Promblem 4.1.6.

Sol. Multiply the equations by $y^T = [1 \ \ 1 \ \ -1]$

so that $0 = y^TAx = y^Tb = 1$.

y is in left nullspace.

so that $y^TAx = (y^TA)x = 0 \cdot x = 0$

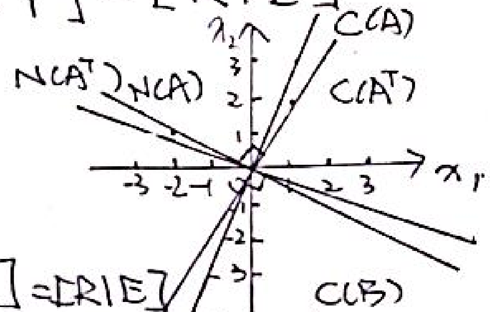
and $y^Tb = 1$ which lead to $0 = 1$.

Problem 4.1.11.

Sol. $[A \mid I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right] = [R \mid E]$

so $C(A) = \left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$, $N(A) = \left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

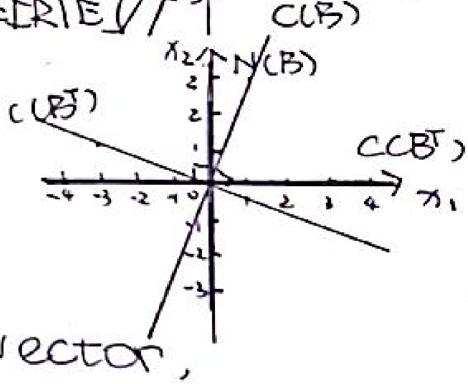
$C(A^T) = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, $N(A^T) = \left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$



$[B \mid I] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right] = [R \mid E]$

so $C(B) = \left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$, $N(B) = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$C(B^T) = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, $N(B^T) = \left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$



Problem 4.1.17.

Sol. • If S ~~soto~~ contains only zero vector, $S^\perp$ is whole R^3.

• If S is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $S^\perp$ is spanned by $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

• If S is spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, the basis of S is $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

Problem 4.1.30.

Pf. Let x be a vector s.t $\|x\| = 5$.

$AB = 0 \Rightarrow A(Bx) = 0$

Bx can present all vector in $C(B)$.

So $C(B) = N(A)$

$\text{rank}(A) + \text{rank}(B)$

$= \dim C(A^T) + \dim C(B)$

$= \dim C(A^T) + \dim N(A)$

$= r + n - r = n = 4$.

Problem 4.2.5.

Sol. $P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{\begin{bmatrix}-1\\2\\2\end{bmatrix}\begin{bmatrix}-1 & 2 & 2\end{bmatrix}}{(-1)^2 + 2^2 + 2^2} = \begin{bmatrix}\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9}\\ -\frac{2}{9} & \frac{4}{9} & \frac{4}{9}\\ -\frac{2}{9} & \frac{4}{9} & \frac{4}{9}\end{bmatrix}$

$P_2 = \frac{a_2 a_2^T}{a_2^T a_2} = \frac{\begin{bmatrix}2\\2\\-1\end{bmatrix}\begin{bmatrix}2 & 2 & -1\end{bmatrix}}{2^2 + 2^2 + (-1)^2} = \begin{bmatrix}\frac{4}{9} & \frac{4}{9} & -\frac{2}{9}\\ \frac{4}{9} & \frac{4}{9} & -\frac{2}{9}\\ -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9}\end{bmatrix}$

$P_1 P_2 = 0$ since a_1 and a_2 are perpendicular.

Problem 4.2.6.

Sol. $p_1 = P_1 b = \frac{1}{9}\begin{bmatrix}1 & -2 & -2\\ -2 & 4 & 4\\ -2 & 4 & 4\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1\\-2\\-2\end{bmatrix}$

$p_2 = P_2 b = \frac{1}{9}\begin{bmatrix}4 & 4 & -2\\ 4 & 4 & -2\\ -2 & -2 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4\\4\\-2\end{bmatrix}$

$p_3 = P_3 b = \frac{\begin{bmatrix}2\\-1\\2\end{bmatrix}\begin{bmatrix}2 & -1 & 2\end{bmatrix}}{2^2 + (-1)^2 + 2^2}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4 & -2 & 4\\ -2 & 1 & -2\\ 4 & -2 & 4\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4\\-2\\4\end{bmatrix}$

$p_1 + p_2 + p_3 = \frac{1}{9}\left(\begin{bmatrix}1\\-2\\-2\end{bmatrix} + \begin{bmatrix}4\\4\\-2\end{bmatrix} + \begin{bmatrix}4\\-2\\4\end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix}9\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} = b$.

Problem 4.2.7

Sol. $P_3 = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$

and $P_1 + P_2 + P_3 = \frac{1}{9}\left(\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}\right) = I$

Problem 4.2.19.

Sol. Choosing $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & 2 \\ 5 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Problem 4.2.20

Sol. $e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ $Q = \frac{ee^T}{e^Te} = \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$

$$P = I - Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Problem 4.2.25

Pf. Since the projection Pb fill the subspace S
$\text{rank}(P) = \dim C(P) = \dim S = n$. QED.

Problem 4.2.34

Pf. Sufficiency:

$P_1P_2 = P_2P_1$

$\Rightarrow (P_1P_2)^2 = P_1(P_2P_1)P_2 = (P_1P_1)(P_2P_2) = P_1P_2$ ①

$(P_1P_2)^T = P_2^TP_1^T = P_2P_1 = P_1P_2$. ②

By ①② P_1P_2 is a projection matrix.

Necessity:

$(P_1P_2)^2 = P_1P_2 = (P_1P_2)^T$

$\Rightarrow P_2P_1 = P_2^TP_1^T = (P_1P_2)^T = P_1P_2$. ③

By ③ $P_1P_2 = P_2P_1$.

Graded Problem

Problem I.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 1 \\ 4 & 5 & 2 \\ 5 & 6 & 3 \end{bmatrix}$, $V = C(A)$.

then $V^\perp = N(A^T)$

$A^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$ $\quad x_n = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Hence, $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $V^\perp$.

Problem 2.

Sol. Let $A=\begin{bmatrix}1&2\\0&-1\\1&2\\0&-1\end{bmatrix}$

$$A^TA=\begin{bmatrix}1&0&1&0\\2&-1&2&-1\end{bmatrix}\begin{bmatrix}1&2\\0&-1\\1&2\\0&-1\end{bmatrix}=\begin{bmatrix}2&4\\4&10\end{bmatrix}$$

$$[A^TA\ \ I]=\left[\begin{array}{cc|cc}2&4&1&0\\4&10&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}2&4&1&0\\0&2&-2&1\end{array}\right]$$

$$\to\left[\begin{array}{cc|cc}2&0&5&-2\\0&2&-2&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&\frac{5}{2}&-1\\0&1&-1&\frac{1}{2}\end{array}\right]=[I\ \ (A^TA)^{-1}]$$

$$P=A(A^TA)^{-1}A^T=\begin{bmatrix}1&2\\0&-1\\1&2\\0&-1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}5&-2\\-2&1\end{bmatrix}\begin{bmatrix}1&0&1&0\\2&-1&2&-1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1&0\\2&-1\\1&0\\2&-1\end{bmatrix}\begin{bmatrix}1&0&1&0\\2&-1&2&-1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}=\begin{bmatrix}1/2&0&1/2&0\\0&1/2&0&1/2\\1/2&0&1/2&0\\0&1/2&0&1/2\end{bmatrix}$$

$$p=P\cdot x=\frac{1}{2}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\frac{1}{2}\begin{bmatrix}4\\6\\4\\6\end{bmatrix}=\begin{bmatrix}2\\3\\2\\3\end{bmatrix}$$

$$e=x-p=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}-\begin{bmatrix}2\\3\\2\\3\end{bmatrix}=\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$$

So $\|e\|=\sqrt{(-1)^2+(-1)^2+1^2+1^2}=\sqrt{4}=2.$