Problem 5.3.15

For $n=5$, the cofactor matrix C containing 25 cofactors.

Each 4×4 cofactor containing 24 Terms and each term needs 3 multiplication

total multiplication: $25\times 24\times 3 = 1800$

Problem 5.3.17

Sol. Volumn: $\begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$

Parallelogram: $\begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2i - 2j + 8k$.

length $= 6\sqrt{2}$

Problem 5.3.23

Sol. $A^TA = \begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^Ta & 0 & 0 \\ 0 & b^Tb & 0 \\ 0 & 0 & c^Tc \end{bmatrix}$

$\det A^TA = \|a\|^2 \|b\|^2 \cdot \|c\|^2$

thus $\det A = |a\| \cdot \|b\| \cdot \|c\|$

Problem 6.1.6

Sol. $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$, $\lambda_1 = \lambda_2 = 1$.

$|B-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$. $\lambda_1 = \lambda_2 = 1$

$|AB-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1 = 0$, $\lambda_1 = 2+\sqrt{3}$, $\lambda_2 = 2-\sqrt{3}$

$|BA-\lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1 = 0$. $\lambda_1 = 2+\sqrt{3}$, $\lambda_2 = 2-\sqrt{3}$

(a) Thus, the eigenvalues of $AB \neq$ A's $\times$ B's

$\Rightarrow$ the eigenvalues of AB equal to the eigenvalues of BA

Problem 6.1.12.

Sol. $|P-\lambda I| = \begin{vmatrix} 0.2-\lambda & 0.4 & 0 \\ 0.4 & 0.8-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(0.2-\lambda)(0.8-\lambda)-0.4^2]$

$= (1-\lambda)\lambda(\lambda-1)$

$\Rightarrow \lambda_1 = \lambda_2 = 1,\ \lambda_3 = 0$

$(P-0\cdot I)x = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$(P-I)x = \begin{bmatrix} -0.8 & 0.4 & 0 \\ 0.4 & -0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} x = 0 \Rightarrow x = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So. $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ is also an eigenvector

Problem 6.1.15

Sol. $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$:

$|P-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 1-\lambda^3 = 0$. $\lambda_1 = 1$, $\lambda_2 = \frac{-1+\sqrt{3}i}{2}$, $\lambda_3 = \frac{-1-\sqrt{3}i}{2}$

$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$:

$|P-\lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda-1)^2(\lambda+1)$. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$.

Problem 6.1.16

Sol. $\det A = \det(A-0I) = \lambda_1\lambda_2\cdots\lambda_n$

Problem 6.1.27.

Sol. $\operatorname{rank}(A)=1$, $\operatorname{trace}(A)=4$.

so the eigenvalue of A are 0, 0, 0, 4.

$\operatorname{rank}(C)=2$, so there are two eigenvalues are 0.

As $[1\ 1\ 1\ 1]^T$ is a eigenvector of $\lambda=2$, with $\operatorname{trace}(C)=4$, we got to know that another eigenvalue is 2,

so 0, 0, 2, 2 are the eigenvalues of C

Problem 6.1.32.

(a) $\operatorname{rank}(A)=2$. so $\operatorname{rank}(N(A))=1$

Thus $\{u\}$ is a basis of $N(A)$

$3v, 5w$ are combs of columns of A,

so $\{3v, 5w\}$ is a basis of $C(A)$

(b) $A(\frac{v}{3}+\frac{w}{5}) = \frac{3v}{3}+\frac{5w}{5} = v+w$

so $x_p = \frac{v}{3}+\frac{w}{5}$, $x_n = cu$

$x = x_p + x_n = \frac{v}{3}+\frac{w}{5}+cu$.

(c) If it did, u would be in the $C(A)$

Graded Problem.

Problem 1.

(a) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \end{bmatrix}$

Volume of the box

$= |\det A|$

$$= \left| \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \end{vmatrix} \right| = \left| \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{vmatrix} \right| = \left| \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{vmatrix} \right| = \left| \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{vmatrix} \right|$$

$= |1 \times (-2) \times 3 \times (-12)|$

$= 72$.

(b)

$|\det(AQ)| = |\det Q| \cdot |\det A| = 1 \cdot 72 = 72$.

Problem 2.

Sol. Let $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & -4 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & -3-\lambda & -1-\lambda \\ 2 & -4 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -2 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & -4 & -1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda)^2$$

So the eigenvalues of A are $1, 1, -1$.

for $\lambda = 1$, let $(A - I)x = 0$, then $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda = -1$, Let $(A + I)x = 0$, then $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Thus $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are the eigenvectors of A.

2 vectors can't be a basis of $\mathbb{R}^3$, so $\mathbb{R}^3$ does not have a basis consisting of eigenvectors for A.