Problem 4.3.1

Sol. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

$$A^T[A\,|\,b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 8 \\ 1 & 3 & 8 \\ 1 & 4 & 20 \end{array}\right] = \left[\begin{array}{cc|c} 4 & 8 & 36 \\ 8 & 26 & 112 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} 4C + 8D = 36 \\ 8C + 26D = 112 \end{array} \Rightarrow \begin{array}{l} C = 1 \\ D = 4 \end{array} \Rightarrow \hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix},\ e = b - p = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$$

$$E = e^Te = (-1)^2 + 3^2 + (-5)^2 + 3^2 = 44.$$

Problem. 4.3.10

Sol. $$[A\,|\,b] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 27 & 8 \\ 1 & 4 & 16 & 64 & 20 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 3 & 9 & 27 & 8 \\ 0 & 4 & 16 & 64 & 20 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 12 & 60 & -12 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & 12 & 20 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 1 & 4 & -8/3 \\ 0 & 0 & 0 & 1 & 5/3 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 19/3 \\ 0 & 0 & 1 & 0 & -28/3 \\ 0 & 0 & 0 & 1 & 5/3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 47/3 \\ 0 & 0 & 1 & 0 & -28/3 \\ 0 & 0 & 0 & 1 & 5/3 \end{array}\right] = [I\,|\,d]$$

$$\Rightarrow x = \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 47/3 \\ -28/3 \\ 5/3 \end{bmatrix},\quad p = b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix},\ e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 4.3.12.

Sol. (a) $a^T[a|b] = [1 \cdots 1]\left[\begin{array}{c|c} 1 & b_1 \\ 1 & b_2 \\ \vdots & \vdots \\ 1 & b_m \end{array}\right] = [m \mid \sum_{i=1}^{m} b_i]$

$\Rightarrow \hat{x} = [\frac{\sum_{i=1}^{m} b_i}{m}]$ Thus, $\hat{x}$ is the mean of b's.

(b) $e = b - a\hat{x} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} - \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} = \begin{bmatrix} b_1-\mu \\ b_2-\mu \\ \vdots \\ b_m-\mu \end{bmatrix}$

$\|e\|^2 = \sum_{i=1}^{m}(b_i-\mu)^2$, $\|e\| = \sqrt{\sum_{i=1}^{m}(b_i-\mu)^2} = m\sigma$

(c) $e = b - p = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$

$e^T p = (-6) + (-3) + 9 = 0 \Rightarrow e \perp p$.

$P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

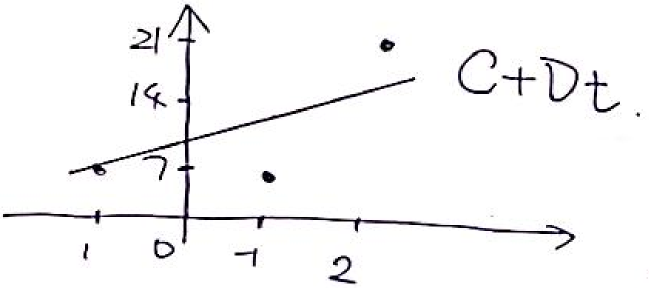
Problem 4.3.17.

Sol. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$,

$A\begin{bmatrix} C \\ D \end{bmatrix} = b \Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$

$A^T[A|b] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\left[\begin{array}{cc|c} 1 & -1 & 7 \\ 1 & 1 & 7 \\ 1 & 2 & 21 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 2 & 35 \\ 2 & 6 & 42 \end{array}\right]$

$\Rightarrow \begin{matrix} 3C + 2D = 35 \\ 2C + 6D = 42 \end{matrix} \Rightarrow \begin{matrix} C = 9 \\ D = 4 \end{matrix} \Rightarrow \hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$



Problem 4.4.2

Sol. $q_1 = \frac{1}{\sqrt{2^2+2^2+(-1)^2}}\begin{bmatrix}2\\2\\-1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}2/3\\2/3\\-1/3\end{bmatrix}$

$q_2 = \frac{1}{\sqrt{(-1)^2+2^2+2^2}}\begin{bmatrix}-1\\2\\2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-1\\2\\2\end{bmatrix} = \begin{bmatrix}-1/3\\2/3\\2/3\end{bmatrix}$

$Q = \begin{bmatrix}q_1 & q_2\end{bmatrix} = \begin{bmatrix}2/3 & -1/3\\2/3 & 2/3\\-1/3 & 2/3\end{bmatrix}$

$Q^TQ = \begin{bmatrix}2/3 & 2/3 & -1/3\\-1/3 & 2/3 & 2/3\end{bmatrix}\begin{bmatrix}2/3 & -1/3\\2/3 & 2/3\\-1/3 & 2/3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$QQ^T = \begin{bmatrix}2/3 & -1/3\\2/3 & 2/3\\-1/3 & 2/3\end{bmatrix}\begin{bmatrix}2/3 & 2/3 & -1/3\\-1/3 & 2/3 & 2/3\end{bmatrix} = \begin{bmatrix}5/9 & 2/9 & -4/9\\2/9 & 8/9 & 2/9\\-4/9 & 2/9 & 5/9\end{bmatrix}$

Problem 4.4.10

(a) Pf. $c_1q_1 + c_2q_2 + c_3q_3 = 0$

$$\Rightarrow \begin{matrix} c_1\|q_1\|^2 + c_2q_1^Tq_2 + c_3q_1^Tq_3 = 0 \\ c_1q_2^Tq_1 + c_2\|q_2\|^2 + c_3q_2^Tq_3 = 0 \\ c_1q_3^Tq_1 + c_2q_3^Tq_2 + c_3\|q_3\|^2 = 0 \end{matrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}$$

Thus the q's are independent.

(b) Pf. $Qx = 0 \Rightarrow Q^TQx = 0 \Rightarrow Ix = x = 0$.

Problem 4.4.18

Sol. $A=a$, $B=b-\frac{A^Tb}{A^TA}A=\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}-\frac{-1}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}1/2\\1/2\\-1\\0\end{bmatrix}$

$C=c-\frac{A^Tc}{A^TA}A-\frac{B^Tc}{B^TB}B=\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}-\frac{0}{2}\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}-\frac{-1}{\frac{3}{2}}\begin{bmatrix}1/2\\1/2\\-1\\0\end{bmatrix}=\begin{bmatrix}1/3\\1/3\\1/3\\-1\end{bmatrix}$

Problem 4.4.22.

Sol. $A=a=\begin{bmatrix}1\\1\\2\end{bmatrix}$, $B=b-\frac{A^Tb}{A^TA}A=\begin{bmatrix}1\\-1\\0\end{bmatrix}-\frac{0}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$

$C=c-\frac{A^Tc}{A^TA}A-\frac{B^Tc}{B^TB}B=\begin{bmatrix}1\\0\\4\end{bmatrix}-\frac{9}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

Problem 4.4.31.

(a) Sol. Let $c=\frac{1}{\sqrt{1^2+1^2+1^2+1^2}}=\frac{1}{2}$

So that Q is an orthogonal matrix.

(b) $p_1=\frac{q_1^Tb}{q_1^Tq_1}q_1=\frac{-1}{1}\times\frac{1}{2}\times\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$

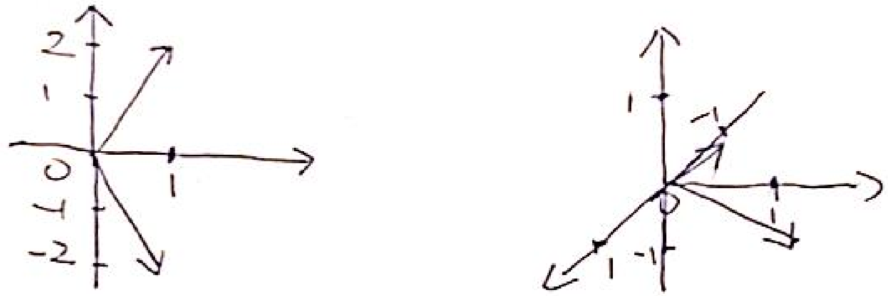
$p_2=\frac{q_1^Tb}{q_1^Tq_1}q_1+\frac{q_2^Tb}{q_2^Tq_2}q_2=\frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}+\frac{-1}{1}\times\frac{1}{2}\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

Problem 4.4.32.

Sol. $Q_1=I-2u_1u_1^T=\begin{bmatrix}1&0\\0&1\end{bmatrix}-2\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0&1\end{bmatrix}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

$Q_2=I-2u_2u_2^T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-2\begin{bmatrix}0\\\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}\begin{bmatrix}0&\sqrt{2}/2&\sqrt{2}/2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$

$\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$ $\begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$



Graded Problem.

Problem 1.

Sol. Let $A=\begin{bmatrix}1&0&0\\1&1&1\\1&2&4\\1&3&9\end{bmatrix}$

$$A^T[A\mid b]=\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&1&4&9\end{bmatrix}\left[\begin{array}{ccc|c}1&0&0&50\\1&1&1&44\\1&2&4&32\\1&3&9&6\end{array}\right]=\left[\begin{array}{ccc|c}4&6&14&132\\6&14&36&126\\14&36&98&226\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{ccc|c}1&3/2&7/2&33\\6&14&36&126\\14&36&98&226\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&3/2&7/2&33\\0&5&15&-72\\0&15&49&-236\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|c}1&3/2&7/2&33\\0&5&15&-72\\0&0&4&-20\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&3/2&7/2&33\\0&1&3&-72/5\\0&0&1&-5\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|c}1&3/2&0&101/2\\0&1&0&3/5\\0&0&1&-5\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&0&0&248/5\\0&1&0&3/5\\0&0&1&-5\end{array}\right]$$

$$\Rightarrow\hat{x}=\begin{bmatrix}C\\D\\E\end{bmatrix}=\begin{bmatrix}248/5\\3/5\\-5\end{bmatrix}\Rightarrow h(t)=\frac{248}{5}+\frac{3}{5}t-5t^2$$

$$-\frac{1}{2}gt^2=-5t^2\Rightarrow g=10$$

Problem 2.

Sol.

$$A=\begin{bmatrix}2&1&2\\1&1&1\\2&1&2\end{bmatrix}\to\begin{bmatrix}1&1/2&1\\1&1&1\\2&1&2\end{bmatrix}\to\begin{bmatrix}1&1/2&1\\0&1/2&0\\0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}$$

so $C(A^T)=\text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

$N(A)=\text{span}\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$

For row space: $a=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $b=\begin{bmatrix}0\\1\\0\end{bmatrix}$

$A=a=\begin{bmatrix}1\\0\\1\end{bmatrix}$, $B=b-\frac{A^Tb}{A^TA}A=\begin{bmatrix}0\\1\\0\end{bmatrix}-\frac{0}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$

$q_1=\frac{A}{\|A\|}=\frac{\sqrt{2}}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}$, $q_2=\frac{B}{\|B\|}=\frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix}$

so $\left\{\begin{bmatrix}\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is an orthonormal basis of $C(A^T)$

For null space: $a=\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$A=a=\begin{bmatrix}-1\\0\\1\end{bmatrix}$, $q_1=\frac{A}{\|A\|}=\frac{\sqrt{2}}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

so $\left\{\begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}\right\}$ is an orthonormal basis of $N(A)$.

Hence, $\left\{\begin{bmatrix}\sqrt{2}/2\\1\\\sqrt{2}/2\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-\sqrt{2}/2\\0\\\sqrt{2}/2\end{bmatrix}\right\}$ is an orthonormal basis for $\mathbb{R}^3$.