§2.3.52.

Proof. $\lim\limits_{x\to c} f(x) = L$

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0$ s.t. if $|x-c|<\delta$ then $|f(x)-L|<\varepsilon$.

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0$ s.t. if $|(x-c)-0|<\delta$ then $|f((x-c)+c)-L|<\varepsilon$.

$\Leftrightarrow \lim\limits_{x-c\to 0} f((x-c)+c) = L$

Let $h=x-c$, $\lim\limits_{h\to 0} f(h+c) = L$.

§2.3.54.

Proof. Let $f(x)=x$, $x_0=0$, $L=10$,

$\forall \varepsilon>0$, choose $x=10$ s.t. $|f(x)-L| = 0 < \varepsilon$.

but $\lim\limits_{x\to 0} f(x) = 0 \neq 10$.

§2.4.5

a. Sol. $\lim\limits_{x\to 0^+} f(x)$ doesn't exist.

Assume there exists a limit L s.t.

$\forall \varepsilon>0, \exists \delta>0$ s.t if $0<x<\delta$ then $|f(x)-L|<\varepsilon$.

① when $x = \dfrac{1}{\frac{\pi}{2}+2k\pi}$ $\sin\frac{1}{x} = 1$.

Let $0<k<\dfrac{\frac{1}{\varepsilon}-\frac{\pi}{2}}{2\pi}$ then $0 < \dfrac{1}{\frac{\pi}{2}+2k\pi} < \varepsilon$

$\Leftrightarrow \exists x\in(0,\varepsilon)$ s.t. $f(x) = 1$.

② when $x = \dfrac{1}{\frac{3\pi}{2}+2k\pi}$, $\sin\frac{1}{x} = -1$.

Let $0<k<\dfrac{\frac{1}{\varepsilon}-\frac{3\pi}{2}}{2\pi}$ then $0 < \dfrac{1}{\frac{3\pi}{2}+2k\pi} < \varepsilon$.

$\Leftrightarrow \exists x\in(0,\varepsilon)$ s.t. $f(x) = -1$.

So we have: $\begin{cases} |1-L|<\varepsilon \\ |-1-L|<\varepsilon \end{cases} \Rightarrow \varepsilon > \frac{1}{2}(|1-L| + |-1-L|)$

$= \frac{1}{2}(|1-L| + |1+L|)$

$\geq \frac{1}{2}\times 2 = 1$

So $\varepsilon > 1$

Thus lead to contradiction.

b. Sol $\lim_{x\to 0^-} f(x) = 0$

$\forall \varepsilon > 0$, choose $\delta = 1$.

s.t. if $0 < -x < \delta = 1$ then $f(x) = 0 < \varepsilon$

so $\lim_{x\to 0^-} f(x) = 0$.

c. Sol. $\lim_{x\to 0} f(x)$ doesn't exist

Since $\lim_{x\to 0^+} f(x)$ doesn't exist., then $\lim_{x\to 0} f(x)$ ~~doesn't~~ exist

$\lim_{x\to a} f(x)$ exists if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

§ 2.4.69

Sol. At most 1 horizontal asymptote

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ then the ratio of f(x)'s and g(x)'s leading coefficents is L

so $\lim_{x\to -\infty} \frac{f(x)}{g(x)} = L$ which means there exists 1 horizontal asymptote at most.

A1. ① $\lim_{x\to 0} x^2 \lfloor \frac{1}{x} \rfloor$

Sol. Since $x^2(\frac{1}{x} - 1) < x^2 \lfloor \frac{1}{x} \rfloor \leq x^2 \cdot \frac{1}{x}$,

$\lim_{x\to 0} x^2(\frac{1}{x} - 1) = \lim_{x\to 0} x - \lim_{x\to 0} x^2 = 0$.

$\lim_{x\to 0} x^2 \cdot \frac{1}{x} = \lim_{x\to 0} x = 0$

then $\lim_{x\to 0} x^2 \cdot \lfloor \frac{1}{x} \rfloor = 1$ (The Sandwich Theorem)

② $\lim_{x\to 0} x\cos(\frac{1}{x})$

Sol. Since $-x \leq x\cos(\frac{1}{x}) \leq x$,

$\lim_{x\to 0}(-x) = 0$, $\lim_{x\to 0}(x) = 0$

then $\lim_{x\to 0} x\cos(\frac{1}{x}) = 0$ (The Sandwich Theorem)

A2.

Pf. Since $\lim_{x\to+\infty} f(x)=L$ and $f(x)$ is an even function.

then $\lim_{x\to+\infty} f(-x)=L$

$\Rightarrow \lim_{-x\to+\infty} f(x)=L$

$\Rightarrow \lim_{x\to-\infty} f(x)=L$

A3.

Pf. $\lim_{x\to+\infty} g(x)=\lim_{x\to+\infty}(f(x)^2-f(x+1)f(x-1))$

$=(\lim_{x\to+\infty} f(x))\cdot(\lim_{x\to+\infty} f(x))-(\lim_{x\to+\infty} f(x+1))(\lim_{x\to+\infty} f(x-1))$

$=L\cdot L-L\cdot L$

$=0$.

A4.

Pf. $\lim_{x\to0}\frac{1-\cos(x)}{x^2}=\lim_{x\to0}\frac{1-(1-2\sin^2(\frac{x}{2}))}{x^2}$

$=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{x^2}$

$=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{4\cdot(\frac{x}{2})^2}$

$=\frac{2}{4}\times\left(\lim_{\frac{x}{2}\to0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$

$=\frac{1}{2}\times1^2$

$=\frac{1}{2}$.