Practice Final Exam

1. (a) Find the unique value of c such that the system of equations has at least one solution:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 3 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= c \\ x_1 - x_2 - 3x_3 - 5x_4 &= 1 \end{aligned}$$

(b) For the value of c you found, find *all* solutions to the system of equations.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 & c \\ 1 & -1 & -3 & -5 & 1 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - \text{Row 1}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 & c-3 \\ 0 & -2 & -4 & -6 & -2 \end{array}\right] \xrightarrow[\text{Row 3} + 2\,\text{Row 2}]{\text{Row 1} - \text{Row 2}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -c+6 \\ 0 & 1 & 2 & 3 & c-3 \\ 0 & 0 & 0 & 0 & 2c-8 \end{array}\right]$$

(a) To get solution(s), need $2c-8=0$, or $\boxed{c=4}$

(b) When $c=4$:

$$\begin{aligned} x_1 \underbrace{- x_3 - 2x_4}_{} &= 2 \\ x_2 \underbrace{+ 2x_3 + 3x_4}_{\text{free variables}} &= 1 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 2 \\ -2x_3 - 3x_4 + 1 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{null space vectors}}$$

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

(b) Find a basis of $\mathbf{R}^3$ consisting of eigenvectors for A.

(c) Find the angles between the eigenvectors in the basis.

Solve $\det(A-\lambda I)=0$:

(a) $$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 1 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & -2 \end{vmatrix}$$

$$= (2-\lambda)\left[(3-\lambda)(3-\lambda)-4\right] - \underbrace{\left[(3-\lambda)+2\right]}_{5-\lambda} + \underbrace{\left[-2-(3-\lambda)\right]}_{\lambda-5}$$

$$= (2-\lambda)\underbrace{(\lambda^2-6\lambda+5)}_{(\lambda-5)(\lambda-1)} + 2(\lambda-5)$$

$$= (\lambda-5)\big(\underbrace{(2-\lambda)(\lambda-1)+2}_{-\lambda^2+3\lambda}\big) = -\lambda(\lambda-3)(\lambda-5) = 0$$

$$\boxed{\lambda = 0, 3, 5}$$

(b) Eigenvectors for $\lambda=0$: Solve $A\vec{x}=\vec{0}$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} \xrightarrow[\text{Row 3} - \frac{1}{2}\text{Row 1}]{\text{Row 2} - \frac{1}{2}\text{Row 1}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & -5/2 \\ 0 & -5/2 & 5/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Row 1} - \frac{1}{2}\text{Row 2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_1 - x_3 = 0 \end{matrix} \longrightarrow \vec{x} = x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda=3}$: Solve $(A-3I)\vec{x}=\vec{0}$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix} \xrightarrow[\text{Row 3+Row 1}]{\text{Row 2+Row 1}} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Row 3+Row 2}]{-\text{Row 1+Row 2}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{array}{l} x_1 - 2x_3 = 0 \\ x_2 - x_3 = 0 \end{array} \longrightarrow \vec{x} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda=5}$: Solve $(A-5I)\vec{x}=\vec{0}$

$$\begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -2 \\ -3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 2+3 Row 1}} \begin{bmatrix} 1 & -2 & -2 \\ 0 & \cancel{-5}\,1 & \cancel{-5}\,1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{array}{l} x_1 = 0 \\ x_2 + x_3 = 0 \end{array} \longrightarrow \vec{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Basis of eigenvectors: $\left\{ \underset{\vec{x}_0}{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{x}_3}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{x}_5}{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}} \right\}$

(c) Angles: $\vec{x}_0 \cdot \vec{x}_3 = (-1)(2) + (1)(1) + (1)(1) = 0$

$\vec{x}_0 \cdot \vec{x}_5 = (-1)(0) + (1)(-1) + (1)(1) = 0$

$\vec{x}_3 \cdot \vec{x}_5 = (2)(0) + (1)(-1) + (1)(1) = 0$

$\downarrow$

The three vectors are all at <u>90° angles</u> to each other.

3. Determine whether the following sets of vectors are bases for $\mathbb{R}^3$. In case a set of vectors is *not* linearly independent, show how to write one of the vectors as a linear combination of the others:

$$\text{(a)} \quad \left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix} \right\} \qquad \text{(b)} \quad \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\9\end{bmatrix}, \begin{bmatrix}1\\5\\25\end{bmatrix} \right\}$$

(a) $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \xrightarrow[\text{Row 3}-7\text{Row 1}]{\text{Row 2}-4\text{Row 1}} \begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix}$

$\xrightarrow{\text{Row 1}-2\text{Row 2}} \underbrace{\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}}_{R \neq I \,\to\, \text{dependent, } \underline{\text{not}} \text{ a basis}} \to \begin{matrix} x_1 = x_3 \\ x_2 = -2x_3 \end{matrix} \longrightarrow \vec{x} = x_3 \begin{bmatrix}1\\-2\\1\end{bmatrix}$

Shows that $1\begin{bmatrix}1\\4\\7\end{bmatrix} - 2\begin{bmatrix}2\\5\\8\end{bmatrix} + 1\begin{bmatrix}3\\6\\9\end{bmatrix} = \vec{0}$, or for exomple,

$$\begin{bmatrix}3\\6\\9\end{bmatrix} = -1\begin{bmatrix}1\\4\\7\end{bmatrix} + 2\begin{bmatrix}2\\5\\8\end{bmatrix}$$

(b) $\begin{bmatrix}1&1&1\\1&3&5\\1&9&25\end{bmatrix} \xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 1}-\text{Row 2}} \begin{bmatrix}1&1&1\\0&2&4\\0&8&24\end{bmatrix} \longrightarrow \begin{bmatrix}1&1&1\\0&1&2\\0&1&3\end{bmatrix} \xrightarrow{\text{Row 3}-\text{Row 2}}$

$\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix} \xrightarrow[\text{elimination}]{\text{more}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$\underline{\text{no}}$ free variobles

$R \overset{3}{=} I \longrightarrow$ independent, these vectors $\underline{\text{are}}$ a bosis

4. Find bases for the null space, column space, row space, and left null space of the matrix:

$$A = \begin{bmatrix} -3 & 1 & 4 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 4 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 5 & 2 \end{bmatrix} \xrightarrow[\text{Row 3} - \text{Row 1}]{3\,\text{Row 2} + \text{Row 1}} \begin{bmatrix} -3 & 1 & 4 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 7 & 1 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & -1/3 & -4/3 & -4/3 \\ 0 & 1 & 1/7 & -2/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{3}\text{Row 2}]{\text{Row 1} +} \begin{bmatrix} 1 & 0 & -9/7 & -10/7 \\ 0 & 1 & 1/7 & -2/7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$\underline{N(A)}$: Solve

$$\begin{aligned} x_1 - \tfrac{9}{7}x_3 - \tfrac{10}{7}x_4 &= 0 \\ x_2 + \underbrace{\tfrac{1}{7}x_3 - \tfrac{2}{7}x_4}_{\text{two free variables, two basis vectors}} &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9/7 \\ -1/7 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $N(A)$.

$\underline{C(A)}$: Basis = pivot columns <u>in A</u>: Cols. 1 and 2

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} \right\}$$

$\underline{R(A) = C(A^T)}$: Basis = non-zero rows <u>in R</u>:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -9/7 \\ -10/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/7 \\ -2/7 \end{bmatrix} \right\}$$

Since $\dim C(A^T) = 2$, you could <u>also</u> take any <u>two</u> <u>independent</u> rows of A to be your basis.

For $N(A^T)$, I'll show three different methods:

1. Solve $A^T \vec{x} = 0$ by elimination:

$$\begin{bmatrix} -3 & 1 & -3 \\ 1 & 2 & 8 \\ 4 & -1 & 5 \\ 4 & -2 & 2 \end{bmatrix} \xrightarrow[\substack{\text{Row 3} - 4\,\text{Row 2} \\ \text{Row 4} - 4\,\text{Row 2}}]{\text{Row 1} + 3\,\text{Row 2}} \begin{bmatrix} 0 & 7 & 21 \\ 1 & 2 & 8 \\ 0 & -9 & -27 \\ 0 & -10 & -30 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-2\,\text{Row 2}]{\text{Row 1}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 + 2x_3 = 0$ $\longrightarrow$ $\vec{x} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

$x_2 + 3x_3 = 0$

Basis vector for $N(A^T)$.

2. Try to solve $A\vec{x} = \vec{b}$ by elimination:

$$\left[\begin{array}{cccc|c} -3 & 1 & 4 & 4 & b_1 \\ 1 & 2 & -1 & -2 & b_2 \\ -3 & 8 & 5 & 2 & b_3 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 1}]{3\,\text{Row 2} + \text{Row 1}} \left[\begin{array}{cccc|c} -3 & 1 & 4 & 4 & b_1 \\ 0 & 7 & 1 & -2 & b_1 + 3b_2 \\ 0 & 7 & 1 & -2 & -b_1 + b_3 \end{array}\right]$$

$$\xrightarrow{\text{Row 3} - \text{Row 2}} \underbrace{\left[\begin{array}{cccc|c} -3 & 1 & 4 & 4 & b_1 \\ 0 & 7 & 1 & -2 & b_1 + 3b_2 \\ 0 & 0 & 0 & 0 & -2b_1 - 3b_2 + b_3 \end{array}\right]}_{\text{Shows that } -2(\text{Row 1}) - 3(\text{Row 2}) + (\text{Row 3}) = \vec{0}} \longrightarrow$$

$$\begin{bmatrix} -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} = \vec{0} \longrightarrow \begin{bmatrix} & A^T & \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \vec{0}$$

Basis vector for $N(A^T)$; it's enough for a basis because $\dim N(A^T) = 3 - \dim C(A) = 1$

3. Remember that $N(A^T) = C(A)^\perp$ = all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad = \text{all solutions to } \begin{cases} x_1 + 2x_2 + 8x_3 = 0 \\ -3x_1 + x_2 - 3x_3 = 0 \end{cases}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 8 \\ -3 & 1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & \cancel{7} & \underset{3}{\cancel{21}} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = -3x_3 \end{array}$$

$$\longrightarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Basis vector

5. (a) Diagonalize the matrix A: write $A = X\Lambda X^{-1}$ where X is invertible and Λ is diagonal.

$$A = \begin{bmatrix} 10 & 12 \\ -6 & -7 \end{bmatrix}$$

(b) Show that the matrix B is *not* diagonalizable:

$$B = \begin{bmatrix} 8 & 9 \\ -4 & -4 \end{bmatrix}$$

(a) $\Lambda \to$ eigenvalues
$X \to$ eigenvectors

$$\begin{vmatrix} 10-\lambda & 12 \\ -6 & -7-\lambda \end{vmatrix} = (10-\lambda)(-7-\lambda)+72$$

$$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0 \to \lambda = 1, 2$$

For $\lambda=1$: Solve $(A-I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 9 & 12 \\ -6 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix} \to \vec{x} = x_2\begin{bmatrix} -4/3 \\ 1 \end{bmatrix} \quad \text{If } x_2=3: \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

For $\lambda=2$: Solve $(A-2I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 8 & 12 \\ -6 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix} \to \vec{x} = x_2\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \quad \text{If } x_2=2: \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\text{So } A = \underbrace{\begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}^{-1}}_{X^{-1}}$$

(b) Eigenvalues: $\begin{vmatrix} 8-\lambda & 9 \\ -4 & -4-\lambda \end{vmatrix} = (8-\lambda)(-4-\lambda)+36 = \lambda^2 - 4\lambda + 4$

$$= (\lambda-2)^2 = 0 \to \lambda = 2, 2$$

Eigenvectors: Solve $(B-2I)\vec{x} = \vec{0}$ $\quad \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$

$\to \vec{x} = x_2\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \leftarrow$ Only one independent eigenvector, not enough for a basis of $\mathbb{R}^2$. So B is not diagonalizable.

6. Consider the (x, y) data points $(-1, 0)$, $(0, 1)$, $(1, 3)$, and $(2, 9)$.

(a) Find the best least squares fit by a linear function $y = mx + b$ to the data.

(b) Plot your linear function from part (a) along the data on a coordinate system.

(c) Find the error of the least squares approximation.

(a) Try to solve

$$\begin{aligned} 1b + (-1)m &= 0 \\ 1b + 0m &= 1 \\ 1b + 1m &= 3 \\ 1b + 2m &= 9 \end{aligned} \longrightarrow \underbrace{\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} b \\ m \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_{\vec{b}}}_{\text{Inconsistent, no solution.}}$$

Instead, solve the "normal equations":

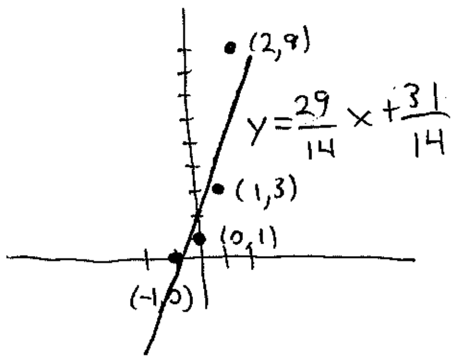
$$\underbrace{A^T A}\hat{x} = A^T \vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 13 \\ 21 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ 21 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 8 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 13 \\ 21 \end{bmatrix} = \begin{bmatrix} 31/14 \\ 29/14 \end{bmatrix} \begin{matrix} b \\ m \end{matrix}$$

$$\boxed{y = \frac{29}{14}x + \frac{31}{14}}$$

(b)



(c) $\text{Error} = \|\vec{e}\| = \|A\hat{x} - \vec{b}\|$

$$= \left\| \begin{bmatrix} 1/7 \\ 31/14 \\ 30/7 \\ 89/14 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} \right\|$$

$$= \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{17}{14}\right)^2 + \left(\frac{9}{7}\right)^2 + \left(-\frac{37}{14}\right)^2}$$

(Probably can't simplify any further without a calculator.)

7. (a) Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 14 \end{bmatrix}$$

(b) Use A^{-1} to solve the linear system of equations $A\mathbf{x} = (1, 2, 1)$.

(a) Use elimination:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ 3 & 7 & 14 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row 3} - 3\,\text{Row 1}]{\text{Row 2} - 2\,\text{Row 1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -3 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row 3}]{\text{Row 2} \leftrightarrow}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & -3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[\text{Row 2} - 5\,\text{Row 3}]{\text{Row 1} - 3\,\text{Row 3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 7 & -3 & 0 \\ 0 & 1 & 0 & 7 & -5 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[-2\,\text{Row 2}]{\text{Row 1}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 7 & -2 \\ 0 & 1 & 0 & 7 & -5 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\underbrace{\begin{matrix} -7 & 7 & -2 \\ 7 & -5 & 1 \\ -2 & 1 & 0 \end{matrix}}_{A^{-1}}$$

(b) $$\overline{x} = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 & 7 & -2 \\ 7 & -5 & 1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$$

8. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{bmatrix}.$$

(a) Find the LU decomposition of A.

(b) Find the volume of the box in $\mathbf{R}^3$ that is spanned by the columns of A.

(a) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{bmatrix} \xrightarrow[\text{Row 3} - \boxed{1}\,\text{Row 1}]{\text{Row 2} - \boxed{1}\,\text{Row 1}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Row 3} - \boxed{-\frac{1}{2}}\,\text{Row 2}]{} \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$

So $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$

(b) Volume $= |\det A| = |(1)(2)(1)| = 2$

9. (a) Find the determinants of the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -3 & 0 & 3 \end{bmatrix}$$

(b) Are the matrices A and B invertible?

(a) $\det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{vmatrix}$

$\overset{\text{Row 4} - \text{Row 2}}{=} - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{vmatrix} \overset{\text{Row 4} - 2\text{Row 3}}{=} - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{vmatrix} = -(1)(-2)(3)(-12)$

$= \boxed{-72}$

$\det B = 2 \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} = 2(3-3) = \boxed{0}$

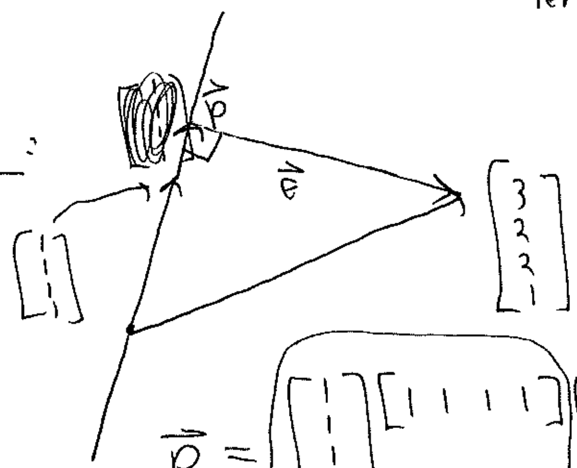
(b) A is invertible, B is not.

10. (a) Find an orthonormal basis for the subspace V of $\mathbf{R}^4$ spanned by $(1,1,1,1)$ and $(3,2,2,1)$.

(b) Find the projection matrix P for the orthogonal projection onto V, and compute the projection of $(0,0,1,1)$ onto V.

(a) First basis vector: $\vec{x}_1 = \frac{1}{\text{length}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{\sqrt{1^2+1^2+1^2+1^2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

Second:



$\vec{x}_2 = \frac{1}{\|\vec{e}\|}\vec{e}$ $\quad \vec{e} = \begin{bmatrix}3\\2\\2\\1\end{bmatrix} - \vec{p}$

$$\vec{p} = \underbrace{\frac{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix}}{\begin{bmatrix}1&1&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}}_{\text{Projection matrix}}\begin{bmatrix}3\\2\\2\\1\end{bmatrix} = \frac{1}{4}(3+2+2+1)\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\\2\end{bmatrix}$$

$$\vec{x}_2 = \frac{1}{\|\vec{e}\|}\left(\begin{bmatrix}3\\2\\2\\1\end{bmatrix} - \begin{bmatrix}2\\2\\2\\2\end{bmatrix}\right) = \frac{1}{\text{length}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$$

Orthonormal basis: $\left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$

(b) $P = A(A^TA)^{-1}A^T = \begin{bmatrix}1&1\\1&0\\1&0\\1&-1\end{bmatrix}\left(\begin{bmatrix}1&1&1&1\\1&0&0&-1\end{bmatrix}\begin{bmatrix}1&1\\1&0\\1&0\\1&-1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&1&1\\1&0&0&-1\end{bmatrix}$

columns = basis for V

Can use this basis, which is orthogonal, though not quite orthonormal

$$= \begin{bmatrix}1&1\\1&0\\1&0\\1&-1\end{bmatrix}\underbrace{\begin{bmatrix}4&0\\0&2\end{bmatrix}^{-1}}_{\begin{bmatrix}1/4&0\\0&1/2\end{bmatrix}}\begin{bmatrix}1&1&1&1\\1&0&0&-1\end{bmatrix}$$

$$= \begin{bmatrix}1&1\\1&0\\1&0\\1&-1\end{bmatrix}\begin{bmatrix}1/4&1/4&1/4&1/4\\1/2&0&0&-1/2\end{bmatrix} = \underbrace{\begin{bmatrix}3/4&1/4&1/4&-1/4\\1/4&1/4&1/4&1/4\\1/4&1/4&1/4&1/4\\-1/4&1/4&1/4&3/4\end{bmatrix}}_{P}$$

$$\boxed{P\begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1/2\\1/2\\1\end{bmatrix}}$$