A1.

Sol. Volume $= 2\int_0^a \left[\pi(b+\sqrt{a^2-y^2})^2 - \pi(b-\sqrt{a^2-y^2})^2\right] dy$

$= 2\pi\int_0^a 4b\sqrt{a^2-y^2}\, dy$

$= 8b\pi \int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\sin^2\theta}\; d\,a\sin\theta$

$= 8b\pi \int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta$

$= 8a^2b\pi \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$= 8a^2b\pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$

$= 8a^2b\pi \left(\int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta + \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta}{4} \cdot d2\theta\right)$

$= 8a^2b\pi\left(\frac{\pi}{4} + 0\right)$

$= 2a^2b\pi^2$.

A2.

Pf.. $y = \sqrt{R^2-x^2}$, $y' = \frac{1}{2}\cdot\frac{-2x}{\sqrt{R^2-x^2}} = \frac{-x}{\sqrt{R^2-x^2}}$

$S = \int_a^{a+h} 2\pi\sqrt{R^2-x^2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$

$= 2\pi\int_a^{a+h}\sqrt{R^2-x^2}\cdot\sqrt{1+\frac{x^2}{R^2-x^2}}\; dx$

$= 2\pi\int_a^{a+h}\sqrt{R^2-x^2+x^2}\; dx$

$= 2\pi(Rx)\Big|_a^{a+h}$

$= 2Rh\pi$

A3.

a) Pf. $y=e^x \Rightarrow y''=e^x>0$

so the graph of e^x is concave up

b) Pf. By the graph.

$$S_{ABCD} < \int_{\ln a}^{\ln b} e^x dx < S_{AEFD}$$

$$S_{ABCD} = \frac{1}{2}(AB+CD)\cdot AD$$
$$= e^{(\ln a+\ln b)/2}(\ln b-\ln a)$$

$$S_{AEFD} = \frac{1}{2}(AE+FD)\cdot AD$$
$$= \frac{1}{2}(e^{\ln a}+e^{\ln b})\cdot(\ln b-\ln a)$$

Thus $e^{(\ln a+\ln b)/2}(\ln b-\ln a)<\int_{\ln a}^{\ln b}e^x dx<\frac{e^{\ln a}+e^{\ln b}}{2}(\ln b-\ln a)$

c) Pf. $\int_{\ln a}^{\ln b} e^x dx = e^x\Big|_{\ln a}^{\ln b} = b-a.$

$$e^{(\ln a+\ln b)/2} = \sqrt{e^{\ln a}\cdot e^{\ln b}} = \sqrt{ab}$$

$$(e^{\ln a}+e^{\ln b})/2 = (a+b)/2.$$

Thus, by (b), we have

$$\sqrt{ab}(\ln b-\ln a)<b-a<\frac{a+b}{2}(\ln b-\ln a)$$

$$\Rightarrow \sqrt{ab} < \frac{b-a}{\ln b-\ln a} < \frac{a+b}{2}$$

A4.

Sol. $\lim\limits_{x\to 0^+} x^x$

$= \lim\limits_{x\to 0^+} e^{x\ln x}$

$= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}}$

$= e^{\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}}$

$= e^{\lim\limits_{x\to 0^+}(-x)}$

$= e^0$

$= 1$

A5.

Pf. $\sec^{-1} x = \sec^{-1}\sec\sec^{-1} x$

$= \sec^{-1}\sec(\pi - \sec^{-1} x)$

$= \pi - \sec^{-1} x.$

B1. ~~For C~~

Pf. For curve C:

$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$

For curve C':

$L' = \int_c^d \sqrt{[\varphi'(u) f'(\varphi(u))]^2 + [\varphi'(u) g'(\varphi(u))]^2}\, du$

$= \int_c^d \varphi'(u)\sqrt{[f'(\varphi(u))]^2 + [g'(\varphi(u))]^2}\, du$

$= \int_c^d \sqrt{[f'(\varphi(u))]^2 + [g'(\varphi(u))]^2}\, d\varphi(u)$

$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = L$

B2.

Pf. Suppose f is not monotonic.

condition ①: there exist a, b, c s.t.

$a<b<c$ and $f(a)<f(b)$, $f(b)>f(c)$.

So there exists k s.t $f(a)<k<f(b)$

$f(c)<k<f(b)$

According to Intermediate value Theorem, there exist x_1, x_2 s.t $x_1 \in (a,b)$, $f(x_1)=k$

$x_2 \in (b,c)$, $f(x_2)=k$

Thus lead to contradiction.

condition ②: there exist a, b, c s.t.

$a<b<c$ and $f(a)>f(b)$ and $f(b)<f(c)$.

So there exists k s.t $f(a)>k>f(b)$

$f(c)>k>f(b)$

According to Intermediate value Theorem, there exist x_1, x_2 s.t. $x_1 \in (a,b)$ $f(x_1)=k$

$x_2 \in (b,c)$ $f(x_2)=k$

Thus lead to contradiction.

By ①②, f must be monotonic