Linear Algebra – Fall 2022
Midterm Exam

NAME:

STUDENT ID:

Instructions:

- Show answers and arguments in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. Answers given without supporting work may receive zero credit.
- This is a closed book exam: no calculators, notes, or formula sheets.

QUESTION	POINTS	SCORE
1	14	
2	12	
3	14	
4	10	
5	10	
6	14	
7	14	
8	12	
TOTAL	100	

1. (a) (10 points) Find all solutions of the system of linear equations:

$$\begin{array}{rcrcrcrcr} x_1 & + & & & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}$$

(b) (4 points) Identify the reduced row echelon form R of the coefficient matrix of the system.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

(a) (6 points) Find the LU decomposition of A.

(b) (6 points) Use the LU decomposition to solve the linear system of equations $A\mathbf{x} = (1, 0, 0)$.

3. (a) (12 points) Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(b) (2 points) Use A^{-1} to solve the system of equations $A\mathbf{x} = (0, 1, 0, 0)$.

4. (a) (5 points) How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

   (b) (5 points) Suppose x, y, and z are any non-zero numbers such that $x + y + z = 0$. Find the angle between $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$.

5. Suppose that Q is any $n \times n$ matrix such that $Q^T = Q^{-1}$.

   (a) (5 points) Show that the columns $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ are all unit vectors, and show that if $\mathbf{q}_i$ and $\mathbf{q}_j$ are two different columns of Q, then $\mathbf{q}_i$ and $\mathbf{q}_j$ are perpendicular.

   (b) (5 points) For any angle θ, find a 2×2 example $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ such that $Q^T = Q^{-1}$ and $q_{11} = \cos\theta$.

6. Let S be the set of all 3×3 matrices A such that

$$A^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A$$

(a) (7 points) Show that S is a subspace of the vector space $\mathbb{R}^{3\times 3}$ of all 3×3 matrices. (Verify all three properties of a subspace.)

(b) (7 points) Find a spanning set for S that contains three matrices.

7. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \end{bmatrix}.$$

(a) (6 points) Find a linear relation on b_1, b_2, b_3 that guarantees that $\mathbf{b} = (b_1, b_2, b_3)$ is a vector in the column space $\mathbf{C}(A)$.

(b) (8 points) Find a spanning set (the special solutions) for the null space $\mathbf{N}(A)$.

8. (a) (8 points) Determine whether the following vectors in $\mathbb{R}^4$ are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 11 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

If the vectors are *not* linearly independent, show how to write one of them as a linear combination of the others.

(b) (4 points) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are non-zero vectors which are all perpendicular to each other, that is, $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

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