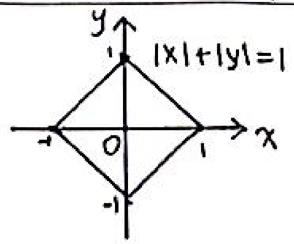
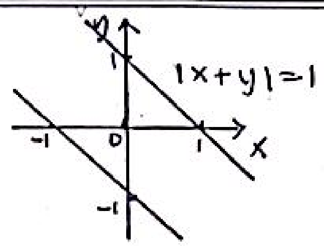
1. a.



b.



They are not graphs of function of x because some of the x in the equtions have not exactly one y.

In (a) we have $(\frac{1}{2}, -\frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$

In (b) we have $(0, 1)$ and $(0, -1)$

2. Sol. $\sqrt{x(x-3)} = \sqrt{3x-5} \Rightarrow x(x-3) = 3x-5$

$\Rightarrow x^2 - 6x + 5 = 0$

$\Rightarrow (x-5)(x-1) = 0$

Since when $x=1$ $x(x-3) = 1\times(-2) < 0$

So $x=1$ is not a solution.

When $x=5$, $x(x-3) > 0$ and $3x-5 > 0$.

So $x=5$ is the only solution of the eqution.

3. Proof. $|a-b| \geq ||a|-|b|| \Leftrightarrow (a-b)^2 \geq (|a|-|b|)^2$

$\Leftrightarrow a^2 - 2ab + b^2 \geq a^2 - 2|a||b| + b^2$

$\Leftrightarrow |a||b| \geq ab$

Since $|a||b| = |ab| \geq ab$, $|a||b| \geq ab$ QED.

4. Sol. The least upper-bound of S is 1.

Suppose there exists a $x_0 < 1$ and x_0 is an upper-bound of S

Let $n = \lceil (\frac{1}{1-x_0})^2 \rceil$, $1 - \frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{\lceil (\frac{1}{1-x_0})^2 \rceil}} \geq 1 - \frac{1}{\sqrt{(\frac{1}{1-x_0})^2}} = x_0$

So x_0 is not an upper-bound of S

It is contradictory.

So 1 is the least upper-bound of S

Bouns Exercises:

1. Proof. $\frac{a+b}{2} \geq \sqrt{ab} \Leftrightarrow a+b \geq 2\sqrt{ab}$

$\Leftrightarrow a^2 + 2ab + b^2 \geq 4ab$

$\Leftrightarrow (a-b)^2 \geq 0$

Since $(a-b)^2 \geq 0$, $\frac{a+b}{2} \geq \sqrt{ab}$ QED.

2. Proof. Suppose there exists $(\frac{p}{q})^2 = 2$ and $\gcd(p,q)=1$

$p^2 = 2q^2$ which means p is even.

Let $p = 2m$ $(m \in \mathbb{Z})$ then $(2m)^2 = 2q^2$

$q^2 = 2m^2$

which means q is even. $\gcd(p,q) = 2$

It is contradictory

So there doesn't exist $x \in \mathbb{Q}$ makes $x^2 = 2$.

3. Proof. If $b-a > 1$

Let $x = [a]+1$, $b > x > a$. ①

If $b-a \in (0,1]$, Because of Archimedean property, ~~exists~~ there exists $n \in \mathbb{N}$ such that $n > \frac{1}{b-a} \Rightarrow nb - na > 1$

Since ① we have m $(m \in \mathbb{Z})$ such that $nb > m > na$

$\Rightarrow b > \frac{m}{n} > a$

~~Since~~ and $\frac{m}{n} \in \mathbb{Q}$. QED.

4. Sol. Let $x = 1-t$. $f(1-t) + (1-t)f(t) = 2-t$

$$\begin{cases} f(x) + xf(1-x) = 1+x \\ (1-x)f(x) + f(1-x) = 2-x \end{cases} \Rightarrow [(1-x)x - 1]f(x) = x(2-x) - (1+x)$$

$\Rightarrow (-x^2+x-1)f(x) = -x^2+x-1$

$\Rightarrow f(x) = 1$ (Since Δ of $-x^2+x-1$ is $5>0$).