Linear Algebra – Homework 2

21 September, 2022
Due: 29 September, 2022

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 1.2.7. (parts (b) and (d) only) Find the angle θ (from its cosine) between these pairs of vectors:

(b) $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(d) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Problem 1.2.16. How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

Problem 1.2.22. Derive the Schwarz inequality $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$ by algebra instead of trigonometry:

(a) Step 1: Multiply out both sides of the inequality, $|\mathbf{v} \cdot \mathbf{w}|^2 = (v_1 w_1 + v_2 w_2)^2$ and $\|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Step 2: Show that the difference between those two sides equals $(v_1 w_2 - v_2 w_1)^2$. This difference cannot be negative since it is a square, so the inequality is true.

Problem 1.2.27. Draw a parallelogram with two sides $\mathbf{v}$ and $\mathbf{w}$. Show that the squared diagonal lengths $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2$ add to the sum of four squared side lengths $2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2$.

Problem 1.2.33. Find 4 unit vectors of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ which are all perpendicular to each other: Choose $+$ or $-$.

Problem 1.3.4. Find a combination $x_1 \mathbf{w}_1 + x_2 \mathbf{w}_2 + x_3 \mathbf{w}_3$ that gives the zero vector with $x_1 = 1$:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Are these vectors dependent or independent? The three vectors lie in a _____.

Problem 1.3.5. Consider the matrix W with columns $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ from Problem 1.3.4. The rows of W are the following vectors (written as columns):

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \qquad \mathbf{r}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Linear algebra says that these vectors must also lie in a plane, so there must be many combinations with $y_1 \mathbf{r}_1 + y_2 \mathbf{r}_2 + y_3 \mathbf{r}_3 = \mathbf{0}$. Find two sets of y's.

Problem 1.3.12 The 4×4 matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

has an inverse. Solve $C\mathbf{x} = (b_1, b_2, b_3, b_4)$ to find its inverse from the solution $\mathbf{x} = C^{-1}\mathbf{b}$.

Graded Problems.

Problem 1. Consider the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Find a unit vector pointing in the same direction as $\mathbf{u}$.

(b) Find nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ that are perpendicular to $\mathbf{u}$ and to each other.

Problem 2. Consider the 2×2 matrix $A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$.

(a) Solve the equation $A\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ for $\mathbf{x}$ in terms of b_1, b_2.

(b) What is the inverse matrix A^{-1}?