Linear Algebra – Fall 2022
Final Exam

NAME:

STUDENT ID:

Instructions:

- Show answers and arguments in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. Answers given without supporting work may receive zero credit.

- This is a closed book exam: no calculators, notes, or formula sheets.

QUESTION	POINTS	SCORE
1	12	
2	10	
3	14	
4	10	
5	10	
6	12	
7	16	
8	16	
TOTAL	100	

1. Consider the system of linear equations

$$x_2 + x_3 + x_4 = b_1$$
$$x_1 + x_2 + 2x_3 + x_4 = b_2$$
$$x_1 + 2x_2 + 2x_3 = b_3$$
$$2x_1 + 3x_3 + 2x_4 = b_4$$

(a) (6 points) Find a linear relation involving b_1, b_2, b_3, and b_4 that guarantees the system has at least one solution.

(b) (6 points) For $(b_1, b_2, b_3, b_4) = (1, 1, 1, 1)$, find *all* solutions of the system of equations.

2. (a) (7 points) Use row operations to find the determinant of A:

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}$$

What conditions on the real numbers a, b, c, d, r, s, and t guarantee that A is invertible?

(b) (3 points) Suppose A is an $n \times n$ invertible matrix and C is its cofactor matrix. Find $\det(C)$ in terms of $\det(A)$.

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3. (14 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}.$$

(a) Find the reduced row echelon form R of A.

(b) Find bases for the null space, row space, column space, and left null space of A.

4. (10 points) Choose a basis for $\mathbf{R}^3$ from among the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix}.$$

Then show how to write the remaining vectors as linear combinations of your basis vectors.

5. (a) (4 points) Without doing any calculation, explain why the two boxes in $\mathbf{R}^3$ determined by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix} \right\}$$

must have equal volumes.

(b) (6 points) Find the area of the triangle in $\mathbf{R}^3$ with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

6. (a) (8 points) Find the best least squares line $C + Dt$ to fit the data points $(0, 1)$, $(1, 3)$, $(2, 2)$, $(3, 4)$, and $(4, 5)$.

   (b) (2 points) Sketch a graph of the data points and your least squares line.

   (c) (2 points) Find the least squares error $\|\mathbf{e}\|$ of the best fit line.

7. Consider the symmetric matrix $A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$.

(a) (11 points) Find all eigenvalues of A and an *orthonormal* basis of $\mathbf{R}^3$ consisting of eigenvectors for A.

(b) (2 points) Show how to write $A = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is diagonal.

(c) (3 points) Show that $\lim\limits_{N \to \infty} A^N = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$.

8. Consider the matrix $A = \begin{bmatrix} 1/2 & -3/2 \\ 3/2 & 7/2 \end{bmatrix}$.

   (a) (5 points) Show that A is not diagonalizable.

   (b) (8 points) Find the singular value decomposition $A = U\Sigma V^T$.

   (c) (3 points) What is the maximum amount by which A stretches vectors in $\mathbb{R}^2$, and what is one vector that A stretches the most? That is, find the maximum value of $\|A\mathbf{x}\|/\|\mathbf{x}\|$, and find one vector $\mathbf{x}$ such that this ratio reaches the maximum value.

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