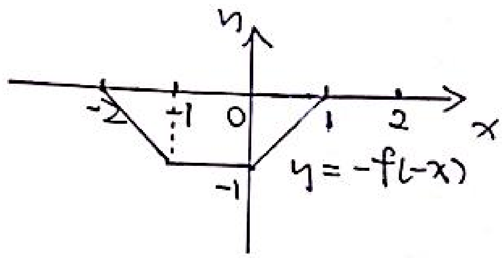
Part 1.

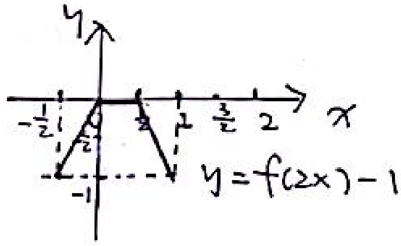
1. B. 2. D. 3. A 4. A

Part 2a.

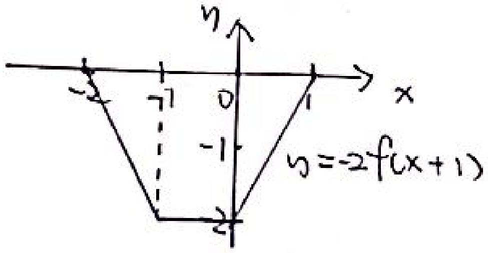
1.



2.



3.



Part 2b.

Sol. $x^2+y^2+2y-6x+1=0$

$\Leftrightarrow x^2-6x+9 \quad + y^2+2y+1=9$

$\Leftrightarrow (x-3)^2+(y+1)^2=9$

so the equation is a circle

the center of the circle is $(3,-1)$

the radius of the circle is $\sqrt{9}=3$.

Part 2C.

Sol. $f(x)=\tan x \Rightarrow f'(x)=\frac{1}{\cos^2 x}$

the tangent line at $x=\frac{\pi}{4}$ is

$y-f(\frac{\pi}{4})=f'(\frac{\pi}{4})(x-\frac{\pi}{4})$

$\Leftrightarrow y-\tan\frac{\pi}{4}=\frac{1}{\cos^2(\frac{\pi}{4})}(x-\frac{\pi}{4})$

$\Leftrightarrow y-1=2(x-\frac{\pi}{4}) \Leftrightarrow y=2x+1-\frac{\pi}{2}$.

Part 2d.

Sol.
$$\lim_{x\to\pi}\frac{\sqrt{1+\cos(\frac{x}{2})}-1}{x-\pi}$$

$$=\lim_{x\to\pi}\frac{(\sqrt{1+\cos(\frac{x}{2})}-1)(\sqrt{1+\cos(\frac{x}{2})}+1)}{(x-\pi)(\sqrt{1+\cos(\frac{x}{2})}+1)}$$

$$=\lim_{x\to\pi}\frac{\cos(\frac{x}{2})}{(x-\pi)(\sqrt{1+\cos(\frac{x}{2})}+1)}$$

$$=\lim_{x\to\pi}\frac{\sin(\frac{\pi}{2}-\frac{x}{2})}{\frac{\pi-x}{2}\times(-2)\times(\sqrt{1+\cos(\frac{x}{2})}+1)}$$

$$=\lim_{x\to\pi}\frac{\sin(\frac{\pi-x}{2})}{\frac{\pi-x}{2}}\times\lim_{x\to\pi}\frac{1}{(-2)\times(\sqrt{1+\cos(\frac{x}{2})}+1)}$$

$$=\left(\lim_{\frac{\pi-x}{2}\to 0}\frac{\sin(\frac{\pi-x}{2})}{\frac{\pi-x}{2}}\right)\times\frac{1}{(-2)\times(\sqrt{1+\cos\frac{\pi}{2}}+1)}$$

$$=1\times\frac{1}{(-2)\times(1+1)}$$

$$=-\frac{1}{4}$$

So the limit exists and is equal to $-\frac{1}{4}$.

Part 2e

Pf. Let $g(x)=f(x)-x$, $x\in[0,1]$

since for all $x\in[0,1]$, $0\le f(x)\le 1$.

we have $0\le f(0)\le 1$, $0\le f(1)\le 1$

$\Rightarrow 0\le f(0)-0\le 1$, $-1\le f(1)-1\le 0$

$\Rightarrow 0\le g(0)\le 1$, $-1\le g(1)\le 0$.

① if $g(0)=0$ then ~~the~~ choose $c=0$, $f(0)=0$.

② if $g(1)=0$ then choose $c=1$, $f(1)=1$

③ if $g(0)\neq 0$, $g(1)\neq 0$

then $0<g(0)\le 1$ and $-1\le g(1)<0$

which means $g(0)>0$ and $g(1)<0$

Since $f(x)$ is continuous and $y=x$ is continuous

$g(x)$ is also continuous.

According to The Intermedian Value Theorem

there exists $c\in(0,1)$ s.t $g(c)=0$

$\Leftrightarrow$ there exists $c\in(0,1)$ s.t $f(c)=c$.

By ①②③, there exists $c\in[0,1]$ s.t $f(c)=c$.

the graph of f intersects the ~~dig diagno~~ diagonal $y=x$.

Part 3. Bonus Question:

$$\lim_{\Delta x\to 0^-}\frac{f(1+\Delta x)-f(1)}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{\frac{1}{2}(1+\Delta x)-\frac{1}{2}\times 1}{\Delta x}$$

$$=\lim_{\Delta x\to 0^-}\frac{\frac{1}{2}\Delta x}{\Delta x}$$

$$=\frac{1}{2}$$

$$\lim_{\Delta x\to 0^+}\frac{f(1+\Delta x)-f(1)}{\Delta x}=\lim_{\Delta x\to 0^+}\frac{a(1+\Delta x)^2+b(1+\Delta x)+1-\frac{1}{2}\times 1}{\Delta x}$$

$$=\lim_{\Delta x\to 0^+}\frac{a(\Delta x^2+2\Delta x+1)+b(1+\Delta x)+\frac{1}{2}}{\Delta x}$$

$$=\lim_{\Delta x\to 0^+}\frac{a\Delta x^2+(2a+b)\Delta x+a+b+\frac{1}{2}}{\Delta x}.$$

For making f differentable at $x=1$.

we must have $\lim\limits_{\Delta x\to 0^-}f(x)=\lim\limits_{\Delta x\to 0^+}f(x)$

and $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^+}f(x)=f(1)$.

$\Leftrightarrow \frac{1}{2}=\lim\limits_{\Delta x\to 0^+}\frac{a\Delta x^2+(2a+b)\Delta x+a+b+\frac{1}{2}}{\Delta x}$ and $\frac{1}{2}=a+b+1$

$\Leftrightarrow \frac{1}{2}=\lim\limits_{\Delta x\to 0^+}(a\Delta x)+\lim\limits_{\Delta x\to 0^+}(2a+b)+\lim\limits_{\Delta x\to 0^+}\frac{a+b+\frac{1}{2}}{\Delta x}$ and $0=a+b+\frac{1}{2}$

$\Leftrightarrow \frac{1}{2}=2a+b+\lim\limits_{\Delta x\to 0^+}\frac{a+b+\frac{1}{2}}{\Delta x}$ and $0=a+b+\frac{1}{2}$

$\Leftrightarrow \begin{cases}2a+b=\frac{1}{2}\\ a+b+\frac{1}{2}=0\end{cases}$ and $0=a+b+\frac{1}{2}$

$\Leftrightarrow \begin{cases}a=1\\ b=-\frac{3}{2}\end{cases}$

So when $a=1$ and $b=-\frac{3}{2}$, f is differentiable at $x=1$.