1. (15 points) Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 3x_2 + 4x_3 &= b_2 \\ 3x_1 + 4x_2 + 5x_3 &= b_3 \end{aligned}$$

(a) Find a linear relation involving b_1, b_2, and b_3 that guarantees the system has at least one solution.

(b) For $(b_1, b_2, b_3) = (1, 1, 1)$, find *all* solutions of the system of equations.

(a) Elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 4 & b_2 \\ 3 & 4 & 5 & b_3 \end{array}\right] \xrightarrow[\text{Row 3}-3\text{Row 1}]{\text{Row 2}-2\text{Row 1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -1 & -2 & -2b_1+b_2 \\ 0 & -2 & -4 & -3b_1+b_3 \end{array}\right] \xrightarrow[\text{Row 2} \to -\text{Row 2}]{\text{Row 3}-2\text{Row 2}}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & 2b_1-b_2 \\ 0 & 0 & 0 & b_1-2b_2+b_3 \end{array}\right] \xrightarrow{\text{Row 1}-2\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3b_1+2b_2 \\ 0 & 1 & 2 & 2b_1-b_2 \\ 0 & 0 & 0 & b_1-2b_2+b_3 \end{array}\right]$$

For solutions to exist, we need $\boxed{b_1 - 2b_2 + b_3 = 0}$

(b) For $(b_1, b_2, b_3) = (1,1,1)$, $1 - 2(1) + (1) = 0$, so solutions exist. Need to solve:

$$\begin{aligned} x_1 - x_3 &= -3b_1 + 2b_2 = -1 \\ x_2 + 2x_3 &= 2b_1 - b_2 = 1 \end{aligned} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{particular solution}} + x_3 \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{\text{null space vectors}}$$

2. (12 points) Find the determinants of A and B:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}, \qquad B = \begin{bmatrix} a & 2 & b \\ 2 & 0 & 2 \\ b & 2 & a \end{bmatrix}.$$

What condition on a, b guarantees that B is *not* invertible?

For A, use row operations:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix} \xrightarrow[\substack{\text{Row 4} - \text{Row 1} \\ \text{(no change in det)}}]{\substack{\text{Row 2} - \text{Row 1} \\ \text{Row 3} - \text{Row 1}}} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{vmatrix} \xrightarrow[\substack{\text{Row 4} - 3\text{Row 2} \\ \text{(no change in det)}}]{\text{Row 3} - 2\text{Row 2}}$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 6 & 42 \end{vmatrix} \xrightarrow[\text{(no change in det)}]{\text{Row 4} - 3\text{Row 3}} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 6 \end{vmatrix} = (1)(1)(2)(6) = 12$$

For B, expand across 2nd row:

$$\begin{vmatrix} a & 2 & b \\ 2 & 0 & 2 \\ b & 2 & a \end{vmatrix} = -2\begin{vmatrix} 2 & b \\ 2 & a \end{vmatrix} - 2\begin{vmatrix} a & 2 \\ b & 2 \end{vmatrix} = -2(2a-2b)-2(2a-2b)$$

$$= -4(a-b)-4(a-b) = -8(a-b)$$

B is <u>not</u> invertible when $a=b$ (then $\det B = 0$)

3. (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 7 & -5 & 2 & 5 \\ -1 & -7 & -8 & 7 \end{bmatrix}.$$

(a) Find the reduced row echelon form R of A.

(b) Find bases for the null space, row space, column space, and left null space of A.

(a) $\begin{bmatrix} 1 & -2 & -1 & 2 \\ 7 & -5 & 2 & 5 \\ -1 & -7 & -8 & +7 \end{bmatrix} \xrightarrow[\text{Row 3 + Row 1}]{\text{Row 2} - 7\text{ Row 1}} \begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & 9 & 9 & -9 \\ 0 & -9 & -9 & 9 \end{bmatrix} \xrightarrow[\frac{1}{9}\text{Row 2}]{\text{Row 3 + Row 2}}$

$\begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1} + 2\text{Row 2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

pivot columns ↑ ↑ ↑ ↑ free variables

(b) $N(A)$ = all solutions to $x_1 + x_3 = 0$, $x_2 + x_3 - x_4 = 0$ $\longrightarrow$

$N(A)$ = all $\begin{bmatrix} -x_3 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

↑ ↑ special solutions, basis vectors for $N(A)$

Basis for row space, $C(A^T)$: non-zero rows in R:

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

Basis for C(A): pivot columns <u>in A</u>: $\begin{bmatrix}1\\7\\-1\end{bmatrix}, \begin{bmatrix}-2\\-5\\-7\end{bmatrix}$

Left nullspace $N(AT) = C(A)^T$ = all solutions to

$$\begin{bmatrix}1\\7\\1\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0, \quad \begin{bmatrix}-2\\-5\\-7\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \longrightarrow$$

$$\begin{aligned} x_1 + 7x_2 + x_3 &= 0 \\ -2x_1 - 5x_2 - 7x_3 &= 0 \end{aligned} \longrightarrow \begin{bmatrix}1 & 7 & -1\\ -2 & -5 & -7\end{bmatrix} \xrightarrow{\text{Row } 2 + 2\text{ Row } 1} \begin{bmatrix}1 & 7 & -1\\ 0 & \underset{1}{\cancel{9}} & \underset{-1}{\cancel{-9}}\end{bmatrix}$$

$$\xrightarrow{\text{Row } 1 - 7\text{ Row } 2} \begin{bmatrix}1 & 0 & 6\\ 0 & 1 & -1\end{bmatrix} \longrightarrow \begin{aligned} x_1 &= -6x_3 \\ x_2 &= x_3 \\ x_3 &\text{ free} \end{aligned} \longrightarrow \vec{x} = x_3\begin{bmatrix}-6\\1\\1\end{bmatrix}$$

Basis for N(AT)

Non-zero rows give <u>another</u> basis for C(A): $\begin{bmatrix}1\\0\\6\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$

4. (a) (7 points) If $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent vectors, show that the sums

$$\mathbf{v}_1 = \mathbf{w}_2 + \mathbf{w}_3, \quad \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_3, \quad \mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2$$

are also independent. (*Hint:* Write $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ in terms of the $\mathbf{w}$'s. Find and solve equations for the c's, to show they are zero.)

(b) (5 points) If $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent vectors, show that the differences

$$\mathbf{v}_1 = \mathbf{w}_2 - \mathbf{w}_3, \quad \mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_3, \quad \mathbf{v}_3 = \mathbf{w}_1 - \mathbf{w}_2$$

are dependent. (Find a linear combination of the $\mathbf{v}$'s that gives zero.)

(a) Need to show that the only solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ is $c_1 = c_2 = c_3 = 0$

$$c_1(\vec{w}_2 + \vec{w}_3) + c_2(\vec{w}_1 + \vec{w}_3) + c_3(\vec{w}_1 + \vec{w}_2) = 0$$

$$\downarrow$$

$$(c_2 + c_3)\vec{w}_1 + (c_1 + c_3)\vec{w}_2 + (c_1 + c_2)\vec{w}_3 = \vec{0}$$

Because $\vec{w}$'s are independent, this only happens if

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \end{cases} \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1} \leftrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{Row 3} - \text{Row 1}}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 3} -} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{more}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow$$

only solution is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, so $\vec{v}$'s are independent.

(b) When is $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$?

$$c_1(\vec{w}_2 - \vec{w}_3) + c_2(\vec{w}_1 - \vec{w}_3) + c_3(\vec{w}_1 - \vec{w}_2) = \vec{0}$$

$$(c_2 + c_3)\vec{w}_1 + (c_1 - c_3)\vec{w}_2 + (-c_1 - c_2)\vec{w}_3 = \vec{0}$$

Since $\vec{w}$'s are independent, this only happens if

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 - c_3 = 0 \\ -c_1 - c_2 = 0 \end{cases} \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row } 2]{\text{Row } 1 \leftrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row } 1]{\text{Row } 3 +} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{Row } 2 + \text{Row } 3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} c_1 = c_3 \\ c_2 = -c_3 \\ c_3 \text{ free} \end{matrix} \longrightarrow \vec{c} = c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

non-zero solutions, so $\vec{v}$'s are dependent.

Take $c_3 = 1$: Then $\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$, so we can write:

$\vec{v}_1 = \vec{v}_2 - \vec{v}_3$,

$\vec{v}_2 = \vec{v}_1 + \vec{v}_3$, and

$\vec{v}_3 = -\vec{v}_1 + \vec{v}_2$

5. (a) (12 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) (3 points) Use A^{-1} to solve the linear system of equations $A\mathrm{x} = (3, 6, 2)$.

(a) Use eliminiation:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row } 3 - \frac{1}{2}\text{Row } 1]{\text{Row } 2 - \frac{1}{2}\text{Row } 1} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 1/2 & -1/2 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row } 2 - \frac{1}{3}\text{Row } 2}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/3 & -1/3 & -1/3 & 1 \end{array}\right] \xrightarrow[\substack{\frac{2}{3}\text{Row } 2 \\ 3 \text{ Row } 3}]{\frac{1}{2}\text{Row } 1} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] \xrightarrow[\text{Row } 2 - \frac{1}{3}\text{Row } 3]{\text{Row } 1 - \frac{1}{2}\text{Row } 3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1 & 1/2 & -3/2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] \xrightarrow{\text{Row } 1 - \frac{1}{2}\text{Row } 2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

(b) $$A\vec{x} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \longrightarrow \vec{x} = A^{-1}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} +1 \\ 4 \\ -3 \end{bmatrix}$$

6. (a) (6 points) Find all eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(b) (10 points) Find an *orthonormal* basis for $\mathbf{R}^3$ consisting of eigenvectors for A.

(a) Eigenvalues = solutions to $\det(A-\lambda I)=0$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= (2-\lambda)\Big(\underbrace{(2-\lambda)(2-\lambda)-1}_{\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)}\Big) + \underbrace{(-2+\lambda-1) - (1+2-\lambda)}_{\lambda-3}$$

Factor out $\lambda-3$:

$$=(\lambda-3)\Big((2-\lambda)(\lambda-1)+1+1\Big) = (\lambda-3)(-\lambda^2+3\lambda) = -\lambda(\lambda-3)=0$$

$$\boxed{\lambda = 0, 3, 3}$$

(b) Eigenvectors for $\lambda=0$: Solve $A\vec{x} = 0\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[\text{Row 3}+\frac{1}{2}\text{Row 1}]{\text{Row2}+\frac{1}{2}\text{Row 1}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \xrightarrow{\text{Row 3}+\text{Row 2}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{2}{3}\text{Row 2}]{\frac{1}{2}\text{Row 1}}$$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{2}\text{Row 2}]{\text{Row 1}+} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix} \longrightarrow \vec{x} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvectors for $\lambda=3$: Solve $A\vec{x} = 3\vec{x} \longrightarrow (A-3I)\vec{x}=0$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x_1+x_2+x_3=0 \longrightarrow \vec{x} = \begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

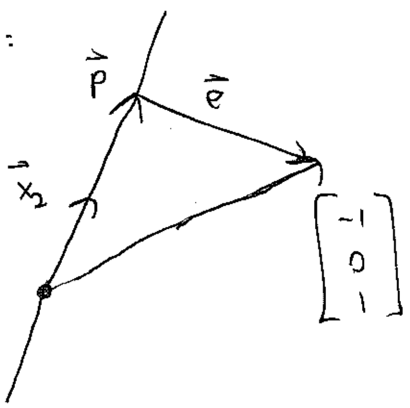
$$\longrightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Here is a basis of eigenvectors: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

But it's <u>not</u> orthonormal. Because A is symmetric, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is already orthogonal to other two, but we need to normalize:

$$\vec{x}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \quad \text{Then take } \vec{x}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

For $\vec{x}_3$:



$$\vec{p} = P \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{e} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\text{Then } \vec{x}_3 = \frac{1}{\|\vec{e}\|} \vec{e} = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{2}{\sqrt{6}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

<u>Orthonormal</u> basis of <u>eigenvectors</u>:

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

7. (a) (4 points) What is the area of the parallelogram in $\mathbf{R}^2$ spanned by $\mathbf{x} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} b \\ d \end{bmatrix}$?

(b) (8 points) Verify that for any 2×2 reflection matrix $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, the area of the parallelogram spanned by $Q\mathbf{x}$ and $Q\mathbf{y}$ is the same as your answer for part (a).

(a) $\text{Area} = \left|\det\begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}\right| = \left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = |ad - bc|$

(b) $\text{Area} = \left|\det\begin{bmatrix} Q\begin{bmatrix} a \\ c \end{bmatrix} & Q\begin{bmatrix} b \\ d \end{bmatrix} \end{bmatrix}\right|$

$$\begin{bmatrix} a\cos\theta + c\sin\theta \\ a\sin\theta - c\cos\theta \end{bmatrix} \qquad \begin{bmatrix} b\cos\theta + d\sin\theta \\ b\sin\theta - d\cos\theta \end{bmatrix}$$

$$= \left| (a\cos\theta + c\sin\theta)(b\sin\theta - d\cos\theta) - (b\cos\theta + d\sin\theta)(a\sin\theta - c\cos\theta) \right|$$

$$= \Big| \cancel{ab\cos\theta\sin\theta} + bc\sin^2\theta - ad\cos^2\theta - \cancel{cd\sin\theta\cos\theta}$$
$$- \cancel{ba\cos\theta\sin\theta} - da\sin^2\theta + bc\cos^2\theta + \cancel{dc\sin\theta\cos\theta} \Big|$$

$$= \left| bc(\sin^2\theta + \cos^2\theta) - ad(\cos^2\theta + \sin^2\theta) \right|$$

$$= |bc - ad| = |ad - bc|$$

8. (a) (10 points) Find the best least squares line $C + Dt$ to fit the data points $(-2,4)$, $(-1,2)$, $(0,-1)$, $(1,0)$, and $(2,0)$.

(b) (2 points) Sketch a graph of the data points and your least squares line.

(c) (3 points) Find the least squares error $\|\mathbf{e}\|$ of the best fit line.

(a) Try to solve:

$$\begin{aligned} C+(-2)D&=4\\ C+(-1)D&=2\\ C+(0)D&=-1\\ C+(1)D&=0\\ C+(2)D&=0 \end{aligned} \longrightarrow \underbrace{\begin{bmatrix}1&-2\\1&-1\\1&0\\1&1\\1&2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}C\\D\end{bmatrix}}_{\hat{x}}=\underbrace{\begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix}}_{\vec{b}}$$

No solution!

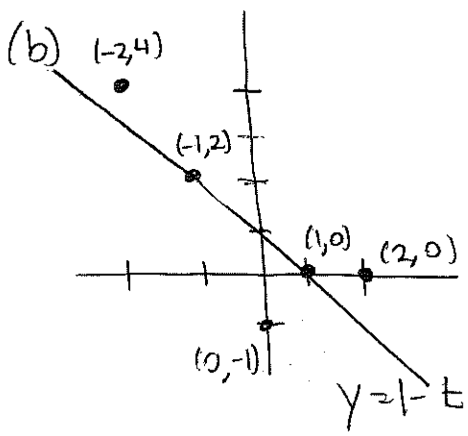
Solve "normal equations" instead: multiply both sides by A^T:

$$A^TA\hat{x}=A^T\vec{b}$$

$$\begin{bmatrix}1&1&1&1&1\\-2&-1&0&1&2\end{bmatrix}\begin{bmatrix}1&-2\\1&-1\\1&0\\1&1\\1&2\end{bmatrix}=\begin{bmatrix}5&0\\0&10\end{bmatrix} \qquad \begin{bmatrix}1&1&1&1&1\\-2&-1&0&1&2\end{bmatrix}\begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}5\\-10\end{bmatrix}$$

$$\begin{bmatrix}5&0\\0&10\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}5\\-10\end{bmatrix} \longrightarrow C=1,\ D=-1 \longrightarrow$$

Best fit line is $\boxed{y=1-t}$

(b) 

(c) Error $\|\vec{e}\| = \|A\hat{x}-\vec{b}\| =$

$$\left\|\begin{bmatrix}1&-2\\1&-1\\1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}-\begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix}\right\|=\left\|\begin{bmatrix}-1\\0\\2\\0\\-1\end{bmatrix}\right\|$$

$$=\sqrt{1+2^2+1}=\sqrt{6}$$

9. Consider 2×2 matrices

$$A = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ 6 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

(a) (8 points) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.

(b) (6 points) Use part (a) to calculate the matrix A^N for any positive integer N. What limit matrix does A^N approach as $N \to \infty$?

(c) (6 points) Show that B is *not* diagonalizable.

(a) Λ comes from eigenvalues, X from eigenvectors.

Eigenvalues: Solve $\det(A-\lambda I) = 0$: $\begin{vmatrix} \frac{5}{2}-\lambda & -\frac{1}{2} \\ 6 & -1-\lambda \end{vmatrix} = \left(\frac{5}{2}-\lambda\right)(-1-\lambda)+3$

$= \lambda^2 - \frac{3}{2}\lambda - \frac{5}{2} + 3 = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda-1)\left(\lambda-\frac{1}{2}\right) = 0 \rightarrow \boxed{\lambda = 1, \frac{1}{2}}$

Eigenvectors for $\lambda=1$: Solve $A\vec{x} = \vec{x} \longrightarrow (A-I)\vec{x} = \vec{0}$

$\begin{bmatrix} 3/2 & -1/2 \\ 6 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/3 \\ 1 & -1/3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \longrightarrow \vec{x} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

$\lambda = \frac{1}{2}$: Solve $(A - \frac{1}{2}I)\vec{x} = \vec{0}$

$\begin{bmatrix} 2 & -1/2 \\ 6 & -3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/4 \\ 1 & -1/4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix} \longrightarrow \vec{x} = x_2 \begin{bmatrix} 1/4 \\ 1 \end{bmatrix}$

Can take $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, $X = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix}$, so

$$A = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix}^{-1}$$

(b) $A^N = (X\Lambda X^{-1})^N = X \Lambda^N X^{-1}$

$\begin{bmatrix} 12 & -3 \\ -12 & 4 \end{bmatrix}$

//

$\begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1^N & 0 \\ 0 & \left(\frac{1}{2}\right)^N \end{bmatrix} \quad \frac{1}{\frac{1}{3}-\frac{1}{4}} \begin{bmatrix} 1 & -1/4 \\ -1 & 1/3 \end{bmatrix}$

$= \frac{1}{1/12} = 12$

$$A^N = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^N \end{bmatrix} \begin{bmatrix} 12 & -3 \\ -12 & +4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 12 & -3 \\ -12/2^N & 4/2^N \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 3(\frac{1}{2})^N & -1 + (\frac{1}{2})^N \\ 12 - 12(\frac{1}{2})^N & -3 + 4(\frac{1}{2})^N \end{bmatrix}$$

As $N \to \infty$, $(\frac{1}{2})^N \to 0$. So $\lim\limits_{N \to \infty} A^N = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

(c) Eigenvalues for B: $\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$

$\rightarrow \lambda = 1, 1$

Eigenvectors: Solve $(B - I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Only <u>one</u> linearly independent eigenvector, not enough for a basis of $\mathbb{R}^2$. So B isn't diagonalizable.

10. (a) (6 points) Find a basis for the subspace S of $\mathbf{R}^4$ spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

(b) (5 points) Find a basis for the orthogonal complement $S^\perp$.

(c) (6 points) Find $\mathbf{b}_1$ in S and $\mathbf{b}_2$ in $S^\perp$ so that $\mathbf{b}_1 + \mathbf{b}_2 = (1, 1, 1, 1)$.

(a) $x_1 + x_2 + x_3 - x_4 = 0$ means

$x_1 = -x_2 - x_3 + x_4$, or $\vec{x} = \begin{bmatrix} -x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Every vector in S is a linear combination of these three, and they are independent. So they are a basis of S.

(b) Since $\dim S = 3$, $\dim S^\perp = 4 - 3 = 1$, so any one non-zero vector in $S^\perp$ is a basis. One vector in $S^\perp$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, since the definition of S is all $\vec{x}$ such that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \vec{x} = 0$.

So $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ is a basis for $S^\perp$.

(c) $\vec{b}_1$ is projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto S, and $\vec{b}_2$ is the projection onto $S^\perp$. Projection matrix for $S^\perp$ is easier:

$$P = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Then $\vec{b}_2 = P\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2\\2\\2\\-2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix}$

Finally, $\vec{b}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix}$

Another solution: write $\begin{bmatrix}1\\1\\1\\+1\end{bmatrix}$ as a linear combination of vectors in a basis for $\mathbb{R}^4$: $\left\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}+1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\right\}$:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \underbrace{x_1\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}1\\0\\0\\1\end{bmatrix}}_{\vec{b}_1} + \underbrace{x_4\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}}_{\vec{b}_2} \longrightarrow \left[\begin{array}{cccc|c}-1 & -1 & 1 & 1 & 1\\ 1 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & -1 & 1\end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 0 & \underset{1}{\cancel{4}} & \underset{1/2}{\cancel{2}}\end{array}\right] \xleftarrow[\text{Row 3}]{\text{Row 4} + \text{Row 1} + \text{Row 2} -} \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & -1 & 1\\ -1 & -1 & 1 & 1 & 1\end{array}\right]$$

$$\Big\downarrow \begin{array}{l}\text{Row 1} - \text{Row 4}\\ \text{Row 2} - \text{Row 4}\\ \text{Row 3} + \text{Row 4}\end{array}$$

$$\left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 1/2\\ 0 & 1 & 0 & 0 & 1/2\\ 0 & 0 & 1 & 0 & 3/2\\ 0 & 0 & 0 & 1 & 1/2\end{array}\right]$$

$$\vec{b}_1 = \frac{1}{2}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\3/2\end{bmatrix}$$

$$\vec{b}_2 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\-1/2\end{bmatrix}$$