A1.

Sol. $x^2+xy+y^2=12$

$\Rightarrow 2x+xy'+y+2y'y=0$

$\Rightarrow (x+2y)y'=-2x-y$

$\Rightarrow y'=\frac{dy}{dx}=\frac{-2x-y}{x+2y}$

$\Rightarrow \frac{dx}{dy}=\frac{x+2y}{-2x-y}=\frac{-x-2y}{2x+y}$

• Horizontal tangent: $y'=0 \Rightarrow -2x-y=0 \Rightarrow y=-2x$.

$x^2+xy+y^2=x^2+x(-2x)+(-2x)^2=3x^2=12$

$\Rightarrow \begin{cases} x=2 \\ y=-2\times2=-4 \end{cases}$ or $\begin{cases} x=-2 \\ y=-2\times(-2)=4 \end{cases}$.

Therefore at points $(2,-4)$, $(-2,4)$ C has horizontal tangent.

• Vertical tangent: $\frac{dx}{dy}=0 \Rightarrow \frac{x+2y}{-2x-y}=0 \Rightarrow x=-2y$.

$x^2+xy+y^2=(-2y)^2+(-2y)y+y^2=3y^2=12$.

$\Rightarrow \begin{cases} y=2 \\ x=-2\times2=-4 \end{cases}$ or $\begin{cases} y=-2 \\ x=(-2)\times(-2)=4 \end{cases}$

Therefore at points $(-4,2)$, $(4,-2)$ C has vertical tangent.

A2.

Pf. 1. Let $x=a$ then $L(a)=f'(a)(a-a)+f(a)=f(a)$

2. $L'(x)=(f'(a)x+f(a)-af'(a))'=f'(a)$.

Assume $g(x)=k\cdot x+b$ and $\begin{cases} g(a)=f(a) \\ g'(a)=f'(a) \end{cases}$.

$\Rightarrow \begin{cases} g(a)=ka+b=f(a) \\ g'(a)=k \quad =f'(a) \end{cases} \Rightarrow \begin{cases} k=f'(a) \\ b=f(a)-f'(a)a \end{cases}$.

$\Rightarrow g(x)=f'(a)x+f(a)-f'(a)a=f'(a)(x-a)+f(a)$

$\Rightarrow g(x)=L(x)$, any other linear function satisfying these two properties must be equal to L.

A3. Pf. Let $P(x) = C_2x^2 + C_1x + C_0$.

$$A(x-a)^2 + B(x-a) + C = Ax^2 + (-2aA+B)x + Aa^2 - aB + C$$

Therefore $\begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ a^2 & -a & 1 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$.

rank of $\begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ a^2 & -a & 1 \end{bmatrix}$ is 3 $\Rightarrow C\left(\begin{bmatrix} 1 & 0 & 0 \\ -2a & 1 & 0 \\ a^2 & -a & 1 \end{bmatrix}\right)$ is all of $\mathbb{R}^3$

$\Rightarrow$ any $\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$ can be obtained by $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$.

$\Rightarrow$ any degree 2 polynomial can be write as $A(x-a)^2 + B(x-a) + C$.

Pf. Let $P(x) = A(x-a)^2 + B(x-a) + C$.

then $P'(x) = 2A(x-a) + B$.

$P''(x) = 2A$.

$$\begin{cases} P(a) = f(a) \\ P'(a) = f'(a) \\ P''(a) = f''(a) \end{cases} \Rightarrow \begin{cases} P(a) = C = f(a) \\ P'(a) = B = f'(a) \\ P''(a) = 2A = f''(a). \end{cases}$$

$$\Rightarrow P(x) = \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)$$

Therefore $P(x)$ is unique.

A4. Sol. $f(x)=|x^3-9x|=|x(x-3)(x+3)|$

①. Absolute Minimum.

$f(x)=|x(x-3)(x+3)|\geq 0$.

and the only 3 zero points are $(0,0)$ $(3,0)$ $(-3,0)$

Therefore absolute minimum of $f(x)$ is 0.

②. Absolute Maximum.

$$f(x)=\begin{cases} x^3-9x, & -3\leq x<0 \text{ or } 3\leq x \\ -x^3+9x, & x<-3 \text{ or } 0\leq x<3. \end{cases}$$

$$\Rightarrow f'(x)=\begin{cases} 3x^2-9, & -3<x<0 \text{ or } 3<x \\ -3x^2+9, & x<-3 \text{ or } 0<x<3. \end{cases}$$

so the absolute maximum occur at $x=\pm\sqrt{3}$

Therefore absolute maximum of $f(x)$ is $|(\sqrt{3})^3-9\sqrt{3}|$

$=|(-\sqrt{3})^3+9\sqrt{3}|$

$=6\sqrt{3}$

By ①, ② The Global Extrema of $f(x)$ is 0 and $6\sqrt{3}$, the global extrema occur at $(0,0)$, $(3,0)$, $(-3,0)$, $(\sqrt{3},6\sqrt{3})$, $(-\sqrt{3},6\sqrt{3})$

A5. Sol. Assume the length of the rectangle is x cm and the width of it is y cm.

we have $x^2+y^2=1$

$$(x-y)^2\geq 0 \Rightarrow x^2+y^2-2xy\geq 0$$
$$\Rightarrow 2xy\leq x^2+y^2$$
$$\Rightarrow xy\leq \frac{1}{2}\times(x^2+y^2)=\frac{1}{2}\times 1=\frac{1}{2}.$$

when $x=y=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$, $xy=\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}=\frac{1}{2}$.

Therefore the rectangle $\frac{\sqrt{2}}{2}$ cm $\times\frac{\sqrt{2}}{2}$ cm has the largest area.

B1

Sol. No. if $f(x)=\begin{cases}0, & x=0\\ x\sin(\frac{1}{x}), & x\in(0,1]\end{cases}$, $f(x)$ is C^0 on $[0,1]$

then for any half-open interval $[0,c)$, $c\le 1$

there exists $x_0\in[0,c)$ s.t $x_0\sin(\frac{1}{x_0})=-x_0<0$

and $x_1\in[0,c)$ s.t $x_1\sin(\frac{1}{x_0})=x_1>0$

Therefore f has no local extrema at $x=0$.

Sol. No. if $f(x)=\begin{cases}0, & x=0\\ x^3\sin(\frac{1}{x}), & x\in(0,1]\end{cases}$.

then $f'(0)=\lim\limits_{\Delta x\to 0^+}\frac{f(0+\Delta x)-f(0)}{\Delta x}$

$=\lim\limits_{\Delta x\to 0^+}\frac{(\Delta x)^3\sin(\frac{1}{\Delta x})}{\Delta x}$

$=\lim\limits_{\Delta x\to 0^+}(\Delta x)^2\sin(\frac{1}{\Delta x})$.

since $-(\Delta x)^2<(\Delta x)^2\sin(\frac{1}{\Delta x})<(\Delta x)^2$

and $\lim\limits_{\Delta x\to 0^+}(-(\Delta x)^2)=\lim\limits_{\Delta x\to 0^+}(\Delta x)^2=0$.

According to The Intermediate Theorem

$f'(0)=\lim\limits_{\Delta x\to 0^+}(\Delta x)^2\sin(\frac{1}{\Delta x})=0$.

$f'(x)=3x^2\sin(\frac{1}{x})-x\cos(\frac{1}{x})$, $x\in[0,1]$

therefore $f(x)$ is differentable on $[0,1]$.

then for any half-open interval $[0,c)$, $c<1$

there exists $x_0\in[0,c)$ s.t $x_0^3\sin(\frac{1}{x_0})=-x_0^3<0$

and $x_1\in[0,c)$ s.t $x_1^3\sin(\frac{1}{x_1})=x_1^3>0$

Therefore f has no local extrema at

$x=0$.