination. The cross-sections which resist these loads also vary. These loads and sections must be carefully identified to permit a meaningful analysis. Special attention is drawn to the forces caused by deflection of formwork, temperature effects, and incompatibility of materials. Deflection calculations should be carried out so that suitable camber can be provided in the reverse direction at the time of construction.

The analysis should take into consideration, where appropriate, the axial, flexural, and torsional stiffnesses for longitudinal girders and transverse medium at various stages of construction. The transverse medium consists of the deck slab and, if present, floor beams and diaphragms. The floor beams may be separate units or they may be monolithic parts of the slab.

• *Simply Supported Spans with Severe Skew* This condition requires, particularly in the case of multi-girder bridges, that recognition be given to the flexibility of supports since the magnitudes of reactions and bending moments may be affected.

The available analytical methods of analysis may be broadly classified into the following main groups:

(a) Analysis by anisotropic or orthotropic plate theory.
(b) Analysis as intersecting longitudinal and transverse members of particular flexural and torsional stiffnesses.
(c) Analysis by considering the longitudinal or primary members as interconnected by some form of cross-section which represents the behaviour of the transverse members.
(d) Analysis by the finite element technique, plane grid, etc.

The most suitable method of analysis should be selected for the solution of each particular problem. The accuracy of the results obtained depends on how closely the actual structure satisfies the assumptions inherent in the employed method of analysis. If analytical methods cannot be used or if available facilities offer a more convenient approach, model testing is recommended. Model testing may be used to advantage when skewed or irregularly shaped superstructures are encountered. A high degree of automation is possible. The modelling material may be plastic, micro-concrete or other materials which adequately approximate the behaviour of the prototype. The effect of scale must be considered. In addition, it should be ascertained that requirements pertaining to fatigue and vibration are satisfied.

• *Semi-continuous Spans* Semi-continuous spans identify structural configurations that consist of precast single-span beams laid consecutively over a number of spans, but where objectionable expansion joints are obviated by the introduction of continuity. This permits the development of support moments due to subsequent dead load and live load, shrinkage and creep. Partial continuity can be achieved as follows:

(a) Casting of transverse beams which fill the spaces between consecutive beam ends and are integral with the deck slab. Reinforcing should be provided within the longitudinal beams, for development of the positive moments generated by shrinkage and creep. Reinforcing, which extends across the joints and is capable of resisting the tension induced by the developed negative moments, should be placed in the slab.

(b) The same as (a) except that the concrete slab is prestressed over supports.

(c) Continuity is confined to the top slab only. Transverse beams are provided at the ends of girders, but the space between consecutive girder ends is not filled. Adjacent girder-ends are provided with separate bearings. Careful detailing is required to prevent slab crushing due to end rotation of girders.

In addition to the forces and effects listed for simply supported spans, effects introduced due to continuity should be considered (e.g., effect of settlement of supports, effect of temperature difference between extreme fibres, etc.).

• *Continuous Spans* Continuous spans identify structural configurations which are supported at intermediate points between the ends of main girders. Continuity may be achieved as follows:

(a) Unprestressed structure: The entire structure is cast in place on stationary form work. Full continuity is achieved for self weight, superimposed dead load, and live load.

(b) Prestressed structure:

- The superstructure is cast in place as in above, and full length post-tensioned tendons are used to prestress the deck. Jacking is done from both ends of the superstructure.
- The superstructure is cast in place, and is prestressed by the use of full length tendons and tendons with intermediate anchorages. This arrangement may be advantageous when the soffits of girders are curved. Jacking may take place from both ends and intermediate locations of the superstructure.
- The superstructure is cast in place, and is post-tensioned by the use of lapped tendons. Such an approach may be advantageous when a large number of spans are prestressed by the use of individual tendons of limited lengths. Lapping is used at supports, in the vicinity of which staggered intermediate anchorages are provided. Supplementary short tendons may also be provided at supports if needed.
- The superstructure is cast in place, and is post-tensioned by the use of coupled tendons. This scheme may be considered as supplementary to the one outlined above.
- The superstructure is built up from consecutively constructed segments, prestressed together. Segments of the superstructure are post-tensioned together by means of tendons threaded through preformed ducts. It is of paramount importance to consider all deformation and resulting stresses which are introduced during construction (particularly due to shrinkage and creep). Special effects due to construction and continuity should be considered in the analysis of continuous spans (settlement of supports, parasitic prestress effects, effect of unequal top and bottom fibre temperatures, etc.)

• *Cantilever Spans* If it is established that economy is achieved by designs which result in high negative moments over piers and relatively small positive moments, cantilever spans of prestressed concrete construction may be used to advantage. They may be cast *in situ* or assembled from precast segments. Cantilevers are fixed to stiff piers. Cantilever bridge structures may be further separated into the following types:

(a) Cantilevers extending from two adjacent piers and interconnected with a shear transfer device within the span length (a hinge).

(b) Cantilevers extending from two adjacent piers and interconnected at midspan in such a manner (by arrangement of tendons) that full continuity is established.

(c) Cantilevers in combination with simply supported suspended (riding) spans.

(d) Cantilevers in combination with suspended spans, which are made continuous with the cantilevers for live loads only.

NOTE (a) and (b) may finally rest on piers through discrete bearings, however.

Past experience indicates that types (a) and (b) are less economical than types (c) and (d), that type (b) is less economical than type (d), and that type (d) is less economical than type (c). Type (c) has more joints and consequently not as good a riding quality.

A considerable degree of engineering skill is required to make cantilever construction fully successful. All deformations and resulting stresses which are introduced during construction must be carefully considered. Deflections of cantilevered spans are influenced greatly by temperature, shrinkage and creep. In general, prestressing of webs is advantageous for increasing their shear capacity, but this is not easy. The analytical procedure should take into consideration axial, flexural and torsional stiffnesses, as described earlier. Also see 'Design of Substructure' given below.

• *Precast Segmental Box Girders* Except as otherwise noted here, the provisions of prestressed concrete apply as much to the analysis and design of precast segmental box girder bridges. Deck slabs without transverse post-tensioning are designed under the applicable provisions as for reinforced concrete deck slabs.

Elastic analysis and beam theory may be used in the design of precast segmental box girder structures. For box girders of unusual proportions, other methods of analysis which consider shear lag (full width of top slab may not be effective) shall be used to determine the portion of the cross-section to be used in resisting longitudinal bending, etc.

1. Design of Superstructure

(i) *Flexure* The transverse design of precast segments for flexure considers the segment as a rigid box frame. Top slabs are analyzed as variable depth sections considering the fillets between the top slab and webs. Wheel loads are positioned to provide maximum moments, and elastic analysis is used to determine the effective longitudinal distribution of wheel loads for each load location. Transverse post-tensioning of top slabs may be adopted. In the analysis of precast segmental box girder bridges no tension is generally permitted in the joints between segments during any stage of erection or service loading.

(ii) *Shear** Shear keys are provided in segment webs to transfer erection shears. Possible reverse shearing stresses in the shear keys shall be investigated, particularly in segments near a pier. At the time of erection, the shear stress (in p.s.i.) carried by the concrete section engaged by the shear keys should preferably not exceed $2\sqrt{f_c'}$(f_c' = 28 day cylinder crushing strength in p.s.i.), unless a more detailed analysis is made.

Design of web reinforcement for precast segmental box girder bridges is to be in accordance with the provisions of design for shear.

(iii) *Torsion** Consideration has to be given to the increase in web shear resulting from eccentric loading or geometry of structure.

(iv) *Deflections* Deflection calculations must consider dead load, prestressing, erection loads, concrete creep and shrinkage, and steel relaxation. Deflections are calculated prior to erection of segments (based on the anticipated segment production and erection schedule) as a guide against which construction deflection measurements are checked. The precamber calculation is necessary in order to give reverse deflection of estimated magnitude at each segment at the time of construction so that eventually (after creep over 2 to 3 years) the deck takes the correct desired profile as closely as possible. Also refer Ch. 37 of this book.

(v) *Details* Epoxy bonding agents for the match cast joints shall be thermosetting, 100% reactive, non-solvent compositions. They shall be formulated to permit erection of match cast segments at site air temperatures from 40°F (5°C) to 122°F (50°C). Epoxy bonding agents shall be relatively insensitive to damp conditions during application and, after curing, shall exhibit high bonding strength to cured concrete, good water resistivity, low creep characteristics, and tensile strength greater than that of concrete. In addition, the epoxy bonding agents shall function as a lubricant during the joining of the match cast segments, as a filler to perfectly match the surfaces of the segments being joined, and to provide a durable water tight bond at the joint.

* For design against Shear and Torsion, refer to Ch. 24 of this book.

2. Design of Substructure In addition to the usual substructure design considerations, unbalanced cantilever moments due to segment weights and erection loads have to be accommodated in pier and foundation design with auxiliary struts or using erection equipment that can eliminate these unbalanced moments. However, a suitable amount of unbalanced moment must be allowed for.

Rigid Frame Spans

Rigid frame is a unit consisting of a longitudinal continuous member rigidly connected with the vertical or inclined members upon which it rests. By *rigid connection* is meant a connection which is designed to resist bending moments, shears, and axial forces without relative displacement among the ends of various members meeting at the connection. Owing to rigid connections the stability of the supports in rigid frames is much greater than that of independent piers.

Each leg member of a frame should be connected with the foundation so as to be able to prevent horizontal movements at the base. Rigid frames should be considered free to sway longitudinally due to the application of vertical dead loads and vertically applied live loads, unless specifically prevented from movement by external restraints.

The effect of continuity of main girders should be included in the calculations of the reactions at the supports. For the analysis of the superstructure column bases should be assumed to be pinned unless they are known to be fully fixed.

The influence of braking forces, changes in temperature, shrinkage in rigid frame members, and yielding of foundations, can be of critical importance. It is also necessary to give proper attention to important changes in the moments of inertia of the frame elements. To determine the elastic properties of the frame, the moment of inertia of the entire superstructure section, excluding rail, curbs, etc., and that of the full cross-section of the bent or pier, should be used.

Provision should be made to prevent the frame from being subjected to a longitudinal horizontal force at the end of the deck due to an expanding road slab or other similar cause.

The members of a rigid frame have to be provided with shear reinforcement as required.

Special attention has to be given to stresses in the rigid frame corners.

Types of Rigid Frames

(i) *Barreled or Solid Slab Frame* The solid slab may be of uniform thickness throughout. Alternatively, the slab thickness may be increased at the supports by the introduction of haunches, or the bottom of the slab may be either segmental or parabolic.

For spans up to 7.5 m it is generally advisable to avoid ribs.

(ii) *Ribbed Rigid Frame* Ribbed frame reduces the dead load bending moments throughout the structure.

(iii) *Hinged Rigid Frame* If the foundation material is of yielding character or the footing is narrow, the frame should be designed on the assumption of hinged conditions of the footing.

(iv) *Fixed Rigid Frame* Full fixity at the footings may be assumed in the design of rigid frames if the foundations are sufficiently rigid and the dead load reaction produces a soil pressure as uniform as possible at the base of the foundations.

(v) *Skew Rigid Frame* In general, a skew frame must be analyzed in two dimensions. The finite element method is recommended for this analysis.
For preliminary design purposes, such a frame may be treated as a right angled frame with the calculated moments and trusts multiplied by a factor equal to the square of the secant of the skew angle.

Skews of more than 50 degrees are impractical for rigid frame type bridges.

The reinforcement at the obtuse corners should receive special consideration.

Arch Spans

Where site conditions are favourable to adoption and construction of an arch span, this type of bridge will usually result in an economically aesthetic solution. Arch spans should not be contemplated at sites where foundation conditions are not ideally suited to this type of bridge. Fixed arch spans should not be considered unless firm foundations prevail at the site for proper resistance to the arch reactions. Pile foundations can be used successfully but the economy of the selection is thus partially diminished. At sites where falsework may be difficult to construct, segmental construction using tie-backs can be considered.

Analysis

• *Short Spans (Elastic Method)* On short spans, the arch geometry should be such that the centroidal axis of the arch ring conforms, as nearly as practicable, to either the equilibrium polygon (line of pressure) for full dead load or the equilibrium polygon for full dead load plus one half live load over the full span, whichever produces the smallest bending stresses. On short spans, the deflection of the arch axis is small and the secondary moments (represented by the thrust times the deflection) may be neglected in the arch design.

• *Long Spans (Deflection Method)* On long spans (over 35 m), it is imperative that the arch centroidal axis coincides with the equilibrium polygon for full dead load.

Any live load system will disturb the above equilibrium polygon and cause an elastic deflection of the rib. This deflection creates additional moments which cause further deflection, thus creating still additional moments. If the rib is too flexible, the arch ring may not be able to regain equilibrium and fail radially by buckling. To prevent this type of failure the arch ring should be designed for the ultimate load and moments, including the elastic and deflection effects. This method of analysis is involved, but by the use of computers the deflection method taking into account the non-linear behaviour may be readily applied and a rapid and accurate analysis is possible. For the analysis, full consideration must be given to plastic flow, creep, temperature, variation in modulus of elasticity, possible geometric errors and other factors that cannot be precisely evaluated.

Span Lengths

The span length of the arch may be determined as follows:

(i) Two and three hinged arches: horizontal distance between the centres of the hinges at the supports.

(ii) Fixed arches: horizontal distance between the spring lines, l', adjusted for possible rotation at the support abutment. The adjusted span length, l, is dependent on the type of foundation material as follows:

$$\text{Solid rock } l = l' + 1.2h_r$$
$$\text{Medium rock } l = l' + 1.8h_r.$$

where h_r is the rib-thickness at the theoretical point of fixity. (On Gravel and sand: arch construction not recommended.)

Resistance to Transverse Buckling

Arch ribs are also investigated for resistance against transverse buckling. For buckling in the transverse direction, it may be assumed that the arch rib is a straight column with a length equal to the span and an axial load equal to the horizontal thrust. The resistance to buckling or the safety factor can then be based on column design.

Types of Arch Spans

• *Spandrel Arches* The spandrel arch may be either the open-spandrel column type or the filled-spandrel type. The arch should be designed for the combination of the dead load, live load, impact, the effect of temperature and rib shortening, and the effect of tractive, wind and centrifugal forces. In lieu of a complete analysis of the entire structure, an open spandrel arch may be designed for concentrated loads applied at the points of spandrel support. The amount of live load concentration at each point should be determined from the live load on the roadway deck, placed to produce maximum stress at the section under consideration. In filled spandrel construction, the equivalent uniform live load or distributed axle load may be used for the arch design.

(i) Arch ribs should have a minimum thickness as governed by architectural appearance as well as facility of construction and be reinforced longitudinally with two main layers of reinforcement. These two layers of reinforcement should have an area preferably not less than 0.2% of the arch section at the crown, and be tied together by a series of stirrups spaced not further apart than the thickness of the rib. Arch ribs and spandrel columns should conform to the column provisions. Transverse walls should be treated as columns with lateral ties.

Bearing seats for columns and transverse walls over arch rings or ribs should be horizontal with suitable dowels to anchor to the spandrel construction.

(ii) *Filled spandrel* The spandrel walls may be of gravity; cantilever or counterfort design or tied back with transverse walls. Counterfort or cantilever construction should be proportioned and reinforced to minimize transmission of torsional stresses to outer section of the arch barrel. The face of the spandrel walls has to be reinforced in two directions for temperature and shrinkage. The arch ring requirements listed in (i) above apply for this type also. The spandrel wall reinforcement in filled type arches shall preferably extend into the barrel with adequate anchorage to develop the strength of the bars in bond.

• *Three Hinged Arches* The three hinged arch is statically determinate and is recommended for short spans. Hinges may be formed by one of the following method:

(a) Structural Steel Pinned Shoe.
(b) Crossing the intradosal and extradosal reinforcement at the hinge centerline and reducing the depth of the ring or rib towards the hinge.

• *Two Hinged Arches* For arch spans founded on soft rock, piled foundations, or any foundation material other than sound rock, the two hinged design is recommended.

The hinge types (a) and (b) listed above are also applicable for two hinged construction.

• *Fixed Arches* Full fixity at the foundations may be assumed if these are sufficiently rigid and the dead load reaction produces a soil pressure as uniform as possible at the base of the foundations.

Truss Spans

To avoid congestion of reinforcement at joints. Vierendeel trusses should be used instead of triangular trusses. Vierendeel trusses may be simply supported, cantilevered, or continuous. They may have parallel chords, inclined upper chords (if through type), or inclined lower chords (if deck type). The deflection of Vierendeel trusses is usually smaller than girder bridges of similar span length. In a Vierendeel truss, inclined end posts with smaller bending stresses are better than upright end posts. Panels having length greater than the truss height are generally more economical than square or upright arrangement. In the analysis of Vierendeel trusses, the entire structure should be treated as a rigid frame to determine the axial, flexural, and shear loads in each member. (Design is rarely governed by shear.) Vierendeel trusses with upper (if through type) or lower (if deck type) lateral systems should be analyzed as rigid space frames.

Members of Vierendeel trusses may be precast in the plant and field-jointed. The availability of high strength concrete, high strength reinforcing bars and prestressing strands makes this type of bridge adaptable to moderately long spans. However, truss type bridges in concrete are seldom built.

Stayed and Suspension Spans

Stayed and suspension structures are best suited for spans exceeding 150 m. The stayed span is characterized by straight inclined cables which support the deck system at one or several locations (Fig. 18.23). The suspension span, on the other hand, is characterized by vertical suspenders and near parabolic suspension cables (Fig. 18.24). (For more details in cable stayed deck refer Ch. 38 of this book.)

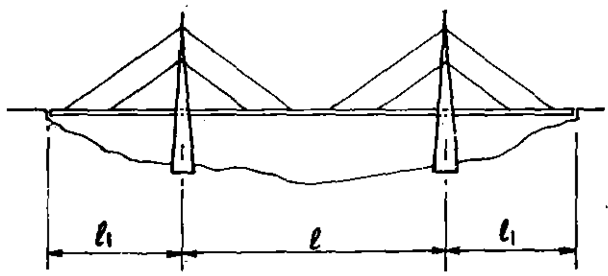


Fig. 18.23 *Stayed span*

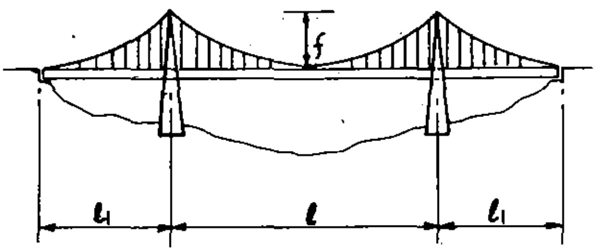


Fig. 18.24 *Suspension span*

Analysis

Short spans may be analyzed by the linear elastic theory. Longer spans, for which the deflections may be substantial, should be analyzed by the deflection theory (non-linear analysis).

Deck and Stiffening System

The deck and the stiffening system may perform either separate or combined functions. In the first case, the deck serves to transfer load to the stiffening and carrying systems. Examples are the conventional steel stiffening girder suspension bridges and some European concrete suspension bridges. In the second case, the deck is merged with the stiffening elements to serve dual purpose. Box girders and solid or voided prestressed concrete slabs are typical examples.

The stiffening system and the cable supporting system share (to various degrees, depending on relative stiffnesses) the live load superimposed on the bridge. In this regard, the stiffness factors must be carefully determined to avoid errors in the stress determination. The following guidelines may be used in the design of the stiffening system:

(i) Under dead load and median temperature, the system may be assumed to be supported by the bearings and cable supports.

(ii) Deflections from live load should not exceed $\frac{l}{800}$, where l = distance between non-yielding supports.

(iii) Live load and temperature effects absorbed by non-yielding supports at the bearings and yielding supports at the cable supports.

(iv) For suspension spans, the stiffening element, being a function of the dead load, becomes less critical and almost unnecessary in very long span structures where dead load exceeds ten times the live load.

(v) For self anchored stayed systems, the added longitudinal thrust in the stiffening element should be considered and can be utilized for the "prestressing".

(vi) The cable sag (f) to span (l) ratio should normally be between 0.09 and 0.14 for suspension spans.

Towers

The towers supporting the main cables may be one of the following types:

(i) Vertical legs with one or several cross girders to provide lateral stability.

(ii) Inclined legs with one or several cross girders for lateral stability. In this instance, the center line of a leg at the top coincides with the cable stay center line. The base width is dependent on the deck width and the lateral stability requirements.

(iii) Inclined legs meeting at a common apex. This type is used only for single cable systems.

In any of the above types, the towers may be hinged or fixed at the base. Fixed bases are more common as temporary support cables are not needed. On the other hand, fixed base results in substantial longitudinal bending moments in the tower legs.

The tower legs may be rectangular or cruciform in shape, their cross-section may be uniform or varied from top to bottom, and they may be prestressed or non-prestressed. Prestressed concrete tower legs are frequently constructed by segmental method. The legs should permit easy access to the top of the tower for maintenance, inspection of the cable saddle and connection, and navigation lights.

Cable Suspension System

The main suspension system on bridges may be one of the following types:

(i) *Parallel wire construction* In this type a series of parallel wires make up the circular sectioned cable.

(ii) *Parallel strand, closed construction* In this type a series of helically wound strands, either singly or more commonly in parallel, make up the circular section of the cable.

(iii) *Parallel strand, open construction* In this type a series of helically wound strands are separated into an open box like shape by means of spacers.

Types (i) and (ii) are nearly always wrapped tightly for protection against water and other corrosive agents. The wrapping usually consists of a skin of red lead paste and a tightly wound galvanized wire encircling the cable. A more recent innovation, which may become the standard method of protecting the cables, is a plastic covering. This utilises plastic filler pieces, zytel nylon film, glass-reinforced acrylic covering and a coating of lucite syrup.

In stayed spans, where the suspension system is virtually straight, concrete encasement of the cable is easier and advantageous. This encasement serves to protect the cable from corrosion, and also to reduce live load deflections, hence providing added safety against fatigue stresses at end connections. However, to fully realize these benefits, the cable must be pretensioned, prior to the concrete encasement, to such a level that tensile stress will not develop in the concrete under the full dead plus live load.

The design of the suspension system takes into account not only the axial dead load, live load and temperature stresses, but also the flexural stresses which could initiate fatigue failure. These flexural stresses normally originate at points of curvature-change and movement.

The ends of all strands are normally embedded in a poured zinc socket designed to develop the full strength of the strand in direct tension.

Suspenders, used in suspension bridges, are normally wire ropes or bridge strands socketed at both ends. The suspenders are fastened to the main cables by means of friction clamps. This design should be carefully checked to prevent slipping of the cable band and crushing of the cable wires by the clamping force.

Aerodynamic Stability

The suspended spans need to be designed so as to minimise the generation of vertical and torsional oscillations. Aerodynamic constants used for design should be carefully evaluated as the bridge site environment may greatly affect the direction and pressure of the wind. Model tests carried out in a wind tunnel for obtaining the most efficient cross-sectional shape of the bridge in its proposed environment are recommended.

NOTE For more details on 'Cantilever Construction' and 'Cable-stayed decks', refer to Chs. 36, 37 and 38 of this book.

CHAPTER 19

Transverse Distribution of Live Load among Deck Longitudinals

19.1 INTRODUCTION

When a concentrated load is applied over a single beam within the width of an open-spaced beam-and-slab deck, some load-sharing clearly takes place with adjacent beams, but the member directly under the load obviously deflects more than the others, and the slab which provides the transverse link between beams is therefore deformed. With a multi-celled box, similar deformations occur at the cross-section when the loads are applied over a single web, but the cross-section here is a closed-frame and the webs of a multi-celled box are not free to rotate in the way that is possible in a beam-and-slab deck, because they are tied laterally, at the top and bottom. The pattern of deformation and the resulting force system are akin to those of a Vierendeel girder.

The term *distortion* which is applied to this pattern of deformation can be misleading because it inevitably is assumed to be asscociated with torsion, which is not necessarily the case. In fact, by considering a twin-celled box subjected to distortion by a concentrated load over the central web, it is evident that the force system which develops can be in equilibrium without imparting any torsional rotation to the deck. Distortion is therefore essentially the effect of differential deflection between adjacent longitudinal members of the deck. This point should be kept in mind.

A bridge deck is basically a platform between piers (and abutments). It could be a slab, or a slab over a grid of longitudinal and transverse beams. Depending on certain considerations, the longitudinal beams could even be interconnected at their bottoms by a soffit slab, in which case the intermediate transverse beams (i.e., other than those at supports and at ends) could be avoided altogether. Whichever the structural scheme for this platform, the live load on it will be distributed among the platform elements in a certain mathematical proportion depending on their flexural and torsional rigidities, their material properties, and the platform geometry and its support conditions. For designing the platform elements it is obviously necessary to analyse their response to the applied load and estimate how much of the latter is apportioned to each one of them. The whole aim of the present chapter is to establish just this in the case of different types of deck-sections, and principally the apportionment of the applied (gravity) live load among the deck longitudinals.

Basically the various methods of analysis of grid and slab structures fall into three main categories. The first category covers those analyses which divide the structure into individual longitudinal and transverse members, each possessing the appropriate flexural and torsional stiffness; for each point of intersection of members, equations of deflection and slope compatibility can be set up and finally a set of governing simultaneous equations must be solved. Lazarides has used a method of this type. A variant of this method depends on the use of moment and torsion distribution or relaxation as a means of obtaining the solution to the simultaneous equations; Janssonius has developed the relaxation approach for bridge structures. This general approach, while of value in certain isolated cases, is extremely cumbersome, involving a great deal of arithmetical work and, of course, cannot be generalised. The advent of electronic digital computers has enabled the abundant numerical problems to be dealt with quickly and accurately, as Lightfoot and Sawko have shown, but it is difficult to produce a simple design procedure based upon this approach.

The second category covers those analyses which separate the longitudinal (or primary) members of the structure and consider some form of secondary cross connection which represents the behaviour of the transverse members. The theories in this category differ in the assumptions made; thus Hetenyi assumed that there was no rotation of individual members at an intersection and used a sine series to represent the load and deflection of the grillage in the direction of the longitudinal members; Pippard and de Waele assumed that the longitudinal members did not rotate and replaced the transverse members by a continuous medium; Leonhardt assumed that the transverse members could be replaced by a single member at mid-span with zero torsional stiffness. Most of these assumptions are invalid in practical bridge structures where the torsional stiffness of members, particularly in reinforced and prestressed concrete, may be considerable. Further, the methods again do not lend themselves to generalisation for an unspecified load position and are, as in the analyses in the first category, very cumbersome to use. Hendry and Jaeger thoroughly developed (during the early fifties) the basic approach outlined above, based upon only one simplifying assumption, which is, that the transverse members can be replaced by a uniform continuous transverse medium of equivalent stiffness. Their approach is to write down the differential equation for the loading on each longitudinal member including, where necessary, the effects of rotation and twisting. *Harmonic analysis* is then used to derive the amplitudes of the deflection and bending moment for each longitudinal member. The approach has been generalised to produce *distribution coefficients* applicable to many practical cases. These coefficients have been derived for bridges with various numbers of longitudinal members. This particular method of analysis is a considerable advance on the previous methods in this category and can be applied to various types of boundary conditions.

The third, and final category, covers those analyses which are based on anisotropic or orthotropic plate theory. These analyses replace the actual bridge deck structure by an equivalent orthotropic plate which is then treated according to the classical elastic theory. Guyon first developed this approach for grillages with members of negligible torsional stiffness and subsequently produced a similar analysis for isotropic slabs. This approach was then generalised by Massonnet to include the effects of torsion.

Extensions and developments of Guyon's and Massonnet's work have been produced, which generalise the use of this method and from which a design procedure has been formulated. This practicalising was done by Little and Morice at the C and CA (London) in the mid-fifties.

This particular approach has the merit that a single set of distribution coefficients for the two extreme cases of a no-torsion grillage and a full-torsion slab, enable the distribution behaviour of many types of deck structure to be found. Further, the implications of the analysis can readily be understood by the designer and hence the calculations become much more meaningful than being just a set of mathematical formulae with no practical applicability.

However, this approach (widely used where appropriate, and loosely referred to as the Little and Morice method) has its limitations—i.e., the deck should be either a slab or a pseudo-slab or a beam-and-slab and should preferably be simply supported with no skew. It cannot accommodate curved geometry. Nevertheless the method is applied for skew angle up to about 20° up to which the skew effect can be ignored without any significant sacrifice of engineering accuracy. Even continuous superstructures are handled by assuming the distance between the two consecutive contraflexure points in a span in the dead load bending moment diagram as the simple span length required in the analysis. (Details are given ahead.)

In 1950, M Courbon came up with an extremely simple approach for estimating the apportionment of live load among deck longitudinals in a beam-and-slab type of deck, but the deck had to be straight in plan, had to have an effective span-to-width ratio of between 2 and 4, had to have enough cross beams in it to make it behave rigidly in order to enable the live load transverse distribution on a rivet group analogy, etc. Courbon's method continues to be very widely used in certain countries, of course within its proclaimed limitations. (Details are given ahead.)

However, a practical method, powerful enough to tackle the load distribution among the curved and the skewed decks, even with box-deck section and variable moment of inertia, had still to appear on the bridge analysis scene.

The advent of electronic computers livened the interest, bringing in its wake the finite strip, finite beam-element and finite plate-element methods of global analyses. With time, these methods were streamlined, debugged and perfected for commercial use. Of these, the one that is good and powerful enough for use in bridge analysis is the *finite beam-element method*. This may also be referred to as the *grillage method* in a two-dimensional set-up. Here the deck is idealised into a grillage of linear beam members (i.e., the beam-elements), generally in two orthogonal directions, crossing each other at the nodes or junctions. Compatibility of deformations of the beam elements is ensured at these nodes. (In the more powerful *finite plate-element method* this compatibility would be ensured all along each edge of each plate-element, and hence the finite plate-element method is more time-consuming (and costly) both in formulating the input data as well as in meaningfully eking out the results from the printout.) More details are given ahead for the grillage method as it is today the most widely used method for the deck analysis where the appropriate computer programs and services are available.

Finite-Strip Method

This is a particular version of finite-element technique. The bridge deck is divided into strips which may all be in the horizontal plane for a plate deck, or in a three dimensional arrangement, as would be required for box construction, with the strips in the horizontal plane representing the slabs, and those in a vertical (or inclined) plane the webs.

This is a useful method, economical in computer time and giving good solutions for those structures which lie within its limitations. It can only represent bridge decks having constant cross-sections and with 'right' end supports. Irregular intermediate supports, can, however, be catered for by the technique of superimposing results to give zero deflections at the points of support on the beam originally considered as spanning between the end supports. (As already stated, it is not a popular method among the bridge analysts.)

Finite Plate-Elements Method

This analytical method is reputed to be very versatile, and is capable of representing complex structures acting in a complex manner. This flexibility inevitably leads to complex computer programs. It is expensive to use compared with the other methods, and its complexities mean that it is not the method for a design engineer who has not been steeped in computer lore and in the mathematics involved in the behaviour of plates — a highly complex mathematical drill.

The method was developed for the design of aircraft structures, where design and development costs dominate the total cost of a project. In that context such sophisticated methods make more sense. For the design of bridgeworks, this method is too sophisticated for regular use.

The ACI Committee 343 report (based mainly on AASHTO specifications) gives very simple rules for transverse distribution of live load among the deck longitudinals (subject to certain restricted dimensions in certain cases). These rules, despite being very simple to apply, obviously deserve to be highly respected, because a very large number of bridges have not only been and are being designed and built to these recommendations in the US, but they are also successfully standing in good service. Big computer programs and complicated formulae need not be necessary in situations where simple formulae simply applied can do for an otherwise complicated job. The details of these ACI recommendations are given ahead, and are strongly recommended for use.

A Word for the 'Practical' Designer

The subject of load distribution in bridge decks in fact can be very complicated. It is deeply rooted in the domains of higher mathematics and ideally involves orthotropic plate analysis. Fourier series is made use of for expressing the applied loads as continuous functions, etc. Rowe has reported on this very extensively, and so have many others. The reader is advised to study the literature indicated in the references (listed at the chapter end) for understanding what is involved.

However, it is important for the practical designer to maintain the realization that any mathematical analysis carried out is only an analogy that is not always truly representative of the way in which the real structure would behave. Too great a reliance on the quantitative answers arising from some form of mathematical analysis may even be indicative of a lack of engineering judgement.

Methods of analysis are evaluated by comparing the results with those given by laboratory tests carried out on models. Considerable progress has been made towards making models representative of prototype structures, but heavy reliance is still placed on measuring the deflected shape under load because of the difficulties in attempting to measure strains which may be more directly related to the stresses developed. The fact that a good comparison is achieved between the displacements obtained on a laboratory model and on an analytical mathematical model does not, in itself, mean that they are both predicting the same pattern of stresses. So the results of mathematical analyses still need to be approached with caution, intuition and (a resolute does of somewhat uncommon) common sense.

Grillage analysis is the most widely used mathematical tool, and the type of deck structure most difficult to represent in this way is a cellular deck, whether this is a voided slab or a box form of construction. The difficulties arise from the fact that the grillage is two-dimensional only, whereas a cellular deck behaves more clearly in a three-dimensional manner. A consideration of these differences can be a valuable aid to understanding both the cellular deck and the analytical limitations of a grillage. There is nothing wrong with making practical idealisations and, where necessary, it is better to go for an approximate solution to an exact problem rather than for an exact solution to an approximated problem. *Time-bound and result-oriented professional practice does not have much room for a frustrated engineer turned mathematician (he has his own place, but elsewhere).*

Consequently, restricting our jurisdiction to workman-like result-oriented professional practice, given in the remainder of this chapter are first certain basic features that influence the load distribution and must be understood, and then the details about the transverse distribution of live load among the deck longitudinals (various methods and deck sections).

19.2 BASIC FEATURES RELATED TO TRANSVERSE LOAD DISTRIBUTION

Shear Lag

Refer to Fig. 19.1. According to the basic assumptions of simple beam theory, where cross-sections are assumed to remain plane even after flexure, the distribution of stress across the top flange of a beam is uniform. In a broad-flanged T or I section, this assumption is not true except for sections which are far from a point of contraflexure. At a point of contraflexure the section is subjected to shearing force but no bending moment. Zero moment implies that there is no 'direct stress' in the flanges, while transverse shear on the section indicates that there are horizontal shearing stresses reducing in intensity toward the extremities of the section. In the case of a broad flanged I-section this means that the horizontal shear flow diminishes to zero at the outer-edges of the flange. Away from the point of contraflexure, direct stresses are present because of the moment on the section and the shearing stresses get modified. As with the case of simple bending theory for beams, the horizontal shear flow and direct stresses are inter-related, and what is happening may be visualized in terms of shear flow injecting direct stresses into the flange. The build up of these direct stresses resulting from the shear flow is not uniform across the width of the wide flange, but produces stresses which tail off toward the extremities, until a distance is attained that is far enough from the point of contraflexure for the pattern of stresses to have reached a balance which produces uniform direct stress. The effects associated with this change of distribution of direct stress are known as shear lag, and it consequently reduces the 'effectiveness' of the area of compression flange.

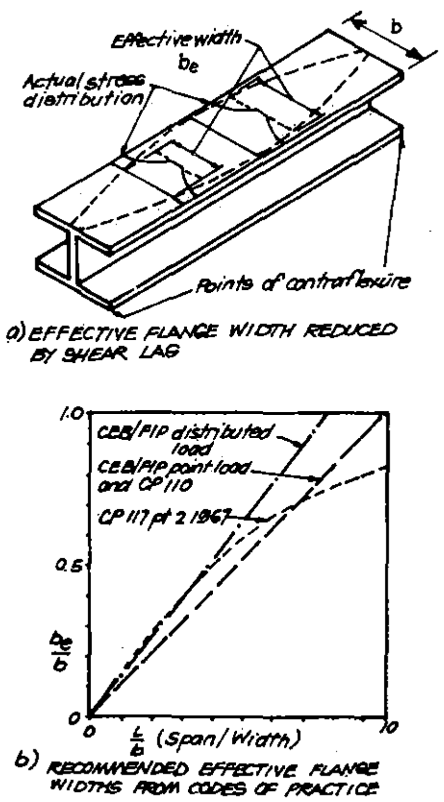


Fig. 19.1 *Shear lag*

In order to assess the peak stresses in such a section, the commonly adopted procedure is to calculate the sectional properties on the basis of a reduced (so-called effective) flange width. BS CP 110 and CP 117 (and indeed other codes too) recommend methods of assessing this reduced width. An exact assessment of the effects of shear lag involves recourse to plate theory, because the effect is a function of each particular loading case as well as of the plate dimensions.

With continuous bridge decks the effects of shear lag are most significant at and near the intermediate supports. The recommendation of CP 110 (i.e., that the effective flange widths are assessed on the assumption of points of contraflexure at distances of 0.7 of the span apart) can produce substantial errors at the support section. A reasonable compromise is to evaluate the effects of shear lag on the basis of the points of contraflexure which arise on a given structure under a uniformly distributed load (although, strictly speaking, these positions should be re-evaluated for each loading case, but that is impractical).

Since shear lag reduces the effective stiffness of a member in addition to modifying the distribution of stresses across it, improved accuracy can be obtained from a grillage analysis if the effective sectional properties, arising from shear lag, are used in the grillage input. This may redistribute some moment away from those sections modified by shear lag, thereby reducing the resultant stresses.

Where it is needed, a good picture of the distribution of stresses across the width of a plate can be obtained by plotting those stresses which arise from the net effective sectional properties, together with those arising from the gross sectional properties. The curved distribution of stress across the flange can then be sketched.

When applying shear lag to a box deck, the sectional properties of the reduced section can imply a different centroidal axis position to that which applies to the gross section. For a prestressed beam having varying sectional properties, it is well known that if the centroidal axis changes in level, this in itself, modifies the moment applied by the prestress. However, in the case of a change in the position of the centroidal axis due to shear lag, this is not the case.

The forces produced by the prestress can be considered as two separate elements — the horizontal force applied to the deck, and the bending moments produced by the eccentric cable profile. In the case of a beam with varying sectional properties a change in the position of the centroidal axis modifies the stresses produced by the horizontal force in the same way that a direct load on a column produces varying stresses if the sectional properties change — because of the changing eccentricity of the load at each section. This analogy cannot be adopted in shear lag calculations because differing rates of shear lag apply to the horizontal load due to the prestress, and to the bending moments. For the horizontal load, shear lag affects only the regions adjacent to the anchorages where the horizontal force is applied. The fact that the stresses induced by bending moments arising from the vertical force applied by changes in direction of the prestressing cable are subject to shear lag does not, in itself, modify the stresses produced by the horizontal force.

Not much adjustment is, therefore, needed to the stresses arising from axial prestress, even though the shear lag phenomenon suggests a change in the position of the centroidal axis when considering moments.

Torsion

Refer to Figs. 19.2 and 19.3. There are fundamental differences between a grillage and a cellular bridge deck in relation to the forces and stresses which arise due to torsion. Elementary considerations of equilibrium demand that a shearing stress in one plane can only co-exist with a shearing stress of equal intensity in the complementary plane. In the case of a simple beam this means that shearing stresses of equal intensity are present over a vertical cross-section and in the complementary horizontal plane. In the case of the top or bottom slab of a cellular deck, the implication is that shear flow arising from torsion is of equal intensity over both transverse and longitudinal sections through the deck, at any one point.

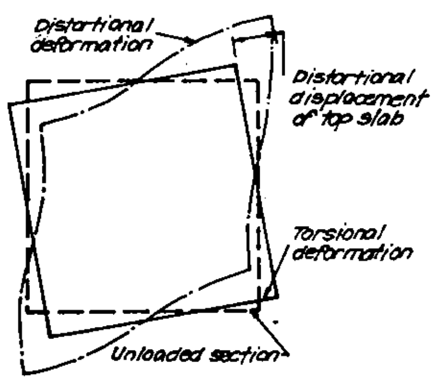


Fig. 19.2 *Torsional and distortional deformations of a box cell*

In contrast with this, there is no interdependence between the longitudinal and transverse torsional moments within a grillage. Balanced results can only be achieved by carefully evaluating the sectional properties. To obtain accurate values for torsion, using a grillage analysis, which might be directly applied to the real structure, theoretically it is necessary to re-assess the torsional stiffnesses of the component members for each individual loading case in the light of the pattern of deformation that is anticipated.

The torsional resistance of a cellular deck results from two primary components, shear flow in the webs and shear flow in the top and bottom slabs. A rectangular cell subjected to torsion develops shear flow of a pattern which means that the vertical (web) members make a contribution to the torsional resistance which is equal to that afforded by the top and bottom slabs.

A grillage subjected to torsion develops a system of reacting forces consisting of three components:

- a torsional shear in the longitudinal members in association with torsion in the transverse members
- a differential flexural shear in the longitudinal members, due to non-uniform load sharing
- torsional moments in the longitudinal members with associated flexure of the transverse members.

The torsional moments evaluated in the longitudinal members by using a grillage are inevitably underestimated and, assuming the stiffnesses of transverse members have been correctly evaluated, an appropriate amount of shear flow will develop in the webs. To this will be added the differential flexural shear reflecting the load-sharing pattern of the longitudinal members. A grillage will only use the torsional stiffness of the longitudinal members to make up the deficiency in equilibrium. If use is made of these longitudinal torsions to evaluate the flow of shear in the top and bottom plates, this will inevitably give a value which is less than that in the vertical webs, which must be equal in terms of force per unit perimeter.

Stiffnesses assigned to transverse members of the grillage have a twofold effect—they give rise to torsional moments which can be used to evaluate the flow of shear in the top and bottom slabs, and they modify the bending moments in the longitudinal grillage members, including shears which represent the flow of shear within the webs of the box. It is this which gives the grillage bending-moment diagrams their characteristic 'saw-toothed' shape.

In a grillage representing a multicelled box, the pattern of torsion in the transverse members within the various 'bays' of the grillage shows the build-up of torsion from the edges of the box inward. Intermediate longitudinal members are influenced only by the change in torsion along a transverse member, on each side of the node at which they intersect.

In considering a particular cross-section of deck, the transverse members reflect the torsion on the basis of the twist along that line. But the shear flow in the webs will be an outcome of the twist in the adjoining transverse members also, which may be some distance away and possibly subject to a significantly different twist, because of the deflected shape of the deck. To this extent, the shear flows evaluated are misrepresented.

It follows that the transverse members are a better source of values for the torsion within a grillage. However, if shear flexibility has been incorporated in the grillage sectional properties this will have the effect of increasing the differential deflection between the longitudinal members and, therefore, correspondingly increasing the twist in the transverse members connecting them. The torsion in the transverse members is therefore overstated.

Clearly, the prime contributors to torsional stiffness are

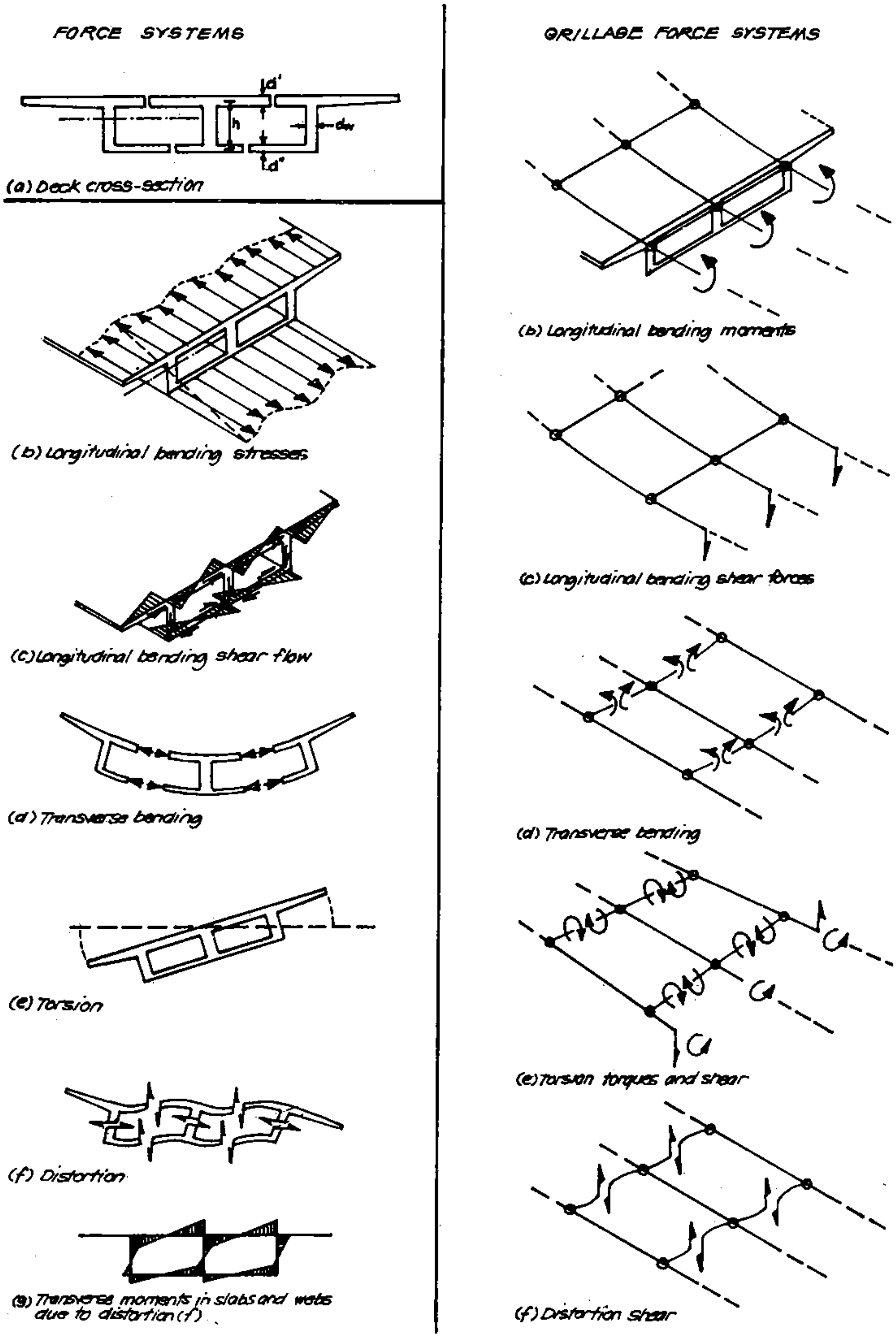


Fig. 19.3 *Grillage force system in a twin-cell box section*

the top and bottom slabs. Where diaphragms are present within the depth of a deck, a consideration of membrane analogy for torsion will readily show that the flow of torsional shear in a diaphragm member is only the difference in the flow of shear in the top and bottom slabs as they cross the diaphragm.

Distortion

Refer to Figs. 19.2 and 19.3. As explained earlier distortion of a cell section is essentially caused by the differential deflection between adjacent longitudinal members of the cell section, and is not necessarily associated with torsion.

Because of distortion, a cellular deck has a dual stiffness transversely. Circular bending across the deck results in direct tension and compression in the top and bottom slabs, thereby forming a couple to resist the bending moments. No distortion is involved. On the other hand, where there is a transfer of shear across the deck, it gives rise to distortion and the stiffness of the deck is now totally different from that which applies to circular bending. Most real loading cases produce a combination of these two effects. It is therefore essential that a grillage should be capable of handling this duality of stiffness.

In calculating the deflections of a beam, it is normal to consider only those deformations which arise due to bending. A full mathematical expression of the deflection in a beam also includes the deformations arising as a result of shear, in addition to those due to flexure. But shear deformations* are normally neglected because they are very small by comparison with those due to bending. To enable the transverse members of a grillage to take account of the dual stiffnesses applicable to bending and to shear, grillage programs have been written that include the full deflection equations, including shear. It is, therefore, possible to take account of the dual stiffness of a cellular section by calculating a shear area for the transverse members which would give the same rate of deformation as that which, in reality, arises from distortion.

The calculation of this 'effective shear area' is based on assumptions of how the cross-section deforms. This will, in fact, differ according to the loading case applied. In the case of a heavy load applied at one edge of a deck section, all the cells deform in a similar manner, including the flexure of the webs themselves. In contrast, a heavy load placed in the middle of the deck section, which is such that the force systems are in balance on each side of the point of application, means that the web under the load does not transversely flex (because there is no moment in it).

In the case of thin-webbed (i.e., steel) boxes, the presence or otherwise of flexure in the webs may significantly affect the stiffness of the cell. The shear area assigned to the transverse members can therefore only approximate the distortional stiffness, but in most practical bridge-deck sections the top and bottom slab elements are considerably more flexible than the webs, so this error is not serious.

Where there are variations in the shape of the cross-section, such as sloping webs, which complicate the assessment of the distortional flexibility, this can be assessed with the aid of a frame representing the cross-sectional shape. The deformation of the frame under the load provides a basis for assessing the 'effective shear area'.

Having obtained the grillage output, the force system arising from deformation is calculated by assuming that the points of contraflexure develop midway between the webs, and that the total shear from the grillage is shared between the top and bottom slabs in proportion to their stiffness.

In designing reinforcement or determining the prestress required to cater for the forces developing around the cross-section, the values are dominated by those forces which arise from local wheel loads. These are best evaluated by applying Pücher's (or a similar) design method for concentrated loads (details regarding the analysis and design of transverse section of deck are given in a separate chapter in this book). The distortional stresses are significant in thin walled (e.g. steel) boxes, and are small in comparison with the effects of local wheel loads.

19.3 TRANSVERSE DISTRIBUTION OF LOADS (AS PER ACI COMMITTEE 343 REPORT AND AASHTO SPECIFICATIONS)

Analysis based on the elastic theory is recommended to find the distribution in the transverse direction of the bending moment in the direction of the span. For the analysis, the structure may be idealized in one of the following ways:

(i) a system of interconnected beams forming a grid
(ii) an orthotropic plate
(iii) an assemblage of thin plate elements or thin plate elements and beams.

Several methods of analysis are available which can be applied with the use of a computer. In addition to the moments in the direction of the span, computer aided analyses can give moments in the transverse members also. A theoretical analysis is particularly recommended for bridges which have large skews or sharp curvatures.

• Beam-and-Slab (T-Beam or Precast I-Girder) and Box Girder Bridge Decks

In lieu of an analysis (based on elastic theory) for the distribution of live loads among longitudinal beams, the following empirical method authorized by AASHTO may be used for T-beam or precast I-girder bridges [see Figs. 19.4 (a) and (c)], and for box-girder bridges [Fig. 19.4 (b)]. The

* For details refer to Ch. 20 of this book.

distribution of shear should be determined by the method prescribed for moment.

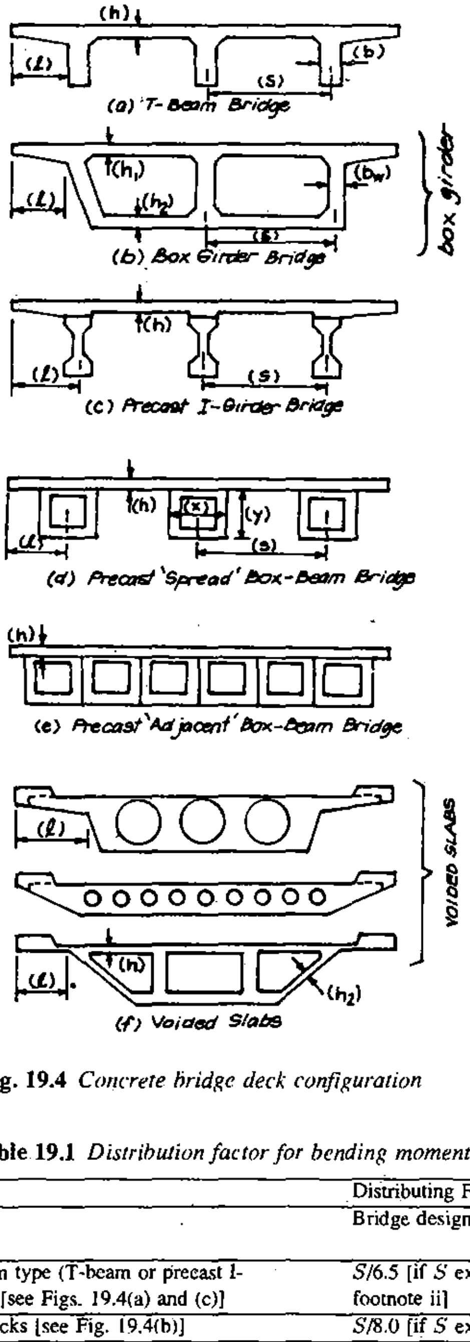


Fig. 19.4 *Concrete bridge deck configuration*

Interior Beams in Cases of Beam-and-Slab and Also Box Girder Decks

The live load bending moment for each interior longitudinal beam should be determined by applying to the beam the fraction (DF) of the wheel load (both front and rear) as defined in Table 19.1. In the present context wheel load (both front and rear) shall mean the effect of one wheel line i.e., half the effect of one truck.

Exterior Beams in Case of Beam-and-Slab Decks Only

The dead load considered as supported by the outside roadway beam is that portion of the floor slab as is carried by that stringer or beam. Curbs, railings and wearing surface, if placed after the slab has cured, may be considered equally distributed to all roadway stringers or beams.

The live load bending moment for outside roadway beams is to be determined by applying to the beam the reaction of the wheel loads obtained by assuming the flooring to act as simple spans transversely between the beams.

When the outside roadway beam supports the sidewalk live load as well as traffic live load and impact, the allowable stress in the beam may be increased 25% for the combination of dead load, traffic live load, and impact, provided the beam is of no less carrying capacity than would be required if there were no sidewalks.

In no case should an exterior beam have less carrying capacity than an interior beam.

Exterior Beams in Case of Box Girder Decks Only

The dead load considered as supported by the exterior girder here should be determined in the same manner as already given.

The wheel load distribution factor to the exterior girder should be taken as $w_e/7$. The width w_e to be used in determining the *wheel line distribution* to the exterior girder is the top slab width as measured from the midpoint between girders to the outside edge of the slab. The cantilever

Table 19.1 *Distribution factor for bending moments in interior main beams (for beam and slab and box-girder decks)*

	Distributing Factor (DF)	
Kind of floor	Bridge designed for one traffic lane	Bridge designed for two or more traffic lanes
Slab and beam type (T-beam or precast I-girder) decks [see Figs. 19.4(a) and (c)]	$S/6.5$ [if S exceeds 6 ft, use footnote ii]	$S/6.0$ [if S exceeds 10 ft, use footnote (ii)]
Box-girder decks [see Fig. 19.4(b)]	$S/8.0$ [if S exceeds 12 ft, use footnote (ii)]	$S/7.0$ [if S exceeds 16 ft, use footnote (ii)]
Multi-precast spread box-beam decks [see Fig. 19.4(d)]	(Given above in this chapter)	
Multi-precast adjacent box-beam decks [see Fig. 19.4(e)]	(Given ahead in this chapter)	

NOTE

(i) S = average beam spacing feet.

(ii) In this case the load on each beam will be the reaction of the wheel loads, assuming the flooring between the beams to act as simple beams.

dimension of any slab extending beyond the exterior girder should not exceed $S/2$, where S is the girder spacing in feet.

Total Capacity of Longitudinal Beams

The combined design load capacity of all the beams and stringers in a span should not be less than that required to support the total load in the span.

In Multi-Precast Spread Box-Beam Decks

For spread box beam superstructures [Fig. 19.4(d)], the lateral distribution of live load for bending moment can be determined as given here:

1. *In interior beams* The live load bending moment in each interior beam can be determined by applying to the beam the fraction (DF) of the wheel load (both front and rear) determined by the following equation:

$$DF = \frac{2n_l}{n_b} + k_l \frac{S}{l}$$

where n_l = number of design traffic lanes
n_b = number of beams ($4 \leq n_b \leq 10$)
S = beam spacing in feet ($6.75 \leq S \leq 11.00$)
l = span length in feet; for continuous structures use lengths to approximate points of zero moment under uniform load
$k_l = 0.07w_c - n(0.1n_l - 0.26) - 0.20n_b - 0.12$
w_c = roadway width between curbs in feet ($32 \leq w_c \leq 66$).

2. *In exterior beams* The live load bending moment in the exterior beams should be determined by applying to the beams the reaction of the wheel loads obtained by assuming the flooring to act as a simple span between beams, but the reaction should not be less than $2n_l/n_b$.

In Multi-Precast 'Adjacent' Box-Beam Decks

This type of deck [Fig. 19.4(e)], is constructed with precast reinforced or prestressed concrete beams which are placed in contact side by side on the supports. The interaction between the beams is developed by continuous longitudinal shear keys and lateral bolts which may, or may not, be prestressed.

In calculating bending moments in these beams, conventional or prestressed, no longitudinal distribution of wheel load should be assumed. The lateral distribution should be determined as given below.

The live load bending moment for each section should be determined by applying to the beam the fraction of a wheel load (both front and rear) determined by the following relations:

$$DF = \frac{(12n_l + 9)/n_b}{(5 + n_l/10) + (3 - 2n_l/7)(1 - C/3)^2}$$

or when C is 3 or greater, use

$$DF = \frac{(12n_l + 9)/n_b}{(5 + n_l/10)}$$

where n_l = number of design traffic lanes
n_b = number of beams
$C = K(w_c/l)$, a stiffness parameter
w_c = roadway width between curbs (in ft)
l = span length (in ft)

VALUES OF K TO BE USED IN $C = K(W_c/l)$

Beam Type and Deck Material	K
Nonvoided rectangular beams	0.7
Rectangular beams with circular voids	0.8
Box section beams	1.0
Channel beams	2.2

• In Solid Slab Type Decks (with main Reinforcement Parallel to Traffic)

In Simple Span Slab Deck

S = effective span length, in ft
E = width of slab in ft over which a wheel load is distributed

For wheel loads the distribution width E shall be $(4 + 0.06S)$ but shall not exceed 7.0 ft. Lane loads are distributed over a width of $2E$.

For simple spans, the maximum live load moment per foot width of slab, without impact, is closely approximated by the following formulae:

(i) *HS 20 loading*

- for spans up to and including 50 ft LLM = $900S$ ft pounds/ft width
- for spans 50 to 100 ft LLM = $1000\,(1.3S - 20)$ ft pounds/ft width
  (LLM = live load moment)

(ii) *HS 15 loading* Use 3/4 of the values obtained from the formulae for HS 20 loading, mentioned above.

Moments in continuous spans shall be determined by suitable analysis using the truck or appropriate lane loading.

• In Cantilevering Slab Deck (Cantilevering Parallel to Traffic Direction) The distribution width for each wheel load on the cantilevering slab (cantilevering parallel to traffic) shall be as given below:

$E = 0.35X + 3.2$, but shall not exceed 7.0 ft and the live load moment per foot width of slab shall be $(P/E)\,X$ foot-pounds, where X is the distance in feet from load to point of support, and P is the load in pounds from one rear wheel of the truck (rear axle being the heaviest axle in the truck).

19.4 COURBON'S METHOD FOR ESTIMATING TRANSVERSE DISTRIBUTION OF LIVE LOAD AMONG DECK LONGITUDINALS IN A BEAM-AND-SLAB TYPE DECK

Reference may be made to J Courbon's *Application de la Resistance des Materiaux au Calculdes Ponts*, published by Dunod, Paris, in 1950.

In simple words this method for transverse load distribution among the deck longitudinals is applicable mainly to beam and slab type decks which are straight in plan (no skew, no curve). The longitudinal beams must be interconnected by full-depth (or almost full-depth) rigid cross beams that are at least five in number (one above each support, and, at least three intermediate cross beams, equally spaced) such that they are not more than about 9 m apart. The cross beams should preferably be cast monolithically with the longitudinals or should be cast at least before any other gravity loads (besides the self weight of the main beams) comes on. The longitudinal effective span ($2a$) should preferably be simply supported, and its ratio with the effective width ($2b$, equal to the product of the number of main beams and the spacing between main beams) should be between 2 and 4. However, the method can be applied even to longitudinally continuous decks in which case $2a$ must be taken as the distance between the two consecutive contraflexure points in a span in a uniformly loaded continuous condition. In addition, the factor ϕ should be less than 0.5, where

$$\phi = \frac{b}{2a}\sqrt[4]{\frac{a}{b}\frac{\Sigma(EI)\text{of main beams}}{\Sigma(EI)\text{of cross beams}}}$$

EI being the flexural rigidity.

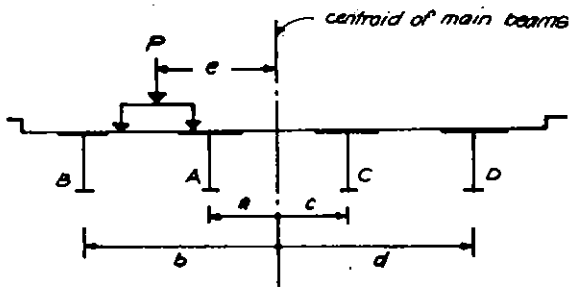
In principle, advantage is taken of the action of short (because of $2a/2b$ restriction) and deep diaphragms, action considered somewhat akin to that of a thick footing resting of piles and assumed to remain straight at all times. Thus, under a truck load P applied as shown in Fig. 19.5 on a deck section that satisfies the above specifications, the transverse distribution of the load P among the various main beams will be such that the reaction felt by a main beam will be equal to

$$\frac{P}{n} \pm \frac{Pe}{I}x$$

i.e.,

$$P\left(\frac{1}{n} \pm \frac{ex}{\Sigma x^2}\right)$$

where x is the distance of that beam from the centroid of the main beams, Σx^2 is the sum of the squares of the distances of various main beams from the said centroid, and n the number of main beams; +ve sign if the beam is on the load-side relative to the centroid of the main beams, and −ve sign otherwise.



$I = \Sigma x^2$
$= (a^2 + b^2 + c^2 + d^2)$
in this case,
n = no. of main beams
= 4 in this case

Fig. 19.5

In Fig. 19.5. if the main beams are equally spaced, say at y centres, then

$$I = \Sigma x^2 = [(1.5y)^2 + (y/2)^2]2 = 5y^2$$

So that,

$$\text{Reaction on beam } B = P\left(\frac{1}{4} + \frac{e(1.5y)}{5y^2}\right) = P(0.25 + 0.3e/y)$$

$$\text{Reaction on beam } A = P\left(\frac{1}{4} + \frac{e(0.5y)}{5y^2}\right) = P(0.25 + 0.1e/y)$$

$$\text{Reaction on beam } C = P(0.25 - 0.1e/y)$$

$$\text{Reaction on beam } D = P(0.25 - 0.3e/y)$$

and, for example, if $e = 1.5y$, then,

Beam B takes: $P(0.25 + 0.3 \times 1.5) = 0.7\ P$
Beam A takes: $P(0.25 + 0.1 \times 1.5) = 0.4\ P$
Beam C takes: $P(0.25 - 0.1 \times 1.5) = 0.1\ P$
Beam D takes: $P(0.25 - 0.3 \times 1.5) = -0.2\ P$(i.e., upward effect)

Total = $1.0\ P$

Instead, if $e = 0$, i.e., load is centrally placed, then each beam takes 0.25 P, i.e., the beams are equally loaded.

However, the latter is not quite true since the central beams will tend to take more load under a centrally placed load.

Thus it is clear that Courbon's method slightly under-estimates the load on interior beams and over-estimates the load in the outer beams. However, deeper, the cross beams and more their number, lesser will be this inaccuracy. Nevertheless, within the restrictions specified earlier, Courbon's method is acceptable and many load tests carried out on completed decks have borne this out. Hundreds of decks have been analysed (and constructed) by this method and more continue to be analysed (and constructed). Test results vouch for the engineering accuracy of this method of transverse load distribution, limited to its stipulations stated earlier.

Caution

What has been described above is valid if the applied load is at least about 5 m away from the supports. For loads placed anywhere within the central $(2a - 10)$ m of the span ($2a$ being the said longitudinal span in meters), the reaction factors as already calculated can be used for apportioning moment, shear, reactions, etc., to the longitudinals. But for loads placed within about 5 m from the supports, the reaction factors for their transverse distribution (i.e., apportionment of moment, shear, reactions, etc., from them) to the various deck longitudinals should be done using the *reaction influence lines* for respective main-beam-support points, assuming the deck section is a continuous beam on rigid supports (supports being the longitudinal beams). This is so since the deck section tends to behave rather rigidly at and close to a support.

Analysis for Cross Beams

Instead of going into too much theoretical detail here, it is basically enough to remember the precaution as pointed out earlier. After working out the live load reaction attractable by the particular cross beam in question, the intermediate-one or the end-one, (for which an approximation may be made that the live load is simply supported longitudinally in between the successive cross beams), for analysing the moments and shears in the cross beam at its various sections. Courbon's live load reaction factors may be adopted for establishing the reactions from the supporting main beams in the case of intermediate cross beams and ordinary continuous beam reaction influence lines may be adopted in the case of (rigidly supported) end cross beams. Once the governing moments and shears at various sections in a cross beam have been established, then follows its section design. In the latter it is necessary to first establish the width of top slab that can be assumed to act along with the stem of the cross beam section in its sagging moment regions. (This will be helped by adequately detailing the flexural and shear reinforcements in order to achieve the combined action of the stem and the slab.) In its hogging moment regions the stem may be considered on its own as being the effective section of the cross beam.

19.5 LITTLE AND MORICE METHOD (FOR ESTIMATING TRANSVERSE DISTRIBUTION OF LIVE LOAD AMONG THE DECK LONGITUDINALS IN A SOLID-SLAB TYPE OR A PSEUDO-SLAB TYPE OR A BEAM-AND-SLAB TYPE DECK)

This is also called the method of distribution coefficients. The necessary background for this method has already been covered to the necessary extent earlier. The working details of the method have already been well documented by the C and CA (London) through their various publications (e.g., their publication Db. 11, first published in July, 1956, from which has gratefully been taken the information quoted ahead and in the associated tables and graphs). Its range of application is limited—it is only applicable to slab, pseudo-slab, and beam-and-slab types of construction having prismatic cross-sections, the spans being simply supported, with line supports and right spans only. In practice this analytical tool is of much wider use than what these limitations first imply. Its results can generally be accepted for skews of up to 20°, and a series of discrete supports can be accepted as representing line supports, provided there is no significant overhang beyond the outer bearings and the spanning effect between the bearings does not become dominant in terms of the behaviour of the deck.

Even where it is felt that this load-distribution, method would not provide a suitable final analysis for a deck, it can still be a useful tool for making a reasonable approximation of the load-distribution characteristics at an early stage in the calculations.

Until recently, this has been a method that was applied only by hand calculation, but the theory has now been reformulated in a manner which lends itself to computerized calculation, with the resulting benefit of improved accuracy and speed.

The basis of this method of distribution coefficients is the study of an equivalent elastic system obtained by transforming the stiffness of a number of beams, which may be considered as concentrated at nodal points [Fig. 19.6(a)], into a uniformly distributed system of the same overall stiffness. The effect of producing a distributed system is to introduce a structure width, $2b$, which is given by the number of original main beams multiplied by their spacing. This results in an 'equivalent width' greater than the original width in certain cases, (Fig. 19.6). The method may also be applied to slabs when the equivalent system is the same as the original system.

Since the equivalent systems are distributed, the section properties too must be expressed per unit width.

Thus, the longitudinal second moment of area per unit

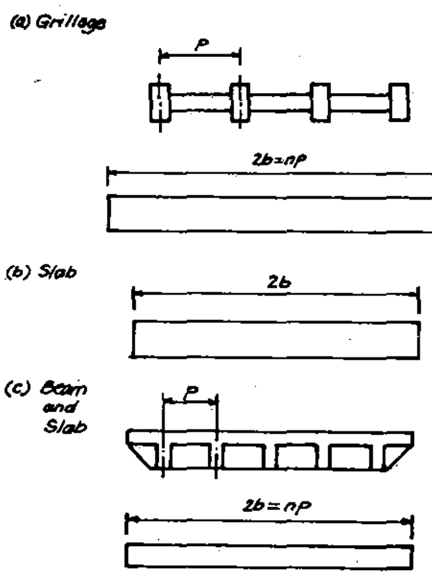


Fig. 19.6 *Actual and effective widths of various types of bridge deck structure*

width is,

$$i = \frac{I}{p}$$

where I = the second moment of area of a longitudinal beam section

and p = the longitudinal beam spacing.

Also, the transverse second moment of area per unit length is,

$$j = \frac{J}{q}$$

where J = the second moment of area of section of a transverse stiffener (i.e., diaphragm or cross beam)

q = the stiffener spacing.

In the case of a slab these per unit width section properties i and j are equal, and are given by

$$i = j = \frac{d^3}{12}$$

where d is the slab depth.

It is convenient to combine these geometrical properties into one parameter defining the relation between longitudinal and transverse properties, since this one parameter has a primary effect upon the distribution coefficients. The parameter, denoted by θ, is defined by,

$$\theta = \frac{b}{2a}\sqrt[4]{\frac{i}{j}} \tag{19.1}$$

where $2a$ is the actual effective longitudinal span.

In the case of a slab the expression for θ is seen to reduce to $\frac{b}{2a}$.

In addition to the effects of θ, the distribution coefficients are also sensitive to the degree of torsional stiffness exhibited by the bridge deck.

The torsional properties, per unit width, are determined in the same way as those for bending (but see ahead), i_0 being the value for a transverse section and j_0 for a longitudinal section. These are combined in an overall torsional parameter α which is defined by,

$$\alpha = \frac{G(i_0 + j_0)}{2E\sqrt{ij}} \tag{19.2}$$

where E = Young's modulus, G = torsional modulus } of the material of the bridge deck.

In the case of a slab α reduces to 1.0. (Refer to the section under *Calculation of torsional stiffness* ahead in this chapter.) So far the actual bridge deck structure has been replaced by a 'quasi' slab which has a width $2b$, not necessarily equal to the actual width of the structure, and a span $2a$, equal to the original span, and with its relative stiffness in bending and torsion specified by the non-dimensional parameters θ and α.

Now it is postulated, that within the accuracy of normal engineering design, the transverse deflection profile of the 'quasi' slab under any given loading is of constant form for all positions along the span of the slab.

This is the crux of the method, since it means that one set of relative arithmetical coefficients may be used for defining the deflected shape of all transverse sections. Thus all longitudinal deflection profiles are of the same shape, and it follows that the longitudinal bending moments are of the same form for any point across the width of the bridge. The set of relative arithmetical coefficients may therefore be used unaltered for deflections, longitudinal moments and longitudinal bending stresses. However, the maximum calculated longitudinal moments should be increased by 10% to correct them for the effect of having included only the first term in the harmonic analysis which forms the basis of the theory. When the arithmetical coefficients are expressed in units of the mean deflection (moment or stress), i.e., the deflection of the deck if the same load were uniformly distributed across the whole effective width, they are in the required form and are normally called *distribution coefficients, K*. The distribution coefficients are normally specified for single concentrated loads and will obviously depend upon the transverse eccentricity of the load. By the above postulation, they will be unaffected by the longitudinal or spanwise position of the load.

The merit of the method for *design office use* lies in the fact that these distribution coefficients (transverse deflection profiles) have been tabulated.

In order to make the system practical, the profile shape is given by distribution coefficients relative to the mean deflection of the whole section for stations at nine equally distributed standard positions across the width (Fig. 19.7), and the loading eccentricities are also given at these nine discrete positions $(-b, -3b/4, -b/2, -b/4, 0, b/4, b/2, 3b/4$ and $b)$.

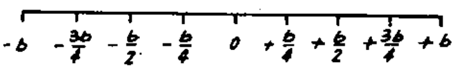


Fig. 19.7 *The nine standard positions on the effective width of a bridge*

The effect of θ upon these distribution coefficients K, is given by plotting them against θ.

The effect of α is introduced by providing two sets of curves, one for $\alpha = 0$. (Graphs 19.1 to 19.6, and the other for $\alpha = 1$, Graphs 19.7 to 19.11.) Any intermediate value of α is catered for by the interpolation expression for the distribution coefficients,

$$K_\alpha = K_0 + (K_1 - K_0)\sqrt{\alpha} \qquad (19.3)$$

where K_α = the distribution coefficient for general values of α

K_0 = the distribution coefficient for $\alpha = 0$

K_1 = the distribution coefficient for $\alpha = 1$

As a result of symmetry of the structure about the centre of a transverse section it is unnecessary to include four of the nine sets of curves, since a change of sign of both load position and reference station leaves the distribution coefficient unchanged. Thus,

K (load position $-3b/4$, reference station $+b/2$)
$= K$ (load position $+3b/4$, reference station $-b/2$)

Step by Step Procedure

The analysis starts with the determination of the elastic constants of the equivalent 'quasi' slab which corresponds to the actual structure (Fig. 19.8). This yields values for $2b$, θ and α; while the value of the span $(2a)$ and the total second moments of area are already known (see Table 19.2).

Next the loadings must be considered. In the first place the total bending moment must be determined (Fig. 19.10). Secondly, the actual loads vis-a-vis their positions must be reduced to statically equivalent sets (λP) acting at the standard positions (Fig. 19.11). These are recorded in Tables 19.3 and 19.4.

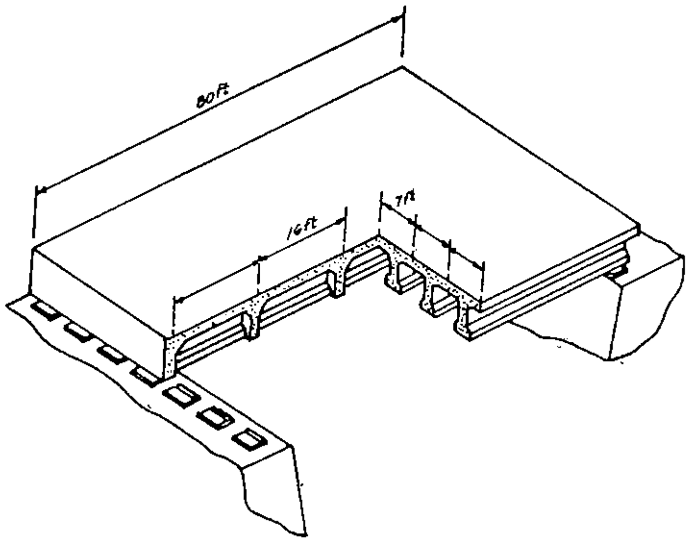


Fig. 19.8 *Actual form of beam and slab bridge structure*

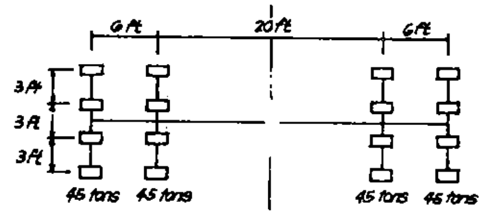


Fig. 19.9 *The wheel positions of the BS, HB loading in plan*

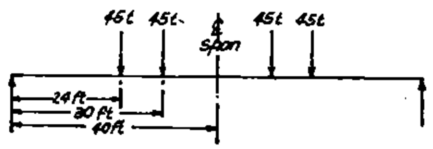


Fig. 19.10 *Longitudinal disposition of loading vehicle on beam and slab bridge*

The procedure may now be carried out in one of the two ways. Either the distribution coefficients may be read from the curves and immediately multiplied by the equivalent load factors, or a table of unit distribution coefficients may be prepared. The latter system is chosen here because it enables a symmetry check to be applied. Also, the calculation may be carried further before the loading is introduced. Thus the consideration of several loading conditions requires less labour.

For a deck of intermediate stiffness $(0 < \alpha < 1)$ the unit distribution coefficients will have to be prepared for

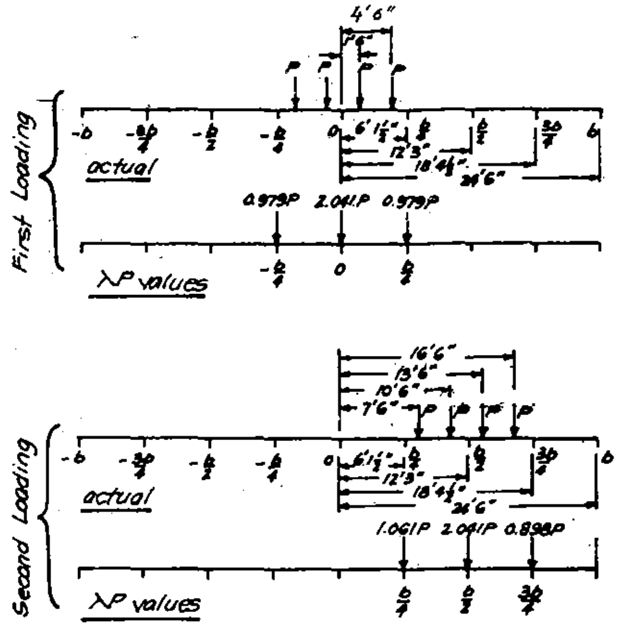


Fig. 19.11 *Transverse dispositions of loading vehicle on beam and slab bridge*

both $\alpha = 0$ and $\alpha = 1$. Two tables are obtained by reading off the values of K from the curves for the value of θ which has been calculated in Table 19.2. Graphs 19.1 to 19.6 give Table 19.5, Graph 19.1 giving the column for reference station 0. Graph 19.2 the column for reference station $+b/4$, and so on. It is now seen that the columns headed $-b$, $-3b/4$, $-b/2$ and $-b/4$ cannot be filled-in by direct reading from the curves. However, it has already been noted that the table possesses the property of symmetry about the diagonals, so that the values for the unfilled spaces may be obtained from those already filled-in. Thus K (reference station $-3b/4$ load position $+b/4$) is equal to K (reference station $+b/4$, load position $-3b/4$). Similarly Graphs 19.7 to 19.11 give Table 19.6, Graph 19.7 giving the fifth column, Graph 19.8 the sixth column, and so on. In practice it will often be found convenient to omit the rows of coefficients for the load positions $-b$ to $-b/4$. A check for these tables exists in the fact that if Simpson's rule is applied to the values in each row, the interval being taken as unity, the sum should equal 8.0. A tolerance in the sum from 7.8 to 8.2 can normally be permitted.

A combined table, Table 19.7, is then prepared from the previous two by use of the interpolation formula given in Eq. (19.3), for the appropriate value of α which has been calculated in Table 19.2. A table has not been prepared here for the computation, but, if required, one may be drawn up for the arithmetical steps K_0, K_1, (K_1-K_0), $(K_1-K_0)\sqrt{\alpha}$ and $K_0 + (K_1 - K_0)\sqrt{\alpha}$ with a space for each of the different values occurring in Table 19.7.

It is at this stage that the loadings are introduced by multiplying the unit coefficients of Table 19.7 by the appropriate loading values λP, which appeared in Table 19.3 for each of the loadings considered; in this case, two loading conditions are introduced. The final coefficients K' for each standard position may now be obtained by summing the value due to each load and dividing by the total load. This working for each loading condition is shown in Tables 19.8 and 19.9, respectively.

In fact we require the moment values in the beams, i.e., at the actual beam positions, and these are best obtained by plotting a curve (Fig. 19.12) of the transverse profile and inserting the actual beam positions found in Table 19.2, and reading the ordinates at these points. These results have been recorded in Table 19.10 for both loading conditions.

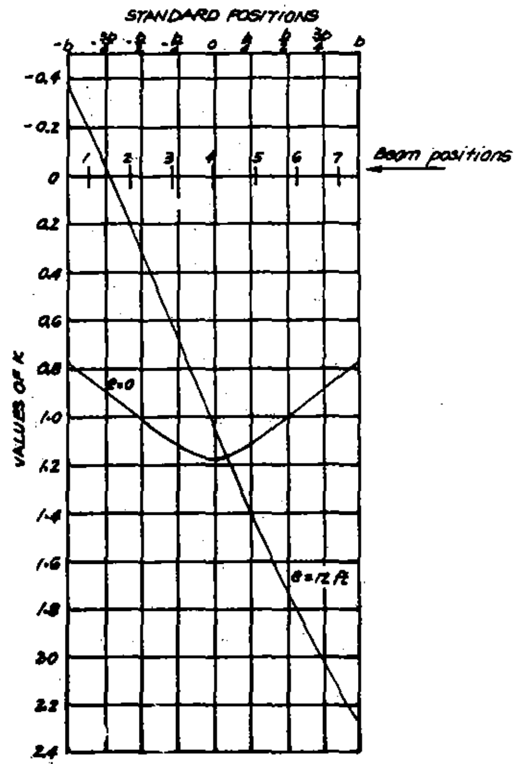


Fig. 19.12 *Transverse distribution profiles for beam and slab bridge*

Calculation of Torsional Stiffness

Equation (19.2), which defines the torsional parameter α, was based on the properties of a grillage for which it is sufficient to use the normal well-known formulae for determining torsional stiffness.

However, if these normal methods are applied to a slab, for example, using the thin rectangular section formula $i_0 = j_0 = d^3/3$ where d is the depth of the slab, it will be found that the resulting value of α is equal to 2 and not unity. This anomaly arises from the fact that the overall continuity of the slab in the longitudinal and transverse directions has been neglected. Actually the values of i_0 and j_0 to be used are equal to $d^3/6$ which results in a value of α equal to unity.

In a T-beam bridge deck, in each individual T-beam only the shear stresses parallel to the top surface can exist and if an individual T-beam is isolated, for convenience in determining the parameters, then the vertical shear stresses are not present. This means that only 50% of the torsional stiffness or inertia contributes to the torsional parameter α.

Thus the rule is that for that portion of the section forming part of the continuous slab portion of the structure, only half of the calculated torsional inertia may be considered in deducing α. This extends to structures intermediate in form between a grillage and a slab and a general rule is that in the determination of i_0 and j_0 the values for a continuous member, as calculated by the normal methods, should be halved, while the values for the non-continuous members should be retained. The total value of i_0 or j_0 is then obtained by summation. As an example in the beam-and-slab bridge deck analysed previously, the torsional stiffness of the web, being non-continuous, was calculated using normal formulae, but the torsional stiffness of the top flange, being continuous, was taken as half of that calculated by these formulae. The total value of i_0 and j_0 in each of the two principal directions was then found by summation of the component stiffness. The interpolation formula $K_\alpha = K_0 + (K_1 - K_0)\sqrt{\alpha}$ could then be applied directly.

For non-thin sections the relaxation method is recommended, the torsional stiffness of the slab again being halved.

A multi-webbed box-section deck is a unique case and i_0 should be calculated by the single cell formula:

$$i_0 = \frac{4A^2}{p\left\{\dfrac{p-2t_3}{t_1} + \dfrac{p-2t_3}{t_2} + \dfrac{2(d-t_1-t_2)}{t_3}\right\}}$$

where A = the area of the hole (Fig. 19.13).

Bending Moments in Cross Beams

A very detailed method has been described in the Db.11 Publication, quoted earlier, to which the reader may refer.

However, the method detailed earlier in *'Analysis for Cross Beams'* (under Courbon's Method) is good enough for the design of cross beams and may be adopted instead.

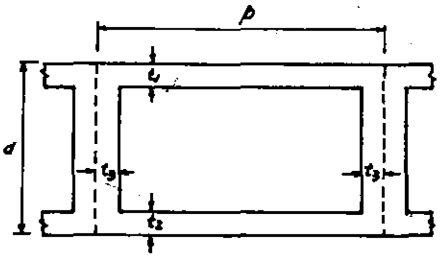


Fig. 19.13 *Nomenclature for torsional stiffness calculations of box section bridge beams*

19.6 GRILLAGE METHOD (FOR ESTIMATING THE APPORTIONMENT OF THE APPLIED LOAD* EFFECT IN THE LONGITUDINAL AND THE TRANSVERSE MEMBERS OF THE DECK)

The Grillage background has already been explained in Sec. 19.1 of this chapter, which may again be referred to before proceeding with this section.

Where a bridge deck is formed of a small number of cells, it is appropriate to place the longitudinal grillage members along the axes of the web members. Where there are sloping webs, the grillage members should be placed along the lines of intersection between the web and the top or bottom slab.

Excessive preoccupation with torsion might lead to the conclusion that longitudinal members would best be placed along the centre lines of the *cells*, but in fact it is better to place such members along the *web* axes to obtain the best representation of the transverse flexural characteristics of the deck. The torsional values included in the output are rarely directly applicable, in any case.

For a multi-celled deck which approaches the characteristics of an orthotropic plate in its behaviour, there is no need to align the grillage members with individual webs. Suggested guidelines for the selection of the number of longitudinal grillage members are that five or more members should be considered, and that the spacing of the longitudinal members should generally not exceed one-half of the width of a traffic lane, or one-and-a-half times the overall depth of the deck, whichever is lesser.

Transverse grillage members must be placed along the line of each diaphragm in a structure. Additional transverse members are also needed to reflect the load-sharing characteristics of the deck. The frequency of these members should be such that the differences between the analytical model and the real structure do not dominate the output, and thereby obscure interpretation. West has suggested that transverse members should be placed at

* Only those loads which appear after the 'grillage structural deck' commences to act as a 'grillage'.

Table 19.2 *Structural data*

Span	$2a$	80 ft.
Number of main beams	n	7
Main beam spacing	p	7 ft.
Effective width	$2b = np$	49 ft.
Number of cross-beams	m	6
Cross-beam spacing	q	16 ft.
Second moment of area of each main beam	I	720,251 in.^{4}
Distributed longitudinal stiffness	$i = I/p$	8,574 in.^{4}/in.
Second moment of area of each cross-beam	J	452,003 in.^{4}
Distributed transverse stiffness	$j = J/q$	2,354 in.^{4}/in.
Bending stiffness parameter	$\theta = \dfrac{b}{2a}\sqrt[4]{\dfrac{i}{j}}$	0.4230
Torsional stiffness of a main beam	I_0	13,329 in.^{4}
Distributed stiffness	$i_0 = \dfrac{I_0}{p}$	159 in.^{4}/in.
Torsional stiffness of a cross-beam	J_0	15,656 in.^{4}
Distributed stiffness	$j_0 = \dfrac{J_0}{q}$	82 in.^{4}/in.
Young's modulus	E	5×10^6 lb/sq. in.
Rigidity modulus	G	2×10^6 lb/sq. in.
Torsional parameter	$\alpha = \dfrac{G(i_0 + j_0)}{2E\sqrt{ij}}$	0.0107 ($\sqrt{\alpha} = 0.1034$)

Actual beam positions in terms of effective width

Beam	1	2	3	4	5	6	7
	$\left(\dfrac{n-1}{n}\right)b$	$\left(\dfrac{n-3}{n}\right)b$	$\left(\dfrac{n-5}{n}\right)b$	$\left(\dfrac{n-7}{n}\right)b$	$\left(\dfrac{n-9}{n}\right)b$	$\left(\dfrac{n-11}{n}\right)b$	$\left(\dfrac{n-13}{n}\right)b$
Beam location	$-\frac{6}{7}b$	$-\frac{4}{7}b$	$-\frac{2}{7}b$	0	$+\frac{2}{7}b$	$+\frac{4}{7}b$	$+\frac{6}{7}b$

Table 19.3 *Applied loading*

Total vehicle load, w	180 tons
Number of wheels	16
Number of axles	4
Wheel load, P	11.25 tons
Total maximum longitudinal moment (on entire deck, all beams)	65.32×10^6 lb in.
Transverse wheel spacing	3 ft
• Eccentricity of vehicle centre of gravity with respect to effective width — 1st loading	0 ft
• Eccentricity of vehicle centre of gravity with respect to effective width — 2nd loading	12 ft

Equivalent loads λP at the nine standard positions — 1st loading

$-b$	$-\frac{3b}{4}$	$-\frac{b}{2}$	$-\frac{b}{4}$	0	$+\frac{b}{4}$	$+\frac{b}{2}$	$+\frac{3b}{4}$	$+b$
0	0	0	$0.978P$	$2.044P$	$0.978P$	0	0	0

Equivalent loads λP at the nine standard positions — 2nd loading

$-b$	$-\frac{3b}{4}$	$-\frac{b}{2}$	$-\frac{b}{4}$	0	$+\frac{b}{4}$	$+\frac{b}{2}$	$+\frac{3b}{4}$	$+b$
0	0	0	0	0	$1.061P$	$2.041P$	$0.898P$	0

Table 19.4 *Mean value of live load moment, M_{av}*

Average of the maximum longitudinal bending moment per beam : Mav.

$$= \frac{\text{total maximum longitudinal moment} : 65.32 \times 10^6}{\text{number of main beams} : 7} = 9{,}331{,}200 \text{ lb in.}$$

Table 19.5 *Beam-and-slab-bridge—unit load distribution coefficient K_0 ($\theta = 0.423$, $\alpha = 0$)*

Load position	Reference station									Row integral*
	$-b$	$-\frac{3b}{4}$	$-\frac{b}{2}$	$-\frac{b}{4}$	0	$+\frac{b}{4}$	$+\frac{b}{2}$	$+\frac{3b}{4}$	$+b$	
$-b$	4.35	3.39	2.37	1.55	0.69	0.10	−0.54	−1.06	−1.62	7.90
$-\frac{3b}{4}$	3.39	2.78	2.08	1.43	0.86	0.33	−0.17	−0.58	−1.06	7.90
$-\frac{b}{2}$	2.37	2.08	1.77	1.39	1.00	0.63	0.23	−0.17	−0.54	7.85
$-\frac{b}{4}$	1.55	1.43	1.39	1.30	1.14	0.92	0.63	0.33	0.10	7.96
0	0.69	0.86	1.00	1.14	1.23	1.14	1.00	0.86	0.69	7.95
$+\frac{b}{4}$	0.10	0.33	0.63	0.92	1.14	1.30	1.39	1.43	1.55	7.96
$+\frac{b}{2}$	−0.54	−0.17	0.23	0.63	1.00	1.39	1.77	2.08	2.37	7.85
$+\frac{3b}{4}$	−1.06	−0.58	−0.17	0.33	0.86	1.43	2.08	2.78	3.39	7.90
$+b$	−1.62	−1.06	−0.54	0.10	0.69	1.55	2.37	3.39	4.35	7.90

Table 19.6 *Beam-and-slab-bridge—unit load distribution coefficient K_1 ($\theta = 0.423, \alpha = 1$)*

Load position	Reference station									Row integral*
	$-b$	$-\frac{3b}{4}$	$-\frac{b}{2}$	$-\frac{b}{4}$	0	$+\frac{b}{4}$	$+\frac{b}{2}$	$+\frac{3b}{4}$	$+b$	
$-b$	1.91	1.59	1.31	1.08	0.90	0.75	0.64	0.55	0.47	7.99
$-\frac{3b}{4}$	1.59	1.44	1.28	1.10	0.95	0.82	0.71	0.63	0.55	7.99
$-\frac{b}{2}$	1.31	1.28	1.23	1.11	1.00	0.89	0.79	0.71	0.64	7.98
$-\frac{b}{4}$	1.08	1.10	1.11	1.11	1.06	0.97	0.89	0.82	0.75	7.98
0	0.90	0.95	1.00	1.06	1.08	1.06	1.00	0.95	0.90	8.01
$+\frac{b}{4}$	0.75	0.82	0.89	0.97	1.06	1.11	1.11	1.10	1.08	7.98
$+\frac{b}{2}$	0.64	0.71	0.79	0.89	1.00	1.11	1.23	1.28	1.31	7.98
$+\frac{3b}{4}$	0.55	0.63	0.71	0.82	0.95	1.10	1.28	1.44	1.59	7.99
$+b$	0.47	0.55	0.64	0.75	0.90	1.08	1.31	1.59	1.91	7.99

Table 19.7 *Beam-and-slab-bridge—unit load distribution coefficient K_α ($\theta = 0.423, \sqrt{\alpha} = 0.1034$), $K_\alpha = K_0 + (K_1 - K_0)\sqrt{\alpha}$*

Load position	Reference station									Row integral*
	$-b$	$-\frac{3b}{4}$	$-\frac{b}{2}$	$-\frac{b}{4}$	0	$+\frac{b}{4}$	$+\frac{b}{2}$	$+\frac{3b}{4}$	$+b$	
$-b$	4.10	3.20	2.26	1.50	0.71	0.17	−0.42	−0.89	−1.40	7.91
$-\frac{3b}{4}$	3.20	2.64	2.00	1.40	0.87	0.38	−0.08	−0.45	−0.89	7.92
$-\frac{b}{2}$	2.26	2.00	1.71	1.36	1.00	0.66	0.29	−0.08	−0.42	7.87

(Contd.)

* The row integral should theoretically be 8.00; since the values are taken from curves, however, some error is bound to occur and it is suggested that these values are typical of the tolerance which may be permitted.

$-\frac{b}{4}$	1.50	1.40	1.36	1.28	1.13	0.92	0.66	0.38	0.17	7.96
0	0.71	0.87	1.00	1.13	1.21	1.13	1.00	0.87	0.71	7.95
$+\frac{b}{4}$	0.17	0.38	0.66	0.92	1.13	1.28	1.36	1.40	1.50	7.96
$+\frac{b}{2}$	−0.42	−0.08	0.29	0.66	1.00	1.36	1.71	2.00	2.26	7.87
$+\frac{3b}{4}$	−0.89	−0.45	−0.08	0.38	0.87	1.40	2.00	2.64	3.20	7.92
$+b$	−1.40	−0.89	−0.42	0.17	0.71	1.50	2.26	3.20	4.10	7.91

Table 19.8 *Beam-and-slab-bridge—distribution coefficient K' for first loading*

Load position	Equivalent load multiplier λ from Table 19.3	Reference station								
		$-b$	$-\frac{3b}{4}$	$-\frac{b}{2}$	$-\frac{b}{4}$	0	$+\frac{b}{4}$	$+\frac{b}{2}$	$+\frac{3b}{4}$	$+b$
$-b$	0									
$-\frac{3b}{4}$	0									
$-\frac{b}{2}$	0									
$-\frac{b}{4}$	0.978	1.47	1.37	1.33	1.25	1.10	0.90	0.65	0.37	0.17
0	2.044	1.45	1.78	2.04	2.31	2.47	2.31	2.04	1.78	1.45
$+\frac{b}{4}$	0.978	0.17	0.37	0.65	0.90	1.10	1.25	1.33	1.37	1.47
$+\frac{b}{2}$	0									
$+\frac{3b}{4}$	0									
$+b$	0									
$\sum \lambda K\alpha$		3.09	3.52	4.02	4.46	4.67	4.46	4.02	3.52	3.09
	$K' = \frac{\sum \lambda K\alpha}{4}$	0.77	0.88	1.00	1.11	1.17	1.11	1.00	0.88	0.77

NOTE

(i) Throughout only two decimal places have been retained, as these give the desired degree of accuracy.

(ii) The denominator in the expression for K' is equal to the number of loads on the transverse section, i.e., 4 for the HB Loading under consideration.

Table 19.9 *Beam-and-slab-bridge—distribution coefficient K' for second loading*

Load position	Equivalent load multiplier λ from Table 19.3	Reference station								
		$-b$	$-\frac{3b}{4}$	$-\frac{b}{2}$	$-\frac{b}{4}$	0	$+\frac{b}{4}$	$+\frac{b}{2}$	$+\frac{3b}{4}$	$+b$
$-b$	0									
$-\frac{3b}{4}$	0									
$-\frac{b}{2}$	0									
$-\frac{b}{4}$	0									
0	0									

(Contd.)

$+\frac{b}{4}$	1.061	0.18	0.40	0.70	0.97	1.20	1.36	1.44	1.48	1.59
$+\frac{b}{2}$	2.041	−0.86	−0.16	0.59	1.35	2.04	2.77	3.49	4.08	4.62
$+\frac{3b}{4}$	0.898	−0.80	−0.40	−0.07	0.34	0.78	1.26	1.80	2.37	2.87
$+b$	0									
$\sum \lambda K \alpha$		−1.48	−0.16	1.22	2.66	4.02	5.39	6.73	7.93	9.08
$K' = \frac{\sum \lambda K \alpha}{4}$		−0.37	−0.04	0.30	0.66	1.00	1.40	1.68	1.98	2.27

Table 19.10 *Beam-and-slab-bridge — distribution coefficient at actual beam positions*

Beam	1	2	3	4	5	6	7
1st loading	0.83	0.97	1.10	1.17	1.10	0.97	0.83
2nd loading	−0.18	0.20	0.63	1.00	1.42	1.77	2.11

NOTE

Maximum bending moment is in beam 7, = 1.1 Mav. × 2.11 = 21,670,000 lb in.

(Mav. being the 'total' bending moment due to total live load on deck divided by the number of beams, Table 19.4).

intervals not exceeding twice the spacing of the longitudinal members. In any event, the adoption of less than five intermediate transverse members produces results in which the interpretation of the values at the intersection points of the grillage members could obscure the results.

Sectional Properties of Grillage Members

In assigning stiffnesses to the grillage members, it can appear for many decks that the arbitrary division of the deck into separate members by intersection lines at the mid-points between the members will give an adequate result. In the case of a deck having sloping webs, however, it is apparent that such an intersection line would result in the outer members having very low moments of inertia, with the position of the centroidal axis changing abruptly from one member to another, whereas it has been demonstrated that such changes in the level of the centroidal axis do not actually take place. Obviously the total stiffness should be correct, and this stiffness could be distributed in proportion to the respective areas of the members or purely as a matter of judgement.

Depending on the proportions of a structure, it may be necessary to take shear-lag (explained earlier) into account in assigning stiffnesses (effective section properties) to the grillage members, effective flange width in boxes, explained ahead.

The torsional stiffness is evaluated from the opposing shearing action of the top and bottom slabs, giving a torsional constant of one-half of that which arises from considering the closed section as a thin-walled box. This halving was found many years ago to best fit the experimental results, but no really satisfactory explanation has been given as to why this should be the case for a cellular deck when the full calculated figure is used for other forms of construction. The explanation* may be in the equal contributions of torsion in the members, and the adjusted shears in the intersecting members.

Incorporation of Skew

Because of the difficulties which arise in evaluating torsional parameters, the values obtainable for a grillage with members intersecting other than at right angles are subject to error. The skewed deck should, therefore, be represented by an orthogonal grillage, apart from the need to show the diaphragms in their real location.

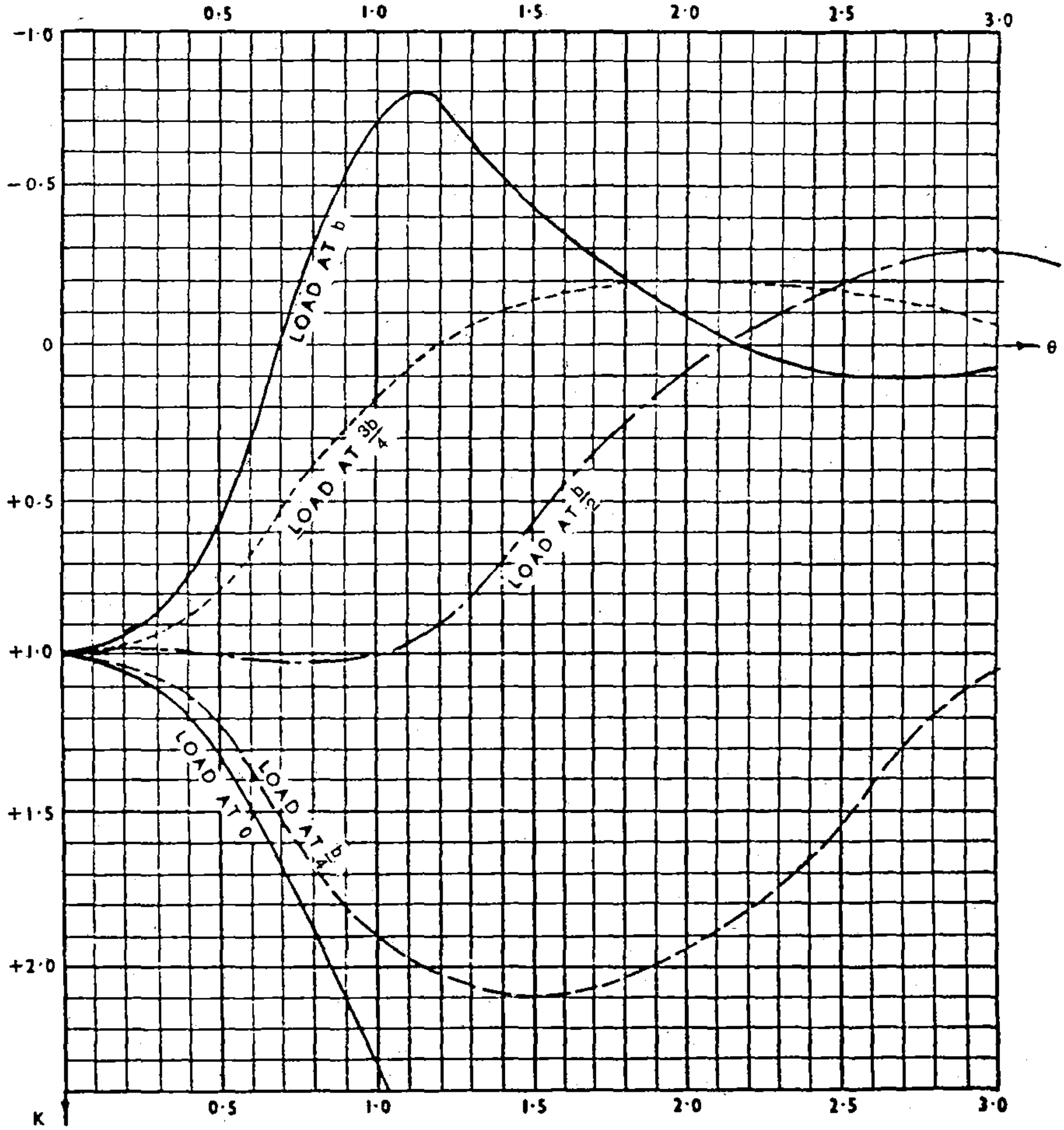
In many instances it is likely that the dominant difference in stiffness between the diaphragm and the other parts of the deck is the increased shearing stiffness. Although difficulties inevitably arise from overlapping members within a skewed deck, it is often sufficient to allow the orthogonal grillage members to represent the stiffness of the cellular section, and to superimpose on this a diaphragm — usually in a skewed direction — which is assigned a flexural stiffness equal to its own net dimensions, and a shear area equal to its actual cross-sectional area.

It is within a skewed deck that the accurate assessment of the torsional stiffness becomes more significant as otherwise the torsions introduced in the zones of skew support will not be evaluated realistically.

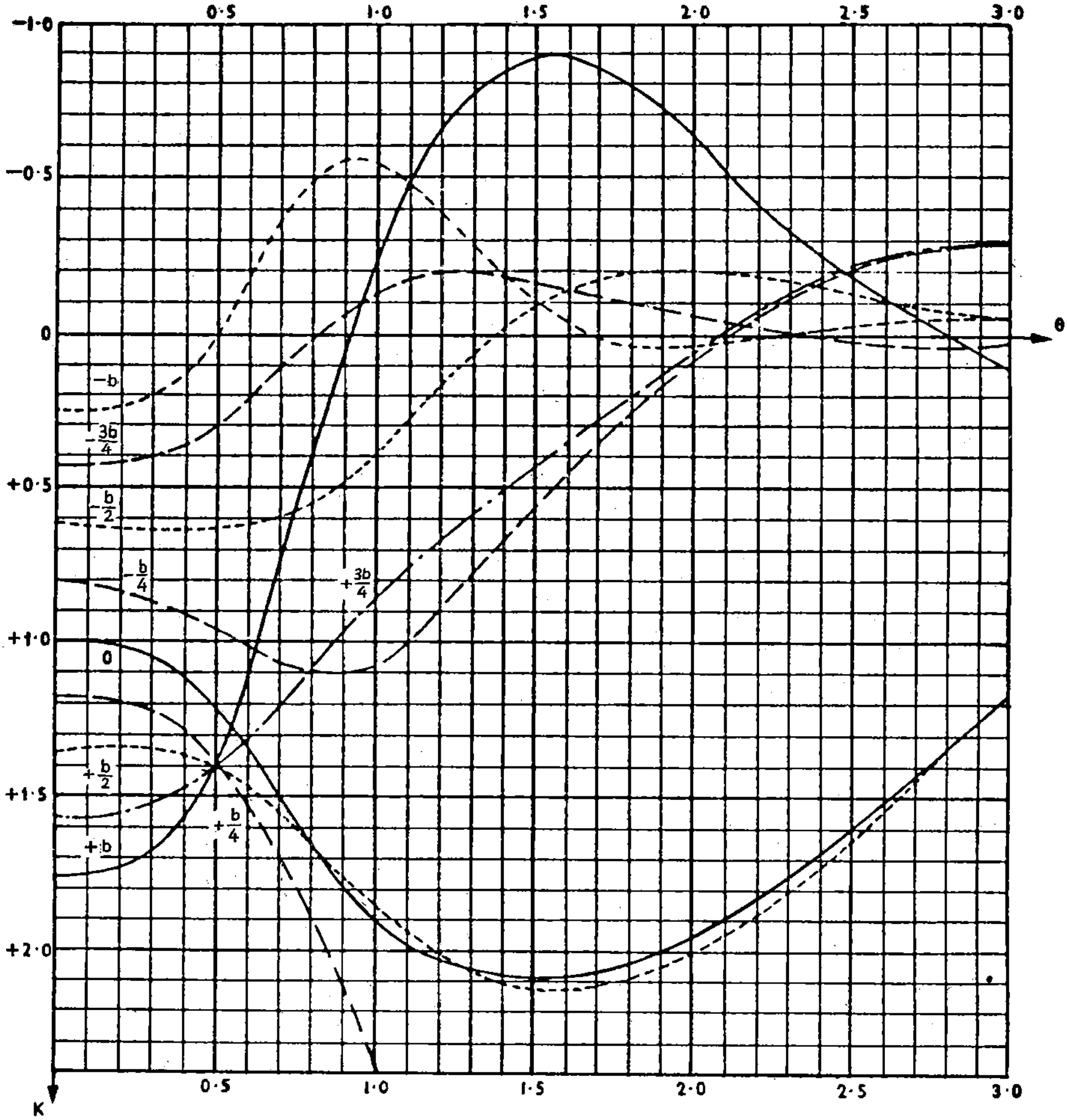
Variations in the plan geometry of a deck can introduce problems of tapering cells, which make the adoption of an orthogonal grillage impossible.

In a majority of cellular structures diaphragm members are contained within the depth of the deck. The force

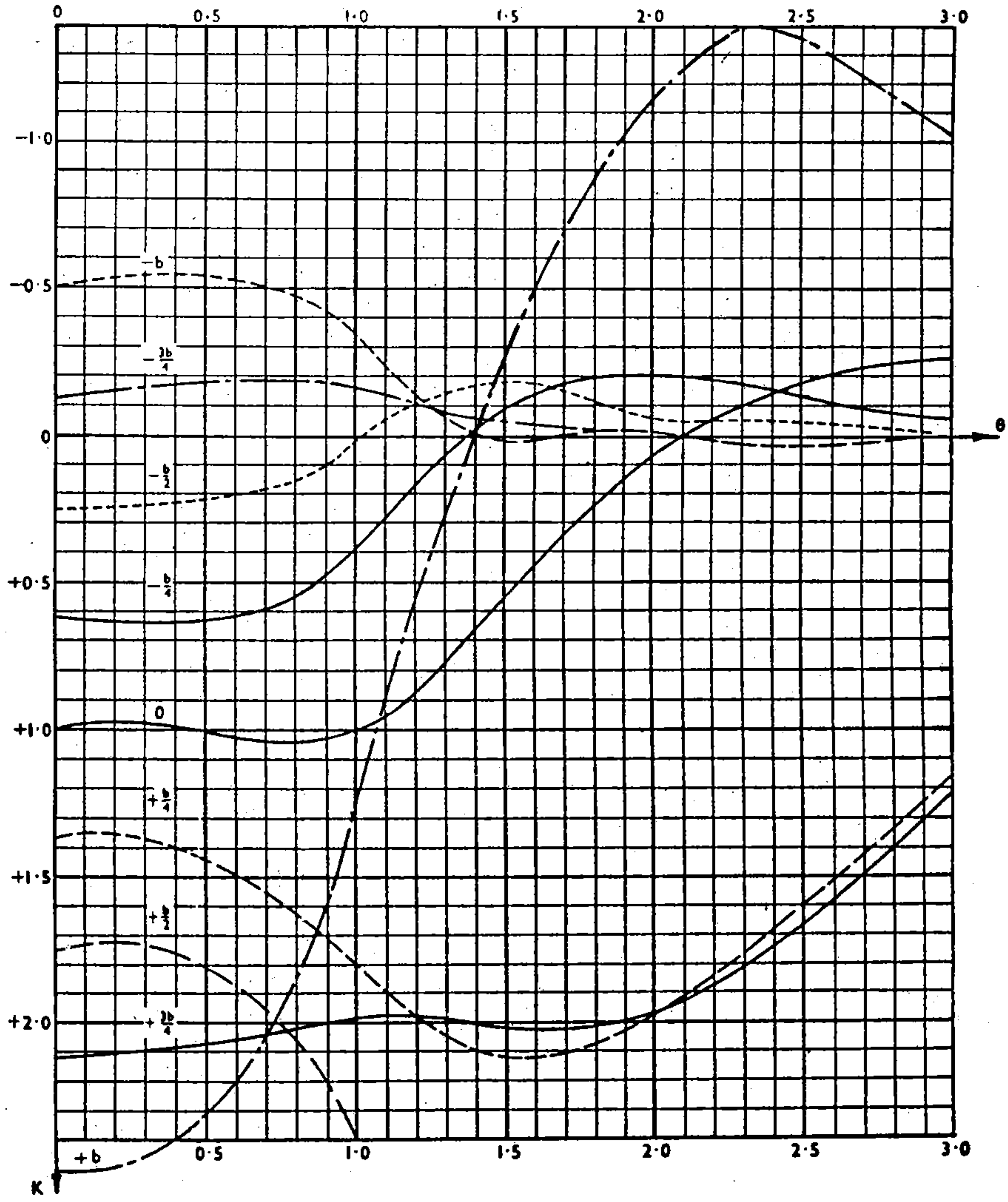
* Also refer to the explanation for 'Torsional Stiffness' in Section 19.5 earlier.



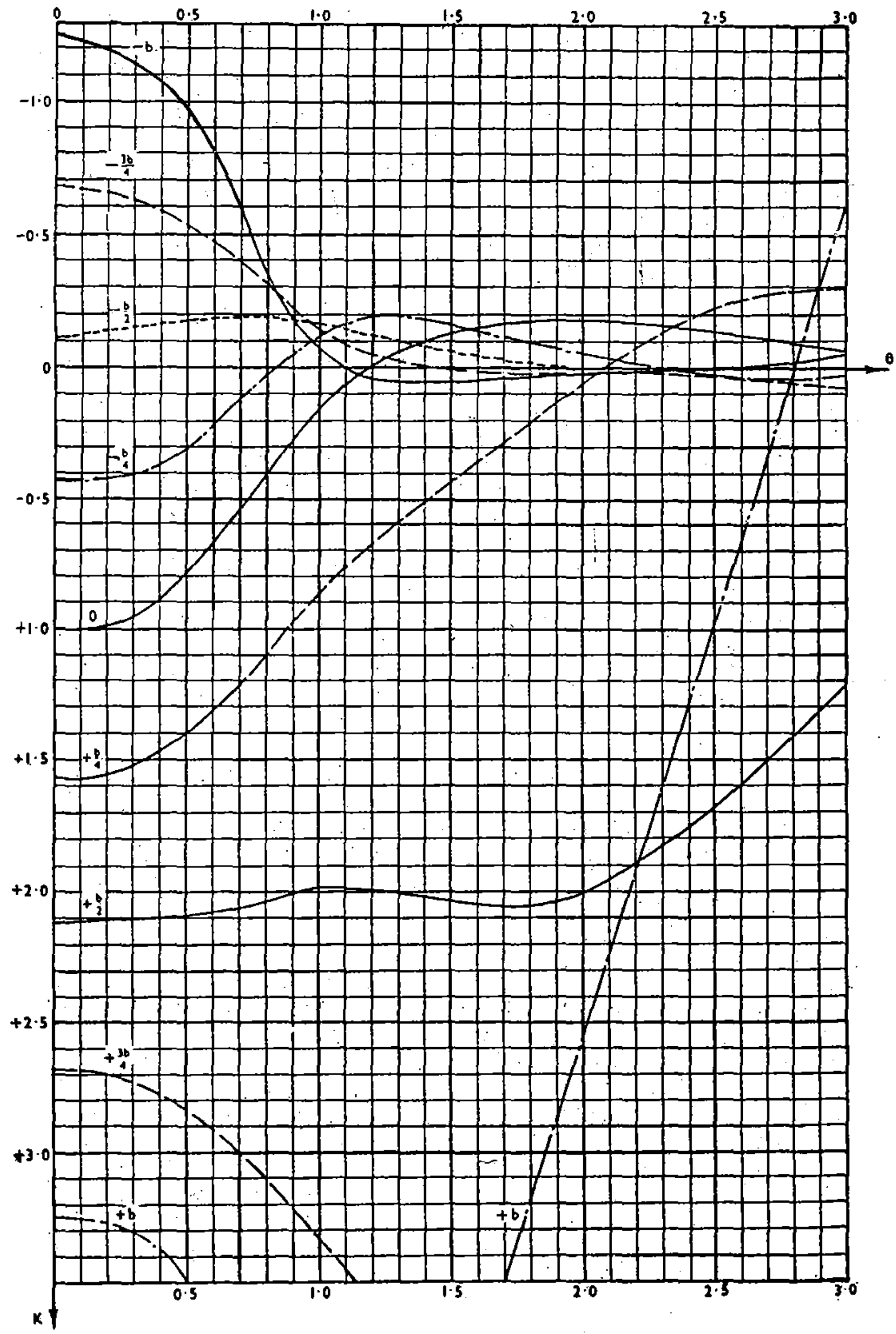
Graph 19.1 *Distribution coefficients K_0 at reference station 0 for various load eccentricities*



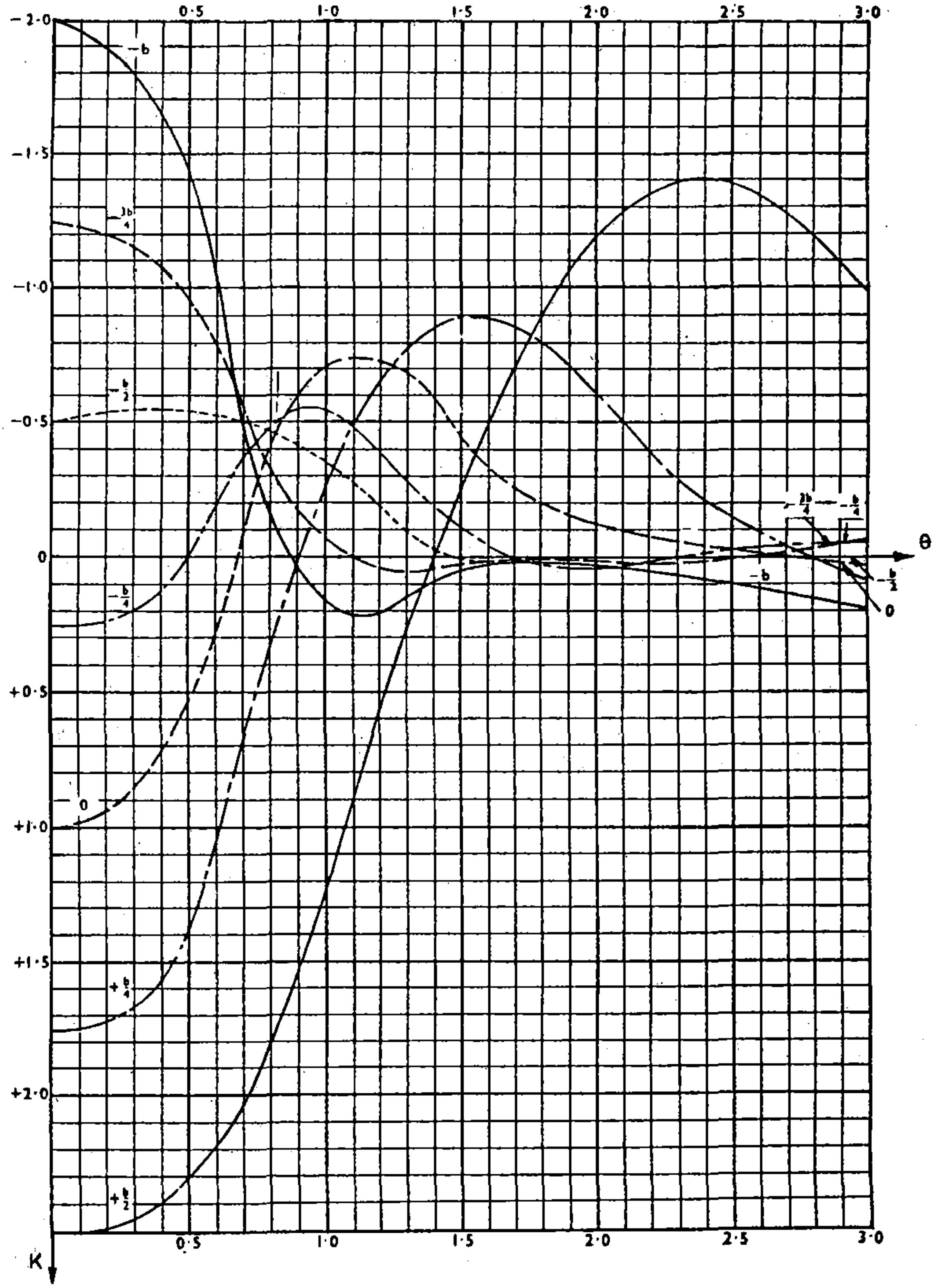
Graph 19.2 *Distribution coefficients K_0 at reference station $\frac{b}{4}$ for various load eccentricities*



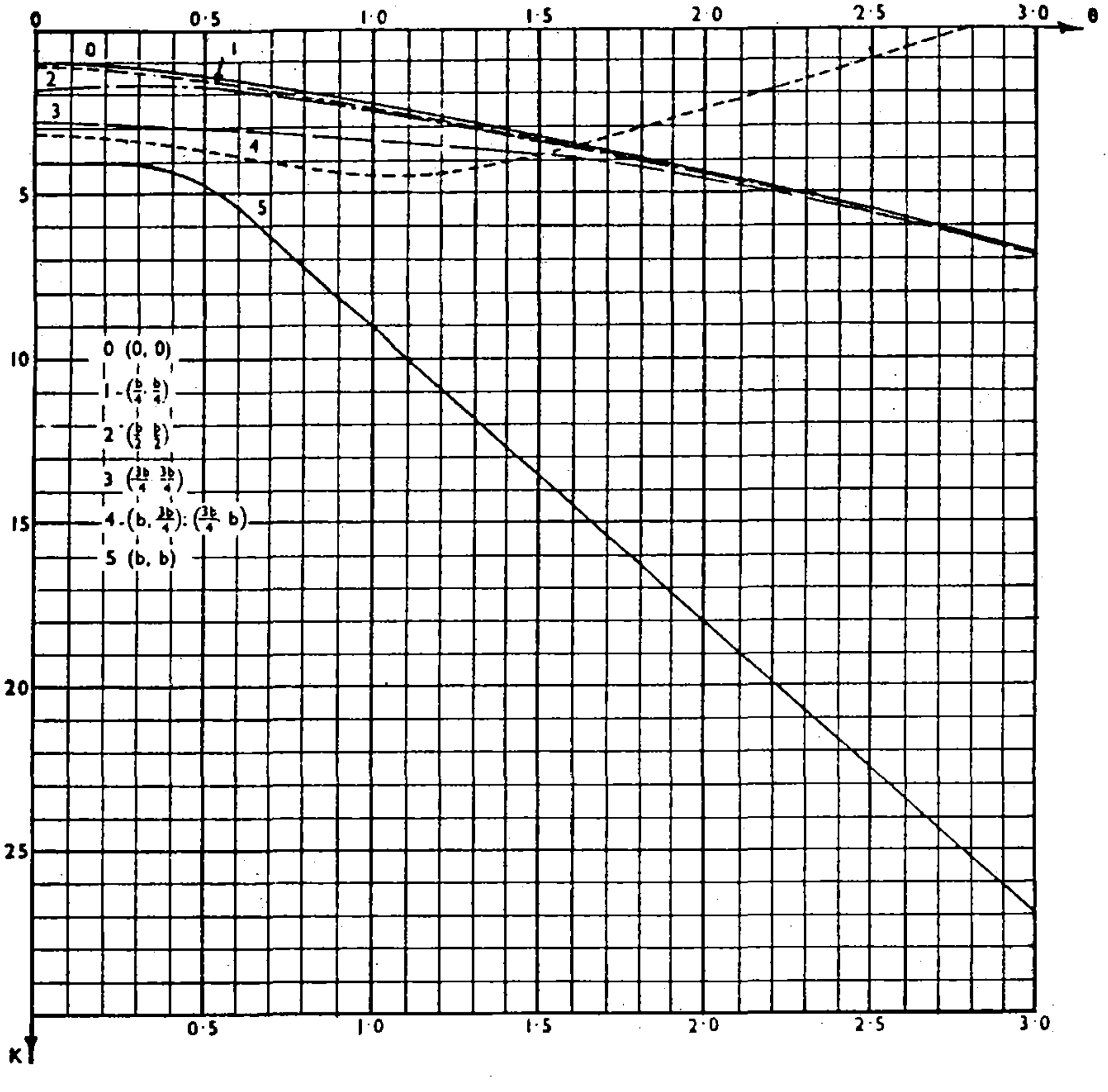
Graph 19.3 *Distribution coefficients K_0 at reference station $\frac{b}{2}$ for various load eccentricities*



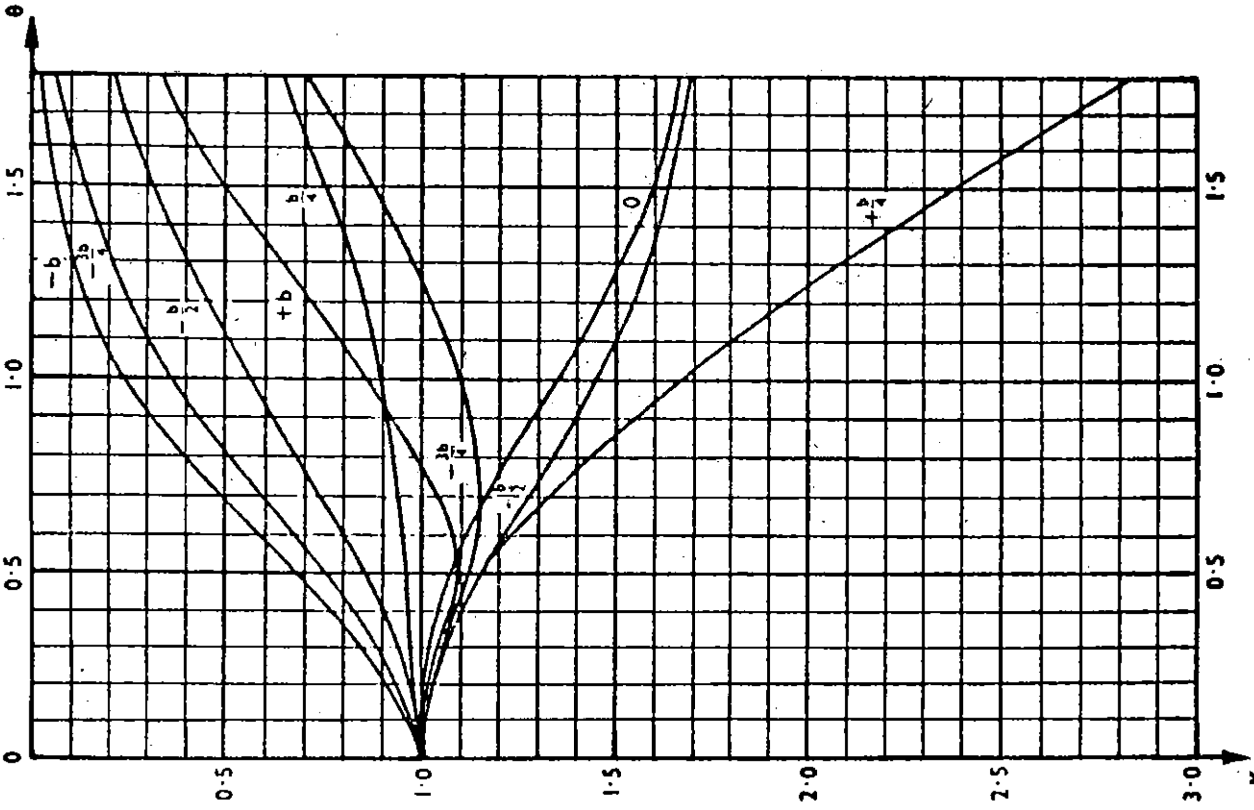
Graph 19.4 *Distribution coefficients K_0 at reference station $\frac{3b}{4}$ for various load eccentricities*



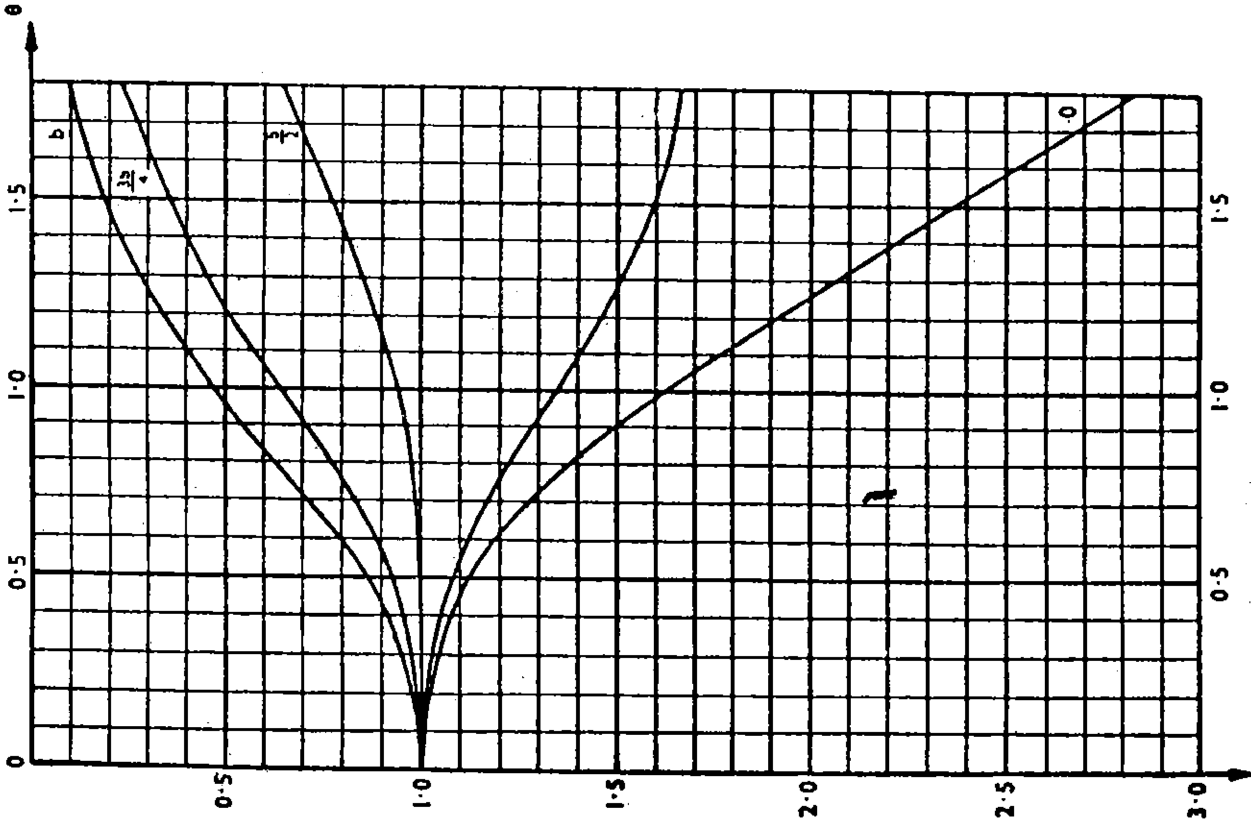
Graph 19.5 *Distribution coefficients K_0 at reference station b for various load eccentricities*



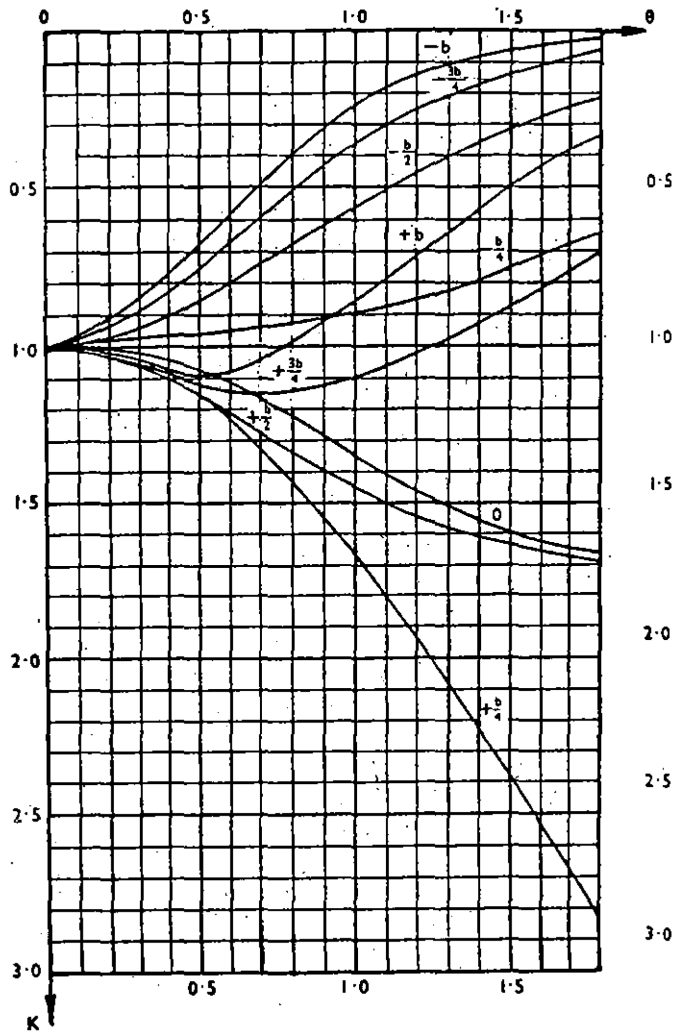
Graph 19.6 *Large range distribution coefficients* K_0



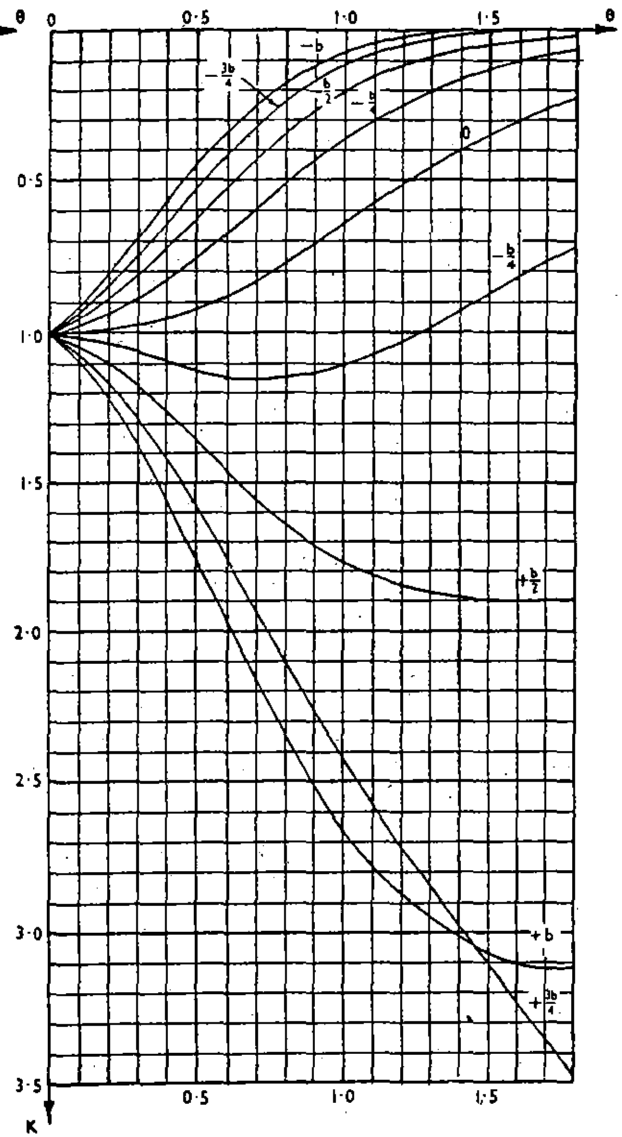
Graph 19.8 *Distribution coefficients K_1 at reference station $\frac{b}{4}$ for various load eccentricities*



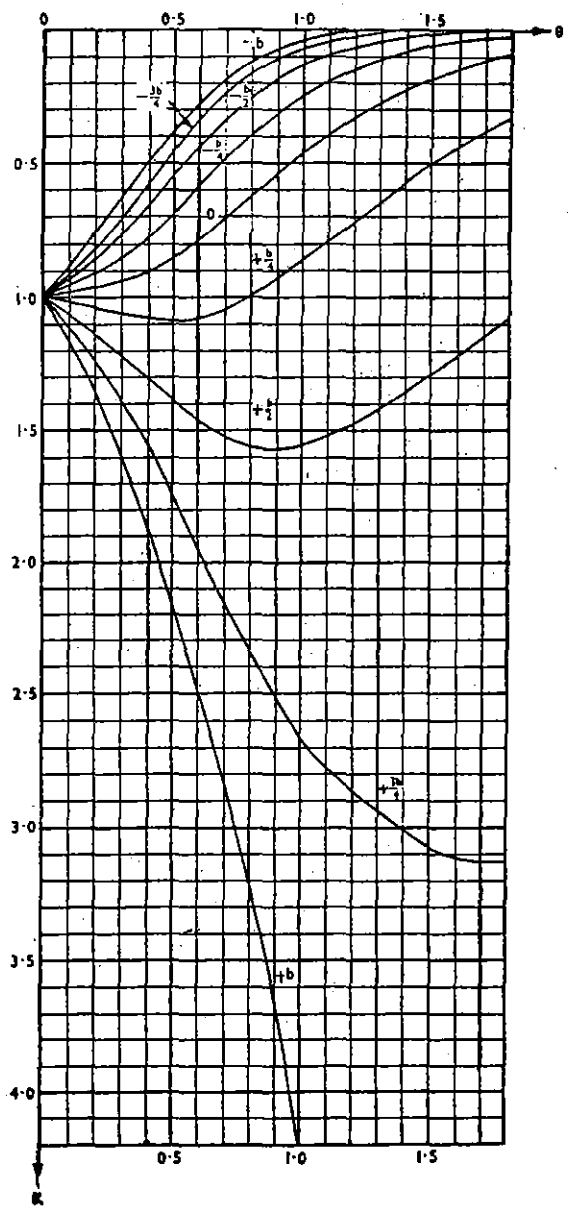
Graph 19.7 *Distribution coefficients K_1 at reference station 0 for various load eccentricities*



Graph 19.9 *Distribution coefficients K_1 at reference station $\frac{b}{2}$ for various load eccentricities.*



Graph 19.10 *Distribution coefficients K_1 at reference station $\frac{3b}{4}$ for various load eccentricities*



Graph 19.11 *Distribution coefficients K_1 at reference station b for various load eccentricities*

systems resulting from the grillage at the intersections between the diaphragm and the web members are not necessarily directly applicable to these members themselves. They represent the force field in the total deck and, in most instances, the web part of the diaphragm members will not be significantly affected by the torsions, which will be resisted by the top and bottom slabs. Any attempt to design an intersection between such members for the torsion which a grillage analysis implies is transmitted through the connection is, therefore, a misdirected effort.

Loading Cases, and Discretising the Grillage Elements

In any structure which is prestressed, the forces developed by the prestressing force are intended to oppose those arising from applied loading. Whenever the geometry of the deck is such that it will induce a complex system of forces, the intensity of these resultant forces in the real structure will still be reduced by the opposition between the prestress and the applied loading. It is, therefore, of fundamental importance to include the prestress as a loading case in any analysis.

In bridge decks having simple geometry it can be sufficient to use the grillage merely to get a picture of the distribution of moments across the width of the deck, and then to resort to continuous beam calculations to obtain the actual design figures.

In many instances the optimum value can be obtained from a grillage analysis by applying axle loads at intervals along the span as individual loading cases, which then form the basis of influence lines as it were. The benefit to be obtained from this is that any local effects can be more closely identified and evaluated.

In all cases it is recommended that the loads should (as far as possible) be applied to the longitudinal members forming the grillage, so that the transverse members reflect the forces arising from distortion, without adding the complication of the local effects.* This, however, is almost impossible in practice. In practice it becomes necessary to locally distribute to the longitudinal as well as to the transverse grillage elements those of the loads (and moments) that physically do not directly sit on them. Consequently, in such cases, it then becomes essential to combine the effects of the global (grillage) analysis with the local wheel-load-effects. This summation of the two effects is uncalled for where the grillage-grid is made small (additional longitudinal beam-elements are incorporated in between the physical longitudinal web members) so that the wheel loads nearly always fall *on* the grillage members rather than *between* them.

As a first step in a grillage analysis, the continuum of the deck must be idealized into a series of discrete elements. These elements are connected at joints (nodes) and it is at these joints that restraints to movement, i.e., supports or fixity, and loads, can be applied. Restraints may be applied at any joint. The members framing into a joint can be at any angle. It is thus possible to analyse a deck with any support conditions — simply supported, or skew. With all the available computer programs† it is possible to include some or all of the restraints as *elastic restraints*, thus simulating for instance the rubber-bearing deformation or the elastic shortening of support columns. It has been found from the analysis of experimental work that quite small movements of supports can have pronounced effects upon the moments and reactions in the slab. It is therefore desirable to have, in the grillage simulation, a true representation of the whole structure, including the supports.

An additional use of the option to apply elastic rotation restraints exists in a continuous structure. Rather than idealizing all the spans as grillage beams, the restraining effect of the deck which is more than one span away from the position at which moments are required can be simulated by an *elastic rotation restraint*.

Application of Loads to the Grillage

Computer programs vary regarding the types of load that it is possible to apply to the structure. All will permit the application of point loads and moments at the joints and some will allow point loads, distributed loads and moments on the members. Since any member-loading can be replaced by point loads and moments at the ends of the member, it is possible to apply any form of loading with any of the programs (but it is in estimating this distribution that the first approximation enters the computation, the other approximations come in while idealising the deck into the grillage and fixing the section properties of its members).

However, be that as it may, when a bridge deck, loaded with uniformly distributed loads or with a vehicle, is being analysed, it is sufficiently accurate to consider the wheel loads as *point loads* acting at the joints, e.g., for a point load acting within a quadrilateral formed by grillage members (Fig. 19.14), consider it statically proportioned to a pair of opposite members, then in the same way from these members to the joints as point loads.

* A load falling on slab in between the longitudinal and the transverse members of the grillage will first create a *local wheel load effect* by deforming the slab, and only then will it get 'conveyed' to the surrounding grillage members. There would be no such local effect if only there was a grillage member directly under the load.

† There are many computer programs in the market, commercially available (some are even suitable for personal computers, e.g., the Graphical Interactive Finite Element Total System — GIFTS/CASA. Tucson. AZ, USA. etc.). Three of the more powerful ones are the IBM-STRUDL (M.I.T.) program, the C and CA (London) program, and the LEAP program.

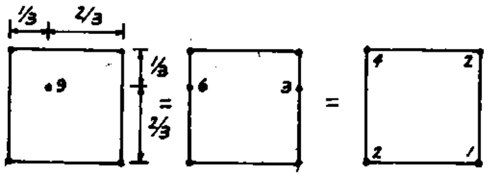


Fig. 19.14 *Distribution of load from a panel to the surrounding nodes*

Interpretation of Results From the Computer Output

The computer output should contain first a summary of the structure and also of the loadings applied. In programs where this is an option, it is advisable to request it in order to again check the input data. This is followed by lists of deflections and rotations at the joints and by shears and moments at the beam ends.

(i) *Bending moments* When examining the longitudinal and transverse bending moments, the user should bear in mind all the time the sign convention used by the program, which will be fully explained in the manual relating to the particular program. Where a grillage beam continues across a joint, the values of moment from end 2 of one member and end 1 of the continuation member will be different. If the two moments are of the same 'sense', the signs will be opposite. The method of dealing with these moments depends upon the actual structure. Where all the members meeting at the node are physical beams, there will be a genuine 'step' in the bending moments at this point and the actual values, output from the program, should be used. This also applies if the longitudinal grillage beams replace more than one physical beam. This method will always cause a slight over-estimate of the moment, because with every deck there will be some continuous slab present. If any of the grillage beams are hypothetical and represent sections of slab, the two moments may be averaged, since in the slab structure no step would occur in its moment diagram.

(ii) *Shear and reactions* Shear at any node should be evaluated from the output results in the same manner as the bending moments. If the reactions are not printed automatically by the program, they may be evaluated by summing the shear forces at the supported node.

(iii) *Twisting moments* The way in which twisting moments are catered for depends upon the type of deck under consideration. Full details are always given in the manual (*the user's manual*).

19.7 TRANSVERSE DISTRIBUTION OF LIVE LOAD IN BOX SECTIONS

A box section (i.e., a closed cell section) has better flexural and torsional properties than an open cell section, other things being equal. Therefore the applied load is distributed much more efficiently (and hence evenly) among the longitudinals of a box (single or multi-cell type, or even in multiple boxes that are interconnected by a common top-slab).

While diaphragms (cross girders or cross beams) in concrete boxes are necessary above their supports and at their ends, they are not required in between because concrete boxes always tend to be relatively more thick-walled (compared to the thin-walled steel box sections). However, intermediate diaphragms may be provided even in concrete boxes if the diaphragm spacing exceeds about 45 m., unless either a thorough (finite element) analysis or a model test is carried out or there is a successful precedence of a similar construction in the past.

In the usually adopted types of the single-cell concrete box decks, the applied load may be assumed to be shared equally by their longitudinal members. However, in the multi-cell concrete box decks the method outlined in Sec. 19.3 may be followed for estimating the apportionment of the applied load effect among the outer and inner longitudinal members. Alternatively, the total live load effect on the box (moment, shear and reactions) may be increased by the factor to which otherwise the heaviest apportioned longitudinal is to be designed, and then the total box section designed for this increased load effect.

Transverse Distribution (of Applied Load) among Various Boxes in the Decks Composed of Individual Boxes that are Interconnected by a Common Top-Slab and by the Diaphragms at the Ends and Over the Supports

Wide decks can conveniently be composed of two or three (and even more) separate boxes which are transversely interconnected by a common top-slab (and by the diaphragms at the ends and over the supports). Podolny and Muller, in their *Construction and Design of Prestressed Concrete Segmental Bridges* (John Wiley and Sons) report basically as follows on this subject:

A. A detailed analysis was made of such decks with regard to the distribution of live load between the various boxes. It was found that in normal structures of this type the combined effect of the flexural rigidity of the roadway slab acting transversely as a rigid frame with the webs and bottom slab of the various box girders, on one hand, and the torsional rigidity of such box girders on the other hand, would result in a very satisfactory transverse distribution of live loads between box girders. There is no need for diaphragms between girders as normally provided for I-girder bridges.

B. Comprehensive programs of load testing of several bridges, including accurate measurements of deflections for eccentric loading, fully confirmed the results of theoretical analysis. This analysis has been reported in various technical documents, and only selected results are presented here.

C. The first bridge analysed in this respect was the Choisy-le-Roi Bridge. A knife-edge load P is considered with a uniform longitudinal distribution along the span, Fig. 19.15. When this load travels crosswise from kerb to kerb, each position may be analysed with respect to the proportion of vertical load carried by each box girder, together with the corresponding torsional moment and transverse moment in the deck slab. These analyses have made it possible to draw transverse influence lines for each effect considered, such as longitudinal bending moments (over the support or at midspan), torsional moments, or transverse moments.

D. For the longitudinal moments it is convenient to use a dimensionless coefficient, Fig. 19.15(c), which represents the increase or decrease of the load carried by one box girder in comparison with the average load assuming an even distribution between both girders. Numerical results show that the transverse distribution of a knife-edge load placed on one side (next to the kerb) of a twin box deck produces bending moments in each box that are 1.4 and 0.6 times the average bending moment, i.e., 1.4 ($^1/_2$ Mp) and 0.6 ($^1/_2$ Mp), i.e., 0.7 and 0.3 Mp, where Mp = the bending moment due to full load P on the total deck of both boxes, and alternatively this would mean *Box Reaction-Factors* of 0.7 and 0.3 (operated on the effect of full P.) For the same configuration, a typical deck with 4 T-girders would have Beam Reaction-Factors of $0.7P$ and $0.4P$ for the two beams on loadside, meaning that these 2 beams together would take $1.1P$ effect which is very high compared to $0.7P$ that a two-webbed box is taking.

E. There are however, two side effects to such encouraging behaviour, which relate to torsion stresses and transverse bending of the deck slab.

F. *Torsional moments in the box girder* An unsymmetrical distribution of live loads in the transverse direction tends to warp the box girders and cause warping shear stresses but they are low in concrete boxes and it is the high torsional rigidity of boxes which produces a favourable distribution of loads in them. However, the maximum torsional moments usually

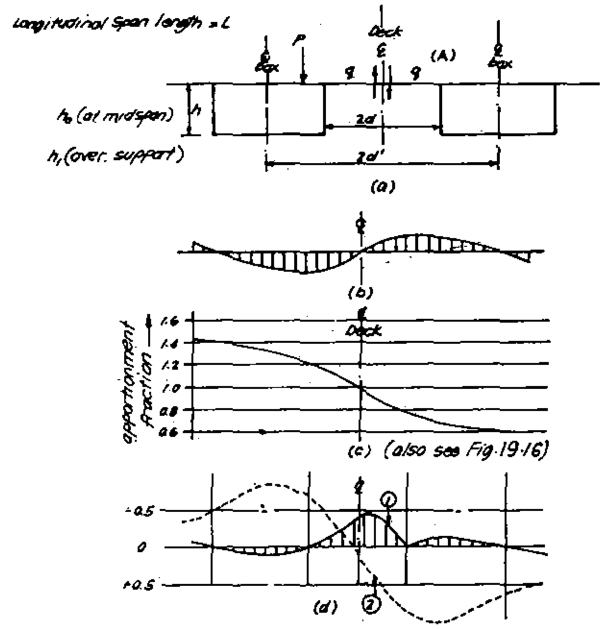


Fig. 19.15 *Principle of transverse distribution of loads between box-girders, (a) Dimensions, (b) Influence line of the shear in the connecting slab, (c) Transverse influence line of longitudinal moment, (d) Transverse bending influence line at section A*

occur when only one-half the structure (in cross-section) is loaded, and the resulting stresses do not cumulate with the shear stresses produced by the full live-load placement for governing shear force.

G. *Transverse Moments in the Deck Slab* The deck slab cannot be considered as a 'continuous beam on fixed supports' because of the relative displacements of the two boxes due to unsymmetrical loading. Figure 19.15(d) shows the consequence. If the slab were resting on fixed supports, the influence line for the moment in a section such as, A in Fig. 19.15(a), would be the typical line 1, in Fig. 19.15(d). Because the box girders undergo certain deflections and rotations, the effect is to superimpose the coordinates of another line such as 2 on line 1.

Numerically, the difference is not as great as may be expected at the first sight, because line 1 pertains to the effect of local concentrated truck loads, while line 2, being the result of differential movements between box girders, pertains to the effect of uniformly distributed loads. In summary, deck slab moments may be increased by only 20 to 30% over their normal values if flexibility of the box girders is ignored.* As a matter of practical interest, the actual numerical values for several bridges in France with

* However, it should be noted that owing to the actual rotational behaviour of these boxes, sagging moments can develop in the top-slab at its supporting webs. For more details refer to Ch. 31.

two box girders, that have all shown excellent performance for more than 15 years so far, are presented in Fig. 19.16.

a)

	Bridge	Spans	2d (ft)	2d′ (ft)	h_o/h_i	Eccn. Coeff.
1	Givors	360′ (110m)	15.7	20.5	6.6/18.0	1.14
2	D/S Paris Ring Park-way	300′ (92m)	13.1	26.2	11.1/18.0	1.10
3	U/S Paris Ring Park-way	(90m) 295′	15.4	33.9	9.2/15.7	1.21
4	Corde	(79m) 260′	19.1	23.3	5.9/14.7	1.22
5	Juvisy	220′ (66m)	14.6	24.0	5.2/10.7	1.23
6	Choisy-le-Roi	180′ (55m)	11.1	22.3	8.2 constant	1.28

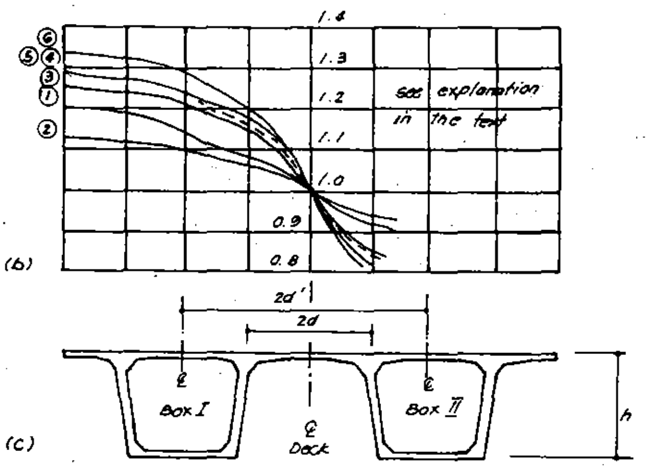


Fig. 19.16 *Transverse distribution of loads between box girders, numerical values for several two-box girders marked 1 to 6*

Explanation of the Influence Lines Shown in Fig. 19.16(b) and Fig. 19.15(c)

These Influence Lines show the *apportionment fraction* of applied load-effect in a box beam for various transverse placements of the load [in bridges 1 to 6 shown in Fig. 19.16(a)].

Example If load P is placed concentrically on deck (i.e., at zero eccentricity wrt ℄ of deck section), then the apportionment factor reads 1.0, meaning that each box takes as its share, $1.0 \times (P/2) = 0.5P$. However, if P is placed on ℄ of Box I (in case of Bridge 2 for example) then its Box I takes $1.1 \times (P/2)$, leaving the Box II to take the balance of $0.9 \times (P/2)$; and so on. The *transverse distribution* is somewhat like what simple statics would give—but not so severe, as can be noticed.

The reader may also refer to Ch. 31 (the analysis and design of certain types of deck cross-sections) and compare the above with the remarks given there regarding the transverse load distribution among the two boxes in a deck of the type shown in Fig. 19.16(c).

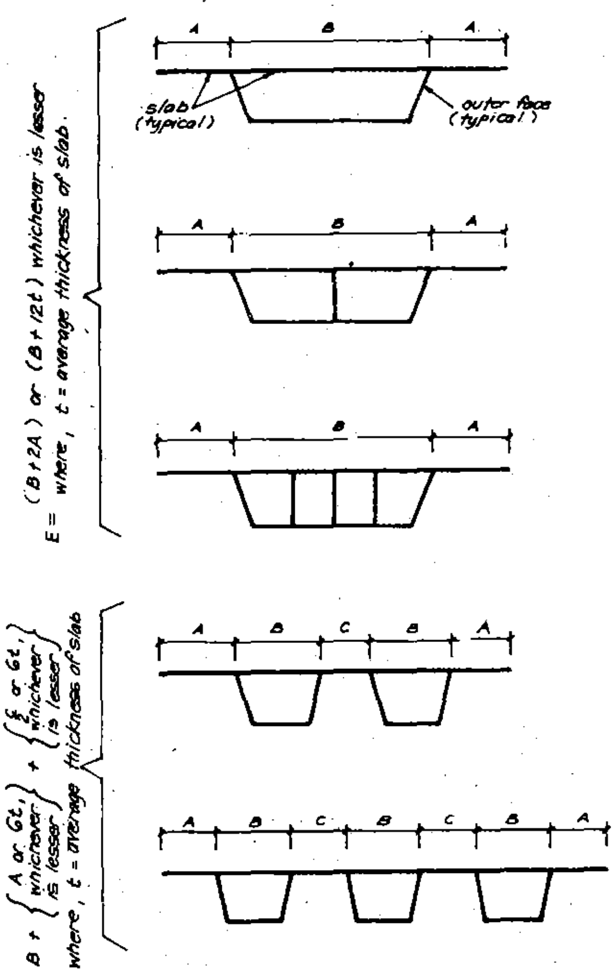


Fig. 19.17

Effective Flexural Section Properties of Box Sections

Owing to shear lag effect (explained earlier) the effective width of the top flange, particularly at and near supports, and the effective width of the compression flange, can turn out to be less than the physical width actually contributing to dead load of the structure. Various codes of practice give various formulae for estimating this effective width and the practising professional knows that some of these impressive looking formulae give values that are only as accurate (or inaccurate) as the very assumptions they are based on (refer Fig. 19.1). On the basis of various box girders actually designed (all of which are serving successfully), some of which were even either actually test-loaded or model studied. Fig. 19.17 gives simple but a good approach for working out this effective flange width E in various types of practically dimensioned box sections. In this, full-depth or almost full-depth diaphragms are assumed at ends and all supports.

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CHAPTER 20

Practical Structural Analysis

20.1 AIM

The final aim being to 'design' the structure (i.e., to decide on the basis of allowable material stresses the section sizes, concrete strengths and amounts and detailing of the 'untensioned' and 'tensioned' reinforcement), the first step obviously is to estimate the load-effects that can be caused at each critical section in the structure under design loads. These load effects are more commonly referred to as bending moment (BM), shear force (SF), thrust (N), torsion (T), reaction (R), deflection (Y) and rotation (θ). Estimation of these load effects, in simple words, is called *structural analysis*, and the methods employed for this analysis are the tools of the structural analyst. The aim, therefore, is first to know these tools, then use them to carry out the structural analysis in a practical and workman-like manner in order to do the *structural design*. (This is then followed by the preparation of *working drawings*, from which are prepared the *Bill of Quantities* of each item of work, which then is 'priced' to estimate the cost of the structure.)

This chapter deals briefly with structural analysis tools in simple language without trying to mystify the reader into numbness as it were, because classical theories of structural analysis can be profusely mathematical and mind-boggling. While the reader is free to go into these realms of mathematical gymnastics (for a deeper understanding of the subject), the aim of this chapter is only to assist him into practical application of the tools in day-to-day practice where results are required relatively quickly by the pressures of work and one has little desire to end up as a 'frustrated engineer turned mathematician.''

NOTE While certain other aspects of structural analysis and various aspects of structural design have been covered in their respective contexts in some of the other chapters of this book, the author presupposes that the reader is conversant with and up-to-date on the mundane matters of structural analysis and design. Certain facets may, therefore, look like already treaded paths in which case they are referenced merely as quick refreshers.

20.2 STRUCTURAL ANALYSIS—FUNDAMENTAL CONCEPTS

The bending moments and shearing forces on freely-supported beams and simple cantilevers are readily determined from simple statical rules but the solution of continuous beams and statically-indeterminate frames is more complex. Until fairly recently the techniques of structural analysis required to solve such problems were presented and employed as independent self-contained methods, the relationships between them being ignored or considered relatively unimportant. The choice of method used depended on its suitability to the type of problem concerned and also to some extent on its appeal to the particular designer involved.

Recently, the underlying inter-relationships between various analytical methods have become clearer. It is now realized that there are two basic types of method—*flexibility methods* (otherwise known as action methods, compatibility methods or force methods) where the behaviour of the structure is considered in terms of unknown forces, and *displacement methods* (otherwise known as stiffness methods or equilibrium methods) where the behaviour is considered in terms of unknown displacements. In each case, the complete solution consists of combining a particular solution, obtained by modifying the structure to make it statically determinate and then analysing it, with a complementary solution, in which the effects of each individual modification are determined. For example, for a continuous-beam system, with flexibility methods, the particular solution involves removing the redundant actions (*i.e.*, the continuity between the individual members) to leave a series of disconnected spans: with displacement methods the particular solution involves violating joint equilibrium by restricting the rotation and/or displacement that would otherwise occur at the joints.

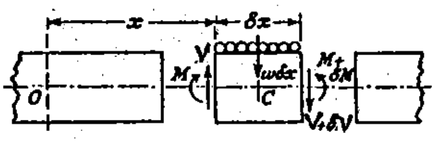
To clarify further the basic differences between the types of method, consider a propped cantilever. With the flexibility approach the procedure is to first remove the prop and calculate the deflection at the position of the prop due to the action of the load only: this gives the particular solution. Next, calculate the concentrated load that must be applied at the prop position to achieve an equal and opposite deflection: this is the complementary solution. The force obtained is the reaction in the prop and, when this is known, all the moments and forces in the propped cantilever can be calculated by combining the two solutions. If displacement methods are used, the span is considered fixed at both supports and the resulting moment acting at the end at which the prop occurs is found: this is the particular solution. The next step is to release this support and determine the moment that must then be applied at the pinned end of the cantilever to negate the fixing moment. Lastly, by summing both resulting moment diagrams the final moments are obtained and the reactions can be calculated.

In practical problems there are a number of unknowns and, irrespective of the method of solution adopted, the preparation and solution of a series of simultaneous equations is normally necessary. Whichever basic method of analysis is employed, the resulting relationship between forces and displacements embodies a series of coefficients which can be set out concisely in matrix form. If flexibility methods are used the resulting *flexibility matrix* is built up of *flexibility coefficients*, each of which represents a displacement produced by a unit action. Similarly, stiffness methods lead to the preparation of a *stiffness matrix* formed of *stiffness coefficients*, each of which represents an action produced by a unit displacement.

The solution of matrix equations, either by inverting the matrix or by a systematic elimination procedure, is ideally handled by machine aid and to this end, methods have been devised (so-called matrix stiffness and matrix flexibility methods) by means of which, not only are the necessary equations solved by computer but the machine is also used to set them up.

Relations between w, V and M

Figure 20.1 shows a short length δx imagined to be a slice cut out from a loaded beam at a distance x from a fixed origin O



w = uniform load, V = shear force
M = bending moment

Fig. 20.1

Let the shearing force at the section x be V, and at $x+\delta x$ be $V+\delta V$. Similarly, the bending moment is M at x, and $M+\delta M$ at $x+\delta x$. If w is the mean rate of loading on the length δx, the total load is $w\delta x$, acting approximately (exactly, if uniformly distributed) through the centre C. The element must be in equilibrium under the action of these forces and couples, and the following equations are obtained:

Taking moments about C.

$$M + V\delta x/2 + (V+\delta V)\delta x/2 = M + \delta M$$

Neglecting the product $\delta V\delta x$, and taking the limit, gives

$$V = dM/dx \qquad (20.1)$$

Resolving vertically.

$$w\delta x + V + \delta V = V$$

or

$$w = -dV/dx \qquad (20.2)$$

$$= -d^2M/dx^2 \text{ from Eq. (20.1)} \qquad (20.3)$$

From Eq. (20.1) it can be seen that, if M is varying continuously, zero shearing force corresponds to maximum or minimum bending moment, the latter usually indicating the greatest value of negative bending moment. It will be seen later, however, that 'peaks' in the bending moment diagram frequently occur at concentrated loads or reactions, and are not then given by $V = dM/dx = 0$, although they may represent the greatest bending moment on the beam. Consequently it is not always sufficient to investigate the points of zero shearing force when determining the maximum bending moment.

At a point on the beam where the type of bending is changing from sagging to hogging, the bending moment must be zero, and this is called a *point of inflection* or *contraflexure*.

By integrating Eq. (20.1) between two values of $x = a$ and b, then

$$M_b - M_a = \int_a^b V dx$$

showing that the increase in bending moment between two sections is given by the area under the shearing force diagram.

Similarly, integrating Eq. (20.2),

$$V_a - V_b = \int_a^b w dx$$

= the area under the load distribution diagram.
Integrating Eq. (20.3) gives

$$M_a - M_b = \int \int_a^b w dx dx$$

These relations prove very valuable when the rate of loading cannot be expressed in an algebraic form, and provide a means of graphical solution.

Deflection by Calculus

With usual symbols, the general equation of bending is,

$$M/EI = 1/R \tag{20.4}$$

and in terms of co-ordinates x and y.

$$\frac{1}{R} = \frac{\pm d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{20.5}$$

where the sign depends on the convention for axes. For beams met with in engineering practice the slope dy/dx is everywhere small; and may be neglected in comparison with 1 in the denominator.

Taking Y positive upwards, under the action of a positive bending moment the curvature of the beam is as shown in Fig. 20.2. It can be seen that dy/dx is increasing as x increases, i.e., d^2y/dx^2 is positive, and $1/R = d^2y/dx^2$ from Eq. (20.5).

Hence $\quad M/EI = d^2y/dx^2$ from Eq. (20.4)

or $$EId^2y/dx^2 = M \tag{20.6}$$

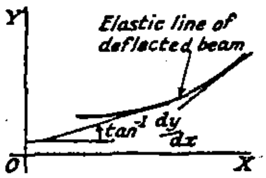


Fig. 20.2

Provided M can be expressed as a function of x, Eq. (20.6) can be integrated to give the slope dy/dx, and the deflection y, of the beam for any value of x. Two constants of integration will be involved, and these can be obtained by substituting known values of slope or deflection at particular points. A mathematical expression is thus obtained for the form of the deflected beam, or *elastic lines*.

- *Notes on application*
  - (i) Take the X axis through the level of the supports
  - (ii) Take the origin at one end, or at a point of zero slope
  - (iii) For a built-in or fixed end, or where the deflection is a maximum, the slope $dy/dx = 0$
  - (iv) For points on the X axis, usually supports, the deflection $y = 0$

Differentiating Eq. (20.6)

$$EId^3y/dx^3 = dM/dx = V$$

and $$EId^4y/dx^4 = dV/dx = -w$$

These forms are of use in some cases, though generally the bending moment relation is the most convenient.

EXAMPLE Obtain expressions for the maximum slope and deflection of a cantilever of length l carrying (a) a concentrated load W at its free end, (b) a uniformly distributed load w along its whole length.

(a) Taking the origin at the free end, the X axis through the fixed end, then at a distance x from the origin $M = -Wx$ (Fig. 20.3) and

$$EId^2y/dx^2 = M = -Wx \text{ from Eq. (20.6)}$$

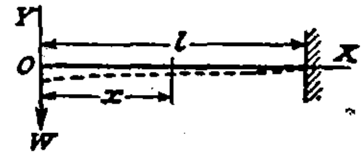


Fig. 20.3

Integrating $\quad EIdy/dx = -Wx^2/2 + A$

But $dy/dx = 0$ at $x = l$

$\therefore \quad A = Wl^2/2$

Integrate again

$$EIy = -Wx^3/6 + Wl^2x/2 + B$$

At $x = l, y = 0$

$$\therefore \quad B = Wl^3/6 - Wl^3/2 = -Wl^3/3$$

The slope and deflection at the free end (where they are a maximum) are given by the values of dy/dx and y when $x = 0$. i.e.,

$$\text{slope} = Wl^2/2EI$$

deflection $= -Wl^3/3EI$ (indicating downward). The deflected shape is shown dotted, Fig. 20.3.

(b) $EId^2y/dx^2 = M = -wx^2/2$ (Fig. 20.4)

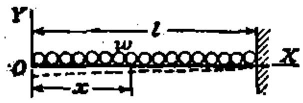


Fig. 20.4

Integrating $EIdy/dx = -wx^3/6 + A$

when $x = l, dy/dx = 0$

$\therefore \quad A = wl^3/6$

Integrating,

$EIy = -wx^4/24 + (wl^3/6)x + B$

when $x = l, y = 0$

$\therefore \quad B = -wl^4/8$

Putting $x = 0$, maximum slope $= wl^3/6EI$
and maximum deflection $= -wl^4/8EI$

Macaulay's Method

In applying the above method normally a separate expression for bending moment must be obtained for each section of the beam between adjacent concentrated loads or reactions, each producing a different equation with its own constants of integration. It will be appreciated that in any but the simplest cases the work involved will be laborious, the separate equations being linked together by equating slopes and deflections given by the expressions on either side of each 'junction' point. However, a method devised by Macaulay enables one continuous expression for bending moment to be obtained, and provided certain rules are followed the constants of integration will be the same for all sections of the beam.

It is advisable to deal with the different types of loading separately.

(i) *Concentrated loads:* Measuring x from one end, write down an expression for the bending moment in the last section of the beam, enclosing all distances less than x in square brackets, i.e.,

$$EId^2y/dx^2 = M = -W_1x + R[x-a] - W_2[x-b] - W_3[x-c] \quad \text{(Fig. 20.5)}$$

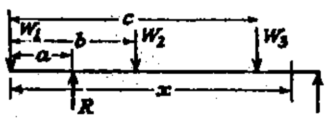


Fig. 20.5

Subject to the condition that all terms for which the quantity inside the square brackets is negative are omitted (i.e., given a value zero), this expression may be said to represent the bending moment for all values of x. If x is less than c the last term is omitted, if x is less than b then both the last two terms are omitted, and so on.

The brackets are to be integrated as a whole, i.e.,

$$EIdy/dx = -W_1x^2/2 + (R/2)[x-a]^2 - (W_2/2)[x-b]^2 - (W_3/2)[x-c]^2 + A$$

$$EIy = -W_1x^3/6 + (R/6)[x-a]^3 - (W_2/6)[x-b]^3 - (W_3/6)[x-c]^3 + Ax + B$$

By so doing it can be shown that the constants of integration are common to all sections of the beam, e.g., if $x = b - \Delta$

$$EIdy/dx = -(W_1/2)(b-\Delta)^2 + (R/2)(b-\Delta-a)^2 + A$$

and $EIy = -(W_1/6)(b-\Delta)^3 + (R/6)(b-\Delta-a)^3 + A(b-\Delta) + B$

and if $x = b + \Delta$

$$EIdy/dx = -(W_1/2)(b+\Delta)^2 + (R/2)(b+\Delta-a)^2 - (W_2/2)\Delta^2 + A'$$

and

$$EIy = -(W_1/6)(b-\Delta)^3 + (R/6)(b+\Delta-a)^3 - (W_2/6)\Delta^3 + A'(b+\Delta) + B'$$

Now as $\Delta \to 0$ these slope and deflection values must correspond (i.e., at $x = b$), from which it is seen that $A = A'$ and $B = B'$.

The values of A and B are found as before.

(ii) *Uniformly distributed loads:* Supposing a load w is stretched from a distance a to a distance b from one end (Fig. 20.6). Then in order to obtain an expression for the bending moment at a distance x from the end, which will apply for all values of x, it is necessary to continue the loading up to the section x, compensating with an equal negative load from b to x, i.e.,

$$M = Rx - (w/2)[x-a]^2 + (w/2)[x-b]^2$$

each length of loading acting at its centre of gravity, square brackets being interpreted as before.

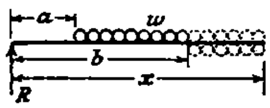


Fig. 20.6

For $x > a$ but $< b$, omit $[x-b]$, and $M = Rx - (w/2)(x-a)^2$, which is clearly correct.

The remaining steps of integration and constant enumeration are as before.

(iii) *Concentrated bending moment:* As shown in Fig. 20.7, write

$$EId^2y/dx^2 = M = -Rx + M_0[x-a]^0$$

then $\quad EIdy/dx = -Rx^2/2 + M_0[x-a] + A$, etc.

EXAMPLE A simply supported beam of length L carries a load W at a distance a from one end, b from the other $(a > b)$. Find the position and magnitude of the maximum deflection and show that the position is always within $L/13$, approximately, of the centre.

The maximum deflection (i.e., zero slope) will occur on the length a, since $a > b$.

Taking the axes as shown in Fig. 20.8,

$$EId^2y/dx^2 = M = (Wb/L)x - W[x-a]$$

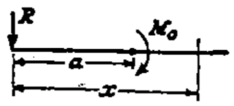


Fig. 20.7

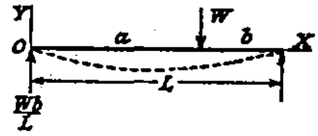


Fig. 20.8

$EIdy/dx = (Wb/L)(x^2/2) - (W/2)[x-a]^2 + A$ (i)
$EIy = (Wb/L)(x^3/6) - (W/6)[x-a]^3 + Ax + B$ (ii)
At $x = 0, y = 0,$ $\therefore B = 0$
At $x = L, y = 0,$ $\therefore AL = -(Wb/L)(L^3/6) + (W/6)b^3$
giving $\quad A = -(Wb/6L)(L^2 - b^2)$
$dy/dx = 0$ at a value of x given by
$(Wb/L)(x^2/2) - (Wb/6L)(L^2 - b^2) = 0$, from (i), omitting $[x-a]$ since $x < a$ for zero slope when $a > b$.

This gives $x = \sqrt{[(L^2 - b^2)/3]}$ at the point of maximum deflection.

Substituting in (ii) to find the value of the maximum deflection:

$$EIy = \frac{Wb}{L}\frac{(L^2-b^2)^{3/2}}{6 \times 3\sqrt{3}} - \frac{Wb}{6L}\frac{(L^2-b^2)^{3/2}}{\sqrt{3}}$$

giving $y = -\dfrac{Wb(L^2-b^2)^{3/2}}{(9\sqrt{3})EIL}$

Distance of point of maximum deflection from centre $= \sqrt{[(L^2 - b^2)/3]} - L/2$
which has a maximum value of $L/\sqrt{3} - L/2$, or approximately $L/13$.

Bending Moments and Shearing Forces: Basic Data

Basic Relationships

At any section:

(i) Shearing force

$$V = \sum \begin{bmatrix} \text{loads and reactions on} \\ \text{one side of section} \end{bmatrix} = \begin{matrix} \text{rate of} \\ \text{change of } M \end{matrix}$$

(ii) Bending moment

$$M = \sum \begin{bmatrix} \text{moments of loads and reactions} \\ \text{on one side of section} \end{bmatrix}$$
$$= \text{rate of change of } EI\theta$$

(iii) Slope

$$\theta = \int \frac{M}{EI} = \text{rate of change of } a \text{ or } y$$

(iv) Deflection (elastic)

$$y = a = \int \theta$$

where a or y = deflection
I = moment of inertia of member at section
E = elastic modulus of material

Material Constants

(i) Poisson's ratio $= 1/m$ = ratio of lateral strain to longitudinal strain, produced by a single stress.

(ii) Bulk Modulus $K = \dfrac{\text{fluid pressure } p}{\text{volumetric strain}}$, fluid pressure symbolising that three dimensional state of stress (equal in all directions) which causes the volumetric strain.

(iii) $E = 3K(1-2/m)$ = Young's Modulus of Elasticity

(iv) $U = p^2/2K$ per unit volume, U being the strain energy per unit volume

(v) $E = 9CK/(C+3K)$, C being the shear modulus of rigidity (shear stress/shear strain), also written as G.

(vi) $E = 2C(1 + 1/m)$

Deflection due to Shear

It is well known that shear stress is set up on transverse sections of a beam, and the accompanying shear strain will cause a distortion of the cross-section, and, since the shear stress varies from zero at the extreme fibres to a maximum at the neutral axis, cross-sections can no longer remain plane after bending.

In fact the warping will be of the form shown in Fig. 20.9, the left-hand view being for positive shear and the right-hand for negative shear. These strains are incompatible with the theory of pure bending, but nevertheless a good approximation to the deflection due to shear can be obtained by strain energy methods. It should also be noted that the shear distribution near to the application point of a concentrated load must differ considerably from that given by the theory since there can be no sudden change of shear strain from one type to the other, as would be implied for a simply supported beam with a central load, across the load position.

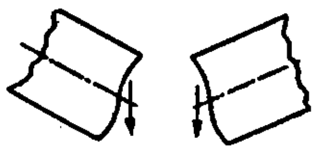


Fig. 20.9

Strain energy due to shear $= (s^2/2C) \times$ volume
where s = shear stress and C as defined earlier.

For the whole beam.

$$U_s = (1/2C)\int\int s^2 dA dx \qquad (20.7)$$

where dA is an element of cross-section and dx an element of length.

The integration can only be performed for particular cross-sections over which the variation of s is known, and rectangular and I-sections will be dealt with below.

Rectangular Section

From the usual formula: $s = \dfrac{VA\bar{Y}}{Ib}$ (shear stress) it follows that, $s = (6V/bd^3)(d^2/4-y^2)$ where y is the distance from the neutral axis, $dA = bdy$, then

$$U_s = \frac{1}{2C}\int\left[\int_{-d/2}^{d/2} \frac{36V^2}{b^2d^6}\left(\frac{d^4}{16} - \frac{d^2y^2}{2} + y^4\right) bdy\right] dx$$

from Eq. (20.7)

$$= \frac{1}{2C}\int \frac{36V^2}{bd^6}\left[\frac{d^4y}{16} - \frac{d^2y^3}{6} + \frac{y^5}{5}\right]_{-d/2}^{d/2} dx$$

$$= \frac{18}{Cbd}\int V^2 2\left(\frac{1}{32} - \frac{1}{48} + \frac{1}{160}\right) dx$$

$$= \frac{3}{5Cbd}\int_0^l V^2 dx \qquad (20.8)$$

Cantilever with Load W at Free End

$$V = W$$

$$\therefore U_s = \frac{3W^2l}{5Cbd} \text{ from Eq. (20.8)}$$

But $U_s = 1/2W\delta_s$, where δ_s is the deflection due to shear
$\therefore \quad \delta_s = 6Wl/5Cbd$

Cantilever with Uniformly Distributed Load The load $w\delta x$, on a length δx at a distance x from the fixed end, treated as a concentrated load, will produce a deflection due to shear $= (6w\delta xx)/5Cbd$ at this point. For this load alone the distortion produced is indicated in Fig. 20.10, being uniform shear force over the length x and zero over $l-x$, hence the total deflection due to shear for all the distributed load.

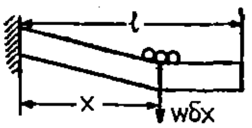


Fig. 20.10

$$= \int_0^l \frac{6wxdx}{5Cbd}$$
$$= 3wl^2/5Cbd$$

Simply Supported Beam with Central Load W

$$V = \pm W/2$$

$$U_s = \int_0^l \frac{3(W^2/4)}{5Cbd} dx \text{ from Eq. (20.8)}$$

$$= 3W^2l/20Cbd$$
$$= 1/2W\delta_s$$
$$\therefore \quad \delta_s = 3Wl/10Cbd$$

The 'simplified' deflection is as shown in the upper diagram of Fig. 20.11 and since the shearing force is constant over each half, this case is equivalent to a cantilever of length $l/2$ carrying an end load of $W/2$.

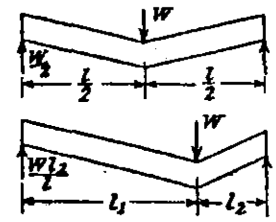


Fig. 20.11

If the load is not centrally applied, but divides the length into l_1 and l_2, then treating either section as a cantilever with an end load equal to the reaction on that side.

$$\delta_s = \frac{6(Wl_2/l)l_1}{5Cbd}$$

$= 6Wl_1l_2/5Cbdl$ under the load (Fig. 20.11)

Simply Supported Beam with Uniformly Distributed Load Due to a load $w\delta x$ only, at a distance x from one end ($x < l/2$), the shear deflection at the load $= 6w\delta x(l-x)x/5Cbdl$ just proved.

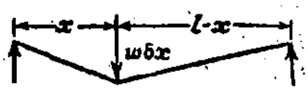


Fig. 20.12

By proportion, the deflection at the centre of the beam

$$= \frac{6w\delta x(l-x)}{5Cbdl}x\left(\frac{l/2}{l-x}\right) \text{ (Fig. 20.12)}$$

Then the total central deflection due to shear

$$= 2\int_0^{l/2} \frac{3wx\,dx}{5Cbd}$$

$$= 3wl^2/20Cbd$$

I-Section

Treating the shearing force as uniformly distributed over the web area bd,

$$s = V/bd \text{ and } \int dA = bd$$

$$\therefore \quad U_s = (1/2C)\int (V^2/b^2d^2)bd\,dx \text{ from Eq. (20.7)}$$

$$= \left(\int V^2 dx\right)/2Cbd \qquad (20.9)$$

By methods similar to those employed for a rectangular section the deflections due to shear may be obtained as follows:

- Cantilever with end load, $\delta_s = Wl/Cbd$
- Cantilever with distributed load, $\delta_s = Wl/2Cbd$
- Simply supported beam with central load, $\delta_s = Wl/4Cbd$
- Simply supported beam with distributed load, $\delta_s = Wl/8Cbd$

The strain energy method known as *'Castigliano's Theorem'* (described ahead) may be used where a number of loads exist concurrently, or to find the deflection due to a distributed load by imposing a concentrated load at the deflection point and later giving it a value zero (i.e., $\delta_s = (\partial U_s/\partial P)_{P=0}$.

EXAMPLE For a given cantilever of rectangular cross-section, length l, and depth d, show that, if δ_s and δ_b are the deflections due to shear and bending due to a concentrated load at the free end, $\delta_s/\delta_b = k(d/l)^2$, and find the value of k for steel. $E = 30 \times 10^6$ lb/sq.in.; $C = 11.5 \times 10^6$ lb/sq.in.

Hence find the least value of l/d if the deflection due to shear is not to exceed 1% of the total.

It has been shown that,

$$\delta_s = 6Wl/5Cbd$$

and $\quad \delta_b = Wl^3/3EI = 4Wl^3/Ebd^3$

for a rectangular section.

$$\delta_s/\delta_b = [6/(5 \times 4)](E/C)(d/l)^2 = k(d/l)^2$$

where $\quad k = (3/10)(E/C) = (3/10)(30/11.5) = 0.783$

If $\quad \delta_s/(\delta_b + \delta_s) = 0.01$

$\therefore \quad \delta_s/\delta_b = 0.01/0.99$

and $\quad \equiv 0.783(d/l)^2 \quad$ from above,

i.e., Least value of $l/d = \sqrt{(0.783 \times 99)}$
= 8.8, which is an important conclusion for practice.

EXAMPLE A 10-in. by 6-in. RSJ with web 0.4 in., flanges 0.7 in. thick, acts as a horizontal cantilever 12ft. long and carries a load of 2 tons at 6 ft from the end. Assuming the shear force is carried by the web and is uniformly distributed, calculate the deflection at the end. $E = 12,800$ tons/sq.in.; $C = 5000$ tons/sq. in.

$$I = (6 \times 10^3 - 5.6 \times 8.6^3)/12 = 204 \text{ in.}^4$$

By the moment-area method (see later) the end deflection due to bending,

$$= \left(\frac{1}{2} \times 12 \times 6\right) \frac{10 \times 1728}{12,800 \times 204} = 0.238 \text{ in. (Fig. 20.13)}$$

Deflection due to shear at the load is given by

$$\frac{Wl}{Cbd} = \frac{2 \times 6 \times 12}{5000 \times 0.4 \times 8.6} = 0.0084 \text{ in.}$$

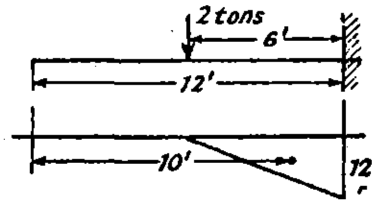


Fig. 20.13

But since the shearing force is zero beyond the load this is also the deflection due to shear at the free end (see also Fig. 20.10).

$$\text{Combined deflection at free end} = 0.238 + 0.0084 = 0.2464 \text{ in.}$$

Theories of Failure

The theory of elasticity and formulae derived are based on the assumption that the material obeys Hooke's law. Consequently no information can be derived from them if the material has passed beyond its elastic limit at any point in the member. In fact, when permanent (non-recoverable) deformations occur the material is said to have *failed*. Note that failure does not imply rupture or collapse.

It is natural to consider that in a simple tensile test, the elastic limit is associated with a certain value of the tensile stress; but at this stage other quantities, such as shear stress and strain energy, also attain definite values, and any one of these may be the deciding factor in the physical cause of failure.

In a complex stress system these quantities can be calculated from the known stresses and material constants, and the problem is to decide which quantity is the criterion of failure, i.e. the cause of the material passing beyond its elastic limit and taking up a permanent set. Having decided, the actual value of that particular factor which corresponds to the onset of failure is usually taken to be the value it reaches in the simple tension case at the elastic limit.

The principal theories of failure are outlined in detail below, in which f is the tensile stress at the elastic limit in simple tension, and f_1, f_2, f_3 the principal stresses in any complex system.

(i) *Maximum Principal Stress Theory (due to Rankine).* According to this theory failure will occur when the maximum principal stress in the complex system reaches the value of the maximum stress at the elastic limit in simple tension, i.e.,

$$f_1 = \frac{1}{2}(f_x + f_y) + \frac{1}{2}\sqrt{[(f_x - f_y)^2 + 4s^2]}$$

$$f_2 \text{ being } \frac{1}{2}(f_x + f_y) - \frac{1}{2}\sqrt{[(f_x - f_y)^2 + 4s^2]}$$

$= f$ in simple tension

where f_x, f_y and s are the stresses on given planes in the complex system.

(ii) *Maximum Shear Stress or Stress Difference Theory (due to Guest and Tresca)* This implies that failure will occur when the maximum shear stress q in the complex system reaches the value of the maximum shear stress in simple tension at the elastic limit. i.e.

$$q = \frac{1}{2}(f_2 - f_1) = \frac{1}{2}\sqrt{[(f_x - f_y)^2 + 4s^2]}$$

on the assumption that the maximum shear is greatest in the XY plane $= \frac{1}{2}f$ in simple tension

or $$f_2 - f_1 = f$$

(iii) *Strain Energy Theory (due to Haigh)*: This theory is based on the argument that as the strains are reversible up to the elastic limit, the energy absorbed by the material should be a single-valued function at failure, independent of the stress system causing it, i.e. strain energy per unit volume causing failure is equal to the strain energy at the elastic limit in simple tension

$$(1/2E)[f_1^2 + f_2^2 + f_3^2 - (2/m)(f_1f_2 + f_2f_3 + f_3f_1)] = f^2/2E$$

or $$[f_1^2 + f_2^2 + f_3^2 - (2/m)(f_1f_2 + f_2f_3 + f_3f_1)] = f^2$$

(iv) *Shear Strain Energy Theory (due to Mises and Hencky)*: At failure the shear strain energy in the complex system and in simple tension are equal, i.e.,

$$(1/12C)[(f_1 - f_2)^2 + (f_2 - f_3)^2 + (f_3 - f_1)^2] = f^2/6C$$

or $$(f_1 - f_2)^2 + (f_2 - f_3)^2 + (f_3 - f_1)^2 = 2f^2$$

(The value in the simple tension case is found by putting the principal stresses equal to $f, 0, 0$.)

(v) *Maximum Principal Strain Theory (due to St. Venant)*: If e_1 is the maximum strain the complex stress system, then according to this theory,

$$e_1 = (1/E)(f_1 - F_2/m - F_3/m)$$
$$= f/E \text{ in simple tension}$$

or $$f_1 - F_2/m - F_3/m = f$$

Other theories have been put forward, but have not proved to be nearer the truth except perhaps for particular types of loading.

Conclusions

Considerable experimental work has been done on various stress systems, such as tubes under the action of internal pressure, end loads, and torsion; also on different materials. So far, however, no conclusive evidence has been produced in favour of any one theory.

It must be admitted that the cause of failure depends not only on the properties of the material but also on the stress system to which it is subjected, and it may not be possible to embody the results for all cases in one comprehensive formula. The following general conclusions may be used as a guide to design.

In the case of brittle materials such as cast iron the maximum principal stress theory should be used. For ductile materials the maximum shear stress or strain energy theories give a good approximation, but the shear strain energy theory is to be preferred, particularly when the mean principal stress is compressive. The maximum strain theory should not be used in general, as it only gives reliable results in particular cases.

It should be noted that, since the shear stress and shear strain energy theories depend only on stress differences, they are independent of the value of the mean stress and imply that a material will not fail under a hydrostatic stress system (i.e. $f_1 = f_2 = f_3$). In practice the effect of such a stress system is to produce a brittle type fracture in a normally ductile material, no plastic deformation having taken place.

EXAMPLE If the principal stresses at a point in an elastic material are $2f$ tensile, f tensile, and $\frac{1}{2}f$ compressive, calculate the value of f at failure according to five different theories.

The elastic limit in simple tension is 20 tons/sq. in and Poisson's ration = 0.3

(i) *Maximum Principal Stress Theory*

In the complex system, maximum stress = $2f$.

In simple tension, maximum stress = 20 tons/sq. in. Equating gives $f = 10$ tons/sq.in.

(ii) *Maximum Shear Stress Theory*

$$\text{Maximum shear stress} = \text{Half difference between principal stresses}$$
$$= \frac{1}{2}\left[2f - (-\tfrac{1}{2}f)\right]$$
$$= \frac{5}{4}f$$

In simple tension, principal stresses are 20, 0, 0 and maximum shear stress $= \frac{1}{2} \times 20$

$$= 10 \text{ tons/sq. in.}$$

Equating gives $f = 8$ tons/sq. in.

(iii) *Strain Energy Theory*

In the complex system

$$U = (1/2E)[(2f)^2 + f^2 + (-\tfrac{1}{2}f)^2 - 2 \times 0.3 \; (2ff - ff/2 - f/2.2f)]$$
$$= 4.95f^2/2E$$

In simple tension $U = 20^2/2E$

Equating gives: $f = 20/\sqrt{4.95}$

$$= 8.98 \text{ tons/sq. in.}$$

(iv) *Shear Strain Energy Theory*

In the complex system

$$U_s = (1/12C)\left[(2f - f)^2 + \left(f + \frac{1}{2}f\right)^2 + \left(-\frac{1}{2}f - 2f\right)^2\right]$$
$$= 9.5f^2/12C$$

In simple tension (principal stresses 20, 0, 0)

$$U_s = 20^2/6C$$

Equating gives $\quad f = 20/\sqrt{4.75}$

$$= 9.17 \text{ tons/sq. in.}$$

(v) *Maximum Strain Theory*

Equating the maximum strain the complex and simple tension cases

$$(1/E)(2f - 0.3f + 0.3f/2) = 20/E$$

or $\quad f = 20/1.85$

$$= 10.8 \text{ tons/sq. in.}$$

EXAMPLE The load on a bolt consists of an axial pull of 1 ton together with a transverse shear force of $\frac{1}{2}$ ton. Estimate the diameter of bolt required according to (i) maximum principal stress theory, (ii) maximum shear stress theory, (iii) strain energy theory, (iv) shear strain energy theory. Elastic limit in tension is 18 tons/sq. in., and a factor of safety of 3 is to be applied. Poisson's ratio = 0.3.

The permissible simple tensile stress is 18/(Factor of safety = 6 tons/sq. in.

Let required diameter be d in., then the applied stresses are,

$$f = \frac{1}{\pi d^2/4} = \frac{4}{\pi d^2} \text{ tons/sq. in. tension}$$

and $s = \dfrac{1}{2\pi d^2/4} = \dfrac{2}{\pi d^2}$ tons/sq. in. shear (Fig. 20.14)

assuming uniform distribution over the cross-section.

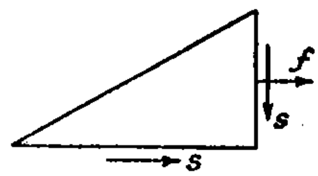


Fig. 20.14

(i) *Maximum Principal Stress in Bolt,*

$$= \frac{1}{2}f + \frac{1}{2}\sqrt{(f^2 + 4s^2)} \qquad (f_x = f, f_y = 0)$$
$$= \frac{1}{2} \cdot 4/\pi d^2 + \frac{1}{2}\sqrt{[(4/\pi d^2)^2 + 4(2/\pi d^2)^2]}$$
$$= (2/\pi d^2)[1 + \sqrt{(1+1)}]$$
$$= 4.829/\pi d^2$$

Maximum stress in simple tension = 6. T/sq. in.
Equating to above gives

$$d = \sqrt{(4.829/6\pi)}$$
$$= 0.506 \text{ in. say } \frac{1}{2} \text{ in.}$$

(ii) *Maximum Shear Stress* $= \frac{1}{2}\sqrt{(f^2 + 4s^2)}$

$$= 2.829/\pi d^2$$

and $\quad \equiv 3$ in simple tension (T/sq. in.)

$\therefore \; d \quad = \sqrt{(2.829/3\pi)}$

$$= 0.548 \text{ in. say } \frac{9}{16} \text{ in.}$$

(iii) *Principal Stresses are* $\frac{1}{2}f \pm \frac{1}{2}\sqrt{(f^2+4s^2)}, 0$ i.e.,

$$4.829/\pi d^2, \quad -0.829/\pi d^2, 0$$

$$\begin{aligned}\text{Strain energy } &= (1/2E)(4.829^2 + 0.829^2 + 2 \times 0.3 \times \\ &\quad 4.829 \times 0.829)/\pi^2 d^4 \\ &= 26.4/(2E\pi^2 d^4)\end{aligned}$$

and $\quad \equiv 6^2/2E$ in simple tension (T/sq. in.)

$$\begin{aligned}\therefore d &= \sqrt[4]{(26.4/36\pi^2)} \\ &= 0.523 \text{ in. say } \frac{9}{16} \text{ in.}\end{aligned}$$

(iv) *Shear Strain Energy*

$$\begin{aligned}&= (1/12C)[(4.829 + 0.829)^2 + 0.829^2 \\ &\quad +4.829^2]/(\pi^2 d^4)\end{aligned}$$

and $\quad \equiv 6^2/6C$ in simple tension (T/sq. in.)

$$\begin{aligned}\therefore \quad d &= \sqrt[4]{[(56.0 \times 6)/(36\pi^2 \times 12)]} \\ &= 0.53 \text{ in., say } \frac{9}{16} \text{ in.}\end{aligned}$$

20.3 'AREA MOMENTS' METHOD OF ANALYSIS

The simplest method of analyzing the stresses in a bent beam, which is statically indeterminate to a low degree, is that of area moments. This method forms a good introduction to methods of wider application.

Principle of Area-Moments (Fig. 20.15)

(a) In any portion, AB, of a bent beam, the angle, ϕ, between the tangents to the beam at A and B is numerically equal to the area of the *M/EI* diagram between these points.

(b) In any portion, AB, of a bent beam, the displacement of A from the tangent to the beam at B is equal to the moment of the area of the *M/EI* diagram between A and B, taken about A.

Since the displacement, *d*, is always very small in relation to the length of the beam, it is immaterial whether it is assumed to be measured at right angles to the beam or at right angles to the tangent.

Two points of importance should be noted and remembered. When calculating displacement, take the moment of the *M/EI* area about the point where the displacement is required. The figures found by following the principle (b) above, do not necessarily indicate the deflection of the beam from its original position.

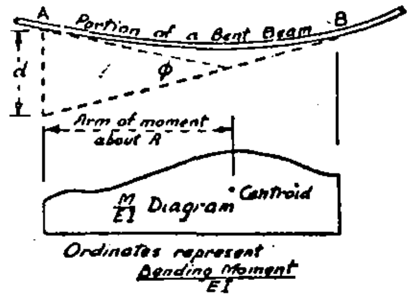


Fig. 20.15

Procedure

(i) Remove the statically indeterminate forces and moments so that the structure is left in a statically determinate condition.

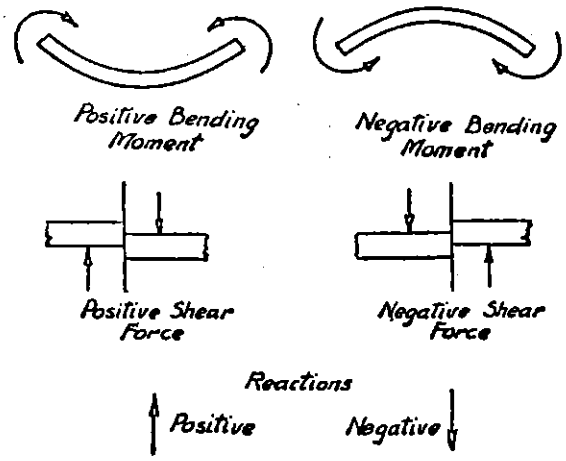
(ii) Apply the external loading and draw the bending moment diagram. This is often known as the free bending moment diagram.

(iii) Divide each ordinate of the bending moment diagram by the term EI relevant to each section and draw the *M/EI* diagram.

(iv) Remove the external loading from the beam and apply the statically indeterminate forces and moments. Draw the *M/EI* diagram for these forces and moments only.

(v) From the conditions of support the nett slope or deflection of the beam at one or more sections is usually known. Find the slope and/or deflection at these sections from (iii) in terms of the known loading, and from (iv) in terms of the unknown reactions or moments, and by comparison, determine the unknowns.

Convention of Signs



Moment-Area Method for Built-in Beams

A beam is said to be built-in or encastre when both its ends are rigidly fixed so that the slope remains horizontal. Usually also the ends are at the same level.

It follows from the moment-area method that, since the change of slope from end to end and the intercept are both zero:

$$\Sigma A = 0 \qquad (20.10)$$

and

$$\Sigma A\bar{x} = 0 \qquad (20.11)$$

It will be found convenient to show the bending moment diagram due to any loading such as [Fig 20.16(a)] as the algebraic sum of two parts, one due to the loads, treating the beam as simply supported [Fig. 20.16(b)], and the other due to the end moments introduced to bring the slopes back to zero [Fig. 20.16(c)].

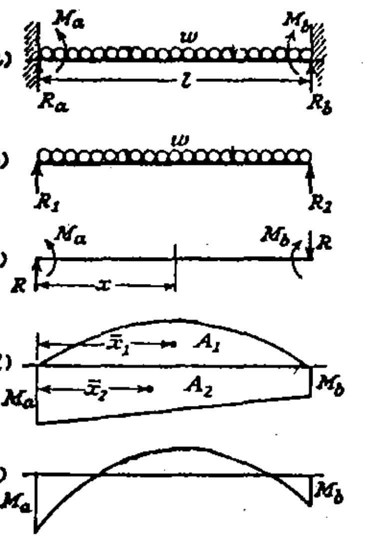


Fig. 20.16

The area and end reactions obtained it freely supported will be referred to as the *free moment diagram* and the *free reactions*, A_1, R_1 and R_2 respectively.

The fixing moments at the ends are M_a and M_b, and in order to maintain equilibrium when M_a and M_b are unequal, the reactions $R = (M_a - M_b)/l$ are introduced, being upwards at the left-hand end and downwards at the right-hand end. Due to M_a, M_b, and R, the bending moment at a distance x from the left-hand end

$$= -M_a + R \cdot x = M_a + [(M_a - M_b)/l]x$$

This gives a straight line going from a value $-M_a$ at $x = 0$ to $-M_b$ at $x = l$, and hence the *fixing moment diagram*, A_2 [Fig. 20.16(d)].

For downward loads, A_1 is a positive area (sagging BM), and A_2 a negative area (hogging BM) consequently the Eqs. (20.10) and (20.11) reduce to,

$$A_1 = A_2 \text{ from Eq. (20.10)}$$

and $$A_1\bar{x}_1 = A_2\bar{x}_2 \text{ (numerically) from Eq. (20.11)}$$

i.e., Area of free moment diagram = Area of fixing moment diagram.

And moments of areas of free and fixing diagrams are equal.

It may be necessary to break down the areas still further to obtain convenient triangles and parabolas.

These two equations enable M_a and M_b to be found, and the total reactions at the ends, are,

$$R_a = R_1 + R$$
$$= R_1 + (M_a - M_b)/l$$

and

$$R_b = R_2 - R$$
$$= R_2 - (M_a - M_b)/l$$

Finally, the combined bending moment diagram is shown in Fig. 20.16 (e) as the algebraic sum of the two components.

EXAMPLE Obtain expressions for the maximum bending moment and deflection of a beam of length l and flexural rigidity EI, fixed horizontally at both ends, carrying a load W (a) Concentrated at midspan, (b) uniformly distributed over the whole beam.

Case (a): By symmetry $M_a = M_b = M$, say (Fig. 20.17)

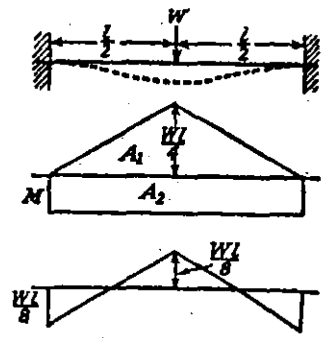


Fig. 20.17

The free moment diagram is a triangle with maximum ordinate $Wl/4$,

$$\therefore \quad \text{Area } A_1 = \frac{1}{2}(Wl/4)l$$
$$= Wl^2/8$$
$$\text{Area } A_2 = Ml$$

Equating $A_1 = A_2$ from Eq. (20.10) gives

$$M = Wl/8$$

The combined bending moment diagram is therefore as shown in the lower diagram, Fig. 20.17, and the maximum bending moment is $Wl/8$ occurring at the end (hogging), and the centre (sagging).

By taking moment-areas about one end for half the beam, the intercept gives the deflection, i.e.,

$$\delta = \frac{\left[\frac{1}{2}(Wl/4)(l/2)\right]\frac{2}{3}l/2 - M(l/2)l/4}{EI}$$
$$= Wl^3/192EI$$

Case (b): Free moment area

$$A_1 = \frac{2}{3}(wl^2/8)l = wl^3/12 \quad \text{(Fig. 20.18)}$$

Fixing moment area,

$$A_2 = Ml$$

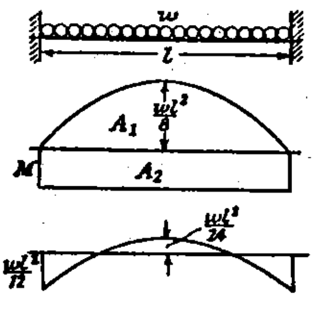


Fig. 20.18

Equating gives

$$M = wl^2/12$$

and this is the maximum bending moment.

Again, for half the beam, the intercept about one end gives the deflection, i.e.

$$\delta = \frac{\left[\frac{2}{3}(wl^2/8)(l/2)\right]\frac{5}{8}l/2 - M(l/2)l/4}{EI}$$
$$= wl^4/384EI$$

EXAMPLE A beam of span l ft has its ends fixed horizontally at the same level and carries a load W at a distance a ft from one end and b ft from the other. Deduce expressions for the fixing moments at the ends. Hence show that, for a distributed load on the same beam, the fixing moment at one end is given by $\int_0^l \frac{px(l-x)^2}{l^2}dx$

where p = load/ft. run at a distance of x ft from the end considered.

Apply the above result to find the fixing moments when l = 20 ft and p varies uniformly from zero at one end to 2 tons/ft at the other.

The free moment diagram is a triangle of height Wab/l, and the fixing moments are M_a and M_b (Fig. 20.19).

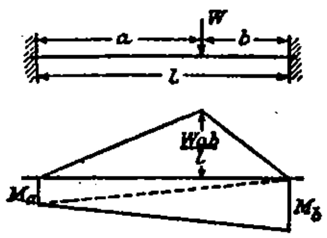


Fig. 20.19

Equating areas

$$\frac{1}{2}(M_a + M_b)l = \frac{1}{2}(Wab/l)l$$

i.e., $$M_a + M_b = Wab/l \qquad (20.12)$$

By moment-areas about the left-hand end, splitting each figure into two triangles

$$\left(\frac{1}{2}M_a l\right) l/3 + \left(\frac{1}{2}M_b l\right) 2l/3 = \left[\left(\frac{1}{2}Wab/l\right) a\right] 2a/3 + \left[\left(\frac{1}{2}Wab/l\right) b\right](a + b/3)$$

i.e., $(M_a + 2M_b)l^2/3 = 2Wa^3b/3l + (Wab^2/l)(a + b/3)$

or $$M_a + 2M_b = (Wab/l^3)(2a^2 + 3ab + b^2) \qquad (20.13)$$

Subtract Eq. (20.12), giving,

$$M_b = (Wab/l^3)(2a^2 + 3ab + b^2 - l^2)$$
$$= (Wab/l^3)(a^2 + ab), l = a + b$$
$$= (Wab/l^3)a(a + b)$$
$$= Wa^2b/l^2$$

From Eq. (20.12) $M_a = Wab/l - Wa^2b/l^2 = Wab^2/l^2$

For a distributed load the fixing moment δM_a due to the $p\delta x$ on a short length at a distance x from that end = $p\delta x x(l - x)^2/l^2$ from above.

Integrating for all the load

$$\therefore \quad M_a = \int_0^l \frac{px(l-x)^2dx}{l^2}$$

$$p = 2x/20 \text{ tons/ft}$$

$$\therefore \quad M_a = \int_0^{20} \frac{2x}{20} \frac{x(20-x)^2}{20^2} dx$$

$$= \frac{1}{4000} \int_0^{20} (400x^2 - 40x^3 + x^4) dx$$

$$= 26\frac{2}{3} \text{ tons-ft.}$$

Similarly

$$M_b = \int_0^{20} \frac{2x}{20} \frac{x^2(20-x)}{20^2} dx$$

$$= \frac{1}{4000} \int_0^{20} (20x^3 - x^4) dx$$

$$= 40 \text{ tons-ft}$$

It will be seen, therefore, that for standard cases the minimum bending moment occurs at one of the fixed ends. More complicated loadings may be built up by superposition and it may be accepted in general that, *for any combination of downward loads the maximum bending moment is given by the greater fixing moment.*

Moment Area Method for Continuous Beams

When a beam is carried on more than two supports it is said to be continuous. It is possible to employ an extension of the moment-area method to obtain a relation between the bending moments at three points (usually supports).

In Fig. 20.20 the areas A_1 and A_2 are "free" bending moment areas, treating the beam as simply supported over two separate spans l_1 and l_2. If the actual bending moments at these points are M_1, M_2 and M_3, a fixing moment diagram consisting of two trapezia will be introduced, the actual BM being the algebraic sum of the two diagrams.

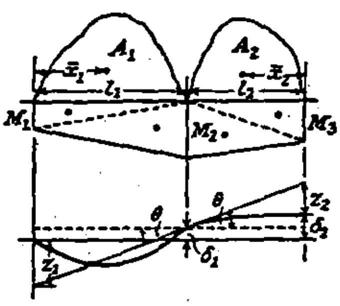


Fig. 20.20

In the lower figure the elastic line of the deflected beam is shown, the deflections δ_1 and δ_2 being relative to the left-hand support and positive upwards, θ is the slope of the beam over the centre support, and z_1 and z_2 the intercepts for l_1 and l_2.

Then $\quad \theta = (z_1 + \delta_1)/l_1 = (z_2 + \delta_2 - \delta_1)/l_2$

(slopes being everywhere small)

i.e., $$\frac{A_1\bar{x}_1 + (M_1 l_1/2)(l_1/3) - (M_2 l_1/2)(2l_1/3)}{EI_1 l_1} + \frac{\delta_1}{l_1}$$

$$= -\frac{A_2\bar{x}_2 + (M_3 l_2/2)(l_2/3) - (M_2 l_2/2)(2l_2/3)}{EI_2 l_2} + \frac{\delta_2 - \delta_1}{l_2}$$

(note that z_2 is a negative intercept)

or $$M_1 l_1/I_1 + 2M_2(l_1/I_1 + l_2/I_2) + M_3 l_2/I_2$$
$$= 6(A_1\bar{x}_1/I_1 l_1 + A_2\bar{x}_2/I_2 l_2) + 6E[\delta_1/l_1 + (\delta_1 - \delta_2)l_2] \quad (20.14)$$

If $I_1 = I_2$

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2$$
$$= 6(A_1\bar{x}_1/l_1 + A_2\bar{x}_2/l_2) + 6EI[\delta_1/l_1 + (\delta_1 - \delta_2)/l_2] \quad (20.15)$$

If the supports are at the same level

$$M_1 l_1 + 2M_2(l_1 + l_2) + M_3 l_2 = 6(A_1\bar{x}_1/l_1 + A_2\bar{x}_2/l_2) \quad (20.16)$$

and if the ends are simply supported ($M_1 = M_3 = 0$)

$$M_2(l_1 + l_2) = 3(A_1\bar{x}_1/l_1 + A_2\bar{x}_2/l_2) \quad (20.17)$$

Equation (20.14) is the most general form of the equation of three moments, also called *Clapeyron's equation*. The others are simplifications to meet particular cases. Eq. (20.16) being the form in which it is most frequently required.

EXAMPLE A beam AD, 60 ft long, rests on supports at A, B and C at the same level. AB = 24 ft; BC = 30 ft. The loading is 1 ton/ft throughout and in addition a concentrated load of 5 tons acts at the mid-point of AB and a load of 2 tons acts at D. Draw the SF and BM diagrams.

$M_a = 0$
$M_c = 2 \times 6 + 6 \times 3 = 30$ tons-ft

Applying Eq. (20.16) to the spans ABC (Fig. 20.21).

$$2M_b \times 54 + 30 \times 30$$

$$= 6\left(\frac{1}{2} \times \frac{5 \times 24}{4} \times 24\right) \times \frac{12}{24} + \left(\frac{2}{3} \times \frac{24^2}{8} \times 24\right) \times \frac{12}{24}$$

$$+ \left(\frac{2}{3} \times \frac{30^2}{8} \times 30\right) \times \frac{15}{30}$$

$$= 6 \times 1881$$

$$\therefore \quad M_b = 96.2 \text{ tons-ft.}$$

BM at mid-point of AB

$$= 5 \times 24/4 + 24^2/8 - M_b/2$$
$$= 53.9 \text{ tons-ft.}$$

BM at mid-point of BC

$$= 30^2/8 - \frac{1}{2}(M_b + 30)$$
$$= 49.4 \text{ tons-ft.}$$

To find the reactions at the supports, note that

$$M_b = -R_a \times 24 + 24 \times 12 + 5 \times 12 \text{ for } AB$$
$$= -R_c \times 30 + 36 \times 18 + 2 \times 36 \text{ for } BCD$$
$$\therefore R_a = (288+60-96.2)/24 = 10.49 \text{ tons, say } 10.5 \text{ tons.}$$

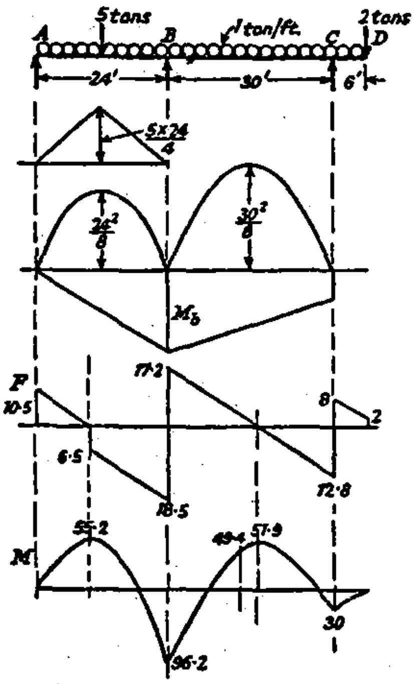


Fig. 20.21

and $R_c = (648 + 72 - 96.2)/30 = 20.8$ tons

By difference $R_b = 60 + 5 + 2 - 10.5 - 20.8$
$= 35.7$ tons

From the shear force diagram it can be worked out that the maximum bending moments occur.

(i) at a distance of 12.8 ft from C, where
$M = 21.5 \times 13.5 - 19.5^2/2 - 2 \times 19.5 = 51.9$ tons-ft

(ii) at a distance of 10.5 from A where
$M = 10.5 \times 10.5 - 10.5^2/2 = 55.2$ tons-ft

20.4 STRAIN ENERGY METHOD OF ANALYSIS

General Principles

The amount of strain energy of work stored in a loaded structure depends on the magnitude of the direct, shear and bending stresses imposed on the various parts of the structure. In pin-jointed frames, for example, where the members are in tension or compression, the work stored depends on direct forces only.

In rigid structures, direct stress, shear stress and bending stress may all occur at any section, and the total strain energy stored in the beam or frame depends on the magnitudes of the three types of stress. It is generally conceded, however, that the work done by the direct and shear forces is so small in comparison with that done by bending that only the latter need be considered when calculated statically indeterminate reactions or moments.

It must be remembered, however, that although the work done by shear and direct forces may be considered negligible, yet the values of the shear and direct stresses must be included in the final stress values when the strength of the structure is being checked.

In any member of a structure subjected to bending the total internal work or strain energy is

$$U = \int_0^l \frac{M^2 ds}{2EI}$$

Where M is the bending moment at any point on the member caused by the combined effect of the imposed loads and the supporting forces and moments. Whether statically determinate or not. The integration must be taken over the whole length of the member, of which ds is an element of length.

Castigliano showed that the partial differential coefficient of the strain energy in a structure, with respect to a load P acting on the structure, is equivalent to the displacement of P along its line of action.

$$\Delta_P = \frac{\partial U}{\partial P} = \int_0^l \frac{2M}{2EI}\frac{\partial M}{\partial P} ds = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial P} ds$$

Similarly, the partial differential coefficient of the strain energy in a structure with respect to a moment acting on the structure is equivalent to the angle through which that portion of the structure rotates when the moment is applied.

$$\theta = \frac{\partial U}{\partial M_x} = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial M_x} ds$$

It very often occurs that the forces P and the moments

M_x, are the supporting forces and moments of a statically indeterminate structure, and if the supports of the structure do not give way under the action of the loading, then there is no deformation of the structure at the points of support and the expressions just quoted can be equated to zero. If the differential coefficient of the strain energy is thus zero, then the strain energy itself is a minimum. The term *Method of Least Work* may therefore be found applied to Strain Energy determinations. Where there is rotation of the support the corresponding $\partial U/\partial M_x$ can be equated to that, and similarly if there is a movement in the direction of the reaction at the support the corresponding $\partial U/\partial P$ (or $\partial U/\partial H$) can be equated to that.

Beams and Frames Having One Redundant Reaction

EXAMPLE If the support R were removed, the structure (Fig. 20.22) would become a cantilever, and it is therefore statically indeterminate to the first degree. A single equation, in addition to those of statics, is sufficient to determine the stresses in the beam.

From the principle of Strain Energy the equation required is

$$\frac{1}{EI}\int_0^l M\frac{\partial M}{\partial R}dx = 0$$

EI is a constant whose value is not required and which can be cancelled from subsequent calculations.

The integration must be made in two sections, since the bending moment expression changes at half span. The work is most conveniently done in tabular form. Working from the right-hand end of the beam, the table is as follows:

	Bending Moment (M)	$\partial M/\partial R$	Limits
Right half	Rx	$+x$	0 to 5
Left half	$Rx - \frac{w}{2}(x-5)^2$	$+x$	5 to 10

It will be found easier to work each integration separately, than to try to evaluate all terms in one operation. By this means there is less chance of a mistake being made in the evaluation of the definite integrals.

$$\int_0^5 M\frac{\partial M}{\partial R}dx = \int_0^5 (Rx)x\,dx = 41.67R$$

$$\int_5^{10} M\frac{\partial M}{\partial R}dx = \int_5^{10}\{Rx - \frac{w}{2}(x-5)^2\}x\,dx =$$

$$291.7R - 364.6$$

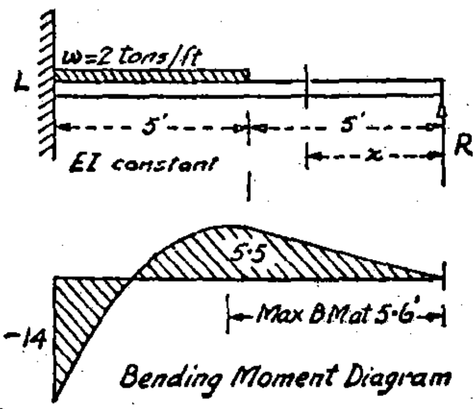


Fig. 20.22

summing and equating to zero,

$$\int_0^{10} M\frac{\partial M}{\partial R}dx = 333R - 364.6 = 0$$

$$R = +1.1 \text{ tons}$$

The bending moment diagram can now be drawn by substituting this value R in the bending moment column of the table.

EXAMPLE If any one of the three supports of the beam shown in Fig. 20.23 were to be removed, the beam would be statically determinate. Only one equation is therefore, required to find one of the reactions. Thereafter the methods of statics can be used to determinate the others.

By taking moments, two of the reactions are first expressed in terms of the third, so that only one unknown appears in the equation

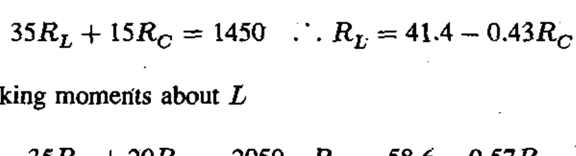


Fig. 20.23

Taking moments about R

$$35R_L + 15R_C = 1450 \quad \therefore R_L = 41.4 - 0.43R_C$$

Taking moments about L

$$35R_R + 20R_C = 2050 \quad R_R = 58.6 - 0.57R_C$$

It must be remembered that the integration must be taken over the whole length of the beam, and in this problem it is easier to work from both ends towards the centre than from one end only. EI is again constant over the whole length.

Member	Bending Moment	Moment rewritten	$\partial M/\partial R_C$	Limits
RC	$R_R x - 2x^2$	$58.6x - 0.57xR_C - 2x^2$	$-0.57x$	0 to 15
LC	$R_L x - x^2$	$41.4x - 0.43xR_C - x^2$	$-0.43x$	0 to 20

$$\int_0^{15} M\frac{\partial M}{\partial R_C}dx = \int_0^{15}(58.6x - 0.57xR_C - 2x^2)(-0.57x)dx = 370R_C - 23,200$$

$$\int_0^{20} M\frac{\partial M}{\partial R_C}dx = \int_0^{20}(41.4x - 0.43xR_C - x^2)(-0.43x)dx = 490R_C - 30,300$$

Summing and equating to zero

$$\int_0^{l} M\frac{\partial M}{\partial R_C}dx = 860R_C - 53,500 = 0 \quad R_C = +62.2 \text{ tons}$$

EXAMPLE The frame $ABCD$ in Fig. 20.24 has rigid joints at the corners B and C, and is pin-jointed at the supports A and D. This type of frame is used sometimes in bridge construction in steel and reinforced concrete, and is usually known as a Portal Frame.

When the frame is loaded as shown in Fig. 20.24, the points A and D have a tendency to move apart, and the horizontal force H is called into play. The vertical reactions V_A and V_D can be evaluated by the method of statics and H is the statically indeterminate force. The frame is indeterminate to the first degree.

The writing of the bending moment equations for this type of frame sometimes present difficulties to the beginner. A sheet of paper should be used to cover the frame except the portion to the right or left of the section being considered. The moments (about the edge of the paper) of all the forces which can be seen must then be written down, the values of any moments applied to the structure being also included.

$$V_A = V_D = 4.5 \text{ tons, } EI \text{ is constant}$$

Member	Bending Moment (M)	$\partial M/\partial H$	Limits
AB	$-Hy$	$-y$	0 to 15
CD	$-Hy$	$-y$	0 to 15
BC(left half)	$-15H + 4.5x$	-15	0 to 10
BC(right half)	$-15H + 4.5x$	-15	0 to 10

$$\frac{2}{EI}\int_0^{15} M\frac{\partial M}{\partial H}dy = \frac{2}{EI}\int_0^{15} Hy^2dy = \frac{2250}{EI}H$$

$$\frac{2}{EI}\int_0^{10} M\frac{\partial M}{\partial H}dx = \frac{2}{EI}\int_0^{10}(4.5x - 15H)(-15)dx = \frac{4500}{EI}H - \frac{6750}{EI}$$

Summing and equating to zero,

$$\int_0^{l} M\frac{\partial M}{\partial H}dx = 6750H - 6750 = 0, H = 1 \text{ ton in the}$$

direction assumed.

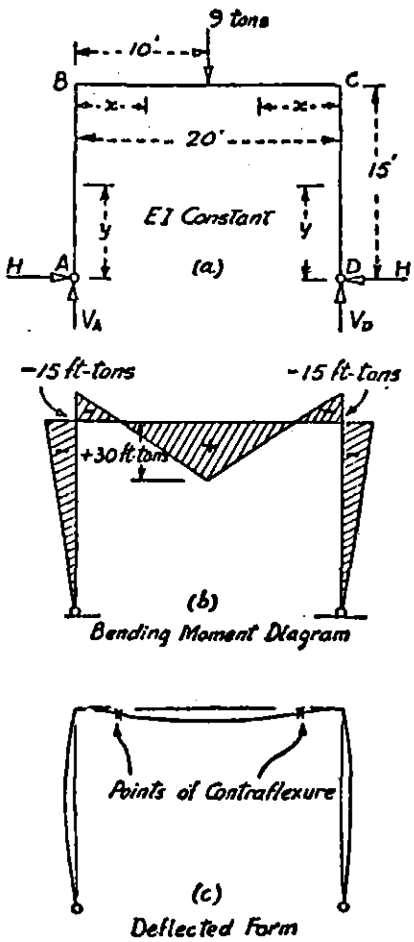


Fig. 20.24

Since the joints B and C are considered to be rigid, the bending moments in the beam and column at these points must be equal. The value of the bending moment of B and C is (from the table) $-15H = -15$ ft tons.

It is usual to plot the bending moment diagram on that 'side' of the beam or column which is in tension.

The reader should make a practice of drawing both the bending moment diagram and a diagram showing, to an exaggerated degree, the deflected form of the frame. Points of contraflexure occur where the bending moment diagram crosses the outline of the frame.

The work done by direct stress has not been included in the total strain energy, but the stress at any section is the sum of that due to the bending moment. [Fig. 20.24(b)] and that due to the direct stress, which must not be forgotten when stresses are being evaluated. At B, for example, the bending stress is obtained from the flexure formula, and the direct stress in the column AB is V_A divided by the cross-sectional area of AB.

Deflection from the Strain Energy (Castigliano's Theorem)

Theorem If U is the total strain energy of any structure due to the application of external loads $W_1, W_2, \ldots$ at $O_1, O_2 \ldots$ in the directions $O_1X_1, O_2X_2 \ldots$, and to couples $M_1, M_2 \ldots$ then the deflections at $O_1, O_2 \ldots$ in the directions $O_1X_1, O_2X_2 \ldots$, are $\partial U/\partial W_1, \partial U/\partial W_2 \ldots$, and the angular rotations of the couples are $\partial U/\partial M_1, \partial U/\partial M_2 \ldots$ at their applied points.

Proof for concentrated loads If the total displacements (in the direction of the loads) produced by gradually applied loads $W_1, W_2, W_3 \ldots$ and $X_1, X_2, X_3 \ldots$ then

$$U = \frac{1}{2}W_1x_1 + \frac{1}{2}W_2x_2 + \frac{1}{2}W_3x_3 + \ldots \qquad (20.18)$$

Let W_1 alone be increased by δW_1, then δU =increase in external work done

$$= (W_1 + \delta W_1/2)\delta x_1 + W_2\delta x_2 + W_3\delta x_3 + \ldots$$

(where $\delta x_1, \delta x_2, \delta x_3$ are increases in x_1, x_2 and x_3 due to application of δW_1)

$$= W_1\delta x_1 + W_2\delta x_2 + W_3\delta x_3 + \ldots \qquad (20.19)$$

neglecting the product $\frac{1}{2}\delta W_1\delta x_1$.

But if the loads $W_1 + \delta W_1, W_2, W_3 \ldots$ were applied gradually from zero, the total strain energy would have been:

$$U + \delta U = \frac{1}{2}(W_1 + \delta W_1)(x_1 + \delta x_1) + \frac{1}{2}W_2(x_2 + \delta x_2) + \frac{1}{2}W_3(x_3 + \delta x_3) + \ldots$$

Subtracting Eq. (20.18) and neglecting products of small quantities:

$$\delta U = \frac{1}{2}W_1\delta x_1 + \frac{1}{2}\delta W_1 x_1 + \frac{1}{2}W_2\delta x_2 + \frac{1}{2}W_3\delta x_3 + \ldots \qquad (20.20)$$

or $\quad 2\delta U = W_1\delta x_1 + \delta W_1 x_1 + W_2\delta x_2 + W_3\delta x_3 + \ldots$

Subtract Eq. (20.19), then $\delta U = \delta W_1 x_1$

and in the limit $\quad \partial U/\partial W_1 = x_1$

Similarly for x_2 and x_3, and the proof can be extended to incorporate couples.

It is important to stress that U is the total strain energy, expressed in terms of the loads and not including statically determinate reactions, and that the partial derivative with respect to each load in turn (treating the others as constant) gives the deflection at the load point in the direction of the load.

The following principles should be observed in applying this theorem:

(i) In finding the deflection of curved beams and similar problems, only strain energy due to bending need normally be taken into account (i.e. $\int M^2ds/2EI$)

(ii) Treat all the loads as variables initially, carry out the partial differentiation and integration, putting in numerical values at the final stage.

(iii) If the deflection is to be found at a point where, or in a direction in which, there is no load, a load may be put in where required and given a value zero in the final reckoning (i.e. $x = (\partial U/\partial W)_{W=0}$).

Generally it will be found that the strain energy method requires less thought in application than the direct method, it being only necessary to obtain an expression for the bending moment; also there is no difficulty over the question of sign, as the strain energy is bound to be positive, and deflection is positive in the direction of the load. The only disadvantage occurs when a case such as note (iii) above has to be dealt with, when the direct method sometimes will probably be shorter.

EXAMPLE Obtain expression for the vertical displacement at A of the beam showing in Fig. 20.25.

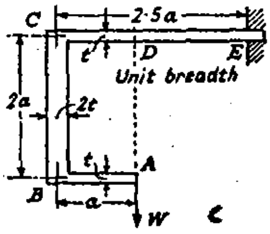


Fig. 20.25

If $a = 2$ in., $t = \frac{1}{4}$ in. find the displacement when $W =$ 5 lb $E = 30 \times 10^6$ lb/. sq.in. and section width is 1 in.

The bending moments in the various sections can be written as follows:

$AB, M = W \cdot x,$ (at x from A)

$BC, M = W \cdot a,$ constant

$CD, M = W \cdot x'$, at x' from D)

$DE, M = W \cdot x''$ at x'' from D)

$U = M^2 ds/2EI$

$$= \int_0^a \frac{W^2x^2dx}{2E \times t^3/12} + \int_0^{2a} \frac{W^2a^2ds}{2E \times (2t)^3/12} +$$

$$\int_0^a \frac{W^2x'^2dx'}{2E \times t^3/12} + \int_0^{1.5a} \frac{W^2x''^2 \cdot dx''}{2E \times t^3/12}$$

$$= (6W^2/Et^3)[a^3/3 + 2a^3/8 + a^3/3 + 1.5^3a^3/3]$$

$$= 24.5W^2a^3/2Et^3$$

Displacement of load at A $= \partial U/\partial W$ vertically

$$= 24.5Wa^3/Et^3$$

$$= (24.5 \times 5 \times 2^3)/ \left[30 \times 10^6 \times \left(\frac{1}{4}\right)^3\right]$$

$$= 0.0021 \text{ in.}$$

An allowance could be made for the linear extension of the portion BC

$(W2a)/(2tE) = (5 \times 2)/\left(\frac{1}{4} \times 30 \times 10^6\right) = \frac{4}{3} \times 10^{-6}$ in. which is clearly negligible compared with the deflection due to bending.

20.5 'MOMENT DISTRIBUTION' METHOD OF ANALYSIS

Moment Distribution is a mechanical process of dealing with indeterminate structures by means of successive approximations in which the moments themselves are treated directly, the calculations involved being purely arithmetical. It was developed by Hardy-Cross.

The method is unique in that all joints are initially considered to be fixed against rotation. The fixed end moments are determined for each member as though it were an encastre beam and then the joints are allowed to rotate, either separately or all at once, the moments induced by the rotations being distributed among the members until the algebraic sum of the moments at each internal joint is zero.

The sign convention most commonly adopted for Moment Distribution is that all moments acting on individual members from supports or other members of a frame are positive if clockwise in application and negative if anti-clockwise. Before a BM diagram is drawn, this convention must be translated into the normal convention whereby in continuous beams, for example, sagging moments are positive and hogging moments are negative. The two conventions are compared in Fig. 20.26.

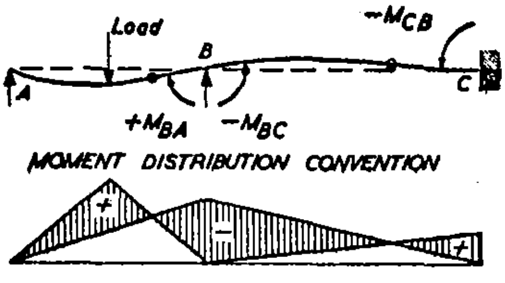


Fig. 20.26

It will be found that the operations of Moment Distribution are more readily understood and checked if the reader considers initially how the structure deflects under load. Consequently, deflection diagrams are incorporated in many of the examples.

Although the structural principles on which Moment Distribution is based are well known, it is advisable to consider them in a definite sequence.

Figure 20.27 shows a beam AB of constant cross-section, i.e. a prismatic beam, fixed in position and direction at A and fixed in position, but not in direction, at B. When the moment M_{BA} is applied at B a moment M_{AB} is induced at A. It can be shown that

$$M_{AB} = \frac{1}{2}M_{BA} \tag{20.21}$$

and

$$M_{BA} = 4E \tan\theta \frac{I}{L}$$

$$= 4E\theta\frac{I}{L} \text{(for small values of } \theta) \tag{20.22}$$

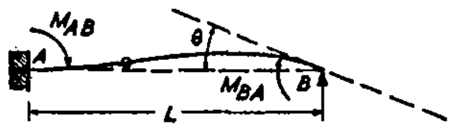


Fig. 20.27

Similarly, if A is fixed in position but not in direction, as in Fig. 20.28, then

$$M_{BA} = 3E\theta\frac{I}{L} \text{ (for small values of } \theta) \tag{20.23}$$

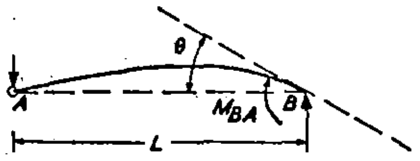


Fig. 20.28

These equations give the three fundamental principles of moment distribution applicable to continuous beams on unyielding supports.

- *Principle I*, Eq. (20.21) When a moment is applied at one end of a prismatic beam, that end remaining fixed in position but not in direction, the other end being fixed both in position and direction, a moment of half the amount and the same sign is induced at the second end.
- *Principle II*, Eq. (20.22) When one end of a beam remains fixed in position and direction, the moment required to produce a rotation of a given angle at the other end of the beam, which remains fixed in position, is proportional to the value I/L for the beam, provided that E is constant. The value I/L, known by the symbol K, is the stiffness factor for the particular beam in question.
- *Principle III*, (Eq. 20.23) When one end of a beam is rotated through a given angle, remaining fixed in position, and the other end remains fixed in position but not in direction, the moment required at the first end of $\frac{3}{4}$ of that required if the second end were fixed both in position and direction, i.e., the equivalent stiffness factor for the beam is $\frac{3}{4}I/L = \frac{3}{4}K$.

The three foregoing principles alone are applied when the supports do not yield. However, when the joints change their positions BMs have to be modified accordingly.

Beams with Support at Different Levels

The ends are assumed, as before, to be horizontal. The bent form of the unloaded beam as shown in Fig. 20.29 is similar to the bent form of two simple cantilevers which can be achieved by cutting the beam at the centre C, and placing downward and upward loads at the free ends of the cantilevers such that the deflection at the end of each cantilever is $d/2$.

Therefore, $\frac{d}{2} = \frac{P(L/2)^3}{3EI}$ (being the standard deflection formula)

or $$P = \frac{12EId}{L^3}$$

This load would cause a BM at A or B equal to

$$P \times \frac{L}{2} = \frac{12EId}{L^3} \times \frac{L}{2} = \frac{6EId}{L^2}$$

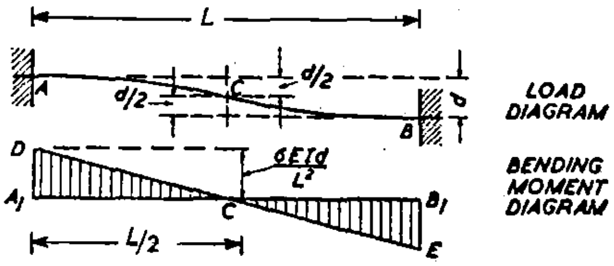


Fig. 20.29

The solution in any given case consists of adding to the ordinary diagram of BMs, the BM diagram A_1DCEB_1.

Shear Forces in Fixed Beams

It must be noted that in the case of fixed beams, it is necessary to evaluate the BMs before the SFs can be determined. This is the converse of the procedure in the case of simply supported beams.

The SF at the end of a beam is found in the following manner:

SF_A = the simple support reaction at $A + \frac{M_A - M_B}{L}$

SF_B = the simple support reaction at $B + \frac{M_B - M_A}{L}$

Consider Fig. 20.30 in which the end A of an encastre beam AB, of span L, has settled an amount d, the ends A and B remaining parallel in direction. As shown above.

$$M_{AB} = M_{BA} = \frac{6EId}{L^2} \qquad (20.24)$$

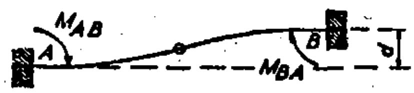


Fig. 20.30

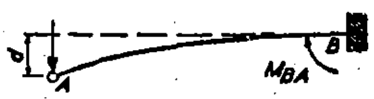


Fig. 20.31

Similarly, in Fig. 20.31 where the end A is hinged, i.e., not fixed in direction,

$$M_{BA} = \frac{3EId}{L^2} = \frac{6EId}{2L^2} \qquad (20.25)$$

From these equations, the following further principles may be derived.

- *Principle IV*, Eq. (20.24) When one end of a beam is deflected through a given distance, that end remaining parallel to its original position and the other remaining fixed in position and direction, equal moments of the same sign are induced at each end, proportional to the I/L^2 value of the beam.
- *Principle V*, Eq. (20.25) When a hinged end of a beam is deflected through a given distance, the other end remaining fixed in position and direction, a moment is induced at the second end, proportional to the $I/2L^2$ value of the beam.

Having stated the principles, the moment distribution processes may be explained by considering some simple examples.

Continuous Beams

EXAMPLE Figure 20.32 shows a continuous beam ABC, of constant cross-section, which is fixed in position and direction at A and C and simply supported at B and which carries uniformly distributed loads of 2 tons per ft on AB and 1 ton per ft on BC.

Under these loads the beam will rotate in an anti-clockwise direction at B and, as it is a fundamental assumption in the theory of continuous beams that the slope does not change over a support, the beam will rotate the same amount θ on either side of B. However, assume that the beam does not rotate at B, but through some locking device remains horizontal after the loads are applied. Then AB and BC are in effect two separate encastre beams and the moments at the end of each span are fixed-end moments (FEMs) depending only on the functions of the span and the loading.

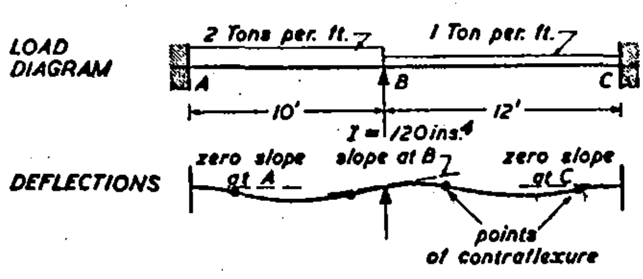


Fig. 20.32

The functions of a span are (i) L, its length, and (ii) I, moment of inertia of its sections.

When I is constant throughout a span, the process is straightforward. Hence, in this example I may be ignored, assuming it constant throughout (as also E).

DISTRIBUTION TABLE

	A	B	B	C
Distribution Factors		0.545	0.455	
Fixed End Moments	−16.67	+16.67	−12.00	+12.00
Distribution		−2.55	−2.12	
Carry Over	−1.27			−1.06
Final Moments	−17.94	+14.12	−14.12	+10.94

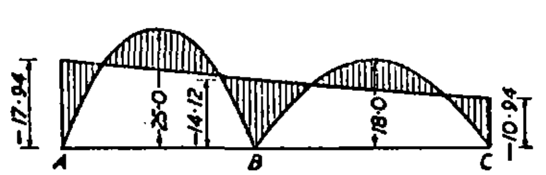


Fig. 20.33 *Final bending moments*

Consider the FEMs for the span AB. From the table given ahead in Fig. 20.37:

$$\begin{aligned}
\text{FEM}_{AB} \text{ (being anti-clockwise)} &= -\frac{WL}{12} \\
&= \frac{-2 \times 10 \times 10}{12} \\
&= -16.67 \text{ tons ft} \\
\text{FEM}_{BA} \text{ (being clockwise)} &= +16.67 \text{ tons ft}
\end{aligned}$$

Similarly, for the span BC,

$$\begin{aligned}
\text{FEM}_{BC} &= -\frac{WL}{12} = \frac{1 \times 12 \times 12}{12} \\
&= -12 \text{ tons ft} \\
\text{FEM}_{CB} &= +12 \text{ tons ft}
\end{aligned}$$

Having made these calculations the beam can be released at B and allowed to rotate in an anti-clockwise direction. Now the algebraic sum of the moments on either side of this support must be zero. However, when the beam was horizontal at B, the value of M_{AB} was +16.67 tons-ft and that of M_{BC} was −12 tons-ft. Therefore, to produce equilibrium at B, the total moment induced by the rotation of the beam there must be −4.67 tons-ft, since the moments are out of balance by +16.67 − 12 = +4.67 tons-ft.

But the span ends meeting at B rotate through the same angle θ. Consequently, the moments induced by rotation on either side of B are proportional to the stiffnesses of AB and BC (Principle II). In other words, the moment of −4.67 tons-ft is distributed between AB and BC in proportion to their stiffnesses, i.e., in the proportion

$K_{AB}/(K_{AB}+K_{BC})$ to the left and $K_{BC}/(K_{AB}+K_{BC})$ to the right.

These proportions are known as the distribution factors (DF) for the spans. Although it may sometimes be more accurate to employ fractions, these factors are usually expressed in decimals, but, in any case, the factors for a support or joint must always add up to unity.

Now $I = 120$ in.^{4}, $AB = 120.$ in. and $BC = 144$ in.

$$\text{Hence} \quad K_{AB} = \frac{120}{120} = 1$$

$$\text{and} \quad K_{BC} = \frac{120}{144} = 0.833$$

$$\text{Therefore} \quad DF_{AB} = \frac{1}{1+0.833} = 0.545$$

$$DF_{BC} = \frac{0.833}{1+0.833} = 0.455$$

The operation of moment distribution is shown in the distribution table in Fig. 20.33, $0.545 \times -4.67 = -2.55$ tons-ft being added to the end BA and $0.455 \times -4.67 = -2.12$ tons-ft being added to the end BC.

From a consideration of Principle I moments are induced at the outer ends of the beam at A and C, equal to half the moments distributed between the spans at B and of the same signs.

Hence $0.5 \times -2.55 = -1.27$ tons-ft must be transferred to the end A and $0.5 \times -2.12 = -1.06$ tons-ft to the end C. This process, which is known as the *carry-over process* is shown in Fig. 20.33.

The final moments in the beam are found by adding each column algebraically. When constructing a BM diagram it is convenient to remember that the moment to the right of a support in a distribution table bears the same sign as the support moment in the BM diagram (in the normal sign convention). Therefore, the final moments at A, B and C are respectively -17.94, -14.12 and -10.94 tons-ft

The maximum static or 'free' BMs for AB and BC are obtained from the formula + $WL/8$ and equal 25 and 18 tons ft, respectively, and the net moment diagram is as shown shaded.

EXAMPLE The continuous beam *ABCDE*, which is shown in Fig. 20.34 is simply supported at *A* and over hangs the other outside support *D*. Consequently, the beam is free to rotate at A and D, although it is restrained to a certain extent at D by the load at E, and when deriving FEMs the beam is assumed to be fixed in a horizontal position at B and C only. Therefore, Principle III applies to spans AB and CD, and the stiffness factors for these spans equal $\frac{3}{4}$K. The stiffens factor for $BC = K$.

Now the moment of inertia I differs for each span, although it is constant throughout a span, as shown in Fig. 20.34.

$$\text{Hence,} \quad \frac{3}{4}K_{AB} = \frac{3 \times 205}{4 \times 168} = 0.915$$

$$K_{BC} = \frac{146}{144} = 1.014$$

$$\frac{3}{4}K_{CD} = \frac{3 \times 122}{4 \times 144} = 0.635$$

$$\text{DF}_{BA} = \frac{3}{4}K_{AB}\Big/\left(\frac{3}{4}K_{AB} + K_{BC}\right) = \frac{0.915}{0.915 + 1.014} = 0.474$$

$$\text{DF}_{BC} = K_{BC}\Big/\left(\frac{3}{4}K_{AB} + K_{BC}\right) = \frac{1.014}{0.915 + 1.014} = 0.526$$

$$\text{DF}_{CB} = K_{BC}/(K_{BC} + \frac{3}{4}K_{CD}) = \frac{1.014}{1.014 + 0.635} = 0.615$$

$$\text{DF}_{CD} = 1 - 0.615 = 0.385$$

Now AB and CD are treated as 'fixed at one end only', and accordingly from the tables of FEMs (given ahead in Figs. 20.37 to 20.40).

$$\text{FEM}_{BA} = +\frac{3PL}{16} = \frac{3 \times 12 \times 14}{16} = 31.50 \text{ tons ft}$$

$$\text{FEM}_{BC} = -\frac{2PL}{9} = -\frac{2 \times 7 \times 12}{9} = -18.67 \text{ tons ft}$$

$$\text{FEM}_{CB} = 18.67 \text{ tons ft}$$

$$\text{FEM}_{CD} = -\frac{WL}{8} = -\frac{22 \times 12}{8} = -33.00 \text{ tons ft}$$

Now under any circumstances in the remainder of the beam the moment of D can only be that due to the load on the cantilever.

The cantilever moment

$$M_{DE} = -M_{DC} = -5 \times 4 = -20.00 \text{ tons ft.}$$

Sufficient data have been accumulated now to analyse the beam by moment distribution as shown in Fig. 20.34.

The inexperienced M_{DE} and M_{DC} are inserted in the appropriate columns, and half M_{DC} is carried over to the other end of the span CD. Subsequently the support D is ignored until the final moments are summated.

The preliminary operations at support C demand some explanation. When the beam is released the unbalanced moment

$$= +10 + (18.67 - 33.00) = -4.33 \text{ tons ft}$$

To balance this a moment + 4.33 tons ft must be distributed between the ends CB and CD.

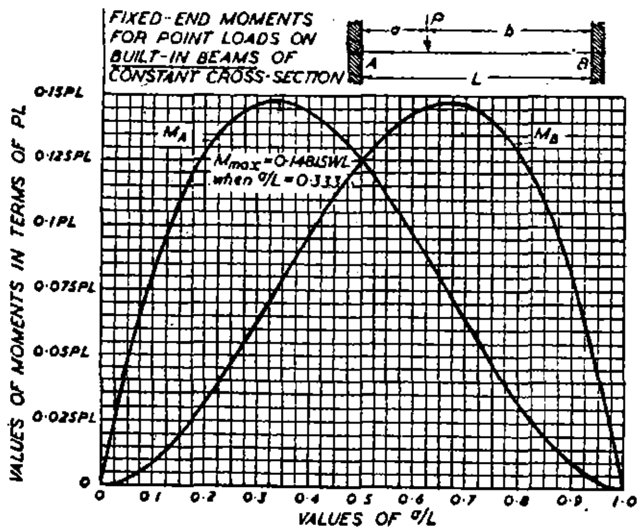


Fig. 20.38

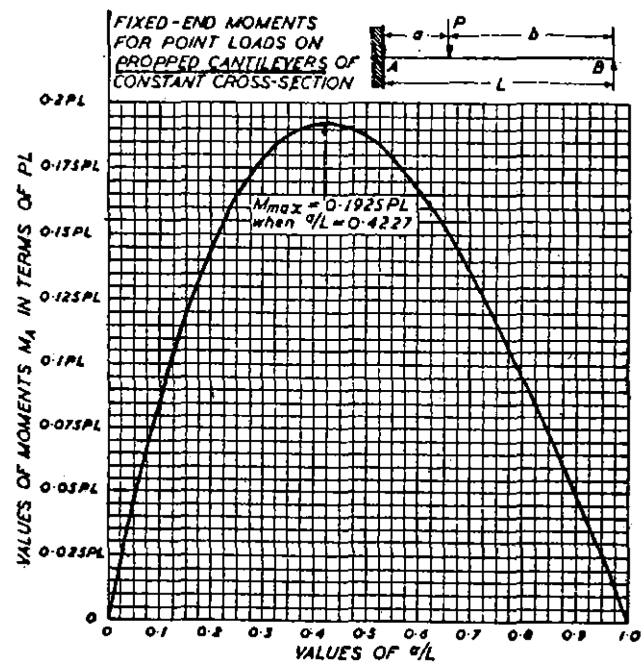
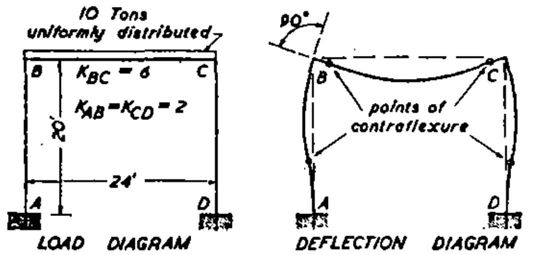


Fig. 20.40

It is appropriate at this stage to include a practical 'short cut' to reduce the amount of work in the distribution table. Just as it is convenient to consider that the equivalent stiffness of a simply supported end span is $3/4K$, so can one modify the stiffness factors of other members under certain conditions.



DISTRIBUTION TABLE

	A	B		C		D
D.F.		0·25	0·75	0·75	0·25	
F.E.M.			−20·00	+20·00		
Dist.		+5·00	+15·00	−15·00	−5·00	
C.O.	+2·50		−7·50	+7·50		−2·50
Dist.		+1·87	+5·63	Figures can be repeated here with opposite signs		
C.O.	+0·94		−2·81			
Dist.		+0·70	+2·11			
C.O.	+0·35		−1·05			
Dist.		+0·26	+0·79			
C.O.	+0·13		−0·39			
Dist.		+0·10	+0·29			
C.O.	+0·05					
Final Moments	+3·97 (Say 4)	+7·93 (8)	−7·93 (−8)	+ 7·93 (8)	−7·93 (−8)	−3·97 (−4)

Fig. 20.41

The portal frame of Fig. 20.41 and its load are symmetrical about the centre of the beam BC.

If, in this case, the stiffness factor for BC is taken as $K/2$, then there is no need to carry-over between B and C.

As $\quad K_{AB} : K_{BC}/2 : K_{CD} = 1 : 1.5 : 1,$

$$\text{DF}_{BA} = \text{DF}_{CD} = \frac{1}{1 + 1.5} = 0.4$$

and $\quad \text{DF}_{BC} = \text{DF}_{CB} = 1 - 0.4 = 0.6$

Employing this method a very short distribution table results as shown in Fig. 20.42.

The final moments are shown in Fig. 20.42, as before.

The above procedure may be adopted for all members which are subject to *equal* end rotations in opposite directions.

DISTRIBUTION TABLE

	A	B		C		D
D.F.		0·4	0·6	0·6	0·4	
F.E.M.			−20·00	+20·00		
Dist.		+8·00	+12·00	−12·00	−8·00	
C.O	+4·00					−4·00
Final Moments	+4·00	+8·00	−8·00	+8·00	−8·00	−4·00

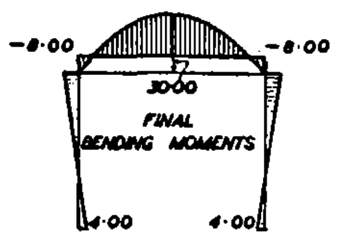


Fig. 20.42

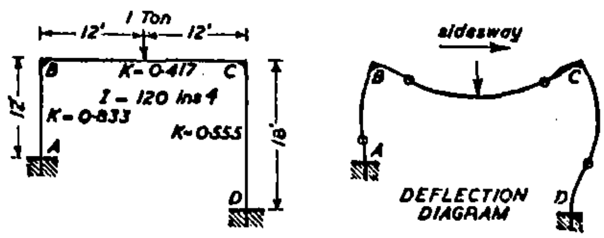
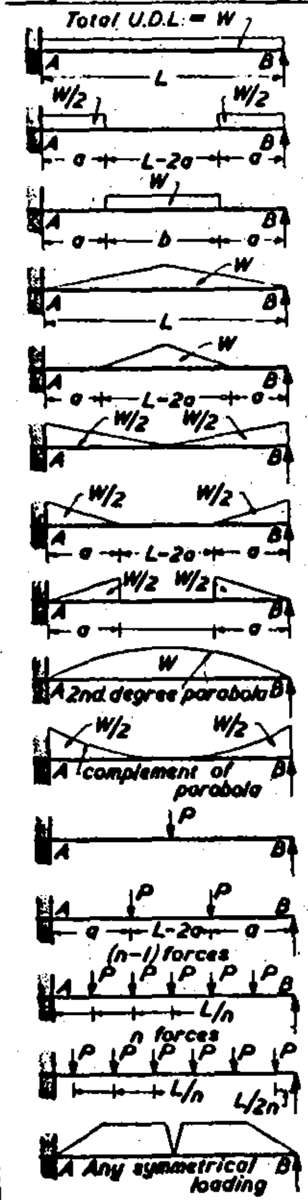


Fig. 20.43

FIXED-END MOMENTS

PROPPED CANTILEVERS OF CONSTANT CROSS-SECTION

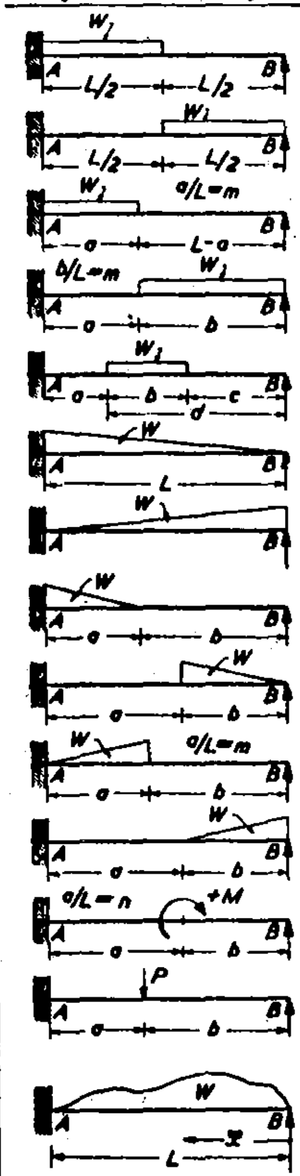


Symmetrical loadings	Values of fixing moment M_A*
Total U.D.L. = W	$-\frac{WL}{8}$
	$-\frac{Wa}{8L}(3L-2a)$
	$-\frac{W}{16L}(3L^2-b^2)$
	$-\frac{5WL}{32}$
	$-\frac{W}{32L}(5L^2+4aL-4a^2)$
	$-\frac{3WL}{32}$
	$-\frac{Wa}{8L}(2L-a)$
	$-\frac{Wa}{8L}(4L-3a)$
2nd degree parabola	$-\frac{3WL}{20}$
complement of parabola	$-\frac{3WL}{40}$
	$-\frac{3PL}{16}$
	$-\frac{3Pa}{2L}(L-a)$
(n−1) forces	$-\frac{PL}{8n}(n^2-1)$
n forces	$-\frac{PL}{16n}(2n^2+1)$
Any symmetrical loading	$-3A_s/2L$ Where A_s is the area of the 'free' bending moment diagram

* For cantilevers of opposite hand, the fixing moments M_B are of opposite sign.

FIXED-END MOMENTS

PROPPED CANTILEVERS OF CONSTANT CROSS-SECTION



Asymmetrical loadings	Values of fixing moment M_A**
	$-\frac{9}{64}WL$
	$-\frac{7}{64}WL$
$a/L=m$	$-\frac{Wa}{8}(2-m)^2$
$b/L=m$	$-\frac{Wb}{8}(2-m^2)$
	$-\frac{W}{8L^2b}(d^2-c^2)(2L^2-c^2-d^2)$
	$-\frac{2}{15}WL$
	$-\frac{7}{60}WL$
	$-\frac{Wa}{60L^2}(3a^2-15aL+20L^2)$
	$-\frac{Wb}{15L^2}(5L^2-3b^2)$
$a/L=m$	$-Wa\left(\frac{m^2}{5}-\frac{3m}{4}+\frac{2}{3}\right)$
	$-\frac{Wb}{60L^2}(10L^2-3b^2)$
$a/L=n$	$+\frac{M}{2}(2-6n+3n^2)$
	$-\frac{Pb}{2L^2}(L^2-b^2)$
	$-\frac{3A_s\bar{x}}{L^2}$ Where A_s is the area of the 'free' B.M. diagram, considering AB as a beam, and $\bar{x}$ is the distance from B to its centroid

** For cantilevers of opposite hand, the fixing moments M_B are of opposite sign.

Fig. 20.39

EXAMPLE when a portal frame is asymmetrical in shape or is asymmetrically loaded, it tends to sway to one side, and analysis by moment distribution has to be carried out in two stages. In the first stage the moments are derived assuming that the frame is propped against sway, while in the second moments induced by the sway are calculated.

Consider the frame shown in Fig. 20.43, which has a constant I of 120 in.^{4}

$$K_{AB} = \frac{120}{144} = 0.833$$
$$K_{BC} = \frac{120}{288} = 0.417$$
$$K_{CD} = \frac{120}{216} = 0.555$$

Hence
$$\mathrm{DF}_{BA} = \frac{0.833}{0.833 + 0.417} = 0.67$$
$$\mathrm{DF}_{BC} = 0.33$$
$$\mathrm{DF}_{CB} = \frac{0.417}{0.417 + 0.555} = 0.43$$
$$\mathrm{DF}_{CD} = 0.57$$

Now
$$\mathrm{FEM}_{BC} = -FEM_{CB} = -\frac{WL}{8} = -\frac{1 \times 24}{8} = -3.00 \text{ tons-ft}$$

When the frame is prevented from swaying, the moments are those obtained in Fig. 20.44 (Stage I).

Distribution Table for Stage I Moments

	A	B		C		D
DF		0.67	0.33	0.43	0.57	
FEM			−3.00	+3.00		
Dist.		+2.00	+1.00	−1.29	−1.71	
CO	+1.00		−0.64	+0.50		−0.85
Dist.		+0.43	+0.21	−0.21	−0.29	
CO	+0.21		−0.11	+0.11		−0.15
Dist.		+0.07	+0.04	−0.05	−0.06	
CO	+0.04					−0.03
Stage I Moments	+1.25	+2.50	−2.50	+2.06	−2.06	−1.03

Fig. 20.44

Now, a frame sways because of unbalanced horizontal thrust.

$$\text{The thrust at A} = \frac{+1.25 + 2.50}{12} = 0.31 \text{ ton}$$

$$\text{while the thrust at D} = \frac{-2.06 - 1.03}{18} = -0.17 \text{ ton}$$
$$\text{difference} = 0.14\text{T}$$

Hence, the propping force equals 0.14 ton and is required in a horizontal direction from C towards B.

The second stage of the calculations is to find what moments result when a force of 0.14 ton acts in a horizontal direction from B towards C.

Unfortunately, there is no direct method of achieving this object. Nevertheless, within the elastic range of the material, the moments produced in a frame are proportional to the applied forces. Hence, if it can be calculated that a certain BM produces a known lateral force, then the bending moment resulting from another lateral force in the same place may be calculated by proportion.

Let the frame sway an amount d along the line BC, the joints B and C being prevented from rotation, as shown in Fig. 20.45.

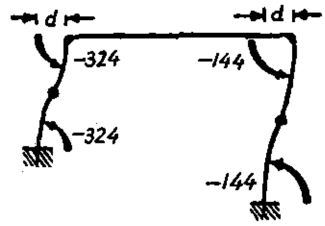


Fig. 20.45

Distribution Table for Sideway

	A	B		C		D
DF		0.67	0.33	0.43	0.57	
FEM	−324	−324	0	0	−144	−144
Dist.		+216	+108	+62	+82	
CO	+108		+31	+54		+41
Dist.		−20	−11	−23	−31	
CO	−10		−11	−5		−15
Dist.		+7	+4	+2	+3	
CO	+3					+1
Final Moments	−223	−121	+121	+90	−90	−117

Fig. 20.46

By Principle IV the moments induced in AB and CD are proportional to their I/L^2 values.

Hence,

$\text{FEM}_{AB} : \text{FEM}_{BA} : \text{FEM}_{CD} : \text{FEM}_{DC} = 1/12^2 : 1/12^2 : 1/18^2 : 1/18^2$
$= -324 : -324 : -144 : -144.$

Using these values as the arbitrary moments, release the joints B and C and calculate the resulting moments in the frame, as shown in Fig. 20.46.

The resulting shears sum up to

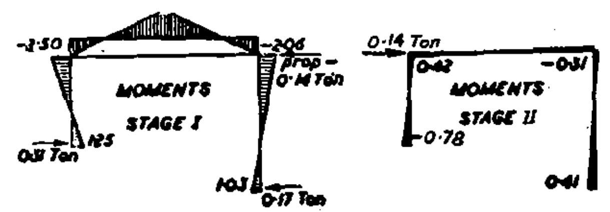
$$= \frac{-223 - 121}{12} + \frac{-90 - 117}{18}$$
$$= -40.2 \text{ tons}$$

This force is 40.2/0.14 = 287 times as greater as the propping force in Stage I. Hence, the Stage II moments are 1/287 of those calculated ahead.

The Stage I and Stage II moments are shown in Figs. 20.47 and 20.48. When added algebraically they provide the final BMs after including the simply-supported moments.

		A	B	C	D
Stage I	Moments	+1.25	−2.50	−2.06	+1.03
Stage II	Moments	−0.78	+0.42	−0.31	+0.41
Final	Moments	+0.47	−2.08	−2.37	+1.44

Fig. 20.47



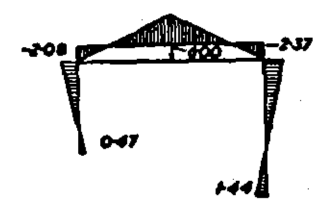


Fig. 20.48 *Final bending moments*

20.6 'SLOPE DEFLECTION' METHOD OF ANALYSIS

This method was made widely known by Maney and Wilson of Minnesota University in 1915, although it was based on the work of Mohr.

Joint rotations and deflections are treated as unknown quantities and, once these have been evaluated, the moments follow automatically by substituting the values in standard equations.

Suppose that the member AB in Fig. 20.49 is one unloaded span of a continuous beam and that the member is of constant moment of inertia. Then, for the conditions shown, the following fundamental equations hold,

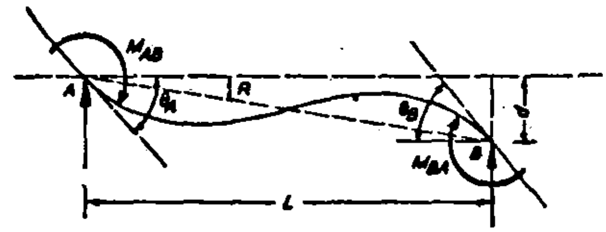


Fig. 20.49 *Slope deflection symbols*

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R)$$

and $$M_{BA} = 2EK(2\theta_B + \theta_A - 3R)$$

where E and K have the normal significance (K being I/L, the stiffness of the member).

θ_A and θ_B are the angles the joints make with the horizontal and R is the angle of rotation of B with respect to A when B sinks an amount d (i.e., $R = d/L$)

With regard to sign convention, following may be assumed here:

θ is positive when the tangent to the beam rotates in a clockwise direction

R is positive when the beam rotates in a clockwise direction

M is positive when the moment acts in a clockwise direction on the beam.

Therefore, the various values of M, θ and R in Fig. 20.49 are all positive.

Suppose that the span AB carries a load acting downwards in the normal fashion. Then,

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) - \text{FEM}_{AB}$$
$$M_{BA} = 2EK(2\theta_B + \theta_A - 3R) + \text{FEM}_{BA}$$

FEM_{AB} and FEM_{BA} are the fixed-end moments which would exist if AB were a fixed-end beam. The values and signs used are precisely the same as in the moment distribution method and the tables given there (Figs 20.37 to 20.40) are equally of use for the Slope Deflection Method.

When the end A of a beam AB is hinged, the formula for the moment at the other end is modified as follows:

$M_{BA} = EK(3\theta_B - 3R)$ (unloaded condition)
or $M_{AB} = EK(3\theta_B - 3R) + \text{FEM}_{BA}$ (loaded condition)

An analogy for this modification exists in moment distribution where the stiffness factor for a beam hinged at one end is reduced to $3/4K$. It should be noted that the value of the FEM is that applicable to beams hinged at one end and fixed in direction and position at the other.

The standard formulae will be applied to some of the examples which appeared in the section on moment distribution. The reader should examine especially the signs which are given to the rotation R.

As in most other methods of analysis the value of the modulus of elasticity E can be ignored in nearly every example.

When calculating the value of FEMs for loads acting downwards in the normal fashion the appropriate signs can be ignored because the fundamental formulae automatically provide the correct signs.

The final BM diagram is prepared by considering all hogging moments as negative and all sagging moments as positive.

When the method of slope deflection is used to find the moments in a continuous beam, the slope of the beam over each internal support is calculated. The value of the slopes may be useful in calculation the deflections in interior spans, but great care is needed with signs. In the slope-deflection calculations a positive value for the slope means that the beam has rotated in a clockwise direction. In the purely mathematical sense a positive slope is one 'going upwards to the right', i.e., dy/dx is positive.

Furthermore, it is essential that the units employed throughout the slope deflection calculations should be the same as those for E and I. Otherwise, the values of the slopes will not be related to the units employed in the deflection calculations.

Continuous Beams

EXAMPLE Consider the continuous beam in Fig. 20.50. Under the action of the loads, the joint B rotates in a anti-clockwise direction. When this rotation θ_B is calculated the whole beam can be analysed.

Let the suffixes 1 and 2 be applied to AB and BC respectively.

A and C are fixed in direction as well as position. Hence,

$$\theta_A = 0 = \theta_C$$

A, B and C are on the same level. Therefore,

$$R_1 = 0 = R_2$$

$$\text{FEM}_{AB} = \text{FEM}_{BA} = \frac{WL}{12} = 200 \text{ tons in.}$$

$$\text{FEM}_{BC} = \text{FEM}_{CB} = \frac{WL}{12} = 144 \text{ tons in.}$$

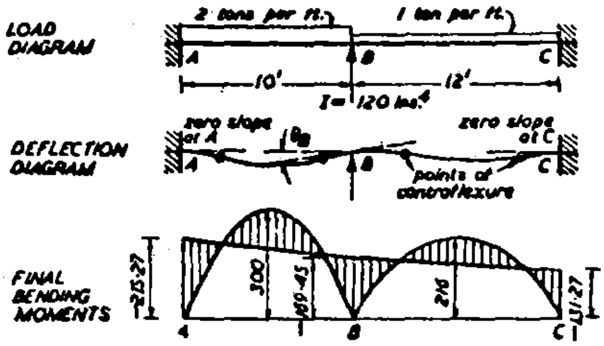


Fig. 20.50

Now $\qquad M_{BA} + M_{BC} = 0$

Hence, $2EK_1(2\theta_B + \theta_A - 3R_1) + 200 + 2EK_2(2\theta_B + \theta_C - 3R_2) - 144 = 0$

But

$$\theta_A = 0 = \theta_C$$

$$R_1 = 0 = R_2$$

$$K_1 = \frac{I}{L} = \frac{120}{120}$$

and

$$K_2 = \frac{I}{L} = \frac{120}{144}$$

Therefore substitution gives,

$$\left(2E \times \frac{120}{120} \times 2\theta_B\right) + \left(2E \times \frac{120}{144} \times 2\theta_B\right) = 144 - 200$$

so that $E\theta_B = -7.6364$

Using the basic formulae,

$$\begin{aligned}
M_{AB} &= 2EK_1(\theta_B) - 200 \\
&= (-2 \times 1 \times 7.6364) - 200 \\
&= -215.27 \text{ tons in.} \\
M_{BA} &= -M_{BC} \\
&= 2EK_1(2\theta_B) + 200 \\
&= (-2 \times 1 \times 2 \times 7.6364) + 200) \\
&= +169.45 \text{ tons in.} \\
M_{CB} &= 2EK_2(\theta_B) + 144 \\
&= (-2 \times 0.833 \times 7.6364) + 144 \\
&= 131.27 \text{ tons in.}
\end{aligned}$$

Symmetrical Portal Frames

EXAMPLE Rigid frames which are symmetrical in shape and symmetrically loaded are easy to analyse by the slope deflection method. The portal frame in Fig. 20.51 will be analysed as an example.

$$FEB_{BC} = FEM_{CB} = \frac{10 \times 24 \times 12}{12}$$
$$= 240 \text{ tons in.}$$

Let the suffixes 1, 2 and 3 be applied to the members AB, BC and CD respectively.
Then
$K_1 : K_2 : K_3 = 1 : 3 : 1$ (seeing the K values given in Fig. 20.51)

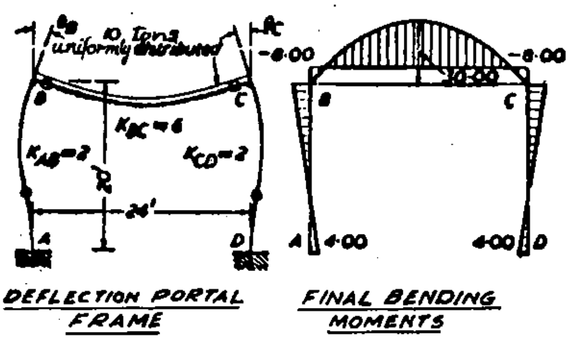


Fig. 20.51

Being symmetrical, the frame does not sway under load. In addition, it is assumed that the lengths of the members do not change.

Hence,
$$R_1 = 0 = R_2 = R_3$$
$$\theta_B = -\theta_C$$

As A and D are fixed in direction as well as position,

$$\theta_A = 0 = \theta_D$$

Now
$$M_{BA} + M_{BC} = 0,$$

i.e.,
$$2EK_1(2\theta_B) + 2EK_2(2\theta_B + \theta_C) - FEM_{BC} = 0$$
$$2EK_1(2\theta_B) + 2E(3K_1)(\theta_B) - 240 = 0$$
$$10EK_1\theta_B = 240$$
$$EK_1\theta_B = 24$$

Using the fundamental formula

$$M_{AB} = 2EK_1(\theta_B)$$
$$= 48 \text{ tons in.} = 4 \text{ tons ft.}$$

and
$$M_{BA} = 2EK_1(2\theta_B)$$
$$= 96 \text{ tons in.} = 8 \text{ tons ft.}$$

Similarly, from the condition that $M_{CD} + M_{CB} = 0$, proceeding as in above we get,

$$M_{CD} = -8 \text{ tons ft}$$
$$M_{DC} = -4 \text{ tons ft}$$

The final bending moment diagram, after including the 'simply supported' moments, is drawn in Fig. 20.51.

Asymmetrical Portal Frames

EXAMPLE As the frame shown in Fig. 20.52 is not symmetrical it is less Easy to analyse than the previous example.

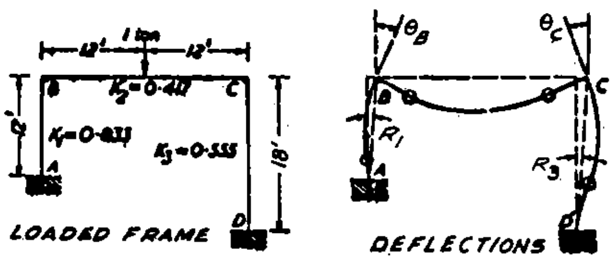


Fig. 20.52

Let the suffixes 1, 2 and 3 be applied to the members AB, BC and CD respectively.
From the data there is no slope at A and D, i.e.,

$$\theta_A = 0 = \theta_D$$

B and C are at the same height and under load it is assumed that they retain their positions relative to one another, i.e.,

$$R_2 = 0$$

$FEM_{BC} = FEM_{CB} = 36$ tons in.
The conditions of equilibrium require that,

$$\Sigma M_B = 0, \Sigma M_C = 0, \text{ and } \Sigma H = 0$$

Hence,
$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CD} = 0$$

and
$$\frac{M_{AB} + M_{BA}}{L_1} + \frac{M_{CD} + M_{DC}}{L_3} = 0$$

The side-sway of B equals that of C. Hence,

$$R_1 L_1 = R_3 L_3$$

Using the basic formulae

$$2EK_1(2\theta_B - 3R_1) + 2EK_2(2\theta_B + \theta_C) - 36 = 0$$
$$2EK_2(2\theta_C - \theta_B) + 36 + 2EK_3(2\theta_C - 3R_3) = 0,$$

and

$$\frac{2EK_1(\theta_B - 3R_1) + 2EK_1(2\theta_B - 3R_1)}{L_1} + \frac{2EK_3(2\theta_C - 3R_3) + 2EK_3(\theta_C - 3R_3)}{L_3} = 0$$

Now $K_1 = 0.833, K_2 = 0.417, K_3' = 0.555$
and $R_1 = 1.5R_3$
Hence,

$$2\times0.833E(2\theta_B - 4.5R_3) + 2\times0.417E(2\theta_B + \theta_C) - 36 = 0$$

$$2 \times 0.417E(2\theta_C + \theta_B) + 2 \times 0.555E(2\theta_C - 3R_3) + 36 = 0, \text{ and}$$

$$\frac{2 \times 0.833E}{12}(3\theta_B - 9R_3) + \frac{2 \times 0.555E}{18}(3\theta_C - 6R_3) = 0$$

That is

$$5E\theta_B + 0.833E\theta_C - 7.5ER_3 - 36 = 0 \quad \text{(i)}$$

$$0.833E\theta_B + 3.888E\theta_C - 3.33ER_3 + 36 = 0 \quad \text{(ii)}$$

$$\text{and } 0.417E\theta_B + 0.185E\theta_C - 1.62ER_3 = 0 \quad \text{(iii)}$$

Multiplying (ii) by 2.25,

$$1.875E\theta_B + 8.75E\theta_C - 7.5R_3 + 81 = 0 \quad \text{(iv)}$$

Subtracting (iv) from (i),

$$3.125E\theta_B - 7.917E\theta_C - 117 = 0 \quad \text{(v)}$$

Multiplying (iii) by 4.63.

$$1.93E\theta_B + 0.858E\theta_C - 7.5ER_3 = 0 \quad \text{(vi)}$$

Subtracting (vi) from (i)

$$3.07E\theta_B - 0.025E\theta_C - 36 = 0 \quad \text{(vii)}$$

Multiplying (vii) by 1.018,

$$3.125E\theta_B - 0.025E\theta_C - 36.65 = 0 \quad \text{(viii)}$$

Subtracting (viii) from (v),

$$-7.892E\theta_C - 80.35 = 0$$

Whence, $E\theta_C = -10.181$
$E\theta_B = 11.643$
$ER_3 = 1.831$
and $ER_1 = 2.746$

Using these values in the basic formulae

$$\begin{aligned}
M_{AB} &= 2EK_1(\theta_B - 3R_1) \\
&= 2 \times 0.833(11.643 - 8.238) \\
&= 5.675 \text{ tons in} = 0.47 \text{ tons ft} \\
M_{BA} &= 2EK_1(2\theta_B - 3R_1) \\
&= 2 \times 0.833(23.286 - 8.238) \\
&= 25.08 \text{ tons in.} = 2.09 \text{ tons ft} \\
M_{CD} &= 2EK_3(2\theta_C - 3R_3) \\
&= 2 \times 0.555(-20.362 - 5.493) \\
&= -28.715 \text{ tons in.} = -2.39 \text{ tons ft} \\
M_{DC} &= 2EK_3(\theta_C - 3R_3) \\
&= 2 \times 0.555(-10.181 - 5.493) \\
&= -17.414 \text{ tons in.} = -1.45 \text{ tons ft}
\end{aligned}$$

The final BM diagram is shown in Fig. 20.53 after including the 'simply-supported moments'

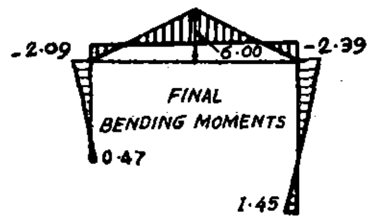


Fig. 20.53

20.7 'FLEXIBILITY' METHOD OF ANALYSIS (δ_{ik} OR V_{rs} METHOD)

This method of elastic analysis developed by Müller-Breslau and his contemporaries, known as the δ_{ik} method, provided continental engineers with a most powerful tool in structural analysis many years before it was accepted in Britain. With its basis in Influence-Coefficients, this method is becoming increasingly popular, particularly for computerised calculations. Jenkins' work on the use of matrices in the analysis of shells probably paved the way for considerable research in the use of this algebraic form in the analysis of beam and frame structures.

The result of all this work has been to produce a most versatile and comprehensive aid to structural analysis, a synthesis of δ_{ik} and matrix algebra. Briefly, the method may be explained physically as follows:

For an n times statically indeterminate structure behaving elastically under a given loading, the n compatibility equations of elastic stability are

$$V_{10} + V_{11}p_1 + V_{12}p_2 + \ldots + V_{1n}p_n = 0$$

$$V_{20} + V_{21}p_1 + V_{22}p_2 + \ldots + V_{2n}p_n = 0$$

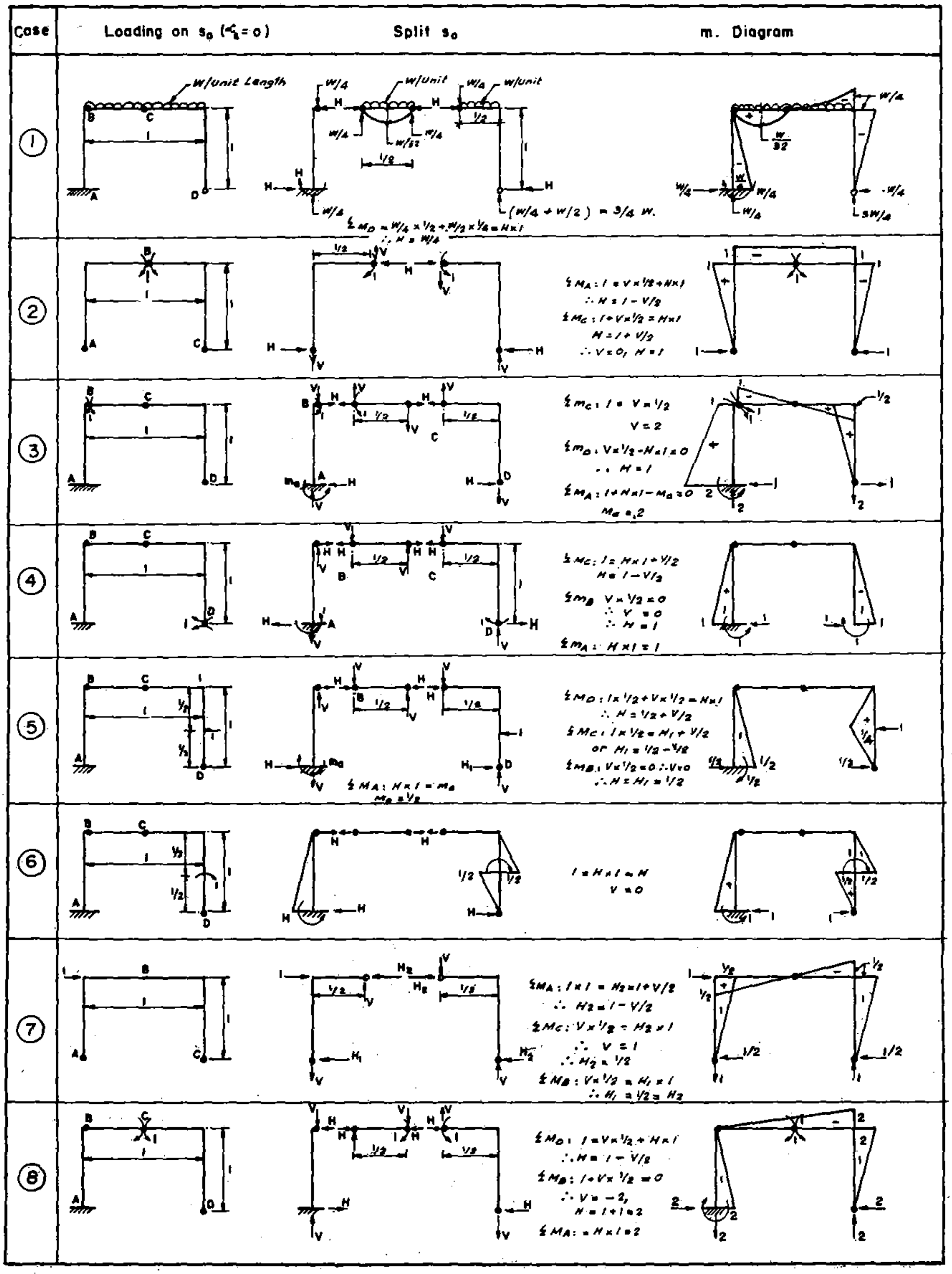


Fig. 20.55

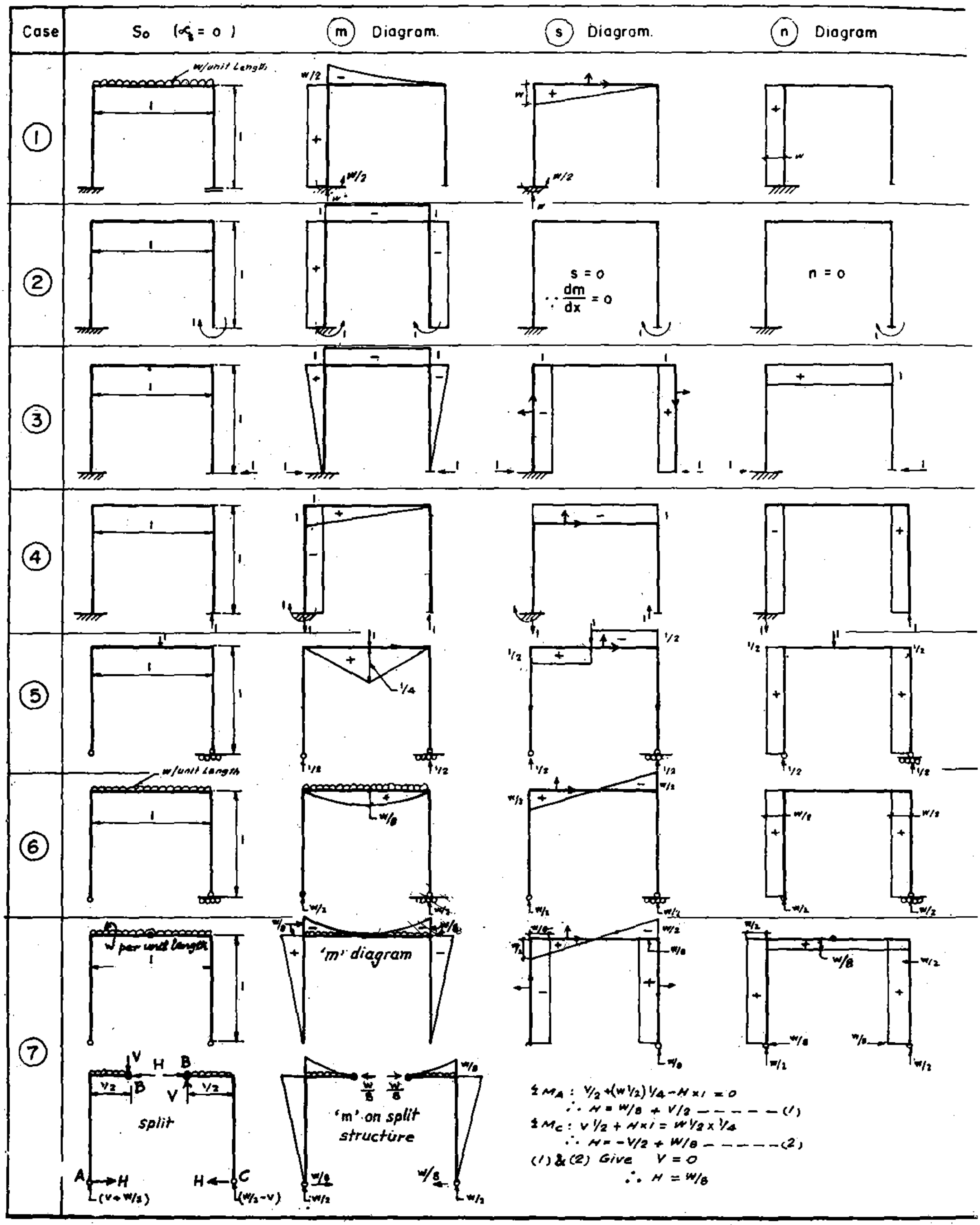

Case
So (α = 0)
(m) Diagram.
(s) Diagram.
(n) Diagram
w/unit Length
s = 0
∴ dm/dx = 0
n = 0
w/unit Length
w per unit length
'm' diagram
split
'm' on split structure
ΣM_A : V/2 + (wl/2) l/4 − H × l = 0
∴ H = wl/8 + V/2 ------ (1)
ΣM_C : V l/2 + H × l = wl/2 × l/4
∴ H = −V/2 + wl/8 ------ (2)
(1) & (2) Give V = 0
∴ H = wl/8

Fig. 20.56

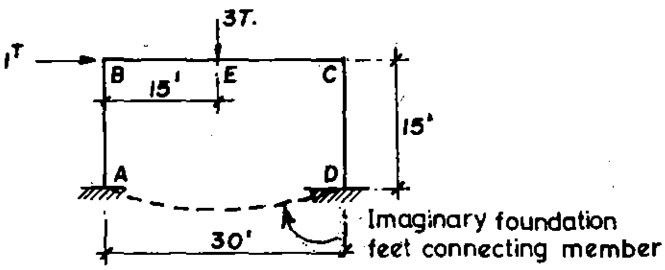


Fig. 20.57

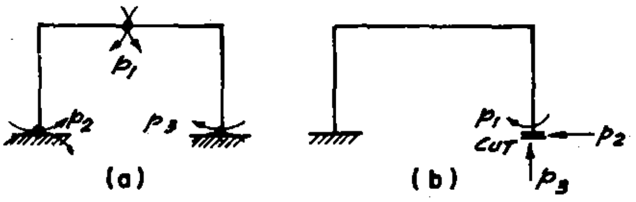


Fig. 20.58

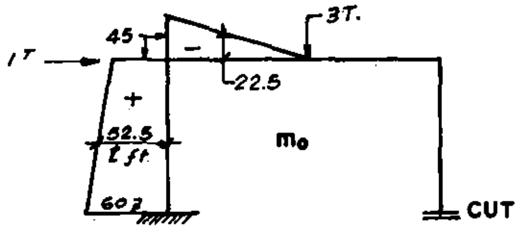


Fig. 20.59

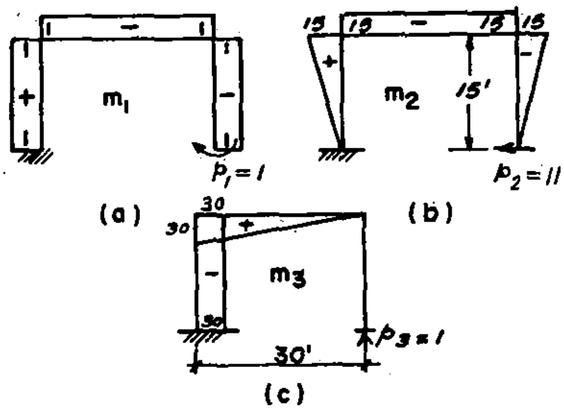


Fig. 20.60

$V_{10}, V_{20}, V_{30}, V_{11}, V_{12}, V_{13}, V_{21}, V_{22}, V_{23}, V_{31}, V_{32}$, and V_{33}.

for: $V_{10} + V_{11}p_1 + V_{12}p_2 + V_{13}p_3 = 0$

$$V_{20} + V_{21}p_1 + V_{22}p_2 + V_{23}p_3 = 0$$

$$V_{30} + V_{31}p_1 + V_{32}p_2 + V_{33}p_3 = 0$$

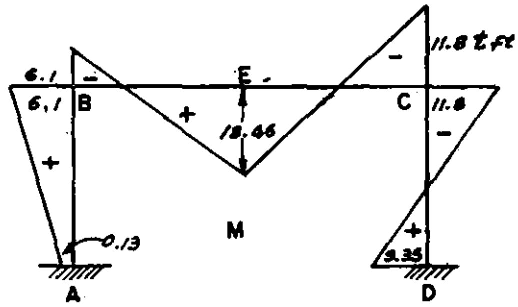


Fig. 20.61 *Final BMD*

Using Simpson's Rule (explained earlier), and retaining all units in Tons and Feet, we have,

$$V_{10} = \frac{1}{EI}\int m_1 m_0 ds = \frac{1}{EI}\Sigma m_1 m_0 ds$$

$$= \frac{1}{EI}\left[\frac{15/2}{3}\{60 \times 1 + 45 \times 1 + 52.5 \times 1\}\right.$$

$$+ \frac{15/2}{3}\{-1 \times -45 + 0 + 4 \times$$

$$(-22.5) \times (-1)\} + 0]$$

$$= 1125/EI$$

$$V_{20} = \frac{1}{EI}\int m_2 m_0 ds = \frac{1}{EI}\Sigma m_2 m_0 ds$$

$$= \frac{I}{EI}\left[\frac{15/2}{3}\{0 + 15 \times 45 + 4 \times 15/2 \times 52.5\}\right.$$

$$+ \frac{15/2}{3}\{-15 \times -45 + 0 + 4 \times (-15) \times$$

$$(-22.5)\} + 0]$$

$$= 10,687.5/EI$$

$$V_{30} = \frac{1}{EI}\int m_3 m_0 ds = \frac{1}{EI}\Sigma m_3 m_0 ds$$

$$= \frac{1}{EI}\left[\frac{15/2}{3}\{-30 \times 60 + (-30) \times 45 + 4 \times\right.$$

$$(-30) \times 52.5\} + \frac{15/2}{3}\{30 \times (-45) + 0 + 4 \times$$

$$22.5 \times (-22.5)\} + 0]$$

$$= -32062.5/EI$$

$$V_{11} = \frac{1}{EI}\int m_1 m_1 ds = \frac{1}{EI}\Sigma m_1 m_1 ds$$

$$= \frac{1}{EI}\left[\frac{15/2}{3}\{1 \times 1 + 1 \times 1 + 4 \times 1 \times 1\}\right. +$$

$$\frac{30/2}{3}\{-1\times-1+(-1)\times(-1)+4(-1)\times(-1)\}+\frac{15/2}{3}\{-1\times-1+(-1)\times(-1)+4(-1)\times(-1)\}]$$

$$=60/EI$$

$$V_{12}=\frac{1}{EI}\int m_1m_2ds=\frac{1}{EI}\Sigma m_1m_2ds$$

$$=\frac{1}{EI}\left[\frac{15/2}{3}\left\{0+1\times 15+4\times 1\times\frac{15}{2}\right\}+\frac{30/2}{3}\{-1\times-15+(-1)\times(-15)+4(-1)\times(-15)\}+\frac{15/2}{3}\left\{0+(-15)\times(-1)+4(-1)\times\left(-\frac{15}{2}\right)\right\}\right]$$

$$=675/EI$$

$$V_{13}=\frac{1}{EI}\int m_1m_3ds=\frac{1}{EI}\Sigma m_1m_3ds$$

$$=\frac{1}{EI}\left[\frac{15/2}{3}\{-30\times 1+(-30)\times 1+4(-30)\times 1\}+\frac{30/2}{3}\{30\times(-1)+0+4\times\frac{30}{2}\times(-1)\}+0\right]$$

$$=-900/EI$$

$V_{21}\equiv V_{12}$, already found above.

$$V_{22}=\frac{1}{EI}\int m_2m_2ds=\frac{1}{EI}\Sigma m_2m_2ds$$

$$=\frac{1}{EI}\left[\frac{15/2}{3}\{0+15\times 15+4\times\frac{15}{2}\times\frac{15}{2}\}\times 2+\frac{30/2}{3}\{(-15)(-15)+(-15)\times(-15)+4(-15)(-15)\}\right]$$

$$=9000/EI$$

$$V_{23}=\frac{1}{EI}\int m_2m_3ds=\frac{1}{EI}\Sigma m_2m_3ds$$

$$=\frac{1}{EI}\left[\frac{15/2}{3}\{0+15(-30)+4(-30)\times\frac{15}{2}\}+\frac{30/2}{3}\{30(-15)+0+4(30/2)\times(-15)\}+0\right]$$

$$=10125/EI$$

$V_{31}\equiv V_{13}$ already found above.

$V_{32}\equiv V_{23}$ already found above.

$$V_{33}=\frac{1}{EI}\int m_3m_3ds=\frac{1}{EI}\Sigma m_3m_3ds$$

$$=\frac{1}{EI}\left[\frac{15/2}{3}\{(-30)\times(-30)+(-30)\times(-30)+4(-30)(-30)\}+\frac{30/2}{3}\{30\times 30+0+4\times\frac{30}{2}\times\frac{30}{2}\}+0\right]$$

$$=22500/EI$$

Step 4 Substitute the values of influence coefficients in the three compatibility equations described in step 3 above:

$$1125+60p_1+675p_2+(-900)p_3=0$$
$$10,687.5+675p_1+9000p_2+(-10,125)p_3=0$$
$$-32,062.5+(-900)p_1+(-10,125)p_2+22500p_3=0$$

(Note that $1/EI$ is a common factor, hence removed.) Solving these three simultaneous equations, we get:

$$p_1=-9.35$$
$$p_2=1.41$$
$$p_3=1.6875$$

Step 5 Evaluate bending moment at any section from $m_1=m_o+(m_1p_1+m_2p_2+m_3p_3)$ at the section

$$m_{t_A}=60+1\times(-9.35)+0+(-30)\times 1.6875=0.13\text{ t ft}$$

$$m_{t_{BB-A}}=45+1\times(-9.35)+15\times 1.41+(-30)\times 1.6875=6.1\text{ t ft}$$

$$m_{t_{CC-B}}=0+(-1)(-9.35)+(-15)\times 1.41+0=-11.8\text{ t ft}$$

$$m_{t_{CC-D}}=m_{t_{CC-B}}=-11.8\text{ t ft}$$

$$m_{t_D}=0+(-1)(-9.35)+0+0=-9.35\text{ t ft}$$

$$m_{t_E}=0+(-1)(-9.35)+(-15)\times 1.41+\frac{30}{2}\times 1.6875=13.46\text{ t ft}$$

The final BMD is shown in Fig. 20.61.

Check

If computations are all right then $\int\frac{Mm_1}{EI}ds\equiv\int\frac{Mm_2}{EI}ds\equiv\int\frac{Mm_3}{EI}ds\equiv 0$. Check any one of the three, let us say the first one,

$$\frac{1}{EI}\int Mm_1 ds = \frac{1}{EI}\begin{bmatrix} AB: & \frac{15/2}{3}\left\{0.13\times 1+6.1\times 1+4\times 1\times\left(\frac{6.1+0.13}{2}\right)\right\}+ \\ BE: & \frac{15/2}{3}\left\{-6.1\times(-1)+13.46\times(-1)+4\times(-1)\times\left(\frac{13.46-6.1}{2}\right)\right\}+ \\ EC: & \frac{15/2}{3}\left\{13.46\times(-1)+(-11.8)(-1)+4\times(-1)\times\left(\frac{13.46-11.8}{2}\right)\right\}+ \\ CD: & \frac{15/2}{3}\left\{9.35\times(-1)+(11.8)(-1)+4\times(-1)\times\left(\frac{-11.8+9.35}{2}\right)\right\} \end{bmatrix}$$

$=-2.5/EI \simeq 0$, hence computations practically OK.

(Very slight deviation from zero indicates slight inaccuracy in the mensuration work done above.)

Note that total shear and thrust at any section can be easily estimated from $s_t = s_0 + s_1p_1 + s_2p_2 + s_3p_3$ and $n_1 = n_0 + n_1p_1 + n_2p_2 + n_3p_3$ after drawing the $s_1, s_2, s_3, s_0, n_1, n_2, n_3$ and n_0 diagrams as explained earlier.

Deflection δ and Rotation θ at Any Point

As per Castigliano, partial derivative of strain energy with respect to a force gives displacement (or deflection) due to the force, at the point of its application and in the direction of force

i.e. $$\delta = \frac{\partial U}{\partial P}$$

And, partial derivative of strain energy with respect to a moment gives rotation due to the moment, at the point of its application, and in direction of moment

i.e. $$\theta = \frac{\partial U}{\partial M}$$

If in any structure S_α deflection δ at any point in any direction (usually vertical or horizontal) is to be found, then apply a load P at that point in the direction of δ. With m_δ being the BM diagram on S_0 due to unit P(S_0 being the statically made determinate structure), the total BM at that point = $M = (m_0 + m_1p_1 + m_2p_2 + \ldots + m_np_n) + m_\delta P$, so that $\dfrac{\partial M}{\partial P} = m_\delta$

$$U = \int \frac{M^2}{2EI}ds,$$

$$\delta_{\text{due to}P} = \frac{\partial U}{\partial P} = \int \frac{M}{EI}\frac{\partial M}{\partial P}ds = \int \frac{M}{EI}m_\delta ds$$

with $P = 0$ deflection in the structure at the point where it was applied, is $\left.\right\}\ \delta = \displaystyle\int \frac{Mm_\delta}{EI}ds$

where $M = (m_0 + m_1p_1 + m_2p_2 + \ldots + m_np_n)$ only.

Similarly, if rotation θ at any point in any direction in $S\alpha$ is to be found, apply a moment M' at the point in that direction. With m_θ = BMD on S_0 due to unit M', the total BM at that point = M = $(m_0 + m_1p_1 + m_2p_2 + m_3p_3 + \ldots + m_np_n) + m_\theta M'$, so that $\dfrac{\partial M}{\partial M'} = m_\theta$.

$$U = \int \frac{M^2}{2EI}ds,$$

$$\theta_{\text{due to}M'} = \frac{\partial U}{\partial M'} = \int \frac{M}{EI}\frac{\partial M}{\partial M'}ds = \int \frac{M}{EI}m_\theta ds$$

with $M' = 0$ rotation in the structure at the point where M' was applied, is $\left.\right\}\ \theta = \displaystyle\int \frac{Mm_\theta}{EI}ds$

where $M = m_o + m_1p_1 + m_2p_2 + \ldots + m_np_n)$ only.

Hence

(i) Deflection at any point is $\delta = \displaystyle\int \frac{Mm_\delta}{EI}ds$
where m_δ = BMD on S_0 due to a unit load applied at the point of δ, in direction of δ.

(ii) Rotation at any point is $\theta = \displaystyle\int \frac{Mm_\theta}{EI}ds$
where m_θ = BMD on S_0 due to a unit moment applied at the point of θ, in direction of θ.

EXAMPLE A steel tube having outside diameter 2 in., bore 1 1/2 in., is bent into a quadrant of 6 ft radius. One end is rigidly attached to horizontal base plate to which a tangent to that end is perpendicular, and the free end supports a load of 100 lb. Determine the vertical and horizontal deflections of the free end under this load; $E = 30 \times 10^6$ lb./sq. in

$$I = (\pi/64)(2^4 - 1.5^4) = (\pi/64)(4-2.25)(4+2.25) = 0.537 \text{ in.}^4$$

$x = 72\sin\theta$ (Fig. 20.62)

$y = 72(1-\cos\theta)$

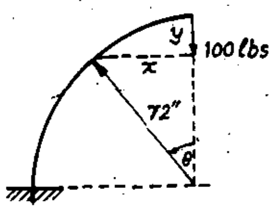


Fig. 20.62

$$M = 100x = 100 \times 72 \sin\theta$$
$$ds = 72 d\theta \text{ (from } R\theta)$$

$m_{\delta_v} = 1 \cdot x = x$, $\therefore$ Vertical deflection $= \int \dfrac{Mm_{\delta_v}}{EI} ds$

$$= \int Mxds/EI = \int_0^{\pi/2} = \frac{100 \times 72^3 \sin^2\theta d\theta}{30 \times 10^6 \times 0.537}$$
$$= 2.32 \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta$$
$$= 2.32 \times \pi/4$$
$$= 1.82 \text{ in.}$$

$m_{\delta_H} = 1 \cdot y = y$, $\therefore$ Horizontal deflection $= \int \dfrac{Mm_{\delta_H}}{EI} ds$

$$= \int Myds/EI = \frac{100 \times 72^3}{30 \times 10^6 \times 0.537}$$

$$\int_0^{\pi/2} \sin\theta(1 - \cos\theta) d\theta$$
$$= 2.32[-\cos\theta + 1/4 \cos 2\theta]_0^{\pi/2}$$
$$= 2.32 \times 1/2$$
$$= 1.16 \text{ in.}$$

EXAMPLE Find moments set up in the frame shown in Fig. 20.63(a) (fixed at A and pinned at D), the horizontal deflection δ_C at C and the rotation θ_D at D. Assume EI is constant throughout.

In any case it will be necessary to first solve the structure and draw the bending moment diagram M whereafter mating this M-diagram with m_δ and $1/EI$ diagrams will give deflection and mating it with m_θ and $1/EI$ diagrams will give rotation

$$\delta_C = \int \frac{Mm_\delta}{EI} ds$$
$$\theta_D = \int \frac{Mm_\theta}{EI} ds$$

where $m\delta$ is the BMD on the released (i.e., statically made determinate) structure S_0 due to a unit load applied at C in the direction of the required deflection (horizontal in the present case).

m_θ is the BMD on S_0 due to a unit couple applied at D in the direction of the required rotation.

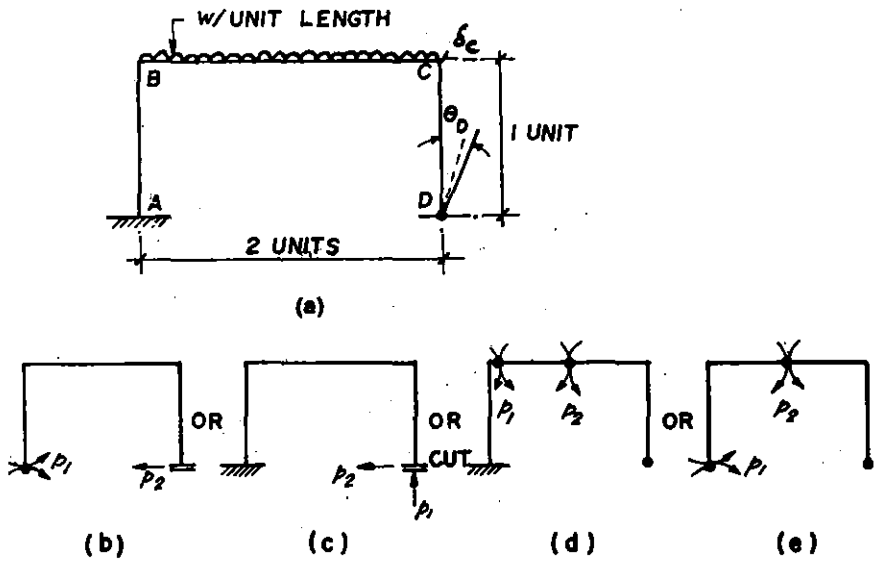


Fig. 20.63

So first we analyse the structure to obtain M, the final BMD, due to the applied loading.

$$\alpha_s = 3(M - N + 1) - R = 3(4 - 4 + 1) - 1 = 2$$

The 2 releases could be introduced in any of the 4 ways shown in Fig. 20.63(b) to 20.63(e) [easiest one is system (c)]. However, for the sake of understanding how to draw the more complicated statical diagrams m_0, m_1 and m_2, let us select the release system shown in Fig. 20.63(d). The corresponding m_0, m_1 and m_2 diagrams on S_0, are shown in Fig. 20.64.

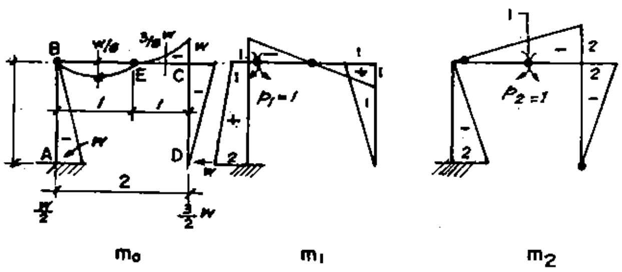


Fig. 20.64

Two compatibility equations are:

$$V_{10} + V_{11}p_1 + V_{12}p_2 = 0$$

and
$$V_{20} + V_{21}p_1 + V_{22}p_2 = 0$$

Now first evaluate the involved influence coefficients

$$V_{10} = \frac{1}{EI}\int m_1 \cdot m_0 \, ds = \frac{1}{EI}\Sigma m_1 m_0 \, ds$$

$$= \frac{1}{EI}\Bigg[\underbrace{\frac{1/2}{3}\left\{-2w + 0 - 4 \times \frac{w}{2} \times \frac{3}{2}\right\}}_{AB} + \overbrace{\frac{1/2}{3}\left\{0 + 0 - 4 \times \frac{w}{8} \times \frac{1}{2}\right\}}^{BE}$$

$$+ \overbrace{\frac{1/2}{3}\left\{0 - 4 \times \frac{3}{8}w \times \frac{1}{2} - w\right\}}^{EC}$$

$$+ \overbrace{\frac{1/2}{3}\left\{-w + 0 - 4 \times \frac{w}{2} \times \frac{1}{2}\right\}}^{CD}\Bigg]$$

$$= -\frac{9w}{6EI}$$

$$V_{20} = \frac{1}{EI}\int m_2 m_0 \, ds = \frac{1}{EI}\Sigma m_2 m_0 \, ds$$

$$= \frac{1}{EI}\Bigg[\overbrace{\frac{1/2}{3}\left\{2w + 0 + 4 \times \frac{w}{2} \times 1\right\}}^{AB}$$

$$+ \underbrace{\frac{1/2}{3}\left\{0 + 0 - 4 \times \frac{w}{8} \times \frac{1}{2}\right\}}_{BE}$$

$$+ \underbrace{\frac{1/2}{3}\left\{0 + 2w + 4 \times \frac{3}{8}w \times \frac{3}{2}\right\}}_{EC}$$

$$+ \underbrace{\frac{1/2}{3}\left\{2w + 0 + 4 \times \frac{w}{2} \times 1\right\}}_{CD}\Bigg]$$

$$= \frac{12w}{6EI}$$

$$V_{11} = \frac{1}{EI}\int m_1 m_1 \, ds = \frac{1}{EI}\Sigma m_1 m_1 \, ds$$

$$= \frac{1}{EI}\Bigg[\overbrace{\frac{1/2}{3}\left\{2 \times 2 + 1 \times 1 + 4 \times \frac{3}{2} \times \frac{3}{2}\right\}}^{AB}$$

$$+ \underbrace{\frac{2/2}{3}\{1 \times 1 + 0 + 1\}}_{BC}$$

$$+ \underbrace{\frac{1/2}{3}\left\{1 + 0 + 4 \times \frac{1}{2} \times \frac{1}{2}\right\}}_{CD}\Bigg]$$

$$= \frac{20}{6EI}$$

$$V_{12} = \frac{1}{EI}\int m_1 m_2 \, ds = \frac{1}{EI}\Sigma m_1 m_2 \, ds$$

$$= \frac{1}{EI}\Bigg[\underbrace{\frac{1/2}{3}\left\{-4 + 0 - 4 \times \frac{3}{2}\right\}}_{AB} + \overbrace{\frac{2/2}{3}\{0 + 0 - 2\}}^{BC}$$

$$+ \underbrace{\frac{1/2}{3}\left\{-2 + 0 - 4 \times 1 \times \frac{1}{2}\right\}}_{CD}\Bigg]$$

$$= -\frac{18}{6EI} \equiv V_{21}$$

$$V_{22} = \frac{1}{EI}\int m_2 m_2 \, ds = \frac{1}{EI}\Sigma m_2 m_2 \, ds$$

$$= \frac{1}{EI}\left[\frac{1/2}{3}\{2\times 2+0+4\times 1\times 1\}\right.$$
$$+\frac{2/2}{3}\{0+2\times 2+4\times 1\times 1\}$$
$$\left.+\frac{1/2}{3}\{2\times 2+0+4\times 1\times 1\}\right]$$
$$=\frac{32}{6EI}$$

Substituting in the aforementioned two compatibility equations:

$$\left.\begin{aligned} -\frac{9w}{6}+\frac{20}{6}p_1-\frac{18}{6}p_2 &= 0\\ \frac{12w}{6}-\frac{18}{6}p_1+\frac{32}{6}p_2 &= 0\end{aligned}\right\}\quad \text{whence} \quad p_1=\frac{18w}{79},\quad p_2=\frac{-39w}{158}$$

Hence total moments:

$$m_{tA} = -w+2\times\frac{18w}{79}-2\times\frac{-39w}{158}$$
$$= -\frac{4w}{79}$$
$$m_{tB_{B-A}} = 0+1\times\frac{18w}{79}+0=\frac{18w}{79}$$
$$m_{t_C} = -w+1\times\frac{18w}{79}-2\times-\frac{39w}{158}=\frac{22w}{79}$$
$$m_{t_D} = 0+0+0=0$$
$$m_{t_E} = 0+0+(-1)\times\frac{-39w}{158}$$
$$=\frac{39w}{158}$$

The final BMD is therefore as shown in Fig. 20.65, and this represents the M diagram for the subsequent matings in the deflection and rotation computations as shown below.

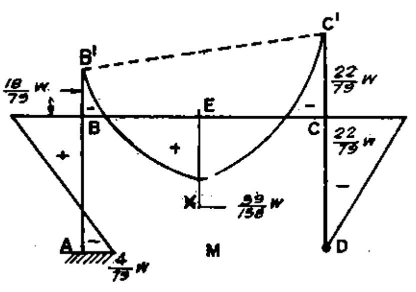


Fig. 20.65 *Final BMD*

Deflection at C in horizontal direction (δ_c) Figure 20.66 shows the m_δ diagram due to a unit horizontal force applied at C in horizontal direction, in the released structure S_0 (a simpler release system is considered here, the one shown in Fig. 20.63(c), hence:

$$\delta_c = \int\frac{Mm\delta}{EI}ds=\frac{1}{EI}\Sigma Mm\delta\, ds$$
$$=\frac{1}{EI}\left[\frac{1/2}{3}\left\{1\times\left(\frac{-4}{79}w\right)\right.\right.$$
$$\left.\left.+\,0+4\times\frac{1}{2}\times\left(\frac{18-4}{79\times 2}\right)w\right\}\right]$$
$$=\frac{5w}{237EI}\ \text{in ft. units if } w \text{ in } t/ft \text{ and}$$

E and I in t and ft. units.

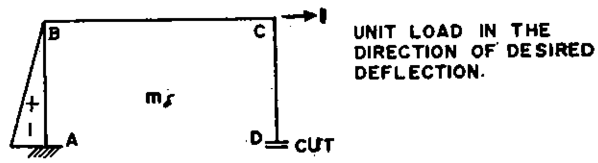


Fig. 20.66

Rotation at D in clock-wise direction: θ_D Figure 20.67 shows the m_θ diagram due to a unit couple applied at D in the clockwise direction, in the released structure (release system as in Figure 20.63(c), hence:

$$\theta_D=\int\frac{Mm_\theta}{EI}ds=\frac{1}{EI}\Sigma Mm_\theta\, ds$$
$$=\frac{1}{EI}\left[\frac{1/2}{3}\left\{1\times\left(\frac{-4}{79}w\right)+1\times\frac{18w}{79}\right.\right.$$
$$\left.+\,4\times 1\times\left(\frac{18-4}{79\times 2}\right)w\right\}$$
$$+\frac{2/2}{3}\left\{1\times\frac{18w}{79}+1\times\frac{22w}{79}\right.$$
$$\left.+\,4\times 1\times\left(\frac{18+22}{79\times 2}\right)w\right\}\ \text{with } BB'C'C$$
$$-\frac{2/2}{3}\{-1\times 0-1\times 0+4\times(-1)\times$$
$$\left.\left(\frac{39}{158}w+\frac{20}{79}w\right)\right\}\ \text{with parabola } B'XC'$$
$$\left.+\frac{1/2}{3}\left\{0+\frac{22}{79}w+4\times 1\times\frac{22}{79\times 2}w\right\}\right]$$
$$=\frac{16w}{237EI}\ \text{radians (if } w \text{ in t/ft, and } E \text{ and } I \text{ in}$$

t and ft. units, respectively)

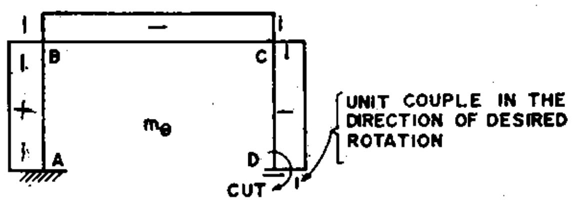


Fig. 20.67

COMBINED FLEXURE AND DIRECT STRESS IN STRUCTURES

• ***Trussed Beam***

A structure may have its ability to resist the effects of external loading conditions reinforced by the inclusion of some element or elements functioning in a different way from the original structure. As an example, a simply supported beam may be strengthened by adding a simple truss (Fig. 20.68). The added elements can take direct stress only, and the beam, while having less bending moment to resist, now has a compressive thrust as well. An extension of this early reinforcement is that which allows an initial prestress to be placed in the truss causing reverse bending to occur in the beam, see Fig. 20.69.

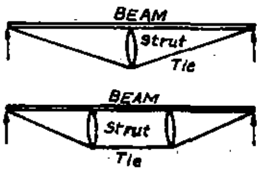


Fig. 20.68

Fig. 20.69

EXAMPLE Consider a timber beam, which has had its live load increased beyond that which it alone can safely carry. It is proposed that it be strengthened and converted to a trussed beam using steel components. Find the bending moments and forces in the resulting structures shown in Fig. 20.70.

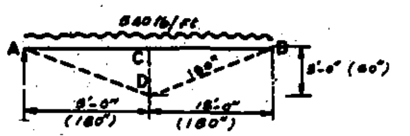


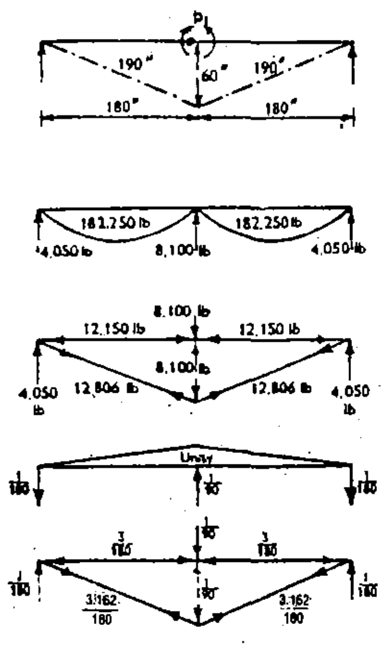
Fig. 20.70

$$E_T = 1.5 \times 10^6 \text{ p.s.i. (timber)} \qquad E_S = 30 \times 10^6 \text{ p.s.i. (Steel)}$$

$$I_T = 144 \text{ in}^4 \qquad A_{ST} = 0.5 \text{ in}^2 \qquad \frac{E_S}{E_T} = 20$$

$$A_T = 32 \text{ in}^2 \qquad A_{SC} = 2.0 \text{ in}^2$$

$$V_{10} = \int_A^B \frac{m_1 m_0}{EI}\,dx + \Sigma\frac{n_1 n_0 L}{EA} \text{ (see Fig. 20.71)}$$



Primary structure S_0, with one needed release introduced (biaction p_1)

diagram m_0 (BMD due to applied loading on released structure S_0)

diagram n_0 (due to m_0)

diagram m_1 (due to unit p_1 on S_0)

diagram n_1 (due to unit p_1 on S_0, i.e., due to m_1)

Fig. 20.71

A common factor may be removed from the term $1/E$, and the modular ratio used, but actual value for $1/I$ and $1/A$ must be used in the calculations.

$$\int_A^B \frac{m_1 m_0}{I}\,dx = -2 \times \frac{180}{3} \times \frac{182{,}250}{144} = -151{,}800$$

$$\text{and } \Sigma\frac{n_1 n_0 L}{20A} =$$

$$\left(\begin{array}{c}\text{one}\\\text{strut}\end{array}\right) 8{,}100 \times \frac{1}{90} \times \frac{60}{2 \times 20} = + \quad 135$$

$$+\left(\begin{array}{c}\text{two}\\\text{ties}\end{array}\right) \frac{2 \times 12{,}806 \times 3{,}162 \times 190}{0.5 \times 180 \times 20} = + \quad 8{,}610$$

$$+\left(\begin{array}{c}\text{two}\\\text{beams}\end{array}\right) \frac{2 \times 12{,}150 \times 3 \times 180}{180 \times 32} = + \quad 2{,}278$$

$$V_{10} = -140{,}777$$

Table 20.1 *Area Integrals*

$\int M_i \; M_k \, dx = (\text{tabulated value}) \times l$

Nr.	TYP (l)	k — k (rectangle)	triangle, k at right	K_1 — K_2 (trapezoid)	k — $-k$	k — $-\frac{k}{2}$	K, αl / βl (triangle)	$\int j^2 \, dx$
1	j — j	jK	$\frac{1}{2} jK$	$\frac{1}{2} j(K_1+K_2)$	0	$\frac{1}{4} jK$	$\frac{1}{2} jK$	j^2
2	triangle, j at right	$\frac{1}{2} jK$	$\frac{1}{3} jK$	$\frac{1}{6} j(K_1+2K_2)$	$-\frac{1}{6} jK$	0	$\frac{1}{6} jK(1+\alpha)$	$\frac{1}{3} j^2$
3	triangle, j at left	$\frac{1}{2} jK$	$\frac{1}{6} jK$	$\frac{1}{6} j(2K_1+K_2)$	$\frac{1}{6} jK$	$\frac{1}{4} jK$	$\frac{1}{6} jK(1+\beta)$	$\frac{1}{3} j^2$
4	j_1 — j_2	$\frac{1}{2} K(j_1+j_2)$	$\frac{1}{6} K(j_1+2j_2)$	$\frac{1}{6} [j_1(2K_1+K_2)$ $+ j_2(K_1+2K_2)]$	$\frac{1}{6} K(j_1-j_2)$	$\frac{1}{4} j_1 K$	$\frac{1}{6} K[j_1(1+\beta)$ $+ j_2(1+\alpha)]$	$\frac{1}{3}(j_1^2 +$ $+ j_1 j_2 + j_2^2)$
5	j — $-j$	0	$-\frac{1}{6} jK$	$\frac{1}{6} j(K_1-K_2)$	$\frac{1}{3} jK$	$\frac{1}{4} jK$	$\frac{1}{6} jK(1-2\alpha)$	$\frac{1}{3} j^2$
6	j — $-\frac{j}{2}$	$\frac{1}{4} jK$	0	$\frac{1}{4} jK_1$	$\frac{1}{4} jK$	$\frac{1}{4} jK$	$\frac{1}{4} jK\beta$	$\frac{1}{4} j^2$
7	$-\frac{j}{2}$ — j	$\frac{1}{4} jK$	$\frac{1}{4} jK$	$\frac{1}{4} jK_2$	$-\frac{1}{4} jK$	$-\frac{1}{8} jK$	$\frac{1}{4} jK\alpha$	$\frac{1}{4} j^2$
8	$\frac{l}{2}$, j	$\frac{1}{2} jK$	$\frac{1}{4} jK$	$\frac{1}{4} j(K_1+K_2)$	0	$\frac{1}{8} jK$	$\frac{jK}{12\beta}(3-4\alpha^2)$	$\frac{1}{3} j^2$
9	γl, δl, j	$\frac{1}{2} jK$	$\frac{1}{6} jK(1+\gamma)$	$\frac{1}{6} j[K_1(1+\delta)$ $+ K_2(1+\gamma)]$	$\frac{1}{6} jK(1-2\gamma)$	$\frac{1}{4} jK\delta$	$\frac{jK}{6\beta\gamma}(2\gamma-\gamma^2-\alpha^2)$ $\gamma \geq \alpha$	$\frac{1}{3} j^2$
10	j Quadr Parabola	$\frac{2}{3} jK$	$\frac{1}{3} jK$	$\frac{1}{3} j(K_1+K_2)$	0	$\frac{1}{6} jK$	$\frac{1}{3} jK(1+\alpha\beta)$	$\frac{8}{15} j^2$
11	j — j Quadr Parabola	$\frac{1}{3} jK$	$\frac{1}{6} jK$	$\frac{1}{6} j(K_1+K_2)$	0	$\frac{1}{12} jK$	$\frac{1}{6} jK(1-2\alpha\beta)$	$\frac{1}{5} j^2$
12	j Quadr Parabola	$\frac{2}{3} jK$	$\frac{1}{4} jK$	$\frac{1}{12} j(5K_1+3K_2)$	$\frac{1}{6} jK$	$\frac{7}{24} jK$	$\frac{1}{12} jK(5-\alpha-\alpha^2)$	$\frac{8}{15} j^2$
13	j Quadr Parabola	$\frac{2}{3} jK$	$\frac{5}{12} jK$	$\frac{1}{12} j(3K_1+5K_2)$	$-\frac{1}{6} jK$	$\frac{1}{24} jK$	$\frac{1}{12} jK(5-\beta-\beta^2)$	$\frac{8}{15} j^2$

14	Quadr Parabola, j	$\frac{1}{3}jK$	$\frac{1}{4}jK$	$\frac{1}{12}j(K_1+3K_2)$	$-\frac{1}{6}jK$	$-\frac{1}{24}jK$	$\frac{1}{12}jK(1+\kappa+\kappa^2)$	$\frac{1}{5}j^2$
15	Quadr. Parabola, j	$\frac{1}{3}jK$	$\frac{1}{12}jK$	$\frac{1}{12}j(3K_1+K_2)$	$\frac{1}{6}jK$	$\frac{5}{24}jK$	$\frac{1}{12}jK(1+\beta+\beta^2)$	$\frac{1}{5}j^2$
16	Quadr Parabola, $-j$, $l/2$, j	$\frac{1}{6}jK$	$\frac{1}{6}jK$	$\frac{1}{6}jK_2$	$-\frac{1}{6}jK$	$-\frac{1}{12}jK$	$\frac{1}{6}jK\kappa(1+2\beta)$	$\frac{1}{5}j^2$
17	Quadr Parabola, j, $l/2$, $-j$	$\frac{1}{6}jK$	0	$\frac{1}{6}jK_1$	$\frac{1}{6}jK$	$\frac{1}{6}jK$	$\frac{1}{6}jK\beta(1+2\alpha)$	$\frac{1}{5}j^2$
18	Quadr Parabola, j_1, $l/2$, j_2, j_3	$\frac{1}{6}K(j_1+4j_2+j_3)$	$\frac{1}{6}K(2j_2+j_3)$	$\frac{1}{6}[j_1K_1+2j_2(K_1+K_2)+j_3K_2]$	$\frac{1}{6}K(j_1-j_3)$	$\frac{1}{12}K(2j_1+2j_2-j_3)$	$\frac{1}{6}K[j_1\beta+2j_2+j_3\kappa-\kappa\beta(j_1-2j_2+j_3)]$	$\frac{1}{15}[2(j_1^2+4j_2^2+j_3^2)+2j_1j_2+2j_2j_3-j_1j_3]$
19	Cubic Parabola, j	$\frac{1}{4}jK$	$\frac{1}{5}jK$	$\frac{1}{20}j(K_1+4K_2)$	$-\frac{3}{20}jK$	$-\frac{1}{20}jK$	$\frac{1}{20}jK(1+\kappa)(1+\kappa^2)$	$\frac{1}{7}j^2$
20	Cubic Parabola, j	$\frac{1}{4}jK$	$\frac{1}{20}jK$	$\frac{1}{20}j(4K_1+K_2)$	$\frac{3}{20}jK$	$\frac{7}{40}jK$	$\frac{1}{20}jK(1+\beta)(1+\beta^2)$	$\frac{1}{7}j^2$
21	Cubic Parabola, j	$\frac{1}{4}jK$	$\frac{2}{15}jK$	$\frac{1}{60}j(7K_1+8K_2)$	$-\frac{1}{60}jK$	$\frac{1}{20}jK$	$\frac{1}{20}jK(1+\alpha)\left(\frac{7}{3}-\alpha^2\right)$	$\frac{8}{105}j^2$
22	Cubic Parabola, j	$\frac{1}{4}jK$	$\frac{7}{60}jK$	$\frac{1}{60}j(8K_1+7K_2)$	$\frac{1}{60}jK$	$\frac{3}{40}jK$	$\frac{1}{10}jK(1+\beta)\left(\frac{7}{3}-\beta^2\right)$	$\frac{8}{105}j^2$

		origin, M_0, M_1, ℓ
23	m, ℓ	$\frac{1}{3}m_1(2M_0-M_1)$
24	m_0, ℓ	$\frac{1}{12}m_0(5M_0-M_1)$
25	m_1, ℓ	$\frac{1}{4}m_1(M_0-M_1)$
26	m_0, m_1, ℓ	$\frac{1}{12}[m_0(3M_0-M_1)+3m_1(M_0-M_1)]$

	a, ℓ	d, $d+c$, ℓ
Quadr Parabola, a, b, ℓ	$\frac{c}{3}(2a+b)$	$\left\{\frac{d}{3}(2d+b)+\frac{c}{4}(a+b)\right\}$
	27	28

$\int M_j \; M_k \; dx = (\text{tabulated value}) \times \ell$

$$V_{11} = \int_A^B \frac{m_1 m_1}{EI}\,dx + \Sigma\frac{n_1 n_1 L}{AE}$$

$$\int_A^B \frac{m_1 m_1}{I}\,dx = 2 \times \frac{180}{3} \times \frac{1^2}{144} = 0.8333$$

$$\text{and } \Sigma\frac{n_1 n_1 L}{20A} = \left(\begin{array}{c}\text{one}\\ \text{strut}\end{array}\right)\frac{1}{90^2} \times \frac{60}{2 \times 20} = 0.0002$$

$$+\left(\begin{array}{c}\text{two}\\ \text{ties}\end{array}\right)\frac{2 \times 3.162^2}{180^2} \times \frac{190}{0.5 \times 20} = 0.0117$$

$$+\left(\begin{array}{c}\text{two}\\ \text{beams}\end{array}\right)\frac{2 \times 1}{60^2} \times \frac{180}{32} = 0.0313$$

$$V_{11} = 0.8765$$

$$p_1 = \frac{140{,}780}{0.8765} = 160{,}700 \text{ lb in.}$$

By making the appropriate substitutions the final bending moment diagram and final forces in the truss members can be found now.

Cable Stayed Beam

EXAMPLE A beam ABC is fixed at A and supported at B and C by steel ties anchored at a point D. Find the bending moments at A and B, and the forces in the ties (Fig. 20.72) E is constant, $I = 100$ in^4 and $A = 6.0$ in^2 for Beam. For Ties: A = 2.0 in^2.

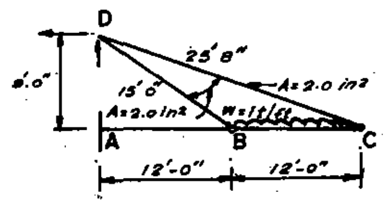


Fig. 20.72

$$V_{11} = \int_A^B \frac{m_1 m_1}{EI}\,dx + \Sigma\frac{n_1 n_1 L}{AE} \text{ (see Fig 20.73)}$$

$$\int_A^B \frac{m_1 m_1}{I}\,dx = \frac{144}{3} \times \frac{1}{100} = 0.48$$

$$\text{and } \Sigma\frac{n_1 n_1 L}{A} : (BD) = \frac{1.67^2}{144^2} \times \frac{180}{2} = 0.0121$$

$$(AB) = \frac{1.33^2}{144^2} \times \frac{144}{6} = 0.002$$

$$V_{11} = 0.4941$$

$$V_{12} = \int_A^B \frac{m_1 m_1}{EI}\,dx + \Sigma\frac{n_1 n_1 L}{AE}$$

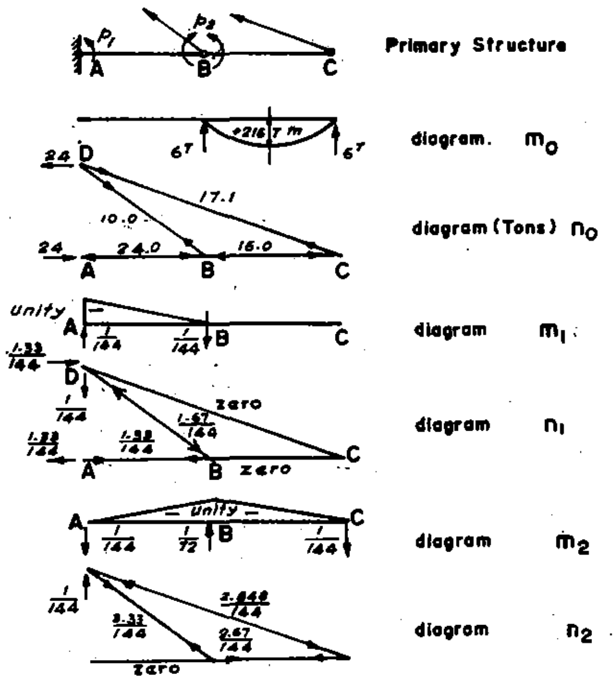


Fig. 20.73

$$\int_A^B \frac{m_1 m_2}{I}\,dx = \frac{144}{6} \times \frac{1}{100} = 0.24$$

$$\text{and } \Sigma\frac{n_1 n_2 L}{A} : (BD) = -\frac{1.67 \times 3.33}{144^2} \times \frac{180}{2} = -0.0241$$

$$V_{12} = 0.2159$$

$$V_{22} = \int_A^C \frac{m_2 m_2}{EI}\,dx + \Sigma\frac{n_2 n_2 L}{AE}$$

$$\int_A^C \frac{m_2 m_2}{I}\,dx = 2 \times \frac{144}{3} \times \frac{1}{100} = 0.96$$

$$\text{and } \Sigma\frac{n_2 n_2 L}{A} : (BD) = \frac{3.33^2}{144^2} \times \frac{180}{2} = 0.0482$$

$$(CD) = \frac{2.848^2}{144^2} \times \frac{308}{2} = 0.0595$$

$$(BC) = \frac{2.67^2}{144^2} \times \frac{144}{6} = 0.0082$$

$$V_{22} = 1.0759$$

$$V_{10} = \int_L \frac{m_1 m_0}{EI}\,dx + \Sigma\frac{n_1 n_0 L}{AE}$$

$$\int_L \frac{m_1 m_0}{I}\, dx = 0$$

$$\text{and } \Sigma\frac{n_1 n_0 L}{A} : (BD) = -10 \times \frac{1.67}{144} \times \frac{180}{2} = -10.417$$

$$(AB) = -24 \times \frac{1.33}{144} \times \frac{144}{6} = -\ 5.333$$

$$V_{10} = -15.75$$

$$V_{20} = \int_B^C \frac{m_2 m_0}{EI}\, dx + \Sigma\frac{n_2 n_0 L}{AE}$$

$$\int_B^C \frac{m_2 m_0}{I}\, dx = -\frac{144}{3} \times \frac{216}{100} = -103.68$$

$$\text{and } \Sigma\frac{n_2 n_0 L}{A} : (BD) = \frac{3.33}{144} \times 10 \times \frac{180}{2} = +\ 20.833$$

$$(CD) = -\frac{2.848}{144} \times 17.1 \times \frac{308}{2} = -\ 52.1$$

$$(BC) = -\frac{2.67}{144} \times 16 \times \frac{144}{6} = -\ 7.111$$

$$V_{20} = -142.06$$

Compatibility Equations

$$0.494p_1 + 0.216p_2 = 15.75$$
$$0.216p_1 + 1.076p_2 = 142.1$$

From which $p_1 = -13.95$ ton in.
$p_2 = 137.5$ ton in.

Again, by substitution the final forces and bending moments can be derived. It would seem as if in this case, the combination of flexural stress and direct stress in the beam would necessitate a review of that member's sizes.

20.8 BEAMS ON ELASTIC FOUNDATIONS

There are many problems in which a beam is supported on a compressible foundation which exerts a distributed reaction on the beam, of intensity proportional to the compressibility. In some cases the foundation can exert upward forces only, and the beam may, if sufficiently long, lose considerable contact with the foundation; in others pressure may be exerted either way. Again, the support may not be truly continuous (such as holding down a railway line) but can be replaced by an equivalent distributed support.

If y is the upward deflection of the foundation at any point and the rate of its upward reaction is $-ky$, then from $EId^2y/dx^2 = M$, $V = \dfrac{dM}{dx}$, and $w = \dfrac{dV}{dx}$, we have,

$$EId^4y/dx^4 = -ky$$

or

$$d^4y/dx^4 = -4\alpha^4 y \qquad (1)$$

where $\alpha^4 = k/4EI$, and k is the reaction force per unit length of beam per unit of its deflection from unloaded position.

A number of standard cases will now be considered.

(a) Long Beam Carrying Central Load W [Fig. 20.74(a)]

Assuming that the foundation can exert upward forces only, let $2l$ be the length of beam in contact with the foundation, and take the origin O at the left-hand end.

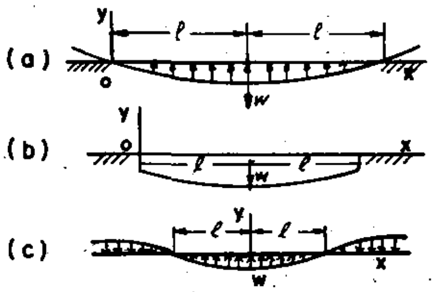


Fig. 20.74

The solution to (1) can be written as,

$$y = A\sin\alpha x\sinh\alpha x + B\cos\alpha x\sinh\alpha x + C\sin\alpha x\cosh\alpha x + D\cos\alpha x\cosh\alpha x$$

At $x = 0, y = 0$ $\quad \therefore D = 0$

and $M = EId^2y/dx^2 = 0$ $\quad \therefore A = 0$

also $V = EId^3y/dx^3 = 0$

giving

$$EI2\alpha^3[B(-\cos 0\cosh 0 - \sin 0\sinh 0) + C(-\sin 0\sinh 0 + \cos 0\cosh 0)] = 0$$

i.e. $\quad C = B$

The equation is now reduced to

$$y = B(\cos\alpha x\sinh\alpha x + \sin\alpha x + \sin\alpha x\cosh\alpha x)$$

At $x = l$, $dy/dx = 0$

$$\therefore B\alpha\cos\alpha l\cosh\alpha l = 0$$

The least solution of this is $\alpha l = \pi/2$ which determines the length in contact with the ground. The value of the constant B is obtained from the condition that the shear force at the centre is $W/2$, since by symmetry it must be numerically the same on either side of the load and it must change by an amount W on passing through the load. Hence,

$$W/2 = EId^3y/dx^3 \text{ at midspan}$$
$$(\text{where } x = l \text{ and } \alpha x = \pi/2)$$
$$= -EI4\alpha^3 B \sin \alpha l \sinh \alpha l$$
$$\text{or } B = -W\alpha/2k \sinh \pi/2$$

The maximum deflection and bending moment are at the centre, $\alpha x = \pi/2$, and are

$$\widehat{y} = -(W\alpha/2k) \coth \pi/2$$
$$M = EI(W\alpha^3/k) \coth \pi/2$$
$$= (W/4\alpha) \coth \pi/2$$

(b) Short Beam Carrying Central Load W [Fig. 20.74(b)]

If $\alpha l < \pi/2$ in case (a), the beam will sink below the unstressed level of the foundation at all points. Again taking the origin at the left-hand end and the overall length of beam as $2l$, the following conditions are obtained for the constants of integration of the general solution of the previous paragraph.

At $x = 0, d^2y/dx^2 = 0 \qquad \therefore A = 0$

and $d^3y/dx^3 = 0 \qquad \therefore B = C$

and

$$y = B(\cos \alpha x \sinh \alpha x + \sin \alpha x \cosh \alpha x) + D \cos \alpha x \cosh \alpha x$$

At $x = l, dy/dx = 0$ giving

$$B \cdot 2 \sin \alpha l \sinh \alpha l + D(\sin \alpha l \cosh \alpha l + \cos \alpha l \sinh \alpha l) = 0$$

and $\qquad EId^3y/dx^3 = W/2$

giving

$$-B \cdot 2 \sin \alpha l \sinh \alpha l - D(\sin \alpha l \cosh \alpha l + \cos \alpha l + \sinh \alpha l) = W/4EI\alpha^3 = W\alpha/k$$

Solving for B and D gives,

$$B = -\frac{W\alpha}{k} \cdot \frac{\sin \alpha l \cosh \alpha l + \cos \alpha l \cdot \sinh \alpha l}{\sin 2\alpha l + \sinh 2\alpha l}$$

$$\text{and} \quad D = -\frac{2W\alpha}{k} \cdot \frac{\cos \alpha l + \cosh \alpha l}{\sin 2\alpha l + \sinh 2\alpha l}$$

The complete solution for y is now known, the maximum deflection and bending moment being under the load.

(c) Infinite Beam Carrying Load W [Fig. 20.74(c)]

Assuming that the support can exert pressure either upwards or downwards, and taking the Y axis through the load and X axis at the undeformed level, a solution of equation (1) can be written in the form

$$y = e^{\alpha x}(A \sin \alpha x + B \cos \alpha x) + e^{-\alpha x}(C \sin \alpha x + D \cos \alpha x)$$

For the length to the right of W, since $y \to 0$ as $x \to \infty$, $A = B = 0$.

At $\qquad x = 0, dy/dx = 0 \quad \therefore C = D$

and $\qquad EId^3y/dx^3 = -W/2$

giving $\quad C = -W/8\alpha^3 EI = -W\alpha/2k$

and $\quad y = -(W\alpha/2k)e^{-\alpha x}(\sin \alpha x + \cos \alpha x)$

The distance from the load at which $y = 0$ is given by

$$\sin \alpha l + \cos \alpha l = 0$$

the least solution being $\alpha l = 3\pi/4$ (giving $2l = 3\pi/2\alpha$)

The maximum deflection and bending moment are at $x = 0$,

$$\widehat{y} = -W\alpha/2k$$

and $\quad \widehat{M} = EIW\alpha^3/k = W/4\alpha$

EXAMPLE A steel railway track is supported on timber sleepers which exert an equivalent load of 400 lb./in.length of rail per inch deflection from its unloaded position. For each rail $I = 30$ in.^{4} $Z = 10$ in.^{4} and $E = 30 \times 10^6$ lb/sq.in. If a point load of 10 tons acts on each rail, find the length of rail over which the sleepers are depressed and the maximum bending stress in the rail:

$$\alpha^4 = k/4EI$$
$$= \frac{400}{4 \times 30 \times 10^6 \times 30}$$

giving $\quad \alpha = 1/54.8$

Each rail can be treated as an infinitely long beam, for which the length over which downward deflection occurs is given in (c) above:

$$2l = 3\pi/2\alpha$$
$$= 3\pi \times 54.8/2$$
$$= 258 \text{ in.}$$

and

$$\widehat{M} = W/4\alpha$$
$$= 10 \times 54.8/4$$
$$= 137 \text{ tons-in.}$$
$$f = \widehat{M}/Z$$
$$= 13.7 \text{ tons/sq. in.}$$

20.9 SIMPLIFIED AIDS FOR RAPID HAND-ANALYSIS

Given here (in Tables 20.2 to 20.44) are self-explanatory sets of certain formulae, and influence lines for enabling a rapid hand-analysis of various load-effects in different types of cantilevers and beams under various types of loadings. (In propped cantilevers, fixed-beams and continuous-beams, I the moment of inertia is assumed to be constant throughout; in frames it is constant within each individual member but can be different in different members, as clearly indicated there. Young's modulus of elasticity, E, is assumed constant throughout in each case.)

Summary of these tables is given below:

- Tables 20.2 and 20.3 Free and propped cantilevers.
- Tables 20.4 and 20.5 Simply supported and fixed beams.
- Tables 20.6 and 20.7 Simple beams ~ max. moments and deflections under various loadings.
- Table 20.8 Fixed end moment coefficients (general data).
- Table 20.9 and 20.10 Fixed end moment coefficients (various loadings).
- Table 20.11 Effects of moments applied at end-supports in equal-span continuous beams
- Tables 20.12 to 20.14 Continuous beams—moments and shears from equal loads on equal spans
- Table 20.15 Rectangular box culvert (as a rigid frame cell) put to different loads, sitting on compressible or rigid ground.
- Tables 20.16 to 20.20 Three types of single-story single-bay portal frames, put to different loadings.
- Tables 20.21 to 20.44 Influence lines for bending moment, shear force and reactions for 2, 3 and 4 span continuous beams of various span ratios.

Influence Lines (Tables 20.21 to 20.44)

By Maxwell's Theorem, if a unit load at YY causes a displacement of δ at XX, then a unit load at XX causes the same displacement, δ, at YY. Influence Lines for various load effects can be evaluated by solving the structure for the load effect for each placement of a unit load on its spans one by one. (For example: the unit load may be placed at each tenth point of each span, turn by turn, and the structure analysed for each placement.) Flexibility Method of analysis is recommended.

Given in Tables 20.21 to 20.44 are coefficients by means of which the moment (positive or negative) at any interior support or at any tenth point along all spans, produced by a unit load P placed at the same or any other tenth point in the spans, can be computed. In order to list all possible values of these coefficients a horizontal tabulation is given opposite all support and tenth points for all spans. However, due to the symmetry of the structures, all moment values can be tabulated in a lesser number of vertical columns.

Values given along any one horizontal line are ordinates to the bending moment diagram produced by a unit load placed at the load point, shown at the left of the table, opposite which they are tabulated, considering the length of the shorter span as equal to unity and that of the longer spans as equal to N. Taken vertically, the values in any one column are ordinates to the influence line for the point under which they are tabulated.

Values shown in the heavily outlined frames are the largest possible at the point on the continuous beam under which they are tabulated and are produced when the load P is placed at this point. Hence, these values are ordinates to an envelope of the maximum positive moments produced by a single moving concentrated load.

The lowest line in these tables designated "Total Area", gives ordinates to the moment diagram produced by a load, uniformly distributed along the entire structure and having a value of unity per unit of shorter span length. The two lines immediately above give, respectively, the largest positive and negative moments produced by partial distribution of the unit uniform load w.

Also included in these tables are influence coefficients for all reactions, and shears adjacent to these reactions.

The following rules for use of these tables are summarized:

(i) Reactions and Shears Due to Concentrated Load—Multiply the tabulated coefficient by the weight of the concentrated load.

(ii) Reactions and Shears Due to Uniform Load—Multiply the tabulated coefficient by the product of the weight per unit length of the uniform load and the length of the *Shorter* span.

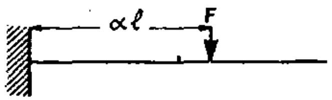
(iii) Moments Due to Concentrated Load—Multiply the tabulated moment coefficient by the product of the weight of the concentrated load and the length of the *Shorter* span.

(iv) Moments Due to Uniform Load—Multiply the tabulated moment area coefficient by the product of the weight per unit length of the uniform load and the square of the length of the *Shorter* span.

NOTE Moment of inertia is assumed constant throughout.

Table 20.2 *Moments: shears: deflections: General cases for cantilevers*

Concentrated load



Shearing forces: when $x < \alpha$, $V_x = F$;

when $x > \alpha$, $V_x = 0$

Bending moments: when $x \leqslant \alpha$, $M_x = -Fl(\alpha - x)$;

when $x \geqslant \alpha$, $M_x = 0$

Deflections: when $x \leqslant \alpha$, $a_x = -\frac{Fl^3}{6EI}x^2(3\alpha - x)$;

when $x \geqslant \alpha$, $a_x = -\frac{Fl^3}{6EI}\alpha^2(3x - \alpha)$

Partial triangular load *apex at l.h. end*

Shearing forces: when $x \leqslant \alpha$, $V_x = F$;

when $\alpha \leqslant x \leqslant (1-\beta)$, $V_x = F\left[1 - \frac{(x-\alpha)}{(1-\alpha-\beta)}\left(2 - \frac{(x-\alpha)}{(1-\alpha-\beta)}\right)\right]$; when $x \geqslant (1-\beta)$, $V_x = 0$

Bending moments: when $x \leqslant \alpha$, $M_x = -\frac{1}{3}Fl(1 + 2\alpha - \beta - 3x)$; when $x \geqslant (1-\beta)$, $M_x = 0$

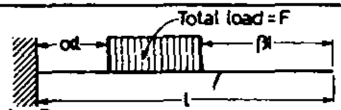
when $\alpha \leqslant x \leqslant (1-\beta)$, $M_x = -\frac{1}{3}Fl\left[(1 + 2\alpha - \beta - 3x) + \frac{(x-\alpha)^2}{(1-\alpha-\beta)}\left(3 - \frac{(x-\alpha)}{(1-\alpha-\beta)}\right)\right]$;

Deflections: when $x \leqslant \alpha$, $a_x = (1 + 2\alpha - \beta - x)Fl^3x^2/6EI\,(1 + 2\alpha - \beta - x)$;

when $\alpha \leqslant x \leqslant (1-\beta)$, $a_x = -\frac{Fl^3}{60EI}\left\{10x^2(1 + 2\alpha - \beta - x) + \frac{(x-\alpha)^4}{(1-\alpha-\beta)}\left[5 - \frac{(x-\alpha)}{(1-\alpha-\beta)}\right]\right\}$;

when $x \geqslant (1-\beta)$, $a_x = -\frac{Fl^3}{60EI}\{\alpha[(1-\beta)^2 + 10x\alpha - 3\alpha^2] - (1+\alpha-\beta)^2(1+\alpha-\beta-5x)\}$

Partial uniform load



Shearing forces: when $x \leqslant \alpha$, $V_x = F$;

When: $\alpha \leqslant x \leqslant (1-\beta)$, $V_x = F\left[1 - \frac{(x-\alpha)}{(1-\alpha-\beta)}\right]$; when $x \geqslant (1-\beta)$, $V_x = 0$

Bending movements when. $x \leqslant \alpha$, $M_x = -Fl[\frac{1}{2}(1 + \alpha - \beta) - x]$;

When $\alpha \leqslant x \leqslant 1-\beta$, $M_x = -\frac{1}{2}Fl\left[(1 + \alpha - \beta - 2x) + \frac{(x-\alpha)^2}{(1-\alpha-\beta)}\right]$; when $x \geqslant (1-\beta)$, $M_x = 0$

Deflections: when $x \leqslant \alpha$, $a_x = -\frac{Fl^3x^2}{12EI}[3(1 + \alpha - \beta) - 2x]$;

when $\alpha \leqslant x \leqslant (1-\beta)$, $a_x = -\frac{Fl^3}{24EI}\left[6x^2(1 + \alpha - \beta - \frac{2}{3}x) + \frac{(x-\alpha)^4}{(1-\alpha-\beta)}\right]$;

when $x \geqslant (1-\beta)$,

$$a_x = -\frac{Fl^3}{24EI}\left[2\alpha(1-\beta)(1 + \alpha - \beta - 2x) - (1 + \alpha - \beta)^2(1 + \alpha - \beta - 4x)\right]$$

Partial triangular load *apex at r.h. end*

Shearing forces: when $x \leqslant \alpha$, $V_x = F$;

when $\alpha \leqslant x \leqslant (1-\beta)$, $V_x = F\left[1 - \frac{(x-\alpha)^2}{(1-\alpha-\beta)^2}\right]$; when $x \geqslant (1-\beta)$, $V_x = 0$

Bending moments: when $x \leqslant \alpha$, $M_x = -\frac{1}{3}Fl(2 + \alpha - 2\beta - 3x)$;

when $\alpha \leqslant x \leqslant (1-\beta)$, $M_x = -\frac{1}{3}Fl\left[(2 + \alpha - 2\beta - 3x) + \frac{(x-\alpha)^3}{(1-\alpha-\beta)^2}\right]$

when $x \geqslant (1-\beta)$, $M_x = 0$

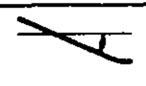
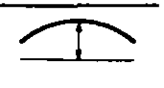
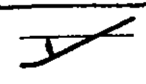
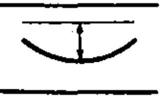
Deflections: when $x \leqslant \alpha$, $a_x = -\frac{Fl^3x^2}{6EI}(2 + \alpha - 2\beta - x)$;

when $\alpha \leqslant x \leqslant (1-\beta)$, $a_x = -\frac{Fl^3}{60EI}\left[10x^2(2 + \alpha - 2\beta - x) + \frac{(x-\alpha)^5}{(1-\alpha-\beta)^2}\right]$;

when $x \geqslant (1-\beta)$,

$$a_x = -(Fl^3/60EI)\{(1-\beta)[\alpha^2 + 10x(1-\beta) - 3(1-\beta)^2] - (1+\alpha-\beta)^2(1+\alpha-\beta-5x)\}$$

Key to sign convention for Tables 20.2 to 20.5

	Reaction	Shearing force	Bending moment	Slope	Deflection
Positive	↑	↑ ↓			
Negative	↓	↓ ↑			

For Tables 20.2 to 20.5 ℓ = span

F total load

x distance of point considered from left-hand support in terms of l

Members with fixed ends To determine deflection, moment, etc., for member with one or both ends fixed or continuous, first calculate deflection, moment, etc., for freely-supported span. Next, determine deflection, moment, etc., throughout span due to action of support moments only. Lastly, obtain final values of deflection, moment, etc., by summing foregoing results algebraically.

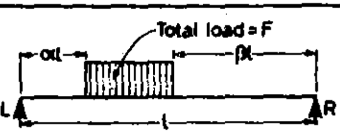
Slope To determine slope at any point, distance xl from left-hand support, differentiate expression for deflection with respect to x.

Table 20.3 *Moments: shears: deflections: Special cases for cantilevers*

	Simple cantilever (fixed at left-hand end)		Propped cantilever (fixed at left-hand support)	
Uniform load	Shearing forces: $V_x = F(1-x)$ Bending moments: $M_x = -\frac{1}{2}Fl(1-x)^2$ $M_{x\,max} = -\frac{1}{2}Fl$ at $x = 0$ Slopes: $\theta_L = 0$; $\theta_R = -\frac{Fl^2}{6EI}$ Deflections: $a_x = -\frac{Fl^3x^2}{24EI}(6 - 4x + x^2)$ $\quad a_{x\,max} = -\frac{Fl^3}{8EI}$ at $x = 1$		Reactions: $R_L = \frac{5}{8}F$; $R_R = \frac{3}{8}F$ Shearing forces: $V_x = F(\frac{5}{8} - x)$ Bending moments: $M_L = -\frac{Fl}{8}$, $M_R = 0$ $\quad M_x = \frac{Fl}{8}\{x(5-4x) - 1\}$ $M_{x\,max} = \frac{9Fl}{128}$ at $x = \frac{5}{8}$ Slopes: $\theta_L = 0$; $\theta_R = \frac{Fl^2}{48EI}$ Deflections: $a_x = -\frac{Fl^3}{48EI}x^2(1-x)(3-2x)$ $\quad a_{x\,max} \simeq -\frac{Fl^3}{185EI}$ at $x = 0.5785$ from L	
Triangular load	*Apex at l.h. end* Shearing forces: $V_x = F(1-x)^2$ Bending moments: $M_x = -\frac{1}{3}Fl(1-x)^3$ $M_{x\,max} = -\frac{1}{3}Fl$ at $x = 0$ Slopes: $\theta_L = 0$; $\theta_R = -Fl^2/12EI$ Deflections: $a_x = -\frac{Fl^3x^2}{60EI}(10 - 10x + 5x^2 - x^3)$ $a_{x\,max} = -\frac{Fl^3}{15EI}$ at $x = 1$	*Apex at r.h. end* Shearing forces: $V_x = F(1-x^2)$ Bending moments: $M_x = -\frac{1}{3}Fl(1-x)^2(2+x)$ $M_{x\,max} = -\frac{2}{3}Fl$ at $x = 0$ Slopes: $\theta_L = 0$: $\theta_R = -Fl^2/4EI$ Deflections: $a_x = -\frac{Fl^3x^2}{60EI}(20 - 10x + x^3)$ $a_{x\,max} = -\frac{11Fl^3}{60EI}$ at $x = 1$	*Apex at l.h. end* Reactions: $R_L = \frac{4}{5}F$; $R_R = \frac{1}{5}F$ Shearing forces: $V_x = F(\frac{4}{5} - 2x + x^2)$ Bending moments: $M_L = -2Fl/15$: $M_R = 0$ $M_x = -(5x^2 - 10x + 2)(1-x)Fl/15$ $M_{x\,max} \simeq Fl/16.77$ at $x = 0.5528$ from L Slopes: $\theta_L = 0$; $\theta_R = Fl^2/60EI$ Deflections: $a_x = -x^2(1-x)(2-x)^2Fl^3/60EI$ $a_{x\,max} \simeq -\frac{Fl^3}{209.6EI}$ at $x = 0.5528$ from L	*Apex at r.h. end* Reactions: $R_L = \frac{9}{20}F$; $R_R = \frac{11}{20}F$ Shearing forces: $V_x = F(\frac{9}{20} - x^2)$ Bending moments: $M_L = -7Fl/60$; $M_R = 0$. $M_x = -(20x^3 - 27x + 7)Fl/60$ $M_{x\,max} \simeq Fl/11.82$ at $x = \sqrt{0.45}$ from L Slopes: $\theta_L = 0$; $\theta_R = Fl^2/40EI$ Deflections: $a_x = -x^2(1-x)(7 - 2x - 2x^2)Fl^3/120EI$ $a_{x\,max} \simeq -\frac{Fl^3}{164EI}$ at $x = 0.5975$ from L
Concentrated load	Load at unsupported end Shearing forces: $V_x = F$ Bending moments: $M_L = -Fl$ $M_x = -F(l-x)$ at $x = 0$ Slope: $\theta_L = 0$; $\theta_R = -\frac{Fl^2}{2EI}$ Deflections: $a_x = -\frac{Fl^3x^2}{6EI}(3-x)$ $\quad a_{x\,max} = -\frac{Fl^3}{3EI}$ at $x = 1$		Load at centre Reactions: $R_L = \frac{11}{16}F$; $R_R = \frac{5}{16}F$ Shearing forces: when $x < \frac{1}{2}$, $V_x = \frac{11}{16}F$; when $x > \frac{1}{2}$, $V_x = -\frac{5}{16}F$ Bending moments: $M_L = -\frac{3}{16}Fl$. $M_R = 0$. $M_{x\,max} = \frac{5}{32}Fl$ beneath load Slopes: $\theta_L = 0$; $\theta_R = Fl^2/32EI$ Deflections: when $x < \frac{1}{2}$, $a_x = -x^2(9 - 11x)Fl^3/96EI$ when $x > \frac{1}{2}$, $a_x = -\frac{Fl^3}{96EI}(x-1)(5x^2 - 10x + 2)$ $a_{x\,max} = -\frac{Fl^3}{48\sqrt{5}EI}$ at $x = \left(1 - \frac{1}{\sqrt{5}}\right)$ from L	

Table 20.4 *Moments: shears: deflections: Special cases for beams*

Partial uniform load



Reactions: $R_L = \frac{1}{2}F(1 - \alpha + \beta)$; $R_R = \frac{1}{2}F(1 + \alpha - \beta)$

Shearing forces: when $x < \alpha$, $V_x = \frac{1}{2}F(1 - \alpha + \beta)$;

when $\alpha < x < (1 - \beta)$, $V_x = F\left[\frac{1}{2}(1 - \alpha + \beta) + \frac{(\alpha - x)}{(1 - \alpha - \beta)}\right]$;

when $x > (1 - \beta)$, $V_x = -\frac{1}{2}F(1 + \alpha - \beta)$

Bending moments: when $x < \alpha$, $M_x = \frac{1}{2}Flx(1 - \alpha + \beta)$;

when $\alpha < x < (1 - \beta)$, $M_x = \frac{1}{2}Fl\left[x(1 - \alpha + \beta) - \frac{(x - \alpha)^2}{(1 - \alpha - \beta)}\right]$;

when $x > (1 - \beta)$, $M_x = \frac{1}{2}Fl(1 - x)(1 + \alpha - \beta)$

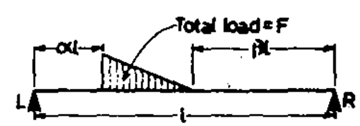
$M_{x\,max} = \frac{1}{8}(1 + \alpha + \beta)(1 + \alpha - \beta)(1 - \alpha + \beta)Fl$ at $\frac{1}{2}(1 + \alpha^2 - \beta^2)l$

Deflections: when $x < \alpha$, $a_x = -\frac{Fl^3(1 - \alpha + \beta)}{24EI}\left[(1 + 2\alpha - \alpha^2 - \beta^2)x - 2x^3\right] = a_1$;

when $\alpha < x < (1 - \beta)$, $a_x = a_1 - \frac{Fl^3(x - \alpha)^4}{24EI(1 - \alpha - \beta)}$;

when $x > (1 - \beta)$, use formula for a_1, transpose α and β and substitute $(1 - x)$ for x

Partial triangular load



Reactions: $R_L = \frac{1}{3}F(2 - 2\alpha + \beta)$; $R_R = \frac{1}{3}F(1 + 2\alpha - \beta)$

Shearing forces: when $x < \alpha$, $V_x = \frac{1}{3}F(2 - 2\alpha + \beta)$;

when $\alpha < x < (1 - \beta)$, $V_x = -F\left[\frac{1}{3}(1 + 2\alpha - \beta) - \frac{(1 - x - \beta)^2}{(1 - \alpha - \beta)^2}\right]$;

when $x > (1 - \beta)$, $V_x = -\frac{1}{3}F(1 + 2\alpha - \beta)$

Bending moments: when $x < \alpha$, $M_x = \frac{1}{3}Flx(2 - 2\alpha + \beta)$;

when $\alpha < x < (1 - \beta)$, $M_x = \frac{1}{3}Fl\left[(1 + 2\alpha - \beta)(1 - x) - \frac{(1 - x - \beta)^3}{(1 - \alpha - \beta)^2}\right]$;

when $x > (1 - \beta)$, $M_x = \frac{1}{3}Fl(1 - x)(1 + 2\alpha - \beta)$

Deflections: when $x < \alpha$,

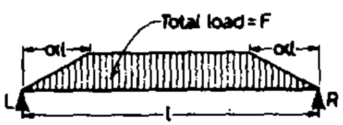
$$a_x = -\frac{Fl^3x(2 - 2\alpha + \beta)}{162EI}\left[9(1 + x)(1 - x) - (2 - 2\alpha + \beta)^2 - (1 - \alpha - \beta)^2\left(\frac{3}{2} - \frac{(1 - \alpha - \beta)}{5(2 - 2\alpha + \beta)}\right)\right] = a_1;$$

when $\alpha < x < (1 - \beta)$, $a_x = a_1 - \frac{Fl^3(x - \alpha)^4}{60EI(1 - \alpha - \beta)}\left(5 - \frac{(x - \alpha)}{(1 - \alpha - \beta)}\right)$;

when $x > (1 - \beta)$,

$$a_x = -\frac{Fl^3(1 - x)(1 + 2\alpha - \beta)}{162EI}\left[9x(2 - x) - (1 + 2\alpha - \beta)^2 - (1 - \alpha - \beta)^2\left(\frac{3}{2} + \frac{(1 - \alpha - \beta)}{5(1 + 2\alpha - \beta)}\right)\right]$$

Trapezoidal load



Reactions: $R_L = R_R = \frac{1}{2}F$

Shearing forces:

when $x < \alpha$, $V_x = \frac{1}{2}F\left(1 - \frac{x^2}{\alpha(1 - \alpha)}\right)$; when $\alpha < x \leqslant \frac{1}{2}$, $V_x = \frac{1}{2}F\frac{(1 - 2x)}{(1 - \alpha)}$

Bending moments: when $x < \alpha$, $M_x = \frac{1}{6}Flx\left(3 - \frac{x^2}{\alpha(1 - \alpha)}\right)$;

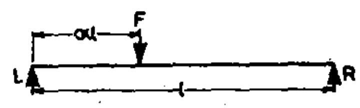
when $\alpha < x \leqslant \frac{1}{2}$, $M_x = \frac{1}{6}Fl\left[\frac{3x(1 - x) - \alpha^2}{(1 - \alpha)}\right]$ $\quad M_{x\,max} = \frac{(3 - 4\alpha^2)}{24(1 - \alpha)}Fl$ at midspan

Deflections: when $x < \alpha$, $a_x = -\frac{Fl^3x}{24EI}\left[(1 + \alpha - \alpha^2) - 2x^2 + \frac{x^4}{5\alpha(1 - \alpha)}\right]$;

when $\alpha < x \leqslant \frac{1}{2}$, $a_x = -\frac{Fl^3}{24EI(1 - \alpha)}\left\{[1 - 2\alpha^2 + x(1 - x)]x(1 - x) + \frac{\alpha^4}{5}\right\}$

$a_{x\,max} = -\frac{(4\alpha^2 - 5)^2Fl^3}{1920(1 - \alpha)EI}$ at midspan

Concentrated load



Reactions: $R_L = F(1 - \alpha)$; $R_R = F\alpha$

Shearing forces: when $x < \alpha$, $V_x = F(1 - \alpha)$; when $x > \alpha$, $V_x = -F\alpha$

Bending moments: when $x < \alpha$, $M_x = Fl(1 - \alpha)x$;

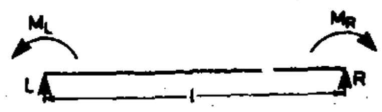
when $x > \alpha$, $M_x = Fl\alpha(1 - x)$; $M_{x\,max} = Fl\alpha(1 - \alpha)$ beneath load

Deflections:

when $x < \alpha$, $a_x = -\frac{Fl^3(1 - \alpha)x}{6EI}\left[\alpha(2 - \alpha) - x^2\right]$; when $x > \alpha$, $a_x = -\frac{Fl^3\alpha(1 - x)}{6EI}\left[x(2 - x) - \alpha^2\right]$

when $\alpha \leqslant \frac{1}{2}$, $a_{x\,max} = -\frac{Fl^3\alpha(1 - \alpha^2)^{3/2}}{9\sqrt{3}EI}$ at $\sqrt{\frac{1 - \alpha^2}{3}}\,l$ from R

Support moments



Reactions: $R_L = \frac{M_L - M_R}{l}$; $R_R = \frac{M_R - M_L}{l}$

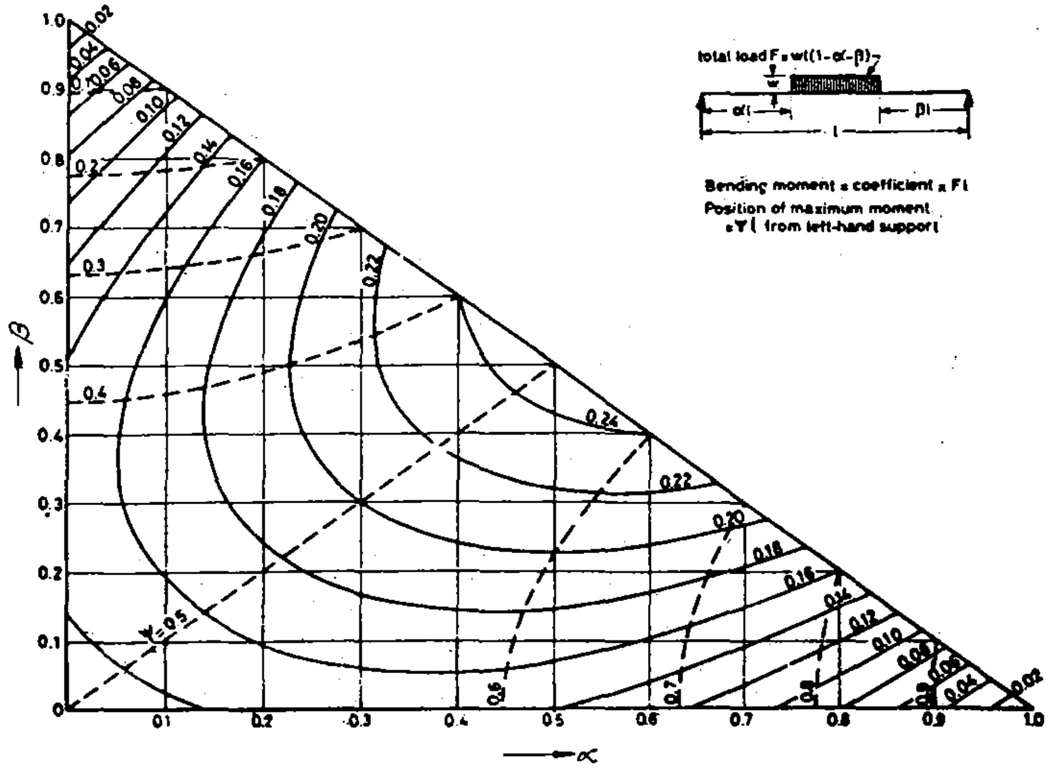
Shearing force: $V_x = \frac{M_L - M_R}{l}$ $\quad$ **Bending moments:** $M_x = M_L(1 - x) + M_Rx$

Deflections: $a_x = -\frac{x(1 - x)l^2}{6EI}\left[(2 - x)M_L + (1 + x)M_R\right]$

Table 20.5 *Moments: shears: deflections: Special cases for beams.*

	Freely supported span	Fully fixed span
Uniform load	 Reactions: $R_L = R_R = \frac{1}{2}F$ Shearing forces: $V_x = F(\frac{1}{2} - x)$ Bending moments: $M_L = M_R = 0$ $M_x = \frac{1}{2}x(1-x)Fl$ $M_{x\,max} = \frac{1}{8}Fl$ at $\frac{1}{2}l$ Slopes: $\theta_L = \theta_R = \pm \frac{Fl^2}{24EI}$ Deflections: $a_x = -\frac{Fl^3x}{24EI}(1-x)(1+x-x^2)$ $a_{x\,max} = -\frac{5Fl^3}{384EI}$ at midspan	 Reactions: $R_L = R_R = \frac{1}{2}F$ Shearing forces: $V_x = F\left(\frac{1}{2} - \frac{x}{l}\right)$ Bending moments: $M_L = M_R = -\frac{1}{12}Fl$ $M_x = \frac{1}{2}Fl[x(1-x) - \frac{1}{6}]$ $M_{x\,max} = \frac{1}{24}Fl$ at midspan Slopes: $\theta_L = \theta_R = 0.$ Deflections: $a_x = -\frac{Fl^3x^2(1-x)^2}{24EI}$ $a_{x\,max} = -\frac{Fl^3}{384EI}$ at midspan
Triangular load	*Apex at l.h. end*  Reactions: $R_L = \frac{2}{3}F$; $R_R = \frac{1}{3}F$ Shearing forces: $V_x = \frac{F}{3}(2 - 6x + 3x^2)$ Bending moments: $M_L = M_R = 0$ $M_x = \frac{1}{3}Flx(1-x)(2-x)$ $M_{x\,max} = \frac{2Fl}{9\sqrt{3}}$ at $\left(1 - \frac{1}{\sqrt{3}}\right)l$ from L Slopes: $\theta_L = -\frac{2Fl^2}{45EI}$; $\theta_R = +\frac{7Fl^2}{180EI}$ Deflections: $a_x = -x(1-x)(2-x)(4+6x-3x^2)Fl^3/180EI$ $a_{x\,max} \simeq -(Fl^3/76.7EI)$ at $x \simeq 0.4807\,l$ from L	*Apex at l.h. end*  Reactions: $R_L = \frac{7}{10}F$; $R_R = \frac{3}{10}F$ Shearing forces: $V_x = \frac{F}{10}(7 - 20x + 10x^2)$ Bending moments $M_L = -\frac{Fl}{10}$ $M_R = -\frac{Fl}{15}$ $M_x = \frac{Fl}{30}(10x^3 - 30x^2 + 21x - 3)$ $M_{x\,max} \simeq \frac{Fl}{23.32}$ at $(1 - \sqrt{0.3})l$ from L Slopes: $\theta_L = \theta_R = 0$ Deflections: $a_x = -\frac{Fl^3}{60EI}x^2(1-x)^2(3-x)$ $a_{x\,max} \simeq -\frac{Fl^3}{382EI}$ at $x \simeq 0.4753\,l$ from L
Central concentrated load	 Reactions: $R_L = R_R = \frac{1}{2}F$ Shearing forces: when $x < \frac{1}{2}$, $V_x = \frac{1}{2}F$; when $x > \frac{1}{2}$, $V_x = -\frac{1}{2}F$ Bending moments: $M_L = M_R = 0.$ $M_{x\,max} = \frac{1}{4}Fl$ at midspan Slopes: $\theta_L = -\frac{Fl^2}{16EI}$; $\theta_R = +\frac{Fl^2}{16EI}$ Deflections: when $x \leqslant \frac{1}{2}$ $a_x = -\frac{Fl^3}{48EI}x(3-4x^2)$ $a_{x\,max} = -\frac{Fl^3}{48EI}$ at midspan	 Reactions: $R_L = R_R = \frac{1}{2}F$ Shearing forces: when $x < \frac{1}{2}$, $V_x = \frac{1}{2}F$; when $x > \frac{1}{2}$, $V_x = -\frac{1}{2}F$ Bending moments: $M_L = M_R = -\frac{1}{8}Fl$ when $x \leqslant \frac{1}{2}$, $M_x = \frac{Fl}{8}(4x-1)$; when $x \geqslant \frac{1}{2}$, $M_x = \frac{Fl}{8}(3-4x)$ $M_{x\,max} = \frac{1}{8}Fl$ at midspan Slopes: $\theta_L = \theta_R = 0$ Deflections: when $x \leqslant \frac{1}{2}$, $a_x = -\frac{Fl^3}{48EI}x^2(3-4x)$ $a_{x\,max} = -\frac{Fl^3}{192EI}$ at midspan

Table 20.6 *Freely-supported beams: Maximum moments*



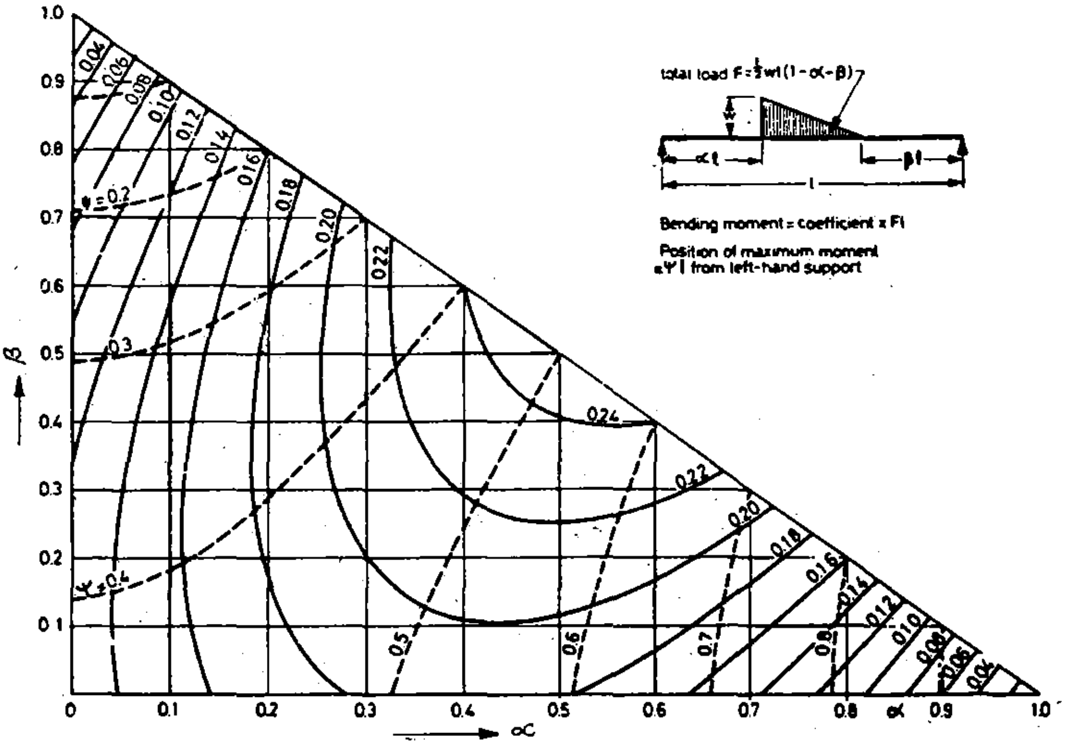
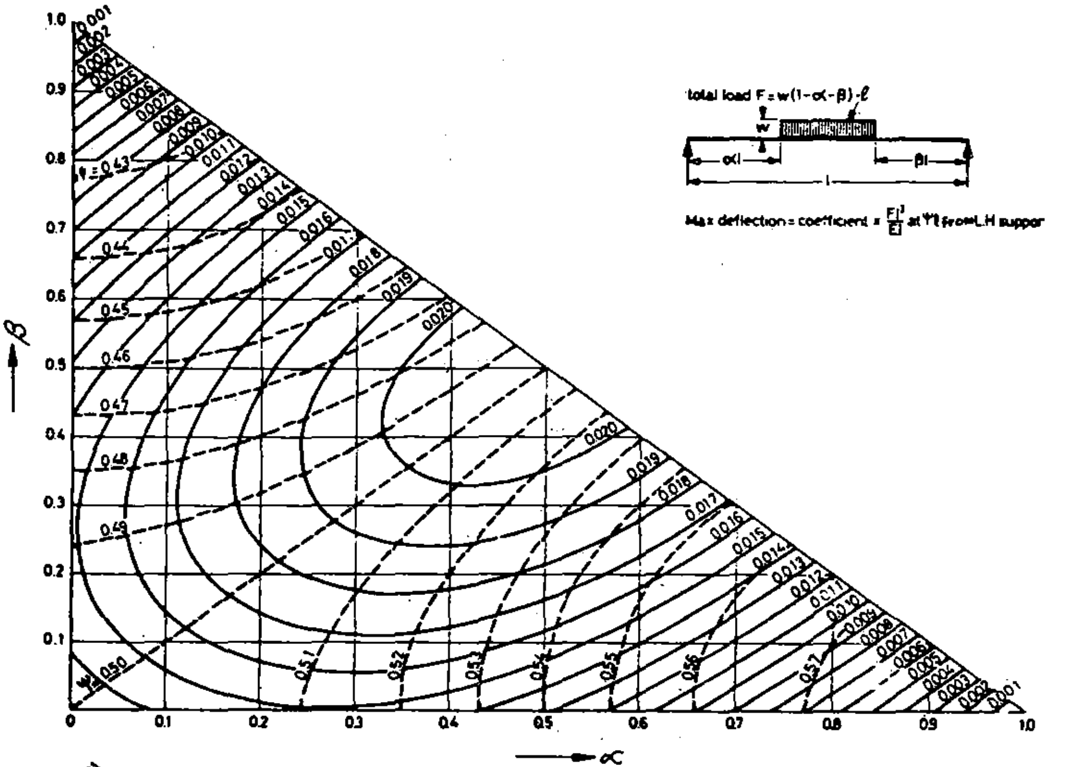


Table 20.7 *Freely-supported beams: Maximum deflections*



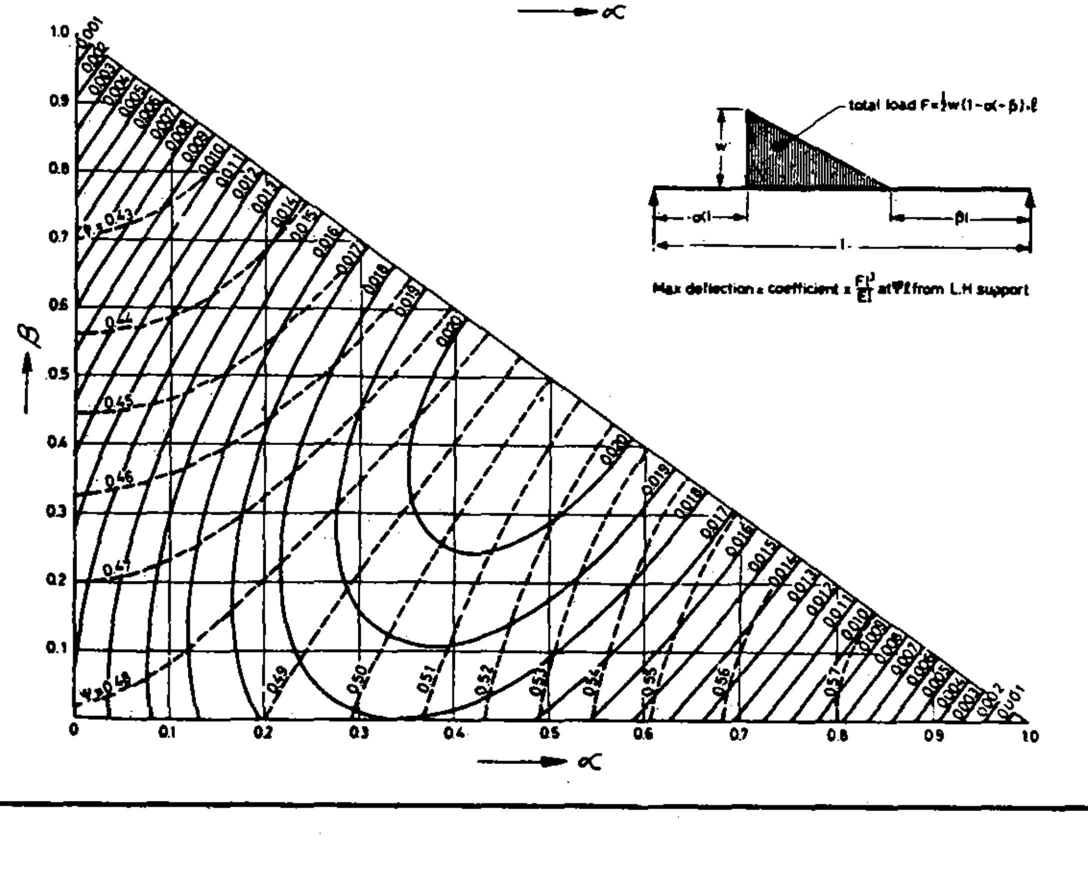
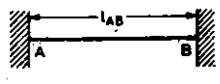
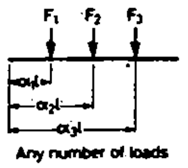
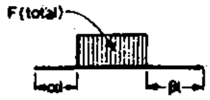
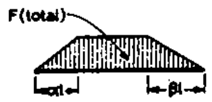


Table 20.8 *Fixed-end moment coefficients: General data*

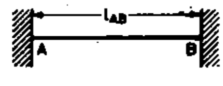
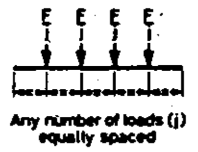
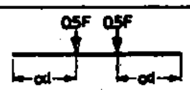
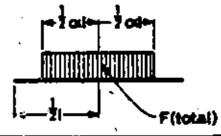
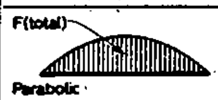
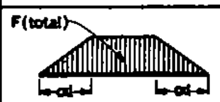
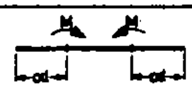
The fixed end moment coefficients C_{AB} and C_{BA} can be used as follows.
To obtain bending moments at supports of single-span beams fixed at both ends

$M_{AB} = -C_{AB}\,l_{AB}$ $M_{BA} = -C_{BA}\,l_{AB}$ With symmetrical load $M_{AB} = M_{BA}$

Unsymmetrical loading

	Fixed-end moment coefficients C_{AB}	Fixed-end moment coefficients C_{BA}
	$\sum \alpha(1-\alpha)^2 F$	$\sum \alpha^2(1-\alpha)F$
	$\alpha(1-\alpha)^2 F$	$\alpha^2(1-\alpha)F$
	Read values from Table 20·10	
	Read values from Table 20·9	
	Read values from Table 20·10	
	$\frac{M}{l}(3\alpha-1)(\alpha-1)$	$\frac{M\alpha}{l}(2-3\alpha)$

Symmetrical loading

	Fixed-end moment coefficients $C_{AB} = C_{BA}$
	$\frac{(j+2)}{12(j+1)}F$ j | factor 1 | 0.125F 2 | 0.111F 3 | 0.104F 4 | 0.100F
	$\frac{\alpha}{2}(1-\alpha)F$
	$\frac{1}{8}\left(1-\frac{\alpha^2}{3}\right)F$
	$\frac{1}{12}F$
	$\frac{5}{48}F$
	$\frac{1}{10}F$
	$\frac{(1+\alpha-\alpha^2)}{12}F$
	$\frac{M}{l}(1-2\alpha)$

Other loadings can generally be considered by combining tabulated cases, thus:

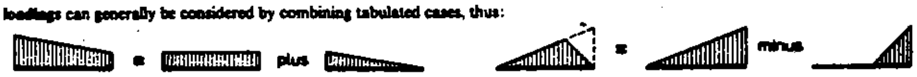


Table 20.9 *Fixed-end moments coefficients: Partial triangular loads*

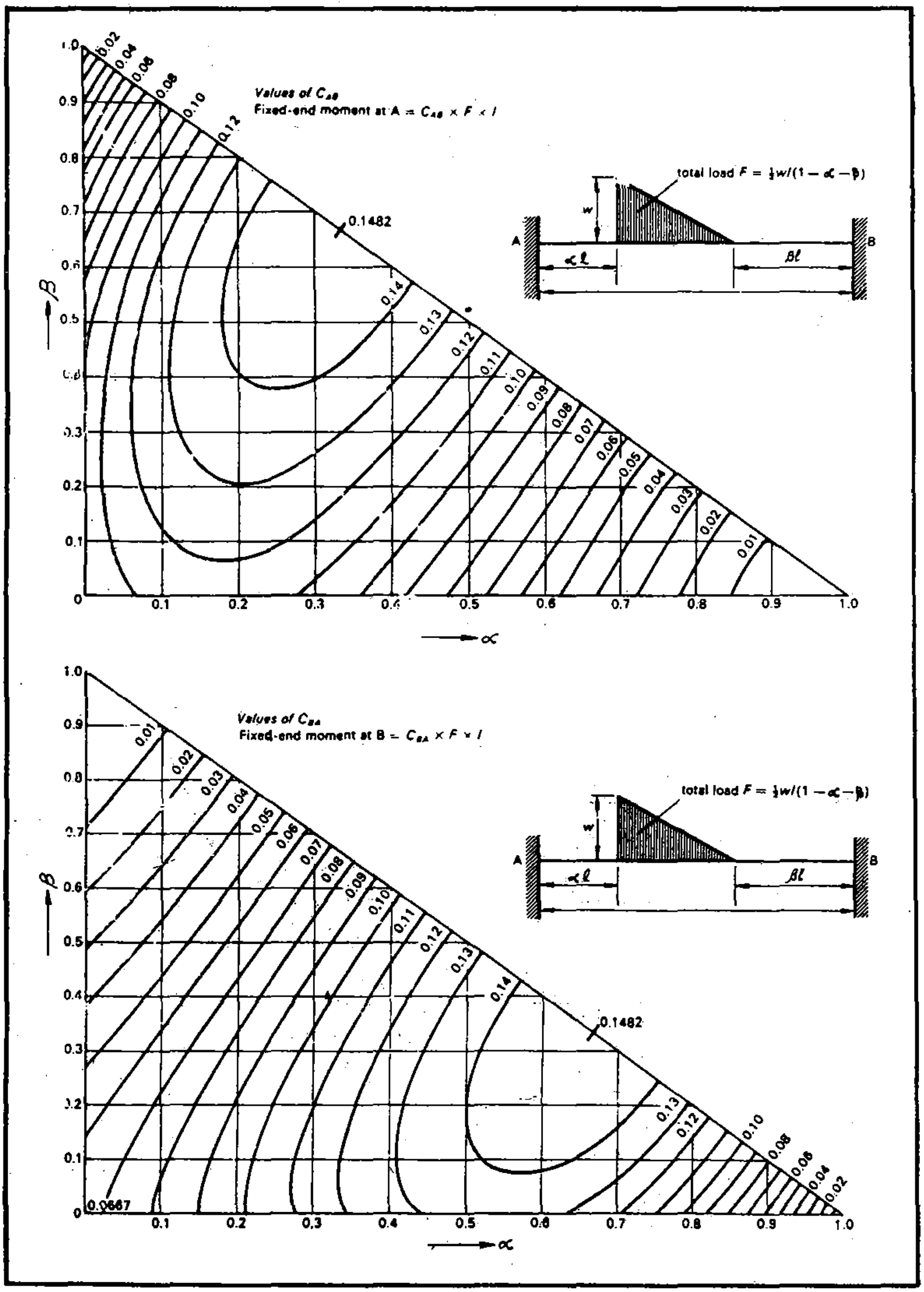
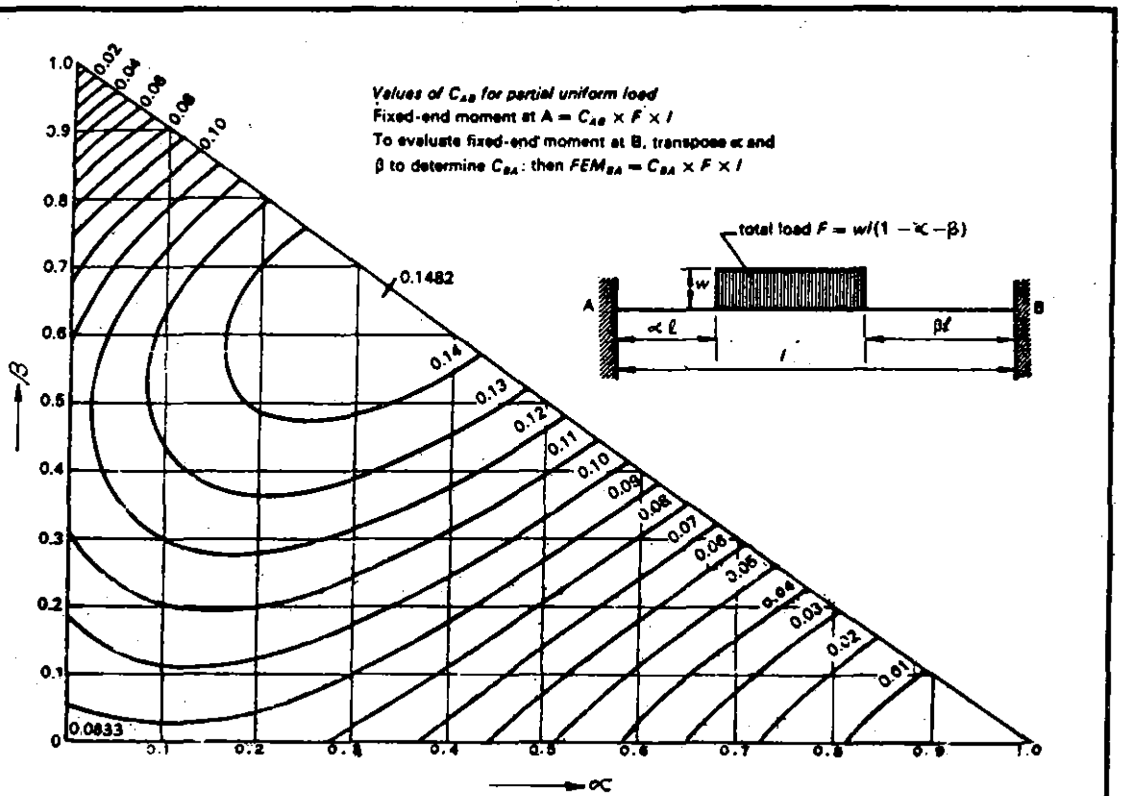


Table 20.10 *Fixed-end moment coefficients: Partial uniform and trapezoidal loads*



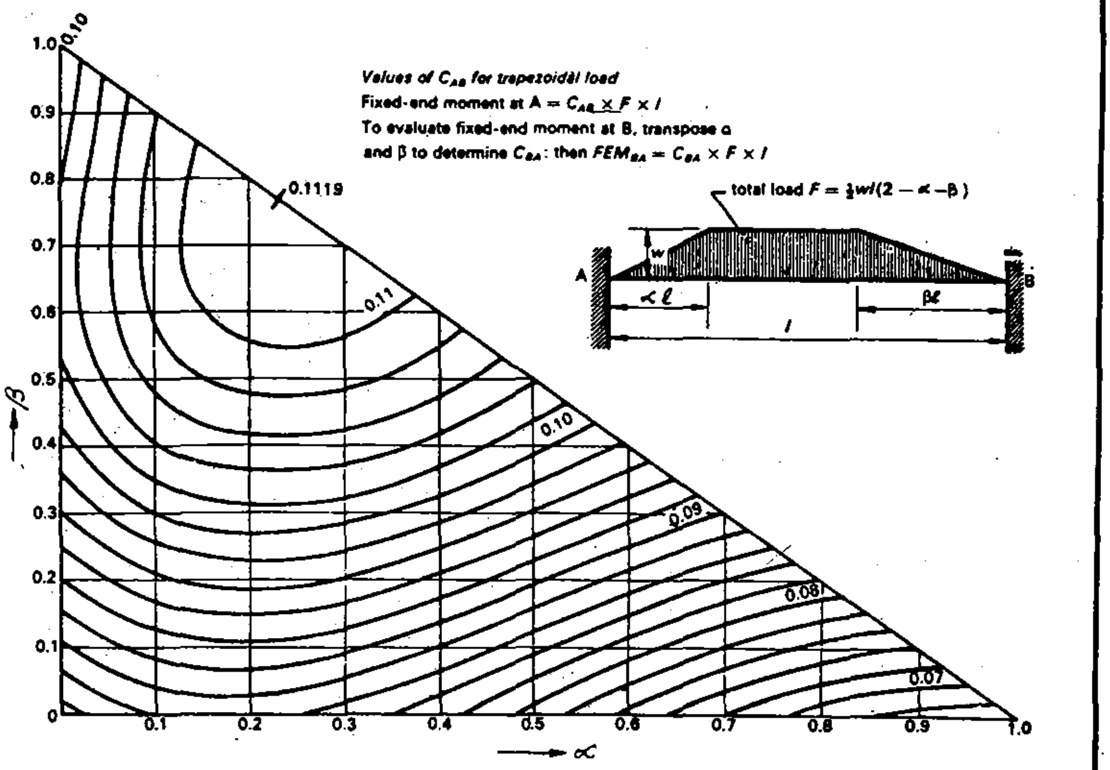
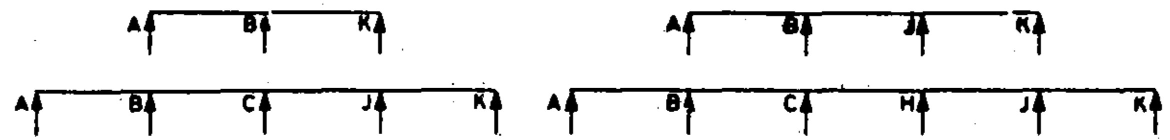


Table 20.11 *Effects of moments applied at end-supports in continuous beams of equal spans*

		Bending moment applied at A only				Equal bending moments applied at A and K			
Number of spans		2	3	4	5	2	3	4	5
Bending moment	M_A	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000
	M_B	+0.250	+0.267	+0.268	+0.268	+0.500	+0.200	+0.286	+0.263
	M_C	—	—	−0.071	−0.072	—	—	−0.143	−0.053
	M_H	—	—	—	+0.019	—	—	—	−0.053
	M_J	—	−0.067	+0.018	−0.005	—	+0.200	+0.286	+0.263
	M_K	0	0	0	0	−1.000	−1.000	−1.000	−1.000
Shearing force	V_{AR}	+1.250	+1.267	+1.268	+1.268	+1.500	+1.200	+1.286	+1.263
	V_{BL}	−1.250	−1.267	−1.268	−1.268	−1.500	−1.200	−1.286	−1.263
	V_{BR}	−0.250	−0.333	−0.339	−0.340	−1.500	0	−0.429	−0.316
	V_{CL}	—	—	+0.339	+0.340	—	—	+0.429	+0.316
	V_{CR}	—	—	+0.089	+0.091	—	—	—	0
	V_{HL}	—	—	—	−0.091	—	—	—	0
	V_{HR}	—	—	—	−0.024	—	—	+0.429	+0.316
	V_{JL}	—	+0.333	−0.089	+0.024	—	0	−0.429	−0.316
	V_{JR}	—	+0.067	−0.018	+0.005	—	−1.200	−1.286	−1.263
	V_{KL}	+0.250	−0.067	+0.018	−0.005	+1.500	+1.200	+1.286	+1.263

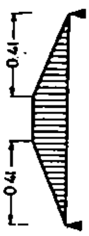
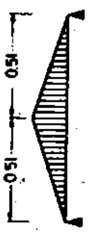
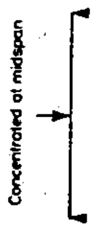
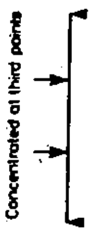
Key:



Notes: Adjustment to bending moment = M-coefficient × applied bending moment

$$\text{Adjustment to shearing force} = \frac{V\text{-coefficient} \times \text{applied bending moment}}{\text{span}}$$

Table 20.12 *Continuous beams: Moments from equal loads on equal spans—I*

Load	All spans loaded (e.g. dead load)		Imposed load (sequence of loaded spans to give max. bending moment)	
	0.155	0.094, 0.094	0.155	0.127, 0.127
	0.124, 0.124	0.107, 0.040, 0.107	0.145, 0.145	0.134, 0.102, 0.134
	0.133, 0.089, 0.133	0.103, 0.054, 0.054, 0.103	(0.144), (0.133), (0.144) 0.149, 0.133, 0.149	0.132, 0.109, 0.109, 0.132
	0.131, 0.098, 0.096, 0.131	0.104, 0.050, 0.066, 0.050, 0.104	(0.144), (0.132), (0.132), (0.144) 0.149, 0.138, 0.138, 0.149	0.133, 0.107, 0.115, 0.107, 0.133
	0.156	0.095, 0.095	0.156	0.129, 0.129
	0.125, 0.125	0.108, 0.042, 0.108	0.146, 0.146	0.136, 0.104, 0.136
	0.134, 0.089, 0.134	0.104, 0.056, 0.056, 0.104	(0.145), (0.134), (0.145) 0.151, 0.134, 0.151	0.134, 0.111, 0.111, 0.134
	0.132, 0.099, 0.099, 0.132	0.105, 0.051, 0.068, 0.051, 0.105	(0.145), (0.133), (0.133), (0.145) 0.150, 0.139, 0.139, 0.150	0.135, 0.109, 0.117, 0.109, 0.135
	0.188	0.156, 0.156	0.188	0.203, 0.203
	0.150, 0.150	0.175, 0.100, 0.175	0.175, 0.175	0.213, 0.175, 0.213
	0.161, 0.107, 0.161	0.170, 0.116, 0.116, 0.170	(0.174), (0.161), (0.174) 0.181, 0.161, 0.181	0.210, 0.183*, 0.183, 0.210
	0.158, 0.118, 0.118, 0.158	0.171, 0.112, 0.132, 0.112, 0.171	(0.174), (0.160), (0.160), (0.174) 0.179, 0.167, 0.167, 0.179	0.211, 0.181, 0.191, 0.181, 0.211
	0.167	0.111, 0.111	0.167	0.139, 0.139
	0.133, 0.133	0.122, 0.033, 0.122	0.156, 0.156	0.144, 0.100, 0.144
	0.143, 0.095, 0.143	0.119, 0.056, 0.056, 0.119	(0.155), (0.143), (0.155) 0.160, 0.144, 0.160	0.143, 0.111, 0.111, 0.143
	0.140, 0.105, 0.105, 0.140	0.120, 0.050, 0.061, 0.050, 0.120	(0.155), (0.142), (0.142), (0.155) 0.159, 0.148, 0.148, 0.159	0.144, 0.108, 0.115, 0.108, 0.144

Bending moment = (coefficient) × (total load on one span) × (span).
Bending moment coefficients:
above line apply to negative bending moment at supports.
below line apply to positive bending moment in span.
Coefficients apply when all spans are equal (or shortest ≯ 15% less than longest). Loads on each loaded span are equal. Moment of inertia same throughout all spans.
Bending moments coefficients in brackets (imposed load) apply if two spans only are loaded.

Table 20.13 *Continuous beams: Moments from equal loads on equal spans – 2*

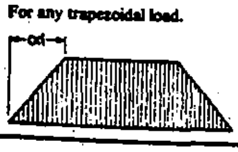
(NOTE: See note in Table 20·12

Load	All spans loaded (e.g. dead load)	Imposed load (sequence of loaded spans to give max. bending moment)
Uniformly distributed	0.125 0.070 0.070	0.125 0.096 0.096
	0.100 0.100 0.080 0.025 0.080	0.117 0.117 0.101 0.075 0.101
	0.107 0.071 0.107 0.077 0.036 0.036 0.077	(0.116) (0.107) (0.116) 0.121 0.107 0.121 0.099 0.081 0.061 0.099
	0.105 0.079 0.079 0.105 0.078 0.033 0.046 0.033 0.078	(0.116) (0.106) (0.106) (0.116) 0.120 0.111 0.111 0.120 0.100 0.079 0.086 0.079 0.100
0.1l / 0.1l	0.136 0.077 0.077	0.136 0.105 0.105
	0.109 0.109 0.088 0.028 0.088	0.127 0.127 0.111 0.083 0.111
	0.117 0.078 0.117 0.085 0.040 0.040 0.085	(0.127) (0.117) (0.127) 0.131 0.117 0.131 0.109 0.089 0.089 0.109
	0.115 0.086 0.086 0.115 0.086 0.037 0.051 0.037 0.086	(0.12[illegible]) (0.116) (0.116) (0.126) 0.131 0.121 0.121 0.131 0.110 0.087 0.094 0.087 0.110
0.2l / 0.2l	0.145 0.084 0.084	0.145 0.114 0.114
	0.116 0.116 0.095 0.032 0.095	0.135 0.135 0.120 0.090 0.120
	0.124 0.083 0.124 0.092 0.045 0.045 0.092	(0.135) (0.124) (0.135) 0.140 0.124 0.140 0.118 0.096 0.096 0.118
	0.122 0.092 0.092 0.122 0.093 0.041 0.056 0.041 0.093	(0.135) (0.123) (0.123) (0.135) 0.139 0.129 0.129 0.139 0.119 0.095 0.102 0.095 0.119
0.3l / 0.3l	0.151 0.090 0.090	0.151 0.121 0.121
	0.121 0.121 0.102 0.036 0.102	0.141 0.141 0.128 0.097 0.128
	0.130 0.086 0.130 0.098 0.050 0.050 0.098	(0.140) (0.130) (0.140) 0.146 0.130 0.146 0.126 0.103 0.103 0.126
	0.127 0.096 0.096 0.127 0.099 0.046 0.062 0.046 0.099	(0.140) (0.129) (0.129) (0.140) 0.145 0.135 0.135 0.145 0.127 0.102 0.109 0.102 0.127

Table 20.14 *Continuous beams: Shears from equal loads on equal spans*

Load	All spans loaded (e.g. dead load)	Imposed load (sequence of loaded spans to give max. shearing force)
Uniformly-distributed	0.375 0.625 0.625 0.375	0.438 0.625 0.625 0.438
	0.400 0.500 0.600 0.600 0.500 0.400	0.450 0.583 0.617 0.617 0.583 0.450
	0.393 0.536 0.464 0.607 0.607 0.464 0.536 0.393	0.446 0.603 0.571 0.621 0.621 0.571 0.603 0.446
	0.395 0.526 0.500 0.474 0.605 0.605 0.474 0.500 0.526 0.395	0.447 0.598 0.591 0.576 0.620 0.620 0.576 0.591 0.598 0.447
Triangularly-distributed	0.344 0.656 0.656 0.344	0.422 0.656 0.656 0.422
	0.375 0.500 0.625 0.625 0.500	0.437 0.605 0.646 0.646 0.605 0.437
	0.366 0.545 0.455 0.634 0.634 0.455 0.545 0.366	0.433 0.628 0.589 0.651 0.651 0.589 0.628 0.433
	0.369 0.532 0.500 0.468 0.631 0.631 0.468 0.500 0.532 0.369	0.434 0.622 0.614 0.595 0.649 0.649 0.595 0.514 0.622 0.434
Concentrated at midspan	0.313 0.688 0.688 0.313	0.406 0.688 0.688 0.406
	0.350 0.500 0.650 0.650 0.500 0.350	0.425 0.625 0.675 0.675 0.625 0.425
	0.339 0.554 0.446 0.661 0.661 0.446 0.554 0.339	0.420 0.654 0.607 0.681 0.681 0.607 0.6[illegible]4 0.420
	0.342 0.540 0.500 0.460 0.658 0.658 0.460 0.500 0.540 0.342	0.421 0.647 0.636 0.615 0.679 0.679 0.615 0.636 0.647 0.421
Concentrated at third points	0.333 0.667 0.667 0.333	0.417 0.667 0.667 0.417
	0.367 0.500 0.633 0.633 0.500 0.367	0.433 0.611 0.656 0.656 0.611 0.433
	0.357 0.548 0.452 0.643 0.643 0.452 0.548 0.357	0.429 0.637 0.595 0.661 0.661 0.595 0.637 0.429
	0.360 0.535 0.500 0.465 0.640 0.640 0.465 0.500 0.535 0.360	0.430 0.631 0.621 0.602 0.659 0.659 0.602 0.621 0.631 0.480

For any trapezoidal load.



S F Coefficient $= (k - \frac{1}{2})(1 + \alpha - \alpha^2) + \frac{1}{2}$ where k is S F coefficient for uniform load, read from above table.
e.g. if $\alpha = 0.5$, coefficient at central support of two-span beam is $(0.625 - 0.5)(1 + 0.5 - 0.25) + 0.5 = 0.656$.

Table 20.15 *Rectangular box culverts*

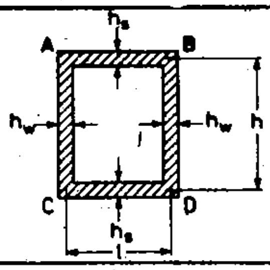
Bending moments **(per unit length of culvert)**

$M_A = M_B \qquad M_C = M_D$

Pressures and uniform loads are per unit area of walls or slab.

Loads F and G are total loads per unit length of culvert. h and l are measured between centres of walls or slabs.

q_1 = pressure transferred to soil.



$k = \frac{h}{l}\left(\frac{h_s}{h_w}\right)^3$	$K_4 = 4k + 9$
	$K_5 = 2k + 3$
$K_1 = k + 1$	$K_6 = k + 6$
$K_2 = k + 2$	$K_7 = 2k + 7$
$K_3 = k + 3$	$K_8 = 3k + 8$

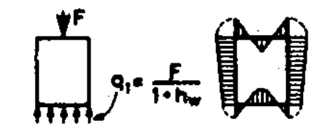
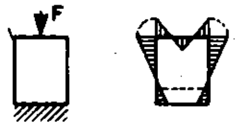
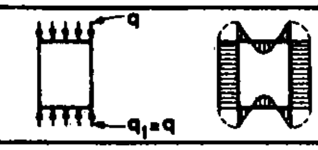
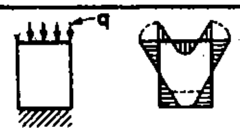
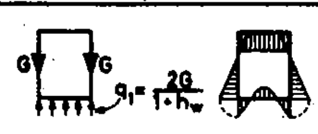
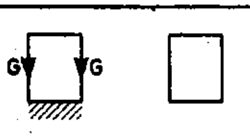
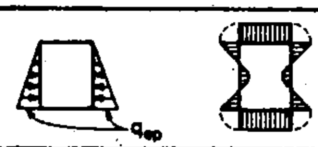
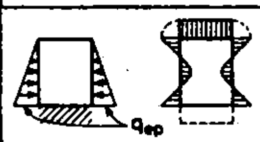
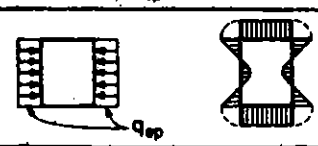
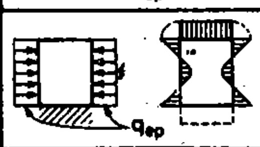
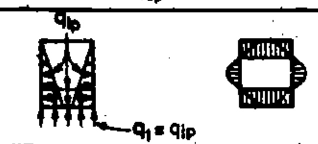
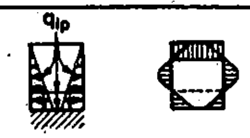
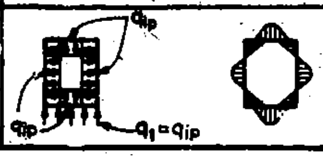
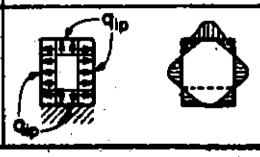
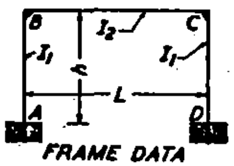
Loading	Condition of supporting ground (limiting cases)			
	Highly-compressible		Non-compressible	
Concentrated load on roof	 $q_1 = \frac{F}{l + h_w}$	$M_A = -\frac{FlK_4}{24K_1K_3}$ $M_C = \frac{K_6}{K_4} M_A$	 F	$M_A = -\frac{Fl}{4K_2}$ $M_C = -\frac{M_A}{2}$
Uniform load on roof	 q $q_1 = q$	$\left.\begin{matrix}M_A\\M_C\end{matrix}\right\} = -\frac{ql^2}{12K_1}$	 q	$M_A = -\frac{ql^2}{6K_2}$ $M_C = -\frac{M_A}{2}$
Weight of walls	 G G $q_1 = \frac{2G}{l + h_w}$	$M_A = +\frac{q_1 l^2 k}{12K_1K_3}$ $M_C = -\frac{K_6}{k} M_A$	 G G	$M_A = M_C = 0$
Earth pressure on walls	 q_{ep}	$M_A = -\frac{q_{ep}h^2kK_7}{60K_1K_3}$ $M_C = \frac{K_8}{K_7} M_A$	 q_{ep}	$M_A = -\frac{q_{ep}h^2k}{30K_2}$ $M_C = \frac{K_8}{2k} M_A$
Earth (surcharge) pressure on walls	 q_{ep}	$\left.\begin{matrix}M_A\\M_C\end{matrix}\right\} = -\frac{q_{ep}h^2k}{12K_1}$	 q_{ep}	$M_A = -\frac{q_{ep}h^2k}{12K_2}$ $M_C = \frac{K_3}{k} M_A$
Hydrostatic (internal) pressure	 q_{ip} $q_1 = q_{ip}$	$M_A = +\frac{q_{ip}h^2kK_7}{60K_1K_3}$ $M_C = \frac{K_8}{K_7} M_A$	 q_{ip}	$M_C = +\frac{q_{ip}h^2k}{30K_2}$ $M_C = \frac{K_8}{2k} M_A$
Excess hydrostatic (internal) pressure	 q_{ip} q_{ip} $q_1 = q_{ip}$	$M_A = +\frac{q_{ip}(h^2kK_3 + l^2K_6)}{12K_1K_3}$ $M_C = +\frac{q_{ip}k(h^2K_3 - l^2)}{12K_1K_3}$	 q_{ip} q_{ip}	$M_A = +\frac{q_{ip}(h^2k + 2l^2)}{12K_2}$ $M_C = +\frac{q_{ip}(h^2K_3 - l^2)}{12K_2}$

Table 20.16 *Formulae for rigid frames*

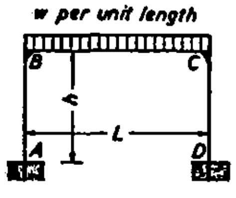
Frame _ I

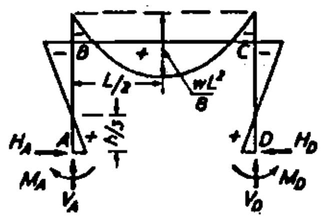


Coefficients:

$k = \frac{I_2}{I_1} \frac{h}{L}$

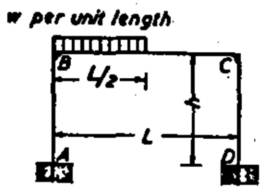
$N_1 = k + 2 \qquad N_2 = 6k + 1$

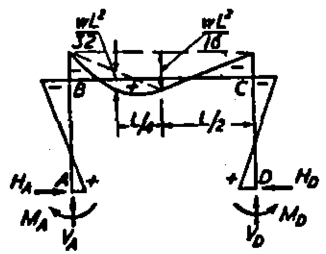




$$M_A = M_D = \frac{wL^2}{12N_1} \qquad M_B = M_C = -\frac{wL^2}{6N_1} = -2M_A$$

$$M_{max} = \frac{wL^2}{8} + M_B \qquad V_A = V_D = \frac{wL}{2} \qquad H_A = H_D = \frac{3M_A}{h}$$

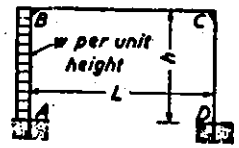


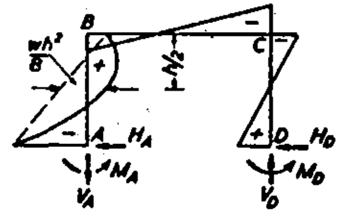


$$M_A = \frac{wL^2}{8}\left[\frac{1}{3N_1} - \frac{1}{8N_2}\right] \qquad M_B = -\frac{wL^2}{8}\left[\frac{2}{3N_1} + \frac{1}{8N_2}\right]$$

$$M_D = \frac{wL^2}{8}\left[\frac{1}{3N_1} + \frac{1}{8N_2}\right] \qquad M_C = -\frac{wL^2}{8}\left[\frac{2}{3N_1} - \frac{1}{8N_2}\right]$$

$$V_D = \frac{wL}{8}\left[1 - \frac{1}{4N_2}\right] \qquad V_A = \frac{wL}{2} - V_D \qquad H_A = H_D = \frac{wL^2}{8hN_1}$$

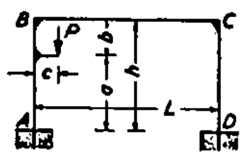


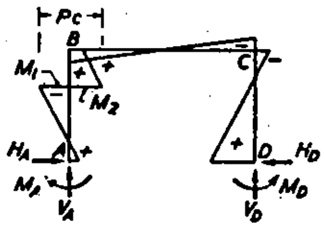


$$M_A = \frac{wh^2}{4}\left[-\frac{k+3}{6N_1} - \frac{4k+1}{N_2}\right] \qquad M_B = \frac{wh^2}{4}\left[-\frac{k}{6N_1} + \frac{2k}{N_2}\right]$$

$$M_D = \frac{wh^2}{4}\left[-\frac{k+3}{6N_1} + \frac{4k+1}{N_2}\right] \qquad M_C = \frac{wh^2}{4}\left[-\frac{k}{6N_1} - \frac{2k}{N_2}\right]$$

$$H_D = \frac{wh(2k+3)}{8N_1} \qquad H_A = -(wh - H_D) \qquad V_A = -V_D = -\frac{wh^2k}{LN_2}$$





Constants: $a_1 = \frac{a}{h} \qquad b_1 = \frac{b}{h}$

$$X_1 = \frac{Pc}{2N_1}[1 + 2b_1k - 3b_1^2(k+1)] \qquad X_2 = \frac{Pcka_1(3a_1 - 2)}{2N_1}$$

$$X_3 = \frac{3Pcka_1}{N_2}$$

$$M_A = +X_1 - \left(\frac{Pc}{2} - X_3\right) \qquad M_B = +X_2 + X_3$$

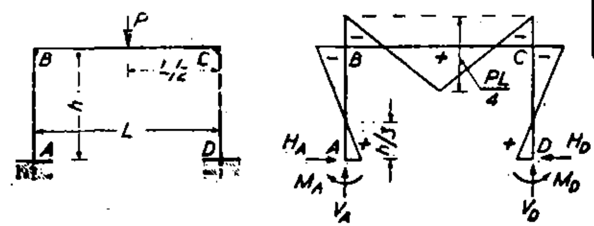
$$M_D = +X_1 + \left(\frac{Pc}{2} - X_3\right) \qquad M_C = +X_2 - X_3$$

$$H_A = H_D = \frac{Pc}{2h} + \frac{X_1 - X_2}{h} \qquad V_D = \frac{2X_3}{L} \qquad V_A = P - V_D$$

$$M_1 = M_A - H_A a \qquad M_2 = M_B + H_D b$$

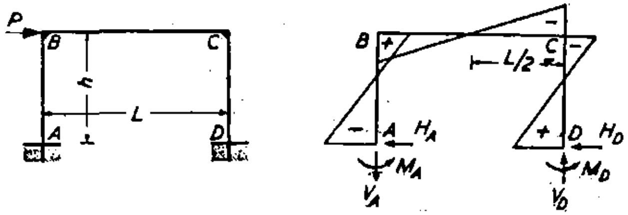
Table 20.17 *Formulae for rigid frames*

Frame-I contd.



$$M_A = M_D = +\frac{PL}{8N_1} \qquad M_B = M_C = -2M_A$$

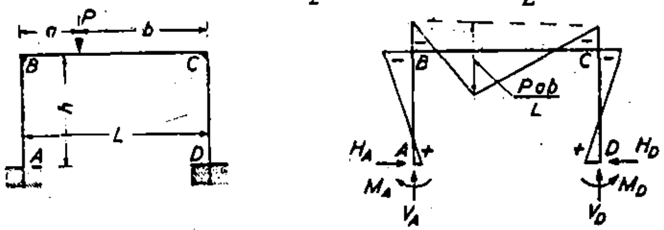
$$V_A = V_D = \frac{P}{2} \qquad H_A = H_D = \frac{3M_A}{h}$$



$$M_A = -\frac{Ph}{2}\,\frac{3k+1}{N_2} \qquad M_B = +\frac{Ph}{2}\,\frac{3k}{N_2}$$

$$M_D = +\frac{Ph}{2}\,\frac{3k+1}{N_2} \qquad M_C = -\frac{Ph}{2}\,\frac{3k}{N_2}$$

$$H_A = -H_D = -\frac{P}{2} \qquad V_A = -V_D = -\frac{2M_B}{L}$$

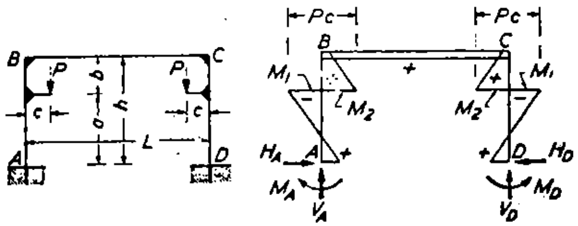


Constants: $a_1 = a/L \qquad b_1 = b/L$

$$M_A = +\frac{Pab}{L}\left[\frac{1}{2N_1} - \frac{b_1 - a_1}{2N_2}\right] \qquad M = -\frac{Pab}{L}\left[\frac{1}{N_1} + \frac{b_1 - a_1}{2N_2}\right]$$

$$M_D = +\frac{Pab}{L}\left[\frac{1}{2N_1} + \frac{b_1 - a_1}{2N_2}\right] \qquad M_C = -\frac{Pab}{L}\left[\frac{1}{N_1} - \frac{b_1 - a_1}{2N_2}\right]$$

$$V_A = Pb_1\left[1 + \frac{a_1(b_1 - a_1)}{N_2}\right] \qquad V_D = P - V_A \qquad H_A = H_D = \frac{3Pab}{2LhN_1}$$



Constants: $a_1 = \frac{a}{h} \qquad b_1 = \frac{b}{h}$

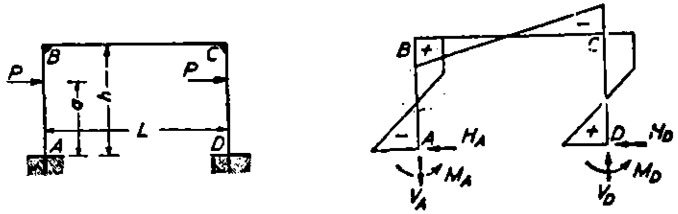
$$X_1 = \frac{Pc}{2N_1}[1 + 2b_1k - 3b_1^2(k+1)] \qquad X_2 = \frac{Pcka_1(3a_1 - 2)}{2N_1}$$

$$M_A = M_D = \frac{Pc}{N_1}[1 + 2b_1k - 3b_1^2(k+1)] = 2X_1$$

$$M_B = M_C = \frac{Pcka_1(3a_1 - 2)}{N_1} = 2X_2$$

$$V_A = V_D = P \qquad H_A = H_D = \frac{Pc + M_A - M_B}{h}$$

$$M_1 = M_A - H_A a \qquad M_2 = M_B + H_D b$$



Constants: $a_1 = \frac{a}{h} \qquad X_1 = \frac{3Paa_1k}{N_2}$

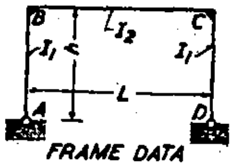
$$M_A = -Pa + X_1 \qquad M_B = X_1$$

$$M_D = +Pa - X_1 \qquad M_C = -X_1$$

$$V_A = -V_D = -\frac{2X_1}{L} \qquad H_A = -H_D = -P$$

Table 20.18 *Formulae for rigid frames*

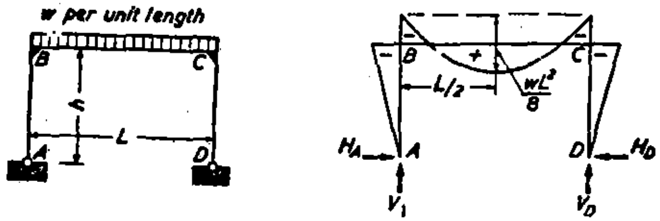
Frame - II



Coefficients:

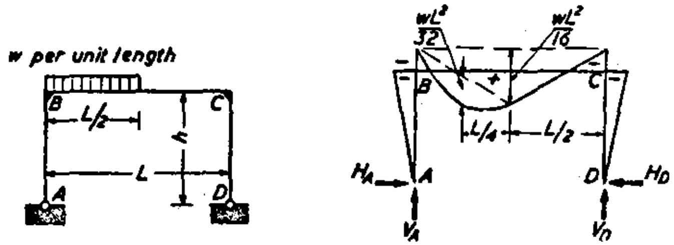
$$k = \frac{I_2}{I_1}\,\frac{h}{L}$$

$$N = 2k + 3$$



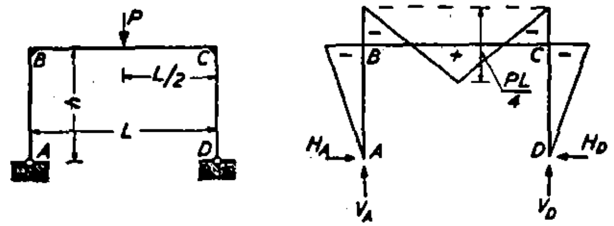
$$M_B = M_C = -\frac{wL^2}{4N} \qquad M_{max} = \frac{wL^2}{8} + M_B$$

$$V_A = V_D = \frac{wL}{2} \qquad H_A = H_D = -\frac{M_B}{h}$$

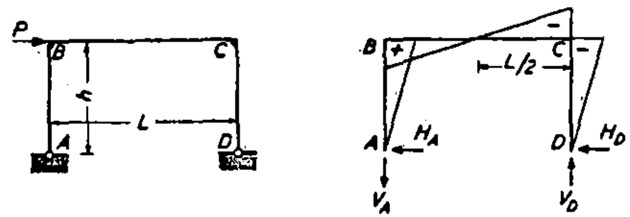


$$M_B = M_C = -\frac{wL^2}{8N}$$

$$V_A = \frac{3wL}{8} \qquad V_D = \frac{wL}{8} \qquad H_A = H_D = -\frac{M_B}{h}$$

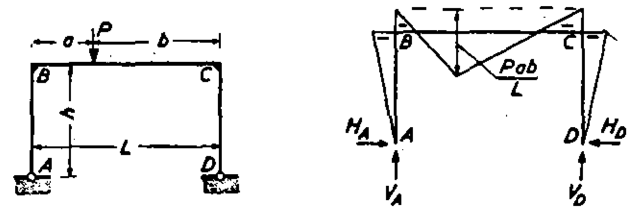


$$M_B = M_C = -\frac{3PL}{8N} \qquad V_A = V_D = \frac{P}{2} \qquad H_A = H_D = -\frac{M_B}{h}$$



$$M_B = -M_C = +\frac{Ph}{2}$$

$$V_A = -V_D = -\frac{Ph}{L} \qquad H_A = -H_D = -\frac{P}{2}$$

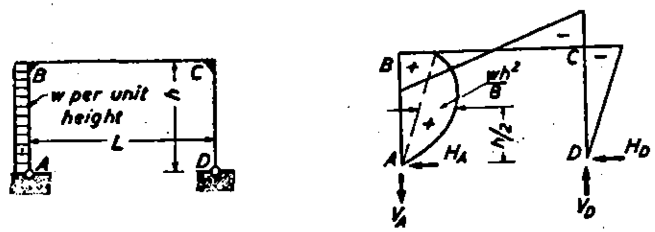


$$M_B = M_C = -\frac{Pab}{L}\,\frac{3}{2N}$$

$$V_A = \frac{Pb}{L} \qquad V_D = \frac{Pa}{L} \qquad H_A = H_D = -\frac{M_B}{h}$$

Table 20.19 *Formulae for rigid frames*

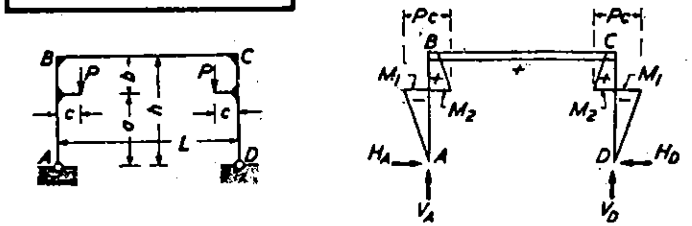
Frame – II contd.



$$M_B = \frac{wh^2}{4}\left[-\frac{k}{2N}+1\right] \qquad H_D = -\frac{M_C}{h}$$

$$M_C = \frac{wh^2}{4}\left[-\frac{k}{2N}-1\right] \qquad H_A = -(wh - H_D)$$

$$V_A = -V_D = -\frac{wh^2}{2L}$$

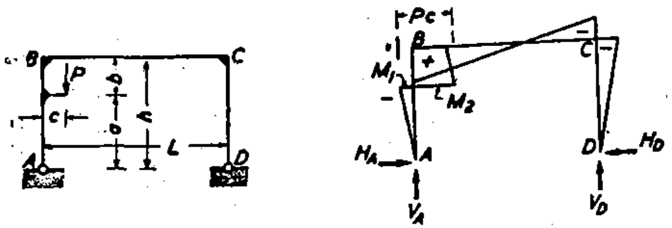


Constant: $a_1 = \frac{a}{h}$

$$M_B = M_C = \frac{Pc(3a_1^2-1)k}{N}$$

$$H_A = H_D = \frac{Pc - M_B}{h} \qquad V_A = V_D = P$$

$$M_1 = -H_A a \qquad M_2 = Pc - H_A a$$



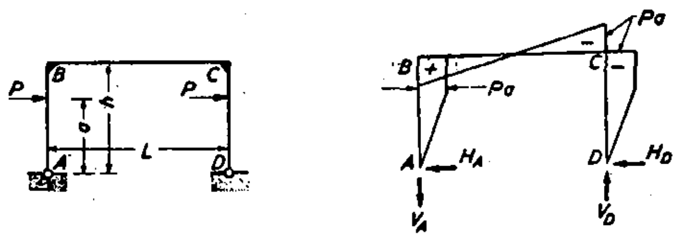
Constant: $a_1 = \frac{a}{h}$

$$M_B = \frac{Pc}{2}\left[\frac{(3a_1^2-1)k}{N}+1\right] \qquad H_A = H_D = -\frac{M_C}{h}$$

$$M_C = \frac{Pc}{2}\left[\frac{(3a_1^2-1)k}{N}-1\right]$$

$$V_D = \frac{Pc}{L} \qquad V_A = P - V_D$$

$$M_1 = -H_A a \qquad M_2 = Pc - H_A a$$



$$M_B = -M_C = Pa \qquad H_A = H_D = P$$

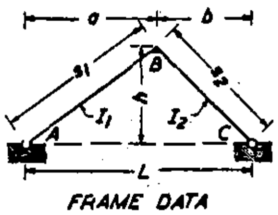
$$V_A = -V_D = -\frac{2Pa}{L}$$

Moment at loads $= \pm Pa$

Table 20.20 *Formulae for rigid frames*

(Extracts: Kleinlogel, Rahmenformeln 11. Auflage Berlin—Verlag von Wilhelm Ernest and Sohn)

FRAME - III



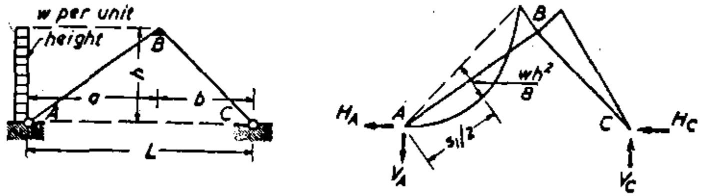
FRAME DATA

Coefficients:

$k = \frac{I_1}{I_2} \cdot \frac{s_2}{s_1}$ $\quad N = 1 + k$

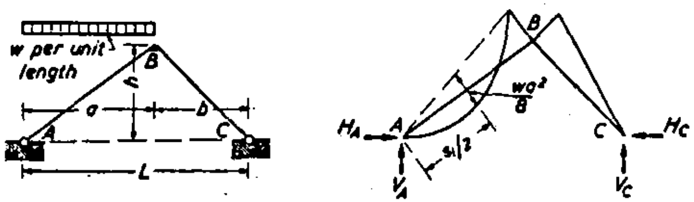
$a_1 = \frac{a}{L}$ $\quad b_1 = \frac{b}{L}$

$(a_1 + b_1 = 1)$



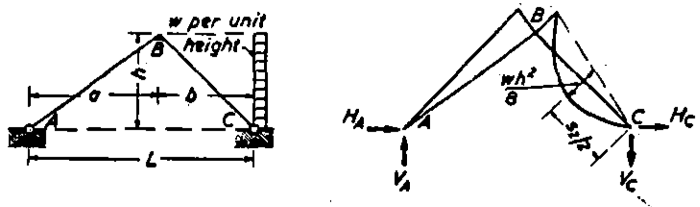
$M_B = -\frac{wh^2}{8N}$ $\quad V_A = -V_C = -\frac{wh^2}{2L}$

$H_C = \frac{whb_1}{2} - \frac{M_B}{h}$ $\quad H_A = -(wh - H_C)$



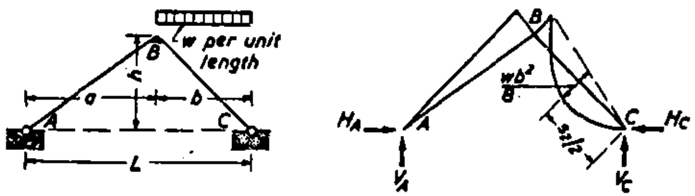
$M_B = -\frac{wa^2}{8N}$ $\quad V_C = \frac{wa^2}{2L}$ $\quad V_A = wa - V_C$

$H_A = H_C = \frac{wa^2 b_1}{2h} - \frac{M_B}{h}$



$M_B = -\frac{wh^2 k}{8N}$ $\quad V_A = -V_C = \frac{wh^2}{2L}$

$H_A = \frac{wha_1}{2} - \frac{M_B}{h}$ $\quad H_C = -(wh - H_A)$



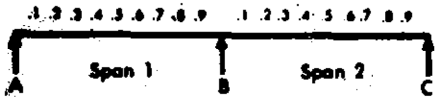
$M_B = -\frac{wb^2 k}{8N}$ $\quad V_A = \frac{wb^2}{2L}$ $\quad V_C = wb - V_A$

$H_A = H_C = \frac{wb^2 a_1}{2h} - \frac{M_B}{h}$

Table 20.21

Unit load at		MOMENTS/PL																				
		SPAN 1											SPAN 2									
		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0875	.0751	.0626	.0501	.0376	.0252	.0127	.0002	-.0123	-.0248	-.0223	-.0198	-.0173	-.0149	-.0124	-.0099	-.0074	-.0050	-.0025	0
	.2	0	.0752	.1504	.1256	.1008	.0760	.0512	.0264	.0016	-.0232	-.0480	-.0432	-.0384	-.0336	-.0288	-.0240	-.0192	-.0144	-.0096	-.0048	0
	.3	0	.0632	.1264	.1895	.1527	.1159	.0791	.0422	.0054	-.0314	-.0683	-.0614	-.0546	-.0478	-.0410	-.0341	-.0273	-.0205	-.0137	-.0068	0
	.4	0	.0516	.1032	.1548	.2064	.1580	.1096	.0612	.0128	-.0356	-.0840	-.0756	-.0672	-.0588	-.0504	-.0420	-.0336	-.0252	-.0168	-.0084	0
	.5	0	.0406	.0813	.1219	.1625	.2031	.1438	.0844	.0250	-.0344	-.0938	-.0844	-.0750	-.0656	-.0563	-.0469	-.0375	-.0281	-.0188	-.0094	0
	.6	0	.0304	.0608	.0912	.1216	.1520	.1824	.1128	.0432	-.0264	-.0960	-.0864	-.0768	-.0672	-.0576	-.0480	-.0384	-.0288	-.0192	-.0096	0
	.7	0	.0211	.0422	.0632	.0843	.1054	.1265	.1475	.0686	-.0103	-.0893	-.0803	-.0714	-.0625	-.0536	-.0446	-.0357	-.0268	-.0179	-.0089	0
	.8	0	.0128	.0256	.0384	.0512	.0640	.0768	.0896	.1024	.0152	-.0720	-.0648	-.0576	-.0504	-.0432	-.0360	-.0288	-.0216	-.0144	-.0072	0
	.9	0	.0057	.0115	.0172	.0229	.0286	.0344	.0401	.0458	.0515	-.0428	-.0385	-.0342	-.0299	-.0257	-.0214	-.0171	-.0128	-.0086	-.0043	0
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0043	-.0086	-.0128	-.0171	-.0214	-.0257	-.0299	-.0342	-.0385	-.0428	.0515	.0458	.0401	.0344	.0286	.0229	.0172	.0115	.0057	0
	.2	0	-.0072	-.0144	-.0216	-.0288	-.0360	-.0432	-.0504	-.0576	-.0648	-.0720	.0152	.1024	.0896	.0768	.0640	.0512	.0384	.0256	.0128	0
	.3	0	-.0089	-.0179	-.0268	-.0357	-.0446	-.0536	-.0625	-.0714	-.0803	-.0893	-.0103	.0686	.1475	.1265	.1054	.0843	.0632	.0422	.0211	0
	.4	0	-.0096	-.0192	-.0288	-.0384	-.0480	-.0576	-.0672	-.0768	-.0864	-.0960	-.0264	.0432	.1128	.1824	.1520	.1216	.0912	.0608	.0304	0
	.5	0	-.0094	-.0188	-.0281	-.0375	-.0469	-.0563	-.0656	-.0750	-.0844	-.0938	-.0344	.0250	.0844	.1438	.2031	.1625	.1219	.0813	.0406	0
	.6	0	-.0084	-.0168	-.0252	-.0336	-.0420	-.0504	-.0588	-.0672	-.0756	-.0840	-.0356	.0128	.0612	.1096	.1580	.2064	.1548	.1032	.0516	0
	.7	0	-.0068	-.0137	-.0205	-.0273	-.0341	-.0410	-.0478	-.0546	-.0614	-.0683	-.0314	.0054	.0422	.0791	.1159	.1527	.1895	.1264	.0632	0
	.8	0	-.0048	-.0096	-.0144	-.0192	-.0240	-.0288	-.0336	-.0384	-.0432	-.0480	-.0232	.0016	.0264	.0512	.0760	.1008	.1256	.1504	.0752	0
	.9	0	-.0025	-.0050	-.0074	-.0099	-.0124	-.0149	-.0173	-.0198	-.0223	-.0248	-.0123	.0002	.0127	.0252	.0376	.0501	.0626	.0751	.0875	0
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0388	.0675	.0863	.0950	.0938	.0825	.0613	.0300	.0061	0	.0061	.0300	.0613	.0825	.0938	.0950	.0863	.0675	.0388	0
− Area		0	-.0063	-.0125	-.0188	-.0250	-.0313	-.0375	-.0438	-.0500	-.0736	-.1250	-.0736	-.0300	-.0438	-.0375	-.0313	-.0250	-.0188	-.0125	-.0063	0
Total Area		0	.0325	.0550	.0675	.0700	.0625	.0450	.0175	-.0200	-.0675	-.1250	-.0675	-.0200	.0175	.0450	.0625	.0700	.0675	.0550	.0325	0

Unit load at		REACTIONS/P			SHEARS/P			
		R_A	R_B	R_C	V_{AB}	V_{BA}	V_{BC}	V_{CB}
SPAN 1	A	1.0	0	0	1.0	0	0	0
	.1	.8753	.1495	-.0248	.8753	-.1247	.0248	.0248
	.2	.7520	.2960	-.0480	.7520	-.2480	.0480	.0480
	.3	.6318	.4365	-.0683	.6318	-.3682	.0683	.0683
	.4	.5160	.5680	-.0840	.5160	-.4840	.0840	.0840
	.5	.4063	.6875	-.0938	.4063	-.5937	.0938	.0938
	.6	.3040	.7920	-.0960	.3040	-.6960	.0960	.0960
	.7	.2108	.8785	-.0893	.2108	-.7892	.0893	.0893
	.8	.1280	.9440	-.0720	.1280	-.8720	.0720	.0720
	.9	.0573	.9855	-.0428	.0573	-.9427	.0428	.0428
	B	0	1.0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 2	.1	-.0428	.9855	.0573	-.0428	-.0428	.9427	-.0573
	.2	-.0720	.9440	.1280	-.0720	-.0720	.8720	-.1280
	.3	-.0893	.8785	.2108	-.0893	-.0893	.7892	-.2108
	.4	-.0960	.7920	.3040	-.0960	-.0960	.6960	-.3040
	.5	-.0938	.6875	.4063	-.0938	-.0938	.5937	-.4063
	.6	-.0840	.5680	.5160	-.0840	-.0840	.4840	-.5160
	.7	-.0683	.4365	.6318	-.0683	-.0683	.3682	-.6318
	.8	-.0480	.2960	.7520	-.0480	-.0480	.2480	-.7520
	.9	-.0248	.1495	.8753	-.0248	-.0248	.1247	-.8753
	C	0	0	1.0	0	0	0	-1.0
+ Area		.4375	1.2500	.4375	.4375	0	.6250	.0625
− Area		-.0625	0	-.0625	-.0625	-.6250	0	-.4375
Total Area		.3750	1.2500	.3750	.3750	-.6250	.6250	-.3750



Influence coefficients — Two continuous spans.

L = Length of SHORTER span; length of LONGER span = NL.

N = 1.0

Table 20.22

Moments/PL — Span 1 columns

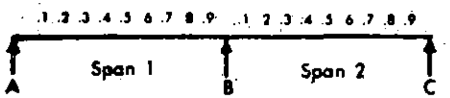
Unit load at	A.	.1	.2	.3	.4	.5	.6	.7	.8	.9	B
Span 1 A.	0	0	0	0	0	0	0	0	0	0	0
Span 1 .1	0	.0876	.0753	.0629	.0506	.0382	.0259	.0135	.0011	-.0112	-.0236
Span 1 .2	0	.0754	.1509	.1263	.1017	.0771	.0526	.0280	.0034	-.0211	-.0457
Span 1 .3	0	.0635	.1270	.1905	.1540	.1175	.0810	.0445	.0080	-.0285	-.0650
Span 1 .4	0	.0520	.1040	.1560	.2080	.1600	.1120	.0640	.0160	-.0320	-.0800
Span 1 .5	0	.0411	.0821	.1232	.1643	.2054	.1464	.0875	.0286	-.0304	-.0893
Span 1 .6	0	.0309	.0617	.0926	.1234	.1543	.1851	.1160	.0469	-.0223	-.0914
Span 1 .7	0	.0215	.0430	.0645	.0860	.1075	.1290	.1505	.0720	-.0065	-.0850
Span 1 .8	0	.0131	.0263	.0394	.0526	.0657	.0789	.0920	.1051	.0183	-.0686
Span 1 .9	0	.0059	.0119	.0178	.0237	.0296	.0356	.0415	.0474	.0534	-.0407
B	0	0	0	0	0	0	0	0	0	0	0
Span 2 .1	0	-.0049	-.0099	-.0148	-.0197	-.0246	-.0296	-.0345	-.0394	-.0443	-.0493
Span 2 .2	0	-.0083	-.0166	-.0249	-.0332	-.0415	-.0498	-.0581	-.0664	-.0747	-.0830
Span 2 .3	0	-.0103	-.0206	-.0309	-.0411	-.0514	-.0617	-.0720	-.0823	-.0926	-.1028
Span 2 .4	0	-.0111	-.0221	-.0332	-.0443	-.0553	-.0664	-.0774	-.0885	-.0996	-.1106
Span 2 .5	0	-.0108	-.0216	-.0324	-.0432	-.0540	-.0648	-.0756	-.0864	-.0972	-.1080
Span 2 .6	0	-.0097	-.0194	-.0290	-.0387	-.0484	-.0581	-.0678	-.0774	-.0871	-.0968
Span 2 .7	0	-.0079	-.0157	-.0236	-.0315	-.0393	-.0472	-.0551	-.0629	-.0708	-.0786
Span 2 .8	0	-.0055	-.0111	-.0166	-.0221	-.0277	-.0332	-.0387	-.0443	-.0498	-.0553
Span 2 .9	0	-.0029	-.0057	-.0086	-.0114	-.0143	-.0171	-.0200	-.0228	-.0257	-.0285
C.	0	0	0	0	0	0	0	0	0	0	0
+Area	0	.0390	.0681	.0871	.0962	.0952	.0843	.0633	.0324	.0067	0
−Area	0	-.0079	-.0158	-.0236	-.0317	-.0396	-.0475	-.0555	-.0634	-.0865	-.1387
Total Area	0	.0311	.0523	.0633	.0645	.0556	.0368	-.0078	-.0310	-.0798	-.1387

Moments/PL — Span 2 columns

Unit load at	.1	.2	.3	.4	.5	.6	.7	.8	.9	C.
Span 1 A.	0	0	0	0	0	0	0	0	0	0
Span 1 .1	-.0212	-.0189	-.0165	-.0141	-.0118	-.0094	-.0071	-.0047	-.0024	0
Span 1 .2	-.0411	-.0366	-.0320	-.0274	-.0229	-.0183	-.0137	-.0091	-.0046	0
Span 1 .3	-.0585	-.0520	-.0455	-.0390	-.0325	-.0260	-.0195	-.0130	-.0065	0
Span 1 .4	-.0720	-.0640	-.0560	-.0480	-.0400	-.0320	-.0240	-.0160	-.0080	0
Span 1 .5	-.0804	-.0714	-.0625	-.0536	-.0446	-.0357	-.0268	-.0179	-.0089	0
Span 1 .6	-.0823	-.0731	-.0640	-.0549	-.0457	-.0366	-.0274	-.0183	-.0091	0
Span 1 .7	-.0765	-.0680	-.0595	-.0510	-.0425	-.0340	-.0255	-.0170	-.0085	0
Span 1 .8	-.0617	-.0549	-.0480	-.0411	-.0343	-.0274	-.0206	-.0137	-.0069	0
Span 1 .9	-.0366	-.0326	-.0285	-.0244	-.0204	-.0163	-.0122	-.0081	-.0041	0
B	0	0	0	0	0	0	0	0	0	0
Span 2 .1	.0547	.0486	.0425	.0364	.0304	.0243	.0182	.0121	.0061	0
Span 2 .2	.0133	.1096	.0959	.0822	.0685	.0548	.0411	.0274	.0137	0
Span 2 .3	-.0156	.0717	.1590	.1363	.1136	.0909	.0681	.0454	.0227	0
Span 2 .4	-.0336	.0435	.1206	.1976	.1647	.1317	.0988	.0659	.0329	0
Span 2 .5	-.0422	.0236	.0894	.1552	.2210	.1768	.1326	.0884	.0442	0
Span 2 .6	-.0431	.0106	.0642	.1179	.1716	.2253	.1690	.1126	.0563	0
Span 2 .7	-.0378	.0031	.0439	.0848	.1257	.1665	.2074	.1383	.0691	0
Span 2 .8	-.0278	-.0003	.0273	.0548	.0823	.1090	.1374	.1649	.0825	0
Span 2 .9	-.0147	-.0008	.0130	.0269	.0407	.0546	.0684	.0823	.0961	0
C.	0	0	0	0	0	0	0	0	0	0
+Area	.0068	.0336	.0716	.0977	.1116	.1135	.1033	.0810	.0465	0
−Area	-.0772	-.0478	-.0417	-.0357	-.0298	-.0238	-.0179	-.0119	-.0060	0
Total Area	-.0704	-.0142	.0299	.0620	.0819	.0897	.0854	.0691	.0405	0

Reactions/P and Shears/P

Unit load at	R_A	R_B	R_C	V_{AB}	V_{BA}	V_{BC}	V_{CB}
Span 1 A.	1.0000	0	0	1.0000	0	0	0
Span 1 .1	.8764	.1450	-.0214	.8764	-.1236	.0214	.0214
Span 1 .2	.7543	.2873	-.0416	.7543	-.2457	.0416	.0416
Span 1 .3	.6350	.4241	-.0591	.6350	-.3650	.0591	.0591
Span 1 .4	.5200	.5527	-.0727	.5200	-.4800	.0727	.0727
Span 1 .5	.4107	.6705	-.0812	.4107	-.5893	.0812	.0812
Span 1 .6	.3086	.7745	-.0831	.3086	-.6914	.0831	.0831
Span 1 .7	.2150	.8623	-.0773	.2150	-.7850	.0773	.0773
Span 1 .8	.1314	.9309	-.0623	.1314	-.8686	.0623	.0623
Span 1 .9	.0593	.9777	-.0370	.0593	-.9407	.0370	.0370
B.	0	1.0000	0	0	-1.0 / 0	0 / 1.0	0
Span 2 .1	-.0493	.9941	.0552	-.0493	-.0493	.9446	-.0552
Span 2 .2	-.0830	.9584	.1246	-.0830	-.0830	.8754	-.1246
Span 2 .3	-.1028	.8963	.2065	-.1028	-.1028	.7935	-.2065
Span 2 .4	-.1106	.8112	.2994	-.1106	-.1106	.7006	-.2994
Span 2 .5	-.1080	.7062	.4018	-.1080	-.1080	.5982	-.4018
Span 2 .6	-.0968	.5848	.5120	-.0968	-.0968	.4880	-.5120
Span 2 .7	-.0786	.4501	.6285	-.0786	-.0786	.3715	-.6285
Span 2 .8	-.0553	.3056	.7497	-.0553	-.0553	.2503	-.7497
Span 2 .9	-.0285	.1545	.8741	-.0285	-.0285	.1259	-.8741
C.	0	0	1.0000	0	0	0	-1.0000
+Area	.4405	1.3149	.4780	.4405	0	.6761	.0541
−Area	-.0792	0	-.0541	-.0792	-.6388	0	-.4780
Total Area	.3613	1.3149	.4239	.3613	-.6388	.6761	-.4239



Influence coefficients — Two continuous spans.

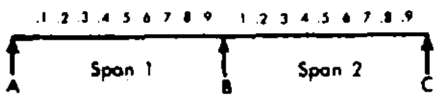
L = Length of SHORTER span   length of LONGER span = NL.

N = 1.1

Table 20.23

Unit load at		MOMENTS/PL																				
		SPAN 1										SPAN 2										
		A.	.1	.2	.3	.4	.5	.6	.7	.8	.9	B.	.1	.2	.3	.4	.5	.6	.7	.8	.9	C.
SPAN 1	A.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0877	.0755	.0633	.0510	.0388	.0265	.0143	.0020	-.0102	-.0225	-.0203	-.0180	-.0157	-.0135	-.0113	-.0090	-.0067	-.0045	-.0023	0
	.2	0	.0756	.1513	.1269	.1025	.0782	.0538	.0295	.0051	-.0193	-.0436	-.0393	-.0349	-.0305	-.0262	-.0218	-.0175	-.0131	-.0087	-.0044	0
	.3	0	.0638	.1276	.1914	.1552	.1190	.0828	.0466	.0104	-.0258	-.0620	-.0558	-.0496	-.0434	-.0372	-.0310	-.0248	-.0186	-.0124	-.0062	0
	.4	0	.0524	.1047	.1571	.2095	.1618	.1142	.0665	.0189	-.0287	-.0764	-.0687	-.0611	-.0535	-.0458	-.0382	-.0305	-.0229	-.0153	-.0076	0
	.5	0	.0415	.0830	.1244	.1659	.2074	.1489	.0903	.0318	-.0267	-.0852	-.0767	-.0682	-.0597	-.0511	-.0426	-.0341	-.0256	-.0170	-.0085	0
	.6	0	.0313	.0625	.0938	.1251	.1564	.1876	.1189	.0502	-.0185	-.0873	-.0785	-.0698	-.0611	-.0524	-.0436	-.0349	-.0262	-.0175	-.0087	0
	.7	0	.0219	.0438	.0657	.0875	.1094	.1313	.1532	.0751	-.0030	-.0811	-.0730	-.0649	-.0568	-.0487	-.0406	-.0325	-.0243	-.0162	-.0081	0
	.8	0	.0135	.0269	.0404	.0538	.0673	.0807	.0942	.1076	.0211	-.0655	-.0589	-.0524	-.0458	-.0393	-.0327	-.0262	-.0196	-.0131	-.0065	0
	.9	0	.0061	.0122	.0183	.0245	.0306	.0367	.0428	.0489	.0550	-.0389	-.0350	-.0311	-.0272	-.0233	-.0194	-.0155	-.0117	-.0078	-.0039	0
	B.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0056	-.0112	-.0168	-.0224	-.0280	-.0336	-.0392	-.0448	-.0504	-.0560	.0576	.0512	.0448	.0384	.0320	.0256	.0192	.0128	.0064	0
	.2	0	-.0094	-.0189	-.0283	-.0377	-.0471	-.0566	-.0660	-.0754	-.0848	-.0943	.0112	.1166	.1020	.0874	.0729	.0583	.0437	.0291	.0146	0
	.3	0	-.0117	-.0234	-.0351	-.0467	-.0584	-.0701	-.0818	-.0935	-.1052	-.1168	-.0212	.0745	.1702	.1459	.1216	.0973	.0729	.0486	.0243	0
	.4	0	-.0126	-.0251	-.0377	-.0503	-.0628	-.0754	-.0880	-.1005	-.1131	-.1257	-.0411	.0435	.1280	.2126	.1772	.1417	.1063	.0709	.0354	0
	.5	0	-.0123	-.0245	-.0368	-.0491	-.0614	-.0736	-.0859	-.0982	-.1105	-.1227	-.0505	.0218	.0941	.1664	.2388	.1909	.1432	.0955	.0477	0
	.6	0	-.0110	-.0220	-.0330	-.0440	-.0550	-.0660	-.0770	-.0880	-.0990	-.1100	-.0510	.0080	.0670	.1260	.1850	.2440	.1830	.1220	.0610	0
	.7	0	-.0089	-.0179	-.0268	-.0357	-.0447	-.0536	-.0625	-.0715	-.0804	-.0893	-.0444	.0005	.0455	.0904	.1353	.1803	.2252	.1501	.0751	0
	.8	0	-.0063	-.0126	-.0189	-.0251	-.0314	-.0377	-.0440	-.0503	-.0566	-.0628	-.0326	.0023	.0280	.0583	.0886	.1189	.1491	.1794	.0897	0
	.9	0	-.0032	-.0065	-.0097	-.0130	-.0162	-.0194	-.0227	-.0259	-.0292	-.0324	-.0172	.0019	.0133	.0286	.0438	.0590	.0743	.0895	.1048	0
	C.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+Area		0	.0393	.0686	.0880	.0973	.0966	.0859	.0652	.0345	.0072	0	.0075	.0372	.0825	.1139	.1309	.1335	.1217	.0956	.0550	0
−Area		0	-.0098	-.0196	-.0295	-.0393	-.0491	-.0589	-.0687	-.0785	-.1017	-.1550	-.0822	-.0460	-.0398	-.0341	-.0284	-.0227	-.0170	-.0114	-.0057	0
Total Area		0	.0295	.0490	.0585	.0580	.0475	.0270	-.0035	-.0440	-.0945	-.1550	-.0747	-.0088	.0427	.0798	.1025	.1108	.1047	.0842	.0493	0

Unit load at		REACTIONS/P			SHEARS/P			
		R_A	R_B	R_C	V_{AB}	V_{BA}	V_{BC}	V_{CB}
SPAN 1	A.	1.0000	0	0	1.0000	0	0	0
	.1	.8775	.1413	-.0187	.8775	-.1225	.0187	.0187
	.2	[illegible]	[illegible]	-.0364	.7564	-.2436	.0384	.0364
	.3	.6380	.4138	-.0517	.6380	-.3620	.0517	.0517
	.4	.5236	.5400	-.0636	.5236	-.4764	.0636	.0636
	.5	.4148	.6563	-.0710	.4148	-.5852	.0710	.0710
	.6	.3127	.7600	-.0727	.3127	-.6873	.0727	.0727
	.7	.2189	.8487	-.0676	.2189	-.7811	.0676	.0676
	.8	.1345	.9200	-.0545	.1345	-.8655	.0545	.0545
	.9	.0611	.9713	-.0324	.0611	-.9389	.0324	.0324
	B.	0	1.0000	0	0	-1.0 / 0	0 / 1.0	0
SPAN 2	.1	-.0560	1.0026	.0534	-.0560	-.0560	.9466	-.0534
	.2	-.0943	.9728	.1215	-.0943	-.0943	.8785	-.1215
	.3	-.1166	.9142	.2026	-.1168	-.1168	.7974	-.2026
	.4	-.1257	.8304	.2953	-.1257	-.1257	.7047	-.2953
	.5	-.1227	.7250	.3977	-.1227	-.1227	.6023	-.3977
	.6	-.1100	.6016	.5084	-.1100	-.1100	.4916	-.5084
	.7	-.0893	.4638	.6255	-.0893	-.0893	.3745	-.6255
	.8	-.0628	.3152	.7476	-.0628	-.0628	.2524	-.7476
	.9	-.0324	.1594	.8730	-.0324	-.0324	.1270	-.8730
	C.	0	0	1.0000	0	0	0	-1.0000
+Area		.4432	1.3842	.5182	.4432	0	.7292	.0473
−Area		-.0982	0	-.0473	-.0982	-.6550	0	-.5182
Total Area		.3450	1.3842	.4709	.3450	-.6550	.7292	-.4709



Influence coefficients — Two continuous spans.

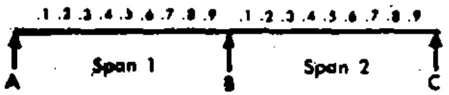
L = Length of SHORTER span length of LONGER span = NL

N = 1.2

Table 20.24

Unit load at		MOMENTS/PL SPAN 1 A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	SPAN 2 .1	.2	.3	.4	.5	.6	.7	.8	.9	C.
SPAN 1	A.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0878	.0757	.0635	.0514	.0392	.0271	.0149	.0028	-.0094	-.0215	-.0194	-.0172	-.0151	-.0129	-.0108	-.0086	-.0065	-.0043	-.0022	0
	.2	0	.0758	.1517	.1275	.1033	.0791	.0550	.0308	.0066	-.0176	-.0417	-.0376	-.0334	-.0292	-.0250	-.0209	-.0167	-.0125	-.0083	-.0042	0
	.3	0	.0641	.1281	.1922	.1563	.1203	.0844	.0485	.0125	-.0234	-.0593	-.0534	-.0475	-.0415	-.0356	-.0297	-.0237	-.0178	-.0119	-.0059	0
	.4	0	.0527	.1054	.1581	.2108	.1635	.1162	.0689	.0216	-.0257	-.0730	-.0657	-.0584	-.0511	-.0438	-.0365	-.0292	-.0219	-.0146	-.0073	0
	.5	0	.0418	.0837	.1255	.1674	.2092	.1511	.0929	.0348	-.0234	-.0815	-.0734	-.0652	-.0571	-.0489	-.0408	-.0326	-.0245	-.0163	-.0082	0
	.6	0	.0317	.0633	.0950	.1266	.1583	.1899	.1216	.0532	-.0151	-.0835	-.0751	-.0668	-.0584	-.0501	-.0417	-.0334	-.0250	-.0167	-.0083	0
	.7	0	.0222	.0445	.0667	.0890	.1112	.1334	.1557	.0779	.0002	-.0776	-.0698	-.0621	-.0543	-.0466	-.0388	-.0310	-.0233	-.0155	-.0078	0
	.8	0	.0137	.0275	.0412	.0550	.0687	.0824	.0962	.1099	.0237	-.0626	-.0563	-.0501	-.0438	-.0376	-.0313	-.0250	-.0188	-.0125	-.0063	0
	.9	0	.0063	.0126	.0188	.0251	.0314	.0377	.0440	.0503	.0565	-.0372	-.0335	-.0297	-.0260	-.0223	-.0186	-.0149	-.0112	-.0074	-.0037	0
	B.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0063	-.0126	-.0188	-.0251	-.0314	-.0377	-.0440	-.0503	-.0565	-.0628	.0605	.0537	.0470	.0403	.0336	.0269	.0202	.0134	.0067	0
	.2	0	-.0106	-.0212	-.0317	-.0423	-.0529	-.0635	-.0741	-.0846	-.0952	-.1058	.0088	.1234	.1079	.0925	.0771	.0617	.0463	.0308	.0154	0
	.3	0	-.0131	-.0262	-.0393	-.0525	-.0656	-.0787	-.0918	-.1049	-.1180	-.1312	-.0270	.0771	.1812	.1553	.1294	.1035	.0777	.0518	.0259	0
	.4	0	-.0141	-.0282	-.0423	-.0564	-.0705	-.0846	-.0988	-.1129	-.1270	-.1411	-.0490	.0431	.1352	.2274	.1895	.1516	.1137	.0758	.0379	0
	.5	0	-.0138	-.0276	-.0413	-.0551	-.0689	-.0827	-.0964	-.1102	-.1240	-.1378	-.0590	.0198	.0986	.1773	.2561	.2049	.1537	.1024	.0512	0
	.6	0	-.0123	-.0247	-.0370	-.0494	-.0617	-.0741	-.0[illegible]34	-.0988	-.1111	-.1234	-.0591	.0052	.0696	.1339	.1983	.2626	.1970	.1313	.0657	0
	.7	0	-.0100	-.0201	-.0301	-.0401	-.0501	-.0602	-.0702	-.0802	-.0903	-.1003	-.0513	-.0022	.0468	.0958	.1449	.1939	.2429	.1619	.0810	0
	.8	0	-.0071	-.0141	-.0212	-.0282	-.0353	-.0423	-.0494	-.0564	-.0635	-.0705	-.0375	-.0044	.0286	.0617	.0947	.1278	.1608	.1939	.0969	0
	.9	0	-.0036	-.0073	-.0109	-.0145	-.0182	-.0218	-.0255	-.0291	-.0327	-.0364	-.0197	-.0031	.0135	.0302	.0468	.0635	.0801	.0967	.1134	0
	C.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+Area		0	.0396	.0691	.0887	.0983	.0978	.0874	.0670	.0365	.0078	.0	.0082	.0410	.0939	.1312	.1515	.1550	.1416	.1113	.0641	0
-Area		0	-.0119	-.0239	-.0358	-.0478	-.0597	-.0716	-.0836	-.0955	-.1192	-.1737	-.0885	-.0448	-.0380	-.0326	-.0272	-.0217	-.0163	-.0109	-.0054	0
Total Area		0	.0277	.0452	.0529	.0505	.0381	.0158	-.0166	-.0590	-.1114	-.1737	-.0803	-.0038	.0559	.0986	.1243	.1333	.1253	.1004	.0587	0

Unit load at		REACTIONS/P R_A	R_B	R_C	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}
SPAN 1	A.	1.0000	0	0.	1.0000	0	0	0
	.1	.8785	.1381	-.0166	.8785	-.1215	.0166	.0166
	.2	.7583	.2738	-.0321	.7583	-.2417	.0321	.0321
	.3	.6407	.4050	-.0457	.6407	-.3593	.0457	.0457
	.4	.5270	.5292	-.0562	.5270	-.4730	.0562	.0562
	.5	.4185	.6442	-.0627	.4185	-.5815	.0627	.0627
	.6	.3165	.7477	-.0642	.3165	-.6835	.0642	.0642
	.7	.2224	.8373	-.0597	.2224	-.7776	.0597	.0597
	.8	.1374	.9108	-.0482	.1374	-.8626	.0482	.0482
	.9	.0628	.9658	-.0286	.0628	-.9372	.0286	.0286
	B.	0	1.0000	0	0	-1.0 / 0	0 / 1.0	0
SPAN 2	.1	-.0628	1.0112	.0517	-.0628	-.0628	.9483	-.0517
	.2	-.1058	.9872	.1186	-.1058	-.1058	.8814	-.1186
	.3	-.1312	.9320	.1991	-.1312	-.1312	.8009	-.1991
	.4	-.1411	.8496	.2915	-.1411	-.1411	.7085	-.2915
	.5	-.1378	.7438	.3940	-.1378	-.1378	.6060	-.3940
	.6	-.1234	.6184	.5050	-.1234	-.1234	.4950	-.5050
	.7	-.1003	.4774	.6228	-.1003	-.1003	.3772	-.6228
	.8	-.0705	.3248	.7457	-.0705	-.0705	.2543	-.7457
	.9	-.0364	.1644	.8720	-.0364	-.0364	.1280	-.8720
	C.	0	0	1.0000	0	0	0	-1.0000
+Area		.4457	1.4574	.5582	.4457	0	.7837	.0418
-Area		-.1194	0	-.0418	.1194	-.6737	0	-.5582
Total Area		.3263	1.4574	.5164	.3263	-.6737	.7837	-.5164



Influence coefficients — Two continuous spans.

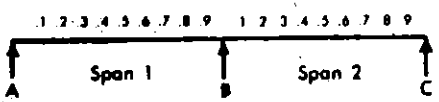
L = Length of SHORTER span ; length of LONGER span = NL

N = 1.3

Table 20.25

Unit load at		MOMENTS/PL																				
		SPAN 1										SPAN 2										
		A.	.1	.2	.3	.4	.5	.6	.7	.8	.9	B.	.1	.2	.3	.4	.5	.6	.7	.8	.9	C.
SPAN 1	A.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0879	.0759	.0638	.0518	.0397	.0276	.0156	.0035	-.0086	-.0206	-.0186	-.0165	-.0144	-.0124	-.0103	-.0082	-.0062	-.0041	-.0021	0
	.2	0	.0760	.1520	.1280	.1040	.0800	.0560	.0320	.0080	-.0160	-.0400	-.0360	-.0320	-.0280	-.0240	-.0200	-.0160	-.0120	-.0080	-.0040	0
	.3	0	.0643	.1286	.1929	.1573	.1216	.0859	.0502	.0145	-.0212	-.0569	-.0512	-.0455	-.0398	-.0341	-.0284	-.0227	-.0171	-.0114	-.0057	0
	.4	0	.0530	.1060	.1590	.2120	.1650	.1180	.0710	.0240	-.0230	-.0700	-.0630	-.0560	-.0490	-.0420	-.0350	-.0280	-.0210	-.0140	-.0070	0
	.5	0	.0422	.0844	.1266	.1688	.2109	.1531	.0953	.0375	-.0203	-.0781	-.0703	-.0625	-.0547	-.0469	-.0391	-.0312	-.0234	-.0156	-.0078	0
	.6	0	.0320	.0640	.0960	.1280	.1600	.1920	.1240	.0560	-.0120	-.0800	-.0720	-.0640	-.0560	-.0480	-.0400	-.0320	-.0240	-.0160	-.0080	0
	.7	0	.0226	.0451	.0677	.0903	.1128	.1354	.1579	.0805	.0031	-.0744	-.0669	-.0595	-.0521	-.0446	-.0372	-.0297	-.0223	-.0149	-.0074	0
	.8	0	.0140	.0280	.0420	.0560	.0700	.0840	.0980	.1120	.0260	-.0600	-.0540	-.0480	-.0420	-.0360	-.0300	-.0240	-.0180	-.0120	-.0060	0
	.9	0	.0064	.0129	.0193	.0257	.0322	.0386	.0451	.0515	.0579	-.0356	-.0321	-.0285	-.0249	-.0214	-.0178	-.0143	-.0107	-.0071	-.0036	0
	B.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0070	-.0140	-.0209	-.0279	-.0349	-.0419	-.0489	-.0559	-.0628	-.0698	.0632	.0561	.0491	.0421	.0351	.0281	.0211	.0140	.0070	0
	.2	0	-.0118	-.0235	-.0353	-.0470	-.0588	-.0706	-.0823	-.0941	-.1058	-.1176	.0062	.1299	.1137	.0974	.0812	.0650	.0487	.0325	.0162	0
	.3	0	-.0146	-.0292	-.0437	-.0583	-.0729	-.0875	-.1020	-.1166	-.1312	-.1458	-.0332	.0794	.1920	.1645	.1371	.1097	.0823	.0548	.0274	0
	.4	0	-.0157	-.0314	-.0470	-.0627	-.0784	-.0941	-.1098	-.1254	-.1411	-.1568	-.0571	.0426	.1422	.2419	.2016	.1613	.1210	.0806	.0403	0
	.5	0	-.0153	-.0306	-.0459	-.0612	-.0766	-.0919	-.1072	-.1225	-.1378	-.1531	-.0678	.0175	.1028	.1881	.2734	.2188	.1641	.1094	.0547	0
	.6	0	-.0137	-.0274	-.0412	-.0549	-.0686	-.0823	-.0960	-.1098	-.1235	-.1372	-.0675	.0022	.0720	.1417	.2114	.2811	.2108	.1406	.0703	0
	.7	0	-.0111	-.0223	-.0334	-.0446	-.0557	-.0669	-.0780	-.0892	-.1003	-.1115	-.0583	-.0052	.0480	.1011	.1543	.2074	.2606	.1737	.0869	0
	.8	0	-.0078	-.0157	-.0235	-.0314	-.0392	-.0470	-.0549	-.0627	-.0706	-.0784	-.0426	-.0067	.0291	.0650	.1008	.1366	.1725	.2083	.1042	0
	.9	0	-.0040	-.0081	-.0121	-.0162	-.0202	-.0243	-.0283	-.0323	-.0364	-.0404	-.0224	-.0043	.0137	.0317	.0498	.0678	.0859	.1039	.1220	0
	C.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+Area		0	.0398	.0696	.0894	.0992	.0990	.0888	.0685	.0383	.0083	0	.0089	.0448	.1058	.1495	.1735	.1780	.1629	.1282	.0739	0
−Area		0	-.0143	-.0286	-.0429	-.0572	-.0715	-.0858	-.1000	-.1143	-.1388	-.1950	-.0962	-.0440	-.0365	-.0312	-.0260	-.0208	-.0156	-.0104	-.0052	0
Total Area		0	.0255	.0410	.0465	.0420	.0275	.0030	-.0315	-.0760	-.1305	-.1950	-.0873	.0008	.0693	.1183	.1475	.1572	.1473	.1178	.0687	0

Unit load at		REACTIONS/P			SHEARS/P			
		R_A	R_B	R_C	V_{AB}	V_{BA}	V_{BC}	V_{CB}
SPAN 1	A.	1.0000	0	0	1.0000	0	0	0
	.1	.8794	-.1354	-.0147	.8794	-.1206	.0147	.0147
	.2	.7600	.2686	-.0286	.7600	-.2400	.0286	.0286
	.3	.6431	.3975	-.0406	.6431	-.3569	.0406	.0406
	.4	.5300	.5200	-.0500	.5300	-.4700	.0500	.0500
	.5	.4219	.6339	-.0558	.4219	-.5781	.0558	.0558
	.6	.3200	.7371	-.0571	.3200	-.6800	.0571	.0571
	.7	.2256	.8275	-.0531	.2256	-.7744	.0531	.0531
	.8	.1400	.9029	-.0429	.1400	-.8600	.0429	.0429
	.9	.0644	.9611	-.0254	.0644	-.9356	.0254	.0254
	B.	0	1.0000	0	0	-1.0 / 0	0 / 1.0	0
SPAN 2	.1	-.0698	1.0197	.0501	-.0698	-.0698	.9498	-.0501
	.2	-.1176	1.0016	.1160	-.1176	-.1176	.8840	-.1160
	.3	-.1458	.9499	.1959	-.1458	-.1458	.8041	-.1959
	.4	-.1568	.8688	.2880	-.1568	-.1568	.7120	-.2880
	.5	-.1531	.7625	.3906	-.1531	-.1531	.6094	-.3906
	.6	-.1372	.6352	.5020	-.1372	-.1372	.4980	-.5020
	.7	-.1115	.4911	.6204	-.1115	-.1115	.3796	-.6204
	.8	-.0784	.3344	.7440	-.0784	-.0784	.2560	-.7440
	.9	-.0404	.1693	.8711	-.0404	-.0404	.1289	-.8711
	C.	0	0	1.0000	0	0	0	-1.0000
+Area		.4479	1.5343	.5979	.4479	0	.8393	.0372
−Area		-.1429	0	-.0372	-.1429	-.6950	0	-.5979
Total Area		.3050	1.5343	.5607	.3050	-.6950	.8393	-.5607



Influence coefficients — Two continuous spans.

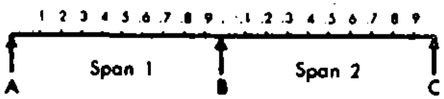
L = Length of SHORTER span ; length of LONGER span = NL

N = 1.4

Table 20.26

Unit load at		MOMENTS/PL																				
		SPAN 1											SPAN 2									
		A.	.1	.2	.3	.4	.5	.6	.7	.8	.9	B.	.1	.2	.3	.4	.5	.6	.7	.8	.9	C.
SPAN 1	A.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0880	.0760	.0641	.0521	.0401	.0281	.0161	.0042	-.0078	-.0198	-.0178	-.0158	-.0139	-.0119	-.0099	-.0079	-.0059	-.0040	-.0020	0
	.2	0	.0762	.1523	.1285	.1046	.0808	.0570	.0331	.0093	-.0146	-.0384	-.0346	-.0307	-.0269	-.0230	-.0192	-.0154	-.0115	-.0077	-.0038	0
	.3	0	.0645	.1291	.1936	.1582	.1227	.0872	.0518	.0163	-.0191	-.0546	-.0491	-.0437	-.0382	-.0328	-.0273	-.0218	-.0164	-.0109	-.0055	0
	.4	0	.0533	.1066	.1598	.2131	.1664	.1197	.0730	.0262	-.0205	-.0672	-.0605	-.0538	-.0470	-.0403	-.0336	-.0269	-.0202	-.0134	-.0067	0
	.5	0	.0425	.0850	.1275	.1700	.2125	.1550	.0975	.0400	-.0175	-.0750	-.0675	-.0600	-.0525	-.0450	-.0375	-.0300	-.0225	-.0150	-.0075	0
	.6	0	.0323	.0646	.0970	.1293	.1616	.1939	.1262	.0586	-.0091	-.0708	-.0691	-.0614	-.0538	-.0461	-.0384	-.0307	-.0230	-.0154	-.0077	0
	.7	0	.0229	.0457	.0686	.0914	.1143	.1372	.1600	.0829	.0057	-.0714	-.0643	-.0571	-.0500	-.0428	-.0357	-.0286	-.0214	-.0143	-.0071	0
	.8	0	.0142	.0285	.0427	.0570	.0712	.0854	.0997	.1139	.0282	-.0576	-.0518	-.0461	-.0403	-.0346	-.0288	-.0230	-.0173	-.0115	-.0058	0
	.9	0	.0066	.0132	.0197	.0263	.0329	.0395	.0461	.0526	.0592	-.0342	-.0308	-.0274	-.0239	-.0205	-.0171	-.0137	-.0103	-.0068	-.0034	0
	B.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0077	-.0154	-.0231	-.0308	-.0385	-.0462	-.0539	-.0616	-.0693	-.0770	.0657	.0584	.0511	.0438	.0365	.0292	.0219	.0146	.0073	0
	.2	0	-.0130	-.0259	-.0389	-.0518	-.0648	-.0778	-.0907	-.1037	-.1166	-.1296	.0034	.1363	.1193	.1022	.0852	.0682	.0511	.0341	.0170	0
	.3	0	-.0161	-.0321	-.0482	-.0643	-.0803	-.0964	-.1125	-.1285	-.1446	-.1607	-.0396	.0815	.2025	.1736	.1447	.1157	.0868	.0579	.0289	0
	.4	0	-.0173	-.0346	-.0518	-.0691	-.0864	-.1037	-.1210	-.1382	-.1555	-.1728	-.0655	.0418	.1490	.2563	.2136	.1709	.1282	.0854	.0427	0
	.5	0	-.0169	-.0337	-.0506	-.0675	-.0844	-.1013	-.1181	-.1350	-.1519	-.1688	-.0769	.0150	.1069	.1988	.2906	.2325	.1744	.1163	.0581	0
	.6	0	-.0151	-.0302	-.0454	-.0605	-.0756	-.0907	-.1058	-.1210	-.1361	-.1512	-.0761	-.0010	.0742	.1493	.2244	.2995	.2246	.1498	.0749	0
	.7	0	-.0123	-.0246	-.0369	-.0491	-.0614	-.0737	-.0860	-.0983	-.1106	-.1228	-.0656	-.0083	.0490	.1063	.1636	.2209	.2781	.1854	.0927	0
	.8	0	-.0086	-.0173	-.0259	-.0346	-.0432	-.0518	-.0605	-.0691	-.0778	-.0864	-.0478	-.0091	.0295	.0682	.1068	.1454	.1841	.2227	.1114	0
	.9	0	-.0045	-.0089	-.0134	-.0178	-.0223	-.0267	-.0312	-.0356	-.0401	-.0445	-.0251	-.0056	.0138	.0333	.0527	.0722	.0916	.1111	.1305	0
	C.	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+Area		0	.0400	.0700	.0900	.1000	.1000	.0900	.0700	.0400	.0069	0	.0096	.0486	.1181	.1688	.1969	.2025	.1856	.1463	.0844	0
-Area		0	-.0169	-.0337	-.0506	-.0675	-.0844	-.1012	-.1181	-.1350	-.1608	-.2188	-.1052	-.0438	-.0350	-.0300	-.0250	-.0200	-.0150	-.0100	-.0050	0
Total Area		0	.0231	.0363	.0394	.0325	.0156	-.0112	-.0481	-.0950	-.1519	-.2188	-.0956	.0050	.0831	.1388	.1719	.1825	.1706	.1363	.0794	0

Unit load at		REACTIONS/P			SHEARS/P			
		R_A	R_B	R_C	V_{AB}	V_{BA}	V_{BC}	V_{CB}
SPAN 1	A.	1.0000	0	0	1.0000	0	0	0
	.1	.8802	.1330	-.0132	.8802	-.1198	.0132	.0132
	.2	.7616	.2640	-.0256	.7616	-.2384	.0256	.0256
	.3	.6454	.3910	-.0364	.6454	-.3546	.0364	.0364
	.4	.5328	.5120	-.0448	.5328	-.4672	.0448	.0448
	.5	.4250	.6250	-.0500	.4250	-.5750	.0500	.0500
	.6	.3232	.7280	-.0512	.3232	-.6768	.0512	.0512
	.7	.2286	.8190	-.0476	.2286	-.7714	.0476	.0476
	.8	.1424	.8960	-.0384	.1424	-.8576	.0384	.0384
	.9	.0658	.9570	-.0228	.0658	-.9342	.0228	.0228
	B.	0	1.0000	0	0	-1.0 / 0	0 / 1.0	0
SPAN 2	.1	-.0770	1.0282	.0487	-.0770	-.0770	.9513	-.0487
	.2	-.1296	1.0160	.1136	-.1296	-.1296	.8864	-.1136
	.3	-.1607	.9677	.1929	-.1607	-.1607	.8071	-.1929
	.4	-.1728	.8880	.2848	-.1728	-.1728	.7152	-.2848
	.5	-.1688	.7813	.3875	-.1688	-.1688	.6125	-.3875
	.6	-.1512	.6520	.4992	-.1512	-.1512	.5008	-.4992
	.7	-.1229	.5047	.6181	-.1229	-.1229	.3819	-.6181
	.8	-.0864	.3440	.7424	-.0864	-.0864	.2576	-.7424
	.9	-.0446	.1743	.8703	-.0446	-.0446	.1297	-.8703
	C.	0	0	1.0000	0	0	0	-1.0000
+Area		.4500	1.6146	.6375	.4500	.0	.8958	.0333
-Area		-.1688	0	-.0333	-.1688	-.7188	0	-.6375
Total Area		.2812	1.6146	.6042	.2812	-.7188	.8958	-.6042



Influence coefficients — Two continuous spans.

L = Length of SHORTER span ; length of LONGER span = NL

N = 1.5

Table 20.27

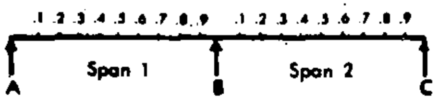
MOMENTS/PL — SPAN 1

Unit load at		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0881	.0762	.0643	.0524	.0405	.0286	.0167	.0048	-.0071	-.0190
	.2	0	.0763	.1526	.1289	.1052	.0815	.0578	.0342	.0105	-.0132	-.0369
	.3	0	.0648	.1295	.1943	.1590	.1238	.0885	.0533	.0180	-.0172	-.0525
	.4	0	.0535	.1071	.1606	.2142	.1677	.1212	.0748	.0283	-.0182	-.0646
	.5	0	.0428	.0856	.1284	.1712	.2139	.1567	.0995	.0423	-.0149	-.0721
	.6	0	.0328	.0652	.0978	.1305	.1631	.1957	.1283	.0609	-.0065	-.0738
	.7	0	.0231	.0463	.0694	.0925	.1157	.1388	.1619	.0851	.0082	-.0687
	.8	0	.0145	.0289	.0434	.0578	.0723	.0868	.1012	.1157	.0302	-.0554
	.9	0	.0067	.0134	.0201	.0268	.0336	.0403	.0470	.0537	.0604	-.0329
	B	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0084	-.0168	-.0253	-.0337	-.0421	-.0505	-.0589	-.0673	-.0758	-.0842
	.2	0	-.0142	-.0284	-.0425	-.0567	-.0709	-.0851	-.0992	-.1134	-.1276	-.1418
	.3	0	-.0176	-.0352	-.0527	-.0703	-.0879	-.1055	-.1230	-.1406	-.1582	-.1758
	.4	0	-.0189	-.0378	-.0567	-.0756	-.0945	-.1134	-.1323	-.1512	-.1701	-.1890
	.5	0	-.0185	-.0369	-.0554	-.0738	-.0923	-.1108	-.1292	-.1477	-.1662	-.1846
	.6	0	-.0165	-.0331	-.0496	-.0662	-.0827	-.0992	-.1158	-.1323	-.1489	-.1654
	.7	0	-.0134	-.0269	-.0403	-.0538	-.0672	-.0806	-.0941	-.1075	-.1210	-.1344
	.8	0	-.0095	-.0189	-.0284	-.0378	-.0473	-.0567	-.0662	-.0756	-.0851	-.0945
	.9	0	-.0049	-.0097	-.0146	-.0195	-.0244	-.0292	-.0341	-.0390	-.0439	-.0487
	C	0	0	0	0	0	0	0	0	0	0	0
+Area		0	.0402	.0704	.0906	.1008	.1010	.0912	.0713	.0415	.0094	.0
-Area		0	-.0197	-.0394	-.0591	-.0788	-.0985	-.1182	-.1378	-.1575	-.1849	-.2450
Total Area		0	.0205	.0310	.0315	.0220	.0025	-.0270	-.0665	-.1160	-.1755	-.2450

MOMENTS/PL — SPAN 2

Unit load at		.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0
	.1	-.0171	-.0152	-.0133	-.0114	-.0095	-.0076	-.0057	-.0038	-.0019	0
	.2	-.0332	-.0295	-.0258	-.0222	-.0185	-.0148	-.0111	-.0074	-.0037	0
	.3	-.0472	-.0420	-.0367	-.0315	-.0262	-.0210	-.0157	-.0105	-.0052	0
	.4	-.0582	-.0517	-.0452	-.0388	-.0323	-.0258	-.0194	-.0129	-.0065	0
	.5	-.0649	-.0577	-.0505	-.0433	-.0361	-.0288	-.0216	-.0144	-.0072	0
	.6	-.0665	-.0591	-.0517	-.0443	-.0369	-.0295	-.0222	-.0148	-.0074	0
	.7	-.0618	-.0549	-.0481	-.0412	-.0343	-.0275	-.0206	-.0137	-.0069	0
	.8	-.0498	-.0443	-.0388	-.0332	-.0277	-.0222	-.0166	-.0111	-.0055	0
	.9	-.0296	-.0263	-.0230	-.0197	-.0164	-.0132	-.0099	-.0066	-.0033	0
	B	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	.0682	.0607	.0531	.0455	.0379	.0303	.0227	.0152	.0076	0
	.2	.0004	.1426	.1248	.1069	.0891	.0713	.0535	.0356	.0178	0
	.3	-.0462	.0834	.2130	.1825	.1521	.1217	.0913	.0608	.0304	0
	.4	-.0741	.0408	.1557	.2706	.2255	.1804	.1353	.0902	.0451	0
	.5	-.0862	.0123	.1108	.2092	.3077	.2462	.1846	.1231	.0615	0
	.6	-.0849	-.0043	.0762	.1568	.2373	.3178	.2384	.1589	.0795	0
	.7	-.0730	-.0115	.0499	.1114	.1728	.2342	.2957	.1971	.0986	0
	.8	-.0531	-.0116	.0298	.0713	.1127	.1542	.1956	.2371	.1185	0
	.9	-.0279	-.0070	.0139	.0348	.0556	.0765	.0974	.1183	.1391	0
	C	0	0	0	0	0	0	0	0	0	0
+Area		.0103	.0528	.1310	.1890	.2215	.2284	.2097	.1654	.0955	0
-Area		-.1136	-.0440	-.0337	-.0288	-.0240	-.0192	-.0144	-.0096	-.0048	0
Total Area		-.1053	.0088	.0973	.1602	.1975	.2092	.1953	.1558	.0907	0

Unit load at		REACTIONS/P R_A	R_B	R_C	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}
SPAN 1	A	1.0000	0	0	1.0000	0	0	0
	.1	.8810	.1309	-.0119	.8810	-.1190	.0119	.0119
	.2	.7631	.2600	-.0231	.7631	-.2369	.0231	.0231
	.3	.6475	.3853	-.0328	.6475	-.3525	.0328	.0328
	.4	.5354	.5050	-.0404	.5354	-.4646	.0404	.0404
	.5	.4279	.6172	-.0451	.4279	-.5721	.0451	.0451
	.6	.3262	.7200	-.0462	.3262	-.6738	.0462	.0462
	.7	.2313	.8116	-.0429	.2313	-.7687	.0429	.0429
	.8	.1446	.8900	-.0346	.1446	-.8554	.0346	.0346
	.9	.0671	.9534	-.0206	.0671	-.9329	.0206	.0206
	B	0	1.0000	0	0	-1.0 / 0	0 / 1.0	0
SPAN 2	.1	-.0842	1.0368	.0474	-.0842	-.0842	.9526	-.0474
	.2	-.1418	1.0304	.1114	-.1418	-.1418	.8886	-.1114
	.3	-.1758	.9856	.1902	-.1758	-.1758	.8098	-.1902
	.4	-.1890	.9072	.2818	-.1890	-.1890	.7182	-.2818
	.5	-.1846	.8000	.3846	-.1846	-.1846	.6154	-.3846
	.6	-.1654	.6688	.4966	-.1654	-.1654	.5034	-.4966
	.7	-.1344	.5184	.6160	-.1344	-.1344	.3840	-.6160
	.8	-.0945	.3536	.7409	-.0945	-.0945	.2591	-.7409
	.9	-.0487	.1792	.8695	-.0487	-.0487	.1305	-.8695
	C	0	0	1.0000	0	0	0	-1.0000
+Area		.4519	1.6981	.6769	.4519	.0	.9531	.0300
-Area		-.1969	0	-.0300	-.1969	-.7450	0	-.6769
Total Area		.2550	1.6981	.6469	.2550	-.7450	.9531	-.6469



Influence coefficients — Two continuous spans.

L = Length of SHORTER span ; length of LONGER span = NL

N = 1.6

Table 20.28

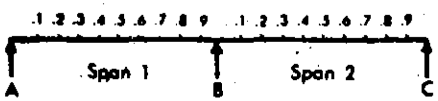
MOMENTS/PL — SPAN 1

	Unit load at	A.	.1	.2	.3	.4	.5	.6	.7	.8	.9	B.
SPAN 1	A.	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0882	.0763	.0645	.0527	.0408	.0290	.0172	.0053	-.0065	-.0183
	.2	0	.0764	.1529	.1293	.1058	.0822	.0587	.0351	.0116	-.0120	-.0356
	.3	0	.0649	.1299	.1948	.1598	.1247	.0897	.0546	.0196	-.0135	-.0506
	.4	0	.0538	.1076	.1613	.2151	.1689	.1227	.0764	.0302	-.0160	-.0622
	.5	0	.0431	.0861	.1292	.1722	.2153	.1583	.1014	.0444	-.0125	-.0694
	.6	0	.0329	.0658	.0987	.1316	.1644	.1973	.1302	.0631	-.0040	-.0711
	.7	0	.0234	.0468	.0702	.0936	.1169	.1403	.1637	.0871	.0105	-.0661
	.8	0	.0147	.0293	.0440	.0587	.0733	.0880	.1027	.1173	.0320	-.0533
	.9	0	.0068	.0137	.0205	.0273	.0342	.0410	.0478	.0547	.0615	-.0317
	B.	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0092	-.0183	-.0275	-.0366	-.0458	-.0549	-.0641	-.0732	-.0824	-.0915
	.2	0	-.0154	-.0308	-.0462	-.0617	-.0771	-.0925	-.1079	-.1233	-.1387	-.1541
	.3	0	-.0191	-.0382	-.0573	-.0764	-.0955	-.1146	-.1337	-.1528	-.1720	-.1911
	.4	0	-.0206	-.0411	-.0617	-.0822	-.1028	-.1233	-.1439	-.1644	-.1850	-.2055
	.5	0	-.0201	-.0401	-.0602	-.0803	-.1003	-.1204	-.1405	-.1606	-.1806	-.2007
	.6	0	-.0180	-.0360	-.0539	-.0719	-.0899	-.1079	-.1259	-.1439	-.1618	-.1798
	.7	0	-.0146	-.0292	-.0438	-.0584	-.0731	-.0877	-.1023	-.1169	-.1315	-.1461
	.8	0	-.0103	-.0206	-.0308	-.0411	-.0514	-.0617	-.0719	-.0822	-.0925	-.1028
	.9	0	-.0053	-.0106	-.0159	-.0212	-.0265	-.0318	-.0371	-.0424	-.0477	-.0530
	C.	0	0	0	0	0	0	0	0	0	0	0
+Area		0	.0404	.0707	.0911	.1015	.1019	.0922	.0726	.0430	.0100	0
-Area		0	-.0227	-.0455	-.0682	-.0910	-.1137	-.1365	-.1592	-.1820	-.2114	-.2737
Total Area		0	.0177	.0252	.0229	.0105	-.0118	-.0443	-.0866	-.1390	-.2014	-.2737

MOMENTS/PL — SPAN 2

	Unit load at	.1	.2	.3	.4	.5	.6	.7	.8	.9	C.
SPAN 1	A.	0	0	0	0	0	0	0	0	0	0
	.1	-.0165	-.0147	-.0128	-.0110	-.0092	-.0073	-.0055	-.0037	-.0018	0
	.2	-.0320	-.0284	-.0249	-.0213	-.0178	-.0142	-.0107	-.0071	-.0036	0
	.3	-.0455	-.0404	-.0354	-.0303	-.0253	-.0202	-.0152	-.0101	-.0051	0
	.4	-.0560	-.0498	-.0436	-.0373	-.0311	-.0249	-.0187	-.0124	-.0062	0
	.5	-.0625	-.0556	-.0486	-.0417	-.0347	-.0278	-.0208	-.0139	-.0069	0
	.6	-.0640	-.0569	-.0498	-.0427	-.0356	-.0284	-.0213	-.0142	-.0071	0
	.7	-.0595	-.0529	-.0463	-.0397	-.0331	-.0264	-.0198	-.0132	-.0066	0
	.8	-.0480	-.0427	-.0373	-.0320	-.0267	-.0213	-.0160	-.0107	-.0053	0
	.9	-.0285	-.0253	-.0222	-.0190	-.0158	-.0127	-.0095	-.0063	-.0032	0
	B.	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	.0706	.0628	.0549	.0471	.0392	.0314	.0235	.0157	.0078	0
	.2	-.0027	.1487	.1301	.1115	.0929	.0743	.0558	.0372	.0186	0
	.3	-.0530	.0852	.2233	.1914	.1595	.1276	.0957	.0638	.0319	0
	.4	-.0830	.0398	.1621	.2847	.2372	.1898	.1423	.0949	.0474	0
	.5	-.0956	.0094	.1145	.2196	.3247	.2597	.1948	.1299	.0649	0
	.6	-.0938	-.0079	-.0781	.1641	.2801	.3361	.2521	.1680	.0840	0
	.7	-.0805	-.0149	.0507	.1163	.1819	.2476	.3132	.2088	.1044	0
	.8	-.0585	-.0142	.0301	.0743	.1186	.1629	.2072	.2514	.1257	0
	.9	-.0307	-.0084	.0139	.0362	.0585	.0808	.1031	.1254	.1477	0
	C.	0	0	0	0	0	0	0	0	0	0
+Area		.0111	.0570	.1442	.2103	.2475	.2558	.2352	.1857	.1073	0
-Area		-.1274	-.0448	-.0324	-.0278	-.0231	-.0185	-.0139	-.0093	-.0046	0
Total Area		-.1163	.0122	.1118	.1825	.2244	.2373	.2213	.1764	.1027	0

	Unit load at	REACTIONS/P			SHEARS/P			
		R_A	R_B	R_C	V_{AB}	V_{BA}	V_{BC}	V_{CB}
SPAN 1	A.	1.0000	0	0	1.0000	0	0	0
	.1	.8817	.1291	-.0108	.8817	-.1183	.0108	.0108
	.2	.7644	.2565	-.0209	.7644	-.2356	.0209	.0209
	.3	.6494	.3803	-.0297	.6494	-.3506	.0297	.0297
	.4	.5378	.4988	-.0366	.5378	-.4622	.0366	.0366
	.5	.4306	.6103	-.0408	.4306	-.5694	.0408	.0408
	.6	.3289	.7129	-.0418	.3289	-.6711	.0418	.0418
	.7	.2339	.8050	-.0389	.2339	-.7661	.0389	.0389
	.8	.146[illegible]	.8847	-.0314	.1467	-.8533	.0314	.0314
	.9	.0683	.9503	-.0186	.0683	-.9317	.0186	.0186
	B.	0	1.0000	0	0	-1.0 / 0	0 / 1.0	0
SPAN 2	.1	-.0915	1.0454	.0462	-.0915	-.0915	.9538	-.0462
	.2	-.1541	1.0448	.1093	-.1541	-.1541	.8907	-.1093
	.3	-.1911	1.0035	.1876	-.1911	-.1911	.8124	-.1876
	.4	-.2055	.9264	.2791	-.2055	-.2055	.7209	-.2791
	.5	-.2007	.8187	.3819	-.2007	-.2007	.6181	-.3819
	.6	-.1798	.6856	.4942	-.1798	-.1798	.5058	-.4942
	.7	-.1461	.5320	.6141	-.1461	-.1461	.3859	-.6141
	.8	-.1028	.3632	.7396	-.1028	-.1028	.2604	-.7396
	.9	-.0530	.1841	.8688	-.0530	-.0530	.1312	-.8688
	C.	0	0	1.0000	0	0	0	-1.0000
+Area		.4537	1.7848	.7162	.4537	.0	1.0110	.0272
-Area		-.2275	0	-.0272	-.2275	-.7738	0	-.7162
Total Area		.2263	1.7848	.6890	.2263	-.7738	1.0110	-.6890



Influence coefficients — Two continuous spans.

L = Length of SHORTER span ; length of LONGER span = NL

N = 1.7

Table 20.29

MOMENTS/PL — SPAN 1

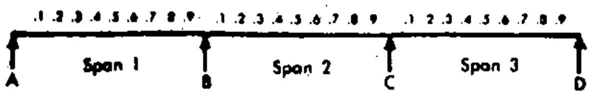
Unit load at		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0874	.0747	.0621	.0494	.0368	.0242	.0115	-.0011	-.0138	-.0264
	.2	0	.0749	.1498	.1246	.0995	.0744	.0493	.0242	-.0010	-.0261	-.0512
	.3	0	.0627	.1254	.1882	.1508	.1136	.0763	.0390	.0018	-.0355	-.0728
	.4	0	.0510	.1021	.1531	.2042	.1552	.1062	.0573	.0083	-.0406	-.0896
	.5	0	.0400	.0800	.1200	.1600	.2000	.1400	.0800	.0200	-.0400	-.1000
	.6	0	.0298	.0595	.0893	.1190	.1488	.1786	.1084	.0381	-.0322	-.1024
	.7	0	.0205	.0410	.0614	.0819	.1024	.1229	.1434	.0638	-.0157	-.0952
	.8	0	.0123	.0246	.0370	.0493	.0616	.0739	.0862	.0986	.0109	-.0768
	.9	0	.0029	.0058	.0088	.0210	.0272	.0326	.0381	.0435	.0490	-.0456
	B	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0039	-.0078	-.0117	-.0156	-.0195	-.0234	-.0273	-.0312	-.0351	-.0390
	.2	0	-.0064	-.0128	-.0192	-.0256	-.0320	-.0384	-.0448	-.0512	-.0576	-.0640
	.3	0	-.0077	-.0154	-.0231	-.0308	-.0385	-.0462	-.0539	-.0616	-.0693	-.0770
	.4	0	-.0080	-.0160	-.0240	-.0320	-.0400	-.0480	-.0560	-.0640	-.0720	-.0800
	.5	0	-.0075	-.0150	-.0225	-.0300	-.0375	-.0450	-.0525	-.0600	-.0675	-.0750
	.6	0	-.0064	-.0128	-.0192	-.0256	-.0320	-.0384	-.0448	-.0512	-.0576	-.0640
	.7	0	-.0049	-.0098	-.0147	-.0196	-.0245	-.0294	-.0343	-.0392	-.0441	-.0490
	.8	0	-.0032	-.0064	-.0096	-.0128	-.0160	-.0192	-.0224	-.0256	-.0288	-.0320
	.9	0	-.0015	-.0030	-.0045	-.0060	-.0075	-.0090	-.0105	-.0120	-.0135	-.0150
	C	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0011	.0023	.0034	.0046	.0057	.0068	.0080	.0091	.0103	.0114
	.2	0	.0019	.0038	.0058	.0077	.0096	.0115	.0134	.0154	.0173	.0192
	.3	0	.0024	.0048	.0071	.0095	.0119	.0143	.0167	.0190	.0214	.0238
	.4	0	.0026	.0051	.0077	.0102	.0128	.0154	.0179	.0205	.0230	.0256
	.5	0	.0025	.0050	.0075	.0100	.0125	.0150	.0175	.0200	.0225	.0250
	.6	0	.0022	.0045	.0067	.0090	.0112	.0134	.0157	.0179	.0202	.0224
	.7	0	.0018	.0036	.0055	.0073	.0091	.0109	.0127	.0146	.0164	.0182
	.8	0	.0013	.0026	.0038	.0051	.0064	.0077	.0090	.0102	.0115	.0128
	.9	0	.0007	.0013	.0020	.0026	.0033	.0040	.0046	.0053	.0059	.0066
	D	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0400	.0700	.0900	.1000	.1000	.0900	.0700	.0402	.0204	.0167
− Area		0	-.0050	-.0100	-.0150	-.0200	-.0250	-.0300	-.0350	-.0402	-.0654	-.1167
Total Area		0	.0350	.0600	.0750	.0800	.0750	.0600	.0350	.0000	-.0450	-.1000

MOMENTS/PL — SPAN 2

Unit load at		.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0
	.1	-.0231	-.0198	-.0165	-.0132	-.0099	-.0066	-.0033	.0000	.0033	.0066
	.2	-.0448	-.0384	-.0320	-.0256	-.0192	-.0128	-.0064	.0000	.0064	.0128
	.3	-.0637	-.0546	-.0455	-.0364	-.0273	-.0182	-.0091	.0000	.0091	.0182
	.4	-.0784	-.0672	-.0560	-.0448	-.0336	-.0224	-.0112	.0000	.0112	.0224
	.5	-.0875	-.0750	-.0625	-.0500	-.0375	-.0250	-.0125	.0000	.0125	.0250
	.6	-.0896	-.0768	-.0640	-.0512	-.0384	-.0256	-.0128	.0000	.0128	.0256
	.7	-.0833	-.0714	-.0595	-.0476	-.0357	-.0238	-.0119	.0000	.0119	.0238
	.8	-.0672	-.0576	-.0480	-.0384	-.0288	-.0192	-.0096	.0000	.0096	.0192
	.9	-.0399	-.0342	-.0285	-.0228	-.0171	-.0114	-.0057	.0000	.0057	.0114
	B	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	.0534	.0458	.0382	.0306	.0230	.0154	.0073	.0002	-.0074	-.0150
	.2	.0192	.1024	.0856	.0688	.0520	.0352	.0184	.0016	-.0152	-.0320
	.3	-.0042	.0686	.1414	.1142	.0870	.0598	.0326	.0054	-.0218	-.0490
	.4	-.0184	.0432	.1048	.1664	.1280	.0896	.0512	.0128	-.0256	-.0640
	.5	-.0250	.0250	.0750	.1250	.1750	.1250	.0750	.0250	-.0250	-.0750
	.6	-.0256	.0128	.0512	.0896	.1280	.1664	.1048	.0432	-.0184	-.0800
	.7	-.0218	.0054	.0326	.0598	.0870	.1142	.1414	.0686	-.0042	-.0770
	.8	-.0152	.0016	.0184	.0352	.0520	.0688	.0856	.1024	.0192	-.0640
	.9	-.0074	.0002	.0078	.0154	.0230	.0306	.0382	.0458	.0534	-.0390
	C	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	.0057	.0000	-.0057	-.0114	-.0171	-.0228	-.0285	-.0342	-.0399	-.0456
	.2	.0096	.0000	-.0096	-.0192	-.0288	-.0384	-.0480	-.0576	-.0672	-.0768
	.3	.0119	.0000	-.0119	-.0238	-.0357	-.0476	-.0595	-.0714	-.0833	-.0952
	.4	.0128	.0000	-.0128	-.0256	-.0384	-.0512	-.0640	-.0768	-.0896	-.1024
	.5	.0125	.0000	-.0125	-.0250	-.0375	-.0500	-.0625	-.0750	-.0875	-.1000
	.6	.0112	.0000	-.0112	-.0224	-.0336	-.0448	-.0560	-.0672	-.0784	-.0896
	.7	.0091	.0000	-.0091	-.0182	-.0273	-.0364	-.0455	-.0546	-.0637	-.0728
	.8	.0064	.0000	-.0064	-.0128	-.0192	-.0256	-.0320	-.0384	-.0448	-.0512
	.9	.0033	.0000	-.0033	-.0066	-.0099	-.0132	-.0165	-.0198	-.0231	-.0264
	D	0	0	0	0	0	0	0	0	0	0
+ Area		.0152	.0300	.0550	.0700	.0750	.0700	.0550	.0300	.0152	.0167
− Area		-.0702	-.0500	-.0500	-.0500	-.0500	-.0500	-.0500	-.0500	-.0702	-.1167
Total Area		-.0550	-.0200	.0050	.0200	.0250	.0200	.0050	-.0200	-.0550	-.1000

REACTIONS/P and SHEARS/P

Unit load at		R_A	R_B	R_C	R_D	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
SPAN 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8736	.1584	-.0396	.0066	.8736	-.1264	.0330	.0330	-.0066	-.0066
	.2	.7488	.3152	-.0768	.0128	.7488	-.2512	.0640	.0640	-.0128	-.0128
	.3	.6272	.4638	-.1092	.0182	.6272	-.3728	.0910	.0910	-.0182	-.0182
	.4	.5104	.6016	-.1344	.0224	.5104	-.4896	.1120	.1120	-.0224	-.0224
	.5	.4000	.7250	-.1500	.0250	.4000	-.6000	.1250	.1250	-.0250	-.0250
	.6	.2976	.8304	-.1536	.0256	.2976	-.7024	.1280	.1280	-.0256	-.0256
	.7	.2048	.9142	-.1428	.0238	.2048	-.7952	.1190	.1190	-.0238	-.0238
	.8	.1232	.9728	-.1152	.0192	.1232	-.8768	.0960	.0960	-.0192	-.0192
	.9	.0544	1.0026	-.0684	.0114	.0544	-.9456	.0570	.0570	-.0114	-.0114
	B	0	1.0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 2	.1	-.0390	.9630	.0910	-.0150	-.0390	-.0390	.9240	-.0760	.0150	.0150
	.2	-.0640	.8960	.2000	-.0320	-.0640	-.0640	.8320	-.1680	.0320	.0320
	.3	-.0770	.8050	.3210	-.0490	-.0770	-.0770	.7280	-.2720	.0490	.0490
	.4	-.0800	.6960	.4480	-.0640	-.0800	-.0800	.6160	-.3840	.0640	.0640
	.5	-.0750	.5750	.5750	-.0750	-.0750	-.0750	.5000	-.5000	.0750	.0750
	.6	-.0640	.4480	.6960	-.0800	-.0640	-.0640	.3840	-.6160	.0800	.0800
	.7	-.0490	.3210	.8050	-.0770	-.0490	-.0490	.2720	-.7280	.0770	.0770
	.8	-.0320	.2000	.8960	-.0640	-.0320	-.0320	.1680	-.8320	.0640	.0640
	.9	-.0150	.0910	.9630	-.0390	-.0150	-.0150	.0760	-.9240	.0390	.0390
	C	0	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 3	.1	.0114	-.0684	1.0026	.0544	.0114	.0114	-.0570	-.0570	.9456	-.0544
	.2	.0192	-.1152	.9728	.1232	.0192	.0192	-.0960	-.0960	.8768	-.1232
	.3	.0238	-.1428	.9142	.2048	.0238	.0238	-.1190	-.1190	.7952	-.2048
	.4	.0256	-.1536	.8304	.2976	.0256	.0256	-.1280	-.1280	.7024	-.2976
	.5	.0250	-.1500	.7250	.4000	.0250	.0250	-.1250	-.1250	.6000	-.4000
	.6	.0224	-.1344	.6016	.5104	.0224	.0224	-.1120	-.1120	.4896	-.5104
	.7	.0182	-.1092	.4638	.6272	.0182	.0182	-.0910	-.0910	.3728	-.6272
	.8	.0128	-.0768	.3152	.7488	.0128	.0128	-.0640	-.0640	.2512	-.7488
	.9	.0066	-.0396	.1584	.8736	.0066	.0066	-.0330	-.0330	.1264	-.8736
	D	0	0	0	1.0	0	0	0	0	0	-1.0
+ Area		.4500	1.2000	1.2000	.4500	.4500	.0167	.5833	.0833	.6167	.0500
− Area		-.0500	-.1000	-.1000	-.0500	-.0500	-.6167	-.0833	-.5833	-.0167	-.4500
Total Area		.4000	1.1000	1.1000	.4000	.4000	-.6000	.5000	-.5000	.6000	-.4000



Influence coefficients — Three continuous spans.

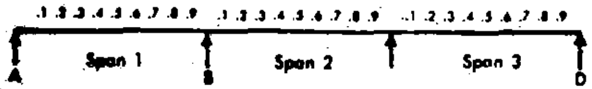
L = Length of EXTERIOR spans; length of interior span = NL

N = 1.0

Table 20.30

Unit load at		MOMENTS/PL																				
		SPAN 1											SPAN 2									
		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0875	.0749	.0624	.0499	.0373	.0248	.0123	-.0002	-.0128	-.0253	-.0221	-.0189	-.0157	-.0125	-.0093	-.0061	-.0030	.0002	.0034	.0066
	.2	0	.0751	.1502	.1253	.1004	.0755	.0506	.0256	.0007	-.0242	-.0491	-.0429	-.0367	-.0305	-.0243	-.0181	-.0119	-.0057	.0005	.0067	.0129
	.3	0	.0630	.1260	.1891	.1521	.1151	.0781	.0411	.0042	-.0328	-.0698	-.0610	-.0522	-.0434	-.0346	-.0258	-.0169	-.0081	.0007	.0095	.0183
	.4	0	.0514	.1028	.1542	.2056	.1571	.1085	.0599	.0113	-.0373	-.0859	-.0751	-.0642	-.0534	-.0425	-.0317	-.0209	-.0100	.0008	.0117	.0225
	.5	0	.0404	.0808	.1212	.1617	.2021	.1425	.0829	.0233	-.0363	-.0959	-.0838	-.0717	-.0596	-.0475	-.0354	-.0233	-.0112	.0009	.0130	.0251
	.6	0	.0302	.0604	.0906	.1207	.1509	.1811	.1113	.0415	-.0283	-.0982	-.0858	-.0734	-.0610	-.0486	-.0362	-.0238	-.0115	.0009	.0133	.0257
	.7	0	.0209	.0417	.0626	.0835	.1044	.1252	.1461	.0670	-.0121	-.0913	-.0797	-.0682	-.0567	-.0452	-.0337	-.0222	-.0106	.0001	.0124	.0239
	.8	0	.0126	.0253	.0379	.0506	.0632	.0758	.0885	.1011	.0137	-.0736	-.0643	-.0550	-.0458	-.0365	-.0272	-.0179	-.0086	.0007	.0100	.0193
	.9	0	.0056	.0113	.0169	.0225	.0281	.0338	.0394	.0450	.0507	-.0437	-.0382	-.0327	-.0272	-.0216	-.0161	-.0106	-.0051	.0004	.0059	.0114
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0045	-.0090	-.0135	-.0179	-.0224	-.0269	-.0314	-.0359	-.0404	-.0449	.0569	.0487	.0406	.0324	.0242	.0160	.0078	-.0004	-.0086	-.0168
	.2	0	-.0074	-.0147	-.0221	-.0294	-.0368	-.0441	-.0515	-.0588	-.0662	-.0735	.0182	.1100	.0917	.0735	.0552	.0370	.0187	.0004	-.0178	-.0361
	.3	0	-.0088	-.0177	-.0265	-.0353	-.0442	-.0530	-.0618	-.0706	-.0795	-.0883	-.0080	.0722	.1525	.1228	.0931	.0634	.0336	.0039	-.0258	-.0555
	.4	0	-.0092	-.0183	-.0275	-.0366	-.0458	-.0549	-.0641	-.0732	-.0824	-.0916	-.0237	.0442	.1121	.1799	.1378	.0957	.0536	.0114	-.0307	-.0728
	.5	0	-.0086	-.0171	-.0257	-.0342	-.0428	-.0514	-.0599	-.0685	-.0771	-.0856	-.0306	.0244	.0794	.1344	.1894	.1344	.0794	.0244	-.0306	-.0856
	.6	0	-.0073	-.0146	-.0218	-.0291	-.0364	-.0437	-.0510	-.0583	-.0655	-.0728	-.0307	.0114	.0536	.0957	.1378	.1799	.1121	.0442	-.0237	-.0916
	.7	0	-.0055	-.0111	-.0167	-.0222	-.0278	-.0333	-.0389	-.0444	-.0500	-.0555	-.0258	.0039	.0336	.0634	.0931	.1228	.1525	.0722	-.0080	-.0883
	.8	0	-.0036	-.0072	-.0108	-.0144	-.0180	-.0216	-.0252	-.0288	-.0325	-.0361	-.0178	.0004	.0187	.0370	.0552	.0735	.0917	.1100	.0182	-.0735
	.9	0	-.0017	-.0034	-.0050	-.0067	-.0084	-.0101	-.0117	-.0134	-.0151	-.0168	-.0086	-.0004	.0078	.0160	.0242	.0324	.0406	.0487	.0569	-.0449
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0011	.0023	.0034	.0046	.0057	.0068	.0080	.0091	.0103	.0114	.0059	-.0004	-.0051	-.0106	-.0161	-.0216	-.0272	-.0327	-.0382	-.0437
	.2	0	.0019	.0039	.0058	.0077	.0097	.0116	.0135	.0154	.0174	.0193	.0100	.0007	-.0086	-.0179	-.0272	-.0365	-.0458	-.0550	-.0643	-.0736
	.3	0	.0024	.0048	.0072	.0096	.0120	.0143	.0167	.0191	.0215	.0239	.0124	.0009	-.0106	-.0222	-.0337	-.0452	-.0567	-.0682	-.0797	-.0913
	.4	0	.0026	.0051	.0077	.0103	.0129	.0154	.0180	.0206	.0231	.0257	.0133	.0009	-.0115	-.0238	-.0362	-.0486	-.0610	-.0734	-.0858	-.0982
	.5	0	.0025	.0050	.0075	.0100	.0126	.0151	.0176	.0201	.0226	.0251	.0130	.0009	-.0112	-.0233	-.0354	-.0475	-.0596	-.0717	-.0838	-.0959
	.6	0	.0023	.0045	.0068	.0090	.0113	.0135	.0158	.0180	.0203	.0225	.0117	.0008	-.0100	-.0209	-.0317	-.0425	-.0534	-.0642	-.0751	-.0859
	.7	0	.0018	.0037	.0055	.0073	.0092	.0110	.0128	.0146	.0165	.0183	.0095	.0007	-.0081	-.0169	-.0258	-.0346	-.0434	-.0522	-.0610	-.0698
	.8	0	.0013	.0026	.0039	.0052	.0065	.0077	.0090	.0103	.0116	.0129	.0067	.0005	-.0057	-.0119	-.0181	-.0243	-.0305	-.0367	-.0429	-.0491
	.9	0	.0007	.0013	.0020	.0026	.0033	.0040	.0046	.0053	.0059	.0066	.0034	.0002	-.0030	-.0061	-.0093	-.0125	-.0157	-.0189	-.0221	-.0253
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0403	.0705	.0908	.1011	.1015	.0917	.0720	.0423	.0209	.0167	.0163	.0346	.0643	.0824	.0865	.0824	.0643	.0346	.0163	.0167
− Area		0	-.0063	-.0125	-.0188	-.0251	-.0315	-.0377	-.0440	-.0503	-.0749	-.1287	-.0718	-.0478	-.0472	-.0472	-.0472	-.0472	-.0472	-.0478	-.0718	-.1287
Total Area		0	.0340	.0580	.0720	.0760	.0700	.0540	.0280	-.0080	-.0540	-.1100	-.0555	-.0132	.0171	.0352	.0413	.0352	.0171	-.0132	-.0555	-.1100

Unit load at		REACTIONS/P				SHEARS/P					
		R_A	R_B	R_C	R_D	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
SPAN 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8747	.1543	-.0356	.0066	.8747	-.1253	.0290	.0290	-.0066	-.0066
	.2	.7509	.3054	-.0692	.0129	.7509	-.2491	.0563	.0563	-.0129	-.0129
	.3	.6302	.4498	-.0983	.0183	.6302	-.3698	.0800	.0800	-.0183	-.0183
	.4	.5141	.5844	-.1210	.0225	.5141	-.4859	.0985	.0985	-.0225	-.0225
	.5	.4041	.7058	-.1350	.0251	.4041	-.5959	.1099	.1099	-.0251	-.0251
	.6	.3018	.8108	-.1383	.0257	.3018	-.6982	.1126	.1126	-.0257	-.0257
	.7	.2087	.8960	-.1286	.0239	.2087	-.7913	.1047	.1047	-.0239	-.0239
	.8	.1264	.9581	-.1038	.0193	.1264	-.8736	.0845	.0845	-.0193	-.0193
	.9	.0563	.9939	-.0616	.0114	.0563	-.9437	.0502	.0502	-.0114	-.0114
	B	0	1.0	0	0	0	-1.0/0	0/1.0	0	0	0
SPAN 2	.1	-.0449	.9704	.0913	-.0168	-.0449	-.0449	.9255	-.0745	.0168	.0168
	.2	-.0735	.9076	.2020	-.0361	-.0735	-.0735	.8341	-.1659	.0361	.0361
	.3	-.0833	.8181	.3257	-.0555	-.0883	-.0883	.7298	-.2702	.0555	.0555
	.4	-.0916	.7086	.4558	-.0728	-.0916	-.0916	.6170	-.3830	.0728	.0728
	.5	-.0856	.5856	.5856	-.0856	-.0856	-.0856	.5000	-.5000	.0856	.0856
	.6	-.0728	.4558	.7086	-.0916	-.0728	-.0728	.3830	-.6170	.0916	.0916
	.7	-.0555	.3257	.8181	-.0883	-.0555	-.0555	.2702	-.7298	.0883	.0883
	.8	-.0361	.2020	.9076	-.0735	-.0361	-.0361	.1659	-.8341	.0735	.0735
	.9	-.0168	.0913	.9704	-.0449	-.0168	-.0168	.0745	-.9255	.0449	.0449
	C	0	0	1.0	0	0	0	0	-1.0/0	0/1.0	0
SPAN 3	.1	.0114	-.0616	.9939	.0563	.0114	.0114	-.0502	-.0502	.9437	-.0563
	.2	.0193	-.1038	.9581	.1264	.0193	.0193	-.0845	-.0845	.8736	-.1264
	.3	.0239	-.1286	.8960	.2087	.0239	.0239	-.1047	-.1047	.7913	-.2087
	.4	.0257	-.1383	.8108	.3018	.0257	.0257	-.1126	-.1126	.6982	-.3018
	.5	.0251	-.1350	.7058	.4041	.0251	.0251	-.1099	-.1099	.5959	-.4041
	.6	.0225	-.1210	.5844	.5141	.0225	.0225	-.0985	-.0985	.4859	-.5141
	.7	.0183	-.0983	.4498	.6302	.0183	.0183	-.0800	-.0800	.3698	-.6302
	.8	.0129	-.0692	.3054	.7509	.0129	.0129	-.0563	-.0563	.2491	-.7509
	.9	.0066	-.0356	.1543	.8747	.0066	.0066	-.0290	-.0290	.1253	-.8747
	D	0	0	0	1.0	0	0	0	0	0	0
+ Area		.4528	1.2500	1.2500	.4528	.4528	.0167	.6233	.0733	.6267	.0628
− Area		-.0628	-.0900	-.0900	-.0628	-.0628	-.6267	-.0733	-.6233	-.0167	-.4528
Total Area		.3900	1.1600	1.1600	.3900	.3900	-.6100	.5500	-.5500	.6100	-.3900



Influence coefficients — Three continuous spans.

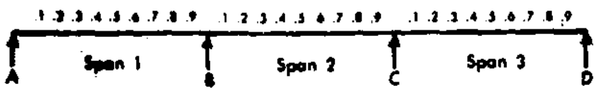
L = Length of EXTERIOR spans; length of interior span = NL

N = 1.1

Table 20.31

Unit load at		MOMENTS/PL																				
		SPAN 1										SPAN 2										
		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0876	.0751	.0627	.0503	.0378	.0254	.0130	.0006	-.0119	-.0243	-.0212	-.0181	-.0150	-.0119	-.0088	-.0057	-.0027	.0004	.0035	.0066
	.2	0	.0753	.1506	.1258	.1011	.0764	.0517	.0270	.0023	-.0224	-.0471	-.0411	-.0351	-.0291	-.0231	-.0171	-.0111	-.0051	.0009	.0069	.0129
	.3	0	.0633	.1266	.1899	.1532	.1165	.0798	.0431	.0064	-.0303	-.0670	-.0585	-.0500	-.0414	-.0329	-.0244	-.0158	-.0073	.0012	.0098	.0183
	.4	0	.0518	.1035	.1553	.2070	.1588	.1105	.0623	.0140	-.0343	-.0825	-.0720	-.0615	-.0510	-.0405	-.0300	-.0195	-.0090	.0015	.0120	.0225
	.5	0	.0408	.0816	.1224	.1632	.2040	.1448	.0855	.0263	-.0329	-.0921	-.0804	-.0686	-.0569	-.0452	-.0335	-.0218	-.0100	.0017	.0134	.0251
	.6	0	.0306	.0611	.0917	.1223	.1529	.1834	.1140	.0446	-.0249	-.0943	-.0823	-.0703	-.0583	-.0463	-.0343	-.0223	-.0103	.0017	.0137	.0257
	.7	0	.0212	.0425	.0637	.0849	.1062	.1274	.1486	.0699	-.0089	-.0877	-.0765	-.0653	-.0542	-.0430	-.0319	-.0207	-.0096	.0016	.0128	.0239
	.8	0	.0129	.0259	.0388	.0517	.0646	.0776	.0905	.1034	.0164	-.0707	-.0617	-.0527	-.0437	-.0347	-.0257	-.0167	-.0077	.0013	.0103	.0193
	.9	0	.0058	.0116	.0174	.0232	.0290	.0348	.0406	.0464	.0522	-.0420	-.0366	-.0313	-.0260	-.0206	-.0153	-.0099	-.0046	.0008	.0061	.0115
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0051	-.0102	-.0153	-.0204	-.0255	-.0305	-.0356	-.0407	-.0458	-.0509	.0603	.0516	.0428	.0340	.0253	.0165	.0078	-.0010	-.0098	-.0185
	.2	0	-.0083	-.0167	-.0250	-.0333	-.0417	-.0500	-.0583	-.0667	-.0750	-.0833	.0170	.1173	.0976	.0780	.0583	.0386	.0189	-.0008	-.0204	-.0401
	.3	0	-.0100	-.0200	-.0300	-.0400	-.0500	-.0599	-.0699	-.0799	-.0899	-.0999	-.0121	.0757	.1634	.1312	.0990	.0668	.0346	.0023	-.0298	-.0621
	.4	0	-.0103	-.0207	-.0310	-.0413	-.0517	-.0620	-.0724	-.0827	-.0930	-.1034	-.0292	.0449	.1191	.1933	.1474	.1016	.0557	.0099	-.0359	-.0818
	.5	0	-.0096	-.0193	-.0289	-.0386	-.0482	-.0579	-.0675	-.0771	-.0868	-.0964	-.0364	.0236	.0836	.1436	.2036	.1436	.0836	.0236	-.0364	-.0964
	.6	0	-.0082	-.0164	-.0245	-.0327	-.0409	-.0491	-.0572	-.0654	-.0736	-.0818	-.0359	.0099	.0557	.1016	.1474	.1933	.1191	.0449	-.0292	-.1034
	.7	0	-.0062	-.0124	-.0186	-.0248	-.0311	-.0373	-.0435	-.0497	-.0559	-.0621	-.0298	.0023	.0346	.0668	.0990	.1312	.1634	.0757	-.0121	-.0999
	.8	0	-.0040	-.0080	-.0120	-.0160	-.0201	-.0241	-.0281	-.0321	-.0361	-.0401	-.0204	-.0008	.0189	.0386	.0583	.0780	.0976	.1173	.0170	-.0833
	.9	0	-.0019	-.0037	-.0056	-.0074	-.0093	-.0111	-.0130	-.0148	-.0167	-.0185	-.0098	-.0010	.0078	.0165	.0253	.0340	.0428	.0516	.0603	-.0509
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0012	.0023	.0035	.0046	.0058	.0069	.0081	.0092	.0104	.0115	.0061	.0008	-.0046	-.0099	-.0153	-.0206	-.0260	-.0313	-.0366	-.0420
	.2	0	.0019	.0039	.0058	.0077	.0097	.0116	.0135	.0154	.0174	.0193	.0103	.0013	-.0077	-.0167	-.0257	-.0347	-.0437	-.0527	-.0617	-.0707
	.3	0	.0024	.0048	.0072	.0096	.0120	.0143	.0167	.0191	.0215	.0239	.0128	.0016	-.0096	-.0207	-.0319	-.0430	-.0542	-.0653	-.0765	-.0877
	.4	0	.0026	.0051	.0077	.0103	.0129	.0154	.0180	.0206	.0231	.0257	.0137	.0017	-.0103	-.0223	-.0343	-.0463	-.0583	-.0703	-.0823	-.0943
	.5	0	.0025	.0050	.0075	.0100	.0126	.0151	.0176	.0201	.0226	.0251	.0134	.0017	-.0100	-.0218	-.0335	-.0452	-.0569	-.0686	-.0804	-.0921
	.6	0	.0023	.0045	.0068	.0090	.0113	.0135	.0158	.0180	.0203	.0225	.0120	.0015	-.0090	-.0195	-.0300	-.0405	-.0510	-.0615	-.0720	-.0825
	.7	0	.0018	.0037	.0055	.0073	.0092	.0110	.0128	.0146	.0165	.0183	.0098	.0012	-.0073	-.0158	-.0244	-.0329	-.0414	-.0500	-.0585	-.0670
	.8	0	.0013	.0026	.0039	.0052	.0065	.0077	.0090	.0103	.0116	.0129	.0069	.0009	-.0051	-.0111	-.0171	-.0231	-.0291	-.0351	-.0411	-.0471
	.9	0	.0007	.0013	.0020	.0026	.0033	.0040	.0046	.0053	.0059	.0066	.0035	.0004	-.0027	-.0057	-.0088	-.0119	-.0150	-.0181	-.0212	-.0243
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0406	.0710	.0916	.1022	.1027	.0932	.0737	.0443	.0214	.0167	.0174	.0394	.0740	.0956	.1028	.0956	.0740	.0394	.0174	.0167
− Area		0	-.0078	-.0154	-.0231	-.0309	-.0386	-.0463	-.0539	-.0617	-.0860	-.1385	-.0744	-.0480	-.0446	-.0446	-.0446	-.0446	-.0446	-.0480	-.0744	-.1385
Total Area		0	.0328	.0556	.0685	.0713	.0641	.0469	.0198	-.0174	-.0646	-.1218	-.0570	-.0086	.0294	.0510	.0582	.0510	.0294	-.0086	-.0570	-.1218

Unit load at		REACTIONS/P				SHEARS/P					
		R_A	R_B	R_C	R_D	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
SPAN 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8757	.1501	-.0324	.0066	.8757	-.1243	.0258	.0258	-.0066	-.0066
	.2	.7529	.2871	-.0529	.0129	.7529	-.2471	.0400	.0400	-.0129	-.0129
	.3	.6330	.4381	-.0894	.0183	.6330	-.3670	.0711	.0711	-.0183	-.0183
	.4	.5175	.5700	-.1100	.0225	.5175	-.4825	.0875	.0875	-.0225	-.0225
	.5	.4079	.6897	-.1227	.0251	.4079	-.5921	.0976	.0976	-.0251	-.0251
	.6	.3057	.7943	-.1257	.0257	.3057	-.6943	.1000	.1000	-.0257	-.0257
	.7	.2123	.8806	-.1168	.0239	.2123	-.7877	.0929	.0929	-.0239	-.0239
	.8	.1293	.9457	-.0943	.0193	.1293	-.8707	.0750	.0750	-.0193	-.0193
	.9	.0580	.9865	-.0560	.0115	.0580	-.9420	.0445	.0445	-.0115	-.0115
	B	0	1.0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 2	.1	-.0509	.9779	.0915	-.0185	-.0509	-.0509	.9270	-.0730	.0185	.0185
	.2	-.0833	.9193	.2041	-.0401	-.0833	-.0833	.8360	-.1640	.0401	.0401
	.3	-.0999	.8314	.3306	-.0621	-.0999	-.0999	.7315	-.2685	.0621	.0621
	.4	-.1034	.7214	.4638	-.0818	-.1034	-.1034	.6180	-.3820	.0818	.0818
	.5	-.0964	.5964	.5964	-.0964	-.0964	-.0964	.5000	-.5000	.0964	.0964
	.6	-.0818	.4638	.7214	-.1034	-.0818	-.0818	.3820	-.6180	.1034	.1034
	.7	-.0621	.3306	.8314	-.0999	-.0621	-.0621	.2685	-.7315	.0999	.0999
	.8	-.0401	.2041	.9193	-.0833	-.0401	-.0401	.1640	-.8360	.0833	.0833
	.9	-.0185	.0915	.9779	-.0509	-.0185	-.0185	.0730	-.9270	.0509	.0509
	C	0	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 3	.1	.0115	-.0560	.9865	.0580	.0115	.0115	-.0445	-.0445	.9420	-.0580
	.2	.0193	-.0943	.9457	.1293	.0193	.0193	-.0750	-.0750	.8707	-.1293
	.3	.0239	-.1168	.8806	.2123	.0239	.0239	-.0929	-.0929	.7877	-.2123
	.4	.0257	-.1257	.7943	.3057	.0257	.0257	-.1000	-.1000	.6943	-.3057
	.5	.0251	-.1227	.6897	.4079	.0251	.0251	-.0976	-.0976	.5921	-.4079
	.6	.0225	-.1100	.5700	.5175	.0225	.0225	-.0875	-.0875	.4825	-.5175
	.7	.0183	-.0894	.4381	.6330	.0183	.0183	-.0711	-.0711	.3670	-.6330
	.8	.0129	-.0529	.2871	.7529	.0129	.0129	-.0500	-.0500	.2471	-.7529
	.9	.0066	-.0324	.1501	.8757	.0066	.0066	-.0258	-.0258	.1243	-.8757
	D	0	0	0	1.0	0	0	0	0	0	-1.0
+ Area		.4554	1.3036	1.3036	.4554	.4554	.0167	.6851	.0651	.6385	.0772
− Area		-.0772	-.0818	-.0818	-.0772	-.0772	-.6385	-.0651	-.6851	-.0167	-.4554
Total Area		.3782	1.2218	1.2218	.3782	.3782	-.6218	.6000	-.6000	.6218	-.3782



Influence coefficients — Three continuous spans.

L = Length of EXTERIOR spans; length of interior span = NL

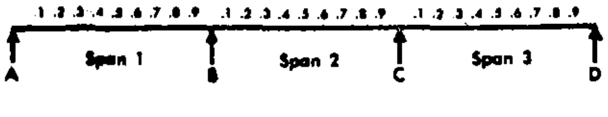
N = 1.2

Table 20.32

MOMENTS/PL

Unit load at		SPAN 1 A	SPAN 1 .1	SPAN 1 .2	SPAN 1 .3	SPAN 1 .4	SPAN 1 .5	SPAN 1 .6	SPAN 1 .7	SPAN 1 .8	SPAN 1 .9	B	SPAN 2 .1	SPAN 2 .2	SPAN 2 .3	SPAN 2 .4	SPAN 2 .5	SPAN 2 .6	SPAN 2 .7	SPAN 2 .8	SPAN 2 .9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0877	.0753	.0630	.0508	.0383	.0260	.0138	.0013	-.0111	-.0234	-.0204	-.0174	-.0144	-.0114	-.0084	-.0054	-.0024	.0006	.0036	.0066
	.2	0	.0755	.1509	.1384	.1019	.0773	.0528	.0282	.0037	.0208	-.0454	-.0395	-.0337	-.0279	-.0221	-.0163	-.0105	-.0046	.0012	.0070	.0128
	.3	0	.0636	.1271	.1907	.1542	.1178	.0813	.0449	.0084	-.0280	-.0645	-.0562	-.0480	-.0397	-.0314	-.0231	-.0149	-.0066	.0017	.0100	.0182
	.4	0	.0521	.1041	.1562	.2082	.1603	.1124	.0644	.0165	-.0314	-.0794	-.0692	-.0590	-.0488	-.0387	-.0285	-.0163	-.0081	.0021	.0123	.0224
	.5	0	.0411	.0823	.1234	.1646	.2057	.1468	.0880	.0291	-.0297	-.0886	-.0772	-.0659	-.0545	-.0431	-.0318	-.0204	-.0091	.0023	.0137	.0250
	.6	0	.0309	.0619	.0928	.1237	.1546	.1856	.1165	.0474	-.0217	-.0907	-.0791	-.0675	-.0558	-.0442	-.0325	-.0209	-.0093	.0024	.0140	.0256
	.7	0	.0216	.0431	.0647	.0863	.1078	.1294	.1510	.0725	-.0059	-.0843	-.0735	-.0627	-.0519	-.0411	-.0303	-.0194	-.0086	.0022	.0130	.0238
	.8	0	.0132	.0264	.0396	.0528	.0660	.0792	.0924	.1056	.0188	-.0680	-.0593	-.0506	-.0419	-.0331	-.0244	-.0157	-.0070	.0018	.0105	.0192
	.9	0	.0060	.0119	.0179	.0238	.0298	.0358	.0417	.0477	.0536	-.0404	-.0352	-.0300	-.0249	-.0197	-.0145	-.0093	-.0041	.0011	.0062	.0114
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0057	-.0114	-.0171	-.0228	-.0286	-.0343	-.0400	-.0457	-.0514	-.0571	.0636	.0543	.0450	.0356	.0263	.0170	.0077	-.0016	-.0109	-.0202
	.2	0	-.0093	-.0187	-.0280	-.0373	-.0467	-.0560	-.0653	-.0747	-.0840	-.0933	.0156	.1249	.1034	.0823	.0613	.0402	.0191	-.0020	-.0231	-.0442
	.3	0	-.0112	-.0223	-.0335	-.0447	-.0559	-.0670	-.0782	-.0894	-.1006	-.1117	-.0164	.0789	.1742	.1395	.1048	.0701	.0354	.0007	-.0340	-.0687
	.4	0	-.0115	-.0231	-.0346	-.0462	-.0577	-.0692	-.0808	-.0923	-.1039	-.1154	-.0350	.0455	.1260	.2064	.1569	.1073	.0578	.0083	.0413	-.0908
	.5	0	-.0107	-.0215	-.0322	-.0430	-.0537	-.0644	-.0752	-.0859	-.0967	-.1074	-.0424	.0226	.0876	.1526	.2176	.1526	.0876	.0226	-.0424	-.1074
	.6	0	-.0091	-.0182	-.0272	-.0363	-.0454	-.0545	-.0636	-.0727	-.0817	-.0908	-.0413	.0083	.0578	.1073	.1569	.2064	.1260	.0455	-.0350	-.1154
	.7	0	-.0069	-.0137	-.0206	-.0275	-.0344	-.0412	-.0481	-.0550	-.0618	-.0687	-.0340	.0007	.0354	.0701	.1048	.1395	.1742	.0789	-.0164	-.1117
	.8	0	-.0044	-.0088	-.0132	-.0177	-.0221	-.0265	-.0309	-.0353	-.0397	-.0442	-.0231	-.0020	.0191	.0409	.0613	.0823	.1034	.1245	.0156	-.0933
	.9	0	-.0020	-.0040	-.0061	-.0081	-.0101	-.0121	-.0142	-.0162	-.0182	-.0202	-.0109	-.0016	.0077	.0170	.0263	.0356	.0450	.0543	.0636	-.0571
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0011	.0023	.0034	.0046	.0057	.0068	.0080	.0091	.0103	.0114	.0062	.0011	-.0041	-.0093	-.0145	-.0197	-.0249	-.0300	-.0352	.0404
	.2	0	.0019	.0038	.0058	.0077	.0096	.0115	.0134	.0154	.0173	.0192	.0105	.0018	-.0070	-.0157	-.0244	-.0331	-.0419	-.0506	-.0593	-.0680
	.3	0	.0024	.0048	.0071	.0095	.0119	.0143	.0167	.0190	.0214	.0238	.0130	.0022	-.0086	-.0194	-.0303	-.0411	-.0519	-.0627	-.0735	-.0843
	.4	0	.0026	.0051	.0077	.0102	.0128	.0154	.0179	.0205	.0230	.0256	.0140	.0024	-.0093	-.0209	-.0325	-.0442	-.0558	-.0675	-.0791	-.0907
	.5	0	.0025	.0050	.0075	.0100	.0125	.0150	.0175	.0200	.0225	.0250	.0137	.0023	-.0091	-.0204	-.0318	-.0431	-.0545	-.0659	-.0772	-.0886
	.6	0	.0022	.0045	.0067	.0090	.0112	.0134	.0157	.0179	.0202	.0224	.0123	.0021	-.0081	-.0183	-.0285	-.0387	-.0488	-.0590	-.0692	-.0794
	.7	0	.0018	.0036	.0055	.0073	.0091	.0109	.0127	.0146	.0164	.0182	.0100	.0017	-.0066	-.0149	-.0231	-.0314	-.0397	-.0480	-.0562	-.0645
	.8	0	.0013	.0026	.0038	.0051	.0064	.0077	.0090	.0102	.0115	.0128	.0070	.0012	-.0046	-.0105	-.0163	-.0221	-.0279	-.0337	-.0395	-.0454
	.9	0	.0007	.0013	.0020	.0026	.0033	.0040	.0046	.0053	.0059	.0066	.0036	.0006	-.0024	-.0054	-.0084	-.0114	-.0144	-.0174	-.0204	-.0234
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0408	.0715	.0923	.1031	.1038	.0946	.0754	.0461	.0218	.0167	.0184	.0441	.0844	.1097	.1182	.1097	.0844	.0441	.0184	.0167
− Area		0	-.0093	-.0186	-.0279	-.0373	-.0465	-.0559	-.0652	-.0745	-.0987	-.1522	-.0778	-.0444	-.0424	-.0424	-.0424	-.0424	-.0424	-.0444	-.0778	-.1522
Total Area		0	.0315	.0529	.0644	.0658	.0573	.0387	.0102	-.0284	-.0769	-.1355	-.0594	-.0003	.0420	.0673	.0758	.0673	.0420	-.0003	-.0594	-.1355

Unit load at		REACTIONS/P R_A	R_B	R_C	R_D	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
SPAN 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8766	.1465	-.0297	.0066	.8766	-.1234	.0231	.0231	-.0066	-.0066
	.2	.7546	.2901	-.0575	.0128	.7546	-.2454	.0447	.0447	-.0128	-.0128
	.3	.6355	.4281	-.0818	.0182	.6355	-.3645	.0636	.0636	-.0182	-.0182
	.4	.5206	.5577	-.1007	.0224	.5206	-.4794	.0783	.0783	-.0224	-.0224
	.5	.4114	.6760	-.1124	.0250	.4114	-.5886	.0874	.0874	-.0250	-.0250
	.6	.3093	.7802	-.1151	.0256	.3093	-.6907	.0895	.0895	-.0256	-.0256
	.7	.2157	.8676	-.1071	.0238	.2157	-.7843	.0833	.0833	-.0238	-.0238
	.8	.1320	.9352	-.0864	.0192	.1320	-.8680	.0672	.0672	-.0192	-.0192
	.9	.0596	.9803	-.0513	.0114	.0596	-.9404	.0399	.0399	-.0114	-.0114
	B	0	1.0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 2	.1	-.0571	.9855	.0918	-.0202	-.0571	-.0571	.9284	-.0716	.0202	.0202
	.2	-.0933	.9311	.2064	-.0442	-.0933	-.0933	.8378	-.1622	.0442	.0442
	.3	-.1117	.8448	.3356	-.0687	-.1117	-.1117	.7331	-.2669	.0687	.0687
	.4	-.1154	.7343	.4719	-.0908	-.1154	-.1154	.6189	-.3811	.0908	.0908
	.5	-.1074	.6074	.6074	-.1074	-.1074	-.1074	.5000	-.5000	.1074	.1074
	.6	-.0908	.4719	.7343	-.1154	-.0908	-.0908	.3811	-.6189	.1154	.1154
	.7	-.0687	.3356	.8448	-.1117	-.0687	-.0687	.2669	-.7331	.1117	.1117
	.8	-.0442	.2064	.9311	-.0933	-.0442	-.0442	.1622	-.8378	.0933	.0933
	.9	-.0202	.0918	.9855	-.0571	-.0202	-.0202	.0716	-.9284	.0571	.0571
	C	0	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 3	.1	.0114	-.0513	.9803	.0596	.0114	.0114	-.0399	-.0399	.9404	-.0596
	.2	.0192	-.0864	.9352	.1320	.0192	.0192	-.0672	-.0672	.8680	-.1320
	.3	.0238	-.1071	.8676	.2157	.0238	.0238	-.0833	-.0833	.7843	-.2157
	.4	.0256	-.1151	.7802	.3093	.0256	.0256	-.0895	-.0895	.6907	-.3093
	.5	.0250	-.1124	.6760	.4114	.0250	.0250	-.0874	-.0874	.5886	-.4114
	.6	.0224	-.1007	.5577	.5206	.0224	.0224	-.0783	-.0783	.4794	-.5206
	.7	.0182	-.0818	.4281	.6355	.0182	.0182	-.0636	-.0636	.3645	-.6355
	.8	.0128	-.0575	.2901	.7546	.0128	.0128	-.0447	-.0447	.2454	-.7546
	.9	.0066	-.0297	.1465	.8766	.0066	.0066	-.0231	-.0231	.1234	-.8766
	D	0	0	0	1.0	0	0	0	0	0	-1.0
+ Area		.4576	1.3604	1.3604	.4576	.4576	.0167	.7083	.0583	.6522	.0931
− Area		-.0931	-.0749	-.0749	-.0931	-.0931	-.6522	-.0583	-.7083	-.0167	-.4576
Total Area		.3645	1.2855	1.2855	.3645	.3645	-.6355	.6500	-.6500	.6355	-.3645



Influence coefficients — Three continuous spans.

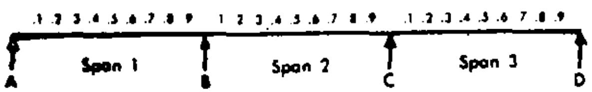
L = Length of EXTERIOR spans; length of interior span = NL

N = 1.3

Table 20.33

Unit load at		MOMENTS/PL																				
		SPAN 1										SPAN 2										
		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0877	.0755	.0632	.0510	.0387	.0265	.0142	.0020	-.0103	-.0225	-.0196	-.0167	-.0138	-.0109	-.0080	-.0051	-.0022	.0008	.0037	.0066
	.2	0	.0756	.1513	.1269	.1025	.0781	.0538	.0294	.0050	-.0193	-.0437	-.0381	-.0324	-.0268	-.0211	-.0155	-.0098	-.0042	.0015	.0071	.0128
	.3	0	.0638	.1276	.1914	.1551	.1189	.0827	.0465	.0103	-.0259	-.0622	-.0541	-.0461	-.0381	-.0300	-.0220	-.0140	-.0060	.0021	.0101	.0181
	.4	0	.0523	.1047	.1570	.2094	.1617	.1141	.0664	.0188	-.0289	-.0765	-.0666	-.0567	-.0469	-.0370	-.0271	-.0172	-.0073	.0026	.0124	.0223
	.5	0	.0415	.0829	.1244	.1658	.2073	.1488	.0902	.0317	-.0269	-.0854	-.0744	-.0633	-.0523	-.0413	-.0302	-.0192	-.0082	.0028	.0139	.0249
	.6	0	.0313	.0625	.0938	.1250	.1563	.1875	.1188	.0500	-.0187	-.0874	-.0761	-.0649	-.0536	-.0423	-.0310	-.0197	-.0084	.0029	.0142	.0255
	.7	0	.0219	.0437	.0656	.0875	.1094	.1312	.1531	.0750	-.0032	-.0813	-.0708	-.0603	-.0498	-.0393	-.0288	-.0183	-.0078	.0027	.0132	.0237
	.8	0	.0134	.0269	.0403	.0538	.0672	.0807	.0941	.1075	.0210	-.0656	-.0571	-.0486	-.0402	-.0317	-.0232	-.0148	-.0063	.0022	.0107	.0191
	.9	0	.0061	.0122	.0183	.0244	.0305	.0366	.0427	.0489	.0530	-.0389	-.0339	-.0289	-.0238	-.0188	-.0138	-.0088	-.0037	.0013	.0063	.0114
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0063	-.0127	-.0190	-.0254	-.0317	-.0381	-.0444	-.0508	-.0571	-.0634	.0667	.0568	.0470	.0372	.0273	.0175	.0076	-.0022	-.0121	-.0219
	.2	0	-.0104	-.0207	-.0311	-.0414	-.0518	-.0621	-.0725	-.0828	-.0932	-.1035	.0140	.1315	.1091	.0866	.0641	.0417	.0192	-.0033	-.0257	-.0482
	.3	0	-.0124	-.0248	-.0371	-.0495	-.0619	-.0743	-.0867	-.0990	-.1114	-.1238	-.0210	.0819	.1847	.1476	.1104	.0733	.0361	-.0011	-.0382	-.0754
	.4	0	-.0128	-.0255	-.0383	-.0511	-.0638	-.0766	-.0893	-.1021	-.1149	-.1276	-.0409	.0459	.1327	.2194	.1662	.1130	.0597	.0065	-.0467	-.1000
	.5	0	-.0119	-.0237	-.0356	-.0474	-.0593	-.0711	-.0830	-.0948	-.1067	-.1185	-.0485	.0215	.0915	.1615	.2315	.1615	.0915	.0215	-.0485	-.1185
	.6	0	-.0100	-.0200	-.0300	-.0400	-.0500	-.0600	-.0700	-.0800	-.0900	-.1000	-.0467	.0065	.0597	.1130	.1662	.2194	.1327	.0459	-.0409	-.1276
	.7	0	-.0075	-.0151	-.0226	-.0301	-.0377	-.0452	-.0528	-.0603	-.0678	-.0754	-.0382	-.0011	.0361	.0733	.1104	.1476	.1847	.0819	-.0210	-.1238
	.8	0	-.0048	-.0096	-.0145	-.0193	-.0241	-.0289	-.0337	-.0386	-.0434	-.0482	-.0257	-.0033	.0192	.0417	.0641	.0866	.1091	.1315	.0140	-.1035
	.9	0	-.0022	-.0044	-.0066	-.0088	-.0110	-.0132	-.0153	-.0175	-.0197	-.0219	-.0121	-.0022	.0076	.0175	.0273	.0372	.0470	.0569	.0667	-.0634
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0011	.0023	.0034	.0046	.0057	.0068	.0080	.0091	.0103	.0114	.0063	.0013	-.0037	-.0088	-.0138	-.0188	-.0238	-.0289	-.0339	-.0389
	.2	0	.0019	.0038	.0057	.0076	.0096	.0115	.0134	.0153	.0172	.0191	.0107	.0022	-.0063	-.0148	-.0232	-.0317	-.0402	-.0486	-.0571	-.0656
	.3	0	.0024	.0047	.0071	.0095	.0119	.0142	.0166	.0190	.0213	.0237	.0132	.0027	-.0078	-.0183	-.0288	-.0393	-.0498	-.0603	-.0708	-.0813
	.4	0	.0026	.0051	.0077	.0102	.0128	.0153	.0179	.0204	.0230	.0255	.0142	.0029	-.0084	-.0197	-.0310	-.0423	-.0536	-.0649	-.0761	-.0874
	.5	0	.0025	.0050	.0075	.0100	.0125	.0149	.0174	.0199	.0224	.0249	.0139	.0028	-.0082	-.0192	-.0302	-.0413	-.0523	-.0633	-.0744	-.0854
	.6	0	.0022	.0045	.0067	.0089	.0112	.0134	.0156	.0178	.0201	.0223	.0124	.0026	-.0073	-.0172	-.0271	-.0370	-.0469	-.0567	-.0666	-.0765
	.7	0	.0018	.0036	.0054	.0072	.0091	.0109	.0127	.0145	.0163	.0181	.0101	.0021	-.0060	-.0140	-.0220	-.0300	-.0381	-.0461	-.0541	-.0622
	.8	0	.0013	.0026	.0038	.0051	.0064	.0077	.0090	.0102	.0115	.0128	.0071	.0015	-.0042	-.0098	-.0155	-.0211	-.0268	-.0324	-.0381	-.0437
	.9	0	.0007	.0013	.0020	.0026	.0033	.0040	.0046	.0053	.0059	.0066	.0037	.0008	-.0022	-.0051	-.0080	-.0109	-.0138	-.0167	-.0196	-.0225
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0410	.0719	.0929	.1038	.1048	.0958	.0768	.0477	.0221	.0166	.0194	.0490	.0952	.1246	.1344	.1246	.0952	.0490	.0194	.0166
− Area		0	-.0111	-.0221	-.0332	-.0442	-.0553	-.0664	-.0775	-.0885	-.1130	-.1676	-.0822	-.0432	-.0404	-.0404	-.0404	-.0404	-.0404	-.0432	-.0822	-.1676
Total Area		0	.0299	.0498	.0597	.0596	.0495	.0294	-.0007	-.0408	-.0909	-.1510	-.0628	.0058	.0548	.0842	.0940	.0842	.0548	.0058	.0628	.1510

Unit load at		REACTIONS/P				SHEARS/P					
		R_A	R_B	R_C	R_D	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
SPAN 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8774	.1433	-.0273	.0066	.8774	-.1226	.0207	.0207	-.0066	-.0066
	.2	.7563	.2841	-.0532	.0128	.7563	-.2437	.0404	.0404	-.0128	-.0128
	.3	.6378	.4195	-.0754	.0181	.6378	-.3622	.0573	.0573	-.0181	-.0181
	.4	.5235	.5471	-.0929	.0223	.5235	-.4765	.0706	.0706	-.0223	-.0223
	.5	.4146	.6642	-.1037	.0249	.4146	-.5854	.0788	.0788	-.0249	-.0249
	.6	.3126	.7681	-.1062	.0255	.3126	-.6874	.0807	.0807	-.0255	-.0255
	.7	.2187	.8563	-.0987	.0237	.2187	-.7813	.0750	.0750	-.0237	-.0237
	.8	.1344	.9261	-.0796	.0191	.1344	-.8656	.0605	.0605	-.0191	-.0191
	.9	.0611	.9749	-.0474	.0114	.0611	-.9389	.0360	.0360	-.0114	-.0114
	B	0	1.0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 2	.1	-.0634	.9931	.0922	-.0219	-.0634	-.0634	.9297	-.0703	.0219	.0219
	.2	-.1035	.9431	.2086	-.0482	-.1035	-.1035	.8396	-.1604	.0482	.0482
	.3	-.1238	.8584	.3408	-.0754	-.1238	-.1238	.7346	-.2654	.0754	.0754
	.4	-.1276	.7474	.4802	-.1000	-.1276	-.1276	.6198	-.3802	.1000	.1000
	.5	-.1185	.6185	.6185	-.1185	-.1185	-.1185	.5000	-.5000	.1185	.1185
	.6	-.1000	.4802	.7474	-.1276	-.1000	-.1000	.3802	-.6198	.1276	.1276
	.7	-.0754	.3408	.8584	-.1238	-.0754	-.0754	.2654	-.7346	.1238	.1238
	.8	-.0482	.2086	.9431	-.1035	-.0482	-.0482	.1604	-.8396	.1035	.1035
	.9	-.0219	.0922	.9931	-.0634	-.0219	-.0219	.0703	-.9297	.0634	.0634
	C	0	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 3	.1	.0114	-.0474	.9749	.0611	.0114	.0114	-.0360	-.0360	.9389	-.0611
	.2	.0191	-.0796	.9261	.1344	.0191	.0191	-.0605	-.0605	.8656	-.1344
	.3	.0237	-.0987	.8563	.2187	.0237	.0237	-.0750	-.0750	.7813	-.2187
	.4	.0255	-.1062	.7681	.3126	.0255	.0255	-.0807	-.0807	.6874	-.3126
	.5	.0249	-.1037	.6642	.4146	.0249	.0249	-.0788	-.0788	.5854	-.4146
	.6	.0223	-.0929	.5471	.5235	.0223	.0223	-.0706	-.0706	.4765	-.5235
	.7	.0181	-.0754	.4195	.6378	.0181	.0181	-.0573	-.0573	.3622	-.6378
	.8	.0128	-.0532	.2841	.7563	.0128	.0128	-.0404	-.0404	.2437	-.7563
	.9	.0066	-.0273	.1433	.8774	.0066	.0066	-.0207	-.0207	.1226	-.8774
	D	0	0	0	1.0	0	0	0	0	0	-1.0
+ Area		.4597	1.4201	1.4201	.4597	.4597	.0166	.7828	.0528	.8676	.1107
− Area		-.1107	-.0691	-.0691	-.1107	-.1107	-.8676	-.0528	-.7828	-.0166	-.4597
Total Area		.3490	1.3510	1.3510	.3490	.3490	-.6510	.7000	-.7000	.6510	-.3490



Influence coefficients — Three continuous spans.

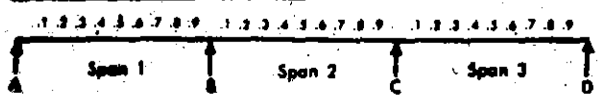
L = Length of EXTERIOR spans, length of interior span = NL

N = 1.4

Table 20.34

Span	Unit load at	MOMENTS/PL SPAN 1 A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	SPAN 2 .1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0878	.0756	.0635	.0513	.0391	.0269	.0148	.0026	-.0096	-.0218	-.0189	-.0161	-.0133	-.0104	-.0076	-.0048	-.0020	.0009	.0037	.0065
	.2	0	.0758	.1516	.1273	.1031	.0789	.0547	.0305	.0062	-.0180	-.0422	-.0367	-.0312	-.0257	-.0203	-.0148	-.0093	-.0038	.0017	.0072	.0127
	.3	0	.0640	.1280	.1920	.1560	.1200	.0840	.0480	.0120	-.0240	-.0600	-.0522	-.0444	-.0366	-.0288	-.0210	-.0132	-.0054	.0024	.0102	.0180
	.4	0	.0526	.1052	.1578	.2105	.1631	.1157	.0683	.0209	-.0265	-.0738	-.0643	-.0546	-.0450	-.0354	-.0258	-.0162	-.0066	.0030	.0126	.0222
	.5	0	.0418	.0835	.1253	.1670	.2088	.1505	.0923	.0341	-.0242	-.0824	-.0717	-.0610	-.0503	-.0396	-.0288	-.0181	-.0074	.0033	.0140	.0247
	.6	0	.0316	.0631	.0947	.1262	.1578	.1894	.1209	.0525	-.0160	-.0844	-.0734	-.0625	-.0515	-.0406	-.0295	-.0186	-.0076	.0034	.0143	.0253
	.7	0	.0222	.0443	.0665	.0886	.1108	.1329	.1551	.0772	-.0006	-.0785	-.0683	-.0581	-.0479	-.0377	-.0275	-.0173	-.0071	.0031	.0133	.0235
	.8	0	.0137	.0273	.0410	.0547	.0684	.0820	.0957	.1094	.0230	-.0633	-.0551	-.0468	-.0386	-.0304	-.0222	-.0139	-.0057	.0025	.0108	.0190
	.9	0	.0062	.0125	.0187	.0250	.0312	.0375	.0437	.0499	.0562	-.0376	-.0327	-.0278	-.0229	-.0180	-.0132	-.0083	-.0034	.0015	.0064	.0113
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0070	-.0140	-.0210	-.0279	-.0349	-.0419	-.0489	-.0559	-.0629	-.0699	.0698	.0594	.0490	.0386	.0283	.0179	.0075	-.0028	-.0132	-.0236
	.2	0	-.0114	-.0228	-.0342	-.0456	-.0570	-.0684	-.0798	-.0911	-.1025	-.1139	.0122	.1384	.1146	.0908	.0669	.0431	.0193	-.0046	-.0284	-.0522
	.3	0	-.0136	-.0272	-.0408	-.0544	-.0680	-.0816	-.0952	-.1088	-.1224	-.1360	-.0256	.0848	.1952	.1556	.1160	.0764	.0368	-.0028	-.0424	-.0820
	.4	0	-.0140	-.0280	-.0420	-.0560	-.0700	-.0840	-.0980	-.1120	-.1260	-.1400	-.0470	.0461	.1392	.2323	.1754	.1185	.0616	.0046	-.0523	-.1092
	.5	0	-.0130	-.0260	-.0389	-.0519	-.0649	-.0779	-.0909	-.1038	-.1168	-.1298	-.0548	.0202	.0952	.1702	.2452	.1702	.0952	.0202	-.0548	-.1298
	.6	0	-.0109	-.0218	-.0328	-.0437	-.0546	-.0655	-.0764	-.0873	-.0983	-.1092	-.0523	.0046	.0616	.1185	.1754	.2323	.1392	.0461	-.0470	-.1400
	.7	0	-.0082	-.0164	-.0246	-.0328	-.0410	-.0492	-.0574	-.0656	-.0738	-.0820	-.0424	-.0028	.0368	.0764	.1160	.1556	.1952	.0848	-.0256	-.1360
	.8	0	-.0052	-.0104	-.0157	-.0209	-.0261	-.0313	-.0366	-.0418	-.0470	-.0522	-.0284	-.0046	.0193	.0431	.0669	.0908	.1146	.1384	.0122	-.1139
	.9	0	-.0024	-.0047	-.0071	-.0094	-.0118	-.0142	-.0165	-.0189	-.0212	-.0236	-.0132	-.0028	.0075	.0179	.0283	.0386	.0490	.0594	.0698	-.0699
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0011	.0023	.0034	.0045	.0057	.0068	.0079	.0090	.0102	.0113	.0064	.0015	-.0034	-.0083	-.0132	-.0180	-.0229	-.0278	-.0327	-.0376
	.2	0	.0019	.0038	.0057	.0076	.0095	.0114	.0133	.0152	.0171	.0190	.0108	.0025	-.0057	-.0139	-.0222	-.0304	-.0386	-.0468	-.0551	-.0633
	.3	0	.0024	.0047	.0071	.0094	.0118	.0141	.0165	.0188	.0212	.0235	.0133	.0031	-.0071	-.0173	-.0275	-.0377	-.0479	-.0581	-.0683	-.0785
	.4	0	.0025	.0051	.0076	.0101	.0127	.0152	.0177	.0202	.0228	.0253	.0143	.0034	-.0076	-.0186	-.0295	-.0406	-.0515	-.0625	-.0734	-.0844
	.5	0	.0025	.0049	.0074	.0099	.0124	.0148	.0173	.0198	.0222	.0247	.0140	.0033	-.0074	-.0181	-.0288	-.0396	-.0503	-.0610	-.0717	-.0824
	.6	0	.0022	.0044	.0067	.0089	.0111	.0133	.0155	.0178	.0200	.0222	.0126	.0030	-.0066	-.0162	-.0258	-.0354	-.0450	-.0546	-.0642	-.0738
	.7	0	.0018	.0036	.0054	.0072	.0090	.0108	.0126	.0144	.0162	.0180	.0102	.0024	-.0054	-.0132	-.0210	-.0288	-.0366	-.0444	-.0522	-.0600
	.8	0	.0013	.0025	.0038	.0051	.0064	.0076	.0089	.0102	.0114	.0127	.0072	.0017	-.0038	-.0093	-.0148	-.0203	-.0257	-.0312	-.0367	-.0422
	.9	0	.0007	.0013	.0020	.0026	.0033	.0039	.0046	.0052	.0059	.0065	.0037	.0009	-.0020	-.0048	-.0076	-.0104	-.0133	-.0161	-.0189	-.0218
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0412	.0723	.0934	.1046	.1057	.0969	.0781	.0492	.0225	.0188	.0204	.0539	.1065	.1402	.1515	.1402	.1065	.0539	.0204	.0188
− Area		0	-.0130	-.0260	-.0389	-.0519	-.0648	-.0779	-.0909	-.1038	-.1289	-.1846	-.0874	-.0422	-.0385	-.0385	-.0385	-.0385	-.0385	-.0422	-.0874	-.1846
Total Area		0	.0282	.0463	.0545	.0527	.0409	.0190	-.0128	-.0546	-.1064	-.1882	-.0670	.0117	.0680	.1017	.1130	.1017	.0680	.0117	-.0670	-.1882

Span	Unit load at	REACTIONS/P R_A	R_B	R_C	R_D	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
SPAN 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8782	.1406	-.0253	.0065	.8782	-.1218	.0188	.0188	-.0065	-.0065
	.2	.7576	.2788	-.0491	.0127	.7576	-.2424	.0366	.0366	-.0127	-.0127
	.3	.6400	.4120	-.0700	.0180	.6400	-.3600	.0520	.0520	-.0180	-.0180
	.4	.5262	.5378	-.0862	.0222	.5262	-.4738	.0640	.0640	-.0222	-.0222
	.5	.4176	.6538	-.0961	.0247	.4176	-.5824	.0714	.0714	-.0247	-.0247
	.6	.3156	.7575	-.0984	.0253	.3156	-.6844	.0731	.0731	-.0253	-.0253
	.7	.2215	.8465	-.0915	.0235	.2215	-.7785	.0680	.0680	-.0235	-.0235
	.8	.1367	.9182	-.0739	.0190	.1367	-.8633	.0549	.0549	-.0190	-.0190
	.9	.0624	.9702	-.0439	.0113	.0624	-.9376	.0326	.0326	-.0113	-.0113
	B	0	1.0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 2	.1	-.0699	1.0007	.0928	-.0236	-.0699	-.0699	.9308	-.0692	.0236	.0236
	.2	-.1139	.9551	.2110	-.0522	-.1139	-.1139	.8412	-.1588	.0522	.0522
	.3	-.1360	.8720	.3460	-.0820	-.1360	-.1360	.7360	-.2640	.0820	.0820
	.4	-.1400	.7606	.4886	-.1092	-.1400	-.1400	.6206	-.3794	.1092	.1092
	.5	-.1298	.6298	.6298	-.1298	-.1298	-.1298	.5000	-.5000	.1298	.1298
	.6	-.1092	.4886	.7606	-.1400	-.1092	-.1092	.3794	-.6206	.1400	.1400
	.7	-.0820	.3460	.8720	-.1360	-.0820	-.0820	.2640	-.7360	.1360	.1360
	.8	-.0522	.2110	.9551	-.1139	-.0522	-.0522	.1588	-.8412	.1139	.1139
	.9	-.0236	.0928	1.0007	-.0699	-.0236	-.0236	.0692	-.9308	.0699	.0699
	C	0	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 3	.1	.0113	-.0439	.9702	.0624	.0113	.0113	-.0326	-.0326	.9376	-.0624
	.2	.0190	-.0739	.9182	.1367	.0190	.0190	-.0549	-.0549	.8633	-.1367
	.3	.0235	-.0915	.8465	.2215	.0235	.0235	-.0680	-.0680	.7785	-.2215
	.4	.0253	-.0984	.7575	.3156	.0253	.0253	-.0731	-.0731	.6844	-.3156
	.5	.0247	-.0961	.6538	.4176	.0247	.0247	-.0714	-.0714	.5824	-.4176
	.6	.0222	-.0862	.5378	.5262	.0222	.0222	-.0640	-.0640	.4738	-.5262
	.7	.0180	-.0700	.4120	.6400	.0180	.0180	-.0520	-.0520	.3600	-.6400
	.8	.0127	-.0493	.2788	.7576	.0127	.0127	-.0366	-.0366	.2422	-.7576
	.9	.0065	-.0253	.1406	.8782	.0065	.0065	-.0188	-.0188	.1218	-.8782
	D	0	0	0	1.0	0	0	0	0	0	-1.0
+ Area		.4615	1.4824	1.4824	.4615	.4615	.0165	.7976	.0476	.8847	.1298
− Area		-.1298	-.0641	-.0641	-.1298	-.1298	-.8847	-.0476	-.7976	-.0165	-.4615
Total Area		.3317	1.4183	1.4183	.3317	.3317	-.8682	.7500	-.7500	.8682	-.3317



Influence coefficients — Three continuous spans.

L = Length of EXTERIOR spans; length of interior span = NL

N = 1.5

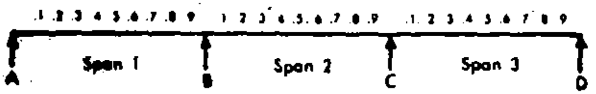
Table 20.35

MOMENTS/PL

Unit load at		Span 1: A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B
Span 1	A	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0879	.0758	.0637	.0516	.0395	.0274	.0153	.0032	-.0089	-.0210
	.2	0	.0759	.1518	.1278	.1037	.0796	.0555	.0315	.0074	-.0167	-.0408
	.3	0	.0642	.1284	.1926	.1568	.1210	.0852	.0494	.0136	-.0222	-.0580
	.4	0	.0529	.1057	.1586	.2115	.1643	.1172	.0700	.0229	-.0242	-.0714
	.5	0	.0420	.0841	.1261	.1681	.2102	.1522	.0942	.0363	-.0217	-.0797
	.6	0	.0318	.0637	.0955	.1274	.1592	.1911	.1229	.0547	-.0134	-.0816
	.7	0	.0224	.0448	.0673	.0897	.1121	.1345	.1569	.0793	.0018	-.0756
	.8	0	.0139	.0278	.0416	.0555	.0694	.0833	.0972	.1111	.0249	-.0612
	.9	0	.0064	.0127	.0191	.0255	.0318	.0382	.0446	.0509	.0573	-.0363
	B	0	0	0	0	0	0	0	0	0	0	0
Span 2	.1	0	-.0076	-.0153	-.0229	-.0306	-.0382	-.0459	-.0535	-.0611	-.0688	-.0764
	.2	0	-.0124	-.0249	-.0373	-.0498	-.0622	-.0747	-.0871	-.0996	-.1120	-.1245
	.3	0	-.0148	-.0297	-.0445	-.0594	-.0742	-.0891	-.1039	-.1188	-.1336	-.1485
	.4	0	-.0153	-.0305	-.0458	-.0610	-.0763	-.0916	-.1068	-.1221	-.1373	-.1526
	.5	0	-.0141	-.0282	-.0424	-.0565	-.0706	-.0847	-.0988	-.1129	-.1270	-.1412
	.6	0	-.0118	-.0237	-.0355	-.0474	-.0592	-.0711	-.0829	-.0948	-.1066	-.1183
	.7	0	-.0089	-.0177	-.0266	-.0355	-.0444	-.0532	-.0621	-.0710	-.0798	-.0887
	.8	0	-.0056	-.0112	-.0169	-.0225	-.0281	-.0337	-.0394	-.0450	-.0506	-.0562
	.9	0	-.0025	-.0050	-.0076	-.0101	-.0126	-.0151	-.0177	-.0202	-.0227	-.0252
	C	0	0	0	0	0	0	0	0	0	0	0
Span 3	.1	0	.0011	.0022	.0034	.0045	.0056	.0067	.0078	.0090	.0101	.0112
	.2	0	.0019	.0038	.0056	.0075	.0094	.0113	.0132	.0150	.0169	.0188
	.3	0	.0023	.0047	.0070	.0093	.0117	.0140	.0163	.0186	.0210	.0233
	.4	0	.0025	.0050	.0075	.0100	.0126	.0151	.0176	.0201	.0226	.0251
	.5	0	.0025	.0049	.0074	.0098	.0123	.0147	.0172	.0196	.0221	.0245
	.6	0	.0022	.0044	.0066	.0088	.0110	.0132	.0154	.0176	.0198	.0220
	.7	0	.0018	.0036	.0053	.0071	.0089	.0107	.0125	.0142	.0160	.0178
	.8	0	.0013	.0025	.0038	.0050	.0063	.0075	.0088	.0100	.0113	.0125
	.9	0	.0007	.0013	.0020	.0026	.0033	.0039	.0046	.0052	.0059	.0065
	D	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0413	.0727	.0940	.1033	.1066	.0979	.0793	.0506	.0228	.0163
− Area		0	-.0150	-.0302	-.0452	-.0602	-.0753	-.0903	-.1054	-.1205	-.1464	-.2037
Total Area		0	.0263	.0425	.0488	.0431	.0313	.0076	-.0261	-.0699	-.1236	-.1874

Unit load at		Span 2: .1	.2	.3	.4	.5	.6	.7	.8	.9	C
Span 1	A	0	0	0	0	0	0	0	0	0	0
	.1	-.0183	-.0155	-.0128	-.0100	-.0073	-.0045	-.0018	.0010	.0037	.0065
	.2	-.0355	-.0301	-.0248	-.0195	-.0141	-.0088	-.0035	.0019	.0072	.0125
	.3	-.0504	-.0428	-.0352	-.0277	-.0201	-.0125	-.0049	.0027	.0103	.0178
	.4	-.0620	-.0527	-.0434	-.0340	-.0247	-.0154	-.0060	.0033	.0126	.0220
	.5	-.0692	-.0588	-.0484	-.0380	-.0276	-.0172	-.0067	.0037	.0141	.0245
	.6	-.0709	-.0602	-.0496	-.0389	-.0282	-.0176	-.0069	.0038	.0144	.0251
	.7	-.0659	-.0560	-.0461	-.0362	-.0262	-.0163	-.0064	.0035	.0134	.0233
	.8	-.0532	-.0452	-.0372	-.0292	-.0212	-.0132	-.0052	.0028	.0108	.0188
	.9	-.0316	-.0268	-.0221	-.0173	-.0125	-.0078	-.0031	.0017	.0064	.0112
	B	0	0	0	0	0	0	0	0	0	0
Span 2	.1	.0727	.0618	.0509	.0401	.0292	.0183	.0074	-.0035	-.0143	-.0252
	.2	.0103	.1452	.1200	.0948	.0696	.0445	.0193	-.0059	-.0310	-.0562
	.3	-.0305	.0875	.2055	.1634	.1214	.0794	.0374	-.0047	-.0467	-.0887
	.4	-.0532	.0462	.1456	.2451	.1845	.1239	.0633	.0027	-.0579	-.1183
	.5	-.0612	.0188	.0988	.1788	.2588	.1788	.0988	.0188	-.0612	-.1412
	.6	-.0579	.0027	.0633	.1239	.1845	.2451	.1456	.0462	-.0532	-.1526
	.7	-.0467	-.0047	.0374	.0794	.1214	.1634	.2055	.0875	-.0305	-.1485
	.8	-.0310	-.0059	.0193	.0445	.0696	.0948	.1200	.1452	.0103	-.1245
	.9	-.0143	-.0035	.0074	.0183	.0292	.0401	.0509	.0618	.0727	-.0764
	C	0	0	0	0	0	0	0	0	0	0
Span 3	.1	.0064	.0017	-.0031	-.0078	-.0125	-.0173	-.0221	-.0268	-.0316	-.0363
	.2	.0108	.0028	-.0052	-.0132	-.0212	-.0292	-.0372	-.0452	-.0532	-.0612
	.3	.0134	.0035	-.0064	-.0163	-.0262	-.0362	-.0461	-.0560	-.0659	-.0758
	.4	.0144	.0038	-.0069	-.0176	-.0282	-.0389	-.0496	-.0602	-.0709	-.0816
	.5	.0141	.0037	-.0067	-.0172	-.0276	-.0380	-.0484	-.0588	-.0692	-.0797
	.6	.0126	.0033	-.0060	-.0154	-.0247	-.0340	-.0434	-.0527	-.0620	-.0714
	.7	.0103	.0027	-.0049	-.0125	-.0201	-.0277	-.0352	-.0428	-.0504	-.0580
	.8	.0072	.0019	-.0035	-.0088	-.0141	-.0195	-.0248	-.0301	-.0355	-.0408
	.9	.0037	.0010	-.0018	-.0045	-.0073	-.0100	-.0128	-.0155	-.0183	-.0210
	D	0	0	0	0	0	0	0	0	0	0
+ Area		.0214	.0589	.1182	.1566	.1694	.1566	.1182	.0589	.0214	.0163
− Area		-.0935	-.0415	-.0368	-.0368	-.0368	-.0368	-.0368	-.0415	-.0935	-.2037
Total Area		-.0721	-.0174	.0814	.1198	.1326	.1198	.0814	.0174	-.0721	-.1874

Unit load at		REACTIONS/P: R_A	R_B	R_C	R_D	SHEARS/P: V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
Span 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8790	.1362	-.0237	.0065	.8790	-.1210	.0172	.0172	-.0065	-.0065
	.2	.7592	.2741	-.0458	.0125	.7592	-.2408	.0333	.0333	-.0125	-.0125
	.3	.6420	.4054	-.0652	.0178	.6420	-.3580	.0474	.0474	-.0178	-.0178
	.4	.5286	.5297	-.0803	.0220	.5286	-.4714	.0583	.0583	-.0220	-.0220
	.5	.4203	.6448	-.0896	.0245	.4203	-.5797	.0651	.0651	-.0245	-.0245
	.6	.3134	.7482	-.0917	.0251	.3184	-.6816	.0666	.0666	-.0251	-.0251
	.7	.2242	.8376	-.0653	.0233	.2242	-.7758	.0620	.0620	-.0233	-.0233
	.8	.1388	.9112	-.0688	.0188	.1388	-.8612	.0500	-.0500	-.0188	-.0188
	.9	.0637	.9660	-.0409	.0112	.0637	-.9363	.0297	.0297	-.0112	-.0112
	B	0	1.0	0	0	0	-1.0/0	0/1.0	0	0	0
Span 2	.1	-.0764	1.0084	.0932	-.0252	-.0764	-.0764	.9320	-.0680	.0252	.0252
	.2	-.1245	.9672	.2135	-.0562	-.1245	-.1245	.8427	-.1573	-.0562	.0562
	.3	-.1485	.8858	.3514	-.0887	-.1485	-.1485	.7373	-.2627	.0887	.0887
	.4	-.1526	.7739	.4972	-.1185	-.1526	-.1526	.6213	-.3787	.1185	.1185
	.5	-.1412	.6412	.6412	-.1412	-.1412	-.1412	.5000	.5000	.1412	.1412
	.6	-.1185	.4972	.7739	-.1526	-.1185	.1185	.3787	-.6213	.1526	.1526
	.7	-.0887	.3514	.8858	-.1485	-.0887	-.0887	.2627	-.7373	.1485	.1485
	.8	-.0562	.2135	.9672	-.1245	-.0562	-.0562	.1573	-.8427	.1245	.1245
	.9	-.0252	.0932	1.0084	-.0764	-.0252	-.0252	.0680	-.9320	.0764	.0764
	C	0	0	1.0	0	0	0	0	-1.0/0	0/1.0	0
Span 3	.1	.0112	-.0409	.9660	.0637	.0112	.0112	-.0297	-.0297	.9363	-.0637
	.2	.0188	-.0688	.9112	.1388	.0188	.0188	-.0500	-.0500	.8612	-.1388
	.3	.0233	-.0653	.8376	.2242	.0233	.0233	-.0620	-.0620	.7758	-.2242
	.4	.0251	-.0917	.7482	.3184	.0251	.0251	-.0666	-.0666	.6816	-.3184
	.5	.0245	-.0896	.6448	.4203	.0245	.0245	-.0651	-.0651	.5797	-.4203
	.6	.0220	-.0803	.5297	.5286	.0220	.0220	-.0583	-.0583	.4714	-.5286
	.7	.0178	-.0652	.4054	.6420	.0178	.0178	-.0474	-.0474	.3580	-.6420
	.8	.0125	-.0458	.2741	.7592	.0125	.0125	-.0333	-.0333	.2408	-.7592
	.9	.0065	-.0237	.1382	.8790	.0065	.0065	-.0172	-.0172	.1210	-.8790
	D	0	0	0	1.0	0	0	0	0	0	-1.0
+ Area		.4632	1.5471	1.5471	.4632	.4632	.0163	.8434	.0434	.7037	.1506
− Area		-.1506	-.0597	-.0597	-.1506	-.1506	-.7037	-.0434	-.8434	-.0163	-.4632
Total Area		.3126	1.4874	1.4874	.3126	.3126	-.6874	.8000	-.8000	.6874	-.3126



Influence coefficients — Three continuous spans.

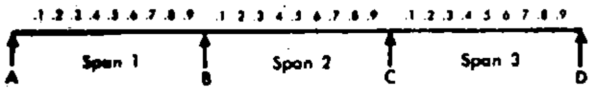
L = Length of EXTERIOR spans: length of interior span = NL

N = 1.6

Table 20.36

Span	Unit load at	MOMENTS/PL SPAN 1 A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	SPAN 2 .1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0880	.0759	.0639	.0519	.0398	.0278	.0158	.0037	-.0083	-.0204	-.0177	-.0150	-.0123	-.0096	-.0070	-.0043	-.0016	.0011	.0037	.0064
	.2	0	.0761	.1521	.1282	.1042	.0803	.0563	.0324	.0084	-.0155	-.0395	-.0343	-.0281	-.0239	-.0187	-.0135	-.0083	-.0031	.0020	.0072	.0124
	.3	0	.0644	.1288	.1932	.1576	.1219	.0863	.0507	.0151	-.0205	-.0561	-.0487	-.0414	-.0340	-.0266	-.0192	-.0118	-.0045	.0029	.0103	.0177
	.4	0	.0531	.1062	.1593	.2124	.1655	.1186	.0717	.0247	-.0222	-.0691	-.0600	-.0509	-.0418	-.0327	-.0237	-.0146	-.0055	.0036	.0127	.0217
	.5	0	.0423	.0846	.1269	.1692	.2115	.1537	.0960	.0383	-.0194	-.0771	-.0669	-.0563	-.0467	-.0365	-.0264	-.0163	-.0061	.0040	.0141	.0243
	.6	0	.0321	.0642	.0963	.1284	.1605	.1926	.1247	.0569	-.0110	-.0789	-.0686	-.0582	-.0478	-.0374	-.0270	-.0167	-.0063	.0041	.0145	.0248
	.7	0	.0227	.0453	.0680	.0906	.1133	.1360	.1586	.0813	.0040	-.0734	-.0637	-.0541	-.0444	-.0348	-.0251	-.0155	-.0058	.0038	.0135	.0231
	.8	0	.0141	.0282	.0422	.0563	.0704	.0845	.0986	.1126	.0267	-.0592	-.0514	-.0436	-.0358	-.0281	-.0203	-.0125	-.0047	.0031	.0109	.0186
	.9	0	.0065	.0130	.0195	.0259	.0324	.0389	.0454	.0519	.0584	-.0352	-.0305	-.0259	-.0213	-.0167	-.0120	-.0074	-.0028	.0018	.0064	.0111
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0083	-.0166	-.0249	-.0332	-.0415	-.0498	-.0581	-.0665	-.0748	-.0831	.0756	.0642	.0528	.0414	.0300	.0187	.0073	-.0041	-.0155	-.0268
	.2	0	-.0135	-.0270	-.0406	-.0541	-.0676	-.0811	-.0946	-.1081	-.1217	-.1352	.0083	.1518	.1253	.0988	.0723	.0458	.0193	-.0072	-.0337	-.0602
	.3	0	-.0161	-.0322	-.0483	-.0644	-.0805	-.0966	-.1127	-.1288	-.1449	-.1610	-.0355	.0901	.2157	.1712	.1268	.0823	.0379	-.0065	-.0510	-.0954
	.4	0	-.0165	-.0331	-.0496	-.0661	-.0826	-.0992	-.1157	-.1322	-.1488	-.1653	-.0595	.0462	.1520	.2577	.1935	.1292	.0650	.0007	-.0635	-.1278
	.5	0	-.0153	-.0305	-.0458	-.0611	-.0763	-.0916	-.1068	-.1221	-.1374	-.1526	-.0678	.0174	.1024	.1874	.2724	.1874	.1024	.0174	-.0676	-.1526
	.6	0	-.0128	-.0256	-.0383	-.0511	-.0639	-.0767	-.0895	-.1022	-.1150	-.1278	-.0635	.0007	.0650	.1292	.1935	.2577	.1520	.0462	-.0595	-.1653
	.7	0	-.0095	-.0191	-.0286	-.0382	-.0477	-.0572	-.0668	-.0763	-.0859	-.0954	-.0510	-.0065	.0379	.0823	.1268	.1712	.2157	.0901	-.0355	-.1610
	.8	0	-.0060	-.0120	-.0181	-.0241	-.0301	-.0361	-.0421	-.0482	-.0542	-.0602	-.0337	-.0072	.0193	.0458	.0723	.0988	.1253	.1518	.0083	-.1352
	.9	0	-.0027	-.0054	-.0080	-.0107	-.0134	-.0161	-.0188	-.0215	-.0241	-.0268	-.0155	-.0041	.0073	.0187	.0300	.0414	.0528	.0642	.0756	-.0831
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0011	.0022	.0033	.0044	.0056	.0067	.0078	.0089	.0100	.0111	.0084	.0018	-.0028	-.0074	-.0120	-.0167	-.0213	-.0259	-.0305	-.0352
	.2	0	.0019	.0037	.0056	.0074	.0093	.0112	.0130	.0149	.0167	.0186	.0109	.0031	-.0047	-.0125	-.0203	-.0281	-.0358	-.0436	-.0514	-.0592
	.3	0	.0023	.0046	.0069	.0092	.0116	.0139	.0162	.0185	.0208	.0231	.0135	.0038	-.0058	-.0155	-.0251	-.0348	-.0444	-.0541	-.0637	-.0734
	.4	0	.0025	.0050	.0074	.0099	.0124	.0149	.0174	.0198	.0223	.0248	.0145	.0041	-.0063	-.0167	-.0270	-.0374	-.0478	-.0582	-.0686	-.0789
	.5	0	.0024	.0049	.0073	.0097	.0122	.0146	.0170	.0194	.0219	.0243	.0141	.0040	-.0061	-.0163	-.0264	-.0365	-.0467	-.0568	-.0669	-.0771
	.6	0	.0022	.0043	.0065	.0087	.0109	.0130	.0152	.0174	.0195	.0217	.0127	.0036	-.0055	-.0146	-.0237	-.0327	-.0418	-.0509	-.0600	-.0691
	.7	0	.0018	.0035	.0053	.0071	.0089	.0106	.0124	.0142	.0159	.0177	.0103	.0029	-.0045	-.0118	-.0192	-.0266	-.0340	-.0414	-.0487	-.0561
	.8	0	.0012	.0025	.0037	.0050	.0062	.0074	.0087	.0099	.0112	.0124	.0072	.0020	-.0031	-.0083	-.0135	-.0187	-.0239	-.0291	-.0343	-.0395
	.9	0	.0006	.0013	.0019	.0026	.0032	.0038	.0045	.0051	.0058	.0064	.0037	.0011	-.0016	-.0043	-.0070	-.0096	-.0123	-.0150	-.0177	-.0204
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0415	.0729	.0945	.1060	.1074	.0989	.0804	.0518	.0231	.0162	.0224	.0641	.1304	.1738	.1882	.1738	.1304	.0641	.0224	.0162
− Area		0	-.0173	-.0345	-.0520	-.0693	-.0865	-.1038	-.1211	-.1384	-.1655	-.2244	-.1006	-.0411	-.0352	-.0352	-.0352	-.0352	-.0352	-.0411	-.1006	-.2244
Total Area		0	.0242	.0384	.0425	.0367	.0209	-.0049	-.0407	-.0866	-.1424	-.2082	-.0782	.0230	.0952	.1386	.1530	.1386	.0952	.0230	-.0782	-.2082

Span	Unit load at	REACTIONS/P R_A	R_B	R_C	R_D	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}
SPAN 1	A	1.0	0	0	0	1.0	0	0	0	0	0
	.1	.8796	.1361	-.0221	.0064	.8796	-.1204	.0157	.0157	-.0064	-.0064
	.2	.7605	.2700	-.0429	.0124	.7605	-.2395	.0308	.0308	-.0124	-.0124
	.3	.6439	.3995	-.0611	.0177	.6439	-.3561	.0434	.0434	-.0177	-.0177
	.4	.5309	.5225	-.0751	.0217	.5309	-.4691	.0534	.0534	-.0217	-.0217
	.5	.4229	.6387	-.0839	.0243	.4229	-.5771	.0596	.0596	-.0243	-.0243
	.6	.3211	.7400	-.0859	.0248	.3211	-.6789	.0611	.0611	-.0248	-.0248
	.7	.2264	.8301	-.0798	.0231	.2266	-.7734	.0567	.0567	-.0231	-.0231
	.8	.1408	.9050	-.0644	.0186	.1408	-.8592	.0458	.0458	-.0186	-.0186
	.9	.0648	.9623	-.0382	.0111	.0648	-.9352	.0271	.0271	-.0111	-.0111
	B	0	1.0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 2	.1	-.0831	1.0161	.0938	-.0268	-.0831	-.0831	.9330	-.0670	.0268	.0268
	.2	-.1352	.9793	.2161	-.0602	-.1352	-.1352	.8441	-.1559	.0602	.0602
	.3	-.1610	.8996	.3568	-.0954	-.1610	-.1610	.7386	-.2614	.0954	.0954
	.4	-.1653	.7874	.5057	-.1278	-.1653	-.1653	.6221	-.3779	.1278	.1278
	.5	-.1526	.6526	.6526	-.1526	-.1526	-.1526	.5000	-.5000	.1526	.1526
	.6	-.1278	.5057	.7874	-.1653	-.1278	-.1278	.3779	-.6221	.1653	.1653
	.7	-.0954	.3568	.8996	-.1610	-.0954	-.0954	.2614	-.7386	.1610	.1610
	.8	-.0602	.2161	.9793	-.1352	-.0602	-.0602	.1559	-.8441	.1352	.1352
	.9	-.0268	.0938	1.0161	-.0831	-.0268	-.0268	.0670	-.9330	.0831	.0831
	C	0	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 3	.1	.0111	-.0382	.9623	.0648	.0111	.0111	-.0271	-.0271	.9352	-.0648
	.2	.0186	-.0644	.9050	.1408	.0186	.0186	-.0458	-.0458	.8592	-.1408
	.3	.0231	-.0798	.8301	.2266	.0231	.0231	-.0567	-.0567	.7734	-.2266
	.4	.0248	-.0859	.7400	.3211	.0248	.0248	-.0611	-.0611	.6789	-.3211
	.5	.0243	-.0839	.6367	.4229	.0243	.0243	-.0596	-.0596	.5771	-.4229
	.6	.0217	-.0751	.5225	.5309	.0217	.0217	-.0534	-.0534	.4691	-.5309
	.7	.0177	-.0611	.3995	.6439	.0177	.0177	-.0434	-.0434	.3561	-.6439
	.8	.0124	-.0429	.2700	.7605	.0124	.0124	-.0305	-.0305	.2395	-.7605
	.9	.0064	-.0221	.1361	.8796	.0064	.0064	-.0157	-.0157	.1204	-.8796
	D	0	0	0	1.0	0	0	0	0	0	-1.0
+ Area		.4648	1.6141	1.6141	.4648	.4648	.0162	.8897	.0397	.7244	.1730
− Area		-.1730	-.0559	-.0559	-.1730	-.1730	-.7244	-.0397	-.8897	-.0162	-.4648
Total Area		.2918	1.5582	1.5582	.2918	.2918	-.7082	.8500	-.8500	.7082	-.2918



Influence coefficients — Three continuous spans.

L = Length of EXTERIOR spans; length of interior span = NL

N = 1.7

Table 20.37

Unit load at		Span 1 A	Span 1 .1	Span 1 .2	Span 1 .3	Span 1 .4	Span 1 .5	Span 1 .6	Span 1 .7	Span 1 .8	Span 1 .9	B	Span 2 .1	Span 2 .2	Span 2 .3	Span 2 .4	Span 2 .5	Span 2 .6	Span 2 .7	Span 2 .8	Span 2 .9	C
		MOMENTS/PL																				
	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 1	.1	0	.0873	.0747	.0620	.0494	.0367	.0241	.0114	-.0012	-.0139	-.0265	-.0232	-.0198	-.0164	-.0131	-.0097	-.0064	-.0030	.0004	.0037	.0071
	.2	0	.0749	.1497	.1246	.0994	.0743	.0491	.0240	-.0011	-.0263	-.0514	-.0449	-.0384	-.0319	-.0254	-.0189	-.0123	-.0058	.0007	.0072	.0137
	.3	0	.0627	.1254	.1881	.1508	.1134	.0761	.0388	.0015	-.0358	-.0731	-.0639	-.0546	-.0453	-.0361	-.0268	-.0176	-.0083	.0010	.0102	.0195
	.4	0	.0510	.1020	.1530	.2040	.1550	.1060	.0570	.0080	-.0410	-.0900	-.0786	-.0672	-.0558	-.0444	-.0330	-.0216	-.0102	.0012	.0126	.0240
	.5	0	.0400	.0799	.1199	.1598	.1998	.1397	.0797	.0196	-.0404	-.1004	-.0877	-.0750	-.0623	-.0496	-.0368	-.0241	-.0114	.0013	.0141	.0268
	.6	0	.0297	.0594	.0891	.1189	.1486	.1783	.1080	.0377	-.0326	-.1029	-.0898	-.0768	-.0638	-.0507	-.0377	-.0247	-.0117	.0014	.0144	.0274
	.7	0	.0204	.0409	.0613	.0818	.1022	.1226	.1431	.0635	-.0161	-.0956	-.0835	-.0714	-.0593	-.0472	-.0351	-.0230	-.0108	.0013	.0134	.0255
	.8	0	.0123	.0246	.0369	.0491	.0614	.0737	.0860	.0983	.0106	-.0771	-.0674	-.0576	-.0478	-.0381	-.0283	-.0185	-.0087	.0010	.0108	.0206
	.9	0	.0054	.0108	.0163	.0217	.0271	.0325	.0379	.0434	.0488	-.0458	-.0400	-.0342	-.0284	-.0226	-.0168	-.0110	-.0052	.0006	.0064	.0122
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0038	-.0077	-.0116	-.0155	-.0194	-.0232	-.0271	-.0310	-.0349	-.0387	.0535	.0458	.0381	.0303	.0226	.0149	.0071	-.0006	-.0083	-.0161
	.2	0	-.0063	-.0127	-.0190	-.0254	-.0317	-.0381	-.0444	-.0507	-.0571	-.0634	.0195	.1024	.0853	.0682	.0511	.0341	.0170	-.0001	-.0172	-.0343
	.3	0	-.0076	-.0152	-.0228	-.0305	-.0381	-.0457	-.0533	-.0609	-.0685	-.0761	-.0038	.0686	.1410	.1133	.0857	.0581	.0304	.0028	-.0249	-.0525
	.4	0	-.0079	-.0158	-.0237	-.0315	-.0394	-.0473	-.0552	-.0631	-.0710	-.0789	-.0178	.0432	.1042	.1653	.1263	.0873	.0483	.0094	-.0296	-.0686
	.5	0	-.0074	-.0147	-.0221	-.0295	-.0368	-.0442	-.0516	-.0589	-.0663	-.0737	-.0243	.0250	.0743	.1237	.1730	.1223	.0717	.0210	-.0297	-.0804
	.6	0	-.0063	-.0125	-.0188	-.0250	-.0313	-.0375	-.0438	-.0501	-.0563	-.0626	-.0249	.0128	.0505	.0882	.1259	.1635	.1012	.0389	-.0234	-.0857
	.7	0	-.0048	-.0095	-.0143	-.0191	-.0238	-.0286	-.0333	-.0381	-.0429	-.0476	-.0211	.0054	.0319	.0584	.0848	.1113	.1380	.0645	-.0090	-.0825
	.8	0	-.0031	-.0062	-.0093	-.0123	-.0154	-.0185	-.0216	-.0247	-.0278	-.0309	-.0146	.0016	.0178	.0341	.0503	.0665	.0827	.0990	.0152	-.0686
	.9	0	-.0014	-.0029	-.0043	-.0057	-.0072	-.0086	-.0100	-.0114	-.0129	-.0143	-.0071	.0002	.0075	.0147	.0220	.0292	.0365	.0437	.0510	-.0418
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0010	.0021	.0031	.0042	.0052	.0063	.0073	.0084	.0094	.0104	.0052	.0000	-.0052	-.0104	-.0157	-.0209	-.0261	-.0313	-.0366	-.0418
	.2	0	.0017	.0034	.0051	.0069	.0086	.0103	.0120	.0137	.0154	.0171	.0086	.0000	-.0086	-.0171	-.0257	-.0343	-.0429	-.0514	-.0600	-.0686
	.3	0	.0021	.0041	.0062	.0083	.0103	.0124	.0144	.0165	.0186	.0206	.0103	.0000	-.0103	-.0206	-.0309	-.0413	-.0516	-.0619	-.0722	-.0825
	.4	0	.0021	.0043	.0064	.0086	.0107	.0129	.0150	.0171	.0193	.0214	.0107	.0000	-.0107	-.0214	-.0321	-.0429	-.0536	-.0643	-.0750	-.0857
	.5	0	.0020	.0040	.0060	.0080	.0100	.0121	.0141	.0161	.0181	.0201	.0100	.0000	-.0100	-.0201	-.0301	-.0402	-.0502	-.0603	-.0703	-.0804
	.6	0	.0017	.0034	.0051	.0069	.0086	.0103	.0120	.0137	.0154	.0171	.0086	.0000	-.0086	-.0171	-.0257	-.0343	-.0429	-.0514	-.0600	-.0686
	.7	0	.0013	.0026	.0039	.0053	.0066	.0079	.0092	.0105	.0118	.0131	.0066	.0000	-.0066	-.0131	-.0197	-.0263	-.0328	-.0394	-.0459	-.0525
	.8	0	.0009	.0017	.0026	.0034	.0043	.0051	.0060	.0069	.0077	.0086	.0043	.0000	-.0043	-.0086	-.0129	-.0171	-.0214	-.0257	-.0300	-.0343
	.9	0	.0004	.0008	.0012	.0016	.0020	.0024	.0028	.0032	.0036	.0040	.0020	.0000	-.0020	-.0040	-.0060	-.0080	-.0100	-.0121	-.0141	-.0161
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0024	-.0027	-.0031	-.0015	.0000	.0015	.0031	.0046	.0061	.0076	.0092	.0107	.0122
	.2	0	-.0005	-.0010	-.0015	-.0021	-.0026	-.0031	-.0036	-.0041	-.0046	-.0051	-.0026	.0000	.0026	.0051	.0077	.0103	.0129	.0154	.0180	.0206
	.3	0	-.0006	-.0013	-.0019	-.0026	-.0032	-.0038	-.0045	-.0051	-.0057	-.0064	-.0032	.0000	.0032	.0064	.0096	.0128	.0159	.0191	.0223	.0255
	.4	0	-.0007	-.0014	-.0021	-.0027	-.0034	-.0041	-.0048	-.0055	-.0062	-.0069	-.0034	.0000	.0034	.0069	.0103	.0137	.0171	.0206	.0240	.0274
	.5	0	-.0007	-.0013	-.0020	-.0027	-.0033	-.0040	-.0047	-.0054	-.0060	-.0067	-.0033	.0000	.0033	.0067	.0100	.0134	.0167	.0201	.0234	.0268
	.6	0	-.0006	-.0012	-.0018	-.0024	-.0030	-.0036	-.0042	-.0048	-.0054	-.0060	-.0030	.0000	.0030	.0060	.0090	.0120	.0150	.0180	.0210	.0240
	.7	0	-.0005	-.0010	-.0015	-.0020	-.0024	-.0029	-.0034	-.0039	-.0044	-.0049	-.0024	.0000	.0024	.0049	.0073	.0098	.0122	.0146	.0171	.0195
	.8	0	-.0003	-.0007	-.0010	-.0014	-.0017	-.0021	-.0024	-.0027	-.0031	-.0034	-.0017	.0000	.0017	.0034	.0051	.0069	.0086	.0103	.0120	.0137
	.9	0	-.0002	-.0004	-.0005	-.0007	-.0009	-.0011	-.0012	-.0014	-.0016	-.0018	-.0009	.0000	.0009	.0018	.0027	.0035	.0044	.0053	.0062	.0071
	E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0396	.0693	.0889	.0986	.0982	.0878	.0675	.0374	.0174	.0134	.0135	.0300	.0588	.0736	.0804	.0771	.0639	.0417	.0311	.0357
− Area		0	-.0053	-.0107	-.0161	-.0214	-.0268	-.0321	-.0375	-.0431	-.0488	-.1205	-.0721	-.0500	-.0482	-.0464	-.0447	-.0429	-.0411	-.0403	-.0611	-.1071
Total Area		0	.0343	.0586	.0728	.0772	.0714	.0557	.0300	-.0057	-.0314	-.1071	-.0586	-.0200	.0086	.0272	.0357	.0343	.0228	.0014	-.0300	-.0714

Unit load at		R_A	R_B	R_C	R_D	R_E	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
		REACTIONS/P					**SHEARS/P**							
	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
SPAN 1	.1	.8735	.1601	-.0424	.0106	-.0018	.8735	-.1265	.0336	.0336	-.0088	-.0088	.0018	.0018
	.2	.7486	.3166	-.0823	.0206	-.0034	.7486	-.2514	.0651	.0651	-.0171	-.0171	.0034	.0034
	.3	.6269	.4658	-.1170	.0292	-.0049	.6269	-.3731	.0926	.0926	-.0244	-.0244	.0049	.0049
	.4	.5100	.6040	-.1440	.0360	-.0060	.5100	-.4900	.1140	.1140	-.0300	-.0300	.0060	.0060
	.5	.3996	.7277	-.1607	.0402	-.0067	.3996	-.6004	.1272	.1272	-.0335	-.0335	.0067	.0067
	.6	.2971	.8331	-.1646	.0411	-.0069	.2971	-.7029	.1303	.1303	-.0343	-.0343	.0069	.0069
	.7	.2044	.9168	-.1530	.0383	-.0064	.2044	-.7956	.1211	.1211	-.0319	-.0319	.0064	.0064
	.8	.1229	.9749	-.1234	.0309	-.0051	.1229	-.8771	.0977	.0977	-.0257	-.0257	.0051	.0051
	.9	.0542	1.0038	-.0733	.0183	-.0031	.0542	-.9458	.0580	.0580	-.0153	-.0153	.0031	.0031
	B	0	1.0	0	0	0	0	-1.0/0	0/1.0	0	0	0	0	0
SPAN 2	.1	-.0387	.9614	.0974	-.0241	.0040	-.0387	-.0387	.9227	-.0773	.0201	.0201	-.0040	-.0040
	.2	-.0634	.8926	.2137	-.0514	.0086	-.0634	-.0634	.8291	-.1709	.0429	.0429	-.0086	-.0086
	.3	-.0761	.7958	.3420	-.0768	.0131	-.0761	-.0761	.7236	-.2764	.0656	.0656	-.0131	-.0131
	.4	-.0789	.6801	.4754	-.1029	.0171	-.0789	-.0789	.6103	-.3897	.0857	.0857	-.0171	-.0171
	.5	-.0737	.5670	.6071	-.1205	.0201	-.0737	-.0737	.4933	-.5067	.1004	.1004	-.0201	-.0201
	.6	-.0626	.4384	.7303	-.1286	.0214	-.0626	-.0626	.3769	-.6231	.1071	.1071	-.0214	-.0214
	.7	-.0476	.3128	.8380	-.1238	.0206	-.0476	-.0476	.2651	-.7349	.1031	.1031	-.0206	-.0206
	.8	-.0309	.1931	.9234	-.1029	.0171	-.0309	-.0309	.1623	-.8377	.0857	.0857	-.0171	-.0171
	.9	-.0143	.0868	.9797	-.0627	.0104	-.0143	-.0143	.0725	-.9275	.0522	.0522	-.0104	-.0104
	C	0	0	1.0	0	0	0	0	0	-1.0/0	0/1.0	0	0	0
SPAN 3	.1	.0104	-.0627	.9797	.0868	-.0143	.0104	.0104	-.0522	-.0522	.9275	-.0725	.0143	.0143
	.2	.0171	-.1029	.9234	.1931	-.0309	.0171	.0171	-.0857	-.0857	.8377	-.1623	.0309	.0309
	.3	.0206	-.1238	.8380	.3128	-.0476	.0206	.0206	-.1031	-.1031	.7349	-.2651	.0476	.0476
	.4	.0214	-.1286	.7303	.4394	-.0626	.0214	.0214	-.1071	-.1071	.6231	-.3769	.0626	.0626
	.5	.0201	-.1205	.6071	.5670	-.0737	.0201	.0201	-.1004	-.1004	.5067	-.4933	.0737	.0737
	.6	.0171	-.1029	.4754	.6891	-.0789	.0171	.0171	-.0857	-.0857	.3897	-.6103	.0789	.0789
	.7	.0131	-.0788	.3420	.7998	-.0761	.0131	.0131	-.0656	-.0656	.2764	-.7236	.0761	.0761
	.8	.0086	-.0514	.2137	.8926	-.0634	.0086	.0086	-.0429	-.0429	.1709	-.8291	.0634	.0634
	.9	.0040	-.0241	.0974	.9614	-.0387	.0040	.0040	-.0201	-.0201	.0773	-.9227	.0387	.0387
	D	0	0	0	1.0	0	0	0	0	0	0	-1.0/0	0/1.0	0
SPAN 4	.1	-.0031	.0183	-.0733	1.0038	.0542	-.0031	-.0031	.0153	.0153	-.0580	-.0580	.9458	-.0542
	.2	-.0051	.0309	-.1234	.9749	.1229	-.0051	-.0051	.0257	.0257	-.0977	-.0977	.8771	-.1229
	.3	-.0064	.0383	-.1530	.9168	.2044	-.0064	-.0064	.0319	.0319	-.1211	-.1211	.7956	-.2044
	.4	-.0069	.0411	-.1646	.8331	.2971	-.0069	-.0069	.0343	.0343	-.1303	-.1303	.7029	-.2971
	.5	-.0067	.0402	-.1607	.7277	.3996	-.0067	-.0067	.0335	.0335	-.1272	-.1272	.6004	-.3996
	.6	-.0060	.0360	-.1440	.6040	.5100	-.0060	-.0060	.0300	.0300	-.1140	-.1140	.4900	-.5100
	.7	-.0049	.0292	-.1170	.4658	.6269	-.0049	-.0049	.0244	.0244	-.0926	-.0926	.3731	-.6269
	.8	-.0034	.0206	-.0823	.3166	.7486	-.0034	-.0034	.0171	.0171	-.0651	-.0651	.2514	-.7486
	.9	-.0018	.0106	-.0424	.1601	.8735	-.0018	-.0018	.0088	.0088	-.0336	-.0336	.1265	-.8735
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4464	1.2232	1.1429	1.2232	.4464	.4464	.0134	.6027	.1071	.5714	.0670	.6205	.0536
− Area		-.0536	-.0804	-.2143	-.0804	-.0536	-.0536	-.6205	-.0670	-.5714	-.1071	-.6027	-.0134	-.4464
Total Area		.3928	1.1428	.9286	1.1428	.3928	.3928	-.6071	.5357	-.4643	.4643	-.5357	.6071	-.3928

.1 .2 .3 .4 .5 .6 .7 .8 .9 .1 .2 .3 .4 .5 .6 .7 .8 .9 .1 .2 .3 .4 .5 .6 .7 .8 .9 .1 .2 .3 .4 .5 .6 .7 .8 .9

Span 1 B Span 2 C Span 3 D Span 4 E

Influence coefficients — Four continuous spans.

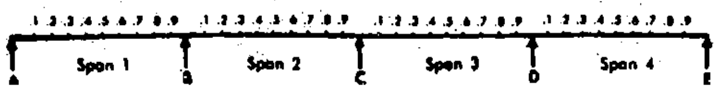
L = Length of EXTERIOR spans; length of interior spans = NL

N = 1.0

Table 20.38

Unit load at		MOMENTS/PL SPAN 1 A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	SPAN 2 .1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0875	.0749	.0624	.0499	.0373	.0248	.0123	-.0003	-.0128	-.0253	-.0221	-.0189	-.0157	-.0125	-.0093	-.0061	-.0029	.0004	.0036	.0068
	.2	0	.0751	.1502	.1253	.1003	.0754	.0505	.0256	.0007	-.0242	-.0492	-.0429	-.0367	-.0305	-.0242	-.0180	-.0118	-.0055	.0007	.0069	.0132
	.3	0	.0630	.1260	.1890	.1520	.1151	.0781	.0411	.0041	-.0329	-.0699	-.0610	-.0522	-.0433	-.0345	-.0256	-.0167	-.0079	.0010	.0098	.0187
	.4	0	.0514	.1028	.1542	.2056	.1570	.1084	.0598	.0112	-.0374	-.0860	-.0751	-.0643	-.0533	-.0424	-.0315	-.0206	-.0097	.0012	.0121	.0230
	.5	0	.0404	.0808	.1212	.1616	.2020	.1424	.0828	.0232	-.0364	-.0960	-.0838	-.0717	-.0595	-.0473	-.0352	-.0230	-.0108	.0013	.0135	.0257
	.6	0	.0302	.0603	.0905	.1207	.1508	.1810	.1112	.0413	-.0285	-.0983	-.0859	-.0734	-.0609	-.0485	-.0360	-.0235	-.0111	.0014	.0138	.0263
	.7	0	.0209	.0417	.0626	.0834	.1043	.1252	.1460	.0669	-.0123	-.0914	-.0798	-.0682	-.0566	-.0451	-.0335	-.0219	-.0103	.0013	.0129	.0245
	.8	0	.0126	.0253	.0379	.0506	.0631	.0758	.0884	.1010	.0136	-.0737	-.0644	-.0550	-.0457	-.0364	-.0270	-.0177	.0083	-.0010	.0104	.0197
	.9	0	.0056	.0112	.0169	.0225	.0281	.0337	.0394	.0450	.0506	-.0438	-.0382	-.0327	-.0271	-.0216	-.0160	-.0105	-.0049	.0006	.0062	.0117
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0045	-.0090	-.0134	-.0179	-.0224	-.0269	-.0313	-.0358	-.0403	-.0448	.0370	.0488	.0405	.0323	.0240	.0158	.0076	-.0007	-.0089	-.0172
	.2	0	-.0073	-.0147	-.0220	-.0293	-.0367	-.0440	-.0513	-.0586	-.0660	-.0733	.0183	.1100	.0916	.0733	.0549	.0365	.0182	-.0002	-.0185	-.0369
	.3	0	-.0088	-.0176	-.0264	-.0352	-.0440	-.0528	-.0616	-.0704	-.0792	-.0880	-.0079	.0723	.1524	.1225	.0926	.0627	.0328	.0030	-.0269	-.0568
	.4	0	-.0091	-.0182	-.0273	-.0364	-.0456	-.0547	-.0638	-.0729	-.0820	-.0911	-.0235	.0442	.1119	.1795	.1372	.0949	.0525	.0102	-.0322	-.0745
	.5	0	-.0085	-.0170	-.0255	-.0340	-.0425	-.0511	-.0596	-.0681	-.0766	-.0851	-.0303	.0244	.0792	.1339	.1887	.1334	.0782	.0229	-.0323	-.0876
	.6	0	-.0072	-.0145	-.0217	-.0289	-.0361	-.0434	-.0506	-.0578	-.0650	-.0723	-.0304	.0115	.0533	.0952	.1370	.1789	.1108	.0426	-.0255	-.0937
	.7	0	-.0055	-.0110	-.0165	-.0220	-.0275	-.0330	-.0385	-.0440	-.0495	-.0550	-.0255	.0334	.0334	.0629	.0923	.1218	.1513	.0707	-.0098	-.0903
	.8	0	-.0036	-.0071	-.0107	-.0142	-.0178	-.0214	-.0249	-.0285	-.0321	-.0356	-.0176	.0005	.0185	.0365	.0546	.0726	.0907	.1087	.0167	-.0752
	.9	0	-.0016	-.0033	-.0049	-.0066	-.0082	-.0099	-.0115	-.0132	-.0148	-.0165	-.0084	-.0004	.0077	.0157	.0238	.0319	.0399	.0480	.0560	-.0459
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0012	.0024	.0036	.0048	.0060	.0072	.0084	.0096	.0108	.0120	.0062	.0004	-.0054	-.0111	-.0169	-.0227	-.0285	-.0343	-.0401	-.0459
	.2	0	.0020	.0039	.0059	.0079	.0099	.0118	.0138	.0158	.0177	.0197	.0102	.0007	-.0088	-.0183	-.0278	-.0373	-.0467	-.0562	-.0657	-.0752
	.3	0	.0024	.0047	.0071	.0095	.0118	.0142	.0166	.0189	.0213	.0237	.0123	.0009	-.0105	-.0219	-.0333	-.0447	-.0561	-.0675	-.0789	-.0903
	.4	0	.0025	.0049	.0074	.0098	.0123	.0147	.0172	.0196	.0221	.0245	.0127	.0009	-.0109	-.0227	-.0346	-.0464	-.0582	-.0700	-.0818	-.0937
	.5	0	.0023	.0046	.0069	.0092	.0115	.0138	.0161	.0184	.0206	.0229	.0119	.0008	-.0102	-.0213	-.0323	-.0434	-.0544	-.0655	-.0765	-.0876
	.6	0	.0020	.0039	.0058	.0078	.0098	.0117	.0137	.0156	.0176	.0195	.0101	.0007	-.0087	-.0181	-.0275	-.0369	-.0463	-.0557	-.0651	-.0745
	.7	0	.0015	.0030	.0045	.0060	.0074	.0089	.0104	.0119	.0134	.0149	.0077	.0005	-.0066	-.0138	-.0210	-.0281	-.0353	-.0425	-.0496	-.0568
	.8	0	.0010	.0019	.0029	.0039	.0048	.0058	.0068	.0077	.0087	.0097	.0050	.0004	-.0043	-.0090	-.0136	-.0183	-.0229	-.0276	-.0322	-.0369
	.9	0	.0004	.0009	.0013	.0018	.0022	.0027	.0031	.0036	.0040	.0045	.0023	.0002	-.0020	-.0042	-.0063	-.0085	-.0107	-.0128	-.0150	-.0172
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0025	-.0028	-.0031	-.0016	-.0001	.0014	.0028	.0043	.0058	.0073	.0088	.0102	.0117
	.2	0	-.0005	-.0010	-.0015	-.0021	-.0026	-.0031	-.0036	-.0041	-.0046	-.0052	-.0027	-.0002	.0023	.0048	.0073	.0098	.0123	.0147	.0172	.0197
	.3	0	-.0006	-.0013	-.0019	-.0026	-.0032	-.0038	-.0045	-.0051	-.0058	-.0064	-.0033	-.0002	.0029	.0059	.0090	.0121	.0152	.0183	.0214	.0245
	.4	0	-.0007	-.0014	-.0021	-.0028	-.0034	-.0041	-.0048	-.0055	-.0062	-.0069	-.0036	-.0002	.0031	.0064	.0097	.0130	.0163	.0197	.0230	.0263
	.5	0	-.0007	-.0013	-.0020	-.0027	-.0034	-.0040	-.0047	-.0054	-.0061	-.0067	-.0035	-.0002	.0030	.0062	.0095	.0127	.0160	.0192	.0224	.0257
	.6	0	-.0006	-.0012	-.0018	-.0024	-.0030	-.0036	-.0042	-.0048	-.0054	-.0060	-.0031	-.0002	.0027	.0056	.0085	.0114	.0143	.0172	.0201	.0230
	.7	0	-.0005	-.0010	-.0015	-.0020	-.0024	-.0029	-.0034	-.0039	-.0044	-.0049	-.0025	-.0002	.0022	.0045	.0069	.0093	.0116	.0140	.0163	.0187
	.8	0	-.0003	-.0007	-.0010	-.0014	-.0017	-.0021	-.0024	-.0028	-.0031	-.0034	-.0018	-.0001	.0015	.0032	.0049	.0065	.0082	.0098	.0115	.0132
	.9	0	-.0002	-.0004	-.0005	-.0007	-.0009	-.0011	-.0012	-.0014	-.0016	-.0018	-.0009	-.0001	.0008	.0016	.0025	.0034	.0042	.0051	.0059	.0068
	E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0403	.0706	.0908	.1011	.1014	.0917	.0720	.0423	.0210	.0168	.0164	.0347	.0661	.0882	.0942	.0902	.0740	.0467	.0313	.0343
− Area		0	-.0067	-.0134	-.0200	-.0267	-.0334	-.0401	-.0468	-.0535	-.0787	-.1309	-.0740	-.0480	-.0472	-.0471	-.0471	-.0471	-.0471	-.0481	-.0730	-.1285
Total Area		0	.0336	.0572	.0708	.0744	.0680	.0516	.0252	-.0112	-.0577	-.1141	-.0576	-.0133	.0189	.0391	.0471	.0431	.0269	-.0014	-.0417	-.0942

Unit load at		REACTIONS/P R_A	R_B	R_C	R_D	R_E	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
SPAN 1	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
	.1	.8747	.1546	-.0370	.0096	-.0018	.8747	-.1253	.0292	.0292	-.0078	-.0078	.0018	.0018
	.2	.7508	.3059	-.0717	.0186	-.0034	.7508	-.2492	.0566	.0566	-.0151	-.0151	.0034	.0034
	.3	.6301	.4504	-.1020	.0264	-.0049	.6301	-.3699	.0805	.0805	-.0216	-.0216	.0049	.0049
	.4	.5140	.5852	-.1255	.0324	-.0060	.5140	-.4860	.0991	.0991	-.0264	-.0264	.0060	.0060
	.5	.4040	.7066	-.1401	.0362	-.0067	.4040	-.5960	.1106	.1106	-.0296	-.0296	.0067	.0067
	.6	.3017	.8116	-.1435	.0371	-.0069	.3017	-.6983	.1133	.1133	-.0302	-.0302	.0069	.0069
	.7	.2086	.8947	-.1334	.0345	-.0064	.2086	-.7914	.1053	.1053	-.0281	-.0281	.0064	.0064
	.8	.1263	.9587	-.1076	.0278	-.0052	.1263	-.8737	.0850	.0850	-.0226	-.0226	.0052	.0052
	.9	.0562	.9942	-.0639	.0165	-.0031	.0562	-.9438	.0505	.0505	-.0134	-.0134	.0031	.0031
	B	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0	0	0
SPAN 2	.1	-.0448	.9099	.0946	-.0242	.0045	-.0448	-.0448	.9251	-.0749	.0197	.0197	-.0048	-.0048
	.2	-.0733	.9064	.2092	-.0320	.0097	-.0733	-.0733	.8331	-.1669	.0423	.0423	-.0097	-.0097
	.3	-.0880	.8163	.3368	-.0800	.0149	-.0880	-.0880	.7283	-.2717	.0652	.0652	-.0149	-.0149
	.4	-.0911	.7082	.4704	-.1050	.0195	-.0911	-.0911	.6151	-.3849	.0855	.0855	-.0195	-.0195
	.5	-.0851	.5828	.6027	-.1234	.0229	-.0851	-.0851	.4977	-.5023	.1005	.1005	-.0229	-.0229
	.6	-.0723	.4528	.7268	-.1320	.0245	-.0723	-.0723	.3806	-.6195	.1075	.1075	-.0245	-.0245
	.7	-.0550	.3228	.8358	-.1273	.0237	-.0550	-.0550	.2679	-.7321	.1036	.1036	-.0237	-.0237
	.8	-.0356	.1996	.9223	-.1060	.0197	-.0356	-.0356	.1640	-.8360	.0863	.0863	-.0197	-.0197
	.9	-.0165	.0898	.9794	-.0647	.0120	-.0165	-.0165	.0733	-.9267	.0527	.0527	-.0120	-.0120
	C	0	0	1.0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 3	.1	.0120	-.0647	.9794	.0898	-.0165	.0120	.0120	-.0527	-.0527	.9267	-.0733	.0165	.0165
	.2	.0197	-.1060	.9223	.1996	-.0356	.0197	.0197	-.0863	-.0863	.8360	-.1640	.0356	.0356
	.3	.0237	-.1273	.8358	.3228	-.0550	.0237	.0237	-.1036	-.1036	.7321	-.2679	.0550	.0550
	.4	.0245	-.1320	.7269	.4528	-.0723	.0245	.0245	-.1075	-.1075	.6195	-.3806	.0723	.0723
	.5	.0229	-.1234	.6027	.5828	-.0851	.0229	.0229	-.1005	-.1005	.5023	-.4977	.0851	.0851
	.6	.0195	-.1050	.4704	.7062	-.0911	.0195	.0195	-.0855	-.0855	.3849	-.6151	.0911	.0911
	.7	.0149	-.0800	.3368	.8163	-.0880	.0149	.0149	-.0652	-.0652	.2717	-.7283	.0880	.0880
	.8	.0097	-.0520	.2092	.9064	-.0733	.0097	.0097	-.0423	-.0423	.1669	-.8331	.0733	.0733
	.9	.0045	-.0242	.0946	.9699	-.0448	.0045	.0045	-.0197	-.0197	.0749	-.9251	.0448	.0448
	D	0	0	0	1.0	0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 4	.1	-.0031	.0165	-.0639	.9942	.0562	-.0031	-.0031	.0134	.0134	-.0505	-.0505	.9438	-.0562
	.2	-.0052	.0278	-.1076	.9587	.1263	-.0052	-.0052	.0226	.0226	-.0850	-.0850	.8737	-.1263
	.3	-.0064	.0345	-.1334	.8967	.2086	-.0064	-.0064	.0281	.0281	-.1053	-.1053	.7914	-.2086
	.4	-.0069	.0371	-.1435	.8116	.3017	-.0069	-.0069	.0302	.0302	-.1133	-.1133	.6983	-.3017
	.5	-.0067	.0362	-.1401	.7066	.4040	-.0067	-.0067	.0295	.0295	-.1106	-.1106	.5960	-.4040
	.6	-.0060	.0324	-.1255	.5852	.5140	-.0060	-.0060	.0264	.0264	-.0991	-.0991	.4860	-.5140
	.7	-.0049	.0264	-.1020	.4504	.6301	-.0049	-.0049	.0215	.0215	-.0805	-.0805	.3699	-.6301
	.8	-.0034	.0185	-.0717	.3058	.7508	-.0034	-.0034	.0151	.0151	-.0566	-.0566	.2492	-.7508
	.9	-.0018	.0096	-.0370	.1546	.8747	-.0018	-.0018	.0078	.0078	-.0292	-.0292	.1253	-.8747
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4528	1.2726	1.2507	1.2726	.4528	.4528	.0168	.6417	.0934	.6253	.0737	.6309	.0669
− Area		-.0669	-.0905	-.1868	-.0905	-.0669	-.0669	-.6309	-.0737	-.6253	-.0934	-.6417	-.0168	-.4528
Total Area		.3859	1.1821	1.0639	1.1821	.3859	.3859	-.6141	.5680	-.5319	-.5319	-.5680	.6141	-.3859



Influence coefficients — Four continuous spans.

L = Length of EXTERIOR spans; length of interior spans = NL

Table 20.39

MOMENTS/PL — SPAN 1

Unit load at		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B
	A	0	0	0	0	0	0	0	0	0	0	0
SPAN 1	.1	0	.0876	.0751	.0627	.0503	.0379	.0254	.0130	.0006	-.0119	-.0243
	.2	0	.0753	.1506	.1259	.1012	.0765	.0518	.0270	.0023	-.0224	-.0471
	.3	0	.0633	.1266	.1899	.1532	.1165	.0798	.0431	.0064	-.0303	-.0669
	.4	0	.0518	.1035	.1553	.2070	.1588	.1106	.0623	.0141	-.0342	-.0824
	.5	0	.0408	.0816	.1224	.1632	.2040	.1448	.0856	.0264	-.0328	-.0920
	.6	0	.0306	.0612	.0918	.1223	.1529	.1835	.1141	.0447	-.0247	-.0942
	.7	0	.0212	.0425	.0637	.0850	.1062	.1275	.1487	.0700	-.0088	-.0875
	.8	0	.0129	.0258	.0388	.0518	.0647	.0776	.0906	.1035	.0164	-.0706
	.9	0	.0058	.0116	.0174	.0232	.0290	.0348	.0406	.0465	.0523	-.0419
	B	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0051	-.0102	-.0153	-.0204	-.0255	-.0306	-.0357	-.0408	-.0459	-.0510
	.2	0	-.0084	-.0167	-.0251	-.0334	-.0418	-.0501	-.0585	-.0668	-.0752	-.0835
	.3	0	-.0100	-.0200	-.0301	-.0401	-.0501	-.0601	-.0701	-.0802	-.0902	-.1002
	.4	0	-.0104	-.0208	-.0311	-.0415	-.0519	-.0623	-.0726	-.0830	-.0934	-.1038
	.5	0	-.0097	-.0194	-.0291	-.0388	-.0484	-.0581	-.0678	-.0775	-.0872	-.0969
	.6	0	-.0082	-.0165	-.0247	-.0329	-.0411	-.0494	-.0576	-.0658	-.0740	-.0823
	.7	0	-.0083	-.0125	-.0188	-.0250	-.0313	-.0375	-.0438	-.0501	-.0563	-.0626
	.8	0	-.0041	-.0081	-.0122	-.0162	-.0203	-.0243	-.0284	-.0324	-.0365	-.0405
	.9	0	-.0019	-.0038	-.0056	-.0075	-.0094	-.0113	-.0131	-.0150	-.0169	-.0188
	C	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0014	.0027	.0041	.0055	.0068	.0082	.0095	.0109	.0123	.0136
	.2	0	.0022	.0045	.0067	.0089	.0112	.0134	.0156	.0178	.0201	.0223
	.3	0	.0027	.0054	.0080	.0107	.0134	.0161	.0187	.0214	.0241	.0268
	.4	0	.0028	.0055	.0083	.0111	.0138	.0166	.0194	.0222	.0249	.0277
	.5	0	.0026	.0052	.0078	.0103	.0129	.0155	.0181	.0207	.0233	.0258
	.6	0	.0022	.0044	.0066	.0088	.0110	.0131	.0153	.0175	.0197	.0219
	.7	0	.0017	.0033	.0050	.0067	.0083	.0100	.0116	.0133	.0150	.0166
	.8	0	.0011	.0021	.0032	.0043	.0054	.0064	.0075	.0086	.0097	.0107
	.9	0	.0005	.0010	.0015	.0020	.0025	.0030	.0035	.0040	.0045	.0050
	D	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0025	-.0028	-.0031
	.2	0	-.0005	-.0010	-.0016	-.0021	-.0026	-.0031	-.0036	-.0041	-.0047	-.0052
	.3	0	-.0006	-.0013	-.0019	-.0026	-.0032	-.0038	-.0045	-.0051	-.0058	-.0064
	.4	0	-.0007	-.0014	-.0021	-.0028	-.0034	-.0041	-.0048	-.0055	-.0062	-.0069
	.5	0	-.0007	-.0013	-.0020	-.0027	-.0034	-.0040	-.0047	-.0054	-.0061	-.0067
	.6	0	-.0006	-.0012	-.0018	-.0024	-.0030	-.0036	-.0042	-.0048	-.0054	-.0060
	.7	0	-.0005	-.0010	-.0015	-.0020	-.0024	-.0029	-.0034	-.0039	-.0044	-.0049
	.8	0	-.0003	-.0007	-.0010	-.0014	-.0017	-.0021	-.0024	-.0028	-.0031	-.0034
	.9	0	-.0002	-.0004	-.0005	-.0007	-.0009	-.0011	-.0012	-.0014	-.0016	-.0018
	E	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0409	.0719	.0828	.1037	.1047	.0956	.0766	.0475	.0249	.0207
− Area		0	-.0082	-.0164	-.0246	-.0328	-.0410	-.0492	-.0574	-.0656	-.0903	-.1433
Total Area		0	.0327	.0555	.0582	.0709	.0637	.0464	.0192	-.0181	-.0654	-.1226

MOMENTS/PL — SPAN 2

Unit load at		.1	.2	.3	.4	.5	.6	.7	.8	.9	C
	A	0	0	0	0	0	0	0	0	0	0
SPAN 1	.1	-.0212	-.0181	-.0150	-.0120	-.0089	-.0058	-.0027	.0004	.0034	.0065
	.2	-.0411	-.0351	-.0292	-.0232	-.0172	-.0113	-.0053	.0007	.0067	.0126
	.3	-.0585	-.0500	-.0415	-.0330	-.0245	-.0160	-.0075	.0010	.0095	.0180
	.4	-.0719	-.0615	-.0510	-.0406	-.0301	-.0197	-.0092	.0012	.0117	.0221
	.5	-.0803	-.0686	-.0570	-.0453	-.0336	-.0220	-.0103	.0013	.0130	.0247
	.6	-.0822	-.0703	-.0583	-.0464	-.0345	-.0225	-.0106	.0014	.0133	.0253
	.7	-.0764	-.0653	-.0542	-.0431	-.0320	-.0209	-.0098	.0013	.0124	.0235
	.8	-.0617	-.0527	-.0438	-.0348	-.0258	-.0169	-.0079	.0010	.0100	.0189
	.9	-.0366	-.0313	-.0260	-.0207	-.0153	-.0100	-.0047	.0006	.0059	.0112
	B	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	.0603	.0516	.0428	.0341	.0254	.0167	.0080	-.0008	-.0095	-.0182
	.2	.0169	.1173	.0977	.0781	.0585	.0390	.0194	-.0002	-.0198	-.0394
	.3	-.0123	.0756	.1636	.1315	.0994	.0673	.0352	.0032	-.0289	-.0610
	.4	-.0294	.0449	.1193	.1936	.1480	.1023	.0566	.0110	-.0347	-.0803
	.5	-.0367	.0235	.0838	.1440	.2042	.1444	.0846	.0248	-.0350	-.0947
	.6	-.0362	.0099	.0559	.1020	.1481	.1942	.1202	.0463	-.0276	-.1016
	.7	-.0301	.0023	.0348	.0672	.0996	.1321	.1645	.0770	-.0106	-.0981
	.8	-.0206	-.0008	.0191	.0390	.0588	.0787	.0985	.1194	.0183	-.0819
	.9	-.0099	-.0010	.0079	.0167	.0256	.0345	.0434	.0522	.0611	-.0500
	C	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	.0073	.0009	-.0055	-.0118	-.0182	-.0246	-.0309	-.0373	-.0437	-.0500
	.2	.0119	.0015	-.0089	-.0193	-.0298	-.0402	-.0506	-.0610	-.0714	-.0819
	.3	.0143	.0018	-.0107	-.0232	-.0357	-.0482	-.0607	-.0732	-.0857	-.0981
	.4	.0148	.0018	-.0111	-.0240	-.0369	-.0499	-.0628	-.0757	-.0886	-.1016
	.5	.0138	.0017	-.0103	-.0224	-.0345	-.0465	-.0586	-.0706	-.0827	-.0947
	.6	.0117	.0015	-.0088	-.0190	-.0292	-.0394	-.0497	-.0599	-.0701	-.0803
	.7	.0089	.0011	-.0067	-.0144	-.0222	-.0300	-.0377	-.0455	-.0532	-.0610
	.8	.0057	.0007	-.0043	-.0093	-.0143	-.0193	-.0244	-.0294	-.0344	-.0394
	.9	.0028	.0003	-.0020	-.0043	-.0066	-.0089	-.0112	-.0136	-.0159	-.0182
	D	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	-.0016	-.0002	.0012	.0027	.0041	.0055	.0070	.0084	.0098	.0112
	.2	-.0028	-.0003	.0021	.0045	.0069	.0093	.0117	.0141	.0165	.0189
	.3	-.0034	-.0004	.0026	.0056	.0085	.0115	.0145	.0175	.0205	.0235
	.4	-.0037	-.0005	.0028	.0060	.0092	.0124	.0156	.0188	.0220	.0253
	.5	-.0036	-.0004	.0027	.0058	.0090	.0121	.0153	.0184	.0215	.0247
	.6	-.0032	-.0004	.0024	.0052	.0080	.0109	.0137	.0165	.0193	.0221
	.7	-.0026	-.0003	.0020	.0042	.0065	.0088	.0111	.0134	.0157	.0180
	.8	-.0018	-.0002	.0014	.0030	.0046	.0062	.0078	.0094	.0110	.0126
	.9	-.0009	-.0001	.0007	.0015	.0024	.0032	.0040	.0049	.0057	.0065
	E	0	0	0	0	0	0	0	0	0	0
+ Area		.0195	.0396	.0760	.0999	.1093	.1044	.0850	.0523	.0317	.0329
− Area		-.0769	-.0463	-.0462	-.0481	-.0500	-.0519	-.0537	-.0566	-.0860	-.1516
Total Area		-.0574	-.0067	.0298	.0518	.0593	.0525	.0313	-.0043	-.0543	-.1187

REACTIONS/P and SHEARS/P

Unit load at		R_A	R_B	R_C	R_D	R_E	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
SPAN 1	.1	.8757	.1499	-.0326	.0087	-.0018	.8757	-.1243	.0257	.0257	-.0069	-.0069	.0018	.0018
	.2	.7529	.2968	-.0632	.0169	-.0034	.7529	-.2471	.0498	.0498	-.0134	-.0134	.0034	.0034
	.3	.6331	.4377	-.0898	.0239	-.0049	.6331	-.3669	.0708	.0708	-.0190	-.0190	.0049	.0049
	.4	.5176	.5695	-.1105	.0295	-.0060	.5176	-.4824	.0871	.0871	-.0234	-.0234	.0060	.0060
	.5	.4080	.6891	-.1234	.0329	-.0067	.4080	-.5920	.0972	.0972	-.0262	-.0262	.0067	.0067
	.6	.3058	.7937	-.1263	.0337	-.0069	.3058	-.6942	.0995	.0995	-.0268	-.0268	.0069	.0069
	.7	.2125	.8801	-.1174	.0313	-.0064	.2125	-.7875	.0925	.0925	-.0249	-.0249	.0064	.0064
	.8	.1294	.9453	-.0947	.0253	-.0052	.1294	-.8706	.0746	.0746	-.0201	-.0201	.0052	.0052
	.9	.0581	.9843	-.0543	.0150	-.0031	.0581	-.9419	.0443	.0443	-.0119	-.0119	.0031	.0031
	B	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0	0	0
SPAN 2	.1	-.0510	.9783	.0919	-.0242	.0050	-.0510	-.0510	.9273	-.0727	.0193	.0193	-.0050	-.0050
	.2	-.0835	.9202	.2051	-.0525	.0107	-.0835	-.0835	.8367	-.1633	.0418	.0418	-.0107	-.0107
	.3	-.1002	.8329	.3320	-.0813	.0166	-.1002	-.1002	.7327	-.2673	.0647	.0647	-.0166	-.0166
	.4	-.1038	.7233	.4657	-.1071	.0219	-.1038	-.1038	.6195	-.3805	.0852	.0852	-.0219	-.0219
	.5	-.0969	.5987	.5987	-.1263	.0258	-.0969	-.0969	.5018	-.4982	.1005	.1005	-.0258	-.0258
	.6	-.0823	.4662	.7238	-.1354	.0277	-.0823	-.0823	.3839	-.6161	.1077	.1077	-.0277	-.0277
	.7	-.0626	.3329	.8337	-.1309	.0268	-.0626	-.0626	.2704	-.7296	.1041	.1041	-.0268	-.0268
	.8	-.0405	.2061	.9213	-.1091	.0223	-.0405	-.0405	.1656	-.8344	.0868	.0868	-.0223	-.0223
	.9	-.0188	.0927	.9781	-.0667	.0136	-.0188	-.0188	.0739	-.9261	.0531	.0531	-.0136	-.0136
	C	0	0	1.0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 3	.1	.0136	-.0667	.9781	.0927	-.0188	.0136	.0136	-.0531	-.0531	.9261	-.0739	.0188	.0188
	.2	.0223	-.1091	.9213	.2061	-.0405	.0223	.0223	-.0868	-.0868	.8344	-.1656	.0405	.0405
	.3	.0268	-.1309	.8337	.3329	-.0626	.0268	.0268	-.1041	-.1041	.7296	-.2704	.0626	.0626
	.4	.0277	-.1354	.7238	.4662	-.0823	.0277	.0277	-.1077	-.1077	.6161	-.3839	.0823	.0823
	.5	.0258	-.1263	.5987	.5987	-.0969	.0258	.0258	-.1005	-.1005	.4982	-.5018	.0969	.0969
	.6	.0219	-.1071	.4657	.7233	-.1038	.0219	.0219	-.0852	-.0852	.3805	-.6195	.1038	.1038
	.7	.0166	-.0813	.3320	.8329	-.1002	.0166	.0166	-.0647	-.0647	.2673	-.7327	.1002	.1002
	.8	.0107	-.0525	.2051	.9202	-.0835	.0107	.0107	-.0418	-.0418	.1633	-.8367	.0835	.0835
	.9	.0050	-.0242	.0919	.9783	-.0510	.0050	.0050	-.0193	-.0193	.0727	-.9273	.0510	.0510
	D	0	0	0	1.0	0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 4	.1	-.0031	.0150	-.0563	.9863	.0581	-.0031	-.0031	.0119	.0119	-.0443	-.0443	.9419	-.0581
	.2	-.0052	.0253	-.0947	.9453	.1294	-.0052	-.0052	.0201	.0201	-.0746	-.0746	.8706	-.1294
	.3	-.0064	.0313	-.1174	.8801	.2125	-.0064	-.0064	.0249	.0249	-.0925	-.0925	.7875	-.2125
	.4	-.0069	.0337	-.1263	.7937	.3058	-.0069	-.0069	.0268	.0268	-.0995	-.0995	.6942	-.3058
	.5	-.0067	.0329	-.1234	.6891	.4080	-.0067	-.0067	.0262	.0262	-.0972	-.0972	.5920	-.4080
	.6	-.0060	.0295	-.1105	.5695	.5176	-.0060	-.0060	.0234	.0234	-.0871	-.0871	.4824	-.5176
	.7	-.0049	.0239	-.0898	.4377	.6331	-.0049	-.0049	.0190	.0190	-.0708	-.0708	.3669	-.6331
	.8	-.0034	.0169	-.0632	.2968	.7529	-.0034	-.0034	.0134	.0134	-.0498	-.0498	.2471	-.7529
	.9	-.0018	.0087	-.0326	.1499	.8757	-.0018	-.0018	.0069	.0069	-.0257	-.0257	.1243	-.8757
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4594	[illegible]	1.3579	1.3270	.4594	.4594	.0207	.6837	.0822	.6789	.0804	.6433	.0820
− Area		-.0820	-.1011	-.1645	-.1011	-.0820	-.0820	-.6433	-.0804	-.6789	-.0822	-.6837	-.0207	-.4594
Total Area		.3774	1.2259	1.1934	1.2259	.3774	.3774	-.6226	.6033	-.5967	.5967	-.6033	.6226	-.3774

.1 .2 .3 .4 .5 .6 .7 .8 .9   .1 .2 .3 .4 .5 .6 .7 .8 .9   .1 .2 .3 .4 .5 .6 .7 .8 .9   .1 .2 .3 .4 .5 .6 .7 .8 .9

A — Span 1 — B — Span 2 — C — Span 3 — D — Span 4 — E

Influence coefficients — Four continuous spans.

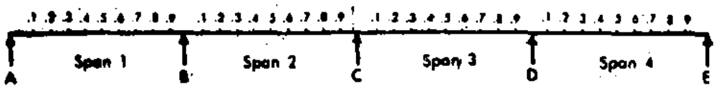
L = Length of EXTERIOR spans; length of interior spans = NL

N = 1.2

Table 20.40

Unit load at		MOMENTS/PL Span 1 A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	Span 2 .1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0877	.0753	.0630	.0507	.0384	.0260	.0137	.0014	-.0110	-.0233	-.0203	-.0174	-.0144	-.0115	-.0085	-.0056	-.0026	.0004	.0033	.0063
	.2	0	.0755	.1510	.1264	.1019	.0774	.0529	.0284	.0039	-.0207	-.0452	-.0394	-.0337	-.0280	-.0222	-.0165	-.0108	-.0050	.0007	.0064	.0122
	.3	0	.0636	.1272	.1907	.1543	.1178	.0815	.0450	.0086	-.0278	-.0642	-.0561	-.0479	-.0398	-.0316	-.0235	-.0153	-.0072	.0010	.0091	.0173
	.4	0	.0521	.1042	.1563	.2084	.1603	.1126	.0647	.0168	-.0311	-.0791	-.0690	-.0590	-.0490	-.0388	-.0289	-.0189	-.0088	.0012	.0112	.0213
	.5	0	.0412	.0824	.1235	.1647	.2059	.1471	.0882	.0294	-.0294	-.0882	-.0770	-.0658	-.0546	-.0434	-.0322	-.0211	-.0099	.0013	.0125	.0237
	.6	0	.0310	.0619	.0929	.1239	.1548	.1858	.1168	.0477	-.0213	-.0903	-.0788	-.0674	-.0560	-.0445	-.0330	-.0216	-.0101	.0014	.0128	.0243
	.7	0	.0218	.0432	.0648	.0864	.1080	.1296	.1512	.0728	-.0056	-.0840	-.0733	-.0627	-.0520	-.0414	-.0307	-.0200	-.0094	.0013	.0119	.0226
	.8	0	.0132	.0264	.0397	.0529	.0661	.0793	.0926	.1058	.0190	-.0678	-.0592	-.0506	-.0420	-.0334	-.0248	-.0162	-.0076	.0010	.0096	.0182
	.9	0	.0060	.0120	.0179	.0239	.0298	.0359	.0418	.0478	.0538	-.0402	-.0351	-.0300	-.0248	-.0198	-.0147	-.0096	-.0045	.0008	.0057	.0108
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0057	-.0115	-.0172	-.0230	-.0287	-.0344	-.0402	-.0459	-.0517	-.0574	.0834	.0542	.0451	.0359	.0267	.0175	.0084	-.0008	-.0100	-.0192
	.2	0	-.0094	-.0188	-.0282	-.0376	-.0470	-.0564	-.0658	-.0752	-.0846	-.0940	.0152	.1244	.1037	.0829	.0621	.0413	.0205	-.0003	-.0211	-.0419
	.3	0	-.0113	-.0225	-.0338	-.0451	-.0564	-.0676	-.0789	-.0902	-.1015	-.1127	-.0170	.0788	.1745	.1403	.1061	.0718	.0376	.0033	-.0309	-.0651
	.4	0	-.0117	-.0233	-.0350	-.0467	-.0584	-.0700	-.0817	-.0934	-.1051	-.1167	-.0357	.0454	.1264	.2075	.1586	.1096	.0607	.0118	-.0372	-.0861
	.5	0	-.0109	-.0218	-.0327	-.0436	-.0545	-.0654	-.0763	-.0872	-.0981	-.1090	-.0433	.0224	.0882	.1539	.2196	.1553	.0910	.0267	-.0375	-.1018
	.6	0	-.0093	-.0185	-.0278	-.0370	-.0463	-.0555	-.0648	-.0740	-.0833	-.0925	-.0422	.0081	.0584	.1087	.1590	.2094	.1297	.0500	-.0297	-.1094
	.7	0	-.0070	-.0141	-.0211	-.0281	-.0352	-.0422	-.0493	-.0563	-.0633	-.0704	-.0349	.0005	.0360	.0714	.1069	.1423	.1777	.0832	-.0114	-.1059
	.8	0	-.0046	-.0091	-.0137	-.0182	-.0228	-.0273	-.0319	-.0364	-.0410	-.0455	-.0238	-.0021	.0196	.0413	.0630	.0847	.1064	.1281	.0198	-.0685
	.9	0	-.0021	-.0042	-.0063	-.0084	-.0105	-.0126	-.0148	-.0169	-.0190	-.0211	-.0114	-.0017	.0080	.0177	.0274	.0371	.0468	.0565	.0662	-.0541
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0015	.0031	.0046	.0061	.0076	.0092	.0107	.0122	.0138	.0153	.0084	.0014	-.0055	-.0125	-.0194	-.0264	-.0333	-.0402	-.0472	-.0541
	.2	0	.0025	.0050	.0075	.0100	.0125	.0150	.0175	.0200	.0225	.0250	.0137	.0023	-.0090	-.0204	-.0317	-.0431	-.0544	-.0658	-.0771	-.0885
	.3	0	.0030	.0060	.0090	.0120	.0150	.0180	.0210	.0239	.0269	.0299	.0163	.0228	-.0108	-.0244	-.0380	-.0516	.0652	-.0787	-.0923	-.1059
	.4	0	.0031	.0062	.0093	.0124	.0155	.0186	.0216	.0247	.0278	.0309	.0169	.0028	-.0112	-.0252	-.0392	-.0533	-.0673	-.0813	-.0954	-.1094
	.5	0	.0029	.0058	.0086	.0115	.0144	.0173	.0201	.0230	.0259	.0288	.0157	.0027	-.0104	-.0235	-.0365	-.0496	-.0626	-.0757	-.0888	-.1018
	.6	0	.0024	.0049	.0073	.0097	.0122	.0146	.0170	.0195	.0219	.0243	.0133	.0022	-.0088	-.0198	-.0309	-.0419	-.0530	-.0640	-.0751	-.0861
	.7	0	.0018	.0037	.0055	.0074	.0092	.0110	.0129	.0147	.0166	.0184	.0101	.0017	-.0067	-.0150	-.0234	-.0317	-.0401	-.0484	-.0568	-.0651
	.8	0	.0012	.0024	.0035	.0047	.0059	.0071	.0083	.0095	.0106	.0118	.0065	.0011	-.0043	-.0096	-.0150	-.0204	-.3258	-.0311	-.0365	-.0419
	.9	0	.0005	.0011	.0016	.0022	.0027	.0033	.0038	.0043	.0049	.0054	.0030	.0005	-.0020	-.0044	-.0069	-.0093	-.0118	-.0143	-.0167	-.0192
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0024	-.0028	-.0031	-.0017	-.0003	.0011	.0025	.0039	.0053	.0067	.0080	.0094	.0108
	.2	0	-.0005	-.0010	-.0015	-.0021	-.0026	-.0031	-.0036	-.0041	-.0046	-.0052	-.0028	-.0005	.0019	.0042	.0065	.0089	.0112	.0136	.0159	.0182
	.3	0	-.0008	-.0013	-.0019	-.0026	-.0032	-.0038	-.0045	-.0051	-.0057	-.0064	-.0035	-.0006	.0023	.0052	.0081	.0110	.0139	.0168	.0197	.0226
	.4	0	-.0007	-.0014	-.0021	-.0027	-.0034	-.0041	-.0048	-.0055	-.0062	-.0069	-.0038	-.0006	.0025	.0056	.0087	.0118	.0150	.0181	.0212	.0243
	.5	0	-.0007	-.0013	-.0020	-.0027	-.0034	-.0040	-.0047	-.0054	-.0060	-.0067	-.0037	-.0006	.0024	.0055	.0085	.0116	.0146	.0176	.0207	.0237
	.6	0	-.0006	-.0012	-.0018	-.0024	-.0030	-.0036	-.0042	-.0048	-.0054	-.0060	-.0033	-.0006	.0022	.0049	.0076	.0104	.0131	.0158	.0185	.0213
	.7	0	-.0005	-.0010	-.0015	-.0020	-.0024	-.0029	-.0034	-.0039	-.0044	-.0049	-.0027	-.0005	.0018	.0040	.0062	.0084	.0106	.0128	.0151	.0173
	.8	0	-.0003	-.0007	-.0010	-.0014	-.0017	-.0021	-.0024	-.0027	-.0031	-.0034	-.0019	-.0003	.0012	.0028	.0044	.0059	.0075	.0090	.0106	.0122
	.9	0	-.0002	-.0004	-.0005	-.0007	-.0009	-.0011	-.0012	-.0014	-.0016	-.0018	-.0010	-.0002	.0006	.0014	.0022	.0031	.0039	.0047	.0055	.0063
	E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0416	.0732	.0948	.1065	.1081	.0997	.0813	.0529	.0293	.0249	.0229	.0448	.0865	.1145	.1256	.1198	.0971	.0585	.0324	.0316
− Area		0	-.0099	-.0198	-.0297	-.0398	-.0495	-.0594	-.0693	-.0791	-.1038	-.1577	-.0808	-.0448	-.0455	-.0493	-.0532	-.0570	-.0609	-.0657	-.1000	-.1765
Total Area		0	.0317	.0534	.0651	.0669	.0586	.0403	.0120	-.0262	-.0745	-.1328	-.0579	.0000	-.0410	.0652	.0724	.0628	.0362	-.0072	-.0676	-.1449

Unit load at		REACTIONS/P R_A	R_B	R_C	R_D	R_E	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
SPAN 1	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
	.1	.8767	.1460	-.0289	.0079	-.0018	.8767	-.1233	.0227	.0227	-.0062	-.0062	.0018	.0018
	.2	.7548	.2883	-.0561	.0154	-.0034	.7548	-.2452	.0441	.0441	-.0120	-.0120	.0034	.0034
	.3	.6358	.4269	-.0797	.0219	-.0049	.6358	-.3642	.0627	.0627	-.0170	-.0170	.0049	.0049
	.4	.5209	.5542	-.0981	.0270	-.0060	.5209	-.4791	.0772	.0772	-.0210	-.0210	.0060	.0060
	.5	.4118	.6744	-.1095	.0301	-.0067	.4118	-.5882	.0861	.0861	-.0234	-.0234	.0067	.0067
	.6	.3097	.7785	-.1123	.0308	-.0068	.3097	-.6903	.0882	.0882	-.0240	-.0240	.0068	.0068
	.7	.2160	.8660	-.1043	.0287	-.0064	.2160	-.7840	.0820	.0820	-.0223	-.0223	.0064	.0064
	.8	.1322	.9339	-.0841	.0231	-.0052	.1322	-.8678	.0661	.0661	-.0180	-.0180	.0052	.0052
	.9	.0598	.9795	-.0500	.0137	-.0031	.0598	-.9402	.0393	.0393	-.0107	-.0107	.0031	.0031
	B	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0	0	0
SPAN 2	.1	-.0574	.9868	.0898	-.0243	.0054	-.0574	-.0574	.9294	-.0706	.0189	.0189	-.0054	-.0054
	.2	-.0940	.9341	.2013	-.0531	.0118	-.0940	-.0940	.8401	-.1599	.0413	.0413	-.0118	-.0118
	.3	-.1127	.8494	.3276	-.0827	.0184	-.1127	-.1127	.7366	-.2634	.0643	.0643	-.0184	-.0184
	.4	-.1167	.7403	.4814	-.1093	.0243	-.1167	-.1167	.6236	-.3764	.0849	.0849	-.0243	-.0243
	.5	-.1090	.6145	.5949	-.1292	.0288	-.1090	-.1090	.5055	-.4945	.1005	.1005	-.0288	-.0288
	.6	-.0925	.4795	.7209	-.1389	.0309	-.0925	-.0925	.3870	-.6130	.1079	.1079	-.0309	-.0309
	.7	-.0704	.3430	.8318	-.1344	.0299	-.0704	-.0704	.2727	-.7273	.1045	.1045	-.0299	-.0299
	.8	-.0455	.2125	.9203	-.1123	.0250	-.0455	-.0455	.1670	-.8330	.0873	.0873	-.0250	-.0250
	.9	-.0211	.0956	.9788	-.0687	.0153	-.0211	-.0211	.0746	-.9254	.0534	.0534	-.0153	-.0153
	C	0	0	1.0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 3	.1	.0153	-.0687	.9788	.0956	-.0211	.0153	.0153	-.0534	-.0534	.9254	-.0746	.0211	.0211
	.2	.0250	-.1123	.9203	.2125	-.0455	.0250	.0250	-.0873	-.0873	.8330	-.1670	.0455	.0455
	.3	.0299	-.1344	.8318	.3430	-.0704	.0299	.0299	-.1045	-.1045	.7273	-.2727	.0704	.0704
	.4	.0309	-.1389	.7209	.4795	-.0925	.0309	.0309	-.1079	-.1079	.6130	-.3870	.0925	.0925
		.0288	-.1292	.5949	.6145	-.1090	.0288	.0288	-.1005	-.1005	.4945	-.5055	.1090	.1090
	[illegible]	.0243	-.1093	.4814	.7403	-.1167	.0243	.0243	-.0849	-.0849	.3764	-.6236	.1167	.1167
	.7	.0184	-.0827	.3276	.8494	-.1127	.0184	.0184	-.0643	-.0643	.2634	-.7366	.1127	.1127
	.8	.0118	-.0531	.2012	.9341	-.0940	.0118	.0118	-.0413	-.0413	.1599	-.8401	.0940	.0940
	.9	.0054	-.0243	.0895	.9868	-.0574	.0054	.0054	-.0189	-.0189	.0706	-.9294	.0574	.0574
	D	0	0	0	1.0	0	00	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 4	.1	-.0031	.0137	-.0500	.9795	.0598	-.0031	-.0031	.0107	.0107	-.0393	-.0393	.9402	-.0598
	.2	-.0052	.0231	-.0841	.9339	.1322	-.0052	-.0052	.0180	.0180	-.0661	-.0661	.8678	-.1322
	.3	-.0064	.0287	-.1043	.8660	.2160	-.0064	-.0064	.0223	.0223	-.0820	-.0820	.7840	-.2160
	.4	-.0069	.0308	-.1122	.7785	.3097	-.0069	-.0069	.0240	.0240	-.0882	-.0882	.6903	-.3097
	.5	-.0067	.0301	-.1095	.6744	.4118	-.0067	-.0067	.0234	.0234	-.0861	-.0861	.5882	-.4118
	.6	-.0060	.0270	-.0981	.5562	.5209	-.0060	-.0060	.0210	.0210	-.0772	-.0772	.4791	-.5209
	.7	-.0049	.0219	-.0797	.4269	.6358	-.0049	-.0049	.0170	.0170	-.0627	-.0627	.3642	-.6358
	.8	-.0034	.0154	-.0561	.2883	.7548	-.0034	-.0034	.0120	.0120	-.0441	-.0441	.2452	-.7548
	.9	-.0018	.0079	-.0289	.1460	.8767	-.0018	-.0018	.0062	.0062	-.0227	-.0227	.1233	-.8767
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4661	1.3856	1.4646	1.3856	.4661	.4661	.0249	.7276	.0730	.7323	.0871	.8577	.0989
− Area		-.0989	-.1120	-.1461	-.1120	-.0989	-.0989	-.8577	-.0871	-.7323	-.0730	-.7278	-.0249	-.4661
Total Area		.3672	1.2736	1.3185	1.2736	.3672	.3672	-.8328	.6407	-.6593	.6593	-.6407	.8328	-.3672



Influence coefficients — Four continuous spans.

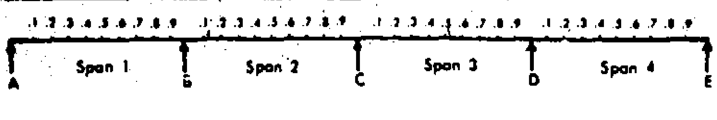
L = Length of EXTERIOR spans; length of interior spans = NL

N = 1.3

Table 20.41

Unit load at		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	.1	.2	.3	.4	.5	.6	.7	.8	.9	C
		MOMENTS/PL																				
		SPAN 1											SPAN 2									
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0878	.0755	.0633	.0510	.0388	.0266	.0143	.0021	-.0101	-.0224	-.0195	-.0167	-.0139	-.0110	-.0082	-.0053	-.0025	.0004	.0032	.0060
	.2	0	.0757	.1513	.1270	.1026	.0783	.0540	.0296	.0053	-.0191	-.0434	-.0379	-.0324	-.0269	-.0214	-.0159	-.0103	-.0048	.0007	.0062	.0117
	.3	0	.0638	.1277	.1915	.1553	.1191	.0830	.0468	.0106	-.0256	-.0617	-.0539	-.0461	-.0382	-.0304	-.0225	-.0147	-.0069	.0010	.0088	.0166
	.4	0	.0524	.1048	.1572	.2096	.1620	.1144	.0668	.0192	-.0284	-.0760	-.0663	-.0567	-.0470	-.0374	-.0277	-.0181	-.0085	.0012	.0108	.0205
	.5	0	.0415	.0830	.1246	.1661	.2076	.1491	.0906	.0322	-.0263	-.0848	-.0740	-.0633	-.0525	-.0417	-.0310	-.0202	-.0094	.0013	.0121	.0229
	.6	0	.0313	.0626	.0940	.1253	.1566	.1879	.1192	.0505	-.0181	-.0868	-.0758	-.0648	-.0538	-.0427	-.0317	-.0207	-.0097	.0014	.0124	.0234
	.7	0	.0219	.0438	.0658	.0877	.1096	.1316	.1535	.0754	-.0027	-.0807	-.0705	-.0602	-.0500	-.0397	-.0295	-.0192	-.0090	.0013	.0115	.0218
	.8	0	.0135	.0270	.0405	.0540	.0674	.0809	.0944	.1079	.0214	-.0651	-.0569	-.0486	-.0403	-.0321	-.0238	-.0155	-.0072	.0010	.0093	.0176
	.9	0	.0061	.0123	.0184	.0245	.0307	.0368	.0429	.0491	.0552	-.0367	-.0338	-.0288	-.0239	-.0190	-.0141	-.0092	-.0043	.0006	.0055	.0104
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0064	-.0128	-.0192	-.0256	-.0320	-.0384	-.0448	-.0512	-.0576	-.0640	.0664	.0568	.0472	.0376	.0280	.0183	.0087	-.0009	-.0105	-.0201
	.2	0	-.0105	-.0209	-.0314	-.0419	-.0523	-.0628	-.0733	-.0838	-.0942	-.1047	.0134	.1314	.1094	.0875	.0655	.0436	.0216	-.0003	-.0223	-.0443
	.3	0	-.0126	-.0251	-.0377	-.0502	-.0628	-.0754	-.0879	-.1005	-.1130	-.1256	-.0220	.0817	.1853	.1490	.1126	.0762	.0399	.0035	-.0328	-.0692
	.4	0	-.0130	-.0260	-.0390	-.0520	-.0650	-.0780	-.0910	-.1040	-.1170	-.1300	-.0422	.0456	.1334	.2213	.1691	.1169	.0647	.0126	-.0396	-.0918
	.5	0	-.0121	-.0243	-.0364	-.0486	-.0607	-.0728	-.0850	-.0971	-.1092	-.1214	-.0501	.0211	.0924	.1636	.2349	.1661	.0974	.0287	-.0401	-.1088
	.6	0	-.0103	-.0206	-.0309	-.0412	-.0515	-.0618	-.0721	-.0824	-.0927	-.1030	-.0484	.0061	.0607	.1153	.1699	.2245	.1391	.0536	-.0318	-.1172
	.7	0	-.0078	-.0157	-.0235	-.0313	-.0392	-.0470	-.0548	-.0627	-.0705	-.0783	-.0399	-.0014	.0371	.0755	.1140	.1525	.1909	.0894	-.0121	-.1137
	.8	0	-.0051	-.0101	-.0152	-.0203	-.0253	-.0304	-.0355	-.0405	-.0456	-.0507	-.0271	-.0036	.0200	.0436	.0671	.0907	.1142	.1378	.0214	-.0951
	.9	0	-.0023	-.0047	-.0070	-.0094	-.0117	-.0141	-.0164	-.0188	-.0211	-.0234	-.0129	-.0024	.0081	.0186	.0292	.0397	.0502	.0607	.0712	-.0582
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0017	.0034	.0051	.0068	.0085	.0102	.0119	.0136	.0153	.0170	.0095	.0019	-.0056	-.0131	-.0206	-.0281	-.0357	-.0432	-.0507	-.0582
	.2	0	.0028	.0055	.0083	.0111	.0139	.0166	.0194	.0222	.0250	.0277	.0154	.0032	-.0091	-.0214	-.0337	-.0459	-.0582	-.0705	-.0828	-.0951
	.3	0	.0033	.0066	.0099	.0133	.0166	.0199	.0232	.0265	.2098	.0331	.0185	.0038	-.0109	-.0256	-.0403	-.0549	-.0696	-.0843	-.0990	-.1137
	.4	0	.0034	.0068	.0103	.0137	.0171	.0205	.0239	.0273	.0308	.0342	.0190	.0039	-.0112	-.0264	-.0415	-.0566	-.0718	-.0869	-.1021	-.1172
	.5	0	.0032	.0063	.0095	.0127	.0159	.0190	.0222	.0254	.0286	.0317	.0177	.0036	-.0104	-.0245	-.0385	-.0526	-.0667	-.0807	-.0948	-.1088
	.6	0	.0027	.0054	.0080	.0107	.0134	.0161	.0187	.0214	.0241	.0268	.0149	.0031	-.0088	-.0207	-.0325	-.0444	-.0562	-.0681	-.0799	-.0918
	.7	0	.0020	.0040	.0061	.0081	.0101	.0121	.0141	.0161	.0182	.0202	.0112	.0023	-.0066	-.0156	-.0245	-.0334	-.0424	-.0513	-.0603	-.0692
	.8	0	.0013	.0026	.0039	.0052	.0065	.0077	.0090	.0103	.0116	.0129	.0072	.0015	-.0042	-.0100	-.0157	-.0214	-.0271	-.0328	-.0385	-.0443
	.9	0	.0006	.0012	.0018	.0023	.0029	.0035	.0041	.0047	.0053	.0059	.0033	.0007	-.0019	-.0045	-.0071	-.0097	-.0123	-.0149	-.0175	-.0201
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0024	-.0027	-.0030	-.0017	-.0003	.0010	.0023	.0037	.0050	.0064	.0077	.0091	.0104
	.2	0	-.0005	-.0010	-.0015	-.0020	-.0026	-.0031	-.0036	-.0041	-.0046	-.0051	-.0029	-.0006	.0017	.0040	.0062	.0085	.0108	.0130	.0153	.0176
	.3	0	-.0006	-.0013	-.0019	-.0025	-.0032	-.0038	-.0044	-.0051	-.0057	-.0063	-.0035	-.0007	.0021	.0049	.0077	.0105	.0133	.0161	.0190	.0218
	.4	0	-.0007	-.0014	-.0020	-.0027	-.0034	-.0041	-.0048	-.0055	-.0061	-.0068	-.0038	-.0008	.0022	.0053	.0083	.0113	.0143	.0174	.0204	.0234
	.5	0	-.0007	-.0013	-.0020	-.0027	-.0033	-.0040	-.0047	-.0053	-.0060	-.0067	-.0037	-.0008	.0022	.0051	.0081	.0111	.0140	.0170	.0199	.0229
	.6	0	-.0006	-.0012	-.0018	-.0024	-.0030	-.0036	-.0042	-.0048	-.0054	-.0060	-.0033	-.0007	.0020	.0046	.0073	.0099	.0125	.0152	.0178	.0205
	.7	0	-.0005	-.0010	-.0015	-.0019	-.0024	-.0029	-.0034	-.0039	-.0044	-.0049	-.0027	-.0006	.0016	.0037	.0059	.0080	.0102	.0123	.0145	.0166
	.8	0	-.0003	-.0007	-.0010	-.0014	-.0017	-.0020	-.0024	-.0027	-.0031	-.0034	-.0019	-.0004	.0011	.0026	.0041	.0057	.0072	.0087	.0102	.0117
	.9	0	-.0002	-.0004	-.0005	-.0007	-.0009	-.0011	-.0012	-.0014	-.0016	-.0018	-.0010	-.0002	.0006	.0014	.0021	.0028	.0037	.0045	.0053	.0060
	E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0423	.0746	.0969	.1092	.1116	.1039	.0862	.0585	.0340	.0296	.0266	.0503	.0975	.1300	.1430	.1363	.1100	.0652	.0332	.0305
− Area		0	-.0118	-.0235	-.0353	-.0471	-.0589	-.0707	-.0824	-.0942	-.1191	-.1742	-.0858	-.0437	-.0447	-.0507	-.0566	-.0626	-.0685	-.0755	-.1149	-.2032
Total Area		0	.0305	.0511	.0616	.0621	.0527	.0332	.0038	-.0357	-.0851	-.1446	-.0592	.0066	.0528	.0793	.0864	.0737	.0415	-.0103	-.0817	-.1727

Unit load at		R_A	R_B	R_C	R_D	R_E	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
		REACTIONS/P					SHEARS/P							
SPAN 1	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
	.1	.8776	.1427	-.0256	.0073	-.0017	.8776	-.1224	.0203	.0203	-.0058	-.0058	.0017	.0017
	.2	.7566	.2828	-.0503	.0142	-.0034	.7566	-.2434	.0394	.0394	-.0108	-.0108	.0034	.0034
	.3	.6383	.4177	-.0713	.0202	-.0049	.6383	-.3617	.0560	.0560	-.0154	-.0154	.0049	.0049
	.4	.5240	.5449	-.0878	.0249	-.0060	.5240	-.4760	.0689	.0689	-.0189	-.0189	.0060	.0060
	.5	.4152	.6617	-.0980	.0278	-.0067	.4152	-.5848	.0769	.0769	-.0211	-.0211	.0067	.0067
	.6	.3132	.7654	-.1003	.0284	-.0068	.3132	-.6868	.0787	.0787	-.0218	-.0218	.0068	.0068
	.7	.2193	.8539	-.0933	.0264	-.0063	.2193	-.7807	.0732	.0732	-.0201	-.0201	.0063	.0063
	.8	.1349	.9242	-.0753	.0213	-.0051	.1349	-.8651	.0591	.0591	-.0162	-.0162	.0051	.0051
	.9	.0613	.9737	-.0447	.0127	-.0030	.0613	-.9387	.0351	.0351	-.0096	-.0096	.0030	.0030
	B	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0	0	0
SPAN 2	.1	-.0640	.9853	.0273	-.0244	.0059	-.0640	-.0640	.9313	-.0687	.0186	.0186	-.0059	-.0059
	.2	-.1047	.9479	.1977	-.0537	.0129	-.1047	-.1047	.8432	-.1568	.0408	.0408	-.0129	-.0129
	.3	-.1256	.8658	.3236	-.0840	.0202	-.1256	-.1256	.7403	-.2597	.0638	.0638	-.0202	-.0202
	.4	-.1300	.7373	.4574	-.1115	.0268	-.1300	-.1300	.6273	-.3727	.0847	.0847	-.0268	-.0268
	.5	-.1214	.6303	.5915	-.1322	.0317	-.1214	-.1214	.5090	-.4910	.1004	.1004	-.0317	-.0317
	.6	-.1030	.4929	.7182	-.1423	.0342	-.1030	-.1030	.3899	-.6101	.1081	.1081	-.0342	-.0342
	.7	-.0783	.3531	.8301	-.1380	.0331	-.0783	-.0783	.2748	-.7252	.1049	.1049	-.0331	-.0331
	.8	-.0507	.2190	.9194	-.1154	.0277	-.0507	-.0507	.1683	-.8317	.0877	.0877	-.0277	-.0277
	.9	-.0234	.0966	.9786	-.0707	.0170	-.0234	-.0234	.0751	-.9249	.0537	.0537	-.0170	-.0170
	C	0	0	1.0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 3	.1	.0170	-.0707	.9786	.0986	-.0234	.0170	.0170	-.0537	-.0537	.9249	-.0751	.0234	.0234
	.2	.0277	-.1154	.9194	.2190	-.0507	.0277	.0277	-.0877	-.0877	.8317	-.1683	.0507	.0507
	.3	.0331	-.1380	.8301	.3531	-.0783	.0331	.0331	-.1049	-.1049	.7252	-.2748	.0783	.0783
	.4	.0342	-.1423	.7182	.4929	-.1030	.0342	.0342	-.1081	-.1081	.6101	-.3899	.1030	.1030
	.5	.0317	-.1322	.5915	.6303	-.1214	.0317	.0317	-.1004	-.1004	.4910	-.5090	.1214	.1214
	.6	.0268	-.1115	.4574	.7573	-.1300	.0268	.0268	-.0847	-.0847	.3727	-.6273	.1300	.1300
	.7	.0202	-.0840	.3236	.8659	-.1256	.0202	.0202	-.0638	-.0638	.2597	-.7403	.1256	.1256
	.8	.0129	-.0537	.1977	.9479	-.1047	.0129	.0129	-.0408	-.0408	.1568	-.8432	.1047	.1047
	.9	.0059	-.0244	.0873	.9953	-.0640	.0059	.0059	-.0186	-.0186	.0687	-.9313	.0640	.0640
	D	0	0	0	1.0	0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 4	.1	-.0030	.0127	-.0447	.9737	.0613	-.0030	-.0030	.0096	.0096	-.0351	-.0351	.9387	-.0613
	.2	-.0051	.0213	-.0753	.9242	.1349	-.0051	-.0051	.0162	.0162	-.0591	-.0591	.8651	-.1349
	.3	-.0063	.0264	-.0933	.8539	.2193	-.0063	-.0063	.0201	.0201	-.0732	-.0732	.7807	-.2193
	.4	-.0068	.0284	-.1003	.7656	.3132	-.0068	-.0068	.0218	.0218	-.0787	-.0787	.6868	-.3132
	.5	-.0067	.0278	-.0980	.6817	.4152	-.0067	-.0067	.0211	.0211	-.0769	-.0769	.5848	-.4152
	.6	-.0060	.0249	-.0878	.5449	.5240	-.0060	-.0060	.0189	.0189	-.0689	-.0689	.4760	-.5240
	.7	-.0049	.0202	-.0713	.4177	.6383	-.0049	-.0049	.0154	.0154	-.0560	-.0560	.3617	-.6383
	.8	-.0034	.0142	-.0502	.2828	.7566	-.0034	-.0034	.0108	.0108	-.0394	-.0394	.2434	-.7566
	.9	-.0017	.0073	-.0259	.1427	.8776	-.0017	-.0017	.0056	.0056	-.0203	-.0203	.1224	-.8776
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4731	1.4460	1.3707	1.4460	.4731	.4731	.0296	.7737	.0653	.7854	.0937	.6743	.1177
− Area		-.1177	-.1234	-.1306	-.1234	-.1177	-.1177	-.6743	-.0937	-.7854	-.0653	-.7737	-.0296	-.4731
Total Area		.3554	1.3246	1.4401	1.3246	.3554	.3554	-.6447	.6800	-.7201	.7201	-.6800	.6447	-.3554



Influence coefficients — Four continuous spans.

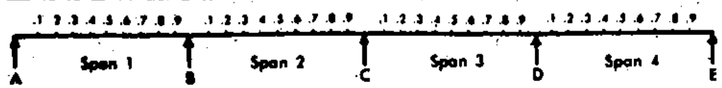
L = Length of EXTERIOR spans; length of interior spans = NL

N = 1.4

Table 20.42

Unit load at		MOMENTS/PL SPAN 1 A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	SPAN 2 .1	.2	.3	.4	.5	.6	.7	.8	.9	C
	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 1	.1	0	.0878	.0757	.0638	.0514	.0392	.0271	.0149	.0028	-.0094	-.0215	-.0188	-.0161	-.0133	-.0106	-.0079	-.0051	-.0024	.0003	.0031	.0058
	.2	0	.0756	.1516	.1275	.1033	.0791	.0549	.0307	.0066	-.0176	-.0418	-.0365	-.0312	-.0258	-.0206	-.0152	-.0099	-.0046	.0007	.0060	.0113
	.3	0	.0641	.1281	.1922	.1542	.1203	.0843	.0484	.0125	-.0235	-.0594	-.0519	-.0443	-.0368	-.0292	-.0217	-.0141	-.0066	.0010	.0085	.0161
	.4	0	.0527	.1054	.1581	.2107	.1634	.1161	.0688	.0215	-.0258	-.0731	-.0638	-.0546	-.0453	-.0360	-.0267	-.0174	-.0081	.0012	.0105	.0198
	.5	0	.0418	.0837	.1255	.1674	.2092	.1510	.0929	.0347	-.0235	-.0816	-.0713	-.0609	-.0506	-.0401	-.0298	-.0194	-.0090	.0013	.0117	.0221
	.6	0	.0316	.0633	.0949	.1266	.1582	.1898	.1215	.0531	-.0152	-.0836	-.0730	-.0623	-.0517	-.0411	-.0306	-.0199	-.0093	.0014	.0120	.0226
	.7	0	.0223	.0445	.0667	.0889	.1112	.1334	.1556	.0778	.0001	-.0777	-.0676	-.0580	-.0481	-.0383	-.0284	-.0185	-.0086	.0013	.0111	.0210
	.8	0	.0137	.0275	.0412	.0549	.0687	.0824	.0961	.1099	.0236	-.0627	-.0547	-.0468	-.0388	-.0308	-.0229	-.0149	-.0069	.0010	.0090	.0169
	.9	0	.0063	.0126	.0188	.0251	.0314	.0377	.0439	.0502	.0565	-.0372	-.0325	-.0276	-.0230	-.0183	-.0136	-.0089	-.0041	.0006	.0053	.0101
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0071	-.0141	-.0212	-.0283	-.0353	-.0424	-.0494	-.0565	-.0636	-.0706	.0893	.0593	.0492	.0392	.0292	.0191	.0091	-.0010	-.0110	-.0210
	.2	0	-.0116	-.0231	-.0347	-.0462	-.0578	-.0694	-.0809	-.0925	-.1041	-.1156	.0113	.1362	.1151	.0920	.0689	.0458	.0227	-.0004	-.0235	-.0466
	.3	0	-.0139	-.0277	-.0416	-.0555	-.0693	-.0832	-.0971	-.1110	-.1249	-.1387	-.0271	.0844	.1960	.1575	.1191	.0806	.0422	.0037	-.0347	-.0732
	.4	0	-.0144	-.0287	-.0431	-.0574	-.0718	-.0861	-.1005	-.1149	-.1292	-.1436	-.0490	.0457	.1403	.2349	.1795	.1241	.0687	.0134	-.0420	-.0974
	.5	0	-.0134	-.0268	-.0402	-.0536	-.0670	-.0804	-.0938	-.1072	-.1206	-.1340	-.0572	.0196	.0965	.1733	.2501	.1769	.1037	.0306	-.0426	-.1158
	.6	0	-.0114	-.0227	-.0341	-.0455	-.0568	-.0682	-.0796	-.0910	-.1023	-.1137	-.0548	.0040	.0629	.1218	.1807	.2395	.1484	.0573	-.0338	-.1249
	.7	0	-.0086	-.0173	-.0259	-.0346	-.0432	-.0518	-.0605	-.0692	-.0778	-.0864	-.0449	-.0034	.0381	.0796	.1211	.1626	.2041	.0956	-.0129	-.1214
	.8	0	-.0056	-.0112	-.0168	-.0224	-.0280	-.0335	-.0391	-.0447	-.0503	-.0559	-.0303	-.0051	.0204	.0458	.0712	.0966	.1221	.1475	.0229	-.1016
	.9	0	-.0026	-.0052	-.0078	-.0103	-.0129	-.0155	-.0181	-.0207	-.0233	-.0258	-.0143	-.0031	.0082	.0196	.0309	.0423	.0536	.0650	.0763	-.0623
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0019	.0037	.0056	.0075	.0094	.0112	.0131	.0150	.0168	.0187	.0108	.0025	-.0056	-.0137	-.0218	-.0299	-.0380	-.0461	-.0542	-.0623
	.2	0	.0030	.0061	.0091	.0122	.0152	.0183	.0213	.0244	.0274	.0305	.0173	.0041	-.0091	-.0224	-.0356	-.0488	-.0620	-.0752	-.0884	-.1016
	.3	0	.0036	.0073	.0109	.0146	.0182	.0218	.0255	.0291	.0328	.0364	.0206	.0047	-.0109	-.0267	-.0425	-.0583	-.0740	-.0898	-.1056	-.1214
	.4	0	.0037	.0075	.0112	.0150	.0187	.0225	.0262	.0300	.0337	.0375	.0212	.0050	-.0112	-.0275	-.0437	-.0600	-.0762	-.0925	-.1087	-.1249
	.5	0	.0035	.0069	.0104	.0139	.0174	.0208	.0243	.0278	.0313	.0347	.0197	.0046	-.0104	-.0255	-.0405	-.0556	-.0706	-.0857	-.1008	-.1158
	.6	0	.0029	.0058	.0088	.0117	.0146	.0175	.0205	.0234	.0263	.0292	.0168	.0039	-.0089	-.0214	-.0341	-.0468	-.0594	-.0721	-.0847	-.0974
	.7	0	.0022	.0044	.0066	.0088	.0110	.0132	.0154	.0176	.0198	.0220	.0124	.0029	-.0066	-.0161	-.0256	-.0351	-.0446	-.0542	-.0637	-.0732
	.8	0	.0014	.0028	.0042	.0056	.0070	.0084	.0098	.0112	.0126	.0140	.0079	.0019	-.0042	-.0102	-.0163	-.0224	-.0284	-.0345	-.0405	-.0466
	.9	0	.0006	.0013	.0019	.0025	.0032	.0038	.0044	.0051	.0057	.0063	.0036	.0008	-.0019	-.0046	-.0074	-.0101	-.0128	-.0156	-.0183	-.0210
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0024	-.0027	-.0030	-.0017	-.0004	.0009	.0022	.0035	.0048	.0061	.0074	.0088	.0101
	.2	0	-.0005	-.0010	-.0015	-.0020	-.0025	-.0030	-.0036	-.0041	-.0046	-.0051	-.0029	-.0007	.0015	.0037	.0059	.0081	.0103	.0125	.0147	.0169
	.3	0	-.0006	-.0013	-.0019	-.0025	-.0032	-.0038	-.0044	-.0050	-.0057	-.0063	-.0036	-.0008	.0019	.0046	.0074	.0101	.0128	.0155	.0183	.0210
	.4	0	-.0007	-.0014	-.0020	-.0027	-.0034	-.0041	-.0047	-.0054	-.0061	-.0068	-.0038	-.0009	.0020	.0050	.0079	.0108	.0138	.0167	.0197	.0226
	.5	0	-.0007	-.0013	-.0020	-.0026	-.0033	-.0040	-.0046	-.0053	-.0060	-.0066	-.0037	-.0009	.0020	.0049	.0077	.0106	.0135	.0163	.0192	.0221
	.6	0	-.0006	-.0012	-.0018	-.0024	-.0030	-.0036	-.0042	-.0047	-.0053	-.0059	-.0034	-.0008	.0018	.0043	.0069	.0095	.0121	.0146	.0172	.0198
	.7	0	-.0005	-.0010	-.0014	-.0019	-.0024	-.0029	-.0034	-.0039	-.0043	-.0048	-.0027	-.0006	.0014	.0035	.0056	.0077	.0098	.0119	.0140	.0161
	.8	0	-.0003	-.0007	-.0010	-.0014	-.0017	-.0020	-.0024	-.0027	-.0030	-.0034	-.0019	-.0005	.0010	.0025	.0040	.0054	.0069	.0084	.0098	.0113
	.9	0	-.0002	-.0003	-.0005	-.0007	-.0009	-.0010	-.0012	-.0014	-.0016	-.0017	-.0010	-.0002	.0005	.0013	.0020	.0028	.0036	.0043	.0051	.0058
	E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0430	.0761	.0991	.1121	.1152	.1082	.0912	.0643	.0390	.0347	.0306	.0560	.1090	.1465	.1613	.1540	.1240	.0725	.0347	.0294
- Area		0	-.0138	-.0277	-.0415	-.0554	-.0692	-.0831	-.0969	-.1108	-.1383	-.1926	-.0918	-.0429	-.0441	-.0522	-.0604	-.0685	-.0767	-.0859	-.1306	-.2316
Total Area		0	.0292	.0484	.0576	.0567	.0460	.0251	-.0057	-.0465	-.0973	-.1581	-.0612	.0131	.0649	.0943	.1011	.0855	.0473	-.0134	-.0966	-.2022

Unit load at		REACTIONS/P R_A	R_B	R_C	R_D	R_E	SHEARS/P V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
SPAN 1	.1	.8785	.1398	-.0233	.0060	-.0017	.8785	-.1215	.0182	.0182	-.0050	-.0050	.0017	.0017
	.2	.7582	.2772	-.0452	.0132	-.0034	.7582	-.2418	.0354	.0354	-.0098	-.0098	.0034	.0034
	.3	.6406	.4087	-.0643	.0187	-.0048	.6406	-.3594	.0503	.0503	-.0139	-.0139	.0048	.0048
	.4	.5269	.5331	-.0791	.0230	-.0059	.5269	-.4731	.0619	.0619	-.0171	-.0171	.0059	.0059
	.5	.4184	.6507	-.0882	.0257	-.0066	.4184	-.5816	.0691	.0691	-.0191	-.0191	.0066	.0066
	.6	.3164	.7544	-.0903	.0263	-.0068	.3164	-.6836	.0708	.0708	-.0196	-.0196	.0068	.0068
	.7	.2223	.8435	-.0840	.0245	-.0063	.2223	-.7777	.0658	.0658	-.0182	-.0182	.0063	.0063
	.8	.1373	.9158	-.0678	.0198	-.0051	.1373	-.8627	.0531	.0531	-.0147	-.0147	.0051	.0051
	.9	.0628	.9667	-.0402	.0117	-.0030	.0628	-.9372	.0318	.0318	-.0087	-.0087	.0030	.0030
	B	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0	0	0
SPAN 2	.1	-.0706	1.0037	.0852	-.0246	.0063	-.0706	-.0706	.9331	-.0669	.0182	.0182	-.0063	-.0063
	.2	-.1156	.9616	.1944	-.0544	.0140	-.1156	-.1156	.8460	-.1540	.0404	.0404	-.0140	-.0140
	.3	-.1387	.8824	.3196	-.0854	.0220	-.1387	-.1387	.7437	-.2563	.0634	.0634	-.0220	-.0220
	.4	-.1436	.7744	.4336	-.1136	.0292	-.1436	-.1436	.6308	-.3692	.0844	.0844	-.0292	-.0292
	.5	-.1340	.6461	.5882	-.1351	.0347	-.1340	-.1340	.5121	-.4879	.1004	.1004	-.0347	-.0347
	.6	-.1137	.5062	.7138	-.1438	.0375	-.1137	-.1137	.3925	-.6075	.1063	.1063	-.0375	-.0375
	.7	-.0864	.3631	.8285	-.1416	.0364	-.0864	-.0864	.2767	-.7233	.1052	.1052	-.0364	-.0364
	.8	-.0559	.2254	.9186	-.1186	.0305	-.0559	-.0559	.1695	-.8305	.0881	.0881	-.0305	-.0305
	.9	-.0258	.1015	.9883	-.0757	.0117	-.0258	-.0258	.0757	-.9243	.0640	.0640	-.0117	-.0117
	C	0	0	1.0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 3	.1	.0187	-.0727	.9704	.1015	-.0258	.0187	.0187	-.0540	-.0540	.9243	-.0757	.0258	.0258
	.2	.0305	-.1186	.9186	.2254	-.0559	.0305	.0305	-.0881	-.0881	.8305	-.1695	.0559	.0559
	.3	.0364	-.1416	.8285	.3632	-.0864	.0364	.0364	-.1052	-.1052	.7233	-.2767	.0864	.0864
	.4	.0375	-.1438	.7138	.5062	-.1137	.0375	.0375	-.1063	-.1063	.6075	-.3925	.1137	.1137
	.5	.0347	-.1351	.5882	.6461	-.1340	.0347	.0347	-.1004	-.1004	.4879	-.5121	.1340	.1340
	.6	.0292	-.1136	.4336	.7744	-.1436	.0292	.0292	-.0844	-.0844	.3692	-.6308	.1436	.1436
	.7	.0220	-.0854	.3196	.8824	-.1387	.0220	.0220	-.0634	-.0634	.2563	-.7437	.1387	.1387
	.8	.0140	-.0544	.1944	.9616	-.1156	.0140	.0140	-.0404	-.0404	.1540	-.8460	.1156	.1156
	.9	.0063	-.0246	.0852	1.0037	-.0706	.0063	.0063	-.0182	-.0182	.0669	-.9331	.0706	.0706
	D	0	0	0	1.0	0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 4	.1	-.0030	.0117	-.0402	.9667	.0628	-.0030	-.0030	.0087	.0087	-.0315	-.0315	.9372	-.0628
	.2	-.0051	.0198	-.0678	.9158	.1373	-.0051	-.0051	.0147	.0147	-.0531	-.0531	.8627	-.1373
	.3	-.0063	.0245	-.0840	.8435	.2223	-.0063	-.0063	.0182	.0182	-.0658	-.0658	.7777	-.2223
	.4	-.0068	.0263	-.0903	.7544	.3164	-.0068	-.0068	.0196	.0196	-.0708	-.0708	.6836	-.3164
	.5	-.0066	.0257	-.0882	.6507	.4184	-.0066	-.0066	.0191	.0191	-.0691	-.0691	.5816	-.4184
	.6	-.0059	.0230	-.0791	.5331	.5269	-.0059	-.0059	.0171	.0171	-.0619	-.0619	.4731	-.5269
	.7	-.0048	.0187	-.0643	.4087	.6406	-.0048	-.0048	.0139	.0139	-.0503	-.0503	.3594	-.6406
	.8	-.0034	.0132	-.0452	.2772	.7582	-.0034	-.0034	.0098	.0098	-.0354	-.0354	.2418	-.7582
	.9	-.0017	.0060	-.0233	.1398	.8785	-.0017	-.0017	.0050	.0050	-.0182	-.0182	.1215	-.8785
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4803	1.5138	1.6765	1.5138	.4803	.4803	.0347	.8210	.0588	.8382	.1004	.6928	.1384
- Area		-.1384	-.1351	-.1176	-.1351	-.1384	-.1384	-.6928	-.1004	-.8382	-.0588	-.8210	-.0347	-.4803
Total Area		.3419	1.3787	1.5589	1.3787	.3419	.3419	-.6581	.7206	-.7794	.7794	-.7206	.6581	-.3419



Influence coefficients — Four continuous spans.

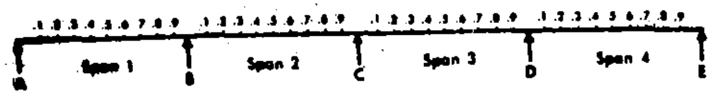
L = Length of EXTERIOR spans; length of interior spans = NL

N = 1.5

Table 20.43

Unit load at		MOMENTS/PL																				
		SPAN 1											SPAN 2									
		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0878	.0758	.0636	.0517	.0396	.0275	.0155	.0034	-.0087	-.0208	-.0181	-.0155	-.0129	-.0102	-.0076	-.0049	-.0023	.0003	.0010	.0036
	.2	0	.0760	.1519	.1279	.1039	.0799	.0558	.0318	.0078	-.0163	-.0403	-.0352	-.0300	-.0248	-.0198	-.0147	-.0096	-.0044	.0007	.0058	.0108
	.3	0	.0643	.1285	.1928	.1571	.1214	.0856	.0499	.0142	-.0215	-.0573	-.0500	-.0427	-.0364	-.0282	-.0209	-.0136	-.0063	.0010	.0082	.0155
	.4	0	.0530	.1059	.1589	.2118	.1648	.1177	.0707	.0236	-.0234	-.0705	-.0615	-.0526	-.0436	-.0347	-.0257	-.0167	-.0078	.0012	.0101	.0191
	.5	0	.0421	.0843	.1264	.1685	.2107	.1528	.0949	.0371	-.0208	-.0787	-.0687	-.0587	-.0487	-.0387	-.0287	-.0187	-.0087	.0013	.0113	.0213
	.6	0	.0319	.0639	.0958	.1278	.1597	.1917	.1236	.0556	-.0125	-.0806	-.0703	-.0601	-.0498	-.0396	-.0294	-.0191	-.0089	.0013	.0116	.0218
	.7	0	.0225	.0450	.0675	.0900	.1126	.1351	.1576	.0801	.0026	-.0749	-.0654	-.0559	-.0463	-.0368	-.0273	-.0178	-.0083	.0012	.0108	.0203
	.8	0	.0140	.0279	.0419	.0558	.0698	.0837	.0977	.1117	.0256	-.0604	-.0527	-.0451	-.0374	-.0297	-.0220	-.0144	-.0067	.0010	.0087	.0164
	.9	0	.0064	.0128	.0192	.0257	.0321	.0385	.0449	.0513	.0577	-.0358	-.0313	-.0268	-.0222	-.0176	-.0131	-.0085	-.0040	.0006	.0052	.0097
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0077	-.0155	-.0232	-.0310	-.0387	-.0465	-.0542	-.0620	-.0697	-.0774	.0721	.0617	.0512	.0408	.0303	.0199	.0094	-.0010	-.0115	-.0219
	.2	0	-.0127	-.0253	-.0380	-.0507	-.0634	-.0760	-.0887	-.1014	-.1141	-.1267	.0090	.1448	.1206	.0964	.0722	.0480	.0238	-.0004	-.0247	-.0489
	.3	0	-.0152	-.0304	-.0456	-.0608	-.0760	-.0912	-.1064	-.1216	-.1368	-.1520	-.0325	.0870	.2064	.1659	.1254	.0849	.0444	.0039	-.0366	-.0771
	.4	0	-.0157	-.0315	-.0472	-.0629	-.0787	-.0944	-.1102	-.1259	-.1416	-.1574	-.0559	.0455	.1470	.2484	.1898	.1313	.0727	.0141	-.0444	-.1030
	.5	0	-.0147	-.0294	-.0441	-.0587	-.0734	-.0881	-.1028	-.1175	-.1322	-.1469	-.0644	.0180	.1004	.1828	.2652	.1876	.1100	.0324	-.0451	-.1227
	.6	0	-.0125	-.0249	-.0374	-.0498	-.0623	-.0748	-.0872	-.0997	-.1121	-.1246	-.0614	.0018	.0650	.1282	.1914	.2546	.1578	.0610	-.0358	-.1327
	.7	0	-.0095	-.0189	-.0284	-.0379	-.0473	-.0568	-.0663	-.0758	-.0852	-.0947	-.0501	-.0056	.0390	.0836	.1281	.1727	.2173	.1018	-.0136	-.1291
	.8	0	-.0061	-.0122	-.0184	-.0245	-.0306	-.0367	-.0429	-.0490	-.0551	-.0612	-.0339	-.0066	.0207	.0480	.0753	.1026	.1299	.1572	.0245	-.1082
	.9	0	-.0028	-.0057	-.0085	-.0113	-.0141	-.0170	-.0198	-.0226	-.0255	-.0283	-.0161	-.0039	.0083	.0204	.0326	.0448	.0570	.0692	.0814	-.0664
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0020	.0041	.0061	.0082	.0102	.0123	.0143	.0164	.0184	.0204	.0118	.0031	-.0056	-.0143	-.0230	-.0317	-.0404	-.0491	-.0577	-.0664
	.2	0	.0033	.0067	.0100	.0133	.0166	.0200	.0233	.0266	.0300	.0333	.0191	.0050	-.0092	-.0233	-.0375	-.0516	-.0658	-.0799	-.0941	-.1082
	.3	0	.0040	.0079	.0119	.0159	.0199	.0238	.0278	.0318	.0357	.0397	.0225	.0060	-.0109	-.0279	-.0447	-.0615	-.0784	-.0953	-.1122	-.1291
	.4	0	.0041	.0082	.0122	.0163	.0204	.0245	.0286	.0327	.0367	.0408	.0235	.0061	-.0112	-.0286	-.0459	-.0633	-.0806	-.0980	-.1153	-.1327
	.5	0	.0038	.0076	.0113	.0151	.0189	.0227	.0264	.0302	.0340	.0378	.0217	.0057	-.0104	-.0264	-.0425	-.0585	-.0746	-.0906	-.1067	-.1227
	.6	0	.0032	.0063	.0095	.0127	.0158	.0190	.0222	.0253	.0285	.0317	.0182	.0048	-.0087	-.0222	-.0356	-.0491	-.0626	-.0760	-.0895	-.1030
	.7	0	.0024	.0047	.0071	.0095	.0118	.0142	.0166	.0190	.0214	.0237	.0136	.0036	-.0065	-.0166	-.0267	-.0368	-.0469	-.0570	-.0670	-.0771
	.8	0	.0015	.0030	.0045	.0060	.0075	.0090	.0105	.0120	.0135	.0150	.0086	.0023	-.0041	-.0105	-.0169	-.0233	-.0297	-.0361	-.0425	-.0489
	.9	0	.0007	.0013	.0020	.0027	.0034	.0040	.0047	.0054	.0061	.0067	.0039	.0010	-.0019	-.0047	-.0076	-.0105	-.0133	-.0162	-.0191	-.0219
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0024	-.0027	-.0030	-.0017	-.0004	.0008	.0021	.0034	.0046	.0059	.0072	.0084	.0097
	.2	0	-.0005	-.0010	-.0015	-.0020	-.0025	-.0030	-.0035	-.0040	-.0045	-.0050	-.0029	-.0008	.0014	.0035	.0057	.0078	.0099	.0121	.0142	.0164
	.3	0	-.0006	-.0012	-.0019	-.0025	-.0031	-.0037	-.0044	-.0050	-.0056	-.0062	-.0036	-.0009	.0017	.0044	.0070	.0097	.0123	.0150	.0176	.0203
	.4	0	-.0007	-.0013	-.0020	-.0027	-.0034	-.0040	-.0047	-.0054	-.0060	-.0067	-.0039	-.0010	.0018	.0047	.0076	.0104	.0133	.0161	.0190	.0218
	.5	0	-.0007	-.0013	-.0020	-.0026	-.0033	-.0039	-.0046	-.0052	-.0059	-.0066	-.0038	-.0010	.0018	.0046	.0074	.0102	.0129	.0157	.0185	.0213
	.6	0	-.0006	-.0012	-.0018	-.0023	-.0029	-.0035	-.0041	-.0047	-.0053	-.0059	-.0034	-.0009	.0016	.0041	.0066	.0091	.0116	.0141	.0166	.0191
	.7	0	-.0005	-.0010	-.0014	-.0019	-.0024	-.0029	-.0033	-.0038	-.0043	-.0048	-.0027	-.0007	.0013	.0033	.0054	.0074	.0094	.0115	.0135	.0155
	.8	0	-.0003	-.0007	-.0010	-.0013	-.0017	-.0020	-.0023	-.0027	-.0030	-.0034	-.0019	-.0005	.0009	.0024	.0038	.0052	.0066	.0081	.0095	.0109
	.9	0	-.0002	-.0003	-.0005	-.0007	-.0009	-.0010	-.0012	-.0014	-.0016	-.0017	-.0010	-.0003	.0005	.0012	.0019	.0027	.0034	.0042	.0049	.0056
	E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0438	.0778	.1012	.1151	.1189	.1127	.0965	.0703	.0445	.0403	.0349	.0620	.1211	.1639	.1811	.1728	.1398	.0803	.0354	.0284
− Area		0	-.0161	-.0322	-.0483	-.0644	-.0805	-.0966	-.1127	-.1288	-.1554	-.2135	-.0989	-.0424	-.0435	-.0540	-.0644	-.0749	-.0853	-.0969	-.1476	-.2618
Total Area		0	.0277	.0454	.0530	.0507	.0384	.0161	-.0162	-.0585	-.1109	-.1732	-.0640	.0196	.0776	.1099	.1167	.0979	.0535	-.0166	-.1122	-.2334

Unit load at		REACTIONS/P					SHEARS/P							
		R_A	R_B	R_C	R_D	R_E	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
SPAN 1	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
	.1	.8792	.1372	-.0211	.0063	-.0017	.8792	-.1208	.0165	.0165	-.0046	-.0046	.0017	.0017
	.2	.7597	.2723	-.0406	.0123	-.0034	.7597	-.2403	.0320	.0320	-.0089	-.0089	.0034	.0034
	.3	.6427	.4028	-.0582	.0175	-.0048	.6427	-.3573	.0455	.0455	-.0127	-.0127	.0048	.0048
	.4	.5295	.5269	-.0718	.0215	-.0059	.5295	-.4705	.0560	.0560	-.0156	-.0156	.0059	.0059
	.5	.4213	.6412	-.0799	.0240	-.0066	.4213	-.5787	.0625	.0625	-.0174	-.0174	.0066	.0066
	.6	.3194	.7465	-.0818	.0245	-.0067	.3194	-.6806	.0640	.0640	-.0178	-.0178	.0067	.0067
	.7	.2251	.8344	-.0761	.0228	-.0062	.2251	-.7749	.0595	.0595	-.0166	-.0166	.0062	.0062
	.8	.1396	.9084	-.0614	.0184	-.0050	.1396	-.8604	.0480	.0480	-.0134	-.0134	.0050	.0050
	.9	.0641	.9644	-.0364	.0109	-.0030	.0641	-.9359	.0285	.0285	-.0079	-.0079	.0030	.0030
	B	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0	0	0
SPAN 2	.1	-.0774	1.0121	.0832	-.0247	.0067	-.0774	-.0774	.9347	-.0653	.0179	.0179	-.0067	-.0067
	.2	-.1267	.9754	.1813	-.0550	.0150	-.1267	-.1267	.8487	-.1513	.0399	.0399	-.0150	-.0150
	.3	-.1520	.8986	.3162	-.0868	.0237	-.1520	-.1520	.7468	-.2532	.0630	.0630	-.0237	-.0237
	.4	-.1574	.7913	.4502	-.1159	.0317	-.1574	-.1574	.6340	-.3660	.0842	.0842	-.0317	-.0317
	.5	-.1469	.6619	.5852	-.1381	.0378	-.1469	-.1469	.5151	-.4849	.1003	.1003	-.0378	-.0378
	.6	-.1246	.5196	.7135	-.1492	.0408	-.1246	-.1246	.3950	-.6050	.1084	.1084	-.0408	-.0408
	.7	-.0947	.3732	.8270	-.1452	.0397	-.0947	-.0947	.2785	-.7215	.1055	.1055	-.0397	-.0397
	.8	-.0612	.2319	.9178	-.1217	.0333	-.0612	-.0612	.1706	-.8294	.0884	.0884	-.0333	-.0333
	.9	-.0283	.1045	.9781	-.0747	.0204	-.0283	-.0283	.0762	-.9238	.0543	.0543	-.0204	-.0204
	C	0	0	1.0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 3	.1	.0204	-.0747	.9781	.1045	-.0283	.0204	.0204	-.0543	-.0543	.9283	-.0762	.0283	.0283
	.2	.0333	-.1217	.9178	.2319	-.0612	.0333	.0333	-.0884	-.0884	.8294	-.1706	.0612	.0612
	.3	.0397	-.1452	.8270	.3732	-.0947	.0397	.0397	-.1055	-.1055	.7215	-.2785	.0947	.0947
	.4	.0408	-.1492	.7135	.5196	-.1246	.0408	.0408	-.1084	-.1084	.6050	-.3950	.1246	.1246
	.5	.0378	-.1381	.5852	.6619	-.1469	.0378	.0378	-.1003	-.1003	.4849	-.5151	.1469	.1469
	.6	.0317	-.1159	.4502	.7913	-.1574	.0317	.0317	-.0842	-.0842	.3660	-.6340	.1574	.1574
	.7	.0237	-.0868	.3162	.8986	-.1520	.0237	.0237	-.0630	-.0630	.2532	-.7468	.1520	.1520
	.8	.0150	-.0550	.1913	.9754	-.1267	.0150	.0150	-.0399	-.0399	.1513	-.8487	.1267	.1267
	.9	.0067	-.0247	.0832	1.0121	-.0774	.0067	.0067	-.0179	-.0179	.0653	-.9347	.0774	.0774
	D	0	0	0	1.0	0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 4	.1	-.0030	.0109	-.0364	.9644	.0641	-.0030	-.0030	.0079	.0079	-.0285	-.0285	.9359	-.0641
	.2	-.0050	.0184	-.0614	.9084	.1396	-.0050	-.0050	.0134	.0134	-.0480	-.0480	.8604	-.1396
	.3	-.0062	.0228	-.0761	.8344	.2251	-.0062	-.0062	.0166	.0166	-.0595	-.0595	.7749	-.2251
	.4	-.0067	.0245	-.0818	.7445	.3194	-.0067	-.0067	.0178	.0178	-.0640	-.0640	.6806	-.3194
	.5	-.0066	.0240	-.0799	.6412	.4213	-.0066	-.0066	.0174	.0174	-.0625	-.0625	.5787	-.4213
	.6	-.0059	.0215	-.0716	.5265	.5295	-.0059	-.0059	.0156	.0156	-.0560	-.0560	.4705	-.5295
	.7	-.0048	.0175	-.0582	.4028	.6427	-.0048	-.0048	.0127	.0127	-.0455	-.0455	.3573	-.6427
	.8	-.0034	.0123	-.0409	.2723	.7597	-.0034	-.0034	.0089	.0089	-.0320	-.0320	.2403	-.7597
	.9	-.0017	.0063	-.0211	.1373	.8792	-.0017	-.0017	.0046	.0046	-.0165	-.0165	.1208	-.8792
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4878	1.5826	1.7818	1.5826	.4878	.4878	.0403	.8694	.0533	.8909	.1070	.7135	.1610
− Area		-.1610	-.1473	-.1065	-.1473	-.1610	-.1610	-.7135	-.1070	-.8909	-.0533	-.8694	-.0403	-.4878
Total Area		.3268	1.4355	1.6753	1.4353	.3268	.3268	-.6732	.7624	-.8376	.8376	-.7624	.6732	-.3268



Influence coefficients — Four continuous spans.

L = Length of EXTERIOR spans; length of interior spans = NL

N = 1.6

Table 20.44

Unit load at		MOMENTS/PL																				
		SPAN 1											SPAN 2									
		A	.1	.2	.3	.4	.5	.6	.7	.8	.9	B	.1	.2	.3	.4	.5	.6	.7	.8	.9	C
SPAN 1	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	.1	0	.0880	.0760	.0640	.0520	.0400	.0280	.0160	.0040	-.0080	-.0200	-.0175	-.0149	-.0124	-.0099	-.0073	-.0048	-.0022	.0003	.0029	.0054
	.2	0	.0761	.1522	.1283	.1044	.0806	.0567	.0328	.0089	-.0150	-.0389	-.0339	-.0290	-.0240	-.0191	-.0142	-.0092	-.0043	.0007	.0056	.0106
	.3	0	.0645	.1289	.1934	.1579	.1224	.0868	.0513	.0158	-.0198	-.0553	-.0483	-.0412	-.0342	-.0272	-.0201	-.0131	-.0061	.0009	.0080	.0150
	.4	0	.0532	.1064	.1596	.2128	.1660	.1192	.0724	.0256	-.0212	-.0680	-.0594	-.0507	-.0421	-.0334	-.0248	-.0161	-.0075	.0012	.0098	.0185
	.5	0	.0424	.0848	.1272	.1696	.2120	.1544	.0968	.0393	-.0183	-.0759	-.0663	-.0566	-.0470	-.0373	-.0277	-.0180	-.0084	.0013	.0110	.0206
	.6	0	.0322	.0644	.0967	.1289	.1611	.1933	.1256	.0578	-.0100	-.0778	-.0679	-.0580	-.0481	-.0382	-.0283	-.0184	-.0086	.0013	.0112	.0211
	.7	0	.0228	.0455	.0683	.0911	.1139	.1366	.1594	.0822	.0049	-.0723	-.0631	-.0539	-.0447	-.0355	-.0263	-.0171	-.0080	.0012	.0104	.0196
	.8	0	.0142	.0283	.0425	.0567	.0708	.0850	.0992	.1133	.0275	-.0583	-.0509	-.0435	-.0361	-.0287	-.0212	-.0138	-.0064	.0010	.0084	.0158
	.9	0	.0066	.0131	.0196	.0262	.0327	.0392	.0458	.0523	.0588	-.0346	-.0302	-.0258	-.0214	-.0170	-.0126	-.0082	-.0038	.0006	.0050	.0094
	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 2	.1	0	-.0084	-.0169	-.0253	-.0337	-.0422	-.0506	-.0590	-.0675	-.0759	-.0843	.0748	.0640	.0531	.0423	.0314	.0206	.0098	-.0011	-.0119	-.0228
	.2	0	-.0138	-.0276	-.0414	-.0552	-.0690	-.0828	-.0966	-.1104	-.1242	-.1380	.0067	.1513	.1260	.1007	.0754	.0501	.0248	-.0005	-.0258	-.0511
	.3	0	-.0166	-.0331	-.0497	-.0662	-.0828	-.0993	-.1159	-.1324	-.1490	-.1656	-.0381	.0894	.2168	.1743	.1317	.0892	.0466	.0041	-.0385	-.0810
	.4	0	-.0171	-.0343	-.0514	-.0685	-.0857	-.1028	-.1199	-.1371	-.1542	-.1714	-.0631	.0452	.1535	.2618	.2001	.1384	.0766	.0149	-.0468	-.1085
	.5	0	-.0160	-.0320	-.0480	-.0640	-.0799	-.0959	-.1119	-.1279	-.1439	-.1599	-.0719	.0162	.1042	.1922	.2803	.1983	.1163	.0343	-.0476	-.1296
	.6	0	-.0136	-.0271	-.0407	-.0543	-.0678	-.0814	-.0950	-.1085	-.1221	-.1356	-.0681	-.0006	.0669	.1345	.2020	.2695	.1671	.0646	-.0379	-.1403
	.7	0	-.0103	-.0206	-.0309	-.0412	-.0515	-.0618	-.0721	-.0825	-.0928	-.1031	-.0554	-.0078	.0398	.0875	.1351	.1827	.2304	.1080	-.0144	-.1367
	.8	0	-.0067	-.0133	-.0200	-.0266	-.0333	-.0400	-.0466	-.0533	-.0600	-.0666	-.0374	-.0083	.0209	.0501	.0793	.1085	.1377	.1669	.0260	-.1148
	.9	0	-.0031	-.0062	-.0092	-.0123	-.0154	-.0185	-.0215	-.0246	-.0277	-.0308	-.0178	-.0047	.0083	.0213	.0343	.0474	.0604	.0734	.0864	-.0705
	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 3	.1	0	.0022	.0044	.0067	.0089	.0111	.0133	.0155	.0178	.0200	.0222	.0129	.0037	-.0056	-.0149	-.0242	-.0334	-.0427	-.0520	-.0613	-.0706
	.2	0	.0036	.0072	.0108	.0145	.0181	.0217	.0253	.0289	.0325	.0361	.0210	.0060	-.0091	-.0242	-.0393	-.0544	-.0695	-.0846	-.0997	-.1148
	.3	0	.0043	.0086	.0129	.0172	.0215	.0258	.0301	.0344	.0387	.0430	.0251	.0071	-.0109	-.0289	-.0468	-.0648	-.0828	-.1008	-.1187	-.1367
	.4	0	.0044	.0088	.0133	.0177	.0221	.0265	.0309	.0353	.0398	.0442	.0257	.0073	-.0112	-.0296	-.0481	-.0665	-.0850	-.1034	-.1219	-.1403
	.5	0	.0041	.0082	.0122	.0163	.0204	.0245	.0286	.0326	.0367	.0408	.0238	.0067	-.0103	-.0274	-.0444	-.0614	-.0785	-.0955	-.1126	-.1296
	.6	0	.0034	.0068	.0103	.0137	.0171	.0205	.0239	.0273	.0307	.0342	.0199	.0056	-.0086	-.0229	-.0372	-.0514	-.0657	-.0800	-.0942	-.1085
	.7	0	.0026	.0051	.0077	.0102	.0128	.0153	.0179	.0204	.0230	.0255	.0149	.0042	-.0065	-.0171	-.0278	-.0384	-.0491	-.0597	-.0704	-.0810
	.8	0	.0016	.0032	.0048	.0064	.0080	.0097	.0113	.0129	.0145	.0161	.0094	.0026	-.0041	-.0108	-.0175	-.0242	-.0310	-.0377	-.0444	-.0511
	.9	0	.0007	.0014	.0022	.0029	.0036	.0043	.0050	.0057	.0085	.0072	.0042	.0012	-.0018	-.0048	-.0078	-.0108	-.0138	-.0168	-.0198	-.0228
	D	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
SPAN 4	.1	0	-.0003	-.0006	-.0009	-.0012	-.0015	-.0018	-.0021	-.0024	-.0027	-.0030	-.0017	-.0005	.0007	.0020	.0032	.0045	.0057	.0069	.0082	.0094
	.2	0	-.0005	-.0010	-.0015	-.0020	-.0025	-.0030	-.0035	-.0040	-.0045	-.0050	-.0029	-.0008	.0013	.0033	.0054	.0075	.0096	.0117	.0137	.0158
	.3	0	-.0006	-.0012	-.0019	-.0025	-.0031	-.0037	-.0043	-.0049	-.0056	-.0062	-.0036	-.0010	.0016	.0041	.0067	.0093	.0119	.0145	.0170	.0196
	.4	0	-.0007	-.0013	-.0020	-.0027	-.0033	-.0040	-.0046	-.0053	-.0060	-.0066	-.0039	-.0011	.0017	.0045	.0072	.0100	.0128	.0156	.0183	.0211
	.5	0	-.0006	-.0013	-.0019	-.0026	-.0032	-.0039	-.0045	-.0052	-.0058	-.0065	-.0038	-.0011	.0016	.0044	.0071	.0098	.0125	.0152	.0179	.0206
	.6	0	-.0006	-.0012	-.0017	-.0023	-.0029	-.0035	-.0041	-.0046	-.0052	-.0058	-.0034	-.0010	.0015	.0039	.0063	.0088	.0112	.0136	.0160	.0185
	.7	0	-.0005	-.0009	-.0014	-.0019	-.0024	-.0028	-.0033	-.0038	-.0042	-.0047	-.0027	-.0008	.0012	.0032	.0051	.0071	.0091	.0111	.0130	.0150
	.8	0	-.0003	-.0007	-.0010	-.0013	-.0017	-.0020	-.0023	-.0027	-.0030	-.0033	-.0019	-.0005	.0008	.0022	.0036	.0050	.0064	.0078	.0092	.0106
	.9	0	-.0002	-.0003	-.0005	-.0007	-.0009	-.0010	-.0012	-.0014	-.0015	-.0017	-.0010	-.0003	.0004	.0011	.0019	.0026	.0033	.0040	.0047	.0054
	E	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
+ Area		0	.0446	.0791	.1037	.1182	.1228	.1174	.1019	.0765	.0503	.0462	.0395	.0683	.1336	.1822	.2019	.1927	.1546	.0887	.0367	.0275
- Area		0	-.0186	-.0371	-.0557	-.0742	-.0928	-.1113	-.1299	-.1484	-.1762	-.2362	-.1070	-.0423	-.0430	-.0559	-.0688	-.0816	-.0945	-.1085	-.1653	-.2938
Total Area		0	.0260	.0420	.0480	.0440	.0300	.0061	-.0280	-.0718	-.1259	-.1900	-.0675	.0260	.0906	.1263	.1331	.1111	.0601	-.0198	-.1286	-.2663

Unit load at		REACTIONS/P					SHEARS/P							
		R_A	R_B	R_C	R_D	R_E	V_{AB}	V_{BA}	V_{BC}	V_{CB}	V_{CD}	V_{DC}	V_{DE}	V_{ED}
SPAN 1	A	1.0	0	0	0	0	1.0	0	0	0	0	0	0	0
	.1	.8800	.1350	-.0192	.0059	-.0017	.8800	-.1200	.0150	.0150	-.0042	-.0042	.0017	.0017
	.2	.7611	.2680	-.0372	.0115	-.0033	.7611	-.2389	.0291	.0291	-.0082	-.0082	.0033	.0033
	.3	.6447	.3986	-.0529	.0163	-.0047	.6447	-.3553	.0413	.0413	-.0116	-.0116	.0047	.0047
	.4	.5320	.5189	-.0652	.0201	-.0058	.5320	-.4680	.0509	.0509	-.0143	-.0143	.0058	.0058
	.5	.4241	.6327	-.0727	.0224	-.0065	.4241	-.5759	.0568	.0568	-.0159	-.0159	.0065	.0065
	.6	.3222	.7359	-.0745	.0230	-.0066	.3222	-.6778	.0581	.0581	-.0163	-.0163	.0066	.0066
	.7	.2277	.8263	-.0692	.0213	-.0062	.2277	-.7723	.0541	.0541	-.0152	-.0152	.0062	.0062
	.8	.1417	.9019	-.0558	.0172	-.0050	.1417	-.8583	.0436	.0436	-.0122	-.0122	.0050	.0050
	.9	.0654	.9605	-.0332	.0102	-.0030	.0654	-.9346	.0259	.0259	-.0073	-.0073	.0030	.0030
	B	0	1.0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0	0	0
SPAN 2	.1	-.0843	1.0206	.0814	-.0249	.0072	-.0843	-.0843	.9362	-.0638	.0176	.0176	-.0072	-.0072
	.2	-.1380	.9892	.1884	-.0556	.0161	-.1380	-.1390	.8511	-.1489	.0395	.0395	-.0161	-.0161
	.3	-.1656	.9153	.3129	-.0882	.0255	-.1656	-.1656	.7497	-.2503	.0627	.0627	-.0255	-.0255
	.4	-.1714	.8083	.4469	-.1181	.0342	-.1714	-.1714	.6370	-.3630	.0839	.0839	-.0342	-.0342
	.5	-.1599	.6777	.5624	-.1410	.0408	-.1599	-.1599	-.5178	-.4822	.1002	.1002	-.0408	-.0408
	.6	-.1356	.5329	.7113	-.1527	.0442	-.1356	-.1356	.3972	-.6028	.1085	.1085	-.0442	-.0442
	.7	-.1031	.3833	.8255	-.1488	.0430	-.1031	-.1031	.2802	-.7198	.1087	.1087	-.0430	-.0430
	.8	-.0666	.2383	.9171	-.1249	.0361	-.0666	-.0666	.1717	-.8283	.0888	.0888	-.0361	-.0361
	.9	-.0308	.1074	.9779	-.0768	.0222	-.0308	-.0308	.0766	-.9234	.0546	.0546	-.0222	-.0222
	C	0	0	1.0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0	0	0
SPAN 3	.1	.0222	-.0768	.9779	.1074	-.0308	.0222	.0222	-.0546	-.0546	.9234	-.0766	.0308	.0308
	.2	.0361	-.1249	.9171	.2383	-.0666	.0361	.0361	-.0888	-.0888	.8283	-.1717	.0666	.0666
	.3	.0430	-.1488	.8255	.3833	-.1031	.0430	.0430	-.1057	-.1057	.7198	-.2802	.1031	.1031
	.4	.0442	-.1527	.7113	.5329	-.1356	.0442	.0442	-.1085	-.1085	.6028	-.3972	.1356	.1356
	.5	.0408	-.1410	.5624	.6777	-.1599	.0408	.0408	-.1002	-.1002	.4822	-.5178	.1599	.1599
	.6	.0342	-.1181	.4469	.8083	-.1714	.0342	.0342	-.0839	-.0839	.3630	-.6370	.1714	.1714
	.7	.0255	-.0882	.3129	.9153	-.1656	.0255	.0255	-.0627	-.0672	.2503	-.7497	.1656	.1656
	.8	.0161	-.0556	.1884	.9892	-.1380	.0161	.0161	-.0395	-.0395	.1489	-.8511	.1380	.1380
	.9	.0072	-.0249	.0814	1.0206	-.0843	.0072	.0072	-.0176	-.0176	.0638	-.9362	.0843	.0843
	D	0	0	0	1.0	0	0	0	0	0	0	-1.0 / 0	0 / 1.0	0
SPAN 4	.1	-.0030	.0102	-.0332	.9605	.0654	-.0030	-.0030	.0073	.0073	-.0259	-.0259	.9346	-.0654
	.2	-.0050	.0172	-.0558	.9019	.1417	-.0050	-.0050	.0122	.0122	-.0436	-.0436	.8583	-.1417
	.3	-.0062	.0213	-.0692	.8263	.2277	-.0062	-.0062	.0152	.0152	-.0541	-.0541	.7723	-.2277
	.4	-.0066	.0230	-.0745	.7359	.3222	-.0066	-.0066	.0163	.0163	-.0581	-.0581	.6778	-.3222
	.5	-.0065	.0224	-.0727	.6327	.4241	-.0065	-.0065	.0159	.0159	-.0568	-.0568	.5759	-.4241
	.6	-.0058	.0201	-.0652	.5189	.5320	-.0058	-.0058	.0143	.0143	-.0509	-.0509	.4680	-.5320
	.7	-.0047	.0163	-.0529	.3986	.6447	-.0047	-.0047	.0116	.0116	-.0413	-.0413	.3553	-.6447
	.8	-.0033	.0115	-.0372	.2680	.7611	-.0033	-.0033	.0082	.0082	-.0291	-.0291	.2389	-.7611
	.9	-.0017	.0059	-.0192	.1350	.8800	-.0017	-.0017	.0042	.0042	-.0150	-.0150	.1200	-.8800
	E	0	0	0	0	1.0	0	0	0	0	0	0	0	-1.0
+ Area		.4956	1.6548	1.8868	1.6548	.4956	.4956	.0462	.8187	.0485	.9434	.1136	.7361	.1855
- Area		-.1855	-.1598	-.0970	-.1598	-.1855	-.1855	-.7361	-.1136	-.9434	-.0485	-.8187	-.0462	-.4956
Total Area		.3101	1.4950	1.7898	1.4950	.3101	.3101	-.6899	.8051	-.8949	.8949	-.8051	.6899	-.3101

.1 .2 .3 .4 .5 .6 .7 .8 .9 .1 .2 .3 .4 .5 .6 .7 .8 .9 .1 .2 .3 .4 .5 .6 .7 .8 .9 .1 .2 .3 .4 .5 .6 .7 .8 .9

A Span 1 B Span 2 C Span 3 D Span 4 E

Influence coefficients — Four continuous spans.

L = Length of EXTERIOR spans; length of interior spans = NL.

N = 1.7

CHAPTER 21

Baker's Method for Ultimate Load Analysis of Indeterminate Concrete Structures

Use may be made of the flexibility method of structural analysis extended to loading beyond elastic behaviour of the structure but allowing plastic rotations at the hinges to within certain empirically predetermined permissible values. Concrete, being far less ductile as compared to steel, makes the estimation of allowable plastic rotations at hinges and of the average effective flexural rigidity of the cracked members over their lengths very difficult. Consequently, the solution to the problem, being only as accurate as the empirical evaluation of the above items, may not always be significantly close to truth. This is an established fact in the case of concrete.

For an n times statically indeterminate structure loaded to beyond its elastic behavioural capacity, assuming all n hinges are formed, the n compatibility equations of inelastic stability just prior to collapse of the structure are

$$\begin{aligned} V_{10} + V_{11}p_1 + V_{12}p_2 + \ldots + V_{1n}p_n &= -\theta_{p_1} \\ V_{20} + V_{21}p_1 + V_{22}p_2 + \ldots + V_{2n}p_n &= -\theta_{p_2} \\ V_{n0} + V_{n1}p_1 + V_{n2}p_2 + \ldots + V_{nn}p_n &= -\theta_{p_n} \end{aligned}$$

which may also be simply written as

$$V_{r0} + V_{rs}\,p_s = -\theta_{pr}$$

r ranging from 1 to n

s ranging from 1 to n

where

$V_{r0} = \int \dfrac{m_r m_0}{EI}\,ds$ = the movement at release no: r due to applied loading (designated with suffix zero)

$V_{rs} = \int \dfrac{m_r m_s}{EI}\,ds$ = the movement at release no: r due to unit value of plastic movement (bi-action) p applied at (release) hinge no: s

θ_{pr} = permissible plastic rotation at release (or hinge) no: r. $p_1, p_2, p_3, \ldots, p_n$, $m_1, m_2, m_3, \ldots, m_n$ and m_0 have already been defined in Ch. 20 (flexibility method of structural analysis).

EI is the effective modulus of rigidity (average of section to section) in a member assumed cracked as the structure is considered near a state of collapse.

Procedure

Step 1 Calculate the number of statical indeterminacies in the structure. Introduce as many releases into the structure, seeing that it remains statically stable. Draw $m_1, m_2, \ldots, m_n$ diagrams, the bending moment diagrams due to unit values of unknown biactions $p_1, p_2, \ldots, p_n$, respectively, the latter representing the plastic moments at hinges $1, 2, \ldots, n$, respectively, in this case on the statically made determinate structure. Draw m_0 diagram, the bending moment diagram due to applied loading, on the statically made determinate structure. (For all this follow the same procedure as in steps 1–4 in Ch. 20.)

Step 2 Work out θ_p values (i.e., permissible plastic rotations) for each (release) hinge from the empirical formulae suggested by Baker.[1]

Step 3 Work out EI values for each beam-member and column-member from the empirical formulae suggested by Baker.[1]

Step 4 Work out the influence coefficients V_{r0} and V_{rs} (r = 1 to n, s = 1 to n).

Step 5 Work out the first-trial values of the plastic moments $p_1, p_2, \ldots, p_n$, from the empirical formulae suggested by Baker.[1]

Step 6 Substituting the values worked out in steps 4 and 5, evaluate the left hand sides of the n compatibility equations mentioned earlier above. If these fall within their respective permissible values established in step 2 above, it is OK, if not, change one or some or (if necessary) all the assumed first trial values of $p_1, p_2, \ldots, p_n$, until the values worked out from the left hand sides of the said n compatibility equations fall within their respective θ_p values (established in step 2). The compatible $p_1, p_2, \ldots, p_n$ values are thus established (representing one possible compatible solution).

Step 7 Total ultimate moment at any section (immediately prior to collapse of the structure) can now be readily estimated from

$$m_t = m_0 + (m_1 p_1 + m_2 p_2 + \cdots + m_n p_n)$$

where $m_0, m_1, m_2, \ldots, m_n$ are the ordinates of these diagrams at that particular section.

REFERENCES

1. Baker, ALL, *Ultimate Load Design*. Concrete Publications, London.
2. Raina, VK,'Analysis of a Multistorey Multibay Reinforced Concrete Frame by Baker's Ultimate Load Method". Internal Teach-In, Unpublished, Imperial College, London.

CHAPTER 22

Effect of Differential Settlement of Supports in a Statically Indeterminate Structure

22.1 EFFECTS OF (AN ASSUMED) PIER SETTLEMENT ON THE MOMENTS IN THE SUPERSTRUCTURE

In the simple case shown in Fig. 22.1 where a continuous beam of constant depth with a large number of identical spans is subjected to the settlement of one pier by a given amount δ, one may easily derive the effect in terms of moments and stresses in the superstructure. Taking the fixed end moment $M' = 6EI\delta/l^2$, the moments over the piers, at midspan and at quarter-span sections are

Over the pier subjected to settlement	$+0.732M'$
Over the adjacent piers	$-0.464M'$
Midspan moment	$+0.134M'$
Quarter-span amount	$+0.433M'$

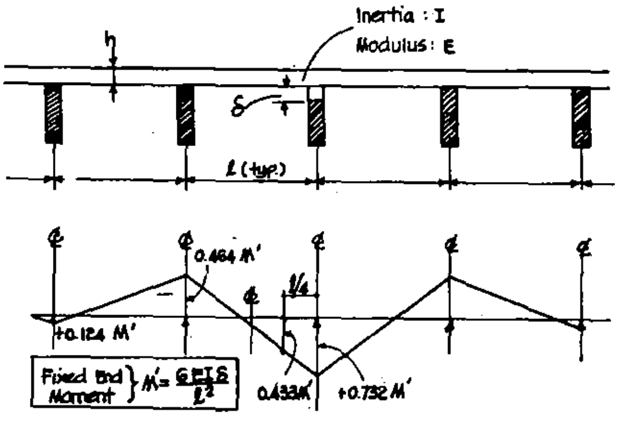


Fig. 22.1 *Effect of differential settlement on a continuous beam with equal spans and constant depth*

The stress produced at an extreme fibre at a section in the superstructure where moment is M, is $f = Mc/I$, where c is the distance between the centroid and the extreme fibre. If the moment is expressed as $M = kM'$, the stress becomes,

$$f = k\frac{6Ec\delta}{l^2}$$

which can be rewritten as follows

$$f = 6kE\frac{c}{h}\frac{h}{l}\frac{\delta}{l}$$

The value of c/h varies between 0.4 and 0.6 and that of h/l between $\frac{1}{18}$ and $\frac{1}{22}$.

Considering the quarter-span section close to the pier where settlement occurred, the stress in the bottom fibre in the superstructure will be, with $k = 0.433$ and E for concrete as 300,000 kips/ft^2 (for longterm loading), and assuming $\frac{c}{h} = 0.55$ and $\frac{h}{l} = \frac{1}{18.32}$,

$$f = 23,400\frac{\delta}{l} \qquad \text{(approx.)}$$

For a settlement $\delta = \frac{l}{1000}$, the stress is equal to 23.4 kips/ft^2 (11.4 kg/cm^2) at the bottom fiber, a very nominal value!

For a 100 ft span, the corresponding settlement is $\delta = 0.1$ ft = 1.2 inches (3 cm approx.).

The amount of settlement to be considered actually is only the part taking place after continuity is achieved in the deck.

In conclusion it can be readily seen that for the same stress effect, a longer span can take bigger differential settlement and conversely even a relatively small differential settlement can produce a significant stress in the deck in a relatively short span. In practice, aggregate additive moments (and shears) should be considered at each critical section due to possible settlement of each support individually.

22.2 CALCULATING THE EFFECT OF DIFFERENTIAL SETTLEMENT OF SUPPORTS IN A STATICALLY INDETERMINATE STRUCTURE BY THE FLEXIBILITY METHOD

If the structure is statically determinate the support settlements do not set up any moments.

Let an n times statically indeterminate structure be called S_n and the statically made determinate structure be called S_0. Consider a 2-span continuous beam ABC in which support B has sunk by δ vertically, [Fig. 22.2(a)] causing an angular movement (at B) of $\theta = \theta_1 + \theta_2 = \frac{2\delta}{l}$. There are two ways of finding the bending moments caused by support sinking.

First method Imagine a shear-type biaction p_1 [Fig. 22.2(b)] is introduced at B as shown, to make $\delta = 0$. Thus, (movement at B due to p_1) + $\delta = 0$, or $V_{11}p_1 + \delta = 0$, where $V_{11} = \int \frac{m_1 m_1}{EI} ds$, m_1 being the BMD due to $p_1 = 1$ on S_0.

$\therefore p_1$ can be known and total moment caused at any section (by the sinking of the support) found out from the product $m_1 p_1$ at that section.

Second method Imagine, instead, a moment-type biaction p_1 [Fig. 22.2(c)] is introduced at B.

For compatibility of deformations:

$$V_{11}\, p_1 + \theta = 0,$$

where

$$V_{11} = \int \frac{m_1 m_1}{EI} ds, m_1 \text{ being BMD on } S_0 \text{ due to } p_1 = 1.$$

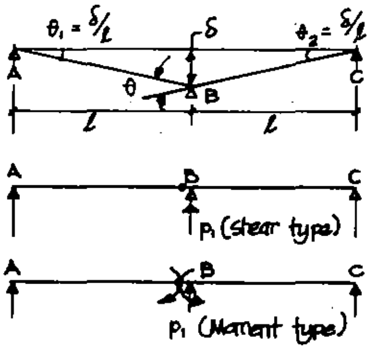


Fig. 22.2

$\therefore p_1$ can be known, and total moment caused at any section (by the sinking of support) found out from the product $m_1 p_1$ at that section.

Now, let us consider a 3-span continuous beam $ABCD$ where supports B and C sink by amounts δ_1 and δ_2 as shown in Fig. 22.3 causing angular movements θ_1 and θ_2 at B and C.

$$\left.\begin{aligned} \theta_1 &= \alpha - \beta = \frac{\delta_1}{l_1} - \frac{\delta_2 - \delta_1}{l_2} \\ \theta_2 &= \gamma + \beta = \frac{\delta_2}{l_3} + \frac{\delta_2 - \delta_1}{l_2} \end{aligned}\right\}, \text{ hence known.}$$

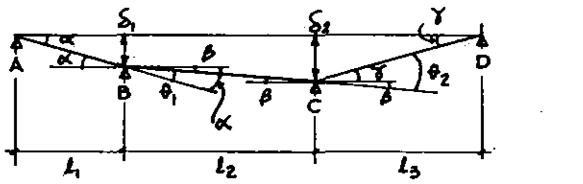


Fig. 22.3

Here again, if we introduce moment-biactions p_1 and p_2 at B and C, causing angular movements of $(V_{11}p_1 + V_{12}p_2)$ at B and $(V_{21}p_1 + V_{22}p_2)$ at C
where

$$V_{11} = \int \frac{m_1 m_1}{EI} ds, \quad V_{22} = \int \frac{m_2 m_2}{EI} ds,$$

$$V_{12} = V_{21} = \int \frac{m_1 m_2}{EI} ds,$$

m_1 being BMD on S_o due to unit p_1, etc.

Then, for total movements at B and C individually to be zero for compatibility of deformations, we have
at B: $(V_{11}p_1 + V_{12}p_2) + \theta_1 = 0$
at C: $(V_{21}p_1 + V_{22}p_2) + \theta_2 = 0$
whence p_1 and p_2 can be found, and total moment (due to support-sinking) at any section evaluated from

$$(m_1 p_1 + m_2 p_2)$$

Numerical Example

EI constant, 2 span continuous beam, $n = 1$ (Fig. 22.4) due to p_1, angular movement at B is

$$V_{11}p_1 = \left(\int \frac{m_1 m_1}{EI} ds\right) p_1$$

where

$$\int \frac{m_1 m_1}{EI} ds = \frac{2}{EI}\left[\frac{20}{6}\left(0 + 4 \times \frac{1}{2} \times \frac{1}{2} + 1\right)\right]$$

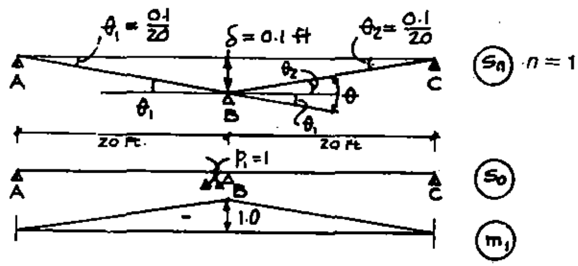


Fig. 22.4 *Numerical example*

$$= \frac{2}{3} \times \frac{20}{EI}$$

$$\theta \text{ (due to } \delta) = \frac{0.1}{20} + \frac{0.1}{20} = 0.01 \text{ radian}$$

from $(V_{11}p_1 + \theta) = 0$, we get $\frac{2}{3} \times \frac{20}{EI} p_1 + 0.01 = 0$

$$p_1 = -\frac{3EI}{4000}$$

Hence total moment (due to support sinking), m^t, at different sections:

$$m_A^t = m_1 p_1 = 0$$

$$m_B^t = m_1 p_1 = (-1) \times \left(-\frac{3EI}{4000}\right) = \frac{3EI}{4000}$$

$$m_C^t = m_1 p_1 = 0$$

CHAPTER 23

Reinforced Concrete Design

23.1 GENERAL BACKGROUND AND PRINCIPLE OF REINFORCED CONCRETE DESIGN

As explained earlier, the load-deformation curve for reinforced concrete is linear during the early stages of its loading history — which is the elastic phase — and thereafter, the deformation increases much more rapidly through the elastoplastic phase to the plastic phase.

The design of reinforced concrete based on the elastic behaviour (i.e., corresponding only to the early elastic phase) is based on five basic assumptions, viz., the stress strain curve is a straight line, plane sections remain plane even after bending (i.e., strains at various fibre levels down any section remain linearly compatible with each other), and the material is homogeneous, isotropic and crackfree. The first assumption is not really true if the total load-deformation curve is kept in view (in case of any unforeseen distress, the deformational response may well go past the linear elastic phase). The third, the fourth and the fifth assumptions are unrealistic in the context of structural concrete, more so in reinforced concrete in which cracking (within limits) is incumbent by virtue of the neutral axis concept. Micro-cracking in concrete commences right from the instant of its initial setting, and as the applied load effect increases, visible cracking sets in even in prestressed concrete, invalidating the rationality of the perpetuality of the above assumptions. This being so, it is very important to ponder whether using the formulae which are derived on the basis of the aforementioned assumptions can hence yield results which may be irrational vis-a-vis the actual stresses in the structure. It is, therefore, more realistic to accept the fact that structural concrete is partially cracked and design it for a loading that is higher than the working load, even if higher material stresses have to be allowed. This higher load could be a certain factored value of the working load, but still lower than the ultimate load that would cause the physical collapse of the structure.

This approach is particularly necessary for designing against those of the load effects under which the collapse could be sudden (for example shear and torsion) because these effects are associated with very little redistribution. This is why there is no increase in allowable stress under such load effects. Unlike shear (and torsion) fortunately the bending moment undergoes redistribution, and unless the sections are over-reinforced (which is unusual), the structure will accept enough visible deformation in high moment zones (Which behave like hinges) and thereby, give enough warning of impending failure. This, however, will depend upon the ductility of the structure in these high moment zones. Otherwise these zones will not 'rotate' enough, thus leading to shortfall in moment redistribution and an earlier collapse without the other hinging zones in the structure maturing fully. (Full maturity of a hinge means that its critical section develops its full ultimate moment capacity).

'Elastic' Versus 'Load-Factor' Approach

(i) In *elastic* (i.e., working load) analysis of reinforced concrete, flexural compressive stress in concrete is always assumed to be a certain fraction of its crushing strength and tensile stress in steel is assumed to be a certain fraction of its yield strength. The saftey factor is then defined as the ratio between failure stress and working stress. This, however, does not mean that if stress in concrete is slightly exceeded, the concrete will crush. Nor does it mean that concrete will not crush until load is increased by the ratio between the crushing strength of concrete and this stress. The former is true since reasonable over-stressing is found to be possible. The latter is true, since, beyond elastic range, the stress-strain relation is non-linear, whereby ultimate compressive strain in concrete develops while the compressive strain is still less than its crushing strength. Thus it is apparent that elastic analysis is conservative in its outlook and does not consider the structural performance beyond the elastic range.

(ii) In *plastic* (i.e., ultimate load) analysis it is assumed that the member is behaving plastically or non-linearly (past the elastic range), and that it is loaded to its ultimate such that it is about to collapse and excessive deformation has set in certain critical zones. This excessive deformation constitutes yield lines in slabs and plastic hinges in columns and beams. Thus, it is apparent that plastic (i.e., ultimate load) analysis is perhaps too broad and risky an outlook.

(iii) In the *load factor approach* the assumed applied load, while still a certain fraction of the ultimate load, is assumed to be a certain factor times the working load. This type of analysis, in which the permissible stresses in concrete and steel are also 'load-factored' values of the elastic stresses permitted in them hitherto, is an analysis intermediate between elastic and ultimate concepts. This type of design analysis, called Load Factor Analysis is, therefore, far more logical and reasonable to apply.

(iv) The basic difference between the elastic and load-factor methods of analysis is that with the elastic method we ensure that the structure will behave satisfactorily at working load but only assume that hopefully there will be a satisfactory factor of safety against failure, whereas with the load factor method we ensure that we have a satisfactory factor against failure but only assume that the structure will behave satisfactorily at working load.

(v) It follows then, that if a Load Factor approach is followed (when the section is designed for a load effect equal to suitably factored working load value, adopting load-factored material stresses, and the two sets of load factors may be different), at least certain serviceability criteria must additionally be ensured in order to guard against any unsightly deformations like large crack-widths and deflections.

(vi) This load factor method, intermediate between working load and ultimate load, (sometimes loosely referred to as ultimate strength method by some) should be adopted as much for its consequence after construction as for its refinements in logic and practical precision during design.

Acceptance of the Concepts

It took decades to accept the load factor approach in the design of reinforced concrete building elements. It is taking longer to accept this approach for the design of bridge elements, possibly because the bridge school feels, that unless the values for load factors on various loads and material stresses are established beyond any shadow of doubt, any such refinement in design method—without a matching assurance on quality control and possible load-variation, may lead to collapse of some bridges, and that this can be much more catastrophic than the partial collapse of a building. This may, however, be only part of the reason for resistance to acceptance, the other reasons being the average designer's limited understanding and design capability coupled with lack of inertial impulse needed to get out of the beaten-track. Fortunately, at least for shear (and torsion), the load factor approach has now been accepted by many a bridge-code, perhaps more because shear (and torsion) failures can be ruthlessly sudden, so much so that the design authorities cannot afford to be complacent much longer.

Having thus briefly introduced the subject of philosophy of design of reinforced concrete, the remainder of the current chapter will now very briefly attempt a quick refresher for practical design of reinforced concrete elements by the two stated methods. It will be presumed that the reader is reasonably conversant with the basic subject-matter and has himself already carried out some reinforced concrete design.

23.2 ELASTIC DESIGN METHOD

This method of design is also referred to as *working load method of design* or *modular-ratio method of design*. Let us consider this in two sections, one being *beams and slabs* and the other *columns*.

Beams and Slabs

• ***Bending***

(See Fig. 23.1)

(i) M = applied moment, MR = moment of resistance of section as singly reinforced section

A_s = total tensile steel A'_s = compression steel

E_s and E_c = modulii of elasticity of steel and concrete

f_s = tensile stress in steel, f_c = maximum flexural compressive stress in concrete (permissible values).

Now MR for Type A cross-section = (area × stress) × (lever arm) = $(bndf_c/2)(d - nd/3)$

$$= \left[n\frac{f_c}{2}(1 - n/3)\right] bd^2 = Rbd^2,$$

where $$R = n\frac{f_c}{2}(1 - n/3)$$

or $$n \cdot \frac{f_c}{2} \cdot j,$$

where $j = (1 - n/3)$, and n from strain compatibility, is obtained from:

$$\frac{f_c}{E_c}/nd = \frac{f_s}{E_s}/(d - nd),$$

i.e., $$(mf_c/f_s) = n/1 - n,$$

so that $$n = \frac{1}{1 + \dfrac{f_s}{mf_c}}, \text{ where } m = E_s/E_c \text{ (modular ratio)}$$

and MR for Type B cross-section singly reinforced ≈ $b \cdot h_t \cdot \frac{f_c}{2} \cdot \left(d - \frac{h_t}{2}\right)$

(ii) If the cross-section is of Type A or Type B (see Fig. 23.1)

(a) if $M <$ the respective MR then the section is O.K. as singly reinforced; then reinforcement is as follows:

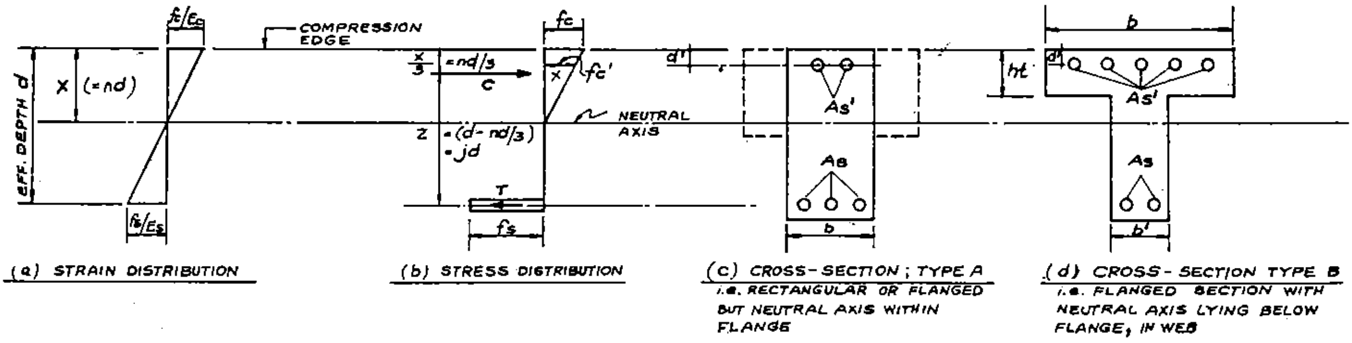


Fig. 23.1 *Bending*

- In Type A cross-section

$$A_s = \frac{M}{f_s jd} \text{ and } As' \text{ is not required.}$$

- In Type B cross-section: Ignoring the web-portion in compression

$$A_s = \frac{M}{f_s(d - h_t/2)} \text{ and } As' \text{ is not required.}$$

(b) if $M >$ the respective MR then the section requires to be reinforced in compression also (i.e. doubly reinforced) with additional tension reinforcement to balance the compression steel, then the reinforcement is as follows:

- In Type A cross-section

$$A_s = \frac{(MR)}{f_s jd} + \frac{M - (MR)}{f_s(d - d')} \text{ and } A_s' = \frac{M - (MR)}{mf_c'(d - d')}$$

where $f_c' = f_c \dfrac{(nd - d')}{nd}$

- In Type B cross-section ignoring the web-portion in compression:

$$A_s = \frac{(MR)}{f_s(d - h_t/2)} + \frac{M - (MR)}{f_s(d - d')}, \text{ and}$$

$$A_s' = \frac{M - (MR)}{mf_c'(d - d')}$$

where $f_c' = f_c \dfrac{(nd - d')}{nd}$

(iii) If the cross-section is (a) other than of types A or B, or (b) even if it is of type A or type B but it is required to do an exact elastic design (e.g., either not ignore the part of the web in compression or, alternatively, assume values for A_s and the section dimensions by trials and then check stresses in concrete and steel, until stresses are found satisfactory), or (c) the section and reinforcement are known and exact stresses in concrete and steel are required to be found under a given M, then in these cases:

- locate the neutral axis by equating about it the moments of 'effective areas' in compression and tension (i.e., equate the 'moments of areas' of 'concrete and steel in the compression zone' and that of 'steel in tension zone') about the unknown location

- ***Shear***

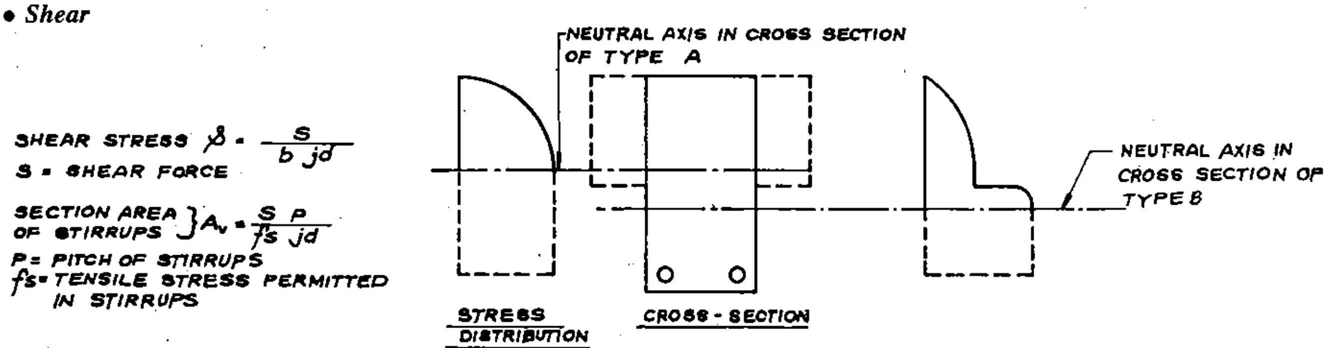


Fig. 23.2 *Shear*

NOTE: For design against SHEAR (and TORSION) refer to Chapter 24 for details of the neutral axis. (Note that in reinforced concrete the location of the neutral axis depends on f_c, f_s and m, and, therefore, not directly on the applied moment M. Although the actual values of f_c and f_c obviously depend on M their 'ratio'—whether of their actual values (which can be known only after the design) or of their limiting values—that enters the formula for n, does not affect the value of n very much).

- estimate the moment of inertia I (second moment of area) of the 'effective section' about the neutral axis.
- then compute the maximum flexural compressive stress in concrete and maximum tensile stress in steel from $\left(\frac{M}{I}y_c\right)$ and $\left(\frac{M}{I}y_t m\right)$ respectively, y_c and y_t being the distances of neutral axis from the extreme compression and tension fibres.

- ***Local Bond***

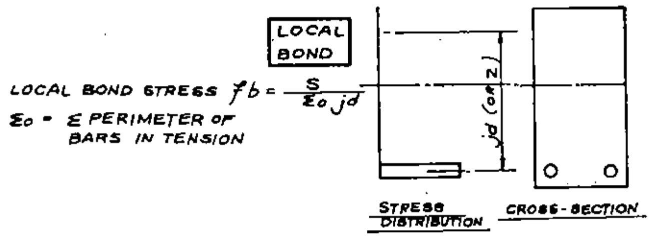


Fig. 23.3 *Local bond*

- ***Crack Control***

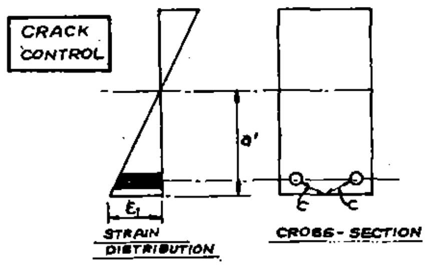


Fig. 23.4 *Crack-control*

For Crack Control, refer to "Serviceability Limit State of Cracking" given ahead in this chapter.

Columns

These may be either 'concentrically loaded' or 'eccentrically loaded' in service.

Concentrically Loaded Columns

In these the longitudinal (i.e., the main) reinforcement is bound together either by independent links or by continuous (helical) binders. The design and detailing are fairly simple in each case and reference may be made to any standard code for this purpose. Reference may also be made to ch. 10 in this book so as to first establish whether the column is short or long. If short, then design the section for the load-value by the pertinent formula in the code. However, if the column is 'long* then first establish the enhanced value of its design load and then design its section by the pertinent codal formula. In case of really long* columns, alternatively establish the design load and possible moments by the second order theory taking into account the buckling effects and then design the section for this load by the pertinent codal formula.

Eccentrically Loaded Columns (Bending about any axis)

As in the case of concentrically loaded columns, here too it is necessary to first establish whether the column is short or long. If short, the load and moment values remain as established from the first order theory, and sections can be designated for these by the regular method.**. However, if the column is long*, then first establish the design values of the enhanced load and moment as for a long column and then design its sections by the regular method**. In case of really long* columns, alternatively establish the design values of the load and the moment by the second order theory taking into account the buckling effects, and then, for the so established design values of load and moment at various sections, design these sections by the regular method**

23.3 LOAD-FACTOR DESIGN METHOD

This method of design is also referred to as the *ultimate strength design method,* and, some even go as far as (wrongly) calling it 'ultimate load design method'. We shall consider this subject in three sections, one being the background information in the run up to the second section that deals with beams and slabs and the third that deals with the columns.

Background Information

Significance of the Load-Factor and its Assessment

When a structure is designed on an elasto-plastic load-characteristic basis, it is analysed for a load equal to the working load multiplied by appropriate load factors. Since a great deal of care is usually taken when analysing the structure, it is only reasonable to assume that a similar

* *Long Column*: Details are given in Ch. 10 under reference.
** *Regular method*: Details of this elastic design method are given in Ch. 25 and it covers biaxial and indeed any axis bending.

Table 23.1 *Elastic Design—permissible stresses in concrete and steel bars as per BS CP 110*

	Concrete (f_{cu} = 28-day standard cube strength) (N/mm^2)				
	Concrete → (f_{cu})	30.0	37.5	45.0	
Permissible	Bending stress	10.0	12.5	12.5	
Permissible	Direct stress	7.6	9.5	9.5	
Permissible	Shear : beams	0.87	0.87	0.87	
Permissible	: slabs	Related to percentage of reinforcement			
Permissible	Average bond†	1.00	1.00	1.00	†plus 25% for deformed bars
Permissible	Local bond†	1.47	1.47	1.47	
	Modulus of elasticity	28kN/mm^2	30.25kN/mm^2	35.5kN/mm^2	
	Reinforcement (N/mm^2)	**Mild Steel**		**High-yield**	
		$\ngtr$ 40mm dia.	> 40mm dia.	steel	
Permissible	Tension	140	125	230	Subject to crack-width control
Permissible	Compression	125	110	175	
Permissible	Range of variation	265	235	325	
Permissible	Shear reinforcement	140	125	175	
	Yield stress	200	200	330	
	Characteristic stress	250	250	425	
	Modulus of elasticity	200kN/mm^2	200kN/mm^2	200kN/mm^2	

degree of care should also be taken in assessing the load factor.

The term load factor tends to imply that it is only a factor against possible overload, but, this is not strictly true. In order to see what this factor covers, it is convenient to trace the steps an engineer treads before obtaining his final design.

His first task is to assess his working load, and this implies that this working load is the one associated with the length of life of the structure. He, therefore, has to view, not necessarily the working load today, but also that which will occur in the next fifty years or so. It is only reasonable to suppose that this estimate is subject to a certain amount of doubt, and the degree of doubt will clearly depend on his source of information. His best assessment will be obtained from information based on a careful statistical study over a large number of years; his worst assessment will be when he has to purely guess. Part of the load factor should, therefore, bear a relation to the accuracy of the assessment of the maximum working load.

The next step in the design is to analyse the structure under a given loading configuration. Any analysis is based on assumptions which rarely are a representation of all the characteristics of the structure in the field. Certain aspects are usually neglected because the effects are small, but they still exist and, therefore, the analysis is not usually a complete picture of the true behaviour. Coupled with the analysis are further assumptions concerning the dimensions, and with each successive assumption the structure behaves in a slightly different way from its supposed replica. Having obtained from the analysis the numerical values of the forces and moments in the structure, the beams, slabs and columns may then be designed to 'fail' at these moments. The design equations are again based on assumptions, usually neglecting creep, neglecting thermal stresses and neglecting foundation settlements. It is, therefore, realistic that some part of the load factor should also include an item which represents the expected inaccuracy of the particular analysis that has been used.

In the ultimate strength equations, values for the strength of concrete, the yield stress of steel. The section dimensions, and the value for the steel percentage are involved. Engineers should also remember the days when concrete is poured in the rain or in hot weather or at near freezing temperatures, the days shutters slip, and the days when there is no one to supervise or inspect the work in progress. Certainly, part of the load factor must include a provision against all these, because these things do happen!

The value of the load factor should also include an item which tries to measure the seriousness of collapse, since it is clearly more important to prevent failure if hundreds of people will be killed and the mishap will cause an economic crisis! Consequently, the load factor must be modified and increased if this is likely to happen (degree of seriousness of the possible mishap).

In a philosophical approach to structural design, mainly due to the work of the international committees, the CEB published its *Recommendations for an International Code of Practice for Reinforced Concrete* in 1963, generally known as the *Blue Book*, and later in conjunction with the FIP, a complementary report dealing with prestressed concrete. Further to these, there was published in 1970 *The Interna-*

tional Recommendations for the Design and Construction of Concrete Structures, giving the principles and recommendations, and generally known as the *Red Book*.

In design the following points have to be taken into consideration

(i) Variations in materials in the structure and in the test specimens
(ii) Variations in loading
(iii) Constructional inaccuracies
(iv) Accuracy of design calculations
(v) Safety and serviceability of the structure

The various criteria required to define the serviceability or usefulness of any structure can be described keeping in mind the following headings as being 'unfit for use':

(i) *Collapse* Failure of one or more critical sections; overturning or buckling.
(ii) *Deflection* The deflection of the structure or any part of the structure adversely affecting the apearance or efficiency of the structure.
(iii) *Cracking* Cracking of the concrete which may adversely affect the appearance or efficiency of the structure.
(iv) *Vibration* Vibration, from forces due to wind or earthquake or machinery, may cause discomfort or alarm, damage to the structure, or may interfere with its proper function.
(v) *Fatigue* Where loading is predominantly cyclic in character the fatigue effects have to be considered.
(vi) *Durability* Porosity of concrete, leading to ingress of deleterious materials and
(vii) *Fire Resistance* Insufficient resistant to fire leading to (i), (ii) and (iii) above.

When any structure is rendered unfit for use for its designed function by one or more of the above causes, it is said to have entered a *limit state*.

(a) *Ultimate limit state* Ultimate limit state is preferred to collapse.
(b) *Serviceability limit states* Deflection, cracking, vibration.
(c) *Other limit states* Special requirements for unusual or special functions of a structure.
(d) *Other considerations* Fatigue, durability, fire resistance, lightning, etc.

The purpose of design then, is to ensure that the structure being designed will not become unfit for the use for which it is required, i.e., that it will not cross a particular limit state. The essential basis of the design method, therefore, is to consider each limit state and to provide a suitable margin of safety correspondingly.

Accepting the fact that the strengths of constructional materials vary, as also do the loads on the structure, two partial safety factors may be used. One will be for mateials and is designated γ_m, and the other, for loading, is termed γ_f. These factors will vary for the various limit states. As new knowledge on either materials or loading becomes available, the factors can be amended quite easily without the complicated procedure to amend one overall factor as in the earlier days.

Characteristic Strength of Materials

For both concrete and reinforcement, the codes use the term characteristic strength instead of the 28-day works cube (or cylinder) strength and yield stress, although it is still related to these. The characteristic strength for all materials has the notation f_k and is defined as the value of the strength of concrete, and of the yield or proof-stress of reinforcement, below which not more than 5% of the test results may be expected to fall.

The value, therefore, is

$$f_k = (f_m - 1.64\ s)$$

where f_m is the mean strength of actual test results determined in accordance with a standard procedure, s is the standard deviation, 1.64 is the value of the constant required to comply with not more than 5% of the test results falling below the characteristic.

(a) Concrete The strength of the concrete is based on tests made on cubes (UK practice) at an age of 28 days unless there is satisfactory evidence that a particular testing regime is capable of predicting the 28 days strength at an earlier age. For concrete the characteristic strength has the notation f_{cu}. The quantity 1.64 s is called the 'current margin' and the concrete mix should be designed to have a mean strength greater than the required characteristic strength by at least this current margin. If there is insufficient data from cube tests to enable a value for the standard deviation to be used, then the margin for the initial mix design must be given an actual numerical value depending on the required characteristic strength.

Concrete is classified into grades, where the grade number is the chaacteristic cube strength. This is shown in Table 23.2 which also indicates the lowest grade for compliance with appropriate use.

From this table it will be seen that the lowest grade of concrete for reinforced concrete with dense aggregate is 20.

So, assuming that this grade of concrete is being used as a designed mix and, from test data the standard deviation is taken as 7.5 N/mm^2 then, a mean strength of (20 + 1.64 $\times$ 7.5) = 32.3 N/mm^2 would be the target strength. With sufficient data the current margin for this grade of concrete is 15 N/mm^2 and the mean strength would be (20 + 15) = 35 N/mm^2.

Table 23.2 *Grades of Concrete (BS CP 110)*

Grade	Characteristic strength N/mm^2	Lowest grade for compliance with appropriate use
7	7.0	—Plain concrete
10	10.0	
15	15.0	—Reinforced concrete with light weight aggregate
20	20.0	—Reinforced concrete with dense aggregate
25	25.0	
30	30.0	—Concrete with post-tensioned tendons
40	40.0	—Concrete with pre-tensioned tendons
50	50.0	
60	60.0	

Strength compliance is judged by test results from a suitable testing regime where the rate of sampling and testing depends on the nature of the work and the volume of concrete at risk. For example, a higher rate of sampling would be required for highly stressed structural members. Also, it would be appropriate to have higher rates of sampling and testing at the start of the work to establish the level of quality quickly. The actual rate of sampling will fluctuate, but at least one set of samples should be taken from each days' concrete of each particular grade, subject to a specified minimum number per batch or per 50 m^3 of poured concrete. Compliance with the specified characteristic strength may be assumed if,

(i) The average strength from any group of four consecutive test cubes exceeds the specified characteristic by not less then 0.5 times the current margin,

(ii) Each individual test result is greater than 85% of the specified characteristic strength

If only one cube result fails to meet the second requirement, then only that particular batch from which the cube was taken does not comply. But if the average strength of any group of four consecutive cubes fails to meet the first requirement then all the batches between the including those from which the first and last samples were taken do not comply. In this case, the mix proportions shall be modified to increase the strength and the engineer will then determine the action to be taken. (These 'criteria' specifications can vary depending on which code governs. In this connection, reference may also be made to the relevant chapter in the author's other book, *Concrete for Construction—Facts and Practice.*)

(b) Reinforcement The reinforcement may comply with BS 4449, BS 4461 or BS 4483, all of which specify the tests for compliance to obtain the characteristic strength which has the notation f_y. The designation of the reinforcement with its specified characteristic strength is shown in Table 23.3.

From the table it will be seen that the characteristic strength of high yield bars depends on whether they are

Table 23.3

Designation	Nominal sizes. mm.	Specified characteristic strength (f_y) N/mm^2
Hot rolled mild steel (BS4449)	All sizes	250
Hot rolled high yield (BS4449)	All sizes	410
Cold worked high yield (BS4461)	Up to and including 16	460
	Over 16	425
Hard drawn steel wire	Up to and including 12	485

'hot rolled' or 'cold rolled' worked. A further sub-division is made in BS CP 110 to determine the bond characteristics.

Characteristic Loads

For loading we use the 'characteristic' load (F_k) as the basis. Ideally, this should be determined from the mean load and its standard deviation from the mean, and using the same probability as for the materials we should say that $F_k = F_m + 1.64s$. The characteristic load would be that value of loading so that not more than 5% of the spectrum of loading throughout the life of the structure will lie above the value of the characteristic load.

Design Strength of Materials

The design strengths of the materials can be obtained by dividing the characteristic strengths by the partial safety factor γ_m i.e. design strength = f_k/γ_m.

γ_m takes account of possible differences between the material in the actual structure and the strength derived from test specimens.

In concrete, this would cover such items as insufficient compaction, differences in curing, etc. For reinforcement, such items as the difference between the assumed and the actual cross-sectional areas (caused by rolling tolerances, corrosion, etc.)

The values of γ_m for each material will be different for the different limit states by virtue of the different probabilities that can be accepted for each individually.

Table 23.4 sets out these values, as per BS CP 110.

Table 23.4 *Values as per BS CP 110*

Limit state	Values of γ_m	
	Concrete	Steel
Ultimate	1.5	1.15
Deflection	1.0	1.0
Cracking	1.3	1.0

Considering the values for both materials, the factor for the ultimate limit state is higher than the others, because, not only must the probability of failure be decreased, but failure could be localised. So the γ_m factor also contains an allowance for this and as a compressive failure in concrete is sudden and without warning the factor for concrete is higher than for reinforcement.

Deflection is related to the whole member and the factor for both materials is 1.0.

For cracking, only parts of the member are affected and a factor in between 1.0 and 1.5 for concrete has been selected, but kept at 1.0 for reinforcement.

When analysing any cross-section within the structure, the properties of the materials should be assumed to be those associated with their design strength appropriate to the limit state being considered.

The short-term design stress-strain curve for concrete is shown in Fig. 23.5 and by putting-in the relevant value of γ_m, depending on the limit state being considered, the appropriate design stress-strain curve can be obtained, as shown in Fig. 23.6.

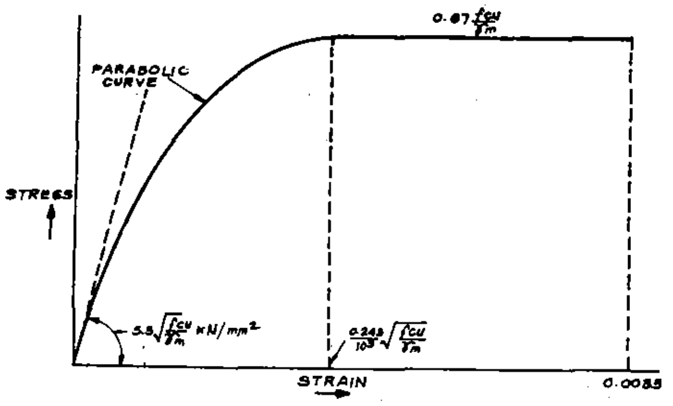


Fig. 23.5 *Short-term design stress-strain relation for normal weight concrete (f_{cu} in N/mm^2) (BS CP 110)*

The design strength has been defined as characteristic strength devided by γ_m and yet the maximum stress value is given as 0.67 $\frac{f_{cu}}{\gamma_m}$ (BS CP 110). The reason for this is that the characteristic strength has been derived from tests on cubes. It is well established from tests that the maximum compressive stress at failure in a member of the same concrete as a cube has a value in the region of 0.8 f_{cu}. This is a peak value, and as an additional safety factor against compressive failure this value has been reduced to 0.67 f_{cu}, which agrees with the present design methods using ultimate load. If a cylinder was to be used in determining the characteristic strength, the factor would be of the order of 0.85 as the cylinder strength is nearer the actual behaviour and is approximately 0.8 × cube strength.

For the serviceability limit states Poisson's Ratio may be taken as 0.2.

For reinforcement the short term stress strain relationship is shown in Fig. 23.7.

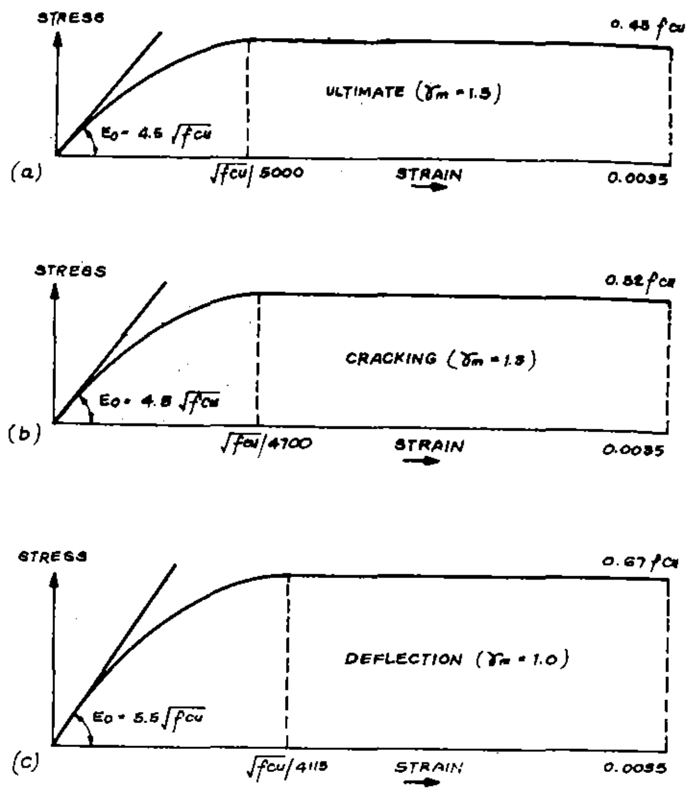


Fig. 23.6 *Short-term design stress-strain curves for concrete ultimate and serviceability limit states (BS CP110)*

NOTE: (1) Ultimate Limit State — Alternative Mean Stress 0.4 f_{cu}
(2) Serviceability Limit States — Alternative Linear Stress-Strain Relationship with Specified Value for E_0 Dependent on f_{cu}
(3) f_{cu} in N/mm^2 and E in kN/mm^2

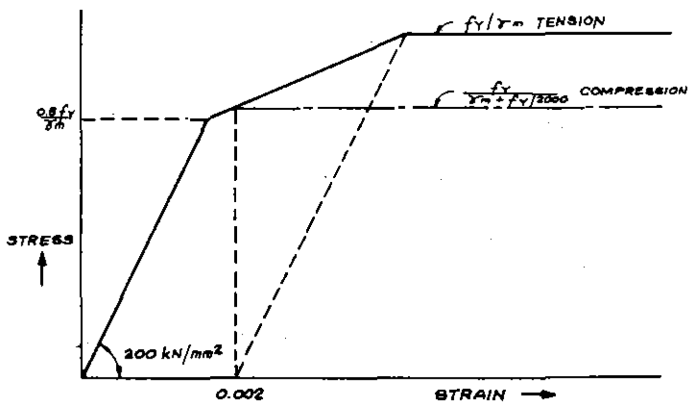


Fig. 23.7 *Short-term design stress-strain relation for reinforcement (f_y in N/mm^2)*

The relationship, which is trilinear, is for all grades of reinforcement, and again, by putting-in the relevant factors for γ_m, the appropriate curves for the different limit states can be obtained and these are shown in Fig. 23.8. The elastic modulus for all types of loading may be taken as 2000 kN/mm^2.

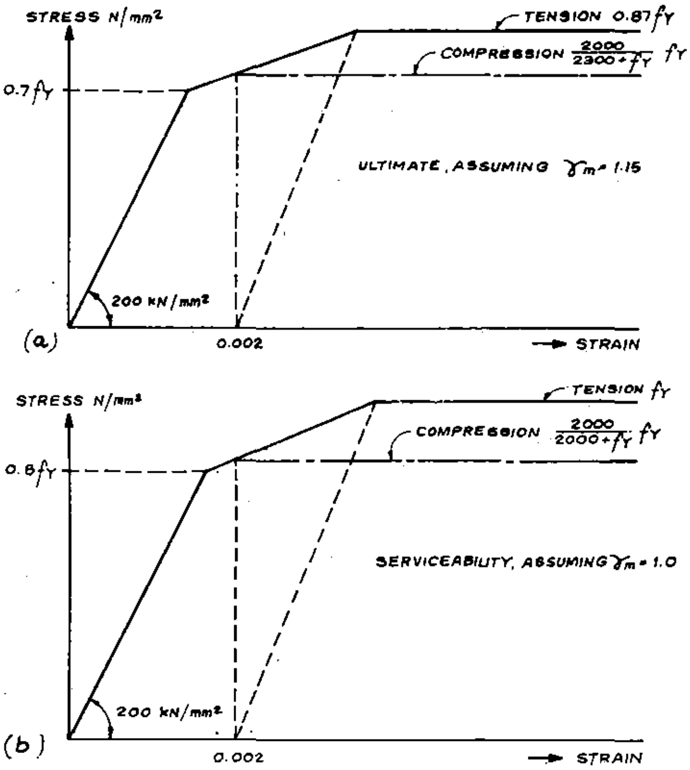


Fig. 23.8 *Short-term design stress-strain curves for reinforcement (ultimate and serviceability limit states)*

Design Loads

The design load is obtained by multiplying the characteristic load by the other partial safety factor γ_f. This factor γ_f is introduced to take account of,

(i) Possible unusual increases in the load beyond those in deriving the characteristic load
(ii) Inaccurate assessment of effects of loading
(iii) Variations in dimensional accuracy achieved in construction,
(iv) The importance of the limit-states being considered. γ_f varies for the different limit-states and values as per CP110 are set out in Tables 23.5 and 23.6.

The criteria to be complied with for the various limit states are broadly as follows.

(i) *Ultimate limit state* The strength of the structure should be sufficient to withstand the design loads.
(ii) *Serviceability limit states*
  (a) *Deflection* The engineer must be satisfied that deflections are not excessive having regard to the particular structure, but reasonable limits may be set by the pertinent code of practice.
  (b) *Cracking* The assessed surface width of cracks should not, in general, exceed 0.3 mm and, for particularly aggressive environments, the assessed surface crack widths at points nearest the main reinforcement should not, in general, exceed 0.004 times the nominal cover. The BS CP 110 points out that it is not possible to predict an absolute maximum crack width and the possibility of some cracks being wider than the above must be accepted unless special precautions are taken.
  (c) *Vibration* Limits for this are generally not given in the codes. Reference may be made to a separate chapter in this book on this subject.
  (d) *Other limit states* Special structures will comply with additional limit states, as considered necessary by the engineer.

Table 23.5 *Values of γ_f: Ultimate Limit State*

Load combination	Dead load		Imposed load		Wind load
	max.	min.	max.	min.	
1. Dead and imposed load	1.4	1.0	1.6	0	0
2. Dead and wind load	1.4	0.9	0		1.4
3. Dead and imposed and wind load	1.2		1.2		1.2

Table 23.6 *Values of γ_f: Serviceability Limit State*

Load combination	Dead load	Imposed load	Wind load
1. Dead and imposed load	1.0	1.0	0
2. Dead and wind load	1.0	0	1.0
3. Dead and imposed and wind load	1.0	0.8	0.8

Beams and Slabs

• In estimating the ultimate (capacity) moment of resistance of a reinforced concrete section, the following assumptions can, therefore, be made. (BS CP 110):

In Concrete in Flexural Compression

1. The strain distribution in the concrete in compression is derived from the assumption that plane sections remain plane, and

2. The stress distribution in the concrete in compression is derived from the stress strain curve in Fig. 23.5 with γ_m = 1.5 (refer to Fig. 23.6(a) and Fig. 23.10). Or, it may be taken as a rectangle with a stress value of $0.4f_{cu}$ over the whole compression zone (Fig. 23.9).

In both cases, the strain at the outermost compression

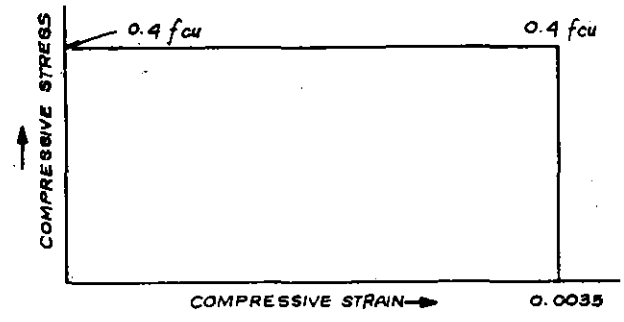


Fig. 23.9

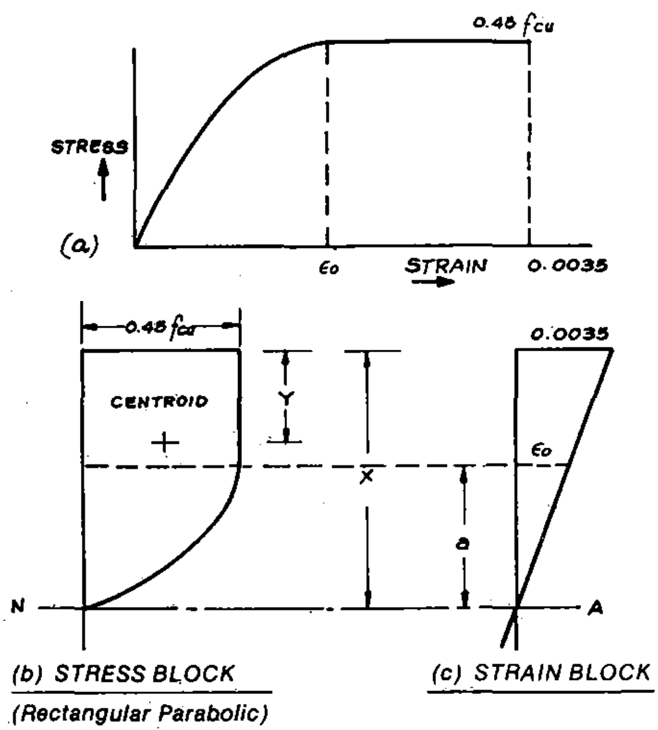


Fig. 23.10

fibre at failure is taken as 0.0035. Where beams are reinforced in tension only, the depth of the concrete in compression is limited to half the effective depth, i.e., $x \leq 0.5d$. The first diagram is usually referred to as the rectangular-parabolic stress block. From the co-ordinates given, the properties of this stress block for a particular concrete grade can be worked out. A chart relating the properties is shown in Fig. 23.11 and it can be seen that the constant stress value of 0.4 f_{cu} for the equivalent rectangle case is a fairly good approximation in the lower grades, but decreases with the higher grades of concrete (see line for coefficient K_1).

The values in Fig. 23.11 can be obtained basically by setting up two equations, one for the area of the stress block, and the other by equating the moments of the total stress-block area and of its component areas about (say) the extreme compression edge. This is done as follows:

From strain block $\dfrac{a}{x} = \dfrac{\varepsilon_0}{0.0035}$

$$\begin{aligned}\text{Area of stress block} &= 0.45 f_{cu} x - \frac{0.45 f_{cu} a}{3} \\ &= 0.45 f_{cu} x \left(1 - \frac{\varepsilon_0}{3 \times 0.0035}\right) \\ &= \frac{0.45 f_{cu}}{0.0035}\left(0.0035 - \frac{\varepsilon_0}{3}\right) x \\ &= f_m x = K_1 f_{cu} x\end{aligned}$$

Taking moments about compressive face and if y is the distance of the centroid of the total rectangular-parabolic stress block from the compressed face, then

$$\begin{aligned} y &= \frac{0.45 f_{cu} x \dfrac{x}{2} - \dfrac{0.45}{3} f_{cu} a \left(x - \dfrac{a}{4}\right)}{0.45 f_{cu}\left(x - \dfrac{a}{3}\right)} \\ &= \frac{6x^2 - a(4x - a)}{4(3x - a)} \\ &= \left\{\frac{\left(2 - \dfrac{a}{x}\right)^2 + 2}{4\left(3 - \dfrac{a}{x}\right)}\right\} x \\ &= \left\{\frac{\left(2 - \dfrac{\varepsilon_0}{0.0035}\right)^2 + 2}{4\left(3 - \dfrac{\varepsilon_0}{0.0035}\right)}\right\} x \\ &= K_2 x\end{aligned}$$

In deriving values for f_m, K_1 and K_2 substitute $\dfrac{\sqrt{f_{cu}}}{5000}$ for ε_0 (with f_{cu} in N/mm^2)

3. The tensile strength of the concrete is ignored.

In Reinforcement

In reinforcement,

1. The strain in the reinforcement is derived from the assumption that plane sections remain plane, and
2. The stress in the reinforcement is derived from the stress-strain curve in Fig. 23.7 with $\gamma_m = 1.15$, i.e., Fig. 23.8(a). There is no simplified version of this diagram. A chart giving the actual values for the various types of reinforcement is shown in Fig. 23.12.

• The above are the basic assumptions and, in the actual design to find the amount of reinforcement required, we can use,

(a) Design charts, or
(b) Design formulae, or

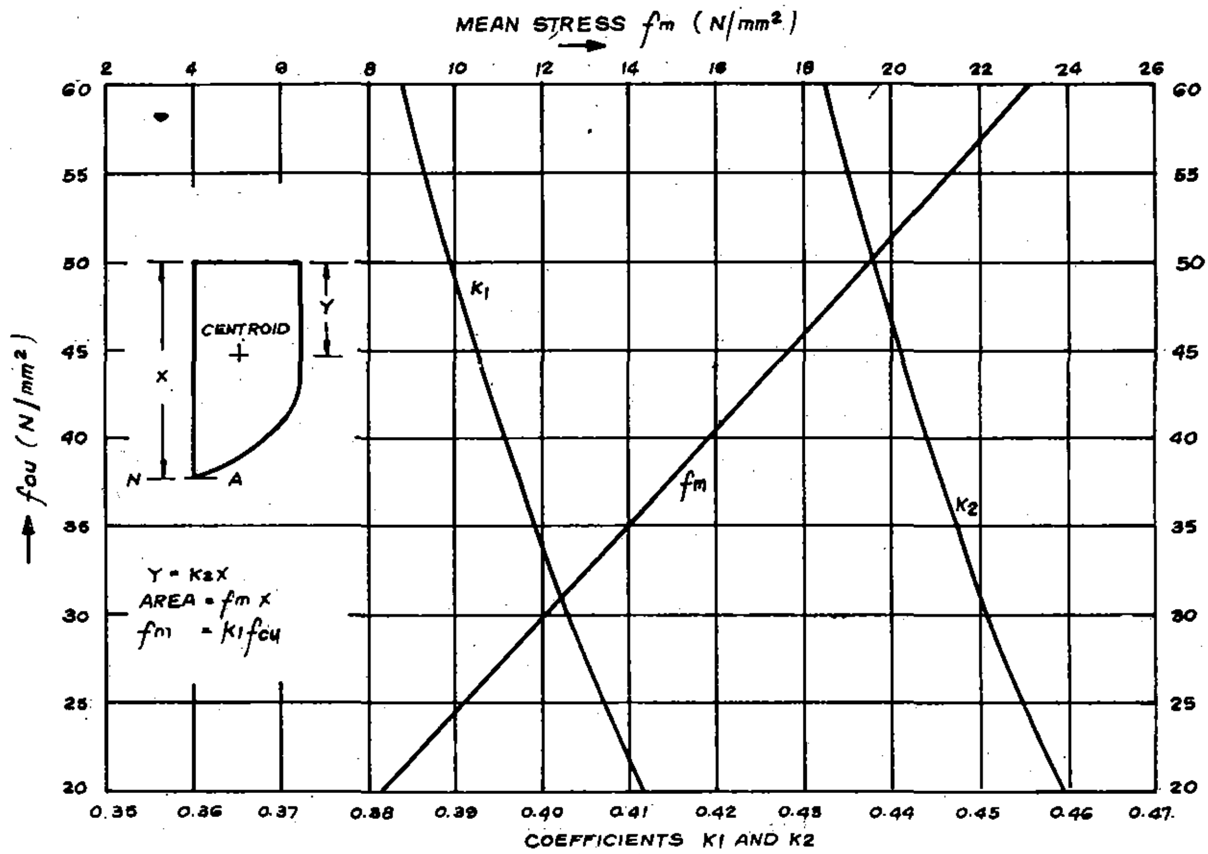


Fig. 23.11 *Properties of rectangular-parabolic stress block*

(c) Strain compatibility

(a) Using Design Charts These have been prepared using the *rectangular-parabolic* stress block for concrete, and the stress-strain curves for the reinforcement. The charts for rectangular sections, reinforced in tension only, or in tension and compression, are given in Part 2 of BS CP 110, which also gives the derivation. Each chart is for a particular grade of concrete, a particular strength of reinforcement, and, in the case of beams reinforced in compression too, for particular value of d'/d.

(b) Using Design Formulae (listed in Table 23.7) There are seven equations as listed ahead in Table 23.7 which can be derived quite simply. The first five deal with rectangular sections or flanged sections where the neutral axis lies within the compression flange, and the remaining two deal with flanged sections where the neutral axis is below the compression flange. In deriving the formulae, a rectangular stress block of maximum depth $0.5d$ and a uniform compressive stress of $0.4f_{cu}$ are assumed. There is a formula for the ultimate resistance moment based on the strength of the concrete and singly reinforced section, above which compression reinforcement is required.

Tensile stress in reinforcement is assumed as $0.87f_y$ as per Fig. 23.8(a). Compressive stress in reinforcement $\left[\dfrac{2000}{2300+f_y}f_y \text{ in Fig. 23.8(a)}\right]$ is assumed as $0.72f_y$. A formula is given for the lever arm z which again depends on the amount of tension reinforcement, but unlike in elastic design as the lever arm is no longer required for shear, etc. in the load factor method, it is not important.

In using the formulae for rectangular sections, it is most important to note that they only apply for (a maximum redistribution of about 10%) the neutral axis in the limit at half the effective depth. With $x/d = 0.5$ (i.e., $n_u = 0.5$), the lever arm, $\left(d - \dfrac{x}{2}\right)$ or $\left(d - \dfrac{n_u d}{2}\right)$ i.e. z, must be 0.75 d. Then, based on the concrete in compression (i.e., based on concrete side), $M_u = 0.4f_{cu}(b(0.5d))(0.75d) = 0.15f_{cu}bd^2$, which is Eq. 23.2 in Table 23.7. For this particular value, sometimes referred to as the balance point, M_u, based on the reinforcement in tension, $= (0.87f_y)A_s z$, and substituting 0.75 d for z we get (by equation the two M_u values),

$$A_s = 0.23bdf_{cu}/f_y$$

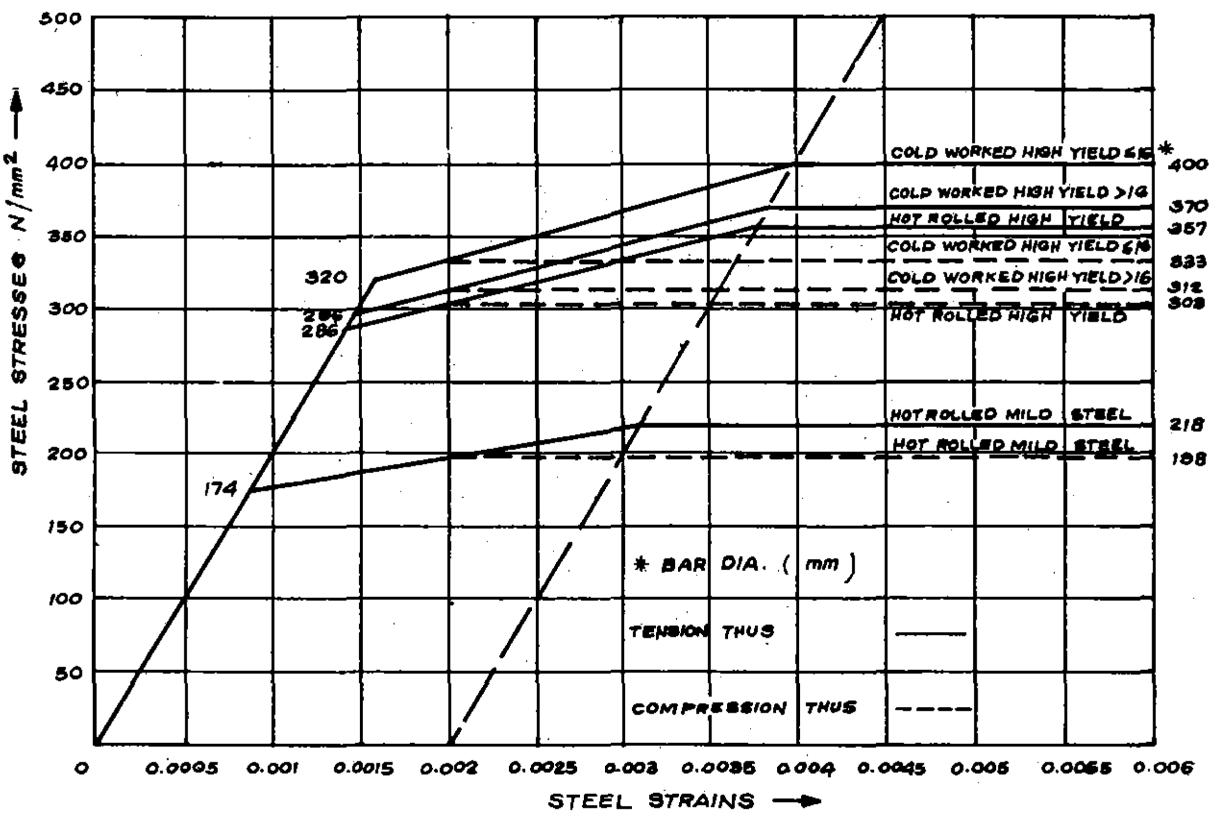


Fig. 23.12 *Design stress-strain curves for ultimate limit state*

If we have more than 10% redistribution, $\frac{x}{d}$ must be less than 0.5, and then although z is greater than $0.75d$ the value of $\frac{M_u}{bd^2}$, based on the concrete side, will get reduced. With 20% redistribution: $\frac{x}{d} \simeq 0.4$, $z \simeq 0.8\ d$, so $M_u \simeq 0.128 f_{cu} bd^2$. With 30% redistribution: $\frac{x}{d} \simeq 0.3$, $z \simeq 0.85\ d$, so $M_u \simeq 0.102\ f_{cu} bd^2$.

So as with the design charts, the more the redistribution, the sooner the compression steel is required.

There is also the condition that the lever arm shall not exceed 0.95 d, which means that for small steel percentages (near the minimum percentages) the value of the lever arm obtained from Eq. (23.5) in Table 23.7 cannot be used. The restriction is not great in any case since the maximum value of z is approximately 0.96 d (using the minimum steel percentage and Grade 20 concrete).

When compression steel is required, Eqs. (23.3) and (23.4) in Table 23.7 should be used. In these equations, the design stress in the compression steel is taken as 0.72 f_y, which is a simplification of $\frac{2000 f_y}{2300 + f_y}$ [Fig. 23.8(a)]. An important condition in the use of Eq. (23.3) is that d'/d is not greater than 0.2. This is to ensure that with a neutral axis depth of 0.5 d, the strain in the compression steel has reached 0.002, i.e., the design stress in compression has been reached. If more than 10% redistribution has been carried out, the values for the resistance moment and force in the concrete should be modified, as explained above, for sections reinforced in tension only. The values for the resistance moment in Eq. (23.3) will be as noted earlier, and the force in Eq. (23.4) will become $(0.16 f_{cu} bd)$ and $(0.12 f_{cu} bd)$ for 20% and 30% redistributions, respectively. The maximum values of d'/d will be 0.16 and 0.12, respectively.

When we have flanged beams with the neutral axis below the flange, the ultimate resistance moment is given by Eqs. (23.6) and (23.7) in Table 23.7. From these equations, it can be seen that any concrete below the flange is ignored. Where redistribution has been carried out, reducing the moment in the span, we have the requirement concerning the neutral axis depth. If this is greater than the depth of the flange, we can still use Eqs. (23.6) and (23.7) but if it is less than or equal to the depth of the flange, we can substitute the actual-depth for the flange-depth in these equations.

If we need the resistance of the concrete in compression to be greater than that given by Eq. (23.7), we have to revert

Table 23.7 *(based on BS CP110): Formulae for Ultimate Resistance Against Moment: Beams and Slabs*

Design formulae: Provided that the amount of redistribution of the elastic ultimate moments has been less than 10%, the following formulae may be used to calculate the ultimate moment of resistance of a solid slab, or of a rectangular beam or of a flanged beam, a ribbed slab or a voided slab section, when the neutral axis lies within the compression flange depth.

- For sections without compression reinforcement the ultimate momennt of resistance may be taken as the lesser of the values obtained from Eqs. (23.1) and (23.2) below.
- Eqs. (23.3) and (23.4) may be used for sections with compression reinforcement.

A rectangular stress block of maximum depth 0.5d and a uniform compressive stress of $0.4f_{cu}$ has been assumed.

$M_u = (0.87f_y)A_s z$ (based on tension failure) (23.1)

$M_u = (0.15f_{cu}bd^2)$ (based on compression failure) (23.2)

$M_u = (0.15f_{cu}bd^2) + (0.72f_y)A'_s(d - d')$ (adding the effect of compression steel) (23.3)

$(0.87f_y)A_s = 0.2f_{cu}bd + 0.72f_yA'_s$ (equating tension to total compression) (23.4)

where M_u is the ultimate resistance moment
A_s is the area of tension reinforcement
$A_{s'}$ is the area of compression reinforcement
b is the width of the section.
d is the effective depth to the tension reinforcement (from compression edge)
d' is the depth to the compression reinforcement (from compression edge)
f_y is the characteristic strength of the reinforcement
z is the lever arm (*i.e.*, $\left(d - \frac{n_u d}{2}\right)$ where n_u is the neutral axis factor at ultimate and $n_u \not> 0.50$, so that $z = 0.75d$ and the section remains under-reinforced.
f_{cu} is the characteristic strength of the concrete

- when d'/d is greater than 0.2 Eq. (23.3) should not be used and the resistance moment should be calculated with the aid of strain compatibility approach (explained earlier in the text).
- The term $0.72f_y$ in Eqs. (23.3) and (23.4) is a simplification of expression $\frac{f_y}{(r_m + f_y/2000)}$ shown in Fig. 23.8(a)
- The lever arm, z in Eq. (23.1) may be calculated from the equation

$$z = \left(1 - \frac{1.1f_yA_s}{f_{cu}bd}\right)d \qquad (23.5)$$

(solving this equation in the limit when $z = 0.75d$ corresponds to tension = compression)

- The value of z shall not be taken as greater than 0.95d
- The ultimate resistance moment of a flanged beam may be taken as the lesser of the values given by Eqs. (23.6) and (23.7) where h_l is the thickness of the flange.

$$M_u = (0.87f_y)A_s\left(d - \frac{h_t}{2}\right) \text{ (based on tension failure)} \qquad (23.6)$$

$$M_u = 0.4f_{cu}bh_t\left(d - \frac{h_l}{2}\right) \text{ (based on compression failure)} \qquad (23.7)$$

- Where it is necessary for the resistance moment to exceed the value given by Eq. (23.7) for instance if neutral axis lies in the web and/or the applied factored moment is higher, the section should be analysed with the aid of strain-compatibility approach (explained earlier in the text).

to the basic assumptions and use strain compatibility.

(c) Using Strain-Compatibility Approach In a section which is non-rectangular in the compression zone (call it above neutral axis), we cannot use design charts or formulae directly, and so we cannot quickly find the area of reinforcement required for a given ultimate moment or vice versa. We, therefore, have to revert to first principles and use the 'strain-compatibility' principle. But even with this method, it is only possible to find the ultimate resistance moment based on an assumed steel area. The basic principle of strain-compatibility method is that for a given section (including the assumed reinforcement), we can assume the neutral axis depth, by trial and error, and work out the total compression force C and the total tension force T for each trial unit the two forces are equal, then we can work out the ultimate moment of resistance of the assumed section by taking moments of these C and T forces about any fibre level in the section and check whether this M_u is adequate.

As for flexural compressive stress block in concrete, one can use a rectangular-parabolic stress block and proceed by dividing the section into segments. But at this would be a long and tedious process, we may simply assume a rectangular stress block with a uniform compression stress of $0.4f_{cu}$.

As for steel, the stress-strain curve for reinforcement is as shown in Fig. 23.8(a).

Using a linear strain-profile, determine the strains in the reinforcements and, from the stress-strain profile for the particular reinforcement, find the stress in the reinforcement and hence the force in it.

We only have to assume a trial neutral axis depth and draw the strain block with the maximum compression strain in the concrete as 0.0035 at the top edge and work out the strains in various reinforcement bars, by proportion. Then work out tension force in each bar (= its area × stress corresponding to strain in it) and hence the total T. Also work out the compression force C in concrete in (flexural) compression (= its area $\times 0.4f_{cu}$). Neutral axis location is OK when T equals C, otherwise repeat the procedure with a fresh trial for neutral axis depth until T equals C. Then take moments of C and T forces; as explained above, in order to find M_u.

- ***Interesting Comparison between the British and the ACI/AASHTO Approaches***

(i) In Appendix 6 (given at the end of this book) are quoted with courtesy relevant extracts on this subject from the ACI and AASHTO stipulations as a matter of comparative companion reading. This American practice is based on 'cylinder-crushing-strength of concrete', a 'rectangular-flexural-compressive stress-block' and certain reduction (i.e. ϕ) factors applied on the limit state strength computation.

"(ii) • The B.S. flexural compressive stress block is assumed rectangular all the way in Neutral axis, the stress value is taken a constant of 0.4 of cube strength of concrete ($0.4f_{cu}$), and the distance to neutral axis from the compression edge is taken equal to $d/2$ in the limit but actually is estimated from $C \equiv T$, i.e. from: $(b.n_u d)(0.4f_{cu}) \equiv (A_s 0.87f_y)$ for a rectangular section as an example, which gives $n_u = \dfrac{2.175A_s \cdot f_y}{bd \cdot f_{cu}}$, sothat the lever arm between C and T is: $(d-n_u d/2)$, i.e. $(1-n_u/2)d$, i.e. $(1 - \dfrac{1.1A_s \cdot f_y}{bd \cdot f_{cu}})d$.

And, then, $M_u = (C \text{ or } T) \cdot (d - n_u d/2)$.

• Although the ACI/AASHTO stress block, after Whitney, is also assumed rectangular, but its depth is taken to range from 0.85 to 0.65 of $n_u d$ (not full $n_u d$), depending on if the concrete cylinder strength is upto 4000 p.s.2 or more. The stress value is taken a constant of 0.85 of Cylinder Strength of Concrete (i.e. $0.85f_{cyl.}$, which is nearly $0.71f_{cu}$). The depth of the stress block, 'a', is then estimated from $C \equiv T$, i.e. from: $(b.a)(0.85f_{cyl.}) \equiv (A_s \cdot f_y)$, which gives: $a = \dfrac{A_s \cdot f_y}{b.(0.85f_{cyl.})}$, sothat the lever arm between C and T is then:

$$(d-a/2) = [d-\frac{0.59A_s \cdot f_y}{b \cdot f_{cyl.}}], \text{ i.e. } (1-\frac{0.71A_s \cdot f_y}{bd \cdot f_{cu}})d$$

And, then $M_u = 0.9[(C \text{ or } T \cdot (d-a/2)]$, 0.9 factor is as stated in the ACI/AASHTO stipulations

• It is clear therefore that the lever arm is bigger in the ACI/AASHTO approach than in the B.S. approach and is likely to give slightly larger M_u relatively. However, the compression force C in the two approaches may not show much difference because the stress block depth relatively is more in the British approach while the stress value is relatively more in the American approach ($0.7f_{cu}$ vs. $0.4f_{cu}$)"

The above procedure for estimating M_u from compatibility of strains has also been explained in rigorous detail in Ch. 27 in this book, using a 'rectangular-parabolic' stress block and higher values for the maximum flexural compressive stress and strain in concrete. But these higher values correspond to the collapse limit state, not an intermediate load-factored limit state, and therefore, the M_u so calculated is the limiting capacity value. [These higher values of stress and strain in concrete are not codified and are based on earlier research done by the author at Imperial College, London, under Prof. ALL Baker (1964).] It will be noted in the said chapter that the averaging stress factor β for the part-parabolic stress block and its centroid-locating factor η have been worked out in terms of concrete strain at the top of the part-parabola in a manner similar to that explained earlier. These β and η factors allow general applicability of that method even if the section is of any shape, not merely rectangular.

• ***Comparison Between Various Flexural Compressive Stress Blocks***

(a) With a rigorous limit state analysis, the resistance moment from the concrete side obtained when a rectangular stress block is assumed, is 0.4 $f_{cu}bx$, and thus ranges from $8bx$ when f_{cu} is equal to 20 N/mm^2 to $16bx$ when f_{cu} equals 40 N/mm^2. These values compare with resistances of $8.14bx$ and $15.66bx$ respectively when a parabolic-rectangular stress block is assumed. It can in fact be shown that for values of f_{cu} of less than 28.14 N/mm^2, the choice of a parabolic-rectangular stress block gives a greater moment resistance, while for higher values of f_{cu} the resistance given by a rectangular stress block is greater. Also, the depth to the centroid of a parabolic-rectangular stress block varies between $0.458x$ and $0.444x$ as f_{cu} increases from 20 to 40 N/mm^2 compared with the constant value of $0.5x$ for a rectangular stress block. The relationship between the moments of resistance provided by the alternative assumptions depends on the ratio of x/d but typical comparative figures are as given in Table 23.8.

The values in Table 23.8 indicate that, while normally showing slight advantage over a rectangular distribution of stress, the choice of parabolic-rectangular stress distribution in the concrete is most advantageous for low values of f_{cu} and high ratios of x/d.

For other than simple rectangular sections the calculations with a parabolic-rectangular stress block are often complex and the choice of a rectangular stress block here is most desirable.

For sections reinforced in tension only, it is sometimes slightly advantageous and never disadvantageous (except when x/d is less than 0.1) to use the CP 110 simplified expressions rather than a rigorous analysis with a rectangular stress block although a slight advantage, in terms of achieving an increased resistance moment and a slight reduction in reinforcement, may be obtained by employing a parabolic-rectangular stress block, especially with low values of f_{cu}.

Table 23.8

Concrete strength f_{cu}	Neutral axis-depth factor x/d(i.e., $n_u d/d$)	Resistance moment M_u (i) Parabolic rectangular stress-block	(ii) CP 110 rectangular stress-block	(iii) Modified* rectangular stress-block	Percentage increase in M_u provided by (i) over (ii)	over (iii)
20 N/mm^2	0.3	2.105 bd^2	2.040 bd^2	2.076 bd^2	+3.2%	+ 1.4%
	0.6†	3.536 bd^2	3.360 bd^2	3.504 bd^2	+5.2%	+ 0.9%
40 N/mm^2	0.3	4.071 bd^2	4.080 bd^2	4.152 bd^2	−0.2%	− 2.0%
	0.6†	6.890 bd^2	6.720 bd^2	7.008 bd^2	+2.5%	− 1.7%

* See (b) above.
† Such value of x/d can only be adopted if compression reinforcement is provided.

(b) The assumption of a uniform rectangular stress block is slightly disadvantageous when considering sections subjected to combined bending and thrust where the latter predominates. This is because the assumed shape of the parabolic-rectangular stress block provides some resistance to bending, whereas in such a condition, a uniform rectangular stress block does not. The purpose of the uniform rectangular stress block is to provide a simple, yet fairly accurate representation of the parabolic-rectangular distribution for use in calculations which would otherwise be unnecessarily complex. One simple means of improving this correspondence is to employ a uniform stress of 0.45 f_{cu} over a depth of 0.85 x (instead of 0.40 f_{cu} over a depth of x). The resulting resistance moments given by the latter assumption are also set out in Table 23.8 and show clearly the improved correspondence obtained.

• ***Serviceability Limit State of Cracking***

Following the reasonable conditions for this purpose:

(i) The assessed surface crack width shall not in general exceed 0.3 mm.

(ii) When members are exposed to a particularly aggressive environment, such as the 'very severe' category (i.e., exposed to sea water and with abrasion), the assessed surface crack widths at points nearest the main reinforcement should not exceed 0.004 times the nominal cover to the main reinforcement.

In a 'normal environment' condition (i) will suffice, while in the 'very severe category' the more critical of conditions (i) and (ii) will govern. For this category, the minimum grade of concrete is 40 N/mm^2 and the nominal cover is 60 mm. Or we can have 50 mm cover using grade 50 concrete. Using the latter, we should have the requirement that at points nearest the main reinforcement the crack width should not exceed 0.2 mm whilst it must not exceed 0.3 mm at any other position.

With large covers to the main bars it will generally be found that unless the bars are at fairly close centres it will be the 0.3 mm criteria that will govern the design.

The widths of flexural cracks at a particular point on the surface of a member depend primarily on three factors:

(i) The proximity to the point under consideration, of reinforcing bars perpendicular to the cracks,

(ii) The proximity of the neutral axis to the point under consideration,

(iii) The average surface strain at the point under consideration.

Referring to Fig. 23.4 the surface crack width in reinforced concrete under flexure may be taken as equal to ($3.3c\epsilon_1$) for deformed bars or ($3.8c\epsilon_1$) for plain bars. If $a' < c$, then substitute a' for c.

The formulae give acceptably accurate results in most normal design circumstances, but it should be emphasized that cracking is a random phenomenon and that an absolute maximum crack width cannot be predicted. The formula is designed to give a width of crack which has an acceptably small chance of being exceeded. Thus an occasional crack slightly larger than the predicted width should not be considered as cause for concern. But if a significant number of the cracks in a structure exceed the calculated width, reasons other than the structural nature of the phenomenon should be sought to explain their presence. (In certain cases, the occurrence of cracks due to plastic shrinkage of setting concrete, much before any load comes on the structure, could be a possible confusing reason.)

Columns

The following is applicable for columns which could be under 'only direct load' or 'direct load and bending moment'.

After estimating the load-factored design values* of

* The 'design values' are obtained after taking into account whether the

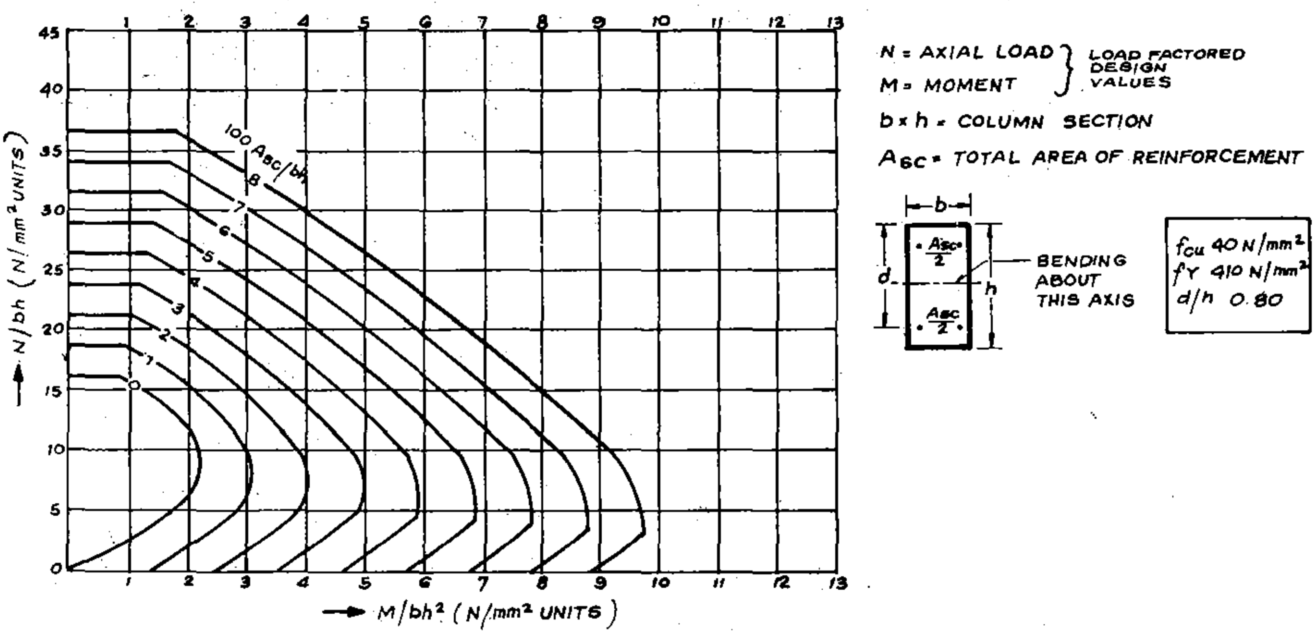


Fig. 23.13 *Rectangular columns*

the axial load and the appropriate moment, the necessary column reinforcement can be calculated by using:

(a) *design charts* for symmetrically reinforced rectangular or circular columns, or

(b) *formulae* for symmetrically or unsymmetrically reinforced rectangular columns, or

(c) *Strain-compatibility* for non-rectangular columns with symmetrical or unsymmetrical reinforcement.

(a) By Using Design Charts These have been prepared using the rectangular parabolic stress block for concrete and the trilinear stress-strain curves for reinforcement as for beams. The charts for rectangular columns are given in Part 2 of BS CP 110 and a typical chart is shown in Fig. 23.13. Charts for circular columns are given in Part 3 of BS CP 110. Each chart is for a particular grade of concrete, a particular characteristic strength of reinforcement, and a particular d/h ratio (i.e., the positioning of the reinforcement).

Knowing N/bh and M/bh^2, the area of reinforcement can be found from the appropriate chart.

It should be noted that A_{sc} is the total area of reinforcement and this is divided equally between the faces parallel to the axis of bending. Any reinforcement in the depth faces of the section is not taken into account.

column is short or long as referred to earlier and as explained in Ch. 10 of this book. If the column is long, then the load and moment values obtained from the first order theory are either directly enhanced by dividing them by the appropriate reduction factor or are modified by doing the buckling analysis (the theory of second order). All this has been very clearly explained in Ch. 10.

On the design chart itself it will be seen that the lines for the percentages of reinforcement flatten off when they approach N/bh axis. This cut off line is to ensure that the accidental moment of 0.05 Nh is automatically allowed for. The 'kinks' in the chart correspond to those in the stress-strain curve for steel.

Above the point B (Fig. 23.14) in the design chart the section is controlled by compression in the concrete and below point B by tension in the reinforcement.

Fig. 23.14 *Relationship of "kinks" on design chart to stress-strain curve of steel*

Note Even for biaxial bending the said design charts can be used with certain additional effort and understanding. For more details see reference 2 and CP 110.

(b) By Using Formulae (listed in Table 23.9) CP 110 gives design formulae (listed in Table 23.9 ahead) which can be derived quite simply and used for rectangular columns with symmetrical or asymmetrical reinforcement in the faces parallel to the axis of bending.

With symmetrical reinforcement, a design chart would generally be used, but with asymmetrical reinforcement the charts mentioned earlier cannot be used. The formulae give a trial and error method (which is a modified form of using 'strain compatibility method', described ahead).

If the ultimate axial load on a column does not exceed the value given by Eq. (1) in Table 23.9

$$N = 0.4F_{cu}b(h - 2e)$$

then, only nominal reinforcement is required, provided the resultant eccentricity $e = M/N$ does not exceed $(h/2-d')$, where h is the depth of the section in the plane of bending, d' is the depth from the surface to the reinforcement in the more highly compressed face.

When the ultimate axial load is greater than that given by Eq. (1), the strength of the section may be assessed using Eqs. (2) and (3) in Table 23.9. The diagrams for the different conditions are shown in Fig. 23.15.

The procedure is to calculate e, assess which of the five cases is appropriate, use Eq. (2) to calculate areas of reinforcement in each face to give the required value of N, and then use this area of reinforcement in Eq. (3) to find the value of M, which should not be less than the ultimate design moment. If the values are not satisfactory, then try a different case, and the results obtained in the first trial should indicate which case to try next (see Examples 1 and 2 ahead).

Use of these Design Formulae

Example 1 Rectangular column 400 × 300 mm carrying a load of 1200 kN and a moment of 120 kNm about the major axis (load factored). Concrete is grade 30(f_{cu} = 30 N/mm^2) and the cover to the main reinforcement is 30 mm. Reinforcement has a characteristic strength of 410 N/mm^2 $h = 400$ mm, $b = 300$ mm, $N = 1200$ kN, $M = 120$ kNm

$$e = \frac{120 \times 10^6}{1200 \times 10^3} = 100 \text{ mm}$$

Assuming $d' = 40, \quad \frac{h}{2} - d' = 200 - 40 = 160$

So, $\quad e < \frac{h}{2} - d'$

From Eq. (1)

$$N = 0.4 \times 30 \times 300(400 - 200) \times 10^{-3}$$
$$= 720 \text{ kN, i.e., < axial load}$$

Table 23.9 *(reference: BS CP 110) Formulae for ultimate resistance against axial load and moment—Rectangular Columns*

Following formulae may be used as appropriate, for the design of a rectangular column section having its longitudinal reinforcement in the two faces parallel to the axis of bending whether that reinforcement is symmetrical or not.

(i) In a column where the ultimate axial load does not exceed the value N given by Eq. (1) only nominal reinforcement is required

$$N = 0.4f_{cu}b(h - 2e) \tag{1}$$

provided the resultant eccentricity $e = M/N$ does not exceed $(h/2 - d')$.

where f_{cu} is the characteristic strength of the concrete

M is the maximum moment due to ultimate loads about the axis considered

b is the breadth of the section

h is the depth of the section in plane of bending,

d' is the depth from the surface to the reinforcement in the more highly compressed face

(ii) When the ultimate axial load N is greater than that given by Eq. (1) the strength of the section may be assessed using Eqs. (2) and (3).

$$N = 0.4f_{cu}bd_c + 0.72f_yA'_{s1} + f_{s2}A_{s2} \tag{2}$$

$$M = 0.2f_{cu}bd_c(h - d_c) + 0.72f_yA'_{s1}(h/2 - d') - f_{s2}A_{s2}(h/2 - d_2) \tag{3}$$

where f_{cu}, M, b, h and d' are as given above.

A'_{s1} is the area of compression reinforcement in the more highly compressed face.

A_{s2} is the area of reinforcement in the other face which may be considered as being either in compression, inactive, or in tension as the resultant eccentricity of load increases and d_c decreases from h to $2d'$.

f_{s2} is the stress in this reinforcement which should not exceed $0.72f_y$ when compressive or $-0.87f_y$ when tensile.

d_2 is the depth to this reinforcement from this face

d_c is the depth of concrete in compression which may be chosen as appropriate, but shall not be taken as less than $2d'$ (the equivalent of neutral axis, which, here, because of direct load and moment, is not the same as effective centroidal axis)

(iii) As an alternative to (ii) when the resultant eccentricity e is not less than $(h/2 - d_2)$, the axial load may be ignored and the column section may be designed to resist purely an increased moment M_a,

$$M_a = M + N(h/2 - d_2) \tag{4}$$

The area of tension reinforcement necessary to provide resistance to this increased moment may be reduced by the amount $N/0.87F_y$.

Note: As in the case of beams and slabs (Table 23.7), here also.

- Maximum compressive strain in concrete = 0.0035
- Maximum compressive stress in concrete = $0.4f_{cu}$ and its distribution is assumed to be rectangular within depth d_c
- Maximum compressive stress in reinforcement: $0.72f_y$
- Maximum Tensile stress in reinforcement: $0.87f_y$

Try Case $(C) d_c = h$ (Fig. 23.15)

$$N = (0.4 \times 30 \times 300 \times 400 + 0.72 \times 410 \times A'_{s1}) \times 10^{-3}$$
$$= 1440 + 0.3A'_{s1}$$

This is greater than the actual load even without taking A'_{s1} into account. But we must have minimum of 1% longitudinal reinforcement, i.e., 1200 mm^2, and minimum bar size of 12 mm,

Try $A'_{s1} = 2/25(982 \text{ mm}^2)$, $A_{s2} = 2/12(226 \text{ mm}^2)$

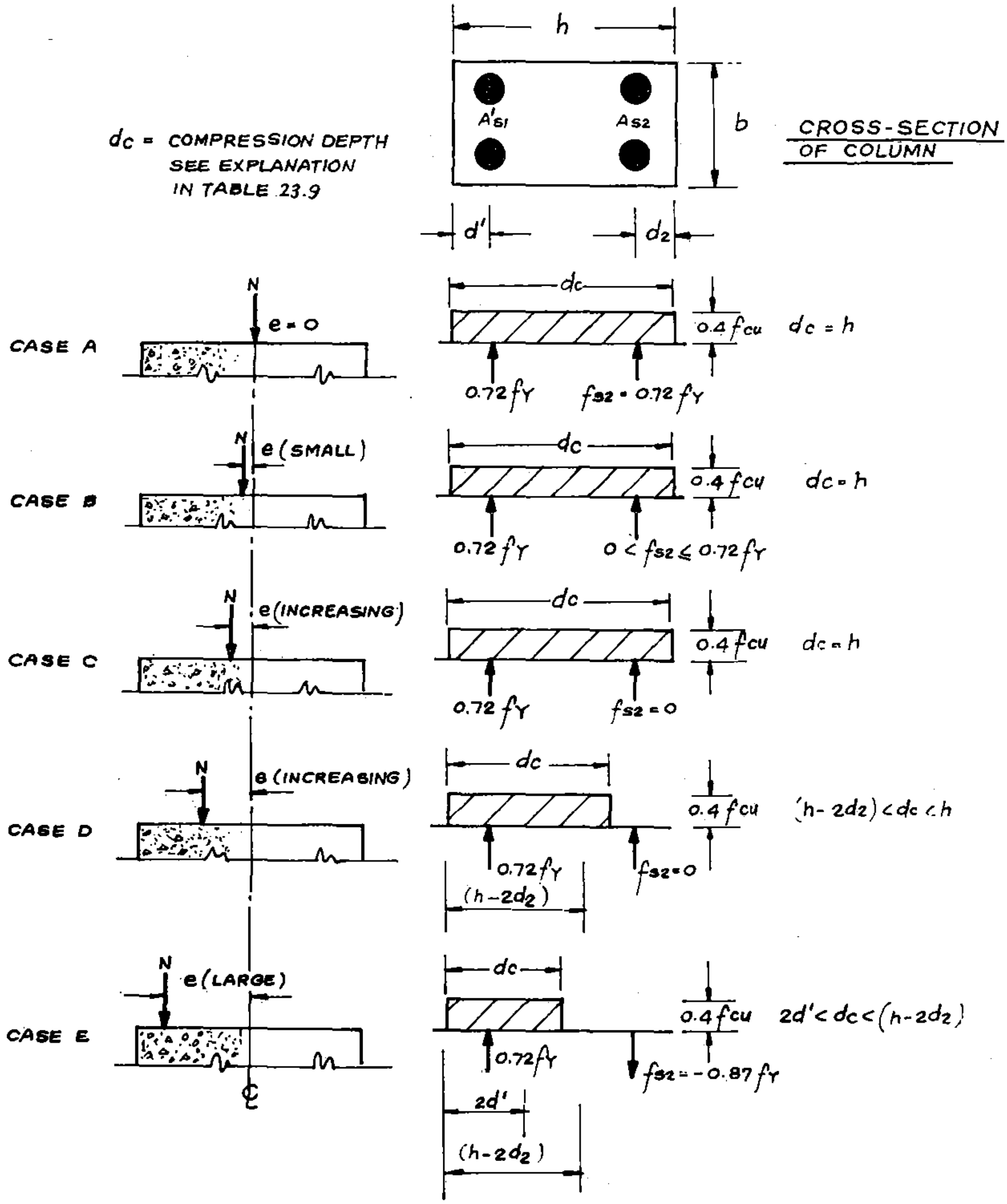


Fig. 23.15 *Diagrams illustrating stress conditions for satisfying Eqs. (2) and (3) in Table 23.9*

$$M = 0 + 0.72 \times 410 \times 982(200 - 40) \times 10^{-6}$$
$$= 46 \text{ kNm}$$

N is too large and M is too small.
Try Case (D) with $d_c = h - 2d_2 = 320$ mm
$f_{s2} = 0$, $A'_{s1} = 982$ mm^2

$$N = (0.4 \times 30 \times 300 \times 320 + 0.72 \times 410 \times 982) \times 10^{-3}$$
$$= 1152 + 290 = 1442 \text{ kN}, > \text{actual}$$
and $M = 0.2 \times 30 \times 300 \times 320(400 - 320) + 0.72 \times 410 \times 982(200 - 40) \times 10^{-6}$
$$= 46 + 46 = 92 \text{ kNm}$$

As N is too large and M is too small we can go to case (E) and we should have to take f_{s2} as $-0.87f_y$ which would reduce N and increase M,
Try Case (E) with $d_c < (h - 2d_2)$ say 300 mm
$f_{s2} = -0.87 \times 410$, $A_{s1} = 982$ mm^2, $A_{s2} = 226$ mm^2

$$N = (0.4 \times 30 \times 300 \times 300 + 0.72 \times 410 \times 982 - 0.87 \times 410 \times 226) \times 10^{-3}$$
$$= 1080 + 290 - 80 = 1290 \text{ kN}$$
and $M = 0.2 \times 30 \times 300 \times 300(100) + 0.72 \times 410 \times 982(160) + 0.87 \times 410 \times 226 \times 160 \times 10^{-6}$
$$= 54 + 46 + 13 = 113 \text{ kN}$$

If we now increase A_{s2} to 2/16 (402 mm^2)

$$N = 1080 + 290 - 143 = 1207 \text{ kN}$$
and $M = 54 + 46 + 23 = 123$ kNm

This is satisfactory, and the assumed reinforcement OK.

Example 2 (See Fig. 23.16)

LOAD FACTORED VALUES:

ULT Design Load = 144 kN
ULT Design Moment = 160 kNm

Assume $f_y = 425$ N/mm^2 $f_{cu} = 30$ N/mm^2

$e = \dfrac{160}{144} = 1.11$ m. This is obviously large for the assumed section dimensions and certainly the A_{s2} will be in tension, so assume a value of d_c less than $(h - 2d_2)$ but more than $2d'$ (i.e., Case E in Fig. 23.15),
$(h - 2d_2) = 500 - 85 = 415$ mm
$2d' = 2 \times 38 = 76$ mm
Try $d_c = 200$ mm and A_{s1} and A_{s2} as shown in Fig. 23.16, hence;

$$N = \{0.4 \times 30 \times 200 \times 200 + 0.72 \times 425 \times 402 + (-0.87 \times 425 \times 981)\} \times 10^{-3}$$
$$= 480 + 123 - 364 = 239 \text{ kN}$$

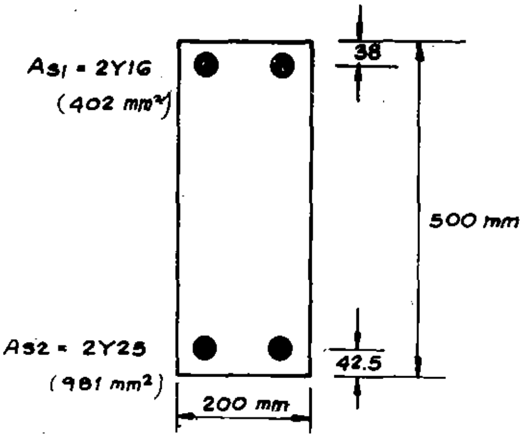


Fig. 23.16

This is then greater than design N of 144 so reduce value of d_c to

$$\frac{385}{0.4 \times 30 \times 200} = 161 \text{ mm}$$

This now gives value for N as 144 kN, hence

$$M = \{0.2 \times 30 \times 200 \times 161(500 - 161) + 0.72 \times 425 \times 402(250 - 38) - (-0.87) \times 425(250 - 42.5)981\} \times 10^{-6}$$
$$= 65.5 + 26.1 + 75.5 = 167.1 \text{ kNm, satisfactory.}$$

Hence assumed section and reinforcement OK.

(c) By Using Strain Compatibility

As with the beams and slabs, if the column has an irregular section, the best method is that of strain-compatibility, using the rectangular stress block for the concrete in compression (value = $0.4F_{cu}$) and maximum compressive strain equal to 0.0035. Again, this can only be done if a first-trial area of reinforcement is assumed first.

Assume a trial neutral axis depth, draw the strain block (as in the case of a beam or a slab, as explained earlier), work out the strain values in the reinforcements (from the strain block, by proportion) and hence tension forces in them and hence the Total Tension, T. Also work out the total compression in concrete and compression steel (if any). If T, C and the externally applied axial load N are in equilibrium them trial is OK, otherwise try again with a more suitable depth of neutral axis, until finally $(T + N) = C$. Then, as in the case of beams and slabs, the ultimate moment of resistance of the assumed section can be found by taking moments of T and C forces about any fibre-level in the section, and its magnitude should be ensured against the applied (load-factored) moment.

• ***Crack Control in Columns***

As per CP110 cracks due to bending in a column designed for an ultimate axial load greater than $0.2f_{cu}A_c$ are unlikely to occur and, therefore, no check is required. A more lightly loaded column subject to bending should be considered as a beam for the purpose of crack control.

• ***Concentrated Loads on Slabs***

The critical section is taken on a perimeter 1.5 h from the boundary of the loaded area as shown in Fig. 23.17.

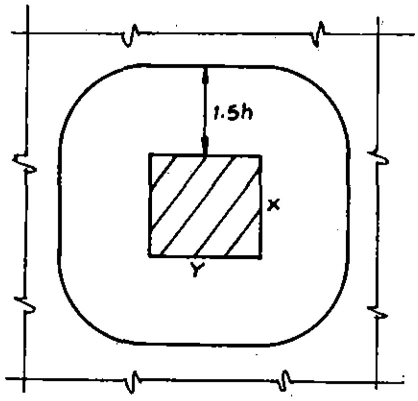


Fig. 23.17 *Perimeter for shear stress in solid slabs (plan)*

The critical perimeter has a length l and if the dimensions of the load contact area are x and y, then:

$$l = 2(x + y) + 2\pi(1.5h) = 2(x + y) + 3\pi h$$

where h is the overall slab thickness.

It is assumed that the shear stress, v, has a constant value throughout the effective depth and length of this critical perimeter. No shear reinforcement is required when this shear stress is within the permissible value stipulated in clause 3.4.5 of BS CP110 to which reference may be made. Generally, shear reinforcement is required in slabs only in special circumstances (e.g., in flat-slabs).

23.4 DETAILING

The UK Concrete Society's report on *Standard Details* and their joint report with the Institution of Structural Engineers on *The Detailing of Reinforced Concrete* may be referred to for workmanlike detailing. The detailing of reinforcement is no less important than designing the sections. For this purpose, due note should be taken of the relevant local stipulations and workmanlike details. Some relevant information is also given in a separate chapter in this book which may be referred to. Practical detailing is learnt only by actually working on the 'drawing board' in a practising office, not by merely computing the stresses, much less by merely proof reading others' calculations and drawings in the name of checking.

REFERENCES

1. Classical works by authorities like KW Johansen, LL Jones, RH Wood, ALL Baker, Hognestad, Hansen, McHenry, Hajnal Konyi, FN Pannell, and Laupa, Seiss and Newmark.
2. "RC Design to CP 110 Simply explained", by AH Allen (from which some material has been taken with grateful thanks).
3. UK Concrete Society Report on 'Standard Details'.
4. 'The Detailing of Reinforced Concrete', joint Report by the UK. Concrete Society and the UK Institution of Structural Engineers, London.

CHAPTER 24

Practical Design against Shear and Torsion and Design of Short-cantilevers and Deep-beams

24.1 PRINCIPLE OF DESIGN AGAINST SHEAR

Concrete subjected to (bending and) shear will fail in tension when the principal tensile stress (diagonal tension) exceeds its modulus of rupture. This would suggest that principal tensile stress should obviously be guarded against. This stress t at a particular fibre level is calculatable from the following expression derived on the basis of classical elastic theory assuming that the material is homogeneous, crack-free, isotropic and elastic,

$$t = -\frac{f}{2} - \sqrt{\left(\frac{f}{2}\right)^2 + s^2} \text{ (−ve sign for tensile stress)}$$

where f = the magnitude of the bending tensile stress
s = the magnitude of the transverse shear stress

at the particular fibre level under consideration. Greatest t will occur where a greatest s and greatest f coexist. In reinforced concrete, greatest permissible f being zero, t then equals s and this is why in reinforced concrete, traditionally only shear stress s used to be checked in the elastic analysis. However, in prestressed concrete, at the fibre where s is greatest, f could be compressive, zero or tensile since some flexural tension may be permitted. This would suggest to calculate t value at various critical fibre levels (at each critical section) in such a case, and this is what used to be done traditionally (and the greatest t was restricted to a permissible fraction of the modulus of rupture).

However, since shear does not accept any significant redistribution (unlike flexure) and, therefore, its failure can be rather sudden (which is why seldom did any code permit any increase in shear stress, unlike in bending stress), we have to be more careful in handling shear. Since the aforementioned elastic formula is based on the stated assumptions none of which are really accurately true in case of concrete (because concrete at least has micro cracks in it and is not truly elastic, homogeneous or isotropic) it is, therefore, wholly irrational to design for shear by trying to calculate and control principal tension in this manner. It would be more rational in case of shear to somehow directly consider the concrete in its cracked state (the load-factor approach), even if one took into account the shear which the unreinforced concrete could take till it cracked (reinforcement taking only the balance). This would be more realistic than using the purely elastic approach for shear under the realistic conditions that prevail. (In structural engineering it is more rational to attempt an approximate solution to an exact problem than a solution to an approximated problem, for obvious reasons.)

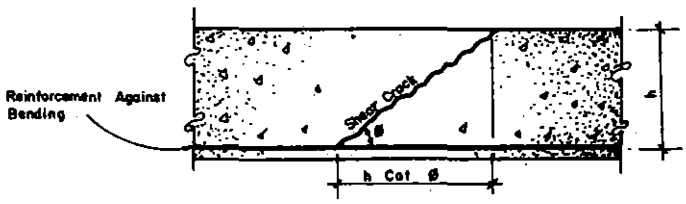


Fig. 24.1

Explained in simple words, if vertical shear reinforcement is A_v at pitch p, then the shear force V it can take within the spread of the shear crack in a concrete member shown in Fig. 24.1 is,

$$V = \frac{A_v}{p} h \cot\phi f_s$$

where f_s is the tensile stress in the shear reinforcement.

In the limit, ultimate shear on reinforcement would be,
= (Ultimate shear on the section) − (shear capacity of unreinforced concrete)
$= V_u - V_c$
with f_s reaching the value of tensile yield stress in reinforcement, f_{sy}
And assuming approximately that $\phi = 45°$ and h is the lever arm d between longitudinal tension and compression forces in the section, then,

$$V_u - V_c = \frac{A_v}{p} d f_{sy}$$

so that shear reinforcement required would simply be,

$$A_v = \frac{(V_u - V_c)p}{f_{sy}d}$$

In this formula,

- V_c, in kg, may be assumed equal to $(10bd)$, where b (the breadth of rib) and d (as defined below) are in cm
- d = 0.8 of effective depth in case of reinforced concrete
  = effective depth or 0.8 of overall depth, whichever greater, in case of prestressed concrete,
- V_u = load factored dead and live load shears, less any relief in shear due to prestress and sloping soffit (assuming unit load factor on prestress)
- f_{sy} = yield stress in vertical stirrups whose cross sectional area is A_v placed at a horizontal pitch p.

However, if $\frac{V_u}{bd}$ exceeds 0.15 of the standard 28-day concrete cylinder crushing strength, the section should be increased in order not to exceed this limit and only then A_v calculated. (This condition is from an overall philosophical standpoint to avoid shear-compression failure and to cover the effects of approximations made in converting the structure into the approximated mathematical model.)

Also note that it is preferable to provide vertical stirrups for shear rather than crank the flexural reinforcement since in the latter detail the resultant vectorial outward force at the bend in the cranked reinforcement has the tendency to wedge-out the concrete lying within the bend.

24.2 DESIGN OF SHORT-CANTILEVERS, CORBELS AND BRACKETS

The design procedure for short cantilevers, brackets and corbels recognizes the deep beam or simple truss action of these short shear-span members, as illustrated in Fig. 24.2. Four possible failure modes must be controlled: (a) Direct shear failure at the interface between bracket or corbel and supporting member; (b) Yielding of the tension tie due to moment and direct tension; (c) Crushing of the internal compression strut; and (d) Localized bearing or shear failure under the loaded area.

The design provisions given here apply only to members having a shear span-to-depth ratio of unity or less ($a/d' \leq 1$) since, for longer spans, diagonal tension cracks may form and the use of 'horizontal shear reinforcement' may not suffice. Furthermore, the method has not been validated by tests beyond $a/d = 1$. (For cases with $a > d'$, the usual design procedures for flexure and for shear should be applied.)

The critical section is designed to resist simultaneously a

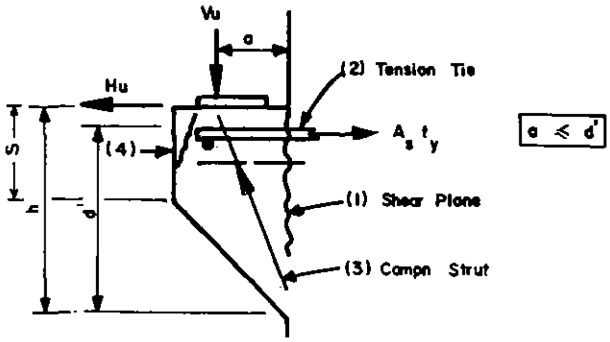


Fig. 24.2 *Structural action of short cantilevers*

shear V_u, a moment $[V_u a + H_u(h - d')]$, and a horizontal tensile force H_u; latter being caused either due to friction at the bearing area (and) or due to restrained shrinkage, creep and thermal deformation.

For design purposes the total reinforcement required is divided into three parts, with each determined separately — (A_{vf}) area of shear-friction reinforcement to resist direct shear V_u; (A_f) area of flexural reinforcement to resist moment $V_u a + H_u$, $(h - d')$; and (A_t) area of tensile reinforcement to resist direct tensile force H_u.

Once the separate areas of reinforcement A_{vf}, A_f and A_t have been determined, the actual reinforcement to be provided, A_s and A_h, may be sized, where A_s will act as the primary tension reinforcement and A_h will act as shear reinforcement (placed as horizontal stirrups, one below the other, below A_s).

Design Steps

(See Fig. 24.3)

Condition 'Shear-span to effective depth ratio'; a/d', is ≤ 1. (If $a > d'$, design for flexure and shear as per usual procedures.)

Step 1 Ensure $S/d' \geq 0.5$

Step 2 Ensure $\frac{V_u}{bd} \leq 0.15 f'_c$, otherwise revise section dimensions

V_u = ultimate shear value
b = width of cantilever/bracket/corbel
f'_c = 28-day standard cylinder strength of concrete used.
d = 0.8 of effective depth $d'(d = 0.8d')$

Step 3 Calculate shear-friction reinforcement A_{vf}:

$$A_{vf} = \frac{V_u}{0.85 f_{sy}\mu}$$

f_{sy} = yield stress value of the reinforcement used
μ = 1.4 for concrete placed monolithically across interface

1.0 for concrete placed against hardened concrete but with roughened surface
0.7 for concrete anchored to structural steel
0.6 for concrete placed against hardened concrete but with surface not roughened.

NOTE Only monolithic construction recommended

Step 4 Calculate direct-tension reinforcement A_t:

$$A_t = \frac{H_u}{0.85 f_{sy}}$$

H_u = 1.7 × actual horizontal force in working load condition if clearly defined
but $\nless 0.2V_u$

Step 5 Calculate flexural-tension reinforcement A_f:

$$A_f = \frac{[V_u a + H_u(h - d')]}{0.85 f_{sy} d}$$

$d = 0.8d'$ (as already defined), all other symbols as already defined or shown in Figs. 24.2 and 24.3.

Step 6 Compute total primary tensile reinforcement A_s

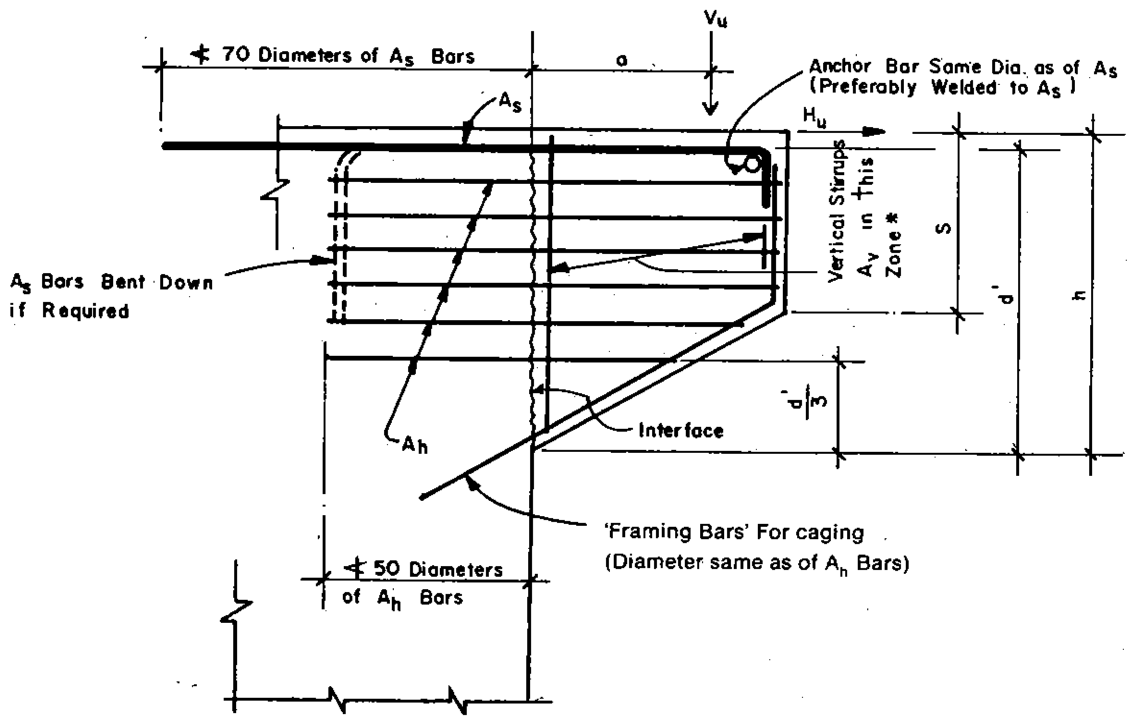
$$\left.\begin{aligned} A_s \quad & \nless (A_f + A_t) \\ & \nless \left(\frac{2}{3}A_{vf} + A_t\right) \\ & \nless (0.04 f_c'/f_{sy}) bd' \end{aligned}\right\} \begin{array}{l}\text{Provide largest of} \\ \text{these three magnitudes} \\ \text{as } A_s\end{array}$$

Step 7 Calculate total section area A_h of stirrups (closed ties) to be provided horizontally, one below other, below and next to A_s:

$A_h \nless 0.5A_f$ and $\nless 0.333A_{vf}$, provide larger of the two. These stirrups shall be provided below A_s and within a depth of $2/3d'$ below A_s, as indicated in Fig. 24.3.

24.3 DESIGN OF AN ARTICULATION (i.e., A HALVING JOINT)

A cut-out 'recessed' seating arrangement whereby one deck sits on another, may be referred to as an 'articulation' or a 'halving joint'. An articulation is obviously a very important part of the whole and it needs special attention



$$* \; A_v = 0.50 \frac{(V_u - V_c)\,p}{f_{sy}\, d}$$

symbols as explained earlier.

Fig. 24.3 *Detailing of A_s and A_h reinforcements in a short cantilever (or bracket or corbel) (any other reinforcement not shown).*

in design, construction and maintenance. Heavy shear and bending stresses on account of abrupt reduction of depth of section occur at the articulations. The design of an ordinary reinforced concrete articulation is generally done by superimposing the bending stresses due to vertical and horizontal reactions at the bearing and the shear stresses along a plane emanating from the end of the articulation block. The assumption is that the concrete in the articulation will crack in an undesirable manner. These cracks should be sealed with a rigid filler (rich non-shrink cement grout or, preferably, a suitable epoxy resin formulation) soon after their appearance and ceasing to widen further.

Design Steps

(See Fig. 24.4)

Condition Shear span to effective depth ratio, a/d', is ≤ 0.6. (If $a/d' > 0.6$, then redimension to suit.)

Step 1 Ensure that $S \nless 0.4$ of overall depth of main part of beam.

Step 2 Ensure that $\frac{V_u}{bd} \leq 0.15 f_c'$.

(otherwise revise section dimensions)

V_u = ultimate shear value

b = width of articulation section

f_c' = 28-day standard cylinder strength of concrete used

d = 0.8 of effective depth $d'(d = 0.8d')$

Step 3 Calculate total horizontal steel A_s:

$$A_s = A_{vf} + A_t$$

$$\text{where } A_{vf} = \frac{V_u}{0.85 f_{sy} \mu}$$

$$A_t = \frac{H_u}{0.85 f_{sy}}$$

A_{vf} = shear-friction reinforcement

A_t = direct-tension reinforcement

f_{sy} = yield stress value of reinforcement used

μ = 1.4 for concrete placed monolithically across interface, and only monolithic concreting recommended

H_u = 1.7 $\times$ actual horizontal force in working load condition, if clearly defined, but $\nless 0.2 V_u$ for this calculation

Step 4 Calculate total vertical stirrups A_v:

$$A_v = A_s/\mu$$

Step 5 Calculate total horizontal stirrups A_h:

$$A_h = 0.5 A_s$$

Step 6 Calculate total inclined stirrups A_i and provide these inclined stirrups at angle θ to horizontal (preferably 45°) such that they intersect the line of action of V_u, going well past outer edge of the bearing.

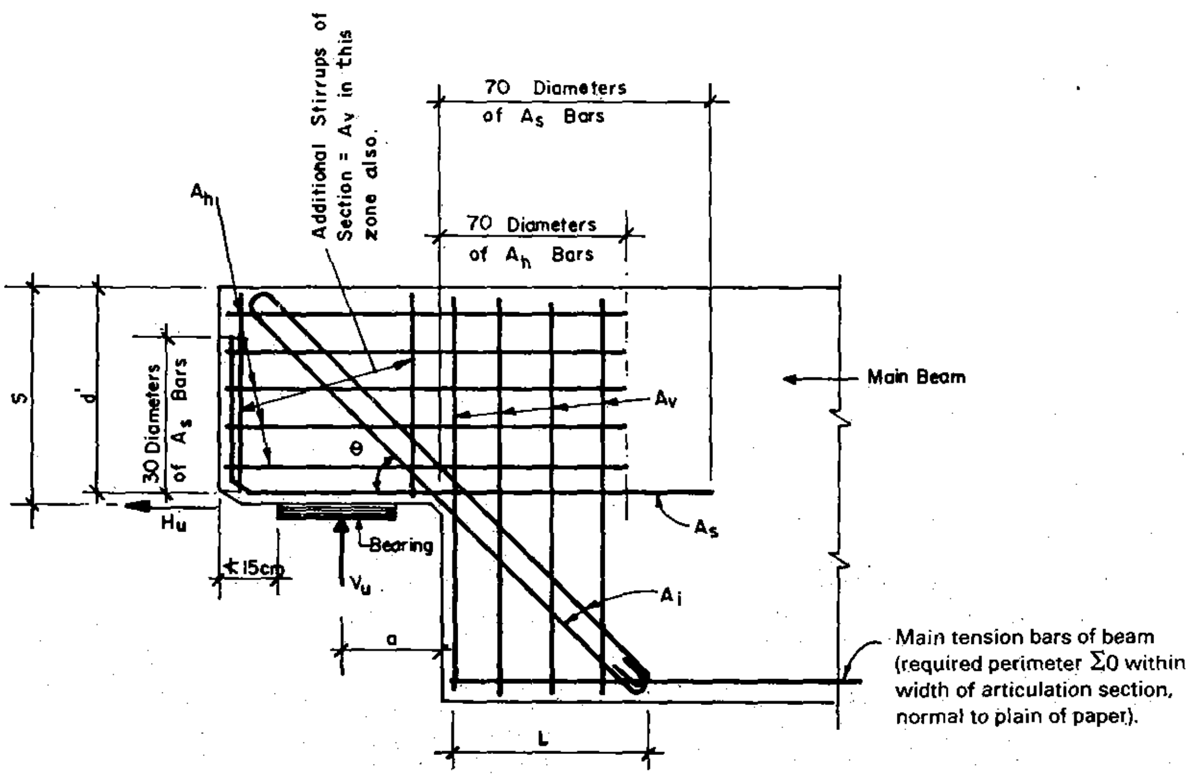


Fig. 24.4 *Detailing of A_i, A_s, A_y and A_h reinforcements in an articulation (other reinforcements not shown, main tension bars of beam only indicated).*

A_i shall be larger of the two values obtained from the following two equations:

(i) $A_i f_{sy} \sin\theta = \dfrac{V_u}{0.85}$ (for resisting V_u)

(ii) $A_i \cos\theta = \dfrac{[V_u a + H_u(S - d')]}{0.85 f_{sy} d}$

(for resisting moments)

Step 7 Ensure that $l \nless \dfrac{0.5V_u}{\Sigma 0 f_{bu}}$

$\Sigma 0$ = perimeter of the main tension reinforcement of the main beam at the articulation (see Fig. 24.4)

f_{bu} = ultimate anchorage bond stress between the aforementioned main tension reinforcement and the concrete (ranges from 14–19 kg/cm^2 for plain bars and 19–28 kg/cm^2 deformed bars, for concrete grades 200–350 kg/cm^2 standard concrete cylinder strengths, respectively).

24.4 DESIGN OF DEEP BEAMS

Design of deep beams (depth > about half the clear span) is a special subject. The design proposals produced by Kong, Sharp and others are based on the results of several hundred tests and, unlike most other procedures, are also applicable to deep beams with web openings. Details of the method are presented ahead with slight modifications on coefficients k_1 and k_2 to suit partial load factors and the assumed load factor approach. As the depth of a beam becomes greater in proportion to its span, the distribution of stress differs from that assumed for a normal beam. In addition, the particular arrangement of the applied loads and of the supports has an increasing influence on this stress distribution. Thus if the ratio of clear span to depth is less than 2 to 3 for a freely-supported beam, or 2$^1/_2$ to 4 for a continuous system, it should be designed as a deep beam.

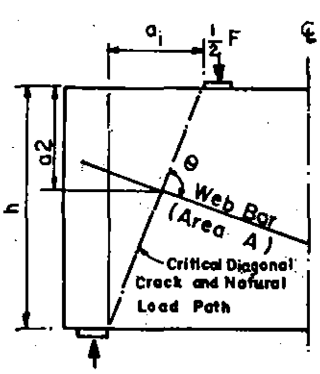


Fig. 24.5

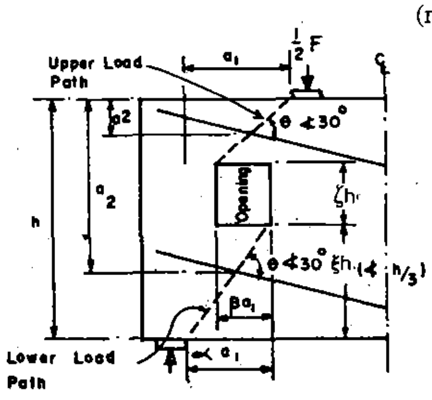


Fig. 24.6

Notation (Figs. 24.5 and 24.6)

A	area of individual web bar
$A_{s_{req}}, A_{s_{prov.}}$	minimum area of main steel required and actual area provided
a_1	clear distance from edge of load to face of support
αa_1	distance from inner edge of opening to face of support
βa_1	width of opening
a_2	depth at which web bar intersects critical diagonal crack
b	breadth of beam
d	effective depth to main steel
f_t	cylinder splitting tensile strength of concrete (see Table 24.1 ahead)
f_{sy}	yield strength of reinforcement
h	overall depth of beam
k_1, k_2	empirical coefficients for concrete and reinforcement. Take k_1 as 0.7 for normal-weight concrete and 0.5 for light-weight concrete: take k_2 as 100 for plain round bars and 225 for deformed bars.
l	span of beam between centres of supports
M	ultimate moment
V	ultimate shearing force
V_1	shearing force resisted by concrete and main reinforcement only
θ	angle between bar being considered and critical diagonal crack
ξ	distance of bottom of opening from beam soffit expressed as proportion of total depth of beam
ζ	depth of opening expressed as proportion of total depth of beam

Design Procedure (Steps)

(refer to table ahead)

(i) Calculate ultimate bending moment M acting on beam

(ii) Calculate area of main reinforcement required from formula (A)

(iii) Calculate ultimate shearing force V acting on beam

(iv) Calculate suitable minimum breadth of beam (or check, if breadth is specified) from formula (B)

(v) Sketch elevation of beam and compute angle θ for main steel

(vi) Calculate shearing resistance V_1 for beam with main reinforcement only from formula (C); thus determine shearing resistance $(V - V_1)$ to be provided by web reinforcement

(vii) From sketch of beam, measure values of θ and a_2 for each individual web bar

(viii) Calculate area of web bars required from formula (D).

Design formula	Without openings in beam	With openings in beam
(A)	$A_{s_{req}} \nless \frac{1.9M}{f_{sy}l}$ or $\frac{1.9M}{f_{sy}h}$	$A_{s_{req}} \nless \frac{1.9M}{f_{sy}l}$ or $\frac{1.55M}{f_{sy}\xi h}$
(B)	$b \simeq \frac{0.65V}{k_1(h-0.35a_1)f_t}$	$b \simeq \frac{0.55V}{k_1(\xi h - 0.35\alpha a_1)f_t}$
(C)	$V_1 = k_1(h-0.35a_1)f_t b + k_2 A_{s_{prov}} d\sin^2\theta/h$	$V_1 = k_1(\xi h - 0.35\alpha a_1)f_t b + k_2 A_{s_{prov}} d\sin^2\theta/h$
(D)	$V - V_1 = k_2 \Sigma A a_2 \sin^2\theta/h$	$V - V_1 = 1.5k_2 \Sigma A a_2 \sin^2\theta/h$

NOTES

(i) The formulae are only known to be applicable if the following conditions apply: $l/h \ngtr 2$. Static loads only occur and these are applied to top of beam only a_1/h is not greatly outside range of 0.23 to 0.70. Positive anchorage is provided to main reinforcement.

(ii) Restrictions to θ and ξh shown in diagrams only apply when opening intersects line of critical diagonal crack. If opening is reasonably clear of this line, the effect of the opening may be disregarded completely when considering shearing resistance.

(iii) For distributed loads, substitute statically-equivalent twin concentrated loads (i.e., replace uniform load F by two concentrated loads of $\frac{1}{2}F$ at distances of $\frac{1}{4}l$ from supports.

(iv) The more nearly perpendicular a web bar is to the principal diagonal crack, the more effective it is in resisting shearing and limiting cracking: its effectiveness also increases with increasing depth a_2. However, inclined web reinforcement may be more expensive to bend and fix.

(v) If openings are present, web reinforcement must pass both above and below them.

Table 24.1

If cylinder splitting tensile strength is not known, estimate as follows:

Cube strength f_{cu} (N/mm^2)	Cylinder splitting tensile Strength f_t(N/mm^2)
20	2.24
25	2.50
30	2.74
40	3.16
50	3.54

Example Design the reinforcement for the deep beam shown in Fig. 24.7 which supports an ultimate load that (including self-weight) can be represented by the twin concentrated loads shown of 625 kN, using 25 grade concrete and mild steel bars (cube strength = 25 N/mm^2). The concentrated loads exert a bending moment of $625 \times 10^3 \times 400 = 250 \times 10^6$ N-mm on the beam. Thus the area of main reinforcement required is the greater of either

$$\frac{1.9M}{f_{sy}l} = \frac{1.9 \times 250 \times 10^6}{250 \times 1{,}650} = 1{,}152 \text{ mm}^2$$

or

$$\frac{1.55M}{f_{sy}\xi h} = \frac{1.55 \times 250 \times 10^6}{250 \times 800} = 1{,}938 \text{ mm}^2$$

Provide four 25 mm bars ($A_s = 1{,}963$ mm^2) against bending. Since $f_{cu} = 25$ N/mm^2, take $f_t = 2.5$ N/mm^2. Then, since $V = 625 \times 10^3$,

breadth of section required $\simeq \frac{0.55V}{k_1(\xi h - 0.35\alpha a_1)f_t} = \frac{0.55 \times 625 \times 10^3}{0.7(800 - 0.35 \times 300)2.5} = 283$ mm;

say $b = 300$ mm. Thus the shearing resistance provided by the concrete together with the main reinforcement only is

$$\begin{aligned} V_1 &= k_1(\xi h - 0.35\alpha a_1)f_t b + k_2 A_{s_{prov}} d\sin^2\theta/h \\ &= 0.7(800 - 0.35 \times 300)2.5 \times 300 + \\ &\quad 100 \times 1{,}963 \times 1{,}425 \times 0.877/1{,}500 \\ &= 528 \times 10^3 \text{ N}, \end{aligned}$$

(note that $\tan\theta = 800/300$, hence $\sin^2\theta = 0.877$)

Thus the balance of $625 \times 10^3 - 528 \times 10^3 = 97 \times 10^3$ must be provided by the web reinforcement. If horizontal links are provided at the depths shown in Fig. 24.7 $\sin^2\theta = 0.877$ for each link and since $\Sigma A a_2 \sin^2\theta/h = (200 + 350 + 750 + 900 + 1{,}050 + 1{,}200 + 1{,}350)A \times 0.877/1{,}500 = 3.39A$, thus

$A = (V - V_1)/(1.5k_2 \times 3.39) = 97 \times 10^3/(3.39 \times 100 \times 1.5) = 191$ mm^2 (2 legs of a link). Provide 12 mm dia. 2 legged links horizontally and vertically as shown.

NOTE The Europeon Concrete Committee recommends that the section areas of vertical bars and horizontal bars should not be less than 0.25% in case of mild steel/0.20% in case of deformed bars, of horizontal and vertical cross-sectional areas of the beam, respectively.

24.5 DESIGN AGAINST COMBINED SHEAR AND TORSION

The basic philosophy explained earlier remains valid here also. Since transverse shear (vertical or horizontal) as well as torsion, both cause shear stresses, it is only rational to consider them simultaneously and suggest a combined approach for total shear-torsion design. Where torsion is absent, the terms corresponding to torsion can be ignored without affecting the combined approach.

Behaviour Under Torsion, Flexure and Associated Shear

When a section is subjected to pure torsion, the torsional shear stress thus caused can have different values at different fibres in the section depending on the shape of the section. But, as the torque approaches its ultimate value, a certain

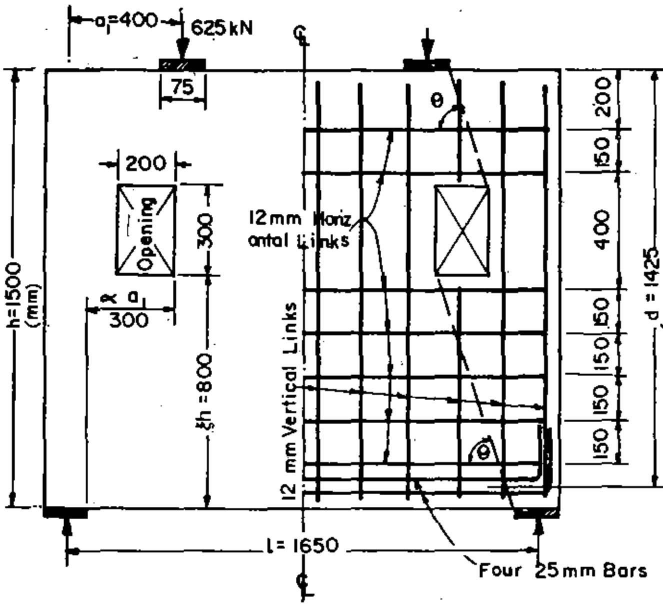


Fig. 24.7 *(Based on Fig. 24.6)*

amount of redistribution of stress takes place among the fibres in that section with the result that, ultimately a more or less unifrom stress exists over a large part of the section. For practical purposes, the ultimate torsional shear stress is, therefore, assumed constant throughout the section.

Flexure creates internal longitudinal compression (and tension) in a member. This compression helps in halting the propagation of spiral-cracking caused by torsion. For this reason, the presence of bending moment does not appear to decrease pure torsional strength of a member except when the applied bending moment is greater that about 80% of the ultimate moment of resistance of the section.[1,2]

Even when a section, which is under combined torsion, shear and moment, has cracked, it still has torsional strength owing to the presence of internal longitudinal compression which locks the torsional cleavage-fracture in the compression zone. This allows the unreinforced torsion capacity (T_c) of a concrete section to be calculated by simply limiting the principal tensile stress at the centroidal level to a suitable value.

Torsion failure, unlike most flexural failures, is sudden (as is a transverse shear failure), and is particularly violent in prestressed concrete owing to unleashing of stored energy in tendons. Torsional moment is associated with very little redistribution, as is the case with shear.

For pure or predominant torsion, cracks first appear on the longer faces of the section and at about 45° to the axis of member. But in pure or predominant bending, the cracks usually appear first on the face stretched by bending, then spread to the side faces, and finally to the compression face, by when concrete in the compression zone may also show distress and disintegration owing to crushing.

Under combined bending and torsion, the cracking is intermediate between about 45° spiral-form and emanating inclined form, and greater the effect of bending moment the steeper is the direction of cracks.

Effect of Reinforcement

Provision of (closed) hoops and longitudinal steel crossing the lines of potential cracks increases the torsional strength of a concrete member, delays the collapse, and lessens the violence of failure. Torsion causes diagonal tension on all faces of a member and hence needs closed hoops and longitudinal bars all round. Transverse vertical shear causes shear stress which varies from zero at top and bottom fibres to a maximum in between, correspondingly causes diagonal tension only on vertical faces and, therefore, needs only vertical stirrups and/or bentup bars.

Transfer of Shear after Cracking

In reinforced concrete, where sections are cracked even under working load, shear is transferred from one (cracked) section to another as a moment caused by the lever arm between the differentials in longitudinal (flexural) compression and tension across the section. The shear, of course, can be the transverse shear or the torsional shear, or both. Transfer of this shear will be accomplished so long as there is bending moment to create the aforesaid differentials. In prestressed concrete, during the elastic phase the sections are theoretically assumed crackfree (by allowing no or little tension) and as such, the shear is assumed to be transferred from section to section as in a homogeneous body. In the ulitmate stage, when cracks have occurred and, therefore, the flexural tensile stress in concrete has been relieved, the situation becomes akin to that in cracked *RC*, the only addition being the presence of a direct stress. Consequently, the shear-transfer from one (cracked) section to another is accomplished in no way less than it is in reinforced concrete.

Design Approach

The upper limit to the usefulness of torsional reinforcement is reached when principal tensile stress approaches a ceiling.[2] Although longitudinal and hoop steel help limit the spreading of diagonal cracks, a stage comes when the concrete between the diagonal cracks gives-in as a compression strut. Any additional reinforcement will obviously be useless. Hence the limitation on the sum of the transverse and torsional shear stresses (see step 3 ahead).

The diagonal tension due to torsion may be taken by the vectors of the tensions in the hoops and the longitudinals, while the diagonal compression due to torsion may be