Problem 5.1.2.

Sol. $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = \frac{1}{8} \times (-1) = -\frac{1}{8}$

$\det(-A) = (-1)^3 \det A = (-1)^3 \times (-1) = 1$

$\det(A^2) = (\det A)^2 = (-1)^2 = 1$

$\det(A^{-1}) = \dfrac{1}{\det A} = -1$

Problem 5.1.7

Sol. $\det Q = \cos^2\theta + \sin^2\theta = 1$

$$\begin{aligned}\det Q &= (1-2\cos^2\theta)(1-2\sin^2\theta) - 4\cos^2\theta\sin^2\theta \\ &= 1 + 4\cos^2\theta\sin^2\theta - 2\cos^2\theta - 2\sin^2\theta - 4\cos^2\theta\sin^2\theta \\ &= -1\end{aligned}$$

Problem 5.1.13.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$\det A = \det U = 1 \times 1 \times 1 = 1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -3/2 \end{bmatrix} = U$$

$\det A = 1 \times (-2) \times (-\frac{3}{2}) = 3$

Problem 5.1.18.

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)$$

Problem 5.1.30.

$$\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$

Problem 5.2.1.

$$\begin{aligned} \det A &= 1\times1\times1 + 2\times2\times3 + 3\times3\times2 - 3\times1\times3 - 2\times2\times1 - 1\times2\times3 \\ &= 1+12+18-9-4-6 \\ &= 12 \quad \Rightarrow \text{the rows of } A \text{ are independent.} \end{aligned}$$

$$\begin{aligned} \det B &= 1\times4\times7 + 2\times4\times5 + 4\times6\times3 - 3\times4\times5 - 1\times4\times6 - 2\times4\times7 \\ &= 28 \overset{68}{+} 40 \overset{140}{+} 72 \overset{80}{-} 60 \overset{56}{-} 24 - 56 \\ &= 0 \quad \Rightarrow \text{the rows of } B \text{ are dependent.} \end{aligned}$$

$\det C = -1\times1\times1 = -1 \Rightarrow$ the rows of C are independent

Problem 5.2.15

(a) Pf. $E_n = a_{11}E_{n-1} - a_{12}\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots 0 & 1 \end{vmatrix} = E_{n-1} - E_{n-2}$

(b) $E_1 = 1$, $E_2 = 0$, $E_3 = E_2 - E_1 = -1$, $E_4 = E_3 - E_2 = -1$.

$E_5 = E_4 - E_3 = 0$, $E_6 = E_5 - E_4 = 1$, $E_7 = E_6 - E_5 = 1$, $E_8 = E_7 - E_6 = 0$

(c) Notice the period of E's is 6.

$E_{100} = E_4 = -1$.

Problem 5.2.19

(a) Sol. $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} x^3 - \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} x^2 + \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} x - \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

Thus V_4 is a cubic polynomial in the variable x.

(b) $r_1 = a$, $r_2 = b$, $r_3 = c$.

they can make row 4 equal to one of row 1-3.

(c) $V_4 = A(x-a)(x-b)(x-c)$ since a, b, c are the roots of cubic polynomial V_4.

$$V_4 = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{vmatrix} = \begin{vmatrix} 0 & a-x & (a-x)(a+x) & (a-x)(a^2+ax+x^2) \\ 0 & b-x & (b-x)(b+x) & (b-x)(b^2+bx+x^2) \\ 0 & c-x & (c-x)(c+x) & (c-x)(c^2+cx+x^2) \\ 1 & x & x^2 & x^3 \end{vmatrix}$$

$$= -\begin{vmatrix} a-x & (a-x)(a+x) & (a-x)(a^2+ax+x^2) \\ b-x & (b-x)(b+x) & (b-x)(b^2+bx+x^2) \\ c-x & (c-x)(c+x) & (c-x)(c^2+cx+x^2) \end{vmatrix} = \begin{vmatrix} 1 & a+x & a^2+ax+x^2 \\ 1 & b+x & b^2+bx+x^2 \\ 1 & c+x & c^2+cx+x^2 \end{vmatrix} (x-a)(x-b)(x-c)$$

$$\Rightarrow A = \begin{vmatrix} 1 & a+x & a^2+ax+x^2 \\ 1 & b+x & b^2+bx+x^2 \\ 1 & c+x & c^2+cx+x^2 \end{vmatrix} = \begin{vmatrix} 1 & a+x & a^2 \\ 1 & b+x & b^2 \\ 1 & c+x & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

(d) Thus $V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$

Problem 5.2.31.

Sol. $\det P = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$. $|P^2| = |P|^2 = 1$.

there are 3 exchanges reorder 4, 1, 2, 3 into 1, 2, 3, 4.

Problem 5.2.34.

(a) Because row 1 = row 2, the matrix is singular.

(b) Because there are $5! = 120$ permutations for 1, 2, 3, 4, 5.

Graded Problem.

Problem 1.

Sol. $$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1\times1\times1\times1\times1 = 1$$

Problem 2.

Sol. $$\det A = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 & 0 \end{vmatrix} = (-1)^{(3+2)}\times(-1)\begin{vmatrix} 1 & 1 & -1 & 1 \\ 1 & 3 & 4 & 5 \\ 4 & -2 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{vmatrix}$$

$$= (-1)^{(4+3)}\times 2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= -2\times(3 \overset{1}{+} (-2) \overset{21}{+} 20 \overset{9}{-} 12 \overset{17}{-} (-10) \overset{18}{-} 1)$$

$$= -2\times 18$$

$$= -36$$