Problem 3.3.18

Sol. $A=\begin{bmatrix}1&0&1\\1&1&2\\1&1&q\end{bmatrix}\rightarrow\begin{bmatrix}1&0&1\\0&1&1\\0&1&q-1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&1\\0&1&1\\0&0&q-2\end{bmatrix}$

if $q-2=0$ then rank of A is 2.

if $q-2\neq 0$ then rank of A is 3.

$\operatorname{rank}(A^T)=\operatorname{rank}(A)=\begin{cases}2, & q-2=0\\3, & q-2\neq 0\end{cases}$.

Problem 3.3.24.

Sol. (a) $A=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}$ (b) $A=\begin{bmatrix}1&0&1&1\\0&1&1&1\end{bmatrix}$

(c) $A=\begin{bmatrix}1&0&1&1\\0&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ (d) $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Problem 3.4.2.

Sol. Let $A=[v_1\ v_2\ v_3\ v_4\ v_5\ v_6]$

$A=\begin{bmatrix}1&1&1&0&0&0\\-1&0&0&1&1&0\\0&-1&0&-1&0&1\\0&0&-1&0&-1&-1\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&0&0&0\\0&1&1&1&1&0\\0&-1&0&-1&0&1\\0&0&-1&0&-1&-1\end{bmatrix}\rightarrow$

$\begin{bmatrix}1&1&1&0&0&0\\0&1&1&1&1&0\\0&0&1&0&1&1\\0&0&-1&0&-1&-1\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&0&0&0\\0&1&1&1&1&0\\0&0&1&0&1&1\\0&0&0&0&0&0\end{bmatrix}$

so rank of A is 3.

Therefore, the largest possible number of independent vectors is 3.

Problem 3.4.8

Pf. Assume the v's are dependent.

$\Rightarrow$ there exists c's (c's are not all zero) s.t.

$c_1v_1 + c_2v_2 + c_3v_3 = 0$.

$\Rightarrow c_1(w_1+w_2) + c_2(w_1+w_3) + c_3(w_2+w_3) = 0$

$\Rightarrow (c_1+c_2)w_1 + (c_1+c_3)w_2 + (c_2+c_3)w_3 = 0$

Since w's are independent,

$c_1+c_2 = c_1+c_3 = c_2+c_3 = 0$.

$\Rightarrow \begin{cases} c_2 = c_3 \\ c_2+c_3=0 \end{cases}$ and $\begin{cases} c_1 = c_3 \\ c_1+c_3=0 \end{cases}$

$\Rightarrow c_1 = c_2 = c_3 = 0$

Thus lead to contradiction.

Therefore v's are ~~d~~ independent.

Problem 3.4.11.

Sol. (a) a line in $\mathbb{R}^3$ (b) a plane in $\mathbb{R}^3$.

(c) All of $\mathbb{R}^3$ (d) All of $\mathbb{R}^3$

Problem 3.4.20.

Sol. Let $A = [1\ \ -2\ \ 3]$ (also $N(A)$)

then $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$ is a basis for $x-2y+3z=0$.

the intersection of xy plane and $x-2y+3z=0$

is $x-2y=0$.

Let $B = [1\ \ -2\ \ 0]$

then $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}\right\}$ is a basis for the intersection

$\left\{\begin{bmatrix}1\\-2\\3\end{bmatrix}\right\}$ is a basis for $C(A^T)$ and also all vectors perpendicular to the plane.

Problem 3.5.2.

Sol. $[B\ I] = \begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 2 & 5 & 8 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix} = [R\ E]$

so rank of B is 2.

	dim	basis.
$C(B)$	2	$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$
$N(B)$	$3-2=1$	$\left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$
$C(B^T)$	2	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$
$N(B^T)$	$2-2=0$	$\emptyset$

Problem 3.5.11.

Sol (a) $r < m$ and $r \leq n$.

(b) dimension of $N(A^T)$ is $m-r$.

Since $r<m$, we have $m-r>0$.

~~while~~ Thus $A^Ty=0$ has solutions other than $y=0$.

Problem 3.5.18

Sol. the combination is $1\times \text{row}1 + (-2)\times\text{row}2 + 1\times\text{row}3 = 0$

$c\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ are in $N(A^T)$

$c\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ are in $N(A)$

Problem 3.5.24.

Sol. If b is in $C(A^T)$ then $A^Ty=d$ is solvable.

y is unique when $N(A^T)$ contains only zero vector.

Graded Problem.

Problem 1.

Sol. $[A\ b] = \left[\begin{array}{cccc|c} 2 & 3 & -1 & 2 & -1 \\ 4 & 6 & 2 & 2 & 1 \\ 6 & 9 & 1 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 2 & 3 & -1 & 2 & -1 \\ 0 & 0 & 4 & -2 & 3 \\ 0 & 0 & 4 & -4 & 2 \end{array}\right]$

$\rightarrow \left[\begin{array}{cccc|c} 2 & 3 & -1 & 2 & -1 \\ 0 & 0 & 4 & -2 & 3 \\ 0 & 0 & 0 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 2 & 3 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & -2 & -1 \end{array}\right]$

$\rightarrow \left[\begin{array}{cccc|c} 1 & 3/2 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right] = [R\ d]$

$x_p = \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 1/2 \end{bmatrix} \quad x_n = x_2 \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Therefore the solution $x = x_p + x_n = \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 1/2 \end{bmatrix} + x_2 \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Problem 2.

Sol. (a) Let $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 3/2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & 4/3 & -2/3 \\ 0 & 0 & -2/3 & 4/3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1 & -2/3 & -2/3 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R.$

Therefore rank of A is 4 which means

$\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$

(b) Let $A=\begin{bmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{bmatrix}$

$$A=\begin{bmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{bmatrix}\to\begin{bmatrix}-1&0&-1&2\\0&-1&2&-1\\2&-1&0&-1\\-1&2&-1&0\end{bmatrix}\to\begin{bmatrix}1&0&1&-2\\0&1&-2&1\\2&-1&0&-1\\-1&2&-1&0\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&1&-2\\0&1&-2&1\\0&-1&-2&3\\0&2&0&-2\end{bmatrix}\to\begin{bmatrix}1&0&1&-2\\0&1&-2&1\\0&0&-4&4\\0&0&4&-4\end{bmatrix}\to\begin{bmatrix}1&0&1&-2\\0&1&-2&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&0&-1\\0&1&0&-1\\0&0&1&-1\\0&0&0&0\end{bmatrix}=R.\quad x_n=x_4\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Therefore rank of A is 3, which means means $\left\{\begin{bmatrix}2\\-1\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\2\\-1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\2\\-1\end{bmatrix},\begin{bmatrix}-1\\0\\-1\\2\end{bmatrix}\right\}$ is dependent.

$$-\begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}-\begin{bmatrix}-1\\2\\-1\\0\end{bmatrix}-\begin{bmatrix}0\\-1\\2\\-1\end{bmatrix}=\begin{bmatrix}-1\\0\\-1\\2\end{bmatrix}$$

Problem 3

Sol. $[A\ I]=\begin{bmatrix}-1&2&-3&4&1&0&0\\3&4&-1&0&0&1&0\\2&1&1&-2&0&0&1\end{bmatrix}\to\begin{bmatrix}-1&&-3&4&1&0&0\\0&10&-10&12&3&1&0\\0&5&-5&6&2&0&1\end{bmatrix}$

$$\to\begin{bmatrix}-1&2&-3&4&1&0&0\\0&10&-10&12&3&1&0\\0&0&0&0&1/2&-1/2&1\end{bmatrix}\to\begin{bmatrix}1&-2&3&-4&-1&0&0\\0&1&-1&6/5&3/10&1/10&0\\0&0&0&0&1&-1&2\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&1&-8/5&-2/5&1/5&0\\0&1&-1&6/5&3/10&1/10&0\\0&0&0&0&1&-1&2\end{bmatrix}=[R\ E]$$

Therefore basis of $C(A)$: $\left\{\begin{bmatrix}-1\\3\\2\end{bmatrix},\begin{bmatrix}2\\4\\1\end{bmatrix}\right\}$

basis of $N(A)$: $\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}8/5\\-6/5\\0\\1\end{bmatrix}\right\}$

basis of $C(A^T)$: $\left\{\begin{bmatrix}1\\0\\1\\-8/5\end{bmatrix},\begin{bmatrix}0\\1\\-1\\6/5\end{bmatrix}\right\}$

basis of $N(A^T)$: $\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$