A1.

a) $U = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$ since f is increasing on $[a,b]$.

$L = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$ since f is increasing on $[a,b]$.

so $U - L = f(x_n)\Delta x - f(x_0)\Delta x = (f(b) - f(a))\Delta x$

b) Since f is increasing on $[a,b]$,

$U = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n$

$L = f(x_0)\Delta x_1 + f(x_1)\Delta x_2 + \cdots + f(x_{n-1})\Delta x_n$

$$\begin{aligned}\Rightarrow U - L &= (f(x_1) - f(x_0))\Delta x_1 + (f(x_2) - f(x_1))\Delta x_2 + \cdots + (f(x_n) - f(x_{n-1}))\Delta x_n \\ &\leq (f(x_1) - f(x_0))\Delta x_{max} + (f(x_2) - f(x_1))\Delta x_{max} + \cdots + (f(x_n) - f(x_{n-1}))\cdot\Delta x_{max} \\ &= (f(b) - f(a))\Delta x_{max} \\ &= |f(b) - f(a)|\Delta x_{max}\end{aligned}$$

Thus $\lim_{\|P\|\to 0}(U - L) = \lim_{\|P\|\to 0}(f(b) - f(a))\Delta x_{max} = 0$ since $\Delta x_{max} = \|P\|$.

A2.

a) True : $h'(x) = f(x)$, $h(x)$ is twice-differentiable for all x since $f(x)$ is differentiable for all x.

b) True : they are both continuous since they are both differentiable.

c) True : $h'(1) = f(1) = 0$.

d) True : $h'(1) = 0$ & $h''(1) = f'(1) < 0$

e) False : $h''(1) = f'(1) < 0$

f) False : $h''(x) = f'(x) < 0$. the sign never changes.

g) True : $h'(x)$ is decreasing since $f'(x) < 0$. and $h'(1) = f(1) = 0$, so $h'(x)$ crosses the x-axis at $x = 1$

A3.

Sol. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}} d\theta$

$$d(\cos\sqrt{\theta}) = (\cos\sqrt{\theta})' d\theta = \frac{\sin\sqrt{\theta}}{-2\sqrt{\theta}} d\theta$$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}} = \int \frac{\sin\sqrt{\theta}}{-2\sqrt{\theta}} \cdot \frac{-2}{\sqrt{\cos^3\sqrt{\theta}}} \cdot d\theta$$

$$= \int \frac{-2}{\sqrt{\cos^3\sqrt{\theta}}} \cdot d(\cos\sqrt{\theta})$$

$$= -2\int (\cos\sqrt{\theta})^{-\frac{3}{2}} d(\cos\sqrt{\theta})$$

$$= -2 \cdot (-2)(\cos\sqrt{\theta})^{-\frac{1}{2}} + C$$

$$= \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C$$

A4.

Sol. $(y-1)^2 = 3 - y \Rightarrow y^2 - 2y + 1 = 3 - y$

$$\Rightarrow y^2 - y - 2 = 0$$

$$\Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow y_1 = 2,\ y_2 = -1.$$

since $y > 0$, the intersection of two lines is $(1, 2)$

$$3-y=2\sqrt{y} \Rightarrow 9-6y+y^2=4y$$
$$\Rightarrow y^2-10y+9=0$$
$$\Rightarrow (y-9)(y-1)=0$$
$$\Rightarrow y_1=9,\ y_2=1$$

Since $x>0$, the intersection of the two lines is $(2,1)$.

$$\text{area}=\int_0^1 2\sqrt{y}\,dy+\int_1^2[(3-y)-(y-1)^2]dy$$
$$=\frac{4}{3}y^{\frac{3}{2}}\Big|_0^1+\left(-\frac{1}{3}y^3+\frac{1}{2}y^2+2y\right)\Big|_1^2$$
$$=\frac{4}{3}+\frac{7}{6}$$
$$=\frac{15}{6}=\frac{5}{2}$$

A5.

Sol. Let $u=a-x$, $du=(a-x)'dx=-dx$.

$$I=\int_0^a\frac{f(x)dx}{f(x)+f(a-x)}=\int_a^0\frac{-f(a-u)du}{f(a-u)+f(u)}=\int_0^a\frac{f(a-x)dx}{f(a-x)+f(x)}$$

Thus
$$I=\frac{1}{2}\int_0^a\frac{f(x)dx}{f(x)+f(a-x)}+\frac{1}{2}\int_0^a\frac{f(a-x)dx}{f(x)+f(a-x)}$$
$$=\frac{1}{2}\int_0^a\frac{f(x)+f(a-x)}{f(x)+f(a-x)}\cdot dx$$
$$=\frac{1}{2}\int_0^a 1\cdot dx$$
$$=\frac{1}{2}(x)\Big|_0^a$$
$$=\frac{a}{2}$$

B1.

Sol. the derivative of the left side:

$$\frac{d}{dx}\left(\int_0^x\left(\int_0^u f(t)dt\right)du\right)=\int_0^x f(t)dt.$$

the derivative of the right side:

$$\frac{d}{dx}\left(\int_0^x f(u)(x-u)du\right)$$

$$=\frac{d}{dx}\left(\int_0^x f(u)x\cdot du\right)-\frac{d}{dx}\left(\int_0^x f(u)u\,du\right)$$

$$=\frac{d}{dx}\left(x\int_0^x f(u)du\right)-f(x)\cdot x$$

$$=\int_0^x f(u)du+x\left[\frac{d}{dx}\left(\int_0^x f(u)du\right)\right]-xf(x)$$

$$=\int_0^x f(u)du+xf(x)-xf(x)$$

$$=\int_0^x f(u)du.$$

When $x=0$, both side must be 0, so the constant must be 0.

Thus. $\int_0^x\left[\int_0^u f(t)dt\right]du=\int_0^x f(u)(x-u)du$.