A1

Sol. $\lim\limits_{x\to0}\frac{8x-4\sin2x}{x^3}$

$=\lim\limits_{x\to0}\frac{8-8\cos2x}{3x^2}$

$=\lim\limits_{x\to0}\frac{16\sin2x}{6x}$

$=\frac{8}{3}\lim\limits_{2x\to0}\frac{\sin2x}{2x}$

$=\frac{8}{3}$

A2.

Sol. $f'(x)=0\Rightarrow x^2(x-1)(x-2)^3=0$

Intervals	$x<0$	$0<x<1$	$1<x<2$	$2<x$
Sign of f'	$+$	$+$	$-$	$+$
Behavior of f'	increasing	increasing	decreasing	increasing

Hence, f has local extrema at $x=1$ and $x=2$, the local maxima is $f(1)$, the local minima is $f(2)$

$f''(x)=(x^2(x-1)(x-2)^3)'$

$=(x^2)'(x-1)(x-2)^3+x^2(x-1)'(x-2)^3+x^2(x-1)[(x-2)^3]'$

$=2x(x-1)(x-2)^3+x^2(x-2)^3+3x^2(x-1)(x-2)^2$

$=(x-2)^2(2x^3-6x^2+4x+x^3-2x^2+3x^3-3x^2)$

$=(x-2)^2(6x^3-11x^2+4x)$

$=x(x-2)^2(2x-1)(3x-4)$

Hence $f''(1)=1\times(-1)^2\times1\times(-1)<0$, f is concave down at $x=1\Rightarrow f$ has local maxima at $x=1$, we can apply the second derivative test at $x=1$.

$f''(2) = 2\times 0^2 \times 3 \times 2 = 0$, we could not apply the second derivative test at $x=2$.

A3.

Pf. Since f is differentiable and convex,
f' is increasing on $\mathbb{R}$. ①

Since f has a local minimum at $x=c$ and f is differentiable on $\mathbb{R}$, $f'(c)=0$. ②

By ①②, we have $\begin{cases} \forall x<c, f'(x)<0 \\ \forall x>c, f'(x)>0 \end{cases}$.

According to First Derivative Text,
Since $f'(x)<0$ at each point $x\in(-\infty, c)$,
f is decreasing on $(-\infty, c]$. ③
Since $f'(x)>0$ at each point $x\in(c,+\infty)$
f is increasing on $[c,+\infty)$ ④.

By ③④, we have $\forall x\neq c, f(x)>f(c)$

Hence, f has a global minimum at $x=c$ and this global minimum is attained only at $x=c$.

A4.

Pf. $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{e^x f(x)}{e^x} = \lim\limits_{x\to+\infty} \frac{e^x(f(x)+f'(x))}{e^x}$

$= \lim\limits_{x\to+\infty} (f(x)+f'(x))$

Since $\lim\limits_{x\to+\infty} f(x) = 0$ and $\lim\limits_{x\to+\infty} f'(x)$ exists.

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}(f(x)+f'(x)) = \lim\limits_{x\to+\infty} f(x) + \lim\limits_{x\to+\infty} f'(x)$

$\Rightarrow \quad 0 = 0 + \lim\limits_{x\to+\infty} f'(x)$

$\Rightarrow \lim\limits_{x\to+\infty} f'(x) = 0$

A5.

Pf. $L_a(x) - f(a) = f'(a)(x-a)$

$\Rightarrow L_a(x) = f'(a)(x-a) + f(a)$

① $\forall x_0 < a$.

According to MVT:

$\exists c \in (x_0, a)$ s.t. $\frac{f(x_0)-f(a)}{x_0-a} = f'(c)$

Since f is convex $\Leftrightarrow$ f' is increasing.

$\exists c \in (x_0, a)$ s.t $\frac{f(x_0)-f(a)}{x_0-a} = f'(c) < f'(a)$

$\Rightarrow f(x_0) - f(a) > f'(a)(x-a)$

$\Rightarrow f(x_0) > f'(a)(x-a) + f(a)$.

② $\forall x_0 > a$.

According to MVT

$\exists c \in (a, x_0)$ s.t. $\frac{f(x_0)-f(a)}{x_0-a} = f'(c) > f'(a)$

$\Rightarrow f(x_0)-f(a) > f'(a)(x_0-a)$

$\Rightarrow f(x_0) > f'(a)(x_0-a) + f(a) = L_a(x_0)$

By ①②, $\forall x \neq a$, $f(x) > L_a(x)$, the graph of f is above all its tangents.

B1.

Pf. for any $a \in \mathbb{R}$, $\forall x \neq a$, $f(x) > L_a(x)$.

$\Rightarrow \forall x \neq a$, $f(x) > L_a(x) = f'(a)(x-a) + f(a)$

Similarly, $\forall a \neq x$, $f(a) > L_x(a) = f'(x)(a-x) + f(x)$

So $f(x) > f'(a)(x-a) + f(a) > f'(a)(x-a) + f'(x)(a-x) + f(x)$

$\Rightarrow f'(x)(x-a) > f'(a)(x-a)$

if $x > a$ then $f'(x) > f'(a)$

if $x < a$ then $f'(x) < f'(a)$

Thus, $f'(x)$ is increasing.

B2.

Pf. Let $g(x) = f(x) - \dfrac{(x-a)(x-b)}{(d-a)(d-b)} f(d)$, $g''(x) = f''(x) - 2\dfrac{f(d)}{(d-a)(d-b)}$

Since $f(a) = f(b) = 0$, $\therefore g(a) = g(b) = 0$.

According to Rolle's Theorem,

$\exists \eta_1 \in (a,d)$ s.t. $g'(\eta_1) = 0$. $\exists \eta_2 \in (d,b)$ s.t. $g'(\eta_2) = 0$

$\exists c \in (\eta_1, \eta_2)$ s.t. $g''(c) = f''(c) - \dfrac{2f(d)}{(d-a)(d-b)} = 0$

$\Rightarrow \exists c \in (\eta_1, \eta_2) \subseteq (a,b)$ s.t. $\dfrac{f''(c)}{2}(d-a)(d-b) = f(d)$