Solution to Problem 3.1

$$\begin{aligned}
&Min\ 20x_1 + 24x_2 + 10x_3 + 6y \\
&S.t.\quad x_1 + 2x_2 + x_3 + 2y \geq 15 \\
&\qquad 4x_1 + 4x_2 + x_3 + y \geq 18 \\
&\qquad x_1, x_2, x_3 \geq 0,\ y \in \{0,1,2,\ldots,10\}
\end{aligned}$$

$$\mathbf{c^T} = \begin{bmatrix} 20 & 24 & 10 \end{bmatrix} \quad \mathbf{d^T} = \begin{bmatrix} 6 \end{bmatrix} \quad \mathbf{A} = [\ ] \ \mathbf{b} = [\ ] \ \mathbf{E} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{h} = \begin{bmatrix} 15 \\ 18 \end{bmatrix}$$

Iteration 1

<u>Solve MP1</u>

MP1 is formulated as.

$$\begin{aligned}
&Min\ z_{lower} \\
&S.t.\quad z_{lower} \geq 6y \\
&\qquad y \in \{0,1,2,\ldots,10\}
\end{aligned}$$

which results in $\hat{y} = 0$ and $\hat{z}_{lower} = 0$.

<u>Solve SP1</u>

With $\hat{y} = 0$, SP1 is formulated as

$$\begin{aligned}
&Min\ 20x_1 + 24x_2 + 10x_3 \\
&S.t.\quad x_1 + 2x_2 + x_3 \geq 15 - 2\hat{y} \qquad\qquad u_1 \\
&\qquad 4x_1 + 4x_2 + x_3 \geq 18 - \hat{y} \qquad\qquad u_2 \\
&\qquad x_1, x_2, x_3 \geq 0
\end{aligned}$$

Solve SP1, we get the optimal solution of 156 with $x_1 = 0, x_2 = 1.5, x_3 = 12$ and dual multipliers $u_1 = 8, u_2 = 2$. Thus, the upper bound optimal solution of the original problem is $\hat{z}_{upper} = 6\hat{y} + 156 = 0 + 156 = 156$ (based on $\hat{z}_{upper} = \mathbf{d^T}\hat{\mathbf{y}} + \mathbf{c^T}\hat{\mathbf{x}}$). Because $\hat{z}_{upper} = 156 > \hat{z}_{lower} = 0$, we will continue with the next iteration.

The feasibility cut is $z_{lower} \geq 6y + 156 - 18*(y-0) = -12y + 156$ (based on $z \geq \mathbf{d^T y} + w(\hat{\mathbf{y}}) - (\mathbf{y} - \hat{\mathbf{y}})^{\mathbf{T}} \mathbf{F^T u^P}$). Note that $\boldsymbol{\pi}^{\mathbf{P}} \equiv -\mathbf{F^T u^P} = -\begin{bmatrix} 2 \\ 1 \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} 8 \\ 2 \end{bmatrix} = -18$

Iteration 2:

<u>Solve MP2</u>

Add $z_{lower} \geq -12y + 156$ to MP2, we have

$$\begin{aligned}
&Min\ z_{lower} \\
&S.t. \quad z_{lower} \geq 6y \\
&\qquad\ \ z_{lower} \geq -12y + 156 \\
&\qquad\ \ y \in \{0,1,2,\ldots,10\}
\end{aligned}$$

which results in $\hat{y} = 9$ and $\hat{z}_{lower} = 54$.

<u>Solve SP1</u>

With $\hat{y} = 9$, SP1 is formulated as

$$\begin{aligned}
&Min\ 20x_1 + 24x_2 + 10x_3 \\
&S.t. \quad x_1 + 2x_2 + x_3 \geq 15 - 2\hat{y} \qquad\qquad u_1 \\
&\qquad 4x_1 + 4x_2 + x_3 \geq 18 - \hat{y} \qquad\qquad u_2 \\
&\qquad x_1, x_2, x_3 \geq 0
\end{aligned}$$

Solve SP1, we get the optimal solution of 45 with $x_1 = 2.25, x_2 = 0, x_3 = 0$ and dual multipliers $u_1 = 0, u_2 = 5$. Thus, the upper bound optimal solution of the original problem is $\hat{z}_{upper} = 6\hat{y} + 45 = 6*9 + 45 = 99$ (based on $\hat{z}_{upper} = \mathbf{d}^{\mathbf{T}}\hat{\mathbf{y}} + \mathbf{c}^{\mathbf{T}}\hat{\mathbf{x}}$). Because $\hat{z}_{upper} = 99 > \hat{z}_{lower} = 54$, we will continue with the next iteration.

The feasibility cut is $z_{lower} \geq 6y + 45 - 5*(y-9) = y + 90$ (based on $z \geq \mathbf{d}^{\mathbf{T}}\mathbf{y} + w(\hat{\mathbf{y}}) - (\mathbf{y} - \hat{\mathbf{y}})^{\mathbf{T}}\mathbf{F}^{\mathbf{T}}\mathbf{u}^{\mathbf{P}}$). Note that $\boldsymbol{\pi}^{\mathbf{P}} \equiv -\mathbf{F}^{\mathbf{T}}\mathbf{u}^{\mathbf{P}} = -\begin{bmatrix}2\\1\end{bmatrix}^{\mathbf{T}}\begin{bmatrix}0\\5\end{bmatrix} = -5$

Iteration 3:

<u>Solve MP2</u>

Add $z_{lower} \geq y + 90$ to MP2, we have

$$\begin{aligned}
&Min\ z_{lower} \\
&S.t. \quad z_{lower} \geq 6y \\
&\qquad\ \ z_{lower} \geq -12y + 156 \\
&\qquad\ \ z_{lower} \geq y + 90 \\
&\qquad\ \ y \in \{0,1,2,\ldots,10\}
\end{aligned}$$

which results in $\hat{y} = 5$ and $\hat{z}_{lower} = 96$.

<u>Solve SP1</u>

With $\hat{y} = 5$, SP1 is formulated as

$$
\begin{aligned}
&Min\ 20x_1 + 24x_2 + 10x_3 \\
&S.t.\quad x_1 + 2x_2 + x_3 \ge 15 - 2\hat{y} \qquad\qquad u_1 \\
&\quad 4x_1 + 4x_2 + x_3 \ge 18 - \hat{y} \qquad\qquad u_2 \\
&\quad x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Solve SP1, we get the optimal solution of 72 with $x_1 = 1.5,\ x_2 = 1.75, x_3 = 0$ and dual multipliers $u_1 = 4, u_2 = 4$. Thus, the upper bound optimal solution of the original problem is $\hat{z}_{upper} = 6\hat{y} + 45 = 6*5 + 72 = 102$ (based on $\hat{z}_{upper} = \mathbf{d}^{\mathbf{T}}\hat{\mathbf{y}} + \mathbf{c}^{\mathbf{T}}\hat{\mathbf{x}}$). Because $\hat{z}_{upper} = 102 > \hat{z}_{lower} = 96$, we will continue with the next iteration.

The feasibility cut is $z_{lower} \ge 6y + 72 - 12*(y-5) = -6y + 132$ (based on $z \ge \mathbf{d}^{\mathbf{T}}\mathbf{y} + w(\hat{\mathbf{y}}) - (\mathbf{y} - \hat{\mathbf{y}})^{\mathbf{T}}\mathbf{F}^{\mathbf{T}}\mathbf{u}^{\mathbf{P}}$). Note that $\boldsymbol{\pi}^{\mathbf{P}} \equiv -\mathbf{F}^{\mathbf{T}}\mathbf{u}^{\mathbf{P}} = -\begin{bmatrix}2\\1\end{bmatrix}^{\mathbf{T}}\begin{bmatrix}4\\4\end{bmatrix} = -12$

Iteration 4:

Solve MP2

Add $z_{lower} \ge -6y + 132$ to MP2, we have

$$
\begin{aligned}
&Min\ z_{lower} \\
&S.t.\quad z_{lower} \ge 6y \\
&\qquad\quad z_{lower} \ge -12y + 156 \\
&\qquad\quad z_{lower} \ge y + 90 \\
&\qquad\quad z_{lower} \ge -6y + 132 \\
&\qquad\quad y \in \{0,1,2,\ldots,10\}
\end{aligned}
$$

which results in $\hat{y} = 6$ and $\hat{z}_{lower} = 96$.

Solve SP1

With $\hat{y} = 6$, SP1 is formulated as

$$
\begin{aligned}
&Min\ 20x_1 + 24x_2 + 10x_3 \\
&S.t.\quad x_1 + 2x_2 + x_3 \ge 15 - 2\hat{y} \qquad\qquad u_1 \\
&\quad 4x_1 + 4x_2 + x_3 \ge 18 - \hat{y} \qquad\qquad u_2 \\
&\quad x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Solve SP1, we get the optimal solution of 60 with $x_1 = 3,\ x_2 = 0, x_3 = 0$ and dual multipliers $u_1 = 0, u_2 = 5$. Thus, the upper bound optimal solution of the original problem is $\hat{z}_{upper} = 6\hat{y} + 45 = 6*6 + 60 = 96$ (based on $\hat{z}_{upper} = \mathbf{d}^{\mathbf{T}}\hat{\mathbf{y}} + \mathbf{c}^{\mathbf{T}}\hat{\mathbf{x}}$). Because $\hat{z}_{upper} = \hat{z}_{lower} = 96$, the problem is converged and we stop the iteration.

Figure 1 shows the convergence of the optimization problem.

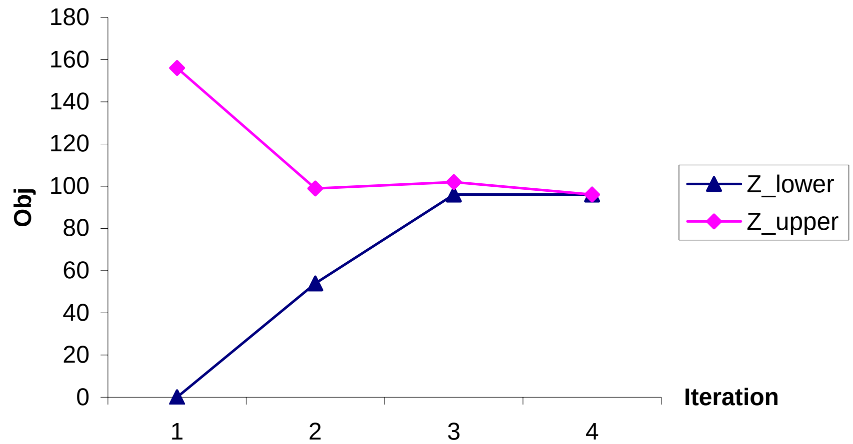

Figure 1

Solution to Problem 3.2

$$\begin{aligned} &Min\ x + y \\ &S.t. \quad 2x - y \le 3 \\ &\qquad\quad x \ge 0,\ y \in \{-5,-4,\ldots,3,4\} \end{aligned}$$

$\mathbf{c}^{\mathbf{T}} = [1] \quad \mathbf{d}^{\mathbf{T}} = [-1] \quad \mathbf{A} = [\ ] \quad \mathbf{b} = [\ ] \quad \mathbf{E} = [2] \quad \mathbf{F} = [1] \quad \mathbf{h} = [3]$

Iteration 1:

Solve MP1

MP1 is formulated as

$$\begin{aligned} &\min \quad z_{\text{lower}} \\ &S.t. \quad z_{\text{lower}} \ge y \\ &\qquad\quad y \in \{-5,-4,\ldots,3,4\} \end{aligned}$$

which results in $\hat{y} = \text{-}5,\ \hat{z}_{\text{lower}} = \text{-}5$.

Solve SP1

With $\hat{y} = \text{-}5$, SP1 is formulated as

$$\begin{array}{lll} \min & x & \\ St. & 2x \leq 3 + \hat{y} & u_1 \\ & x \geq 0 & \end{array}$$

This SP1 is infeasible at $\hat{y} = -5$. So, we have to solve SP2.

<u>Solve SP2</u>

With $\hat{y} = -5$ and adding slack variables, SP2 is formulated as

$$\begin{array}{lll} \min & s & \\ St. & 2x - s \leq 3 + \hat{y} & u_1 \\ & x \geq 0, s \geq 0 & \end{array}$$

The optimal solution is 2 with $x = 0, s = 2$ and its dual multipliers are $u_1 = 1$. The Benders cut is $2 - 1 * (y - (-5)) \leq 0$ (based on $v(\hat{\mathbf{y}}) - (\mathbf{y} - \hat{\mathbf{y}})^{\mathbf{T}} \mathbf{F}^{\mathbf{T}} \mathbf{u}^{\mathbf{r}} \leq \mathbf{0}$), that is $y \geq -3$ at $\hat{y} = -5$. Note that $\boldsymbol{\pi}^{\mathbf{r}} \equiv -\mathbf{F}^{\mathbf{T}} \mathbf{u}^{\mathbf{r}} = -[1]^{\mathbf{T}} [1] = [-1]$

Iteration 2:

<u>Solve MP2</u>

Add $y \geq -3$ to MP2, we have

$$\begin{array}{ll} \min & z_{\text{lower}} \\ S.t. & z_{\text{lower}} \geq y \\ & y \geq -3 \\ & y \in \{-5, -4, \ldots, 3, 4\} \end{array}$$

which results in $\hat{y} = -3, \quad \hat{z}_{\text{lower}} = -3$.

<u>Solve SP1</u>

With $\hat{y} = -3$, SP1 is formulated as

$$\begin{array}{lll} \min & x & \\ St. & 2x \leq 3 + \hat{y} & u_1 \\ & x \geq 0 & \end{array}$$

SP1 is feasible. The optimal solution is 0 with $x = 0$ and dual multipliers $\hat{u}_1 = 1.736$. Accordingly, the upper-bound solution of the original problem is $\hat{z}_{upper} = \hat{y} + 0 = -3$ (based on $\hat{z}_{upper} = \mathbf{d}^{\mathbf{T}} \hat{\mathbf{y}} + \mathbf{c}^{\mathbf{T}} \hat{\mathbf{x}}$). Because $\hat{z}_{upper} = \hat{z}_{lower} = -3$, the problem is converged and we stop the iteration.