Problem 6.2.7

Sol. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{cases} Ax_1=\lambda_1 x_1 \\ Ax_2=\lambda_2 x_2 \end{cases} \Rightarrow \begin{cases} \begin{bmatrix} a+b \\ c+d \end{bmatrix}=\begin{bmatrix} \lambda_1 \\ \lambda_1 \end{bmatrix} \\ \begin{bmatrix} a-b \\ c-d \end{bmatrix}=\begin{bmatrix} \lambda_2 \\ -\lambda_2 \end{bmatrix} \end{cases} \Rightarrow \begin{cases} a+b=c+d \\ a-b=d-c \end{cases} \Rightarrow \begin{cases} a=d \\ b=c \end{cases}$$

So $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is all 2x2 matrices that have eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Problem 6.2.9.

Sol. (a) $A=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

(b) $\det(A-\lambda I)=\begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ 1 & -\lambda \end{bmatrix}=(\frac{1}{2}-\lambda)(-\lambda)-\frac{1}{2}$

$$=\frac{1}{2}(2\lambda+1)(\lambda-1)=0$$

$\lambda_1=-\frac{1}{2}$.

$$(A+\tfrac{1}{2}I)x=\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}x=0 \Rightarrow x_1=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\lambda_2=1$

$$(A-I)x=\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix}x=0 \Rightarrow x_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X=\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, X^{-1}=\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

when $n\to\infty$, $A^n=X\Lambda^n X^{-1}$

$$=\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} (-\frac{1}{2})^n & 0 \\ 0 & 1^n \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

(c)

Let $\begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = c_1 x_1 + c_2 x_2$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 1/3 \\ c_2 = 2/3 \end{cases}$$

$$\begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix} = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = \frac{1}{3} \times \left(-\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{2}{3} \times 1^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow G_k = \frac{1}{3} \times \left(-\frac{1}{2}\right)^k \times (-2) + \frac{2}{3}$$

when $k \to \infty$, $G_k = \frac{2}{3}$

Problem 6.2.15.

~ iff every λ has absolute value less than 1.

Sol. $A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} .6-\lambda & .9 \\ .4 & .1-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 0.3) = 0$$

$\Rightarrow \lambda_1 = 1, \lambda_2 = -0.3$. $A^k \not\to 0 \ (k \to \infty)$

$A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} .6-\lambda & .9 \\ .1 & .6-\lambda \end{vmatrix} = (0.3 - \lambda)(0.9 - \lambda) = 0$$

$\Rightarrow \lambda_1 = 0.3, \lambda_2 = 0.9$, $A^k \to 0 \ (k \to \infty)$

Problem 6.2.16.

A is $X \Lambda X^{-1}$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, (k \to \infty)$

$A^k = X \Lambda^k X^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ as $k \to \infty$ It is steady state.

Problem 6.2.30.

(a) Sol.

$$(A-aI)(B-bI)=\begin{bmatrix}0 & b\\ 0 & d-a\end{bmatrix}\begin{bmatrix}a-d & b\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

(b)

$$A^2-A-I=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^2-\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}-\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

Problem 6.3.4.

Sol. $A=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} \Rightarrow \begin{matrix}\lambda_1=0\\ \lambda_2=-2\end{matrix} \Rightarrow \begin{matrix}x_1=\begin{bmatrix}1\\ 1\end{bmatrix}\\ x_2=\begin{bmatrix}1\\ -1\end{bmatrix}\end{matrix}$.

$$\begin{bmatrix}v(0)\\ w(0)\end{bmatrix}=\begin{bmatrix}30\\ 10\end{bmatrix}=20\begin{bmatrix}1\\ 1\end{bmatrix}+10\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\Rightarrow \begin{matrix}v(1)=20+10e^{-2}\\ w(1)=20-10e^{-2}\end{matrix} \quad \& \quad \begin{matrix}v(\infty)=20\\ w(\infty)=20.\end{matrix}$$

Problem 6.3.21.

Sol. $A=\begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix} \Rightarrow \begin{matrix}\lambda_1=1\\ \lambda_2=0\end{matrix} \Rightarrow \begin{matrix}x_1=\begin{bmatrix}1\\ 0\end{bmatrix}\\ x_2=\begin{bmatrix}4\\ -1\end{bmatrix}\end{matrix}$

$$\text{So } A=\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}=X\Lambda X^{-1}.$$

$$e^{At}=Xe^{\Lambda t}X^{-1}=\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}\begin{bmatrix}e^t & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & -1\end{bmatrix}=\begin{bmatrix}e^t & 4e^t-4\\ 0 & 1\end{bmatrix}$$

Problem 6.4.8

Sol. $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 25 \end{matrix} \Rightarrow \begin{matrix} x_1 = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \\ x_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{matrix}$.

$Q = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$, or we can exchange the columns or exchange the signs of ~~column~~ any columns.

Problem 6.4.21.

Sol. $S = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda = 1, d, -d$.

$\Rightarrow X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix}$, $\lambda = 1, d, -d$

$\Rightarrow X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$

Perpendicnlar for S but not perpendicnlar for B since $B^T \neq B$.

Graded Problem.

$$A=\begin{bmatrix}2&1&1\\1&3&-2\\1&-2&3\end{bmatrix}$$

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda&1&1\\1&3-\lambda&-2\\1&-2&3-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda&1&0\\1&3-\lambda&\lambda-5\\1&-2&5-\lambda\end{vmatrix}$$

$$=\begin{vmatrix}2-\lambda&1&0\\2&1-\lambda&0\\1&-2&5-\lambda\end{vmatrix}=(5-\lambda)\begin{vmatrix}2-\lambda&1\\2&1-\lambda\end{vmatrix}=(5-\lambda)[(2-\lambda)(1-\lambda)-2]$$

$$=(5-\lambda)\lambda(\lambda-3)=0$$

$$\Rightarrow \lambda=0,3,5$$

$$\lambda_1=0 \Rightarrow (A-0I)x=\begin{bmatrix}2&1&1\\1&3&-2\\1&-2&3\end{bmatrix}x=0.$$

$$\begin{bmatrix}2&1&1\\1&3&-2\\1&-2&3\end{bmatrix}\rightarrow\begin{bmatrix}2&1&1\\2&6&-4\\2&-4&6\end{bmatrix}\rightarrow\begin{bmatrix}2&1&1\\0&5&-5\\0&-5&5\end{bmatrix}\rightarrow\begin{bmatrix}2&1&1\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1&0&1\\0&1&-1\\0&0&0\end{bmatrix}\rightarrow \vec{x}=x_3\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\lambda_2=3\Rightarrow(A-3I)x=\begin{bmatrix}-1&1&1\\1&0&-2\\1&-2&0\end{bmatrix}x=0$$

$$\begin{bmatrix}-1&1&1\\1&0&-2\\1&-2&0\end{bmatrix}\rightarrow\begin{bmatrix}-1&1&1\\0&1&-1\\0&-1&1\end{bmatrix}\rightarrow\begin{bmatrix}-1&0&2\\0&1&-1\\0&0&0\end{bmatrix}\rightarrow\vec{x}=x_3\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\lambda_3 = 5 \Rightarrow (A-5I)x = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -5/2 & 0 & 0 \\ 1 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \vec{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

so orthonormal basis for $\mathbb{R}^3$ of eigenvectors:

$$\vec{x}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{x}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{x}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A = Q \Lambda Q^T = \begin{bmatrix} -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A^N = Q \Lambda^N Q^T = \begin{bmatrix} -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3^N & 0 \\ 0 & 0 & 5^N \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$