Problem 2.7.5

Sol. (a) $x^TAy=[0\ \ 1]\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=[4\ 5\ 6]\begin{bmatrix}0\\1\\0\end{bmatrix}=[5]$

(b) This is the row $x^TA=[4\ 5\ 6]$ times the column $y=(0,1,0)$

(c) This is the row $x^T=[0\ \ 1]$ times the column $Ay=\begin{bmatrix}2\\5\end{bmatrix}$

Problem 2.7.11

Sol. As $P=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$, $PA=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&2&3\\0&4&5\\0&0&6\end{bmatrix}=U$

P_2 exchange columns of A.

As $P_1=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$, $P_2=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$, $P_1AP_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&6\\1&2&3\\0&4&5\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$

$=\begin{bmatrix}6&0&0\\5&4&0\\3&2&1\end{bmatrix}=L$

Problem 2.7.16.

Sol. (a) $(A^2-B^2)^T=(A^2)^T-(B^2)^T=(A^T)^2-(B^T)^2=A^2-B^2$

which means A^2-B^2 is symmetric

(b) $[(A+B)(A-B)]^T=(A-B)^T(A+B)^T=(A^T-B^T)(A^T+B^T)$

$=(A-B)(A+B)$

which means $(A+B)(A-B)$ is not symmetric.

(c) $(ABA)^T=A^TB^TA^T=ABA$

which means ABA is symmetric.

(d) $(ABAB)^T=B^TA^TB^TA^T=BABA$

Which means $ABAB$ is not symmetric.

Problem 2.7.22.

Sol. Let $P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, $PA=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&1\\2&3&4\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&1\\2&3&4\end{bmatrix}$

$[PA\ \ I]=\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&1&0&1&0\\2&3&4&0&0&1\end{array}\right]\xrightarrow{r_3-2r_1}\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&1&0&1&0\\0&3&2&-2&0&1\end{array}\right]\xrightarrow{r_3-3r_2}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & -3 & 1 \end{array}\right] = [U \ L^{-1}]$$

so $PA = LU \Leftrightarrow \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix}\begin{bmatrix} 0&1&1\\1&0&1\\2&3&4 \end{bmatrix} = \begin{bmatrix} 1&0&0\\0&1&0\\2&3&1 \end{bmatrix}\begin{bmatrix} 1&0&1\\0&1&1\\0&0&-1 \end{bmatrix}$

Let $P = \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}$, $PA = \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}\begin{bmatrix} 1&2&0\\2&4&1\\1&1&1 \end{bmatrix} = \begin{bmatrix} 1&1&1\\1&2&0\\2&4&1 \end{bmatrix}$

$$[PA \ I] = \left[\begin{array}{ccc|ccc} 1&1&1&1&0&0\\1&2&0&0&1&0\\2&4&1&0&0&1 \end{array}\right] \xrightarrow[r_3-2r_1]{r_2-r_1} \left[\begin{array}{ccc|ccc} 1&1&1&1&0&0\\0&1&-1&-1&1&0\\0&2&-1&-2&0&1 \end{array}\right] \xrightarrow{r_3-2r_2}$$

$$\left[\begin{array}{ccc|ccc} 1&1&1&1&0&0\\0&1&-1&-1&1&0\\0&0&1&0&-2&1 \end{array}\right] = [U \ L^{-1}]$$

so $PA = LU \Leftrightarrow \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}\begin{bmatrix} 1&2&0\\2&4&1\\1&1&1 \end{bmatrix} = \begin{bmatrix} 1&0&0\\1&1&0\\2&2&1 \end{bmatrix}\begin{bmatrix} 1&1&1\\0&1&-1\\0&0&1 \end{bmatrix}$

Problem 2.7.39.

Proof.. (a) Since $Q^TQ = I$, $\sum_{k=1}^{n} Q_{ki}Q_{kj} = 0$ (if $i \neq j$)

$\sum_{k=1}^{n} Q_{ki}Q_{kj} = 1$ (if $i = j$).

$\|q_i^2\| = \sum_{k=1}^{n} Q_{ki}Q_{ki} = 1$, so. q_i is unit vector.

(b) $q_1^Tq_2 = \|q_1 \cdot q_2\| = \sum_{k=1}^{n} Q_{ki}Q_{kj} = 0$ ($i \neq j$).

Sol. (c) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is the 2x2 matrix.

Problem 3.1.4.

Sol. zero vector $\begin{bmatrix} 0&0\\0&0 \end{bmatrix}$ $\frac{1}{2}A = \begin{bmatrix} 1&-1\\1&-1 \end{bmatrix}$ $-A = \begin{bmatrix} -2&2\\-2&2 \end{bmatrix}$

the smallest subspace containing A is cA

Problem 3.1.10.

Sol. (a) $c(n, n, m) + d(p, p, q) = (cn+dp, cn+dp, cm+dq)$

can also write as (b_1, b_2, b_3) $(b_1 = b_2)$

so. ~~(a)~~ it is a subspace.

(b) $(0,0,0)$ is not in this space.

so it is not a subspace.

(c) It is the space for all vector (b_1,b_2,b_3) (at least one of b's is zero), $(0,1,1)$, $(1,1,0)$ are in the space

$(0,1,1)+(1,1,0)=(1,2,1)$ which is not in the space.

So it is not a subspace.

(d) All linear combs of $v=(1,4,0)$ and $w=(2,2,2)$ is column space of $\begin{bmatrix}1&2\\4&2\\0&2\end{bmatrix}$ so it is a subspace.

(e) (p_1,p_2,p_3) and (q_1,q_2,q_3) are two vectors in the space. $c(p_1,p_2,p_3)+d(q_1,q_2,q_3)=(cp_1+dq_1,cp_2+dq_2,cp_3+dq_3)$

and $cp_1+dq_1+cp_2+dq_2+cp_3+dq_3$

$=c(p_1+p_2+p_3)+d(q_1+q_2+q_3)=0.$

which means $(cp_1+dq_1,cp_2+dq_2,cp_3+dq_3)$ is also in the space.

So it is a subspace.

(f) $(1,2,3)$ and $(4,5,7)$ is two vectors in the space. $(1,2,3)+(-1)(4,5,7)=(-3,-3,-4)$ which is not a vector in this space.

So it is not a subspace.

Problem 3.1.15.

(a) The intersection of two planes through $(0,0,0)$ is probably a line in R^3 but it could be a space. It can't be Z!

(b) The intersection of a plane through (0,0,0) with a line through (0,0,0) is probably a point. but it could be a line.

(c) Since S and T are subspaces of R^5

If v, w are the vectors in both S and T

the cv+dw are in both S and T

which means $S \cap T$ is a subspace of R^5.

Problem 3.1.20.

Sol. (a) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \xrightarrow[r_3+r_1]{r_2-2r_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3+b_1 \end{array}\right]$

So these system solvable only if $b_2=2b_1$ and $b_3=-b_1$

(b) $\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow[r_3+r_1]{r_2-2r_1} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2-2b_1 \\ 0 & 0 & b_3+b_1 \end{array}\right] \Rightarrow \begin{array}{l} b_1 = x_1+4x_2 \\ b_2-2b_1 = x_2 \\ b_3 = -b_1 \end{array}$

So these system solvable only if $b_3=-b_1$.

Problem 3.1.25

Sol. Solution to $Ax=b+b^*$ is $z=x+y$.

If b and b^* are in $C(A)$ so is $b+b^*$.

Graded Problem.

Problem 1.

(a) Proof. Let P be the subset of symmetric matrices.

Let A, B be symmetric matrices.

then A, B are in P.

Let $E=cA+dB$

then $E_{ij}=cA_{ij}+dB_{ij}=cA_{ji}+dB_{ji}=E_{ji}$

so E is also symmetric matrix

so E is also in P which means P is a subspace of M.

(b) Proof. $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} A_{ki}B_{jk} = (AB)_{ji}$

So AB is also a symmetric matrix and subset of symmetric matrices closed under matrix multiplication.

Problem 2.

Proof. Let $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

$B(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$

$A(x)$, $B(x)$ are all in P_n.

$cA(x) + dB(x) = ca_0 + db_0 + (ca_1 + db_1)x + (ca_2 + db_2)x^2 + \cdots + (ca_n + db_n)x^n$

So $cA(x) + dB(x)$ is a polynomial with degree $\le n$ which means $cA(x) + dB(x)$ is also in P_n.

So P_n is a subspace of the vector space F.